THE ELEMENTS OF OPTICS:

DESIGNED

FOR THE USE OF STUDENTS

IN THE

UNIVERSITY.

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CONTENTS.

	PAGE
On the Nature of Light - - -	1
On the Laws of Reflection - - -	7
On the Laws of Refraction - - -	9
On the Reflection of Rays at plane and spherical Surfaces - - - -	15
On Images formed by Reflection -	31
On the Refraction of Rays at plane and spherical Surfaces - - - -	46
On Images formed by Refraction -	107
On the Eye - - - -	123
On Optical Instruments - -	147
On Aberrations produced by the unequal Refrangibility of different Rays -	196
On Aberrations produced by the spherical form of reflecting and refracting Surfaces -	209
On the Rainbow - - -	225
On Caustics - - - -	241

THE

ELEMENTS OF OPTICS.

SECTION I.

ON THE NATURE OF LIGHT AND THE LAWS OF REFLECTION AND REFRACTION.

Art. 1. BY *Optics* we underſtand that branch of Natural Philoſophy which treats of the nature and properties of Light, and the Theory of Viſion.

2. Modern Philoſophers have made two hypotheſes to explain the manner in which viſion is produced by luminous objects. DES CARTES, HUYGENS and EULER, ſuppoſe that there is a ſubtile, elaſtic medium which penetrates all bodies, and fills all ſpace; and that vibrations, excited in this fluid by the luminous body, are propagated thence to the eye, and produce the ſenſation of viſion, in the ſame manner that the vibrations of the air, ſtriking againſt the ear, produce the ſenſation of ſound.

It has been objected to this hypotheſis, and the objection has never been anſwered, that the vibrations of an elaſtic fluid are propagated in every direction, and into every corner to which the fluid extends; on the ſuppoſition therefore that light is nothing more than the effect of the vibrations of ſuch a fluid, there could be no ſhadow, or darkneſs.

If it be ſaid that the fluid, by means of which viſion is excited, is different from all other elaſtic fluids, the effect is aſcribed to a cauſe, the nature of which is unknown; and the hypotheſis amounts to nothing more than a confeſſion, that we are ignorant in what manner viſion is produced.

The other hypotheſis, adopted by Sir I. NEWTON and his followers, is, that light conſiſts of very ſmall particles of matter, which are conſtantly thrown off from luminous bodies, and which produce the ſenſation of viſion by actual impact upon the proper organ.

In favour of this hypotheſis, it is obſerved that the motion of light is conformable to the laws which regulate the motions of ſmall bodies, under the ſame circumſtances: Thus, where it meets with no impediment, it moves uniformly forward in right lines*; and in it's paſſage into, and reflection from different mediums, the direction of it's motion is changed as it would be, did it conſiſt of ſmall particles of matter, attracted towards, or repelled from the ſurfaces upon which they are incident †.

Whether light has other properties of matter or not, is a queſtion which does not appear to have been fairly decided; we may however be allowed to conſider it as material, and to ſpeak of it as conſiſting of particles of *matter*, till a more ſatisfactory hypotheſis can be framed; eſpecially, as we deduce no concluſions from the ſuppoſition; — we build no theory upon this foundation. Thoſe properties of light from which our theory of viſion is derived, are diſcovered by experiment, and they

* Vid. Mech. Art. 27. † Newt. Princip. Prop. 94, 96.

[T]here is this objection also ag^st Sir. I. N's hypo[-]thesis, that these particles of matter of w^h [l]ight is said to consist, pass thro' less [p]orous substances as glass, &c yet cannot [p]ass thro' those that are more porous [a]s wood, brick wall &c —

Rays of light are repreſented by lines, drawn in the directions in which the particles move.

6. Def. Whatever affords a paſſage to the rays of light is called a *Medium*; as glaſs, water, air, &c. and in this ſenſe, a vacuum is called a medium.

7. Def. The *denſity* of light is meaſured by the number of parts, or particles uniformly diffuſed over a *given* ſurface.

Cor. If the ſurface be not given, the denſity varies as the number of particles directly, and inverſely as the area over which they are diffuſed.

8. Rays of light are not lines of *contiguous* particles.

For, rays proceed from every viſible point in the univerſe to every other point; and, in their progreſs, paſs freely through torrents of light iſſuing in all directions from different ſuns, and different ſyſtems; but were the particles in each ray contiguous, one ray could not croſs another without producing ſome confuſion and irregularity in each; and thus viſion would be rendered indiſtinct and precarious. Neither is ſuch contiguity of the particles of light neceſſary to produce conſtant viſion; for, if a burning coal be made to deſcribe a circle, with a ſufficient velocity, the whole circumference appears luminous; which ſhews that the impreſſion made by the light upon the ſenſorium, when the coal is in any one point of the circumference, remains till the coal returns again to the ſame point *.

9. There

* It is obſerved that if the revolution of the coal be performed in 7‴, the whole circle appears luminous; that is, if the particles ſucceed each other at an interval which does not exceed that time, conſtant viſion is produced: and ſince light paſſes over rather more than 22,000 miles in 7‴, if the diſtance of the particles in a ray be not greater

9. There is ſomething extremely ſubtile in the nature of light; and it's properties can with difficulty be explained, either on the ſuppoſition of it's materiality, or on that of it's being only a quality of an elaſtic medium. The facility and regularity with which it is tranſmitted through bodies of conſiderable denſity, cannot be accounted for on either hypotheſis. If it conſiſt of particles of matter, which is much the more probable ſuppoſition, their minuteneſs greatly exceeds the limits of our faculties, even the power of human imagination. Notwithſtanding the aſtoniſhing velocity of theſe particles (Art. 3), their momentum is not ſo great as to diſcompoſe the delicate texture of the eye; and when they are collected in the focus of a powerful burning glaſs, it ſeems doubtful, whether they are capable of communicating motion to the thinneſt lamina of metal that can be expoſed to their impact.

Prop. I.

10. *A ray of light, whilſt it continues in the ſame uniform medium *, proceeds in a ſtraight line.*

For, objects cannot be ſeen through bent tubes; and the ſhadows of bodies are terminated by ſtraight lines. Alſo, the concluſions, drawn from calculations made on this ſuppoſition, are found by experience to be true.

11. Cor. 1. Hence it follows, that the denſity of light varies inverſely as the ſquare of the diſtance from a luminous point; ſuppoſing no particles to be ſtopped in their progreſs.

For,

greater than 22,000 miles, they are ſufficiently near to anſwer the purpoſes of conſtant viſion. Sir Iſaac Newton ſuppoſes the impreſſion to continue about one ſecond of time. Vid. Opt. Qu. 16.

* In ſpeaking of a medium, we always ſuppoſe it to be uniform, unleſs the contrary be expreſſed.

For, if the point from which the light proceeds be conſidered as the common center of two ſpherical ſurfaces, the ſame particles, which are uniformly diffuſed over the firſt, will afterwards be diffuſed, in the ſame manner, over the latter; and ſince the denſity of light varies, in general, as the number of particles directly, and inverſely as the ſpace over which they are uniformly diffuſed (Art. 7), in this caſe, it varies inverſely as the ſpace over which they are diffuſed, becauſe the number of particles is the ſame; therefore the denſity at the firſt ſurface : the denſity at the latter :: the area of the latter ſurface : the area of the former, that is, :: the ſquare of the diſtance in the latter caſe : the ſquare of the diſtance in the former *.

12. Def. When a ray of light, incident upon any ſurface, is turned back into the medium in which it was moving, it is ſaid to be *reflected*.

13. Def. When a ray of light paſſes out of one medium into another, and has it's direction changed at the common ſurface of the two mediums, it is ſaid to be *refracted*.

14. Def. The angle contained between the incident ray and the perpendicular to the reflecting, or refracting ſurface at the point of incidence, is called the *angle of incidence*.

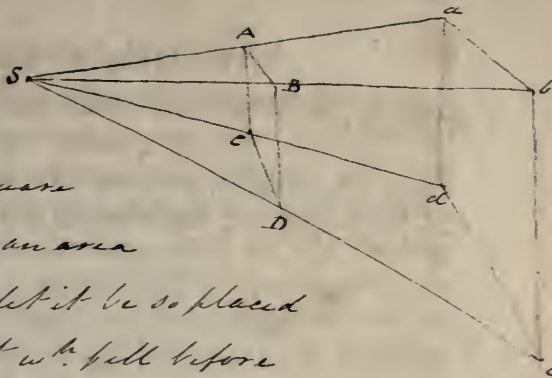
15. Def. The angle contained between the reflected ray and the perpendicular to the reflecting ſurface at the point of incidence, is called *the angle of reflection*.

16. Def. The angle contained between the refracted ray and the perpendicular to the refracting ſurface, at the point of incidence, is called *the angle of refraction*.

17. Def. The angle contained between the incident ray

* Flux. page 89.

Supposing the [illegible] light issuing [illegible] S, were [illegible] uniformly diffused [illegible] ABCD, w.t is a square [illegible] area – Let abcd be an area [illegible] & ∥ ABCD & let it be so placed [illegible] all the light w.t fell before [illegible] ABCD may now be uniformly diffused [illegible] abcd – Then since the w.t of particles is [illegible] same in both cases,



[illegible] density in ABCD : dens.y in abcd :: $\frac{1}{\text{area ABCD}}$: $\frac{1}{\text{area abcd}}$

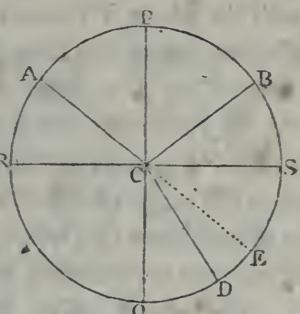
:: $\frac{1}{AB^2}$: $\frac{1}{ab^2}$

:: $\frac{1}{SB^2}$: $\frac{1}{Sb^2}$

[illegible] same proof applies to all other sim.r figures –

ray produced and the reflected or refracted ray, is called *the angle of deviation.*

If RS reprefent the reflecting furface, AC a ray incident upon it, GB the reflected ray, and PCQ be



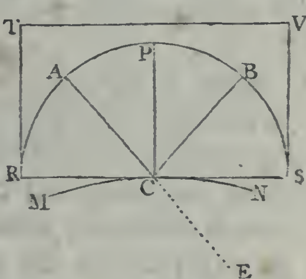
drawn, through C, perpendicular to RS, and AC be produced to E; then ACP is the angle of incidence, PCB the angle of reflection, and BCE the angle of deviation.

If RS be a refracting furface, and CD the refracted ray; then QCD is the angle of refraction, and ECD the angle of deviation.

Prop. II.

18. *The angles of incidence and reflection are in the fame plane, and they are equal to each other.*

Let a ray of light AC, admitted through a fmall hole



into a dark chamber, be incident upon the reflecting

 furface

ſurface *RS* at the point *C*; and let *CB* be the reflected ray; draw *CP* perpendicular to the reflector. Then, if the plane ſurface of a board *TS* be made to coincide with *CA* and *CP*, the reflected ray *CB* is found alſo to coincide with the plane *TS*; or the angles of incidence and reflection are in the ſame plane.

Again, if from *C* as a center, with any radius *CA*, the circle *RPS* be deſcribed, the arc *AP* is found to be equal to the arc *PB*; therefore the angle of incidence, *ACP*, is equal to the angle of reflection, *BCP*.

The angles of incidence and reflection are alſo found to be equal when rays are reflected at a curve ſurface.

19. Cor. 1. The angles *ACR*, *BCS*, which are the complements of the angles of incidence and reflection, are alſo equal.

20. Cor. 2. If *BC* be the incident ray, *CA* will be the reflected ray. For, the angle *PCA* is equal to the angle *PCB*, and in the ſame plane; therefore *CA* is the reflected ray.

21. Cor. 3. If the ray *PC* be incident perpendicularly upon the reflecting ſurface, it will be reflected in the perpendicular *CP*.

22. Cor. 4. If *AC* be produced to *E*, the angle *BCE*, which meaſures the deviation of the ray *AC* from it's original courſe, is $180° - \angle ACB$; or $180° - 2\angle$ of incidence.

23. Cor. 5. A ray of light will be reflected at a curve ſurface, in the ſame manner as at a plane which touches the curve at the point of incidence.

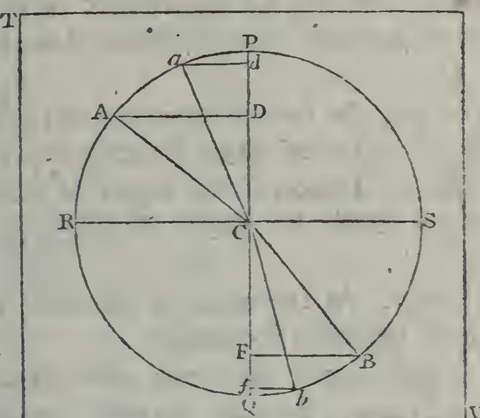
For, the angle of incidence, and conſequently the angle of reflection is the ſame, whether we ſuppoſe the reflection to take place at the curve, or at the plane.

PROP.

Prop. III.

24. *The angles of incidence and refraction are in the ſame plane; and, whilſt the mediums are the ſame, the ſine of the angle of incidence is to the ſine of the angle of refraction, in a given ratio*.*

Upon the plane ſurface of a board *TV*, with the center *C* and any radius *CA*, deſcribe a circle *PRQ*,



draw the diameters *RS*, *PQ* at right angles to each other, and immerſe the board into a veſſel of water, in ſuch a manner that *PQ* may be perpendicular to, and *RS* coincide with the ſurface of the water. Then, if a ray of light, admitted through a ſmall hole into a dark chamber, be incident upon the ſurface *RS* in the direction *AC*, coincident with the plane of the board, *CB*,

* The latter part of this propoſition is only to be underſtood of rays of the ſame kind. At preſent it is not neceſſary to take into conſideration the unequal refrangibility of differently coloured rays.

The ſines of the angles of incidence and refraction are uſually, for the ſake of conciſeneſs, called *the ſines of incidence and refraction.*

CB, the direction of the refracted ray, is found to coincide with that plane; that is, the angles of incidence and refraction are in the same plane.

Also, if *AD* and *BF* be drawn at right angles to *PQ*, they are the sines of incidence and refraction, to the radius *CA*; and it is found that *AD* has to *BF* the same ratio, whatever be the inclination of the incident ray to the refracting surface. That is, if *aC* be any other incident ray, *Cb* the refracted ray, *ad* and *bf* the sines of incidence and refraction, then *AD* : *BF* :: *ad* : *bf*.

The ratio of the sines of incidence and refraction is the same, when the refracting surface is curved.

25. Cor. 1. Hence, if the angles of incidence of two rays be equal, the angles of refraction are also equal.

26. Cor. 2. As the angle of incidence increases, the angle of refraction increases.

For, if the angle of incidence, which is always less than a right angle, increase, it's sine increases; and therefore the sine of refraction, which bears an invariable ratio to the sine of incidence, increases; and consequently the angle of refraction increases.

27. Cor. 3. When the angle of incidence vanishes, the angle of refraction vanishes also. In this case the ray suffers no refraction.

28. Cor. 4. A ray of light is refracted at a curve surface, in the same manner as at a plane which touches the curve at the point of incidence.

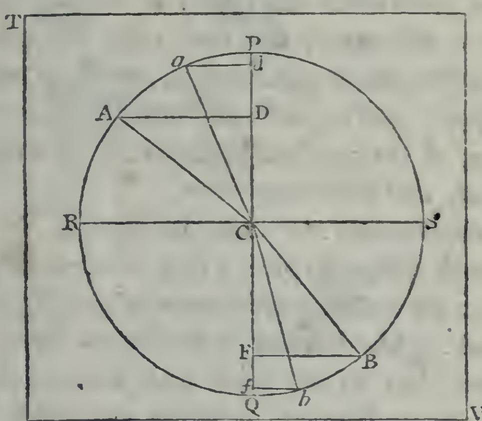
For, the angle of incidence, and consequently the angle of refraction is the same, whether we suppose the refraction to take place at the curve, or at the plane, supposing them to be mediums of the same kind.

PROP.

Prop. IV.

29. *If a ray* AC *be refracted at the surface* RS *in the direction* CB, *then a ray* BC, *coming the contrary way, will be refracted in the direction* CA.

The construction being made as before, let a small object be placed upon the board at *B*; and when the board is immersed perpendicularly in water, till *RS* coincides with the surface, the object *B* will be seen



from *A*, in the direction *AC*; and since the motion of light, in the same medium, is rectilinear (Art. 10), the ray, by which the object is seen, is incident at *C*, and refracted in the direction *CA*.

30. Cor. 1. The angle of deviation of the ray *AC*, is equal to the angle of deviation of the ray *BC*, which is incident in the contrary direction.

31. Cor. 2. When a ray of light passes out of air into water, the sine of incidence : the sine of refraction :: 4 : 3; consequently, when a ray passes out of water into

into air, the ſine of incidence : the ſine of refraction :: 3 : 4*.

In the ſame manner, out of air into glaſs, the ſine of incidence : the ſine of refraction :: 3 : 2; therefore out of glaſs into air, the ſine of incidence : the ſine of refraction :: 2 : 3*.

SCHOLIUM.

32. The preceding propoſitions, which are uſually called the Laws of Reflection and Refraction, are the principles upon which the theory of viſion is founded. They were diſcovered, and their truth has been eſtabliſhed by repeated experiments, made expreſſly for this purpoſe; and it is alſo confirmed by the conſtant agreement of the concluſions derived from them, with each other, and with experience.

The experiments we have mentioned, are rather choſen with a view to give a clear illuſtration of theſe laws, than for practical application in proving, exactly, their truth. The laws of reflection are indeed eaſily eſtabliſhed, but to determine with accuracy the proportion of the ſines of incidence and refraction, in different caſes, recourſe muſt be had to expedients which cannot, in this place, be explained. The learner, when a little farther advanced in the ſubject, may conſult on this head, Sir I. Newton's Optics, Sect. 2. and the Encyclopædia Britannica, Art. Teleſcopes, p. 356.

33. When a ray of light paſſes out of a rarer medium into a denſer, that is, out of one which is ſpecifically

* Theſe numbers do not expreſs the exact proportions, as will be ſeen hereafter; but they are ſufficiently accurate for our preſent purpoſe.

ſpecifically lighter, into one which is ſpecifically heavier, it is, in general, turned towards the perpendicular; and the contrary.

Though this is not univerſally the caſe*, yet in the ſubſequent part of the work, when we have occaſion to ſpeak of a denſer medium, we ſhall always ſuppoſe it to have a greater refracting power.

34. When light is reflected or refracted at a poliſhed ſurface, the motion of the general body of the rays is conformable to the laws above laid down; ſome are indeed thrown to the eye in whatever ſituation it is placed; and conſequently, a part of the light is diſperſed, in all directions, by the irregularity of the medium upon which it is incident. This diſperſion is, however, much leſs than would neceſſarily be produced, were the rays reflected or refracted by the ſolid parts of bodies; becauſe, the moſt poliſhed ſurfaces, that human art can produce, muſt have inequalities incomparably greater than the particles of light. This, and other conſiderations, led Sir I. NEWTON to conclude that theſe effects are produced by ſome power, or medium, which is evenly diffuſed over every ſurface, and extends to a ſmall, though finite diſtance from it†.

That bodies do act upon light before it comes into contact with them, is manifeſt from the ſhadows of hairs, ſmall needles, &c. which are much larger than they ought to be, on ſuppoſition that rays paſs by them in ſtraight lines. In order to examine this phenomenon more minutely, Sir I. NEWTON admitted a ſmall beam of light into a darkened chamber, and cauſing it to paſs near the edge of a ſharp knife, he found

* NEWT. Opt. Book 2. Part 3. Prop. 10.

† NEWT. Opt. Book 2. Prop. 8.

found that the rays were turned confiderably from their rectilinear courfe, and that thofe rays were more inflected, or bent, which paffed at a lefs diftance from the edge, than thofe which were more remote. He alfo obferved, that fome of the rays were turned towards the edge, and others from it; fo that rays of light, at different diftances from the furfaces of bodies, are apparently acted upon by two different powers, one of which attracts, and the other repels them*.

The laws, according to which thefe powers vary, have not yet been difcovered; but fuppofing the effects produced by them, at the fame diftance from a given furface, to be always the fame, Sir I. NEWTON has fhewn, that if fmall bodies were reflected and refracted by them, the angles of incidence and reflection would be equal; and the fines of the angles of incidence and refraction, in a given ratio to each other †. Thefe conclufions leave us little room to doubt but that reflection and refraction are produced by fuch powers; and they afford fome ground for prefuming that the particles of light are material.

* NEWT. Opt. Book 3.

† Princip. Prop. 94, 96. Opt. Book 1. Part 1. Prop. 6.

SECTION

SECTION II.

ON THE REFLECTION OF RAYS AT PLANE AND SPHERICAL SURFACES.

DEFINITIONS.

35 BY a *pencil of rays* we underſtand a number of rays taken collectively, and diſtinct from the reſt.

Theſe pencils conſiſt either of *parallel*, *converging*, or *diverging* rays.

Converging rays are ſuch as approach to each other in their progreſs, and, if not intercepted, at length meet.

Diverging rays are ſuch as recede from each other, and whoſe directions meet if produced backwards.

36. The *focus* of a pencil of rays is that point towards which they converge, or from which they diverge.

If the rays in a pencil, after reflection, or refraction, do not meet exactly in the ſame point, the pencil muſt be diminiſhed; and the focus is the limit of the interſections of the extreme rays, when they approach nearer and nearer to each other, and at length coincide. In this caſe, the focus is uſually called the *Geometrical focus.*

The

The focus is *real*, when the rays actually meet in that point; and *imaginary*, or *virtual*, when their directions must be produced to meet.

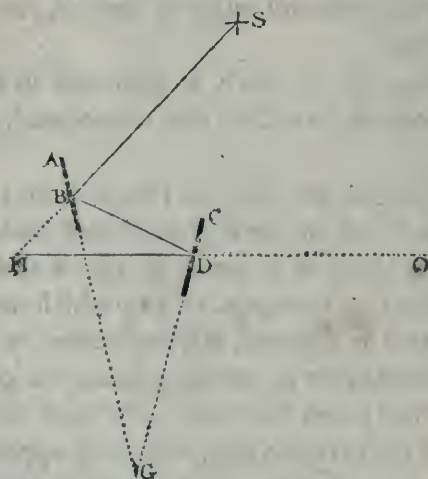
37. The *axis* of a pencil is that ray which is incident perpendicularly upon the reflecting or refracting surface.

38. The *principal focus* of a reflector, or refractor, is the geometrical focus of parallel rays incident nearly perpendicularly upon it.

Prop. V.

39. *If a ray of light be reflected once by each of two plane surfaces, and in a plane which is perpendicular to their common intersection, the angle contained between the first and last directions of the ray, is equal to twice the angle at which the reflectors are inclined to each other.*

Let *AB*, *CD* be two plane reflectors, inclined at the angle *AGD*; *SB*, *BD*, *DH*, the course of a ray



reflected by them. Produce *HD* to *O*, and *SB* till it meets *DH* in *H*. Then, because the $\angle HBG =$ the $\angle$ *ABS*

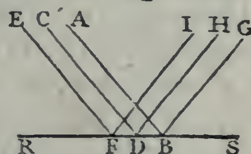
$ABS =$ the $\angle DBG$ (Art. 19), the whole angle $DBH = 2 \angle DBG$. In the same manner, the $\angle BDO = 2 \angle BDC$. And since the $\angle BGD =$ the $\angle BDC -$ the $\angle DBG$*, we have $2 \angle BGD = 2 \angle BDC - 2 \angle DBG =$ the $\angle BDO -$ the $\angle DBH =$ the $\angle BHD$*.

Prop. VI.

40. *Parallel rays, reflected at a plane surface, continue parallel.*

Case 1. When the angles of incidence are in the *same* plane.

Let RS be the reflecting surface; AB, CD the

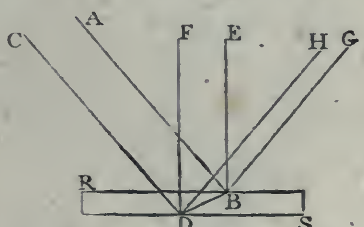


incident, BG, DH the reflected rays.

Then the $\angle ABR =$ the $\angle GBS$, and the $\angle CDR =$ the $\angle HDS$ (Art. 19); but, since AB and CD are parallel, the $\angle ABR =$ the $\angle CDR$; therefore the $\angle GBS =$ the $\angle HDS$, and BG, DH are parallel (Euc. 28. 1).

Case 2. When the angles of incidence are in *different* planes.

Let AB, CD be the incident rays; BE, DF per-



pendiculars to the reflecting surface at the points of incidence;

* Euc. 32. 1.

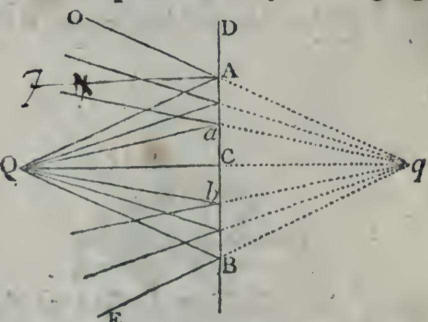
incidence; join *BD*; and let *AB* be reflected in the direction *BG*; alſo, let *DH* be the interſection of the planes *CDF*, *GBD*.

Then, ſince *BE*, *DF*, which are perpendicular to the ſame plane, are parallel (Euc. 6. 11), and *AB*, *CD* are parallel, by the ſuppoſition, the angles of incidence *ABE*, *CDF* are equal (Euc. 10. 11); therefore the angles of reflection are equal. Again, ſince *EB* and *FD* are parallel, as alſo *AB* and *CD*, the planes *ABG*, *CDH* are parallel (Euc. 15. 11), and they are interſected by the plane *GBDH*; conſequently, *DH* is parallel to *BG* (Euc. 16. 11); therefore the angles *EBG*, *FDH* are equal (Euc. 10. 11); but the angle *EBG* is the angle of reflection of the ray *AB*; therefore the angle *FDH* is equal to the angle of reflection of the ray *CD*; and ſince *DH* is in the plane *CDF*, *CD* is reflected in the direction *DH* (Art. 18), which has before been ſhewn to be parallel to *BG*.

Prop. VII.

41. *If diverging or converging rays be reflected at a plane ſurface, the foci of incident and reflected rays are on contrary ſides of the reflector, and equally diſtant from it.*

Let *Q AB* be a pencil of rays diverging from *Q*,



and incident upon the plane reflector *ACB*; draw *QC* perpendicular

perpendicular to the ſurface; then will $\mathcal{Q}C$ be reflected in the direction $C\mathcal{Q}$ (Art. 21). Let $\mathcal{Q}A$ be any other ray; and ſince a perpendicular to the ſurface at A, is in the ſame plane with $\mathcal{Q}C$ and $\mathcal{Q}A$ (Euc. 6. and 7. 11), $\mathcal{Q}A$ will be reflected in this plane (Art. 18). Produce CA to D, and make the angle DAO equal to the angle $\mathcal{Q}AC$, then will AO be the reflected ray (Art. 19). Produce OA, $\mathcal{Q}C$ till they meet in q. Then, ſince the $\angle qAC =$ the $\angle OAD =$ the $\angle \mathcal{Q}AC$, and alſo the $\angle qCA =$ the $\angle \mathcal{Q}CA$, and the ſide CA is common to the two triangles $\mathcal{Q}CA$, CAq, the ſide $\mathcal{Q}C$ is equal to Cq. In the ſame manner it may be ſhewn, that every other reflected ray in the pencil, will, if produced backwards, meet the axis in q; that is, the rays, after reflection, diverge from the focus q.

If $OABE$ be a pencil of rays converging to q, they will, after reflection at the ſurface ACB, converge to $\mathcal{Q}$ (Art. 20); therefore, in this caſe alſo, the foci of incident and reflected rays are on contrary ſides of the reflector, and equally diſtant from it.

42. Cor. 1. The divergency, or convergency of rays, is not altered by reflection at a plane ſurface.

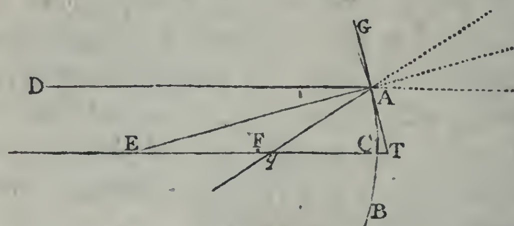
43. Cor. 2. In the triangles $\mathcal{Q}AC$, CAq, Aq is equal to $\mathcal{Q}A$; if therefore any reflected ray AO be produced backwards to q, making $Aq = A\mathcal{Q}$, q is the focus of reflected rays.

44. Cor. 3. If the incident rays $\mathcal{Q}A$, $\mathcal{Q}a$, be parallel, or the diſtance of $\mathcal{Q}$ from the reflector be increaſed without limit with reſpect to Aa, the diſtance of q is increaſed without limit, or the reflected rays are parallel (Vid. Art. 40).

PROP. VIII.

45. *If parallel rays be incident nearly perpendicularly upon a ſpherical reflector, the geometrical focus of reflected rays is the middle point in the axis between the ſurface and center.*

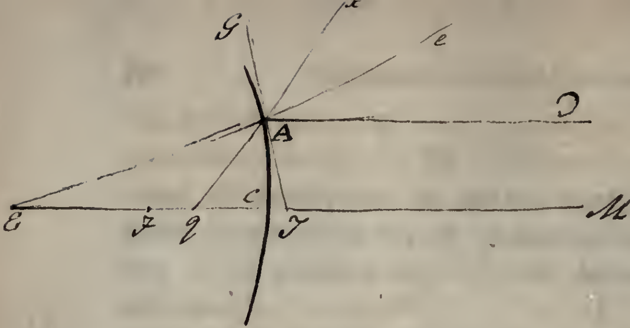
Let *ACB* be a ſpherical reflector whoſe center is *E*; *DA*, *EC*, two rays of a parallel pencil incident upon it,



of which, *EC* paſſes through the center, and is therefore reflected in the direction *CE*; join *EA*, and in the plane *DACE*, make the angle *EAq* equal to the angle *DAE*, and *DA* will be reflected in the direction *Aq* (Art. 18); draw *GAT*, in the ſame plane, touching the reflector in *A*, and let it meet *EC* produced in *T*. Then, ſince the $\angle EAq =$ the $\angle DAE =$ the $\angle AEq$ (Euc. 29. 1), $Eq = Aq$; alſo, the $\angle qAT =$ the $\angle DAG$ (Art. 19) $=$ the $\angle ATq$ (Euc. 29. 1); therefore $Aq = qT$; conſequently $Eq = qT$; that is, q biſects *ET* the ſecant of the arc *AC*. Now let *DA* approach to *EC*, and the arc *AC* will decreaſe, and it's ſecant, at length, become equal to the radius; conſequently the limit of the interſections of *Aq* and *CE* is *F*, the middle point between *E* and *C*†.

If

† It is manifeſt that the rays, incident nearly perpendicularly, do not meet accurately in *F*; but when the arc is ſmall in compariſon of



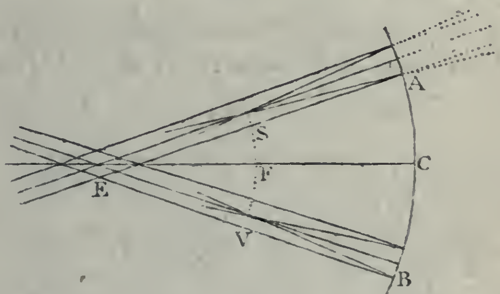
When the rays fall on the convex side of the reflector

Let DA, MC be the // incident rays – Produce EA to e & in the plane DAe make the ∠ eAx = ∠ DAe & DA will be reflected in the direction Ax – Produce xA to meet the axis in q – Then ∠ DAe = ∠ xAe & ∠ DAe = ∠ qEA & ∠ xAe = ∠ qAE ∴ ∠ qEA = ∠ qAE and qA = qE – Again ∠ xAG = ∠ DAT and ∠ xAG = ∠ qAT & ∠ DAT = ∠ qTA ∴ ∠ qAT = ∠ qTA and qA = qT but qA = qE ∴ qE = qT as before

If the rays be incident upon the convex ſide of the reflector, the reflected rays muſt be produced backwards to meet the axis; and in this caſe, F, the middle point between E and C, may be ſhewn to be the limit of the interſections of CE and Aq, as before.

46. Cor. 1. As the arc AC decreaſes, Eq, or Fq, decreaſes. Thus, when AC is 60°, $Eq = EC$; and when AC is 45°, Aq is perpendicular to EC, and $Eq : EC :: 1 : \sqrt{2}$.

47. Cor. 2. If different pencils of parallel rays be reſpectively incident, nearly perpendicularly, upon the



reflector, the foci of reflected rays will lie in the ſpherical ſurface SFV, whoſe center is E and radius EF.

48. Cor. 3. If the axes, EA, EC, EB, of theſe pencils, lie in the ſame plane, the foci will lie in the circular arc SFV.

49. Cor. 4. If any point S, in the arc SFV, whoſe radius EF is one half of EC, be the focus of a pencil of

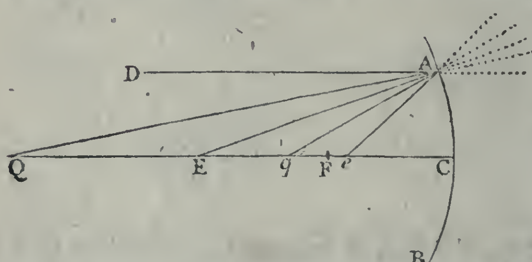
of the radius, in all calculations, made for the conſtruction of optical inſtruments, F and q may be conſidered as coincident. Thus, if the arc AC be 20′, and the radius be divided into 100,000 equal parts, Fq is leſs than one of thoſe parts; and all the rays which are incident upon the ſurface generated by the revolution of the arc AC about the axis EC, after reflection, cut the axis between F and q.

of rays incident nearly perpendicularly upon the reflector, theſe rays will be reflected parallel to each other, and to *EA* the axis of that pencil (Art. 20).

Prop. IX.

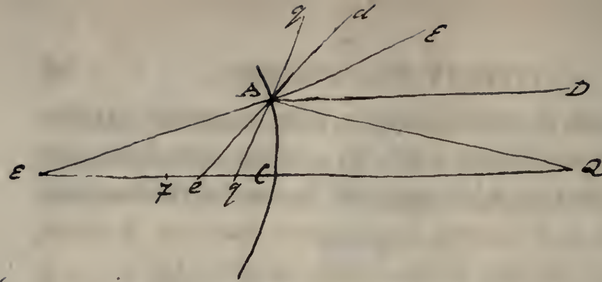
50. *When diverging or converging rays are incident nearly perpendicularly upon a ſpherical reflector, the diſtance of the focus of incident rays from the principal focus, meaſured along the axis of the pencil, is to the diſtance of the principal focus from the center, as this diſtance is to the diſtance of the principal focus from the geometrical focus of reflected rays.*

Let *ACB* be the ſpherical reflector, whoſe center is *E*; *Q* the focus of incident rays; *QA*, *QC* two rays of



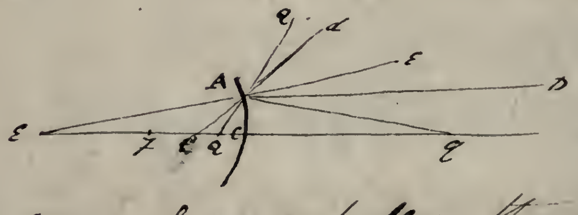
the pencil, of which *QC* paſſes through the center *E*, and is therefore reflected in the direction *CQ*; join *EA*; and, in the plane *QACE*, make the angle *EAq* equal to the angle *EAQ*; then the ray *QA* will be reflected in the direction *Aq*.

Draw *DA* parallel to *QC*, and make the angle *EAe* equal to the angle *EAD*; biſect *EC* in *F*. Then, ſince the $\angle DAE =$ the $\angle EAe$, and the $\angle QAE$ $=$ the $\angle EAq$, the $\angle DAQ$, or it's equal AQe, is equal to the $\angle eAq$; alſo, the $\angle qeA$ is common to the two triangles *AQe*, *Aqe*; therefore they are ſimilar,



When diverging rays are incident on the convex side of a spherical reflector —

DA is reflected in the direction Ad, & QA in direct^n Aq, & Ad, Aq are produced backwards to meet the axis in e, q — And ∠ AQe =(∠ QAD = ∠ qAd) = ∠ eAq the rest as before —

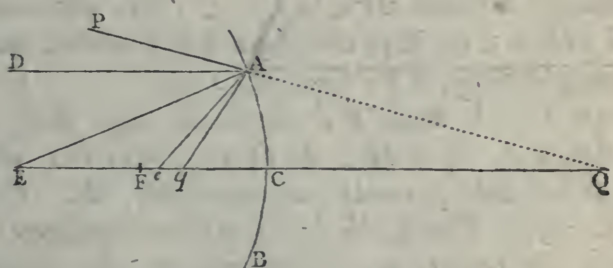


When converg^g rays fall on the convex surface – Here ∠ QAe = (∠ QAd = ∠ DAq =) ∠ Aqe &c

When converg^g rays fall on the concave side, Q and q change places on the axis.

ſimilar, and $Qe : eA :: eA : eq$; or, ſince $eA = eE$ (Art. 45), $Qe : eE :: eE : eq$. Now let the arc AC be diminiſhed without limit, or the ray QA be incident nearly perpendicularly, then e coincides with F (Art. 45); and the limit of the interſections, of CQ and Aq, is determined by the proportion $QF : FE :: FE : Fq$*.

The preceding figure is conſtructed for the caſe in which diverging rays are incident upon a concave ſpherical ſurface, and the ſame demonſtration is applicable when the incident rays converge, as is repreſented in the annexed figure.



If the lines DA, QA, EA, eA, qA be produced, the figures ſerve for thoſe caſes in which the rays are incident upon the convex ſurface.

51. Cor. 1. If q be the focus of incident, Q will be the focus of reflected rays (Art. 20); and Q and q are called *conjugate* foci.

52. Cor. 2. If the diſtance QF be very great when compared with FE, Fq is very ſmall when compared with

* When QA is incident *nearly* perpendicularly upon the reflector, q may be conſidered as coincident with that point which is determined by the proportion $QF : FE : FE : Fq$ (Vid. note page 20); and all other rays, incident nearly perpendicularly, will, after reflection, cut the axis in the ſame point; therefore that point is the focus of reflected rays.

with it. Thus, if the rays diverge from a point in the ſun's diſc, and fall upon a reflector whoſe radius does not exceed a few feet, F and q may, for all practical purpoſes, be conſidered as coincident.

53. Cor. 3 When $\mathcal{Q}$ coincides with E, all the rays are incident perpendicularly upon the reflector, and therefore they are reflected perpendicularly (Art. 21), or q coincides with E.

54. Cor. 4. The point e biſects the ſecant of the arc AC (Art. 45).

55. Cor. 5. Since $\mathcal{Q}e : eE :: eE : eq$, by compoſition, or diviſion, $\mathcal{Q}e : \mathcal{Q}E :: eE : Eq$; alternately, $\mathcal{Q}e : eE :: \mathcal{Q}E : Eq$; and, when $\mathcal{Q}A$ is incident nearly perpendicularly, $\mathcal{Q}F : FE :: \mathcal{Q}E : Eq$.

56. Cor. 6. Since EA biſects the angle $\mathcal{Q}Aq$ (or PAq), $\mathcal{Q}A : Aq :: \mathcal{Q}E :: Eq$ (Euc. 3. 6); and, when $\mathcal{Q}A$ is incident nearly perpendicularly, $\mathcal{Q}C : Cq :: \mathcal{Q}E : Eq$. That is, the diſtances of the conjugate foci from the center, are proportional to their diſtances from the ſurface.

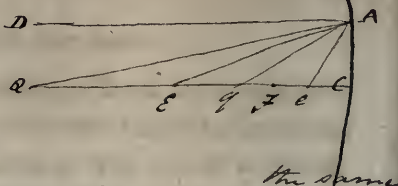
57. Cor. 7. Since $\mathcal{Q}E : Eq :: \mathcal{Q}F : FE$ (Art. 55), and $\mathcal{Q}E : Eq :: \mathcal{Q}C : Cq$ (Art. 56), we have, ultimately, $\mathcal{Q}F : FE :: \mathcal{Q}C : Cq$.

58. Cor. 8. As the arc AC decreaſes, Eq, the diſtance of the interſection of the reflected ray and the axis from the center, decreaſes; unleſs $\mathcal{Q}$ coincide with E, or lie between E and e.

For, $\mathcal{Q}e :: eE :: \mathcal{Q}E : Eq$ (Art. 55), and as AC decreaſes Ee decreaſes (Art. 54); therefore, when $\mathcal{Q}$ is in eE produced, the terms of the ratio of greater inequality, $\mathcal{Q}e : Ee$, are equally diminiſhed, and that ratio, or it's equal $\mathcal{Q}E : Eq$, increaſes (Alg. Art. 163); and, ſince $\mathcal{Q}E$ is invariable, Eq decreaſes.

When

Prop. X. aliter –

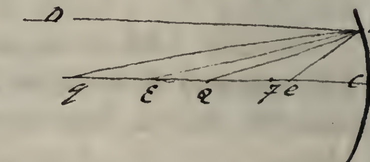


Let F be the principal focus, QA an oblique incident ray & reflected to q — — Q and q are always on ~~opposite~~ the same side of F, and they move in opposite directns. –

Case. 1. Let Q lie to the left of E —

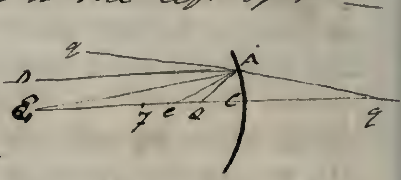
Now ∠ DAE = ∠ EAe
& ∠ QAE = ∠ EAq, but ∠ DAE is g^t. than ∠ QAE since Q is below D ∴ ∠ EAe is g^t. than ∠ EAq ∴ q lies between E and e, i.e between E and F. ∴ Q and q are measur'd the same way from F —

Case. 2. Let Q be between E and F –



Then this case coincides with the former except that Q and q have changed places ∴ as before, Q and q will be measur'd to the left of F —

Case. 3. Let Q be between F and C —



Now ∠ QAE is g^t. than ∠ FAE
∴ ∠ QAE is g^t. than ∠ DAE
∴ reflected ray is above DA or q is in EC, produced.
∴ Q and q lie the same way from F

Case 4. Let Q be to the r^t. of C —

Then as in the second case, Q and q

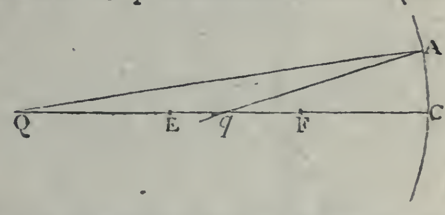
When Q is in Ee produced, as AC decreaſes Qe increaſes, and Ee decreaſes; therefore the ratio of Qe : Ee, or of $QE : Eq$ increaſes; and conſequently, as before, Eq decreaſes.

But when Q lies between E and e, as AC decreaſes the terms of a ratio of leſs inequality $Qe : Ee$ are equally diminiſhed; therefore that ratio, or it's equal $QE : Eq$, decreaſes (Alg. Art. 163); and ſince QE is invariable, Eq increaſes. When Q coincides with E, q alſo coincides with it, whatever be the magnitude of the arc AC.

Prop. X.

59. *The conjugate foci,* Q *and* q, *lie on the ſame ſide of the principal focus; they move in oppoſite directions, and meet at the center and ſurface of the reflector.*

Since $QF : FE :: FE : Fq$, we have $QF \times Fq = \overline{FE}|^2$; that is, Q and q are ſo ſituated that the rectangle under QF and Fq is invariable. Alſo, when Q coincides with E, q coincides with it (Art. 53); in



this caſe then, QF and Fq are meaſured in the ſame direction from F; and, ſince their rectangle is invariable, they muſt always be meaſured in the ſame direction (Alg. Art. 473).

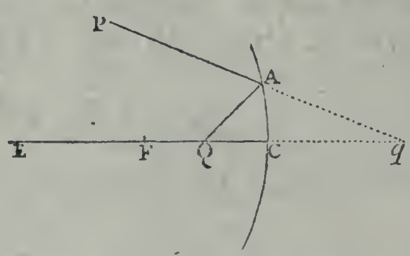
That Q and q move in oppoſite directions may thus be proved: the rectangle $QF \times Fq$ is invariable; and therefore as one of theſe quantities increaſes, increafes, the other decreafes; alfo, Q and q lie the fame way from the fixed point F; they muft therefore move in oppofite directions.

Having given the place of Q, and FE the focal length of the reflector, to determine the place of the conjugate focus q, we muft take $QF : FE :: FE : Fq$, and meafure FQ and Fq in the fame direction from F.

Thus, when Q, the focus of incident rays, is farther from the reflector than E, and on the fame fide of it, FQ is greater than FE, therefore FE is greater than Fq; or q, the focus of reflected rays, lies between F and E.

When Q is between E and F, q lies the other way from E; and whilft Q moves from E to F, q moves in the oppofite direction from E to an infinite diftance.

When Q is between F and C, QF is lefs than FE or FC; therefore FC is lefs than Fq; and, fince FQ



and Fq are meafured in the fame direction from F, q is on the convex fide of the reflector.

When Q coincides with C, QF is equal to FC; therefore FC is equal to Fq; or q coincides with C.

When converging rays are incident upon the concave furface of the reflector, QF is greater than FC; therefore FC is greater than Fq; or q lies between F and C.

60. Cor.

Again Q and q move in opposite directions.

For $QF : FE :: FE : Fq$

and if Q approaches to F the ratio of $QF : FE$ is decreas'd

$\therefore$ the r.° of $FE : Fq$ is decreas'd & FE is constant

$\therefore Fq$ increases or q recedes from F –

So also when Q recedes from F, q approaches to it –

60. Cor. 1. A concave ſpherical reflector leſſens the divergency, or increaſes the convergency of all pencils of rays incident nearly perpendicularly upon it.

For, if the rays diverge from a point farther from the reflector than the principal focus, they are made to converge.

If they diverge from F, they are reflected parallel to CE.

If the focus of incidence lie between F and C, q is on the other ſide of the ſurface; or the rays diverge after reflection; and becauſe $\mathcal{Q}F : FE :: \mathcal{Q}C : Cq$ (Art. 57), and $\mathcal{Q}F$ is leſs than FE, $\mathcal{Q}C$ is leſs than Cq; alſo, the ſubtenſe AC is common; therefore the angle contained between the incident rays $\mathcal{Q}A$, $\mathcal{Q}C$, is greater than the angle contained between the reflected rays AP, $C\mathcal{Q}$; or the reflected rays diverge leſs than the incident rays.

If converging rays fall upon the reflector, $\mathcal{Q}F$ (Fig. page 23) is greater than FE, therefore $\mathcal{Q}C$ is greater than Cq; or the reflected rays converge to a focus nearer to the reflector than the focus of incident rays; and their convergency is increaſed.

61. Cor. 2. In the ſame manner it may be ſhewn, that a convex ſpherical reflector increaſes the divergency, or diminiſhes the convergency of all rays incident nearly perpendicularly upon it.

SCHOLIUM.

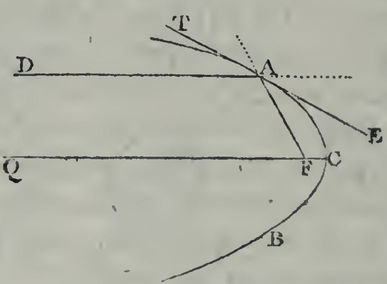
62. It appears from Art. 58, that unleſs the focus coincide with the center, a ſpherical reflector does not cauſe all the rays in a pencil either to converge or diverge accurately. This circumſtance produces ſome confuſion in viſion, when theſe reflectors are made uſe of; and, by increaſing the breadth of each pencil, or,

or, which is the ſame thing, by enlarging the aperture of the reflecting ſurface, in order to increaſe the quantity of light, the indiſtinctneſs thus produced is increaſed, as we ſhall have occaſion to obſerve hereafter.

To remedy this inconvenience, it has been propoſed to make uſe of reflecting ſurfaces formed by the revolution of conic ſections about their axes; and it may be proper to ſhew that ſuch ſurfaces will, in particular caſes, cauſe rays to converge or diverge accurately.

63. *Parallel rays may be made to converge, or diverge accurately, by means of a parabolic reflector.*

Let ACB be a parabola, by the revolution of which about it's axis QC, a parabolic reflector is generated;



take F the focus; let DA, which is parallel to QC, be a ray of light incident upon the concave ſide of this reflector; and join AF. Draw TAE in the plane DAF, and touching the paraboloid in A. Then ſince the angle TAD is equal to the angle EAF, from the nature of the parabola, the ray DA will be reflected in the direction AF (Art. 19). In the ſame manner it may be ſhewn, that any other ray, parallel to QC, will be reflected to F; and therefore the reflected rays converge accurately to this point.

If DA, FA be produced, it is manifeſt that rays, incident upon the convex ſurface of the paraboloid, parallel

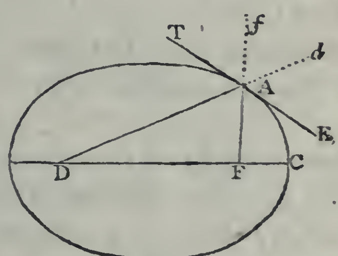
parallel to the axis, will, after reflection, diverge accurately from F.

The advantage, however, of a parabolic reflector is not ſo great as might, at firſt, be expected; for, if the pencil be inclined to the axis of the parabola, the rays will not be made to converge or diverge accurately; and the greater this inclination is, the greater will the error become.

Cor. If F be the focus of incidence, the rays will be reflected parallel to the axis.

64. *Diverging or converging rays may be made to converge or diverge accurately, by a reflector in the form of a ſpheroid; and to diverge or converge accurately, by one in the form of an hyperboloid.*

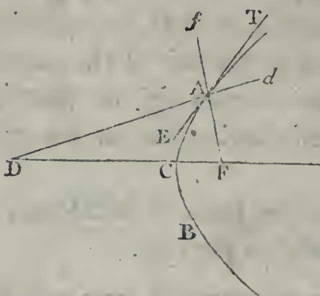
Let F and D be the foci of the conic ſection, by the



revolution of which, about it's axis, the reflecting ſurface is formed; F the focus of incident rays; then will D be the focus of reflected rays.

For,

For, let FA be an incident ray; join DA, and produce it to d; draw TAE in the plane DAF, and



touching the reflector in A; then the angle EAF is equal to the angle DAT, in the ellipse, and to dAT in the hyperbola; therefore AD is the reflected ray in the former case, and Ad in the latter; thus D is the focus of reflected rays.

If FA be produced to f, the figures serve for the cases in which rays are incident upon the convex surfaces.

We may here remark, as in the preceding article, that if rays fall upon the reflector converging to, or diverging from any other point than one of the foci, they will not converge or diverge accurately after reflection.

SECTION

SECTION III.

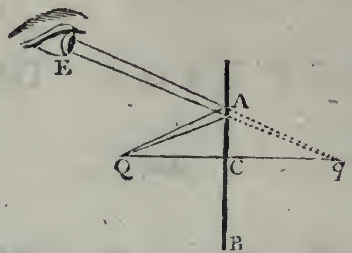
ON IMAGES FORMED BY REFLECTION.

65. THE rays of light which diverge from any point in an object, and fall upon the eye, excite a certain sensation in the mind, corresponding to which, as we know by experience, there exists an external substance in the place from which the rays proceed; and whenever the same impression is made upon the organ of vision, we expect to find a similar object, and in a similar situation. It is also evident, that if the rays belonging to any pencil, after reflection or refraction, converge to, or diverge from, a point, they will fall upon the eye, placed in a proper situation, as if they came from a real object; and therefore the mind, insensible of the change which the rays may have undergone in their passage, will conclude that there is a real object corresponding to that impression.

In some cases indeed, chiefly in reflections, the judgment is corrected by particular circumstances which have no place in naked vision, as the diminution of light, or the presence of the reflecting surface, and we are sensible of the illusion; but still the impression is made, and a representation, or *image* of the object, from which the rays originally proceeded, is formed.

Thus,

Thus, the rays which diverge from Q, after reflection at the plane ſurface ACB, enter an eye, placed



at E, as if they came from q; or q is the image of Q.

If then the rays, which diverge from any viſible point in an object, fall upon a reflecting or refracting ſurface, the focus of the reflected or refracted rays is the *image* of that point.

66. The image is ſaid to be *real*, or *imaginary*, according as the foci of the rays by which it is formed are *real*, or *imaginary*.

67. The image of a phyſical line is determined by finding the images of all the points in the line; and of a ſurface, by finding the images of all the lines in the ſurface, or into which we may ſuppoſe the ſurface to be divided.

Prop. XI.

68. *The image of a ſtraight line, formed by a plane reflector, is a ſtraight line, on the other ſide of the reflector; the image and object are equally diſtant from, and equally inclined to, the reflecting plane; and they are equal to each other.*

Let PR be a ſtraight line*, placed before the plane reflector

* It is almoſt unneceſſary to remind the reader that the lines, which are conſidered as objects, muſt be phyſical lines, of ſufficient thickneſs to reflect as many rays as are neceſſary for the purpoſes of viſion.

(It is better in these propositions that relate the images of lines &c not to take the extre-
-ities of the figures but any point in them —
The construction as in Wood —
r if any point Q be taken in PR & Qq be drawn
r to AB & produced to meet pr in q,

by Δs RB : BA :: QC : CA
and BA : rB :: CA : Cq

ex aequo RB : rB :: QC : Cq

but RB = rB ∴ QC = Cq

∴ q is the image of Q —

ut Q was taken any point in PR ∴ pqr is
e image of PQR ——

Again, since BR = Br and AB is common to the
two Δs ARB, ABr and the angles at B are
rt angles, AR = Ar & ∠ RAB = ∠ rAB & these
e the angles of inclination — — ——

Again since Pp is ∥ Rr, AR : Ar :: AP : Ap

but AR = Ar ∴ AP = Ap

∴ rem.r PR = rem.r pr

or the image = object ——

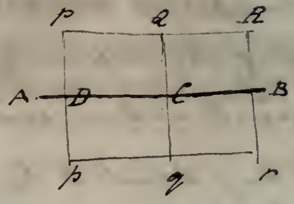
69. When the object is // mirror —

Draw $RBr \perp^r AB$

make $Br = BR$ —

thro' r draw $rp \parallel RP$

and draw $PDp \perp^r AB$ —



In PR take any point Q

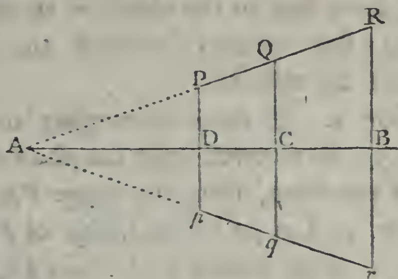
and draw $QCq \perp^r AB$ meet^g. rp in q —

Then $Cq = Br = BR = QC$. ∴ q is the image

of Q and Q is any point in PR ∴

pr is the image of PR, and they are // —

reflector AB; produce RP, if neceſſary, till it meets the ſurface in A; draw RBr at right angles to AB,



and make Br equal to RB; join Ar; and from P draw PDp perpendicular to AB, meeting Ar in p; then will pr be the image of PR.

Since RBr is perpendicular to AB, and Br is equal to BR, r is the image of R (Art. 41).

Alſo, from the ſimilar triangles ABR, ADP, $RB : AB :: PD : AD$; and from the ſimilar triangles ABr, ADp, $AB : Br :: AD : Dp$; ex æquo, $RB : Br :: PD : Dp$; and ſince RB is equal to Br, PD is equal to Dp; or p is the image of P. In the ſame manner it may be ſhewn, that the image of every other point in PQR is the correſponding point in pqr; that is, pr is the whole image of PR.

Again, ſince BR is equal to Br, and AB common to the two triangles ABR, ABr, and alſo the angles at B are right angles, the angles of inclination RAB, BAr are equal, and AR is equal to Ar. In the ſame manner, AP is equal to Ap; therefore PR is equal to pr.

69. Cor. 1. If the object PR be parallel to the reflector, the image, pr will alſo be parallel to it.

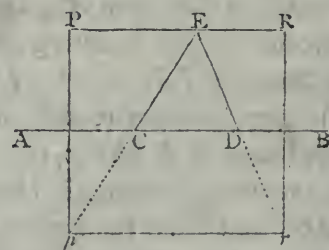
70. Cor. 2. If PR be a curve, pr will be a curve, ſimilar and equal to PR, and ſimilarly ſituated on the other ſide of the reflector.

C 71. Cor.

71. Cor. 3. Whatever be the form of the object, the image will be similar and equal to it. For, the image of every line in the object is an equal and corresponding line, equally inclined to, and equally distant from the reflector.

72. Cor. 4. If rays, converging to the several points in *pr*, be received upon the plane reflector, they will, after reflection, form the image *PR* (Art. 20).

73. Cor. 5. Let *pr* be the image of *PR*; and suppose an eye to be placed at *E*; join *pE*, *rE*, cutting



the reflector in *C* and *D*; then, considering the pupil as a point, the image will be seen in the part *CD* of the reflector; and it will subtend the angle *CED* at the eye; because all the rays enter the eye as if they came from a real object *pr* (Art. 65).

74. Cor. 6. When *PR* is parallel to *AB*, and *E* is situated in *PR*, *CD* is the half of *pr*, or *PR*.

For, in this case, *pr* is parallel to *AB* (Art. 69), and therefore $CD : pr :: ED : Er :: 1 : 2$.

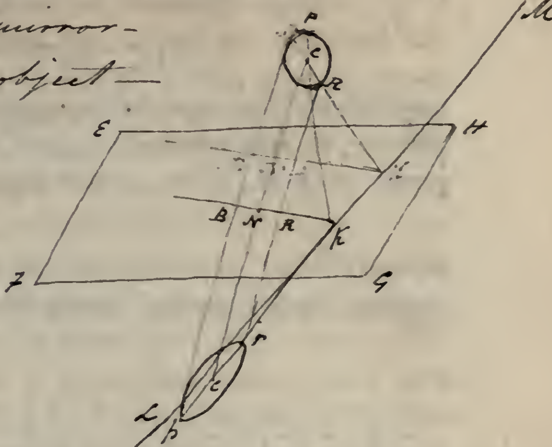
No rays enter the eye from any other part of the reflector.

PROP.

aving given a plane circle for the object,
to construct the image ——

t EFGH be the mirror -

R the circle or object —



t LM be the intersection of the planes of
mirror and circle - PK in the plane of
⊙ ⊥r to this intersection —

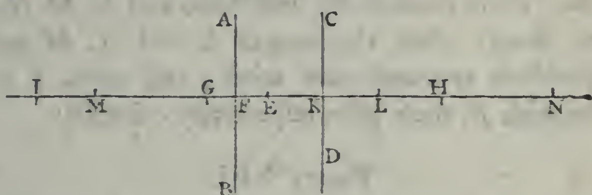
en ∠BKP = ∠ of inclinat. of the plane of
object to the mirror — From the centre C
aw CNc ⊥r plane of the mirror, make Nc = NC
c is the image of C ——

w suppose a plane to pass thro LM & the
int c, & with centre c and pc = PC describe
⊙ pcr in this plane, then pcr is the image
PCR — For pcr is a ⊙ = ⊙PCR — & is equidistant
om the mirror & its plane inclin'd to the
ane of the mirror at ∠ pKB = ∠PKB —

Prop. XII.

75. *When an object is placed between two parallel plane reflectors, a row of images is formed which are gradually fainter as they are more remote, and at length they become invisible.*

Let AB, CD be two plane reflectors, parallel to each other; E an object placed between them; through



E draw the indefinite right line NEI perpendicular to AB, or CD. Take $FG=FE$; $KH=KG$; $FI=FH$, &c. Alfo, take $KL=KE$; $FM=FL$; $KN=KM$, &c.

Then, the rays which diverge from E and fall upon AB, will, after reflection, diverge from G (Art. 41); or G will be an image of E. Alfo, thefe rays, after reflection at AB, will fall upon CD as if they proceeded from a real object at G, and after reflection at CD they will diverge from H; that is, H will be an image of G; or a fecond image of E, &c. In the fame manner, the rays which diverge from E, and fall upon CD, will form the images L, M, N, &c.

It is found by experiment, that all the light incident upon any furface, however well polifhed, is not regularly reflected from it. A part is difperfed in all directions (Art. 34); and a confiderable portion enters the furface, and feems to be abforbed by the body.

 In

In the paſſage alſo of light through any uniform medium, ſome rays are continually diſperſed, or abſorbed; and thus, as it is thrown backward and forward through the plate of air contained between the two reflectors *AB*, *CD*, it's quantity is diminiſhed. On all theſe accounts, therefore, the ſucceeding images become gradually fainter, and, at length, wholly inviſible.

76. Cor. If *E* move towards *F*, the images *G*, *H*, *I*, &c. move towards the reflectors, and *L*, *M*, *N*, &c. from them; thus the images *L* and *H*, *M* and *I*, reſpectively approach each other; and when *E* coincides with *F*, theſe pairs reſpectively coincide.

Prop. XIII.

77. *If an object be placed between two plane reflectors inclined to each other, the images formed will lie in the circumference of a circle, whoſe center is the interſection of the two planes, and radius the diſtance of the object from that interſection.*

Let *AB*, *AC* be two plane reflectors inclined at the

angle *BAC*; *E* an object placed between them. Draw

EF

EF perpendicular to AB, and produce it to G, making $FG = EF$; then the rays which diverge from E and fall upon AB, will, after reflection, diverge from G; or G will be an image of E. From G, draw GH perpendicular to AC, and produce it to I, making $HI = GH$, and I will be a second image of E, &c. Again, draw ELM perpendicular to AC, and make $LM = EL$; alſo, draw MNO perpendicular to AB, and make $NO = MN$, &c. and M, O, &c. will be images of E, formed on the ſuppoſition that it is placed before AC. Let K, V; P, Q be the other images, determined in the ſame manner.

Then, ſince EF is equal to FG, and AF common to the triangles AFG, AFE, and the angles at F are right angles, AG is equal to AE (Euc. 4. 1). In the ſame manner it appears, that AI, AK, &c. AM, AO, AP, &c. are equal to each other, and to AE; that is, all the images lie in the circumference of the circle $EMIK$ whoſe center is A, and radius AE.

78. Cor. If the angle BAC be finite, the number of images is limited. For, BA and CA being produced to S and R, the rays which are reflected by either ſurface, diverging from any point Q between S and R, will not meet the other reflector; that is, no image of Q will be formed.

Prop. XIV.

79. *Having given the inclination of two plane reflectors, and the ſituation of an object between them, to find the number of images.*

It appears from the conſtruction in the laſt propoſition, that the lines EG, MO, IK, PQ, &c. are parallel, as alſo EM, GI, OP, KV, &c. Hence it follows, that the arcs EG, MI, OK, PV, &c. are equal; as

 alſo,

alſo, the arcs EM, GO, IP, KQ, &c. Let $BC=a$, $EB=b$, $EC=c$; then, the arc $EG=2b$; $EM=2c$;

$EO=EG+GO=EG+EM=2b+2c=2a$; $EK=EO+OK=EO+EG=2a+2b$; $EOQ=EK+KQ=EK+EM=2a+2b+2c=4a$, &c. Thus there is one ſeries of images, formed by the reflections at AB, whoſe diſtances from E, meaſured along the circular arc EOR, are $2b$, $2a+2b$, $4a+2b$, ... $2na-2a+2b$ $(2na-2c)$, where n is the number of images; this ſeries will be continued as long as $2na-2a+2b$, or $2na-2c$ is leſs than the arc EOR, or $180°+b$; and conſequently n, the number of images in this ſeries, is that whole number which is next inferior to $\frac{180+b+2c}{2a}$, or to $\frac{180+a+c}{2a}$. There is alſo a ſecond ſeries of images, formed by reflections at the ſame ſurface, whoſe diſtances from E are $2a$, $4a$, $6a$, ... $2ma$, continued as long as $2ma$ is leſs than $180+b$, and therefore m, the number of theſe images, is that whole number which is next inferior to $\frac{180+b}{2a}$.

In

In the ſame manner, the number of images formed by reflections at the ſurface *AC*, is found by taking the whole numbers next inferior to $\frac{180+a+b}{2a}$, and $\frac{180+c}{2a}$.

80. Cor. 1. If a be a meaſure of 180, the number of images formed will be $\frac{360}{a}$.

For, if a be contained an even number of times in 180, or $2a$ be a meaſure of 180, the number of images in each ſeries is $\frac{180}{2a}$*; and the number upon the whole is $4 \times \frac{180}{2a} = \frac{360}{a}$. If a be contained an odd number of times in 180, $2a$ is a meaſure of $180+a$, or $180-a$; and the number of images is $\frac{180+a}{2a} + \frac{180-a}{2a}† + \frac{180+a}{2a} + \frac{180-a}{2a} = \frac{360}{a}$.

81. Cor. 2. When a is a meaſure of 180, two images coincide.

For, if a be contained an even number of times in 180, then the number of images in the ſecond ſeries, formed by reflections at the ſurface *AB*, is $\frac{180}{2a}$; and the diſtance EOQ, $(2ma)$, of the laſt image from E, is 180°. In the ſame manner, the diſtance EIV, of the laſt image in the ſecond ſeries formed by reflections at

* $\frac{180+a+c}{2a} = \frac{180}{2a} + \frac{a+c}{2a}$; the latter part, $\frac{a+c}{2a}$, being leſs than unity, is neglected.

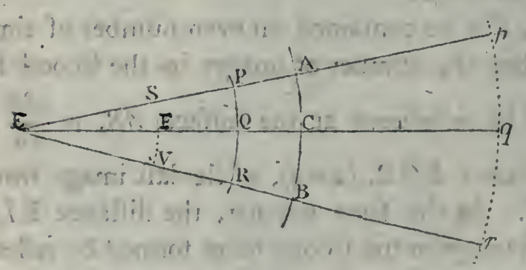
† $\frac{180+b}{2a} = \frac{180-a+a+b}{2a} = \frac{180-a}{2a} + \frac{a+b}{2a}$; the latter part, $\frac{a+b}{2a}$, being leſs than unity, is neglected.

at AC, is 180°; therefore the two images, $\mathcal{Q}$ and P, coincide in EA produced. If a be contained an odd number of times in 180, then the number of images, in the firſt ſeries, formed by reflections at AB, is $\frac{180+a}{2a}$; and the diſtance EOK, of the laſt of theſe images from E, is $\frac{180+a}{2a} \times 2a - 2c$; or $180° + a - 2c$. Alſo, the diſtance EMP, of the laſt image in the firſt ſeries formed by reflections at AC, is $180° + a - 2b$; therefore $EOK + EMP = 360° + 2a - 2c - 2b = 360°$; that is, K and P coincide.

Prop. XV.

82. *If the object placed before a ſpherical reflector be a circular arc concentric with it, the image will alſo be a circular arc concentric with, and ſimilar to the object.*

Let ACB be the ſpherical reflector, $P\mathcal{Q}R$ the circular arc; E their common center. Take any points P, $\mathcal{Q}$, R, in the object; join EP, $E\mathcal{Q}$, ER, and



produce theſe lines, if neceſſary; biſect EC in F, and take $F\mathcal{Q} : FE :: FE : Fq$, meaſuring Fq and $F\mathcal{Q}$ in the ſame direction from F; with the center E and radii Eq, EF, deſcribe the circular arcs pqr, SFV,

cutting

The same object^n may be made to this
prop as in Art: 68 ——

Let AB be the spherical surface. E its centre.
Let its principal foci lie in the ⌒ SFO =
PQR an object placed before it — Take

PS : SE :: SE : Sp, then p is the image of
P — Join ER & produce it indefinitely : with
centre E & rad. Ep describe an ⌒ pr — Then
shall pr be the image of PR —

For in PR take any point Q & produce EQ to
meet the arc pr in q — Then since

$FE = SE$ & $EQ = EP$ ∴ $QF = SP$ and since
$Eq = Ep$ & $ES = EF$ ∴ $Fq = Sp$

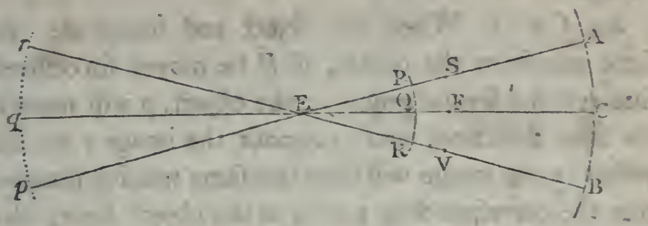
But $PS : SE :: SE : Sp$

∴ $QF : FE :: FE : Fq$ ∴ q is y^e image of Q —

& Q was taken any point in PQR

∴ pqr is the image of PQR — —

cutting EP, ER, or thoſe lines produced, in p, r, and



S, V; then will pqr be the image of PQR.

For, ſince $FQ : FE :: FE : Fq$, and FQ and Fq are meaſured in the ſame direction from F; q is the image of Q (Art. 50). Alſo, ſince EP, is equal to EQ, ES to EF, and Ep to Eq, SP is equal to FQ, and Sp to Fq; therefore $SP : SE :: SE : Sp$, and p is the image of P. In the ſame manner it may be proved, that the image of every other point in PQR, is the correſponding point in pqr; that is, pqr is the image of PQR.

Again, the image and object, ſince their extremities are determined by ſtraight lines which paſs through the center E, ſubtend the ſame, or equal angles at that center; therefore they are ſimilar arcs.

83. Cor. 1. In the ſame manner it appears that, if pqr be the object, PQR will be it's image; and if rays are incident the contrary way, converging to the ſeveral points in PQR, pqr will be the image of PQR.

84. Cor. 2. Becauſe ſimilar arcs are proportional to their radii, $PR : pr :: QE : Eq$.

85. Cor. 3. Since $QF : FE :: QE : Eq$ (Art. 55), $PR : pr :: QF : FE$.

86. Cor. 4. If the object be placed any where between E and C, QF is leſs than FE; therefore the object is leſs than the image. If it be placed in EC produced,

duced, or in CE produced, the image is leſs than the object.

87. Cor. 5. When the object and image lie the ſame way from the center, if P be moved ſucceſſively through the ſeveral points in the object, p will move in the ſame direction, and trace out the image; or, the points in the image will have the ſame relative ſituation that the correſponding points in the object have; that is, the image will be erect. But if P and p be on contrary ſides of E, they will move in oppoſite directions; or the image will be inverted.

88. Cor. 6. When the object is in FC, or in FC produced, the image is erect (Art. 87). When it is in FE, or FE produced, the image is inverted (Vid. Art. 59).

89. Cor. 7. Since $Q E : Eq :: QC : Cq$, the object and image are in the ratio of their diſtances from the reflector.

90. Cor. 8. If QC and qC be known, the radius of the reflector may be found. For, $QF : FE :: QC : qC$; therefore $QF \mp FE : FE :: QC \mp qC : qC$, or $QC : FE :: QC \mp qC : qC$; hence FE is known, and $2FE$ is the radius ſought.

The upper ſign is to be uſed when Q and q are on different ſides of the reflector, and the lower, when they are on the ſame ſide.

91. Cor. 9. If the object be a ſpherical *ſurface*, generated by the revolution of PQR about the axis EC, the image will be a ſimilar ſurface, and the magnitude of the object : the magnitude of the image :: $\overline{EQ}|^2 : \overline{Eq}|^2$.

92. Cor. 10. If O, the place of the eye, be given, the part of the object ſeen in a given portion AB, of the reflector, may be thus determined:

Join

the image of R; in the line PED take ED equal to EP, and draw DH at right angles to PD; from r draw rH, rS, respectively parallel to DN, and RP.

Then, becaufe r is the image of R, $RF : FE :: ER : Er$ (Art. 55), alternately, $RF : ER :: FE : Er$. Alfo, in the fimilar triangles EPR, ESr, $ER : Er :: EP : ES :: ED : ES$, therefore $RF : FE :: ED : ES$, and by compofition, or divifion, $RF : RF \mp FE :: ED : ED \mp ES$; that is, $RF : ER :: ED : DS\ (Hr)$; confequently, $ED : Hr :: FE : Er$; alternately, $ED : FE :: Hr : Er$; that is, Hr bears an invariable ratio to Er, and therefore r is a point in the conic fection whofe focus is E and directrix DH.

94. Cor. 1. When EP is equal to EF, Hr is equal to Er, and the conic fection is a *Parabola*. It is an *Ellipfe* or *Hyperbola*, according as EP is *greater* or *lefs* than EF.

95. Cor. 2. When the diftance of the object is fo great that the rays which come from any point in it may be confidered as parallel, the image is a circle whofe radius is EF.

96. Cor. 3. If ET be drawn perpendicular to EN, when r coincides with T, Hr becomes equal to ED; and fince $ED : EF :: Hr : Er$, Er becomes equal to EF. This is, ET, half the latus rectum of the conic fection, is equal to EF, half the radius of the reflector.

97. Cor. 4. Since the radius of curvature at the vertex of the figure is equal to half the latus rectum, the curvature of the image at N is the fame, wherever the object is placed.

98. Cor. 5. If the radius of the reflector be finite, the evanefcent arc rN is equal to the ordinate rS (Newt.

(Newt. Lem. 7); and therefore *RP* : *rN* :: *EP* : *ES* :: *EP* : *EN*.

Alſo, whilſt the angle *REP*, which the ſtraight line ſubtends at the center of the reflector is ſmall, though finite, the arc *rN* will, as to ſenſe, be a right line.

99. Cor. 6. In the preceding figure, where the object is ſuppoſed to lie between the principal focus and the ſurface *ACB*, *TNx* is the erect image of the line *RP* indefinitely produced both ways, and *TMx* it's inverted image. The part *ANB* is formed by reflection from the concave ſurface *ACB*; *AT* and *Bx*, by reflection from the convex ſurfaces *AK*, *BL*; and *TMx* by reflection from the concave ſurface *KGL*.

SECTION

SECTION IV.

ON THE REFRACTION OF RAYS AT PLANE AND SPHERICAL SURFACES.

PROP. XVII.

100. *WHEN a ray of light paſſes out of one medium into another, as the angle of incidence increaſes, the angle of deviation alſo increaſes.*

Let A and B be the angles of incidence and refraction; and let ſin. A : ſin. B :: m : n. Then by compoſition and diviſion, ſin. A+ſin. B : ſin. A∼ſin. B :: $m+n$: $m \sim n$; but ſin. A+ſin. B : ſin. A∼ſin. B :: tang. $\frac{A+B}{2}$: tang. $\frac{A \sim B}{2}$*; therefore tang. $\frac{A+B}{2}$: tang. $\frac{A \sim B}{2}$ :: $m+n$: $m \sim n$. Now let A increaſe, then B alſo

* This is deducible from the common principles of trigonometry. The ſides of a triangle are proportional to the ſines of the oppoſite angles; therefore the ſum of two ſides : their difference :: the ſum of the ſines of the oppoſite angles : the difference of their ſines; and the ſum of the ſides : their difference :: the tangent of half the ſum of the oppoſite angles : the tangent of half their difference; conſequently, the ſum of the ſines of the angles : the difference of their ſines :: the tangent of half their ſum : the tangent of half their difference.

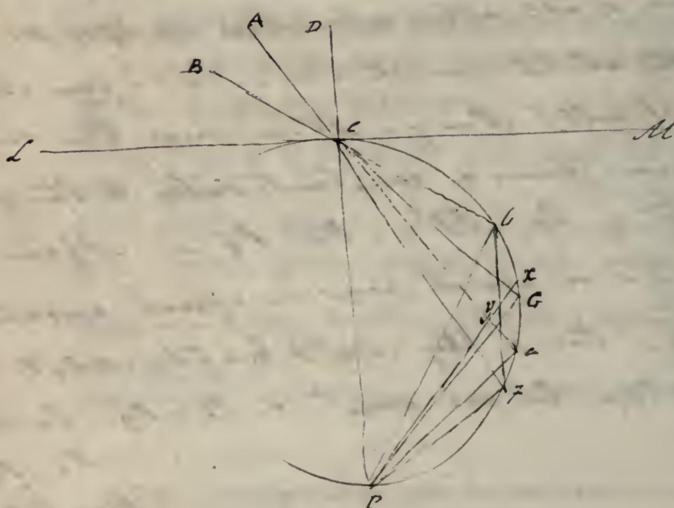
alſo increaſes (Art. 26); therefore $\frac{A+B}{2}$ increaſes; and $\frac{A+B}{2}$ is leſs than a quadrant, therefore tang. $\frac{A+B}{2}$ increaſes; hence, tang. $\frac{A\sim B}{2}$, and conſequently $\frac{A\sim B}{2}$, and alſo, $A\sim B$, the angle of deviation, increaſes.

Prop. XVIII.

LM be the refracting surface, DCP ⊥ to LM
& $\frac{\odot}{2}$ describ'd on CP, ACa, BCb incid^t rays. CF
direction of AC after refraction; & CG of BC –
in bP, GP, aP, FP, make ∠ bx = ∠ aF – join xP
wh also bF cutts xP in y – Now since the sine
incidence bears a constant ratio to the sine
refract^n, Pa : PF :: Pb : PG — Again
since the ∠s on equal arcs in equal circles
are equal, the ∠ bPy = ∠ aPF & ∠ Pby = ∠ PaF
the Δs Pby, PaF are sim^r ∴ Pb : Py :: Pa : PF
:: Pb : PG
∴ Py = PG —
and Py is less than Px since by F falls within ⊙.
PG is less than Px ∴ the point G falls below x
& bG is > (arc bx =) ∠ aF ∴ ∠ bCG is g^t than ∠ aCF,
that is deviation of ray BC is g^t than deviat^n
of ray AC ...

SECTION IV.

ON THE REFRACTION OF RAYS AT

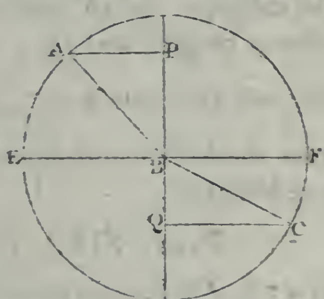


alſo increaſes (Art. 26); therefore $\frac{A+B}{2}$ increaſes; and $\frac{A+B}{2}$ is leſs than a quadrant, therefore tang. $\frac{A+B}{2}$ increaſes; hence, tang. $\frac{A\sim B}{2}$, and conſequently $\frac{A\sim B}{2}$, and alſo, $A\sim B$, the angle of deviation, increaſes.

Prop. XVIII.

101. *A ray of light cannot paſs out of a denſer into a rarer medium if the angle of incidence exceed a certain limit.*

Let a ray of light AB be incident upon the ſurface EBF of a rarer medium, and refracted in the direction BC. Through B, draw PBQ perpendicular to the ſurface; take BC equal to AB, and draw AP, CQ



at right angles to PQ. Then, ſince the ray paſſes out of a denſer medium into a rarer, the ſine of refraction is greater than the ſine of incidence (Art. 33); and theſe ſines are always in a given ratio (Art. 24); therefore the ſine of refraction will become equal to the radius ſooner than the ſine of incidence; let the angle of

incidence

incidence be increaſed till the ſine of refraction is equal to the radius, and let the angle of incidence, and conſequently the ſine of incidence, be farther increaſed; then, if the ray be refracted, the ſine of refraction muſt alſo be increaſed, which is impoſſible; therefore the ray cannot be refracted, when the angle of incidence exceeds this limit.

To determine the limit, let the ſine of incidence be to the ſine of refraction, out of the denſer medium into the rarer, $:: n : m$; and let x be the ſine of incidence when the correſponding ſine of refraction is r, the radius; then $n : m :: x : r$, and $x = \frac{nr}{m}$; and the ſine of incidence being known, the correſponding angle may be found from a table of ſines.

Thus, if the two mediums be water and air, $n : m :: 3 : 4$; therefore $x = \frac{3r}{4}$; and the angle, whoſe ſine is $\frac{3}{4}$ of the radius, is 48° 36′, nearly.

If the mediums be glaſs and air, $n : m : 2 : 3$, and $x = \frac{2r}{3}$; in this caſe, the angle is 41°. 49′; and the rays which fall upon the ſurface at a greater angle of incidence will be reflected *.

Prop. XIX.

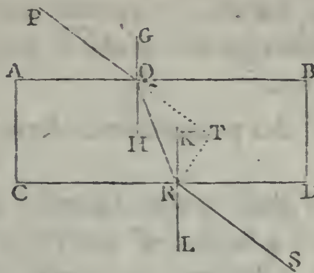
102. *When a ray of light paſſes through a medium contained by two parallel plane ſurfaces, the directions in which it is incident and emergent are parallel to each other* †.

Let $ABDC$ be the medium, PQ, QR, RS, the courſe of

* The reflection thus made, is much ſtronger than can be produced by any poliſhed metallic ſurface, or glaſs ſpeculum.

† In this, and ſimilar caſes, the ambient medium is ſuppoſed to be uniform.

of a ray refracted through it; GQH, KRL perpendiculars to AB and CD at the points Q and R. Now,



the effect of the refraction is the same, whether $PQRS$ be the course of the ray, or QR pass both ways out of the medium (Art. 30); let the latter supposition be made; then, since AB is parallel to CD, the alternate angles BQR, QRC are equal; and therefore their complements RQH, QRK, which are the angles of incidence, are equal; hence, the angles of refraction PQG, SRL, are equal (Art. 25); therefore the angles PQA, SRD are equal; and if to these the equal angles AQR, QRD be added, the whole angles PQR, QRS are equal, and therefore PQ, RS are parallel.

103. Cor. 1. Whatever be the form of the surfaces, if the planes which touch them at the points Q and R be parallel, PQ and RS will be parallel (Art. 28).

104. Cor. 2. If RT be drawn perpendicular to PQ produced, $QR : RT ::$ rad. : sin. RQT; wherefore, when the angle of incidence PQG, and consequently the angle of deviation RQT, is small, and the thickness of the medium also small, $PQRS$ may, without sensible error, be considered as a straight line.

Prop. XX.

105. *When a ray of light is refracted through two contiguous mediums, contained by parallel plane surfaces, it emerges in a direction parallel to that in which it is incident upon the first surface.*

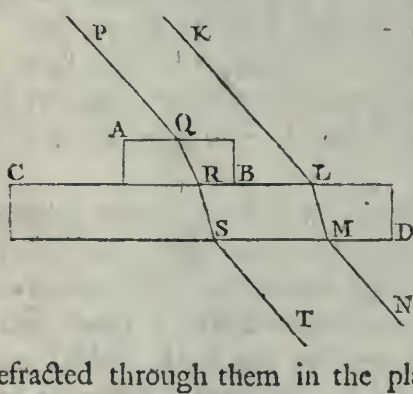
The truth of this proposition is derived from experiment.

Let a plate of glass be placed parallel to the horizon, and covered with a lamina of water; then the surface of the water will become horizontal, and therefore parallel to the surfaces of the glass plate; and if the sun's light be refracted through the two mediums, the incident and emergent rays are found to be parallel *.

Prop. XXI.

106. *A ray of light is as much refracted in passing through one medium into another, when they are terminated by parallel plane surfaces, as it is in passing immediately into the latter medium.*

Let *AB*, *CD* be the mediums; *PQRST* the course



of a ray refracted through them in the plane of the paper;

* Newt. Lectiones Opticæ. Par. I. Sect. II.

paper; *KLMN* the courſe of a ray which is incident parallel to *PQ*, and refracted, in the ſame plane, through the medium *CD*.

Then, ſince *PQ* and *ST* are parallel (Art. 105), and alſo *KL* and *MN* (Art. 102), and *PQ* is parallel to *KL* by the ſuppoſition, *ST* is parallel to *MN*; and if *TS* and *NM* be ſuppoſed to be the incident rays, *SR* and *ML* will be the refracted rays (Art. 29); alſo, ſince the angles of incidence, of the rays *TS*, *NM*, are equal, the angles of refraction are alſo equal (Art. 25); hence, the complements of theſe angles are equal, and *SR*, *ML* are parallel; that is, the ſum, or difference, of the deviations at *Q* and *R*, is equal to the deviation at *L*.

107. Cor. Hence it follows, that if a ray paſs through any number of mediums, contained by parallel plane ſurfaces, it will be as much bent from it's original courſe as if it paſſed immediately out of the firſt medium into the laſt *.

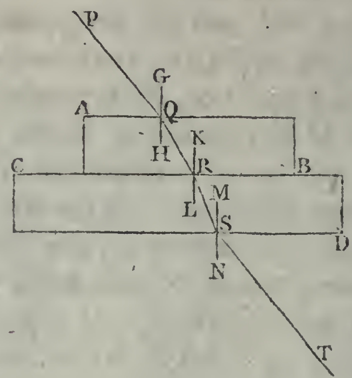
Prop. XXII.

108. *Having given the ratio of the ſines of incidence and refraction, when a ray paſſes out of one medium into each of two others, to find the ratio of the ſine of incidence to the ſine of refraction out of one of the latter mediums into the other.*

Let *AB*, *CD* be two contiguous mediums contained by parallel plane ſurfaces; *PQRST* the courſe of a ray refracted through them; through the points *Q*, *R*, *S*, draw *GQH*, *KRL*, *MSN* at right angles to the ſurfaces;

* Newton's Optics, Book II. Part III. Prop. X.

surfaces; and let $a : b ::$ sin. incidence : sin. refraction out of the surrounding medium into AB; $c : d ::$ sin.



incidence : sin. refraction out of the surrounding medium into CD.

Then, since PQ is parallel to ST (Art. 105), and GH to MN, the angles PQG, NST are equal; also, the angles HQR, LRS, are respectively equal to the angles QRK, RSM. Now, from the hypothesis, sin. PQG : sin. $HQR :: a : b$; inversely, sin. HQR : sin. $PQG :: b : a$; or

sin. QRK : sin. $NST :: b : a$; also,

sin. NST : sin. $RSM\ (SRL) :: c : d$; by composition, sin. QRK : sin. $SRL :: bc : ad$.

Ex. When a ray passes out of air into water, sin. incidence : sin. refraction $:: 4 : 3 :: a : b$; out of air into glass, sin. incidence : sin. refraction $:: 3 : 2 :: c : d$; therefore, out of water into glass, sin. incidence : sin. refraction $:. bc : ad :: 9 : 8$.

109. Def.

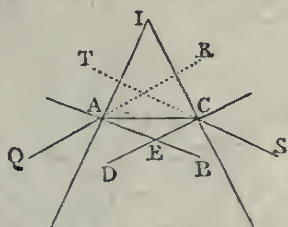
109. Def. A *Priſm*, in optics, is a ſolid terminated by three rectangular parallelograms, and two ſimilar, equal and parallel triangles.

110. A line drawn through the center of gravity of the priſm, parallel to the interſections of the parallelograms, is called the *axis* of the priſm.

Prop. XXIII.

111. *A ray of light which paſſes through a priſm, in a plane perpendicular to it's axis, is turned* towards *the thicker part, or* from *it, according as the priſm is* denſer, *or* rarer, *than the ſurrounding medium.*

Let AIC repreſent a ſection of the priſm, made by a plane which is perpendicular to it's axis, and therefore



to it's ſurfaces (Euc. 8. and 18. 11); QA a ray incident in the plane AIC; AC, CS the courſe of the refracted ray, in that plane (Art. 24). Then, the effect of the refraction is the ſame, whether we ſuppoſe the ray to paſs thus through the priſm, or AC to paſs both ways out of the priſm (Art. 30); let this latter ſuppoſition be made, and the propoſition reſolves itſelf into the three following caſes:

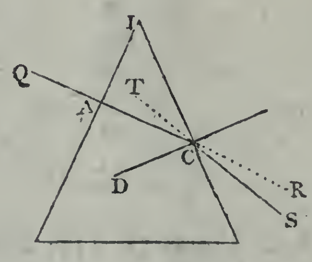
Caſe 1. When AC makes two acute angles with the ſides of the priſm.

 Draw

Draw *AB*, *CD* at right angles to *IA*, *IC*, and let them meet in *E*. Then, ſince the ∠ *CAI* is leſs than the ∠ *EAI*, *CA* is nearer to the vertex *I* than *EA*; and as they croſs each other at *A*, *CA* produced is farther from the vertex than *EA* produced; alſo, the ray *CA*, when the priſm is denſer than the ſurrounding medium, is turned from the perpendicular; that is, from *EA* produced, or towards the thicker part of the priſm. In the ſame manner it may be proved, that the ray *AC* is refracted at *C* towards the thicker part of the priſm; conſequently, the bending upon the whole is in that direction.

Caſe 2. When *AC* makes a right angle with one ſide of the priſm.

Let the angle at *A* be a right angle; then, ſince there is no refraction at *A* (Art. 27), the whole



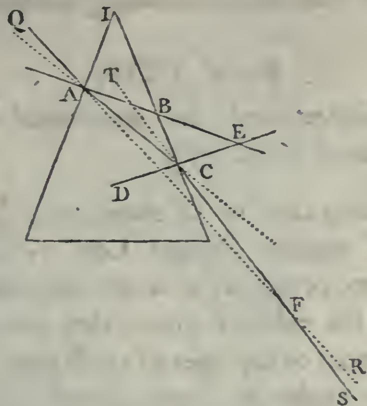
bending is at *C*, which may be ſhewn, as before, to be towards the thicker part of the priſm.

Caſe 3. When *AC* makes an obtuſe angle with one ſide of the priſm.

Let the angle *IAC* be obtuſe, and the conſtruction being made as before, *CA* lies nearer to the baſe of the priſm than *EA*; and *CA* produced lies farther from the baſe than *EA* produced; alſo, the ray *CA*, in

paſſing

paſſing from a denſer medium into a rarer, is turned from the perpendicular; that is, from *EA* produced,



or from the baſe. Thus then, the refraction at *A* is from the thicker part of the priſm, and the refraction at *C*, as appears from the firſt caſe, is the contrary way; therefore the refraction, upon the whole, is the difference of the refractions at *A* and *C*. Now, the angles *IAC*, *ICA* being together leſs than two right angles, the angles *BAC*, *ACB* are together leſs than one right angle; to theſe add the right angle *BCE*, and the angles *BAC*, *ACB*, *BCE*, or *BAC*, *ACE*, are together leſs than two right angles; therefore *AB* and *CE* will meet, if produced, above *AC* (Euc. Ax. 12); let them meet in *E*. Then, ſince the exterior angle *DCA*, of the triangle *CAE*, is greater than the interior and oppoſite angle *CAE*, the angle of incidence of the ray *AC* is greater than the angle of incidence of the ray *CA*, and therefore the deviation at *C* is greater than the deviation at *A* (Art. 100); or the exceſs of the deviation is towards the thicker part of the priſm.

In the ſame manner it may be proved, that a ray of light will be bent *from* the thicker part of a priſm which is *rarer* than the ſurrounding medium.

Prop. XXIV.

112. *Evaneſcent angles are proportional to their ſines, when the radius is given.*

The limiting ratio of an evaneſcent arc to it's ſine is a ratio of equality (Newt. Lem. 7. Cor. 1); and ſince angles are proportional to the arcs which ſubtend them, when the radius is given, they are, in this caſe, alſo proportional to the ſines of thoſe arcs.

When the angles are ſmall, though of finite magnitude, the ſame propoſition is nearly true; and ſufficiently accurate, if the concluſions drawn from it be conſidered in a practical light, and applied to the conſtruction of optical inſtruments, or the explanation of the phænomena of refraction.

Prop. XXV.

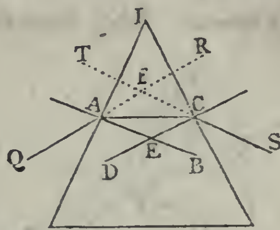
113. *When a ray of light paſſes through a priſm, in a plane which is perpendicular to it's axis, and the angles of incidence are ſmall, the vertical angle of the priſm is to the angle of deviation, as the ſine of incidence, out of the priſm into the ambient medium, is to the difference of the ſines of incidence and refraction.*

The ſame conſtruction and ſuppoſition being made as in Art. 111, the propoſition will in like manner reſolve itſelf into three caſes.

Caſe 1. When *AC* makes two acute angles with the ſides of the priſm.

Let

Let $m : n ::$ ſin. incidence : ſin. refraction, out of the priſm into the ambient medium; produce QA, SC, to R and T; then the $\angle CAE$ is the angle of in-

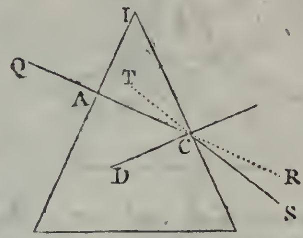


cidence of the ray CA, and the $\angle EAR$ is equal to the angle of refraction of the ſame ray; alſo the $\angle ECA$ is the angle of incidence, and the $\angle ECT$ equal to the angle of refraction of the ray AC; and ſince the angles of incidence, and conſequently the angles of refraction are ſmall, they are proportional to their ſines (Art. 112); therefore $EAC : EAR :: m : n$; and $EAC : EAC \sim EAR$ $(CAR) :: m : m \sim n$. In the ſame manner, $ECA : ACT :: m : m \sim n$; hence $EAC : CAR :: ECA : ACT$, and $EAC + ECA : CAR + ACT :: ECA : ACT :: m : m \sim n$ (Euc. 12. 5). Again, ſince the four angles of the quadrilateral figure $IAEC$ are equal to four right angles, and the angles IAE, ICE are right angles, the two angles AEC, AIC are together equal to two right angles, or to the two angles AEC, AED; conſequently, the angle AIC is equal to the angle AED; and AED is equal to the ſum of the angles EAC, ECA; therefore the angle AIC is equal to the ſum of thoſe angles; alſo, the ſum of the angles CAR, ACT is, in this caſe, the whole deviation (Prop. 23. Caſe 1); therefore, from the laſt proportion, AIC : the whole deviation $:: m : m \sim n$.

Caſe

Case 2. When AC is at right angles to one side of the prism.

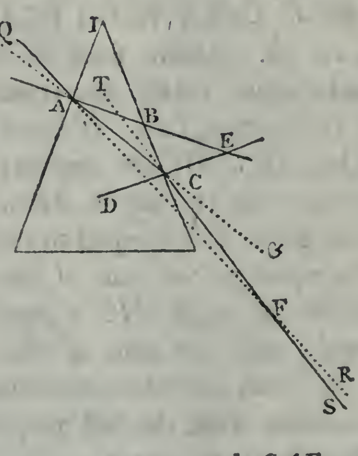
Let the angle CAI be a right angle; then the whole refraction is at C; and in this case, as before, $DCA : ACT :: m : m \sim n$. Also, since the right angle



DCI is equal to the sum of the two ACI, AIC, take away the common angle ACI, and the remaining angles DCA, AIC are equal; consequently $AIC : ACT :: m : m \sim n$.

Case 3. When AC makes an obtuse angle with one side of the prism.

It may be shewn in the same manner as in Case 1, that



$ACD : ACT :: m : m \sim n$; and $CAE : CAF :: m : m \sim n$.

ACT + ACS > CAR + ACS & ACT + ACS = 2 rt. ∠s

∴ ACS + CAR < than 2 rt. ∠s . ∴ TC & AF meet.

Hence also ACT is gt. than CAF

$m \sim n$. Hence, $ACD - CAE : ACT - CAF :: m : m \sim n$ (Euc. 19. 5.); and ſince ACD is the exterior angle of the triangle ACE (Prop. 23. Caſe 3), $ACD - CAE = CEA$; alſo, $ACT - CAF$ is the whole deviation; therefore CEA : the deviation :: $m : m \sim n$. Again, ſince the triangles AIB, BCE, have vertical angles at B, and right angles IAB, BCE, the angles AIB, BEC are equal; therefore, from the laſt proportion, AIB : the whole deviation :: $m : m \sim n$.

114. Cor. 1. It appears from the demonſtration of the foregoing propoſition, that when the ray makes two acute angles with the ſides of the priſm, the angle at the vertex is equal to the *ſum* of the angles of incidence; and when it makes an obtuſe angle with one ſide, the angle at the vertex is equal to the *difference* of the angles of incidence.

115. Cor. 2. Hence it follows, that the angles of incidence cannot be ſmall, unleſs the angle at the vertex of the priſm be alſo ſmall.

Ex. 1. Let the angle at the vertex of a glaſs priſm, placed in air, be $1°$; then $m : n :: 2 : 3$, and $m : m \sim n :: 2 : 1 :: 1° : \frac{1}{2}°$, the angle of deviation when the angles of incidence are ſmall.

Ex. 2. Let the ſame priſm be placed in water, then $m : n :: 8 : 9$; and $m : m \sim n :: 8 : 1 :: 1° : \frac{1}{8}°$, the angle of deviation in this caſe.

116. Cor. 3. When the angle at the vertex of the priſm vaniſhes, or the ſides become parallel, the angle of deviation alſo vaniſhes (Vid. Art. 102.)

117. Cor.

117. Cor. 4. If the quantity and direction of the deviation, and the angle at the vertex of the prism be known, the ratio of the sine of incidence to the sine of refraction may be determined.

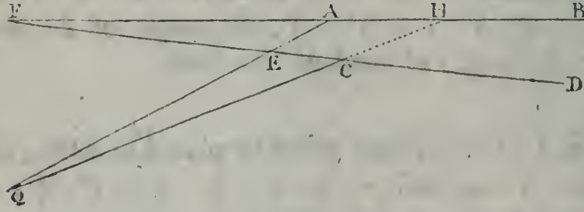
Thus, if the deviation be $\frac{1}{3}$ of the angle at the vertex, and towards the thicker part of the prism, $m : n-m :: 3 : 1$; and by composition, $m : n :: 3 : 4$.

If the deviation be towards the thinner part of the prism, $m : m-n :: 3 : 1$; therefore $m : n :: 3 : 2$. In these, as in the former cases, the angles of incidence and refraction are supposed to be proportional to their sines.

Prop. XXVI.

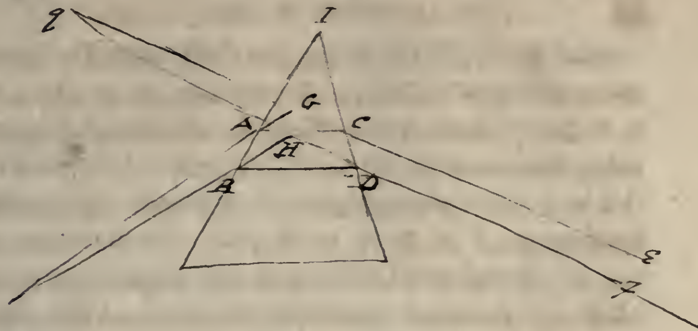
118. *When two rays are refracted through a prism, in the same plane, and the angles of incidence on each surface are small, the emergent rays are inclined to each other at an angle equal to that which is contained between the incident rays.*

Let QA, QC be the incident rays; AB, CD the directions of the refracted rays; and, if possible, let



AB, CD be parallel. Produce QC till it meets FAB in H; then since QH falls upon the parallel lines AB, ED, the angles ECQ and AHC are equal; and the exterior angle FAE, of the triangle AHQ, is greater than the interior and opposite angle AHQ; therefore it

1. aliter –



Let 2 be the focus of incid.t rays – 2ACE, 2BDF the course of two rays thro' the prism – produce EC, FC towards q; produce 2A to meet qE in G – & produce 2B to meet qF in H – Then since the vertical ∠ of the prism, & the ∠s of incid.: are small, ∠ qGA (or whole ∠ of dev.n of 2A = $\frac{m-n}{m} . \angle I$

& ∠ qHB (or whole dev.n of 2B = $\frac{m-n}{m} . \angle I$

∴ ∠ qGA = ∠ 2HA & ∠ qAG = ∠ 2AH

∴ ∠ qGA + ∠ qAG = ∠ 2HA + ∠ 2AH

& the two latter are less than two r.t angles

∴ the two former : . . . . . . two r.t angles –

∴ EC, FD meet in some point q, if produced –

the two ∠s in each of the △s qAG, 2AH have been shewn to be equal ∴ the △s are sim.r

∴ ∠ Gq A = ∠ H2A — QED –

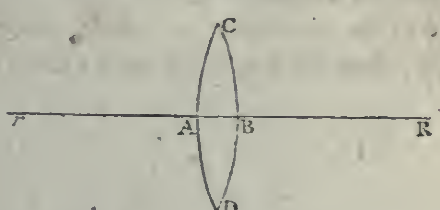
it is alſo greater than the angle ECQ; but, becauſe the angle at the vertex of the priſm is invariable, the angles FAE, ECQ, which are equal to the angles of deviation of the rays QA, QC, are equal to each other (Art. 113), which is abſurd; therefore AB, and CD are not parallel *. Let them meet in F; then ſince the vertical angles FEA, QEC are equal, and alſo the angles FAE, ECQ, the remaining angles AFE, EQC are equal.

119. Def. A *Lens* is a thin piece of glaſs, or other tranſparent ſubſtance, whoſe ſurfaces are either both ſpherical, or one plane and the other ſpherical.

This definition compriſes the ſix following ſorts of lenſes: the *double convex*, the *double concave*, the *plano convex*, the *plano concave*, the *meniſcus*, and the *concavo-convex* lens.

1. A *double convex lens* is bounded by two convex ſpherical ſurfaces.

Let R and r be the centers of two circular arcs CAD, CBD which are concave towards each other,



and which meet in C and D; join Rr, and ſuppoſe the figure CD to revolve about Rr as an axis; the ſolid, thus generated, is called a *double convex lens*.

The line Rr is called the *axis* of the lens.

The

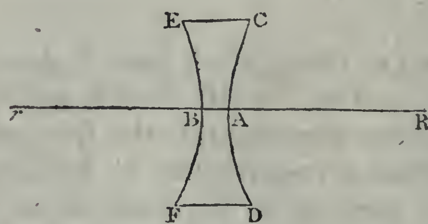
* See alſo Prop. 31.

The figure *CADB* repreſents a ſection of the lens, made by a plane which paſſes through the axis.

If *CD* be joined, this line is called the *diameter*, or *linear aperture* of the lens.

2. A *double concave lens* has both it's ſurfaces concave.

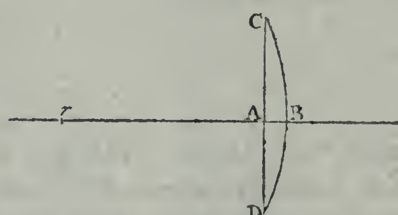
Let *R* and *r* be the centers of two circular arcs which are convex towards each other; join *Rr*; draw



EC, *FD* parallel to *Rr*, and equally diſtant from it; then the ſolid generated by the revolution of the figure *ECDF*, about the axis *Rr*, is called *a double concave lens*.

3. A *plano- convex lens* is bounded by a plane, and a convex ſpherical ſurface.

Let *CBD* be a circular arc whoſe center is *r*, and chord *CD*; draw *rAB* at right angles to *CD*; and the

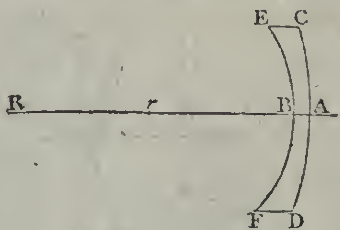


ſolid generated by the revolution of the figure *CD*, about the axis *rB*, is called a *plano-convex lens*.

4. A

6. A *concavo-convex lens* has alſo one ſurface concave and the other convex, but the convex ſurface, which has the leſs curvature, does not, if continued, meet the concave ſurface.

The manner in which the lens is generated is ſuffi-



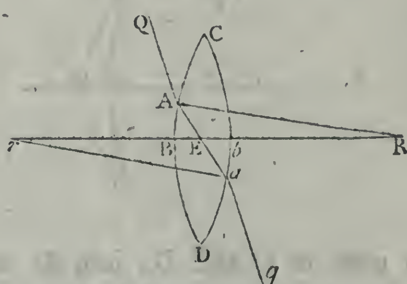
ciently evident from the preceding deſcriptions.

The thickneſs of theſe lenſes is ſuppoſed to be inconſiderable, unleſs the contrary be ſpecified.

Prop. XXVII.

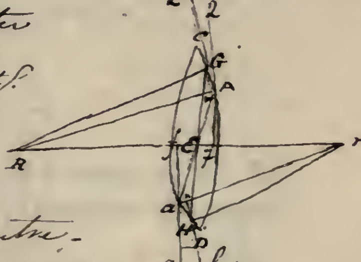
120. *If the radii* RA, ra, *of the ſurface of a lens, be drawn parallel to each other, the incident and emergent parts of a ray of light which paſſes through the lens in the direction* Aa, *will be parallel.*

Join Aa, and ſuppoſe two rays Aa, aA to paſs out



of the lens, in the directions aq, AQ (Art. 29), which are

11. There is a point in every lens thro' wh. if a ray of light passes, the incident & emergent parts shall be ∥ – This point is called the centre of the lens; & it is thus determin'd — Draw the ∥ radii RA, ra, Join Aa & the point in wh. Aa (produced if necessary) cuts the axis is the centre of y^e lens.



Let CD be the lens – Draw the radii RA, ra – Join Aa cutts the axis in E – then E is call'd the centre of the lens –

Now let Aa pass thro' the centre.
Then since RA is ∥ ra, ∠ aAR = ∠ Aar that is, the ∠s of incidence at A, Aa are equal ∴ the ∠s of ref.n are equal ∴ fE is ∥ FQ —

Also if any other ray pass thro' E the incid.t & emergent rays shall be ∥ —

Let GH pass thro' E, LG be the incid.t ray, Hl the emergent ray – join rH, RG – Then LG is ∥ Hl —

By Δs RAE, raE, RA : RE :: ra : rE

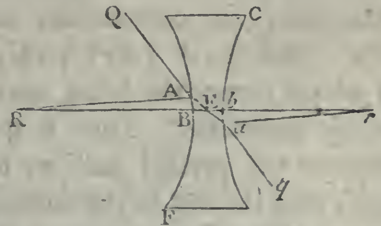
alt.° RA : ra :: RE : rE

or RG : rH :: RE : rE

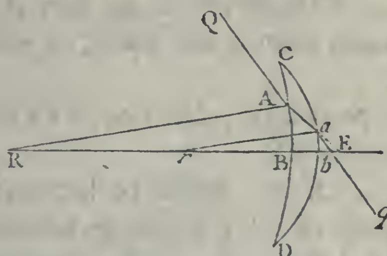
alt.° RG : RE :: rH : rE

and ∠ REG = ∠ rEH ∴ Δs RGE, rEH are sim.r ∴ ∠ EGR = ∠ EHr, that is the ∠s of inc.e are equal ∴ the ∠s of dev.n are equal ∴ ∠ mGE = ∠ nHE ∴ nl is ∥ mL that is LG & Hl are ∥ —

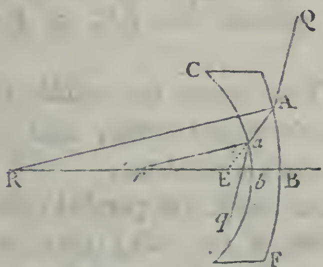
are on opposite sides of *Aa* produced (Art. 33), and in the plane which passes through *RA*, *ra* (Art. 24). Then,



the angles of incidence at *A* and *a* being equal, the angles of deviation are equal; therefore the angles



QAa, *Aaq*, which are the supplements of the angles of



deviation, are equal; and these are alternate angles; consequently, *AQ* and *aq* are parallel.

121. Def. The point *E*, where *Aa*, or *Aa* produced, cuts the axis, is called the *center* of the lens.

 122. Cor.

122. Cor. 1. The center E, of the ſame lens, is a fixed point.

In the ſimilar triangles RAE, raE, $RA : ra :: RE : rE$; therefore, by compoſition or diviſion, $RA \mp ra : ra :: RE \mp rE$ $(Rr) : rE$; the three firſt terms in which proportion being invariable, the fourth, rE, is alſo invariable. Thus it appears, that in whatever manner the parallel radii are drawn, Aa, or Aa produced, cuts the axis in the ſame point.

123. Cor. 2. In the ſame triangles, $AE : aE :: RA : ra$; and when A and a coincide with the axis in B and b, $BE : bE :: RB : rb$.

124. Cor. 3. The center of the lens is nearer to that ſurface which has the leſs radius, or which is the more curved.

For, in the preceding proportion, if rb be leſs than RB, bE is leſs than BE.

125. Cor. 4. If one radius be increaſed without limit, the ſurface to which it belongs becomes plane; and the center of the lens lies in the other ſurface.

For, if RB be indefinitely greater than rb, BE becomes indefinitely greater than bE, or E coincides with b.

126. Cor. 5. The center lies *within* the double convex and double concave lenſes, and *without* the meniſcus and concavo convex lens.

In the two former caſes, the parallel radii RA, ra, lie on oppoſite ſides of the axis; therefore the line which joins the points A and a, cuts the axis. In the two latter caſes, the centers R, r, are on the ſame ſide of the lens; therefore the parallel radii RA, ra lie on the

3. $RA : RE :: rA : rE$

$RB : RE :: rb : rE$

$\therefore RB : rb :: RE : rE$

$RB : rb :: (RB - RE)\, EB : (rb - rE) : Eb$.

127. al. By art. 123, RB : rb :: EB : Eb

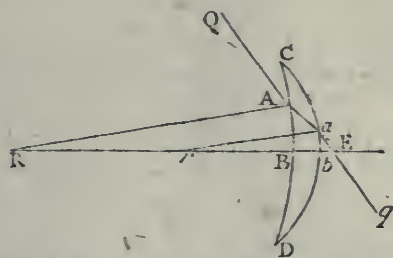
∴ RB : RB - rb :: EB : (EB - Eb) Bb

∴ if RB be infinitely gt. compared with the diff: of the radii, EB is infinitely gt. com=pared with Bb —

the ſame ſide of the axis; conſequently Aa muſt be produced to meet the axis.

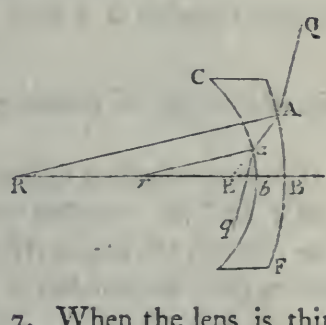
127. Cor. 6. The center of a meniſcus may be at any diſtance from it's ſurface.

In the ſimilar triangles RAE, raE, $RA : ra :: AE : aE$; and by diviſion, $RA : RA - ra :: AE : AE - aE$



(Aa); conſequently, when A and a coincide with B and b, $RB : RB - rb :: BE : Bb$. If, then, the difference of the radii decreaſe with reſpect to one of them RB, the diſtance BE increaſes with reſpect to the thickneſs of the lens; and when RB and rb are equal, BE is indefinitely great.

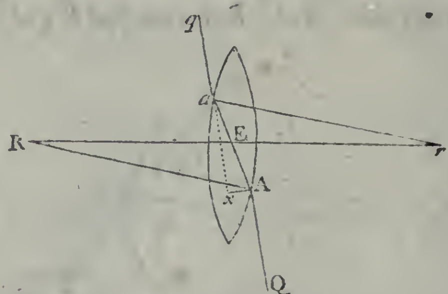
In the concavo convex lens, when R and r coincide, E coincides with them.



128. Cor. 7. When the lens is thin, and the ray QA is incident nearly in the direction of the axis,

QAaq may, without ſenſible error, be conſidered as a ſtraight line.

From *A* draw *Ax* perpendicular to *qa* produced; then, *Ax* : *Aa* :: ſin. *Aax* : rad. and when the angle



of incidence at *a* is ſmall, the angle of deviation, *xaA*, is alſo ſmall; therefore ſin. *xaA* is ſmall with reſpect to the radius; and conſequently, *Ax* is ſmall when compared with the thickneſs of the lens.

129. Cor. 8. The ray *QAaq* may alſo be conſidered as coincident with a line drawn through *E*, parallel to *QA*, when *E* is within, or very near to the lens.

Prop. XXVIII.

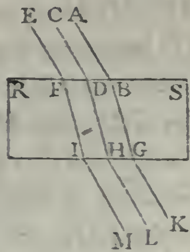
130. *Parallel rays, refracted at a plane ſurface, continue parallel* *.

Caſe 1. When the angles of incidence are in the ſame plane.

Let *RS* be the plane refracting ſurface; *AB*, *CD* the incident, *BG*, *DH* the refracted rays. Then, ſince *AB*, *CD* are parallel, the angles *ABR*, *CDR* are equal; and therefore the complements of theſe, or the angles

* In this, and the following propoſitions, the rays are ſuppoſed to be equally refrangible.

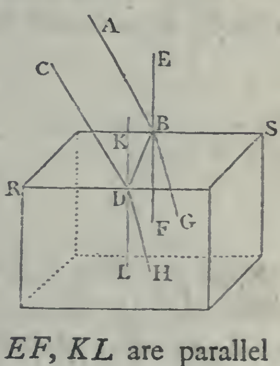
angles of incidence are equal; hence, the angles of refraction are equal, and consequently the complements



of the angles of refraction, that is, the angles *SBG*, *SDH* are equal; and therefore *BG* and *DH* are parallel.

Case 2. When the angles of incidence are not in the same plane.

Let *AB*, *CD* be the incident rays; *EBF*, *KDL* perpendiculars to the refracting surface *RS*, at the points of incidence *B* and *D*; join *BD*; and let *AB* be refracted in the direction *BG*, which lies in the plane *ABF* (Art. 24); also, let *DH* be the intersection of the planes *GBD*, *CDL*.



Then, since *EF*, *KL* are parallel (Euc. 6. 11), as also *AB*, *CD*, by the supposition, the angles of incidence *ABE*, *CDK* are equal (Euc. 10. 11); conse-

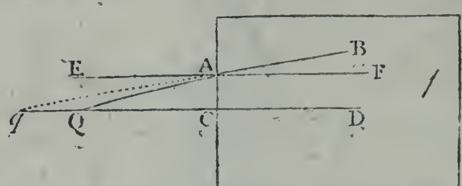
 quently

quently the angles of refraction are equal. Again, becauſe *EF* and *KL* are parallel, and alſo *AB* and *CD*, the planes *ABF*, *CDL* are parallel (Euc. 15. 11); and the plane *GBD* interſects them; hence it follows, that *BG* and *DH* are parallel (Euc. 16. 11); and therefore the angles *GBF* and *HDL* are equal (Euc. 10. 11); but the angle *GBF* is the angle of refraction of the ray *AB*; therefore *HDL* is equal to the angle of refraction of the ray *CD*; and ſince *DH* is in the plane *CDL*, *CD* is refracted in the direction *DH* (Art. 24), which has before been ſhewn to be parallel to *BG*.

Prop. XXIX.

131. *When diverging or converging rays are incident nearly perpendicularly upon a plane refracting ſurface, the diſtance of the focus of incident rays from the ſurface, is to the diſtance of the geometrical focus of refracted rays from the ſurface, as the ſine of refraction to the ſine of incidence.*

Let *AC* be the plane refracting ſurface; *QA*, *QC* two of a pencil of rays diverging from *Q*, of which *QC* is perpendicular to the ſurface, and therefore ſuffers no



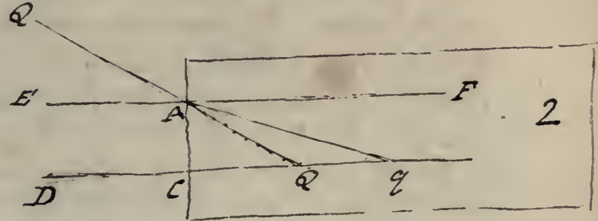
refraction. Through *A*, draw *EAF* parallel to *QC*; and let *QA* be refracted in the direction *AB*; produce *BA* till it meets *CQ*, or *CQ* produced, in *q* *. Then, the

* If they do not meet, *AB* coincides with *AF*; that is, the angle of refraction vaniſhes with reſpect to the angle of incidence; or the refracting power is infinite.

rop. 29. In this there are 4 cases —.

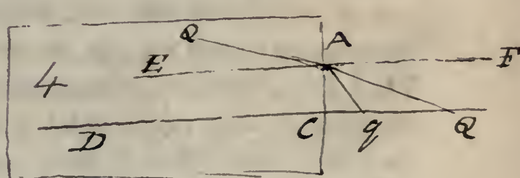
When diverging rays fall on the plane surface of a denser medium. Fig. 1 —

When con[verg]. rays fall ... Fig. 2.

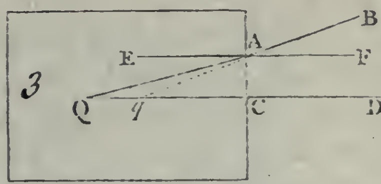


When diverg. rays are incident on the plane surface of a rarer medium — Fig. 3 —

. When conver[g]ing rays are [inci]dent on the plane [sur]face of a rarer medium. Fig. 4 —



the ∠ EAQ is the angle of incidence of the ray QA, and the ∠ BAF the angle of refraction; alſo, the ∠



EAQ is equal to the alternate angle AQC, and the ∠ BAF is equal to the interior and oppoſite angle AqC; therefore, ſin. EAQ, or ſin. incidence, = ſin. AQC = ſin. AQq; and ſin. BAF, or ſin. refraction, = ſin. AqC = ſin. AqQ; hence, $QA : qA$ :: (ſin. AqQ : ſin. AQq ::) ſin. refraction : ſin. incidence. Now, let A approximate to C, and QA is ultimately equal to QC, and qA to qC; therefore, the proportion becomes, $QC : qC$ :: ſin. refraction : ſin, incidence *.

Let q be the focus of rays incident the contrary way; then BAF is the angle of incidence, and EAQ the angle of refraction (Art. 29); and it may be proved as before, that, when Q is the limit of the interſections of the refracted rays and the axis, $qC : QC$ :: ſin. refraction : ſin. incidence.

Ex. 1. When diverging rays paſs out of air into water, according to the hypotheſis made in the propoſition, $QC : qC :: 3 : 4$; and $QC : Qq :: 3 : 1$.

Ex. 2. When diverging rays paſs out of water into air, $QC : qC :: 4 : 3$; and $QC : Qq :: 4 : 1$.

Ex. 3.

* Vid. Note, page 20.

Ex. 3. When converging rays paſs, in the ſame manner, out of air into glaſs, $\mathcal{Q}C : qC :: 2 : 3$; when they paſs out of glaſs into air, $\mathcal{Q}C : qC :: 3 : 2$.

132. Cor. 1. A plane refracting ſurface of a denſer medium, diminiſhes the divergency of diverging rays, and the convergency of converging rays, incident nearly perpendicularly upon it. A plane refracting ſurface of a rarer medium produces the contrary effect.

133. Cor. 2. As the point A approaches to C, q approaches to $\mathcal{Q}$.

Let ſin. incidence : ſin. refraction :: $m : n$; then $\mathcal{Q}A : qA :: n : m$; and $\mathcal{Q}A^2 : qA^2 :: n^2 : m^2$; or $\mathcal{Q}C^2 + CA^2 : qC^2 + CA^2 :: n^2 : m^2$; hence $\mathcal{Q}C^2 + CA^2 : \mathcal{Q}C^2 \sim qC^2 :: n^2 : n^2 \sim m^2$; and ſince the point $\mathcal{Q}$ is fixed, and $\mathcal{Q}C$ invariable, as alſo the ratio of $n^2 : n^2 \sim m^2$, when CA decreaſes, $\mathcal{Q}C^2 + CA^2$ decreaſes; and therefore $\mathcal{Q}C^2 \sim qC^2$ decreaſes; conſequently $\mathcal{Q}q$ decreaſes.

Prop. XXX.

134. *When diverging or converging rays paſs, nearly perpendicularly, through a medium contained by parallel plane ſurfaces, the diſtance of the foci of incident and emergent rays, is to the thickneſs of the medium, as the difference of the ſines of incidence and refraction, to the ſine of incidence upon the firſt ſurface.*

Let $ACcB$ be the medium, $\mathcal{Q}A$, $\mathcal{Q}C$ two rays of a pencil incident upon it, of which $\mathcal{Q}C$ is perpendicular to the ſurface AC, and therefore paſſes through the medium without ſuffering any refraction; let $\mathcal{Q}A$ be

refracted

(134). In this prop.ⁿ also there are 4 cases.

1. When diverg. Rays are incident upon the surface of a denser medium. Fig

2. When Converging Rays are incident on it

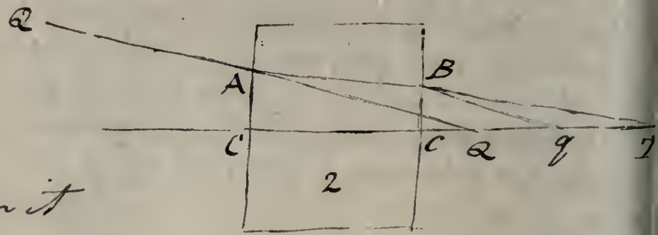
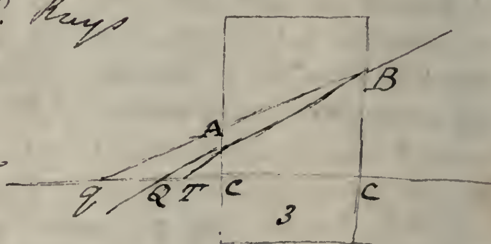


Fig. 2.

3. When diverg. Rays are incident upon the plane Surface of a rarer medium — Fig. 3 —



4. When converging rays are incident on it —

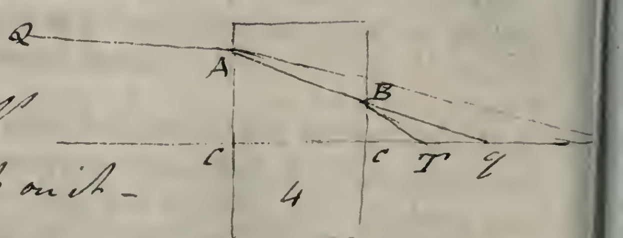
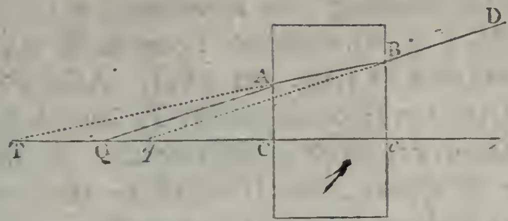


Fig. 4.

refracted in the direction AB, and emergent in the



direction BD. Produce BA and DB, till they meet the axis in T and q.

Then, because QA and qB are parallel (Art. 102), $TA : AB :: TQ : Qq$ (Euc. 2. 6); and because AC is parallel to Bc, $TA : AB :: TC : Cc$; therefore $TQ : Qq :: TC : Cc$; and alternately, $TQ : TC :: Qq : Cc$. Now let A approximate to C, and T is, ultimately, the geometrical focus of the rays after the first refraction; therefore $QC : TC ::$ sin. refraction : sin. incidence (Art. 131); and by division, $QT : TC ::$ sin. incidence $\sim$ sin. refraction : sin. incidence; consequently, $Qq : Cc ::$ sin. incidence $\sim$ sin. refraction : sin. incidence.

If rays, incident the contrary way, converge to q, they will, after both refractions, converge to Q (Art. 29); therefore, as before, $Qq : Cc ::$ sin. incidence $\sim$ sin. refraction : sin. incidence on the first surface.

Ex. If the medium be glass, placed in air, $Qq : Cc :: 1 : 3$; if water, placed in air, $Qq : Cc :: 1 : 4$.

135. Cor. 1. When the incident rays diverge, the geometrical focus of emergent rays is nearer to c, or farther from it than the focus of incident rays, according as $ACcB$ is denser, or rarer than the ambient medium.

Let

Let *ACcB* be denſer than the ambient medium; and let ſin. incidence : ſin. refraction :: $m : n$. Then, $TC : QC :: m : n$; therefore Tc has to Qc a leſs ratio than that of $m : n$ (Alg. Art. 162). Alſo, $Tc : qc :: m : n$ (Art. 131); conſequently, Tc has a leſs ratio to Qc than to qc; or, qc is leſs than Qc. In the ſame manner it appears, that if *ACcB* be rarer than the ambient medium, qc is greater than Qc.

136. Cor. 2. Since *CT* is greater, or leſs than *CQ*, according as *ACcB* is denſer, or rarer than the ambient medium (Art. 131), it is manifeſt that *T* and *q* lie on oppoſite ſides of *Q*.

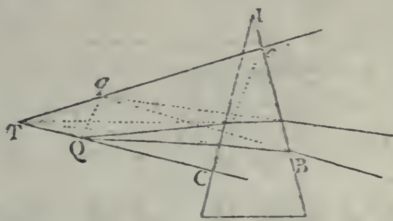
137. Cor. 3. When *ACcB* is denſer than the ſurrounding medium, as *A* approaches to *C*, *T* approaches to *Q* (Art. 133); and in conſequence of this motion of *T*, *q* approaches to *c*. Again, as *A* approaches to *C*, *B* approaches to *c*; and on this account *q* approaches to *T* (Art. 133), or recedes from *c*. Thus the two motions of *q* are in oppoſite directions, and the aberration of oblique rays, from the geometrical focus, is leſs than when the rays are refracted at a ſingle ſurface. The ſame may be ſhewn when *ACcB* is rarer than the ambient medium.

Prop. XXXI.

138. *Having given the focus of incidence of a pencil of rays which paſſes nearly perpendicularly through the ſides of a priſm, and alſo the ratio of the ſine of incidence to the ſine of refraction, out of the ambient medium into the priſm, to find the focus of emergent rays.*

Let *CIB* be the priſm; *Q* the focus of incident rays; take $m : n$ :: ſin. incidence : ſin. refraction out of

of the ambient medium into the prifm. From Q, draw QC perpendicular to IC; and in CQ, or CQ



produced, take $TC : QC :: m : n$; then will T be the focus after refraction at the furface IC (Art. 131), or the focus of rays incident upon the furface IB. From T, draw Tc perpendicular to IB, and in cT, or cT produced, take $qc : Tc :: n : m$; and q will be the focus of emergent rays (Art. 131).

139. Cor. Since $QC : TC :: n : m :: qc : Tc$, if Cc and Qq be joined, thefe lines are parallel (Euc. 2. 6); and therefore $Qq : Cc :: TQ : TC :: m \sim n : m$.

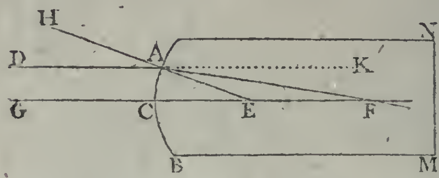
Prop. XXXII.

140. *Parallel rays, refracted at a convex fpherical furface of a denfer, or a concave of a rarer medium, into which they pafs, are made to converge; and refracted at a concave fpherical furface of a denfer, or convex of a rarer medium, they are made to diverge.*

1. Let DA, GC be two rays of a parallel pencil, paffing out of a rarer medium into a denfer, and incident upon the convex fpherical furface ACB, whofe center is E. Let GCE pafs through the center of the furface, and it fuffers no refraction. Join EA, and produce it to H; alfo, produce DA to K; and let DA be refracted in the direction AF; then, DAH

is

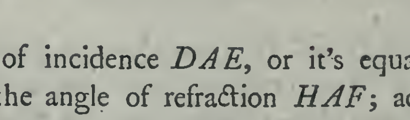
is the angle of incidence, and *EAF* the angle of refraction of this ray; and ſince it paſſes out of a rarer



medium into a denſer, the ∠ *EAF* is leſs than the ∠ *HAD*, and therefore it is leſs than the ∠ *KAE*; add to each the ∠ *AEF*, and the two angles *FAE*, *AEF* are together leſs than the two *KAE*, *AEF*; and therefore they are leſs than two right angles (Euc. 29. 1); conſequently, *AF*, and *GE*, if produced, will meet.

2. When the rays paſs out of a denſer medium into a rarer, and the ſurface of the medium into which they are refracted is ſpherically concave.

The conſtruction being made as before, ſince the ray *DA* paſſes out of a denſer medium into a rarer,



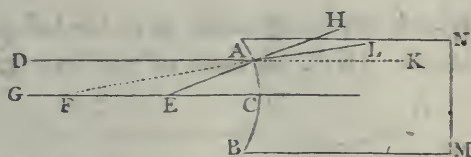
the angle of incidence *DAE*, or it's equal *AEC*, is leſs than the angle of refraction *HAF*; add to each the angle *EAF*, and the two *EAF*, *AEF*, are together leſs than the two *EAF*, *HAF*; that is, they are together leſs than two right angles; therefore *AF* and *EC*, if produced, will meet.

3. When

3. When the rays paſs out of a rarer medium into a denſer, and the ſurface of the medium into which they are refracted is ſpherically concave.

The ſame conſtruction being made, let *DA* be refracted in the direction *AL*, and produce *LA* to *F*.

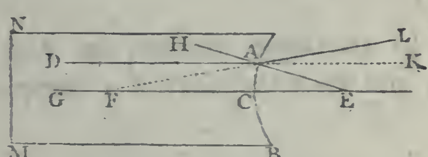
Then, ſince the ray *DA* paſſes out of a rarer medium into a denſer, the ∠*DAE* is greater than the ∠*HAL*,



or *FAE*; add to each the ∠ *AEG*, and the two *FAE*, *AEG*, are together leſs than the two *DAE*, *AEG*; that is, they are leſs than two right angles; therefore *AF* and *EG* will meet.

4. When the rays paſs out of a denſer medium into a rarer, and the ſurface of the medium into which they are refracted is ſpherically convex.

In this caſe, as before, the ∠ *DAH*, or it's equal *AEC*, is leſs than the ∠ *EAL*; add to each the ∠



EAF, and the two *EAF*, *AEC*, are together leſs than the two *EAF*, *EAL*; that is, they are leſs than two right angles; therefore *AF* and *EC*, if produced, will meet.

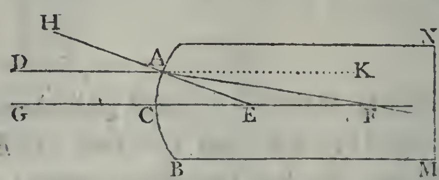
In the two laſt caſes, the refracted rays meet if produced backwards; that is, they diverge.

PROP.

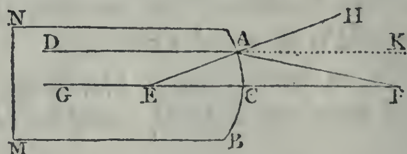
Prop. XXXIII.

141. *When parallel rays are incident nearly perpendicularly upon a ſpherical refracting ſurface, the diſtance of the geometrical focus of refracted rays from the ſurface, is to it's diſtance from the center, as the ſine of incidence to the ſine of refraction.*

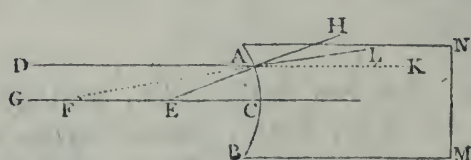
The conſtruction being made as in the laſt propoſition, the angle AEF, in each of the four caſes, is



either equal to the angle of incidence of the ray DA, or to it's ſupplement; therefore ſin. $\angle AEF$ = ſin. in-



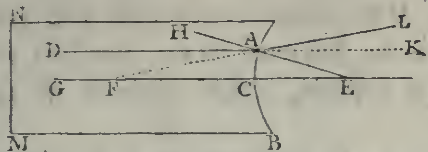
cidence. In the ſame manner, ſin. $\angle EAF$ = ſin. refraction; and ſince the ſides of a triangle are pro-



portional to the ſines of the oppoſite angles, $FA : FE ::$ ſin. $\angle AEF$: ſin. $\angle EAF$ :: ſin. incidence : ſin. refraction. Now let the point A approximate to C, and

FA

FA is, ultimately, equal to *FC*; therefore, the pro-



portion becomes *FC* : *FE* :: ſin. incidence : ſin. refraction.

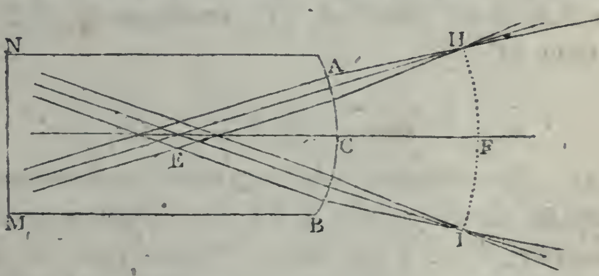
142. Cor. 1. Since *FC* : *FE* :: ſin. incidence : ſin. refraction, by diviſion, *FC* : *CE* :: ſin. incidence : ſin. incidence ∼ ſin. refraction. Alſo, *FE* : *CE* :: ſin. refraction : ſin. incidence ∼ ſin. refraction.

Ex. 1. If parallel rays paſs out of air into the medium *ABMN* of glaſs, *FC* : *FE* :: 3 : 2; alſo, *FC* : *EC* :: 3 : 1; and *FE* : *EC* :: 2 : 1.

Ex. 2. When the rays paſs out of glaſs into air, *FC* : *FE* :: 2 : 3; alſo, *FC* : *EC* :: 2 : 1; and *FE* : *EC* :: 3 : 1.

Cor. 2. If *EC* be diminiſhed whilſt the refracting power remains unaltered, *FC* is alſo diminiſhed.

143. Cor. 3. If the axes of ſeveral pencils of parallel



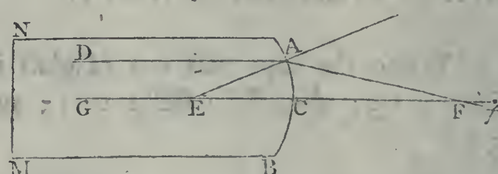
rays be inclined to each other, in the ſame plane, the

the foci of refracted rays will lie in the circumference of a circle, *HFI*, whose center is *E*, and radius *EF*. If the axes be in different planes, the foci will lie in the surface of a sphere, whose center is *E*, and radius *EF*.

144. Cor. 4. If any point *H*, in the arc *HFI*, be the focus of rays incident the contrary way, join *HE*, and those rays of the pencil which are incident nearly perpendicularly, will be refracted parallel to each other, and to *HE* (Art. 29).

145. Cor. 5. The distance *EF*, of the intersection of the refracted ray and the axis, from the center, is the greatest, when the arc *AC* is evanescent.

Let f be the geometrical focus; $m : n$ the ratio of the sine of incidence to the sine of refraction.



Then, $Ef : EC :: n : m \sim n$; also, $EF : AF :: n : m$; therefore $EF : EF \sim AF :: n : m \sim n$; hence $Ef : EC :: EF : EF \sim AF$; but $EF \sim AF$ is less than EA, or EC (Euc. 20. 1); consequently, EF is less than Ef.

Prop. XXXIV.

146. *If parallel rays be incident, nearly perpendicularly, in opposite directions, upon a spherical refracting surface, the distance of one of the foci of refracted rays from the surface, is equal to the distance of the other from the center of the refractor.*

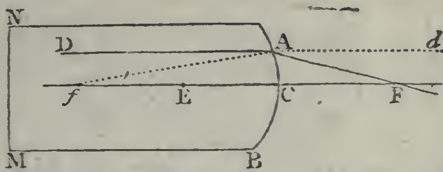
Let

147. Prop. 35 – The most useful proport^n. is that contain'd in the 3^rd. Cor. of this Prop. i.e.

"The distance of the focus of incident rays from the principal focus of rays conv^g. in the contrary direct^n. : the dist of this focus from the centre of the refractor :: distance of the centre of the refractor from the principal focus of rays in the same direct^n. : to the dist^ce of this focus from the geometrical focus of refr-
-ted rays, or $QF : FE :: Ef : fq$ –

– QA will be refracted in the same manner whether it proceed from Q or G – on the latter suppos^n. it will be refracted // GE ∴ Aq is // GE – Again ~~little~~ f the principal focus of rays incid^t. in the same direct^n. with QA. With centre E and rad: Ef describe the arc fg. join gE – Then the ray qA will be refracted in the same manner whether it proceed from q or g – on the latter suppos^n. it will be refracted // gE ∴ QA is // gE. Hence $\angle GQE = \angle gEq$ & $\angle GEQ = \angle gqE$ ∴ Δs GQE, GEq

Let F be the focus when the rays paſs out of the denſer medium into the rarer, and f the focus when

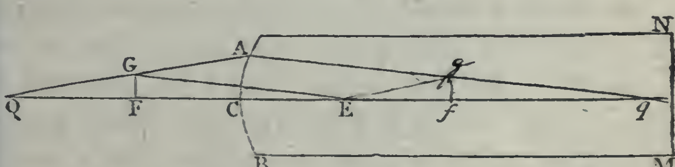


they paſs out of the rarer into the denſer; then $FC : CE :: n : m-n$ (Art. 142); alſo, $fE : CE :: n : m-n$; therefore $FC : CE :: fE : CE$; or $FC = fE$. By adding EC to, or ſubtracting it from each of theſe equal quantities, $FE = fC$.

Prop. XXXV.

147. *When diverging or converging rays are incident nearly perpendicularly upon a ſpherical refracting ſurface, the diſtance of the focus of incident rays from the principal focus of rays coming in the contrary direction, is to it's diſtance from the center of the refractor, as it's diſtance from the ſurface, to it's diſtance from the geometrical focus of refracted rays.*

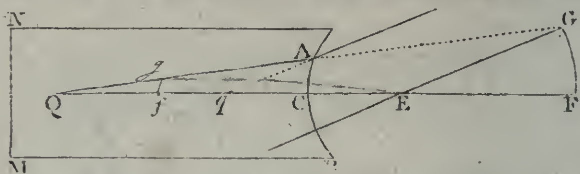
Let ACB be the refracting ſurface; E it's center; Q the focus of incident rays; QA and QC two rays of



the pencil, of which QCE paſſes through the center, and therefore ſuffers no refraction. Take F the prin-

 cipal

cipal focus of rays incident in the contrary direction, parallel to EC; and from the center E, with the



radius EF, describe the arc FG, cutting QA, or QA produced, in G; join EG; draw Aq parallel to EG, and let it meet the axis in q.

1. When diverging rays are incident upon a convex spherical refractor of a denser medium.

The ray QA will be refracted at A, in the same manner and degree, whether it be considered as one of a pencil of rays diverging from Q, or as one of a pencil diverging from G; and, on the latter supposition, it will be refracted parallel to GE (Art. 144); therefore Aq is the refracted ray *. And since the triangles QGE, QAq are similar, $QG : QE :: QA : Qq$; let the point A approximate to C, that the ray QA may be incident nearly perpendicularly upon the refracting surface, and the point G approximates to F; therefore ultimately, $QF : QE :: QC : Qq$.

2. When diverging rays are incident upon a convex spherical surface of a rarer medium.

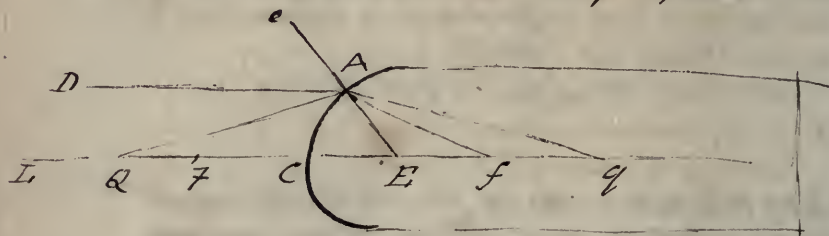
In this case, Q and F are on contrary sides of the refracting surface (Art. 140); and the ray QA will be refracted in the same manner and degree, whether it be considered as one of a pencil of rays diverging from Q, or, as one of a pencil converging to G; consequently,

* This is only true when GA is incident nearly perpendicularly upon AC; and therefore the intersection of the refracted ray and the axis is only determined in that case.

In this prop. there are eight cases —

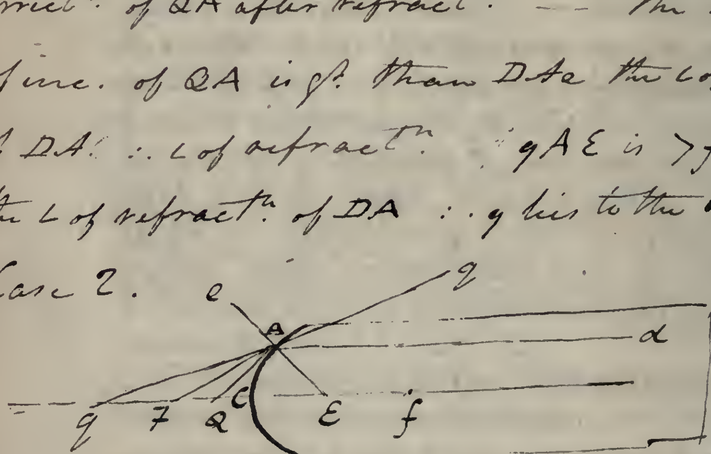
(151). The situations of Q and q may be thus determin'd.

Case. 1. Let Q lie in FL indef.ly prod.d tow.ds



DA a || ray, Af the refracted ray, QA.A two of a pencil of diverging rays. Aq the direct.n of QA after refract.n — The ∠ QAe of inc. of QA is g.r than DAe the ∠ of inc. of DA ∴ ∠ of refract.n qAE is > fAE the ∠ of refract.n of DA ∴ q lies to the R.t of f.

Case 2.



When Q is between F and C — Sin incid. of QA i.e. sin. of ∠ QAe is g.r than sin. of incid. of FA, i.e than sin. ∠ FAe ∴ ∠ of refract.n of QA is g.r than ∠ of refr.n of FA, i.e ∠ qAe is > ∠ dA ∴ (since dA is || QE) qA prod.d backwards.

quently, qA, produced, is the refracted ray (Art. 144). Hence, as before, $QG : QE :: QA : Qq$; and ultimately, $QF : QE :: QC : Qq$. A similar proof is applicable in all the other cases *.

148. Cor. 1. From the same triangles, $QG : GE :: QA : Aq$; therefore, ultimately, $QF : FE :: QC : Cq$.

149. Cor. 2. Since GE is parallel to Aq one side of the triangle QAq, the other sides QA, Qq, or those sides produced, are cut proportionally (Euc. 2. 6); therefore $QG : GA :: QE : Eq$; and ultimately, $QF : FC :: QE : Eq$.

150. Cor. 3. If f be the other principal focus, and q the focus of incident rays, Q is the focus of refracted rays (Art. 29); therefore, $qf : fE :: qC : QC$ (Art. 148); invertendo, $fE : qf :: QC : Cq :: QF : FE$; hence, $QF : FE :: Ef : fq$.

Prop. XXXVI.

151. *The distances* QF *and* Qq *must be measured in the same, or opposite directions from* Q, *according as* QC *and* QE *are measured in the same, or opposite directions from that point.*

Since $QF : QE :: QC : Qq$, we have $QF \times Qq = QE \times QC$; and, measuring these lines from Q, if the rectangles have the same sign in any one case, they will always have the same sign. Now, if QF be very great when compared with FE or EC, qf is very small (Art. 150); therefore all the lines QF, QC, QE, Qq, are measured the same way from Q, and

* Since the incident rays may either *converge* or *diverge*, and fall upon a *convex* or *concave* surface of a *rarer* or *denser* medium, the proposition admits of eight cases.

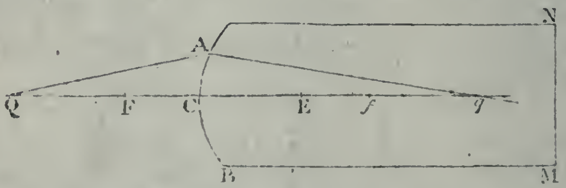
and the rectangles $\mathcal{Q}F \times \mathcal{Q}q$, and $\mathcal{Q}E \times \mathcal{Q}C$, in this caſe, have the ſame ſign; conſequently, they will always be either both poſitive, or both negative; and according as $\mathcal{Q}C$ and $\mathcal{Q}E$ have the *ſame*, or *different* ſigns, $\mathcal{Q}F$ and $\mathcal{Q}q$ muſt have the *ſame*, or *different* ſigns; that is, $\mathcal{Q}F$ and $\mathcal{Q}q$ muſt be meaſured, from $\mathcal{Q}$, in the *ſame*, or *different* directions, according as $\mathcal{Q}C$ and $\mathcal{Q}E$ are meaſured in the *ſame*, or *different* directions (Alg. Art. 473).

152. Nearly in the ſame manner, it may be ſhewn that $\mathcal{Q}F$ and fq muſt always be meaſured in oppoſite directions from F and f.

Prop. XXXVII.

153. *The conjugate foci,* Q *and* q, *move in the ſame direction upon the indefinite line* QCq, *and they coincide at the ſurface and center of the refractor.*

Let the rays be incident nearly perpendicularly on ACB, a convex ſpherical refracting ſurface of a denſer,



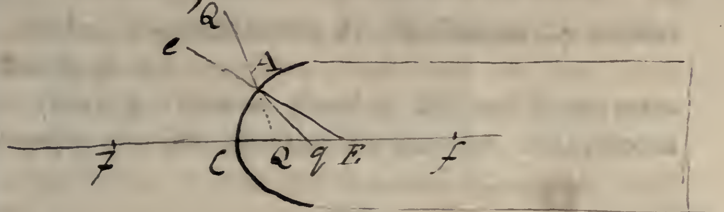
medium; take f the principal focus of rays, thus incident, and F the other principal focus.

When the incident rays are parallel, the refracted rays converge to f.

As $\mathcal{Q}$ approaches towards F, ſince $\mathcal{Q}F : FE :: \mathcal{Q}C : Cq$ (Art. 148), and the ratio of $\mathcal{Q}F : \mathcal{Q}C$ decreaſes (Alg. Art. 163), the ratio of $FE : Cq$ decreaſes; therefore Cq increaſes.

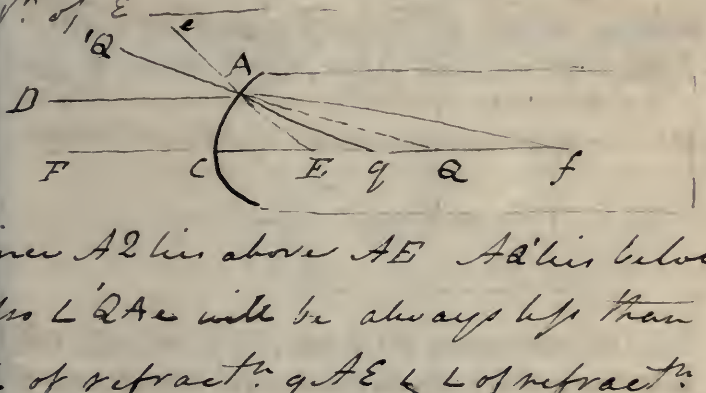
When

Case. 3. Let Q be between C & E —



∠ of refract^n. qAE must be less than ∠ of incidence QAE and both lie on same side of the ⊥r AE ∴ q lies between and E, or q lies to the left of f.

Case. 4. Let Q lie any where to the r. of E —

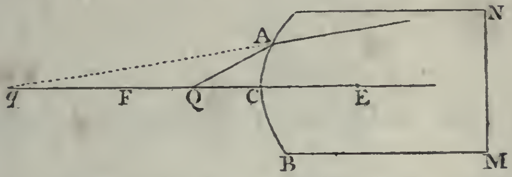


Since AQ lies above AE AQ' lies below Ae — also ∠'QAe will be always less than ∠DAe ∠ of refract^n. qAE < ∠ of refract^n. fAE q lies between E & f ∴ q lies to the left of f — that is, Q and q lie on diff^t. sides of F & f — — — ———

Knowing the situations of Q and q with respect to F and f, it may be shewn that Q & q move in the same direct^n. by means of the proport^n. QF : FE :: Ef : fq — —— —

When Q coincides with F, the diſtance Cq is indefinitely great.

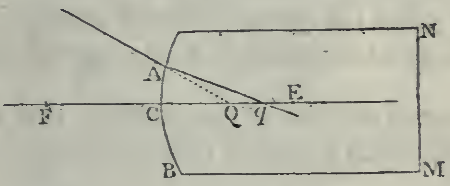
When Q is between F and C, QF and Qq are meaſured in the ſame direction from Q (Art. 151); and as



the ratio of QF to QC increaſes, the ratio of FE to Cq increaſes, or Cq decreaſes.

When Q coincides with C, the ratio of $QF : FE$ is finite; therefore the ratio of $QC : Cq$ is finite, and ſince QC vaniſhes, qC alſo vaniſhes; that is, q coincides with C.

When Q is between C and E, Qq muſt be meaſured from Q towards E (Art. 151); and ſince $QF : QE ::$

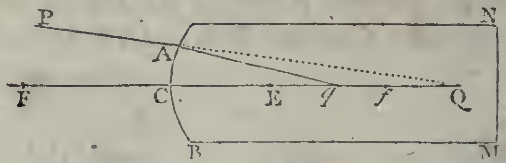


$QC : Qq$ (Art. 147), and QF is greater than QC, QE is greater than Qq; conſequently, q lies between Q and E.

When Q coincides with E, ſince QF is equal to FE, QC is equal to Cq; or, q coincides with E.

When Q is in CE produced, Qq muſt be meaſured from Q towards C; and ſince $QF : QE :: QC : Qq$, and QF is greater than QC, QE is greater than Qq; that is, q lies between Q and E. Alſo, ſince $QF : FE ::$

$\mathcal{Q}C : Cq$ (Art. 184), and as $\mathcal{Q}C$ or $\mathcal{Q}F$ increaſes, the ratio of $\mathcal{Q}F$ to $\mathcal{Q}C$ decreaſes (Alg. Art. 162), the



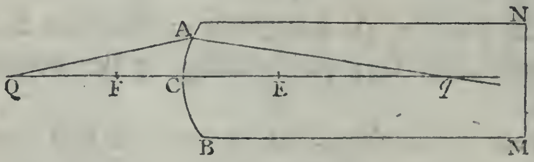
ratio of FE to Cq alſo decreaſes; that is, Cq increaſes.

The ſame demonſtration, mutatis mutandis, may be applied to all the other caſes.

Prop. XXXVIII.

154. *A convex ſpherical refracting ſurface of a denſer, and a concave of a rarer medium, diminiſh the divergency, or increaſe the convergency of all pencils of rays incident nearly perpendicularly upon them, unleſs the focus of incident rays be in the ſurface or center of the refractor, or between thoſe two points; a concave ſpherical ſurface of a denſer, and convex of a rarer medium, have a contrary effect.*

It appears from the laſt propoſition, that when rays are incident upon a convex ſpherical ſurface of a denſer



medium diverging from $\mathcal{Q}$, a point farther from the ſurface than F, they are made to converge.

When the incident rays diverge from F, the refracted rays are parallel.

When

When *Q* is between *F* and *C*, the refracted rays diverge from a point which is farther from the surface than *Q*; therefore the divergency of the rays is diminished *.

When the incident rays converge to any point in *CE* produced, the refracted rays converge to a point which is nearer to the surface than the focus of incident rays; or the convergency is increased.

But when the incident rays converge to a point between *C* and *E*, the refracted rays converge to a point farther from the surface than the focus of incident rays; or the convergency is diminished.

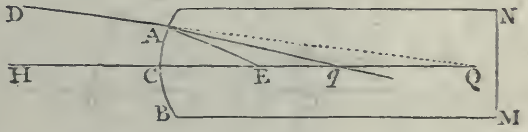
When *Q* coincides with *C* or *E*, the convergency, or divergency, is not altered.

In the same manner, the proposition may be proved in the other cases.

PROP. XXXIX.

155. *If* E *be the center of a spherical refractor* ACB, *and, in* CE *produced,* QE *be taken to* EC :: *sin. incidence : sin. refraction, all the rays converging to* Q, *when the refracting surface is convex, and diverging from* Q, *when that surface is concave, will, after refraction, converge to, or diverge from, one point.*

When the refracting surface is convex, let *DA* and



HC be two rays of the pencil, of which *HC* passes through

* Vid. Art. 69.

through the center, and therefore ſuffers no refraction; and let DA be refracted in the direction Aq; join EA.

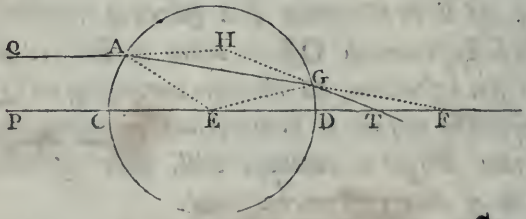
Then, ſince ſin. incidence : ſin. refraction :: QE : EA :: ſin. $\angle QAE$: ſin. $\angle AQE$, and the $\angle QAE$ is equal to the angle of incidence, the angle AQE is equal to the angle of refraction, that is, to the $\angle EAq$; alſo, the $\angle AEQ$ is common to the two triangles QAE, AqE; therefore theſe triangles are ſimilar, and $QE : EA :: EA : Eq$, the three firſt terms of which proportion being invariable, the fourth, Eq, is alſo invariable; that is, all the refracted rays meet the axis in the ſame point.

The propoſition may be proved in the ſame manner when the refracting ſurface is concave.

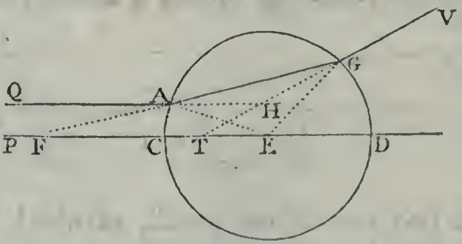
Prop. XL.

156. *To find the principal focus of a ſphere.*

Let a pencil of parallel rays be incident upon the



ſphere ACD, whoſe center is E; and let PCE be that



ray which paſſes through the center, and therefore ſuffers

ſuffers no refraction at either ſurface (Art. 27); alſo, let QA, any other ray of the pencil, be refracted in the direction AG, and emergent in the direction GT, or GV, which, produced backwards, or forwards, as the caſe may require, cuts the axis in T. Produce QA, and TG, or VG, till they meet in H; join EA, EG.

Then, if two rays GA, AG paſs out of the ſphere at A and G, the angles of incidence EAG, EGA are equal, and therefore the angles of deviation are equal; but the deviation at A is the ſame, whether GA or QA be the incident ray (Art. 30); conſequently, when the ray QA is refracted through the ſphere, the deviation at A is equal to the deviation at G; or, the $\angle HAG =$ the $\angle FGT$. Alſo, the $\angle HAG =$ the $\angle GFT$; therefore the $\angle GFT =$ the $\angle FGT$, and $FT = TG$. Now, let the point A approximate to C, and F is, ultimately, the principal focus of rays after the firſt refraction; alſo, the point G approximates to D*; conſequently, TG is ultimately equal to TD, and therefore $FT = TD$; that is, the principal focus biſects the diſtance between the focus after the firſt refraction, and the farther extremity of the diameter in the direction of which the rays are incident.

157. Cor. 1. Since $2TD = FD$, and $2DE = CD$, we have $2TD \mp 2DE = FD \mp CD$; that is, $2TE = FC$.

158. Cor. 2. Since $FC : CE ::$ ſin. incidence : ſin. incidence $\sim$ ſin. refraction (Art. 142), we have $2TE : CE ::$ ſin. incidence : ſin. incidence $\sim$ ſin. refraction;

* Otherwiſe, F would coincide with C, which cannot be the caſe ſo long as the refracting power is finite (Art. 141).

refraction; or, $TE : CE ::$ sin. incidence : 2 sin. incidence ∼ 2 sin. refraction.

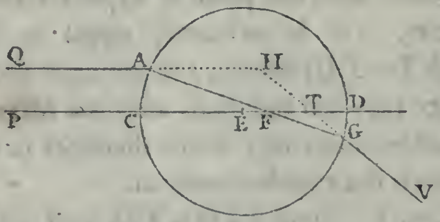
The distance TE is called the *focal length* of the sphere.

Ex. 1. If the sphere be glass, placed in air, sin. incidence : sin. refraction :: 3 : 2; therefore $TE : CE ::$ 3 : 2.

Ex. 2. If the sphere be water, placed in air, sin. incidence : sin. refraction :: 4 : 3; and $TE : CE ::$ 4 : 2 :: 2 : 1.

Ex. 3. If sin. incidence : sin. refraction :: 2 : 1, $TE : CE ::$ 2 : 2; or T coincides with D.

Ex. 4. If the sine of incidence be to the sine of refraction in a greater ratio than that of 2 : 1, T falls



within the sphere; and the rays, after the second refraction, diverge from T.

159. Cor. 3. If the axes of different pencils of parallel rays, incident upon the sphere, lie in the same plane, the foci will lie in the circumference of a circle whose center is E, and radius ET.

160. Cor.

160. Cor. 4. If T be the focus of rays incident nearly perpendicularly upon the ſphere, in the contrary direction, theſe rays will be refracted parallel to each other, and to TE.

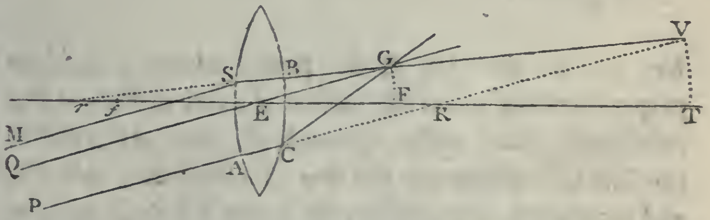
161. Cor. 5. If the radius of the ſphere, and it's focal length be known, the ratio of the ſine of incidence to the ſine of refraction may be determined.

Let $m : n ::$ ſin. incidence : ſin. refraction; then, $2TE : EC :: m : m \sim n$; therefore $2TE : 2TE \mp EC :: m : n$; where the negative ſign is to be uſed when the rays converge after the firſt refraction, and the poſitive ſign, when they diverge.

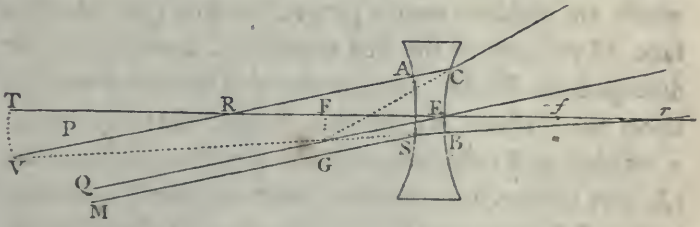
PROP. XLI.

162. *To find the principal focus of a lens whoſe thickneſs is inconſiderable.*

Let AB be a lens, whoſe axis is rT, and center E;

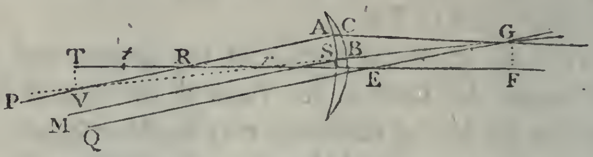


R and r the centers of the ſurfaces A and B; PA, QE,

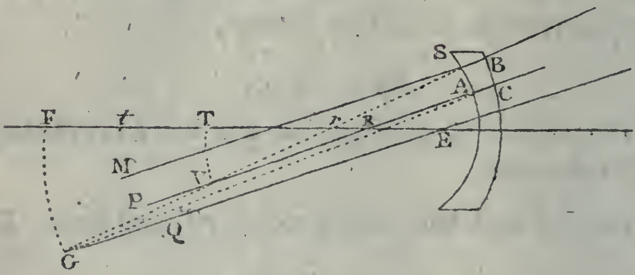


MS a pencil of parallel rays incident upon it; of which

which *QE* paſſes through the center, and may therefore be conſidered as proceeding in that direction after



the ſecond refraction (Art. 129); conſequently, the focus of emergent rays will be in *QE*, or *QE* produced.



Let *PA* be that ray of the pencil which is incident perpendicularly upon the ſurface *A*; and in *PA*, or *PA* produced, which paſſes through *R*, take *AV* : *RV* :: the ſine of incidence : the ſine of refraction; join *Vr*, and produce it, if neceſſary, till it cuts *QE* in *G*, and the ſurface *CB* in *B*. Then, all the rays in the pencil *MSAP*, which are incident nearly perpendicularly upon the ſurface *AS*, will, after the firſt refraction, converge to, or diverge from *V* (Art. 141), and in this ſtate they will fall upon the ſurface *B*; of this pencil, that ray which is incident at *B* coincides with the direction of the radius *rB*, and is therefore incident perpendicularly upon the ſurface *B*; conſequently, it will proceed in the direction *SB* (Art. 27); and the focus of emergent rays will be ſomewhere

ſomewhere in the line BSV, or BSV produced. The focus will alſo, as was before obſerved, be ſomewhere in QEG; therefore G, the interſection of the two lines, QEG and BSV, is the focus of emergent rays.

Now, ſince RV is parallel to EG, the triangles RVr and EGr are ſimilar; and $Rr : RV :: Er : EG$; alternately, $Rr : Er :: RV : EG$, the diſtance of the principal focus from the center, or the *focal length* of the lens *.

163. Cor. 1. Since $RA \mp rB : rB :: Rr : Er$ (Art. 122), we have alſo, $RA \mp rB : rB :: RV : EG$.

164. Cor. 2. If the inclination of the pencil, to the axis of the lens, be continually diminiſhed, the principal focus G will deſcribe the circular arc GF, whoſe center is E, and radius EG.

For, ſince $AV : RV ::$ ſin. incidence : ſin. refraction :: $m : n$, we have $AR : RV :: m \sim n : n$; and, becauſe AR, m and n are invariable, RV is alſo invariable. Again, $Rr : Er :: RV : EG$; and, ſince the three firſt terms in this proportion are invariable, the fourth, EG, is alſo invariable.

165. Cor. 3. If any point G, in the arc FG, be the focus of rays incident in the contrary direction, theſe rays will emerge parallel to each other and to GE (Art. 29).

166. Cor.

* It is neceſſary to obſerve, that we only determine the ultimate interſection of the rays, when they are incident nearly perpendicularly on each ſurface; that is, when their inclination to the axis of the lens is diminiſhed without limit. The concluſion is, however, nearly true, when they are inclined at a *ſmall*, though *finite* angle to that axis.

QEG is alſo ſuppoſed to be a ſtraight line, paſſing through E; this cannot be allowed, unleſs E be within, or very near to the lens. Vid. Arts. 127. 128.

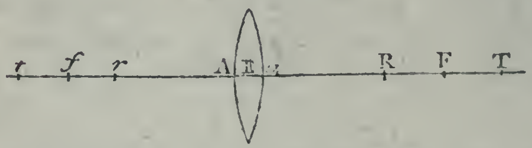
166. Cor. 4. It appears from the conſtruction of the figures, that parallel rays are made to converge, by a double convex lens, a plano convex lens, and a meniſcus, of greater denſity than the ſurrounding medium. And that they are made to diverge, by a double concave, a plano concave, and a concavo convex lens, of the ſame deſcription *.

167. Cor. 5. If R and r be the radii of the ſurfaces, $m : n$ the ratio of the ſine of incidence to the ſine of refraction at the firſt ſurface, the focal length of the lens is $\frac{Rr}{R \mp r} \times \frac{n}{m-n}$.

For, $R \mp r : r :: RV : EG$ (Art. 163); and $m-n : n :: R : RV$ (Art. 142); therefore, by compounding theſe proportions, $\overline{m-n} . \times \overline{R \mp r} : nr :: R : EG$; and $EG = \frac{Rr}{R \mp r} \times \frac{n}{m-n}$.

168. Cor. 6. The diſtance of the principal focus from the center is the ſame on each ſide of the lens.

Let F be the principal focus when the rays are inci-



dent in the direction tA; f the principal focus when they are incident in the contrary direction.

Then,

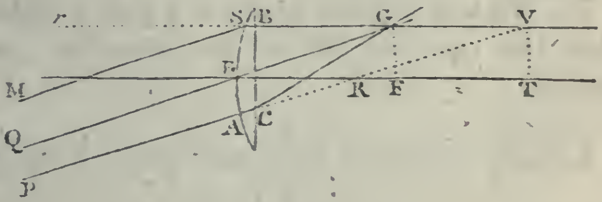
* Lenſes are always ſuppoſed to be denſer than the ſurrounding medium, unleſs the contrary be ſpecified.

Then, ſince m and n are the ſame in both caſes, and R and r are alike concerned in the expreſſion $\frac{Rr}{R \mp r}$, the value of $\frac{Rr}{R \mp r} \times \frac{n}{m-n}$ is the ſame, whether the rays are firſt incident on the ſurface A, or on a; that is, $EF = Ef$.

169. Cor. 7. If the radii of the ſurfaces be given, the focal length varies as $\frac{n}{m-n}$.

170. Cor. 8. If one ſurface be plane, the focal length of the lens is equal to the focal length of the other ſurface.

When the diſtance of r is indefinitely increaſed, Vr



is parallel to ER, and the figure $GERV$ becomes a parallelogram; therefore $EG = RV = R \times \frac{n}{m-n}$.

171. Ex. 1. The focal length of a double convex, or double concave glaſs lens, whoſe ſurfaces are equally curved, is equal to the radius of either ſurface.

In this caſe, $m : n :: 3 : 2$; and $m - n : n :: 1 : 2 :: AR : RV$ (Fig. Art. 162); hence $RV = 2AR$. Alſo, $RA + rB : rB :: 2 : 1 ::$ ($RV : EG ::$) $2AR : EG$; conſequently, $EG = AR$.

Ex. 2.

Ex. 2. If one ſurface of a glaſs lens be plane, the focal length is equal to the diameter of the other ſurface.

Here, $EG = RV$ (Art. 170); that is, $EG = 2AR$.

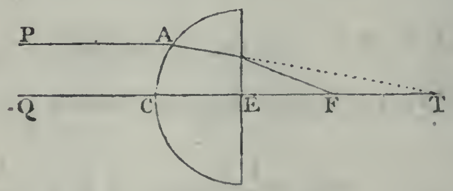
Prop. XLII.

172. *To determine the focal length of a lens whoſe thickneſs is conſiderable.*

Find the focus after the firſt refraction by Prop. 33, and this point being the focus of rays incident upon the ſecond ſurface, the focus of emergent rays may be found by Prop. 29. or 35.

Ex. To find the focal length of a glaſs hemiſphere, when the rays are firſt incident upon the convex ſurface.

Let E be the center of the hemiſphere, T the focus



after the firſt refraction, F the focus of emergent rays.

Then, $TE : EC :: 2 : 1$ (Art. 142);

and $FE : TE :: 2 : 3$ (Art. 131);

hence, $FE : EC :: 4 : 3$.

174. Cor. 1. From the same similar triangles, $QG : GE :: QH : Hq$; therefore, ultimately, $QF : FE : QE : Eq$.

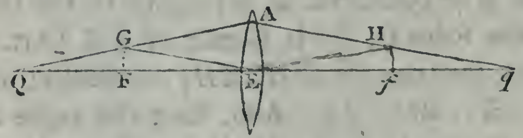
175. Cor. 2. If f be the principal focus of rays incident in the direction QC, then, $QF : FE :: Ef : fq$.

For, if q be the focus of incident rays, Q is the focus of refracted rays (Art. 29); therefore $qf : fE :: qE : EQ$ (Art. 174); invertendo, $Ef : fq :: QE : Eq :: QF : FE$; that is, $QF : FE :: Ef : fq$.

Prop. XLIV.

176. *When diverging or converging rays are incident nearly perpendicularly upon a lens, whose thickness is inconsiderable, the distances of the focus of incident rays from the principal focus of rays coming in the contrary direction, from the center of the lens, and from the geometrical focus of emergent rays, are in continual proportion.*

Let AE be the lens; E its center; Q the focus of incident rays; QA, QE two rays of the pencil, of which QE is coincident, or nearly coincident with the axis of the lens, and therefore suffers no refraction (Art. 128); let QA be emergent in the direction AH; produce AH backwards or forwards as the case requires, till it meets the axis in q; take F the principal focus of

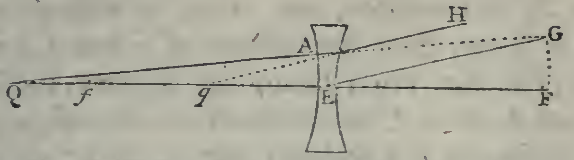


rays incident the contrary way; and from the center E, with the radius EF, describe the circular arc FG, meeting

The two important proportions are

QF : FE :: Ef or FE : fq & QF : FE :: QE : Eq

meeting QA, or QA produced, in G; join GE. Then, since the ray QA will be refracted in the same manner,



and degree, whether it be considered as belonging to the focus Q, or to the focus G, AH is parallel to GE (Art. 165). Hence, the triangles QGE, QAq are similar, and $QG : QE :: QA : Qq$; therefore, ultimately, $QF : QE :: QE : Qq$*.

177. Cor. 1. In the same triangles, $QG : GE :: QA : Aq$; and ultimately, $QF : FE :: QE : Eq$.

178. Cor. 2. Since $QF = QE \pm FE$, we have $QE \pm FE : FE :: QE : Eq$; therefore $Eq = \frac{QE \times FE}{QE \pm FE}$; and $\frac{1}{Eq} = \frac{1}{FE} \pm \frac{1}{QE}$. From this equation, if the nature of the lens be known, any two of the three quantities FE, QE, and Eq being given, the third may be found.

179. Cor. 3. If f be the other principal focus, $QF : FE :: Ef : fq$.

For, if q be the focus of incident rays, Q is the focus of refracted rays (Art. 29); therefore $qf : fE :: qE : EQ$ (Art. 177); invertendo, $Ef : fq :: QE : Eq :: QF : FE$; that is, $QF : FE :: Ef : fq$.

PROP. XLV.

180. *The distances* QF *and* Qq *must always be measured in the same direction from* Q.

Since

* In this demonstration we suppose E to be within the lens, or very near to it. Vid. Note, p. 93.

Since $\mathfrak{Q}F : \mathfrak{Q}E :: \mathfrak{Q}E : \mathfrak{Q}q$ (Art. 176), we have $\mathfrak{Q}F \times \mathfrak{Q}q = \mathfrak{Q}E^2$; therefore the ſign of the rectangle $\mathfrak{Q}F \times \mathfrak{Q}q$ is invariable; and, when the diſtance of $\mathfrak{Q}$ from F is very great, the diſtance of q from f is very ſmall (Vid. Art. 179); meaſuring, therefore, the lines $\mathfrak{Q}F$ and $\mathfrak{Q}q$ from $\mathfrak{Q}$, their rectangle, in this caſe, is poſitive, conſequently it is always poſitive, or $\mathfrak{Q}F$ and $\mathfrak{Q}q$ muſt always be meaſured the ſame way from $\mathfrak{Q}$ (Alg. Art. 473).

181. Nearly in the ſame manner, it may be proved that $E\mathfrak{Q}$ and Eq muſt be meaſured in the ſame, or oppoſite directions from E, according as $F\mathfrak{Q}$ and FE are meaſured in the ſame, or oppoſite directions from F. As alſo, that $\mathfrak{Q}F$ and fq muſt be meaſured in oppoſite directions from F and f.

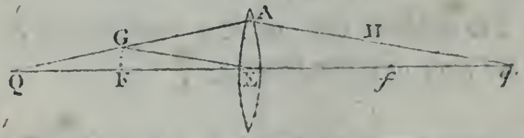
182. Cor. Becauſe $\mathfrak{Q}F \times fq = FE \times Ef$ (Art. 179), $\mathfrak{Q}F$ varies inverſely as fq; and ſince theſe diſtances are meaſured in oppoſite directions from F and f, it is manifeſt that the conjugate foci, $\mathfrak{Q}$ and q, move in the ſame direction upon the indefinite line $\mathfrak{Q}Eq$.

Prop. XLVI.

183. *A convex lens increaſes the convergency, or diminiſhes the divergency of rays incident nearly perpendicularly upon it, unleſs they converge to, or diverge from the center.*

Parallel rays are refracted converging to the principal focus.

When the incident rays diverge from a point farther

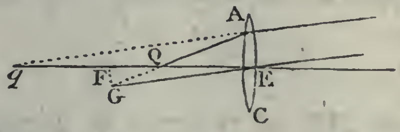


from the lens than it's principal focus, ſince $\mathfrak{Q}F$: $\mathfrak{Q}E$ ::

$QE :: QE : Qq$, and QF is leſs than QE, QE is leſs than Qq; alſo, QF and Qq are always meaſured the ſame way from Q; therefore q is beyond the lens; or the refracted rays converge.

When Q coincides with F, the refracted rays are parallel.

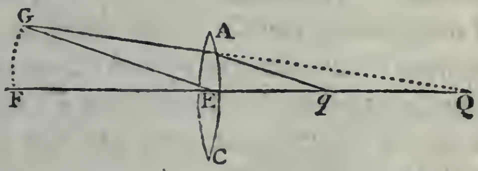
When Q is between F and E, q is on the ſame ſide of the lens, and farther from E than Q is (Art. 180);



therefore, the refracted rays diverge leſs than the incident rays.

When Q coincides with E, q alſo coincides with it, and the convergency, or divergency, is not altered.

When converging rays are incident upon the lens, QF is greater than QE; therefore QE is greater than



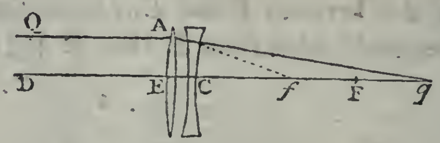
Qq; and q lies between Q and E; conſequently, the refracted rays converge more than the incident rays.

184. In the ſame manner it may alſo be proved, that a concave lens increaſes the divergency, or diminiſhes the convergency of rays incident nearly perpendicularly upon it; except when the focus of incident rays coincides with the center of the lens.

Prop. XLVII.

185. *To find the focal length of a compound lens.*

Let the two lenſes A and C be placed cloſe together, in ſuch a manner that their axes may coincide; and



let QA and DE be two rays of a parallel pencil incident upon them, of which DE is coincident with their common axis. Take f the principal focus of rays incident upon the lens A, in the direction DE; and F the principal focus of rays incident, the contrary way, upon the lens C. Then, after refraction at the lens A, the rays converge to f, and are thus incident upon the lens C; if, therefore, we take $fF : FC :: Cf : Cq$, and meaſure Cq and Cf in the ſame, or oppoſite directions from C, according as Ff and FC are meaſured in the ſame, or oppoſite directions from F (Art. 181), q is the focus of emergent rays, and Cq the focal length of the compound lens.

By proceeding in the ſame manner, we may determine the focal length, when any number of lenſes are combined together.

Cor. When F and f are coincident, the emergent rays are parallel.

SCHOLIUM.

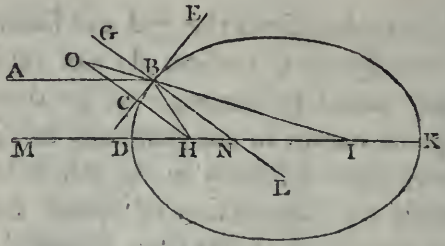
SCHOLIUM.

186. It is evident from the 33d and following propositions, that spherical surfaces do not cause all the rays in any pencil to converge to, or diverge from the same point, except in one particular case; and this will be shewn more distinctly in the 7th section.

To remedy the imperfection of optical instruments arising from this cause, it has been proposed to adopt such refractors as are generated by the revolution of the ellipse or hyperbola. But as these are never resorted to in practice, on account of the great difficulty of giving them the exact form, and because the same effect may, in a great measure, be produced by the proper adjustment of the surfaces of spherical refractors, it will be sufficient to explain the geometrical principles upon which the properties of the proposed refractors depend.

187. *If a prolate spheroid be generated by an ellipse whose major axis is to the distance between it's foci, as the sine of incidence to the sine of refraction out of the ambient medium into the solid, a pencil of parallel rays, incident in the direction of it's axis, will be refracted, converging accurately, to the farther focus.*

Let *BDK* be the ellipse, by the revolution of which, about it's major axis *DK*, the spheroid is generated; *H* and *I* it's foci; then, by the supposition, *DK* : *HI* :: sin. incidence : sin. refraction. Let *AB*, which is parallel to *DK*, be a ray of light incident upon the ſpheroid; join HB, IB; draw EBC touching the generating ellipſe in B; through B and H, draw GBL and

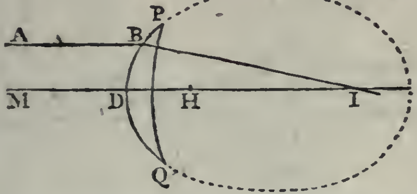


HCO at right angles to EBC; let GBL meet DK in N; and produce IB till it meets HCO in O.

Then, ſince the ∠ HBC is equal to the ∠ IBE, by the nature of the ellipſe, and the ∠ OBC to the ∠ IBE, the angles HBC, OBC are equal; alſo, the angles BCH, BCO are right angles, and BC is common to the two triangles BCH, BCO; therefore, $BO = BH$ (Euc. 26. 1), and $IO = DK$; conſequently, $IO : IH ::$ ſin. incidence : ſin. refraction; and, becauſe BN is parallel to OH, $IB : IN :: IO : IH ::$ ſin. incidence : ſin. refraction; alſo, $IB : IN ::$ ſin. INB : ſin. IBN :: ſin. BNH, or ſin. ABG : ſin. IBL; therefore, ſin ABG : ſin. IBL :: ſin. incidence : ſin. refraction; and, ſince ſin. ABG is the ſine of incidence, ſin. IBL is the ſine of refraction; and becauſe the angle LBI is leſs than a right angle, BI is the refracted ray. In the ſame manner it may be ſhewn, that every other ray in the pencil will be refracted to I.

188. Cor. 1. If from the center I, with any radius leſs than ID, a circular arc PQ be deſcribed, the ſolid generated

generated by the revolution of $PD\mathcal{Q}$ about the axis



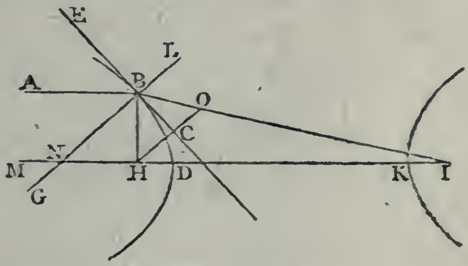
DI, will refract all the rays, incident parallel to DI, accurately to I.

For, after refraction at the surface $PD\mathcal{Q}$ the rays converge to I; and they suffer no refraction at the surface $P\mathcal{Q}$, because they are incident perpendicularly upon it.

189. Cor. 2. Rays diverging from I will be refracted parallel to ID.

190. *If an hyperboloid, whose major axis is to the distance between the foci as the sine of incidence to the sine of refraction out of the solid into the ambient medium, be generated in a similar manner, parallel rays, incident in the direction of the axis, and refracted out of the hyperboloid, will converge to the farther focus.*

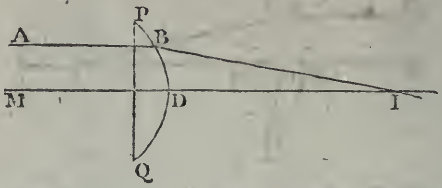
The former proof may be applied to this case, and



nearly in the same words.

191. Cor.

191. Cor. 1. If PQ be drawn perpendicular to the axis of the hyperbola, and meet the curve in P and Q,



the ſolid generated by the revolution of PDQ, about the axis MDI, will refract all the rays, incident parallel to MI, accurately to I.

For, the rays will ſuffer no refraction at the plane ſurface PQ.

192. Cor. 2. Rays diverging from I, and incident upon the ſurface PDQ, will be refracted parallel to ID.

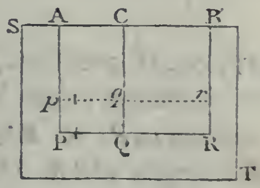
SECTION V.

ON THE IMAGES FORMED BY PLANE AND SPHERICAL REFRACTORS.

Prop. XLVIII.

193. *THE image of a ſtraight line, formed by a plane refracting ſurface, is a ſtraight line.*

Caſe 1. Let *PQR* be a ſtraight line, parallel to the plane refracting ſurface *ACB*; from *P* and *R*, draw



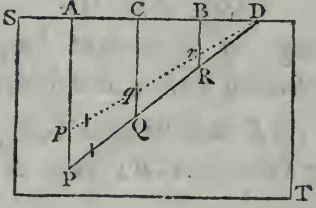
PA, *RB*, at right angles to *AB*; and in *AP*, or *AP* produced, take *PA* : *pA* :: ſin. refraction : ſin. incidence; and *p* is the image of *P* (Art. 131). Draw *pr* parallel to *AB*, or *PR*, and let it meet *BR*, or *BR* produced in *r*; then, ſince the figures *AR*, *Ar*, are parallelograms, *RB* = *PA*, and *rB* = *pA*; therefore *RB* : *rB* :: *PA* : *pA* :: ſin. refraction :

fraction : sin. incidence; consequently r is the image of R. In the same manner it may be shewn, that the image of any other point Q, is q, the corresponding point in pr, determined by drawing QC perpendicular to AB, and producing it, if necessary, till it meets pr; consequently, pr is the whole image of PR.

In this case, since pR is a parallelogram, the image is equal and parallel to the object.

Case 2. When PQR is inclined to the refracting surface.

Produce PR, if necessary, till it meets the surface in D; from P and R, draw PA, RB, at right angles



to AD; and in AP, or AP produced, take $PA : pA ::$ sin. refraction : sin. incidence; then is p the image of P. Join Dp, cutting BR, or BR produced in r; and in the similar triangles DBR, DAP, $RB : BD :: PA : AD$; also, in the similar triangles DBr, DAp, $BD : rB :: AD : pA$; and ex æquo, $RB : rB :: PA : pA ::$ sin. refraction : sin. incidence; therefore r is the image of R (Art. 131). In the same manner it may be shewn, that the image of any other point Q, in PR, is q, the corresponding point in pr, found by drawing QC perpendicular to AD, and producing CQ if necessary; that is, pr is the whole image of PR.

In

In this case, $PQ : pq :: QD : qD :: QR : qr$ (Euc. 2. 6); that is, the corresponding parts of the image and object are proportional.

194. Cor. 1. The image and object are on the same side of the refracting surface; and the image is *nearer* to, or *farther* from the surface than the object, according as the rays pass out of a *denser* into a *rarer*, or out of a *rarer* into a *denser* medium.

Ex. If the medium ST be water, contiguous to air, $PA : pA :: 4 : 3$; and $PA : Pp :: 4 : 1$. Thus, the image of the bed of a river is nearer to the surface than the bed itself, by one fourth part of the whole depth.

195. Cor. 2. Any two points p, r, in the image, have the same relative situation that the corresponding points P, R, of the object have; therefore the image is erect.

196. Cor. 3. If PR, the $\angle PDA$, and the ratio of the sines of incidence and refraction, be known, the $\angle pDA$, and pr may be found.

For, DA being made the radius, tang. PDA : tang. $pDA :: PA : pA ::$ sin. refraction : sin. incidence; therefore the $\angle pDA$ may be found from the tables. Again, $PR : pr :: PD : pD ::$ sec. PDA : sec. pDA.

197. Cor. 4. The image of a straight line inclined to the surface, is *greater*, or *less* than the object, according as the rays pass out of a *rarer* into a *denser*, or out of a *denser* into a *rarer* medium.

198. Cor. 5. If the figure ST move parallel to itself, on a line which is perpendicular to it's plane, PQR, and pqr, will generate planes, the latter of which is the image of the former.

199. Cor.

199. Cor. 6. When the object is a plane, parallel to the refracting surface, the image is equal and parallel to the object.

200. Cor. 7. If the object be a plane, inclined to the refracting surface, the breadths of the object and image, measured by corresponding lines which are parallel to their common intersection, are equal; but their breadths *PR*, *pr*, measured by corresponding lines perpendicular to that intersection, are unequal. In this case, the image and object are not similar.

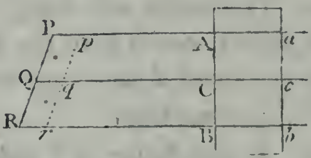
201. Cor. 8. If *pr* be the image of *PR*, and the eye be so placed as to receive the rays which are incident nearly perpendicularly upon the surface *AB*, they will enter the eye as if they came from a real object in the situation *pr*.

Cor. 9. If the rays be refracted at a second surface, *pr* may be considered as an object placed before that surface, and it's image determined in the same manner.

Prop. XLIX.

202. *The image of a straight line, formed by a medium contained by parallel plane surfaces, is a straight line, equal and parallel to the object.*

Let *ABba* be the medium, *PQR* the object placed before it. From *P*, and *R*, draw *PAa*, *RBb* at right



angles to *AB*; and in *AP*, or *AP* produced, according as *Ab* is denser, or rarer than the surrounding medium, take *Pp* : *Aa* :: sin. incidence∼sin. refraction : sin. incidence;

incidence; and p is the image of P (Art. 134). Draw pr parallel to PR, and let it meet BR, or BR produced, in r. Then, ſince Pr and Ab are parallelograms, $Rr=Pp$, and $Bb=Aa$; therefore $Rr : Bb :: Pp : Aa ::$ ſin. incidence ∼ ſin. refraction : ſin. incidence, or r is the image of R. In the ſame manner it may be ſhewn, that the image of any other point Q in the object, is q, the correſponding point in pr, determined by drawing QC perpendicular to AB, and producing it, if neceſſary, till it meets pr. It appears, from the conſtruction, that pr is equal and parallel to PR.

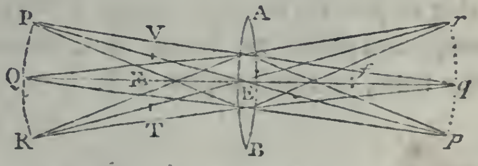
Ex. If the medium Ab be glaſs, ſurrounded by air, $Pp : Aa :: 1 : 3$.

203. Cor. Whatever be the form of the object, the image will be ſimilar and equal to it (Vid. Art. 71).

Prop. L.

204. *If the object placed before a lens, or ſphere, be a circular arc concentric with it, the image will alſo be a circular arc* concentric with, and ſimilar to the object.*

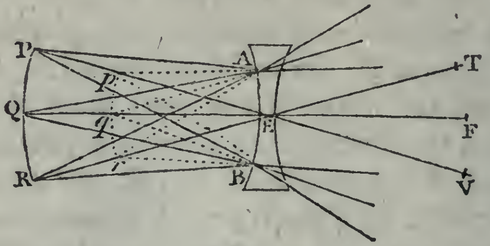
Let AB be the refractor, E it's center; PQR a circular arc whoſe center is E; in PQR take any



point Q, and join QE; let F be the principal focus of

* In the caſe of the lens, the propoſition is not accurately true, as appears by the obſervation contained in the next note.

of rays incident in the oppoſite direction to QE. In QE, or QE produced, take $QF : FE :: QE : Eq$, EQ and Eq being meaſured in the ſame, or oppoſite



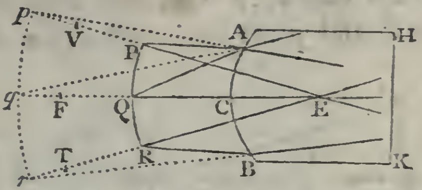
directions from E, according as Q and E lie in the ſame, or oppoſite directions from F; and q will be the image of Q (Art. 181). From the center E, with the radii EF, Eq, deſcribe the circular arcs VFT, pqr; and from the points P and R, in the object, draw PE, and RE, producing them, if neceſſary, till they meet pqr in p and r; then will pr be the image of PR. For, ſince $EP = EQ$, and $EV = EF$, the ſum, or difference of EP and EV, is equal to the ſum, or difference of QE and EF; that is, $PV = QF$; alſo, $Ep = Eq$, by the conſtruction; and $QF : FE :: QE : Eq$; therefore $PV : VE :: PE : Ep$, or p is the image of P*. In the ſame manner it may be ſhewn, that the image of every other point in PQR, is the correſponding point in pqr; that is, pqr is the whole image of PQR.

The image and object are ſimilar arcs, becauſe they ſubtend the ſame, or equal angles at E.

205. If

* Here it is ſuppoſed that the foci q, p of direct, and oblique pencils, are equally diſtant from E. This is not accurately true in the caſe of the lens, and conſequently, the image is not a circular arc; it does not, however, ſenſibly differ from that form, when the angle which PQR ſubtends at E is ſmall.

205. If the refractor be a ſpherical ſurface with



which the object is concentric, it may be ſhewn, nearly in the ſame manner, from the 149th article, that the image is ſimilar to, and concentric with the object.

206. Cor. 1. Since PR and pr are ſimilar arcs, $PR : pr :: EQ : Eq$; hence, in the lens, or ſphere, $PR : pr : QF : FE$ (Arts. 174. 177).

207. Cor. 2. If the figure be ſuppoſed to revolve about the axis Qq, PQR and pqr will generate ſimilar ſpherical ſurfaces, the latter of which is the image of the former.

208. Cor. 3. In this caſe, the magnitude of the object : the magnitude of the image :: $\overline{EQ}|^2 : \overline{Eq}|^2$.

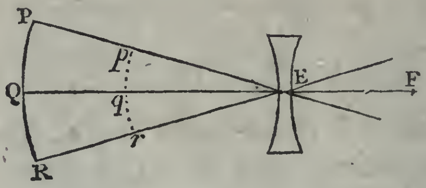
209. Cor. 4. If the half of a lens, cut off by a plane paſſing through the axis, be removed, the image formed by the remaining part will be of the ſame magnitude, and in the ſame ſituation as before; the only alteration produced will be a diminution of the brightneſs.

The ſame may be ſaid, if a part of the lens be cut off by any plane which is parallel to the former.

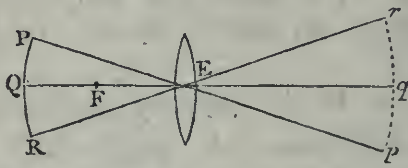
Prop. LI.

210. *When the image and object are on the ſame ſide of the center of the refractor, the image is erect with reſpect to the object; when they are on oppoſite ſides of the center, it is inverted.*

If the line PpE revolve round E, and the point P trace out the object, p will trace out the image. Alſo, when P and p are on the *ſame* ſide of E, they



move in the *ſame* direction, during the rotation of the line PpE; thus, the ſeveral points in the image have the ſame relative poſition that the correſponding points in the object have, or the image is erect. But,



when P and p are on *oppoſite* ſides of the center, they move in *oppoſite* directions, and the ſituation of any two points in the image is inverted, if compared with the ſituation of the correſponding points in the object; conſequently the whole image is inverted.

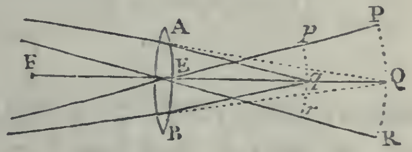
211. Cor. 1. When the object is placed before a convex lens, and QE is greater than FE, the image is inverted. For, QF and FE, in this caſe, are meaſured in oppoſite directions from F; therefore Q and q are on oppoſite ſides of E (Art. 181) When the object is between F and E, the image is erect.

In the former caſe, QF may be greater than, equal to, or leſs than FE; therefore the image may be leſs than,

than, equal to, or greater than the object (Art. 206). In the latter case, the image is always greater than the object.

212. Cor. 2. When the refracted rays actually meet, if a screen be placed at their concourse, an image or picture of the object will be formed upon it. If the screen be placed nearer to the lens, or farther from it, than the focus of refracted rays, the image will be indistinct; because the rays, which proceed from a single point, will be diffused over some space upon the screen, and mixed with the rays which diverge from other points in the object; and this indistinctness will increase as the distance of the screen, from the focus of refracted rays, increases.

213. Cor. 3. When converging rays, which tend to form an image PQR, are received by a convex lens



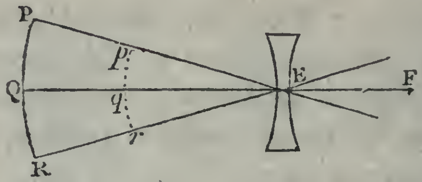
which is concentric with PQR, another image pqr will be formed, nearer to the lens, and erect with respect to the first image.

Let rays, converging to the several points in PQR, be intercepted by the convex lens AB; take F the principal focus of rays incident in the contrary direction.

Then, since QF is greater than FE, QE is greater than Eq; consequently, PQR is greater than pqr. Also, since Q and E are on the same side of F, Q and q are on the same side of the center E; and therefore, pqr is erect with respect to PQR.

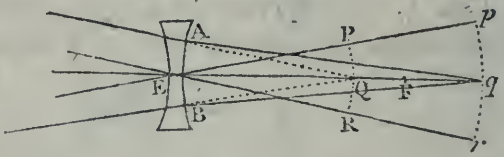
 214. Cor.

214 Cor. 4. When the object is placed before a double concave lens, since $Q F$ and $F E$ are measured



the same way from F, Q and q are on the same side of E; that is, the image is erect. Also, since $Q F$ is greater than $F E$, the object is greater than the image.

215. Cor. 5. If converging rays, which tend to form the image $P Q R$, be intercepted by a double concave lens concentric with $P Q R$, the second image $p q r$



will be *erect*, or *inverted*, with respect to the first, according as Q is *nearer to*, or *farther from*, the lens than F, the principal focus of rays incident in the contrary direction.

When Q is between F and E, $E Q$ and $E q$ are measured in the same direction from E (Art. 181); consequently, $p q r$ is erect with respect to $P Q R$. When F is between Q and E, $P Q R$ and $p q r$ are on opposite sides of the center; and therefore the image $p q r$ is inverted.

216. Cor. 6. When Q lies between F and E, or $p q r$ is erect, $Q F$ is less than $F E$, and consequently $E Q$ is

less

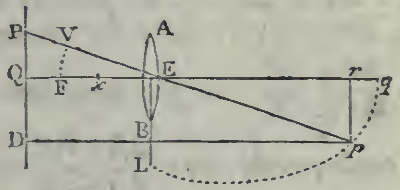
leſs than Eq, or PQR is leſs than pqr. In other caſes, PQR may be greater than, equal to, or leſs than pqr.

217. Cor. 7. When Q coincides with F, the emergent rays, in each pencil, are parallel.

Prop. LII.

218. *The image of a ſtraight line, formed by a lens or ſphere, is the arc of a conic ſection.*

Let AB be a lens, or ſphere, whoſe center is E; PD a ſtraight line placed before it; through E, draw QEq at right angles to PQ; in PD take any point P;



join PE, and produce it. Let F be the principal focus of rays incident in the direction qE; with the center E, and radius EF, deſcribe the circular arc FV, cutting PE in V. In PEp, take $PV : PE :: PE : Pp$, meaſuring PV and Pp in the ſame direction from P; then p is the image of P (Art. 180 *). Draw pD parallel to qQ. Then, ſince the triangles PEQ, PpD are ſimilar, $PE : Pp :: QE : Dp$; conſequently, $PV : PE :: QE : Dp$. Alſo, $PV : VE :: PE : Ep$ (Arts. 174. 177); alternately, $PV : PE :: VE\ (FE) : Ep$; therefore $QE : Dp :: FE : Ep$; and alternately, $QE :: FE :: Dp : Ep$; conſequently the

* Vid. Note, p. 93.

the locus of the point p, is a conic ſection, whoſe focus is E, and directrix PD *.

219. Cor. 1. The curve is an *ellipſe*, *parabola*, or *hyperbola*, according as QE is *greater* than, *equal* to, or *leſs* than FE.

220. Cor. 2. When Ep coincides with EL, that ordinate to the axis which paſſes through the focus, Dp becomes equal to QE, and therefore $EL = EF$; that is, half the latus rectum of the conic ſection is equal to the focal length of the glaſs.

221. Cor. 3. The curvature of the image, at it's vertex, is the ſame, wherever the object is placed.

222. Cor. 4. If xq be the major axis of the conic ſection, $Qq : Eq :: QE : FE$; and by diviſion, or compoſition, $QE : Eq :: QF : FE$; therefore $Eq = \frac{QE \times FE}{QF} = \frac{QE \times FE}{QE \mp FE}$. In the ſame manner, $Ex = \frac{QE \times FE}{QE \pm FE}$; conſequently, $xq = \frac{QE \times FE}{QE \sim FE} \pm \frac{QE \times FE}{QE + FE} = \frac{2QE^2 \times FE}{QE^2 \sim FE^2}$. Alſo, $xE \times Eq$, the ſquare of the ſemi-axis minor, $= \frac{QE^2 \times FE^2}{QE^2 \sim FE^2}$.

223. Cor. 5. If the focal length of the refractor be finite, and pr be drawn perpendicular to the axis, the evaneſcent arc pq is equal to pr; and $QP : qp :: EQ : Eq$.

Alſo, whilſt the angle QEP, which QP ſubtends at the center of the glaſs, is ſmall, though finite, the image pq, when formed at a finite diſtance from the refractor, will, as to ſenſe, be a right line, and $QP : qp :: EQ : Eq$.

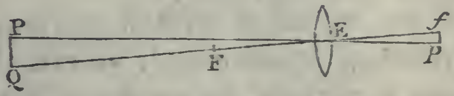
PROP.

* Vid. Art. 93.

Prop. LIII.

224. *The sun's image, formed by a spherical refracting surface, lens or sphere, is a circle, and nearly in the principal focus of the refractor.*

Let E be the center of the refractor; F and f it's principal foci; QP a radius of the sun's disc. Then,



since FE is inconsiderable with respect to QF, the image of Q, may, for all practical purposes, be considered as coincident with the principal focus f of the refractor (Arts. 150. 175. 179); also, since QP subtends a small angle at E*, it's image, fp, may be considered as a straight line (Art. 223). Now, let the figure revolve about Qf as an axis, and whilst QP generates the circle which represents the sun's disc, fp will generate it's image, which is, therefore, a circle.

In the same manner it may be shewn, that the sun's image, formed by a spherical reflector, is a circle, and in the principal focus of the reflector.

225. Cor. 1. Since the angle fEp is given, fp the radius of the image, is proportional to Ef, the focal length of the glass.

226. Cor. 2. The area of the image varies as the square of it's radius; and therefore as the square of the focal length of the reflector, or refractor.

227. Def.

* About 16′.

227. Def. By the reflecting, or refracting *powers* of different ſubſtances, we underſtand the ratio of the number of rays reflected, or tranſmitted by them, if the number of incident rays be the ſame.

Thus, if one ſurface reflect two thirds, and another one third of the incident rays, the reflecting *powers* are ſaid to be as 2 : 1.

228. Cor. The number of rays reflected, or tranſmitted, varies as the number incident, and the reflecting, or refracting power, jointly.

For, if the number of incident rays be given, the number reflected, or tranſmitted, varies as the power; if the power be the ſame, the number of rays reflected, or tranſmitted, varies as the number incident; therefore, when both vary, the number of rays reflected, or tranſmitted, varies as the number incident, and the reflecting, or refracting power, jointly.

Prop. LIV.

229. *The denſity of rays in the ſun's image varies directly as the area of the aperture of the reflector, or refractor by which it is formed, and the reflecting, or refracting power, jointly; and inverſely as the ſquare of the focal length of the reflector, or refractor.*

The denſity of rays in the image varies directly as their number, and inverſely as the ſpace over which they are diffuſed *; that is, directly as the number, and inverſely as the ſquare of the focal length of the reflector,

* Here we ſuppoſe the rays to be *uniformly* diffuſed over the image, which is not the caſe; it is, however, true at points ſimilarly ſituated in images formed by rays which are diffuſed according to the ſame law.

reflector, or refractor (Art. 226). Alſo, the number of rays reflected, or tranſmitted, varies as the number incident, and the reflecting, or refracting power, jointly; that is, as the area of the aperture through which the incident rays paſs, and the power, jointly; conſequently, the denſity of rays in the image, varies, directly, in the compound ratio of the aperture and power; and, inverſely, as the ſquare of the focal length of the reflector, or refractor *.

230. Cor. 1. When the apertures are circular, the denſity varies, directly, in the compound ratio of the ſquare of the linear aperture and power; and inverſely as the ſquare of the focal length of the reflector, or refractor.

231. Cor. 2. If the radii of the ſurfaces of a concave reflector, and a double convex lens of glaſs, be equal, as well as their apertures and powers, ſince the focal length of the reflector : the focal length of the lens :: 1 : 2 (Arts. 45. 171), the denſity of rays in the image formed by the reflector : the denſity in the image formed by the lens :: 4 : 1.

232. Cor. 3. The focal length of a glaſs ſphere, is three times as great as the focal length of a reflector of the ſame radius (Art. 158); therefore, on the former ſuppoſition, the denſity of rays in the image formed by the reflector : the denſity in the image formed by the ſphere :: 9 : 1.

233. Cor. 4. If the rays which tend to form the ſun's image, be received by a double convex lens, another

* The rays loſt in paſſing through the air, are not taken into the account.

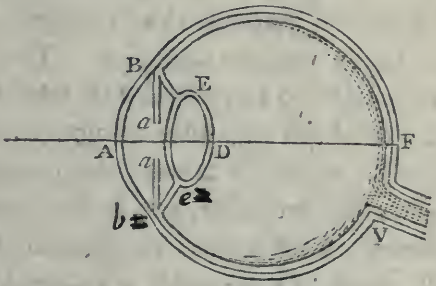
another image, nearer to the lens, and consequently less than the former, will be produced (Art. 213). Hence it appears, that independent of the rays lost in their passage through the lens thus employed, the burning power of a reflector, or refractor may be increased.

SECTION

SECTION VI.

ON THE EYE AND THEORY OF VISION.

234. THE annexed figure reprefents a fection of the human eye, made by a plane which is perpendicular to the furfaces of the coats which contain it's feveral humours, and alfo to the nofe.



It's form is nearly fpherical, and would be exactly fo, were not the forepart a little more convex than the remainder; the parts *BFB*, *BAB*, are, in reality, feg-ments of a greater and a lefs fphere.

The

The humours of the eye are contained in a firm coat *BFBA*, called the *ſclerotica*; the more convex, or protuberant part of which, *BAB*, is tranſparent, and from it's conſiſtency, and horny appearance, it is called the *cornea*. This coat is repreſented by the ſpace contained between the two exterior circles *BFBA*.

Contiguous to the ſclerotica is a ſecond coat of a ſofter ſubſtance, called the *choroeides*. This coat is repreſented by the next white ſpace, and extends, along the back part of the ſclerotica, to the cornea.

From the junction of the choroeides and cornea ariſes the *uvea*, *Ba*, *Ba*, a flat, opaque membrane, in the forepart of which, and nearly in it's center *, is a circular aperture called the *pupil*.

The pupil is capable of being enlarged, or contracted with great readineſs †; by which means, a greater or leſs number of rays may be admitted into the eye, as the circumſtances of viſion require. In weak light, too few rays might render objects indiſtinct; and in ſtrong light, too many might injure the organ. Whilſt the pupil is thus enlarged, or contracted, it's figure remains unaltered. This remarkable effect is thought to be produced by means of ſmall fibres which ariſe from the outer circumference of the uvea, and tend towards it's center; this circumference is alſo *ſuppoſed* to be muſcular, and by it's equal action upon the fibres, on each ſide, the form of the pupil is preſerved, whilſt it's diameter is enlarged, or contracted.

At

* In ſome eyes, the pupil is a little nearer to the noſe than the center of the uvea.

† The limit of it's aperture, in the eyes of adult perſons, appears to be from about $\frac{1}{4}$ to $\frac{1}{10}$ or $\frac{1}{12}$ of an inch. Harris's Optics, p. 94.

At the back part of the eye, a little nearer to the nofe than the point which is oppofite to the pupil, enters the *optic nerve V*, which fpreads itfelf over the whole of the choroeides like a fine net; and from this circumftance is called the *retina*. It is immerfed in a dark mucus which adheres to the choroeides.

Thefe three coats, the *fclerotica*, the *choroeides*, and the *retina*, enter the focket of the eye at the fame place. The fclerotica is a continuation of the *dura mater*, a thick membrane which lies immediately under the fcull. The choroeides is a continuation of the *pia mater*, a fine thin membrane which adheres clofely to the brain. The retina proceeds from the brain.

Within the eye, a little behind the pupil, is a foft tranfparent fubftance *EDE*, nearly of the form of a double convex lens, the anterior furface of which is lefs curved than the pofterior, and rounded off at the edges, *E*, *E*, as the figure reprefents. This humour, which is nearly of the confiftency of hard jelly, decreafing gradually in denfity from the center to the circumference, is called the *cryftalline* humour It is kept in it's place by a mufcle, called the *ligamentum ciliare*, which takes it's rife from the junction of the choroeides and cornea, and is a little convex towards the uvea *.

The cavity of the eye, between the cornea and the cryftalline humour, is filled with a tranfparent fluid like water, called the *aqueous humour*. The cavity between the cryftalline humour and the back part of the eye,

* The anterior furface of this mufcle, and the pofterior furface of the uvea, are covered with a black mucus, evidently defigned to abforb any of the extreme rays which may happen to reach fo far, and which might be reflected to the retina, and produce confufion in the vifion.

eye, is alſo filled with a tranſparent fluid, rather more viſcous than the former, called the *vitreous humour*.

235. It is not eaſy to aſcertain, with great accuracy, the refracting powers of the ſeveral humours; the refracting powers of the aqueous and vitreous humours, are nearly equal to that of water; the refracting power of the cryſtalline humour is ſomewhat greater *.

236. The ſurfaces of the ſeveral humours of the eye are ſo ſituated as to have one line perpendicular to them all. This line *ADF* is called the *axis* of the eye, or the *optic axis* †.

237. The *focal center* of the eye is that point in the axis at which the image upon the retina, and the object, ſubtend equal angles.

This point is not far diſtant from the poſterior ſurface of the cryſtalline lens ‡, though it's ſituation is probably ſubject to a ſmall change, as the figure of the eye, or the diſtance of the object is changed.

238. From the conſideration of the ſtructure of the eye, we may eaſily underſtand how the notices of external objects are conveyed to the brain.

Let *PQR* be an object, towards which the axis of the eye is directed; then, the rays which diverge from any point *Q*, and fall upon the convex ſurface of the aqueous humour §, have a degree of convergency

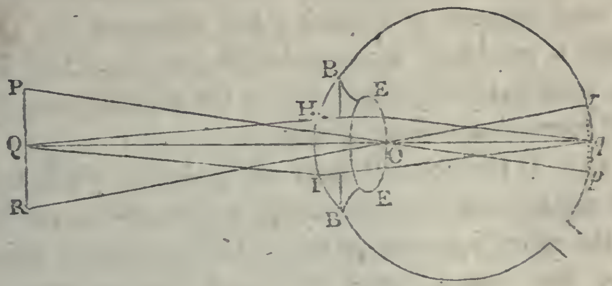
* This is manifeſt from the figure of the cryſtalline humour, and the circumſtance that perſons couched, (in which caſe the cryſtalline lens is taken out) are obliged to uſe convex glaſſes.

† The dimenſions of the ſeveral parts of the eye may be ſeen in Harris's Optics, page 94.

‡ Harris, p. 97.

§ The ſurfaces of the cornea are nearly parallel to each other, and therefore it produces little alteration in the divergency of rays which paſs through it.

gency given them; they are then refracted by a double convex lens, denser than the ambient mediums, which



increases the convergency; and if the extreme rays *QH*, *QI*, have a proper degree of divergency before incidence, the pencil will be again collected upon the retina, at *q*, and there form an image of *Q*. In the same manner, the rays which diverge from any other points, *P*, *R*, in the object, will be collected at the corresponding points *p*, *r*, of the retina, and a complete image, *pqr*, of the object *PQR*, will be formed there. The impression, thus made, is conveyed to the brain by the optic nerve, which originates there, and is evidently calculated to answer this purpose.

239. Since the axes of the several pencils cross each other at *O*, the focal center of the eye, the image upon the retina is inverted with respect to the object *; and if, by any means, the image of an erect object, be erect upon the retina, that object appears inverted.

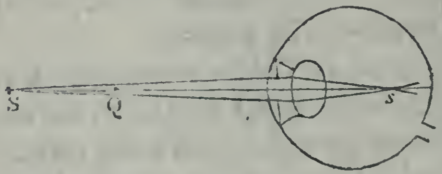
240. It has been objected, that if the images upon the retina be inverted, external objects ought to appear inverted. To which it may be answered, that experience alone

* If the outer coat be taken from an ox's eye, whilst it is warm, the images of external objects are observed to be inverted upon the retina.

alone teaches us, what ſituation of the external object correſponds to a particular impreſſion upon the retina; nor is it of any conſequence what that impreſſion is, or in what manner it is made; but whenever the ſame effect is produced upon the organ, we expect to find the ſame external object, and in the ſituation to which our former experience directs us.

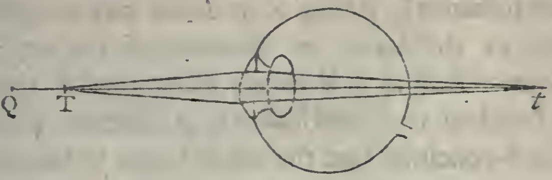
241. If the point P move along the line PQR, from the right to the left, the image p moves from the left to the right, upon the retina. And in general, whenever the image, upon the retina, moves from the left to the right, we are led, by experience, to conclude that the object really moves from the right to the left.

242. If the form of the eye, the ſituation of the ſeveral humours, and their reſpective ſurfaces, remain unaltered, it is manifeſt that thoſe rays only, which diverge from points at a particular diſtance, can be collected upon the retina. Thus, if the image of Q be formed exactly upon the retina, the image of S, a



point farther from the eye than Q, will be formed within the eye; therefore, the rays which proceed from this point, will be diffuſed over ſome ſpace upon the retina; and, if they are mixed with the rays which diverge from other points in the object, neceſſary to be diſtinguiſhed from the former, the viſion will be indiſtinct.

indiſtinct *. The rays which diverge from *T*, a point nearer to the eye than *Q*, will, after refraction, con-



verge to *t*, a point behind the retina; in this caſe alſo, they will be diffuſed over ſome ſpace upon the retina, and the viſion, as before, will be indiſtinct.

243. By what change in the conformation of the eye, we are enabled to ſee objects diſtinctly at different diſtances, is not fully aſcertained. The fact itſelf is ſufficiently manifeſt; but authors differ as to the manner in which the effect is produced. It is ſuppoſed by ſome, that the general figure of the eye is altered; that, when the object to be viewed is near, the length of the eye, meaſured along the axis, is increaſed by the lateral preſſure of external muſcles; and, on the contrary, when the object is remote, that the length of the eye is diminiſhed, by the relaxation of that preſſure. Others ſuppoſe the effect to be produced by a change in the place, or figure of the cryſtalline humour. Others, by an alteration in the diameter of the pupil. Others aſcribe the effect to a change in the curvature of the cornea.

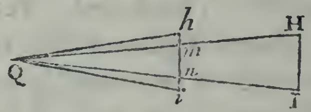
Much ſtreſs cannot be laid upon the firſt of theſe cauſes, as diſtinguiſhed from the laſt, ſince it's exiſtence is

* In many caſes it is not neceſſary to diſtinguiſh very nicely the adjacent parts of objects; as in reading large print, viewing trees, houſes, mountains, &c. and though the rays are not exactly collected upon the retina, the image is ſufficiently well defined for the purpoſe.

is not proved by experiment; and there is no neceſſity for recurring to a bare hypotheſis of this kind. With reſpect to the ſecond, the *ligamentum ciliare* does not appear ſufficiently ſtrong to produce any conſiderable change in the form, or ſituation of the cryſtalline humour. And as it is clearly aſcertained *, that perſons couched can ſee diſtinctly at different diſtances, we muſt conclude that the effect is not to be aſcribed to any change in this humour.

A change in the aperture of the pupil has ſome effect in rendering objects diſtinct at different diſtances.

If the eye of a ſpectator be directed, firſt to a diſtant object, and then to one which is nearer, the diameter of the pupil is obſerved to decreaſe. Let *HI* be the diameter of the pupil; *QH*, *QI*, the extreme rays of a



pencil, diverging from *Q*; and which the humours of the eye are capable of collecting upon the retina. Then, when *Q* is at a leſs diſtance, *Qh*, from the eye than before, if *hi*, the diameter of the pupil, be equal to *HI*, the extreme rays *Qh*, *Qi*, of the pencil which now enters the pupil, diverge more than *QH*, *QI*; and therefore, they will not be collected upon the retina (Art. 242); but if *QH*, *QI*, cut *hi* in *m* and *n*, and the diameter of the pupil be contracted to *mn*, then the extreme rays *Qm*, *Qn*, coincide with *QH*, *QI*, and will, as thoſe rays are, be collected upon the retina. Thus, if the diameter of the pupil, *mn*, be proportional to

* Philoſ. Tranſ, Vol. LXXXV. p. 6.

to the diſtance $2m$, and the object be ſeen diſtinctly at any one diſtance, it will be ſeen diſtinctly at all other diſtances, by the variation of the pupil alone, as far as this variation extends (Vid. note, page 124)*. The diameter of the pupil, however, appears to be much more affected by the quantity of light, than by the diſtance of an object from it; therefore, though it's variation tends to correct the different degrees of divergency of the extreme rays, when objects are at different diſtances, it will not produce the effect ſo conſtantly and regularly as is neceſſary for diſtinct viſion.

The principal change by which the effect is produced, ſeems to be an alteration in the curvature of the cornea. In order to ſhew that ſuch a change takes place, Mr. RAMSDEN fixed the head of a ſpectator ſo ſecurely, that no deception could ariſe from it's motion, and directed him to look at a diſtant object; whilſt the eye was in this ſituation, he placed a microſcope, in ſuch a manner, that the wire, with which it was furniſhed, apparently coincided with the outer ſurface of the cornea; and then directing the ſpectator to look at a nearer object, he found that the cornea immediately projected beyond the wire of the microſcope †.

Now, when the diſtance of an object is diminiſhed, ſuppoſing

* It is on this account that a ſmall hole in a thin plate enables us to view objects at a leſs diſtance than we could with the naked eye, as it anſwers the purpoſe of a farther contraction of the pupil, and excludes thoſe rays in each pencil, which diverge too much. This aſſiſtance cannot be made uſe of to any great extent, becauſe the image upon the retina will ſoon become indiſtinct for want of light; and the inflection of rays at the ſides of the hole, will render it confuſed.

† This experiment is deſcribed in a very ingenious paper by Mr. HOME, Philoſoph. Tranſ. Vol. LXXXV. p. 16.

ſuppoſing no alteration to take place in the eye, the divergency of the extreme rays of the pencil incident upon the pupil, is increaſed; and therefore, if the image of the object in the firſt ſituation, be formed upon the retina, in the latter it will be formed behind it (Art. 242); but an increaſe in the curvature of the cornea will increaſe the convergency of the refracted rays, or bring them ſooner to a focus; and thus, by a proper change in this coat of the eye, the rays will again be brought to a focus upon the retina, and the object be ſtill ſeen diſtinctly.

244. The leaſt diſtance at which objects can be ſeen diſtinctly by common eyes, is about 7 or 8 inches*. The greateſt diſtance cannot be ſo eaſily, or accurately aſcertained. It ſeems that the generality of eyes are capable of collecting parallel rays upon the retina, or ſo near to it as to produce diſtinct viſion; and thus, the greateſt diſtance at which objects can be diſtinctly viewed, is unlimited. For this reaſon, in adapting optical inſtruments to common eyes, and calculating their powers, we ſuppoſe the parts to be ſo arranged, that the rays in *each pencil* may, when they fall upon the cornea, be parallel.

245. If the humours of the eye be too convex, parallel rays, and ſuch pencils as diverge from points at any conſiderable diſtance, are collected before they reach the retina (Art. 242); and objects, to be ſeen diſtinctly, muſt be brought nearer to the eye. This inconvenience may be remedied by a concave glaſs whoſe focal length is ſo adjuſted as to give the rays, proceeding from a diſtant object, ſuch a degree of divergency as the eye requires.

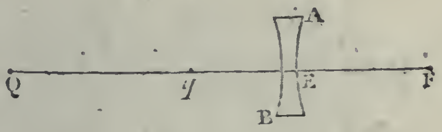
PROP.

* Harris, p. 124.

Prop. LV.

246. *Having given the diſtance at which a ſhort ſighted perſon can ſee diſtinctly, to find the focal length of a glaſs which will enable him to ſee diſtinctly at any other given diſtance.*

If qE be the diſtance at which he can ſee diſtinctly, and $\mathcal{Q}E$ a greater diſtance, at which he wiſhes to view



objects, let AB be a concave lens, whoſe focal length is ſuch, that the rays which are incident upon it, diverging from $\mathcal{Q}$, may, after refraction, diverge from q; then they will have a proper degree of divergency for the eye of this ſpectator. Take F the principal focus of rays incident in the contrary direction; then, ſince $\mathcal{Q}$ and q are conjugate foci, $\mathcal{Q}F : \mathcal{Q}E :: \mathcal{Q}E : \mathcal{Q}q$ (Art. 176); dividendo, $FE : \mathcal{Q}E :: Eq : \mathcal{Q}q$; therefore $FE = \frac{\mathcal{Q}E \times Eq}{\mathcal{Q}q}$.

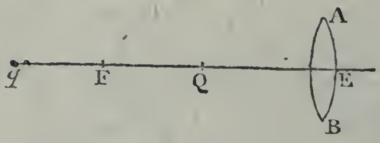
247. Cor. If $\mathcal{Q}E$ be indefinitely great, $FE = Eq$.

248. When the humours of the eye are too flat, the rays which diverge from a point near the eye, converge to a point behind the retina. This imperfection may be remedied by a convex lens, whoſe focal length is adjuſted to the diſtance at which objects are to be viewed, and the degree of convergency in the rays of each pencil, which the eye requires.

Prop. LVI.

249. *Having given the diſtance at which a long ſighted perſon can ſee diſtinctly, to find the focal length of a glaſs which will enable him to ſee diſtinctly at any other given diſtance.*

If qE be the diſtance at which he can ſee diſtinctly, and QE the diſtance at which he wiſhes to view



objects, let AB be a convex lens whoſe focal length FE, is ſuch, that the rays which diverge from Q, may, after refraction, diverge from q. Take F the principal focus of rays incident in the contrary direction; and ſince Q and q are conjugate foci, $QF : QE :: QE : Qq$; componendo, $FE : QE :: Eq : Qq$; and $FE = \frac{QE \times Eq}{Qq}$.

250. Cor. 1. If qE be indefinitely great, or the eye require parallel rays, $FE = QE$.

251. Cor. 2. If the eye require converging rays, q falls on the other ſide of the lens; in this caſe, FE is leſs than QE.

252. In the choice of glaſſes for long, or ſhort ſighted perſons, care ſhould be taken to ſelect ſuch as have the leaſt refracting power that will anſwer the purpoſe. For, the eye has a tendency to retain that conformation

conformation to which it is accuſtomed; and therefore, by the uſe of improper glaſſes, it's imperfection may be increaſed.

Prop. LVII.

253. *If the apparent diſtance of an object be given, and the angle which it ſubtends at the center of the eye be ſmall, it's apparent linear magnitude is nearly proportional to that angle*

When objects are at the ſame diſtance from the eye, and appear to be ſo, we learn by experience to form an eſtimate of their linear magnitudes with conſiderable accuracy. That is, the apparent magnitudes are nearly proportional to the real magnitudes, and the *real* magnitudes are proportional to the angles which the objects ſubtend at the center of the eye, when thoſe angles are ſmall; therefore their *apparent* magnitudes are nearly in that ratio.

254. An object, and it's image upon the retina, ſubtend equal angles at the center of the eye; and ſuppoſing the center fixed, and the angles ſmall, the linear magnitude of the image is nearly proportional to the angle which it ſubtends at that center; therefore the linear magnitude of an object at a given diſtance from the eye, is nearly proportional to the linear magnitude of it's picture upon the retina *.

255. The

* On this account, perhaps, we learn to eſtimate the magnitudes of objects, at a given diſtance, more readily than we ſhould otherwiſe be able to do; but, did the magnitude of the picture upon the retina vary according to any other law, we ſhould ſtill learn by experience to eſtimate magnitudes by the ſight; that is, the apparent and real magnitudes would ſtill be proportional.

When

255. The judgment we form of the magnitude of an object, depends very much upon the notion we have of it's diftance; and fince the apparent diftance depends upon a variety of caufes, which are fubject to no calculation, in fpeaking of apparent magnitude authors generally fuppofe the apparent diftance to be given.

256. Def. By the *vifual angle* of an object, we underftand the angle which the axes of the extreme pencils coming from it, contain at the center of the eye; whether the object is viewed with the naked eye, or with the affiftance of reflecting furfaces, or refracting mediums.

Prop. LVIII.

257. *When a given object is viewed with the naked eye, the denfity of light in the image upon the retina, fuppofing none to be loft in it's paffage through the air, and the diameter of the pupil to be invariable, is nearly the fame at all diftances of the eye from the object.*

The denfity of light, in the image of a fmall portion of the object, varies directly as the number of rays, and inverfely as the fpace over which they are diffufed. The number of rays which pafs through the pupil, fuppofing it's diameter given, and that none are ftopped in their progrefs, varies inverfely as the fquare of the diftance of the object from the eye (Art. 11). Alfo,

When objects fubtend confiderable angles at the center of the eye, we judge of their magnitudes by carrying the optic axes over their feveral parts; and in this cafe alfo, the apparent, and real magnitudes are nearly proportional, if we have had fufficient experience in eftimating magnitudes of this defcription.

Alſo, the linear magnitude of the picture upon the retina, varies as the angle which the object ſubtends at the center of the eye, nearly (Art. 254); that is, nearly in the inverſe ratio of the diſtance of the object from the eye; conſequently, the area of the picture upon the retina, or ſpace over which the rays are diffuſed, varies inverſely as the ſquare of that diſtance, nearly. Hence it follows, that the denſity of rays in the image, varies inverſely as the ſquare of the diſtance of the object from the eye, on one account, and directly as the ſquare of that diſtance on the other; therefore, upon the whole, the denſity is invariable.

What has been proved of the image of one ſmall portion of the object, may be proved of every other; conſequently, the denſity, in every part of the image upon the retina, is invariable.

SCHOLIUM.

258. It may here be obſerved, that a conſiderable quantity of light is loſt, or abſorbed, in it's paſſage through the air; and that the quantity thus loſt, cæteris paribus, increaſes as the diſtance between the object and the eye increaſes, though not in that ratio*. On

* If the ſpaces, through which the light paſſes, increaſe in arithmetical progreſſion, the quantity of light will decreaſe in geometrical progreſſion.

Let the ſpace be divided into equal portions; and let *A*, *B*, *C*, *D*, &c. repreſent the quantity of light which enters the 1ſt, 2d, 3d, 4th, &c. portion, reſpectively; alſo, ſuppoſe $\frac{1}{n}$th part of the whole light to be loſt, or abſorbed in it's paſſage through the 1ſt portion of ſpace; then $\frac{1}{n}$th part of the remainder will be loſt in paſſing through

On this account, therefore, the brightneſs of an object decreaſes, as it's diſtance from the eye increaſes. As the diſtance of the object increaſes, however, the aperture of the pupil is enlarged; and therefore more rays are, by this means, received into the eye; and thus the former effect is, in ſome degree counteracted.

Did the denſity of rays in the picture upon the retina decreaſe conſiderably, as the diſtance of the object increaſes, bodies in the neighbourhood of the ſpectator would, by their ſuperior brightneſs, overpower the impreſſions made by thoſe which are more remote: and the latter would be diſcerned with great difficulty, or not at all. We are indeed able to diſtinguiſh objects in exceedingly different degrees of light, at different times; thus we are able to read a ſmall print by moon light, though it's intenſity does not exceed $\frac{1}{90,000}$th part of the intenſity of common day-light *. But this quantity of light is not ſufficient to render ſuch objects diſcernible as are ſurrounded by others much more luminous; for, the ſtrong light proceeding from the latter bodies, by the powerful impreſſion it makes upon the retina, overcomes the effect produced by the more delicate pencils which flow from the former, as weaker ſounds are not diſtinguiſhable in a hurricane.

through the 2d. and ſo on. Thus $A-\frac{A}{m}$, or $\frac{m-1}{m}\times A=B$. In the ſame manner, $\frac{m-1}{m}\times B=C$; $\frac{m-1}{m}\times C=D$, &c. That is, A, B, C, D, &c. form a decreaſing geometrical progreſſion, whoſe common ratio is $\frac{m-1}{m}$.

* This is Dr. Smith's *calculation;* M. Bouguer concludes from *experiment,* that the ſtrength of moon-light is about $\frac{1}{300,000}$th part of that of day-light.

hurricane. This ſeems to be the reaſon that the flame of a candle is ſcarcely diſcernible in broad day light; and that ſtars become viſible at different times after ſun-ſet, according to their different degrees of brightneſs.

259. The impreſſions made by rays of light upon the retina continue ſome time after the impulſes ceaſe, as appears by the experiment of a burning coal, whirled round in a circle, which was mentioned on a former occaſion (Art. 8). Sir Isaac Newton accounts for this phenomenon by ſuppoſing that the impreſſions of light are conveyed to the brain by vibrations excited in the retina, and propagated, through the optic nerve, to the ſenſorium *; and that the vibrations once produced, continue ſome time, perhaps about 1″, after the exciting cauſe has ceaſed to act.

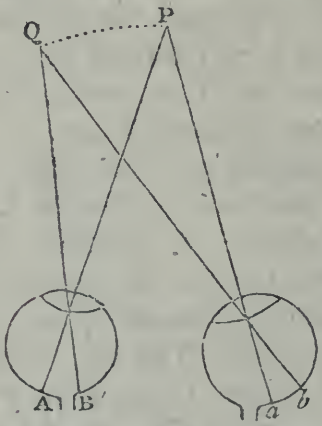
260. In explaining the nature and circumſtances of viſion, we have only to attend to the ſtructure of one eye; for, in whatever manner rays are refracted, and images formed by the humours of one eye, in the ſame manner will the ſame effects be produced by the humours of the other. The only queſtion that can ariſe is, how it happens that in viſion with both eyes, objects appear ſingle. It is not eaſy to decide, whether this effect is produced by the ſimilarity of correſponding parts of the optic nerves, and their union in the brain †, or by habit. In ſupport of the latter opinion, we may be allowed to alledge the following fact, related by Mr. Chesselden: A perſon had one of his eyes diſtorted by a blow; and, for ſome time, every object, to him, appeared double; but by degrees

* Optics, Query 16.

† Op. Query 15.

degrees the moſt familiar ones became ſingle, and in time, all objects became ſo, though the diſtortion continued *. To this we may add, that children ſometimes *learn* to ſquint; and by proper attention, this habit may again, in a great meaſure, be corrected. Under both circumſtances, objects appear ſingle, and it is manifeſt that the images cannot, in both caſes, fall upon correſponding points of the retinas.

261. But, whatever be the cauſe of ſingle viſion, when an object is viewed attentively, the axes of both eyes are, in general †, directed to it. Thus, if *P* be the object, the eyes are moved till the optic axes, *AP*,



aP, meet in *P*; and the images, *A*, *a*, are formed on correſponding points of the retinas ‡. In this poſition of

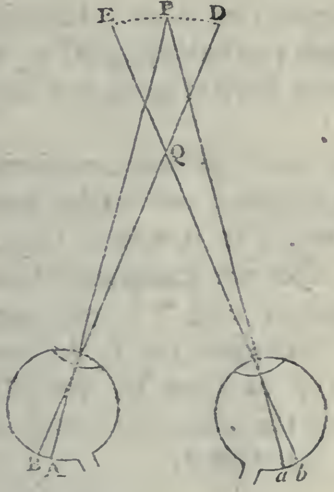
* SMITH's Optics, Art. 137.

† Perſons who ſquint do not direct the optic axes to the object they are looking at.

‡ The retina may perhaps be moſt ſuſceptible of the impreſſions of light where the optic axis meets it; and the images formed near that part of the retina, will be leſs diſtorted and more regularly and diſtinctly defined, than when the rays paſs more obliquely through the

of the images, whether from the correſpondence of the nerves, or from experience, the idea of a ſingle object is ſuggeſted to the mind; ſcarcely differing from the idea excited by one of the images alone, excepting that the object appears ſomewhat brighter when ſeen with both eyes, than when ſeen with one. Alſo, whilſt the eyes remain in the ſame poſition, the images, *B*, *b*, of *Q*, an object near to *P*, and at the ſame diſtance from the eyes, will be formed on the retinas; the eyes having aſſumed a proper conformation for diſtinct viſion at that diſtance; and *B*, *b*, which are both on the right, or both on the left of the reſpective axes, are correſponding points, and ſuggeſt the idea of a ſingle object, as in the former caſe.

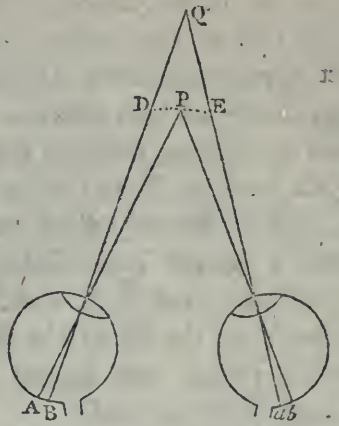
But, if an object *Q* lie between the optic axes, or thoſe



axes produced, it's images will be formed at *B*, *b*, on points

the humours of the eye. For one or both of theſe reaſons, we direct the axis of each eye to an object, when we wiſh to view it to the greateſt advantage.

points upon the retinas which do not correſpond; and thus they will excite the ſenſations, uſually produced



by different objects, *D*, *E*, at that diſtance to which the eyes are adapted for viſion. However, when the attention is more particularly called to the object *Q*, the optic axes are directed to it, and the points, *D*, *E*, coincide.

262. Of apparent diſtance, meaſured in a direct line from the ſpectator, nothing has been ſaid in the foregoing ſection. It is ſubject to no calculation, and therefore, does not immediately fall in with the plan of the preſent work. It may not however be improper to enumerate the cauſes upon which it depends, as they are given by Mr. HARRIS in his Optics, referring the reader, for farther information on the ſubject, to that work *.

1. *The change of conformation in the eye, is one of the means whereby we are enabled to judge of ſmall diſtances.*

In

* Page 154 . . . . 168.

In viewing near objects, or ſuch as are within about an arm's length of us, at every ſenſible change of diſtance, the eye muſt alſo change it's conformation for procuring diſtinct viſion. And thus, if we were accuſtomed to look attentively at objects with one eye only, it is very likely that the changes made on theſe occaſions, would be ſenſible enough, after repeated trials, to enable us to judge pretty accurately of the different degrees of ſmall diſtances.

This method, however, will not ſerve us to a greater extent than, perhaps, about 20 or 30 inches. Beyond which limit, the different degrees of divergency of rays in the different pencils which enter the eye, bear no ſenſible proportion to the different diſtances of the points from which they diverge.

2. *Inclination of the optic axes, is another more certain mean of diſtinguiſhing degrees of ſmall diſtances.*

When an object is viewed attentively with both eyes, the axis of each is directed towards it; and as the diſtance of the object is increaſed or diminiſhed, there is a correſponding diminution or increaſe in the angle at which theſe axes are inclined to each other. The ſenſations which accompany theſe different inclinations, enable us to determine with conſiderable accuracy, the places of objects which are not above five or ſix feet from us. As the diſtance becomes greater, we begin to be more uncertain in our eſtimations of it; and beyond three or four yards, the means, hitherto conſidered, ſeem to be of little or no uſe. For, beyond that extent, the differences of the optic

angles,

angles, arifing from the different diftances, are fo very fmall as to become in a manner infenfible.

3. *The length of the ground plane, or the number of intervening parts perceived in it, is another mean, by which we eftimate diftance.*

When the floor, or ground on which we ftand, is uniformly extended before us, in a line produced directly from us on this plane, we can diftinguifh, that fuch fucceffive parts as form fenfible angles at the eye, are fucceffive, or one behind the other; and the greater the number of vifible parts which the line contains, the greater, confequently, is the vifible extent of the whole.

Again, a row of houfes, columns, or trees regularly planted, appears longer than a plain wall of the fame extent. For, the more vifible and remarkable parts in the former cafe, enable us to correct the eftimate we make when fuch objects do not intervene; and alfo, our previous knowledge that the feveral intermediate objects are difpofed at equal intervals, tends to protract the apparent length of the whole chain ftill farther. A river, at firft, looks not fo broad, as after we have had a fide view of the bridge acrofs it: and indeed, a given extent of water, does not appear fo long as the fame extent of land; as it is more difficult to diftinguifh parts in the furface of the one, than it is in the furface of the other.

4. *Different degrees of apparent diftance are fuggefted by the different appearances of known objects, or by the known magnitudes of their leaft vifible parts.*

A

A building, none of whofe parts are difcernible, appears much farther off than another, whofe windows and doors are vifible; and this latter appears farther off than a houfe having vifible parts which are known to be ftill fmaller; as the bricks in the wall, tiles on the roof, &c. Objects of unufual magnitudes, detached as it were, from others, miflead us in our judgement of diftances; the greater magnitude ufually fuggefting to the mind, the idea of lefs diftance.

5. *All other things being the fame, different colours and degrees of brightnefs of objects, caufe a difference of apparent diftance.*

As objects become more and more remote, the light, which arrives at the eye from their leaft vifible parts, is continually diminifhed (Art. 258); and they appear more faint, languid, and obfcure. Alfo, their colours not only gradually lofe their luftre, but likewife degenerate from their native hue, and participate more of the blueifh colour of the fky, as the rays have paffed through a greater body of air, or as the images upon the retina are tinged with a greater proportion of fky light.

Thefe different appearances are of ufe to us in judging of the real diftances of known objects; and confequently affect the ideas of apparent diftances; thofe objects that are brighteft, and whofe colours are moft vivid, appearing neareft. Thus, in foggy weather all objects appear farther off than ordinary; the diminution of light, in this cafe, producing the effect of that diminution which arifes from greater diftance.

263. When objects, which subtend small angles at the center of the eye, are of the same colour and brightness, and at the same distance, their apparent magnitudes are proportional to those angles (Art. 253). And when they are at different distances, and subtend equal angles at the center of the eye, since their *real* magnitudes are proportional to their *real* distances, it is probable that their *apparent* magnitudes are nearly proportional to their *apparent* distances *. And thus, in general, the apparent magnitudes are as the visual angles, and apparent distances, jointly.

Hence it follows, that any error in our estimate of apparent distance, will produce a proportional error in our estimate of magnitude. Thus, in foggy weather, at the same time that objects appear farther off, they appear larger; and the diameter of the sun, or moon, appears greater, or less, according as we are led by circumstances to suppose it's distance greater or less at one time than at another.

* Any error in this proposition must arise from the limited experience we have of the truth of the former. The magnitude of which we are here speaking, is the *linear* magnitude.

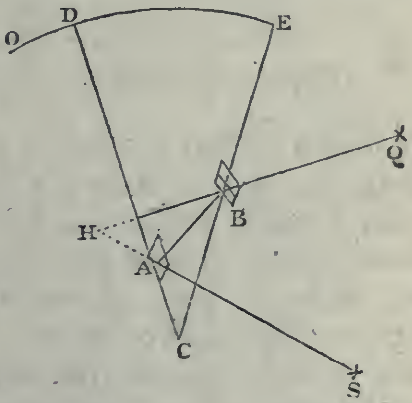
SECTION

SECTION VII.

ON OPTICAL INSTRUMENTS.

On HADLEY's *Quadrant.*

264. UPON the radii *DC*, *EC*, of the quadrant *OEC*, and at right angles to it's plane, are fixed two plane reflectors *A*, *B*, whose surfaces are



parallel when the index *D*, on the moveable radius *CD*, is brought to *O*; and consequently, the arc *OD* will

 measure

meaſure their inclination when the moveable radius *CD* is in any other ſituation. The whole ſurface of the glaſs *B* is not quickſilvered, a part of it being left tranſparent that objects may be ſeen directly through it, and by rays which paſs cloſe to the quickſilvered, or reflecting part.

When the angular diſtance of two objects, *S*, *Q*, is to be taken, the quadrant is held in ſuch a poſition that it's plane paſſes through them both; and the radius *CD* is moved till one of them *S* is ſeen, after two reflections of the incident ray *SA*, in the direction *HQ*; and the other *Q*, by the direct ray *QH*, in the ſame line; that is, till the objects apparently coincide. Then, if *SA* be produced till it meets *QH* in *H*, the angle *SHQ*, contained between the firſt incident, and laſt reflected ray, is equal to twice the angle of inclination of the two reflectors (Art. 39); therefore, the angular diſtance of the two objects is meaſured by twice the arc *DO**.

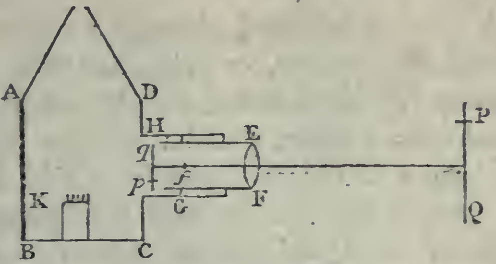
On the Magic Lantern.

265. The figure *ABCD* repreſents a tin box, or lantern, in the fore part of which is a ſliding tube, furniſhed with a double convex lens *EF*. Between the lantern and the lens, a ſmall ſpace, *qp*, is left to admit a thin plate of glaſs, upon which inverted figures are painted in tranſparent colours.

When this inſtrument is uſed, the lamp *K* being lighted, and the room darkened, the tube is moved, till

* The method of adjuſting this inſtrument may be ſeen in Mr. Vince's Practical Aſtronomy, p. 8.

till qp is farther from the lens than it's principal focus f; and consequently PQ, an inverted image of pq, or an



erect image of the figure intended to be represented, is formed at some distance from the lens (Art. 211), and painted in it's proper colours upon a screen placed at the concourse of the refracted rays.

Sometimes a reflector is placed behind the lamp, or a convex lens before it, for the purpose of throwing a greater quantity of light upon pq.

266. Cor. If the screen and lantern be fixed, and their distance exceed four times the focal length of the lens, the image may be thrown upon the screen, by moving the lens nearer to, or farther from pq, as the case requires.

For, $qf : qE :: qE : qQ$ (Art 176), or $qE - fE : qE :: qE : qQ$; in which proportion there is only one unknown quantity, qE, which may be determined by the solution of a quadratic equation whose roots are possible, except the distance qQ be less than four times the focal length of the lens EF*.

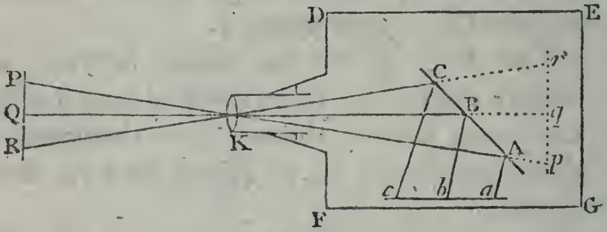
On

* Let $qE = x$; $fE = a$; $qQ = b$. Then, $x - a : x :: x : b$; therefore $x^2 = bx - ba$; or, $x^2 - bx = -ba$; hence, $x^2 - bx + \frac{b^2}{4} = \frac{b^2 - 4ba}{4}$; and $x = \frac{b}{2} \pm \frac{\sqrt{b^2 - 4ba}}{2}$. If $b = 4a$, we have $x = 2a$. If b be less than $4a$, the expression is impossible, which shews that the image cannot, in this case, be formed upon the screen.

On the Camera Obſcura.

267. If light be admitted, through a convex lens, into a darkened chamber, or into a box from which all extraneous light is excluded, and the refracted rays be received upon a ſcreen, placed at a proper diſtance, inverted images of external objects will be formed upon it. And if the lens be fixed in a ſliding tube, the images of objects at different diſtances may ſucceſſively be thrown upon the ſcreen, by moving the lens backwards or forwards, as in the magic lantern.

Let *PQR* be an object at a conſiderable diſtance from the lens, and at right angles to it's axis; the image *pqr*, will be formed, nearly in the principal

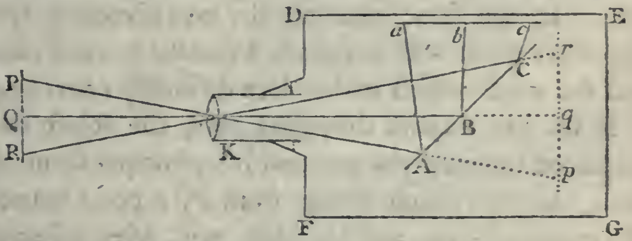


focus of the glaſs, and may be received upon a ſcreen placed there. But, in general, the rays are intercepted before they form the image, by a plane reflector, *AC*, inclined at an angle of 45° to the axis of the lens; or, which is the ſame thing, at an angle of 45° to the image *pqr*; by this means, an image *abc* is formed, ſimilar and equal to *pqr*, and inclined at an equal angle to the reflector (Art. 72), and conſequently, parallel to the axis of the lens. If the lens be moved towards the reflector, the image *pqr*, (which is in the principal

principal focus of the lens,) and consequently, *abc*, will move from the reflector; and the contrary. Thus, the image *abc*, may, by a proper adjustment of the lens, be thrown upon a table, or any surface prepared to receive it.

268. Cor. 1. When the image is thrown downwards, it will appear erect to a spectator between *DF* and the reflector *ABC*; since the point *a*, which corresponds to *p*, or to *P* in the object, is nearest to the reflector. Also, the axis of a pencil of rays which flows from a point to the right of *Q*, will cross the axis *Qq* at the center of the lens, and the rays will form an image which, to this spectator, is to the right of *b*.

269. Cor. 2. When the image is thrown upwards, if it be viewed by a spectator whose face is turned towards the object, it will appear erect as to the top and

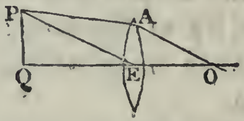


the bottom; for the point *a*, which corresponds to *p*, or to *P* in the object, is the farthest point in the image from the reflector; and therefore, the farthest point from the spectator. But, with respect to the right and the left, the image will be inverted. For, the axis of a pencil of rays which flows from a point in the object to the right of *Q*, will cross the axis *Qq* at the center of the lens, and the rays will be collected at a point which, to this ſpectator, is to the left of *b*.

Prop. LIX.

270. *An object may be ſeen diſtinctly through a convex lens.*

Let *AE* be a convex lens*; *E* it's center; *PQ* an object placed in it's principal focus. Then, the rays



which diverge from any point *P* will be refracted parallel to each other, and to *PE* (Art. 165); and therefore, they will be proper for viſion to common eyes. In the ſame manner, the rays diverging from any other point will be refracted parallel to each other, and the whole object will be ſeen diſtinctly (Art. 244).

If the eye require diverging rays, the object muſt be placed between the lens and it's principal focus; for then, the rays which diverge from *P*, a point between the principal focus and the glaſs, will, after refraction diverge (Art. 182); and therefore be proper for diſtinct viſion in this caſe.

If the eye require converging rays, the object muſt be placed beyond the principal focus; for then, the rays in each pencil will, after refraction, converge.

Prop.

* Of this deſcription are the double convex lens, the plano convex, and the meniſcus.

Prop. LX.

271. *When an object is placed in the principal focus of a convex lens through which it is viewed, it appears erect.*

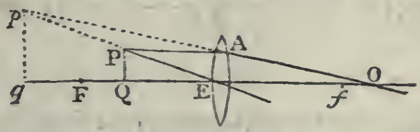
For, if *AO* be drawn parallel to *PE* (Fig. Art. 270), the rays which diverge from *P*, and are received by the eye, enter the pupil in the direction *AO*; and the rays which diverge from *Q*, enter the pupil in the direction *EO*. Thus, the rays which flow from the extreme points *P*, *Q*, of the object, croſs each other at *O*, and therefore the picture upon the retina is inverted; or, the object appears erect (Art. 239).

272. Cor. In the ſame manner, if the object be *near* to the principal focus, and the eye not very diſtant from the glaſs, the image upon the retina will be inverted; and conſequently, the object will appear erect.

Prop. LXI.

273. *To determine the variation of the angle which a given object ſubtends at the center of the eye, when viewed through a convex lens.*

Let *E* be the center of the lens; *F* and *f* it's principal foci; *PQ* the given object; *pq* it's image; *O*



the place of the eye; join *Op*, and let it meet the lens in *A*. Then, the rays which diverge from *P*, enter the

the eye in the direction AO; and thoſe which diverge from Q, enter the eye in the direction EO; therefore, the $\angle AOE$, or pOq, is the angle which the object, thus ſeen, ſubtends at the center of the eye; and this angle varies as $\frac{pq}{Oq}$*. Now, $QF : FE$ ($:: QE : Eq$) $:: QP : qp$, therefore $qp = \frac{FE \times QP}{QF}$. Alſo, $QF : FE :: Ef : fq$ (Art. 179); therefore, $fq = \frac{FE \times Ef}{QF}$, and, when Q is between F and E, and O beyond f, $Oq = fq + Of = \frac{FE \times Ef}{QF} + Of = \frac{FE \times Ef + QF \times Of}{QF} = \frac{FE \times Ef + \overline{FE - QE} \times Of}{QF} = \frac{FE \times \overline{Ef + Of} - QE \times Of}{QF} = \frac{FE \times OE - QE \times Of}{QF}$. Conſequently, $\frac{pq}{Oq} = \frac{FE \times QP}{FE \times OE - QE \times Of}$; therefore, the viſual angle varies as $\frac{FE \times QP}{FE \times OE - QE \times Of}$; or, ſince FE and QP are invariable, inverſely as $FE \times OE - QE \times Of$.

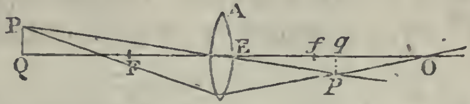
When O is between E and f, Of is negative, and the viſual angle varies inverſely as $FE \times EO + QE \times Of$.

When the image pq is between O and f, the expreſſion, $FE \times EO - QE \times Of$, becomes negative; for, $Oq = Of - fq = \frac{QE \times Of - FE \times EO}{QF}$; and therefore

pq

* In theſe, and other calculations of the ſame kind, the angles contained by the axes of the extreme pencils, at the center of the eye, are ſuppoſed to be ſmall; and our concluſions, though not ſtrictly true, are ſufficiently accurate for the purpoſes to which they are applied.

$\frac{pq}{Oq} = \frac{FE \times \mathcal{Q}P}{\mathcal{Q}E \times Of - FE \times EO}$; and the viſual angle varies inverſely as $\mathcal{Q}E \times Of - FE \times EO$. This ſhews that the angle pOq lies on the other ſide of the



axis $\mathcal{Q}O$; and the image upon the retina, which was before inverted, will now be erect.

274. Cor. 1. When the eye is placed cloſe to the glaſs, the expreſſion, $FE \times EO + \mathcal{Q}E \times Of$, becomes $\mathcal{Q}E \times Ef$; therefore the viſual angle varies inverſely as $\mathcal{Q}E$. In this caſe, the $\angle pOq =$ the $\angle PE\mathcal{Q}$.

275. Cor. 2. When O coincides with f, the expreſſion becomes $FE \times Ef$, which is invariable. That is, when the eye is in the principal focus of the glaſs, the viſual angle is the ſame, whatever be the diſtance of the object from the lens.

276. Cor. 3. When $\mathcal{Q}$ coincides with F, the expreſſion becomes $FE \times EO \pm FE \times Of$, or $FE \times Ef$.

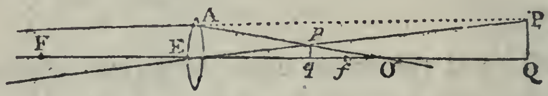
That is, the viſual angle is the ſame, whatever be the diſtance of the eye from the glaſs; and it is equal to the viſual angle when the eye is in the principal focus, and to the angle which the object ſubtends at the center of the lens.

277. Cor. 4. When the eye is farther from the glaſs than the principal focus f, as $\mathcal{Q}E$ decreaſes, $FE \times EO - \mathcal{Q}E \times Of$ increaſes; and, therefore, the viſual angle decreaſes, unleſs the image fall between the eye and the glaſs; in which caſe the viſual angle varies inverſely as $\mathcal{Q}E \times Of - FE \times EO$; and therefore it increaſes as $\mathcal{Q}E$ decreaſes.

278. Cor.

278. Cor. 5. When the eye is between the principal focus and the lens, as QE decreases, the expression $FE \times EO + QE \times Of$ decreases; and, therefore, the visual angle increases.

279. Cor. 6. When the rays, tending to form the image QP, are intercepted by the glass, and afterwards



received by the eye, QE becomes negative; and the visual angle, when EO is greater than Ef, varies inversely as $FE \times EO + QE \times Of$. When EO is less than Ef, the visual angle varies inversely as $FE \times EO - QE \times Of$, or $QE \times Of - FE \times EO$, according as EO is greater, or less than Eq.

Prop. LXII.

280. *The rays which, after reflection or refraction, tend to form an image, may be refracted to the eye by a concave lens, in such a manner as to form a distinct image upon the retina.*

Let the rays which tend to form the image PQ, be received upon the concave lens AE *, whose focal

length is EQ. Then, since P is the principal focus of

* Of this description are the double concave, the plano concave, and the concavo convex lenses.

of the lens, the rays which converge to *P* will, after refraction, be parallel to each other (Art. 165), and therefore proper for vision to common eyes; that is, a distinct image of *P* will be formed upon the retina. In the same manner it appears, that a distinct image of every other point in *PQ* will be formed upon the retina; and thus a complete, and distinct image of the whole object will be formed there.

If the eye require diverging rays, the glass must be moved farther from the image *PQ*. For, then the rays in each pencil converge to a point farther from the glass than the principal focus; and therefore, after refraction they will diverge (Art. 215), and be proper for vision in this case.

If the eye require converging rays, the lens must be moved the contrary way.

Prop. LXIII.

281. *When the image* QP *is in the principal focus of the concave lens, the picture upon the retina is erect with respect to* QP.

Let *E* be the center of the lens *; *O* the place of the eye; *P* the lowest point in the image; join *PE*; and draw *AO* parallel to *EP*. Then, those rays of the pencil belonging to *P*, which are received by the eye, enter the pupil in the direction *AO*, and proceed to the lower part of the retina; and those which belong to *Q*, enter the pupil in the direction *EO*, and proceed to the upper part; consequently, the picture upon the retina is erect with respect to *QP*.

282. Cor.

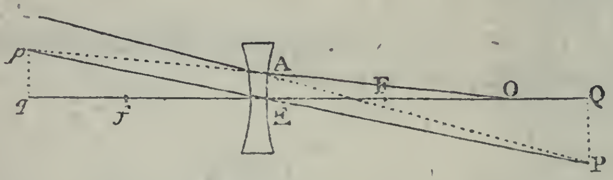
* See the last figure.

282. Cor. In the ſame manner it appears, that when QP is *nearly* in the principal focus, and the eye not very diſtant from the glaſs, the image upon the retina is erect.

Prop. LXIV.

283. *To find how the viſual angle varies, when rays, which tend to form an image in the axis of a concave lens, are refracted to an eye ſituate in that axis.*

Let rays which tend to form the image QP be intercepted by the concave lens AE, whoſe center is E, and axis qFQ; and after refraction let them be received



by an eye placed at O. Take qp the image of QP, and join pO. Then, the viſual angle pOq varies as $\frac{qp}{Oq}$. Now, $QF : FE$ (:: $QE : Eq$) :: $QP : qp$; therefore, $qp = \frac{QP \times FE}{QF}$. Alſo, $QF : FE :: Ef : fq$; whence, $fq = \frac{FE \times Ef}{QF}$; and, when EQ is greater than EF, $Oq = fq + Of$ (Art. 181) $= \frac{FE \times Ef + QF \times Of}{QF} = \frac{FE \times Ef + \overline{QE - FE} \times Of}{QF} = \frac{FE \times \overline{Ef - Of} + QE \times Of}{QF} = \frac{QE \times Of - FE \times EO}{QF}$; therefore, $\frac{qp}{Oq} =$

$QP \times$

$\frac{QP \times FE}{QE \times Of - FE \times EO}$; and ſince QP and FE are given, the viſual angle varies inverſely as $QE \times Of - FE \times EO$.

284. Cor. 1. When the eye is placed cloſe to the glaſs, EO vaniſhes; therefore the viſual angle varies inverſely as $QE \times Of$; that is, inverſely as QE. In this caſe, the angle qOp becomes equal to QEP.

285. Cor. 2. When Q coincides with F, the expreſſion, $QE \times Of - FE \times EO$, becomes $FE \times \overline{Of - EO} = FE \times Ef$, which is invariable; in this caſe, therefore, the viſual angle is the ſame, whatever be the diſtance of the eye from the glaſs.

286. Cor. 3. When the expreſſion, $QE \times Of - FE \times EO$, becomes negative, the image upon the retina, which was before erect, will be inverted with reſpect to QP.

On the aſtronomical Teleſcope.

287. The aſtronomical teleſcope conſiſts of two convex lenſes, whoſe axes are in the ſame right line, and whoſe diſtance is equal to the ſum of their focal lengths.

288. The common axis of the two lenſes is called the *axis* of the teleſcope.

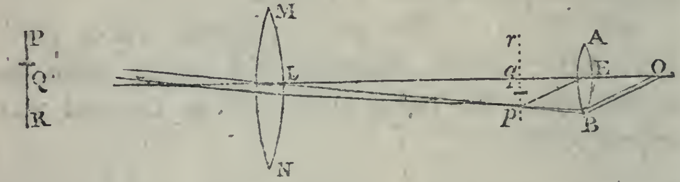
289. The lens which is turned towards the object to be viewed, and which has the greater focal length, is called the *object glaſs*. The lens to which the eye is applied, is called the *eye glaſs*.

PROP.

Prop. LXV.

290. *A diſtant object may be ſeen diſtinctly through the aſtronomical teleſcope; and the angle which it ſubtends at the center of the eye, when thus ſeen, is to the angle which it ſubtends at the naked eye, as the focal length of the object glaſs, to the focal length of the eye glaſs.*

Let L and E be the centers of the two glaſſes; QP an object, towards which the axis of the teleſcope is directed; and ſo diſtant, that the rays which flow from any one point in it, and fall upon the object glaſs L,



may be conſidered as parallel. Then qp, an inverted image of QP, will be formed in the principal focus of the glaſs L, and contained between the lines QLq, and PLp (Art. 204); and, becauſe LE is equal to the ſum of the focal lengths of the two glaſſes, pq is in the principal focus of the glaſs AB; conſequently, pq may be ſeen diſtinctly through this glaſs, if the eye of the obſerver be able to collect parallel rays upon the retina (Art. 270). Produce PLp till it meets the eyeglaſs in B; join pE; and draw BO parallel to pE. Then, the rays which flow from P in the object, or p it's image, enter the eye, placed at O, in the direction BO. Alſo, the rays which flow from Q, enter the eye in the direction EO; thus, the angle which QP ſubtends at the center of the eye, when viewed through the teleſcope, is the angle

angle BOE, which is equal to pEq. The angle which QP ſubtends, when viewed with the naked eye from L, is PLQ, which is equal to pLq. And, ſince the $\angle pEq$: the $\angle pLq$ (when theſe angles are ſmall) * :: $Lq : Eq$, the angle which the object ſubtends at the center of the eye, when ſeen through the teleſcope : the angle which it ſubtends at the center of the naked eye :: $Lq : Eq$.

291. Cor. 1. If the angle which the object ſubtends at the center of the naked eye be given, the angle which it ſubtends at the center of the eye, when ſeen through the teleſcope, varies as $\frac{Lq}{Eq}$. This quantity is uſually called the *magnifying power* of the teleſcope.

292. Cor. 2. If the teleſcope be inverted, the object may be ſeen diſtinctly; and the viſual angle will be as much diminiſhed as it was magnified in the former caſe.

293. Cor. 3. To adapt the teleſcope to a nearer object, the eye glaſs muſt be moved farther from the object glaſs.

For, if QP be brought nearer to the glaſs L, the image qp will be formed at a greater diſtance from it (Art. 182); and therefore, in order that qp may ſtill be in the principal focus of the glaſs E, this glaſs muſt be moved farther from L.

294. Cor. 4. When QP is brought nearer to L, the magnifying power is increaſed. For, Lq is increaſed, and Eq remains the ſame; therefore $\frac{Lq}{Eq}$ is increaſed.

295. Cor. 5. To adjuſt the teleſcope to the eye of a

* Proportions of this kind are to be conſidered only as approximations, which become more accurate as the angles decreaſe.

a ſhort ſighted perſon, the eye glaſs muſt be moved nearer to the object glaſs. If the eye require converging rays, the eye glaſs muſt be moved the contrary way (Art. 270).

296. Cor. 6. If we ſuppoſe the eye to be placed between the glaſs AB and it's principal focus, the viſual angle is increaſed by adjuſting the teleſcope to the eye of a ſhort ſighted perſon (Art. 278). If the eye be farther from the glaſs than it's principal focus, that angle is diminiſhed (Art. 277). The contrary effects are produced when the teleſcope is adjuſted to the eye of a long ſighted perſon.

Prop. LXVI.

297. *Objects, ſeen through the aſtronomical teleſcope, appear inverted.*

The image of pqr is inverted upon the retina (Art. 271); and pqr is inverted with reſpect to PQR (Art. 211); therefore the image upon the retina is erect with reſpect to PQR; and conſequently the object appears inverted (Art. 239).

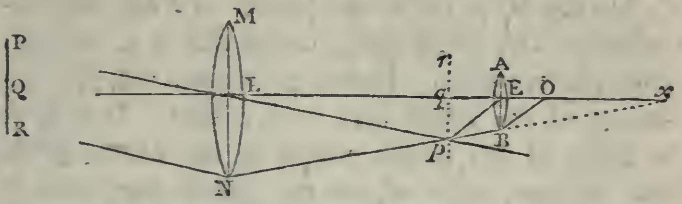
298. Cor. An object moving acroſs the field of the teleſcope from the right to the left, appears to move from the left to the right.

Prop. LXVII.

299. *In the aſtronomical teleſcope, the field of view is the greateſt, when the eye is placed between the eye glaſs and it's principal focus.*

The field of view is the greateſt, when the eye is ſo placed as to receive the extreme rays, refracted from the

the object glaſs to the eye glaſs. Let *MN* be the diameter of the object glaſs; *AB*, the diameter of the



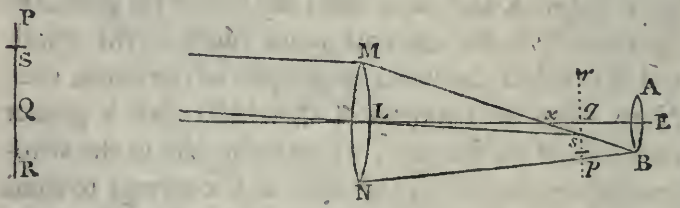
eye glaſs; join *N*, *B*, the correſponding extremities of the glaſſes, and let *NB* cut the image *pqr* in *p*; join *pL*, and ſuppoſe *pL*, *EL* to be produced till they meet the object *QP*, in *P* and *Q*; join alſo, *pE*, *NP*; and draw *BO* parallel to *pE*. Then, if the eye be placed at *O*, it will receive the ray *NBO*. And this is the ray which comes from a point in the object, at the greateſt viſible diſtance from the axis of the teleſcope; for, the rays which flow from any point in the object, above *P*, will, after the firſt refraction, converge to a point below *p*, and all of them fall below the glaſs *AB*; therefore *P* is the extreme point viſible in the object, and *QP* is half the linear magnitude of the viſible area. Now, when the diameter of the object glaſs is greater than that of the eye glaſs, which is the caſe in the aſtronomical teleſcope, the rays *NB*, *LE* converge to ſome point *x* beyond the eye glaſs; and therefore, the point *O* to which they converge after the ſecond refraction, and where the eye muſt be placed to receive them, is between the glaſs *E* and it's principal focus (Vid. Arts. 179. 181.)

300. Cor. 1. If *NB* and *LE* be parallel, *O* is the principal focus of the glaſs *E*; and if they diverge, *O* lies beyond the principal focus.

 301. Cor.

301. Cor. 2. It appears from the preceding figure, that *NBO* is the only ray which comes to the eye from *P*; for, any other ray of the pencil, as *PLp*, after refraction at the object glaſs, croſſes *NB* in *p*, and falls below the eye glaſs. Hence it follows, that the extremity of the viſible area is very faint. If a point be taken in the object, nearer to the center of the field of view, more of the rays which flow from it will be refracted to the eye; and thus, the brightneſs will continually increaſe till all the rays in each pencil, incident upon *MN*, are received by the glaſs *AB*; when this takes place, the brightneſs will become uniform; becauſe the ſame number of rays, or very nearly ſo, is received by the glaſs *MN* from every point in the object *.

302. Cor. 3. To determine the bright part of the field of view, join *MB*, cutting *pqr* in *s*, and the axis *LE* in *x*; join alſo *sL*, and produce it till it meets



the object in *S*. Then, if *xE* be greater than *qE*, all the rays which flow from *S*, and fall upon *MN*, will be refracted to the eye glaſs; for, *SM* is refracted in the direction *MsB*; and any other ray of the pencil, as *SL*, croſſes *MB* at *s*, and falls ſomewhere between *A* and

* Here it is ſuppoſed that the pupil is properly placed, and ſo large as to receive all the rays refracted by the lens *AB*.

A and *B*. In the ſame manner, the rays which flow from any point between *S* and *Q*, and fall upon *MN*, will be refracted to the eye glaſs *AB*.

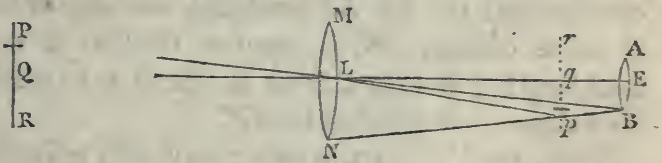
303. Cor. 4. If *q* and *x* coincide, that is, if the linear apertures, *MN*, *AB*, of the glaſſes, be proportional to their focal lengths, the brightneſs of the field increaſes to the center.

304. Cor. 5. If *q* fall between *L* and *x*, the whole of the rays belonging to any one pencil incident upon *MN*, will not be received by the eye glaſs. In this caſe, a leſs aperture of the object glaſs would produce the ſame brightneſs, with leſs aberration *.

Prop. LXVIII.

305. *The linear magnitude of the greateſt viſible area is meaſured by the angle which the diameter of the eye glaſs ſubtends at the center of the object glaſs, increaſed by the difference between the angles which the diameter of the object glaſs ſubtends at the image, and at the eye glaſs.*

Let *pqr*, as in the preceding propoſitions, be the image formed by the object glaſs. Join *NB*, cutting



pqr in *p*; join alſo, *LB*, *Lp*; and ſuppoſe *pL*, *EL* to be produced till they meet the object in *P* and *Q*. Then, *QP*, which is half the linear magnitude of the greateſt viſible

* Vid. Sect. VIII.

visible area (Art. 299), is measured by the angle PLQ, or it's equal pLq; that is, by $BLE + BLp$; or, $BLE + LpN - LBN$; therefore, $2QP$ is measured by $2BLE + 2LpN - 2LBN$.

306. Cor. In the same manner it may be shewn, that the linear magnitude of the brightest part of the visible area, is measured by the angle which the diameter of the eye glass subtends at the center of the object glass, diminished by the difference between the angles which the diameter of the object glass subtends at the image, and at the eye glass.

Prop. LXIX.

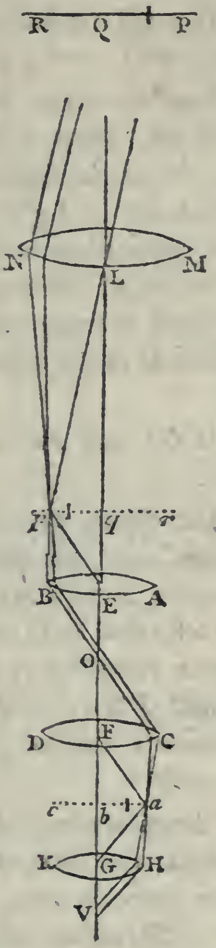
307. *If two convex lenses be added to the former, and placed in a similar manner, a distant object may be seen distinctly through the telescope, and the visual angle will be altered in the ratio of the focal lengths of the additional glasses.*

Let CD, HK, be the additional glasses, whose axes coincide with the line LE produced, and the distance of whose centers, FG, is equal to the sum of their focal lengths; also, let CD be so placed as to receive the extreme rays refracted by AB.

Then, since the rays in each pencil, after refraction at the glass E, are parallel (Art. 290), they will be collected in the principal focus of the glass CD; that is, in the principal focus of the glass HK; and an image, abc, will be formed there, which may be seen distinctly through the lens HK (Art. 270).

Again,

Again, let *BC* be the extreme pencil of rays refracted by *AB*; draw *Fa* parallel to *BC*, and let it meet *abc*



in *a*; join *aG*, *Ca*; and produce *Ca* till it meets the lens *HK* in *H*; draw *HV* parallel to *aG*; and at *V* let the eye be placed. Then, *NBCHV* being the course of the pencil of rays which flows from *P*; and *LEFGV*

the

the course of the pencil which flows from Q, the angle which QP subtends at the center of the eye, when seen through the four glasses: the angle which it subtends there, when seen through the first two :: the $\angle HVG$: the $\angle EOB$ :: the $\angle aGb$: the $\angle aFb$ :: $Fb : Gb$.

308. Cor. 1. Since the angle which QP subtends at the center of the eye, when it is seen through the two first glasses : the angle which it subtends at the center of the naked eye :: $Lq : Eq$ (Art. 290), by compounding this proportion with the last, we have, the angle which the object subtends at the center of the eye, when viewed through the four glasses : the angle which it subtends at the naked eye :: $Lq \times Fb$: $Eq \times Gb$.

309. Cor. 2. If Gb and Fb be equal, the visual angle is not altered.

310. Cor. 3. The object, when viewed through the glasses thus combined, appears erect. For, the axes of the several pencils of rays which flow from the image pqr, cross each other at O; therefore, the image abc is inverted with respect to pqr; or, erect with respect to the object PQR; and consequently, the image upon the retina is inverted (Art. 271); that is, the object appears erect *.

311. Cor. 4. If the apertures of the additional glasses be properly adjusted, the field of view will not be altered. For, the extreme rays, refracted by AB, may be received by CD, and refracted to HK.

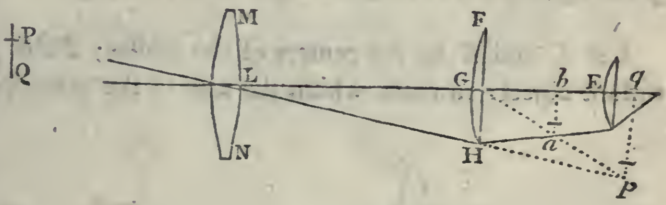
312. Cor.

* This is one advantage gained by the additional glasses. The inconvenience with which they are attended is, that they render objects

312. Cor. 5. This teleſcope may be adjuſted to the eye of a ſhort ſighted perſon, by moving the glaſs *HK* nearer to *CD* (Art. 270); or, if *E*, *F*, *G*, be connected, by moving theſe three glaſſes nearer to *L*. If the eye of the obſerver require converging rays, the glaſſes muſt be moved the contrary way.

313. Sometimes a convex lens is interpoſed between the object glaſs and it's principal focus, in the aſtronomical teleſcope, for the purpoſe of increaſing the field of view, and diminiſhing the aberration of the lateral rays (Sect. 8).

Let *FGH* be ſuch a lens, whoſe axis is coincident with the axis of the teleſcope. Then, the rays which



tend to form the image *qp*, after refraction at the glaſs *FH*, form the image *ab*, between G and *q* (Art. 213); and this image is viewed through the eye glaſs *E*, whoſe focal length is *bE* *.

On

objects more faint, by increaſing the quantity of the refracting medium through which the rays muſt paſs, before they arrive at the eye (Vid. note, p. 137.)

* Other conſtructions, with the explanation of their advantages, may be ſeen in the *Encyclopædia Britannica*, Article *Teleſcope*.

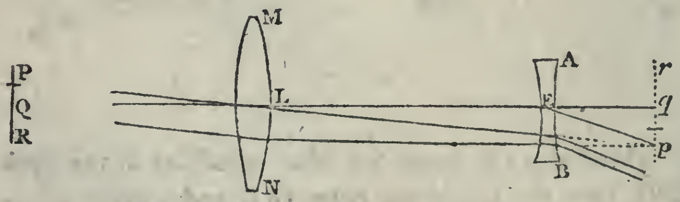
On Galileo's Teleſcope.

314. Galileo's teleſcope conſiſts of a convex and a concave lens, whoſe axes are in the ſame line, and whoſe diſtance is equal to the difference of their focal lengths.

Prop. LXX.

315. *A diſtant object may be ſeen diſtinctly through Galileo's teleſcope; and the angle which it ſubtends at the center of the eye when thus ſeen, is to the angle which it ſubtends at the center of the naked eye, as the focal length of the object glaſs, to the focal length of the eye glaſs.*

Let L and E be the centers of the glaſſes; PQR a diſtant object, towards which the axis of the teleſcope



is directed; pqr it's image in the principal focus of the glaſs L, and therefore in the principal focus of the glaſs E; then, ſince the rays tend to form an image in the principal focus of the concave lens E, after refraction at that lens, they will be proper for viſion (Art. 284); or, a diſtinct image of the object PQR, will be formed upon the retina of a common eye.

Alſo,

Alſo, the angle under which the object QP is ſeen through the teleſcope, is equal to the angle qEp (Art. 284); and the angle under which it is ſeen with the naked eye from L, is QLP, which is equal to qLp; therefore, the viſual angle in the former caſe : the viſual angle in the latter :: $Lq : Eq$.

316. Cor. 1. The magnifying power of the teleſcope is meaſured by $\frac{Lq}{Eq}$ (Vid. Art. 291).

317. Cor. 2. To adapt this teleſcope to a nearer object, the eye glaſs muſt be moved to a greater diſtance from the object glaſs.

For, as QL decreaſes, Lq increaſes (Art. 182); therefore, in order that the principal focus of the glaſs E may coincide with q, LE muſt be increaſed.

This is the conſtruction of a common opera glaſs.

318. Cor. 3. When the teleſcope is adjuſted to a nearer object, the magnifying power is increaſed; for Lq is increaſed, and Eq remains the ſame; therefore $\frac{Lq}{Eq}$ is increaſed.

319. Cor. 4. To adjuſt this teleſcope to the eye of a ſhort ſighted perſon, the eye glaſs muſt be moved nearer to the object glaſs; if the eye require converging rays, the eye glaſs muſt be moved the contrary way (Art. 280).

320. Cor. 5. To take in the greateſt field of view, the eye muſt be placed cloſe to the glaſs AB, as will be ſhewn in a ſubſequent article; and therefore, by adjuſting the teleſcope to the eye of a ſhort ſighted perſon, the viſual angle of a given object is diminiſhed (Art. 284); and on the contrary, by adjuſting it to the eye of a long ſighted perſon, this angle is increaſed.

PROP.

Prop. LXXI.

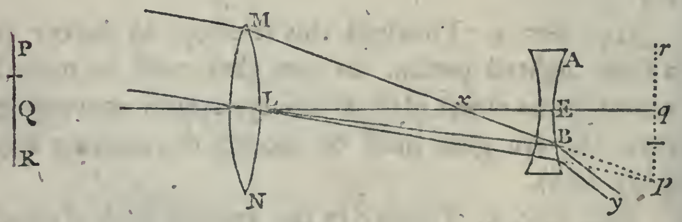
321. *Objects, seen through Galileo's telescope, appear erect.*

For, the image upon the retina is erect, with respect to pqr (Art. 281); therefore it is inverted with respect to the object; or the object appears erect (Art. 239).

Prop. LXXII.

322. *The linear magnitude of the field of view, when the eye is placed close to the concave lens in Galileo's telescope, is measured by the angle which the diameter of the pupil subtends at the center of the object glass, increased by the difference between the angles which the diameter of the object glass subtends at the pupil, and at the image.*

Let MN be the diameter of the object glass; AB, the diameter of the pupil, whose center is in the axis of

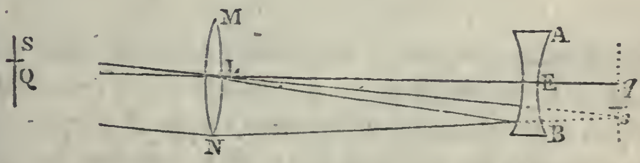


the telescope; join M, B, the opposite extremities of these diameters; and let MB meet the axis in x, and the image in p; join LB, Lp; and suppose pL, qL to be produced till they meet the object in P and Q; also, suppose MP to be joined. Then will PM be refracted

refracted to the pupil in the direction MB; but any other ray in the pencil, as PL, and every ray which flows from a point more distant from the axis of the telescope, and is refracted by MN, will fall below the pupil; therefore QP is half the linear magnitude of the greatest visible area. Also, QP is measured by the angle QLP, or it's equal $pLq = ELB + BLp = ELB + LBM - LpM$; and by doubling these quantities, $2QP$ is measured by $2ELB + 2LBM - 2LpM$.

323. Cor. 1. The rays MxB, LxE, which are incident upon the glass E, diverging from x, after refraction will diverge more (Art. 183); therefore if the eye be moved to any other point in the axis of the telescope, the ray xBy will not enter the pupil; and consequently the visible area will be diminished.

324. Cor. 2. To determine the brightest part of the visible area, join NB, and let it meet the image in s; join BL, sL, and produce sL till it meets the



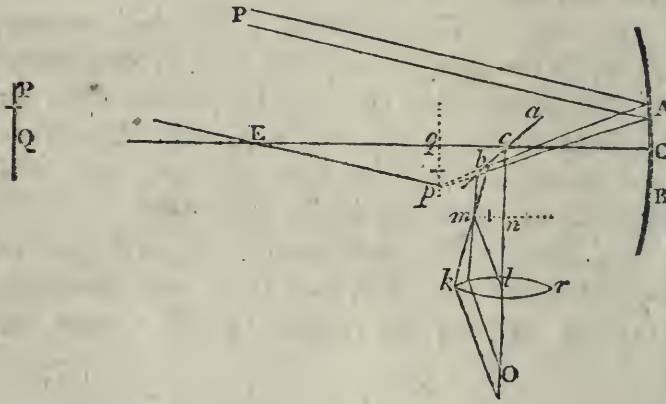
object in S. Then, all the rays which flow from S, or from any point between S and Q, and fall upon the object glass, will be refracted to the eye*; and QS will be the linear magnitude of half the bright part of the field of view. Also, QS is measured by the angle SLQ, which is equal to $sLq = BLE - BLs = BLE - LBN$

* Vid. Art. 301.

$LBN + LsN$; and $2\mathcal{Q}S$ is meaſured by $2BLE - 2LBN + 2LsN$.

On Sir ISAAC NEWTON'S *Teleſcope.*

325. Let ACB be a concave ſpherical reflector, whoſe middle point is C, and center E. Join CE,



this is called the *axis of the reflector*, or of the teleſcope. Let CE be directed to the point $\mathcal{Q}$, in the diſtant object $\mathcal{Q}P$; then, qp, an inverted image of $\mathcal{Q}P$, would be formed in the principal focus of the reflector, at right angles to CE (Art. 47), and terminated by the lines PEp, $\mathcal{Q}Eq$, were the reflected rays ſuffered to proceed thither; but, before they arrive at the focus, they are received upon a plane reflector acb, inclined at an angle of 45° to the axis CE; and thus an image, mn, is formed, ſimilar and equal to pq, and equally inclined to the plane reflector (Art. 72); and conſequently, mn is parallel to EC. This image is viewed through a convex

çonvex eye glaſs *klr*, whoſe axis is perpendicular to *EC*, and whoſe diſtance from the image *mn*, is equal to it's focal length *.

If the reflection be made at *c*, by a right angled priſm whoſe ſides are perpendicular to the incident and emergent rays, much leſs light will be loſt, than if the reflection be made by a plane ſpeculum (Vid. Art. 101).

326. Dr. HERSCHEL has ſo far increaſed the focal lengths and apertures of his reflectors, that the image *qp* can be viewed directly, through an eye glaſs placed between *q* and *E*. The axis of the teleſcope is inclined a little from the object, that the image *qp* may be formed near the ſide of the tube which contains the reflector; and the head of the obſerver does not intercept ſo many rays as materially to affect the brightneſs of the image. By this conſtruction, one reflection of the rays is avoided; and the ſtrongeſt, and moſt effective pencils are preſerved, which, in the Newtonian teleſcope, are ſtopped by the plane reflector.

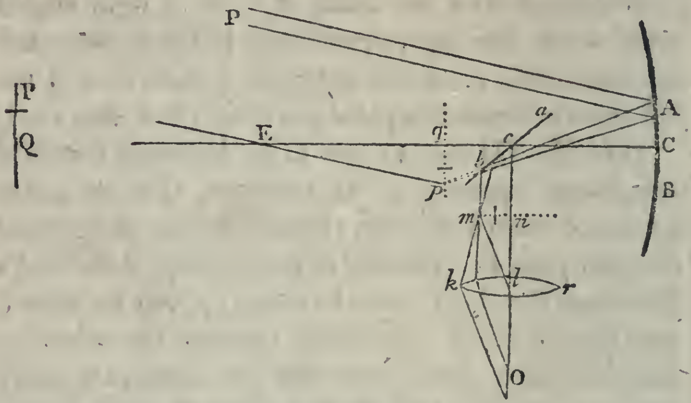
PROP. LXXIII.

327. *When a diſtant object is viewed with* Sir ISAAC NEWTON'S *reflecting teleſcope, the angle which it ſubtends at the center of the eye, is to the angle which it ſubtends at the naked eye, as the focal length of the reflector, to the focal length of the eye glaſs.*

Let *QP* be the object; *qp* it's image, in the principal focus of the reflector; *nm* the image formed by reflection at the plane ſurface *acb*; *QEC* the axis of the teleſcope, cutting *acb* in *c*. Draw *cnlO* perpendicular

* See Art. 270.

pendicular to CE; and at l let a convex eye glaſs klr be placed, whoſe focal length is ln, and whoſe axis coincides



with that line; join ml. Then the image nm, which is equal to qp, and correſponds to QP in the object, is ſeen through the glaſs klr, under an angle which is equal to mln; and QP is ſeen, with the naked eye placed at E, under an angle which is equal to qEp; and ſince theſe angles have equal ſubtenſes nm, and qp, they are to each other inverſely as the radii ln, qE; therefore the angle which the object ſubtends at the center of the eye, when viewed with the teleſcope : the angle which it ſubtends at the naked eye :: $Eq : ln$.

328. Cor. 1. The magnifying power of this teleſcope is meaſured by $\frac{Eq}{ln}$.

329. Cor. 2. To adapt the teleſcope to nearer objects, the reflector acb, to which the eye glaſs is attached, muſt be moved towards E. For, as QE decreaſes, qE decreaſes (Art. 59); and therefore, that

qc,

qc, or *cn* may remain of the ſame magnitude, *acb* muſt be moved nearer to *E*.

330. Cor. 3. When the teleſcope is adjuſted to nearer objects, it's magnifying power is diminiſhed. For, *Eq* decreaſes, and *ln* is invariable; therefore $\frac{Eq}{ln}$ decreaſes.

331. Cor. 4. To adjuſt the teleſcope to the eye of a ſhort ſighted perſon, the reflector *acb* muſt be moved towards *C*. For then the image *mn* will be at a greater diſtance from *c*, or nearer to the eye glaſs; and therefore, the rays in each pencil, after refraction at the glaſs *klr*, will diverge. If the eye require converging rays, the reflector *acb* muſt be moved the contrary way.

Prop. LXXIV.

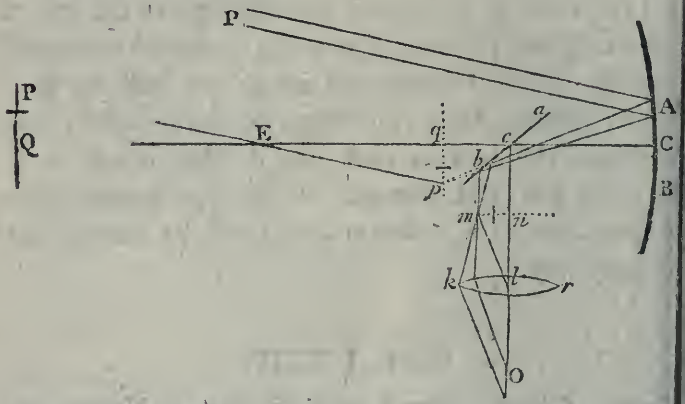
332. *Objects, viewed with* Sir Isaac Newton's *teleſcope, appear inverted.*

Let *QP* lie in the plane of the paper; and, when the eye of the obſerver is applied to the glaſs *kr*, let the plane which paſſes through both his eyes, alſo coincide with the plane of the paper. Then, the rays which flow from *P*, the right ſide of the object, converge to *m*, the left ſide of the image. Alſo, the rays which flow from a point in the object, above the plane of the paper, converge to a point in the image, below that plane; thus, the image *mn* is inverted; and ſince it is in, or near to, the principal focus of the convex lens through which it is viewed, it appears inverted (Art. 271).

Prop. LXXV.

333. *To determine the field of view in Sir* Isaac Newton's *telescope.*

Join b, k, the corresponding extremities of the plane reflector and the eye glass; and let bk cut the image



mn in m; take qp equal to nm; draw pE, and produc it till it meets the object in P.

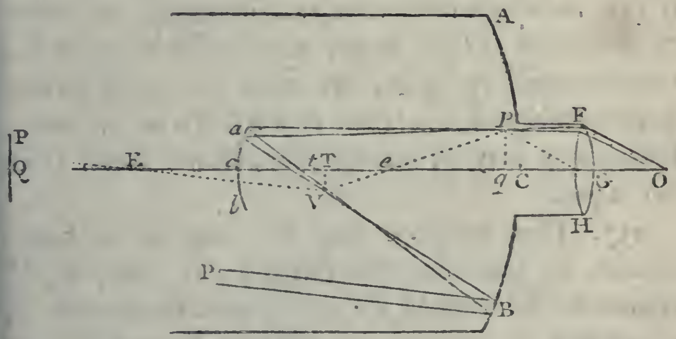
Then, a small pencil of rays flowing from P, will, acb be not so large as to intercept them before the are incident upon the reflector ACB, be reflected at to the extremity of the eye glass, and refracted thence the eye at O; the point P will therefore be visibl Also, this point is the extremity of the field of vie For, if a point be taken in nm, farther from t axis of the lens than m, any straight line drav through it, will either fall without ab, or without k that is, no ray, belonging to a point in the obje QP, beyond P, can be reflected from acb to the e glass.

334. C

334. Cor. In order that the field of view may be circular, *acb* muſt be the tranſverſe ſection of a cone, or cylinder, generated by the revolution of *bk*, about the axis *cl*; and therefore it muſt be an ellipſe, whoſe major and minor axes depend upon the nature of this ſolid.

On the GREGORIAN *Teleſcope.*

335. In the annexed figure, *ACB*, *acb* repreſent two concave ſpherical reflectors, a greater and a



ſmaller, whoſe axes are coincident, and whoſe concave ſurfaces are turned towards each other; *E* and *e*, their centers; *T*, *t*, their principal foci, of which, *T* lies between *e* and *t*. In the middle of the larger reflector is an opening, nearly of the ſame dimenſions with the aperture of the ſmaller, to admit a moveable tube containing a convex eye glaſs *FGH*.

When the axis of the teleſcope is directed to the point *Q* in a diſtant object *QP*, an inverted image *TV*, of this object, terminated by the lines *QET*, *PEV*, is formed in the principal focus of the reflector

 AB.

AB. The rays which diverge thence, and fall upon the concave reflector *acb*, after reflection, form an image *qp*, terminated by the lines *Teq*, *Vep*; which, becauſe *T* is between *e* and *t*, is inverted with reſpect to *TV* (Art. 88); or erect, with reſpect to *QP*.

The rays are then received by the eye glaſs *FGH*, whoſe axis coincides with the axis of the teleſcope, and whoſe focal length is *Gq*; and therefore the image *qp* may be ſeen diſtinctly.

336. If the diſtance *Cc* be diminiſhed, the diſtance *tq* will be increaſed.

For, as *Cc* decreaſes, *Tt* decreaſes, and *Tt* : *te* :: *te* : *tq*, in which proportion *te* is invariable; therefore *tq* increaſes *. By a proper adjuſtment then, of the reflectors and eye glaſs, the image *qp* may be formed in the principal focus of the lens *FGH*; or in ſuch a ſituation as may be neceſſary for diſtinct viſion (Vid. Art. 270).

337. This teleſcope may be adapted to nearer objects, by increaſing the diſtance *Cc*. For, as *QP* approaches towards *E*, *TV* alſo approaches towards *E*, or towards *t*; therefore the diſtance *tq* increaſes; or *q* is nearer to the eye glaſs than before; which inconvenience may be remedied by increaſing the diſtance *Cc* (Art. 336).

338. Objects ſeen through this teleſcope, appear erect. For, *qp* is an erect image of *QP*, and in, or near

* Having given the focal lengths of the reflectors, and the diſtance of *q* from *T*, the diſtance of the reflectors may be found.

Let $Tt = x$; $te = b$; $TC = a$; $Tq = c$. Then, $x : b :: b : x + c$; therefore, $x^2 + cx = b^2$; from which equation we obtain $x = \frac{\sqrt{4b^2 + c^2} - c}{2}$; and $Cc = a + b + x$.

near to the principal focus of the convex lens through which it is viewed; therefore it appears erect (Art. 271).

339. As ſo much has been ſaid upon the field of view in other teleſcopes, the reader will find no difficulty in determining it in this caſe. Join *F*, *a*, correſponding extremities of the ſmall reflector, and the eye glaſs; let *Fa* cut *qp*, in *p*. Draw *pe*, and produce it till it meets *TV* in *V*; join *VE*, and produce this line till it meets the object *QP* in *P*; then is *P* the extremity of the field of view. For, if *aVB* be drawn, the rays which flow from *P*, and fall upon the large mirror at *B*, after reflection converge to *V*, and fall upon the reflector *acb* at *a*; thence they proceed in the direction *apF*, and are refracted to the eye in the direction *FO*, which is parallel to *pG* *. But no ray, which belongs to a point in the object above *P*, can be reflected from *acb* to the eye glaſs *FH*.

Since *qp* is much nearer to the eye glaſs than to the reflector *acb*, the field of view will depend more upon the aperture of the former, than of the latter.

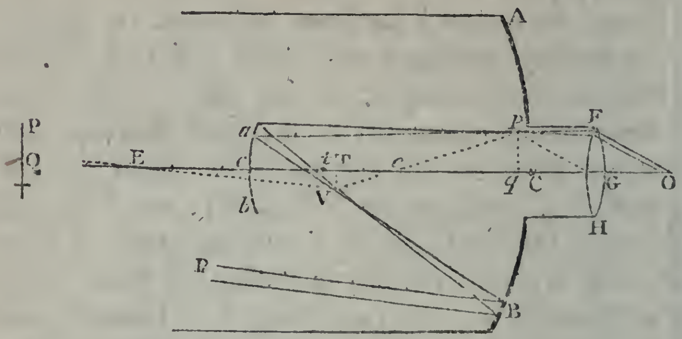
Prop. LXXVI.

340. *To determine the angle which an object ſubtends at the center of the eye, when ſeen through* Gregorie's *teleſcope.*

The conſtruction being made as before, the angle under which *QP* is ſeen through the teleſcope, is equal

* This is on ſuppoſition that *aVB* meets the reflector *AB*; and that the ray *PB*, which is parallel to *EV*, is not intercepted by the reflector *ab*.

equal to pGq; and the angle under which it is ſeen with the naked eye, is equal to TEV. Now, when the angles



are ſmall, the $\angle pGq$: the $\angle peq$ (TeV) :: $eq : qG$; and alſo, the $\angle TeV$: the $\angle TEV$ :: $ET : eT$; by comp. the $\angle pGq$: the $\angle TEV$ :: $eq \times ET : qG \times eT$; that is, the viſual angle, when the object is ſeen through the teleſcope : the viſual angle, when it is ſeen with the naked eye :: $eq \times ET : qG \times eT$.

341. Cor. Since $Tt : te :: eT : eq$ (Art. 54); inverſely, $te : Tt :: eq : eT$; therefore $te \times ET : Tt \times qG :: eq \times ET : qG \times eT$ (Alg. Art. 184); and the viſual angle, when the object is ſeen through the teleſcope : the viſual angle, when it is ſeen with the naked eye :: $te \times ET : Tt \times qG :: \frac{te \times ET}{qG} : Tt$.

On Cassegrain's *Teleſcope.*

342. In this teleſcope, the ſmaller reflector, acb, is convex, and ſo placed, that T falls between t and c. In other reſpects, it is ſimilar to the Gregorian teleſcope (See Art. 335).

Let

Let GCE, the axis of this telescope, be directed to the point Q in a distant object QP; then, an inverted

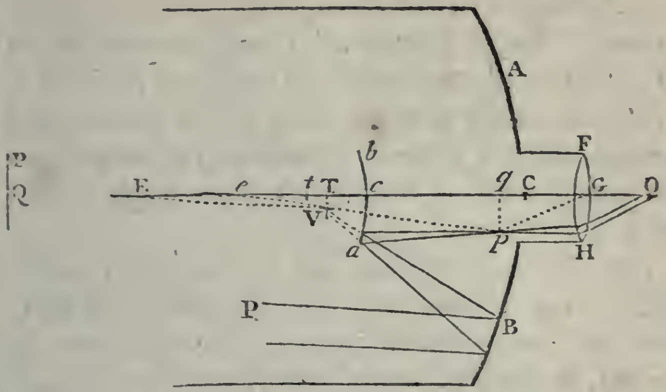


image TV, of this object, would be formed in the principal focus of the large reflector, and terminated by the lines QET, PEV, if the rays were suffered to proceed thither. But, before they reach the focus, they are received upon the convex reflector bca; and since, by the construction, TV falls between the surface c, and principal focus t, of this reflector, an erect image qp, of TV, is formed, and terminated by the lines eTq, eVp. This image is viewed through the lens FGH, whose axis coincides with the axis of the telescope, and whose focal length is Gq.

343. The distance of the image qp, from t, may be determined by the proportion $Tt : et :: et : tq$. Also, by diminishing the distance Cc, the distance Tt is diminished, and since te is given, tq is increased.

Hence it is manifest, that by a proper adjustment of the reflectors and the eye glass, the image qp may be formed in the principal focus of the lens FGH; or

 in

in ſuch a ſituation, that the emergent rays may have a proper degree of divergency, or convergency, for the eye of the obſerver.

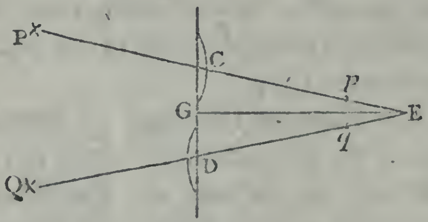
344. Objects viewed through this teleſcope appear inverted. For, the image *qp* is erect with reſpect to *TV*, or inverted with reſpect to the object; and it is in, or near to the principal focus of the convex glaſs through which it is viewed; therefore the image upon the retina is erect, or the object appears inverted (Art. 271).

345. The field of view may be determined exactly in the ſame manner as in the Gregorian teleſcope; and the demonſtration given in Art. 339, may be applied, in the ſame words, to the preceding figure.

This teleſcope may alſo be adapted, in the ſame manner, to nearer objects; and the viſual angle is expreſſed in the ſame terms (See Arts. 337. 340).

On the divided object glaſs Micrometer.

346. This micrometer conſiſts of a convex lens divided into two equal parts, *C*, *D*, by a plane which paſſes through it's axis; and the ſegments



are moveable on a graduated line *CD*, perpendicular to that axis. Let *C*, *D*, be the centers of the ſegments; and *P*, *Q*, two remote objects, images of which

which will be formed in the lines *PCE*, *QDE*, and alſo in the principal foci of the ſegments (Art. 209). Let the glaſſes be moved till theſe images coincide*, as at *E*; then, the angle *PEQ*, which the objects ſubtend at *E* the principal focus of *C*, or *D*, is equal to the angle which *CD*, the diſtance of the centers of the two ſegments, ſubtends at the ſame point; and therefore, by calculating this angle, we may determine the angular diſtance of the bodies *P* and *Q*, as ſeen from *E*. Draw *EG* perpendicular to *CD*; and, becauſe the triangle *CED* is iſoſceles, $CG = GD$, and the $\angle CEG =$ the $\angle GED$; alſo, *GD* is the ſine of the angle *GED*, to the radius *ED*; therefore, knowing *ED*, and *GD*, the angle *GED* may be found by the tables; and conſequently $2GED$, or *CED* may be determined.

347. The angle *CED* is in general ſo ſmall, that it may, without ſenſible error, be conſidered as proportional to the ſubtenſe *CD*. And being determined in one caſe by obſervation, it may be found in any other, by a ſingle proportion.

348. If the objects be at a *given finite* diſtance, the angle *PEQ* will ſtill be proportional to *CD*; for, on this ſuppoſition, the diſtance *CE*, or *DE*, of either image from the correſponding glaſs, will be invariable; therefore, the angle *CED* will be proportional to *CD*.

The divided object glaſs is applied both to reflecting and refracting teleſcopes; and thus ſmall angular diſtances in the heavens, are meaſured with great accuracy.

On

* Two other images are formed, an image of *P* by the ſegment *D*, and an image of *Q* by the ſegment *C*; but, as *CD* increaſes, theſe images always *recede* from each other.

On ſingle Microſcopes.

349. If the angle which an object ſubtends at the center of the eye, when at a proper diſtance for diſtinct viſion, be diminiſhed beyond a certain limit, the image upon the retina is ſo ſmall as to convey to the mind only the idea of a ſingle phyſical point, not diſtinguiſhable into parts; reſpecting which, therefore, no judgment can be formed by the ſight, except what relates to it's colour *. If we endeavour to increaſe the image upon the retina by bringing the object nearer to the eye, the extreme rays which enter the pupil will diverge too much, and the image become confuſed. If the extreme rays be ſtopped, to leſſen the divergency of the pencils which are received by the eye, the image will be indiſtinct for want of light †. But if the object be placed in the principal focus of a glaſs ſpherule, or lens whoſe focal length is ſhort, it may be ſeen diſtinctly; the viſual angle, as well as the quantity of light admitted into the eye, will be increaſed; and thus, the ſeveral parts, of what before appeared only as a ſingle point, will be ſubjected to examination.

Theſe glaſſes are called *ſingle Microſcopes.*

PROP.

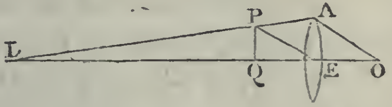
* The leaſt viſible part of an object does not ſubtend at the center of the eye, an angle much leſs than 4′. A ſingle, detached object is perceivable under an angle of about 2′. Harris's Opt. p. 121. Jurin's Eſſay, in Smith's Optics, Art. 164.

† Vid. note, p. 131.

Prop. LXXVII.

350. *The viſual angle of an object when ſeen through a ſingle microſcope, is to it's viſual angle when viewed with the naked eye at the leaſt diſtance of diſtinct viſion, as that leaſt diſtance, to the focal length of the glaſs.*

Let QP be an object placed in the principal focus of the lens, or ſpherule AE, whoſe center is E; LQ



the leaſt diſtance at which it can be ſeen diſtinctly with the naked eye. Join LP, PE. Then, the angle under which the object is ſeen through the glaſs, is equal to PEQ; and the angle under which it is ſeen with the naked eye, is QLP; alſo, when theſe angles are ſmall, ſince they have a common ſubtenſe QP, they are nearly in the inverſe ratio of the radii EQ, LQ; that is, the viſual angle when the object is ſeen through the glaſs : the viſual angle when it is ſeen with the naked eye at the diſtance LQ :: $LQ : EQ$.

Ex. If the focal length of the glaſs be $\frac{1}{50}$ of an inch, and the leaſt diſtance of diſtinct viſion, 8 inches, the viſual angle of the object when viewed through the glaſs : the viſual angle when it is ſeen with the naked eye :: $8 : \frac{1}{50}$ :: 400 : 1.

In this microſcope, the object appears erect (Art. 271).

351. The

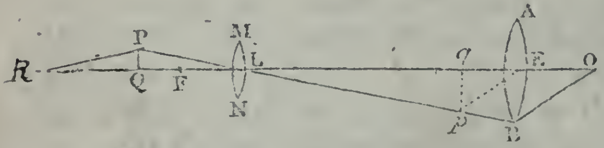
351. The ſolar microſcope is a ſingle convex lens, uſed in the ſame manner as in the mágic lantern (Art. 265). The moveable tube is adjuſted to a hole in a window ſhutter; and the object to be examined, is ſtrongly illuminated, and placed a little farther from the lens than it's principal focus; an inverted image of the object is thus formed, at a conſiderable diſtance from the lens, and received upon a ſcreen placed at the concourſe of the refracted rays.

The angles which the image and object ſubtend at the center of the eye, when viewed at the leaſt diſtance of diſtinct viſion, are proportional to their linear magnitudes, that is, to their diſtances from the center of the glaſs *.

On the double Microſcope.

352. The aſtronomical teleſcope, when adapted to near objects, becomes a double microſcope.

QP is an object, placed a little farther from the lens *MN* than it's principal focus *F*; *qp* the image of *QP*,



formed on the other ſide of the lens, and at a conſiderable diſtance from *L* it's center. *AEB* is a convex eye glaſs, whoſe axis coincides with the axis of the

* On the conſtruction, and uſe of the ſolar microſcope, the reader may conſult Mr. ADAM's work, entitled 'Eſſays on the microſcope.'

the lens MN, and whose distance from L is equal to the sum of Lq, and it's own focal length qE; consequently, the image qp is in the principal focus of the eye glass, and it may therefore be seen distinctly by a spectator whose eyes are able to collect parallel rays.

353. Since the conjugate foci, Q and q, move in the same direction upon the indefinite line QLO (Art. 182), if the glasses be fixed in a tube, or attached to each other in any other way, by moving the object QP, the image qp may be brought into the principal focus of the eye glass; or into such a situation, that the rays may, after the latter refraction, have a proper degree of divergency, or convergency for the eye of the spectator (Vid. Art 270).

Prop. LXXVIII.

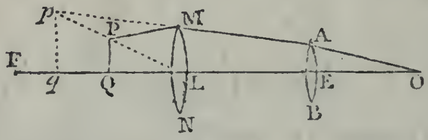
354. *To compare the angle which an object subtends at the center of the eye when seen through the double microscope, with the angle which it subtends at the naked eye when viewed at the least distance of distinct vision.*

Let qp (Fig. Art. 352) be the image of QP, formed in the principal focus of the lens AB, and terminated by the lines QLq, PLp. Then, the visual angle will be increased by the microscope on two accounts; first, because the image qp is greater than the object; and secondly, because this image is seen under a greater angle when viewed through the glass, than when viewed with the naked eye. Now, supposing QP and qp to be viewed with the naked eye, at the least distance of distinct vision, the visual angle of QP : the visual

viſual angle of qp (:: $QP : qp$) :: $QL : Lq$. Alſo, the viſual angle of qp, when thus viewed, : it's viſual angle when ſeen through the glaſs AB :: qE : the leaſt diſtance of diſtinct viſion (Art. 350); therefore, by compounding theſe two proportions, the viſual angle of QP, when viewed with the naked eye at the leaſt diſtance of diſtinct viſion, : the viſual angle, when it is viewed with the microſcope, :: $QL \times qE$: $Lq \times$ the leaſt diſtance of diſtinct viſion.

355. When the glaſſes are thus combined, the object appears inverted (Art. 297).

356. When a great magnifying power is not required, the object is placed between the glaſs MN



and it's principal focus; thus an erect image qp is formed, on the ſame ſide of the lens with the object; and if Eq be the focal length of the eye glaſs AB, the image may be ſeen diſtinctly.

The viſual angle may be determined as in the preceding caſe.

One advantage of this conſtruction is a greater field of view. The object alſo appears erect; and leſs confuſion is produced by the ſpherical ſurfaces of the glaſſes, than would be cauſed by a ſingle glaſs with the ſame magnifying power.

Prop.

Prop. LXXIX.

357. *The denſity of rays in the bright part of the image of a given object, formed upon the retina by a refracting teleſcope, or double microſcope, varies, nearly, as the aperture of the object glaſs directly, and the area of the picture upon the retina inverſely.*

The denſity of rays in the image, varies directly as their number and inverſely as the area over which they are diffuſed (Art. 7); that is, ſuppoſing the tranſmitting power to be given, and all the rays refracted at the object glaſs to be received by the eye, as the area of the aperture of the object glaſs directly, and the area of the image upon the retina inverſely.

358. Cor. 1. The denſity alſo varies according to the ſame law, when reflectors are uſed; but the effects of reflectors and refractors are not here compared, becauſe a much greater quantity of light is loſt in reflection than in refraction.

359. Cor. 2. If F and f be the focal lengths of the object glaſs and eye glaſs, and A the linear aperture of the object glaſs, the denſity of rays in the picture upon the retina varies as $\frac{A^2 f^2}{F^2}$.

For, the viſual angle of the object when ſeen with the naked eye : it's viſual angle when ſeen through the teleſcope :: $f : F$; and ſince the object is given, the firſt term in the proportion is invariable; therefore the viſual angle when the object is ſeen through the teleſcope, or the linear magnitude of the image upon the retina,

retina, varies as $\frac{F}{f}$; and the area of the image, as $\frac{F^2}{f^2}$; therefore, the denſity of rays in that image, varies as $\frac{A^2 f^2}{F^2}$.

360. Cor. 3. In the ſame manner, if F be the focal length of the object metal in the Newtonian teleſcope, A it's linear aperture, and f the focal length of the eye glaſs, the denſity of rays in the picture upon the retina, varies as $\frac{A^2 f^2}{F^2}$.

361. The denſity of rays in the picture upon the retina is uſually taken as a meaſure of the apparent brightneſs; though ſtrictly ſpeaking, apparent brightneſs has no numerical meaſure.

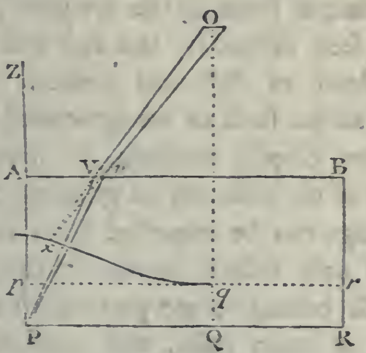
SCHOLIUM.

362. In explaining the conſtruction and effects of optical inſtruments, we have ſuppoſed the images to be ſimilar to the objects, and accurately formed in the geometrical foci of refracted or reflected rays. Were theſe ſuppoſitions true, teleſcopes and microſcopes would be perfect; no limit could be ſet to their magnifying powers, but ſuch as ariſe from the difficulty of forming ſpherical ſurfaces of proper dimenſions; and by their aſſiſtance, objects might be ſeen as diſtinctly as if they were viewed in plane reflectors. The imperfections to which they are ſubject, ariſe from two cauſes; The ſpherical figure of the reflecting and refracting ſurfaces; and the unequal refrangibility

frangibility of the different rays which conſtitute the body of light by which objects are ſeen.

1. When any points in an object are ſeen by oblique pencils, that is, by ſuch pencils as are not nearly perpendicular to the reflecting, or refracting ſurfaces, thoſe points do not appear in the places determined by the conſtructions and calculations, hitherto given *.

This will be eaſily underſtood by conſidering the moſt ſimple caſe of refraction. Suppoſe *AB* to be a plane



refracting ſurface; *PQR* a ſtraight line parallel to it; *pqr* the geometrical image of *PQR*, as determined Art. 193; *O* the place of the eye. Then, the rays which flow from *P*, after refraction at the ſurface *AB*, are diffuſed through all parts of the medium in which the eye is placed; and if thoſe rays of the pencil, which paſs through *O*, diverge accurately from *p*, they enter the eye as if they came from a real object there; and *p* is the *viſible* image of *P*; but, when the eye is at any conſiderable diſtance from the perpendicular *AZ*, the point *P* is ſeen by an oblique portion of the general

* The caſe of objects ſeen by rays reflected at plane ſurfaces is excepted.

general pencil, as *PVO*. Thofe oblique rays do not diverge accurately, from *any* point; and therefore the vifible image will be indiftinct. But, if x be the place where, if produced backwards, they occupy the fmalleft fpace, we may fuppofe this to be the vifible image of P; and the curve which is the locus of x, to be the vifible image of the whole line PR*. Again, if a diftorted image be formed by the object glafs, or reflector of a telefcope, different parts of it lie at different diftances from the principal focus of the eye glafs; and, if one part can be feen diftinctly, the reft will appear confufed. Inftead of fpherical furfaces, it has been propofed to adopt fuch as are generated by the revolution of the ellipfe, parabola, or hyperbola; but, independent of the difficulty of grinding thefe furfaces little advantage can be expected from them, as eacl furface, will only reflect, or refract, thofe rays accurately, which belong to one particular focus, and th aberrations, in other cafes, will generally be greate than thofe produced by fuch furfaces as are of a fphe rical form.

2. Another, and more confiderable caufe of imper fection in optical inftruments, is the unequal refrangi bility of differently coloured rays. We have, in a ou

* There is confiderable difficulty in determining the *vifible* imag of an object, when feen by reflected, or refracted rays. In t preceding cafe, if we fuppofe the impreffion to be made by the rays which are incident in the plane perpendicular to the refracti furface, the equation to the curve which is the locus of x, rifes eight dimenfions. If we fuppofe the impreffion to be made thofe rays which are equally inclined to the refracting furface, equation rifes to four dimenfions. Vid. NEWT. Lect. Op. P. Prop. VIII.

our calculations, ſuppoſed light to be homogeneal; that, whilſt it paſſes out of one given medium into another, the ſine of incidence bears the ſame invariable ratio to the ſine of refraction. But, Sir ISAAC NEWTON diſcovered that the common light by which objects are viewed, conſiſts of rays which differ both in *colour* and *refrangibility*; and, that thoſe rays which differ in colour, always differ in refrangibility; that is, if the ſines of incidence be equal, the ſines of refraction are different, though the mediums remain the ſame. Hence it follows, that if the image of an object be diſtinctly formed by the red, which are the leaſt refrangible rays, at one particular diſtance from a refractor, a diſtinct image will be formed by rays which have a different degree of refrangibility, as the blue rays, at a different diſtance from it; thus, the rays of different colours, which flow from the ſame point, being collected at different diſtances from the refractors, a confuſed, and coloured image of that point, is neceſſarily produced upon the retina; or upon any ſcreen which receives the refracted rays.

We are now to conſider, in what manner theſe imperfections may, in ſome degree, be remedied. And we ſhall begin with the latter, which is of greater importance, as the errors it produces are much more conſiderable than thoſe which ariſe from the ſpherical form of the ſurfaces; we may add moreover, that it's theory is more eaſily explained, and it's effects more likely to be corrected in practice.

SECTION VIII.

ON THE ABERRATIONS PRODUCED BY THE UNEQUAL REFRANGIBILITY OF THE RAYS OF LIGHT; AND BY THE SPHERICAL FORM OF REFLECTING AND REFRACTING SURFACES.

Prop. LXXX.

363. *THE ſun's light conſiſts of rays which differ in refrangibility and colour.*

This important diſcovery was made by Sir Isaac Newton, who deſcribes the experiment by which it is eſtabliſhed, in the following words*.

" In a very dark chamber, at a round hole, about one third part of an inch broad, made in the ſhut of a window, I placed a glaſs priſm, whereby the beam of the ſun's light which came in at the hole, might be refracted upwards toward the oppoſite wall of the chamber, and there form a coloured image of the ſun. The

* Opt. B. I. P. I. Prop. II.

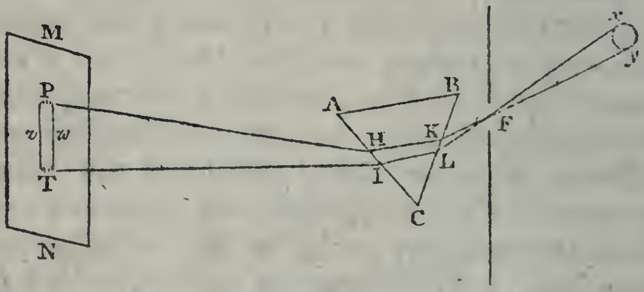
* a faint shade observed between the perfect shadow and full light —

The axis of the prifm was perpendicular to the incident rays. About this axis I turned the prifm flowly, and faw the refracted light on the wall, or coloured image of the fun, firft to defcend, and then to afcend. Between the defcent and afcent, when the image feemed ftationary, I ftopped the prifm, and fixed it in that pofture, that it fhould be moved no more. For in that pofture, the refractions of the light at the two fides of the refracting angle, that is, at the entrance of the rays into the prifm, and at their going out of it, were equal to one another *. The prifm therefore being placed in this pofture, I let the refracted light fall perpendicularly upon a fheet of white paper at the oppofite wall of the chamber, and obferved the figure and dimenfions of the folar image formed on the paper by that light. This image was oblong and not oval, but terminated with two rectilinear and parallel fides, and two femicircular ends. On it's fides it was bounded pretty diftinctly, but on it's ends very confufedly and indiftinctly, the light there decaying and vanifhing by degrees. The breadth of this image anfwered to the fun's diameter, and was about two inches and the eighth part of an inch, including the penumbra. For the image was eighteen feet and an half diftant from the prifm, and at this diftance, that breadth, if diminifhed by the diameter of the hole in the window-fhut, that is by a quarter of an inch, fubtended an angle at the prifm of about half a degree, which is the fun's apparent diameter. But the length of the image was about ten inches and a quarter, and the length of the rectilinear fides about eight inches; and

* Lect. Opt. P. I. Prop. XXV.

and the refracting angle of the prifm, whereby fo great a length was made, was 64 degrees. With a lefs angle the length of the image was lefs, the breadth remaining the fame. It is farther to be obferved, that the rays went on in right lines from the prifm to the image; and therefore, at their very going out of the prifm, had all that inclination to one another from which the length of the image proceeded, that is the inclination of more than two degrees and an half. And yet, according to the laws of optics vulgarly received, they could not poffibly be fo much inclined to one another.

" For let *F* reprefent the hole made in the window-fhut, through which a beam of the fun's light was tranfmitted into the darkened chamber, and *ABC* a



triangular imaginary plane, whereby the prifm is feigned to be cut tranfverfely through the middle of the light; and let *xy* be the fun, *MN* the paper upon which the folar image or fpectrum is caft, and *PT* the image itfelf; whofe fides, towards *v* and *w*, are rectilinear and parallel, and ends, toward, *P* and *T*, femicircular. *yKHP*, and *xLIT* are two rays; the firft of which comes from the lower part of the fun to the

the higher part of the image, and is refracted in the prism at *K* and *H*; and the latter comes from the higher part of the sun to the lower part of the image, and is refracted at *L* and *I*. Since the refractions on both sides the prism are equal to one another, that is the refraction at *K* equal to the refraction at *I*, and the refraction at *L* equal to the refraction at *H*, so that the refractions of the incident rays at *K* and *L* taken together, are equal to the refractions of the emergent rays at *H* and *I* taken together; it follows, by adding equal things to equal things, that the refractions at *K* and *H* taken together, are equal to the refractions at *I* and *L* taken together; and therefore the two rays, being equally refracted, have the same inclination to one another after refraction, which they had before; that is, the inclination of half a degree, answering to the sun's diameter. So then, the length of the image *PT* would, by the rules of vulgar optics, subtend an angle of half a degree at the prism, and by consequence be equal to the breadth *vw*; and therefore the image would be round *. Since then it is found by experience that the image is not round, but about five times longer than broad, the rays which, going to the upper end *P* of the image, suffer the greatest refraction, must be more refrangible than those which go to the lower end *T*.

"The image or spectrum *PT* was coloured, being red at it's least refracted end *T*, and violet at it's most refracted end *P*, and yellow, green and blue in the intermediate spaces."

364. To

* Vid. Lect. Opt. P. I. Sect. IV, &c.

364. To ſhew that the unequal refrangibility of the rays in this experiment, is not accidental, or owing to any new modification produced by the medium through which they paſs, Sir Isaac Newton refracted the rays of each colour ſeparately, and found that they ever after retained both their colour and peculiar degree of refrangibility *.

365. By an experiment ſimilar to the former, it may be ſhewn that common day light conſiſts of rays which differ in colour and refrangibility.

For, if the round hole in the ſhutter receive only light from the clouds, it's image, formed by the priſm, will be oblong, and coloured as in the former caſe.

366. Sir Isaac Newton, with the aſſiſtance of a perſon who had a more critical eye than himſelf, diſtinguiſhed the ſpectrum into ſeven principal colours, proceeding from the leſs to the more refrangible rays, in the following order; red, orange, yellow, green, blue, indigo, violet; of which the yellow and orange were found to be the moſt luminous, and the next in ſtrength were the red and green; the darker colours, eſpecially the indigo and violet, affected the eye much leſs ſenſibly.

367. If, by any method, the priſmatic colours be again united in the proportion which they have in the ſpectrum, they compound a white ſun light; and by the mixture of different ſorts of rays, in different proportions, various colours are produced, according to the quantity and nature of the rays united

Thus,

* Newt. Optics, Part I. Exp. 6.

Thus, a mixture of red and yellow produces an orange; yellow and blue form a green *, &c.

368. From the former part of the laſt article we may conclude, that if a ray of white light be refracted through a medium contained by parallel planes whoſe diſtance is inconſiderable, it will not, as to ſenſe, be ſeparated into diſtinct colours.

For, the ray of each particular colour emerges parallel to the incident white ray; conſequently, the emergent rays of different colours are parallel to each other; and ſince the thickneſs of the medium is inconſiderable, they emerge nearly at the ſame point, and therefore excite only the ſenſation of whiteneſs.

Thus it happens, that objects ſeen through common window glaſs do not appear coloured †.

369. The ſame may be ſaid, if the emergent rays, after ſeveral refractions, be parallel to the incident white ray, and the points of emergence nearly coincide.

Prop. LXXXI.

370. *The more refrangible rays are more reflexible.*

A ray of light cannot, conſiſtently with the general law of refraction, paſs out of a denſer medium into a rarer when the ſine of incidence exceeds the limit determined by this proportion, ſin. refraction : ſin. incidence :: radius : ſin. incidence, which is the limit ſought (Art. 101); therefore, the greater the ratio of the ſine of refraction to the ſine of incidence, the

* Newt. Opt. P. I. Prop. IV.

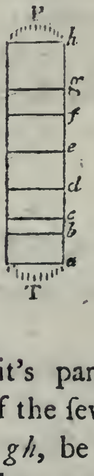
† Vid. Newt. Lect. Op. P. II. Sect. IV.

the lefs will this limit be; and, confequently, the fooner will the rays be reflected *.

Prop. LXXXII.

371. *If a fmall cylindrical beam of white light pafs nearly perpendicularly out of common glafs into air, the difperfion of the differently coloured rays is about* $\frac{2}{55}$ *of the mean refraction.*

Let a fmall beam of the fun's light be refracted by a glafs prifm, in the manner defcribed (Art. 363); and let *PT* reprefent the fpectrum, divided by lines which



are perpendicular to it's parallel fides, and drawn through the confines of the feveral colours. Alfo, let *ab*, *bc*, *cd*, *de*, *ef*, *fg*, *gh*, be the fpaces occupied by the red, orange, yellow, green, blue, indigo and violet rays, refpectively; then if the whole length *ah* be reprefented by unity, *ab* is found to be $\frac{1}{8}$; $ac=\frac{1}{5}$; $ad=\frac{1}{3}$; $ae=\frac{1}{2}$; $af=\frac{2}{3}$; $ag=\frac{7}{9}$; and thefe are nearly proportional

* This propofition may alfo be proved by experiment. Newt. Opt. B. I. P. I. Prop. III.

proportional to the differences of the ſines of refraction of the differently coloured rays, to a common ſine of incidence.

Now, when the rays paſs out of glaſs into air, if the common ſine of incidence be repreſented by 50, the ſines of refraction of the extreme red and violet rays are found to be 77 and 78 reſpectively*; therefore the ſines of refraction of the other rays, are $77\frac{1}{8}$, $77\frac{1}{5}$, $77\frac{1}{3}$, $77\frac{1}{2}$, $77\frac{2}{3}$, $77\frac{7}{9}$. That is, the ſine of incidence of any red ray, is to the ſine of refraction, in a ratio not greater than that of 50 : 77, nor leſs than that of 50 : $77\frac{1}{8}$; but varying, in different ſhades of red, through all the intermediate ratios. In the ſame manner, the ſines of refraction of all the orange rays extend from $77\frac{1}{8}$ to $77\frac{1}{5}$, &c. the rays which are in the confines of the green and blue, have a mean degree of refrangibility, and the ſine of incidence of theſe rays, is to the ſine of refraction, as 50 to $77\frac{1}{2}$.

When the angles of incidence and refraction are ſmall, they are nearly proportional to their ſines; and conſequently, if the common angle of incidence be repreſented by 50, the deviation of the violet rays is $78 - 50$, or 28; the deviation of the red rays is $77 - 50$, or 27; therefore the difference of theſe, or the angle through which the rays of different colours are diſperſed, is $\frac{1}{27}$ of the deviation of the red rays, $\frac{1}{28}$ of the deviation of the violet rays, and $\frac{1}{27\frac{1}{2}}$, or $\frac{2}{55}$ of the deviation of the rays of mean refrangibility, from their original courſe.

372. The

* Newt. Opt. P. I. Prop. VII.

372. The angle through which all the red rays are diſperſed is $\frac{1}{8}$ of $\frac{2}{55}$, or $\frac{1}{220}$ of the mean refraction, &c.

373. In general, if the ſines of refraction of the red and violet rays, in their paſſage out of any given medium into air, be $1+m$ and $1+n$, to the common ſine of incidence 1, then, when the angles of incidence and refraction are ſmall, the diſperſion of the rays is an $\frac{n-m}{m}$th part of the refraction of the red rays; and ſince m and n are invariable, this expreſſion may be properly taken as the meaſure of the *diſperſing* power of the medium.

374. Whilſt the refracting mediums are the ſame, a given refraction of the mean rays is always attended with the ſame diſperſion, which may be deſtroyed by an equal refraction in the oppoſite direction (Art. 29). But if the latter refraction fall ſhort of the former, the diſperſion will not be wholly corrected; if it exceed the former, the diſperſion will be the contrary way; that is, the order of the colours will be changed; and no refraction can finally be produced by mediums of the ſame kind, without colour.

375. Mr. DOLLAND, an eminent optician in London, diſcovered*, about the year 1757, that different ſubſtances have different diſperſing powers; that the ſame diſperſion may be produced, or corrected, by a leſs refraction of the mean rays in one caſe, than in another; and thus refraction may, upon the whole, be produced without colour.

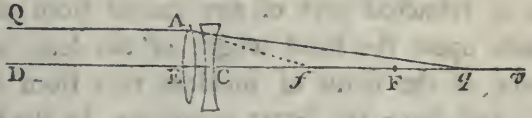
PROP.

* The diſcovery has been aſcribed to others; Mr. DOLLAND was the firſt who made it public.

Prop. LXXXIII.

376. *Having given the refracting powers of two mediums, to find the ratio of the focal lengths of a convex and concave lens, formed of these substances, which, when united, produce images nearly free from colour.*

Let $1+m:1$, and $1+n:1$ be the ratios of the sines of incidence and refraction of the red and violet rays out of air into the convex lens; $1+p:1$, and $1+q:1$, the ratios of those sines, out of air into the concave lens; F and f the focal lengths of the lenses, for red rays. Then $\frac{1}{m}:\frac{1}{n}::F$: the focal length of the convex lens for violet rays (Art. 169); therefore,



the focal length of the convex lens for violet rays is $\frac{mF}{n}$; in the same manner it appears, that $\frac{pf}{q}$ is the focal length of the concave lens for violet rays. Let AC be the compound lens; Eq it's focal length for red rays; Ev it's focal length for violet rays. Then $f-F:f::F:Eq$; and $\frac{pf}{q}-\frac{mF}{n}:\frac{pf}{q}::\frac{mF}{n}:Ev$ (Art. 184); hence $Eq=\frac{Ff}{f-F}$, and $Ev=\frac{mpFf}{npf-mqF}$; and when $Ev=Eq$, the red and violet rays, after both refractions,

refractions, are collected at q, or v. In this case, $\frac{Ff}{f-F} = \frac{mpFf}{npf-mqF}$; or $npf - mqF = mpf - mpF$; whence $p \,.\, \overline{n-m} \,.\, f = m \,.\, \overline{q-p} \,.\, F$; and $F : f :: p \,.\, \overline{n-m} : m \,.\, \overline{q-p} :: \frac{n-m}{m} : \frac{q-p}{p}$. That is, the focal lengths are proportional to the dispersing powers of the two mediums. If the intermediate rays be dispersed according to the same law by the two mediums, it is manifest that the focal length of the compound lens, for these colours, will be Eq or Ev; and thus the image of a distant object will be formed in q or v, free from colour (Art. 367).

When the rays of different colours proceed from a point at a finite distance from this compound lens, after refraction they will converge to, or diverge from a common focus. For, the distance of the focus of refracted rays of any colour from the lens, depends upon the focal length of the lens, and the distance of the focus of incident rays from it (Art. 177); and since the latter quantities, by the supposition, are the same for rays of all colours, the distance of the focus of refracted rays from the lens, is the same; and thus, the image of an object at any finite distance from the compound lens, will be free from colour.

377. Ex. In crown, or common glass, $1+m = 1.54$; and $1+n = 1.56$. In flint glass, $1+p = 1.565$; and $1+q = 1.595$ *; therefore the dispersing

* The refracting and dispersing powers of different kinds of glass are exceedingly various; and the causes upon which they depend are but imperfectly understood. See Dr. BLAIR's experiments on this subject, in the Edinburgh Transactions, Vol. III.

difperfing power of common glafs : the difperfing power of flint glafs :: $\frac{.02}{.54} : \frac{.03}{.565}$ :: $2 \times 565 : 3 \times 540$. To form a compound lens of thefe fubftances which fhall produce a *real* image of a diftant object, nearly free from colour, the convex lens muft have the greater refracting power; and therefore it muft be made of common glafs, which has the lefs difperfing power. In this cafe, $F : f :: 2 \times 565 : 3 \times 540 :: 7 : 10$, nearly.

The focal length of the compound lens, $\frac{Ff}{f-F} = \frac{10F}{3}$.

378. Cor. 1. If the greater refraction be produced by the concave lens, it's focal length : the focal length of the convex lens :: 7 : 10, nearly; and the refracting power of the compound lens correfponds to that of a fingle concave glafs.

379. Cor. 2. It is found by experience, that the extreme and intermediate rays are not difperfed by crown and flint glafs, according to the fame law; therefore, though the red and violet rays are united by the compound lens above defcribed, yet the intermediate rays are not collected at the fame point; and confequently, the images formed are not entirely free from colour.

The difcovery of two forts of glafs, which fhall difperfe the extreme and intermediate rays in the fame proportion, is ftill a defideratum in optics.

To form the moft diftinct image, the lenfes ought to be fo adjufted as to collect the brighteft, and ftrongeft colours, the yellow and orange.

380. Cor.

380. Cor. 3. By a method ſimilar to that employed in the propoſition, two *compound* lenſes, which collect the extreme rays, but diſperſe the intermediate rays in different proportions, might be ſo adjuſted as to collect rays of three different colours, exactly; but the advantage thus gained, would probably not compenſate for the loſs of light.

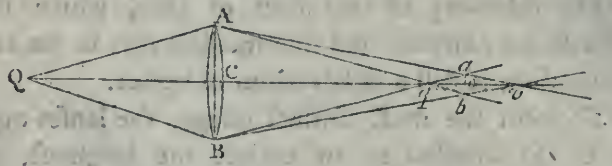
381. Cor. 4. Inſtead of a ſingle convex lens, two are frequently employed, one on each ſide of the concave lens, which, when combined, have the ſame focal length with the ſingle lens for which they are ſubſtituted. This conſtruction leſſens the aberration ariſing from the ſpherical form of the refracting ſurfaces.

Prop. LXXXIV.

382. *Having given the aperture of any lens, ſingle or compound, and the foci to which rays of different colours, belonging to the ſame pencil, converge, to find the leaſt circle of aberration through which theſe rays paſs.*

Let QCv be the axis of the lens; AB, or $2AC$ it's linear aperture; q and v the foci of differently coloured rays. Draw Av, Bv; Aqb, Bqa; join a, b the points of their interſection, and let ab cut the axis QCv, in c.

Then, in the ſimilar and equal triangles ACv, BCv, $Av = Bv$; and the $\angle AvC =$ the $\angle CvB$. In the



ſame manner, the $\angle AqC =$ the $\angle BqC$, or the $\angle bqc =$ the

the $\angle aqc$; therefore, in the triangles aqv, bqv, the angles avq, aqv are reſpectively equal to the angles bvq, bqv, and qv is common to both triangles, conſequently aq is equal to bq. Hence it follows, that the triangles AqB, aqb, as alſo the triangles AqC, bqc, are ſimilar, and that ab is perpendicular to Qcv; therefore ab is the diameter of the leaſt circle of aberration, into which the rays converging to q and v are collected.

Now, from the ſimilar triangles AqB, aqb, $AB : ab :: Aq : qb$; and from the ſimilar triangles ACq, bqc, $Aq : bq :: Cq : cq$; therefore $AB : ab :: Cq : cq$. In the ſame manner, $AB : ab :: Cv : cv$; therefore $AB : ab :: Cv + Cq : cv + cq \ (Cv - Cq)$.

383. Cor. 1. When the ratio of Cv to Cq is given, ab varies as AB; and the area of the leaſt circle of aberration varies as AB^2.

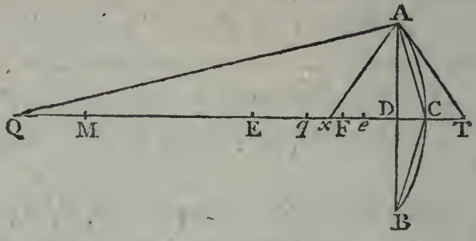
384. Cor. 2. Let parallel rays fall upon a ſingle lens of crown glaſs, to compare the linear aperture of the lens, with the diameter of the leaſt circle into which all the rays, of different colours, are collected.

Here $1+m : 1 :: 1.54 : 1$; and $1+n : 1 :: 1.56 : 1$ (Art. 377); and $Cv : Cq :: 56 : 54$ (Art. 169); therefore, $Cv + Cq : Cv - Cq :: 110 : 2 :: 55 : 1$; that is, $AB : ab :: 55 : 1$; or the diameter of the leaſt circle of aberration into which the extreme rays, and conſequently all the intermediate rays, are collected, is $\frac{1}{55}$ part of the linear aperture.

Prop. LXXXV.

385. *When a ray of light is incident obliquely upon a ſpherical reflector, to determine the interſection of the reflected ray and the axis of the pencil to which it belongs.*

Let ACB be the ſpherical reflector; E it's center QA a ray incident obliquely upon it; QET the axi



of the pencil to which the ray belongs. Draw AT touching the arc ACB in A, and let it meet QT in T biſect ET, EC, in e and F; take q the geometric focus, conjugate to Q; and let QA be reflected in th direction Ax.

Then, $QF : FE :: QE : Eq$;

and $QF+Fe : FE+Fe :: QE : Ex$ (Art. 54)

therefore $Eq = \frac{QE \times FE}{QF}$; and $Ex = \frac{QE \times \overline{FE+Fe}}{QF+Fe}$

whence, by actual diviſion, $Ex = \frac{QE \times FE}{QF}$ +

$\frac{QE^2 \times Fe}{QF \times \overline{QF+Fe}} = \frac{QE \times FE}{QF} + \frac{QE^2}{QF^2} \times Fe$, nearly

therefore $Ex - Eq = qx = \frac{QE^2}{QF^2} \times Fe$, nearly.

Alſo, ſince $ET = 2Ee$, and $EC = 2EF$, CT, c $ET - EC = 2Fe$; conſequently, $qx = \frac{QE^2}{QF^2} \times \frac{C}{2}$ nearly.

386. This quantity, qx, is called the *longitudin* aberration of the oblique ray QA.

387. Co

387. Cor 1. Since the expreſſion $\frac{QE^2}{QF^2} \times Fe$ has always the ſame ſign, Ex is always greater than Eq.

388. Cor. 2. Draw AD perpendicular to QC, and produce it till it meets the ſurface in B; join AC, CB. Then, the $\angle TAC =$ the $\angle CBA =$ the $\angle DAC$, and $CD : CT :: AD : AT$ (Euc. 3. 6); and when the arc AC is evaneſcent, AD is equal to AT; therefore, $CD = CT$; and the longitudinal aberration $qx = \frac{QE^2}{QF^2} \times \frac{CD}{2}$.

389. Cor. 3. When QA is parallel to QC, QE becomes equal to QF, and $qx = \frac{CD}{2}$.

390. Cor. 4. If Q, when in FE, or in FE produced, approach to E, the ratio of QE to QF decreaſes; and therefore, if CD be given, the aberration decreaſes. If Q be in FC, or FC produced, as QF decreaſes, the aberration increaſes.

391. Cor. 5. If the diſtances QE and QF be invariable, the aberration varies as CD.

392. Cor. 6. Let CM be the diameter of the reflector; then, by the property of the circle, $CD : DA :: DA : DM$, and $CD = \frac{DA^2}{DM}$; therefore, when CD is very ſmall, or DM nearly equal to the diameter of the given reflector, CD varies as DA^2, nearly; and conſequently, when QE, QF are given, and the arc AC is very ſmall, the longitudinal aberration varies as DA^2.

393. Cor. 7. When parallel rays are incident upon the reflector, the longitudinal aberration is ultimately

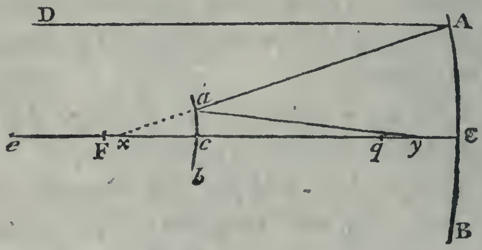
 equal

equal to $\frac{CD}{2} = \frac{DA^2}{2DM} = \frac{DA^2}{4EC} = \frac{DA^2}{8EF}$; and therefore it varies as $\frac{DA^2}{EF}$.

Prop. LXXXVI.

394. *If parallel rays be reflected at a concave, and afterwards fall upon a convex spherical reflector, converging to a point between it's surface and principal focus, as in* Cassegrain's *telescope, the aberration of the lateral rays produced by the first reflection, will, in some measure, be corrected by the latter.*

Let ecC be the axis of the telescope; x the intersection of the axis and lateral ray after the first reflec-



tion; y their intersection after the second reflection; q the geometrical focus after both reflections.

The place of y, with respect to q, will be affected by two causes: 1st. x is nearer to c than F, the geometrical focus after the first reflection (Art. 387); and therefore, on this account, yc is less than qc (Art. 59); 2dly. in consequence of the aberration arising from the

the form of the reflector *acb*, *yc* is greater than *qc* (Art. 387); therefore these causes counteract each other; and, by a proper adjustment of the reflectors, the aberration *qy* may, in a great measure, be destroyed.

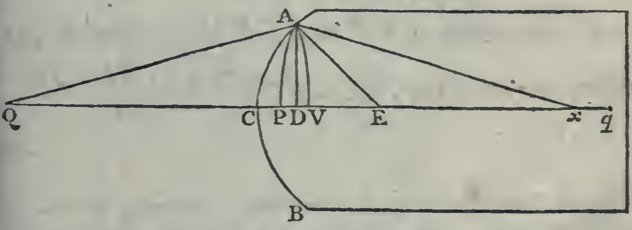
395. Cor. 1. Though the aberration *qy*, of the extreme ray *DA*, should be wholly destroyed, the aberration of the intermediate rays will not be entirely corrected; and this seems to be an insuperable obstacle to the perfection of reflecting telescopes.

396. Cor. 2. If the reflectors be both concave, as in GREGORIE's telescope, the aberrations produced by the two reflections are in the same direction; that is, the second reflection increases the aberration produced by the first.

PROP. LXXXVII.

397. *When a ray of homogeneal light is incident obliquely upon a spherical refracting surface, to determine the intersection of the refracted ray and the axis of the pencil to which it belongs.*

Let *ACB* be the refractor; *E* it's center; *QA* a ray incident obliquely upon it; *QCq* the axis of the



pencil to which *QA* belongs; *Ax* the refracted ray;

 q the

q the geometrical focus, conjugate to Q. Draw AD perpendicular to the axis; and from the centers Q, x, with the radii QA, xA, deſcribe the circular arcs AV, AP, cutting the axis in V and P. Take $m+1 : 1 ::$ ſin. incidence : ſin. refraction.

Then, in the triangle QAE, $QE : QA ::$ ſin. incidence : ſin. $\angle AEQ$; alſo, in the triangle AEx, $Ax : Ex ::$ ſin. $\angle AEx$ (ſin. $\angle AEQ$) : ſin. refraction; therefore, by compounding theſe two proportions, $QE \times Ax : QA \times Ex ::$ ſin. incidence : ſin. refraction $:: m+1 : 1$; hence $\overline{m+1} . QA : QE :: Ax : Ex$; by diviſion, $\overline{m+1} . QA - QE : QE :: Ax - Ex : Ex$*; that is, $\overline{m+1} . QV - QE : QE :: EP : Ex$; or $\overline{m+1} . QC + \overline{m+1} . CV - QE : QE :: EC - CP : Ex$; or $m . QC - EC + \overline{m+1} . CV : QE :: EC - CP : Ex$.

Now, $DC : DV :: QV : EC :: QC : EC$, nearly; and by compoſition, $DC : CV :: QC : QE$; therefore, when the arc AC is ſmall, $CV = \frac{QE \times DC}{QC}$. Alſo, $DC : DP :: Ax : EC$; by diviſion, $DC : CP :: Ax : Ax - EC :: Ax : Ex :: \overline{m+1} . QC : QE$, nearly; therefore $CP = \frac{QE \times DC}{\overline{m+1} . QC}$, nearly. And, by ſubſtituting theſe values of CV and CP in the former proportion, we obtain $m . QC - EC + \frac{\overline{m+1} . QE \times DC}{QC}$: QE

* In this inveſtigation of the aberration, diverging rays are ſuppoſed to fall upon a convex ſpherical ſurface of a denſer medium, and to converge after refraction. The other caſes may be derived from this, by a proper attention to the ſymbols.

$QE :: EC - \frac{QE \times DC}{\overline{m+1} \,.\, QC} : Ex$; hence, $Ex = QE \times$

$$\frac{EC - \frac{QE \times DC}{\overline{m+1} \,.\, QC}}{m \,.\, QC - EC + \frac{\overline{m+1} \,.\, QE \times DC}{QC}}$$; and by actually dividing, and taking the remainder, $Ex =$

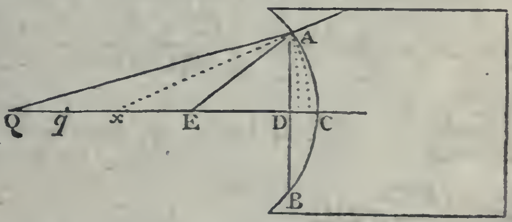
$$\frac{QE \times EC}{m \,.\, QC - EC} - \frac{m \,.\, QE^2}{\overline{m+1} \,.\, QC} \times \frac{QC + \overline{m+2} \,.\, EC}{\overline{m \,.\, QC - EC}|^2} \times DC,$$

nearly. When DC vanishes, $Ex = Eq = \frac{QE \times EC}{m \,.\, QC - EC}$; therefore the aberration $qx = \frac{m \,.\, QE^2}{\overline{m+1} \,.\, QC} \times \frac{QC + \overline{m+2} \,.\, EC}{\overline{m \,.\, QC - EC}|^2} \times DC$.

398. Cor. 1. If the refractor be given, and the situation of the focus of incident rays, the aberration varies as DC, the versed sine of the arc AC.

399. Cor. 2. When the incident rays are parallel, QC becomes equal to QE; and $qx = \frac{DC}{m \,.\, \overline{m+1}}$

400. Cor. 3. When diverging rays are incident upon a concave spherical refracting surface of a denser



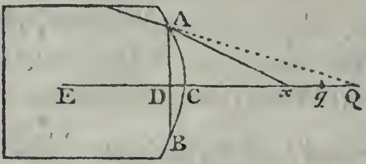
medium, the construction being made as before, EC

 and

and DC are negative; whence, $Ex = -\frac{QE \times EC}{m \cdot \overline{QC+EC}} + \frac{m \cdot QE^2}{\overline{m+1} \cdot QC} \times \frac{QC - \overline{m+2} \cdot EC}{m \cdot \overline{QC+EC}|^2} \times DC$; and $qx = -\frac{m \cdot QE^2}{\overline{m+1} \cdot QC} \times \frac{QC - \overline{m+2} \cdot EC}{m \cdot \overline{QC+EC}|^2} \times DC$; this aberration, therefore, is to be meaſured in an oppoſite direction to the former.

401. Cor. 4. In this laſt caſe, if $QC = \overline{m+2} \cdot EC$, the aberration vaniſhes; that is, if $QC : EC :: m+2 : 1$, or $QE : EC :: m+1 : 1$ :: ſin. incidence : ſin. refraction (Art. 155).

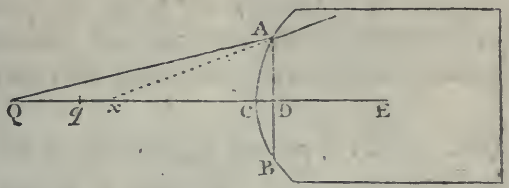
402. Cor. 5. When converging rays are incident upon a concave ſpherical ſurface of a rarer medium,



QC, QE, EC and DC are negative. Alſo, if $1-\mu$: 1 :: ſin. incidence : ſin. refraction, $-\mu$ muſt be ſubſtituted for m, and $Ex = \frac{QE \times EC}{\mu \cdot \overline{QC+EC}} - \frac{\mu \cdot QE^2}{\overline{1-\mu} \cdot QC} \times \frac{QC + \overline{2-\mu} \cdot EC}{\mu \cdot \overline{QC+EC}|^2} \times DC$; hence, $qx = \frac{\mu \cdot QE^2}{\overline{1-\mu} \cdot QC} \times \frac{QC + \overline{2-\mu} \cdot EC}{\mu \cdot \overline{QC+EC}|^2} \times DC$.

403. Cor.

403. Cor. 6. When diverging rays are incident upon a convex ſpherical ſurface of a rarer medium,



$Ex = -\frac{QE \times EC}{\mu . QC + EC} + \frac{\mu . QE^2}{1-\mu . QC} \times \frac{QC + \overline{2-\mu} . EC}{\overline{\mu . QC + EC}|^2} \times$ DC; therefore the aberration $qx = -\frac{\mu . QE^2}{1-\mu . QC} \times$ $\frac{QC + \overline{2-\mu} . EC}{\overline{\mu . QC + EC}|^2} \times DC.$

In the ſame manner, the aberration may be found in the other caſes.

404. Cor. 7. Since $DC = \frac{DA^2}{2EC}$, nearly (Art. 392), by ſubſtituting this value of DC in the foregoing expreſſions, we obtain the aberration in terms of the ſemi-aperture.

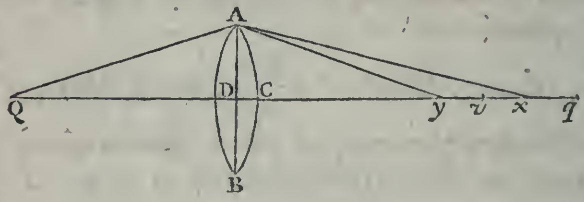
405. Cor. 8. Since $Eq = \frac{QE \times EC}{m . QC - EC}$, if QE be diminiſhed by the ſmall quantity x, Eq will be increaſed by the quantity $\frac{\overline{m+1} . EC^2 \times x}{\overline{m . QC - EC}|^2}$. For on this ſuppoſition, Eq becomes $\frac{\overline{QE - x} . EC}{m . \overline{QC - x} - EC} =$ $\frac{\overline{QE - x} . EC}{m . QC - EC - mx} = \frac{QE \times EC}{m . QC - EC} + \frac{\overline{m+1} . EC^2 \times x}{\overline{m . QC - EC}|^2}$, nearly;

nearly; and therefore $\frac{\overline{m+1} \cdot EC^2 \times x}{m \cdot \overline{\mathcal{Q}C - EC}|^2}$, where x is the decrement of $\mathcal{Q}E$, is the increment of Eq, nearly.

406. Cor. 9. If x vary as DC, the increment of Eq, when the radius of the refractor and the ſituation of the focus of incident rays are given, will alſo vary as DC.

407. Cor. 10. By a proper application of the foregoing rules, the longitudinal aberration, ariſing from the ſpherical form of refracting ſurfaces, may be found in all caſes where the apertures are ſmall.

Ex. Let $\mathcal{Q}q$ be the axis of a lens; $\mathcal{Q}$ the focus of incident rays; q the geometrical focus after the firſt refraction, determined by Art. 147; v the geometrical



focus of emergent rays (Art. 176). Alſo, let $\mathcal{Q}A$ be refracted, at the firſt ſurface, in the direction Ax, and emergent in the direction Ay. Then, the aberration vy ariſes from two cauſes; 1ſt. x does not coincide with the geometrical focus q (Art. 397); and ſince v is determined on ſuppoſition that q is the focus of rays incident upon the ſecond ſurface, an aberration will be produced, which may be determined by Art. 405. 2dly. Ax is incident obliquely upon the latter ſurface, and

and the aberration arifing from this caufe may be determined by Art. 402; therefore the whole aberration vy may be found.

408. Cor. 11. If the lens and place of the focus of incident rays be given, the aberration arifing from each of thefe caufes will vary nearly as AD^2 (Arts. 397. 406. 404); and therefore the final aberration vy, which is the fum or difference of the former, will alfo vary nearly as AD^2.

409. It is not confiftent with the plan of this work to enter farther into thefe calculations; perhaps too much has been faid already. The reader will find little difficulty in the application of the principles, if he wifh to deduce practical rules for the conftruction of object glaffes. Thus much it may be proper to obferve, that the aberrations produced by a convex and concave lens are of contrary affections, and tend to correct each other; by a proper adjuftment therefore, of the radii of the furfaces, a compound lens may be conftructed, which will entirely deftroy the aberration of the extreme rays *.

We may alfo obferve, that the aberration is lefs, when two furfaces, or two lenfes of the fame kind are employed, than when the *fame refraction* is produced by a fingle furface, or lens of the fame defcription, and equal aperture.

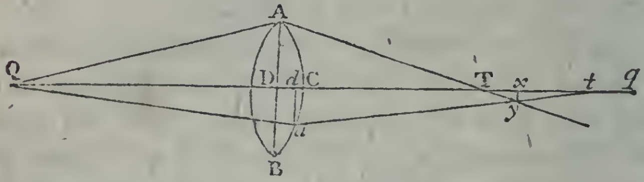
PROP.

* This fubject is treated with great ability in the Encyclopædia Britannica, under the head Telefcopes.

Prop. LXXXVIII.

410. *To find the leaſt circle of aberration into which all the homogeneal rays of the ſame pencil, refracted by a lens or ſingle ſurface, are collected.*

Let AB be the refractor; $\mathcal{Q}Cq$ it's axis; $\mathcal{Q}$ the focus of incident rays; T the interſection of the ex-



treme ray $\mathcal{Q}AT$, and the axis; t the interſection of any other ray $\mathcal{Q}at$, on the other ſide of $\mathcal{Q}C$, and the axis; y the interſection of ATy and at. Draw AD, ad, and yx at right angles to $\mathcal{Q}Cq$; then, if the point a move from C towards B, the perpendicular xy will vary on two accounts; the increaſe of the angle Cta, and the decreaſe of the diſtance Tt; and when xy is a maximum, all the rays incident upon the ſame ſide of $\mathcal{Q}C$ with $\mathcal{Q}a$, will paſs through it; and if the figure revolve about the axis $\mathcal{Q}q$, all the rays incident upon the lens will paſs through the circle generated by xy. It is alſo manifeſt, that the circle thus generated, is leſs than any other circle through which all the refracted rays paſs. To find when xy is the greateſt poſſible, let $Tx = x$; $ad = v$; $AD = a$; $DT = f$; $Tq = b$. Then, ſince AD, ad, when the lens is thin, are the ſemi-apertures through which the rays $\mathcal{Q}AT$, $\mathcal{Q}at$ paſs, $AD^2 : ad^2 :: qT : qt$ (Art. 408); or,

or, $a^2 : v^2 :: b : qt$; whence $qt = \frac{bv^2}{a^2}$; therefore $Tq - qt = Tt = b - \frac{bv^2}{a^2} = \frac{b}{a^2} \times \overline{a^2 - v^2}$. Again, $DT : DA :: Tx : xy$; or, $f : a :: x : xy$; consequently, $xy = \frac{ax}{f}$; also, $da : dt :: xy : tx$; or, $v : f :: \frac{ax}{f} : tx$; therefore $tx = \frac{ax}{v}$; hence, $Tx + xt = Tt = x + \frac{ax}{v} = \frac{b}{a^2} \times \overline{a^2 - v^2}$; or, $\frac{x}{v} \times \overline{a + v} = \frac{b}{a^2} \times \overline{a + v} \times \overline{a - v}$; and $x = \frac{b}{a^2} \times v \times \overline{a - v}$; consequently, x is the greatest possible, and therefore xy is the greatest possible, when $v \times \overline{a - v}$ is the greatest possible; or when $v = \frac{1}{2}a$. Hence it follows, that the greatest value of x is $\frac{b}{4}$; and the corresponding value of $xy = \frac{ab}{4f} = \frac{DA \times qT}{4DT}$.

411. Cor. 1. If the focal length of the refractor, and the focus of incidence, be given, DT is given, and $xy \propto qT \times DA \propto DA^3$ (Art. 408).

412. Cor. 2. On the same supposition, the area of the least circle of aberration varies as DA^6.

413. Cor. 3. Exactly in the same manner, we may find the least circle into which a pencil of rays, reflected by a spherical surface, is collected.

414. Cor. 4. When parallel rays are incident upon a spherical reflector, the longitudinal aberration varies directly as the square of the semi-aperture, and inversely

inverſely as the focal length (Art. 393); therefore, xy, the radius of the leaſt circle of aberration, varies directly as the cube of the ſemi-aperture, and inverſely as the ſquare of the focal length of the reflector.

Prop. LXXXIX.

415. *The area of a circle of aberration in the image formed upon the retina by a teleſcope, or double microſcope, varies directly as the area of the circle of aberration in the focus of the eye glaſs, and inverſely as the ſquare of the focal length of the eye glaſs.*

When the circle of aberration is in the principal focus of the glaſs through which it is viewed, it's viſual angle is equal to the angle which it ſubtends at the center of the glaſs; and therefore, the *linear magnitude* of the circle of aberration upon the retina, varies as this angle (Art. 254); that is, it varies directly as the linear magnitude of the circle of aberration in the principal focus of the glaſs, and inverſely as the focal length of the glaſs; conſequently, the *area* of the circle of aberration on the retina, varies directly as the area of the circle of aberration in the focus of the eye glaſs, and inverſely as the ſquare of the focal length of the eye glaſs.

416. Cor. 1. In a reflecting teleſcope of Sir Isaac Newton's conſtruction, if F be the focal length of the reflector, A it's ſemi-aperture, f the focal length of the eye glaſs, the radius of the circle of aberration in it's principal focus, varies as $\frac{A^3}{F^2}$ (Art. 414); and there-

fore

fore the area of this circle varies as $\frac{A^6}{F^4}$; confequently, the area of the circle of aberration on the retina, varies as $\frac{A^6}{F^4 f^2}$.

417. Cor. 2. The area of the circle of aberration on the retina, has ufually been confidered as a meafure of the apparent indiftinctnefs of vifion. And, though it is manifeft that indiftinctnefs admits of no numerical reprefentation *, yet if the circle of aberration be the fame in two cafes, cæteris paribus, the indiftinctnefs will be the fame; and if the circle of aberration be greater in one cafe than in another, the indiftinctnefs will alfo, cæteris paribus, be greater. For, the rays which proceed from one point in the object, are diffufed over the circle of aberration, and confequently they are mixed with the rays which belong to as many different foci as there are fenfible points in that circle; therefore, the greater the area of the circle, the greater muft be the confufion, or indiftinctnefs arifing from this difperfion of the rays.

Prop. XC.

418. *To find on what fuppofition a given diftant object appears equally bright, and equally diftinct, when viewed with different reflecting telefcopes of* Sir Isaac Newton's *conftruction.*

The notation in the 416th article being retained; fince the

* One degree of indiftinctnefs can no more be faid to be a multiple or part of another, than one degree of tafte, or fmell can be faid to be the double, or half of another.

the brightneſs is given, $\frac{4A^2f^2}{F^2} \propto 1$ (Art. 360); or, $\frac{A^6f^6}{F^6} \propto 1$. Alſo, ſince the indiſtinctneſs is given, $\frac{A^6}{F^4f^2}$ $\propto 1$; therefore $\frac{A^6f^6}{F^6} \propto \frac{A^6}{F^4f^2}$; and $f^8 \propto F^2$; or, $f \propto F^{\frac{1}{4}}$.

Again, $\frac{A^4f^4}{F^4} \propto 1$; that is, $\frac{A^4F}{F^4} \propto 1$; or, $A^4 \propto F^3$; and $A \propto F^{\frac{3}{4}}$.

419. Cor. The magnifying power $\propto \frac{F}{f}$ (Art. 328); that is, as $F^{\frac{3}{4}}$.

SECTION

SECTION IX.

ON THE RAINBOW.

Prop. XCI.

420. *If two quantities bear an invariable ratio to each other, their correſponding increments are in the ſame ratio.*

Let X and Y be the two quantities; x and y their correſponding increments. Then, by the ſuppoſition, $X : Y :: X+x : Y+y$; and alternately, $X : X+x :: Y : Y+y$; by diviſion, $X : x :: Y : y$; therefore, alternately, $X : Y :: x : y$.

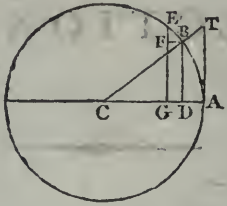
Prop. XCII.

421. *If the ſines of two arcs be always in a given ratio, the evaneſcent increments of the arcs are proportional to their tangents.*

Let AB be a circular arc whoſe radius is CA, ſine BD, and tangent AT; draw EG parallel, and indefinitely near to BD, BF parallel to DG, and join EB.

P Then,

Then, the triangle CBD is similar to the triangle EBF, formed by EF, FB, and the chord BE; for, the angles CDB, BFE are right angles; and the



$\angle EBC$ is a right angle (Newt. Princip. Lem. 6), and therefore equal to the $\angle FBD$; take away the common angle FBC, and the remaining angles, CBD and FBE are equal. Hence, $FE : BE :: CD : CB$; and in the similar triangles CDB, CAT, $CD : CA$ $(CB) :: DB : AT$; therefore, $FE : BE :: DB : AT$; whence $BE = \frac{FE \times AT}{DB}$; and BE is ultimately equal to the increment of the arc AB (Newt. Princip. Lem. 7); consequently, BE, the increment of the arc, $= \frac{FE \times AT}{DB}$; and since FE, the increment of the sine, varies as DB the sine (Art. 420), BE varies as AT.

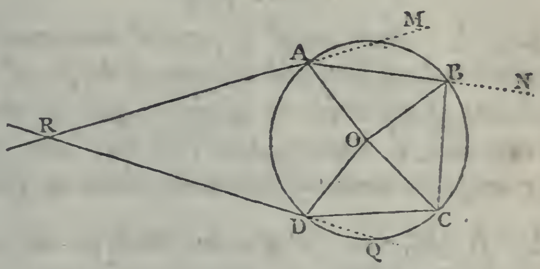
Prop. XCIII.

422. *If a ray of light refracted into a sphere, emerge from it after any given number of reflections, to determine the deviation of the ray, and the angle contained between the directions in which it is incident and emergent.*

Let a ray of light RA, incident upon the sphere $ABCD$ at A, be refracted in the direction AB; at B let

let it be reflected in the direction BC; and at C, in the direction CD; at D let it be refracted out of the ſphere, in the direction DR; produce RA, RD, to M, Q.

Take O the center of the ſphere; join OA, OB, OC, OD; and let A be the angle of incidence of the ray RA; B the angle of refraction; R, a right angle,



Then the $\angle OAM = A$; the $\angle OAB =$ the $\angle OBA$ (Euc. 5. 1) $=$ the $\angle OBC$ (Art. 18) $=$ the $\angle OCB =$ the $\angle OCD =$ the $\angle ODC = B$. Alſo, the angles of deviation at A and D are equal; for if BA be ſuppoſed to be incident at A, the angle of incidence BAO, is equal to the angle of incidence CDO, of the ray CD; therefore the angles of deviation are equal*; and ſince the angle of deviation at A, is $A - B$, the whole deviation ariſing from the two refractions, is $2A - 2B$. Again, the angle of deviation at B is $2R - 2B$; and the angle of deviation, at every other reflection, is the ſame; therefore, if there be p reflections, the whole deviation, ariſing from this cauſe, is $2pR - 2pB$. To this,

* Vid. Art. 25.

this, let the deviation arising from the refractions be added, and the whole deviation of the ray from it's original direction, is $2pR - 2 \cdot \overline{p+1} \cdot B + 2A$.

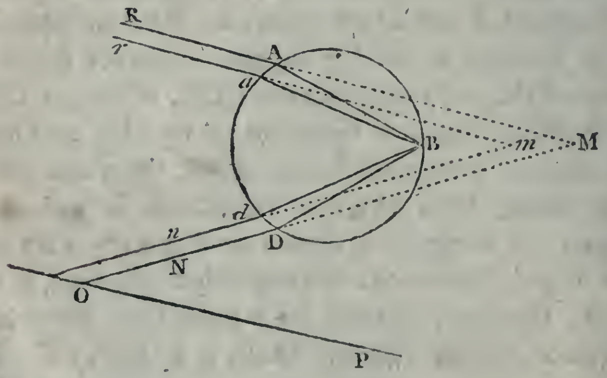
Also, a deviation through the angle $2pR$, which is a multiple of 180°, produces no inclination of the emergent to the incident ray; therefore, the inclination is represented by $2A - 2 \cdot \overline{p+1} \cdot B$; or $2 \cdot \overline{p+1} \cdot B - 2A$.

Prop. XCIV.

423. *If a small pencil of parallel homogeneal rays be refracted into a sphere, and the ratio of the sine of incidence to the sine of refraction be known, to find at what angle the rays must be incident, that they may emerge parallel after any given number of reflections within the sphere.*

Let RAM, ram, be the directions of the incident, DN, dn, the directions of the emergent rays; produce ND, nd, if necessary, till they meet RM, rm, in M and m.

Then, since AM, am, as also DN, dn, are parallel



by the supposition, the angles at M, and m are equal; therefore,

therefore, when the rays are incident at, or near to A, the angle RMN, contained between the incident and emergent ray, ceases to increase, or decrease; and therefore, the notation in the last article being retained, $2 . \overline{p+1} . B - 2A$, and consequently $\overline{p+1} . B - A$, ceases to increase, or decrease; that is, the increment of $\overline{p+1} . B$, is equal to the corresponding increment of A. Also, since sin. A is in a given ratio to sin. B; the increment of B : the increment of A :: tang. B : tang. A (Art. 421); or, multiplying the first and third terms by $p+1$, $\overline{p+1} \times$ increment of B : increment of A :: $\overline{p+1}$. tang. B : tang. A; and $\overline{p+1} \times$ increment of $B =$ increment of $\overline{p+1} . B$ (Art. 420); therefore, the increment of $\overline{p+1} . B$: the increment of A :: $\overline{p+1}$. tang. B : tang. A; and since the increment of $\overline{p+1} . B$ is equal to the increment of A, when the rays emerge parallel, $\overline{p+1}$. tang. $B =$ tang. A; or, tang. A : tang. B :: $p+1$: 1.

To determine the angles A and B, suppose x and y to be their cosines, the radius being unity; and let sin. A : sin. B :: m : n.

Then, $\sqrt{1-x^2} =$ sin. A; $\sqrt{1-y^2} =$ sin. B; $\frac{\sqrt{1-x^2}}{x}$ $=$ tang. A; $\frac{\sqrt{1-y^2}}{y} =$ tang. B. And, from the relation of the required angles, we have the following proportions; $\frac{\sqrt{1-x^2}}{x} : \frac{\sqrt{1-y^2}}{y} :: p+1 : 1$;

and $\sqrt{1-y^2} : \sqrt{1-x^2} :: n : m$;

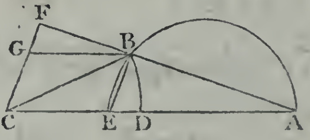
 by

by compoſition, $\frac{1}{x} : \frac{1}{y} :: n . \overline{p+1} : m$; hence, $y = \frac{\overline{p+1} . nx}{m}$; and $y^2 = \frac{\overline{p+1}|^2 . n^2x^2}{m^2}$; therefore, $1 - y^2 : 1 - x^2 :: 1 - \frac{\overline{p+1}|^2 . n^2x^2}{m^2} : 1 - x^2 :: n^2 : m^2$; and by multiplying extremes and means, $n^2 - n^2x^2 = m^2 - \overline{p+1}|^2 . n^2x^2$; hence, $\overline{\overline{p+1}|^2 - 1} . n^2x^2 = m^2 - n^2$; or $\sqrt{p^2+2p} . nx = \sqrt{m^2-n^2}$; conſequently, $1 : x :: \sqrt{p^2+2p} . n : \sqrt{m^2-n^2}$. The coſine of *A* being determined by this proportion, the angle itſelf may be found from the tables.

Alſo, $m : n ::$ ſin. *A* : ſin. *B*; and the three firſt terms in the proportion being known, the fourth is known; that is, ſin. *B* is known; and therefore the angle *B* may alſo be found from the tables.

The angles *A* and *B* may alſo be determined by the following conſtruction:

In the ſtraight line *CEDA*, take *CA* to *CD* as *m* to *n*, and *CA* to *CE* as $p+1$ to 1; with the center *C* and



radius *CD*, deſcribe an arc *DB*, cutting the circle *ABE* whoſe diameter is *AE*, in *B*; draw *ABF*; and join *BC*; then, the ſine of the ∠ *CBF* will be to the ſine of the ∠ *CAF* as *m* to *n*; and the tangent of *CBF* to the tangent of *CAF* as $p+1$ to 1; and conſequently *CBF*, *CAF* will be the angles required.

Join

Join BE, and complete the parallelogram $CEBG$ produce CG till it meets ABF in F. Then, in the triangle CAB, ſin. CBA (ſin. CBF) : ſin. CAB :: CA : CB :: CA : CD :: m : n. Again, ſince CF is parallel to EB, the $\angle BFG$, is equal to the $\angle EBA$, and is therefore a right angle; conſequently, the lines FC, FG are tangents of the angles CBF, GBF (CAF) to the radius BF; and, in the ſimilar triangles FCA, FGB, $FC : FG :: CA : GB :: CA : CE :: p+1 : 1$.

424. Ex. 1. If a ſmall pencil of parallel red rays be incident upon a ſphere of water, at an angle of about $59°.23'$, and ſuffer two refractions and one reflection, the rays will emerge parallel.

Here, $p=1$; and $m : n :: 108 : 81 :: 4 : 3$; therefore, $1 : x :: \sqrt{27} : \sqrt{7}$; or $x=\sqrt{\frac{7}{27}}$; and the angle whoſe coſine, to the radius unity, is $\sqrt{\frac{7}{27}}$, is $59°.23'$, nearly.

The angle of refraction B, whoſe ſine is to the ſine of $59°.23' :: 3 : 4$, is $40°.12'$. Hence, the whole deviation, $2R-4B+2A$ (Art. 422), is $137°.58'$; which ſubtracted from $180°$, gives the inclination of the incident, to the emergent pencil, $42°.2'$.

When violet rays are thus incident and emergent, $m : n :: 109 : 81$, and in this caſe, $A=58°.40'$; $B=39°.24'$; hence, $2R-4B+2A$ is $139°.44'$, and the inclination of the emergent, to the incident pencil, $40°.16'$.

425. Ex. 2. If parallel red rays fall upon a ſphere of water, they will emerge parallel, after two refrac-

 tions

tions and two intermediate reflections, when the angle of incidence is about 71°.50′.

In this case, $p=2$; and $1 : x :: \sqrt{72} : \sqrt{7}$; therefore the cosine of the angle of incidence is $\sqrt{\frac{7}{72}}$, which corresponds to an angle of 71°.50′, nearly.

Also, $B=45°.27'$; and the whole deviation, $4R-6B+2A,=230°.58'$; hence, the inclination of the emergent, to the incident pencil, which is the excess of the whole deviation above 180°, = 50°.58′, nearly.

When violet rays are thus incident and emergent, $A=71°.26'$; $B=44°.47'$; $4R-6B+2A=234°.10'$; and the inclination of the emergent, to the incident pencil = 54°.10′, nearly.

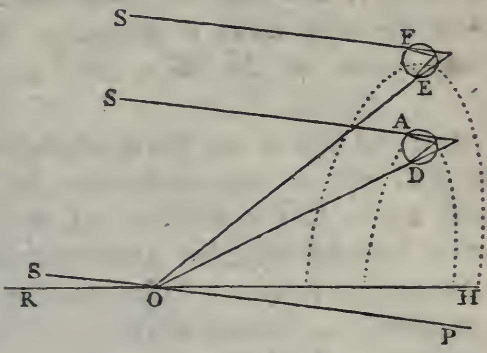
On the formation of the Rainbow.

426. It has long been known that the rainbow is owing to the refraction and reflection of the sun's light by drops of rain. Antonius de Dominis first discovered that the interior, or primary bow, is caused by two refractions of the rays of light at each drop of water, and one reflection between them; and the exterior, or secondary bow, by two refractions and two reflections between them. This discovery he confirmed by experiments, which have been successfully repeated by more modern writers. If glass globes, filled with water, be placed in the sun's light, they may be elevated or depressed till they successively transmit to the eye, the colours of each bow, in their proper order *.

427. To

* Newt. Opt. Book I. Prop. IX.

427. To underſtand how the interior bow is formed, let *O* be the eye of a ſpectator; *SOP* a line paſſing through the eye and the ſun. At the point *O*,



in the line *PO*, make the angle $POE = 42°.2'$; then, when a drop of rain, *FE*, is in ſuch a ſituation that the angle which *OE* makes with a perpendicular to it's ſurface at *E* is 59°.23′, a ſmall pencil of parallel red rays will emerge from it at *E*, and enter the eye in the direction *EO*. For, if *OE* be conſidered as the incident pencil, it will emerge, after two refractions and one reflection, in the direction *FS*, which makes an angle of 137°.58′ with *OE* produced (Art. 424), or, an angle of 42°.2′ with *OE*, and is therefore parallel to *OS*; thus *FS* will paſs through the ſun*. Converſely, out of the beam of ſun-light which falls upon the drop, the red rays incident at, and near to *F*, will, after two refractions and one reflection, emerge parallel, and entering the eye in the direction *EO* (Art. 29), will excite the ſenſation of their proper colour.

In

* The diſtance of the ſun is ſo great, that two lines drawn from any points upon the ſurface of the earth, to a point in his diſc, may be conſidered as parallel, in theſe calculations.

In the ſame manner, if *OE* revolve about the axis *OP*, every drop of water in the ſurface of the cone thus deſcribed, will tranſmit to the eye a ſmall parallel pencil of red rays; and thus a red arc, whoſe radius, *meaſured by the angle which it ſubtends at the eye*, is 42°.2′, will appear in the falling rain, oppoſite to the ſun.

The other red rays of the beam which falls upon the drop *FE*, will, at their emergence, be inclined at different angles to the direction of the incident rays, and be ſo much diſperſed before they reach the eye, and enter it in ſo weak a ſtate, mixed with other rays, as to produce no diſtinct effect.

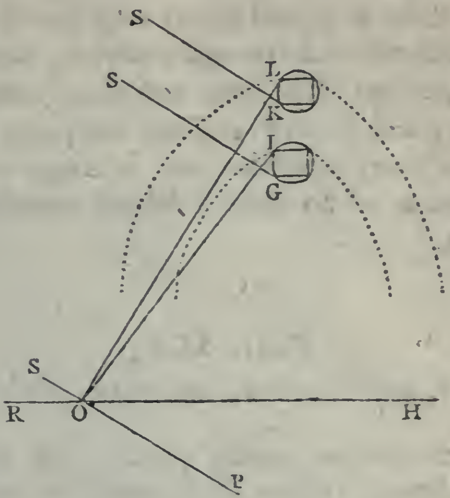
The parallel pencils of red rays, which emerge from other drops, fall above, or below the eye.

If the angle *POD* be 40°.16′, and *OD* revolve about the axis *OP*, every drop of rain in the ſurface of the cone thus deſcribed, will tranſmit to the eye a parallel pencil of violet rays; and thus a violet arc will be formed, whoſe radius is 40° 16′.

The drops between *E* and *D* will tranſmit to the eye parallel pencils of rays of different colours, orange, yellow, green, blue, indigo, in the order which they have in the priſmatic ſpectrum (Art. 366); and the radii of the arcs of theſe reſpective colours may be calculated by the method employed in the 424th Article.

428. Again, let the angle *POI* = 50°.58′; and the angle *POL* = 54°.10′. Alſo, let *OI*, *OL* revolve about the axis *OP*. Then, it may be ſhewn as in the preceding caſe, that every drop of rain in the conical ſurface generated by *OI*, will tranſmit to the eye a ſmall

ſmall parallel pencil of red rays, which has ſuffered



two refractions and two reflections, but ſufficiently ſtrong to excite the ſenſation of it's proper colour.

Alſo, every drop in the conical ſurface generated by *OL* will tranſmit to the eye a ſmall pencil of parallel violet rays; and the intermediate drops, parallel pencils of rays of the intermediate colours. Thus the exterior bow is formed, in which the radii of the red and violet arcs are, reſpectively, 50°.58′, and 54°.10′.

The radii of the intermediate arcs may be determined by the method employed in the 425th Article.

429. Cor. 1. The colours in the two bows lie in a contrary order; the red forming the exterior ring of the primary, and the interior ring of the ſecondary bow.

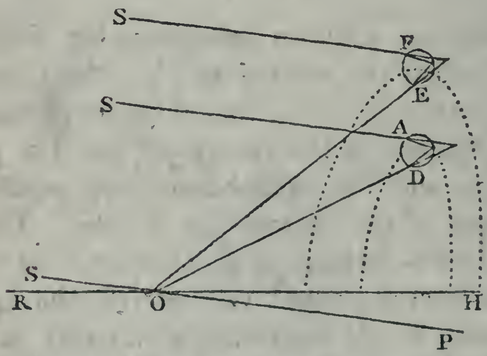
430. Cor. 2. Were the pencils ſufficiently ſtrong, a third bow, formed by two refractions and three reflections of the ſun's rays in drops of rain, might be ſeen.

ſeen. But, when the rays which are refracted into a drop of water, reach the farther ſurface, ſome of them paſs out of the drop, and others are reflected within it. When theſe reflected rays again meet the ſurface, ſome of them paſs out of the drop, and others ſuffer another reflection; and ſo on *. Thus the pencil becomes weaker at every reflection; and at length it contains ſo few rays as not to make a diſtinct impreſſion upon the retina.

Prop. XCV.

431. *To find the altitude, and breadth of the rainbow.*

The conſtruction being made as in the 427th Article, through *O* draw *HOR* parallel to the horizon.



Then the angle *ROS*, or *HOP*, meaſures the altitude of the point *S* above the horizon; and the altitude of

* This is a fact, the cauſe of which has not been ſatisfactorily explained. Sir Isaac Newton ſuppoſes that rays of light, when they arrive at the ſurface of a medium, are ſometimes in a ſtate to be reflected, and ſometimes to be tranſmitted; theſe ſtates he calls fits

of the higheſt point of the red arc above the horizon, in the primary bow, is meaſured by EOH, or $EOP - HOP$, which is equal to $42°.2' - HOP$. Alſo, the altitude of the higheſt point of the violet arc is meaſured by $DOP - HOP$, or $40°.16' - HOP$. Hence it follows, that the breadth of the bow, ſuppoſing it to be formed by the rays which come from one point S, in the ſun's diſc, is $42°.2' - 40°.16'$, or, $1°.46'$.

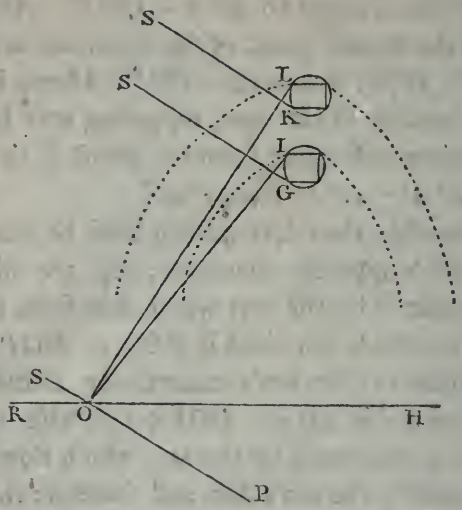
The breadth, thus determined, muſt be increaſed by $30'$, the ſun's apparent diameter; for, the higheſt red arc is produced by the rays which flow from the loweſt point in the ſun's diſc, and if ROS, or HOP, meaſure the altitude of the ſun's center, the altitude of the higheſt red arc is $42°.2' - HOP + 15'$; alſo, the loweſt violet arc is produced by the rays which flow from the higheſt point in the ſun's diſc, and therefore the altitude of this arc, is $40°.16' - HOP - 15'$; conſequently, the breadth of the bow is $1°.46' + 30'$, or $2°.16'$.

In the ſame manner it appears, that the altitude of the violet arc, in the exterior bow, is $54°.10' - SOR$; and the altitude of the red arc, $50°.58' - SOR$; therefore, the breadth of the bow, formed by rays which proceed from any one point in the ſun's diſc, is $3°.12'$.

If

fits of eaſy reflection, and tranſmiſſion; and accounts for them in the following manner: "Nothing more is requiſite for putting the rays of light into fits of eaſy reflection, and eaſy tranſmiſſion, than that they be ſmall bodies, which by their attractive powers, or ſome other force, ſtir up vibrations in what they act upon; which vibrations being ſwifter than the rays, overtake them ſucceſſively, and agitate them, ſo as by turns to increaſe and diminiſh their velocities, and thereby put them into thoſe fits." Opt. Query 29.

If to this we add 30′, the ſun's apparent diameter,



we have the actual breadth of the exterior bow = 3°.42′.

432. Cor. 1. Since the altitude of an arc of any colour in the bow is equal to the radius of this arc diminiſhed by the ſun's altitude, when the ſun is in the horizon, the altitude of the arc is equal to it's radius.

433. Cor. 2. The radius of any arc in the rainbow is equal to the altitude of the arc above the horizon, together with the ſun's altitude.

434. Cor. 3. When the ſun's altitude above the horizon, is equal to, or exceeds 42°.2′, the primary bow cannot be ſeen; nor the ſecondary, when his altitude is equal to, or exceeds 54°.10′.

PROP.

PROP. XCVI.

435. *Having given the radius of an arc of any colour in the primary rainbow, to find the ratio of the ſine of incidence to the ſine of refraction when rays of that colour paſs out of air into water.*

If A be the angle of incidence of the effective rays, B the angle of refraction, the radius of the arc is $4B-2A$ (Art. 422); let the tangent of $2B-A$, half this angle, to the radius unity, be a; z the tangent of B. Then $2z=$ tang. A (Art. 423). Alſo, from the principles of trigonometry, tang. $2B$: $2\times$ tang. B $(2z)$ :: 1^2 : 1^2-z^2; therefore tang. $2B=\frac{2z}{1-z^2}$. Again, tang. $\overline{2B-A}$ (a) : tang. $2B-$ tang. A $\left(\frac{2z}{1-z^2}-2z\right)$ :: $1^2 : 1^2+\frac{4z^2}{1-z^2}$ *; hence, $\frac{2z}{1-z^2}-2z=a+\frac{4az^2}{1-z^2}$; and by reduction, $2z^3-3az^2-a=0$. The value of z being obtained † from this equation, the angles B and A, and conſequently their ſines, may be found from the tables.

436. Cor. In the ſame manner, if p be the number of reflections within the drop, z the tangent of B, Q the

* The propoſitions here referred to are the following;

1ſt. The tangent of the ſum of two arcs, is to the ſum of their tangents, as the ſquare of radius, is to the ſquare of radius diminiſhed by the rectangle under the two tangents. 2d. The tangent of the difference of two arcs, is to the difference of their tangents, as the ſquare of radius, to the ſquare of radius increaſed by the rectangle under the two tangents. Mr. VINCE's Trig. Art. 117.

† This equation has two impoſſible roots. Vid. Alg. Art. 358.

the tangent of $\overline{p+1}\,.\,B$, a the tangent of $\overline{p+1}\,.\,B-A$, then $\overline{p+1}\,.\,z=$ tang. A; and $a:Q-\overline{p+1}\,.\,z::1^2:1^2+\overline{p+1}\,.\,Qz$; therefore $Q-\overline{p+1}\,.\,z=a+\overline{p+1}\,.\,aQz$.

From which equation, the value of z being found, the angles A and B, and consequently their sines, may be determined by the tables.

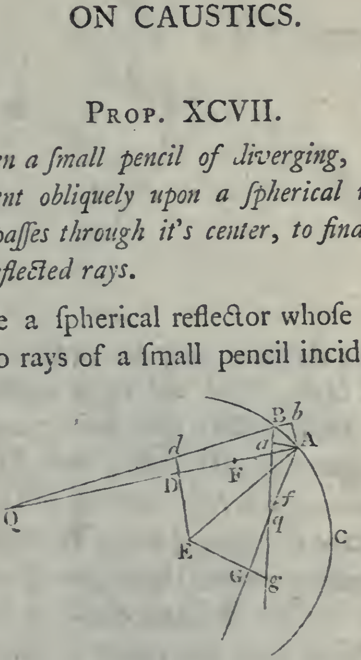
SECTION X.

ON CAUSTICS.

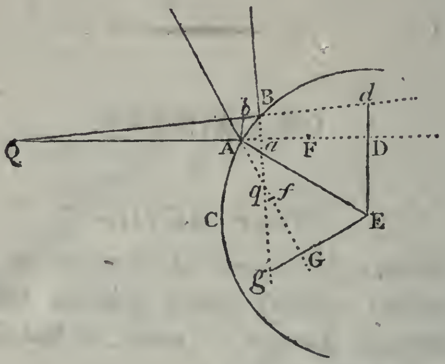
Prop. XCVII.

437. *When a ſmall pencil of diverging, or converging rays is incident obliquely upon a ſpherical reflector, in a plane which paſſes through it's center, to find the geometrical focus of reflected rays.*

Let BC be a ſpherical reflector whoſe center is E; QA, QB two rays of a ſmall pencil incident obliquely



upon it, in the plane QAE; AG, Bg the reflected rays, or thoſe rays produced backwards; q their interſection. From E, draw EDd, EGg at right angles to QA, AG; and when the arc AB is diminiſhed without limit, they are alſo at right angles to QB, Bg; join EA, AB; from A, draw Ab, Aa at right angles to QB, qB, produced if neceſſary; biſect AD, *AG* in *F*, *f*. Then, the angles *EAD*, *EAG* are the angles of incidence and reflection of the ray *QA*, or



equal to them; therefore they are equal to each other; the angles *EDA*, *EGA* are right angles; and the side *EA* is common to the two triangles *EAD*, *EAG*; consequently $AD = AG$; and $ED = EG$. In the same manner, $Ed = Eg$; whence, $Dd = Gg$. Again, in the evanescent triangles *ABa*, *ABb*, the angles *ABb*, *ABa* are complements to the angles of incidence and reflection of the ray *QB*, or equal to those complements; therefore they are equal to each other; also, the angles *AbB*, *AaB* are right angles; and *AB* is common to the two triangles; consequently, $Ab = Aa$.

Now, in the similar triangles *QDd*, *QAb*, $QA : QD :: Ab : Dd :: Aa : Gg$; and in the similar triangles *qAa*, *qGg*, $Aa : Gg :: Aq : qG$; therefore, $QA : QD :: Aq : qG$; whence, by composition, and division, $QA + QD : QA \sim QD :: Aq + qG : Aq \sim qG$; that

that is, $2QF : 2FA :: 2Af : \overline{Af+fq} - \overline{Af-fq}$ $(2qf)$*; or, $QF : FA :: Af : fq$.

438. Cor. 1. In the case represented by the first figure, QF and FA are measured in opposite directions from F; and since $QA : QD :: Aq : qG$, and QA is greater than QD, Aq is greater than qG; and therefore Af and fq are measured in opposite directions from f; hence it follows, that the equal rectangles $QF \times fq$ and $FA \times Af$ have, in this case, the same sign; therefore they will always have the same sign; that is, whenever QF and QA are measured in opposite directions from F, Af and fq are measured in opposite directions from f; and the contrary.

439. Cor. 2. When the incident rays are parallel, FA is evanescent with respect to QF; therefore fq is evanescent with respect to Af; or, q coincides with f. Here, $Aq = \frac{1}{2} AG = \frac{1}{2} AD$.

440. Cor. 3. If D be the focus of incident rays, G will be the focus of reflected rays. In this case, $QF = FA$; therefore $Af = fq$; and since QF and FA are measured in opposite directions from F, Af and fq must be measured in opposite directions from f; consequently, q coincides with G.

441. Cor.

* This conclusion depends upon the supposition that when QA and QD are measured in the same direction from Q, qA and qG are measured in opposite directions from q. If this be not the case, $Aq + qG = 2qf$; and $Aq \sim qG = 2fA = 2FA$; therefore $2qf = 2QF$, and $qf = QF$. Now let the rays be incident nearly perpendicularly upon the reflector, and F and f coincide with the principal focus; therefore Q and q are always equally distant from the principal focus, which is absurd (Vid. Art. 50).

441. Cor. 4. If $\mathcal{Q}$ be a point in the circumference of the circle BC, $FA=\frac{1}{3}\mathcal{Q}F$; therefore $fq=\frac{1}{3}Af=\frac{1}{12}\mathcal{Q}A$; hence, $Aq=\frac{1}{4}\mathcal{Q}A+\frac{1}{12}\mathcal{Q}A=\frac{1}{3}\mathcal{Q}A$.

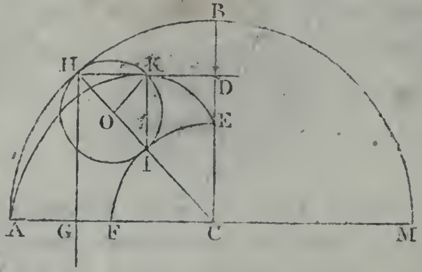
442. Cor. 5. The ſame propoſitions are true of any other reflecting curve, if E be the center of curvature of the evaneſcent arc AB.

443. Def. If an indefinite number of ſmall pencils belonging to the focus $\mathcal{Q}$, be incident, in the ſame manner, upon the reflecting ſurface BC, the curve which is the locus of the geometrical foci of reflected rays, is called the *cauſtic by reflection.*

Prop. XCVIII.

444. *To determine the form of the cauſtic, when the reflecting curve is a circular arc, and parallel rays are incident in the plane of the circle.*

Let C be the center of the propoſed arc; CB, that radius of the circle which is parallel to the incident



rays; and ACM the diameter which is perpendicular to CB. Suppoſe GH, one of the incident rays, to be reflected in the direction HD; join CH; biſect CH in

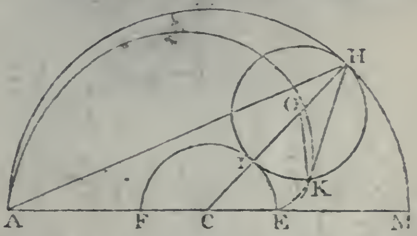
in I, and HI in O; with the centers C, O and radii CI, OH, deſcribe the circles EIF, HKI; and let K be the interſection of HKI and HD; join OK, IK; and from C draw CD perpendicular to HD.

Then, ſince the angle HKI, in a ſemi-circle, is a right angle, the triangles HKI, HDC are ſimilar; whence, $HD : HK :: HC : HI :: 2 : 1$; therefore K is a point in the cauſtic (Art. 439). Alſo, the $\angle KOI = 2 \angle IHK = 2 \angle CHG$ (Art. 18) $= 2 \angle ICE$ (Euc. 29. 1); and ſince circular arcs are as the angles which they ſubtend at their reſpective centers, and their radii jointly, the arc EI : the arc $IK :: 1 \times 2 : 2 \times 1$. Hence it follows, that the locus of the point K is an epicycloid, generated by the rotation of the circle HKI upon the circle EIF, in the plane of incidence AHM.

Prop. XCIX.

445. *To find the nature of the cauſtic, when the reflecting curve is a circular arc, and the focus of incident rays is in the circumference of the circle.*

Let AHM be the reflecting circle, C it's center, A the focus of incident rays; draw the diameter AM;



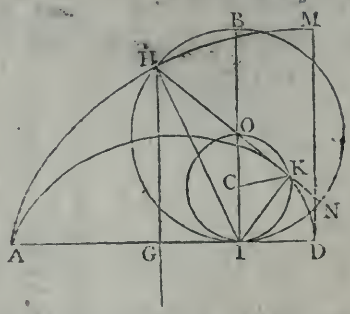
and let the ray AH be reflected in the direction HK; join CH, and divide it into three equal parts CI, IO, OH;

OH; with the centers *C* and *O*, and radii *CI*, *OI*, defcribe the circles *EIF*, *IKH*; let *K* be the interfection of the reflected ray *HK*, and the circle *HKI*; join *OK*. Then, fince the ∠ *OKH*=the ∠ *OHK*= the ∠ *CHA*=the ∠ *CAH*, the triangles *HCA*, *HOK* are fimilar, and *HA* : *HC* :: *HK* : *HO*; alternately, *HA* : *HK* :: *HC* : *HO* :: 3 : 1; therefore *K* is a point in the cauftic (Art. 441). Alfo, fince the ∠ *HOK*=the ∠ *HCA*, the ∠ *ICE*=the ∠ *KOI*; and the radii *CI*, *OI* are equal; therefore the arcs *EI*, *IK*, are equal; and the locus of the point *K* is an epicycloid, generated by the rotation of the circle *HKI* upon the circle *EIF*, in the plane of the reflecting arc *AHM*.

Prop. C.

446. *To find the nature of the cauftic, when the reflecting curve is a common cycloid, and the rays are incident parallel to it's axis.*

Let *AHM* be the reflecting femi-cycloid, whofe bafe is *AD*, and axis *DM*; *GH* a ray of light incident



upon it at *H*; *BHI* the fituation of the generating circle,

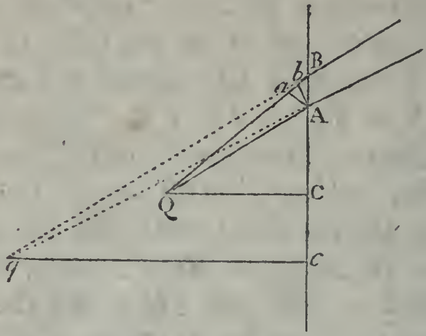
circle, when the point which traces out the cycloid is at H; and let I be the point in contact with the base. Take O the center of this circle, and draw the diameter HON; join OI; bisect the line OI in C, and with the center C and radius CI, describe the circle OKI, cutting HN in K; join IK, CK. Then, since OI is perpendicular to AD, or parallel to HG, the $\angle OIH =$ the $\angle GHI$; and the $\angle OIH =$ the $\angle OHI$; therefore, the $\angle OHI =$ the $\angle GHI$, and the ray GH is reflected in the direction HOK. Also, since IK is perpendicular to HK, and IH is half the radius of curvature of the cycloid at H (Mech. Art. 287), HK is one fourth part of the chord of curvature in the direction of the reflected ray; and therefore K is a point in the caustic (Art. 439). Again, since the $\angle KCI = 2 \angle NOI$, and $OI = 2IC$, the arc $IK =$ the arc $IN = ID$; therefore, the locus of the point K is a common cycloid, whose base is AD, and generating circle OKI.

PROP. CI.

447. *When a small pencil of homogeneal rays falls obliquely upon a plane refracting surface, and in a plane which is perpendicular to that surface, having given the focus of incident rays, and the angles of incidence and refraction, to find the geometrical focus of refracted rays.*

Let BAC be the refracting surface; QA, QB, the extreme rays of the oblique pencil, incident in the plane of the paper; qA, qB produced, the directions in which they are refracted; q the intersection of the refracted rays.

rays. From Q and q draw QC, qc at right angles to BC; and from A, draw, Aa, Ab, at right angles to



QB, qB. Take S and s to represent the sines of incidence and refraction of the ray QA; C and c their cosines; T and t their tangents. Then, since the angles AQC, BQC, are equal to the angles of incidence, and Aqc, Bqc, to the angles of refraction of the rays QA, QB, BQA and BqA are contemporaneous increments of the angles of incidence and refraction of the ray QA; and therefore, the $\angle BQA$: the $\angle BqA : T : t$* $:: \frac{S}{C} : \frac{s}{c}$. Also, the $\angle BQA$: the $\angle BqA :: \frac{Aa}{QA} : \frac{Ab}{qA}$; and since Aa, Ab are the cosines of the angles of incidence and refraction, to the radius BA, $\frac{C}{QA} : \frac{c}{qA} ::$ the $\angle BQA$: the $\angle BqA :: T : t :: \frac{S}{C} : \frac{s}{c}$; whence, $qA : QA :: \frac{T}{C} : \frac{t}{c} :: \frac{S}{C^2} : \frac{s}{c^2}$.

448. Cor. 1. Since $QA : QC :: r$ (radius) $: C$, we have $QA = \frac{r \times QC}{C}$. In the same manner, $qA =$

$r \times qc$

* Art. 421.

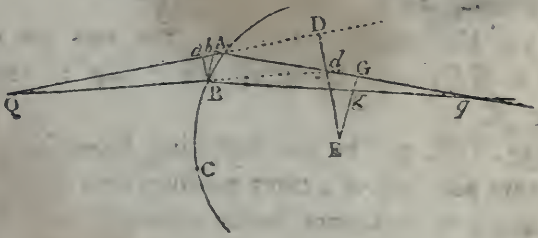
$\frac{r\times qc}{c}$; therefore $\frac{r\times qc}{c} : \frac{r\times \mathcal{Q}C}{C} :: \frac{T}{C} : \frac{t}{c}$; whence, $qc : \mathcal{Q}C :: \frac{T}{C^2} : \frac{t}{c^2}$.

449. Cor. 2. In the ſame manner it may be proved, that $\mathcal{Q}A = \frac{r\times AC}{S}$; and $qA = \frac{r\times Ac}{s}$; whence, $\frac{r\times Ac}{s} : \frac{r\times AC}{S} :: \frac{T}{C} : \frac{t}{c}$; conſequently, $Ac : AC :: \frac{T}{S\times C} : \frac{t}{s\times c} :: \frac{1}{C^2} : \frac{1}{c^2} :: c^2 : C^2$.

Prop. CII.

450. *When a ſmall pencil of homogeneal rays falls obliquely upon a ſpherical reflector, in a plane which paſſes through it's center, having given the focus of incident rays, and the angles of incidence and refraction, to determine the geometrical focus of refracted rays.*

Let $\mathcal{Q}A$, $\mathcal{Q}B$, be the extreme rays of a ſmall pencil incident obliquely upon the ſpherical refractor ABC,



in a plane which paſſes through it's center E; and let Aq, Bq, be the refracted rays. Draw EdD, EgG, and Ba, Bb, at right angles to $\mathcal{Q}A$, qA, or to thoſe lines

 produced;

produced; and when the arc AB is diminiſhed without limit, Ed and Eg, are at right angles to $\mathcal{Q}d$, Bq. Take I and R to repreſent the angles of incidence and refraction of the ray $\mathcal{Q}A$. Then, ſince $ED : EG ::$ ſin. I : ſin. $R :: Ed : Eg$, we have $Dd : Gg ::$ ſin. I : ſin. R (Euc. 19. 5). Alſo, Ba, Bb are the coſines of I and R, to the radius AB.

From theſe two conſiderations, and the ſimilarity of the triangles $\mathcal{Q}Dd$, $\mathcal{Q}aB$; and qbB, qGg; we obtain the following proportions;

$Dd : Gg ::$ ſin. I : ſin. R;
$Gg : Bb :: Gq : bq\ (Aq)$;
$Bb : Ba ::$ cos. R : cos. I;
$Ba : Dd :: \mathcal{Q}a\ (\mathcal{Q}A) : \mathcal{Q}D$;

by compounding which proportions, we have ſin. $I \times Gq \times$ cos. $R \times \mathcal{Q}A =$ ſin. $R \times Aq \times$ cos. $I \times \mathcal{Q}D$; and therefore, $Aq : Gq :: \frac{\text{ſin. } I}{\text{cos. } I} \times \mathcal{Q}A : \frac{\text{ſin. } R}{\text{cos. } R} \times \mathcal{Q}D ::$ tang. $I \times \mathcal{Q}A$: tang. $R \times \mathcal{Q}D$.

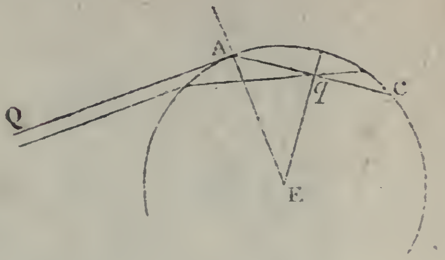
451. Cor. 1. The diſtances qA, qG, muſt be meaſured in the *ſame*, or *oppoſite* directions from q, according as $\mathcal{Q}A$, $\mathcal{Q}D$, are meaſured in the *ſame*, or *oppoſite* directions from $\mathcal{Q}$*.

452. Cor. 2. When the incident rays are parallel, $\mathcal{Q}A = \mathcal{Q}D$, and therefore $Aq : Gq ::$ tang. I : tang. R.

453. Cor. 3. On the foregoing ſuppoſition, when the rays paſs out of a rarer medium into a denſer, and the angle of incidence becomes nearly a right angle, tang,

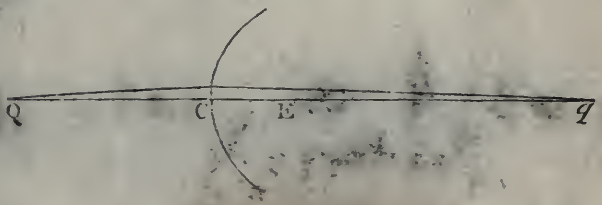
* Vid. Art. 438, and Note, p. 243.

tang. I is indefinitely greater than tang. R; therefore



qG vaniſhes; or q biſects the chord of the arc, cut off by the refracted ray.

454. Cor. 4. When the rays are incident nearly perpendicularly upon the refracting ſurface, tang. I :



tang. R :: ſin. I : ſin. R; alſo, D and G coincide with E; therefore $qC : qE$ :: ſin. $I \times 2C$: ſin. $R \times 2E$.

455. Cor. 5. Similar concluſions may be drawn reſpecting the refraction of a ſmall pencil of rays at any other ſurface, if E be the center of curvature of the refractor at the point of incidence.

On this ſubject, the reader may conſult HAYES's Fluxions, Sect. IX, X. SMITH's Optics, Book II. Chap. IX.

THE END.