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Butler's Elements of Plane Geometry

Butler's Series of Mathematics.

ELEMENTS

OF

PLANE GEOMETRY.

BY

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PREFACE.

THIS little volume has been prepared with a view to furnish a suitable text-book on Plane Geometry for Grammar Schools, Preparatory Schools, etc.

A simple method of designating angles has been adopted, and recognized symbols have been freely used in the demonstrations, thus bringing the several steps closely together and enabling the student to master the argument with ease. The reasons on which the steps of an argument depend are not formally given, but are referred to by numbers indicating the sections in which they are found: it is believed that the pupil will impress the principles most firmly on his mind by frequency of reference.

No valid objection can be offered against the algebraic form of which some of the demonstrations partake, for most of the axioms laid down are nothing more than properties of the equation.

No apology is deemed necessary for the application of the *Infinitesimal* method: it has been employed whenever it gave directness, brevity, and simplicity to the demonstration.

At the close of each book, except the second, a collection of theorems and problems has been placed for the purpose of giving the pupil an opportunity to exercise his originality in demonstration and construction. A proper use of these exercises will do much toward stimulating thought and awakening a spirit of invention in the pupil.

During the preparation of this treatise, DIESTERWEG'S "Elementare Geometrie" and most of our American treatises have been freely consulted.

And now, this little work is respectfully submitted to the educational public, in the hope that it may at least merit a careful perusal.

F. IBACH.

PHILADELPHIA, PA., *May, 1882.*

NOTE TO THE TEACHER.

In recitation, when studying a book for the first time, the pupil should be required to draw the diagram accurately and write the demonstration neatly on the blackboard.

Being called upon to recite, he enunciates the proposition and gives the demonstration, pointing to the parts of the diagram as reference is made to them.

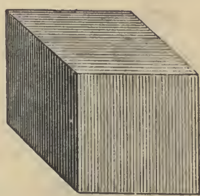
In review, the diagram only should be put on the board.

INTRODUCTION.

SUBJECT-MATTER.

THE accompanying diagram represents a block of granite,—a *physical solid*, of regular form.

Such a block has six flat faces, called *Surfaces*. It has also twelve sharp edges in which these surfaces meet, called *Lines*.



It has, besides, eight sharp corners in which these lines meet, called *Points*.

If the block be removed, we can imagine the space which it filled to have the same shape and size as the block. This limited portion of space, which has length, breadth, and thickness, is called a *Geometrical Solid*. Its boundaries or *surfaces* separate it from surrounding space, and have length and breadth but no thickness. The boundaries of these surfaces are *lines*, and have length only. The limits of these lines are *points*, and have position only. We thus come in three steps from solids to points, which have no magnitude. Having thus acquired notions of solids, surfaces, lines, and points, we can easily conceive of them distinct from one another. It is of such *ideal* solids, surfaces, lines, and points that Geometry treats; and these in various forms, except points, are called *Geometrical Magnitudes* or *Magnitudes of Space*.

ELEMENTS

OF

PLANE GEOMETRY.

BOOK I.

DEFINITIONS.

1. ***Geometry*** is the science which treats of the properties and relations of magnitudes of space.

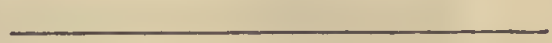
Space has extension in all directions; but for the purpose of determining the size of portions of space, we consider it as having three *dimensions*, namely, *length*, *breadth*, and *thickness*.

2. A ***Point*** is position without size.

3. A ***Line*** is that which has but one dimension, namely, *length*.

A line may be conceived as traced by a moving point. Lines are ***straight*** or ***curved.***

4. A ***Straight Line*** is one which has the same direction at all its points.



5. A ***Curved Line*** is one which changes its direction at all its points. When the sense is obvious, the word ***line,*** alone, is used for *straight line*, and the word ***curve,*** alone, for *curved line*.

6. A ***Surface*** is that which has only two dimensions, *length* and *breadth*.

A surface may be conceived as generated by a moving line. Surfaces are ***plane*** or ***curved.***

7. A ***Plane Surface,*** or a ***Plane,*** is a surface with which a straight line can be made to coincide in any direction.

8. A ***Curved Surface*** is a surface no portion of which is a plane.

9. A ***Solid*** is that which has three dimensions, *length*, *breadth*, and *thickness*. A solid may be conceived as generated by a moving surface.

Points, lines, surfaces, and solids are the *concepts* of Geometry, and may be said to constitute the subject-matter of the science.

10. A ***Figure*** is some definite form of magnitude.

11. Lines, surfaces, and solids are called *figures* when reference is had to their form.

12. A ***Plane Figure*** is one, all of whose points are in the same plane.

13. PLANE GEOMETRY treats of plane figures.

14. ***Equal Figures*** are such as have the same form and size, that is, such as fill exactly the same space.

15. ***Equivalent Figures*** are such as have equal magnitudes.

16. ***Similar Figures*** are such as have the same form, although they may have different magnitudes.

17. A ***Plane Angle,*** or an ***Angle,*** is the opening between two lines which meet each other. The point in which the lines meet is called the *Vertex*, and the lines are called the sides of the angle. A plane angle is a species of surface.

An angle is designated by placing a letter at each end of its sides, and one at its vertex, or by placing a small letter in it near the vertex. The latter is the method employed in this book, whenever it is convenient. In reading, when there is but one angle, we may name the letter at the vertex; but when there are two or more vertices at the same point, we name the three letters, with the one at the vertex *between* the other two: we may, however, in either case, simply name the letter placed in it. Thus, in Fig. 1, we say angle C, or angle a.

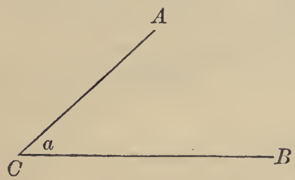


FIG. 1.

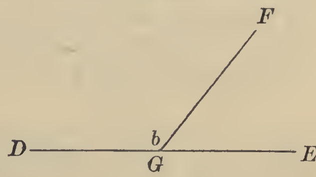
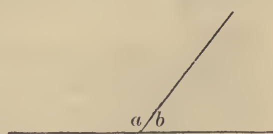


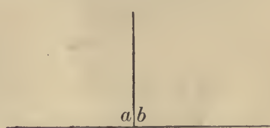
FIG. 2.

In Fig. 2, G being the common vertex, we must say angle DGF, or angle b. The size of an angle depends upon the *extent of opening* of its sides, and not upon the length of the sides.

18. ***Adjacent Angles*** are such as have a common vertex and one common side between them. Thus, the angles a and b are adjacent angles.

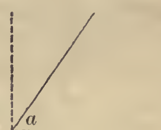


19. A ***Right Angle*** is an angle included between two straight lines which meet each other so as to make the adjacent angles equal. Thus, if the angles a and b are equal, each is a right angle.



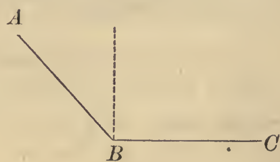
20. ***Perpendicular Lines*** are such as make right angles with each other.

21. An ***Acute Angle*** is one which is less than a right angle; as angle a.



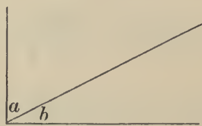
22. An ***Obtuse Angle*** is one which is greater than a right angle; as angle ABC.

Acute and obtuse angles are called ***oblique angles.***

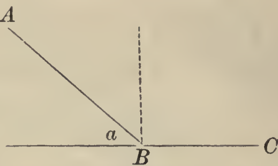


23. ***Oblique Lines*** are lines which are not perpendicular to each other, and which meet if sufficiently produced.

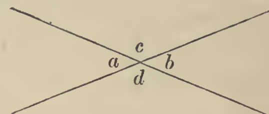
24. Two angles are ***Complements*** of each other when their sum is equal to a right angle. Thus, angle a is the complement of angle b, and angle b is the complement of angle a.



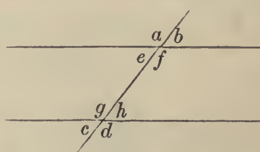
25. Two angles are ***Supplements*** of each other when their sum is equal to two right angles. Thus, angle a is the supplement of angle ABC, and angle ABC is the supplement of angle a.



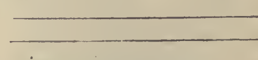
26. ***Vertical Angles*** are such as have a common vertex, and their sides lying in opposite directions. Thus, angles a and b are vertical; also angles c and d.



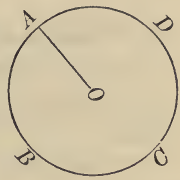
27. If two lines are cut by a third line, eight angles are formed, which are named as follows: Angles a, b, c, and d are ***Exterior Angles.*** Angles e, f, g, and h are ***Interior Angles.*** The pairs of angles a and d, b and c, are ***Alternate Exterior Angles.*** The pairs of angles e and h, f and g, are ***Alternate Interior Angles.*** The pairs of angles a and g, b and h, e and c, f and d, are ***Corresponding Angles.***



28. ***Parallel Straight Lines*** are such as lie in the same plane and cannot meet how far soever they are produced either way. They have the *same direction.*



29. A ***Circle*** is a plane figure bounded by a curve, all the points of which are equally distant from a point within, called the ***Centre.***



The ***Circumference*** of a circle is the curve which bounds it.

A ***Radius*** of a circle is a line extending from the centre to any point in the circumference.

The diagram represents a circle whose centre is O. The curve $ABCD$ is the circumference, and the line OA is a radius.

DEFINITIONS OF MATHEMATICAL TERMS.

30. A ***Demonstration,*** or ***Proof,*** is a course of reasoning by which the truth of a statement is deduced.

31. An ***Axiom*** is a statement of a truth which is self-evident.

32. A ***Theorem*** is a statement of a truth which is to be demonstrated.

33. A ***Problem*** is a statement of something to be done.

34. A ***Postulate*** is a problem whose solution is self-evident.

35. Axioms, theorems, and problems are called ***Propositions.***

36. A ***Corollary*** is a statement of a truth which is a direct inference from a proposition.

37. An ***Hypothesis*** is a supposition made in a proposition or in a demonstration.

38. A ***Scholium*** is a comment on one or more propositions.

39. AXIOMS.

1. Things which are equal to the same thing are equal to each other.

2. If equals are added to equals, the sums are equal.

3. If equals are subtracted from equals, the remainders are equal.

4. If equals are added to unequals, the sums are unequal.

5. If equals are subtracted from unequals, the remainders are unequal.

6. If equals are multiplied by equals, the products are equal.

7. If equals are divided by equals, the quotients are equal.

8. The whole is greater than any of its parts.

9. The whole is equal to the sum of all its parts.

10. Only one straight line can join two points.

11. A straight line is the shortest distance from one point to another.

12. All right angles are equal.

40. POSTULATES.

1. A straight line can be drawn joining any two points.

2. A straight line can be produced to any length.

3. From the greater of two straight lines, a part can be cut equal to the less.

4. In a plane a circumference of a circle can be described, with any point as a centre, and any distance as a radius.

5. Figures can be freely moved in space without change of form or size.

41. DEMONSTRATION.

A Demonstration is a logical process, the premises being definitions and self-evident and previously established truths.

There are two methods of demonstration, called the ***Direct Method*** and the ***Indirect Method.***

The ***Direct Method*** proves a truth by referring to definitions and self-evident and previously deduced truths, and concludes directly with the proof of the truth in question.

The ***Indirect Method*** proves a truth by showing that a supposition of its falsity leads to an absurdity;—called also *reductio ad absurdum.*

Pupils frequently fall into errors of demonstration. Notable among these errors are ***Begging the Question*** and ***Reasoning in a Circle.***

Begging the Question is a form of argument in which the truth to be proved is assumed as established.

Reasoning in a Circle is a form of argument in which a truth is employed to prove another truth on which the former depends for *its* proof.

42. EXPLANATION OF SYMBOLS AND ABBREVIATIONS.

$=$	denotes	equality.
$+$	"	increased by.
$-$	"	diminished by.
$\times$	"	multiplied by.
$\div$	"	divided by.
$\therefore$	"	therefore.
$\parallel$	"	parallel.
$\parallel$s	"	parallels.
$\angle$	"	angle.
$\angle$s	"	angles.
$\llcorner$	"	right angle.
$\llcorner$s	"	right angles.
$\perp$	"	perpendicular.
$\perp$s	"	perpendiculars.
$>$	"	is greater than.
$<$	"	is less than.
$\triangle$	"	triangle.
$\triangle$s	"	triangles.
R$\triangle$	"	right-angled triangle.
R$\triangle$s	"	right-angled triangles.
$\bigcirc$	"	circle.
$\bigcirc$s	"	circles.
▱	"	parallelogram.
▱s	"	parallelograms.

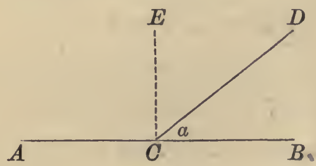
Ax.	denotes	axiom.
Cor.	"	corollary.
Cons.	"	construction.
Hyp.	"	hypothesis.
Q.E.D.	"	quod erat demonstrandum (which was to be demonstrated).
Q.E.F.	"	quod erat faciendum (which was to be done).

PERPENDICULAR AND OBLIQUE STRAIGHT LINES.

THEOREM I.

43. *At a point in a straight line, only one perpendicular can be erected to that line.*

Let C be a point in the line AB.



To prove that only one $\perp$ *can be erected to* AB *at* C.

Draw the oblique line CD.

Revolve CD about C so as to increase $\angle a$ and decrease $\angle ACD$.

It is evident that in one position of CD, as EC, the adjacent $\angle$s are equal.

But then EC is $\perp$ to AB. (20)

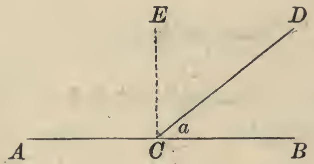
And there can be only one position of the line in which the adjacent $\angle$s are equal;

$\therefore$ only one $\perp$ can be erected to AB at the point C. Q. E. D.

THEOREM II.

44. *The sum of the two adjacent angles formed by two lines which meet equals two right angles.*

Let $\angle$s a and ACD be formed by the line CD meeting AB.



To prove that $\angle a + \angle ACD = 2 \text{ rt. } \angle$s.

Let CE be $\perp$ to AB at C.

$$\angle a + \angle ECD = \text{a rt. } \angle,$$

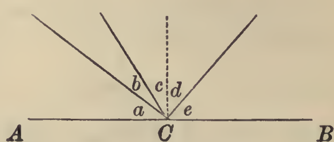
and $\angle ACD - \angle ECD = \text{a rt. } \angle$.

Add; then $\angle a + \angle ACD = 2 \text{ rt. } \angle$s. (Ax. 2) Q. E. D.

45. Cor. 1.—*If one of the two adjacent angles formed by two lines which meet is a right angle, the other is also a right angle.*

46. Cor. 2.—*The sum of all the angles formed at a common point on the same side of a straight line equals two right angles.*

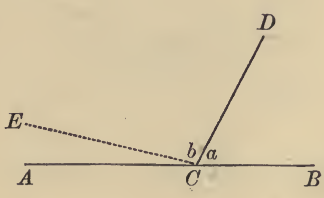
Thus, $\angle a + \angle b + \angle c + \angle d + \angle e = 2 \text{ rt. } \angle$s.



THEOREM III.

47. CONVERSELY.—*If the sum of two adjacent angles equals two right angles, their exterior sides lie in the same straight line.*

Let $\angle a + \angle ACD = 2 \text{ rt. } \angle s.$



To prove that AC and BC lie in the same straight line.

Draw EC.

If EC and BC lie in the same straight line,

$$\angle a + \angle b = 2 \text{ rt. } \angle s. \qquad (44)$$

But $\angle a + \angle ACD = 2 \text{ rt. } \angle s;$ (Hyp.)

$\therefore$ $\angle a + \angle b = \angle a + \angle ACD.$ (Ax. 1)

From each member subtract $\angle a$.

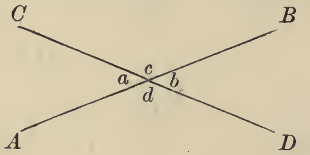
Then $\angle b = \angle ACD$, which is impossible. (Ax. 8)

$\therefore$ AC and BC lie in the same straight line. Q. E. D.

THEOREM IV.

48. *If two straight lines intersect, the opposite or vertical angles are equal.*

Let AB and CD intersect.



To prove that $\angle a = \angle b$.

$$\angle a + \angle c = 2 \text{ L s}, \qquad (44)$$

and $$\angle b + \angle c = 2 \text{ L s}; \qquad (44)$$

$\therefore$ $$\angle a + \angle c = \angle b + \angle c. \qquad (\text{Ax. } 1)$$

From each member subtract $\angle c$.

Then $$\angle a = \angle b.$$

Likewise we can prove that $\angle c = \angle d$. Q. E. D.

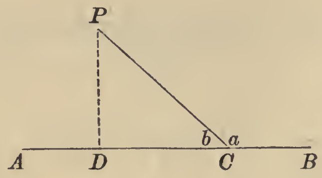
49. COR. 1.—*If two straight lines intersect, the sum of the four angles formed equals four right angles.*

50. COR. 2.—*The sum of all the angles that can be formed at a common point equals four right angles.*

THEOREM V.

51. *From a point without a straight line, only one perpendicular can be drawn to that line.*

Let P be a point without AB.



To prove that only one $\perp$ *can be drawn from* P *to* AB.

Draw the oblique line PC. With the point P fixed, revolve PC so as to decrease $\angle\ a$ and increase $\angle\ b$, while the common vertex moves in the direction CA.

At some position of the line, as PD, the adjacent angles are equal.

Then PD is $\perp$ to AB. (20)

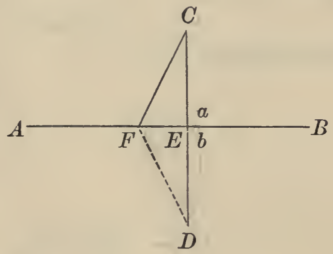
There is *only one* position of the line in which the angles are equal.

$\therefore$ only one $\perp$ can be drawn from P to AB. Q. E. D.

THEOREM VI.

52. *From a point without a straight line, a perpendicular is the shortest distance to that line.*

Let AB be a straight line, C any point without it, CE a $\perp$, and CF any oblique line.



To prove that $CE < CF$.

Produce CE to D, making $ED = CE$, and draw FD.

On AB as an axis, revolve the plane of CEF till it falls in the plane of DEF.

Since $\angle$s a and b are $\llcorner$s, the line CE takes the direction ED, the point C falling on D.

$$CE = ED; \qquad \text{(Cons.)}$$

$\therefore$ $$CE + ED = 2\,CE.$$

$$CF = FD; \qquad \text{(Ax. 10)}$$

$\therefore$ $$CF + FD = 2\,CF.$$

$$CE + ED < CF + FD,$$

or $$2\,CE < 2\,CF. \qquad \text{(Ax. 11)}$$

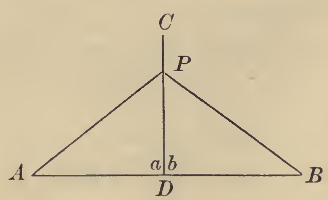
Divide by 2;

then $$CE < CF. \qquad \text{Q. E. D.}$$

THEOREM VII.

53. *Any point in the perpendicular erected at the middle point of a straight line is equally distant from the extremities of that line.*

Let P be any point in CD which is $\perp$ to AB at its middle point D, and let AP and BP be drawn.



To prove that $AP = BP$.

On CD as an axis, revolve APD till it falls in the plane of BPD.

Since $\angle$s a and b are $\llcorner$s, and $AD = BD$, A falls on B.

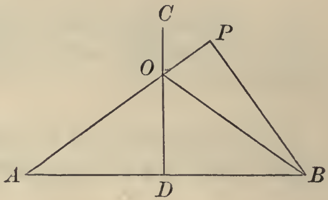
$\therefore$ $AP = BP$. (Ax. 10) Q. E. D.

54. Cor. 1.—*If a point is equally distant from the extremities of a straight line, it lies in the perpendicular erected at the middle point of that line.*

55. Cor. 2.—*If each of two points in a straight line is equally distant from the extremities of another straight line, the former is perpendicular to the latter at its middle point.*

THEOREM VIII.

56. *Any point without the perpendicular erected at the middle point of a straight line is unequally distant from the extremities of that line.*



Let P be any point without the line CD which is $\perp$ to AB at its middle point. Draw AP cutting CD at O, and draw BO and BP.

To prove that $AP > BP$.

$$BO + OP > BP, \qquad \text{(Ax. 11)}$$

and $$AO = BO. \qquad (53)$$

Substitute AO for its equal BO.

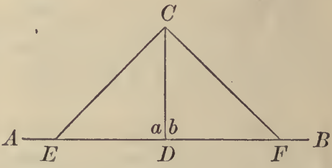
Then $$AO + OP > BP.$$

But $$AO + OP = AP;$$

$\therefore$ $$AP > BP. \qquad \text{Q. E. D.}$$

THEOREM IX.

57. *Two oblique lines drawn from a point in a perpendicular are equal if they cut off equal distances from the foot of the perpendicular.*



Let CD be $\perp$ to AB, and CE and CF oblique lines cutting off $ED = DF$.

To prove that $CE = CF$.

On CD as an axis, revolve CDE till it falls in the plane of CDF.

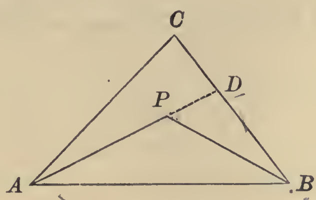
Since $\angle$s a and b are $\llcorner$s, and $ED = DF$, E falls on F.

$\therefore$ $$CE = CF. \qquad \text{(Ax. 10)} \quad \text{Q. E. D.}$$

THEOREM X.

58. *The sum of two lines drawn from a point to the extremities of a straight line is less than the sum of two other lines similarly drawn and enveloping them.*

Let AP and BP be two lines drawn from P to the extremities of AB, and let AC and BC be two lines drawn similarly and enveloping AP and BP.



To prove that $AP + BP < AC + BC$.

Produce AP to D, a point in BC.

$$AP + PD < AC + CD,$$

and $$BP < PD + DB. \qquad \text{(Ax. 11)}$$

Add the inequalities.

Then $AP + BP + PD < AC + CD + DB + PD$.

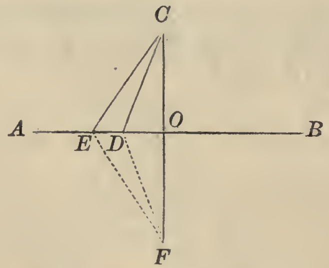
Substitute BC for its equal $CD + DB$, and subtract PD from each member.

Then $$AP + BP < AC + BC. \qquad \text{Q. E. D.}$$

THEOREM XI.

59. *Of two oblique lines drawn from the same point, that is the greater which terminates at the greater distance from the foot of the perpendicular.*

Let CO be $\perp$ to AB, and CE and CD oblique lines drawn so that $EO > DO$.



To prove that $CE > CD$.

Produce CO to F, making $OF = CO$, and draw EF and DF.

Then, as in (52), $CD = DF$, and $CE = EF$.

But $$CE + EF > CD + DF,$$

or $$2\,CE > 2\,CD. \qquad (58)$$

Divide each member by 2.

Then $$CE > CD.$$ Q. E. D.

60. Cor. 1.—*Two equal oblique lines terminate at equal distances from the foot of the perpendicular.*

61. Cor. 2.—*Only two equal straight lines can be drawn from a point to a line; and of two unequal oblique lines, the greater terminates at the greater distance from the foot of the perpendicular.*

PARALLEL STRAIGHT LINES.

THEOREM XII.

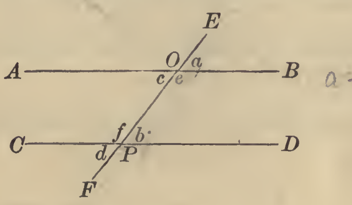
62. *If two parallel lines are cut by a third line—*

I. *The corresponding angles are equal;*

II. *The alternate interior angles are equal;*

III. *The sum of the interior angles on the same side of the secant equals two right angles.*

Let the ||s AB and CD be cut by the line EF.



I. *To prove that* $\angle a = \angle b$.

Since OB and PD are ||, they have the same direction and open equally from the line EF;

$\therefore \qquad \angle a = \angle b.$

Likewise we can prove that $\angle c = \angle d$.

II. *To prove that* $\angle c = \angle b$.

$\angle c = \angle a.$ (48)

But $\angle a = \angle b;$ (Case I.)

$\therefore \qquad \angle c = \angle b.$ (Ax. 1)

Likewise we can prove that $\angle e = \angle f$.

III. *To prove that* $\angle\, b + \angle\, e = 2$ ⊾s.

$$\angle\, a + \angle\, e = 2 \text{ ⊾s.} \qquad (44)$$

Substitute $\angle\, b$ for its equal $\angle\, a$.

Then $\angle b + \angle\, e = 2$ ⊾s.

Likewise we can prove that $\angle\, c + \angle\, f = 2$ ⊾s. Q. E. D.

63. Cor.—*If a straight line lies in the same plane with two parallels, and is perpendicular to one of them, it is perpendicular to the other also.*

THEOREM XIII.

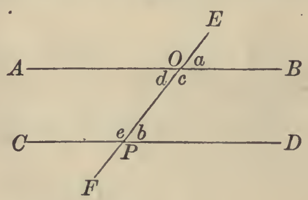
64. *If two straight lines are cut by a third line, these two lines are parallel—*

I. *If the corresponding angles are equal;*

II: *If the alternate interior angles are equal;*

III. *If the sum of the two interior angles on the same side of the secant equals two right angles.*

Let the straight lines AB and CD be cut by EF.



I. *To prove that* AB *and* CD *are* ∥ *if* $\angle\, a = \angle\, b$.

If $\angle\, a = \angle b$, OB and PD open equally from EF, and hence have the same direction, or are ∥. (28)

II. *To prove that* AB *and* CD *are* ∥ *if* $\angle\, d = \angle\, b$.

If $\angle\, a = \angle\, b$, AB and CD are ∥. (Case I.)

But $\angle\, d = \angle\, a$; (48)

$\therefore$ if $\angle\, d = \angle\, b$, AB and CD are ∥.

III. *To prove that AB and CD are* ∥ *if* $\angle c + \angle b = 2 \text{ rt. } \angle s$.

$\angle a + \angle c = 2 \text{ rt. } \angle s$; (44)

$\angle c + \angle b = 2 \text{ rt. } \angle s$; (Hyp.)

$\therefore$ $\angle a + \angle c = \angle c + \angle b$. (Ax. 1)

Subtract $\angle c$ from each member.

Then $\angle a = \angle b$.

But then the lines are ∥; (Case I.)

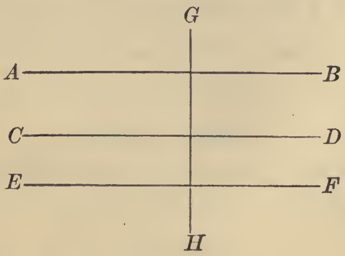
$\therefore$ if $\angle c + \angle b = 2 \text{ rt. } \angle s$, AB and CD are ∥. Q.E.D.

65. Cor.—*If two lines in the same plane are perpendicular to the same line, they are parallel.*

THEOREM XIV.

66. *If two straight lines are parallel to a third line, they are parallel to each other.*

Let AB and CD be ∥ to EF.



To prove that AB is ∥ *to CD.*

Let GH be $\perp$ to EF;

Then GH is $\perp$ to AB. (63)

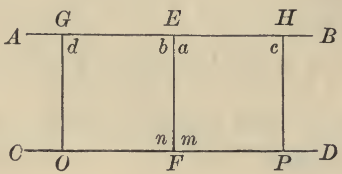
GH is also $\perp$ to CD; (63)

$\therefore$ AB is ∥ to CD. (65) Q.E.D.

THEOREM XV.

67. *Two parallel lines are everywhere equally distant from each other.*

Let AB and CD be $\parallel$, and GO and HP $\perp$s to CD drawn from any two points in AB.



To prove that $GO = HP$.

Let the $\perp$ EF be drawn at the middle point between G and H.

GO, HP, and EF are $\perp$ to both $\parallel$s. (63)

On EF as an axis, revolve the part of the plane on the right of EF till it falls in the part on the left of EF.

$\angle$s a and b are $\llcorner$s, and $EH = EG$;

$\therefore$ H falls on G.

$\angle$s c and d are $\llcorner$s;

$\therefore$ HP takes the direction GO, and P falls in GO, or GO produced.

$\angle$s m and n are $\llcorner$s;

$\therefore$ P falls in the line FC.

Now P falling in both GO and FC, must fall at their intersection O;

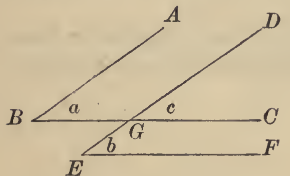
$\therefore$ $GO = HP$. Q. E. D.

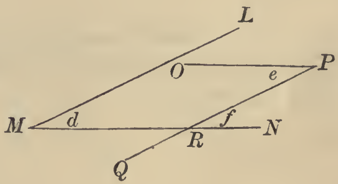
EQUALITY OF ANGLES.

THEOREM XVI.

68. *Two angles are equal if their sides are respectively parallel and lying in the same or in opposite directions from their vertices.*

I. Let AB and BC, the sides of $\angle a$, be respectively $\parallel$ to DE and EF, the sides of $\angle b$.





To prove that $\angle a = \angle b$.

If necessary, produce two sides not $\parallel$ till they intersect, as at G.

Then	$\angle a = \angle c$,	
and	$\angle b = \angle c$;	(62)
$\therefore$	$\angle a = \angle b$.	(Ax. 1)

II. Let LM and MN, the sides of $\angle d$, be respectively $\parallel$ to PQ and OP, the sides of $\angle e$.

To prove that $\angle d = \angle e$.

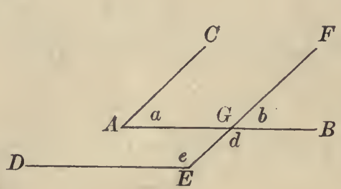
If necessary, produce two sides not $\parallel$ till they intersect, as at R.

Then	$\angle d = \angle f$,	
and	$\angle e = \angle f$;	(62)
$\therefore$	$\angle d = \angle e$.	(Ax. 1) Q. E. D.

THEOREM XVII.

69. *Two angles are supplements of each other if they have two sides parallel and lying in the same direction, and the other sides parallel and lying in opposite directions from their vertices.*

Let $\angle$s a and e have AC and EF $\parallel$ and lying in the same direction, and AB and ED $\parallel$ and lying in opposite directions from their vertices.



To prove that $\angle$s *a and e are supplements of each other.*

If necessary, produce two sides not $\parallel$ till they intersect, as at G.

Then $\angle a = \angle b$, and $\angle d = \angle e$. (62)

But $\angle$s b and d are supplements of each other;

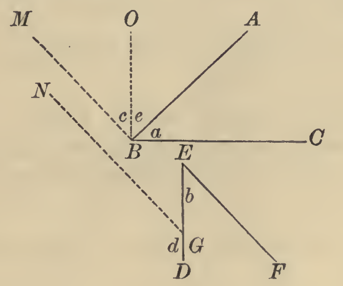
$\therefore$ $\angle$s a and e are supplements of each other.

Q. E. D.

THEOREM XVIII.

70. *Two angles having their sides respectively perpendicular to each other are either equal or supplements of each other.*

Let the $\angle$s a and b have the sides BA and BC respectively $\perp$ to EF and ED.



To prove that $\angle$s a and b are either equal or supplements of each other.

At B, let BO and BM be drawn respectively $\parallel$ to ED and EF, and let GN be drawn $\parallel$ to EF or BM.

Then $\qquad \angle c = \angle b, \qquad$ (68)

and $\qquad \angle c$ is the supplement of $\angle d. \qquad$ (69)

Now $\quad \angle c + \angle e =$ a $\llcorner$, and $\angle e + \angle a =$ a $\llcorner$;

$\therefore \qquad \angle c + \angle e = \angle e + \angle a.$

Subtract $\angle e$ from each member.

Then $\qquad \angle c = \angle a.$

But $\qquad \angle c = \angle b;$

$\therefore \qquad \angle a = \angle b.$

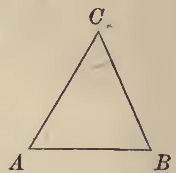
Also, since $\quad \angle c$ is the supplement of $\angle d$,

$\angle a$ is the supplement of $\angle d$. $\qquad$ Q. E. D.

TRIANGLES.

DEFINITIONS.

71. A ***Triangle*** is a plane figure bounded by three straight lines; as ABC.

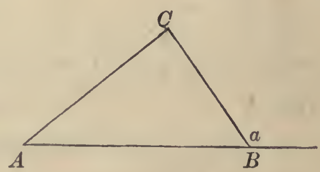


The ***Sides*** of a triangle are the bounding lines.

The ***Angles*** of a triangle are the angles formed by the sides meeting one another. A triangle has six parts,—three sides and three angles.

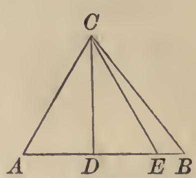
The ***Base*** of a triangle is the side upon which it is supposed to stand.

The ***Vertical Angle*** of a triangle is the angle opposite the base. An ***Exterior Angle*** of a triangle is an angle formed by a side and an adjacent side produced; as $\angle a$.



The ***Vertex*** of a triangle is the angular point at the vertical angle.

The ***Altitude*** of a triangle is the perpendicular distance from the vertex to the base, or to the base produced. Thus, CD is the altitude of both the triangles ABC and EBC.

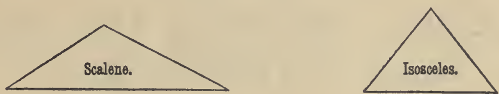


A ***Medial Line*** of a triangle is a line drawn from a vertex to the middle of the opposite side.

72. Triangles are classified as to their sides and angles.

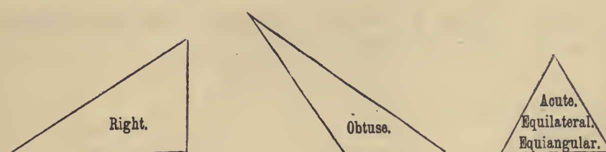
A ***Scalene Triangle*** is one which has no two sides equal.

An ***Isosceles Triangle*** is one which has two sides equal.



An ***Equilateral Triangle*** is one which has all its sides equal.

An ***Acute-Angled Triangle*** is one which has three acute angles.



A ***Right-Angled Triangle*** is one which has *one* right angle.

An ***Obtuse-Angled Triangle*** is one which has *one* obtuse angle.

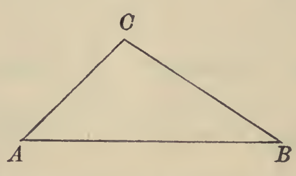
An ***Equiangular Triangle*** is one which has all its angles equal.

RELATION BETWEEN THE SIDES OF A TRIANGLE.

THEOREM XIX.

73. *Any side of a triangle is greater than the difference between the other two sides.*

Let ABC be any $\triangle$.



To prove that $AC > AB - BC$.

$$AB < AC + BC. \qquad \text{(Ax. 11)}$$

Subtract BC from each member.

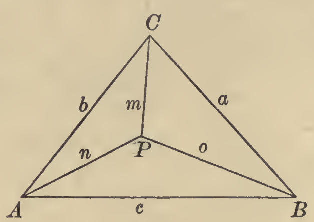
Then $$AB - BC < AC,$$

or $$AC > AB - BC. \qquad \text{Q. E. D.}$$

THEOREM XX.

74. *The sum of the three lines drawn from a point within a triangle to the vertices is greater than half the sum of the sides of the triangle.*

Let m, n, o, be the three lines drawn from any point P in the $\triangle\ ABC$, and a, b, c, the sides of the $\triangle$.



To prove that $m + n + o > \frac{a + b + c}{2}$.

$$m + o > a,$$

$$n + o > c,$$

and $$m + n > b. \qquad \text{(Ax. 11)}$$

Add these inequalities.

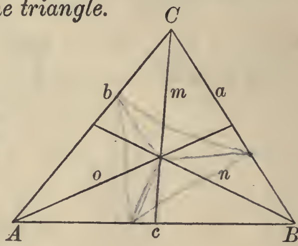
Then $$2m + 2n + 2o > a + b + c,$$

or $$m + n + o > \frac{a + b + c}{2} \qquad \text{Q. E. D.}$$

MEDIAL LINES.

THEOREM XXI.

75. *The sum of the three medial lines of a triangle is greater than half the sum of the sides of the triangle.*



In the $\triangle ABC$, let a, b, c, be the sides, and m, n, o, the medial lines.

To prove that $m + n + o > \frac{a+b+c}{2}$.

$$\left.\begin{array}{l} m > a - \frac{c}{2} \\ m > b - \frac{c}{2} \\ n > c - \frac{b}{2} \\ n > a - \frac{b}{2} \\ o > c - \frac{a}{2} \\ o > b - \frac{a}{2} \end{array}\right\} \qquad (73)$$

Add these inequalities.

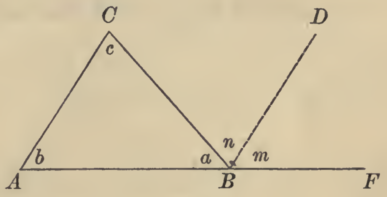
Then $2m + 2n + 2o > a + b + c$,

or $m + n + o > \frac{a+b+c}{2}$. Q. E. D.

ANGLES OF A TRIANGLE.

THEOREM XXII.

76. *The sum of the three angles of a triangle equals two right angles.*



Let ABC be any $\triangle$.

To prove that $\angle a + \angle b + \angle c = 2 \llcorner s.$

Produce AB, and let DB be $\parallel$ to AC.

Then $\quad \angle m + \angle n + \angle a = 2 \llcorner s.$ (46)

But $\quad \angle m = \angle b,$

and $\quad \angle n = \angle c.$ (62)

For $\angle s$ m and n, substitute their equals $\angle s$ b and c.

Then $\quad \angle a + \angle b + \angle c = 2 \llcorner s.$ Q. E. D.

77. Cor. 1.—*The sum of two angles of a triangle being given, the third can be found by subtracting their sum from two right angles.*

78. Cor. 2.—*If two angles of a triangle are respectively equal to two angles of another, the third angles are also equal.*

79. Cor. 3.—*In any triangle, there can be but one right angle, or but one obtuse angle.*

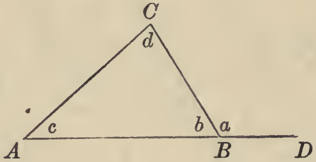
80. Cor. 4.—*In any right-angled triangle, the sum of the acute angles equals a right angle; that is, they are complements of each other.*

81. Cor. 5.—*In an equiangular triangle, each angle equals one-third of two right angles.*

THEOREM XXIII.

82. *An exterior angle of a triangle equals the sum of the two interior non-adjacent angles.*

Let ABC be any $\triangle$, and a an exterior $\angle$.



To prove that $\angle a = \angle c + \angle d$.

$$\angle a + \angle b = 2 \text{ rt. } \angle\text{s}, \qquad (44)$$

and $$\angle b + \angle c + \angle d = 2 \text{ rt. } \angle\text{s}; \qquad (76)$$

$\therefore$ $$\angle a + \angle b = \angle b + \angle c + \angle d.$$

Subtract $\angle b$ from each member;

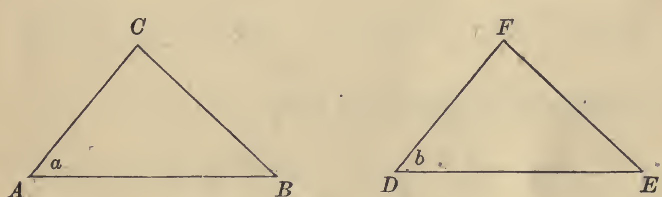
then $$\angle a = \angle c + \angle d. \qquad \text{Q. E. D.}$$

EQUALITY OF TRIANGLES.

THEOREM XXIV.

83. *Two triangles are equal in all their parts if two sides and the included angle of the one are respectively equal to two sides and the included angle of the other.*

In the $\triangle$s ABC and DEF, let $AB = DE$, $AC = DF$, and $\angle a = \angle b$.



To prove that $\triangle ABC = \triangle DEF$.

Place the $\triangle ABC$ upon DEF, applying $\angle a$ to its equal $\angle b$, AB on its equal DE, and AC on its equal DF.

The points C and B fall on the points F and E;

$\therefore$ $BC = EF$. (Ax. 10)

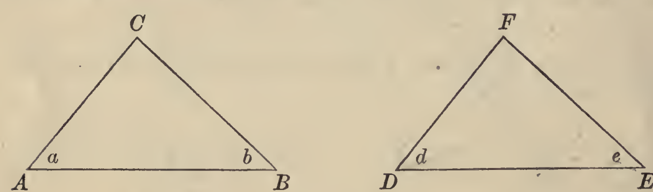
and $\triangle ABC = \triangle DEF$. (14)

Q. E. D.

THEOREM XXV.

84. *Two triangles are equal in all their parts if two angles and the included side of the one are respectively equal to two angles and the included side of the other.*

In the $\triangle$s ABC and DEF, let $\angle a = \angle d$, $\angle b = \angle e$, and $AB = DE$.



To prove that $\triangle ABC = \triangle DEF$.

Place the $\triangle ABC$ upon the $\triangle DEF$, applying AB to its equal DE, the point A on D, and the point B on E.

Since $\angle a = \angle d$, AC takes the direction DF, and C falls somewhere in DF or DF produced.

Since $\angle b = \angle e$, BC takes the direction EF, and C falls in EF or EF produced;

$\therefore$ the point C, falling in both DF and EF, falls at their intersection F;

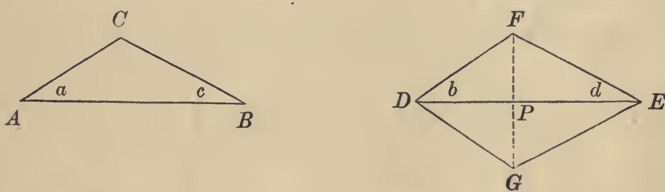
$\therefore$ $\triangle ABC = \triangle DEF$. (14) Q. E. D.

85. Cor.—*If a triangle has a side, its opposite angle, and one adjacent angle, respectively equal to the corresponding parts of another triangle, the triangles are equal.*

THEOREM XXVI.

86. *Two triangles are equal in all their parts if the three sides of the one are respectively equal to the three sides of the other.*

In the $\triangle$s ABC and DEF, let $AC = DF$, $BC = EF$, and $AB = DE$.



To prove that $\triangle\ ABC = \triangle\ DEF$.

Place ABC in the position DEG, AB in its equal DE, and the $\angle$s a and c adjacent to the $\angle$s b and d.

Draw FG cutting DE at P.

$DG = DF$, and $EG = EF$; (Hyp.)

$\therefore$ the points D and E are equally distant from F and G, and DE is $\perp$ to FG at its middle point. (55)

On DE as an axis, revolve DEG till it falls in the plane of DEF.

Then the point G falls on F, since $PG = PF$;

$\therefore$ $\triangle\ ABC = \triangle\ DEF$. (14) (Q. E. D.)

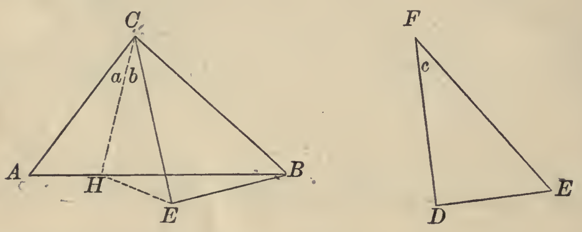
87. Cor.—*In equal triangles, the equal angles lie opposite the equal sides.*

88. Scholium.—The statement, *two triangles are equal*, means that the six parts of the one are respectively equal to the six parts of the other.

THEOREM XXVII.

89. *If two triangles have two sides of the one respectively equal to two sides of the other, and the included angles unequal, the third sides are unequal, and the greater third side is in the triangle having the greater included angle.*

In the $\triangle$s ABC and DEF, let $AC = DF$, $CB = FE$, and $ACB > \angle c$.



To prove that $AB > DE$.

Place the $\triangle$ DEF so that EF falls in its equal BC.

Let CH bisect $\angle$ ECA, and draw EH.

$$CE = CA, \qquad \text{(Hyp.)}$$

$$CH = CH,$$

and $$\angle a = \angle b; \qquad \text{(Cons.)}$$

$\therefore$ $$\triangle CHE = \triangle CHA, \qquad (83)$$

and $$EH = AH.$$

Now $$HB + EH > EB.$$

Substitute AH and DE for their equals EH and EB.

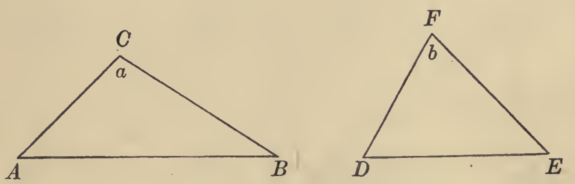
Then $$AH + HB > DE,$$

or $$AB > DE. \qquad \text{Q. E. D.}$$

THEOREM XXVIII.

90. CONVERSELY.—*If two sides of a triangle are respectively equal to two sides of another, and the third sides unequal, the angle opposite the third side is greater in the triangle having the greater third side.*

In the $\triangle$s ABC and DEF, let $AC = DF$, $BC = EF$, and $AB > DE$.



To prove that $\angle\ a > \angle\ b$.

If $\qquad \angle\ a = \angle\ b,\ \triangle\ ABC = \triangle\ DEF,$ (83)

and $\qquad AB = DE.$

If $\qquad \angle\ a < \angle\ b,\ AB < DE.$ (89)

But both these conclusions, being contrary to the hypothesis, are absurd.

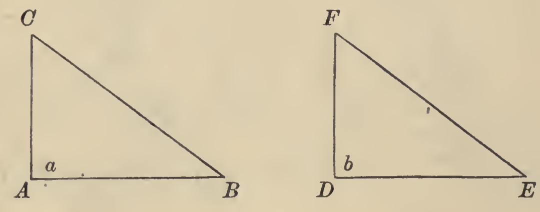
$\therefore\ \angle\ a$ cannot equal $\angle\ b$, and cannot be less than $\angle\ b$.

$\therefore \qquad \angle\ a > \angle\ b.$ Q. E. D.

THEOREM XXIX.

91. *Two right-angled triangles are equal in all their parts if the hypothenuse and one side of the one are respectively equal to the hypothenuse and one side of the other.*

In the R$\triangle$s ABC and DEF, let the hypothenuse $BC = EF$, and $AC = DF$.



To prove that R$\triangle$ ABC = R$\triangle$ DEF.

Place ABC on DEF so that AC falls in its equal DF.

$\angle$s a and b are $\llcorner$s; (Hyp.)

$\therefore$ AB takes the direction DE,

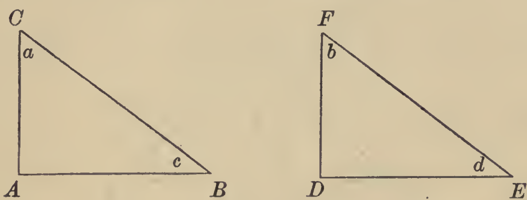
and B falls on E; (60)

$\therefore$ R$\triangle$ ABC = R$\triangle$ DEF. (14) Q. E. D.

THEOREM XXX.

92. *Two right-angled triangles are equal in all their parts if the hypothenuse and one acute angle of the one are respectively equal to the hypothenuse and one acute angle of the other.*

In the R$\triangle$s ABC and DEF, let $BC = EF$, and $\angle\ a = \angle\ b$.



To prove that R$\triangle$ $ABC =$ R$\triangle$ DEF.

$$BC = EF, \qquad \text{(Hyp.)}$$

$$\angle\ a = \angle\ b, \qquad \text{(Hyp.)}$$

and $$\angle\ c = \angle\ d; \qquad (78)$$

$\therefore$ R$\triangle$ $ABC =$ R$\triangle$ DEF. (84) Q. E. D.

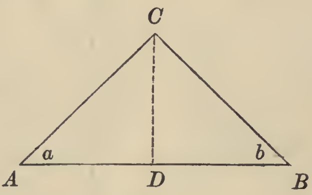
93. Cor.—*Two right-angled triangles are equal if a side and an acute angle of the one are respectively equal to a side and an acute angle of the other.*

RELATION BETWEEN THE PARTS OF A TRIANGLE.

THEOREM XXXI.

94. *In an isosceles triangle, the angles opposite the equal sides are equal.*

In the isosceles $\triangle ABC$, let AC and BC be the equal sides.



To prove that $\angle a = \angle b$.

Let CD be drawn to the middle of AB.

$$AC = BC, \qquad \text{(Hyp.)}$$

$$AD = BD, \qquad \text{(Cons.)}$$

and $$CD = CD;$$

$\therefore$ $$\triangle ADC = \triangle BDC, \qquad (86)$$

and $$\angle a = \angle b. \qquad \text{Q. E. D.}$$

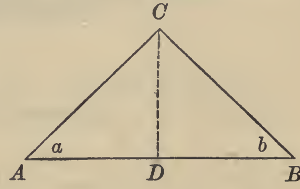
95. Cor. 1.—*The straight line joining the vertex and the middle of the base of an isosceles triangle bisects the vertical angle and is perpendicular to the base.*

96. Cor. 2.—*The straight line which bisects the vertical angle of an isosceles triangle bisects the base at right angles.*

97. Cor. 3.—*Any equilateral triangle is equiangular.*

THEOREM XXXII.

98. Conversely.—*If two angles of a triangle are equal, the sides opposite them are equal, and the triangle is isosceles.*



In the $\triangle\ ABC$, let $\angle\ a = \angle\ b$.

To prove that $AC = BC$.

Let CD be $\perp$ to AB.

$$\angle\ a = \angle\ b, \qquad \text{(Hyp.)}$$

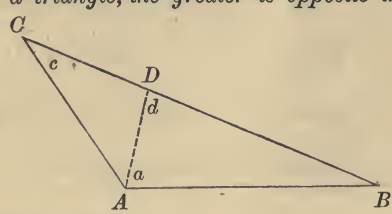
and $$CD = CD;$$

$\therefore$ $$\text{R}\triangle\ ADC = \text{R}\triangle\ BDC, \qquad (93)$$

and $$AC = BC. \qquad \text{Q. E. D.}$$

THEOREM XXXIII.

99. *Of two angles of a triangle, the greater is opposite the greater side.*



In the $\triangle\ ABC$, let $CB > AB$.

To prove that $\angle\ BAC > \angle\ c$.

Cut off $BD = AB$, and draw AD.

$$\angle\ a = \angle\ d. \qquad (94)$$

But $$\angle\ d > \angle\ c; \qquad (82)$$

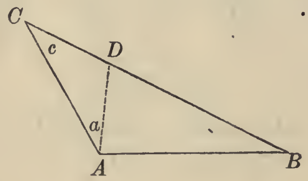
$\therefore$ $$\angle\ a > \angle\ c.$$

And much more is $\angle\ BAC > \angle\ c$. Q. E. D.

THEOREM XXXIV.

100. Conversely.—*Of two sides of a triangle, the greater is opposite the greater angle.*

In the $\triangle ABC$, let $\angle BAC > \angle c$.



To prove that $BC > AB$.

Let AD be drawn so as to make $\angle a = \angle c$.

Then $AD = CD$. (98)

Now $AD + BD > AB$. (Ax. 11)

Substitute CD for its equal AD.

Then $CD + BD > AB$,

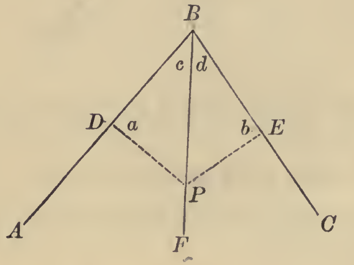
or $BC > AB$. Q. E. D.

BISECTORS OF ANGLES.

THEOREM XXXV.

101. *Any point in the bisector of an angle is equally distant from the sides of the angle.*

Let BF be the bisector of the $\angle ABC$, P any point in it, and PD and PE $\perp$s to AB and BC.



To prove that $PD = PE$.

$$\angle\text{s } a \text{ and } b \text{ are } \llcorner\text{s},$$

$$\angle c = \angle d, \qquad \text{(Hyp.)}$$

and $$BP = BP;$$

$\therefore$ $$R \triangle BDP = R \triangle BEP, \qquad (92)$$

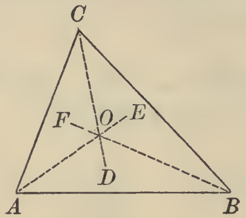
and $$PD = PE. \qquad \text{Q. E. D.}$$

102. Cor.—*Any point in an angle equally distant from the sides lies in the bisector of the angle.*

THEOREM XXXVI.

103. *The bisectors of the angles of a triangle meet in a common point.*

Let AE, BF, and CD be the bisectors of the $\angle$s of the $\triangle ABC$.



To prove that AE, BF, and CD meet in a common point.

Let AE and BF meet in a point, as O.

Then O is equally distant from AB and AC; also from AB and BC; (101)

$\therefore$ O is equally distant from AC and BC, and lies in CD; (102)

$\therefore$ the bisectors AE, BF, and CD meet in a common point.

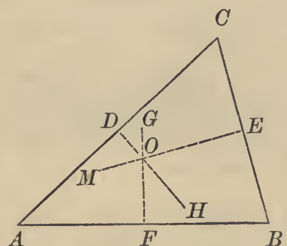
Q. E. D.

104. Cor.—*The point in which the bisectors of the angles of a triangle meet is equally distant from the three sides of the triangle.*

THEOREM XXXVII.

105. *The perpendiculars erected at the middle points of the sides of a triangle meet in a common point.*

In the △ *ABC*, let *DH*, *FG*, and *EM* be respectively ⊥ to *AC*, *AB*, and *BC*, at their middle points.



To prove that DH, FG, and EM meet at a common point.

The ⊥s *DH* and *FG* meet in some point, as *O*, otherwise they would be ∥, and *AC* and *AB*, the ⊥s to these ∥s, would lie in the same straight line, which is impossible.

Now *O* is equally distant from *A* and *C*; also from *A* and *B*; (53)

∴ *O* is equally distant from *C* and *B*, and must lie in *EM* (54). That is, the ⊥ *EM* passes through *O*;

∴ *DH*, *FG*, and *EM* meet in a common point. Q. E. D.

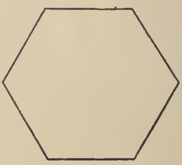
106. Cor.—*The common point of the perpendiculars erected at the middle points of the sides of a triangle is equally distant from the vertices of the triangle.*

POLYGONS.

DEFINITIONS.

107. A ***Polygon*** is a plane figure bounded by straight lines. The bounding lines are the *sides* of the polygon.

The ***Perimeter*** of a polygon is the sum of the bounding lines. The angles which the adjacent sides make with each other are the *angles* of the polygon.



A ***Diagonal*** of a polygon is a line joining two non-adjacent angles.

NOTE.—Let the pupil illustrate.

108. An ***Equilateral Polygon*** is one all of whose sides are equal.

109. An ***Equiangular Polygon*** is one all of whose angles are equal.

Two polygons are *mutually equilateral* when their sides are respectively equal.

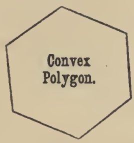
Two polygons are *mutually equiangular* when their angles are respectively equal.

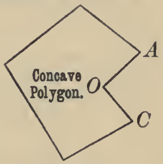
Homologous sides or ***angles*** are those which are similarly placed.

110. A ***Convex Polygon*** is one no side of which when produced can enter the surface bounded by the perimeter.

Each angle of such a polygon is called a *salient* angle.

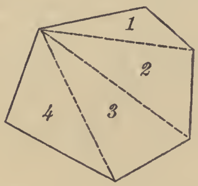
111. A ***Concave Polygon*** is one of which two or more sides, when produced, will enter the space enclosed by the perimeter.





The angle AOC is called a *re-entrant* angle.

112. By drawing diagonals from the vertex of any angle of a polygon, it may be divided into as many triangles as it has sides less two.



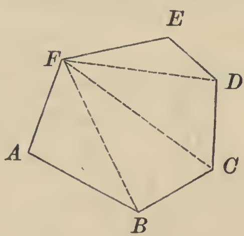
113. A polygon of three sides is a *Triangle;* of four, a *Quadrilateral;* of five, a *Pentagon;* of six, a *Hexagon;* of seven, a *Heptagon;* of eight, an *Octagon;* of nine, a *Nonagon;* of ten, a *Decagon;* of twelve, a *Dodecagon.*

ANGLES OF A POLYGON.

THEOREM XXXVIII.

114. *The sum of all the angles of any polygon equals two right angles taken as many times less two as the polygon has sides.*

Let $ABCDEF$ be a polygon of n sides.



To prove that $\angle\ AFE + \angle\ FED + \angle\ EDC$, *etc.* $= 2 \text{∟s}\ (n-2)$

From any vertex, as F, draw the diagonals FB, FC, and FD.

Then we have $(n-2)$ $\triangle$s. (112)

The sum of the $\angle$s of the $\triangle$s = the sum of the $\angle$s of the polygon.

But the sum of the $\angle$s of a $\triangle = 2$ ∟s; (76)

$\therefore$ the sum of the $\angle$s of the polygon $= 2$ ∟s $(n-2)$.

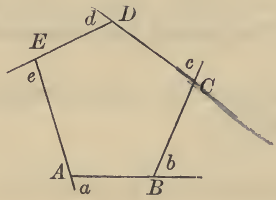
Q. E. D.

115. Cor.—*The sum of the angles of a quadrilateral is* 4 ∟s; *of a pentagon,* 6 ∟s; *of a hexagon,* 8 ∟s, *etc.*

THEOREM XXXIX.

116. *If the sides of a convex polygon are produced so as to form one exterior angle at each vertex, the sum of the exterior angles equals four right angles.*

Let $ABCDE$ be a polygon of n sides, and let the sides be produced so as to form the exterior $\angle$s a, b, c, d, e.



To prove that $\angle a + \angle b + \angle c + \angle d + \angle e = 4$ $\llcorner$s.

At each vertex there are two $\angle$s whose sum $= 2$ $\llcorner$s; (44)

and since there are as many vertices as there are sides, we have $n \times 2$ $\llcorner$s.

But the sum of the interior $\angle$s $= 2 \llcorner$s $(n-2)$; (114)

$$\therefore \angle a + \angle b + \angle c + \angle d + \angle e = n \times 2 \llcorner\text{s} - 2 \llcorner\text{s}\,(n-2)$$

$$= 2n \llcorner\text{s} - 2n \llcorner\text{s} + 4 \llcorner\text{s}.$$

$$= 4 \llcorner\text{s}.$$

Q. E. D.

QUADRILATERALS.

DEFINITIONS.

117. A ***Quadrilateral*** is a polygon of four sides.

118. There are three *classes* of quadrilaterals, namely, ***Trapeziums, Trapezoids,*** and ***Parallelograms.***

119. A ***Trapezium*** is a quadrilateral which has no two of its sides parallel.

120. A ***Trapezoid*** is a quadrilateral which has two of its sides parallel.

The parallel sides are called the *bases.*

121. A ***Parallelogram*** is a quadrilateral which has its opposite sides parallel.

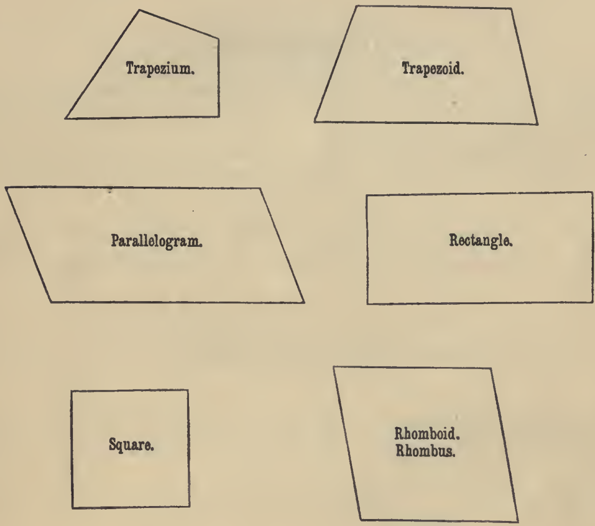
The side upon which a parallelogram is supposed to stand and the opposite side are called the *bases.*

122. A ***Rectangle*** is a parallelogram whose angles are right angles.

123. A ***Square*** is an equilateral rectangle.

124. A ***Rhomboid*** is a parallelogram whose angles are oblique.

125. A ***Rhombus*** is an equilateral rhomboid.



126. A ***Diagonal*** of a parallelogram is a line joining any two opposite vertices.

NOTE.—Let the pupil illustrate.

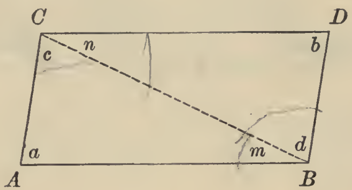
127. The ***Altitude*** of a parallelogram or trapezoid is the perpendicular distance between its *bases*.

PARALLELOGRAMS.

THEOREM XL.

128. *In any parallelogram, the opposite sides and angles are equal.*

Let $ABCD$ be any $\square$.



To prove that $AB = CD$, $AC = BD$, $\angle a = \angle b$, and $\angle ACD = \angle ABD$.

Draw the diagonal BC.

$$\angle m = \angle n, \quad (62)$$

$$\angle c = \angle d, \quad (62)$$

and $$BC = BC;$$

$\therefore$ $$\triangle ABC = \triangle CDB, \quad (84)$$

and $AB = CD$, $AC = BD$, and $\angle a = \angle b$.

From the first two equations,

$$\angle c + \angle n = \angle d + \angle m,$$

or $$\angle ACD = \angle ABD. \quad \text{Q.E.D.}$$

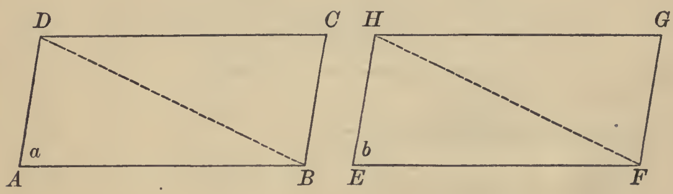
129. Cor. 1.—*A diagonal of a parallelogram divides it into two equal triangles.*

130. Cor. 2.—*Parallels intercepted between parallels are equal.*

THEOREM XLI.

131. *Two parallelograms are equal if two sides and the included angle of the one are respectively equal to two sides and the included angle of the other.*

Let $ABCD$ and $EFGH$ be two ▱s, having $AD = EH$, $AB = EF$, and $\angle a = \angle b$.



To prove that ▱ $ABCD =$ ▱ $EFGH$.

Draw the diagonals BD and FH.

$AD = EH$, (Hyp.)

$AB = EF$, (Hyp.)

and $\angle a = \angle b$; (Hyp.)

$\therefore$ $\triangle ABD = \triangle EFH$. (83)

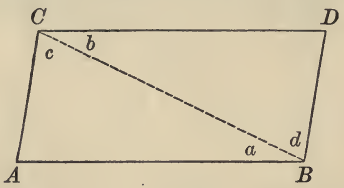
But these $\triangle$s are the halves of the ▱s.

$\therefore$ ▱ $ABCD =$ ▱ $EFGH$. Q. E. D.

THEOREM XLII.

132. *If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.*

In the quadrilateral $ABCD$, let $AB = CD$, and $AC = BD$.



To prove that $ABCD$ is a ▱.

Draw the diagonal BC.

$$AB = CD, \qquad \text{(Hyp.)}$$

$$AC = BD, \qquad \text{(Hyp.)}$$

and $$BC = BC;$$

∴ $$\triangle ABC = \triangle CDB, \qquad (86)$$

and $$\angle a = \angle b;$$

∴ AB and CD are ∥. (64)

Also, $$\angle c = \angle d;$$

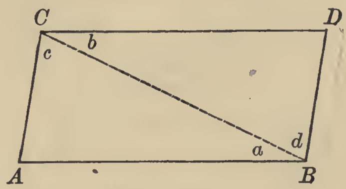
∴ AC and BD are ∥, (64)

and $ABCD$ is a ▱. (121) Q. E. D.

THEOREM XLIII.

133. *If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.*

Let $ABCD$ be a quadrilateral, having AB and CD equal and $\parallel$.



To prove that $ABCD$ is a ▱.

Draw the diagonal BC.

$AB = CD$, (Hyp.)

$BC = BC$,

and $\angle a = \angle b$; (62)

$\therefore$ $\triangle ABC = \triangle CDB$, (83)

and $\angle c = \angle d$;

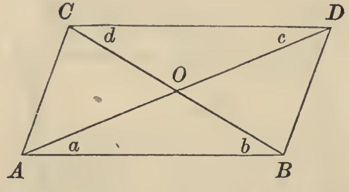
$\therefore$ AC and BD are $\parallel$, (64)

and $ABCD$ is a ▱. (121) Q.E.D.

THEOREM XLIV.

134. *The diagonals of a parallelogram bisect each other.*

Let $ABCD$ be a ▱, and AD, BC, its diagonals.



To prove that $AO = DO$, *and* $BO = CO$.

$$\angle a = \angle c, \qquad (62)$$

$$\angle b = \angle d, \qquad (62)$$

and $$AB = CD; \qquad (128)$$

$\therefore$ $$\triangle ABO = \triangle CDO; \qquad (84)$$

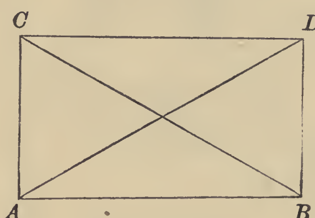
whence $AO = DO$, and $BO = CO$. Q. E. D.

135. Cor.—*The diagonals of a rhombus bisect each other at right angles.*

THEOREM XLV.

136. *The diagonals of a rectangle are equal.*

Let $ABCD$ be a rectangle, and AD, BC, its diagonals.



To prove that $AD = BC$.

$$AB = CD, \qquad (128)$$

$$BD = BD,$$

and $$\angle\ ABD = \angle\ CDB; \qquad (122)$$

$\therefore$ $$R \triangle\ ABD = R \triangle\ CDB, \qquad (83)$$

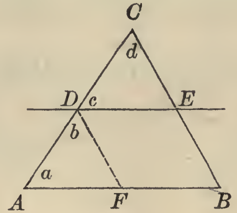
and $$AD = BC. \qquad \text{Q. E. D.}$$

137. Cor.—*The diagonals of a square are equal and bisect each other at right angles.*

THEOREM XLVI.

138. *If a parallel to the base of a triangle bisects one of the sides, it bisects the other also; and the part of it intercepted between the sides equals half the base.*

In the $\triangle ABC$, let DE be $\parallel$ to the base AB and bisect AC at D.



I. *To prove that DE bisects BC.*

Let DF be $\parallel$ to BC.

$$\angle a = \angle c, \qquad (62)$$

$$\angle b = \angle d, \qquad (62)$$

and $$AD = DC; \qquad \text{(Hyp.)}$$

$\therefore$ $$\triangle ADF = \triangle DCE, \qquad (84)$$

and $$DF = CE.$$

$$FBDE \text{ is a } \square; \qquad (121)$$

$\therefore$ $$DF = EB.$$

But $$DF = CE;$$

$\therefore$ $$EB = CE.$$

II. *To prove that $DE = \frac{1}{2} AB$.*

In the equal $\triangle$s ADF and DCE,

$$AF = DE,$$

and in the $\square$ $FBDE$,

$$DE = FB; \qquad (128)$$

$\therefore$ $AF = FB$, or F is the middle point of AB.

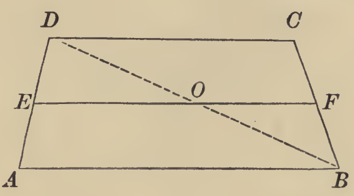
$\therefore$ $DE = \frac{1}{2} AB$. Q. E. D.

139. Cor.—*The straight line which joins the middle points of two sides of a triangle is parallel to the third side and is equal to half that side.*

THEOREM XLVII.

140. *The parallel to the bases of a trapezoid, bisecting one of the non-parallel sides, bisects the other also.*

Let $ABCD$ be a trapezoid, AB and DC its bases, E the middle point of AD, and let EF be $\parallel$ to AB and DC.



To prove that EF bisects BC.

Draw the diagonal BD.

In the $\triangle\ ABD$, BD is bisected at O. (138)

Then in the $\triangle\ DBC$, BC is bisected at F; (138)

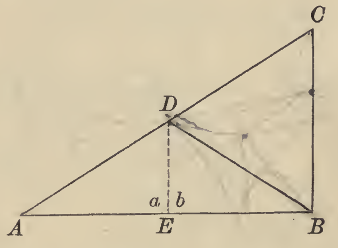
$\therefore$ EF bisects BC. Q.E.D.

141. Cor. *The straight line joining the middle points of the non-parallel sides of a trapezoid, is parallel to the base and equals half their sum.*

THEOREM XLVIII.

142. *The straight line drawn from the vertex of a right angle of a right-angled triangle to the middle of the hypothenuse is equal to half the hypothenuse.*

Let the R△ ABC be right-angled at B, and let BD be drawn to the middle of AC.



To prove that $BD = \frac{1}{2} AC$.

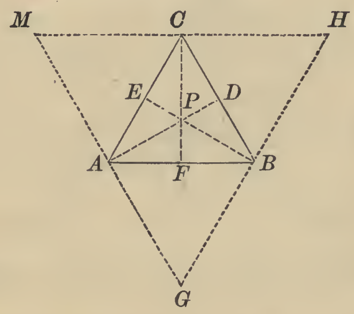
Let ED be ∥ to BC.

	$\angle b$ is a ∟.	(63)
	ED bisects AB;	
or,	$AE = BE$,	(138)
	$ED = ED$,	
and	$\angle a = \angle b$;	(45)
∴	$\triangle AED = \triangle BED$,	(83)
and	$BD = AD = \frac{1}{2} AC$.	Q. E. D.

THEOREM XLIX.

143. *The perpendiculars drawn from the vertices of a triangle to the opposite sides meet in a common point.*

In the $\triangle ABC$, let AD, BE, and CF be the $\perp$s from the vertices to the opposite sides.



To prove that AD, BE, and CF meet in a common point.

Through the vertices, let MH, GM, and GH be drawn respectively $\parallel$ to AB, BC, and AC.

$ABHC$ and $ABCM$ are $\square$s; (121)

$\therefore$ $AB = CH = MC$, (128)

and C is the middle point of MH.

Now CF is $\perp$ to AB and MH; (63)

$\therefore$ CF is $\perp$ to MH at its middle point.

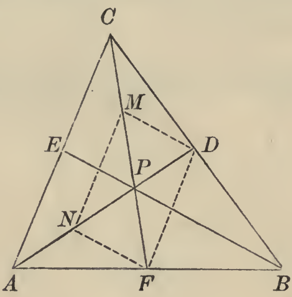
Likewise we can prove that AD and BE are $\perp$s to GM and GH at their middle points;

$\therefore$ the three $\perp$s meet in a common point. (105) Q.E.D.

THEOREM L.

144. *The medial lines of a triangle meet in a common point.*

In the $\triangle ABC$, let AD, BE, and CF be the medial lines.



To prove that AD, BE, and CF meet in a common point.

Let AD and CF meet at P, and let M and N be the middle points of CP and AP.

Draw MN, NF, FD, MD.

MN is $\parallel$ to AC and equals $\frac{1}{2}$ AC, (139)

FD is $\parallel$ to AC and equals $\frac{1}{2}$ AC; (139)

$\therefore$ MN and FD are $\parallel$ (66) and equal,

and $NFDM$ is a $\square$; (133)

$\therefore$ $PF = PM$ (134) $= MC = \frac{1}{3}$ CF.

Or AD intersects CF at P, a point whose distance from F equals $\frac{1}{3}$ CF.

Likewise we can prove that BE intersects CF at a point whose distance from F equals $\frac{1}{3}$ CF.

$\therefore$ AD, BE, and CF meet in a common point. Q. E. D.

EXERCISES IN INVENTION.

THEOREMS.

1. The two straight lines which bisect the two pairs of vertical angles formed by two lines are perpendicular to each other.

2. Two equal straight lines drawn from a point to a straight line make equal angles with that line.

3. If the three sides of an equilateral triangle are produced, all the external acute angles are equal, and all the obtuse angles are equal.

4. If the equal angles of an isosceles triangle are bisected, the triangle formed by the bisectors and the base is an isosceles triangle.

5. The three straight lines joining the middle points of the sides of a triangle divide the triangle into four equal triangles.

6. If one of the acute angles of a right-angled triangle is double the other, the hypothenuse is double the shortest side.

7. If through any point in the base of an isosceles triangle parallels to the equal sides are drawn, a parallelogram is formed whose perimeter equals the sum of the equal sides of the triangle.

8. If the diagonals of a parallelogram bisect each other at right angles, the figure is either a square or a rhombus.

9. The sum of the four lines drawn to the vertices of any quadrilateral from any point except the intersection of the diagonals, is greater than the sum of the diagonals.

10. The straight lines which join the middle points of the adjacent sides of any quadrilateral, form a parallelogram whose perimeter is equal to the sum of the diagonals of the given quadrilateral.

11. Lines joining the middle points of the opposite sides of any quadrilateral, bisect each other.

12. If the four angles of a quadrilateral are bisected, the bisectors form a second quadrilateral whose opposite angles are supplements of each other.

NOTE.—If the figure is a rhombus or a square, there is no second one formed.

PROBLEMS.

1. Show by a diagram that between five points, no three of which lie in the same straight line, $\frac{5 \times 4}{2}$ straight lines can be drawn connecting the points.

2. Between n points, no three of which lie in the same straight line, $\frac{n \times (n - 1)}{2}$ straight lines can be drawn connecting the points.

3. What is the greatest number of angles that can be formed with four straight lines? *Ans.* 24.

4. If the sum of the interior angles of a polygon equals the sum of its exterior angles, how many sides has the polygon?

5. If the sum of the interior angles of a polygon is double the sum of its exterior angles, how many sides has the figure?

6. If the sum of the exterior angles of a polygon is double the sum of its interior angles, how many sides has the figure?

BOOK II.

RATIO AND PROPORTION.

DEFINITIONS.

145. To *measure* a quantity is to find how many times it contains some other quantity of the same kind called the *unit of measure.*

146. Two quantities are *commensurable* if they have a common unit of measure.

Two quantities are *incommensurable* if they have no common unit of measure.

Any two similar *quantities* may be considered as having a common unit of measure *infinitely small.*

147. In Geometry we compare two similar quantities by finding how many times one contains the other; that is, we measure one by the other. The magnitude, therefore, of a quantity is always relative to the magnitude of some other similar quantity.

148. ***Ratio*** is the measure of relation between two similar quantities, and is expressed by the quotient resulting from dividing the first by the second.

The first of the two quantities compared is called the *Antecedent,* and the second the *Consequent.* Taken together they are called the *Terms of the Ratio,* or a *Couplet.*

Ratio is indicated by a colon placed between the quantities compared, or by the fractional form of indicating division; thus, the ratio of a to b is written, $a : b$, or $\frac{a}{b}$.

149. ***A Proportion*** is an expression of equality between two equal ratios.

Thus, $\frac{a}{b} = \frac{c}{d}$. This means that the ratio of a to b equals the ratio of c to d. Usually the proportion is indicated by a double colon placed between the two couplets, thus:

$$a : b :: c : d.$$

This is read, a is to b as c is to d; or, the ratio of a to b is equal to the ratio of c to d.

Of the four *terms* compared, the *first* and *third* are the *Antecedents*, and the *second* and *fourth* are the *Consequents*. The *Extremes* are the *first* and *fourth* terms. The *Means* are the *second* and *third* terms. The *Fourth Proportional* is the *fourth term*. When the means are equal, as in

$$a : b :: b : c,$$

b is said to be the *Mean Proportional* between a and c; and c is said to be the *Third Proportional* to a and b.

150. Four quantities are *Reciprocally Proportional* when the first is to the second as the reciprocal of the third is to the reciprocal of the fourth.

Thus, $$a : b :: \frac{1}{c} : \frac{1}{d}.$$

Two quantities and their reciprocals form a reciprocal proportional.

Thus, $$a : b :: \frac{1}{b} : \frac{1}{a}.$$

151. A proportion is taken by *Alternation*, when antecedent is compared with antecedent, and consequent with consequent.

Thus, if $a : b :: c : d$, we have by alternation either $a : c :: b : d$; or, $d : b :: c : a$.

152. A proportion is taken by *Inversion*, when the antecedents are made consequents and the consequents antecedents.

Thus, if $a : b :: c : d$, we have by inversion $b : a :: d : c$.

153. A proportion is taken by *Composition*, when the sum of antecedent and consequent is compared with either antecedent or consequent.

Thus, if $a : b :: c : d$, we have by composition

$$a + b : a :: c + d : c;$$

or $$a + b : b :: c + d : d.$$

154. A proportion is taken by *Division*, when the difference of antecedent and consequent is compared with either antecedent or consequent.

Thus, if $a : b :: c : d$, we have by division

$$a - b : a :: c - d : c;$$

or $$a - b : b :: c - d : d.$$

155. ***A Continued Proportion*** is a series of equal ratios.

Thus, if $a : b :: b : c :: c : d :: d : e$, we have a continued proportion.

THEOREM I.

156. *In any proportion the product of the extremes is equal to the product of the means.*

Let $a : b :: c : d.$

To prove that $a\,d = b\,c.$

Take the form $\frac{a}{b} = \frac{c}{d}.$

Multiply both members by $b\,d;$

then $a\,d = b\,c.$ Q. E. D.

THEOREM II.

157. *If the product of two quantities equals the product of two others, the quantities in either product can be made the means, and those of the other product the extremes of a proportion.*

Let $a\,d = b\,c.$

To prove that $a : b :: c : d.$

Divide both members by $b\,d.$

Then $\frac{a}{b} = \frac{c}{d},$

or $a : b :: c : d.$ Q. E. D.

THEOREM III.

158. *A mean proportional between two quantities equals the square root of their product.*

Let $$a : b :: b : c.$$

To prove that $b = \sqrt{a\,c}.$

$$b^2 = a\,c. \qquad (156)$$

Extract the square root of both members.

Then $$b = \sqrt{a\,c}.$$ Q. E. D.

THEOREM IV.

159. *The corresponding members of two equations form the couplets of a proportion.*

Let $$a = c,$$
and $$b = d.$$

To prove that $a : b :: c : d.$

Divide.

Then $$\frac{a}{b} = \frac{c}{d},$$

or $$a : b :: c : d.$$ Q. E. D.

THEOREM V.

160. *If four quantities are in proportion, they are in proportion by alternation.*

Let $a : b :: c : d.$

To prove that $a : c :: b : d.$

Take the form $\frac{a}{b} = \frac{c}{d}.$

Multiply both members by $\frac{b}{c}.$

Then $\frac{a}{c} = \frac{b}{d},$

or $a : c :: b : d.$ Q. E. D.

THEOREM VI.

161. *If four quantities are in proportion, they are in proportion by inversion.*

Let $a : b :: c : d.$

To prove that $b : a :: d : c.$

Take the form $\frac{a}{b} = \frac{c}{d}.$

Divide 1 by each member.

Then $\frac{b}{a} = \frac{d}{c},$

or $b : a :: d : c.$ Q. E. D.

THEOREM VII.

162. *If four quantities are in proportion, they are in proportion by composition.*

Let $$a : b :: c : d.$$

To prove that $a + b : b :: c + d : d.$

Take the form $$\frac{a}{b} = \frac{c}{d}.$$

To each member add 1.

Then $$\frac{a}{b} + 1 = \frac{c}{d} + 1;$$

whence $$\frac{a+b}{b} = \frac{c+d}{d},$$

or $$a + b : b :: c + d : d.$$ Q. E. D.

THEOREM VIII.

163. *If four quantities are in proportion, they are in proportion by division.*

Let $$a : b :: c : d.$$

To prove that $a - b : b :: c - d : d.$

Take the form $$\frac{a}{b} = \frac{c}{d}.$$

From each member subtract 1.

Then $$\frac{a}{b} - 1 = \frac{c}{d} - 1;$$

whence $$\frac{a-b}{b} = \frac{c-d}{d},$$

or $$a - b : b :: c - d : d.$$ Q. E. D.

THEOREM IX.

164. *If two proportions have a couplet in each the same, the other couplets form a proportion.*

Let $a : b :: c : d,$

and $a : b :: e : f.$

To prove that $c : d :: e : f.$

Take the forms $\frac{a}{b} = \frac{c}{d}$ and $\frac{a}{b} = \frac{e}{f}.$

Then $\frac{c}{d} = \frac{e}{f},$ (Ax. 1.)

or $c : d :: e : f.$ Q. E. D.

THEOREM X.

165. *Equimultiples of two quantities are proportional to the quantities themselves.*

Let a and b be any two quantities.

To prove that $ma : mb :: a : b.$

$$\frac{a}{b} = \frac{a}{b}.$$

Multiply both terms of the first member by m.

Then $\frac{ma}{mb} = \frac{a}{b},$

or $ma : mb :: a : b.$ Q. E. D.

THEOREM XI.

166. *In any proportion, any equimultiples of the first couplet are proportional to any equimultiples of the second couplet.*

Let $a : b :: c : d.$

To prove that $ma : mb :: nc : nd.$

Take the form $\frac{a}{b} = \frac{c}{d}.$

Multiply both terms of the first member by m, and both terms of the second member by n.

Then $\frac{ma}{mb} = \frac{nc}{nd},$

or $ma : mb :: nc : nd.$ Q. E. D.

THEOREM XII.

167. *If two quantities are increased or diminished by like parts of each, the results are proportional to the quantities themselves.*

Let a and b be any two quantities.

To prove that $a \pm \frac{p}{q} a : b \pm \frac{p}{q} b :: a : b.$

$ma : mb :: a : b.$ (165)

For m, substitute $1 \pm \frac{p}{q}.$

Then $\left(1 \pm \frac{p}{q}\right) a : \left(1 \pm \frac{p}{q}\right) b :: a : b,$

or $a \pm \frac{p}{q} a : b \pm \frac{p}{q} b :: a : b.$ Q. E. D.

THEOREM XIII.

168. *In any continued proportion, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

Let $a : b :: c : d :: e : f :: g : h.$

To prove that $a + c + e + g : b + d + f + h :: a : b.$

Denote the common ratio by r.

Then $$r = \frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{g}{h}.$$

Whence $a = br,\ c = dr,\ e = fr,\ g = hr.$

Add these equals.

Then $a + c + e + g = (b + d + f + h)\, r.$

Divide both members by $(b + d + f + h)$.

Then $$\frac{a + c + e + g}{b + d + f + h} = r = \frac{a}{b},$$

or $a + c + e + g : b + d + f + h :: a : b.$ Q. E. D.

THEOREM XIV.

169. *In two or more proportions, the products of the corresponding terms are proportional.*

Let $$\begin{cases} a : b :: c : d, \\ e : f :: g : h, \\ m : n :: o : p. \end{cases}$$

To prove that $aem : bfn :: cgo : dhp.$

Take the forms $\frac{a}{b} = \frac{c}{d},\ \frac{e}{f} = \frac{g}{h},\ \frac{m}{n} = \frac{o}{p}.$

By multiplication we have

$$\frac{aem}{bfn} = \frac{cgo}{dhp},$$

or $aem : bfn :: cgo : dhp.$ Q. E. D.

THEOREM XV.

170. *Like powers, or like roots, of the terms of a proportion form a proportion.*

Let $$a : b :: c : d.$$

To prove that $a^n : b^n :: c^n : d^n$,

and that $$a^{\frac{1}{n}} : b^{\frac{1}{n}} :: c^{\frac{1}{n}} : d^{\frac{1}{n}}.$$

Take the form $$\frac{a}{b} = \frac{c}{d}.$$

Raise each member to the n^{th} power.

Then $$\frac{a^n}{b^n} = \frac{c^n}{d^n},$$

or $$a^n : b^n :: c^n : d^n.$$

Also extract the n^{th} root of each member.

Then $$\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \frac{c^{\frac{1}{n}}}{d^{\frac{1}{n}}},$$

or $$a^{\frac{1}{n}} : b^{\frac{1}{n}} :: c^{\frac{1}{n}} : d^{\frac{1}{n}}.$$ Q. E. D.

171. Scholium. The product of two quantities implies that at least one is numerical.

In (169) and (170), all the quantities must be numerical.

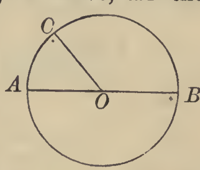
In (160) and (168), all the quantities must be of the same kind.

BOOK III.

THE CIRCLE.

DEFINITIONS.

172. A ***Circle*** is a plane bounded by a curve, all the points of which are equally distant from a point within, called the *centre*.



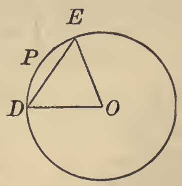
The ***Circumference*** of a circle is the curve which bounds it.

An ***Arc*** is a part of the circumference; as, AC. A *Semi-circumference* is an arc equal to half of the circumference.

A ***Radius*** is a straight line extending from the centre to any point in the circumference; as, OC.

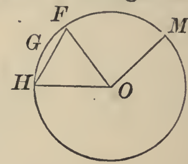
173. A ***Diameter*** of a circle is a straight line passing through the centre and terminating each way in the circumference; as, AB.

174. A ***Chord*** is a straight line joining any two points in the circumference; as, ED.



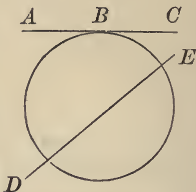
The arc EPD is said to be *subtended* by its chord ED. Every chord subtends two arcs, whose sum equals the whole circumference. Whenever an arc and its chord are spoken of, the less arc is meant.

175. A ***Segment*** of a circle is the portion enclosed by an arc and its chord; as, FGH. A ***Semicircle*** is a segment equal to one-half of the circle.



176. A ***Sector*** of a circle is the portion enclosed by an arc and the radii drawn to its extremities; as, OFM.

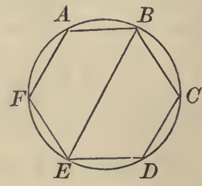
177. A ***Tangent*** is a straight line which touches the circumference but does not intersect it; as, ABC.



The common point B is called the *point of contact*, or the *point of tangency*.

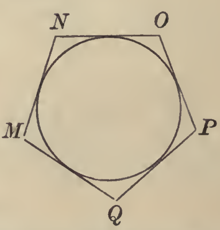
178. A ***Secant*** is a straight line which cuts the circumference in two points; as, DE.

179. An ***Inscribed Angle*** is one whose vertex is in the circumference and whose sides are chords; as, ABE.



180. An ***Inscribed Polygon*** is one whose sides are chords of a circle; as, $ABCDEF$. The circle is then said to be *circumscribed* about the polygon.

181. A ***Polygon*** is circumscribed about a circle when all its sides are tangents to the circumference; as, $MNOPQ$. The circle is then said to be *inscribed*.



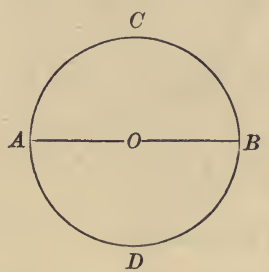
182. By the definition of a circle, all its *radii are equal;* also, all its *diameters are equal.* It also follows from the definition that *circles are equal when their radii are equal.*

CHORDS, ARCS, ANGLES AT THE CENTRE, SECANTS, AND RADII.

THEOREM I.

183. *Any diameter bisects the circle and its circumference.*

Let $ABCD$ be a $\odot$, and AB any diameter.



To prove that $ABC = ABD$.

On AB as an axis, revolve the portion ABC till it falls in the plane of ABD.

Then the curve ACB coincides with the curve ADB, for all the points in each are equally distant from the centre O.

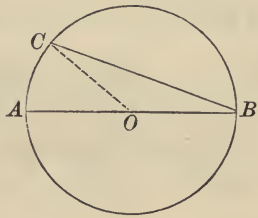
$\therefore$ $$ABC = ABD, \qquad (14)$$

and $$\text{curve } ACB = \text{curve } ADB. \qquad \text{Q. E. D.}$$

THEOREM II.

184. *A diameter of a circle is greater than any other chord.*

In the ○ ACB, let AB be any diameter and BC any other chord.



To prove that $AB > BC$.

From the centre O, draw OC,

$$OA = OC. \qquad (182)$$

$$OC + OB > BC. \qquad (\text{Ax. } 11)$$

Substitute OA for its equal OC.

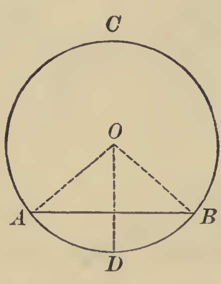
Then $$OA + OB > BC,$$

or $$AB > BC. \qquad (\text{Q. E. D.})$$

THEOREM III.

185. *A straight line cannot cut the circumference of a circle in more than two points.*

In the ○ $ADBC$, let AB cut the circumference at A and B.



To prove that AB cannot cut the circumference in more than two points.

Draw the radii OA, OB, OD,

$$OA = OB = OD. \qquad (182)$$

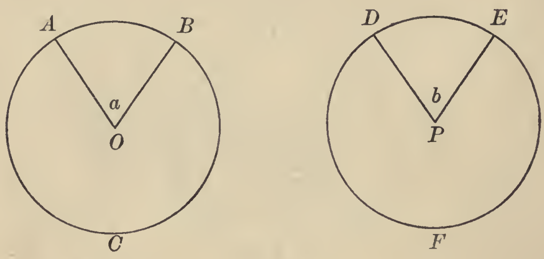
If AB could cut the circumference at A, B, and D, there would be three equal straight lines drawn from the same point to the same straight line, which cannot be. (61)

∴ a straight line cannot cut a circumference in more than two points. Q. E. D.

THEOREM IV.

186. *In equal circles, or in the same circle, equal angles at the centre intercept equal arcs.*

Let O and P be the centres of the equal $\bigcirc$s ABC and DEF, and let $\angle\ a = \angle\ b$.



To prove that arc $AB =$ *arc* DE.

Place the $\bigcirc ABC$ on the $\bigcirc DEF$ so that $\angle\ a$ coincides with $\angle\ b$.

$$OA = OB = PD = PE; \qquad (182)$$

$\therefore$ A falls on D, and B falls on E;

$\therefore$ $$ABO = DEP, \qquad (14)$$

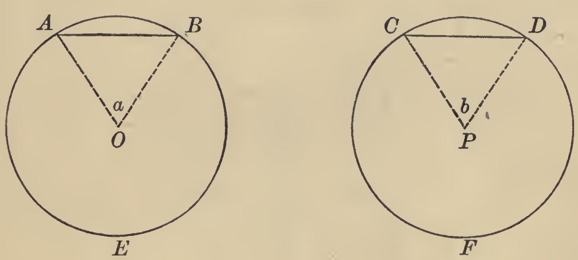
and arc $AB =$ arc DE. Q. E. D.

187. Cor.—*In equal circles, or in the same circle, equal arcs subtend equal angles at the centre.*

THEOREM V.

188. *In equal circles, or in the same circle, equal arcs are subtended by equal chords.*

In the equal ⊙s ABE and CDF, let arc $AB =$ arc CD.



To prove that chord $AB =$ chord CD.

Draw the radii OA, OB, PC, PD.

$$OA = PC, \qquad (182)$$

$$OB = PD, \qquad (182)$$

and $$\angle a = \angle b; \qquad (187)$$

∴ $$\triangle ABO = \triangle CDP, \qquad (83)$$

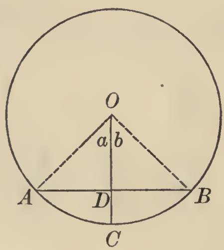
and chord $AB =$ chord CD. Q. E. D.

189. Cor.—*In equal circles, or in the same circle, equal chords subtend equal arcs.*

THEOREM VI.

190. *The radius perpendicular to a chord bisects the chord and the subtended arc.*

Let OC be a radius $\perp$ to the chord AB at D.



To prove that $AD = BD$, *and arc* $AC =$ *arc* BC.

Draw the radii OA and OB.

$$OA = OB, \qquad (182)$$

and $$OD = OD;$$

$\therefore$ $$\text{R}\triangle\ AOD = \text{R}\triangle\ BOD, \qquad (91)$$

and $$AD = BD.$$

Also $$\angle\ a = \angle\ b;$$

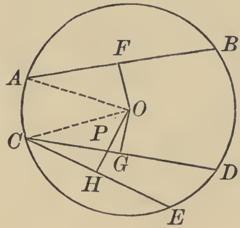
$\therefore$ $$\text{arc } AC = \text{arc } BC. \qquad (186)$$ Q. E. D.

191. Cor.—*The perpendicular erected at the middle point of a chord passes through the centre of the circle and bisects the subtended arc.*

THEOREM VII.

192. *In the same circle, or in equal circles, equal chords are equally distant from the centre; and if two chords are unequal, the less is at the greater distance from the centre.*

Let chord $AB =$ chord CD, and chord $CE <$ chord CD; and let OF, OG, and OH be $\perp$s to these chords from the centre O.



To prove that $OF = OG$, *and* $OH > OG$.

OF and OG bisect the equal chords AB and CD; (190)

$\therefore$ $AF = CG$;

$\therefore$ R$\triangle$ $AOF =$ R$\triangle$ COG, (91)

and $OF = OG$.

Again, $CD > CE$;

$\therefore$ OH cuts CD in some point, as P.

Now $OH > OP$.

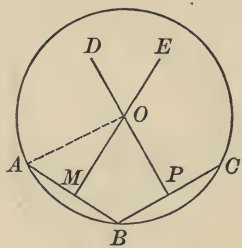
But $OP > OG$; (52)

$\therefore$ still more is $OH > OG$. Q. E. D.

THEOREM VIII.

193. *Through three points not in the same straight line, a circumference of a circle can be passed.*

Let A, B, and C be any three points not in the same straight line.



To prove that through A, B, and C, a circumference of a ○ *can be passed.*

Draw AB and BC, and at their middle points let the ⊥s EM and DP be erected.

AB and BC are not in the same straight line;

∴ the ⊥s EM and DP meet in some point, as O.

Now, O is equally distant from A and B; also from B and C; (53)

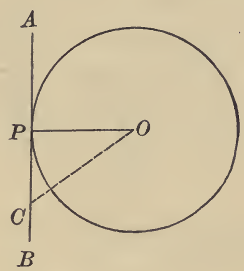
∴ O is equally distant from A, B, and C, and a circumference with OA as a radius passes through these points.

Q. E. D.

THEOREM IX.

194. *A straight line perpendicular to a radius at its extremity is a tangent to the circle.*

Let AB be $\perp$ to the radius OP at P.



To prove that AB is a tangent to the ○ *at the point P.*

From the centre draw any oblique line, as OC.

$$OC > OP; \qquad (52)$$

$\therefore$ the point C is without the circumference, and all points in AB, except P, are without the circumference;

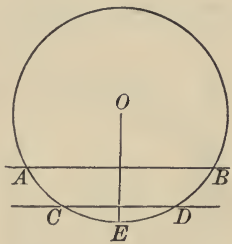
$\therefore$ AB is a tangent to the ○ at P. (177) Q. E. D.

195. Cor.—*A straight line tangent to a circle is perpendicular to the radius drawn to the point of contact.*

THEOREM X.

196. *Two parallel secants intercept equal arcs.*

Let the $\parallel$s AB and CD intercept the arcs AC and BD.



To prove that arc $AC =$ *arc* BD.

Suppose the radius OE to be drawn $\perp$ to AB and CD.

Then arc $AE =$ arc BE,

and arc $CE =$ arc DE. (190)

Subtract;

then arc $AE -$ arc $CE =$ arc $BE -$ arc DE.

or arc $AC =$ arc BD. Q. E. D.

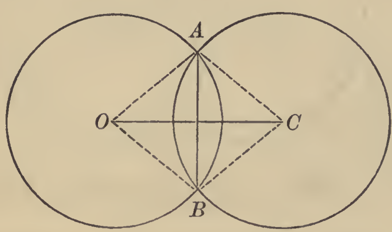
197. Scholium.—This proposition is true for any position of the parallels; hence it is true if one or both become tangents; and the straight line which joins the points of contact of two parallel tangents is a diameter.

RELATIVE POSITION OF CIRCLES.

THEOREM XI.

198. *If two circles cut each other, the straight line joining their centres bisects their common chord at right angles.*

Let AB be a common chord of two ○s which cut each other, and OC join the centres O and C.



To prove that OC is $\perp$ to AB at its middle point.

Draw the radii OA, OB, CA, and CB.

$$OA = OB, \qquad (182)$$

and $$CA = CB; \qquad (182)$$

$\therefore$ O is equally distant from A and B,

and C is equally distant from A and B;

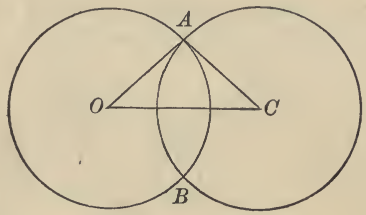
$\therefore$ OC is $\perp$ to AB at its middle point. (55) Q. E. D.

199. Cor.—*If two circles touch each other, either externally or internally, the point of contact is in the line joining their centres.*

THEOREM XII.

200. *If two circles cut each other, the distance between their centres is less than the sum and greater than the difference of their radii.*

Let O and C be the centres of two ○s whose circumferences cut each other at A and B, and draw the radii OA and CA.



To prove that $OC < OA + CA$, *and* $OC > OA - CA$.

The point A does not lie in OC; (199)

$\therefore$ OCA is a $\triangle$.

Now, $OC < OA + CA$, (Ax. 11)

and $OC > OA - CA$. (73) Q. E. D.

201. Cor. 1.—*If two circles touch each other externally, the distance between their centres equals the sum of their radii.*

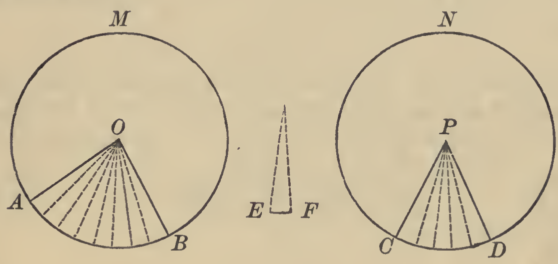
202. Cor. 2.—*If two circles are wholly exterior to each other, the distance between their centres is greater than the sum of their radii.*

THE MEASUREMENT OF ANGLES.

THEOREM XIII.

203. *In equal circles, or in the same circle, angles at the centre are to each other as the arcs which they intercept.*

In the equal ⊙s ABM and CDN, let O and P be the centres, and let the ∠s AOB and CPD intercept the arcs AB and CD.



To prove that $\angle\ AOB : \angle\ CPD ::$ *arc* AB : *arc* CD.

Let EF be a common unit of measure of AB and CD, and suppose it to be contained in AB 8 times, and in CD 5 times.

Then arc AB : arc CD :: 8 : 5.

Draw radii at the several points of division of the arcs.

Then the partial ∠s are equal. (187)

AOB contains 8, and CPD contains 5 equal ∠s;

∴ $\angle\ AOB : \angle\ CPD :: 8 : 5.$

But arc AB : arc CD :: 8 : 5;

∴ $\angle\ AOB : \angle\ CPD ::$ arc AB : arc CD.

The same proportion is found if other numbers than 8 and 5 are taken.

Now, this is true, whatever may be the length of the common unit of measure of the arcs; hence it is true when it is infinitely small, as is the case when the arcs are incommensurable. Hence, in any case, the proposition is true.

Q. E. D.

204. COR.—*In equal circles, or in the same circle, arcs which are intercepted by angles at the centre are to each other as the angles.*

205. SCHOLIUM.—The truth of this proposition gave rise to the method of *measuring angles by arcs.* It will be observed, that if arcs are struck *with the same radius* from the vertices of angles as centres, the angles are to each other as the arcs intercepted by their sides. Hence *the angle is said to be measured by the arc.*

The unit of measure generally adopted is an arc equal to $\frac{1}{360}$ of the circumference of a circle, called a *degree*, and denoted thus (°).

The degree is divided into 60 equal parts, called *minutes*, denoted thus, (′).

The minute is divided into 60 equal parts, called *seconds*, denoted thus, (″).

A right angle, therefore, is measured by 90°; or, as we say, it is an angle of 90°.

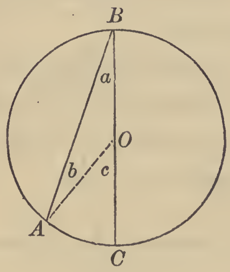
An angle of 45° is $\frac{1}{2}$ of a right angle; an angle of 30° is $\frac{1}{3}$ of a right angle. Thus we have a definite idea of the magnitude of an angle if we know the number of degrees by which it is measured.

THEOREM XIV.

206. *An inscribed angle is measured by half the intercepted arc.*

In the ○ ABC, let AB and CB be the sides of the inscribed $\angle\ a$.

Case I.—*Let the centre O be in one of the sides.*



To prove that $\angle\ a$ *is measured by* $\frac{1}{2}$ *arc* AC.

Draw the radius OA.

$$OA = OB; \qquad (182)$$

$\therefore$ $\triangle\ ABO$ is isosceles,

and $\angle\ a = \angle\ b$. (94)

$\angle\ c = a\ \angle + \angle\ b$. (82)

Substitute $\angle\ a$ for its equal $\angle\ b$.

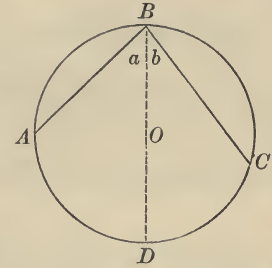
Then $\angle\ c = 2\ \angle\ a$.

Now, $\angle\ c$ is measured by arc AC; (205)

$\therefore$ $2\ \angle\ a$ is measured by arc AC,

and $\angle\ a$ is measured by $\frac{1}{2}\ AC$.

Case II.—*Let the centre O fall within the inscribed* $\angle$.



To prove that $\angle$ *ABC is measured by* $\frac{1}{2}$ *arc AC.*

Draw the diameter BD.

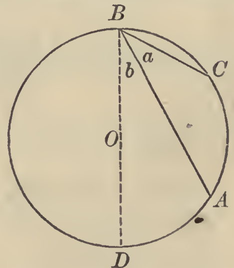
$\angle\ a$ is measured by $\frac{1}{2}$ arc AD,

and $\angle\ b$ is measured by $\frac{1}{2}$ arc DC; (Case I.)

$\therefore$ $\angle\ a + \angle\ b$ is measured by $\frac{1}{2}$ (arc AD + arc DC),

or $\angle\ ABC$ is measured by $\frac{1}{2}\ AC$.

Case III.—*Let the centre O fall without the inscribed* $\angle$.



To prove that $\angle$ *a is measured by* $\frac{1}{2}$ *arc AC.*

Draw the diameter BD.

$\angle\ DBC$ is measured by $\frac{1}{2}$ arc DC,

and $\angle\ b$ is measured by $\frac{1}{2}$ arc DA; (Case I.)

$\therefore$ $\angle\ DBC - \angle\ b$ is measured by $\frac{1}{2}$ (arc DC — arc DA),

or $\angle\ a$ is measured by $\frac{1}{2}$ arc AC. Q. E. D.

207. Cor. 1.—*All angles inscribed in the same segment are equal.*

208. Cor. 2.—*Any angle inscribed in a semicircle is a right angle.*

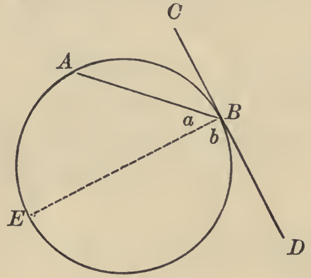
209. Cor. 3.—*Any angle inscribed in a segment less than a semicircle is obtuse.*

210. Cor. 4.—*Any angle inscribed in a segment greater than a semicircle is acute.*

THEOREM XV.

211. *Any angle formed by a tangent and a chord is measured by half the intercepted arc.*

Let the $\angle ABD$ be formed by the tangent CD and the chord AB.



To prove that $\angle ABD$ *is measured by* $\frac{1}{2}$ *arc* AEB.

Draw the diameter BE.

$\angle b =$ a $\llcorner$, (195) and is measured by $\frac{1}{2}$ arc EB, (208)

and $\angle a$ is measured by $\frac{1}{2}$ arc AE; (206)

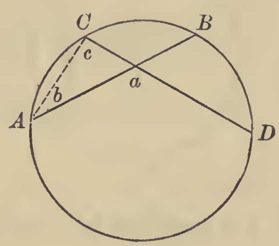
$\therefore$ $\angle a + \angle b$ is measured by $\frac{1}{2}$ (arc AE + arc EB),

or $\angle ABD$ is measured by $\frac{1}{2}$ arc AEB. Q. E. D.

THEOREM XVI.

212. *Any angle formed by two chords intersecting is measured by half the sum of the arcs intercepted between its sides and the sides of its vertical angle.*

Let the $\angle\ a$ be formed by the chords AB and CD.



To prove that $\angle\ a$ *is measured by* $\frac{1}{2}$ *(arc* AD + *arc* CB*).*

Draw AC.

$\angle\ c$ is measured by $\frac{1}{2}$ arc AD,.

and $\angle\ b$ is measured by $\frac{1}{2}$ arc CB. (206)

But $\angle\ a = \angle\ c + \angle\ b$; (82)

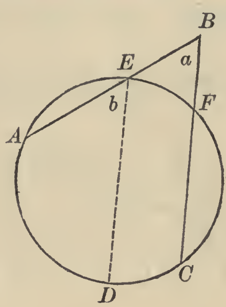
$\therefore$ $\angle\ a$ is measured by $\frac{1}{2}$ (arc AD + arc CB).

Q. E. D.

THEOREM XVII.

213. *Any angle formed by two secants is measured by half the difference of the intercepted arcs.*

Let the $\angle\ a$ be formed by the secants AB and BC.



To prove that $\angle\ a$ is measured by $\frac{1}{2}$ (arc AC — arc EF).

Let DE be ∥ to CB.

$$\text{arc } DC = \text{arc } EF. \tag{196}$$

$$\angle\ b \text{ is measured by } \tfrac{1}{2} \text{ arc } AD.$$

But arc AD = (arc AC — arc DC) = (arc AC — arc EF,)

and $$\angle\ b = \angle\ a; \tag{62}$$

$\therefore$ $\angle\ a$ is measured by $\frac{1}{2}$ (arc AC — arc EF). Q. E. D.

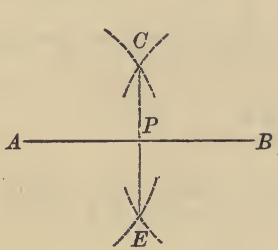
214. Scholium.—This proposition is true for any position of the secants; hence it is true if one or both become tangents.

PROBLEMS IN CONSTRUCTION.

PROBLEM I.

215. *To bisect a given straight line.*

Let AB be the given straight line.



With A and B as centres and a radius greater than half of AB, describe arcs intersecting at C and E.

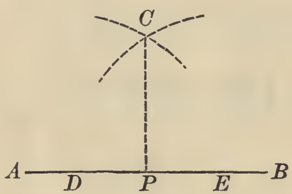
Draw CE.

Then CE bisects AB at the point P. (55) Q. E. F.

PROBLEM II.

216. *At any point in a straight line to erect a perpendicular.*

Let P be any point in the straight line AB.



Cut off $PD = PE$.

With D and E as centres and a radius greater than PD or PE, describe arcs intersecting at C.

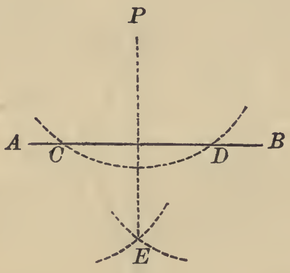
Draw CP.

Then CP is $\perp$ to AB. (55) Q. E. F.

PROBLEM III.

217. *From any point without a straight line to draw a perpendicular to that line.*

Let P be any point without the straight line AB.



With P as a centre and a radius sufficiently great, describe an arc cutting AB at C and D.

With C and D as centres and a radius greater than half of CD, describe arcs intersecting at E.

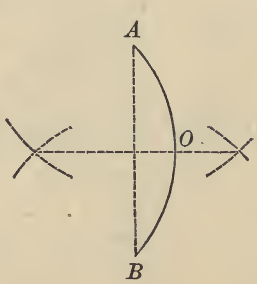
Draw PE.

Then is $PE \perp$ to AB. (55) Q. E. F.

PROBLEM IV.

218. *To bisect a given arc.*

Let AOB be a given arc.



Draw the chord AB.

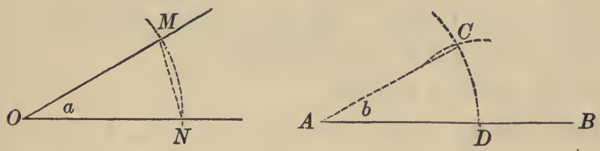
Bisect AB by a $\perp$ as in (215).

This $\perp$ bisects the arc. (191) Q. E. F.

PROBLEM V.

219. *To construct an angle equal to a given angle, at any point in a line.*

Let a be the given $\angle$, and A the point in the line AB.



With the vertex O as a centre, and any radius, describe an arc cutting the sides of $\angle\ a$ at M and N.

With A as a centre and the same radius, describe the indefinite arc CD.

Draw the chord $M\ N$.

With D as a centre and $M\ N$ as a radius, describe an arc cutting the indefinite arc at C.

Draw $A\,C$.

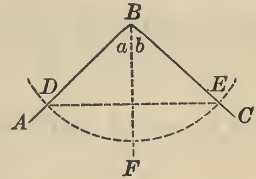
Then arc $CD =$ arc $M\ N$; (189)

$\therefore$ $\angle\ b = \angle\ a$. (187) Q. E. F.

PROBLEM VI.

220. *To bisect a given angle.*

Let ABC be the given $\angle$.



With B as a centre and any radius, describe an arc cutting the sides of the $\angle$ at D and E.

Bisect the arc as in (218).

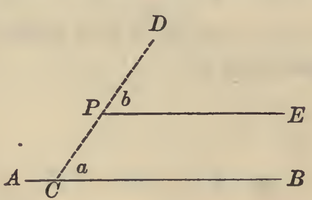
Then, since arc $DF =$ arc FE, $\angle\ a = \angle\ b$. (187)

Q. E. F.

PROBLEM VII.

221. *Through a given point, to draw a straight line parallel to a given straight line.*

Let P be the given point, and AB the given straight line.



From any point in AB, as C, draw the line CD through P.

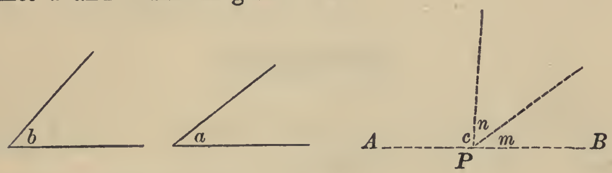
At P construct the $\angle\ b = \angle\ a$ as in (219).

Then PE is $\parallel$ to AB. (64) Q. E. F.

PROBLEM VIII.

222. *Given two angles of a triangle, to find the third angle.*

Let a and b be the given $\angle$s.



Draw the indefinite line AB.

At any point in AB, as P, construct

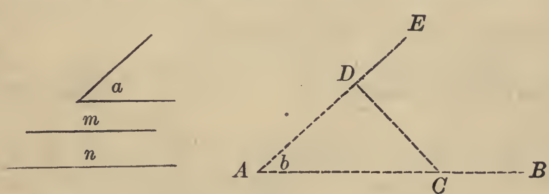
$\angle\ m = \angle\ a$, and $\angle\ n = \angle\ b$, as in (219).

Then $\angle\ c$ is the third $\angle$ of the $\triangle$. (77) Q. E. F.

PROBLEM IX.

223. *Two sides and the included angle of a triangle being given, to construct the triangle.*

Let m and n be the given sides, and a the included $\angle$.



Draw the indefinite line AB.

Cut off $AC = n$.

At A construct $\angle\ b = \angle\ a$.

On AE cut off $AD = m$.

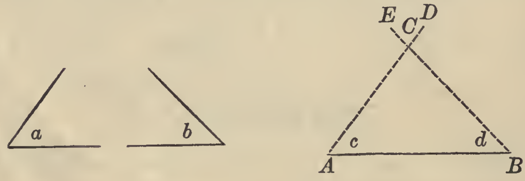
Draw CD.

Then ACD is the required $\triangle$. (83) Q. E. F.

PROBLEM X.

224. *One side and two adjacent angles of a triangle being given, to construct the triangle.*

Let AB be the given side, and a and b the given $\angle$s.



At A construct $\angle\ c = \angle\ a$, and at B construct $\angle\ d = \angle\ b$.

The sides AD and BE intersect at C,

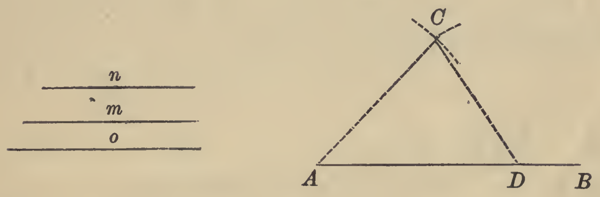
and ABC is the required $\triangle$. (84) Q. E. F.

225. Scholium. This problem is not possible if the sum of the two given angles is equal to or greater than two right angles.

PROBLEM XI.

226. *Given the three sides of a triangle, to construct the triangle.*

Let m, n, and o be the three sides of a $\triangle$.



Draw an indefinite line AB.

Cut off $AD = o$.

With A as a centre and m as a radius, describe an arc; and with D as a centre and n as a radius, describe an arc cutting the first at C.

Draw AC and DC.

Then ADC is the required $\triangle$. (86) Q. E. F.

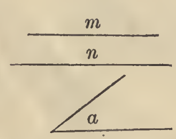
227. Scholium.—This problem is not possible if one side is equal to or greater than the sum of the other two sides.

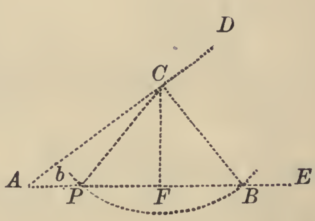
PROBLEM XII.

228. *Given two sides of a triangle and the angle opposite one of them, to construct the triangle.*

Let m and n be the given sides, and a the given $\angle$ opposite the shorter side m.

I. *When the given angle is acute, and the side opposite is less than the other given side.*





Construct $\angle\ b = \angle\ a$, and on AD cut off $AC = n$.

With C as a centre and m as a radius, describe an arc cutting the side AE at B and P.

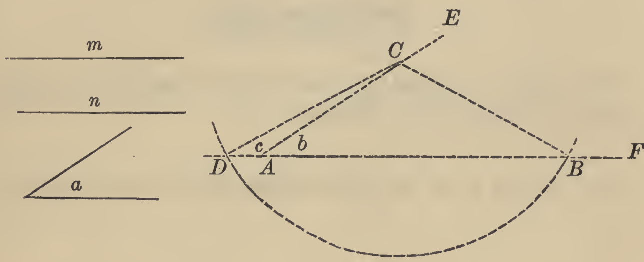
Draw CB and CP.

Then either ABC or APC is the required $\triangle$, and there are two solutions to the problem.

When m equals the $\perp$ CF, there is but one construction.

When m is less than CF, the problem is not possible.

II. *When the given angle is acute, right, or obtuse, and the side opposite is greater than the other given side.*



When the given $\angle\ a$ is acute, construct $\angle\ b = \angle\ a$, and cut off on the side AE, $AC = n$.

With C as a centre and m, the greater side, as a radius, describe an arc cutting the side AF at B, and AF produced at D.

Draw CB and CD.

Then ABC is the required $\triangle$; and there is but one solution. Q. E. F.

When the given $\angle$ is obtuse, as c, the $\triangle\ ACD$ is the required one; and there is but one solution.

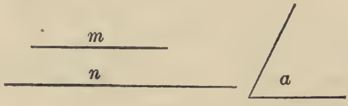
When the given $\angle$ is a $\llcorner$, the problem has two solutions. Let the pupil give the construction.

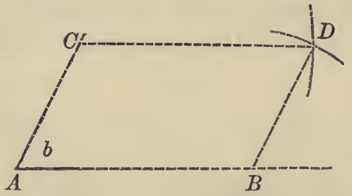
229. Scholium.—The problem is not possible if the given angle is right or obtuse, and the side opposite is less than the other given side.

PROBLEM XIII.

230. *Given two sides and the included angle of a parallelogram, to construct the parallelogram.*

Let m and n be the given sides, and a the included $\angle$.





Construct $\angle\ b = \angle\ a$, and on the sides cut off AC and AB respectively equal to m and n.

With B as a centre and AC as a radius, describe an arc; with C as a centre and AB as a radius, describe an arc cutting the other at D.

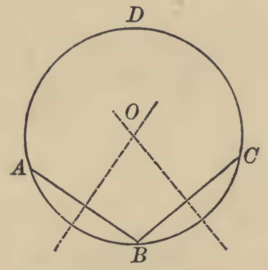
Draw BD and CD.

Then $ABCD$ is the required ▱. (132) Q. E. F.

PROBLEM XIV.

231. *To find the centre of a given circle.*

Let $ABCD$ be a given ○.



From any point in the circumference, as B, draw two chords, AB and BC.

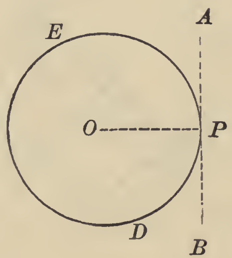
Bisect AB and BC by $\perp$s as in (215).

The point O, the intersection of the $\perp$s, is the required centre. (191) Q. E. F.

PROBLEM XV.

232. *At a given point in the circumference of a circle, to draw a tangent to the circle.*

Let P be the given point in the circumference of the ○ EDP.



If the centre is not given, find it by (231).

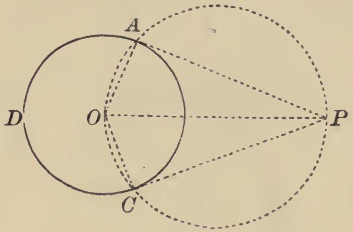
Draw the radius OP, and at P draw $AB \perp$ to OP.

Then AB is the required tangent. (194) Q. E. F.

PROBLEM XVI.

233. *Through a given point without a given circle, to draw a tangent to the circle.*

Let O be the centre of the given ○, and P the given point.



Draw OP, and upon it describe a circumference cutting the given circumference at A and C.

Draw AP and CP; also the radii OA and OC.

$$\angle\ OAP = \text{a} \llcorner; \tag{208}$$

$\therefore$ AP is $\perp$ to OA, and is tangent to the ○. (194)

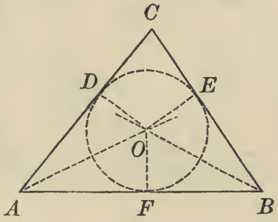
Likewise we can prove that CP is a tangent. Q. E. F.

234. Cor.—*From any point without a given circle, two equal tangents to the circle can be drawn.*

PROBLEM XVII.

235. *To inscribe a circle in a given triangle.*

Let ABC be the given $\triangle$.



Bisect the $\angle$s ABC and CAB.

The bisectors meet in some point, as O.

From O draw OF, OD, and OE, $\perp$ to the sides of the $\triangle$.

The $\perp$s are all equal. (104)

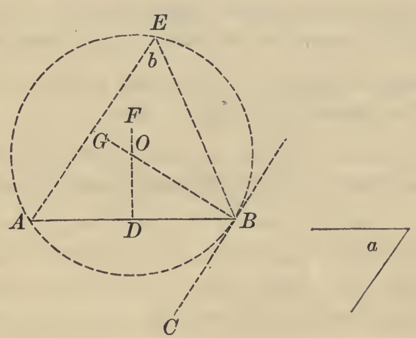
With O as a centre and OF as a radius, describe a circle.

Then FED is the required $\bigcirc$. (181) Q. E. F.

PROBLEM XVIII.

236. *On a given straight line, to describe a segment of a circle which shall contain a given angle.*

Let AB be the given line, and a the given $\angle$.



At B construct the $\angle\ ABC = \angle\ a$.

Draw $GB \perp$ to BC at B.

Bisect AB, and at its middle point erect the $\perp$ DF.

With O, the intersection of DF and GB, as a centre and OB as radius, describe a circumference.

Now, BC is $\perp$ to the radius OB;

$\therefore$ BC is a tangent to the circle. (194)

The $\angle\ ABC$ is measured by $\frac{1}{2}$ arc AB. (211)

But $\angle\ b$, the $\angle$ inscribed in the segment AEB, is measured by $\frac{1}{2}$ arc AB; (206)

$\therefore$ AEB is the required segment. Q.E.F.

EXERCISES IN INVENTION.

THEOREMS.

1. In any circumscribed quadrilateral, the sum of two opposite sides is equal to the sum of the other two sides.

2. A quadrilateral is inscriptible if two of its opposite angles are supplements of each other.

3. The bisectors of the angles formed by producing the opposite sides of an inscribed quadrilateral intersect at right angles.

4. If a circle is described on the radius of another circle, any straight line drawn from the point of contact to the outer circumference is bisected by the interior one.

PROBLEMS.

1. To trisect a right angle.

2. Given two lines that would meet if sufficiently produced, to find the bisector of their included angle without finding its vertex.

3. To draw a common tangent to two given circles.

4. Inscribe a square in a given rhombus.

5. To construct a square, given its diagonal.

6. Construct an angle of 30°, one of 60°, one of 120°, one of 150°, one of 45°, and one of 135°.

7. Construct a triangle, given the base, the angle opposite the base, and the medial line to the base.

8. Construct a triangle, given the vertical angle, and the radius of the circumscribing circle.

9. Construct a triangle, given the base, the vertical angle, and the perpendicular from the extremity of the base to the opposite side.

10. Construct a triangle, given the base, an angle at the base, and the sum or difference of the other two sides.

11. Construct a square, given the sum or difference of its diagonal and side.

12. Describe a circle cutting the sides of a square, so as to divide the circumference at the points of intersection into eight equal arcs.

13. Through any point within a circle, except the centre, to draw a chord which shall be bisected at that point.

BOOK IV.

AREA AND RELATION OF POLYGONS.

DEFINITIONS.

237. ***Similar Polygons*** are polygons which are mutually equiangular, and have their homologous sides proportional.

238. The ***Area*** of a polygon is its quantity of surface; it is expressed by the number of times the polygon contains some other area taken as a unit of measure. The unit of measure usually assumed is a square, a side of which is some linear unit; as, a *square inch*, a *square foot*, etc.

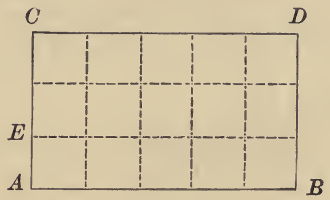
239. ***Equivalent Figures*** are such as have equal areas.

AREAS.

THEOREM I.

240. *The area of a rectangle equals the product of its base and altitude.*

Let $ABCD$ be a rectangle, AB the base, and AC the altitude.



To prove that the area of $ABCD = AB \times AC$.

Let AE be a common unit of measure of the sides AB and AC, and suppose it to be contained in AB 5 times, and in AC 3 times.

Apply AE to AB and AC, dividing them respectively into five and three equal parts.

Through the several points of division draw $\perp$s to the sides.

The rectangle will then be divided into equal squares, as the angles are all $\llcorner$s, and the sides all equal. (130)

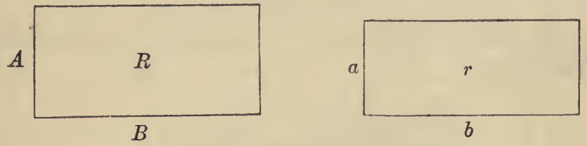
Now, the whole number of these squares is equal to the number in the row on AB multiplied by the number of rows, or the number of linear units in AB multiplied by the number in AC.

Now, this is true, whatever may be the length of the common unit of measure; hence it is true if it is infinitely small, as is the case when the sides are incommensurable. Therefore, in any case, the proposition is true. Q.E.D.

THEOREM II.

241. *Rectangles are to each other as the products of their bases and altitudes.*

Let R and r denote the areas of the rectangles whose bases are B and b, and whose altitudes are A and a.



To prove that $R : r :: A \times B : a \times b$.

$$R = A \times B,$$

and $$r = a \times b; \qquad (240)$$

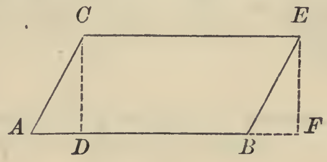
$\therefore \quad R : r :: A \times B : a \times b.$ (159) Q. E. D.

242. Cor.—*Rectangles having equal bases are to each other as their altitudes; rectangles having equal altitudes are to each other as their bases.*

THEOREM III.

243. *The area of a parallelogram equals the product of its base and altitude.*

Let $ABEC$ be a $\square$, AB its base, and CD its altitude.



To prove that the area of $\square\ ABEC = AB \times CD$.

Construct the rectangle $DFEC$ having the same base and altitude as the $\square$.

$$AC = BE, \text{ and } DC = FE;$$

$\therefore$ $$\text{R}\triangle\ ACD = \text{R}\triangle\ BEF. \qquad (91)$$

Remove $\triangle\ ACD$, and rectangle $DFEC$ remains.

Remove $\triangle\ BEF$, and $\square\ ABEC$ remains;

$\therefore$ rectangle $DFEC = \square\ ABEC$.

But the area of the rectangle $DFEC = AB \times CD$;

$\therefore$ the area of the $\square\ ABEC = AB \times CD$. Q. E. D.

244. Cor. 1.—*Parallelograms are to each other as the products of their bases and altitudes.*

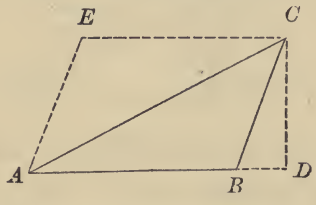
245. Cor. 2.—*Parallelograms having equal bases are to each other as their altitudes; parallelograms having equal altitudes are to each other as their bases.*

246. Cor. 3.—*Parallelograms having equal bases and equal altitudes are equivalent figures.*

THEOREM IV.

247. *The area of a triangle equals half the product of its base and altitude.*

Let ABC be a $\triangle$, AB its base, CD its altitude.



To prove that the area of the $\triangle\ ABC = \frac{1}{2}(AB \times CD)$.

Through C draw $CE \parallel$ to AB, and through A draw $AE \parallel$ to BC.

The $\triangle\ ABC$ is half of the $\square\ ABCE$; (129)

but the area of the $\square\ ABCE = AB \times CD$; (243)

$\therefore$ the area of the $\triangle\ ABC = \frac{1}{2}(AB \times CD)$. Q. E. D.

248. Cor. 1.—*Triangles are to each other as the products of their bases and altitudes.*

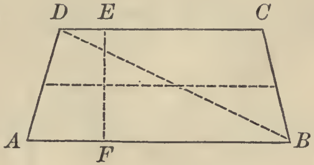
249. Cor. 2.—*Triangles having equal bases are to each other as their altitudes; triangles having equal altitudes are to each other as their bases.*

250. Cor. 3.—*Triangles having equal bases and equal altitudes are equivalent figures.*

THEOREM V.

251. *The area of a trapezoid is equal to half the sum of its parallel sides multiplied by its altitude.*

Let $ABCD$ be a trapezoid, AB and DC its ∥ sides, and EF its altitude.



To prove that the area of $ABCD = \frac{1}{2}(AB + DC)\,EF$.

Draw the diagonal BD, forming △s ABD and CDB.

The area of the △ $ABD = \frac{1}{2} AB \times EF$,

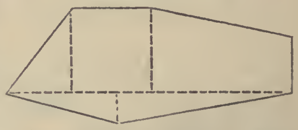
and the area of the △ $CDB = \frac{1}{2} DC \times EF$; (247)

∴ △ ABD + △ CDB = area of $ABCD = \frac{1}{2}(AB + DC)\,EF$.

Q. E. D.

252. Cor.—*The area of a trapezoid is equal to the product of the line joining the middle points of its non-parallel sides and its altitude.*

253. Scholium.—The area of an irregular polygon is found by finding the areas of the several triangles into which it can be divided. In surveying, the method usually resorted to is to draw the longest diagonal, and to draw perpendiculars to this diagonal from the other vertices, thus dividing the polygon into rectangles, right-angled triangles, and trapezoids. The areas of these figures are then readily found.

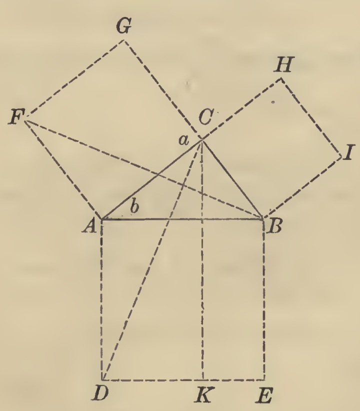


SQUARES ON LINES.

THEOREM VI.

254. *The square described on the hypothenuse of a right-angled triangle is equivalent to the sum of the squares on the other two sides.*

Let ABC be a R$\triangle$, right-angled at C.



To prove that $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$.

On AB, AC, and BC, construct the squares AE, AG, and BH.

Through C draw $CK \parallel$ to BE, and draw BF and CD.

$\angle ACB =$ a $\llcorner$ (Hyp.), and $\angle a =$ a $\llcorner$; (Cons.)

$\therefore$ GCB is a straight line.

For a like reason ACH is a straight line.

$$AB = AD, \text{ and } AF = AC, \qquad \text{(Cons.)}$$

and $\angle FAB = \angle CAD$, each being a $\llcorner + \angle b$;

$$\therefore \qquad \triangle ABF = \triangle ACD. \qquad (83)$$

Square AG and $\triangle ABF$ have a common base AF and a common altitude;

$\therefore$ Square AG is double $\triangle ABF$; (240) and (247)

and for a like reason

$$\square AK \text{ is double } \triangle ACD.$$

But $\triangle ACD = \triangle ABF$;

$$\therefore \qquad \square AK = \text{square } AG.$$

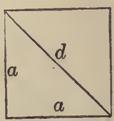
Likewise we can prove $\square BK =$ square BH;

$\therefore$ $\square AK + \square BK = \overline{AB}^2 = \overline{AC}^2 \times \overline{BC}^2$. (Q. E. D.)

255. Cor. 1.—*The square on either side about a right-angled triangle is equivalent to the square on the hypothenuse minus the square on the other side.*

256. Cor. 2.—*The square on the diagonal of a square is double the given square.*

257. Cor. 3.—*The diagonal and the side of a square are incommensurable.*



Let d be the diagonal, and a the side of a square.

Then $d^2 = a^2 + a^2 = 2\,a^2$.

Extract the square root of each member.

Then $d = a\sqrt{2}$.

Divide by a;

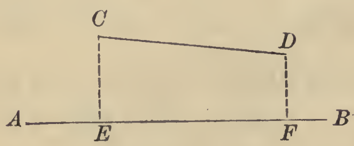
then $$\frac{d}{a} = \sqrt{2} = 1.41421+.$$

PROJECTION.

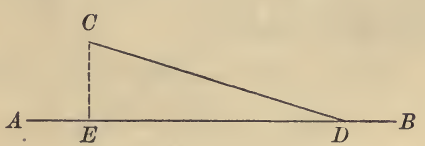
DEFINITIONS.

258. The ***Projection of a Point*** upon an indefinite straight line is the foot of the perpendicular drawn from the point to the line.

Thus, the projection of the point C upon the line AB is the point E.



The ***Projection of a Finite Straight Line*** upon an indefinite one, is the part of the line intercepted between the perpendiculars drawn from the extremities of the finite line. Thus, EF is the projection of CD upon AB.

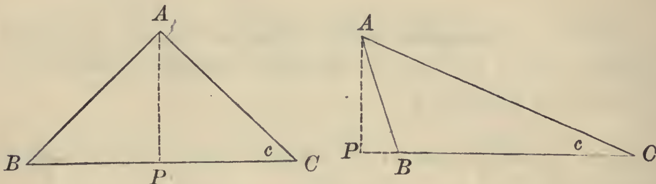


If one extremity of CD rests upon the other line AB, then the projection of CD is ED.

THEOREM VII.

259. *In any triangle, the square on the side opposite an acute angle equals the sum of the squares of the other two sides minus twice the product of one of those sides and the projection of the other upon that side.*

In the $\triangle ABC$, let c be an acute $\angle$, and PC the projection of AC upon BC.



To prove that $\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 - 2\,BC \times PC$.

If P falls on the base,

$$PB = BC - PC.$$

If P falls on the base produced,

$$PB = PC - BC.$$

In either case, by Algebra,

$$\overline{PB}^2 = \overline{BC}^2 + \overline{PC}^2 - 2\,BC \times PC.$$

Add $\overline{AP}^2$ to each member;

then $\overline{AP}^2 + \overline{PB}^2 = \overline{BC}^2 + \overline{AP}^2 + \overline{PC}^2 - 2\,BC \times PC.$

But $\overline{AB}^2 = \overline{AP}^2 + \overline{PB}^2,$

and $\overline{AC}^2 = \overline{AP}^2 + \overline{PC}^2.$ (254)

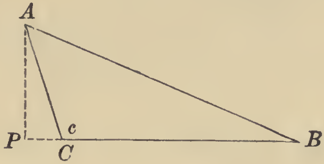
Substitute $\overline{AB}^2$ and $\overline{AC}^2$ for their equals;

then $\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 - 2\,BC \times PC.$ Q. E. D.

THEOREM VIII.

260. *In any obtuse-angled triangle, the square on the side opposite the obtuse angle equals the sum of the squares of the other two sides plus twice the product of one of those sides and the projection of the other upon that side.*

In the $\triangle ABC$, let c be the obtuse $\angle$, and PC the projection of AC upon BC produced.



To prove that $\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 + 2\ BC \times PC.$

$$PB = BC + PC.$$

By Algebra, $\overline{PB}^2 = \overline{BC}^2 + \overline{PC}^2 + 2\ BC \times PC.$

Add $\overline{AP}^2$ to each member.

Then $\overline{AP}^2 + \overline{PB}^2 = \overline{BC}^2 + \overline{AP}^2 + \overline{PC}^2 + 2\ BC \times PC.$

But $\overline{AB}^2 = \overline{AP}^2 + \overline{PB}^2,$

and $$\overline{AC}^2 = \overline{AP}^2 + \overline{PC}^2; \qquad (254)$$

Substitute $\overline{AB}^2$ and $\overline{AC}^2$ for their equals.

Then $\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 + 2\ BC \times PC.$ Q. E. D.

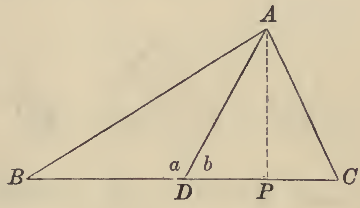
THEOREM IX.

261. *In any triangle, if a medial line is drawn to the base:*

I.—*The sum of the squares of the two sides equals twice the square of half the base plus twice the square of the medial line.*

II.—*The difference of the squares of the two sides equals twice the product of the base and the projection of the medial line upon the base.*

In the $\triangle ABC$, let AD be the medial line, and DP the projection of AD upon the base BC.



To prove

$$\text{I.—That } \overline{AB}^2 + \overline{AC}^2 = 2\,\overline{BD}^2 + 2\,\overline{AD}^2.$$

$$\text{II.—That } \overline{AB}^2 - \overline{AC}^2 = 2\,BC \times DP.$$

If $AB > AC$, $\angle\ a$ is obtuse, and $\angle\ b$ is acute.

Then $\overline{AB}^2 = \overline{BD}^2 + \overline{AD}^2 + 2\,BD \times DP,$ (260)

and $\overline{AC}^2 = \overline{DC}^2 + \overline{AD}^2 - 2\,DC \times DP.$ (259)

Add these equations, observing that $BD = DC$.

Then $\overline{AB}^2 + \overline{AC}^2 = 2\,\overline{BD}^2 + 2\,\overline{AD}^2.$

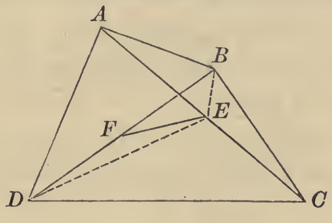
Subtract the second equation from the first.

Then $\overline{AB}^2 - \overline{AC}^2 = 2\,BC \times DP.$ Q. E. D.

THEOREM X.

262. *The sum of the squares of the sides of any quadrilateral equals the sum of the squares of the diagonals plus four times the square of the line joining the middle points of the diagonals.*

In the quadrilateral $ABCD$, let EF be the line joining the middle points of the diagonals BD and AC.



To prove that

$$\overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 + \overline{DA}^2 = \overline{AC}^2 + \overline{BD}^2 + 4\,\overline{EF}^2.$$

Draw DE and BE.

$$\overline{AB}^2 + \overline{BC}^2 = 2\,\overline{AE}^2 + 2\,\overline{BE}^2,$$

and $$\overline{CD}^2 + \overline{DA}^2 = 2\,\overline{AE}^2 + 2\,\overline{DE}^2. \qquad (261)$$

Add these equations.

Then $\overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 + \overline{DA}^2 = 4\,\overline{AE}^2 + 2\,(\overline{BE}^2 + \overline{DE}^2)$

Now, $\overline{BE}^2 + \overline{DE}^2 = 2\,\overline{BF}^2 + 2\,\overline{EF}^2$; (261)

$\therefore\ \overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 + \overline{DA}^2 = 4\,\overline{AE}^2 + 4\,\overline{BF}^2 + 4\,\overline{EF}^2.$

But $$4\,\overline{AE}^2 = (2\,AE)^2 = \overline{AC}^2,$$

and $$4\,\overline{BF}^2 = (2\,BF)^2 = \overline{BD}^2;$$

$\therefore$ substituting, $\overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 + \overline{DA}^2 = \overline{AC}^2 + \overline{BD}^2 + 4\,\overline{EF}^2$. Q. E. D.

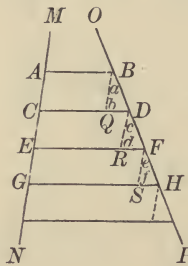
263. Cor.—*In any parallelogram, the sum of the squares of the sides equals the sum of the squares of the diagonals.*

PROPORTIONAL LINES.

THEOREM XI.

264. *If a number of parallels cutting two straight lines, intercept equal parts on one of the lines, they also intercept equal parts on the other.*

Let the ∥s AB, CD, EF, GH, intersect MN and OP, intercepting on MN equal parts AC, CE, EG.



To prove that the ∥s intercept on OP equal parts BD, DF, FH.

Through the points B, D, F, draw BQ, DR, FS, ∥ to MN.

$$AC = BQ,\ CE = DR, \text{ and } EG = FS. \qquad (130)$$

But $\quad AC = CE = EG;$ (Hyp.)

$\therefore \quad BQ = DR = FS.$

Now, in the △s BQD, DRF, and FSH,

$$\angle a = \angle c = \angle e, \text{ and } \angle b = \angle d = \angle f; \qquad (62)$$

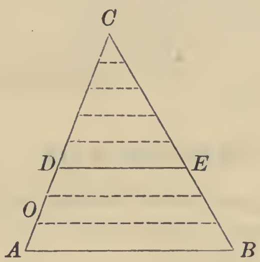
$\therefore \quad \triangle BQD = \triangle DRF = \triangle FSH,$ (84)

and $\quad BD = DF = FH.$ Q. E. D.

THEOREM XII.

265. *A line parallel to the base of a triangle divides the other two sides proportionally.*

In the $\triangle ABC$, let the line DE be parallel to AB.



To prove that $CA : CD :: CB : CE.$

Suppose CA and CD to have a common unit of measure, as AO, and suppose it to be contained in CA eight times, and in CD five time.

Then $CA : CD :: 8 : 5.$

Draw $\parallel$s to AB through the points of division on CA.

Then CB is divided into eight equal parts, (264)

and $CB : CE :: 8 : 5.$

But $CA : CD :: 8 : 5;$

$\therefore$ $CA : CD :: CB : CE.$ (164)

Now, this is true, whatever may be the length of the common unit of measure; hence it is true if it is infinitely small, as is the case when the lines are incommensurable. Hence in any case the proposition is true. Q. E. D.

266. Cor. 1.—By (163), the proportion becomes

$$CA - CD : CA :: CB - CE : CB,$$

or $$DA : CA :: EB : CB.$$

267. Cor. 2.—By (161), the last proportion gives

$$CA : DA :: CB : EB, \text{ which by } \qquad (163)$$

gives $$CA - DA : DA :: CB - EB : EB,$$

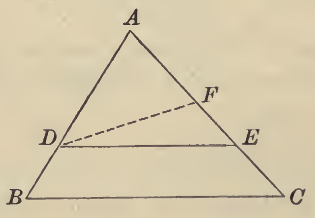
or $$CD : DA :: CE : EB;$$

$\therefore$ $$CD : CE :: DA : EB. \qquad (160)$$

THEOREM XIII.

268. *A straight line which divides two sides of a triangle proportionally is parallel to the third side.*

In the $\triangle ABC$, let DE divide the sides AB and AC proportionally.



To prove that DE is $\parallel$ to BC.

Suppose DF to be drawn $\parallel$ to BC.

Then $AB : AD :: AC : AF$. (265)

But $AB : AD :: AC : AE$; (Hyp.)

$\therefore$ $AC : AF :: AC : AE$; (164)

$\therefore$ $AF = AE$, which cannot be unless DF coincides with DE;

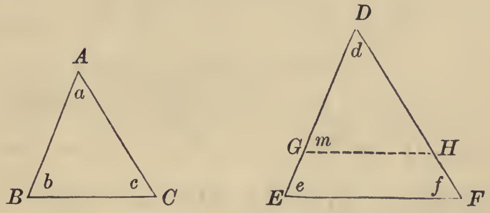
$\therefore$ DE is $\parallel$ to BC. Q.E.D.

SIMILARITY OF POLYGONS.

THEOREM XIV.

269. *Two triangles mutually equiangular are similar.*

Let the $\angle$s a, b, c, of the $\triangle$ ABC be respectively equal to the $\angle$s d, e, f, of the $\triangle$ DEF.



To prove that $\triangle$s *ABC and DEF are similar.*

Place the $\triangle$ ABC on $\triangle$ DEF so as to make $\angle$ a coincide with its equal $\angle$ d.

Then $\triangle$ ABC takes the position DGH.

Since $\angle\ b = \angle\ m = \angle\ e$, GH is $\parallel$ to EF; (64)

$\therefore$ $$DE : DG :: DF : DH,$$

or $$DE : AB :: DF : AC. \quad (265)$$

Likewise we can prove that

$$DE : AB :: EF : BC.$$

$\therefore$ the $\triangle$s are similar. (237) Q. E. D.

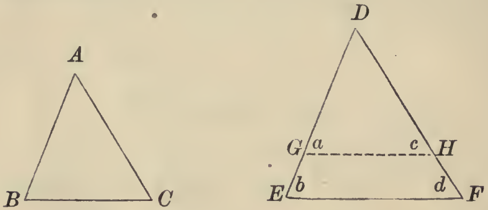
270. Cor.—*Two triangles are similar if two angles of the one are respectively equal to two angles of the other; two right-angled triangles are similar if one has an acute angle equal to an acute angle of the other.*

THEOREM XV.

271. *Two triangles whose homologous sides are proportional are similar.*

In the $\triangle$s ABC and DEF let

$$DE : AB :: DF : AC :: EF : BC.$$



To prove that $\triangle$s ABC and DEF are similar.

On DE and DF, cut off DG and DH respectively equal to AB and AC, and draw GH.

Since $DG = AB$, and $DH = AC$,

$DE : DG :: DF : DH$; (Hyp.)

$\therefore$ GH is $\parallel$ to EF. (268)

$\angle a = \angle b$, and $\angle c = \angle d$; (62)

$\therefore$ $\triangle$s DGH and DEF are similar, (269)

and $DE : DG :: EF : GH$; (237)

also $DE : AB :: EF : BC$. (Hyp.)

Of these two proportions take the forms

$$\frac{DE}{DG} = \frac{EF}{GH}, \text{ and } \frac{DE}{AB} = \frac{EF}{BC}.$$

Divide; then $$\frac{AB}{DG} = \frac{BC}{GH}.$$

But $DG = AB$;

$\therefore$ $GH = BC$;

$\therefore$ $\triangle DGH = \triangle ABC$. (86)

But $\triangle$s DGH and DEF are similar;

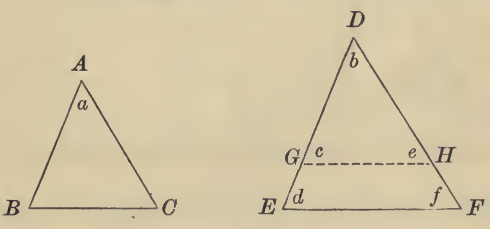
$\therefore$ $\triangle$s ABC and DEF are similar. Q. E. D.

THEOREM XVI.

272. *Two triangles having an angle in each equal and the including sides proportional, are similar.*

In the $\triangle$s ABC and DEF, let

$\angle a = \angle b$, and $DE : AB :: DF : AC$.



To prove that $\triangle$s ABC and DEF are similar.

Place $\triangle ABC$ on $\triangle DEF$ so that $\angle a$ coincides with $\angle b$.

B falls somewhere on DE, as at G, and C falls somewhere on DF, as at H.

Then $DE : DG :: DF : DH$; (Hyp.)

$\therefore$ GH is $\parallel$ to EF. (265)

$\angle c = \angle d$, and $\angle e = \angle f$; (62)

$\therefore$ $\triangle$s DGH and DEF are similar. (269)

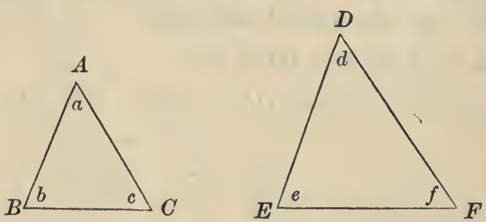
But $\triangle DGH = \triangle ABC$;

$\therefore$ $\triangle$s ABC and DEF are similar. Q. E. D.

THEOREM XVII.

273. *Two triangles having their sides respectively parallel are similar.*

In the $\triangle$s ABC and DEF, let AB, AC, and BC be respectively $\parallel$ to DE, DF, and EF.



To prove that $\triangle$s ABC and DEF *are similar.*

Since the sides are $\parallel$, the corresponding $\angle$s are either equal or supplements of each other. (68) and (69)

$\therefore$ three suppositions can be made:

(1) $\angle a + \angle d = 2$ ∟s, $\angle b + \angle e = 2$ ∟s, $\angle c + \angle f = 2$ ∟s.

(2) $\angle a = \angle d$, $\angle b + \angle e = 2$ ∟s, $\angle c + \angle f = 2$ ∟s.

(3) $\angle a = \angle d$, $\angle b = \angle e$; $\therefore \angle c = \angle f$.

Now, the sum of all the $\angle$s of two $\triangle$s *cannot exceed* 4 ∟s; (76)

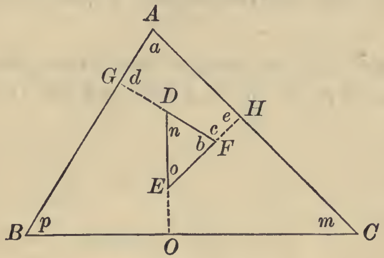
$\therefore$ the third supposition is the only one admissible;

$\therefore$ $\triangle$s ABC and DEF are similar. (269) Q. E. D.

THEOREM XVIII.

274. *Two triangles having their sides respectively perpendicular to each other are similar.*

In the $\triangle$s ABC and DEF, let the sides AB, AC, and BC be respectively $\perp$ to DF, FE, and DE.



To prove that $\triangle$s ABC *and* DEF *are similar.*

Prolong the sides of the $\triangle$ DEF till they meet the sides of the $\triangle$ ABC.

The sum of the $\angle$s of the quadrilateral $AGFH$ equals 4 $\llcorner$s. (115)

But $\angle$s d and e are $\llcorner$s. (Hyp.)

$\therefore$ $\angle a + \angle c = 2 \llcorner$s.

But $\angle b + \angle c = 2 \llcorner$s, (44)

$\therefore$ $\angle b = \angle a$.

Likewise we can prove

$$\angle o = \angle m, \text{ and } \angle n = \angle p.$$

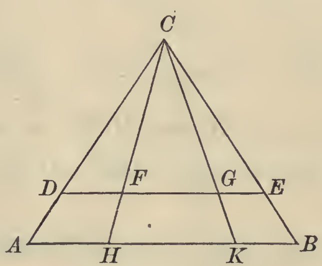
$\therefore$ the $\triangle$s are similar. (269) Q. E. D.

275. Scholium.—In two triangles whose sides are respectively parallel, or perpendicular, the homologous sides are the parallel sides, or the perpendicular sides.

THEOREM XIX.

276. *Straight lines drawn from the vertex of a triangle to the base divide the base and its parallel proportionally.*

Let CH and CK be straight lines drawn from the vertex C to the base AB of the $\triangle ABC$, and let DE be parallel to the base.



To prove that

$$AH : DF :: HK : FG :: KB : GE.$$

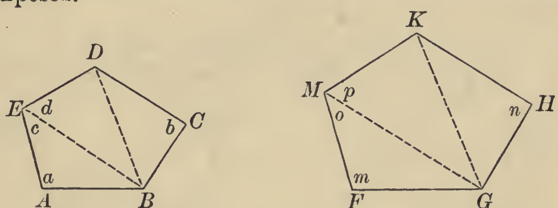
$\triangle$s CAH, CHK, and CKB are respectively similar to $\triangle$s CDF, CFG, and CGE. (269)

$\therefore AH : DF :: CH : CF :: HK : FG :: CK : CG$
$:: KB : GE.$ (237) Q. E. D.

THEOREM XX.

277. *Two polygons are similar if they are composed of the same number of triangles similar each to each and similarly placed.*

Let the $\triangle$s ABE, EBD, and DBC, of which the polygon $ABCDE$ is composed, be respectively similar to the $\triangle$s FGM, MGK, and KGH, of which the polygon $FGHKM$ is composed.



To prove that polygons $ABCDE$ and $FGHKM$ are similar.

$$\angle a = \angle m, \text{ and } \angle b = \angle n. \qquad (237)$$

$$\angle c = \angle o,$$

and $$\angle d = \angle p. \qquad (237)$$

Add these two equations.

Then $$\angle c + \angle d = \angle o + \angle p,$$

or $$\angle AED = \angle FMK.$$

Likewise we can prove $\angle EDC = \angle MKH$,

and $$\angle ABC = \angle FGH;$$

$\therefore$ the polygons are mutually equiangular.

Also $AB : FG :: AE : FM :: EB : MG :: ED : MK$, etc.; (237)

$\therefore$ the polygons have their homologous sides proportional.

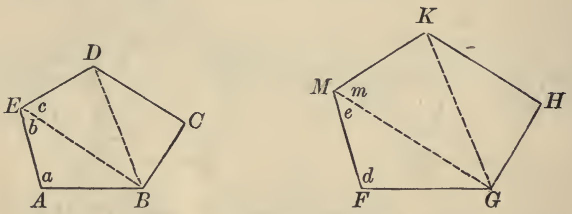
$\therefore$ polygons $ABCDE$ and $FGHKM$ are similar. (237)

Q. E. D.

THEOREM XXI.

278. *Two similar polygons can be divided into the same number of triangles similar each to each and similarly placed.*

Let the polygon $ABCDE$ be similar to the polygon $FGHKM$, and let diagonals be drawn from the vertices B and G.



To prove that the $\triangle$s ABE, EBD, *and* DBC *are respectively similar to the* $\triangle$s FGM, MGK, *and* KGH.

$\angle a = \angle d$, and $AB : FG :: AE : FM$; (237)

$\therefore$ $\triangle ABC$ is similar to $\triangle FGM$. (272)

Also $\angle b + \angle c = \angle e + \angle m$. (237)

But $\angle b = \angle e$;

$\therefore$ $\angle c = \angle m$.

And $EB : MG :: AE : FM :: ED : MK$; (237)

$\therefore$ $\triangle EBD$ is similar to $\triangle MGK$. (272)

Likewise we can prove that $\triangle DBC$ is similar to $\triangle KGH$.

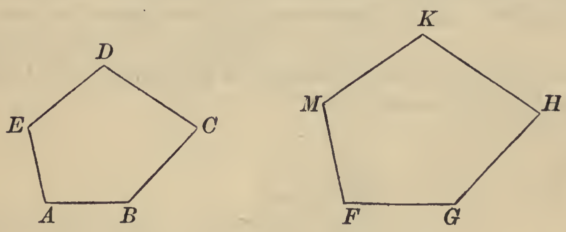
Q. E. D.

279. Cor.—*Two similar polygons can be divided into the same number of triangles similar each to each and similarly placed, by drawing lines to their vertices from any two homologous points.*

THEOREM XXII.

280. *The perimeters of two similar polygons are to each other as any two homologous sides.*

Let the polygon $ABCDE$ be similar to the polygon $FGHKM$, and let p and P denote their perimeters.



To prove that $p : P :: AB : FG$.

$$AB : FG :: BC : GH :: CD : HK, \text{ etc.}; \quad (237)$$

$$\therefore AB + BC + DC, \text{ etc.} : FG + GH + HK, \text{ etc.} :: AB : FG, \quad (168)$$

or $$p : P :: AB : FG. \qquad \text{Q.E.D.}$$

281. Cor.—*The perimeters of two similar polygons are to each other as any two homologous lines.*

THEOREM XXIII.

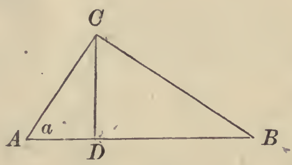
282. *If a perpendicular is drawn from the vertex of the right angle of a right-angled triangle to the hypothenuse:*

I. *The two triangles formed are similar to the given triangle and similar to each other.*

II. *The perpendicular is a mean proportional between the segments of the hypothenuse.*

III. *Each side of the right-angled triangle is a mean proportional between the hypothenuse and the adjacent segment.*

Let ABC be a R$\triangle$, right-angled at C, and let CD be the $\perp$ to the hypothenuse AB.



I. *To prove that the* $\triangle$s ACD, ABC, *and* CBD *are similar to each other.*

$\angle$ a is common to the R$\triangle$s ACD and ABC;

$\therefore$ $\triangle$ ACD and $\triangle$ ABC are similar. (270)

For a like reason $\triangle$s CBD and ABC are similar, and hence similar to ACD.

II. *To prove that* $AD : CD :: CD : DB$.

The $\triangle$s ACD and CBD are similar;

$\therefore$ $AD : CD :: CD : DB$. (237)

III. *To prove that* $AB : AC :: AC : AD$,

and that $AB : BC :: BC : BD$.

The $\triangle$s ACD and ABC are similar;

$\therefore \qquad AB : AC :: AC : AD, \qquad (237)$

And the $\triangle$s CBD and ABC are similar;

$\therefore \qquad AB : BC :: BC : BD. \qquad (237)$ Q. E. D.

283. Cor. 1.—*The squares of the sides about the right angle are to each other as the adjacent segments of the hypothenuse.*

From the last two proportions we have

$$\overline{AC}^2 = AB \times AD,$$

and

$$\overline{BC}^2 = AB \times BD. \qquad (156)$$

Divide, and cancel the common factor AB.

Then

$$\frac{\overline{AC}^2}{\overline{BC}^2} = \frac{AD}{BD},$$

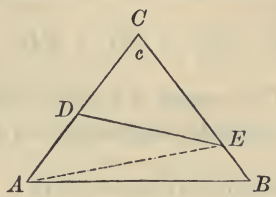
or

$$\overline{AC}^2 : \overline{BC}^2 :: AD : BD.$$

THEOREM XXIV.

284. *Two triangles having an angle in each the same are to each other as the products of the sides including the equal angles.*

In the $\triangle$s ABC and DEC let the angle c be common.



To prove that $\triangle ABC : \triangle DEC :: CA \times CB : CD \times CE.$

Draw the line AE.

$$\triangle ABC : \triangle AEC :: CB : CE,$$

and $$\triangle AEC : \triangle DEC :: CA : CD. \qquad (249)$$

Take the forms

$$\frac{\triangle ABC}{\triangle AEC} = \frac{CB}{CE},$$

and $$\frac{\triangle AEC}{\triangle DEC} = \frac{CA}{CD}.$$

Multiply, and cancel the common factor.

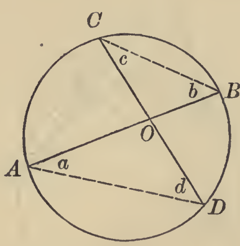
Then $$\frac{\triangle ABC}{\triangle DEC} = \frac{CA \times CB}{CD \times CE},$$

or $\triangle ABC : \triangle DEC :: CA \times CB : CD \times CE.$ Q.E.D.

THEOREM XXV.

285. *If two chords of a circle intersect, their segments are reciprocally proportional.*

Let the chords AB and CD intersect at O.



To prove that $CO : AO :: BO : DO$.

Draw AD and CB.

$\angle\ c = \angle\ a$, and $\angle\ b = \angle\ d$; (206)

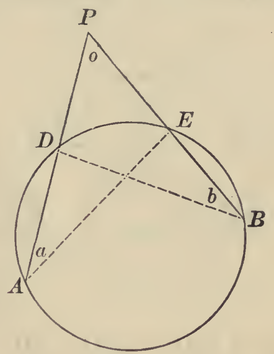
$\therefore$ $\triangle$s COB and AOD are similar, (269)

and $CO : AO :: BO : DO$. (237) Q.E.D.

THEOREM XXVI.

286. *If two secants are drawn from a point without a circle, the secants and their external segments are reciprocally proportional.*

Let the secants PA and PB be drawn from the point P without the ○.



To prove that $PA : PB :: PE : PD$.

Draw AE and BD.

$\angle a = \angle b$, (206)

and $\angle o = \angle o$;

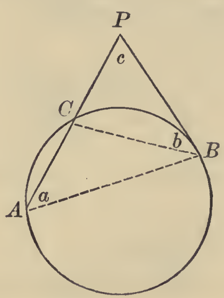
$\therefore$ $\triangle$s PAE and PBD are similar, (270)

and $PA : PB :: PE : PD$. (237) Q. E. D.

THEOREM XXVII.

287. *If a tangent and a secant are drawn from a point without a circle, the tangent is a mean proportional between the secant and its external segment.*

Let the tangent PB and the secant PA be drawn from a point P without the $\bigcirc$.



To prove that $PA : PB :: PB : PC$.

Draw AB and BC.

$\angle a$ is measured by $\frac{1}{2}$ arc CB, (206)

and $\angle b$ is measured by $\frac{1}{2}$ arc CB; (211)

$\therefore$ $\angle a = \angle b$.

Also $\angle c = \angle c$.

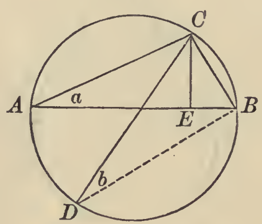
$\therefore$ $\triangle$s PAB and PBC are similar, (270)

and $PA : PB :: PB : PC$. (237) Q. E. D.

THEOREM XXVIII.

288. *In any triangle, the product of two sides equals the product of the diameter of the circumscribed circle and the perpendicular drawn to the third side from the vertex of the opposite angle.*

Let a ○ be circumscribed about the △ ABC, and let CD be a diameter and CE a ⊥ to AB.



To prove that $AC \times CB = CD \times CE$.

Draw DB.

$$\angle a = \angle d; \qquad (206)$$

∴ R△s AEC and CDB are similar, (270)

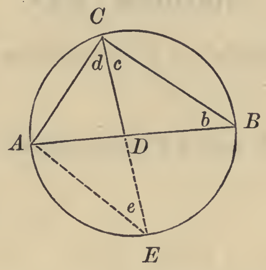
and $AC : CD :: CE : CB$; (237)

∴ $AC \times CB = CD \times CE$. (156) Q. E. D.

THEOREM XXIX.

289. *In any triangle the product of two sides equals the product of the segments of the third side formed by the bisector of the opposite angle plus the square of the bisector.*

Let a ○ be circumscribed about the $\triangle ABC$, and let the $\angle$ opposite AB be bisected by CD.



To prove that $CB \times CA = BD \times DA + \overline{CD}^2$.

Produce CD to E, a point in the circumference, and draw AE.

	$\angle b = \angle e$,	(206)
and	$\angle c = \angle d$;	(Cons.)
$\therefore$	$\triangle$s CBD and CAE are similar,	(270)
and	$CB : CE :: CD : CA$;	(237)
$\therefore$	$CB \times CA = CE \times CD$.	(156)

But $CE \times CD = (DE + CD)\, CD = DE \times CD + \overline{CD}^2$;

$\therefore \quad CB \times CA = DE \times CD + \overline{CD}^2$.

Now, $\quad DE \times CD = BD \times DA$. (285)

Substitute $BD \times DA$ for its equal $DE \times CD$.

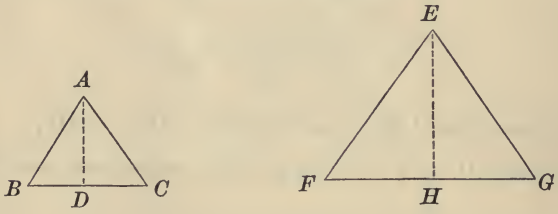
Then $\quad CB \times CA = DB \times DA + \overline{CD}^2$. Q. E. D.

RELATION OF POLYGONS.

THEOREM XXX.

290. *Similar triangles are to each other as the squares of their homologous sides.*

Let the $\triangle$s ABC and EFG be similar.



To prove that $\triangle ABC : \triangle EFG :: \overline{AB}^2 : \overline{EF}^2$.

Draw the altitudes AD and EH.

Then $\triangle ABC : \triangle EFG :: BC \times AD :: FG \times EH$. (248)

But $BC : FG :: AB : EF$,

and $AD : EH :: AB : EF$; (237)

$\therefore$ $BC \times AD : FG \times EH : \overline{AB}^2 : \overline{EF}^2$ (169)

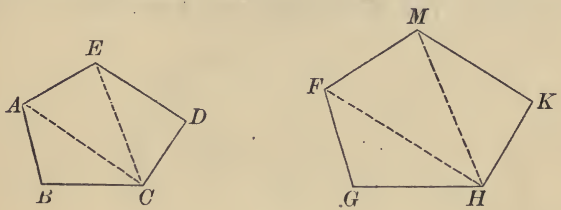
Compare this with the first proportion;

then $\triangle ABC : \triangle EFG :: \overline{AB}^2 : \overline{EF}^2$. (164) Q. E. D.

THEOREM XXXI.

291. *Similar polygons are to each other as the squares of their homologous sides.*

Let the polygons $ABCDE$ and $FGHKM$ be similar, and denote their surfaces by s and S.



To prove that $s : S :: \overline{AB}^2 : \overline{FG}^2$.

Draw the diagonals AC, EC, and FH, MH, dividing the polygons into homologous $\triangle$s. (278)

Then $\triangle ABC : \triangle FGH :: \overline{AC}^2 : \overline{FH}^2$,

and $\triangle ACE : \triangle FHM :: \overline{AC}^2 : \overline{FH}^2$; (290)

$\therefore \quad \triangle ABC : \triangle FGH :: \triangle ACE : \triangle FHM$. (164)

Likewise we get $\triangle ACE : \triangle FHM :: \triangle ECD : \triangle MHK$;

$\therefore \quad \triangle ABC + \triangle ACE + \triangle ECD : \triangle FGH + \triangle FHM + \triangle MHK :: \triangle ABC : \triangle FGH$. (168)

or $s : S :: \triangle ABC : \triangle FGH$.

But $\triangle ABC : \triangle FGH :: \overline{AB}^2 : \overline{FG}^2$; (290)

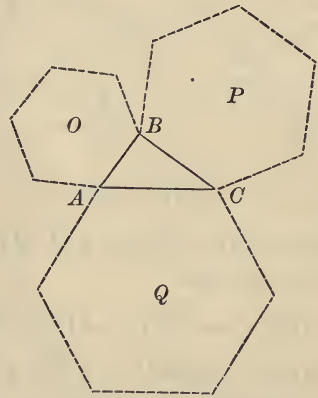
$\therefore \quad s : S :: \overline{AB}^2 : \overline{FG}^2$. (164) Q. E. D.

292. Cor.—*Similar polygons are to each other as the squares of any of their homologous lines.*

THEOREM XXXII.

293. *If similar polygons are constructed on the three sides of a right-angled triangle, the polygon on the hypothenuse is equivalent to the sum of the polygons on the other two sides.*

Let O, P, and Q be the similar polygons constructed respectively on AB, BC, and AC, the three sides of a R△.



To prove that $Q = O + P$.

$$O : P :: \overline{AB}^2 : \overline{BC}^2,$$

and $$Q : P :: \overline{AC}^2 : \overline{BC}^2. \qquad (291)$$

From the first proportion we get

$$O + P : P :: \overline{AB}^2 + \overline{BC}^2 : \overline{BC}^2. \qquad (162)$$

But $$\overline{AB}^2 + \overline{BC}^2 = \overline{AC}^2;$$

$\therefore$ $$O + P : P :: \overline{AC}^2 : \overline{BC}^2.$$

Comparing this with the second proportion, we get

$$Q : P :: O + P : P.$$

Take the form $\frac{Q}{P} = \frac{O + P}{P}$, and multiply each member by P.

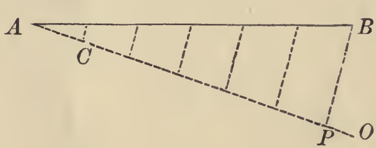
Then $$Q = O + P.$$ Q.E.D.

PROBLEMS IN CONSTRUCTION.

PROBLEM I.

294. *To cut a given straight line into any number of equal parts.*

Let AB be the given straight line.



From one extremity A draw the indefinite straight line AO.

Take any convenient line, as AC, and cut it off on AO as many times as AB is to be cut into equal parts.

Join the last point of division, as P, and the extremity B of the given line.

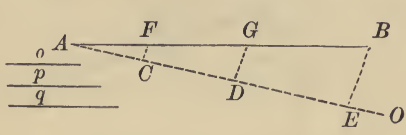
Through all the other points of division on AO, draw $\parallel$s to PB.

Then AB is cut into equal parts. (264) Q. E. F.

PROBLEM II.

295. *To cut a given straight line into parts proportional to given straight lines.*

Let AB, o, p, and q be the given straight lines.



From A draw the indefinite straight line AO.

Cut off $AC = o$, $CD = p$, and $DE = q$.

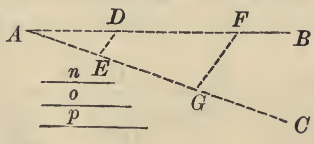
Join EB, and draw DG and $CF \parallel$ to EB.

Then the parts AF, FG, and GB are proportional to o, p, and q. (265) Q. E. F.

PROBLEM III.

296. *To construct a fourth proportional to three given straight lines.*

Let n, o, and p be the three given lines.



Construct any convenient $\angle$, producing the sides AB and AC indefinitely.

Cut off $AD = o$, $AE = n$, and $EG = p$.

Join ED, and draw $GF \parallel$ to ED.

Then $\quad AE : AD :: EG : DF, \quad$ (267)

or $\quad n : o :: p : DF;$

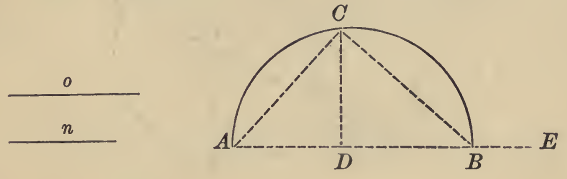
$\therefore \quad DF$ is the required line. Q. E. F.

297. Scholium.—If EG is equal to AD, DF is the third proportional to AE and AD, or to n and o.

PROBLEM IV.

298. *Given two straight lines, to construct a mean proportional between them.*

Let n and o be the two given lines.



Draw the indefinite straight line AE.

Cut off $AD = n$, and $DB = o$.

Describe a semi-circumference on AB, and at D erect the $\perp$ DC.

Draw AC and BC.

$$\angle ACB = \text{a} \llcorner; \qquad (208)$$

$\therefore$ $$AD : DC :: DC : DB, \qquad (282)$$

or $$n : DC :: DC : o;$$

$\therefore$ DC is the required line. Q. E. F.

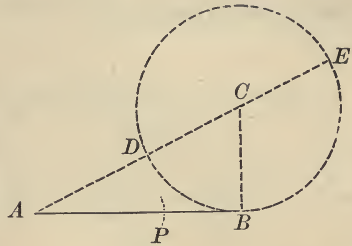
DEFINITION.

299. A straight line is said to be divided *in extreme and mean ratio*, when the greater part is a mean proportional between the whole line and the less part.

PROBLEM V.

300. *To divide a given straight line in extreme and mean ratio.*

Let AB be the given straight line.



At the extremity B erect the $\perp$ BC equal to half of AB.

With C as a centre and CB as a radius, describe a $\bigcirc$.

Draw AC, and produce it till it meets the circumference at E.

On AB cut off $AP = AD$.

Now, AB, being $\perp$ to the radius CB at B, is a tangent;

$\therefore$ $$AE : AB :: AB : AD; \quad (287)$$

$\therefore$ $$AE - AB : AB :: AB - AD : AD. \quad (163)$$

But $$DE = 2\ CB = AB;$$

$\therefore$ $$AE - AB = AD = AP.$$

$$AB - AD = AB - AP = PB.$$

Substitute AP and PB for their equals.

Then $AP : AB :: PB : AP$, which gives

$$AB : AP :: AP : PB; \quad (161)$$

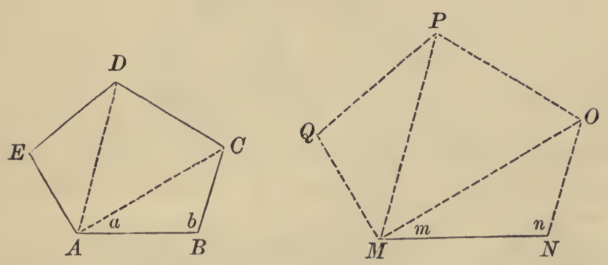
$\therefore$ AB is divided in extreme and mean ratio at P. (299)

Q. E. F.

PROBLEM VI.

301. *On a given straight line to construct a polygon similar to a given polygon.*

Let MN be the given line, and $ABCDE$ the given polygon.



Divide the polygon into $\triangle$s.

At M construct $\angle\, m = \angle\, a$, and at N construct $\angle\, n = \angle\, b$.

Then $\triangle$s MNO and ABC are similar. (270)

Likewise construct $\triangle$s MOP and MPQ similar to $\triangle$s ACD and ADE.

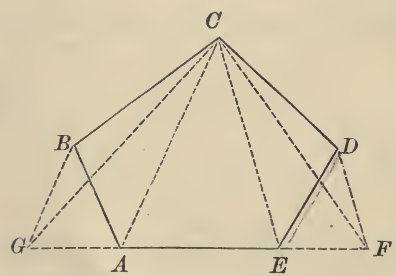
Then polygons $MNOPQ$ and $ABCDE$ are similar; (277)

$\therefore$ $MNOPQ$ is the polygon required. Q. E. F.

PROBLEM VII.

302. *To construct a triangle equivalent to a given polygon.*

Let $ABCDE$ be the given polygon.



Draw the diagonal CA, produce EA, draw $BG \parallel$ to CA, and draw CG.

Then $\triangle$s BAC and GAC have a common base AC and a common altitude;

$\therefore \qquad \triangle BAC = \triangle GAC; \qquad (250)$

$\therefore \qquad$ polygon $GCDE =$ polygon $ABCDE$.

Draw the diagonal CE, produce AE, draw $DF \parallel$ to CE, and draw CF.

Then $\qquad \triangle FCE = \triangle DCE; \qquad (250)$

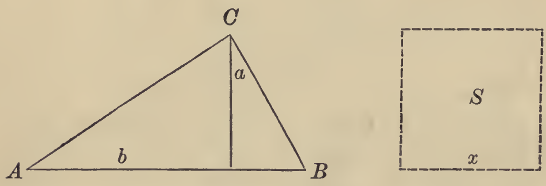
$\therefore \quad \triangle GCF =$ polygon $GCDE =$ polygon $ABCDE$;

$\therefore \qquad GCF$ is the required $\triangle$. Q. E. F.

PROBLEM VIII.

303. *To construct a square equivalent to a given triangle.*

Let ABC be the given $\triangle$, b its base, and a its altitude.



Find the mean proportional x between a and one-half of b, by (298).

On x construct the square S.

Then $$x^2 = \tfrac{1}{2}\, b \times a = \triangle\, ABC; \qquad (247)$$

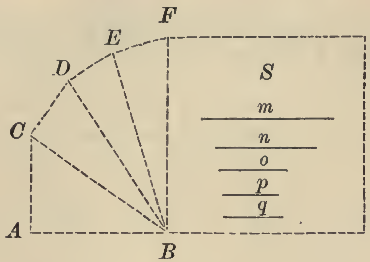
$\therefore$ S is the required square. Q. E. F.

304. Scholium.—By means of this problem and the preceding one, a square can be constructed equivalent to any given polygon.

PROBLEM IX.

305. *To construct a square equivalent to any number of given squares.*

Let m, n, o, p, and q be the sides of the given squares.



Draw $AB = m$.
At A draw $AC = n$ and $\perp$ to AB, and draw BC.
At C draw $CD = o$ and $\perp$ to BC, and draw BD.
At D draw $DE = p$ and $\perp$ to BD, and draw BE.
At E draw $EF = q$ and $\perp$ to EB, and draw BF.
On BF construct the square S.

Now, $\overline{BF}^2 = \overline{EF}^2 + \overline{BE}^2$

$= \overline{EF}^2 + \overline{DE}^2 + \overline{BD}^2$

$= \overline{EF}^2 + \overline{DE}^2 + \overline{CD}^2 + \overline{CB}^2$

$= \overline{EF}^2 + \overline{DE}^2 + \overline{CD}^2 + \overline{AC}^2 + \overline{AB}^2$, (254)

or $S = m^2 + n^2 + o^2 + p^2 + q^2$;

$\therefore$ S is the required square. Q. E. F.

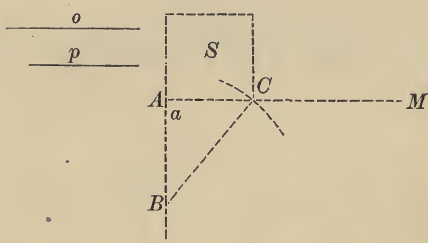
306. SCHOLIUM 1.—By means of this and the two preceding problems, a square can be constructed equivalent to the sum of any number of given polygons.

307. SCHOLIUM 2.—If m, n, o, p, and q are homologous sides of given similar polygons, BF is the homologous side of a similar polygon which is equivalent to the sum of the given polygons (293).

PROBLEM X.

308. *To construct a square equivalent to the difference of two given squares.*

Let o be the side of the larger square, and p the side of the smaller.



Construct a $\llcorner\, a$, and cut $AB = p$.

With B as a centre and o as a radius, describe an arc cutting AM at C.

On AC construct the square S.

Now, $$\overline{BC}^2 - \overline{AB}^2 = \overline{AC}^2 = S, \quad (255)$$

or $$o^2 - p^2 = S;$$

$\therefore$ S is the required square. Q. E. F.

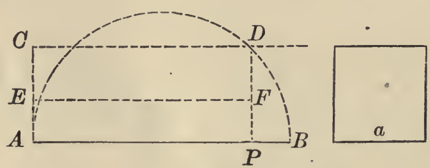
309. Scholium 1.—By means of this problem and the two immediately preceding (305), a square can be constructed equivalent to the difference of any two polygons.

310. Scholium 2.—If o and p are the homologous sides of two given similar polygons, AC is the homologous side of a similar polygon equivalent to the difference of the two given polygons.

PROBLEM XI.

311. *To construct a rectangle, given its area and the sum of the base and altitude.*

Let AB be equal to the given sum of the base and altitude, and let the given area equal that of the square whose side is a.



On AB as a diameter, describe a semicircle, and at A erect the $\perp$ $AC = a$.

Draw $CD \parallel$ to AB, cutting the circumference at D, and from D draw $DP \perp$ to AB.

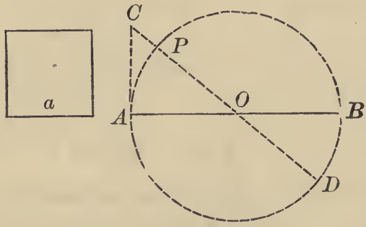
Now, $$\overline{DP}^2 = AP \times PB = a^2; \qquad (282)$$

$\therefore$ AP is the base and PB is the altitude of the required rectangle. And $APFE$, whose altitude $PF = PB$, is the required rectangle. Q. E. F.

PROBLEM XII.

312. *To construct a rectangle, given its area and the difference of the base and altitude.*

Let AB equal the given difference of the base and altitude, and let the given area be that of the square whose side is a.



On AB as a diameter, describe a ○.

At A draw the tangent $AC = a$, and draw the secant CD so as to pass through the centre O.

Now, $$CD \times CP = \overline{CA}^2 = a^2, \qquad (287)$$

and $$CD - CP = PD = AB;$$

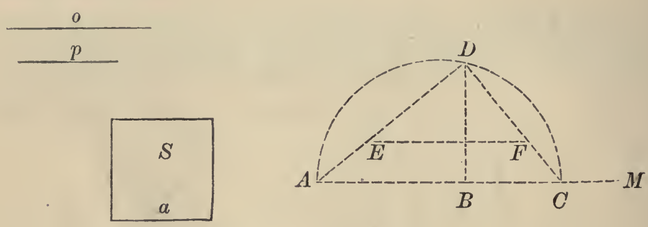
$\therefore$ CD is the base and CP is the altitude of the rectangle required, and the rectangle can readily be constructed.

Q. E. F.

PROBLEM XIII.

313. *To construct a square having a given ratio to a given square.*

Let $o : p$ be the given ratio, and S the given square whose side is a.



On the indefinite line AM, cut off $AB = o$, and $BC = p$.

On AC as a diameter, describe a semicircle.

At B erect a $\perp$ cutting the circumference at D, and draw AD and CD.

Take $DF \doteq a$, and draw $EF \parallel$ to AC.

Now, $DE : DF :: DA : DC$ (267), which gives

$$\overline{DE}^2 : \overline{DF}^2 :: \overline{DA}^2 : \overline{DC}^2. \qquad (170)$$

But $\quad \overline{DA}^2 : \overline{DC}^2 :: AB : BC :: o : p; \qquad (283)$

$\therefore \quad \overline{DE}^2 : \overline{DF}^2 :: o : p, \qquad (164)$

or $\quad \overline{DE}^2 : a^2 :: o : p;$

$\therefore$ the square whose side is DE is the required one. Q.E.F.

314. Scholium.—Since similar polygons are to each other as the squares of their homologous sides, we can find, by means of the above problem, the homologous side of a polygon similar and having a given ratio to a given polygon.

EXERCISES IN INVENTION.

THEOREMS.

1. The square inscribed in a circle is equivalent to half the square described on the diameter.

2. Prove geometrically that the square described on the sum of two lines is equivalent to the sum of the squares described on the two lines *plus* twice the rectangle of the lines.

3. Prove geometrically that the square described on the difference of two lines is equivalent to the sum of the squares described on the two lines *minus* twice the rectangle of the lines.

4. Prove geometrically that the rectangle of the sum and difference of two lines is equivalent to the difference of the squares described on the lines.

5. If a straight line is drawn from the vertex of an isosceles triangle to any point in the base, the square of this line is equivalent to the rectangle of the segments of the base together with the square of either of the equal sides.

6. The area of a circumscribed polygon equals half the product of the perimeter and the radius of the inscribed circle.

7. The triangle formed by drawing straight lines from the extremities of one of the non-parallel sides of a trapezoid to the middle point of the other, is equivalent to half the trapezoid.

8. The two triangles formed by drawing straight lines from any point within a parallelogram to the extremities of either pair of opposite sides, are equivalent to half the parallelogram.

9. The bisector of the vertical angle of a triangle divides the base into parts proportional to the adjacent sides of the triangle.

PROBLEMS.

1. Trisect a given straight line.

2. Bisect a parallelogram by a line passing through any given point in the perimeter.

3. Construct a parallelogram whose surface and perimeter are respectively equal to the surface and perimeter of a given triangle.

4. On a given straight line construct a rectangle equivalent to a given rectangle.

5. Construct a polygon similar to a given polygon and whose area is in a given ratio to that of the given polygon.

BOOK V.

REGULAR POLYGONS AND THE CIRCLE.

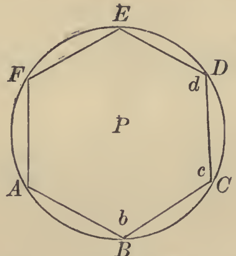
DEFINITION.

315. A ***Regular Polygon*** is one which is both equilateral and equiangular.

THEOREM I.

316. *Any equilateral polygon inscribed in a circle is regular.*

Let P be an equilateral polygon inscribed in a ○.



To prove that P is a regular polygon.

Arc AB = arc BC = arc CD = arc DE, etc.; (189)

∴ arc ABC = arc BCD, etc.; (Ax. 2.)

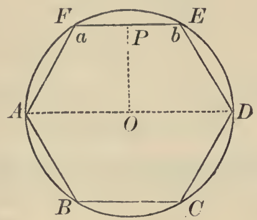
∴ $\angle b = \angle c = \angle d$, etc.; (207)

∴ P is a regular polygon. (315) Q. E. D.

THEOREM II.

317. *A circle can be circumscribed about any regular polygon.*

Let $ABCDEF$ be a regular polygon.



To prove that a ○ *can be circumscribed about* $ABCDEF$.

Describe a circumference through the vertices A, F, and E. (193)

From O, the centre, draw $OP \perp$ to FE, bisecting it at P, and draw OA and OD.

On OP as an axis, revolve the quadrilateral $AOPF$ till it falls in the plane of $ODEP$.

Since OP is $\perp$ to FE, PF falls on its equal PE, F falling on E; and since $\angle a = \angle b$, FA falls on its equal ED, A falling on D;

$\therefore \quad OA = OD$, and the circumference passes through D.

Likewise we can prove that the circumference passing through F, E, and D passes also through the vertex C, and thus through all the successive vertices of the polygon. Q. E. D.

318. Cor.—*A circle can be inscribed in a regular polygon.*

DEFINITIONS.

319. The ***Centre*** of a regular polygon is the common centre of its circumscribed and inscribed circles.

320. The ***Angle at the Centre*** of a regular polygon is the angle formed by two lines drawn from the centre to the extremities of any side.

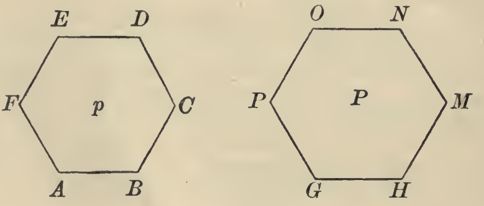
The angles at the centre are equal, any one being equal to four right angles divided by the number of sides of the polygon.

321. The ***Apothem*** of a regular polygon is the perpendicular distance from the centre to any side, and is equal to the radius of the inscribed circle.

THEOREM III.

322. *Regular polygons of the same number of sides are similar figures.*

Let p and P be regular polygons of the same number of sides.



To prove that p and P are similar figures.

Since p and P are regular and have the same number of sides, they are mutually equiangular. (114)

Also $AB : BC :: 1 : 1$,

and $GH : HM :: 1 : 1$; (315)

$\therefore$ $AB : BC :: GH : HM$; (164)

$\therefore$ p and P are similar. (237) Q. E. D.

323. Cor. 1.—*The perimeters of similar regular polygons are to each other as the radii of their circumscribed or inscribed circles* (281).

324. Cor. 2.—*The areas of similar regular polygons are to each other as the squares of the radii of their circumscribed or inscribed circles* (292).

RELATION BETWEEN THE CIRCUMFERENCE AND DIAMETER OF A CIRCLE.

THEOREM IV.

325. *The circumferences of circles are to each other as their radii.*

Let c and C be the circumferences, r and R the radii, and d and D the diameters of the ○s o and O.

To prove that $c : C :: r : R$.

Inscribe in the two circles similar regular polygons, and denote their perimeters by p and P.

Then $$p : P :: r : R. \qquad (323)$$

Now, this is true whatever may be the number of sides of the polygons, if there is the same number in each; hence it is true when the number of sides is infinitely great, in which case $p = c$, and $P = C$, while r and R remain the same;

$\therefore$ $$c : C :: r : R. \qquad \text{Q. E. D.}$$

326. Cor. 1.—*The circumferences of circles are to each other as their diameters.*

By (166) the above proportion becomes

$$c : C :: 2r : 2R,$$

or $$c : C :: d : D.$$

327. Cor. 2.—*The ratio of the circumference of a circle to its diameter is a constant quantity.*

By (160) the last proportion becomes

$$c : d :: C : D,$$

or $$\frac{c}{d} = \frac{C}{D}.$$

This constant ratio is usually denoted by π, the Greek letter *p*, called pi. The numerical value of π can be found only approximately, as can be proved by the higher mathematics.

Hence, in any circle, *the circumference and its diameter are incommensurable.*

328. Cor. 3.—*The circumference of a circle equals the diameter multiplied by π.*

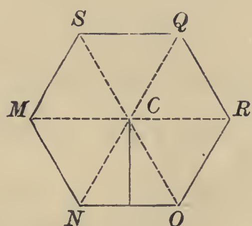
$$\frac{C}{D} = \pi, \text{ whence } C = \pi D,$$

or $$C = 2\pi R.$$

THEOREM V.

329. *The area of a regular polygon equals half the product of its perimeter and apothem.*

Let P be the perimeter and A the apothem of the regular polygon $MNORQS$.



To prove that the area of $MNORQS = \frac{1}{2} P \times A$.

Draw CO, CR, CQ, etc., dividing the polygon into as many $\triangle$s as it has sides.

All the $\triangle$s have the common altitude A, and the sum of their bases equals P;

$\therefore$ the sum of the areas of the $\triangle s = \frac{1}{2} P \times A$, (247)

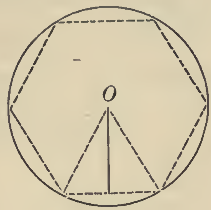
or the area of the polygon $= \frac{1}{2} P \times A$.

Q. E. D.

THEOREM VI.

330. *The area of a circle equals half the product of its circumference and radius.*

Let C be the circumference and R the radius of the ○ O.



To prove that the area of the circle $= \frac{1}{2} C \times R$.

Inscribe a regular polygon, and denote its perimeter by P, and its apothem by A.

Then the area of the polygon $= \frac{1}{2} P \times A$. (329)

Now, this is true whatever may be the number of sides of the polygon; hence it is true when the number is infinitely great, in which case $P = C$, and $A = R$;

$\therefore$ the area of the ○ $= \frac{1}{2} C \times R$. Q.E.D.

331. Cor. 1.—*The area of a* ○ $= \pi R^2$, *R being the radius.*

The area of a ○ $= \frac{1}{2} C \times R$.

But $C = 2 \pi R$; (327)

$\therefore$ the area of a ○ $= \pi R^2$.

332. Cor. 2.—*The area of a sector of a circle equals half the product of its arc and the radius.*

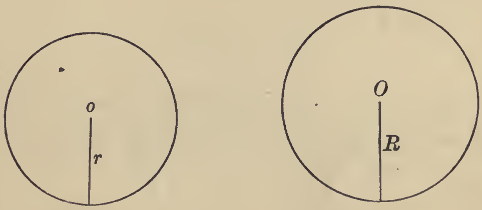
DEFINITION.

333. **Similar Sectors** are sectors of different circles, which have equal angles at the centre.

THEOREM VII.

334. *Circles are to each other as the squares of their radii.*

Let r and R denote the radii of the ○s o and O.



To prove that $o : O :: r^2 : R^2$,

$$o = \pi r^2,$$

and $$O = \pi R^2. \qquad (331)$$

Divide; then

$$\frac{o}{O} = \frac{\pi r^2}{\pi R^2} = \frac{r^2}{R^2},$$

or $$o : O :: r^2 : R^2. \qquad \text{Q.E.D.}$$

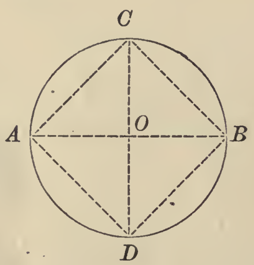
335. Cor.—*Similar sectors are to each other as the squares of their radii.*

PROBLEMS IN CONSTRUCTION.

PROBLEM I.

336. *To inscribe a square in a given circle.*

Let O be the centre of the given ○.



Draw any two diameters, as AB and CD, $\perp$ to each other, and draw AC, CB, BD, and AD.

Now the angles about the centre are equal;

$\therefore$ the circumference is divided into four equal arcs;

$\therefore$ the chords AC, CB, BD, and AD are equal. (188)

The $\angle$s ADB, DBC, BCA, and CAD are $\llcorner$s; (208)

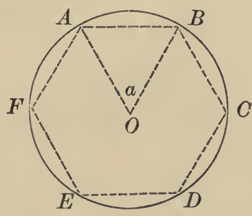
$\therefore$ $ADBC$ is the required square. (123) Q. E. F.

337. Cor.—*To inscribe a regular polygon of 8 sides, bisect the arcs subtended by the sides of an inscribed square and draw chords; and by continuing the process, we can inscribe regular polygons of 16, 32, etc., sides.*

PROBLEM II.

338. *To inscribe a regular hexagon in a circle.*

Let O be the centre of a given ○.



Draw any radius, as OA, and with A as a centre and the radius of the circle describe an arc, cutting the circumference at B.

Draw AB and OB.

Now, the $\triangle\ ABO$ is both equilateral and equiangular; (97)

$\therefore$ $\qquad \angle\ a = \frac{1}{3}$ of 2 ∟s $= \frac{1}{6}$ of 4 ∟s; (81)

$\therefore$ arc $AB = \frac{1}{6}$ of the circumference, and the chord AB is the side of a regular inscribed hexagon;

$\therefore$ $ABCDEF$, which is formed by applying the radius six times as a chord, is the required hexagon. Q. E. F.

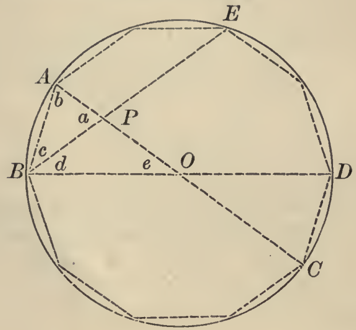
339. Cor. 1.—*To inscribe an equilateral triangle, join the alternate vertices of a regular inscribed hexagon.*

340. Cor. 2.—*To inscribe a regular polygon of 12 sides, bisect the arcs subtended by the sides of a regular inscribed hexagon and draw chords; and by continuing the process, we can inscribe regular polygons of 24, 48, etc., sides.*

PROBLEM III.

341. *In a given circle to inscribe a regular decagon.*

Let O be the centre of the given ○.



Suppose the problem to be solved, and let ABC, etc., be the regular inscribed decagon.

Draw AC and BD.

Now, AC and BD bisect the circumference;

∴ they are diameters and intersect at the centre C.

Draw BE, cutting AC at P.

$\angle a$ is measured by $\frac{1}{2}$ (arc AB + arc EC), or $\frac{1}{2}$ arc BC, (212)

and $\angle b$ is measured by $\frac{1}{2}$ arc BC; (206)

∴ $\triangle APB$ is isosceles, and $AB = BP$.

Also, $\angle\ d$ is measured by $\frac{1}{2}$ arc ED, or arc AB,

and $\quad \angle\ e$ is measured by arc AB; (205)

$\therefore \quad \triangle\ BOP$ is isosceles, and $OP = BP = AB$.

$\angle\ c$ is measured by $\frac{1}{2}$ arc AE, or AB; (206)

$\therefore$ $\triangle$s APB and ABO are mutually equiangular and similar; (269)

$$\therefore \quad AO : AB :: AB : AP,$$

or

$$AO : OP :: OP : AP.$$

But this shows that AO, the radius, is divided in extreme and mean ratio at P, and that OP, the greater part, equals AB, a side of the regular inscribed decagon.

Therefore, to inscribe a regular decagon, divide the radius in extreme and mean ratio, and apply the greater part ten times as a chord. Q. E. F.

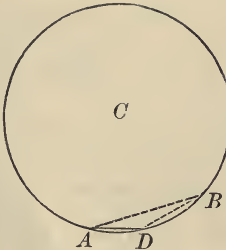
342. Cor. 1.—*To inscribe a regular pentagon, join the alternate vertices of a regular inscribed decagon.*

343. Cor.—*To inscribe a regular polygon of 20 sides, bisect the arcs subtended by the sides of a regular inscribed decagon and draw chords; and by continuing the process, we can inscribe regular polygons of 40, 80, etc., sides.*

PROBLEM IV.

344. *In a given circle to inscribe a regular pentadecagon.*

Let C be the given ○.



Draw the chord AB equal to the side of a regular inscribed hexagon, and the chord BD equal to the side of a regular inscribed decagon, and draw AD.

Now, arc AD = arc AB — arc DB

$= \frac{1}{6}$ of the circumference — $\frac{1}{10}$ of the circumference

$= \frac{1}{15}$ of the circumference.

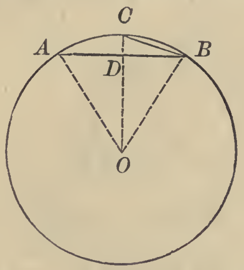
Therefore chord AD = a side of a regular inscribed pentadecagon; and hence if we apply AD fifteen times as a chord we get the required polygon. Q. E. F.

345. Cor.—*To inscribe a regular polygon of 30 sides, bisect the arcs subtended by the sides of a regular inscribed pentadecagon and draw chords; and by continuing the process, we can inscribe regular polygons of 60, 120, etc., sides.*

PROBLEM V.

346. *In a circle whose radius is unity, to find the value of the chord of half an arc in terms of the chord of the whole arc.*

Let O be the centre of a ○ whose radius is 1, AB the chord of an arc, and BC the chord of half the arc.



Draw the radii OB and OC.

In the R△ BDO, $\overline{OB}^2 = \overline{OD}^2 + \overline{BD}^2$.

Whence $OD = \sqrt{\overline{OB}^2 - \overline{BD}^2}$.

But $OB = OC = 1$, and $BD = \dfrac{AB}{2}$;

$\therefore \quad OD = \sqrt{1 - \left(\dfrac{AB}{2}\right)^2}$.

In the R△ CDB, $BC = \sqrt{\overline{BD}^2 + \overline{CD}^2}$.

But $CD = 1 - OD = 1 - \sqrt{1 - \left(\dfrac{AB}{2}\right)^2}$.

Substitute $\dfrac{AB}{2}$ and $1 - \sqrt{1 - \left(\dfrac{AB}{2}\right)^2}$ for their equals BD and CD, and reduce.

Then $BC = \sqrt{2 - \sqrt{4 - \overline{AB}^2}}$. Q. E. F.

PROBLEM VI.

347. *To find the numerical value of* π, *approximately.*

Let C be the circumference, and R the radius of a ○.

$$\pi = \frac{C}{2\,R}. \tag{327}$$

When $$R = 1,\ \pi = \frac{C}{2}.$$

Now, by means of the formula $BC = \sqrt{2 - \sqrt{4 - \overline{AB}^2}}$, established in (346), we make the following computations:

In a regular inscribed polygon of

No. Sides.	*Form of Computation.*	*Length of Side.*	*Perimeter.*
6.	See (338)	1.00000000	6.00000000.
12.	$BC = \sqrt{2 - \sqrt{4 - 1^2}} =$	.51763809	6.21165708.
24.	$BC = \sqrt{2 - \sqrt{4 - (.51763809)^2}} =$	.26105238	6.26525722.
48.	$BC = \sqrt{2 - \sqrt{4 - (.26105238)^2}} =$	.13080626	6.27870041.
96.	$BC = \sqrt{2 - \sqrt{4 - (.13080626)^2}} =$	.06543817	6.28206396.
192.	$BC = \sqrt{2 - \sqrt{4 - (.06543817)^2}} =$	.03272346	6.28290510.
384.	$BC = \sqrt{2 - \sqrt{4 - (.03272346)^2}} =$	.01636228	6.28311544.
768.	$BC = \sqrt{2 - \sqrt{4 - (.01636228)^2}} =$	.00818121	6.28316941.

It will be seen that the first four decimal places remain the same, to whatever extent we increase the number of sides. Hence we can consider 6.28317 as the approximate value of the circumference of a circle whose radius is 1.

$$\therefore \quad \pi = \frac{C}{2} = \frac{6.28317}{2} = 3.1416 \text{ nearly.} \qquad \text{Q. E. F.}$$

EXERCISES IN INVENTION.

THEOREMS.

1. The side of an inscribed equilateral triangle equals half the side of the circumscribed equilateral triangle.

2. The diameter of a circle is a mean proportional between the sides of the equilateral triangle and the regular hexagon circumscribed about the circle.

3. The square inscribed in a circle equals half the square on the diameter.

4. The area of a regular inscribed hexagon equals three-fourths the area of a regular circumscribed hexagon.

5. The area of a regular inscribed hexagon is a mean proportional between the areas of the inscribed and circumscribed equilateral triangles.

6. If the vertices of a square are taken as centres and half the diagonal as a radius and circles be described, the points of intersection of the circumferences and the sides of the square are the vertices of a regular octagon.

7. The area of a regular inscribed octagon equals the area of a rectangle whose adjacent sides equal the sides of the circumscribed and inscribed squares.

8. The area of a regular inscribed dodecagon equals three times the square on the radius.

PROBLEMS.

1. Inscribe in a given circle a regular polygon similar to a given regular polygon.

2. Circumscribe a polygon similar to a given inscribed polygon.

3. In a given circle, inscribe three equal circles, touching each other and the given circle.

4. In a given circle, inscribe four equal circles in mutual contact with each other and the given circle.

5. In a given equilateral triangle, inscribe three equal circles, touching each other, and each touching two sides of the triangle.

6. About a given circle, describe six circles, each equal to the given one and in mutual contact with each other and the given circle.