W. Reginald Hodgkin.
U. C. 1900.

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ELEMENTS

OF

PLANE AND SPHERICAL

TRIGONOMETRY

BY

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CROCKETT. PLANE AND SPHER. TRIGONOM.

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PREFACE.

THIS work has been prepared for the use of beginners in the study of trigonometry. Assuming that a high degree of proficiency cannot be expected from such students, the author has limited himself to the selection of simple proofs of the formulas, not striving after original demonstrations. Geometrical proofs have been added in many cases, experience having shown that the student is assisted by them to a clearer understanding of the subject.

The student is expected, in technical institutions, to acquire facility in the use of the tables. All of the numerical examples have been computed by the author, with special attention to correctness in the last decimal place, and the arrangement of the computations has been carefully considered. Five-place tables have been adopted, and the angles in the examples are given to the nearest tenth of a minute, because the instruments ordinarily used by engineers are read by the vernier only to the nearest minute of arc, while the angle corresponding to a computed function may be found usually to the nearest tenth of a minute by the use of five-place tables.

Credit is due particularly to the works of Chauvenet, Snowball, Beasley, Woodhouse, Newcomb, and Todhunter, although many others have been consulted. A number of the illustrative examples in Art. 111 were taken from Gillespie's "Land Surveying," the numerical values being assigned by the author of this work.

The author cannot hope that among so many examples there are no errors; he therefore requests those finding such to kindly notify him.

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GREEK ALPHABET.

Α, α, α, α	*Alpha*	Ν, ν	*Nu*
Β, β, β, β	*Beta*	Ξ, ξ	*Xi*
Γ, γ, γ, γ	*Gamma*	Ο, ο	*Omicron*
Δ, δ	*Delta*	Π, π	*Pi*
Ε, ϵ	*Epsilon*	Ρ, ρ	*Rho*
Ζ, ζ	*Zeta*	Σ, σ, ς	*Sigma*
Η, η	*Eta*	Τ, τ	*Tau*
Θ, θ	*Theta*	Υ, υ	*Upsilon*
Ι, ι	*Iota*	Φ, ϕ	*Phi*
Κ, κ	*Kappa*	Χ, χ	*Chi*
Λ, λ	*Lambda*	Ψ, ψ	*Psi*
Μ, μ	*Mu*	Ω, ω	*Omega*

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PART ONE.

PLANE AND ANALYTICAL TRIGONOMETRY.

CHAPTER I.

MEASUREMENT OF ANGLES; TRIGONOMETRIC FUNCTIONS OF ANGLES LESS THAN NINETY DEGREES.

1. Analytical Trigonometry treats of the relations of lines and angles by algebraic methods. In Plane and Spherical Trigonometry, these relations are applied to the solution of plane and spherical triangles.

2. Directed Lines; Angles. — *A directed line* is one whose beginning, direction, and length are known. The direction of the line is indicated by the order of the letters in its symbol; for instance, the line AB is drawn from A to B. If one direction along the line is considered positive, the opposite direction will be negative; thus, if the line AB is positive, the line BA will be negative, their numerical measures being equal, or

$$\text{line } AB = -\text{ line } BA.$$

An angle is the figure formed by two intersecting lines, the point of intersection being the *vertex.*

The angle between any two given lines, whether intersecting or not intersecting,* is defined to be the same as the angle formed by two lines drawn through any point parallel to and in the same direction as the given lines. Hence an angle may be defined as the difference in direction of two directed lines.

* That is, parallel or in space.

3. Measurement of Angles. — Two methods of measuring angles are in common use, — the sexagesimal and the circular or natural methods.

4. Sexagesimal Measure.* — The circumference of a circle described about the vertex of the angle as a center is divided into 360 equal parts, and the angle at the center subtended by one of these parts is taken as the unit. The length of one of these divisions of the circle will depend upon its radius; but the corresponding angle at the center will be independent of the radius, since it is $\frac{1}{360}$ of four right angles. This unit angle, called a *degree*, is divided into 60 parts called *minutes*, each of which is subdivided into 60 parts called *seconds*. These are marked $^\circ$, $'$, $''$; thus $43^\circ\,14'\,35''.2$ is read, "43 degrees, 14 minutes, and 35.2 seconds."

How many degrees are there in

1. Two thirds of four right angles? *Ans.* 240°.
2. Two fifths of three right angles? *Ans.* 108°.
3. Five sixths of two right angles? *Ans.* 150°.

5. The Circular or Natural Measure. — From geometry we know that in any two concentric circles the arcs intercepted by any angle at the center are to each other as the radii of the circles. Therefore, if ACB be any central angle, we have

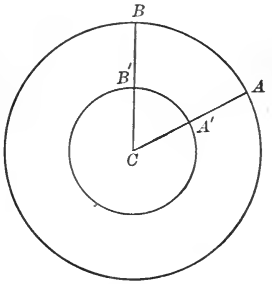


Fig. 1.

$$\frac{\text{arc } AB}{CA} = \frac{\text{arc } A'B'}{CA'}. \qquad (1)$$

Hence the length of the intercepted arc divided by the radius is a number that is always the same for the same angle, no matter what the radius may be.

We also know that in any circle any two central angles are to each other as their intercepted arcs, and therefore as the quotients of their intercepted arcs divided by the radius. We can, then, use these quotients to measure the angles.

* From *sexagesimus*, sixtieth.

The circular measure of an angle is the quotient obtained by dividing the length of its intercepted arc, in a circle whose center is at the vertex of the angle, by the radius of the circle. Thus, if c is the circular measure of the angle, l its intercepted arc, and r the radius, we have

$$c = \frac{l}{r}. \tag{2}$$

If the radius of the circle is unity,

$$c = l. \tag{3}$$

Hence the circular measure is represented by the length of the intercepted arc in the circle whose radius is unity.

The angle whose circular measure is one, that is, whose intercepted arc is equal to the radius, is called the *radian.*

1. The length of the intercepted arc of a central angle is 4 feet in a circle whose radius is 2 feet; the length of the intercepted arc of another central angle is 20 meters in a circle whose radius is 5 meters. Show that the second angle is twice as large as the first.

2. In a circle with a radius of 10 inches, the intercepted arc of a central angle is 5 inches, and that of an angle whose vertex is on the circumference is 10 inches. Find their circular measures. *Ans.* $\frac{1}{2}$.

6. Relation between the Two Measures. — Two right angles are measured by 180°, and also by πr * $\div r = \pi$, since πr is the semicircumference of a circle whose radius is r. Hence, using the equality sign to represent "corresponds to," we have

$$180° = \pi \text{ in circular measure}; \tag{1}$$

$$\therefore 1° = \frac{\pi}{180} \text{ in circular measure.} \tag{2}$$

Again, $$\pi \text{ in circular measure} = 180°; \tag{3}$$

$$\therefore 1 \text{ in circular measure} = \frac{180°}{\pi}. \tag{4}$$

$$\therefore 1 \text{ in circular measure} = 57°.29577\,95 + \tag{5}$$

$$\therefore 1 \text{ in circular measure} = 206\,264''.806. \tag{6}$$

1. What is the circular measure of 120°? *Ans.* $120 \times \frac{\pi}{180} = \frac{2}{3}\pi$.

2. What is the circular measure of 10° 10′ 10″?

The circular measure of 1° is $\frac{\pi}{180}$, and that of 1″ is $\frac{\pi}{180 \times 60 \times 60}$. But $10°\,10'\,10'' = 36610''$. $\therefore$ Circular measure of $10°\,10'\,10'' = \frac{36610\,\pi}{180 \times 60 \times 60}$.

* π denotes the ratio of the circumference of a circle to its diameter, and is the number 3.14159 265+.

3. What is the sexagesimal measure of the angle whose circular measure is $\frac{1}{3}\pi$?

$$\pi = 180°; \quad \therefore \frac{1}{3}\pi = 60°.$$

4. What is the sexagesimal measure of the angle whose circular measure is $\frac{2}{3}$?

$$\text{Unity in circular measure} = \frac{180°}{\pi}; \quad \therefore \frac{2}{3} \text{ corresponds to } \frac{120°}{\pi}.$$

5. What are the sexagesimal and circular measures corresponding to $\frac{2}{9}$ of three right angles? *Ans.* $60°$; $\frac{1}{3}\pi$.

6. The sexagesimal measures of two angles are $22° 30'$ and $43° 14' 3''$. Show that their circular measures are $\frac{1350\pi}{180 \times 60}$ and $\frac{155643\pi}{180 \times 60 \times 60}$.

7. The circular measures of three angles are $\frac{1}{12}\pi$, $\frac{2}{9}\pi$, and $\frac{1}{50}\pi$. Show that their sexagesimal measures are $15°$, $40°$, and $3° 36'$.

8. The circular measures of three angles are $\frac{1}{4}$, $\frac{5}{3}$, and $\frac{2}{9}$. Show that their sexagesimal measures are $\frac{45°}{\pi}$, $\frac{300°}{\pi}$, and $\frac{40°}{\pi}$.

9. Find the sexagesimal and circular measures corresponding to

(*a*) Seven tenths of four right angles. *Ans.* $252°$; $\frac{7}{5}\pi$.

(*b*) Five fourths of two right angles. *Ans.* $225°$; $\frac{5}{4}\pi$.

(*c*) Two thirds of one right angle. *Ans.* $60°$; $\frac{1}{3}\pi$.

7. Centesimal Measure. — In this system, proposed by the French, the right angle is divided into 100 parts called *grades*, each of which is subdivided into 100 parts called *minutes*, each minute being divided into 100 parts called *seconds;* marked ^{g}, `, ``.

8. Trigonometric Ratios. — Let the sides of a right-angled triangle be denoted as shown in Fig. 2. The trigonometric ratios may be defined as follows:

$$\left.\begin{array}{llll}
\text{The } \textit{sine} & \text{of an angle} = \dfrac{\text{side opposite}}{\text{hypotenuse}}; & \text{written} & \mathbf{sin}\, A = \dfrac{o}{h} \\[2ex]
\text{The } \textit{cosine} & \text{of an angle} = \dfrac{\text{side adjacent}}{\text{hypotenuse}}; & \text{written} & \mathbf{cos}\, A = \dfrac{a}{h} \\[2ex]
\text{The } \textit{tangent} & \text{of an angle} = \dfrac{\text{side opposite}}{\text{side adjacent}}; & \text{written} & \mathbf{tan}\, A = \dfrac{o}{a} \\[2ex]
\text{The } \textit{cotangent} & \text{of an angle} = \dfrac{\text{side adjacent}}{\text{side opposite}}; & \text{written} & \mathbf{cot}\, A = \dfrac{a}{o} \\[2ex]
\text{The } \textit{secant} & \text{of an angle} = \dfrac{\text{hypotenuse}}{\text{side adjacent}}; & \text{written} & \mathbf{sec}\, A = \dfrac{h}{a} \\[2ex]
\text{The } \textit{cosecant} & \text{of an angle} = \dfrac{\text{hypotenuse}}{\text{side opposite}}; & \text{written} & \mathbf{cosec}\, A = \dfrac{h}{o}
\end{array}\right\} \quad (1)$$

These fundamental equations should be thoroughly memorized.

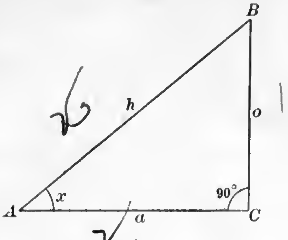


FIG. 2.

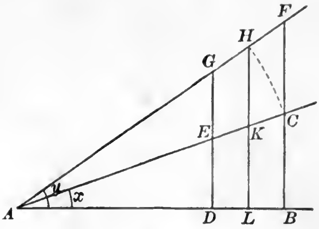


FIG. 3.

9. The Ratios are Constant for Any One Angle. — In Fig. 3 let BAC and BAF be two angles differing by a quantity as small as we please. At any two points B and D on AB, draw BF and DG perpendicular to AB; with A as a center, and radius AC, describe the arc CH, and draw LH perpendicular to AB. The triangles BAC and DAE are similar.

$$\therefore\ \frac{BC}{AC} = \frac{DE}{AE} = \text{a constant} = \frac{\text{side opposite}}{\text{hypotenuse}} = \sin x.$$

$$\frac{BC}{AB} = \frac{DE}{AD} = \text{a constant} = \frac{\text{side opposite}}{\text{side adjacent}} = \tan x.$$

$$\frac{AC}{AB} = \frac{AE}{AD} = \text{a constant} = \frac{\text{hypotenuse}}{\text{side adjacent}} = \sec x.$$

10. The Values of the Ratios differ for Different Angles. — From Fig. 3 we have, since $AH = AC$,

$$\sin x = \frac{BC}{AC} \text{ and } \sin y = \frac{LH}{AH} = \frac{LH}{AC};$$

$$\tan x = \frac{BC}{AB} \text{ and } \tan y = \frac{BF}{AB};$$

$$\sec x = \frac{AC}{AB} \text{ and } \sec y = \frac{AF}{AB}.$$

11. The Angle may be constructed when One of the Ratios is known. — Let $\sin x = \frac{1}{2}$. With any convenient radius AC, describe a circle about A as a center. Draw AD perpendicular to AB, and on it lay off $AD = \frac{1}{2}AC$; draw DC parallel to AB until

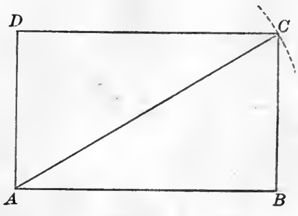


FIG. 4.

it intersects the circle at C; join A and C, and BAC will be the required angle, since

$$\sin BAC = \frac{BC}{AC} = \frac{AD}{AC} = \frac{1}{2}.$$

Let $\tan x = \frac{3}{4}$. Lay off any convenient distance AB; at B draw BC perpendicular to AB, and lay off $BC = \frac{3}{4} AB$; join A and C, and BAC will be the required angle, since

$$\tan BAC = \frac{BC}{AB} = \frac{3}{4}.$$

Let $\sec x = 2$. Lay off any convenient distance AB; erect the perpendicular line BC; with a radius $AC = 2\,AB$ describe an arc cutting BC at C; join A and C, and BAC will be the required angle, since

$$\sec BAC = \frac{AC}{AB} = 2.$$

Let the student construct the angle whose cosine is $\frac{1}{3}$, the angle whose cotangent is 5, and the angle whose cosecant is 4.

12. We therefore conclude that to any one angle there will correspond a special value of each of these ratios, that the value of each ratio will differ for different angles, and that, if any one of these ratios is given, the angle may be constructed.

13. Tables of Sines, Cosines, etc. — The values of these ratios for angles between 0° and 90° have been computed, and are given in tables so arranged that the values corresponding to any angle may be readily found. The tables of *natural sines*, etc., contain the actual values of these ratios; while the tables of *logarithmic sines*, etc., contain their logarithms.

14. Ratios for 30°, 45°, 60°.

(*a*) *Ratios for* 45°. — In Fig. 5 let the angle $A = 45°$; then $B = 90° - A = 45°$.

$$\therefore AC = CB, \text{ since they are opposite equal angles.}$$

Let $AC = a$; then $CB = a$, and $AB = \sqrt{a^2 + a^2} = a\sqrt{2}$.

$$\therefore \sin 45° = \frac{CB}{AB} = \frac{1}{\sqrt{2}}; \quad \tan 45° = \frac{CB}{AC} = 1; \quad \sec 45° = \frac{AB}{AC} = \sqrt{2};$$

$$\cos 45° = \frac{AC}{AB} = \frac{1}{\sqrt{2}}; \quad \cot 45° = \frac{AC}{CB} = 1; \quad \text{cosec } 45° = \frac{AB}{CB} = \sqrt{2}.$$

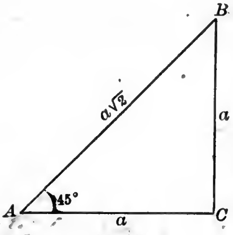


FIG. 5.

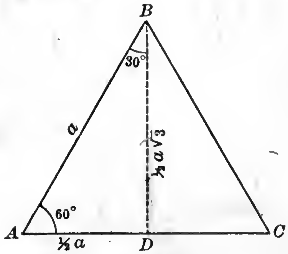


FIG. 6.

(*b*) *Ratios for* 30° *and* 60°. — In the equilateral triangle ABC (Fig. 6), let $AB = a$; draw DB perpendicular to AC; AC will be bisected at D, making $AD = \frac{1}{2}a$, and the angle ABD = angle $DBC = 30°$.

Also $DB = \sqrt{a^2 - \frac{1}{4}a^2} = \frac{1}{2}a\sqrt{3}$.

$$\therefore \sin ABD = \sin 30° = \frac{AD}{AB} = \frac{1}{2}; \quad \tan 30° = \frac{AD}{DB} = \frac{1}{\sqrt{3}}; \quad \sec 30° = \frac{AB}{DB} = \frac{2}{\sqrt{3}};$$

$$\cos 30° = \frac{DB}{AB} = \frac{\sqrt{3}}{2}; \quad \cot 30° = \frac{DB}{AD} = \sqrt{3}; \quad \text{cosec } 30° = \frac{AB}{AD} = 2.$$

$$\sin DAB = \sin 60° = \frac{DB}{AB} = \frac{\sqrt{3}}{2}; \quad \tan 60° = \frac{DB}{AD} = \sqrt{3}; \quad \sec 60° = \frac{AB}{AD} = 2;$$

$$\cos 60° = \frac{AD}{AB} = \frac{1}{2}; \quad \cot 60° = \frac{AD}{DB} = \frac{1}{\sqrt{3}}; \quad \text{cosec } 60° = \frac{AB}{DB} = \frac{2}{\sqrt{3}}.$$

Note that the sines of 30°, 45°, and 60°, are $\frac{1}{2}\sqrt{1}$, $\frac{1}{2}\sqrt{2}$, and $\frac{1}{2}\sqrt{3}$ respectively.

15. The Ratios are not Independent of Each Other; for we have from Fig. 2,

$$h^2 = a^2 + o^2,$$

so that if two of the three quantities h, o, and a, are given, the third can be found. Hence if we know one of the ratios, that is, the relative values of two of the three elements, we can determine the relative value of the third element, and from it the other ratios.

Thus if $\tan x = \frac{3}{4}$, and the other ratios are required, we have

$$\tan x = \frac{o}{a} = \frac{3}{4};\ \text{let } o = 3,\ a = 4;\ \text{then } h = 5.$$

$$\therefore\ \sin x = \frac{o}{h} = \frac{3}{5};\ \cos x = \frac{a}{h} = \frac{4}{5};\quad \cot x = \frac{a}{o} = \frac{4}{3};$$

$$\sec x = \frac{h}{a} = \frac{5}{4};\ \operatorname{cosec} x = \frac{h}{o} = \frac{5}{3}.$$

Having given the ratio on the left, find the ratios on the right:

	sin x.	cos x.	tan x.	cot x.	sec x.	cosec x.
1. $\sin x = \frac{8}{17}$	—	$\frac{15}{17}$	$\frac{8}{15}$	$\frac{15}{8}$	$\frac{17}{15}$	$\frac{17}{8}$
2. $\cos x = \frac{5}{13}$	$\frac{12}{13}$	—	$\frac{12}{5}$	$\frac{5}{12}$	$\frac{13}{5}$	$\frac{13}{12}$
3. $\tan x = \frac{7}{24}$	$\frac{7}{25}$	$\frac{24}{25}$	—	$\frac{24}{7}$	$\frac{25}{24}$	$\frac{25}{7}$
4. $\cot x = 2$	$\frac{1}{\sqrt{5}}$	$\frac{2}{\sqrt{5}}$	$\frac{1}{2}$	—	$\frac{1}{2}\sqrt{5}$	$\sqrt{5}$
5. $\sec x = \frac{29}{20}$	$\frac{21}{29}$	$\frac{20}{29}$	$\frac{21}{20}$	$\frac{20}{21}$	—	$\frac{29}{21}$
6. $\operatorname{cosec} x = 3$	$\frac{1}{3}$	$\frac{2}{3}\sqrt{2}$	$\frac{1}{4}\sqrt{2}$	$2\sqrt{2}$	$\frac{3}{4}\sqrt{2}$	—

16. Measurement of Angles in the Field.—In Fig. 7, *FGHK* represents a fixed graduated circle, and *ABDE* a circle resting on the plate *FGHK*, and capable of moving about a pivot at *C*; *I* and *O* are two small rods fixed to *ABDE*, and perpendicular to the planes of the circles; and *M* is a mark on the circle *ABDE* in the same line with *I*, *C*, and *O*. If we wish to measure the horizontal angle between two distant objects, two church towers, for example, we proceed as follows: first place the circles in a horizontal position; revolve the circle *ABDE*, looking along the line *IO*, until the line of sight passes through one of the objects, and note the reading

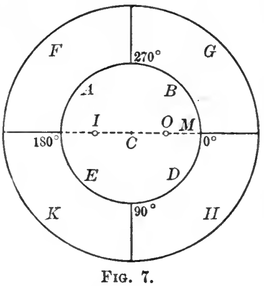


Fig. 7.

of the circle opposite the mark *M*; then revolve the circle *ABDE*, being careful not to move *FGHK*, until the line of sight passes through the second object, and note the new reading of the circle opposite the mark *M*. The difference between the two readings will be the angular distance required.

17. The Engineers' Transit, shown in Fig. 8, is used in measuring horizontal and vertical angles. The lower circle is provided with two levels, by which its horizontality is tested.

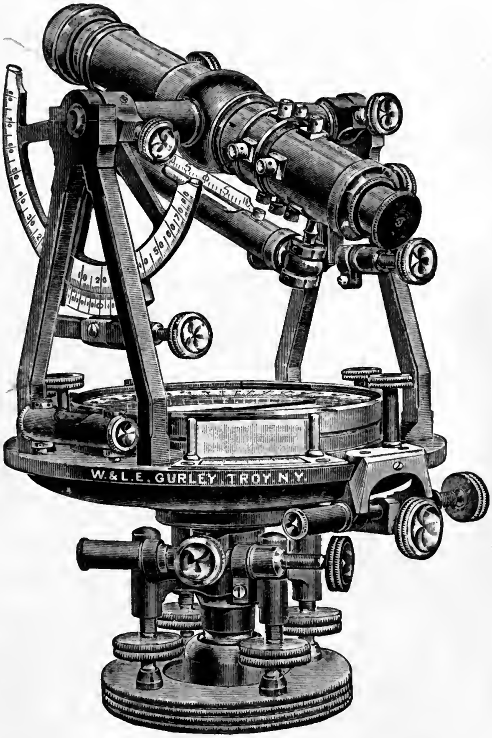


Fig. 8.

The rods *I* and *O* are replaced by the telescope with a system of intersecting wires in the common focus of the object glass and eyepiece, the telescope being capable of rotation about an axis parallel to the horizontal circle. The circle fixed to the axis of the telescope is vertical when the plate bearing the upright supports is horizontal.

18. Illustrations of the Application of the Ratios.*

1. A rope fastened to the top of a vertical pole 60 feet high, and to a stake driven in the ground, is inclined at an angle of 30°. How far is the stake from the bottom of the pole? How long is the rope?

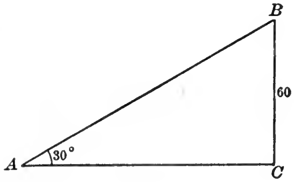


FIG. 9.

$$\frac{CB}{AC} = \tan 30° = \frac{1}{\sqrt{3}}.$$

$$\therefore\ AC = \sqrt{3}\,CB = 60\sqrt{3} \text{ feet.}$$

$$\frac{CB}{AB} = \sin 30° = \frac{1}{2}.$$

$$\therefore\ AB = 2\,CB = 120 \text{ feet.}$$

2. The angle at the vertex of a right circular cone is 60°, and the slant height is 10 inches. What is the altitude and the radius of the base of the cone?

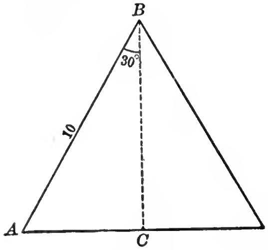


FIG. 10.

$$\frac{CB}{AB} = \cos 30° = \frac{\sqrt{3}}{2}.$$

$$\therefore\ CB = \frac{\sqrt{3}}{2} AB = 5\sqrt{3} \text{ inches.}$$

$$\frac{AC}{AB} = \sin 30° = \frac{1}{2}.$$

$$\therefore\ AC = \tfrac{1}{2} AB = 5 \text{ inches.}$$

3. The top of a ladder 30 feet long rests on the upper edge of a wall 15 feet high. What is the inclination of the ladder?

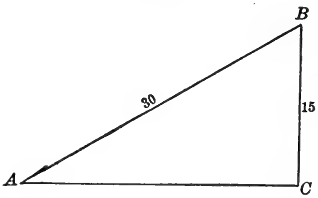


FIG. 11.

$$\sin CAB = \frac{CB}{AB} = \frac{15}{30} = \frac{1}{2};$$

but $\sin 30° = \frac{1}{2}$. $\therefore\ CAB = 30°$.

In these cases the ratios corresponding to the angles were known from Art. 14. Usually it will be necessary to refer to the tables in solving problems involving the ratios.

* It is assumed that the ground is horizontal.

CHAPTER II.

RIGHT PLANE TRIANGLES.

19. It has been shown in Geometry that a right-angled triangle can be constructed when two elements* besides the right angle are known, one of the known elements being a side. We also know that

(1) The hypotenuse is greater than either of the other two sides.

(2) The hypotenuse is less than the sum of the other two sides.

(3) The sum of the two acute angles must be 90°.

(4) The greater side is opposite the greater angle.

(5) The square on the hypotenuse is equal to the sum of the squares on the other two sides.

20. A triangle is said to be *solved* when, having some of the elements given, the others have been found by some process.

21. The Solution of a Right Triangle is effected by means of the trigonometric ratios. Each equation, as $\sin A = \frac{a}{c}$, contains three quantities; and two of them must be known in order that the third may be found. Hence in any particular case we use the equations that contain the two given elements; thus, if a and b are given, we use $\tan A = \frac{a}{b}$ to find A, and then c may be found from either $\sin A = \frac{a}{c}$ or $\cos A = \frac{b}{c}$.

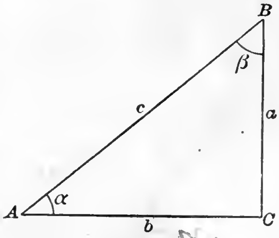


FIG. 12.

* The elements of a triangle are the three sides and the three angles.

The equations used in the solution of right triangles are

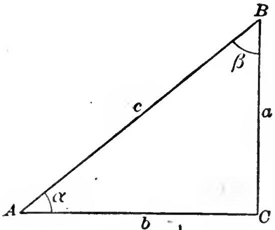


FIG. 13.

$$\left.\begin{aligned}\sin A &= \frac{a}{c} = \cos B.\\ \cos A &= \frac{b}{c} = \sin B.\\ \tan A &= \frac{a}{b} = \cot B.\\ \cot A &= \frac{b}{a} = \tan B.\\ A + B &= 90^\circ.\\ c^2 &= a^2 + b^2.\end{aligned}\right\} \quad (1)$$

22. From the Trigonometric Ratios we have

$$\left.\begin{aligned}\tan A &= \frac{a}{b}; \quad \therefore\ a = b \tan A,\\ \cot B &= \frac{a}{b}; \quad \therefore\ a = b \cot B,\end{aligned}\right\} \quad (1)$$

or, *any side of a right triangle is equal to the other side multiplied by the tangent of the angle opposite, or by the cotangent of the angle adjacent, to the side itself.*

$$\left.\begin{aligned}\sin A &= \frac{a}{c}; \quad \therefore\ a = c \sin A,\\ \cos B &= \frac{a}{c}; \quad \therefore\ a = c \cos B,\end{aligned}\right\} \quad (2)$$

or, *any side is equal to the hypotenuse multiplied by the sine of the opposite angle, or by the cosine of the adjacent angle.*

$$\left.\begin{aligned}\sec A &= \frac{c}{b}; \quad \therefore\ c = b \sec A,\\ \operatorname{cosec} B &= \frac{c}{b}; \quad \therefore\ c = b \operatorname{cosec} B,\end{aligned}\right\} \quad (3)$$

or, *the hypotenuse is equal to a side multiplied by the secant of the adjacent angle, or by the cosecant of the opposite angle.*

NOTE. — The secant of an angle is the reciprocal of its cosine, and the cosecant is the reciprocal of its sine; hence the logarithm of the secant is the arithmetical complement of that of the cosine, and the logarithm of the cosecant is the A. C. of that of the sine, or

$$\log \sec x = \operatorname{colog} \cos x, \text{ and } \log \operatorname{cosec} x = \operatorname{colog} \sin x.$$

23. Case I. Given *c* and *A*.

Formulas: $\begin{cases} a = c \sin A. \\ b = c \cos A. \\ B = 90° - A. \end{cases}$

1. Solve the triangle when $c = 1.0034$, and $A = 42° 10'.3$.

$$\therefore B = 90° - A = 47° 49'.7.$$

(*a*) By natural functions.

$$a = c \sin A = 1.0034 \times 0.67136 = 0.67364.$$
$$b = c \cos A = 1.0034 \times 0.74114 = 0.74366.$$

(*b*) By the use of logarithms.

$$a = c \sin A; \therefore \log a = \log c + \log \sin A.$$
$$b = c \cos A; \therefore \log b = \log c + \log \cos A.$$

Always write first all the formulas that will be used in the problem; then write them in a form adapted to logarithmic computation; then refer to the tables and write the logarithms in their proper places. Thus in this case we arrange the work as follows:

$\log c =$	$\log c =$	or	$+ \log \sin A =$	(1)
$+ \log \sin A =$	$+ \log \cos A =$		$\log c =$	(3)
$\therefore \log a =$	$\therefore \log b =$		$+ \log \cos A =$	(2)
$\therefore a =$	$\therefore b =$		$\therefore \log a = (1) + (3)$	(4)
			$a =$	(6)
			$\therefore \log b = (2) + (3)$	(5)
			$b =$	(7)

The positive signs preceding log sin *A* and log cos *A* indicate that they are to be added to log *c*.

We now find the angle *A* in the table of logarithmic functions and take from the table both log sin *A* and log cos *A*, writing them in their proper places. Then we refer to the table of logarithms of numbers and find log *c*, writing it opposite log *c*. Then we add the proper quantities to find log *a* and log *b*, finally looking in the table of the logarithms of numbers for the numbers corresponding to the computed values of log *a* and log *b*.

The arrangement on the right is preferable, since it saves the writing of one line. The numbers in the parentheses indicate the order in which the quantities should be found.

$$\begin{aligned} \log c &= 0.00147 \\ +\log \sin A &= 9.82695-10 \\ \hline \log a &= 9.82842-10 \\ a &= 0.67363 \end{aligned} \qquad \begin{aligned} \log c &= 0.00147 \\ +\log \cos A &= 9.86990-10 \\ \hline \log b &= 9.87137-10 \\ b &= 0.74365 \end{aligned}$$

or

$$\begin{aligned} \log \sin A &= 9.82695-10 \\ \log c &= 0.00147 \\ \log \cos A &= 9.86990-10 \\ \hline \log a &= 9.82842-10 \\ a &= 0.67363 \\ \log b &= 9.87137-10 \\ b &= 0.74365 \end{aligned}$$

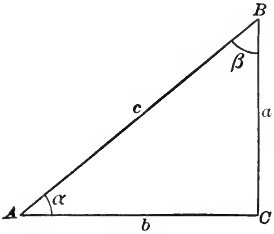


Fig. 14.

Check:

$$\begin{aligned} a^2 &= (c+b)(c-b) \\ c+b &= 1.74705 \\ c-b &= 0.25975 \\ \hline \log(c+b) &= 0.24230 \\ \log(c-b) &= 9.41456-10 \\ \hline \therefore \log a^2 &= 9.65686-10 \\ \log a &= 9.82843-10 \end{aligned}$$

Exact agreement is not expected, since the tables give the values of the functions only to the *nearest* unit in the fifth decimal place. The -10 is usually omitted, and $\sin A$ is written for $\log \sin A$, when there is no danger of confusion.

2. Solve the triangle when $c = 34.687$, and $B = 49° 8'.4$.

Ans. $A = 40° 51'.6$; $b = 26.234$; $a = 22.6925$.

3. Solve the triangle when $c = 305$, and $A = 63° 31'.14$, using the natural functions.

Ans. $a = 273.00$; $b = 136.00$.

4. Solve the triangle when $c = 205$, and $B = 49° 33'.01$, using the natural functions.

Ans. $a = 133.00$; $b = 156.00$.

24. Case II. Given c and a.

Formulas:

$$\begin{cases} \sin A = \dfrac{a}{c}. \\ b = a \cot A = c \cos A. \\ B = 90° - A. \end{cases}$$

1. Solve the triangle when $c = 8.7982$, and $a = 3.1292$.

$\therefore \log \sin A = \log a - \log c$; $\log b = \log a + \log \cot A = \log c + \log \cos A$.

$$\begin{aligned} \log a &= 0.49544 \\ -\log c &= 0.94439 \\ \hline \log \sin A &= 9.55105-10 \\ A &= 20° 50'.1 \\ B &= 69° \; 9'.9 \end{aligned} \qquad \begin{aligned} \log a &= 0.49544 \\ +\log \cot A &= 0.41958 \\ \hline \log b &= 0.91502 \\ b &= 8.2228 \end{aligned} \qquad \begin{aligned} \log c &= 0.94439 \\ +\log \cos A &= 9.97063-10 \\ \hline \log b &= 0.91502 \\ b &= 8.2228 \end{aligned}$$

or * $\log \cot A = 0.41958$ (5)
$\log a = 0.49544$ (1)
$-\log c = 0.94439$ (2)
$\log \cos A = 9.97063$ (6)

$\log \sin A = 9.55105$ (1)−(2)
$A = 20° 50'.1$ (4)
$B = 69° 9'.9$
$\log b = 0.91502$ $\begin{cases}(1)+(5)\\(2)+(6)\end{cases}$
$b = 8.2228$

Check: $b^2 = (c-a)(c+a)$
$c - a = 5.6690$, $\log(c-a) = 0.75351$
$c + a = 11.9274$, $\log(c+a) = 1.07655$
$\log b^2 = 1.83006$
$\log b = 0.91503$

2. Solve the triangle when $c = 369.27$, and $b = 235.64$.
Ans. $A = 50° 20'.9$; $B = 39° 39'.1$; $a = 284.31$.

3. Solve the triangle when $c = 281$, and $a = 160$, using the natural functions.
Ans. $A = 34° 42'.5$; $b = 231.00$ or 231.01.

4. Solve the triangle when $c = 365$, and $b = 76$, using the natural functions.
Ans. $A = 77° 58'.93$; $a = 357.00$.

25. Case III. Given a and b.

Formulas:
$$\begin{cases}\tan A = \dfrac{a}{b}.\\[2ex] c = \dfrac{a}{\sin A} = \dfrac{b}{\cos A}.\\[2ex] B = 90° - A.\end{cases}$$

1. Solve the triangle when $a = 169.03$, and $b = 203.44$.

$\therefore \log \tan A = \log a - \log b$; $\log c = \log a - \log \sin A = \log b - \log \cos A$.

$\log a = 2.22796$
$-\log b = 2.30843$
$\log \tan A = 9.91953 - 10$
$A = 39° 43'.3$
$B = 50° 16'.7$

$\log a = 2.22796$
$-\log \sin A = 9.80555 - 10$
$\log c = 2.42241$
$c = 264.49$

$\log b = 2.30843$
$-\log \cos A = 9.88602 - 10$
$\log c = 2.42241$
$c = 264.49$

or * $\log a = 2.22796$ (1)
$\log \sin A = 9.80555$ (5)
$\log \cos A = 9.88602$ (6)
$\log b = 2.30843$ (2)
$\therefore \log \tan A = 9.91953$ (3)
$A = 39° 43'.3$ (4)
$B = 50° 16'.7$
$\log c = 2.42241$ $\begin{cases}(1)-(5)\\(2)-(6)\end{cases}$
$c = 264.49$

Check: $a^2 = c^2 - b^2$
$c + b = 467.93$
$c - b = 61.05$
$\log(c+b) = 2.67018$
$\log(c-b) = 1.78569$
$\therefore \log a^2 = 4.45587$
$\log a = 2.22794$

* This form is preferable.

2. Solve the triangle when $a = 4.8199$, and $b = 2.6492$.

Ans. $A = 61° 12'.3$; $B = 28° 47'.7$; $c = 5.4999$.

3. Solve the triangle when $a = 60$, and $b = 91$, using the natural functions.

Ans. $A = 33° 23'.9$; $c = 109.00$.

4. Solve the triangle when $a = 72$, and $b = 65$, using the natural functions.

Ans. $A = 47° 55'.5$; $c = 97.000$.

26. Case IV. Given a and A.

Formulas:
$$\begin{cases} b = a \cot A. \\ c = \dfrac{a}{\sin A} = \dfrac{b}{\cos A}. \\ B = 90° - A. \end{cases}$$

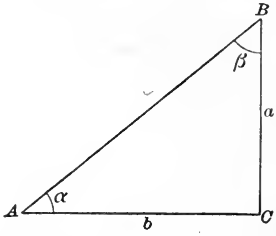


Fig. 15.

1. Solve the triangle when $a = 613.35$, and $A = 40° 12'.6$.

$\therefore\ B = 90° - A = 49° 47'.4.$

$\log b = \log a + \log \cot A.$

$\log c = \log a - \log \sin A = \log b - \log \cos A.$

$$\begin{aligned} \log a &= 2.78770 \\ + \log \cot A &= 0.07295 \\ \hline \log b &= 2.86065 \\ b &= 725.52 \end{aligned} \qquad \begin{aligned} \log a &= 2.78770 \\ - \log \sin A &= 9.80996 - 10 \\ \hline \log c &= 2.97774 \\ c &= 950.04 \end{aligned} \qquad \begin{aligned} \log b &= 2.86065 \\ - \log \cos A &= 9.88291 - 10 \\ \hline \log c &= 2.97774 \\ c &= 950.04 \end{aligned}$$

or
$$\begin{aligned} \log \sin A &= 9.80996 \quad (1) \\ \log a &= 2.78770 \quad (3) \\ \log \cot A &= 0.07295 \quad (2) \\ \hline \log c &= 2.97774 \quad (3)-(1) \\ c &= 950.04 \\ \log b &= 2.86065 \quad (3)+(2) \\ b &= 725.52 \end{aligned}$$

Check: $a^2 = (c+b)(c-b)$

$$\begin{aligned} c + b &= 1675.56, & \log(c+b) &= 3.22416 \\ c - b &= 224.52, & \log(c-b) &= 2.35126 \\ \hline & & \log a^2 &= 5.57542 \\ & & \log a &= 2.78771 \end{aligned}$$

2. Solve the triangle when $a = 3.6378$, and $B = 69° 23'.5$.

Ans. $A = 20° 36'.5$; $b = 9.6738$; $c = 10.335$.

3. Solve the triangle when $b = 160$, and $A = 55° 17'.48$, using the natural functions.

Ans. $c = 281.00$; $a = 231.00$.

4. Solve the triangle when $a = 340$, and $A = 60° 55'.85$, using the natural functions.

Ans. $c = 389.00$; $b = 189.00$.

27. Isosceles Triangles. — If a perpendicular to the base is drawn from the vertex, it will bisect the base and the angle at the vertex, forming two equal right triangles.

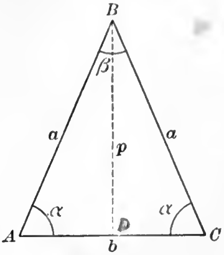


FIG. 16.

$$\angle ABD = \angle DBC = \tfrac{1}{2}\beta\,;\ AB = BC\,;$$

$$AD = DC = \tfrac{1}{2}b.$$

1. Solve the triangle when $b = 2.1452$, and $\beta = 121° \, 14'.6$.

$\therefore\ AD = 1.0726\,;\ ABD = 60° \, 37'.3\,;$

$\alpha = 90° - \tfrac{1}{2}\beta = 29° \, 22'.7.$

$$a = \frac{\tfrac{1}{2}b}{\sin\tfrac{1}{2}\beta}\,;\quad \therefore\ \log a = \log\tfrac{1}{2}b - \log\sin\tfrac{1}{2}\beta.$$

$$p = \tfrac{1}{2}b\cot\tfrac{1}{2}\beta\,;\quad \therefore\ \log p = \log\tfrac{1}{2}b + \log\cot\tfrac{1}{2}\beta.$$

$$\begin{aligned} \log\tfrac{1}{2}b &= 0.03044 \\ -\log\sin\tfrac{1}{2}\beta &= 9.94022 - 10 \\ \hline \log a &= 0.09022 \\ a &= 1.2309 \end{aligned} \qquad \begin{aligned} \log\tfrac{1}{2}b &= 0.03044 \\ +\log\cot\tfrac{1}{2}\beta &= 9.75049 - 10 \\ \hline \log p &= 9.78093 - 10 \\ p &= 0.60385 \end{aligned}$$

2. Solve the triangle when $\alpha = 52° \, 10'.2$, and $a = 600.2$.

Ans. $\beta = 75° \, 39'.6\,;\ \tfrac{1}{2}b = 368.12\,;\ p = 474.07.$

28. Given *c* and *b* (Special Method). — When b nearly equals c, the angle found from the formula $\cos A = \dfrac{b}{c}$ is uncertain, the tabular difference for the cosine being so small that a small error in $\cos A$ would produce a large error in A.

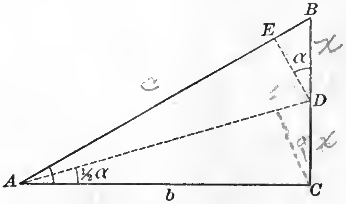


FIG. 17.

In the figure, AD bisects the angle A, and DE is perpendicular to AB; $\therefore\ DE = CD$. Let $CD = x = DE$;

$$\therefore\ \tan\tfrac{1}{2}\alpha = \frac{x}{b} \qquad (1)$$

Also, $$CB = a = CD + DB = CD + DE\sec\alpha\,;$$

$$\therefore\ a = x + x\sec\alpha\,;\quad \therefore\ x = \frac{a}{1+\sec\alpha}\,;$$

$$\therefore\ x = \frac{a}{1+\dfrac{c}{b}} = \frac{ab}{c+b}. \qquad (2)$$

From (1) and (2),

$$\tan \tfrac{1}{2} \alpha = \frac{a}{c+b} = \frac{\sqrt{c^2 - b^2}}{c+b} = \sqrt{\frac{(c+b)(c-b)}{(c+b)^2}};$$

$$\therefore \tan \tfrac{1}{2} \alpha = \sqrt{\frac{c-b}{c+b}}. \tag{3}$$

Suppose that we wish to find the greatest distance at sea at which a mountain 4.3 miles high can be seen, the earth being considered as a sphere with a radius of 3963.3 miles, and the distance being measured as a chord.

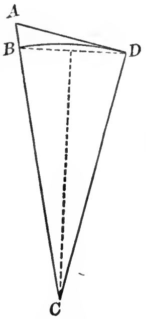


FIG. 18.

Let $BA = 4.3$, and $CB = CD = 3963.3$; BD being the distance required. Then $\cos DCA = \frac{CD}{CA}$, giving $\log \cos DCA = 9.99952$; and DCA as found from the tables might have any value between 2° 40′.5 and 2° 42′.5.

Using (3), we have

$$\begin{aligned}
CA - CD &= \quad 4.3; \ \log = 0.63347 \\
CA + CD &= 7930.9; \ \log = 3.89932 \\
& \qquad 2)\underline{6.73415 - 10} \\
\log \tan \tfrac{1}{2} DCA &= 8.36708 - 10 \\
\text{Cpl. } T' &= 3.53620 \\
\log (\tfrac{1}{2} DCA)' &= 1.90328
\end{aligned}$$

$$\therefore \tfrac{1}{2} DCA = 80'.035; \quad \therefore DCA = 2° 40'.07.$$

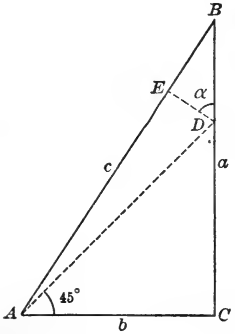
Then $BD = 2 CD \sin \frac{1}{2} DCA$ will give the chord BD. The arc BD is found from the proportion:

$$360° : DCA = 2\pi \times 3963.3 : \text{arc } BD.$$

NOTE. — Eq. (3) follows directly from (4), Art. 69:

$$\tan \tfrac{1}{2} \alpha = \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}}, \text{ where } \cos \alpha = \frac{b}{c}.$$

29. Given a and b (Special Method). — When a and b are nearly equal, the angle α may be determined more accurately, as follows:



Draw AD, making $CAD = 45°$, and DE perpendicular to AB. Then

$$\tan DAE = \tan(\alpha - 45°) = \frac{DE}{AE}.$$

But $DE = DB \cos \alpha = (CB - CD) \cos \alpha$

$$= (a - b) \cos \alpha = \frac{(a-b)b}{c},$$

and

$$AE = AB - EB = AB - DB \sin\alpha = c - \frac{(a-b)a}{c} = \frac{c^2 - a^2 + ab}{c}$$

$$= \frac{b^2 + ab}{c}.$$

$$\therefore \frac{DE}{AE} = \frac{(a-b)b}{ab+b^2} = \frac{a-b}{a+b}.$$

$$\therefore \tan(\alpha - 45°) = \frac{a-b}{a+b}. \qquad (1)$$

If b were greater than a, the formula would be

$$\tan(45° - \alpha) = \frac{b-a}{b+a}. \qquad (2)$$

NOTE. — Eq. (1) may be found from the relation proved in Art. 100:

$\frac{a-b}{a+b} = \frac{\tan\frac{1}{2}(\alpha - \beta)}{\tan\frac{1}{2}(\alpha + \beta)}$, where $\frac{1}{2}(\alpha + \beta) = 45°$, and $\frac{1}{2}(\alpha - \beta) = \alpha - 45°$.

EXAMPLES.

NOTE. — The angle between the line of sight and a horizontal plane is called an *angle of elevation* when the point sighted on is above the horizontal plane, and an *angle of depression* when it is below the horizontal plane.

1. The shadow of a vertical pole 30 feet high is 40 feet long. Find the elevation of the sun above the horizon. *Ans.* 36° 52′.2.

2. The vertical central pole of a circular tent is 20 feet high, and its top is fastened by ropes 40 feet long to stakes set in the ground, the ground being horizontal. How far are the stakes from the foot of the pole, and what is the inclination of the ropes to the ground? *Ans.* 34.641 feet; 30°.

3. The top of a lighthouse is 200 feet above the sea level, and the angle of depression to a buoy is 9° 52′.8. Find the horizontal distance of the buoy from the lighthouse. *Ans.* 1148.3 feet.

4. The horizontal distance from a point to the vertical wall of a tower is 1000 feet, and the angle of elevation of the top is 4° 15′.2. Find the height of the top of the wall above the point. *Ans.* 74.370 feet.

5. Two points A and B are on the opposite banks of a stream. A line AC at right angles to AB is measured 300 feet long, and the angle ACB is found by measurement to be 62° 30′.4. What is the distance from A to B?

Ans. 576.45 feet.

6. From the top of a lighthouse, 150 feet above the sea level, the angle of depression to a buoy was 12° 10′.2, and that to the shore, measured in the same vertical plane with the buoy, was 62° 14′.8. Find the distance in feet of the buoy from the shore. *Ans.* Log. Tables, 616.60; Nat. Tables, 616.61

7. The angle of elevation to the top of the vertical wall of a tower is 20° 10 .4, and the angle of depression to the bottom is 10° 11′.6, the horizontal distance from the observer to the wall being 250 feet. Find the height of the wall. *Ans.* 136.802 feet.

8. We wish to make a ladder that would reach from a point 20 feet in front of a building to the fourth story, a height of 45 feet. Find the length of the ladder and the angle it would make with the ground in this position. *Ans.* 49.244 feet; 66° 2′.2.

9. The ridgepole of a roof is 15 feet above the center of the garret floor, and the garret is 40 feet wide. What is the inclination of the roof to a horizontal plane? *Ans.* 36° 52′.2.

10. A chord of a circle is 20 feet long, and the angle at the center subtended by it is 46° 43′.6. Find the radius of the circle. *Ans.* 25.217 feet.

11. The angle between two lines is 40° 12′.4, and a circle whose radius is 5730 feet is tangent to both lines. Find the distance from the point of tangency to the point of intersection of the two lines when the circle is in the smaller angle, and when it is in the larger angle formed by producing one of the lines. *Ans.* 15655 and 2097.2 feet.

12. The legs of a pair of dividers are set so that the angle between them is 80° 24′.4. What is the distance between the points, the legs being 6 inches long? *Ans.* 7.7460 inches.

13. An equilateral triangle is circumscribed about a circle whose radius is 10 inches. Find the perimeter of the triangle. *Ans.* $60\sqrt{3}$ inches.

14. A wedge measures 12 inches along the side, and its base is 2 inches wide. Find the angle at its vertex. *Ans.* 9° 33′.6.

15. The side of a regular decagon is 2.4304 feet. Find the radii of the inscribed and circumscribed circles. *Ans.* 3.7400 feet; 3.9325 feet.

16. The area of a regular octagon is 24 square feet. Find the radius of the inscribed circle and the length of one of the sides. *Ans.* 2.6912 feet; 2.229$\bar{5}$ feet.

17. The radius of the circumscribing circle of a regular dodecagon is 10 feet. Find the area of the dodecagon. *Ans.* 300.00 square feet.

18. A cord is stretched around two wheels with radii of 7 feet and 1 foot respectively, and with their centers 12 feet apart. Prove that the length of the cord is $12\sqrt{3} + 10\pi$ feet.

19. A cord is stretched around, and crossed between, two wheels whose radii are 5 feet and 1 foot respectively, their centers being 12 feet apart. Prove that the length of the cord is $12\sqrt{3} + 8\pi$ feet.

20. Find the radius and the length of an arc of 1° of the parallel of latitude at a place whose latitude is 42° 43′.9, the earth being regarded as a sphere whose radius is 3963.3 miles. *Ans.* 2911.1 miles; 50.809 miles.

21. The altitude of a right circular cone is 4.1436 feet, and the angle at its vertex is 20° 14′.2. Find its convex surface. *Ans.* 9.7780 square feet.

22. The altitude of a right pyramid with a square base is 14.453 feet, and the sides of the base are each 4.7036 feet. Find its slant height, its lateral edge, and the angle between a face of the pyramid and its base.

Ans. 14.643 feet; 14.831 feet; 80° 45′.5.

23. The base of a trapezoid measured 600.430 feet, and the angles at the ends of the base were found to be 62° 14′.3 and 74° 18′.6. Find the length of the other base, the altitude being 40 feet. *Ans.* 568.138 feet.

24. Find the length of the perpendicular from the vertex of the right angle of a triangle to the hypotenuse, the hypotenuse being 6.4603 inches long, and one of the angles of the triangle being 40° 40′.4.

Ans. 3.1934 inches.

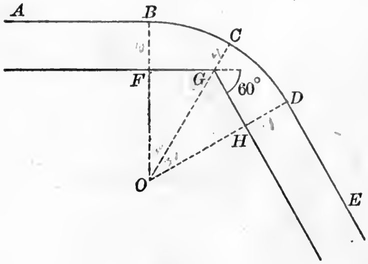


Fig. 20.

25. A street-railway track is 10 feet from the curbstone ($FB = HD = 10$), and in passing a corner where the street is deflected through an angle of 60°, the rail must be 4 feet from the corner ($GC = 4$). Find the radius of the circular curve. *Ans.* $OC = \frac{20 - 4\sqrt{3}}{2 - \sqrt{3}}$.

26. Before paying for a pavement, it was necessary to find the area shaded in Fig. 21. Prove that it is $\frac{28750}{\sqrt{3}} + 7500$ square feet, the streets being 50 feet wide.

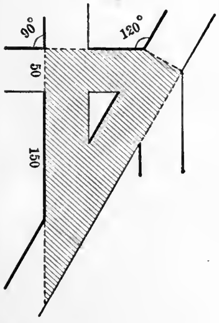


Fig. 21.

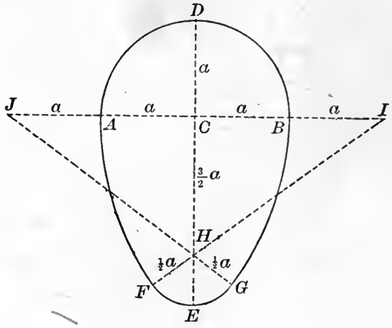


Fig. 22.

27. In the egg-shaped sewer (Fig. 22), C is the center of the arc ADB with a radius a; I and J, of AF and BG respectively with the radii $3a$; and H, of FEG with the radius $\frac{1}{2}a$. Prove that its area is

$$a^2\left(\frac{\pi}{2} + \frac{1}{4}\tan^{-1}\frac{4}{3} + 9\tan^{-1}\frac{3}{4} - 3\right) = a^2\left(\frac{5}{8}\pi + \frac{35}{4}\tan^{-1}\frac{3}{4} - 3\right) = 4.59413\,a^2,$$

where $\tan^{-1}\frac{4}{3}$ is the angle whose tangent is $\frac{4}{3}$.

28. A hill rises 1 foot vertically in a horizontal distance of 30 feet. What is the difference of elevation of two points that are 1000 feet apart, the distance being measured on the ground?

$$\begin{aligned} \log \tan \alpha &= 8.52288 - 10 \\ \text{Cpl. } T' &= 3.53611 \\ \hline \log \alpha' &= 2.05899 \\ S' &= 6.46365 - 10 \\ \hline \log \sin \alpha &= 8.52264 - 10 \\ \log 1000 &= 3. \\ \hline \log \text{diff. of elev.} &= 1.52264 \\ \text{diff. of elev.} &= 33.315 \text{ feet.} \end{aligned}$$

29. The horizontal distance between the two extreme positions of the end of a pendulum 40 inches long is 4 inches. Through what angle does it swing?

Half-angle $= 2° \, 51'.96$. *Ans.* $5° \, 43'.92$.

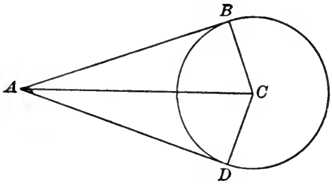


FIG. 23.

30. The angular diameter of the moon is $31'.12$, and its distance is 238 840 miles. Find its diameter in miles.

$BAD = 31'.12$, and $AC = 238\,840$.

Ans. 2162.0 miles.

31. The equatorial horizontal parallax of the sun is $8''.8$, and the radius of the earth is 3963.3 miles. Find the distance of the sun from the earth.

$BAC = 8''.8$, and $BC = 3963.3$. *Ans.* 92 896 000 miles.

32. A circular chimney 100 feet high is 10 feet in diameter at the base, and 8 feet at the top. Find the angle at the vertex of the cone of which it is a frustum.

Half-angle $= 34'.376$. *Ans.* $1° \, 8'.752$.

Solve the following triangles, the first two elements being given:

33. $c = 0.02934$, $A = 31° \, 14'.2$. $\therefore B = 58° \, 45'.8$; $a = 0.015215$; $b = 0.025086$.

34. $c = 4.6136$, $B = 47° \, 15'.6$. $\therefore A = 42° \, 44'.4$; $a = 3.1311$; $b = 3.3885$.

35. $c = 436.53$, $A = 74° \, 10'.6$. $\therefore B = 15° \, 49'.4$; $a = 419.98$; $b = 119.03$.

36. $c = 0.96724$, $B = 40° \, 40'.2$. $\therefore A = 49° \, 19'.8$; $a = 0.73363$; $b = 0.63036$.

37. $c = 110.97$, $a = 67.291$. $\therefore A = 37° \, 19'.8$; $B = 52° \, 40'.2$; $b = 88.236$.

38. $c = 1843.7$, $b = 618.42$. $\therefore A = 70° \, 24'.1$; $B = 19° \, 35'.9$; $a = 1736.9$.

39. $c = 8226.5$, $a = 814.33$. $\therefore A = 81° \, 50'.5$; $B = 8° \, 9'.5$; $b = 116.74$.

40. $c = 0.03672$, $b = 0.01296$. $\therefore A = 69° \, 19'.9$; $B = 20° \, 40'.1$; $a = 0.034357$.

41. $c = 4.8293$, $b = 0.31435$. $\therefore A = 86° \, 16'.1$; $B = 3° \, 43'.9$; $a = 4.8191$.

42. $a = 43.148$, $b = 84.107$. $\therefore A = 27° \; 9'.5$; $B = 62° \; 50'.5$; $c = 94.530$.

43. $a = 759.28$, $b = 51.85$. $\therefore A = 86° \; 5'.6$; $B = 3° \; 54'.4$; $c = 761.05$.

44. $a = 7642.5$, $b = 864.7$. $\therefore A = 83° \; 32'.7$; $B = 6° \; 27'.3$; $c = 7691.3$.

45. $a = 0.04326$, $b = 0.54318$. $\therefore A = 4° \; 33'.2$; $B = 85° \; 26'.8$; $c = 0.54489$.

46. $a = 903.64$, $A = 22° \; 10'.3$. $\therefore B = 67° \; 49'.7$; $b = 2217.4$; $c = 2394.5$.

47. $b = 0.47922$, $A = 62° \; 16'.4$. $\therefore B = 27° \; 43'.6$; $a = 0.91176$; $c = 1.0300$.

48. $a = 8.4642$, $B = 30° \; 16'.4$. $\therefore A = 59° \; 43'.6$; $b = 4.9409$; $c = 9.80075$.

49. $b = 18.436$, $B = 65° \; 15'.6$. $\therefore A = 24° \; 44'.4$; $a = 8.4954$; $c = 20.299$.

Solve the isosceles triangles (Fig. 16) in the following examples, the first two elements being given:

50. $a = 57.906$, $b = 62.736$. $\therefore \alpha = 57° \; 12'.05$; $\beta = 65° \; 35'.9$; $p = 48.673$.

51. $a = 3.4782$, $\alpha = 20° \; 20'.6$. $\therefore \beta = 139° \; 18'.8$; $b = 6.5224$; $p = 1.2091$.

52. $a = 99.674$, $\beta = 40° \; 30'.4$. $\therefore \alpha = 69° \; 44'.8$; $b = 69.008$; $p = 93.510$.

53. $b = 0.96042$, $\alpha = 70° \; 10'.4$. $\therefore \beta = 39° \; 39'.2$; $a = 1.4158$; $p = 1.3319$.

54. $b = 1146.48$, $\beta = 80° \; 36'.4$. $\therefore \alpha = 49° \; 41'.8$; $a = 886.24$; $p = 675.87$.

55. $a = 87.904$, $p = 46.812$. $\therefore \alpha = 32° \; 10'.6$; $\beta = 115° \; 38'.8$; $b = 148.806$.

56. $b = 6.9044$. $p = 5.7806$. $\therefore \alpha = 59° \; 9'.2$; $\beta = 61° \; 41'.6$; $a = 6.7330$.

57. $p = 18.478$, $\alpha = 37° \; 19'.8$. $\therefore \beta = 105° \; 20'.4$; $a = 30.471$; $b = 48.458$.

58. $p = 0.46424$, $\beta = 100° \; 36'.8$. $\therefore \alpha = 39° \; 41'.6$; $a = 0.72690$; $b = 1.11865$.

CHAPTER III.

TRIGONOMETRIC FUNCTIONS OF ANY ANGLE.

30. Generation of Angles. — An angle may be considered as generated by a line revolving about a fixed point, the vertex; thus OA revolving about O in the direction a, to the position OB, describes the angle AOB. The side of the angle *from which* the revolution takes place is called the *initial side*, and that *to which* the describing line moves is called the *terminal side*.

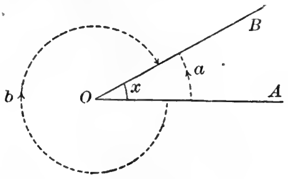


FIG. 24.

The letters describing the initial side are written first in the symbol of the angle, so that the angle AOB is one in which the motion is from OA to OB.

31. Direction of Measurement. — The revolving line can move from OA to OB either in the direction marked a or in that marked b. The former motion, contrary to that of the hands of a watch, is arbitrarily considered positive and the latter negative. Thus if the angle x, between OA and OB, is 30°, the angle AOB is either $+30°$ or $-330°$.

Any angle has two measures less than 360°, one positive and the other negative, their numerical sum being 360°.

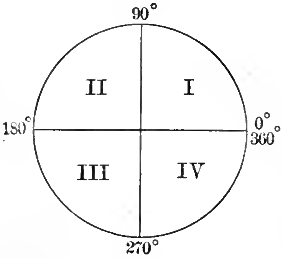


FIG. 25.

32. Quadrants. — For convenience the measuring circle is divided into four parts called *quadrants*, as in the figure. An angle is in the first quadrant when its value lies between 0° and 90°; in the second, between 90° and 180°; in the third, between 180°

and 270°; in the fourth, between 270° and 360°. Angles between 0° and − 90° are in the fourth quadrant; between − 90° and − 180°, in the third; between − 180° and − 270°, in the second; between − 270° and − 360°, in the first.

Also, an angle between zero and $\frac{1}{2}\pi$ is in the first quadrant; between $\frac{1}{2}\pi$ and π, in the second; between π and $\frac{3}{2}\pi$, in the third; and between $\frac{3}{2}\pi$ and 2π, in the fourth.

33. Complement and Supplement. — Two angles are said to be complementary when their algebraic sum is 90°, as 60° and 30°, 120° and − 30°, 260° and − 170°; and supplementary when their algebraic sum is 180°, as 120° and 60°, 230° and − 50°, 300° and − 120°.

NOTE. — In Fig. 2, $\frac{a}{h}$ is the sine of B; that is, it is the sine of the complement of A, and hence it is called the *co*sine of A.

Since $\frac{1}{2}\pi$ corresponds to 90°, and π to 180°, two angles are complementary when the algebraic sum of their circular measures is $\frac{1}{2}\pi$, and supplementary when it is π.

1. The complement of 200° is $90° - 200° = -110°$.

2. The complement of $90° + x$ is $90° - (90° + x) = -x$.

3. The supplement of 200° is $180° - 200° = -20°$.

4. The supplement of $270° + x$ is $180° - (270° + x) = -90° - x$.

5. The complement of $\frac{9}{10}\pi$ is $\frac{1}{2}\pi - \frac{9}{10}\pi = -\frac{2}{5}\pi$.

6. The supplement of $\frac{5}{3}\pi$ is $\pi - \frac{5}{3}\pi = -\frac{2}{3}\pi$.

Show that the complement of the first angle of each of the following pairs is equal to the second angle:

7. 145° and −55°; 300° and −210°; −70° and +160°; −200° and +290°.

8. $180° - x$ and $-90° + x$; $270° - x$ and $-180° + x$; $360° - x$ and $-270° + x$.

9. $\frac{1}{4}\pi$ and $\frac{1}{4}\pi$; $\frac{3}{2}\pi$ and $-\pi$; $\pi - x$ and $x - \frac{1}{2}\pi$; $\frac{2}{3}\pi + x$ and $-\frac{1}{6}\pi - x$.

Show that the supplement of the first angle of each of the following pairs is equal to the second angle:

10. 145° and 35°; 225° and − 45°; − 160° and 340°; − 70° and 250°.

11. $270° - x$ and $-90° + x$; $90° + x$ and $90° - x$; $x - 90°$ and $270° - x$.

12. $\frac{1}{4}\pi$ and $\frac{3}{4}\pi$; $\frac{5}{3}\pi$ and $-\frac{2}{3}\pi$; $x - \pi$ and $2\pi - x$; $\frac{3}{2}\pi + x$ and $-\frac{1}{2}\pi - x$.

34. General Measure of an Angle. —The line OA may be brought into the position OB by revolving either through the small angle x, or through that angle and then through any number of complete revolutions in either direction. The general measure of the angle AOB is then not x, but $x+n\,360°$, where n is any whole number, positive or negative.

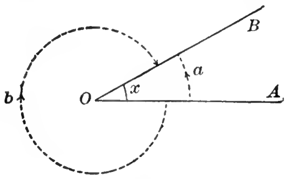


FIG. 26.

The general circular measure of the angle whose circular measure less than 2π is x would be $x+2n\pi$, since 2π corresponds to a complete revolution.

1. Show that 1000° is in the fourth quadrant.*

$1000° = 720° + 280° = 2 \times 360° + 280°$, two complete revolutions and 280° beyond; 280° lies in the fourth quadrant.

2. Show that $-3000°$ is in the third quadrant.

$-3000° = -2880° - 120° = 8(-360°) - 120°$, eight complete revolutions and 120° beyond in the negative direction; $-120°$ lies in the third quadrant.

3. Show that $\frac{\pi}{2}(8n + \frac{3}{5})$ is in the first quadrant.

$\frac{\pi}{2}(8n + \frac{3}{5}) = 2n \times 2\pi + \frac{3}{10}\pi$, $2n$ complete revolutions and $\frac{3}{10}\pi$ beyond; $\frac{3}{10}\pi$ is in the first quadrant.

4. Show that 1500° is in the first quadrant, 2690° in the second, 2720° in the third, 2100° in the fourth.

5. Show that $-910°$ is in the second quadrant, $-1100°$ in the fourth, $-1400°$ in the first, $-1920°$ in the third.

6. Show that $\frac{\pi}{5}(10n + 6)$ is in the third quadrant, $\frac{\pi}{3}(12n + 2)$ in the second, $\frac{\pi}{4}(8n + 7)$ in the fourth, $\frac{2}{3}\pi(3n + 2)$ in the third.

7. Show that $\frac{4}{5}\pi(10n - \frac{1}{2})$ is in the fourth quadrant, $\frac{4}{5}\pi(15n - \frac{2}{3})$ in the third, $\frac{4}{3}\pi(-9n - \frac{2}{3})$ in the third, $\frac{1}{5}\pi(10n - 9)$ in the first.

8. Show that $\frac{\pi}{3}(9n + 1)$ will lie in the third or in the first quadrant, according as n is odd or even.

9. Show that the general circular measure of 0° is $2n\pi$, and not $n\pi$.

10. Show that the general circular measure of 90° is $(2n + \frac{1}{2})\pi$; of 180°, $(2n + 1)\pi$; of 270°, $(2n + \frac{3}{2})\pi$.

11. If $x = 60°$, show that one third of the general measure of x will be 20°, 140°, and 260°, the terminal side of the angle for all values of $\frac{1}{3}x$ greater than 260° falling in one of these positions.

We have, using the general measure, $x + n\,360°$,

$$x = 60°,\ 420°,\ 780°,\ 1140°,\ 1500°,\ 1860°, \cdots$$

$$\therefore\ \tfrac{1}{3}x = 20°,\ 140°,\ 260°,\ 380°,\ 500°,\ 620°, \cdots$$

or

$$\tfrac{1}{3}x = 20°,\ 140°,\ 260°,\ 20°,\ 140°,\ 260°, \cdots$$

if we reduce the values of $\frac{1}{3}x$ that are greater than 360° to others less than 360° by subtracting some multiple of 360°.

* That is, show that when the angle is 1000° the terminal side will lie in the fourth quadrant.

12. If $x = 45°$, show that $\frac{1}{3}x$ will be 15°, 135°, 255°, three values.

13. If $x = 20°$, show that $\frac{1}{4}x$ will be 5°, 95°, 185°, 275°, four values.

14. If $x = 60°$, show that $\frac{1}{6}x$ will be 10°, 70°, 130°, 190°, 250°, 310°, six values.

15. If $x = m°$, show that $\frac{1}{n}x$ will have n values less than 360°, as $\frac{m°}{n}$, $\frac{m°}{n} + \frac{360°}{n}$, $\frac{m°}{n} + \frac{720°}{n}, \cdots$ to $\frac{m°}{n} + \frac{(n-1)360°}{n}$.

35. The definitions of the trigonometric ratios in Art. 8 are applicable only to angles less than 90°. We shall now consider the more general definitions, of which those in Art. 8 are special cases.

36. Map Drawing by Coördinates.* — Let $ABCD$ be a field whose map is wanted. From any point O in the field, measure the distances Oa, Ob, Oc, and Od, and also measure the distances aA, bB, cC, and dD, at right angles to $X'OX$. Lay off on the paper a line $X'X$ of indefinite length, and take on it some point O to represent the point O in the field. Lay off Oa according to some convenient scale; thus if Oa were 200 feet, and the scale were 20 feet to 1 inch, we would on the map make Oa 10 inches long. Then draw the line aA perpendicular to OX on the proper side of OX, and lay off on it the distance corresponding to aA according to the same scale, thus locating the point A. The other points would be located in a similar manner.

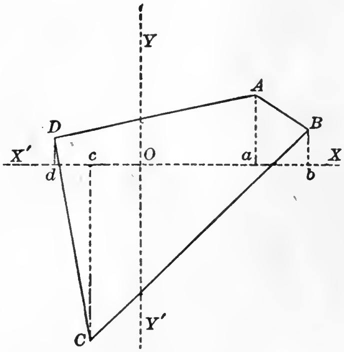


FIG. 27.

Since Oa and Oc are measured from O in contrary directions, and aA and cC are measured on opposite sides of $X'X$, there is danger of laying them off in the wrong direction; hence their directions must be carefully distinguished.

37. Coördinates. — The distance Oa, measured along $X'OX$, is called the *abscissa* of the point A; aA, measured parallel to

* This is called the method of offsets.

$Y'OY$, the *ordinate* of A; and the two distances Oa and aA, the *coördinates* of A. The line $X'OX$ is called the *axis of abscissas;* the line $Y'OY$, the *axis of ordinates;* and the point O, the *origin of coördinates.*

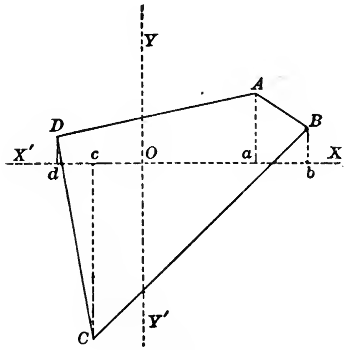


FIG. 28.

The *abscissa* of a point is its distance from the axis of ordinates measured on a line parallel to the axis of abscissas.

The *ordinate* of a point is its distance from the axis of abscissas measured on a line parallel to the axis of ordinates.

The abscissa is *positive* when the point is on the *right* of the axis of ordinates, and *negative* when it is on the *left;* the ordinate is *positive* when the point is *above* the axis of abscissas, and *negative* when it is *below.* If we consider the abscissas as measured from $Y'OY$, and the ordinates from $X'OX$, they will be positive when measured to the right and upward respectively.

Using the customary notation for directed lines,* Oc will represent a line measured from O to c, and cO will be measured from c to O. The line cO measured to the right is positive, and Oc to the left is negative. Hence the coördinates of C are Oc and cC, or $-cO$ and $-Cc$. For brevity, the coördinates of a point are written in a parenthesis with a comma between them, the abscissa being written first; thus the point D is called the point (Od, dD).

The ordinate of any point on $X'OX$ is zero, the abscissa of any point on $Y'OY$ is zero, and both coördinates of the origin are zero.

The signs of the numerical coördinates of points in the different quadrants are as follows:

Quadrant	I.	II.	III.	IV.
Abscissa	+	−	−	+
Ordinate	+	+	−	−

* See Art. 2.

38. Distance of a Point from the Origin. — Represent the abscissa of the point by a, its ordinate by o, and its distance from the origin by h. Then

$$h = \sqrt{a^2 + o^2},$$

since h is the hypotenuse of a right triangle whose sides are a and o. Although h may be either positive or negative, it will be sufficient for our purposes to treat it as being always positive.

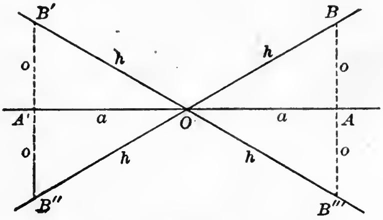


FIG. 29.

39. Trigonometric Ratios. — Take the origin of coördinates at the vertex of the angle and the initial side as the axis of abscissas. From *any* point B on the terminal side of the angle, draw AB perpendicular to the initial side; denote the abscissa OA of the point by a, its ordinate AB by o, and its distance OB from the origin by h. The general definitions of the trigonometric ratios are:

$$\left.\begin{array}{lll}
\text{The } \textit{sine} & \text{of the angle} = \dfrac{\text{ordinate}}{\text{distance}} = \dfrac{o}{h} \\[2ex]
\text{The } \textit{cosine} & \text{of the angle} = \dfrac{\text{abscissa}}{\text{distance}} = \dfrac{a}{h} \\[2ex]
\text{The } \textit{tangent} & \text{of the angle} = \dfrac{\text{ordinate}}{\text{abscissa}} = \dfrac{o}{a} \\[2ex]
\text{The } \textit{cotangent} & \text{of the angle} = \dfrac{\text{abscissa}}{\text{ordinate}} = \dfrac{a}{o} \\[2ex]
\text{The } \textit{secant} & \text{of the angle} = \dfrac{\text{distance}}{\text{abscissa}} = \dfrac{h}{a} \\[2ex]
\text{The } \textit{cosecant} & \text{of the angle} = \dfrac{\text{distance}}{\text{ordinate}} = \dfrac{h}{o}
\end{array}\right\} \quad (1)$$

NOTE.—The origin is always at the vertex of the angle; the axis of abscissas always coincides with the initial side; and the positive direction of the axis of ordinates is along the line that makes an angle of $+90°$ with the initial side.

Prove that the following equations are true, using Eqs. (1):

1. $\dfrac{\sec x}{\sqrt{\sec^2 x - 1}} = \operatorname{cosec} x.$

$$\sec x = \frac{h}{a};$$

$$\therefore \frac{\sec x}{\sqrt{\sec^2 x - 1}} = \frac{\frac{h}{a}}{\sqrt{\frac{h^2}{a^2} - 1}} = \frac{h}{\sqrt{h^2 - a^2}} = \frac{h}{o} = \operatorname{cosec} x.$$

2. $\sec x \cos x = 1.$

3. $\operatorname{cosec} x \sin x = 1.$

4. $\operatorname{cosec}^2 x = 1 + \cot^2 x.$

5. $\dfrac{\tan x}{\sqrt{1 + \tan^2 x}} = \sin x.$

6. $\dfrac{\sqrt{\operatorname{cosec}^2 x - 1}}{\operatorname{cosec} x} = \cos x.$

7. $\tan x \cot x = 1.$

8. $\sin^2 x + \cos^2 x = 1.$

9. $\sec^2 x = 1 + \tan^2 x.$

10. $\dfrac{\sqrt{1 + \cot^2 x}}{\cot x} = \sec x.$

11. $\sqrt{\dfrac{1 + \cot^2 x}{\operatorname{cosec}^2 x - 1}} = \sec x.$

12. $(\tan x - \cot x)(\tan x + \cot x) = \dfrac{h^2(o^2 - a^2)}{o^2 a^2} = \sec^2 x - \operatorname{cosec}^2 x.$

13. $(\tan x + \cot x) \sin x \cos x = 1.$

Construct geometrically the angles, and compute the corresponding ratios in the following examples: *

		Quadrant.	Sin.	Cos.	Tan.	Cot.	Sec.	Cosec.
14.	$\sin x = +\frac{3}{5}$.	I.	—	$+\frac{4}{5}$.	$+\frac{3}{4}$.	$+\frac{4}{3}$.	$+\frac{5}{4}$.	$+\frac{5}{3}$.
		II.	—	$-\frac{4}{5}$.	$-\frac{3}{4}$.	$-\frac{4}{3}$.	$-\frac{5}{4}$.	$+\frac{5}{3}$.
15.	$\sin x = -\frac{1}{3}$.	III.	—	$-\frac{2}{3}\sqrt{2}$.	$+\frac{1}{4}\sqrt{2}$.	$+2\sqrt{2}$.	$-\frac{3}{4}\sqrt{2}$.	-3.
		IV.	—	$+\frac{2}{3}\sqrt{2}$.	$-\frac{1}{4}\sqrt{2}$.	$-2\sqrt{2}$.	$+\frac{3}{4}\sqrt{2}$.	-3.
16.	$\cos x = +\frac{1}{2}$.	I.	$+\frac{1}{2}\sqrt{3}$.	—	$+\sqrt{3}$.	$+\frac{1}{3}\sqrt{3}$.	$+2$.	$+\frac{2}{3}\sqrt{3}$.
		IV.	$-\frac{1}{2}\sqrt{3}$.	—	$-\sqrt{3}$.	$-\frac{1}{3}\sqrt{3}$.	$+2$.	$-\frac{2}{3}\sqrt{3}$.
17.	$\cos x = -\frac{1}{3}$.	II.	$+\frac{2}{3}\sqrt{2}$.	—	$-2\sqrt{2}$.	$-\frac{1}{4}\sqrt{2}$.	-3.	$+\frac{3}{4}\sqrt{2}$.
		III.	$-\frac{2}{3}\sqrt{2}$.	—	$+2\sqrt{2}$.	$+\frac{1}{4}\sqrt{2}$.	-3.	$-\frac{3}{4}\sqrt{2}$.
18.	$\tan x = +\frac{1}{2}$.	I.	$+\frac{1}{5}\sqrt{5}$.	$+\frac{2}{5}\sqrt{5}$.	—	$+2$.	$+\frac{1}{2}\sqrt{5}$.	$+\sqrt{5}$.
		III.	$-\frac{1}{5}\sqrt{5}$.	$-\frac{2}{5}\sqrt{5}$.	—	$+2$.	$-\frac{1}{2}\sqrt{5}$.	$-\sqrt{5}$.
19.	$\tan x = -2$.	II.	$+\frac{2}{5}\sqrt{5}$.	$-\frac{1}{5}\sqrt{5}$.	—	$-\frac{1}{2}$.	$-\sqrt{5}$.	$+\frac{1}{2}\sqrt{5}$.
		IV.	$-\frac{2}{5}\sqrt{5}$.	$+\frac{1}{5}\sqrt{5}$.	—	$-\frac{1}{2}$.	$+\sqrt{5}$.	$-\frac{1}{2}\sqrt{5}$.

* See Arts. 11 and 15. If $\sin x$ is positive, o must be positive, since h is always positive, and the angle lies in quadrants I. and II.

		Quadrant.	Sin.	Cos.	Tan.	Cot.	Sec.	Cosec.
20.	$\cot x = +\frac{4}{3}$.	I.	$+\frac{3}{5}$.	$+\frac{4}{5}$.	$+\frac{3}{4}$.	—	$+\frac{5}{4}$.	$+\frac{5}{3}$.
		III.	$-\frac{3}{5}$.	$\frac{4}{5}$.	$+\frac{3}{4}$.	—	$-\frac{5}{4}$.	$-\frac{5}{3}$.
21.	$\cot x = -3$.	II.	$+\frac{1}{10}\sqrt{10}$.	$-\frac{3}{10}\sqrt{10}$.	$-\frac{1}{3}$.	—	$-\frac{1}{3}\sqrt{10}$.	$+\sqrt{10}$.
		IV.	$-\frac{1}{10}\sqrt{10}$.	$+\frac{3}{10}\sqrt{10}$.	$-\frac{1}{3}$.	—	$+\frac{1}{3}\sqrt{10}$.	$-\sqrt{10}$.
22.	$\sec x = +3$.	I.	$+\frac{2}{3}\sqrt{2}$.	$+\frac{1}{3}$.	$+2\sqrt{2}$.	$+\frac{1}{4}\sqrt{2}$.	—	$+\frac{3}{4}\sqrt{2}$.
		IV.	$-\frac{2}{3}\sqrt{2}$.	$+\frac{1}{3}$.	$-2\sqrt{2}$.	$-\frac{1}{4}\sqrt{2}$.	—	$-\frac{3}{4}\sqrt{2}$.
23.	$\sec x = -\frac{5}{3}$.	II.	$+\frac{4}{5}$.	$-\frac{3}{5}$.	$-\frac{4}{3}$.	$-\frac{3}{4}$.	—	$+\frac{5}{4}$.
		III.	$-\frac{4}{5}$.	$-\frac{3}{5}$.	$+\frac{4}{3}$.	$+\frac{3}{4}$.	—	$-\frac{5}{4}$.
24.	$\operatorname{cosec} x = +\frac{13}{5}$.	I.	$+\frac{5}{13}$.	$+\frac{12}{13}$.	$+\frac{5}{12}$.	$+\frac{12}{5}$.	$+\frac{13}{12}$.	—
		II.	$+\frac{5}{13}$.	$-\frac{12}{13}$.	$-\frac{5}{12}$.	$-\frac{12}{5}$.	$-\frac{13}{12}$.	—
25.	$\operatorname{cosec} x = -\frac{25}{7}$.	III.	$-\frac{7}{25}$.	$-\frac{24}{25}$.	$+\frac{7}{24}$.	$+\frac{24}{7}$.	$-\frac{25}{24}$.	—
		IV.	$-\frac{7}{25}$.	$+\frac{24}{25}$.	$-\frac{7}{24}$.	$-\frac{24}{7}$.	$+\frac{25}{24}$.	—

40. Trigonometric Functions. — One quantity is said to be a function of another when it depends upon the latter for its value. Thus, if $y = \sin x$, y is a function of x, since it depends upon x for its value, any change in the value of x producing a change in the value of y.

The *trigonometric functions* are the sine, cosine, tangent, cotangent, secant, cosecant, versed sine, coversed sine, and suversed sine. The last three are defined by the equations:

$$\left.\begin{array}{ll}\text{The versed sine is} & \mathbf{vers}\,x = 1 - \cos x.\\ \text{The coversed sine is} & \mathbf{covers}\,x = 1 - \sin x.\\ \text{The suversed sine is} & \mathbf{suvers}\,x = 1 + \cos x.\end{array}\right\} \quad (1)$$

41. Geometrical Representation of the Functions. — In Fig. 30 let the radius OB, of the circle described about the vertex O of the angle AOB as a center, be unity, and let the angle AOY be equal to 90°. NM and FD are tangent to the circle at X and Y respectively; the triangles OAB, OXM, and OYD, are right-angled; and the angle YDO is equal to the given angle AOB. Then the trigonometric functions of the angle AOB are represented by the lines shown in the figure. For, in Figs. 2 and 29, B is any point on the terminal side OB of the angle AOB, and therefore we may choose the position of B so that OB, or h, shall be equal to unity. Comparing Fig. 30 with Figs. 2 and 29, and using the definitions in Arts. 8, 39, and 40, we see that

$$\sin AOB = \frac{o}{h} = \frac{AB}{OB} = AB.$$

$$\cos AOB = \frac{a}{h} = \frac{OA}{OB} = OA.$$

$$\tan AOB = \frac{o}{a} = \frac{AB}{OA} = \frac{XM}{OX} = XM.$$

$$\cot AOB = \frac{a}{o} = \frac{OA}{AB} = \frac{CB}{OC} = \frac{YD}{OY} = YD.$$

$$\sec AOB = \frac{h}{a} = \frac{OB}{OA} = \frac{OM}{OX} = OM.$$

$$\text{cosec } AOB = \frac{h}{o} = \frac{OB}{AB} = \frac{OB}{OC} = \frac{OD}{OY} = OD.$$

$$\text{vers } AOB = 1 - \cos AOB = OX - OA = AX.$$

$$\text{covers } AOB = 1 - \sin AOB = OY - OC = CY.$$

$$\text{suvers } AOB = 1 + \cos AOB = X'O + OA = X'A.$$

The trigonometric functions are *ratios*,—pure numbers,—and are represented by these lines in the circle whose radius is unity; that is, they are actually equal to the ratios of these lines to the radius.

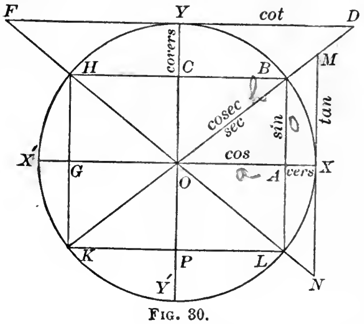


FIG. 30.

If, with a radius of unity and the vertex of the angle as the center, a circle be described and two tangents be drawn, one where the initial side OA cuts the circle, and the other at a distance of $+90°$ from this point (at X and Y respectively), the trigonometric functions will be represented as follows:

The sine of an angle will be the perpendicular distance from the point where the terminal side of the angle cuts the circle, to the initial side, produced if necessary; positive when it is above, and negative when below, the initial side. Thus $\sin AOB = AB$, $\sin AOH = GH$, $\sin AOK = GK$, $\sin AOL = AL$. AB and GH, above $X'OX$, are positive, while GK and AL are negative, being below $X'OX$. The sine is therefore positive when the angle is in the first or second quadrant, and negative when it is in the third or fourth.

*The cosine will be the distance from the center to the foot * of the sine; positive when measured to the right, and negative to the left, of the center.* Thus $\cos AOB = OA$, $\cos AOH = OG$, $\cos AOK = OG$, $\cos AOL = OA$. OA, measured to the right of the center, is positive, while OG, measured to the left, is negative. The cosine is therefore positive when the angle is in the first or fourth quadrant, and negative when it is in the second or third.

The tangent will be the distance along the line tangent to the circle at the point where the initial side cuts the circle, from this point to the point where this tangent is cut by the terminal side of the angle, produced if necessary; positive when measured above, and negative when below, the initial side. Thus $\tan AOB = XM$, $\tan AOH = XN$, $\tan AOK = XM$, $\tan AOL = XN$. XM, above $X'OX$, is positive, and XN, below $X'OX$, is negative. Therefore the tangent is positive when the angle is in the first or third quadrant, and negative when it is in the second or fourth.

The cotangent will be the distance along the second tangent (FYD) *from the point of tangency to the point where this line is cut by the terminal side of the angle, produced if necessary; positive when measured to the right, and negative to the left, of the point of tangency.* Thus $\cot AOB = YD$, $\cot AOH = YF$, $\cot AOK = YD$, $\cot AOL = YF$. YD, measured to the right, is positive, and YF, measured to the left, is negative. Therefore the cotangent is positive when the angle is in the first or third quadrant, and negative when it is in the second or fourth.

NOTE. — The positive directions of measurement are above $X'OX$ and to the right of $Y'OY$, and the negative are below $X'OX$ and to the left of $Y'OY$.

The secant will be the distance from the center along the terminal side of the angle, produced if necessary, to its point of intersection with the tangent at the point of intersection of the initial side with the circle; positive when measured along the side itself, and negative when along the side produced. Thus $\sec AOB = OM$, $\sec AOH = ON$, $\sec AOK = OM$, $\sec AOL = ON$.

* The foot of the sine is the point where the perpendicular line representing the sine cuts the initial side, produced if necessary.

Since sec AOB and sec AOL are measured along the terminal side itself, they are positive. The terminal sides (OH and OK) of the angles AOH and AOK must be produced in order that they may intersect the tangent line NM, and therefore sec AOH and sec AOK are negative. Hence the secant is positive when the angle is in the first or fourth quadrant, and negative when it is in the second or third.

The cosecant will be the distance from the center along the terminal side, produced if necessary, to its intersection with the second tangent, FYD; *positive when measured along the side itself, and negative when along the side produced.* Thus cosec $AOB = OD$, cosec $AOH = OF$, cosec $AOK = OD$, cosec $AOL = OF$. Since cosec AOB and cosec AOH are measured along the terminal side itself, they are positive, while cosec AOK and cosec AOL, measured along the side produced, are negative. Therefore the cosecant is positive when the angle is in the first or second quadrant, and negative when it is in the third or fourth.

The versed sine (1 − *cos* x) *will be the distance from the foot of the sine to the point where the initial side cuts the circle; always positive, because cos* x *can never be greater than the radius, or unity.* Thus vers $AOB = AX$, vers $AOH = GX$, vers $AOK = GX$, vers $AOL = AX$.

The coversed sine (1 − *sin* x) *will be the distance from the point* C *or* P, *where a line drawn through the point of intersection of the terminal side and the circle parallel to the initial side cuts* Y′OY, *to the point* Y; *always positive, since sin* x *can never be greater than the radius, or unity.* Thus covers $AOB = CY$, covers $AOH = CY$, covers $AOK = PY$, covers $AOL = PY$.

The suversed sine (1 + *cos* x) *will be the distance from the point* X′, *where the initial side produced cuts the circle, to the foot of the sine; always positive, since cos* x *can never be algebraically less than minus unity.* Thus suvers $AOB = X'A$, suvers $AOH = X'G$, suvers $AOK = X'G$, suvers $AOL = X'A$.

NOTE. — These lines represent the trigonometric functions, only when the radius of the circle is unity. If the radius differs from unity, the functions are equal to the lengths of these lines divided by the radius.

42. Changes in the Values of the Functions. — Let OX be the initial side of the angle, and let the terminal side first coincide with OX, and then, in revolving about O, come into the positions OM, OY, OH, OX', OK, OY', ON, and OX, and let us consider the resulting changes in the values of the sine and of the tangent.

The sine of 0°, the terminal side coinciding with OX, is zero. As the angle increases, the sine, being positive, also increases ($\sin AOB = AB$), until at 90° it is equal to the radius, or $+1$ ($\sin AOY = OY$). The sine then decreases ($\sin AOH = GH$), still being positive; and at 180° it is zero, the terminal side coinciding with OX'. The sine then becomes negative, and decreases algebraically, increasing numerically ($\sin AOK = GK$), until at 270° it is equal to the radius, or -1 ($\sin AOY' = OY'$). It then increases algebraically, decreasing numerically ($\sin AOL = AL$); and at 360° it again becomes zero.

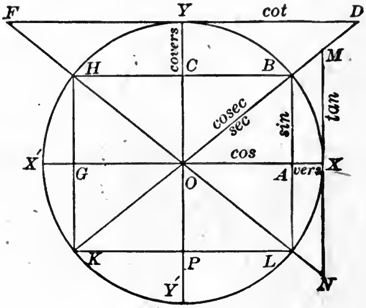


FIG. 31.

The tangent of 0° is zero; the tangent then becomes positive, and at 90° it is infinite, the terminal side being parallel to XM; then negative, and at 180° it is zero; then positive, and at 270° it is infinite; then negative, and at 360° it is zero. Just before the terminal side reaches the position OY, the tangent is positive, and just after, it is negative; therefore the tangent of 90° is $\pm\infty$, the upper sign being that of the function of an angle a little less than 90°.

The table gives the values of the functions of 0°, 90°, 180°, 270°, and 360°, and their signs in quadrants I., II., III., and IV.:

	0°.	I.	90°.	II.	180°.	III.	270°.	IV.	360°.
sin.	0	+	+1	+	0	−	−1	−	0
cos.	+1	+	0	−	−1	−	0	+	+1
tan.	0	+	∞	−	0	+	∞	−	0
cot.	∞	+	0	−	∞	+	0	−	∞
sec.	+1	+	∞	−	−1	−	∞	+	+1
cosec.	∞	+	+1	+	∞	−	−1	−	∞

43. Limiting Values of the Functions. — The sine and cosine may have any value between $+1$ and -1, but they cannot have a value numerically greater than unity.

The tangent and cotangent may have any value between $+\infty$ and $-\infty$; that is, no matter what a number may be, there will always be some angle that will have that number as the value of its tangent, and another having it as its cotangent.

The secant and cosecant may have any value between $+1$ and $+\infty$, or -1 and $-\infty$; but they cannot have a value numerically less than unity.

The versed sine, coversed sine, and suversed sine may have any value between zero and $+2$.

NOTE. — In the first quadrant, all the functions are positive, and the sine, tangent, and secant increase as the angle increases; while the cosine, cotangent, and cosecant decrease as the angle increases.

NOTE. — The functions change signs only when they pass through the values zero and infinity.

44. Graphical Representation of the Functions. — Let the distance OL represent 360°, so that 1° is represented by $\frac{1}{360}OL$. At C, such that $OC = \frac{1}{4}OL$, draw a line perpendicular to OL, and

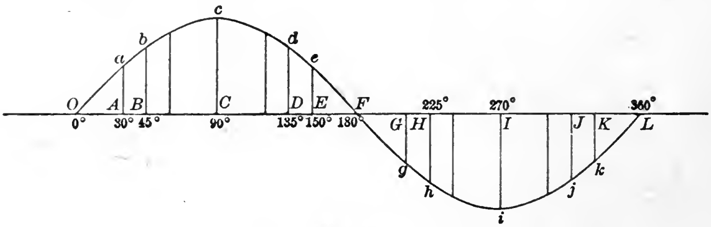


FIG. 32.

lay off on it any convenient distance Cc, to represent the sine of 90°, above the line OL, since $\sin 90° = +1$. At A, such that $OA = \frac{1}{12}OL$, lay off $Aa = \frac{1}{2}Cc$, since $\sin 30° = +\frac{1}{2}$; at B, such that $OB = \frac{1}{8}OL$, lay off $Bb = Cc\sqrt{\frac{1}{2}}$, since $\sin 45° = +\sqrt{\frac{1}{2}}$; at H, such that $OH = \frac{5}{8}OL$, lay off $Hh = Cc\sqrt{\frac{1}{2}}$, below OL, since $\sin 225° = -\sqrt{\frac{1}{2}}$; and so on, locating as many points a, b, c, h, etc., as may be necessary. Draw a smooth curve through O, a, b, c, d, e, F, h, i, j, L, and we have the *sinusoid*, in which the abscissas correspond to the angles, and the ordinates to their sines.

We might have taken OL equal to the circumference of the circle whose radius is unity, and Cc equal to this radius. The scale would then have been the same for both the ordinates and the abscissas.

The graphical representations of the other functions may be constructed in a similar manner.

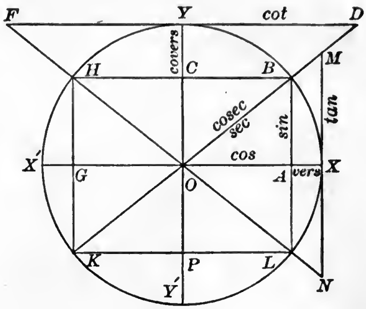


Fig. 33.

45. Two Angles correspond to Any Given Function. — In Fig. 33 let the arcs YB, YH, $Y'K$, and $Y'L$ be equal; therefore the arcs XB, $X'H$, $X'K$, and XL are equal. Hence

$$AB = GH = OC; \quad AL = GK = OP; \quad OM = ON; \quad OD = OF.$$

OC is not equal to OP since they have contrary signs, OC being positive and OP negative on account of their directions.

$$AB = \sin XOB; \quad GH = \sin XOH;$$
$$\therefore \sin XOB = \sin XOH.$$
$$GK = \sin XOK; \quad AL = \sin XOL;$$
$$\therefore \sin XOK = \sin XOL.$$

Therefore two angles that differ by equal amounts from 90°, or from 270°, will have the same sine; thus $\sin(90° + 2°) = \sin(90° - 2°)$, and $\sin(270° + 3°) = \sin(270° - 3°)$.

Note. — The two angles corresponding to a given function may be *identical*; thus, if $\sin x = +1$, the only value of x is 90°, or $90° - 0°$ and $90° + 0°$.

Again $\quad OA = \cos XOB = \cos XOL;$

and $\quad OG = \cos XOH = \cos XOK.$

Therefore two angles differing by equal amounts from 0°, from 180°, or from 360°, will have the same cosine; thus $\cos(-5°) = \cos 5°$, $\cos(180° + 5°) = \cos(180° - 5°)$, and $\cos(360° - 10°) = \cos 10°$.

Also $\quad XM = \tan XOB = \tan XOK;$

and $\quad XN = \tan XOH = \tan XOL.$

Therefore two angles differing from each other by 180° will have the same tangent; thus $\tan 140° = \tan 320°$.

Again $\qquad YD = \cot XOB = \cot XOK$;

and $\qquad YF = \cot XOH = \cot XOL$.

Therefore two angles differing from each other by 180° will have the same cotangent; thus $\cot 200° = \cot 20°$.

Also $\qquad +OM = \sec XOB$; $\; +ON = \sec XOL$;

$$\therefore \sec XOB = \sec XOL.$$

$-OM = \sec XOK$; $\; -ON = \sec XOH$;

$$\therefore \sec XOK = \sec XOH.$$

Therefore two angles differing by equal amounts from 0°, from 180°, or from 360°, will have the same secant; thus $\sec(-5°) = \sec 5°$, $\sec(180° - 3°) = \sec(180° + 3°)$, and $\sec(360° - 5°) = \sec 5°$.

Again $\qquad +OD = \operatorname{cosec} XOB$; $\; +OF = \operatorname{cosec} XOH$;

$$\therefore \operatorname{cosec} XOB = \operatorname{cosec} XOH.$$

$-OD = \operatorname{cosec} XOK$; $\; -OF = \operatorname{cosec} XOL$;

$$\therefore \operatorname{cosec} XOK = \operatorname{cosec} XOL.$$

Therefore two angles differing by equal amounts from 90°, or from 270°, will have the same cosecant; thus $\operatorname{cosec}(90° + 10°) = \operatorname{cosec}(90° - 10°)$, and $\operatorname{cosec}(270° - 60°) = \operatorname{cosec}(270° + 60°)$.

The four angles XOB, XOH, XOK, and XOL, have the same functions numerically. Thus if $\sin x = \pm \frac{1}{2}$, x will be 30°, 150°, 210°, and 330°; the first two corresponding to the value $+\frac{1}{2}$, and the last two to $-\frac{1}{2}$.

EXAMPLES.

1. What angle has the same sine as 140°? *Ans.* 40°.
2. What angle has the same sine as 220°? *Ans.* 320°.
3. What angle has the same cosine as 330°? *Ans.* 30°.
4. What angle has the same cosine as 220°? *Ans.* 140°.
5. What angle has the same tangent as 230°? *Ans.* 50°.
6. What angle has the same tangent as 300°? *Ans.* 120°.
7. What angle has the same cotangent as 240°? *Ans.* 60°.
8. What angle has the same cotangent as 110°? *Ans.* 290°.
9. What angle has the same secant as 315°? *Ans.* 45°.
10. What angle has the same secant as 160°? *Ans.* 200°.
11. What angle has the same cosecant as 110°? *Ans.* 70°.
12. What angle has the same cosecant as 300°? *Ans.* 240°.

~~cos 330~~ = Sin 140 = Sin(90 +
Sin(180 − 40) = cos 40

Find the values of θ less than 360° in Exs. (13–24) : *

13. $\sin\theta = -\sin 200°$. *Ans.* 20°.

14. $\sin\theta = -\sin 100°$. *Ans.* 260°.

15. $\cos\theta = -\cos 150°$. *Ans.* 30°.

16. $\cos\theta = -\cos 300°$. *Ans.* 120°.

17. $\tan\theta = -\tan 350°$. *Ans.* 10°.

18. $\tan\theta = -\tan 230°$. *Ans.* 130°.

19. $\cot\theta = -\cot 105°$. *Ans.* 75°.

20. $\cot\theta = -\cot 205°$. *Ans.* 155°.

21. $\sec\theta = -\sec 140°$. *Ans.* 40°.

22. $\sec\theta = -\sec 325°$. *Ans.* 145°.

23. $\operatorname{cosec}\theta = -\operatorname{cosec} 120°$. *Ans.* 240°.

24. $\operatorname{cosec}\theta = -\operatorname{cosec} 355°$. *Ans.* 5°.

25. $\cos 3\theta = +\frac{1}{2}\sqrt{3}$. Find three values of θ less than 180°.

3θ may be 30°, or 330°, or these values plus any number of circumferences;

$$\therefore\ 3\theta = 30°, 390°, 750°, \cdots, 330°, 690°, 1050°, \cdots$$

$$\therefore\ \theta = 10°, 130°, 250°, \cdots, 110°, 230°, 350°, \cdots$$

Ans. $\theta = 10°, 110°, 130°$.

26. $\sin 2\theta = -\frac{1}{2}$. Find four values of θ less than 360°.

Ans. 105°, 285°, 165°, 345°.

27. $\tan 3\theta = -1$. Find six values of θ less than 360°.

Ans. 45°, 165°, 285°, 105°, 225°, 345°.

28. $\sec 5\theta = -2$. Find five values of θ less than 180°.

Ans. 24°, 96°, 168°, 48°, 120°.

29. $\cot 5\theta = +1$. Find five values of θ less than 180°.

Ans. 9°, 81°, 153°, 45°, 117°.

30. $\cos 4\theta = -\frac{1}{2}$. Find four values of θ less than 180°.

Ans. 30°, 120°, 60°, 150°.

31. $\sin\theta = \frac{1}{2}$. Show that the general measure of θ is $(2n + \frac{1}{2})\pi \pm \frac{1}{3}\pi$. $\theta = 30°$ and 150°, or $90° - 60°$ and $90° + 60°$, or $90° \pm 60°$, or $\frac{1}{2}\pi \pm \frac{1}{3}\pi$. But the general measures of θ are these values increased by any number (n) of circumferences. $\therefore\ \theta = 2n\pi + \frac{1}{2}\pi \pm \frac{1}{3}\pi = (2n + \frac{1}{2})\pi \pm \frac{1}{3}\pi$.

32. $\sin\theta = +\frac{1}{2}\sqrt{2}$, $\tan\theta = -1$; the general measure of θ is $2n\pi + \frac{3}{4}\pi$. Note that θ is in the second quadrant, since its sine is positive and its tangent is negative.

33. $\cos\theta = -\frac{1}{3}$, $\operatorname{cosec}\theta = +\dfrac{3}{2\sqrt{2}}$; the general measure of θ is $2n\pi + \theta'$, where θ' is the value of θ that lies between $\frac{1}{2}\pi$ and π.

34. $\cos\theta = -\frac{1}{2}$; the general measure of θ is $(2n + 1)\pi \pm \frac{1}{3}\pi$.

35. $\sin 2\theta = +\frac{1}{2}$; the general measures of θ are $(2n + \frac{1}{4})\pi \pm \frac{1}{6}\pi$, and $(2n + \frac{5}{4})\pi \pm \frac{1}{6}\pi$.

36. $\cos 3\theta = -\frac{1}{2}$; the general measures of θ are $(2n + \frac{1}{3})\pi \pm \frac{1}{9}\pi$, $(2n + 1)\pi \pm \frac{1}{9}\pi$, and $(2n + \frac{5}{3})\pi \pm \frac{1}{9}\pi$.

Construct geometrically (Art. 11) the two angles when

37. $\sin x = +\frac{1}{3}$.

38. $\sin x = -\frac{1}{4}$.

39. $\cos x = +\frac{2}{5}$.

40. $\cos x = -\frac{2}{3}$.

41. $\tan x = +2$.

42. $\tan x = -\frac{1}{2}$.

43. $\cot x = +\frac{2}{3}$.

44. $\cot x = -\frac{3}{2}$.

45. $\sec x = +3$.

46. $\sec x = -\frac{5}{4}$.

47. $\operatorname{cosec} x = +6$.

48. $\operatorname{cosec} x = -\frac{4}{3}$.

* Only one of the two answers is given.

CHAPTER IV.

RELATIONS BETWEEN THE FUNCTIONS OF ONE ANGLE.

46. Relations between the Functions of One Angle.

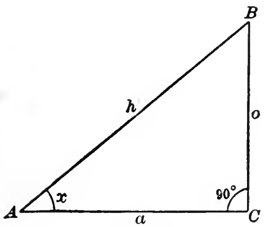


FIG. 34.

$$o^2 + a^2 = h^2; \quad \therefore \frac{o^2}{h^2} + \frac{a^2}{h^2} = 1;$$

$$\therefore \sin^2 x + \cos^2 x = 1. \qquad (1)$$

$$\tan x = \frac{o}{a} = \frac{\frac{o}{h}}{\frac{a}{h}} = \frac{\sin x}{\cos x};$$

$$\therefore \tan x = \frac{\sin x}{\cos x}; \qquad (2)$$

$$\cot x = \frac{a}{o} = \frac{1}{\tan x}; \qquad (3)$$

or,

$$\cot x = \frac{\cos x}{\sin x}. \qquad (4)$$

$$h^2 = a^2 + o^2; \quad \therefore \frac{h^2}{a^2} = 1 + \frac{o^2}{a^2};$$

$$\therefore \sec^2 x = 1 + \tan^2 x. \qquad (5)$$

$$h^2 = o^2 + a^2; \quad \therefore \frac{h^2}{o^2} = 1 + \frac{a^2}{o^2};$$

$$\therefore \text{cosec}^2 x = 1 + \cot^2 x. \qquad (6)$$

$$\sec x = \frac{h}{a} = \frac{1}{\cos x}. \qquad (7)$$

$$\text{cosec}\, x = \frac{h}{o} = \frac{1}{\sin x}. \qquad (8)$$

$$\text{vers}\, x = 1 - \cos x. \qquad (9)$$

$$\text{covers}\, x = 1 - \sin x. \qquad (10)$$

$$\text{suvers}\, x = 1 + \cos x. \qquad (11)$$

Note. — These formulas may be easily remembered by the use of Fig. 30, where

$$AB^2 + OA^2 = OB^2, \quad \text{or} \quad \sin^2 x + \cos^2 x = 1.$$

$$\tan x = \frac{XM}{OX} = \frac{AB}{OA}, \quad \text{or} \quad \tan x = \frac{\sin x}{\cos x}.$$

$$\cot x = \frac{YD}{OY} = \frac{CB}{OC}, \quad \text{or} \quad \cot x = \frac{\cos x}{\sin x}.$$

$$OM^2 = OX^2 + XM^2, \quad \text{or} \quad \sec^2 x = 1 + \tan^2 x.$$

$$OD^2 = OY^2 + YD^2, \quad \text{or} \quad \operatorname{cosec}^2 x = 1 + \cot^2 x.$$

47. To express One Function in Terms of Each of the Others. — Suppose that we wish to find expressions for $\sin x$ that shall contain only $\cos x$, $\tan x$, $\cot x$, $\sec x$, and $\operatorname{cosec} x$ respectively. From the preceding article we have:

$$\sin^2 x + \cos^2 x = 1, \text{ and } \operatorname{cosec} x = \frac{1}{\sin x};$$

$$\therefore \sin x = \pm\sqrt{1 - \cos^2 x}$$

and

$$\sin x = \frac{1}{\operatorname{cosec} x}.$$

The other expressions are derived from these as follows:

$$\sin x = \frac{1}{\operatorname{cosec} x} = \pm\frac{1}{\sqrt{1 + \cot^2 x}}, \text{ from (6).}$$

$$\therefore \sin x = \pm\frac{1}{\sqrt{1 + \cot^2 x}} = \pm\frac{1}{\sqrt{1 + \dfrac{1}{\tan^2 x}}} = \pm\frac{\tan x}{\sqrt{1 + \tan^2 x}}, \text{ from (3).}$$

$$\therefore \sin x = \pm\frac{\tan x}{\sqrt{1 + \tan^2 x}} = \pm\frac{\sqrt{\sec^2 x - 1}}{\sec x}, \text{ from (5).}$$

The double signs are due to the fact that there are two angles corresponding to any given function; thus if $\cos x = \frac{1}{2}$, the angle might be either in the first or in the fourth quadrant, and the sine would be positive in the first case and negative in the second. It will be seen that if any one of the functions is given, all the others found from it will have the double sign, except its reciprocal.

In the same way it may be shown that*

$$\cos x = \sqrt{1-\sin^2 x} = \frac{1}{\sqrt{1+\tan^2 x}} = \frac{\cot x}{\sqrt{1+\cot^2 x}} = \frac{1}{\sec x} = \frac{\sqrt{\operatorname{cosec}^2 x - 1}}{\operatorname{cosec} x}.$$

$$\tan x = \frac{\sin x}{\sqrt{1-\sin^2 x}} = \frac{\sqrt{1-\cos^2 x}}{\cos x} = \frac{1}{\cot x} = \sqrt{\sec^2 x - 1} = \frac{1}{\sqrt{\operatorname{cosec}^2 x - 1}}.$$

$$\cot x = \frac{\sqrt{1-\sin^2 x}}{\sin x} = \frac{\cos x}{\sqrt{1-\cos^2 x}} = \frac{1}{\tan x} = \frac{1}{\sqrt{\sec^2 x - 1}} = \sqrt{\operatorname{cosec}^2 x - 1}.$$

$$\sec x = \frac{1}{\sqrt{1-\sin^2 x}} = \frac{1}{\cos x} = \sqrt{1+\tan^2 x} = \frac{\sqrt{1+\cot^2 x}}{\cot x} = \frac{\operatorname{cosec} x}{\sqrt{\operatorname{cosec}^2 x - 1}}.$$

$$\operatorname{cosec} x = \frac{1}{\sin x} = \frac{1}{\sqrt{1-\cos^2 x}} = \frac{\sqrt{1+\tan^2 x}}{\tan x} = \sqrt{1+\cot^2 x} = \frac{\sec x}{\sqrt{\sec^2 x - 1}}.$$

If any one of the functions is given, the others may be found from these formulas. It is easier in general to find first the sine and cosine, and then to find the others.

48. Find the Unknown Functions in the Following:

1. $\tan x = -\frac{3}{4}$, x being in the fourth quadrant. Compute the numerical values of the ratios by the method of Art. 15, and then select the proper signs for the functions in the fourth quadrant. Thus let

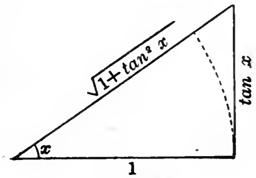


FIG. 35.

$$o = 3, \quad a = 4, \quad h = 5;$$

$$\therefore \sin x = -\frac{3}{5}, \quad \cos x = +\frac{4}{5},$$

$$\cot x = -\frac{4}{3}, \quad \sec x = +\frac{5}{4}, \quad \operatorname{cosec} x = -\frac{5}{3}.$$

2. $\tan x = 2$, x being in the third quadrant. Then

$$\sin x = \pm\frac{\tan x}{\sqrt{1+\tan^2 x}} = -\frac{2}{\sqrt{5}}, \quad \cos x = \pm\frac{1}{\sqrt{1+\tan^2 x}} = -\frac{1}{\sqrt{5}}.$$

These convenient formulas may be easily remembered from Fig. 35. Knowing $\sin x$ and $\cos x$, we have

$$\cot x = \frac{1}{\tan x} = +\frac{1}{2}; \quad \sec x = \frac{1}{\cos x} = -\sqrt{5};$$

$$\operatorname{cosec} x = \frac{1}{\sin x} = -\frac{1}{2}\sqrt{5}.$$

* The radicals should be taken with the double sign.

3. $\cot x = -2$, x being in the second quadrant.

$$\therefore \operatorname{cosec} x = \pm\sqrt{1+\cot^2 x} = +\sqrt{5};\ \sin x = \frac{1}{\operatorname{cosec} x} = +\frac{1}{\sqrt{5}};$$

$$\cos x = \pm\sqrt{1-\sin^2 x} = -\frac{2}{\sqrt{5}};\ \tan x = \frac{1}{\cot x} = -\frac{1}{2};$$

$$\sec x = \frac{1}{\cos x} = -\frac{1}{2}\sqrt{5}.$$

4. $\sec x = -\frac{17}{8}$, x being in the third quadrant.

$$\therefore \cos x = \frac{1}{\sec x} = -\frac{8}{17};\ \sin x = \pm\sqrt{1-\cos^2 x} = -\frac{15}{17};$$

$$\tan x = \frac{\sin x}{\cos x} = +\frac{15}{8};\ \cot x = +\frac{8}{15};\ \operatorname{cosec} x = -\frac{17}{15}.$$

5. $\sin x = -\frac{4}{5}$, x being in the third quadrant.

$\therefore \cos x = -\frac{3}{5}$; $\tan x = +\frac{4}{3}$; $\cot x = +\frac{3}{4}$; $\sec x = -\frac{5}{3}$; $\operatorname{cosec} x = -\frac{5}{4}$.

6. $\cos x = +\frac{2}{3}$, x being in the fourth quadrant.

$\therefore \sin x = -\frac{1}{3}\sqrt{5}$; $\tan x = -\frac{1}{2}\sqrt{5}$; $\cot x = -\frac{2}{5}\sqrt{5}$; $\sec x = +\frac{3}{2}$;
$\operatorname{cosec} x = -\frac{3}{5}\sqrt{5}$.

7. $\tan x = -\frac{5}{12}$, x being in the second quadrant.

$\therefore \sin x = +\frac{5}{13}$; $\cos x = -\frac{12}{13}$; $\cot x = -\frac{12}{5}$; $\sec x = -\frac{13}{12}$; $\operatorname{cosec} x = +\frac{13}{5}$.

8. $\cot x = +\frac{7}{24}$, x being in the third quadrant.

$\therefore \sin x = -\frac{24}{25}$; $\cos x = -\frac{7}{25}$.

9. $\sec x = -\frac{17}{15}$, x being in the second quadrant.

$\therefore \cos x = -\frac{15}{17}$; $\sin x = +\frac{8}{17}$; $\tan x = -\frac{8}{15}$.

10. $\operatorname{cosec} x = -\frac{41}{9}$, x being in the fourth quadrant.

$\therefore \sin x = -\frac{9}{41}$; $\cos x = +\frac{40}{41}$; $\tan x = -\frac{9}{40}$.

11. If $\sin\frac{1}{2}\theta = \sqrt{\frac{(s-b)(s-c)}{bc}}$ where $s = \frac{a+b+c}{2}$, show that

$$\cos\tfrac{1}{2}\theta = \sqrt{\frac{s(s-a)}{bc}}.$$

12. If $\tan\frac{1}{2}\theta = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$ where $s = \frac{a+b+c}{2}$, show that

$$\cos\tfrac{1}{2}\theta = \sqrt{\frac{s(s-c)}{ab}}.$$

13. If $\sec\theta = a$, show that $\sin\theta$ is imaginary if a is numerically less than unity.

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{1}{\sec^2\theta}} = \sqrt{1-\frac{1}{a^2}} = \frac{\sqrt{a^2-1}}{a}.$$

14. If $\tan\theta = a$, show that $\operatorname{cosec}\theta$ is real for all values of a.

15. If $\cos\theta = a$, show that $\operatorname{cosec}\theta$ is imaginary when a is numerically greater than unity.

49. The Signs of the Functions are given by the formulas of Art. 46, so that it is necessary to remember only that the sine is positive in the first and second quadrants and the cosine in the first and fourth. Thus, in the second quadrant,

$$\tan x = \frac{\sin x}{\cos x} = \frac{+}{-} = - \,; \quad \cot x = \frac{\cos x}{\sin x} = \frac{-}{+} = - \,;$$

$$\sec x = \frac{1}{\cos x} = \frac{+}{-} = - \,; \quad \operatorname{cosec} x = \frac{1}{\sin x} = \frac{+}{+} = +.$$

50. Find the Values of the Following Expressions:

1. $\dfrac{\operatorname{vers} x \tan x - 1}{\sec x}$ when $\tan x = 4$, x being in the third quadrant. Find the numerical values of $\operatorname{vers} x$ and $\sec x$, and substitute.

$$\therefore \cos x = -\frac{1}{\sqrt{17}}, \quad \sec x = -\sqrt{17}, \quad \operatorname{vers} x = 1 + \frac{1}{\sqrt{17}}.$$

$$\therefore \frac{\dfrac{\sqrt{17}+1}{\sqrt{17}} \cdot 4 - 1}{-\sqrt{17}} = \frac{4\sqrt{17} + 4 - \sqrt{17}}{-17} = -\frac{3\sqrt{17}+4}{17}.$$

2. $\dfrac{\sin x \sec x}{\cos x \operatorname{cosec} x}$ when $\operatorname{vers} x = \frac{3}{4}$, x in the fourth quadrant. *Ans.* $+15$.

3. $\dfrac{\tan x - \cot x}{\tan x + \cot x}$ when $\operatorname{cosec} x = -\sqrt{5}$, x in the third quadrant. *Ans.* $-\frac{3}{5}$.

4. $\dfrac{\sec x + \sin x}{\operatorname{cosec} x + \cos x}$ when $\cot x = -\frac{1}{2}$, x in the second quadrant. *Ans.* -2.

5. $\dfrac{\sin x + \tan x}{\cos x + \operatorname{vers} x}$ when $\sec x = -\frac{5}{4}$, x in the third quadrant. *Ans.* $+\frac{3}{20}$.

6. $\dfrac{\sec x - \operatorname{vers} x}{\sec x + \operatorname{vers} x}$ when $\cot x = -2$, x in the second quadrant.

Ans. $\dfrac{9+2\sqrt{5}}{1-2\sqrt{5}} = -\dfrac{29+20\sqrt{5}}{19}$.

7. $\dfrac{\sin x + \tan^2 x}{\cos^2 x + \operatorname{vers}^2 x}$ when $\sec x = -\frac{5}{4}$, x in the second quadrant. *Ans.* $\frac{465}{1552}$.

8. $\dfrac{\sec x + \sin x}{1 - \cot x}$ when $\tan x = 2$, x in the third quadrant. *Ans.* $-\frac{14}{5}\sqrt{5}$.

9. $\dfrac{\operatorname{cosec} x + \sec x}{\cot x \cos x}$ when $\sec x = +\sqrt{10}$, x in the fourth quadrant. *Ans.* -20.

10. $\dfrac{\sec x - \operatorname{cosec} x}{\sec x + \operatorname{cosec} x}$ when $\cot x = -2$, x in the second quadrant. *Ans.* -3.

11. $\dfrac{\operatorname{vers} x - \operatorname{covers} x}{\sec x - \operatorname{cosec} x}$ when $\sin x = -\frac{2}{3}$, x in the fourth quadrant. *Ans.* $-\frac{2}{9}\sqrt{5}$.

51. Change the Given Expression to Another containing only One Function:

1. $\dfrac{2\sec^2 x + \sec^2 x \tan^2 x - \sec^4 x}{\sec^2 x - 1}$ to contain only $\operatorname{cosec} x$.

It is best generally to change the expression to another containing only $\sin x$ and $\cos x$, and then to change this into one containing the proper function.

$$\therefore \frac{\dfrac{2}{\cos^2 x} + \dfrac{\sin^2 x}{\cos^4 x} - \dfrac{1}{\cos^4 x}}{\dfrac{1}{\cos^2 x} - 1} = \frac{2\cos^2 x + \sin^2 x - 1}{\cos^2 x(1-\cos^2 x)}$$

$$= \frac{2 - 2\sin^2 x + \sin^2 x - 1}{(1-\sin^2 x)\sin^2 x} = \frac{1 - \sin^2 x}{(1-\sin^2 x)\sin^2 x} = \frac{1}{\sin^2 x} = \operatorname{cosec}^2 x.$$

2. $\dfrac{\sin^2 x - \cos^2 x}{\operatorname{vers} x - \operatorname{covers} x}$ to contain only $\tan x$.

$$\therefore \frac{\sin^2 x - \cos^2 x}{1 - \cos x - 1 + \sin x} = \sin x + \cos x = \pm\frac{\tan x}{\sqrt{1+\tan^2 x}} \pm \frac{1}{\sqrt{1+\tan^2 x}},$$

where the signs used will depend upon the quadrant of x. The true result is $\pm\dfrac{1+\tan x}{\sqrt{1+\tan^2 x}}$, where the positive sign corresponds to x in the first or fourth quadrant, and the negative to x in the second or third.

Use radicals as little as possible.

3. $1 - 2(1 - \operatorname{covers} x)^2 + \dfrac{\tan^4 x}{(1+\tan^2 x)^2}$ to contain only $\cos x$. *Ans.* $\cos^4 x$.

4. $\dfrac{\sec x \operatorname{cosec} x - 4\sin x \cos x}{\sin x \sec x}$ to contain only $\sin x$. *Ans.* $\dfrac{(1-2\sin^2 x)^2}{\sin^2 x}$.

5. $\dfrac{(1 - \operatorname{covers} x)^2 \operatorname{cosec}^4 x}{(\operatorname{cosec}^2 x - 1)\cot^2 x}$ to contain only $\tan x$. *Ans.* $\tan^2 x + \tan^4 x$.

6. $\dfrac{\sec^2 x - \sec^2 x \sin^4 x(1+\cot^2 x)}{\sin^2 x \cos^2 x}$ to contain only $\operatorname{cosec} x$.

Ans. $\dfrac{\operatorname{cosec}^4 x}{\operatorname{cosec}^2 x - 1}$.

7. $\tan^2\theta\sec^2\theta-\sin^2\theta\cos^2\theta$ to contain only $\cot\theta$. *Ans.* $\dfrac{1+3\cot^2\theta+3\cot^4\theta}{\cot^4\theta(1+\cot^2\theta)^2}$.

8. $\dfrac{(1-\tan^2 x)^2}{(1+\tan^2 x)^2}(\cos^4 x-\sin^4 x)$ to contain only $\sin x$. *Ans.* $(1-2\sin^2 x)^3$.

9. $\dfrac{\sec^2 a\sin^2 a}{(\tan a+2\cot a)^2}$ to contain only $\operatorname{cosec} a$. *Ans.* $\dfrac{1}{(2\operatorname{cosec}^2 a-1)^2}$.

10. $\dfrac{\sin^2\theta\tan^2\theta}{\sin^2\theta-\cos^2\theta}$ to contain only $\sec\theta$. *Ans.* $\dfrac{(\sec^2\theta-1)^2}{\sec^2\theta-2}$.

11. $\dfrac{\sec^2\theta\operatorname{cosec}^2\theta+\sec^2\theta-\operatorname{cosec}^2\theta-1}{\tan^2\theta-\operatorname{cosec}^2\theta+1}$ to contain only $\cot\theta$. *Ans.* $\dfrac{\cot^2\theta+2}{1-\cot^4\theta}$.

52. Solution of Trigonometric Equations. — Transform the given equation into one containing only a single function (usually the sine or cosine), because in a single equation we must have only one unknown quantity. Then solve the equation algebraically for this function as the unknown quantity. The corresponding angle may then be found from the tables. Test the angles by substitution in the given equation.

1. $\sin\theta\cos\theta=+\frac{1}{2}$.

$$\therefore\ \sin\theta\sqrt{1-\sin^2\theta}=+\tfrac{1}{2};\quad \therefore\ \sin^2\theta(1-\sin^2\theta)=\tfrac{1}{4};$$

$$\therefore\ \sin^4\theta-\sin^2\theta+\tfrac{1}{4}=0;\quad \therefore\ \sin^2\theta-\tfrac{1}{2}=0;\quad \therefore\ \sin\theta=\pm\sqrt{\tfrac{1}{2}}.$$

$\therefore\ \theta$ might be 45°, 135°, 225°, or 315°. But the given equation shows that the product of the sine and cosine must be positive, and hence that they must have the same sign. Both the sine and cosine are positive in the first quadrant, and negative in the third, but they have contrary signs in the second and fourth quadrants. Hence the only admissible values of θ are 45° and 225°.

2. $\tan\theta\sec\theta=-\sqrt{2}$. $\quad\therefore\ \theta=225°,\ 315°$.
3. $\operatorname{cosec}\theta=\frac{2}{3}\tan\theta$. $\quad\therefore\ \theta=60°,\ 300°$.
4. $\tan\theta+\cot\theta=2$. $\quad\therefore\ \theta=45°,\ 225°$.
5. $\sec^2\theta+\operatorname{cosec}^2\theta=4$. $\quad\therefore\ \theta=45°,\ 135°,\ 225°,\ 315°$.
6. $\sin\theta=\pm\sqrt{3}\operatorname{vers}\theta$. $\quad\therefore\ \theta=0°,\ 60°,\ 300°$.
7. $\sec\theta+\tan\theta=\pm\sqrt{3}$. $\quad\therefore\ \theta=30°,\ 150°$.
8. $\sec^2\theta+\cot^2\theta=\frac{13}{3}$. $\quad\therefore\ \theta=30°,\ 60°,\ 120°,\ 150°,\ 210°,\ 240°,$ [300°, 330°.
9. $\sin x=+\sqrt{3}\cos x$. $\quad\therefore\ x=60°,\ 240°$.
10. $\tan x=-2\sqrt{3}\cos x$. $\quad\therefore\ x=240°,\ 300°$.
11. $\sin x\cos x=-\frac{1}{4}\sqrt{3}$. $\quad\therefore\ x=120°,\ 150°,\ 300°,\ 330°$.
12. $\sin\theta+\operatorname{cosec}\theta=-\frac{5}{2}$. $\quad\therefore\ \theta=210°,\ 330°$.
13. $3\sin x=2\cos^2 x$. $\quad\therefore\ x=30°,\ 150°$.
14. $\sec x\tan x=+2\sqrt{3}$. $\quad\therefore\ x=60°,\ 120°$.

15. $\sec\theta \operatorname{vers}\theta = 1 - \tan\theta.$

$$\therefore \frac{1}{\cos\theta}(1-\cos\theta) = 1 - \frac{\sin\theta}{\cos\theta}; \quad \therefore \sin\theta = 2\cos\theta - 1;$$

$$\therefore \sin^2\theta = 4\cos^2\theta - 4\cos\theta + 1; \ \therefore 1 - \cos^2\theta = 4\cos^2\theta - 4\cos\theta + 1;$$

$$\therefore 5\cos^2\theta - 4\cos\theta = 0; \quad \therefore \cos\theta\,(5\cos\theta - 4) = 0;$$

$$\therefore \cos\theta = 0 \text{ and } 5\cos\theta - 4 = 0.$$

(*a*) $\cos\theta = 0$ gives $\theta = 90°$ or $270°$. These values are rejected for reasons involving the methods of the Differential Calculus.

(*b*) $\cos\theta = \frac{4}{5}$ gives $\sin\theta = \pm\frac{3}{5}$, since this value of the cosine will allow the angle to lie either in the first or in the fourth quadrant. Transposing in the original equation, we have

$$\sec\theta \operatorname{vers}\theta + \tan\theta - 1 = 0,$$

and we test by substitution. For θ in the first quadrant, we have

$$\tfrac{5}{4}\cdot\tfrac{1}{5} + \tfrac{3}{4} - 1 = 0,$$

showing that θ has a value in the first quadrant. For θ in the fourth quadrant, we have

$$\tfrac{5}{4}\cdot\tfrac{1}{5} - \tfrac{3}{4} - 1 = -\tfrac{3}{2},$$

not zero; and hence θ does not have a value in the fourth quadrant.

16. $\sin x \tan x = -\frac{9}{20}$. $\therefore \sin x = \pm\frac{3}{5}$, $\cos x = -\frac{4}{5}$; quadrants II. and III.

17. $\operatorname{vers} x = 2 \operatorname{covers} x$. $\therefore \cos x = \frac{3}{5}$ or -1; first quadrant, and 180°.

18. $\sin x \tan x = 2\cos x$. $\therefore \sin x = \pm\sqrt{\frac{2}{3}}$, $\cos x = \pm\sqrt{\frac{1}{3}}$; four quadrants.

19. $\sec x \operatorname{cosec} x = -2$. $\therefore \sin x = \pm\frac{1}{2}\sqrt{2}$, $\cos x = \mp\frac{1}{2}\sqrt{2}$; 135° and 315°.

20. $\cos x \cot x = -\frac{5}{6}$. $\therefore \sin x = -\frac{2}{3}$; quadrants III. and IV.

21. $\sin x \cos x = -\frac{12}{25}$. $\therefore \sin x = \pm\frac{4}{5}$ or $\pm\frac{3}{5}$; quadrants II. and IV.

22. $\tan x = -\sqrt{20}\cos x$. $\therefore \sin x = -\dfrac{2}{\sqrt{5}}$; quadrants III. and IV.

23. $\sec x + \tan x = 2$. $\therefore \tan x = +\frac{3}{4}$; first quadrant.

24. $\sec x \tan 2x\,(1 - 2\cos x) = 0.$

The values of x are found by placing each factor equal to zero, and solving the resulting equations. Hence we have

$$\sec x = 0, \ \tan 2x = 0, \ 1 - 2\cos x = 0.$$

But $\sec x = 0$ is impossible; $\tan 2x = 0$ gives $2x = 0°$ or $180°$, and, using the general measures of the angles, $x = 0°$, $90°$, $180°$, $270°$, the second and the last values being inadmissible. $1 - 2\cos x = 0$ gives $\cos x = \frac{1}{2}$, and $x = 60°$ or $300°$.

25. $\tan \frac{1}{2}x = 0$. $\therefore x = 0°$.

26. $\text{vers}\, 3x = 0$. $\therefore x = 0°, 120°, 240°$.

27. $\sin x \cos x (1 + 2\cos x) = 0$. $\therefore x = 0°, 90°, 180°, 270°, 120°, 240°$.

28. $\cos 2x (3 - 4\cos^2 x) = 0$. $\therefore x = 45°, 225°, 135°, 315°, 30°, 150°, 210°, 330°$.

29. $(1 + \tan x)(1 - 2\sin x) = 0$. $\therefore x = 45°, 225°, 30°, 150°$.

30. $\tan x = -2\sin x$. $\therefore x = 0°, 120°, 180°, 240°$.

31. $\sin 2x\, \text{vers}\, 3x = 0$. $\therefore x = 0°, 90°, 180°, 120°, 240°, 270°$.

53. The Functions of an Angle Greater than 360° are the same as those of the angle less than 360°, found by increasing or diminishing the given angle by some multiple of 360°; for the position of the terminal side would not be changed by these operations. Thus

$$\tan 1010° = \tan (1010° - 720°) = \tan 290°;$$

$$\cos (-835°) = \cos (-835° + 720°) = \cos (-115°),$$

or

$$\cos (-835°) = \cos (-835° + 1080°) = \cos 245°.$$

54. The Functions of 90° ± x and of 270° ± x are numerically equal to the *co*functions of x, but may differ from them in signs. Let the arcs EB, ED, KJ, KM, and NP be equal,

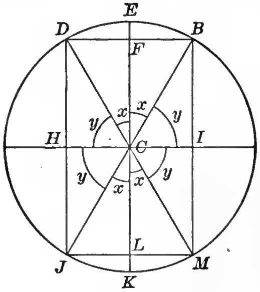


FIG. 36.

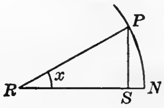


FIG. 37.

the radii CB and RP each being unity. Then the right triangles FCB, FCD, LCJ, LCM, and SRP are equal, having the same hypotenuse (unity) and the angle x the same in each. Therefore

$$FB = CI = LM = DF = HC = JL = SP,$$

and $$CF = IB = HD = LC = MI = JH = RS.$$

$$\therefore \left.\begin{array}{l}\sin(90° - x) = IB = CF = RS = +\cos x; \\ \cos(90° - x) = CI = FB = SP = +\sin x.\end{array}\right\} \quad (1)$$

$$\left.\begin{array}{l}\sin(90° + x) = HD = CF = RS = +\cos x; \\ \cos(90° + x) = CH = FD = -DF^* = -SP = -\sin x.\end{array}\right\} \quad (2)$$

$$\left.\begin{array}{l}\sin(270° - x) = HJ = CL = -LC^* = -RS = -\cos x; \\ \cos(270° - x) = CH = LJ = -JL^* = -SP = -\sin x.\end{array}\right\} \quad (3)$$

$$\left.\begin{array}{l}\sin(270° + x) = IM = CL = -LC^* = -RS = -\cos x; \\ \cos(270° + x) = CI = LM = SP = +\sin x.\end{array}\right\} \quad (4)$$

Thus $\sin 100° = \sin(90° + 10°) = +\cos 10°$;

$\cos 100° = \cos(90° + 10°) = -\sin 10°$.

$\sin 200° = \sin(270° - 70°) = -\cos 70°$;

$\cos 200° = \cos(270° - 70°) = -\sin 70°$.

$\sin 300° = \sin(270° + 30°) = -\cos 30°$;

$\cos 300° = \cos(270° + 30°) = +\sin 30°$.

55. The Functions of 180° ± y and of 360° − y are numerically equal to the *same* functions of y, but may differ from them in signs. From Fig. 36,

$$\left.\begin{array}{l}\sin(180° - y) = HD = IB = +\sin y; \\ \cos(180° - y) = CH = -HC^* = -CI = -\cos y.\end{array}\right\} \quad (1)$$

$$\left.\begin{array}{l}\sin(180° + y) = HJ = -JH^* = -IB = -\sin y; \\ \cos(180° + y) = CH = -HC^* = -CI = -\cos y.\end{array}\right\} \quad (2)$$

$$\left.\begin{array}{l}\sin(360° - y) = IM = -MI^* = -IB = -\sin y; \\ \cos(360° - y) = CI = +\cos y.\end{array}\right\} \quad (3)$$

Thus $\sin 100° = \sin(180° - 80°) = +\sin 80°$;

$\cos 100° = \cos(180° - 80°) = -\cos 80°$.

$\sin 200° = \sin(180° + 20°) = -\sin 20°$;

$\cos 200° = \cos(180° + 20°) = -\cos 20°$.

$\sin 300° = \sin(360° - 60°) = -\sin 60°$;

$\cos 300° = \cos(360° - 60°) = +\cos 60°$.

* See Art. 2.

56. The Functions of a Negative Angle are numerically equal to the same functions of an equal positive angle, but may differ from them in signs.

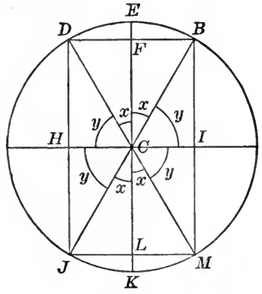


Fig. 38.

$$\left.\begin{array}{l} \sin(-y) = IM = -MI^* = -IB \\ \qquad = -\sin y; \\ \cos(-y) = CI = +\cos y. \end{array}\right\} \quad (1)$$

Thus

$$\sin(x-180^\circ) = \sin[-(180^\circ - x)]$$
$$= -\sin(180^\circ - x) = -\sin x.$$
$$\cos(x-180^\circ) = \cos[-(180^\circ - x)]$$
$$= +\cos(180^\circ - x) = -\cos x.$$

57. Summary. — Using the equations of Art. 46,

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \text{cosec}\, x = \frac{1}{\sin x}.$$

and the results of Arts. 54, 55, and 56, we have

$$\left.\begin{array}{ll} \sin(90^\circ - x) = +\cos x; & \cos(90^\circ - x) = +\sin x; \\ \tan(90^\circ - x) = +\cot x; & \cot(90^\circ - x) = +\tan x; \\ \sec(90^\circ - x) = +\text{cosec}\, x; & \text{cosec}(90^\circ - x) = +\sec x. \end{array}\right\} \quad (1)$$

$$\left.\begin{array}{ll} \sin(90^\circ + x) = +\cos x; & \cos(90^\circ + x) = -\sin x; \\ \tan(90^\circ + x) = -\cot x; & \cot(90^\circ + x) = -\tan x; \\ \sec(90^\circ + x) = -\text{cosec}\, x; & \text{cosec}(90^\circ + x) = +\sec x. \end{array}\right\} \quad (2)$$

$$\left.\begin{array}{ll} \sin(180^\circ - x) = +\sin x; & \cos(180^\circ - x) = -\cos x; \\ \tan(180^\circ - x) = -\tan x; & \cot(180^\circ - x) = -\cot x; \\ \sec(180^\circ - x) = -\sec x; & \text{cosec}(180^\circ - x) = +\text{cosec}\, x. \end{array}\right\} \quad (3)$$

$$\left.\begin{array}{ll} \sin(180^\circ + x) = -\sin x; & \cos(180^\circ + x) = -\cos x; \\ \tan(180^\circ + x) = +\tan x; & \cot(180^\circ + x) = +\cot x; \\ \sec(180^\circ + x) = -\sec x; & \text{cosec}(180^\circ + x) = -\text{cosec}\, x. \end{array}\right\} \quad (4)$$

$$\left.\begin{array}{ll} \sin(270^\circ - x) = -\cos x; & \cos(270^\circ - x) = -\sin x; \\ \tan(270^\circ - x) = +\cot x; & \cot(270^\circ - x) = +\tan x; \\ \sec(270^\circ - x) = -\text{cosec}\, x; & \text{cosec}(270^\circ - x) = -\sec x. \end{array}\right\} \quad (5)$$

$$\left.\begin{array}{ll} \sin(270^\circ + x) = -\cos x; & \cos(270^\circ + x) = +\sin x; \\ \tan(270^\circ + x) = -\cot x; & \cot(270^\circ + x) = -\tan x; \\ \sec(270^\circ + x) = +\text{cosec}\, x; & \text{cosec}(270^\circ + x) = -\sec x. \end{array}\right\} \quad (6)$$

$$\left.\begin{array}{ll} \sin(360^\circ - x) = -\sin x; & \cos(360^\circ - x) = +\cos x; \\ \tan(360^\circ - x) = -\tan x; & \cot(360^\circ - x) = -\cot x; \\ \sec(360^\circ - x) = +\sec x; & \text{cosec}(360^\circ - x) = -\text{cosec}\, x. \end{array}\right\} \quad (7)$$

$$\left.\begin{array}{ll} \sin(-x) = -\sin x; & \cos(-x) = +\cos x; \\ \tan(-x) = -\tan x; & \cot(-x) = -\cot x; \\ \sec(-x) = +\sec x; & \text{cosec}(-x) = -\text{cosec}\, x. \end{array}\right\} \quad (8)$$

* See Art. 2.

These formulas may be remembered from the three facts:

(*a*) Whenever the angle is $90° \pm x$, or $270° \pm x$, the functions of the angle are numerically equal to the corresponding *co*functions of x.

(*b*) Whenever the angle is $180° \pm x$, $360° - x$, or $-x$, the functions of the angle are numerically equal to the *same* functions of x.

(*c*) The sign to be placed before the function of x is that of the original function when x is less than 90°. Thus

$$\sin (270° + x) = -\cos x,$$

since, when $x < 90°$, $270° + x$ will be in the fourth quadrant, and $\sin (270° + x)$ will therefore be negative.

58. General Method of Proof. — In Arts. 54, 55, and 56, both x and y were less than 90°, but the formulas in Art. 57 are true for all values of x. Suppose, for example, that we wish to prove the formulas for $270° + x$ when x is in the fourth quadrant, that is, when x is between 270° and 360°. Let $KAEGJ = x$; then $AEGJKAEGJ = 270° + x$. Let $AEGKJ' = x$. Then in the right triangles JCL and $J'CL'$ the angles JCL and $J'CL'$ are equal, each being $360° - x$; therefore the triangles are equal, and $CL = CL'$ and $LJ = L'J'$ numerically. Algebraically $CL = - CL'$ and $LJ = + L'J'$.

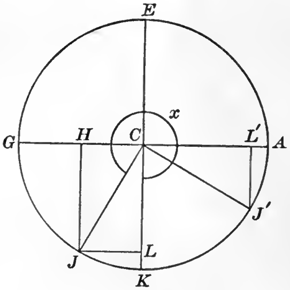


Fig. 39.

$$\therefore \sin (270° + x) = HJ = CL = - CL' = - \cos x;$$
$$\cos (270° + x) = CH = LJ = + L'J' = + \sin x.$$

EXAMPLES.

1. From the preceding equations prove that

(*a*) $\tan (-1200°) = \cot 30°$.
(*b*) $\sec 1000° = \operatorname{cosec} 10°$.
(*c*) $\cos (-890°) = -\cos 10°$.
(*d*) $\cot 1700° = \cot 80°$.
(*e*) $\operatorname{cosec} (-1235°) = -\sec 65°$.
(*f*) $\sin 1340° = -\cos 10°$.
(*g*) $\sin (-3000°) = -\cos 30°$.
(*h*) $\cos 1300° = -\cos 40°$.
(*i*) $\tan 3200° = -\tan 40°$.
(*j*) $\cot (-1300°) = -\cot 40°$.
(*k*) $\sec (-2900°) = +\sec 20°$.
(*l*) $\operatorname{cosec} 2420° = -\sec 10°$.

2. If $\tan\theta = -\cot 140°$, find the two values of θ less than 360°

$\therefore\ \tan\theta = -\cot(90° + 50°) = +\tan 50°$. $\quad\therefore\ \theta = 50°,\ 230°$.

3. Find the values of θ in the following equations:

(a)	$\sin\theta = +\cos 220°$.	$\therefore$ 230°, 310°.
(b)	$\sin\theta = +\cos 310°$.	$\therefore$ 40°, 140°.
(c)	$\sin\theta = -\cos 210°$.	$\therefore$ 60°, 120°.
(d)	$\sin^2\theta = +\cos^2 200°$.	$\therefore$ 70°, 110°, 250°, 290°.
(e)	$\cos\theta = +\sin 150°$.	$\therefore$ 60°, 300°.
(f)	$\cos\theta = +\sin 250°$.	$\therefore$ 160°, 200°.
(g)	$\cos\theta = -\sin 170°$.	$\therefore$ 100°, 260°.
(h)	$\cos\theta = -\sin 275°$.	$\therefore$ 5°, 355°.
(i)	$\cos^2\theta = +\sin^2 100°$.	$\therefore$ 10°, 350°, 170°, 190°.
(j)	$\tan\theta = +\cot 100°$.	$\therefore$ 170°, 350°.
(k)	$\tan\theta = +\cot 200°$.	$\therefore$ 70°, 250°.
(l)	$\tan\theta = -\cot 230°$.	$\therefore$ 140°, 320°
(m)	$\cot\theta = +\tan 260°$.	$\therefore$ 10°, 190°.
(n)	$\cot\theta = +\tan 345°$.	$\therefore$ 105°, 285°.
(o)	$\cot\theta = -\tan 245°$.	$\therefore$ 155°, 335°.
(p)	$\cot\theta = -\tan 305°$.	$\therefore$ 35°, 215°.
(q)	$\sec\theta = -\operatorname{cosec} 100°$.	$\therefore$ 170°, 190°.
(r)	$\sec\theta = +\operatorname{cosec} 130°$.	$\therefore$ 40°, 320°.
(s)	$\sec\theta = +\operatorname{cosec} 310°$	$\therefore$ 140°, 220°.
(t)	$\operatorname{cosec}\theta = +\sec 315°$.	$\therefore$ 45°, 135°.
(u)	$\operatorname{cosec}\theta = +\sec 230°$.	$\therefore$ 220°, 320°.
(v)	$\operatorname{cosec}\theta = -\sec 185°$.	$\therefore$ 85°, 95°.
(w)	$\operatorname{cosec}\theta = -\sec 335°$.	$\therefore$ 245°, 295°.
(x)	$\operatorname{cosec}^2\theta = +\sec^2 250°$.	$\therefore$ 20°, 160°, 200°, 340°.
(y)	$\sec\theta = -\operatorname{cosec} 290°$.	$\therefore$ 20°, 340°.
(z)	$\sin\theta = -\cos 300°$.	$\therefore$ 210°, 330°.

4. $\cos\theta = \sin 2\theta$. Show that one value of θ is 30°.

5. $\tan n\theta = -\cot 120°$. Show that one value of θ is $30° \div n$.

6. $\sec 3\theta = \operatorname{cosec}(n-1)\theta$. Show that one value of θ is $90° \div (n+2)$.

7. If $\cot 309° = -\frac{8}{10}$, find $\sin 219°$.

$\sin 219° = \sin(180° + 39°) = -\sin 39°$.

But $\cot 309° = \cot(270° + 39°) = -\tan 39°$; $\quad\therefore\ \tan 39° = +\frac{8}{10}$.

$$\therefore\ \sin 39° = \frac{\tan 39°}{\sqrt{1+\tan^2 39°}} = \frac{8}{\sqrt{164}} = \frac{4}{\sqrt{41}};\quad \therefore\ \sin 219° = -\frac{4}{\sqrt{41}}.$$

8. If $\sin 217° = -\frac{6}{10}$, prove that $\tan 127° = -\frac{4}{3}$.

9. If $\cos 125° = -a$, prove that $\tan 325° = -\dfrac{a}{\sqrt{1-a^2}}$.

10. If $\cot 260° = +a$, prove that $\cos 350° = +\dfrac{1}{\sqrt{1+a^2}}$.

11. If $\sec 340° = +a$, prove that $\sin 110° = \dfrac{1}{a}$, and $\tan 110° = -\dfrac{1}{\sqrt{a^2-1}}$.

12. If $\cos 300° = +a$, prove that $\cot 120° = -\dfrac{a}{\sqrt{1-a^2}}$.

13. If $\sin 115° = +a$, prove that $\dfrac{\tan 205° \sec 245°}{\operatorname{cosec} 335°} = +\dfrac{\sqrt{1-a^2}}{a}$.

14. If $\cos 200° = -m$, prove that $\tan 110° \operatorname{cosec} 250° \cot 290° = -\dfrac{1}{m}$.

15. If $\operatorname{cosec} 185° = -m$, prove that $\tan 355° \tan 275° \cos 175° = -\dfrac{\sqrt{m^2-1}}{m}$.

16. Show that $\cot \frac{1}{6}(-x - 540°) = \tan \frac{1}{6}x$.

$$\cot \tfrac{1}{6}(-x-540°) = \cot(-\tfrac{1}{6}x - 90°)$$
$$= \cot[-(90° + \tfrac{1}{6}x)] = -\cot(90° + \tfrac{1}{6}x) = +\tan \tfrac{1}{6}x.$$

17. Show that $\sin(y - 90°) = -\cos y$.

18. Show that $\sin(y - 180°) = -\sin y$.

19. Show that $\cos(y - 270°) = -\sin y$.

20. Show that $\sec(-x - 540°) = -\sec x$.

21. Show that $\tan(y - 360°) = +\tan y$.

22. Show that $\cos \frac{1}{3}(x - 270°) = +\sin \frac{1}{3}x$. [Note that 270° in the parenthesis is to be multiplied by $\frac{1}{3}$.]

23. Show that $\cos \frac{1}{3}(-810° + a - b) = -\sin \frac{1}{3}(a-b)$.

24. Show that $\operatorname{cosec} \frac{1}{4}(x - 360°) = -\sec \frac{1}{4}x$.

25. Show that $\sec \frac{1}{5}(-900° - x) = -\sec \frac{1}{5}x$.

26. Show that $\tan \frac{1}{2}(360° + a - b) = +\tan \frac{1}{2}(a-b)$.

27. Show that $\cos(180° - x)$ is equal to the sine of the complementary angle.

Complement $= 90° - (180° - x) = -(90° - x)$; $\sin[-(90° - x)] = -\sin(90° - x) = -\cos x$. But $\cos(180° - x) = -\cos x$. $\therefore \cos(180 - x) = \sin[90° - (180° - x)]$. Q. E. D.

28. Show that $\operatorname{cosec}(270° - x)$ equals the secant of the complementary angle.

29. Show that $\tan(180° + x)$ equals the cotangent of the complementary angle.

30. Show that $\sec(270° + x)$ equals the cosecant of the complementary angle.

31. Show that $\cos(90°+x)$ equals the sine of the complementary angle.

32. Show that $\cot(360°-x)$ equals the tangent of the complementary angle.

33. Show that $\tan(270°+x)$ is equal to the negative of the tangent of the supplementary angle.

Supplement $= 180° - (270° + x) = -(90° + x)$; $\tan[-(90° + x)] = -\tan(90° + x) = +\cot x$. But $\tan(270° + x) = -\cot x$. $\therefore \tan(270° + x) = -\tan[180° - (270° + x)]$. Q. E. D.

34. Show that $\operatorname{cosec}(180° + x)$ is equal to the cosecant of the supplementary angle.

35. Show that $\sin(360° - x)$ is equal to the sine of the supplementary angle.

36. Show that $\sec(90° + x)$ is equal to the negative of the secant of the supplementary angle.

37. Show that $\cos(270° - x)$ is equal to the negative of the cosine of the supplementary angle.

38. Show that $\cot(270° + x)$ is equal to the negative of the cotangent of the supplementary angle.

59. The Trigonometric Tables. — The relations shown in Arts. 53 and 57 enable us to find the functions of any angle, although the tables contain only the sines, cosines, tangents, and cotangents of angles less than 45°. For, since

$$\sin(90° - x) = \cos x,\ \cos(90° - x) = \sin x,$$

$$\tan(90° - x) = \cot x,\ \cot(90° - x) = \tan x,$$

the tables are immediately extended to 90° by writing the proper degrees and minutes at the bottom and on the right of the page respectively.

Then, since the value of any function of an angle greater than 90° can be found in terms of a function of an angle less than 90°, we can find the numerical value of the function from the tables.

1. Find from the tables the logarithmic functions of 580° 42'.4.

$$580° 42'.4 = 360° + 220° 42'.4.$$

$\therefore \sin 580° 42'.4 = \sin 220° 42'.4 = \sin(180° + 40° 42'.4) = -\sin 40° 42'.4$;

$\therefore \log\sin 580° 42'.4 = 9.81437\ n.$

$\cos 580° 42'.4 = -\cos 40° 42'.4$; $\therefore \log\cos 580° 42'.4 = 9.87971\ n.$

$\tan 580° 42'.4 = +\tan 40° 42'.4$; $\therefore \log\tan 580° 42'.4 = 9.93467.$

$\cot 580° 42'.4 = +\cot 40° 42'.4$; $\therefore \log\cot 580° 42'.4 = 0.06533.$

2. Find from the tables the logarithmic functions of the following angles:

Angle.	log sin.	log cos.	log tan.	log cot.
499° 29'.7.	9.81258.	9.88102 *n*.	9.93158 *n*.	0.06842 *n*.
597° 8'.3.	9.92427 *n*.	9.73449 *n*.	0.18978.	9.81022.
689° 27'.6.	9.70598 *n*.	9.93514.	9.77084 *n*.	0.22916 *n*.

3. $\sin b = \tan 250° 15'.5 \cot 278° 17'.3$; find $b =$ 203° 57'.0 or 336° 3'.0.

4. $\cos\beta = \cos 149° 27'.6 \sin 216° 44'.0$; find $\beta =$ 58° 59'.7 or 301° 0'.3.

5. $\tan a = \sin 319° 52'.0 \div \cot 254° 30'.2$; find $a =$ 113° 16'.5 or 293° 16'.5.

6. $\cot c = \cos 216° 44'.0 \div \tan 329° 27'.6$; find $c =$ 36° 21'.6 or 216° 21'.6.

60. Transform the First Member into the Second in the following examples. Usually it is best to change the given expression into one containing the sine and cosine, and then to change this into the required form. Any operation is admissible that does not change the *value* of the expression. Use radicals only when unavoidable. If the expression is factored, it is often advantageous to reduce each factor separately, not multiplying until it becomes necessary.

1. $\frac{\tan x - \sin x}{\sin^3 x} = \frac{\sec x}{1 + \cos x}.$

$$\frac{\tan x - \sin x}{\sin^3 x} = \frac{\frac{\sin x}{\cos x} - \sin x}{\sin^3 x} = \frac{\sin x(1 - \cos x)}{\cos x \sin^3 x} = \frac{1 - \cos x}{\cos x \sin^2 x}$$

$$= \frac{1 - \cos x}{\cos x(1 - \cos^2 x)} = \frac{1}{\cos x(1 + \cos x)} = \frac{\sec x}{1 + \cos x}.$$

2. $\cos x \tan x + \sin x \cot x = \sin x + \cos x.$

3. $(2 - \text{vers}\, x) \text{vers}\, x = \sin^2 x.$

4. $\frac{\cos x}{\sin x \cot^2 x} = \tan x.$

5. $(\tan x + \cot x) \sin x \cos x = 1.$

6. $(\sec^2 x - 1)(\text{cosec}^2 x - 1) = 1.$

7. $\sec x \,\text{cosec}\, x (\cos^2 x - \sin^2 x) = \cot x - \tan x.$

8. $(\sin x + \cos x)(\tan x + \cot x) = \sec x + \text{cosec}\, x.$

9. $\cot x + \frac{\sin x}{1 + \cos x} = \text{cosec}\, x.$

10. $\sin x (\sec x + \text{cosec}\, x) - \cos x (\sec x - \text{cosec}\, x) = \sec x \,\text{cosec}\, x.$

11. $(\text{cosec}\, x - \cot x)^2 = \frac{1 - \cos x}{1 + \cos x}.$

12. $(1 + \tan^2 x)(1 - \cot^2 x) = \sec^2 x - \text{cosec}^2 x.$

13. $\frac{\tan x - \cot x}{\tan x + \cot x} = \frac{2}{\text{cosec}^2 x} - 1.$ [First change to an expression containing only $\sin x$, the reciprocal of $\text{cosec}\, x$.]

14. $\sec^2 x \,\text{cosec}^2 x - 2 = \tan^2 x + \cot^2 x.$ [Substitute for $\sec x$ and $\text{cosec}\, x$ their values in terms of $\tan x$ and $\cot x$ respectively.]

15. $\frac{\tan \alpha + \tan \beta}{\cot \alpha + \cot \beta} = \tan \alpha \tan \beta.$

16. $\cot x - \sec x \,\text{cosec}\, x (1 - 2 \sin^2 x) = \tan x.$ [The expression reduces to $\sin x \div \cos x$.]

17. $\operatorname{cosec} x\,(\sec x - 1) - \cot x\,(1 - \cos x) = \tan x - \sin x$. [Factor as soon as possible, and reduce each factor separately.]

18. $\operatorname{vers} x\,(\sec x + 1) + \operatorname{covers} x\,(\operatorname{cosec} x + 1) = \sin x \tan x + \cos x \cot x$.

19. $\dfrac{\operatorname{vers} x\,(1 + \sec x)}{\sin x} + \dfrac{\operatorname{covers} x\,(1 + \operatorname{cosec} x)}{\cos x} = \sec x \operatorname{cosec} x$.

20. $\sin^2 x\,(\tan^2 x - 1) + \cos^2 x\,(\cot^2 x - 1) = \dfrac{(1 - 2\cos^2 x)^2 \sec^4 x}{\tan^2 x}$.

21. $\sec x \operatorname{cosec} x\,[\operatorname{vers} x(\operatorname{vers} x - 2) - \operatorname{covers} x(\operatorname{covers} x - 2)] = \cot x - \tan x$.

22. $\cos^4 x - \sin^4 x = \cos x\,(1 - \tan x)(\sin x + \cos x)$.

23. $\dfrac{\operatorname{vers} x\,(1 + \cos x) - \operatorname{covers} x\,(1 + \sin x)}{\sec^2 x \operatorname{cosec}^2 x} = \dfrac{\tan^4 x - \tan^2 x}{\sec^6 x}$. [Change to an expression containing only $\sin x$ and $\cos x$, and then substitute their values in terms of $\tan x$.]

24. $\dfrac{\sec^2 x \sin^2 x - \operatorname{cosec}^2 x + \operatorname{cosec}^2 x \cos^2 x}{\sec^2 x \sin^2 x - \operatorname{cosec}^2 x \cos^2 x} = \sin^2 x$.

25. $\tan^2 x - \sin^2 x \cos^2 x = \dfrac{(\sec^2 x + 1)(\sec^2 x - 1)^2}{\sec^4 x}$.

26. $\dfrac{\cos x \cot x - \sin x \tan x}{\operatorname{cosec} x - \sec x} = 1 + \sin x \cos x$.

27. $\dfrac{(\sec x + \operatorname{cosec} x)^2}{\tan x + \cot x} = \dfrac{(1 + \tan x)^2}{\tan x}$.

28. $2 + \dfrac{\sin^4 x + \cos^4 x}{\sin^2 x \cos^2 x} = \sec^2 x + \operatorname{cosec}^2 x$.

29. $\dfrac{\sec x + \operatorname{cosec} x}{\sec x - \operatorname{cosec} x} = \dfrac{\tan x + 1}{\tan x - 1} = \dfrac{1 + \cot x}{1 - \cot x}$.

30. $\dfrac{\sin x - \tan^2 x \operatorname{covers} x}{\operatorname{cosec} x \cot^2 x} = \dfrac{\sin^4 x}{(1 + \sin x)\cos^2 x}$.

31. $\sec^2 x \operatorname{cosec}^2 x\,[\operatorname{vers} x\,(\operatorname{vers} x - 2) - \operatorname{covers} x\,(\operatorname{covers} x - 2)]$
$= \cot^2 x - \tan^2 x$.

32. $\tan x + \cot x = \dfrac{\sec^2 x + \operatorname{cosec}^2 x}{\sec x \operatorname{cosec} x}$. [It is admissible to divide both numerator and denominator by $\sin^2 x \cos^2 x$.]

33. $\tan^2 \alpha \tan^2 \beta - 1 = \dfrac{\sin^2 \alpha - \cos^2 \beta}{\cos^2 \alpha \cos^2 \beta} = \dfrac{\sin^2 \beta - \cos^2 \alpha}{\cos^2 \alpha \cos^2 \beta}$.

34. $\dfrac{1 - \tan^2 \alpha \tan^2 \beta}{\tan^2 \alpha \tan^2 \beta} = \dfrac{\cos^2 \alpha - \sin^2 \beta}{\sin^2 \alpha \sin^2 \beta} = \dfrac{\cos^2 \beta - \sin^2 \alpha}{\sin^2 \alpha \sin^2 \beta}$.

35. $\sin^2 x \tan^2 x + \cos^2 x \cot^2 x = \tan^2 x + \cot^2 x - 1$.

36. $\sin^2 x \tan x + \cos^2 x \cot x + 2 \sin x \cos x = \sec x \operatorname{cosec} x$.

37. $\sec^4 x + \tan^4 x = 1 + 2\sec^2 x \tan^2 x$. [It is admissible to add and subtract $2 \sec^2 x \tan^2 x$.]

38. $(r \cos \phi)^2 + (r \sin \phi \sin \theta)^2 + (r \sin \phi \cos \theta)^2 = r^2$.

$\therefore\ r^2 \cos^2 \phi + r^2 \sin^2 \phi\,(\sin^2 \theta + \cos^2 \theta) = r^2(\cos^2 \phi + \sin^2 \phi) = r^2$,
since $\sin^2 x + \cos^2 x = 1$.

39. $(2r \sin a \cos a)^2 + r^2(\cos^2 a - \sin^2 a)^2 = r^2.$

40. $(a \sin \gamma)^2 + (a \cos \gamma \sin \delta)^2 + (a \cos \gamma \cos \delta)^2 = a^2.$

41. $(\cos a \cos b - \sin a \sin b)^2 + (\sin a \cos b + \cos a \sin b)^2 = 1.$

42. $(\cos a \cos b + \sin a \sin b)^2 + (\sin a \cos b - \cos a \sin b)^2 = 1.$

43. $(x \cos \theta - y \sin \theta)^2 + (x \sin \theta + y \cos \theta)^2 = x^2 + y^2.$

44. $\dfrac{1}{(\cos^2 x - \sin^2 x)^2} - \dfrac{4 \tan^2 x}{(1 - \tan^2 x)^2} = 1.$

45. $\dfrac{1}{4 \sin^2 x \cos^2 x} - \dfrac{(1 - \tan^2 x)^2}{4 \tan^2 x} = 1.$

46. $(3 \sin a \cos^2 a - \sin^3 a)^2 + (\cos^3 a - 3 \sin^2 a \cos a)^2 = 1.$

47. $x^2 + y^2 + z^2 = r^2$ when

$$x = r \cos a \cos \beta + r \cos i \sin a \sin \beta,$$
$$y = r \cos i \cos a \sin \beta - r \sin a \cos \beta,$$
$$z = r \sin i \sin \beta.$$

CHAPTER V.

RELATIONS BETWEEN FUNCTIONS OF SEVERAL ANGLES.

61. Sine and Cosine of the Sum of Two Angles. — Let x and y be the angles, each, as well as their sum, being less than $90°$. QC is perpendicular to OP, BC and DQ are perpendicular to OA, and EC is parallel to OA, the radius of the circle being unity. Then

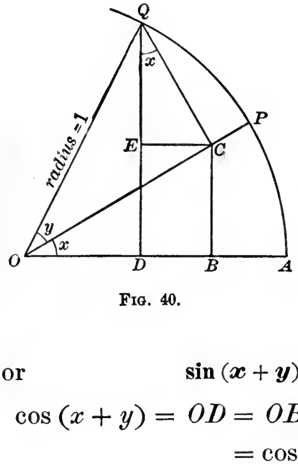


FIG. 40.

$$x + y = AOQ,$$

the angle $EQC = x$, $OC = \cos y$, and $CQ = \sin y$.

$$\sin(x+y) = DQ = BC + EQ$$
$$= OC \sin BOC + CQ \cos EQC$$
$$= \cos y \sin x + \sin y \cos x,$$

or
$$\mathbf{\sin(x+y) = \sin x \cos y + \cos x \sin y.} \quad (1)$$

$$\cos(x+y) = OD = OB - EC = OC \cos BOC - CQ \sin EQC$$
$$= \cos y \cos x - \sin y \sin x,$$

or
$$\mathbf{\cos(x+y) = \cos x \cos y - \sin x \sin y.} \quad (2)$$

1. $\sin 90° = \sin(60° + 30°) = \sin 60° \cos 30° + \cos 60° \sin 30°$

$$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot \frac{1}{2} = 1.$$

2. $\cos 90° = \cos(60° + 30°) = \cos 60° \cos 30° - \sin 60° \sin 30°$

$$= \frac{1}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} \cdot \frac{1}{2} = 0.$$

3. If $\sin \alpha = \frac{3}{5}$, and $\sin \beta = \frac{5}{13}$, find $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$ when $\alpha < 90°$, and $\beta < 90°$

Ans. $\sin(\alpha + \beta) = \frac{56}{65}$, $\cos(\alpha + \beta) = \frac{33}{65}$.

4. If $\tan\alpha = \frac{3}{4}$, and $\tan\beta = \frac{7}{24}$, find $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$ when $\alpha < 90°$, and $\beta < 90°$.

Ans. $\sin(\alpha+\beta) = \frac{4}{5}$, $\cos(\alpha+\beta) = \frac{3}{5}$.

NOTE.—At a point A the angle of elevation DAB to the top of a vertical wall is α, and the angle of depression CAD to its base is β. Find the height CB of the wall, the horizontal distance AD being a feet.

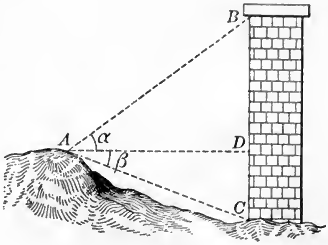


FIG. 41.

$$CB = CD + DB = a\tan\beta + a\tan\alpha$$
$$= a(\tan\alpha + \tan\beta). \qquad (3)$$
$$= a\left(\frac{\sin\alpha}{\cos\alpha} + \frac{\sin\beta}{\cos\beta}\right)$$
$$= a\frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta}$$
$$= a\frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}. \qquad (4)$$

Eq. (3) would be solved by the use of the natural functions, while (4) is adapted to logarithmic computation.

62. Sine and Cosine of the Difference of Two Angles.—Let x and y be the angles, each being less than 90° and x being greater than y. QC is perpendicular to OP, BC and DQ are perpendicular to OA, and EQ is parallel to OA, the radius of the circle being unity. Then $x - y = AOQ$, $ECQ = x$, $OC = \cos y$, and $CQ = \sin y$.

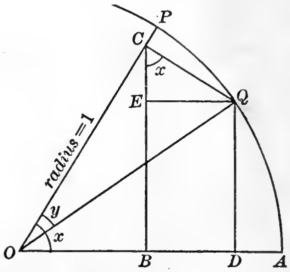


FIG. 42.

$$\sin(x-y) = DQ = BC - EC$$
$$= OC\sin BOC - CQ\cos ECQ$$
$$= \cos y\sin x - \sin y\cos x,$$

or
$$\mathbf{\sin(x-y) = \sin x\cos y - \cos x\sin y.} \qquad (1)$$

$$\cos(x-y) = OD = OB + EQ = OC\cos BOC + CQ\sin ECQ$$
$$= \cos y\cos x + \sin y\sin x,$$

or
$$\mathbf{\cos(x-y) = \cos x\cos y + \sin x\sin y.} \qquad (2)$$

In this proof we have assumed that x is the greater angle, but (1) and (2) are true when y is greater than x. To prove this, let β be greater than α. Then

$$\sin(\alpha-\beta) = \sin[-(\beta-\alpha)] = -\sin(\beta-\alpha),$$

and, developing $\sin(\beta-\alpha)$ by (1),

$$\sin(\alpha-\beta)=-(\sin\beta\cos\alpha-\cos\beta\sin\alpha)$$

$$=\sin\alpha\cos\beta-\cos\alpha\sin\beta. \qquad \text{Q.E.D.}$$

Also $\cos(\alpha-\beta)=\cos[-(\beta-\alpha)]=\cos(\beta-\alpha)$

$$=\cos\beta\cos\alpha+\sin\beta\sin\alpha. \qquad \text{Q.E.D.}$$

1. $\sin 30^\circ=\sin(90^\circ-60^\circ)=\sin 90^\circ\cos 60^\circ-\cos 90^\circ\sin 60^\circ$

$$=1\cdot\frac{1}{2}-0\cdot\frac{\sqrt{3}}{2}=\frac{1}{2}.$$

2. $\cos 30^\circ=\cos(90^\circ-60^\circ)=\cos 90^\circ\cos 60^\circ+\sin 90^\circ\sin 60^\circ$

$$=0\cdot\frac{1}{2}+1\cdot\frac{\sqrt{3}}{2}=\frac{\sqrt{3}}{2}.$$

3. If $\sin\alpha=\frac{5}{13}$, and $\sin\beta=\frac{9}{41}$, find $\sin(\alpha-\beta)$ and $\cos(\alpha-\beta)$ when $\alpha<90^\circ$, and $\beta<90^\circ$.

Ans. $\sin(\alpha-\beta)=\frac{92}{533}$; $\cos(\alpha-\beta)=\frac{525}{533}$.

4. If $\tan\alpha=\frac{4}{3}$, and $\tan\beta=\frac{3}{4}$, find $\sin(\alpha-\beta)$ and $\cos(\alpha-\beta)$ when $\alpha<90^\circ$, and $\beta<90^\circ$.

Ans. $\sin(\alpha-\beta)=\frac{7}{25}$; $\cos(\alpha-\beta)=\frac{24}{25}$.

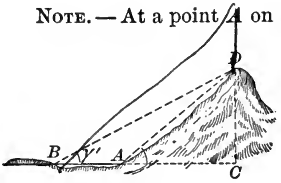


FIG. 43.

NOTE.—At a point A on a horizontal plane, the angle CAD to the top of a crag is γ, and a feet farther away in the same vertical plane (at B), the angle CBD is γ'. Find $AC=x$.

$$CD=AC\tan\gamma=x\tan\gamma.$$

$$CD=BC\tan\gamma'=(a+x)\tan\gamma'.$$

$$\therefore\ x\tan\gamma=(a+x)\tan\gamma'.$$

$$\therefore\ x=\frac{a\tan\gamma'}{\tan\gamma-\tan\gamma'}. \qquad (3)$$

$$=\frac{a\sin\gamma'\cos\gamma}{\sin\gamma\cos\gamma'-\cos\gamma\sin\gamma'}=\frac{a\sin\gamma'\cos\gamma}{\sin(\gamma-\gamma')}. \qquad (4)$$

Eq. (3) would be solved by the use of the natural functions, while (4) is adapted to logarithmic computation.

63. General Proof of the Addition Formulas.—These formulas were shown in Art. 61 to be true when x, y, and $x+y$ were each less than 90°. That they are true for all values of these angles may be shown by proving the special cases separately. Let us consider first the case when

$$x<90^\circ,\ y<90^\circ,\ x+y>90^\circ \text{ and } <180^\circ.$$

Let $x = 90° - \alpha,\ y = 90° - \beta;\ \therefore\ x + y = 180° - (\alpha + \beta).$

$\therefore\ \alpha = 90° - x,\ \beta = 90° - y,\ \alpha + \beta = 180° - (x + y).$

$\therefore\ \alpha < 90°,\ \beta < 90°,\ \alpha + \beta < 90°,$ since $x + y > 90°.$

Then $\sin(\alpha + \beta)$ may be developed by (1), Art. 61, since the conditions of that article are satisfied. But

$$\begin{aligned}\sin(x + y) &= \sin[180° - (\alpha + \beta)] = \sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta\\ &= \sin(90° - x)\cos(90° - y) + \cos(90° - x)\sin(90° - y)\\ &= \cos x\sin y + \sin x\cos y. \qquad \text{Q.E.D.}\end{aligned}$$

Also $$\begin{aligned}\cos(x + y) &= \cos[180° - (\alpha + \beta)] = -\cos(\alpha + \beta) = -\cos\alpha\cos\beta + \sin\alpha\sin\beta\\ &= -\cos(90° - x)\cos(90° - y) + \sin(90° - x)\sin(90° - y)\\ &= -\sin x\sin y + \cos x\cos y. \qquad \text{Q.E.D.}\end{aligned}$$

Hence the formulas are true for $x < 90°,\ y < 90°,\ x + y < 180°.$

To illustrate the proof for any special case, let us take x in the second and y in the fourth quadrant. Place $x = 90° + \alpha$, and $y = 270° + \beta$, so that α and β are each less than 90°. Then

$$\begin{aligned}\sin(x + y) &= \sin[360° + (\alpha + \beta)] = \sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta\\ &= \sin(x - 90°)\cos(y - 270°) + \cos(x - 90°)\sin(y - 270°)\\ &= (-\cos x)(-\sin y) + \sin x\cos y\\ &= \cos x\sin y + \sin x\cos y. \qquad \text{Q.E.D.}\end{aligned}$$

Let the student prove the addition formulas in the following cases:

1. x in the first, and y in the third quadrant.
2. x in the second, and y in the second quadrant.
3. x in the second, and y in the third quadrant.
4. x in the third, and y in the third quadrant.
5. x in the third, and y in the fourth quadrant.
6. x in the fourth, and y in the fourth quadrant.

64. General Proof of the Subtraction Formulas. — These formulas were shown in Art. 62 to be true when x and y were each less than 90°, both for $x > y$ and for $x < y$. That they are true for all values of the angles may be shown by proving the special cases separately. For illustration, let x be in the second, and y in the third quadrant. Place $x = 90° + \alpha$, and $y = 180° + \beta$, so that $\alpha < 90°$, and $\beta < 90°$. Then

$$\begin{aligned}\sin(x - y) &= \sin[90° + \alpha - (180° + \beta)] = \sin[-90° + (\alpha - \beta)] = -\sin[90° - (\alpha - \beta)]\\ &= -\cos(\alpha - \beta) = -\cos\alpha\cos\beta - \sin\alpha\sin\beta\\ &= -\cos(x - 90°)\cos(y - 180°) - \sin(x - 90°)\sin(y - 180°)\\ &= -\sin x(-\cos y) - (-\cos x)(-\sin y)\\ &= \sin x\cos y - \cos x\sin y. \qquad \text{Q.E.D.}\end{aligned}$$

Also $$\begin{aligned}\cos(x - y) &= \cos[-90° + (\alpha - \beta)] = \cos[90° - (\alpha - \beta)] = \sin(\alpha - \beta)\\ &= \sin\alpha\cos\beta - \cos\alpha\sin\beta\\ &= \sin(x - 90°)\cos(y - 180°) - \cos(x - 90°)\sin(y - 180°)\\ &= (-\cos x)(-\cos y) - \sin x(-\sin y)\\ &= \cos x\cos y + \sin x\sin y. \qquad \text{Q.E.D.}\end{aligned}$$

Let the student prove the subtraction formulas in the following cases:

1. x in the fourth, and y in the first quadrant.
2. x in the fourth, and y in the second quadrant.
3. x in the fourth, and y in the third quadrant.
4. x in the third, and y in the third quadrant.
5. x in the third, and y in the fourth quadrant.
6. x in the second, and y in the fourth quadrant.

65. Tangent of the Sum and of the Difference of Two Angles.

$$\tan(x+y) = \frac{\sin(x+y)}{\cos(x+y)} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}.$$

Divide both numerator and denominator by $\cos x \cos y$.

$$\therefore \tan(x+y) = \frac{\dfrac{\sin x \cos y}{\cos x \cos y} + \dfrac{\cos x \sin y}{\cos x \cos y}}{\dfrac{\cos x \cos y}{\cos x \cos y} - \dfrac{\sin x \sin y}{\cos x \cos y}} = \frac{\dfrac{\sin x}{\cos x} + \dfrac{\sin y}{\cos y}}{1 - \dfrac{\sin x \sin y}{\cos x \cos y}};$$

$$\therefore \tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}. \quad (1)$$

In the same way, we may show that

$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}. \quad (2)$$

1. $\tan 75° = \tan(45° + 30°) = \dfrac{\tan 45° + \tan 30°}{1 - \tan 45° \tan 30°} = \dfrac{1 + \dfrac{1}{\sqrt{3}}}{1 - \dfrac{1}{\sqrt{3}}} = \dfrac{\sqrt{3}+1}{\sqrt{3}-1} = 2 + \sqrt{3}.$

2. $\tan 15° = \tan(45° - 30°) = \dfrac{\tan 45° - \tan 30°}{1 + \tan 45° \tan 30°} = \dfrac{1 - \dfrac{1}{\sqrt{3}}}{1 + \dfrac{1}{\sqrt{3}}} = \dfrac{\sqrt{3}-1}{\sqrt{3}+1} = 2 - \sqrt{3}.$

3. If $\sin\alpha = \frac{12}{13}$ and $\sin\beta = \frac{3}{5}$, find $\tan(\alpha+\beta)$ and $\tan(\alpha-\beta)$, when $\alpha < 90°$ and $\beta < 90°$. *Ans.* $\tan(\alpha+\beta) = -\frac{63}{16}$; $\tan(\alpha-\beta) = \frac{33}{56}$.

66. Geometrical Proof. — In Fig. 44, let $OA = 1$, $AOB = x$, $BOC = y$. Draw BC perpendicular to OB, and CD parallel to OA; $\therefore DBC = x$, $DCE = x + y$. Then

$$\tan(x+y) = AE = AB + BD + DE.$$

But $\quad BD = BC \cos x = OB \tan y \cos x = \sec x \tan y \cos x = \tan y,$

and $\quad DE = CD \tan(x+y) = BC \sin x \tan(x+y) = OB \tan y \sin x \tan(x+y)$

$$= \sec x \tan y \sin x \tan(x+y) = \tan x \tan y \tan(x+y).$$

$$\therefore \tan(x+y) = \tan x + \tan y + \tan x \tan y \tan(x+y).$$

$$\therefore \tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}.$$

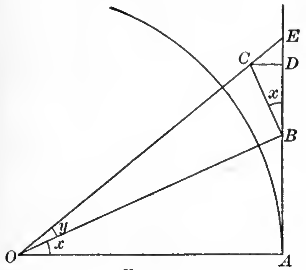


FIG. 44.

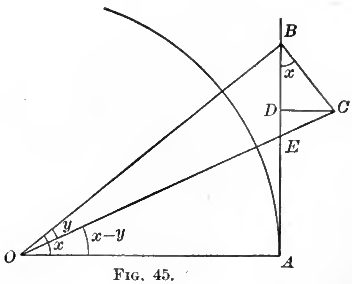


FIG. 45.

In Fig. 45, let $OA = 1$, $AOB = x$, $COB = y$. Draw BC perpendicular to OB, and DC parallel to OA; $\therefore DBC = x$, $DCE = x - y$. Then

$$\tan(x-y) = AE = AB - DB - ED.$$

But $\quad DB = BC \cos x = OB \tan y \cos x = \sec x \tan y \cos x = \tan y,$

and $\quad ED = DC \tan(x-y) = BC \sin x \tan(x-y) = OB \tan y \sin x \tan(x-y)$

$$= \sec x \tan y \sin x \tan(x-y) = \tan x \tan y \tan(x-y).$$

$$\therefore \tan(x-y) = \tan x - \tan y - \tan x \tan y \tan(x-y).$$

$$\therefore \tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}.$$

EXAMPLES.

Find by inspection one value of x in Exs. (1–6):

1. $\sin(n-1)\alpha \cos\alpha + \cos(n-1)\alpha \sin\alpha = \sin x.$ *Ans.* $x = n\alpha.$

2. $\cos(10°+\alpha)\cos(10°-\alpha) + \sin(10°+\alpha)\sin(10°-\alpha) = \cos x.$ *Ans.* $x = 2\alpha.$

3. $\sin(\alpha-\beta+10°)\cos(\beta-\alpha+10°) - \cos(\alpha-\beta+10°)\sin(\beta-\alpha+10°) = \sin x.$
Ans. $x = 2(\alpha - \beta).$

4. $\cos 45° \cos(90° - \alpha) - \sin 45° \sin(90° - \alpha) = \cos x.$ *Ans.* $x = 135° - \alpha.$

5. $\sin(90° + \frac{1}{2}\alpha)\cos(90° - \frac{1}{2}\alpha) + \cos(90° + \frac{1}{2}\alpha)\sin(90° - \frac{1}{2}\alpha) = \sin x.$
Ans. $x = 180°.$

6. $\cos(45°-\alpha)\cos(45°+\alpha) - \sin(45°-\alpha)\sin(45°+\alpha) = \cos x.$ *Ans.* $x = 90°.$

7. Given the functions of 30° and 45°, find those of 75°.

Ans. $\sin 75° = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$; $\cos 75° = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$; $\tan 75° = \dfrac{\sqrt{3}+1}{\sqrt{3}-1} = 2 + \sqrt{3}.$

8. Given the functions of 30° and 45°, find those of 15°.

Ans. $\sin 15° = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$; $\cos 15° = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$; $\tan 15° = 2 - \sqrt{3}$.

9. If $\tan \alpha = \frac{3}{4}$ and $\sin \beta = \frac{12}{13}$, find the functions of $\alpha + \beta$ when α is in the third, and β in the second quadrant.

Ans. $\sin(\alpha + \beta) = -\frac{33}{65}$; $\cos(\alpha + \beta) = \frac{56}{65}$; $\tan(\alpha + \beta) = -\frac{33}{56}$.

10. If $\cos \alpha = -\frac{40}{41}$ and $\sin \beta = -\frac{5}{13}$, find the functions of $\alpha - \beta$ when α is in the third, and β in the fourth quadrant.

Ans. $\sin(\alpha - \beta) = -\frac{308}{533}$; $\cos(\alpha - \beta) = -\frac{435}{533}$; $\tan(\alpha - \beta) = +\frac{308}{435}$.

11. If $\cos \alpha = \frac{3}{5}$ and $\sin \beta = -\frac{3}{5}$, find the functions of $\alpha + \beta$ and of $\alpha - \beta$ when α is in the fourth, and β in the third quadrant.

Ans. $\sin(\alpha + \beta) = +\frac{7}{25}$; $\cos(\alpha + \beta) = -\frac{24}{25}$; $\tan(\alpha + \beta) = -\frac{7}{24}$;
$\sin(\alpha - \beta) = +1$; $\cos(\alpha - \beta) = 0$; $\tan(\alpha - \beta) = \infty$.

Transform the first member into the second (or last) in Exs. (12–32):

12. $\sin(\alpha + \beta)\sin(\alpha - \beta) = \sin^2\alpha - \sin^2\beta = \cos^2\beta - \cos^2\alpha$.

13. $\cos(\alpha + \beta)\cos(\alpha - \beta) = \cos^2\alpha - \sin^2\beta = \cos^2\beta - \sin^2\alpha$.

14. $\sin(60° + \alpha) - \sin\alpha = \sin(60° - \alpha)$.

15. $(r'\cos v' - r\cos v)^2 + (r'\sin v' - r\sin v)^2 = r^2 + r'^2 - 2rr'\cos(v' - v)$.

16. $\cos^2\alpha + \cos^2\beta - 2\cos\alpha\cos\beta\cos\omega = \sin^2\omega$, when $\omega = \alpha + \beta$. [Place $\alpha = \omega - \beta$.]

17. $\tan\alpha + \dfrac{\tan\phi\sec\alpha}{\cos\alpha - \tan\phi\sin\alpha} = \tan(\alpha + \phi)$.

18. $\sin^2\theta + \sin^2(\omega - \theta) + 2\sin\theta\cos\omega\sin(\omega - \theta) = \sin^2\omega$.

19. $\cos^2\theta + \cos^2(\omega - \theta) - 2\cos\theta\cos\omega\cos(\omega - \theta) = \sin^2\omega$.

20. $\tan x \pm \tan y = \dfrac{\sin(x \pm y)}{\cos x \cos y}$.

21. $\cot x \pm \cot y = \dfrac{\sin(y \pm x)}{\sin x \sin y}$.

22. $\cot x \pm \tan y = \dfrac{\cos(x \mp y)}{\sin x \cos y}$.

23. $\tan(30° + x) + \tan(30° - x) = \sin 60° \sec(30° + x)\sec(30° - x)$.

24. $\dfrac{1 - \tan\alpha}{1 + \tan\alpha} = \tan(45° - \alpha)$. [Note that $1 = \tan 45°$.]

25. $\dfrac{1 - \cot\alpha}{1 + \cot\alpha} = -\tan(45° - \alpha)$. [Note that $1 = \cot 45°$.]

26. $\sin(60° + \alpha) - \sin(60° - \alpha) = \sin\alpha$.

27. $\tan(45° + \alpha) - \tan(45° - \alpha) = \dfrac{4\tan\alpha}{1 - \tan^2\alpha}$.

28. $\dfrac{\sin(x + y)}{\cos(x - y)} = \dfrac{\tan x + \tan y}{1 + \tan x \tan y} = \dfrac{\cot x + \cot y}{1 + \cot x \cot y}$.

29. $\sin(a+b+c)=\sin[(a+b)+c]=\sin(a+b)\cos c+\cos(a+b)\sin c$

$=(\sin a\cos b+\cos a\sin b)\cos c+(\cos a\cos b-\sin a\sin b)\sin c$

$=\sin a\cos b\cos c+\cos a\sin b\cos c+\cos a\cos b\sin c-\sin a\sin b\sin c.$

30. $\cos(a+b+c)=\cos a\cos b\cos c-\sin a\sin b\cos c-\sin a\cos b\sin c$
$-\cos a\sin b\sin c.$

31. $\tan(a+b+c)=\dfrac{\tan a+\tan b+\tan c-\tan a\tan b\tan c}{1-\tan a\tan b-\tan a\tan c-\tan b\tan c}.$

32. $\sin(a+b-c)=\sin a\cos b\cos c+\cos a\sin b\cos c-\cos a\cos b\sin c$
$+\sin a\sin b\sin c.$

33. If $\sin\alpha=\frac{3}{5}$, $\sin\beta=-\frac{12}{13}$, $\sin\gamma=-\frac{3}{5}$, find $\sin(\alpha-\beta-\gamma)$, when α, β, and γ are in the second, third, and fourth quadrants respectively. *Ans.* $-\frac{12}{13}$.

34. If $\sin\alpha=\frac{3}{5}$, $\cos\beta=\frac{4}{5}$, $\tan\gamma=\frac{3}{4}$, find $\cos(\alpha-\beta+\gamma)$, when α, β, and γ are in the second, fourth, and third quadrants respectively. *Ans.* $+\frac{4}{5}$.

67. To express the Functions of an Angle in Terms of those of Half the Angle.—In (1) and (2), Art. 61, let $y=x$.

$$\therefore \sin 2x=\sin x\cos x+\cos x\sin x=2\sin x\cos x. \qquad (1)$$

$$\cos 2x=\cos^2 x-\sin^2 x, \qquad (2)$$

$$\cos 2x=1-\sin^2 x-\sin^2 x=1-2\sin^2 x, \qquad (3)$$

or

$$\cos 2x=\cos^2 x-1+\cos^2 x=2\cos^2 x-1. \qquad (4)$$

From (1), Art. 65,

$$\tan 2x=\frac{\tan x+\tan x}{1-\tan x\tan x}=\frac{2\tan x}{1-\tan^2 x}. \qquad (5)$$

1. To express the functions of 40° in terms of those of 20°, we have

$$\sin 40^\circ=2\sin 20^\circ\cos 20^\circ;$$

$$\cos 40^\circ=\cos^2 20^\circ-\sin^2 20^\circ;$$

$$\tan 40^\circ=\frac{2\tan 20^\circ}{1-\tan^2 20^\circ}.$$

2. $\sin\theta=\dfrac{2}{\sqrt{5}}$, θ being in the second quadrant. Find $\sin 2\theta$, $\cos 2\theta$, $\tan 2\theta$.

Ans. $\sin 2\theta=-\frac{4}{5}$; $\cos 2\theta=-\frac{3}{5}$; $\tan 2\theta=+\frac{4}{3}$.

3. $\tan\theta=+2$, θ being in the third quadrant. Find $\sin 2\theta$, $\cos 2\theta$, $\tan 2\theta$.

Ans. $\sin 2\theta=+\frac{4}{5}$; $\cos 2\theta=-\frac{3}{5}$; $\tan 2\theta=-\frac{4}{3}$

4. $\cot\theta=-\frac{3}{4}$, θ being in the fourth quadrant. Find $\sin 2\theta$, $\cos 2\theta$, $\tan 2\theta$.

Ans. $\sin 2\theta=-\frac{24}{25}$; $\cos 2\theta=-\frac{7}{25}$; $\tan 2\theta=+\frac{24}{7}$.

NOTE. — To find the area of a right triangle, given c and α, we have

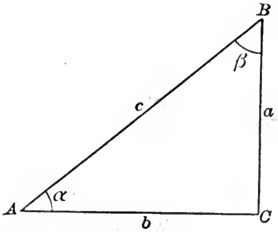


FIG. 46.

$$\text{area} = \tfrac{1}{2}ab.$$

But $a = c\sin\alpha$, and $b = c\cos\alpha$.

$$\therefore\ \text{area} = \tfrac{1}{2}c^2\sin\alpha\cos\alpha; \qquad (6)$$

$$\therefore\ \text{area} = \tfrac{1}{4}c^2\sin 2\alpha. \qquad (7)$$

In (6) we should have to find both $\sin\alpha$ and $\cos\alpha$ from the tables, in (7) we find only $\sin 2\alpha$, so that time is saved by using (7) instead of (6).

68. Geometrical Proof. — Let the radius OA of the circle be unity.

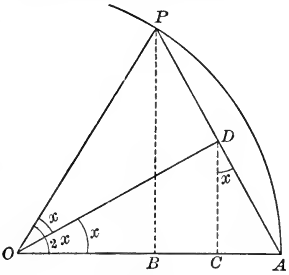


FIG. 47.

$$\sin 2x = BP = 2\,CD = 2\,OD\sin x$$
$$= 2\,OA\cos x\sin x.$$

$$\cos 2x = OB = OC - BC = OC - CA$$
$$= OD\cos x - AD\sin x$$
$$= OA\cos^2 x - OA\sin^2 x$$
$$= OA(\cos^2 x - \sin^2 x).$$

$$\tan 2x = \frac{BP}{OB} = \frac{2\,CD}{OC - CA} = \frac{2\,OC\tan x}{OC - CD\tan x}$$
$$= \frac{2\,OC\tan x}{OC - OC\tan^2 x} = \frac{2\tan x}{1-\tan^2 x}.$$

69. To express the Functions of an Angle in Terms of those of Double the Angle. — From Art. 67,

$$\cos 2x = 1 - 2\sin^2 x;$$

$$\therefore\ \mathbf{2\sin^2 x = 1 - \cos 2x.} \qquad (1)$$

Also
$$\cos 2x = 2\cos^2 x - 1;$$

$$\therefore\ \mathbf{2\cos^2 x = 1 + \cos 2x.} \qquad (2)$$

From (1) and (2),
$$\mathbf{\tan^2 x = \frac{1-\cos 2x}{1+\cos 2x}.} \qquad (3)$$

$$\therefore\ \tan x = \pm\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}. \qquad (4)$$

$$\therefore\ \tan x = \sqrt{\frac{1-\cos 2x}{1+\cos 2x}\cdot\frac{1+\cos 2x}{1+\cos 2x}} = \sqrt{\frac{1-\cos^2 2x}{(1+\cos 2x)^2}};$$

$$\therefore\ \mathbf{\tan x = \frac{\sin 2x}{1+\cos 2x}.} \qquad (5)$$

Also
$$\tan x = \sqrt{\frac{1-\cos 2x}{1+\cos 2x}\cdot\frac{1-\cos 2x}{1-\cos 2x}} = \sqrt{\frac{(1-\cos 2x)^2}{1-\cos^2 2x}};$$

$$\therefore\ \mathbf{\tan x = \frac{1-\cos 2x}{\sin 2x}.} \qquad (6)$$

Note. — The double sign is not used in (5) and (6), for

$$\frac{\sin 2x}{1+\cos 2x} = \frac{2\sin x \cos x}{2\cos^2 x} = \tan x,$$

and

$$\frac{1-\cos 2x}{\sin 2x} = \frac{2\sin^2 x}{2\sin x\cos x} = \tan x.$$

1. To express the functions of 20° in terms of those of 40°, we have

$$2\sin^2 20° = 1 - \cos 40°;$$

$$2\cos^2 20° = 1 + \cos 40°;$$

$$\tan^2 20° = \frac{1-\cos 40°}{1+\cos 40°};$$

$$\tan 20° = \frac{\sin 40°}{1+\cos 40°} = \frac{1-\cos 40°}{\sin 40°}.$$

2. $\tan 2\theta = -2$, 2θ being in the second quadrant. Find the functions of θ.

$$\therefore \cos 2\theta = -\frac{1}{\sqrt{5}}, \text{ and } \sin 2\theta = +\frac{2}{\sqrt{5}}.$$

$$\therefore \sin\theta = \pm\sqrt{\frac{1}{2}\left(1+\frac{1}{\sqrt{5}}\right)}, \text{ from (1)};$$

$$\cos\theta = \pm\sqrt{\frac{1}{2}\left(1-\frac{1}{\sqrt{5}}\right)}, \text{ from (2)};$$

$$\tan\theta = \frac{\frac{2}{\sqrt{5}}}{1-\frac{1}{\sqrt{5}}} = \frac{2}{\sqrt{5}-1}, \text{ from (5)}.$$

Since 2θ is in the second quadrant, θ may be either in the first or in the third quadrant; hence $\sin\theta$ and $\cos\theta$ have the double sign, and $\tan\theta$ is positive.

3. Given the functions of 30°, find those of 15°.

Ans. $\sin 15° = \frac{\sqrt{3}-1}{2\sqrt{2}}$; $\cos 15° = \frac{\sqrt{3}+1}{2\sqrt{2}}$; $\tan 15° = 2-\sqrt{3}$.

4. Given the functions of 45°, find those of $22\frac{1}{2}°$.

Ans. $\sin 22\frac{1}{2}° = \frac{1}{2}\sqrt{2-\sqrt{2}}$; $\cos 22\frac{1}{2}° = \frac{1}{2}\sqrt{2+\sqrt{2}}$; $\tan 22\frac{1}{2}° = \sqrt{2}-1$.

70. Geometrical Proof. — Let the radius CA of the circle be unity.

$$\sin x = \frac{BP}{OP} = \frac{\sqrt{OB\cdot BA}}{\sqrt{OB\cdot OA}} = \sqrt{\frac{BA}{OA}} = \sqrt{\frac{BA}{2}}$$

$$= \sqrt{\frac{CA-CB}{2}} = \sqrt{\frac{1-\cos 2x}{2}}.$$

$$\cos x = \frac{OB}{OP} = \frac{OB}{\sqrt{OB\cdot OA}} = \sqrt{\frac{OB}{OA}} = \sqrt{\frac{OB}{2}}$$

$$= \sqrt{\frac{OC+CB}{2}} = \sqrt{\frac{1+\cos 2x}{2}}.$$

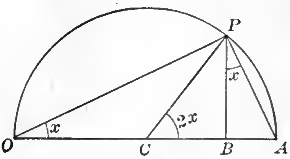


Fig. 48.

$$\tan x = \frac{BP}{OB} = \frac{BP}{OC + CB} = \frac{\sin 2x}{1 + \cos 2x}.$$

$$\tan x = \frac{BA}{BP} = \frac{CA - CB}{BP} = \frac{1 - \cos 2x}{\sin 2x}.$$

71. Multiple Angles. — Suppose that we wish to express $\sin 3x$ in terms of powers of $\sin x$.

$$\begin{aligned}\sin 3x = \sin(2x + x) &= \sin 2x \cos x + \cos 2x \sin x \\ &= 2 \sin x \cos^2 x + (1 - 2\sin^2 x) \sin x \\ &= 2 \sin x - 2 \sin^3 x + \sin x - 2 \sin^3 x \\ &= 3 \sin x - 4 \sin^3 x. \qquad \text{Q.E.I.}\end{aligned}$$

1. Show that $\cos 3x = 4 \cos^3 x - 3 \cos x$.

2. Show that $\tan 3x = \dfrac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}$.

3. Show that $\sin 4x = 4 \sin x \cos x - 8 \sin^3 x \cos x$. [Use $4x = 2x + 2x$.]

4. Show that $\cos 4x = 1 - 8 \sin^2 x + 8 \sin^4 x$.

5. Show that $\tan 4x = \dfrac{4 \tan x (1 - \tan^2 x)}{1 - 6 \tan^2 x + \tan^4 x}$.

6. Show that $\sin 5x = 5 \sin x - 20 \sin^3 x + 16 \sin^5 x$. [Use $5x = 3x + 2x$.]

7. Show that $\cos 5x = 5 \cos x - 20 \cos^3 x + 16 \cos^5 x$.

8. Find the functions of 18°, of 36°, and of 72°.

Place $x = 18°$; then, since $\cos 54° = \sin 36°$, we have

$$\cos 3x = \sin 2x.$$

$$\therefore\ 4 \cos^3 x - 3 \cos x = 2 \sin x \cos x.$$

$$\therefore\ \cos x (4 \cos^2 x - 3 - 2 \sin x) = 0.$$

$$\therefore\ 1 - 4 \sin^2 x - 2 \sin x = 0.$$

$$\therefore\ \sin x = \tfrac{1}{4}(-1 \pm \sqrt{5}).$$

$$\therefore\ \sin 18° = \cos 72° = \tfrac{1}{4}(\sqrt{5} - 1);\ \cos 18° = \sin 72° = \tfrac{1}{4}\sqrt{10 + 2\sqrt{5}}.$$

Hence

$$\sin 36° = 2 \sin 18° \cos 18° = \tfrac{1}{4}\sqrt{10 - 2\sqrt{5}};$$

$$\cos 36° = 1 - 2 \sin^2 18° = \tfrac{1}{4}(\sqrt{5} + 1).$$

72. To change the Product of Functions of Angles into the Sum of Functions. — From Arts. 61 and 62,

$$\sin(x + y) = \sin x \cos y + \cos x \sin y;$$

$$\sin(x - y) = \sin x \cos y - \cos x \sin y.$$

$$\therefore\ \sin(x + y) + \sin(x - y) = 2 \sin x \cos y, \qquad (1)$$

and

$$\sin(x + y) - \sin(x - y) = 2 \cos x \sin y. \qquad (2)$$

Also $\cos(x+y) = \cos x \cos y - \sin x \sin y;$

$\cos(x-y) = \cos x \cos y + \sin x \sin y.$

$\therefore \cos(x+y) + \cos(x-y) = 2 \cos x \cos y,$ (3)

and $\cos(x+y) - \cos(x-y) = -2 \sin x \sin y.$ (4)

Reversing (1), (2), (3), and (4), we have

$$\sin x \cos y = \frac{1}{2}\sin(x+y) + \frac{1}{2}\sin(x-y). \quad (5)$$

$$\cos x \sin y = \frac{1}{2}\sin(x+y) - \frac{1}{2}\sin(x-y). \quad (6)$$

$$\cos x \cos y = \frac{1}{2}\cos(x+y) + \frac{1}{2}\cos(x-y). \quad (7)$$

$$\sin x \sin y = -\frac{1}{2}\cos(x+y) + \frac{1}{2}\cos(x-y). \quad (8)$$

In applying these formulas, let x represent the larger angle.

1. $\sin 4\theta \cos 2\theta = \frac{1}{2}\sin(4\theta + 2\theta) + \frac{1}{2}\sin(4\theta - 2\theta)$, from (5),
$= \frac{1}{2}\sin 6\theta + \frac{1}{2}\sin 2\theta.$

2. $\cos 6\theta \sin 2\theta = \frac{1}{2}\sin(6\theta + 2\theta) - \frac{1}{2}\sin(6\theta - 2\theta)$, from (6),
$= \frac{1}{2}\sin 8\theta - \frac{1}{2}\sin 4\theta.$

3. $\cos 8\theta \cos 2\theta = \frac{1}{2}\cos 10\theta + \frac{1}{2}\cos 6\theta$, from (7).

4. $\sin 6\theta \sin 4\theta = -\frac{1}{2}\cos 10\theta + \frac{1}{2}\cos 2\theta$, from (8).

5. $\cos 2\theta \sin 4\theta = \frac{1}{2}\sin 6\theta - \frac{1}{2}\sin(-2\theta)$, from (6),
$= \frac{1}{2}\sin 6\theta + \frac{1}{2}\sin 2\theta$, as in Ex. 1.

6. $\sin 2\theta \cos 6\theta = \frac{1}{2}\sin 8\theta + \frac{1}{2}\sin(-4\theta)$, from (5),
$= \frac{1}{2}\sin 8\theta - \frac{1}{2}\sin 4\theta$, as in Ex. 2.

7. $\cos 2\theta \cos 8\theta = \frac{1}{2}\cos 10\theta + \frac{1}{2}\cos(-6\theta)$, from (7),
$= \frac{1}{2}\cos 10\theta + \frac{1}{2}\cos 6\theta$, as in Ex. 3.

8. $\sin 4\theta \sin 6\theta = -\frac{1}{2}\cos 10\theta + \frac{1}{2}\cos(-2\theta)$, from (8),
$= -\frac{1}{2}\cos 10\theta + \frac{1}{2}\cos 2\theta$, as in Ex. 4.

9. $\sin^2\theta \cos\theta = \sin\theta[\sin\theta\cos\theta] = \sin\theta[\frac{1}{2}\sin(\theta+\theta) + \frac{1}{2}\sin(\theta-\theta)]$
$= \sin\theta[\frac{1}{2}\sin 2\theta + \frac{1}{2}\sin 0°] = \frac{1}{2}\sin\theta\sin 2\theta$
$= \frac{1}{2}[-\frac{1}{2}\cos 3\theta + \frac{1}{2}\cos\theta] = -\frac{1}{4}\cos 3\theta + \frac{1}{4}\cos\theta.$

10. Reduce $\sin^3 \alpha \cos \alpha$ to $\frac{1}{4}\sin 2\alpha - \frac{1}{8}\sin 4\alpha$.

$$\sin^3\alpha \cos\alpha = \sin^2\alpha \cdot \sin\alpha\cos\alpha;$$

using (8) and (5), or the relations in Arts. 69 and 67, we have

$$\sin^3\alpha\cos\alpha = \frac{1}{2}(1 - \cos 2\alpha)\cdot\frac{1}{2}\sin 2\alpha = \frac{1}{4}\sin 2\alpha - \frac{1}{4}\sin 2\alpha\cos 2\alpha$$
$$= \frac{1}{4}\sin 2\alpha - \frac{1}{8}\sin 4\alpha.$$

11. Reduce $\sin^2\theta\cos^2\theta$ to $\frac{1}{8}(1 - \cos 4\theta)$.

12. Reduce $\sin^2\theta\cos^3\theta$ to $\frac{1}{8}(\cos\theta - \frac{1}{2}\cos 3\theta - \frac{1}{2}\cos 5\theta)$.

13. Reduce $\sin^3\theta\cos^3\theta$ to $\frac{1}{32}(3\sin 2\theta - \sin 6\theta)$.

14. Reduce $\cos^5\theta$ to $\frac{1}{16}(10\cos\theta + 5\cos 3\theta + \cos 5\theta)$.

15. Reduce $\cos^3\theta$ to $\frac{1}{4}(\cos 3\theta + 3\cos\theta)$.

16. Reduce $\sin^5\theta\cos^3\theta$ to $\frac{1}{64}(3\sin 2\theta - \sin 4\theta - \sin 6\theta + \frac{1}{2}\sin 8\theta)$.

73. To change the Algebraic Sum of Functions of Angles into the Product of Functions. — Let $x + y = u$ and $(x - y) = v$.

$$\therefore\ x = \tfrac{1}{2}(u + v) \text{ and } y = \tfrac{1}{2}(u - v).$$

Substituting in (1), (2), (3), and (4), Art. 72, we have

$$\sin u + \sin v = 2 \sin \tfrac{1}{2}(u + v) \cos \tfrac{1}{2}(u - v). \quad (1)$$

$$\sin u - \sin v = 2 \cos \tfrac{1}{2}(u + v) \sin \tfrac{1}{2}(u - v). \quad (2)$$

$$\cos u + \cos v = 2 \cos \tfrac{1}{2}(u + v) \cos \tfrac{1}{2}(u - v). \quad (3)$$

$$\cos u - \cos v = - 2 \sin \tfrac{1}{2}(u + v) \sin \tfrac{1}{2}(u - v). \quad (4)$$

In applying the formulas, let u represent the larger angle.

1. Reduce $\sin 3\theta + \sin \theta$ to $2 \sin 2\theta \cos \theta$.

   Let $u = 3\theta$ and $v = \theta$ in (1).

2. Reduce $\cos \theta - \cos 3\theta$ to $4 \sin^2 \theta \cos \theta$.

   $\cos \theta - \cos 3\theta = -(\cos 3\theta - \cos \theta)$. Let $u = 3\theta$ and $v = \theta$ in (4).

   $\therefore\ -(\cos 3\theta - \cos \theta) = -(-2 \sin 2\theta \sin \theta) = +2 \sin 2\theta \sin \theta$
   $= 4 \sin \theta \cos \theta \sin \theta = 4 \sin^2 \theta \cos \theta$.

3. Reduce $\sin 3\theta + \cos \theta$ to a product.

   $\sin 3\theta + \cos \theta = \sin 3\theta + \sin(90° - \theta) = 2 \sin(45° + \theta) \cos(2\theta - 45°)$
   $= 2 \sin(45° + \theta) \cos(45° - 2\theta)$.

74. Geometrical Proof. — In the figure, OD bisects the angle QOP, and is therefore perpendicular to QP. Using the notation there shown, we have

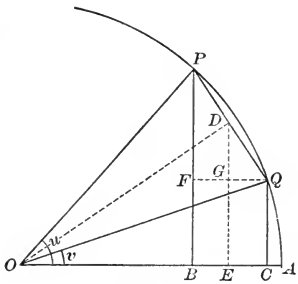


FIG. 49.

$QOP = u - v$; $\therefore\ QOD = DOP = \tfrac{1}{2}(u - v)$;
$AOD = AOQ + QOD = v + \tfrac{1}{2}(u - v) = \tfrac{1}{2}(u + v)$;
$FPQ = GDQ = AOD = \tfrac{1}{2}(u + v)$. Then, if the radius $= 1$,

$$\sin u + \sin v = BP + CQ = 2\,ED = 2\,OD \sin AOD = 2\,OP \cos DOP \sin AOD = 2 \sin \tfrac{1}{2}(u + v) \cos \tfrac{1}{2}(u - v). \quad (1)$$

$$\sin u - \sin v = BP - CQ = 2\,GD = 2\,DQ \cos GDQ = 2\,OQ \sin QOD \cos GDQ = 2 \cos \tfrac{1}{2}(u + v) \sin \tfrac{1}{2}(u - v). \quad (2)$$

$$\cos u + \cos v = OB + OC = 2\,OE = 2\,OD \cos AOD = 2\,OP \cos DOP \cos AOD = 2 \cos \tfrac{1}{2}(u + v) \cos \tfrac{1}{2}(u - v). \quad (3)$$

$$\cos u - \cos v = OB - OC = -2\,GQ = -2\,DQ \sin GDQ = -2\,OQ \sin QOD \sin GDQ = -2 \sin \tfrac{1}{2}(u + v) \sin \tfrac{1}{2}(u - v). \quad (4)$$

EXAMPLES.

Show that the first member of the equation may be reduced to the second (or last) in Exs. (1–7):

1. $\sin(45° + x) + \sin(45° - x) = 2 \sin 45° \cos x = \sqrt{2} \cos x.$

2. $\sin(90° + x) - \sin(180° + x) = 2 \cos(135° + x) \sin(-45°)$
$= -\sqrt{2} \cos(135° + x).$

3. $\cos(180° + x) + \cos(180° - x) = 2 \cos 180° \cos x = -2 \cos x.$

4. $\cos(270° + x) - \cos(270° - x) = -2 \sin 270° \sin x = +2 \sin x.$

5. $\sin 3x + 2 \sin 5x + \sin 7x = 4 \sin 5x \cos^2 x.$

6. $\cos 3x + 2 \cos 5x + \cos 7x = 4 \cos 5x \cos^2 x.$

7. $\cos(b - c) - \cos a = +2 \sin \frac{1}{2}(a + b - c) \sin \frac{1}{2}(a - b + c).$

8. Show that $\sin(\lambda'' - \lambda') - \sin(\lambda'' - \lambda) + \sin(\lambda' - \lambda)$ may be reduced to $4 \sin \frac{1}{2}(\lambda' - \lambda) \sin \frac{1}{2}(\lambda'' - \lambda') \sin \frac{1}{2}(\lambda'' - \lambda)$. [The formula $\sin x = 2 \sin \frac{1}{2}x \cos \frac{1}{2}x$ is used in the process.]

If $\alpha + \beta + \gamma = 180°$, reduce the first member to the second in Exs. (9–14):

9. $\sin \alpha + \sin \beta + \sin \gamma = 4 \sin \frac{1}{2}(\alpha + \beta) \cos \frac{1}{2}\alpha \cos \frac{1}{2}\beta.$

$\gamma = 180° - (\alpha + \beta)$; $\therefore \sin \gamma = \sin(\alpha + \beta)$. Then

$$\sin \alpha + \sin \beta + \sin(\alpha + \beta) = 2 \sin \tfrac{1}{2}(\alpha + \beta) \cos \tfrac{1}{2}(\alpha - \beta) + 2 \sin \tfrac{1}{2}(\alpha + \beta) \cos \tfrac{1}{2}(\alpha + \beta)$$
$$= 2 \sin \tfrac{1}{2}(\alpha + \beta)[\cos \tfrac{1}{2}(\alpha - \beta) + \cos \tfrac{1}{2}(\alpha + \beta)]$$
$$= 2 \sin \tfrac{1}{2}(\alpha + \beta)(2 \cos \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\beta).$$

10. $\cos \alpha + \cos \beta + \cos \gamma = 4 \cos \frac{1}{2}(\alpha + \beta) \sin \frac{1}{2}\alpha \sin \frac{1}{2}\beta + 1$. [Note that $\cos \gamma = -\cos(\alpha + \beta) = -2 \cos^2 \frac{1}{2}(\alpha + \beta) + 1$.]

11. $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = -4 \cos \alpha \cos \beta \cos \gamma - 1.$

12. $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 4 \sin \alpha \sin \beta \sin \gamma.$

13. $2 \sin^2 \alpha + 2 \sin^2 \beta + 2 \sin^2 \gamma = 4 + 4 \cos \alpha \cos \beta \cos \gamma.$

14. $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = -4 \cos \frac{3}{2}\alpha \cos \frac{3}{2}\beta \cos \frac{3}{2}\gamma.$

15. If $\alpha + \beta + \gamma = 360°$, $\sin \alpha + \sin \beta + \sin \gamma = 4 \sin \frac{1}{2}\alpha \sin \frac{1}{2}\beta \sin \frac{1}{2}\gamma.$

16. If $\alpha + \beta + \gamma = 360°$, $\sin \alpha + \sin \beta + 2 \sin \frac{1}{2}\gamma = 4 \sin \frac{1}{2}(\alpha + \beta) \cos^2 \frac{1}{4}(\alpha - \beta).$

75. Circular, or Inverse Trigonometric, Functions. — If y is the sine of the angle or arc x, then x is the arc whose sine is y. This is written $x = \sin^{-1} y$, read "*x is the arc whose* sine is y." So also if $\tan x = m$, then "*x is the arc whose* tangent is m," written $x = \tan^{-1} m$.

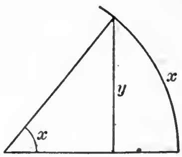


Fig. 50.

In consequence of this notation, if we have $\dfrac{1}{\sin x}$ and wish to bring $\sin x$ into the numerator, we must write it in a parenthesis with the exponent -1;

$\frac{1}{\sin x} = (\sin x)^{-1}$. All other exponents may be written above the name of the functions; $\frac{1}{\sin^2 x} = \sin^{-2} x = (\sin x)^{-2}$.

1. $y = \tan^{-1} m + \tan^{-1} n$. Find $\tan y$.

Let $\tan^{-1} m = a$ and $\tan^{-1} n = b$; $\therefore \tan a = m$, $\tan b = n$.

$$\therefore y = a + b; \therefore \tan y = \frac{\tan a + \tan b}{1 - \tan a \tan b} = \frac{m+n}{1-mn}.$$

2. $\tan^{-1}\frac{1}{m} = \tan^{-1}\frac{1}{m+n} + \tan^{-1} x$. Find x.

$\therefore \tan^{-1} x = \tan^{-1}\frac{1}{m} - \tan^{-1}\frac{1}{m+n}$. Let $a = \tan^{-1}\frac{1}{m}$, $b = \tan^{-1}\frac{1}{m+n}$.

$$\therefore \tan^{-1} x = a - b; \therefore x = \tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a \tan b} = \frac{\frac{1}{m} - \frac{1}{m+n}}{1 + \frac{1}{m(m+n)}};$$

$$\therefore x = \frac{n}{m^2 + mn + 1}.$$

3.* $y = \sin^{-1}\frac{1}{2} + \tan^{-1}\frac{3}{4}$. Find $\sin y$. *Ans.* $\sin y = \frac{1}{10}(4 + 3\sqrt{3})$.

4. $\tan^{-1}\frac{1}{m} = \tan^{-1}\frac{1}{m-n} - \tan^{-1} x$. Find $x = \frac{n}{m^2 - mn + 1}$.

5. $\tan^{-1} a = \tan^{-1}\frac{1}{4} + \tan^{-1}\frac{1}{13}$. Find $a = \frac{1}{3}$.

6.* $y = \sin^{-1} m + \sin^{-1} n$. Find $\sin y = m\sqrt{1-n^2} + n\sqrt{1-m^2}$.

7.* $y = \cos^{-1} m + \cos^{-1} n$. Find $\sin y = n\sqrt{1-m^2} + m\sqrt{1-n^2}$.

8.* $y = \cos^{-1} m - \sin^{-1} n$. Find $\cos y = m\sqrt{1-n^2} + n\sqrt{1-m^2}$.

9. $\tan^{-1} a = \tan^{-1}\frac{1}{2} - \tan^{-1}\frac{1}{7}$. Find $a = \frac{1}{3}$

10.* $m = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$. Find $m = 45°$.

11.* $m = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{4} + \tan^{-1}\frac{1}{13}$. Find $m = 45°$.

12.* $m = 2\tan^{-1}\frac{1}{2} - \tan^{-1}\frac{1}{7}$. Find $m = 45°$.

Let $\tan^{-1}\frac{1}{2} = a$, $\tan^{-1}\frac{1}{7} = b$; $\therefore m = 2a + b$; $\therefore \tan m =$, etc.

13.* $m = 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7}$. Find $m = 45°$.

14.* Show that $\tan^{-1}\frac{1}{2}(1-m) = \sec^{-1}\frac{1}{2}\sqrt{5 - 2m + m^2}$.

Let $\tan^{-1}\frac{1}{2}(1-m) = x$; $\therefore \tan x = \frac{1}{2}(1-m)$; $\sec x = \sqrt{1 + \tan^2 x}$;

$$\therefore \sec x = \tfrac{1}{2}\sqrt{5 - 2m + m^2}. \quad \therefore x = \sec^{-1}\tfrac{1}{2}\sqrt{5 - 2m + m^2}.$$

15. Show that $\tan^{-1} m = \frac{1}{2}\tan^{-1}\frac{2m}{1-m^2}$.

Let $x = \tan^{-1} m$, or $m = \tan x$. If the equation is true, we must have

$$x = \tfrac{1}{2}\tan^{-1}\frac{2m}{1-m^2}, \text{ or } 2x = \tan^{-1}\frac{2\tan x}{1-\tan^2 x}, \text{ or } \tan 2x = \frac{2\tan x}{1-\tan^2 x},$$

a formula proved in Art. 67.

16. Show that $\cos^{-1} m = \frac{1}{2}\cos^{-1}(2m^2 - 1)$.

17.* Show that $\sin^{-1}\frac{2\sqrt{ab}}{a+b} = \tan^{-1}\frac{2\sqrt{ab}}{a-b}$.

* When the angles are less than 90°.

18. Show that $\sin\left(\frac{\pi}{2} - 2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right) = x$.

19.* Show that $\frac{1}{2}\text{vers}^{-1}\frac{1}{2}a^2 - \sin^{-1}\frac{1}{2}a$ is constant for all possible values of a.

Let $\theta = \frac{1}{2}\text{vers}^{-1}\frac{1}{2}a^2 - \sin^{-1}\frac{1}{2}a$, and let $m = \text{vers}^{-1}\frac{1}{2}a^2$, $n = \sin^{-1}\frac{1}{2}a$.

$\therefore\ \theta = \frac{1}{2}m - n$; $\therefore\ 2\theta = m - 2n$; $\therefore\ \cos 2\theta = \cos m \cos 2n + \sin m \sin 2n$.

But $\qquad \sin n = \frac{1}{2}a$; $\therefore\ \cos n = \frac{1}{2}\sqrt{4-a^2}$;

$\therefore\ \sin 2n = \frac{1}{2}a\sqrt{4-a^2}$; $\cos 2n = \frac{1}{2}(2-a^2)$.

Also $\qquad \cos m = 1 - \text{vers}\, m = 1 - \frac{1}{2}a^2$; $\therefore\ \sin m = \frac{a}{2}\sqrt{4-a^2}$.

$$\therefore\ \cos 2\theta = \frac{2-a^2}{2}\cdot\frac{2-a^2}{2} + \frac{a}{2}\sqrt{4-a^2}\,\frac{a}{2}\sqrt{4-a^2} = 1.$$

$\therefore\ 2\theta = 0°$, or $\theta = 0°$.

20.* Show that $\tan^{-1}\frac{\sqrt{1-a^2}}{a} + \sin^{-1}a$ is constant for all possible values of a.

21.* Show that $\text{vers}^{-1}a - 2\cot^{-1}\sqrt{\frac{2-a}{a}}$ is constant for all possible values of a.

22.* Show that $\text{vers}^{-1}\frac{x}{12} - 2\sin^{-1}\sqrt{\frac{x}{24}}$ is constant for all possible values of x.

76. To prove that $\tan x > x > \sin x$ when $x < \frac{\pi}{2}$, x being expressed in Circular Measure.—Let $AOB = BOC = x$, the radius being unity. Evidently $AT > SB$, or $\tan x > \sin x$.

Also, since the shortest distance from a point to a line is perpendicular to the line, $SB < AB$, or $\sin x < x$.

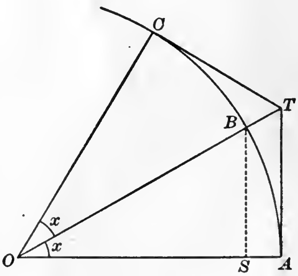


FIG. 51.

The arc AC may be considered as composed of an infinite number of infinitesimal straight lines; hence $AT + TC >$ arc ABC, since ABC is a convex polygon lying in the triangle formed by a chord AC with the tangent lines TA and TC. Then

$$2\,AT > \text{arc } ABC, \text{ or } AT > \text{arc } AB, \text{ or } \tan x > x.$$

Hence $\qquad \tan x > x$, and $x > \sin x$. $\qquad$ Q.E.D.

77. To prove that $\sin x$, $\tan x$, and x approach Equality as the Angle x approaches Zero.—As the angle AOT decreases,

* When the angles are less than 90°.

the points B and T approach A, and hence approach each other. But

$$\frac{SB}{AT} = \frac{\sin x}{\tan x} = \cos x.$$

When the angle x approaches zero as its limit, $\cos x$ approaches unity as its limit. Hence $\frac{SB}{AT}$, or $\frac{\sin x}{\tan x}$, approaches unity as its limit, or $\sin x$ and $\tan x$ approach equality.

The arc x is intermediate in value between $\sin x$ and $\tan x$; hence the three quantities approach equality as the angle becomes smaller. That is, the three ratios

$$\frac{\sin x}{\tan x}, \quad \frac{\sin x}{x}, \quad \frac{\tan x}{x},$$

approach unity as the angle approaches zero.

Hence we may say that when the angle is small, its sine and its tangent are equal to the arc itself, and its cosine is equal to unity. The smaller the angle, the more nearly correct will be the assumption.

78. Development of $\sin x$, of $\cos x$, and of $\tan x$. — Let us assume that

$$\sin x = a + bx + cx^2 + dx^3 + ex^4 + fx^5 + \cdots \qquad (1)$$

is true for all values of x. Then it is true when x has the values $+y$ and $-y$; hence

$$\sin y = a + by + cy^2 + dy^3 + ey^4 + fy^5 + \cdots \qquad (2)$$

and $$\sin(-y) = a - by + cy^2 - dy^3 + ey^4 - fy^5 + \cdots \qquad (3)$$

But $\sin y = -\sin(-y)$, or $\sin y + \sin(-y) = 0$. Adding (2) and (3),

$$2a + 2cy^2 + 2ey^4 + \cdots = 0. \qquad (4)$$

But (4) is true for all values of y, since (1) is true for all values of x. In order that all values of y may reduce the left member of (4) to zero, we must have $a = 0$, $c = 0$, $e = 0$, ... Hence (1) becomes

$$\sin x = bx + dx^3 + fx^5 + \cdots \qquad (5)$$

or $$\frac{\sin x}{x} = b + dx^2 + fx^4 + \cdots \qquad (6)$$

But as x approaches zero, $\frac{\sin x}{x}$ approaches unity, and $b + dx^2 + fx^4 + \cdots$ approaches b. Hence

$$1 = b, \tag{7}$$

and (5) becomes

$$\mathbf{\sin x = x + dx^3 + fx^5 + \cdots} \tag{8}$$

Again, let

$$\cos x = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \cdots \tag{9}$$

Since $\cos x = \cos(-x)$, we have

$$A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \cdots$$
$$= A - Bx + Cx^2 - Dx^3 + Ex^4 - Fx^5 + \cdots \tag{10}$$

or

$$2\,Bx + 2\,Dx^3 + 2\,Fx^5 + \cdots = 0. \tag{11}$$

In order that this may be true for all values of x, we must have $B = 0$, $D = 0$, $F = 0 \cdots$, and (9) becomes

$$\cos x = A + Cx^2 + Ex^4 + \cdots \tag{12}$$

But when $x = 0$, (12) reduces to

$$1 = A, \tag{13}$$

and hence (12) becomes

$$\mathbf{\cos x = 1 + Cx^2 + Ex^4 + \cdots} \tag{14}$$

Substituting from (14) and (8) in the formula

$$\cos 2\,x = \cos^2 x - \sin^2 x,$$

we have $1 + 4\,Cx^2 + 16\,Ex^4 + \cdots = 1 + (2\,C - 1)x^2$

$$+ (2\,E + C^2 - 2\,d)x^4 + \cdots \tag{15}$$

Equating the coefficients of like powers of x,

$$4\,C = 2\,C - 1, \quad \text{or} \quad 2\,C + 1 = 0. \tag{16}$$

$$16\,E = 2\,E + C^2 - 2\,d, \quad \text{or} \quad 14\,E - C^2 + 2\,d = 0. \tag{17}$$

Substituting from (14) and (8) in the formula

$$\sin 2\,x = 2\sin x \cos x,$$

we have $2\,x + 8\,dx^3 + 32\,fx^5 + \cdots = 2\,x + 2\,(C + d)x^3$

$$+ 2\,(E + Cd + f)x^5 + \cdots \tag{18}$$

Equating the coefficients of like powers of x,

$$4\,d = C + d, \qquad \text{or} \qquad 3\,d - C = 0. \tag{19}$$

$$16\,f = E + Cd + f, \quad \text{or} \quad 15\,f - E - Cd = 0. \tag{20}$$

From (16), $$C = -\tfrac{1}{2}. \tag{21}$$

From (19), $$d = -\frac{1}{6} = -\frac{1}{\underline{|3}}. \tag{22}$$

From (17), $$E = +\frac{1}{24} = +\frac{1}{\underline{|4}}. \tag{23}$$

From (20), $$f = +\frac{1}{120} = +\frac{1}{\underline{|5}}. \tag{24}$$

These values, substituted in (8) and (14), give

$$\sin x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \cdots \tag{25}$$

$$\cos x = 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \cdots \tag{26}$$

Dividing (25) by (26),

$$\tan x = x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + \cdots \tag{27}$$

In (25), (26), and (27), which are the required developments, x must be expressed in circular measure.

79. Computation of the Trigonometric Functions (First Method). — The functions may be computed by (25), (26), and (27), Art. 78. Thus, to find $\sin 20°$, we place $x = \frac{1}{9}\pi$, the circular measure of $20°$.

$\log \pi^3 = 1.49145$	$\log \pi^5 = 2.4857$	$x = \frac{\pi}{9} = 0.34906\,59$
$\text{col}\, 9^3 = 7.13727 - 10$	$\text{col}\, 9^5 = 5.2288 - 10$	$\frac{x^3}{\underline{\|3}} = 0.00708\,88$
$\text{col}\, 6 = 9.22185 - 10$	$\text{col}\, 120 = 7.9208 - 10$	$0.34197\,71$
$\log \frac{x^3}{\underline{\|3}} = 7.85057 - 10$	$\log \frac{x^5}{\underline{\|5}} = 5.6353 - 10$	$\frac{x^5}{\underline{\|5}} = 0.00004\,32$
$\therefore\ \frac{x^3}{\underline{\|3}} = 0.0070888$	$\therefore\ \frac{x^5}{\underline{\|5}} = 0.0000432$	$\therefore\ \sin 20° = 0.34202\,03$

In the tables, $\sin 20° = 0.34202$.

80. Computation of the Trigonometric Functions (Second Method). — From (25), Art. 78, it may be shown that

$$\sin 1'' = 0.00000\,48481\,36811\,07637,$$

while $$\text{arc } 1'' = 0.00000\,48481\,36811\,09536.$$

$$\therefore \text{arc } 1'' - \sin 1'' = 0.00000\,00000\,00000\,0\overline{2}.$$

Again, $$\sin 1' = 0.00029\,08882\,04563\,42460,$$

while $$\text{arc } 1' = 0.00029\,08882\,08665\,72160.$$

$$\therefore \text{arc } 1' - \sin 1' = 0.00000\,00000\,04.$$

Again, $$\sin 1^\circ = 0.01745\,24064\,37283\,51282,$$

while $$\text{arc } 1^\circ = 0.01745\,32925\,19943\,29577.$$

$$\therefore \text{arc } 1^\circ - \sin 1^\circ = 0.00000\,0\overline{9}.$$

Also, from (26), Art. 78,

$$\cos 1'' = 0.99999\,99999\,88 = 1 - 0.00000\,00000\,12.$$

$$\cos 1' = 0.99999\,99576\,92 = 1 - 0.00000\,00423\,08.$$

$$\cos 1^\circ = 0.99984\,76952 \quad = 1 - 0.00015.$$

In computing a set of five-place tables, we may assume

$$\sin 1' = \text{arc } 1' = 0.00029\,08882 \text{ with an error of } \overline{5} \times 10^{-12},$$

and $$\cos 1' = 1 \qquad \text{with an error of } 4 \times 10^{-8}.$$

Then $$\sin 2' = 2 \sin 1' \cos 1'\,; \;\; \cos 2' = \cos^2 1' - \sin^2 1'.$$

$$\sin 3' = \sin 2' \cos 1' + \cos 2' \sin 1'\,;$$

$$\cos 3' = \cos 2' \cos 1' - \sin 2' \sin 1'.$$

$$\sin 4' = \sin (3' + 1');\;\; \cos 4' = \cos (3' + 1'),$$

or $$\sin 4' = 2 \sin 2' \cos 2'\,;\;\; \cos 4' = \cos^2 2' - \sin^2 2'.$$

And so on.

This method would be employed until the functions of all angles less than 30° had been computed. Then, since

$$\sin (30^\circ + x) = \cos x - \sin (30^\circ - x),$$

and $$\cos (30^\circ + x) = \cos (30^\circ - x) - \sin x,$$

the functions of angles between 30° and 45° would be found by combining the functions already found. Thus, if $x = 10^\circ$, we have

$$\sin 40^\circ = \cos 10^\circ - \sin 20^\circ,$$

and $$\cos 40^\circ = \cos 20^\circ - \sin 10^\circ.$$

It is possible to compute independently the sine and cosine of 3°, 6°, 9°, ⋯, 39°, 42°, 45°. We have found in this chapter* the sine and cosine of 15°, of 18°, and of 36°, and we have

$$3°=18°-15°,\quad 6°=36°-30°,\quad 9°=45°-36°,\quad 12°=30°-18°,$$
$$21°=36°-15°,\quad 24°=45°-21°,\quad 27°=45°-18°,$$
$$33°=18°+15°,\quad 39°=45°-\ 6°,\quad 42°=45°-\ 3°.$$

The values found from these relations would serve as checks upon the computation.

The computations may also be checked by Euler's and Legendre's verification formulas:

$$\sin(36°+A)-\sin(36°-A)-\sin(72°+A)+\sin(72°-A) = \sin A.$$

$$\cos(36°+A)+\cos(36°-A)-\cos(72°+A)-\cos(72°-A) = \cos A.$$

81. Approximate Assumptions. — It can be shown that

$$\tan 1'' - \text{arc } 1'' = 0.00000\ 00000\ 00000\ 0\bar{4};$$
$$\text{arc } 1'' - \sin 1'' = 0.00000\ 00000\ 00000\ 0\bar{2};$$
$$\tan 1'' - \sin 1'' = 0.00000\ 00000\ 00000\ 0\bar{6}.$$

Hence we may assume that

$$\mathbf{\sin 1'' = \tan 1'' = arc\, 1''.} \qquad (1)$$

In the whole circumference of a circle there are 1296000″, so that the error due to placing arc 1″ = sin 1″ in finding the circumference of a circle with a radius of unity will be only $2\frac{1}{2}$ units in the eleventh decimal place.

In the computation of elliptic orbits there occurs the equation

$$M = E - e \sin E,$$

where M and E are expressed in circular measure. If M'' is the number of seconds in the angle, $M = M''\text{ arc } 1''$, and approximately $M = M'' \sin 1''$ and $E = E'' \sin 1''$.

Hence the equation may be written

$$M'' = E'' - \frac{e}{\sin 1''} \sin E.$$

* Ex. 3, Art. 69, and Ex. 8, Art. 71.

Another assumption that is often made is that for small angles

$$\sin n'' = n \sin 1''. \qquad (2)$$

The error introduced is

for $1'$, $n'' = 60''$, error $= + 0.00000\ 00000\ 04$;

for $1°$, $n'' = 3600''$, error $= + 0.00000\ 0\bar{9}$.

Thus, if $\sin \alpha = 0.4 \sin 2°$, we should have, since α must be small, $\alpha'' \sin 1'' = 0.4 \sin 2°$ or $\alpha'' = \frac{0.4 \sin 2°}{\sin 1''}$.

82. Transform the First Member into the Second (or last) in the following examples:

1. $\frac{\cos \alpha - \sec \alpha}{\sec \alpha} = 4 \cos^2 \frac{1}{2} \alpha (\cos^2 \frac{1}{2} \alpha - 1).$

The first member contains the angle α and the second $\frac{1}{2} \alpha$; hence we must change the angle.

$$\frac{\cos \alpha - \frac{1}{\cos \alpha}}{\frac{1}{\cos \alpha}} = \cos^2 \alpha - 1 = (2 \cos^2 \tfrac{1}{2} \alpha - 1)^2 - 1$$

$$= 4 \cos^4 \tfrac{1}{2} \alpha - 4 \cos^2 \tfrac{1}{2} \alpha = 4 \cos^2 \tfrac{1}{2} \alpha (\cos^2 \tfrac{1}{2} \alpha - 1).$$

2. $\operatorname{cosec} 2\alpha + \cot 2\alpha = \cot \alpha.$

3. $\frac{\operatorname{cosec} 2\alpha - \cot 2\alpha}{\operatorname{cosec} 2\alpha + \cot 2\alpha} = \tan^2 \alpha.$

4. $\cot \alpha - \tan \alpha = 2 \cot 2\alpha.$

We may either reduce the expression as far as possible before changing the angle, or change the angle and then reduce.

(a) $\frac{\cos \alpha}{\sin \alpha} - \frac{\sin \alpha}{\cos \alpha} = \frac{\cos^2 \alpha - \sin^2 \alpha}{\sin \alpha \cos \alpha} = \frac{\cos 2\alpha}{\frac{1}{2} \sin 2\alpha} = 2 \cot 2\alpha.$

(b) $\frac{1 + \cos 2\alpha}{\sin 2\alpha} - \frac{1 - \cos 2\alpha}{\sin 2\alpha} = \frac{2 \cos 2\alpha}{\sin 2\alpha} = 2 \cot 2\alpha.$

Note. — Avoid radicals if possible.

5. $\sec \alpha \operatorname{cosec} \alpha = 2 \operatorname{cosec} 2\alpha.$

6. $(\sin \frac{1}{2} \theta + \cos \frac{1}{2} \theta)^2 = 1 + \sin \theta.$

7. $\frac{1 - \tan^2 \frac{1}{2} v}{1 + \tan^2 \frac{1}{2} v} = \cos v.$

8. $\cot \frac{1}{2} \theta + \tan \frac{1}{2} \theta = 2 \operatorname{cosec} \theta.$

9. $\sin x - 2 \sin^3 x = \sin x \cos 2x.$

10. $\frac{1}{2} (\sec \theta + \sec^2 \theta) = \frac{1 + \tan^2 \frac{1}{2} \theta}{(1 - \tan^2 \frac{1}{2} \theta)^2}.$

11. $\frac{2\tan\frac{1}{2}v}{1+\tan^2\frac{1}{2}v}=\sin v.$

12. $\frac{\sec^2\theta}{2-\sec^2\theta}=\sec 2\theta.$

13. $1+\tan x\tan\frac{1}{2}x=\sec x.$

14. $\frac{1}{2}(1+\tan\frac{1}{2}a)^2=\frac{1+\sin a}{1+\cos a}.$

15. $\tan\frac{1}{2}a+2\sin^2\frac{1}{2}a\cot a=\sin a.$

16. $\frac{\sin x(1-\tan^2 x)}{\sec^2 x}\left(\frac{1}{\cos x-\sin x}+\frac{1}{\cos x+\sin x}\right)=\sin 2x.$

17. $(1-\tan^2\theta)\sin\theta\cos\theta=\cos 2\theta\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}.$

18. $\frac{1+\tan^2 a}{1-\tan^2 a}=\sec 2a.$

19. $\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}=\frac{1-\sin 2\theta}{\cos 2\theta}=\frac{\cos 2\theta}{1+\sin 2\theta}.$

20. $\frac{\sec\theta+\cos\theta+2}{\sec\theta+\cos\theta-2}=\cot^4\frac{1}{2}\theta.$

21.* $\tan\theta\frac{1+\tan\frac{1}{2}\theta}{1-\tan\frac{1}{2}\theta}=\frac{\sin\theta}{1-\sin\theta}.$

22. $\sec 2a+\tan 2a+1=\frac{2}{1-\tan a}.$

23. $(\sqrt{1+\sin a}-\sqrt{1-\sin a})^2=4\sin^2\frac{1}{2}a.$

24. $(\sqrt{1+\sin a}+\sqrt{1-\sin a})^2=4\cos^2\frac{1}{2}a.$

25. $2\sin A-\sin(A-B)-4\sin A\sin^2\frac{1}{2}B=\sin(A+B).$

26.† $\cos(36°+A)+\cos(36°-A)-\cos(72°+A)-\cos(72°-A)=\cos A.$

27.† $\sin(36°+A)-\sin(36°-A)-\sin(72°+A)+\sin(72°-A)=\sin A.$

28. $\frac{\sin x+\sin 2x}{\cos x-\cos 2x}=\cot\frac{1}{2}x.$

29. $1+\cot^2\frac{1}{2}v=\frac{2}{\sin v\tan\frac{1}{2}v}.$

30. $\tan^3\frac{1}{2}v(1+\cot^2\frac{1}{2}v)^3=\frac{8}{\sin^3 v}.$

31. $\frac{\sin a\cos\frac{1}{2}a-2\cos a\sin\frac{1}{2}a}{2\sin\frac{1}{2}a-\sin a}=2\cos^2\frac{1}{4}a.$

32. $\frac{\tan^2\frac{1}{2}x+\cot^2\frac{1}{2}x}{\tan^2\frac{1}{2}x-\cot^2\frac{1}{2}x}=-\frac{1+\cos^2 x}{2\cos x}.$

33. Given $\tan\frac{1}{2}v=\sqrt{\frac{1+e}{1-e}}\tan\frac{1}{2}E$, show that

$$\frac{1}{(1+e)\cos^2\frac{1}{2}v+(1-e)\sin^2\frac{1}{2}v}=\frac{1-e\cos E}{1-e^2}.$$

34. $\tan(45°+A)-\tan(45°-A)=2\tan 2A.$

(a) $$\frac{\tan 45°+\tan A}{1-\tan 45°\tan A}-\frac{\tan 45°-\tan A}{1+\tan 45°\tan A}$$

$$=\frac{1+\tan A}{1-\tan A}-\frac{1-\tan A}{1+\tan A}=\frac{4\tan A}{1-\tan^2 A}=2\tan 2A.$$

* After substituting, multiply both numerator and denominator by the quantity $\sin\theta-1+\cos\theta$.

† $\cos 36°=\frac{1}{4}(1+\sqrt{5})$.

(b) $$\frac{1-\cos(90^\circ+2A)}{\sin(90^\circ+2A)}-\frac{1-\cos(90^\circ-2A)}{\sin(90^\circ-2A)}$$
$$=\frac{1+\sin 2A}{\cos 2A}-\frac{1-\sin 2A}{\cos 2A}=\frac{2\sin 2A}{\cos 2A}=2\tan 2A.$$

35. $\dfrac{\tan(45^\circ+\frac{1}{2}A)+\tan(45^\circ-\frac{1}{2}A)}{\tan(45^\circ+\frac{1}{2}A)-\tan(45^\circ-\frac{1}{2}A)}=\text{cosec } A.$

36. $\tan(45^\circ+\theta)-\cot(45^\circ+\theta)=2\tan 2\theta.$

37. $\tan^2(45^\circ+\theta)+\cot^2(45^\circ+\theta)=2+4\tan^2 2\theta.$

38. $\tan^2(45^\circ+\alpha)-\cot^2(45^\circ+\alpha)=4\tan 2\alpha\sec 2\alpha.$

39. $\dfrac{\tan(45^\circ+\frac{1}{2}\theta)}{\tan(45^\circ-\frac{1}{2}\theta)}=\dfrac{1+\sin\theta}{1-\sin\theta}.$

40. $\tan\theta\tan(45^\circ+\frac{1}{2}\theta)=\dfrac{\sin\theta}{1-\sin\theta}.$

41. $\cot(45^\circ-\frac{1}{2}\alpha)-\tan(45^\circ-\frac{1}{2}\alpha)=2\tan\alpha.$

42. $\tan^2(45^\circ+\frac{1}{2}\alpha)=\dfrac{1+\sin\alpha}{1-\sin\alpha}.$

43. $\tan(45^\circ+\theta)+\tan(45^\circ-\theta)=2\sec 2\theta.$

44. $\dfrac{1-\tan^2(45^\circ-\theta)}{1+\tan^2(45^\circ-\theta)}=\sin 2\theta.$

45. $\tan(45^\circ+\frac{1}{2}x)\dfrac{1+\tan\frac{1}{2}x}{1-\tan\frac{1}{2}x}=\dfrac{1+\sin x}{1-\sin x}.$

46. $\dfrac{\tan(45^\circ+\frac{1}{2}x)}{1+\cot^2(45^\circ+\frac{1}{2}x)}=\frac{1}{2}\cos x\dfrac{1+\sin x}{1-\sin x}.$

47. $\sin(45^\circ-\frac{1}{2}\theta)+\cos(45^\circ-\frac{1}{2}\theta)=\sqrt{2}\cos\frac{1}{2}\theta=\dfrac{\sin\theta}{\sqrt{1-\cos\theta}}.$

CHAPTER VI.

TRIGONOMETRIC EQUATIONS.

83. One Equation Containing Multiple Angles.* — Change the equation so that it shall contain a single angle, and then proceed as in Art. 52.

1. $\cos 3x = \sin 2x$; find x. (See Ex. 8, Art. 71.)

$$4\cos^3 x - 3\cos x = 2\sin x \cos x.$$

$$\therefore \cos x (1 - 4\sin^2 x - 2\sin x) = 0.$$

$$\therefore \cos x = 0, \text{ giving } x = 90° \text{ and } 270°;$$

and $1 - 4\sin^2 x - 2\sin x = 0$, giving $\sin x = \frac{1}{4}(\sqrt{5} - 1)$

and $\sin x = -\frac{1}{4}(\sqrt{5} + 1)$, or $x = 18°, 162°, 234°, 306°$.

2. $\cos 2\theta + \cos\theta = -1$; find θ. *Ans.* 90°, 270°, 120°, 240°.
3. $\cot 2\theta + \tan\theta = -\frac{2}{3}\sqrt{3}$; find θ. *Ans.* 150°, 330°, 120°, 300°.
4. $\cos 2x + \sin x = +1$; find x. *Ans.* 0°, 30°, 150°, 180°.
5. $\sin 3x + \sin 2x = \sin x$; find x. *Ans.* 0°, 180°, 60°, 300°.
6. $\tan 2x = -2\sin x$; find x. *Ans.* 0°, 60°, 180°, 300°.
7. $\tan 2x \tan x = +1$; find x. *Ans.* 30°, 150°, 210°, 330°.
8. $\tan^2 x \tan 2x + 2\tan x = +\sqrt{3}$; find x. *Ans.* 30°, 120°, 210°, 300°.
9. $\sin 4z - 2\sin 2z = 0$; find z. *Ans.* 0°, 90°, 180°, 270°.

The equation may sometimes be solved by the use of the equations of Art. 73.

10. $\cos 3x - \sin 2x = 0$; find x.

$$\cos 3x - \sin 2x = \sin(90° + 3x) - \sin 2x$$

$$= 2\cos(45° + \tfrac{5}{2}x)\sin(45° + \tfrac{1}{2}x) = 0.$$

$\cos(45° + \frac{5}{2}x) = 0$ gives $45° + \frac{5}{2}x = 90°, 270°, 450°, 630°, 810°$,

or $x = 18°, 90°, 162°, 234°, 306°$.

$\sin(45° + \frac{1}{2}x) = 0$ gives $45° + \frac{1}{2}x = 0°, 180°$,

or $x = -90°$ and $270°$.

* See Art. 52 for the solution of equations when only one angle is involved.

11. $\cos\theta - \cos 3\theta = \sin 2\theta$; find θ by both methods.
Ans. 0°, 30°, 90°, 150°, 180°, 270°.

12. $\sin 3\theta + \sin 2\theta + \sin\theta = 0$; find θ by both methods.
Ans. 0°, 90°, 120°, 180°, 240°, 270°.

13. $\cos 2\theta = \sin\theta$; find θ by both methods. *Ans.* 30°, 150°, 270°.

14. $\cos 5\theta - \cos 3\theta + \sin\theta = 0$; find θ. *Ans.* 0°, 180°, $(2n + \frac{1}{2} \pm \frac{1}{3})\frac{\pi}{4}$.

15. $\sin 5\theta + \sin 3\theta + 2\cos\theta = 0$; find θ. *Ans.* 90°, 270°, $(2n + \frac{3}{2})\frac{\pi}{4}$.

16. $\sin(60° - x) - \sin(60° + x) = +\frac{1}{2}\sqrt{3}$; find x. *Ans.* 240°, 300°.

17. $\sin(30° + x) - \cos(60° + x) = -\frac{1}{2}\sqrt{3}$; find x. *Ans.* 210°, 330°.

18. $\cos 4z - \cos 2z = 0$; find z. *Ans.* 0°, 60°, 120°, 180°, 240°, 300°.

84. Find r and ϕ from the Equations

$$\left.\begin{aligned} r\sin\phi &= a, \\ r\cos\phi &= b, \end{aligned}\right\} \quad \begin{aligned} &(1)\\ &(2)\end{aligned}$$

a and b being known.

(1) ÷ (2) gives
$$\tan\phi = \frac{a}{b}. \qquad (3)$$

From (1) and (2)
$$r = \frac{a}{\sin\phi} = \frac{b}{\cos\phi}. \qquad (4)$$

1. Find r and ϕ when $\log a = 0.47141$, and $\log b = 0.63927\,n$, r being positive.

$$\begin{aligned}
\log(r\sin\phi) = \log a &= 0.47141 && (1)\\
\log\sin\phi &= 9.74972 && (5)\\
\log\cos\phi &= 9.91758\,n && (6)\\
\log(r\cos\phi) = \log b &= \underline{0.63927\,n} && (2)\\
(1)-(2) = \log\tan\phi &= 9.83214\,n && (3)\\
\phi &= 145°\,48'.4 && (4)\\
(1)-(5) = (2)-(6) = \log r &= 0.72169 && (7)\\
r &= 5.2685 && (8)
\end{aligned}$$

The numbers on the right indicate the order in which the quantities are found. If the two values of $\log r$ had differed, we should have taken that found from $\log\cos\phi$, as a small error in $\log\tan\phi$ would, for this value of ϕ, affect the logarithmic cosine less than the logarithmic sine. The angle ϕ is placed in the second quadrant, since $r\cos\phi$ is negative and $r\sin\phi$ positive, r being considered positive.

2. Find r and ϕ when $\log a = 0.46843\,n$, and $\log b = 0.43742$, r being positive.

Ans. $\phi = 312°\,57'.4$; $r = 4.0178$.

3. Find r and ϕ when $\log a = 1.46444\,n$, and $\log b = 1.86903\,n$, r being positive.

Ans. $\phi = 201°\,30'.0$; $r = 79.497$.

85. Find r, ϕ, and θ from the Equations

$$\left.\begin{aligned} r\cos\phi\cos\theta &= a, \\ r\sin\phi\cos\theta &= b, \\ r\sin\theta &= c, \end{aligned}\right\} \quad \begin{aligned} (1) \\ (2) \\ (3) \end{aligned}$$

a, b, and c being known.

(2) ÷ (1) gives $\tan\phi = \frac{b}{a}$. (4)

From (1) and (2), $r\cos\theta = \frac{a}{\cos\phi} = \frac{b}{\sin\phi}$. (5)

From (3), $r\sin\theta = c$. (6)

(6) ÷ (5) gives $\tan\theta = \frac{c\cos\phi}{a} = \frac{c\sin\phi}{b}$. (7)

From (5) and (6),

$$r = \frac{a}{\cos\phi\cos\theta} = \frac{b}{\sin\phi\cos\theta} = \frac{c}{\sin\theta}. \quad (8)$$

1. Given $\log a = 0.46472$, $\log b = 0.72413\,n$, $\log c = 0.62817$, find r, ϕ, and θ, θ being numerically less than 90°, and r being positive.

$\log(r\cos\phi\cos\theta) = \log a =$	0.46472		(1)
$\log\cos\phi =$	(9.68314)	Only as a check.	(5)
$\log\sin\phi =$	9.94256 n		(5)
$\log(r\sin\phi\cos\theta) = \log b =$	0.72413 n		(2)
$(2) - (1) = \log\tan\phi =$	0.25941 n		(3)
$\phi =$	298° 49′.4		(4)
$(2)-(5)=(1)-(5)=\log(r\cos\theta) =$	0.78157		(6)
$\log\cos\theta =$	9.91291		(10)
$\log\sin\theta =$	(9.75951)	Only as a check.	(10)
$\log c = \log(r\sin\theta) =$	0.62817		(7)
$(7)-(6) = \log\tan\theta =$	9.84660		(8)
$\theta =$	35° 5′.1		(9)
$(6)-(10)=(7)-(10)=\log r =$	0.86866		(11)
$r =$	7.3903		(12)

The angle ϕ is placed in the fourth quadrant, since $r\cos\theta$ is positive, and therefore $\cos\phi$ must be positive and $\sin\phi$ negative, $r\cos\phi\cos\theta$ being positive and $r\sin\phi\cos\theta$ negative.

2. Given $\log a = 0.26903\,n$, $\log b = 0.32426$, $\log c = 0.36903\,n$, find r, ϕ, and θ, r being positive and θ numerically less than 90°.

Ans. $\phi = 131° \, 22'.0$; $\theta = -39° \, 45'.6$; $r = 3.6572$.

3. Given $\log a = 9.43942\,n$, $\log b = 9.40403\,n$, $\log c = 9.56700\,n$, find r, ϕ, and θ, r being positive and θ numerically less than 90°.

Ans. $\phi = 222° \, 40'.1$; $\theta = -44° \, 36'.4$; $r = 0.525425$ or 0.52544.

86. Find ϕ from the Equation

$$a \sin \phi + b \cos \phi = c \tag{1}$$

by formulas adapted to logarithmic computation, a, b, and c being known.

Let M be an auxiliary angle and m a positive constant, so that

$$\left.\begin{aligned} m \sin M &= a, \\ m \cos M &= b. \end{aligned}\right\} \tag{2}$$

The angle M is always possible, for we have, by division,

$$\tan M = \frac{a}{b}, \tag{3}$$

and since the tangent may have any value between $+\infty$ and $-\infty$, there will always be some angle whose tangent is equal to $\frac{a}{b}$. Also, squaring and adding Eqs. (2), we have

$$m^2 \sin^2 M + m^2 \cos^2 M = m^2 = a^2 + b^2,$$

or

$$m = \sqrt{a^2 + b^2}. \tag{4}$$

Therefore the assumptions in (2) are always possible, since M and m will be real quantities if a and b are real.

Substituting (2) in (1), we have

$$m \sin M \sin \phi + m \cos M \cos \phi = c,$$

or

$$m \cos (\phi - M) = c. \tag{5}$$

Hence, from (2) find M and m by the method of Art. 84; from (5) find $\phi - M$ (two values $< 360°$), and thence find ϕ.

1. Find ϕ when $2 \sin \phi - 3 \cos \phi = 1$.

Ans. $M = 146° 18'.6$; $\phi = 220° 12'.5$, or $72° 24'.7$.

2. Find ϕ when $2 \sin \phi + 4 \cos \phi = -3$.

Ans. $M = 26° 33'.9$; $\phi = 158° 41'.8$, or $254° 26'.0$.

87. Find ϕ from the Equation

$$a \tan \phi + b \cot \phi = c$$

by formulas adapted to logarithmic computation, a, b, and c being known.

Substituting for $\tan\phi$ and $\cot\phi$ in terms of $\sin\phi$ and $\cos\phi$, we have, after reducing,

$$(a-b)\cos 2\phi + c\sin 2\phi = a+b.$$

$$\text{Let}\quad \left.\begin{aligned} m\sin M &= a-b,\\ m\cos M &= c.\end{aligned}\right\}\quad \therefore\ m\sin(M+2\phi)=a+b.$$

1. Find ϕ when $2\tan\phi - \cot\phi = -3$.

Ans. $M=135°$; $\phi = 15° 41'.0,\ 119° 19'.0,\ 195° 41'.0,\ 299° 19'.0.$

2. Find ϕ when $\tan\phi + 3\cot\phi = -2\sqrt{3}$.

Ans. $M=210°$; $\phi = 120°$ or $300°$.

88. Find φ from the Following Equations, a and α being known:

(*a*) $\sin(\phi+\alpha) = a\sin\phi.$ (1)

Expanding, $\sin\phi\cos\alpha + \cos\phi\sin\alpha = a\sin\phi.$

$$\therefore\ \sin\phi\,(a-\cos\alpha) = \cos\phi\sin\alpha.$$

$$\therefore\ \tan\phi = \frac{\sin\alpha}{a-\cos\alpha}. \qquad (2)$$

Eq. (2) is not adapted to logarithmic computation. But from (1) we have

$$\frac{\sin(\phi+\alpha)}{\sin\phi} = \frac{a}{1},$$

and, by composition and division,

$$\frac{\sin(\phi+\alpha)+\sin\phi}{\sin(\phi+\alpha)-\sin\phi} = \frac{a+1}{a-1},$$

and this, from the equations of Art. 73, becomes

$$\frac{\tan(\phi+\tfrac{1}{2}\alpha)}{\tan\tfrac{1}{2}\alpha} = \frac{a+1}{a-1},$$

or

$$\tan(\phi+\tfrac{1}{2}\alpha) = \frac{a+1}{a-1}\tan\tfrac{1}{2}\alpha. \qquad (3)$$

Let $\tan\beta = a$, and note that $\tan 45° = 1$.

$$\therefore\ \tan(\phi+\tfrac{1}{2}\alpha) = \frac{\tan\beta+\tan 45°}{\tan\beta-\tan 45°}\tan\tfrac{1}{2}\alpha$$

$$= \frac{\sin(\beta+45°)}{\sin(\beta-45°)}\tan\tfrac{1}{2}\alpha.$$

$$\therefore\ \tan(\phi+\tfrac{1}{2}\alpha) = \cot(\beta-45°)\tan\tfrac{1}{2}\alpha. \qquad (4)$$

(*b*) $\cos(\phi + \alpha) = a \cos\phi.$

$\therefore \tan(\phi + \frac{1}{2}\alpha) = \tan(45^\circ - \beta)\cot\frac{1}{2}\alpha$, if $\tan\beta = a.$

(*c*) $\sin(\alpha - \phi) = a \sin\phi.$

$\therefore \tan(\phi - \frac{1}{2}\alpha) = \tan(45^\circ - \beta)\tan\frac{1}{2}\alpha$, if $\tan\beta = a.$

(*d*) $\sin(\phi + \alpha) = a\cos\phi.$

$\therefore \sin(\phi + \alpha) = a\sin(90^\circ + \phi);$

$\therefore \tan(45^\circ + \phi + \frac{1}{2}\alpha) = \cot(45^\circ - \beta)\tan(45^\circ - \frac{1}{2}\alpha)$, if $\tan\beta = a.$

(*e*) $\cos(\phi + \alpha) = a\sin\phi.$

$\therefore \cos(\phi + \alpha) = a\cos(90^\circ - \phi);$

$\therefore \tan(\phi + \frac{1}{2}\alpha - 45^\circ) = \tan(45^\circ - \beta)\cot(45^\circ + \frac{1}{2}\alpha)$, if $\tan\beta = a.$

NOTE. — The equation $a\sin(\phi + \alpha) = a'\sin(\phi + \alpha')$ and similar equations may be solved by expansion, the solution of the given equation being

$$\tan\phi = \frac{a'\sin\alpha' - a\sin\alpha}{a\cos\alpha - a'\cos\alpha'}.$$

A solution adapted to logarithmic computation may be found by the method of this article, giving

$$\tan[\phi + \tfrac{1}{2}(\alpha + \alpha')] = \cot(\beta - 45^\circ)\tan\tfrac{1}{2}(\alpha - \alpha'), \text{ if } \tan\beta = \frac{a'}{a}.$$

89. Find ϕ from the Following Equations, *a* and α being known:

(*a*) $\sin(\phi + \alpha)\sin\phi = a.$

From (8), Art. 72,

$$\cos\alpha - \cos(2\phi + \alpha) = 2a.$$

$$\therefore \cos(2\phi + \alpha) = \cos\alpha - 2a. \quad (1)$$

Let

$$\tan\beta = \frac{2a}{\sin\alpha}. \quad (2)$$

$$\therefore \cos(2\phi + \alpha) = \cos\alpha - \sin\alpha\tan\beta = \frac{\cos\alpha\cos\beta - \sin\alpha\sin\beta}{\cos\beta}.$$

$$\therefore \cos(2\phi + \alpha) = \frac{\cos(\alpha + \beta)}{\cos\beta}. \quad (3)$$

(*b*) $\sin(\alpha - \phi)\sin\phi = a.$

$$\therefore \cos(\alpha - 2\phi) - \cos\alpha = 2a;$$

$$\therefore \cos(\alpha - 2\phi) = \cos\alpha + 2a.$$

$$\therefore \cos(\alpha - 2\phi) = \frac{\cos(\alpha - \beta)}{\cos\beta}, \text{ if } \tan\beta = \frac{2a}{\sin\alpha}.$$

(*c*) $\sin(\phi+\alpha)\cos\phi = a.$

$$\therefore \sin(2\phi+\alpha)+\sin\alpha = 2a.$$

$$\therefore \sin(2\phi+\alpha) = 2a - \sin\alpha = \frac{\sin(\beta-\alpha)}{\cos\beta}, \text{ if } \tan\beta = \frac{2a}{\cos\alpha}.$$

(*d*) $\cos(\phi+\alpha)\cos\phi = a.$

$$\therefore \cos(2\phi+\alpha)+\cos\alpha = 2a.$$

$$\therefore \cos(2\phi+\alpha) = 2a - \cos\alpha = -\frac{\cos(\alpha+\beta)}{\cos\beta}, \text{ if } \tan\beta = \frac{2a}{\sin\alpha}.$$

(*e*) $\cos(\phi+\alpha)\sin\phi = a.$

$$\therefore \sin(2\phi+\alpha) = 2a + \sin\alpha = \frac{\sin(\alpha+\beta)}{\cos\beta}, \text{ if } \tan\beta = \frac{2a}{\cos\alpha}.$$

90. Find ϕ from the Following Equations, a, α, and α' being known:

(*a*) $\tan(\phi+\alpha) = a\tan\phi.$

$$\therefore \frac{\tan(\phi+\alpha)}{\tan\phi} = \frac{a}{1}; \therefore \frac{\tan(\phi+\alpha)+\tan\phi}{\tan(\phi+\alpha)-\tan\phi} = \frac{a+1}{a-1};$$

$$\therefore \frac{\sin(2\phi+\alpha)}{\sin\alpha} = \frac{a+1}{a-1}. \qquad (1)$$

Let $$\tan\beta = a; \qquad (2)$$

$$\therefore \frac{a+1}{a-1} = \cot(\beta - 45^\circ),$$

and $$\sin(2\phi+\alpha) = \cot(\beta-45^\circ)\sin\alpha. \qquad (3)$$

Find β from (2) and $2\phi+\alpha$ from (3).

(*b*) $\tan(\phi+\alpha) = a\cot\phi.$

$$\therefore \cos(2\phi+\alpha) = \tan(45^\circ-\beta)\cos\alpha, \text{ if } \tan\beta = a.$$

(*c*) $\cot(\alpha-\phi) = a\cot\phi.$

$$\therefore \sin(2\phi-\alpha) = \tan(\beta-45^\circ)\sin\alpha, \text{ if } \tan\beta = a.$$

(*d*) $\cot(\phi+\alpha) = a\cot(\phi-\alpha).$

$$\therefore \sin 2\phi = \cot(45^\circ-\beta)\sin 2\alpha, \text{ if } \tan\beta = a.$$

(*e*) $\tan(\phi+\alpha) = a\tan(\phi+\alpha').$

$$\therefore \sin(2\phi+\alpha+\alpha') = \cot(\beta-45^\circ)\sin(\alpha-\alpha'), \text{ if } \tan\beta = a.$$

(*f*) $\cot(\phi+\alpha) = a\cot(\phi+\alpha').$

$$\therefore \sin(2\phi+\alpha+\alpha') = \cot(\beta-45^\circ)\sin(\alpha'-\alpha), \text{ if } \tan\beta = a.$$

(*g*) $\cot(\phi+\alpha) = a\tan(\phi+\alpha').$

$$\therefore \cos(2\phi+\alpha+\alpha') = \tan(\beta-45^\circ)\cos(\alpha-\alpha'), \text{ if } \tan\beta = a.$$

91. Find ϕ from the Following Equations, a, α, and α' being known: *

(*a*) $\tan(\phi + \alpha)\tan\phi = a.$

$$\therefore \sin(\phi + \alpha)\sin\phi = a\cos(\phi + \alpha)\cos\phi.$$

From the equations of Art. 72, we have

$$-\cos(2\phi + \alpha) + \cos\alpha = a\cos(2\phi + \alpha) + a\cos\alpha;$$

$$\therefore \cos(2\phi + \alpha) = \frac{1 - a}{1 + a}\cos\alpha.$$

Let $\tan\beta = a.$

$$\therefore \cos(2\phi + \alpha) = \tan(45° - \beta)\cos\alpha.$$

(*b*) $\tan(\phi + \alpha)\cot\phi = a.$

$$\therefore \sin(2\phi + \alpha) = \cot(\beta - 45°)\sin\alpha, \text{ if } \tan\beta = a.$$

(*c*) $\tan(\phi + \alpha)\tan(\phi - \alpha) = a.$

$$\therefore \cos 2\phi = \tan(45° - \beta)\cos 2\alpha, \text{ if } \tan\beta = a.$$

(*d*) $\tan(\phi + \alpha)\cot(\phi + \alpha') = a.$

$$\therefore \sin(2\phi + \alpha + \alpha') = \cot(\beta - 45°)\sin(\alpha - \alpha'), \text{ if } \tan\beta = a.$$

92. Find r and ϕ from the Following Equations, a, b, α, and β being known:

$$\left.\begin{aligned} r\sin(\phi + \alpha) &= a, \\ r\cos(\phi + \beta) &= b. \end{aligned}\right\} \quad \begin{aligned}(1)\\(2)\end{aligned}$$

$$\therefore \frac{\sin(\phi + \alpha)}{\cos(\phi + \beta)} = \frac{a}{b}; \quad \therefore b\sin(\phi + \alpha) = a\cos(\phi + \beta).$$

$$\therefore b\sin\phi\cos\alpha + b\cos\phi\sin\alpha = a\cos\phi\cos\beta - a\sin\phi\sin\beta.$$

$$\therefore b\sin\phi\cos\alpha + a\sin\phi\sin\beta = a\cos\phi\cos\beta - b\cos\phi\sin\alpha.$$

$$\therefore \sin\phi(b\cos\alpha + a\sin\beta) = \cos\phi(a\cos\beta - b\sin\alpha).$$

$$\therefore \tan\phi = \frac{a\cos\beta - b\sin\alpha}{b\cos\alpha + a\sin\beta}, \quad (3)$$

and

$$r = \frac{a}{\sin(\phi + \alpha)} = \frac{b}{\cos(\phi + \beta)}. \quad (4)$$

The quadrant of ϕ will be determined by the sign assigned to r.

* The method of Art. 90 may be used, since $\tan x = \dfrac{1}{\cot x}$ and $\cot x = \dfrac{1}{\tan x}$.

1. If $r \sin(\phi + \alpha) = a$, and $r \sin(\phi + \beta) = b$, show that

$$\tan \phi = \frac{a \sin \beta - b \sin \alpha}{b \cos \alpha - a \cos \beta}.$$

2. If $r \cos(\phi + \alpha) = a$, and $r \cos(\phi + \beta) = b$, show that

$$\tan \phi = \frac{a \cos \beta - b \cos \alpha}{a \sin \beta - b \sin \alpha}.$$

93. Find r and ϕ from the Following Equations, a, b, α, and β being known, and the formulas derived being adapted to logarithmic computation:

$$\left.\begin{aligned} r \sin(\phi + \alpha) &= a, \quad (1)\\ r \sin(\phi + \beta) &= b. \quad (2)\end{aligned}\right\}$$

$$(1)+(2) = \quad r[\sin(\phi + \alpha) + \sin(\phi + \beta)] = a + b;$$

$$\therefore\ 2r \sin[\phi + \tfrac{1}{2}(\alpha + \beta)] \cos \tfrac{1}{2}(\alpha - \beta) = a + b. \qquad (3)$$

$$(1)-(2) = \quad r[\sin(\phi + \alpha) - \sin(\phi + \beta)] = a - b;$$

$$\therefore\ 2r \cos[\phi + \tfrac{1}{2}(\alpha + \beta)] \sin \tfrac{1}{2}(\alpha - \beta) = a - b. \qquad (4)$$

From (3) and (4), we have

$$\left.\begin{aligned} r \sin[\phi + \tfrac{1}{2}(\alpha + \beta)] &= \frac{a + b}{2 \cos \tfrac{1}{2}(\alpha - \beta)},\\ r \cos[\phi + \tfrac{1}{2}(\alpha + \beta)] &= \frac{a - b}{2 \sin \tfrac{1}{2}(\alpha - \beta)},\end{aligned}\right\}$$

from which r and $\phi + \frac{1}{2}(\alpha + \beta)$ are found by the method of Art. 84.

1. If $r \cos(\phi + \alpha) = a$, and $r \cos(\phi + \beta) = b$, show that

$$r \cos[\phi + \tfrac{1}{2}(\alpha + \beta)] = \frac{a + b}{2 \cos \tfrac{1}{2}(\alpha - \beta)},$$

$$r \sin[\phi + \tfrac{1}{2}(\alpha + \beta)] = \frac{b - a}{2 \sin \tfrac{1}{2}(\alpha - \beta)}.$$

2. If $r \sin(\phi + \alpha) = a$, and $r \cos(\phi + \beta) = b$, show that by placing $\cos(\phi + \beta) = \sin(90° + \phi + \beta)$ we may obtain

$$r \sin[\phi + 45° + \tfrac{1}{2}(\alpha + \beta)] = \frac{a + b}{2 \cos[45° - \tfrac{1}{2}(\alpha - \beta)]},$$

$$r \cos[\phi + 45° + \tfrac{1}{2}(\alpha + \beta)] = \frac{b - a}{2 \sin[45° - \tfrac{1}{2}(\alpha - \beta)]}.$$

3. Find r and ϕ when $r \sin(\phi + 100°) = 2$, and $r \sin(\phi + 200°) = 3$, r being positive.

Ans. $\phi = 290° 28'.4$; $r = 3.9436$.

CHAPTER VII.

OBLIQUE PLANE TRIANGLES.

94. It has been shown in Geometry that a triangle can be constructed when three elements, one being a side, are known. If the three angles only are given, there will be an infinite number of triangles satisfying the conditions of the problem, since the data determine the *shape* and not the size of the triangle.

We also know that in any triangle

(1) The sum of the three angles is 180°.

(2) If one angle is 90°, the sum of the other two is 90°.

(3) The greater side is opposite the greater angle, and conversely.

(4) Any side is less than the sum of the other two.

95. The Sine Proportion. — *The sides of a triangle are to each other as the sines of the opposite angles.*

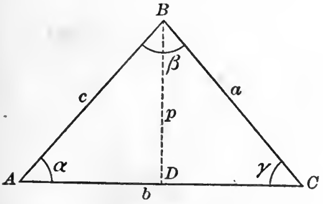


FIG. 52.

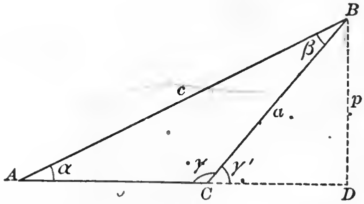


FIG. 53.

In Fig. 52, $p = a \sin \gamma$; $p = c \sin \alpha$.

$$\therefore \ a \sin \gamma = c \sin \alpha. \tag{1}$$

$$\therefore \ \frac{a}{c} = \frac{\sin \alpha}{\sin \gamma}, \text{ or } \frac{a}{\sin \alpha} = \frac{c}{\sin \gamma}. \tag{2}$$

In Fig. 53,

$$p = a \sin \gamma' = a \sin (180^\circ - \gamma) = a \sin \gamma, \text{ and } p = c \sin \alpha.$$

$$\therefore \; a \sin \gamma = c \sin \alpha, \text{ as before.}$$

In the same way, by drawing a line perpendicular to AB from C (Figs. 52 and 53), we can show that

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta}.$$

$$\therefore \; \frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}, \qquad (3)$$

true for both acute and obtuse angled triangles.

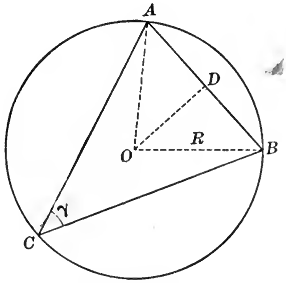


Fig. 54.

NOTE. — The constant quotient $\frac{a}{\sin \alpha}$ is called the *modulus* of the triangle, and is equal to the diameter of the circumscribed circle.

For, in Fig. 54, $c = AB = 2R \sin AOD = 2R \sin \frac{1}{2} AOB = 2R \sin \gamma$.

$$\therefore \frac{c}{\sin \gamma} = 2R.$$

96. The Square of Any Side of a Triangle *is equal to the sum of the squares of the other two sides, diminished by twice the product of the two sides multiplied by the cosine of their included angle.*

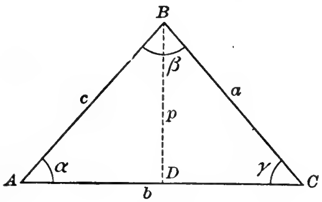


Fig. 55.

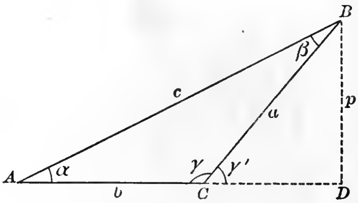


Fig. 56.

From geometry we have, in Fig. 55,

$$c^2 = a^2 + b^2 - 2b \cdot DC = \mathbf{a^2 + b^2 - 2ab\cos\gamma}. \qquad (1)$$

Also, in Fig. 56,

$$c^2 = a^2 + b^2 + 2b \cdot CD = a^2 + b^2 + 2ab\cos\gamma',$$

or $$\mathbf{c^2 = a^2 + b^2 - 2ab\cos\gamma}. \qquad (2)$$

This relation may also be proved as follows :

In Fig. 55, $b = AC = AD + DC = c\cos\alpha + a\cos\gamma.$

In Fig. 56, $b = AC = AD - CD = c\cos\alpha - a\cos\gamma'$

$$= c\cos\alpha + a\cos\gamma.$$

$$\therefore\ b = c\cos\alpha + a\cos\gamma.$$

$$\therefore\ c\cos\alpha = b - a\cos\gamma.$$

$$\therefore\ c^2\cos^2\alpha = a^2\cos^2\gamma + b^2 - 2ab\cos\gamma.$$

But $c^2\sin^2\alpha = a^2\sin^2\gamma,$ from (1), Art. 95.

By addition, $c^2 = a^2 + b^2 - 2ab\cos\gamma,$

since $\sin^2 x + \cos^2 x = 1.$

97. Case I. Given One Side and Two Angles (a, α, β).

Formulas: $\gamma = 180° - (\alpha + \beta)$;

$$b = \frac{a}{\sin\alpha}\sin\beta;$$

$$c = \frac{a}{\sin\alpha}\sin\gamma.$$

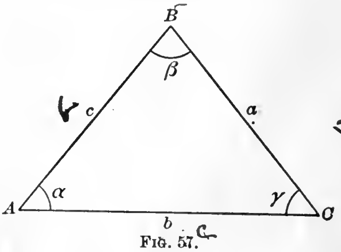


FIG. 57.

1. Solve the triangle when $a = 3.4356$, $\alpha = 17° 43'.4$, $\gamma = 60° 35'.7$.

$\therefore\ \beta = 180° - (\alpha + \gamma) = 101° 40'.9.$

(*a*) By natural functions.

$$b = a \times \sin\beta \div \sin\alpha = 3.4356 \times .97929 \div .30442 = 11.052.$$

$$c = a \times \sin\gamma \div \sin\alpha = 3.4356 \times .87117 \div .30442 = 9.8318.$$

(*b*) By the use of logarithms.

$$\log b = \log a - \log\sin\alpha + \log\sin\beta = \log a + \text{col}\sin\alpha + \log\sin\beta.$$

$$\log c = \log a - \log\sin\alpha + \log\sin\gamma = \log a + \text{col}\sin\alpha + \log\sin\gamma.$$

$\log a = 0.53600$	$\log a = 0.53600$
$\text{col}\sin\alpha = 0.51652$	$\text{col}\sin\alpha = 0.51652$
$\log\sin\beta = 9.99091$	$\log\sin\gamma = 9.94010$
$\log b = 1.04343$	$\log c = 0.99262$
$b = 11.052$	$c = 9.8315$

2. Solve the triangle when $c = 54.376$, $\alpha = 103° 3'.2$, $\beta = 40° 10'.3$.

Ans. $\gamma = 36° 46'.5$; $b = 58.591$; $a = 88.478$.

3. Solve the triangle when $a = 0.14323$, $\alpha = 53° 17'.3$, $\beta = 62° 23'.5$.

Ans. $\gamma = 64° 19'.2$; $b = 0.15832$; $c = 0.16101$.

98. Case II. Given Two Sides and the Angle Opposite One of them (a, c, α). — From the sine proportion, we have

$$\sin \gamma = \frac{c}{a} \sin \alpha. \tag{1}$$

Since γ is found from (1) by means of its sine, it may have two values, one in the first and one in the second quadrant, their sum being 180°. Therefore there *may* be two triangles with the given elements.

If α is obtuse, γ must be acute, since there can be only one obtuse angle in a plane triangle, and there will be only one solution.

If α is acute, and a is greater than c, γ will be acute, since α must be greater than γ, and there will be only one solution.

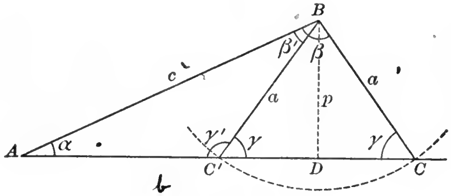


FIG. 58.

If α is acute, and a is equal to c, there will be only one solution, since the points C' and A will coincide.

If α is acute, and a is less than c, γ will be greater than α, and therefore γ may be either in the first or in the second quadrant.

In order that there may be two solutions, the given angle must be acute, and the side opposite it must be less than the side adjacent.

If $a = DB$, the two triangles will be coincident, γ being 90°. If a is less than DB, the triangle will be impossible; this will be shown in the computation where $\sin \gamma$, found from (1), will be greater than unity.

If we use primed letters to represent the unknown elements of one of the triangles, and unprimed letters for those of the other, we have

Formulas: $\sin\gamma = \frac{c}{a}\sin\alpha = \sin\gamma'$;

$$\beta = 180° - (\alpha + \gamma);\quad \beta' = 180° - (\alpha + \gamma');$$

$$b = \frac{a}{\sin\alpha}\sin\beta;\quad b' = \frac{a}{\sin\alpha}\sin\beta';$$

or

$$b = \frac{c}{\sin\gamma}\sin\beta;\quad b' = \frac{c}{\sin\gamma'}\sin\beta'.$$

1. Solve the triangle when $a = 9.4672$, $c = 14.433$, $\alpha = 11° 14'.3$.

In this example $\alpha < 90°$, $a < c$; $\therefore$ two solutions.

$\log\sin\gamma = \log c + \text{col } a + \log\sin\alpha = \log\sin\gamma'$.

$\beta = 180° - (\alpha + \gamma)$; $\beta' = 180° - (\alpha + \gamma')$.

$\log b = \log a + \text{col}\sin\alpha + \log\sin\beta = \log c + \text{col}\sin\gamma + \log\sin\beta$.

$\log b' = \log a + \text{col}\sin\alpha + \log\sin\beta' = \log c + \text{col}\sin\gamma' + \log\sin\beta'$.

$\log c = 1.15936$	$\log a = 0.97622$	$\log c = 1.15936$
$\text{col } a = 9.02378$	$\text{col}\sin\alpha = 0.71021$	$\text{col}\sin\gamma = 0.52707$
$\log\sin\alpha = 9.28979$	$\log\sin\beta = 9.67899$	$\log\sin\beta = 9.67899$
$\log\sin\gamma = 9.47293$	$\log b = 1.36542$	$\log b = 1.36542$
$\gamma = 17° 17'.1$	$b = 23.196$	$b = 23.196$
$\gamma' = 162° 42'.9$		
$\therefore\ \beta = 151° 28'.6$	$\log a = 0.97622$	$\log c = 1.15936$
$\beta' = 6°\ 2'.8$	$\text{col}\sin\alpha = 0.71021$	$\text{col}\sin\gamma' = 0.52707$
	$\log\sin\beta' = 9.02259$	$\log\sin\beta' = 9.02259$
	$\log b' = 0.70902$	$\log b' = 0.70902$
	$b' = 5.1170$	$b' = 5.1170$

2. Solve the triangle when $a = 2.4741$, $c = 1.0003$, $\alpha = 69° 14'.8$.

Ans. $\gamma = 22° 12'.8$; $\beta = 88° 32'.4$; $b = 2.6449$.

3. Solve the triangle when $a = 10.473$, $b = 12.987$, $\alpha = 44° 11'.3$.

Ans. $\begin{cases} \beta = 59° 48'.5;\ \gamma = 76° 0'.2;\ c = 14.579; \\ \beta' = 120° 11'.5;\ \gamma' = 15° 37'.2;\ c' = 4.0456. \end{cases}$

4. Solve the triangle when $a = 0.43477$, $b = 0.40031$, $\alpha = 94° 17'.6$.

Ans. $\beta = 66° 39'.6$; $\gamma = 19° 2'.8$; $c = 0.14228$.

99. Case III. Given the Three Sides (a, b, c).

(a) From Art. 96,

$$a^2 = b^2 + c^2 - 2\,bc\cos\alpha.$$

$$\therefore \cos\alpha = \frac{b^2 + c^2 - a^2}{2\,bc}. \quad (1)$$

From this equation we may find α by means of its *natural* cosine.

(*b*) To adapt (1) to logarithmic computation, subtract each member from unity.

$$\therefore\ 1-\cos\alpha=1-\frac{b^2+c^2-a^2}{2\,bc}=\frac{2\,bc-b^2-c^2+a^2}{2\,bc}=\frac{a^2-(b-c)^2}{2\,bc}.$$

$$\therefore\ 2\sin^2\tfrac{1}{2}\alpha=\frac{[a+(b-c)][a-(b-c)]}{2\,bc}=\frac{(a+b-c)(a-b+c)}{2\,bc}.$$

Let $$a+b+c=2\,s;$$

$$\therefore\ a+b-c=a+b+c-2\,c=2\,s-2\,c=2(s-c);$$

$$a-b+c=a+b+c-2\,b=2\,s-2\,b=2(s-b).$$

$$\therefore\ \sin^2\tfrac{1}{2}\alpha=\frac{2(s-b)\,2(s-c)}{4\,bc}=\frac{(s-b)(s-c)}{bc},$$

where s is half the sum of the three sides, and b and c are the sides adjacent to the angle.

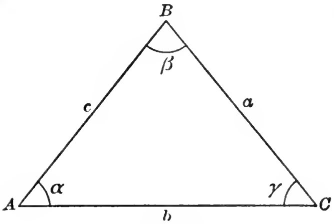


FIG. 59.

$$\left.\begin{aligned}\therefore\ \sin^2\tfrac{1}{2}\alpha&=\frac{(s-b)(s-c)}{bc},\\ \sin^2\tfrac{1}{2}\beta&=\frac{(s-a)(s-c)}{ac},\\ \sin^2\tfrac{1}{2}\gamma&=\frac{(s-a)(s-b)}{ab}.\end{aligned}\right\}\quad(2)$$

(*c*) Again, adding each member of (1) to unity,

$$1+\cos\alpha=1+\frac{b^2+c^2-a^2}{2\,bc}=\frac{2\,bc+b^2+c^2-a^2}{2\,bc}=\frac{(b+c)^2-a^2}{2\,bc}.$$

$$\therefore\ 2\cos^2\tfrac{1}{2}\alpha=\frac{(b+c+a)(b+c-a)}{2\,bc}=\frac{2\,s\cdot 2(s-a)}{2\,bc}.$$

$$\left.\begin{aligned}\therefore\ \cos^2\tfrac{1}{2}\alpha&=\frac{s(s-a)}{bc},\\ \cos^2\tfrac{1}{2}\beta&=\frac{s(s-b)}{ac},\\ \cos^2\tfrac{1}{2}\gamma&=\frac{s(s-c)}{ab}.\end{aligned}\right\}\quad(3)$$

(*d*) Dividing $\sin^2\frac{1}{2}\alpha$ by $\cos^2\frac{1}{2}\alpha$, we have

$$\tan^2\tfrac{1}{2}\alpha=\frac{(s-b)(s-c)}{s(s-a)}.$$

Similarly,

$$\left.\begin{aligned}\tan^2\tfrac{1}{2}\beta&=\frac{(s-a)(s-c)}{s(s-b)},\\ \tan^2\tfrac{1}{2}\gamma&=\frac{(s-a)(s-b)}{s(s-c)}.\end{aligned}\right\}\quad(4)$$

Or

$$\tan^2 \tfrac{1}{2}\alpha = \frac{(s-a)(s-b)(s-c)}{s(s-a)^2} = \frac{(s-a)(s-b)(s-c)}{s} \cdot \frac{1}{(s-a)^2}.$$

$$\therefore \tan \tfrac{1}{2}\alpha = \frac{1}{s-a}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

Let

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} \qquad (5)$$

Similarly,

$$\left.\begin{aligned} \therefore \tan \tfrac{1}{2}\alpha &= \frac{r}{s-a}. \\ \tan \tfrac{1}{2}\beta &= \frac{r}{s-b}, \\ \tan \tfrac{1}{2}\gamma &= \frac{r}{s-c}. \end{aligned}\right\} \qquad (6)$$

The angles of the triangle may be found from (2), (3), (4), or (5) and (6), the computation being checked by

$$\tfrac{1}{2}\alpha + \tfrac{1}{2}\beta + \tfrac{1}{2}\gamma = 90°.$$

In finding all the angles, (5) and (6) should be used.

NOTE. — The tabular difference for $\tan x$ is greater than that for either $\sin x$ or $\cos x$, so that a small error in $\tan x$ will affect the angle x less than would a corresponding error in $\sin x$ or $\cos x$. Hence the angles should be determined by means of their tangents whenever practicable.

Again, when x is less than 45°, the tabular difference for $\sin x$ exceeds that for $\cos x$, and when x is greater than 45°, the tabular difference for $\cos x$ is the greater. Hence the angle should be determined by means of its sine rather than its cosine when the angle is less than 45°, and by its cosine rather than its sine when it is greater than 45°.

NOTE. — r is the radius of the inscribed circle. For, considering the areas,

$$\triangle ABC = \triangle OAC + \triangle OCB + \triangle OBA$$

$$= \frac{AC}{2}OE, + \frac{BC}{2}OF + \frac{AB}{2}OD$$

$$= \tfrac{1}{2}(AC + BC + AB)\,r = sr.$$

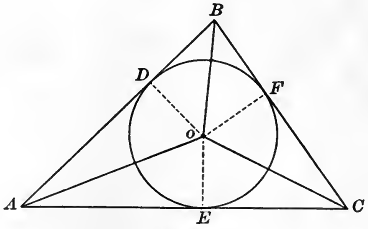


FIG. 60.

But, from Art. 109,

$$\triangle ABC = \sqrt{s(s-a)(s-b)(s-c)}.$$

$$\therefore sr = \sqrt{s(s-a)(s-b)(s-c)}.$$

$$\therefore r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

1. Solve the triangle when $a = 0.0093146$, $b = 0.0176530$, $c = 0.0095768$.

$$\log r = \tfrac{1}{2}[\log(s-a) + \log(s-b) + \log(s-c) + \text{col } s],$$

$$\log \tan \tfrac{1}{2}\alpha = \log r - \log(s-a), \text{ etc.}$$

$a = 0.0093146$	$\log(s-a) = 7.95219 - 10$
$b = 0.0176530$	$\log(s-b) = 6.79183 - 10$
$c = 0.0095768$	$\log(s-c) = 7.93929 - 10$
$2s = 0.0365444$	$\text{col } s = 1.73821$
$s = 0.0182722$	$\log r^2 = 4.42152 - 10$
$s-a = 0.0089576$	$\log r = 7.21076 - 10$
$s-b = 0.0006192$	$\therefore \log\tan\frac{1}{2}\alpha = 9.25857$
$s-c = 0.0086954$	$\frac{1}{2}\alpha = 10^\circ\, 16'.8$
$\text{sum} = 0.0365444$	$\log\tan\frac{1}{2}\beta = 0.41893$
$2s = 0.0365444$	$\frac{1}{2}\beta = 69^\circ\, 8'.2$
a check.	$\log\tan\frac{1}{2}\gamma = 9.27147$
	$\frac{1}{2}\gamma = 10^\circ\, 35'.0$

In finding $\log\tan\frac{1}{2}\alpha$, write $\log r$ on the margin of a slip of paper, place it above $\log(s-a)$, and write the difference of the two logarithms opposite $\log\tan\frac{1}{2}\alpha$; then find $\log\tan\frac{1}{2}\beta$ and $\log\tan\frac{1}{2}\gamma$ in the same way. Find $s-a$, $s-b$, and $s-c$ in a similar manner.

2. Solve the triangle when $a = 32.456$, $b = 41.724$, $c = 53.987$.

Ans. $\frac{1}{2}\alpha = 18^\circ\, 27'.4$; $\frac{1}{2}\beta = 25^\circ\, 16'.3$; $\frac{1}{2}\gamma = 46^\circ\, 16'.4$.

3. Solve the triangle when $a = 0.14679$, $b = 0.10433$, $c = 0.04796$.

Ans. $\frac{1}{2}\alpha = 73^\circ\, 20'.4$; $\frac{1}{2}\beta = 11^\circ\, 29'.4$; $\frac{1}{2}\gamma = 5^\circ\, 10'.2$.

100. Case IV. Given Two Sides and the Included Angle (b, c, α). First Method. — *The sum of any two sides of a triangle is to their difference as the tangent of half the sum of the opposite angles is to the tangent of half their difference.* For we have

$$\frac{b}{c} = \frac{\sin\beta}{\sin\gamma}.$$

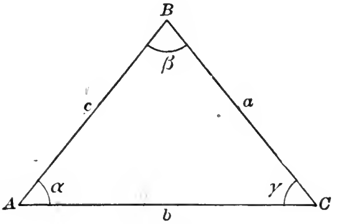


FIG. 61.

By composition and division,

$$\frac{b+c}{b-c} = \frac{\sin\beta + \sin\gamma}{\sin\beta - \sin\gamma}$$

$$= \frac{2\sin\frac{1}{2}(\beta+\gamma)\cos\frac{1}{2}(\beta-\gamma)}{2\cos\frac{1}{2}(\beta+\gamma)\sin\frac{1}{2}(\beta-\gamma)}.$$

(Art. 73.)

$$\therefore \frac{b+c}{b-c} = \frac{\tan\frac{1}{2}(\beta+\gamma)}{\tan\frac{1}{2}(\beta-\gamma)}. \qquad (1)$$

But

$$\beta+\gamma = 180^\circ - \alpha;\ \tfrac{1}{2}(\beta+\gamma) = 90^\circ - \tfrac{1}{2}\alpha. \ \therefore \tan\tfrac{1}{2}(\beta+\gamma) = \cot\tfrac{1}{2}\alpha.$$

$$\therefore \tan\tfrac{1}{2}(\beta-\gamma) = \frac{b-c}{b+c}\cot\tfrac{1}{2}\alpha. \qquad (2)$$

From (2) we find $\frac{1}{2}(\beta-\gamma)$; adding $\frac{1}{2}(\beta-\gamma)$ to $\frac{1}{2}(\beta+\gamma)$, we have β, and subtracting $\frac{1}{2}(\beta-\gamma)$ from $\frac{1}{2}(\beta+\gamma)$, we have γ. Then the third side is found from the sine proportion.

Formulas:
$$\left.\begin{aligned}\tan\tfrac{1}{2}(\beta-\gamma) &= \frac{b-c}{b+c}\cot\tfrac{1}{2}\alpha,\\ \tfrac{1}{2}(\beta+\gamma) &= 90^\circ-\tfrac{1}{2}\alpha,\\ \beta &= \tfrac{1}{2}(\beta+\gamma)+\tfrac{1}{2}(\beta-\gamma),\\ \gamma &= \tfrac{1}{2}(\beta+\gamma)-\tfrac{1}{2}(\beta-\gamma),\\ a &= \frac{b\sin\alpha}{\sin\beta}=\frac{c\sin\alpha}{\sin\gamma}.\end{aligned}\right\}$$

In using (1) or (2) the greater side and the greater angle should be written first; thus, if c were greater than b, we should use $c-b$ and $\gamma-\beta$ instead of $b-c$ and $\beta-\gamma$. If the smaller side is written first, the tangent of half the difference of the two angles will be negative, giving the half-difference as an angle between 0° and − 90°.

1. Solve the triangle when $b=0.14367$, $c=0.11412$, $\alpha=42^\circ\,14'.6$.

$$\therefore\ \tfrac{1}{2}\alpha=21^\circ\,7'.3;\ \tfrac{1}{2}(\beta+\gamma)=90^\circ-\tfrac{1}{2}\alpha=68^\circ\,52'.7.$$
$$\log\tan\tfrac{1}{2}(\beta-\gamma)=\log(b-c)+\text{col}\,(b+c)+\log\cot\tfrac{1}{2}\alpha.$$
$$\log a=\log b+\log\sin\alpha+\text{col}\sin\beta=\log c+\log\sin\alpha+\text{col}\sin\gamma.$$

$b-c=0.02955$
$b+c=0.25779$

$\log(b-c)=8.47056$
$\text{col}\,(b+c)=0.58874$
$\log\cot\frac{1}{2}\alpha=0.41308$

$\log\tan\frac{1}{2}(\beta-\gamma)=9.47238$
$\frac{1}{2}(\beta-\gamma)=16^\circ\,31'.7$
$\frac{1}{2}(\beta+\gamma)=68^\circ\,52'.7$

$\beta=85^\circ\,24'.4$
$\gamma=52^\circ\,21'.0$

$\log b=9.15737$
$\log\sin\alpha=9.82755$
$\text{col}\sin\beta=0.00140$

$\log a=8.98632$
$a=0.096900$

$\log c=9.05737$
$\log\sin\alpha=9.82755$
$\text{col}\sin\gamma=0.10141$

$\log a=8.98633$
$a=0.096902$

2. Solve the triangle when $a=101.47$, $c=99.367$, $\beta=47^\circ\,48'.2$.
Ans. $\alpha=67^\circ\,27'.1$; $\gamma=64^\circ\,44'.7$; $b=81.396$ or 81.394.

3. Solve the triangle when $b=19.937$, $c=62.475$, $\alpha=130^\circ\,9'.4$.
Ans. $\beta=11^\circ\,26'.1$; $\gamma=38^\circ\,24'.5$; $a=76.858$ or 76.860.

101. Case IV. Given b, c, α. Second Method. — To prove the equations

$$a\sin\tfrac{1}{2}(\beta-\gamma)=(b-c)\cos\tfrac{1}{2}\alpha, \quad (1)$$
$$a\cos\tfrac{1}{2}(\beta-\gamma)=(b+c)\sin\tfrac{1}{2}\alpha. \quad (2)$$

$$\frac{b}{c}=\frac{\sin\beta}{\sin\gamma}. \quad \therefore \quad \frac{b+c}{c}=\frac{\sin\beta+\sin\gamma}{\sin\gamma}.$$

$$\therefore \quad \frac{b+c}{\sin\beta+\sin\gamma}=\frac{c}{\sin\gamma}=\frac{a}{\sin\alpha}=\frac{a}{\sin(\beta+\gamma)}.$$

$$\therefore \quad \frac{b+c}{2\sin\frac{1}{2}(\beta+\gamma)\cos\frac{1}{2}(\beta-\gamma)}=\frac{a}{2\sin\frac{1}{2}(\beta+\gamma)\cos\frac{1}{2}(\beta+\gamma)}.$$

$$\therefore \quad a\cos\tfrac{1}{2}(\beta-\gamma)=(b+c)\cos\tfrac{1}{2}(\beta+\gamma)$$

$$=(b+c)\sin\tfrac{1}{2}\alpha. \qquad \text{Q.E.D.}$$

Similarly, $\dfrac{b-c}{c}=\dfrac{\sin\beta-\sin\gamma}{\sin\gamma}$ reduces to

$$a\sin\tfrac{1}{2}(\beta-\gamma)=(b-c)\sin\tfrac{1}{2}(\beta+\gamma)=(b-c)\cos\tfrac{1}{2}\alpha.$$

1. Solve the triangle when $b=0.14367$, $c=0.11412$, $\alpha=42°\,14'.6$.

$b-c=0.02955$	(1)	$(4)+(6)=\log[a\sin\frac{1}{2}(\beta-\gamma)]=8.44036$	(8)
$b+c=0.25779$	(2)	$\log\sin\frac{1}{2}(\beta-\gamma)=9.45404$	(12)
$\frac{1}{2}\alpha=21°\,7'.3$	(3)	$\log\cos\frac{1}{2}(\beta-\gamma)=9.98167$	(12)
		$(5)+(7)=\log[a\cos\frac{1}{2}(\beta-\gamma)]=8.96799$	(9)
$\log(b-c)=8.47056$	(4)		
$\log\cos\frac{1}{2}\alpha=9.96980$	(6)	$\log\tan\frac{1}{2}(\beta-\gamma)=9.47237$	(10)
		$\frac{1}{2}(\beta-\gamma)=16°\,31'.7$	(11)
$\log(b+c)=9.41126$	(5)	$\frac{1}{2}(\beta+\gamma)=68°\,52'.7$	(14)
$\log\sin\frac{1}{2}\alpha=9.55673$	(7)	$\beta=85°\,24'.4$	(15)
		$\gamma=52°\,21'.0$	(16)
		$(8-12)=(9)-(12)=\log a=8.98632$	(13)
		$a=0.096900$	(17)

2. Solve the triangle when $b=2.3671$, $c=1.4345$, $\alpha=112°\,43'.4$.

Ans. $\beta=42°\,54'.5$; $\gamma=24°\,22'.1$; $a=3.2069$.

3. Solve the triangle when $a=101.47$, $c=99.367$, $\beta=47°\,48'.2$.

Ans. $\alpha=67°\,27'.1$; $\gamma=64°\,44'.7$; $b=81.396$.

102. Case IV. Given b, c, α. Third Method.—To find the third side only.

$$a^2=b^2+c^2-2bc\cos\alpha.$$

But

$$\cos\alpha=1-2\sin^2\tfrac{1}{2}\alpha.$$

$$\therefore \quad a^2=b^2+c^2-2bc+4bc\sin^2\tfrac{1}{2}\alpha$$

$$=(b-c)^2+4bc\sin^2\tfrac{1}{2}\alpha$$

$$=(b-c)^2\left[1+\frac{4bc\sin^2\frac{1}{2}\alpha}{(b-c)^2}\right].$$

$$\therefore \quad a=(b-c)\sqrt{1+\frac{4bc\sin^2\frac{1}{2}\alpha}{(b-c)^2}}.$$

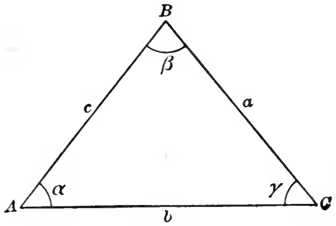


FIG. 62.

Let x be an angle such that

$$\tan^2 x = \frac{4\,bc \sin^2 \frac{1}{2}\alpha}{(b-c)^2};$$

or $$\tan x = \frac{2 \sin \frac{1}{2}\alpha}{b-c}\sqrt{bc}. \qquad (1)$$

This assumption is possible, since the value of the second member of (1) must lie between $+\infty$ and $-\infty$, so that there will always be some angle whose tangent is equal to this quantity.

$$\therefore\ a = (b-c)\sqrt{1+\tan^2 x} = (b-c)\sec x;$$

or $$a = \frac{b-c}{\cos x}. \qquad (2)$$

First find x from (1), and then a from (2). In these equations $b - c$ is replaced by $c - b$ when $c > b$.

1. Find a when $c = 1.4345$, $b = 2.3671$, and $\alpha = 112° 43'.4$.

$$\log \tan x = \tfrac{1}{2}(\log b + \log c) + \log 2 + \log \sin \tfrac{1}{2}\alpha + \text{col}\,(b-c).$$
$$\log a = \log (b-c) - \log \cos x.$$

$\log b = 0.37422$	$\log (b-c) = 9.96970$
$\log c = 0.15670$	$-\log \cos x = 9.46361$
$\log bc = 0.53092$	$\log a = 0.50609$
$\log \sqrt{bc} = 0.26546$	$a = 3.2069$
$\log 2 = 0.30103$	
$\log \sin \frac{1}{2}\alpha = 9.92041$	
$\text{col}\,(b-c) = 0.03030$	
$\log \tan x = 0.51720$	
$x = 73° 5'.6$	

2. Find b when $a = 101.47$, $c = 99.367$, $\beta = 47° 48'.2$.

$\therefore\ x = 88° 31'.17$; $b = 81.396$.

3. Find a when $b = 19.937$, $c = 62.475$, $\alpha = 130° 9'.4$.

$\therefore\ x = 56° 23'.7$; $a = 76.858$.

OBLIQUE TRIANGLES SOLVED BY RIGHT TRIANGLES.

103. Case I. Given a, α, γ. — In Figs. 63 and 64, on the next page, draw DB perpendicular to AC. Considering the first figure, in the triangle BDC we know a and γ, and we compute DB and DC; then in the triangle BDA we know DB and α, and we compute AD and c; then $b = AD + DC$, completing the solution. In the second figure, where γ is obtuse, we know, in the triangle BDC, a and $DCB = 180° - \gamma$, and we compute DB and CD; then in the triangle BDA we know DB and α, and we compute c and AD; then $b = AD - CD$, completing the solution.

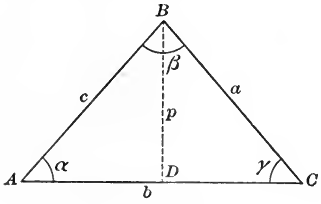


FIG. 63.

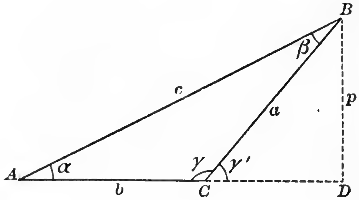


FIG. 64.

1. Solve the triangle when $a = 3.4356$, $\alpha = 17° 43'.4$, $\gamma = 60° 35'.7$.
$\therefore\ \beta = 101° 40'.9$; $DC = 1.6868$; $DB = 2.9929$; $AD = 9.3650$;
$c = 9.8315$; $b = 11.0518$.

2. Solve the triangle when $a = 54.376$, $\gamma = 103° 3'.2$, $\beta = 40° 10'.3$.
Ans. $\alpha = 36° 46'.5$; $c = 88.478$; $b = 58.592$.

3. Solve the triangle when $c = 230.47$, $\alpha = 21° 32'.2$, $\beta = 36° 24'.4$.
Ans. $\gamma = 122° 3'.4$; $a = 99.825$; $b = 161.3975$.

104. Case II. Given a, c, α.—In the right triangle ADB we know c and α, and we compute AD and DB; then in the triangle CBD we know DB and a, and we find DC and γ; then

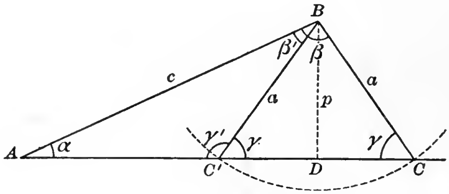


FIG. 65.

$$b = AD + DC; \quad \beta = 180° - (\alpha + \gamma);$$

$$b' = AD - DC; \quad \gamma' = 180° - \gamma; \quad \beta' = 180° - (\alpha + \gamma').$$

Two solutions are possible only when α is acute and a is less than c and greater than DB.

If α is obtuse, as in Fig. 66, we solve first the triangle BAD, then the triangle BCD, and find $b = DC - DA$.

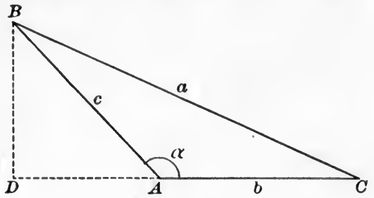


FIG. 66.

1. Solve the triangle when $c = 23.647$, $a = 14.135$, $\alpha = 33° 17'.3$.

$\therefore$ $AD = 19.767$; $DB = 12.979$; $\gamma = 66° 40'.0$; $DC = 5.5986$;

Ans. $\begin{cases} \gamma = 66° 40'.0; & \beta = 80° \ 2'.7; & b = 25.3656; \\ \gamma' = 113° 20'.0; & \beta' = 33° 22'.7; & b' = 14.1684. \end{cases}$

2. Solve the triangle when $a = 2.4741$, $c = 1.0003$, $\alpha = 69° 14'.8$.

$\therefore$ $\gamma = 22° 12'.8$; $\beta = 88° 32'.4$; $AD = 0.35445$; $DC = 2.2905$; $b = 2.64495$.

3. Solve the triangle when $a = 10.473$, $b = 12.987$, $\alpha = 44° 11'.3$.

Ans. $\begin{cases} \beta = 59° 48'.5; & \gamma = 76° \ 0'.2; & c = 14.5793; \\ \beta' = 120° 11'.5; & \gamma' = 15° 37'.2; & c' = 4.0455. \end{cases}$

4. Solve the triangle when $a = 0.43477$, $b = 0.40031$, $\alpha = 94° 17'.6$.

Ans. $\beta = 66° 39'.6$; $\gamma = 19° 2'.8$; $c = 0.142282$.

105. Case III. Given *a*, *b*, *c*. — In Fig. 63,

$$p^2 = c^2 - AD^2; \; p^2 = a^2 - DC^2.$$

$$\therefore \; c^2 - AD^2 = a^2 - DC^2.$$

$$\therefore \; AD^2 - DC^2 = c^2 - a^2.$$

$$\therefore \; AD - DC = \frac{(c+a)(c-a)}{AD + DC} = \frac{(c+a)(c-a)}{b},$$

from which $AD - DC$ may be computed. Then

$$AD = \tfrac{1}{2}[b + (AD - DC)],$$

and

$$DC = \tfrac{1}{2}[b - (AD - DC)].$$

If either AD or DC is negative, it is exterior to the triangle; that is, the point D is on the line AC produced.

Having found AD and DC, the angles are found from the right triangles DBA and DBC.

1. Solve the triangle when $a = 27.103$, $b = 16.432$, $c = 12.511$.

$\therefore$ $c - a = -14.592$; $AD - DC = -35.178$; $AD = -9.373$; $DC = 25.805$.

Ans. $\alpha = 138° 31'.2$; $\gamma = 17° 48'.5$; $\beta = 23° 40'.3$.

In this example D lies to the left of A.

2. Solve the triangle when $a = 32.456$, $b = 41.724$, $c = 53.987$.

$\therefore$ $AD - DC = 44.607$; $AD = 43.1655$; $DC = -1.4415$.

Ans. $\alpha = 36° 54'.7$; $\gamma = 92° 32'.7$; $\beta = 50° 32'.6$.

3. Solve the triangle when $a = 0.14679$, $b = 0.10433$, $c = 0.04796$.

$\therefore$ $AD - DC = -0.18448$; $AD = -0.040075$; $DC = +0.144405$.

Ans. $\alpha = 146° 40'.75$; $\gamma = 10° 21'.0$; $\beta = 22° 58'.25$.

106. Case IV. Given b, c, α. — In the triangle ADB, knowing c and α, find AD and DB. Then in the triangle DBC we know DB and $DC = b - AD$, so that we can compute a and γ.

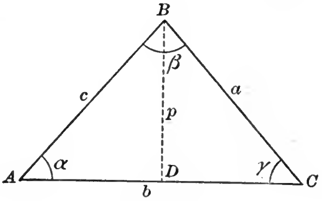


FIG. 67.

1. Solve the triangle when $b = 1143.7$, $c = 1822.4$, $\alpha = 15° 6'.4$.

$\therefore$ $AD = 1759.5$; $DB = 474.96$; $DC = -615.8$.

Ans. $\gamma = 142° 21'.5$; $a = 777.68$; $\beta = 22° 32'.1$.

The negative value of DC shows that D is to the right of C.

2. Solve the triangle when $b = 19.937$, $c = 62.475$, $\alpha = 130° 9'.4$.

$\therefore$ $AD = -40.288$; $DC = 60.225$.

Ans. $\gamma = 38° 24'.5$; $\beta = 11° 26'.1$; $a = 73.857$, or 76.858.

Note that α is obtuse.

3. Solve the triangle when $a = 101.47$, $c = 99.367$, $\beta = 47° 48'.2$.

Ans. $\gamma = 64° 44'.6$; $\alpha = 67° 27'.2$; $b = 81.394$.

AREAS OF TRIANGLES.

107. Given Two Sides and the Included Angle (b, c, α).— Represent the area by A. From geometry, in Fig. 67,

$$A = \tfrac{1}{2} pb.$$

But
$$p = c \sin \alpha.$$

$$\therefore A = \frac{1}{2} bc \sin \alpha, \qquad (1)$$

or, the area of a triangle is equal to half the product of the two sides multiplied by the sine of their included angle.

108. Given One Side and the Three Angles (b, α, β, γ).—Substitute in (1), Art. 107, the value of c found from the sine proportion,

$$c = \frac{b \sin \gamma}{\sin \beta},$$

giving

$$A = \frac{b^2}{2} \cdot \frac{\sin \alpha \sin \gamma}{\sin \beta}. \qquad (1)$$

109. Given the Three Sides (a, b, c).—We have

$$A = \tfrac{1}{2} bc \sin \alpha = bc \sin \tfrac{1}{2} \alpha \cos \tfrac{1}{2} \alpha.$$

From (2) and (3), Art. 99, we have

$$A = bc\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{s(s-a)}{bc}} = \sqrt{s(s-a)(s-b)(s-c)}. \qquad (1)$$

110. Given Two Sides and the Angle Opposite One of them (b, c, β).—First find γ by the formula

$$\sin \gamma = \frac{c}{b} \sin \beta.$$

Then

$$\alpha = 180° - (\beta + \gamma),$$

and

$$A = \tfrac{1}{2} bc \sin \alpha.$$

EXAMPLES.

1. Find the area when $b = 0.14367$, $c = 0.11412$, $\alpha = 42° 14'.6$.

$$\begin{aligned} \log b &= 9.15737 \\ \log c &= 9.05737 \\ \log \sin \alpha &= 9.82755 \\ \text{col } 2 &= 9.69897 \\ \hline \log A &= 7.74126 \\ A &= 0.0055114 \end{aligned}$$

2. Find the area when $a = 3.4356$, $\alpha = 17° 43'.4$, $\gamma = 60° 35'.7$.

$$\therefore\ \beta = 101° 40'.9.$$

$$\begin{aligned} \log a^2 = 2 \log a &= 1.07200 \\ \text{col } 2 &= 9.69897 \\ \log \sin \beta &= 9.99091 \\ \log \sin \gamma &= 9.94010 \\ \text{col} \sin \alpha &= 0.51652 \\ \hline \log A &= 1.21850 \\ A &= 16.539 \end{aligned}$$

3. Find the area when $a = 0.0093146$, $b = 0.0176530$, $c = 0.0095768$.

$$\begin{aligned} 2s &= 0.0365444 \\ \hline s &= 0.0182722 \\ s - a &= 0.0089576 \\ s - b &= 0.0006192 \\ s - c &= 0.0086954 \\ \hline \text{sum} &= 0.0365444 \end{aligned}$$

a check.

$$\begin{aligned} \log s &= 8.26179 \\ \log (s-a) &= 7.95219 \\ \log (s-b) &= 6.79183 \\ \log (s-c) &= 7.93929 \\ \hline 2)\,&10.94510 - 20 \\ \log A &= 5.47255 - 10 \\ A &= 0.000029686 \end{aligned}$$

4. Find the area when $a = 9.4672,\ c = 14.433,\ \alpha = 11° 14'.3$.

$\log c = 1.15936$	$\log a = 0.97622$	$\log a = 0.97622$
$\log \sin \alpha = 9.28979$	$\log c = 1.15936$	$\log c = 1.15936$
$\text{col } a = 9.02378$	$\text{col } 2 = 9.69897$	$\text{col } 2 = 9.69897$
$\log \sin \gamma = 9.47293$	$\log \sin \beta = 9.67899$	$\log \sin \beta' = 9.02259$
$\gamma = 17° 17'.1$	$\log A = 1.51354$	$\log A' = 0.85714$
$\gamma' = 162° 42'.9$	$A = 32.624$	$A' = 7.1968$
$\therefore\ \beta = 151° 28'.6$		
$\beta' = 6° 2'.8$		

Note that $\log A$ and $\log A'$ can be found by adding $\log \sin \beta$ and $\log \sin \beta'$ respectively to $\log a + \log c + \text{col } 2$, a shorter method than that given in this example.

5. Find the area when $a = 0.013456,\ b = 0.023678,\ \alpha = 40° 31'.4$.

Ans. 0.00010351.

6. Find the area when $c = 43.145,\ \alpha = 40° 40'.3,\ \beta = 60° 30'.3$. *Ans.* 538.19.

7. Find the area when $a = 1.4142,\ b = 1.6735,\ c = 2.8533$. *Ans.* 0.83826.

8. Find the area when $a = 14.135,\ c = 23.647,\ \alpha = 33° 17'.3$.

Ans. 164.61 or 91.948.

111. Illustrative Examples. — The *bearing* of a line is the angle it makes with the magnetic meridian, shown by the magnetic needle. The letter indicating whether the line is measured north or south of the point of beginning is written, then the number of degrees and minutes in the angle, and then the letter indicating whether the line lies to the east or to the west of the magnetic meridian. Thus, if the bearing of the line AB is S. 60° W., the line is measured from A to the west of south by an angle of 60°.

The distances and the angles given in the examples are horizontal unless otherwise specified.

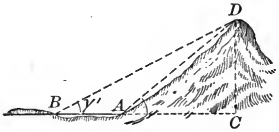


Fig. 68.

1. From a point on a horizontal plane the angle of elevation to the top of a crag is 40° 28'.6, and 4163.2 feet farther away in the same vertical plane the angle is 28° 50'.4. Find the distances from the points to the top of the crag, and its height above the horizontal plane.

$\therefore\ BD = 13399$ feet; $AD = 9956.2$ feet; $CD = 6463.0$ feet;
$BC = 11737$ feet; $AC = 7573.2$ feet.

2. A tower 160.43 feet high is situated at the top of a hill (Fig. 69); 600 feet down the hill the angle between the surface of the hill and a line to the top of the tower is $8° 40'.4$. Find the distance to the top of the tower, and the inclination of the ground to a horizontal plane.

$\therefore\ ABC = 136° 59'.7;\ AC = 725.60$ feet; $DAB = 46° 59'.7$.

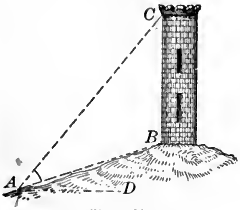


FIG. 69.

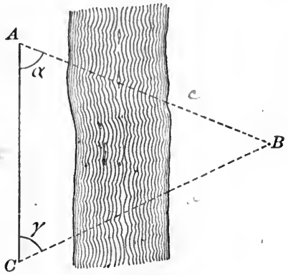


FIG. 70.

3. To find the horizontal distance from a point A to an inaccessible point B (Fig. 70), the horizontal distance AC and the angles α and γ were measured and found to be 1042.3 feet, $72° 9'.4$, and $14° 13'.7$, respectively.

$\therefore\ AB = 256.69$ feet; $CB = 994.15$ feet.

4. To find the distance between two points A and B not visible from each other (Fig. 71). — Select a third point C from which A and B are visible, and measure the distances $CA = 444.38$ feet, $CB = 222.76$ feet, and the angle $ACB = 17° 17'.6$. *Ans.* $AB = 240.97$ feet.

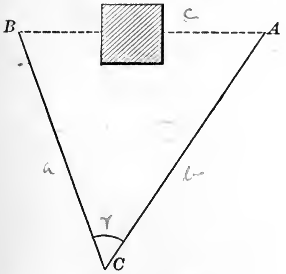


FIG. 71.

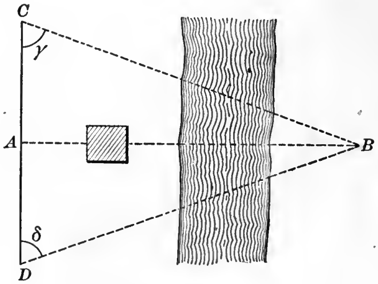


FIG. 72.

5. To find the distance from a point A to another point B, the latter being inaccessible and invisible from A (Fig. 72). — Select two points C and D so that C, A, and D shall be in the same straight line, A and B being visible both from C and from D. From measurement it is found that $CA = 456.72$ feet, $AD = 490.74$ feet, $\gamma = 71° 22'.7$, $\delta = 36° 19'.4$.

$\therefore\ CB = 589.10$ feet; $DB = 942.475$ feet; $AB = 619.51$, or 619.53 feet.

6. To find the elevation of the top of a church steeple D (Fig. 73) above the horizontal plane ACB, and the distances of the steeple from A and B. —Let the horizontal distance $AB = 435.53$ feet, the horizontal angles $\alpha = 140° 40'.2$ and $\beta = 10° 7'.6$, and the vertical angles $\gamma = 32° 45'.6$ and $\gamma' = 10° 7'.3$.

$\therefore$ $AC = 156.95$ feet; $BC = 565.74$ feet; $CD = 100.99$, or 101.00 feet.

The agreement of the values of CD is a check upon the observed angles and upon the computations.

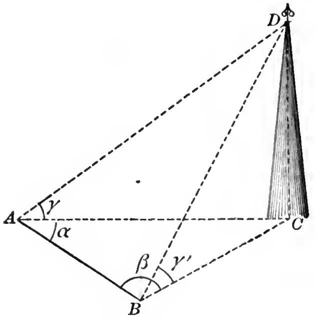


FIG. 73.

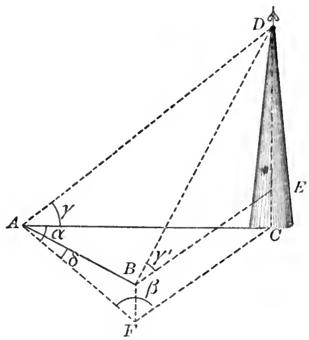


FIG. 74.

7. To find the elevation of the top of a church steeple D (Fig. 74) above the two points A and B, not in the same horizontal plane, the inclined distance from A to B, and its angle of inclination δ to a horizontal plane being measured, as well as the angles α, β, γ, and γ', shown in the preceding example. — Let $AB = 134.70$ feet, $\delta = 3° 2'.7$, $\alpha = 43° 14'.8$, $\beta = 63° 17'.5$, $\gamma = 56° 36'.6$, $\gamma' = 62° 17'.3$.

[First find the horizontal distance AF and the vertical distance FB in the right triangle AFB; then solve the horizontal triangle AFC; and then find CD and ED from the right triangles ACD and BED respectively.]

$\therefore$ $AF = 134.51$ feet; $FB = 7.1553$ feet; $FC = BE = 96.135$ feet;

$AC = 125.34$ feet; $CD = 190.17$ feet; $ED = 183.02$ feet.

Check: $CD = FB + ED.$

8. To find the distance between two inaccessible points A and B. — Select two points C and D from which both A and B can be seen, and measure

$CD = 456.32$ feet, $\alpha = 30° 40'.6$,

$\beta = 40° 14'.8$, $\gamma = 35° 16'.4$,

$\delta = 56° 47'.4$.

$\therefore$ $AD = 449.09$ feet; $AC = 274.41$ feet;

$BD = 398.66$ feet; $BC = 616.66$ feet;

$AB = 405.57$, or 405.58 feet.

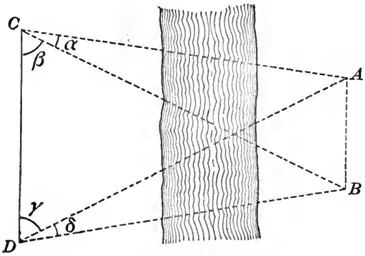


FIG. 75.

9. To find the distance between two inaccessible points A and B, both being visible from only one accessible point C.—Select a point D from which A and C are visible, and another point E from which B and C are visible. From measurement

$CD = 943.37$ feet, $CE = 673.33$ feet,

$\alpha = 72° 9'.3$, $\beta = 60° 17'.9$,

$\gamma = 32° 14'.6$, $\delta = 67° 33'.9$,

$\epsilon = 19° 14'.7$.

$\therefore$ $CA = 1217.0$ feet; $CB = 222.28$ feet;

$AB = 1035.8$ feet.

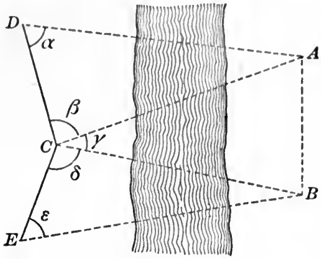


FIG. 76.

10. To find the distance between two inaccessible points A and B, there being no accessible point from which both A and B are visible (Fig. 77). — Select the points C, D, E, and F so that A, C, and E shall be visible from D, and D, F, and B from E. Measure the angles α, β, γ, δ, ϵ, and θ, and the distances CD, DE, and EF. Show how AB may be found from the data thus obtained.

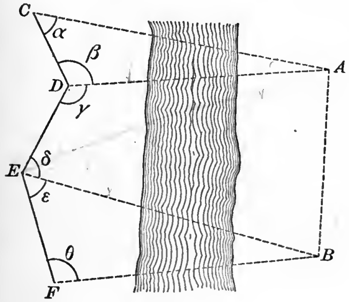


FIG. 77.

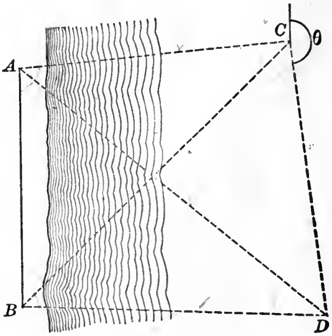


FIG. 78.

11. Two points A and B, 8763.6 feet apart (Fig. 78), are situated at the sea level in the same north and south line; a vessel is seen at C, and an hour later at D. The required quantities are AC, BC, AD, BD, CD, and the angle that CD makes with the north and south line, having measured $BAC = 120° 30'.6$, $BAD = 30° 14'.4$, $ABC = 40° 18'.8$, $ABD = 140° 28'.2$.

$\therefore$ $AC = 17260$ feet; $BC = 22985$ feet; $AD = 34552.5$ feet; $BD = 27340$ feet;

$ACD = 63° 14'.5$; $ADC = 26° 29'.3$; $BCD = 44° 3'.8$; $BDC = 35° 46'.8$;

$CD = 38696$, 38697, or 38699 feet;

$\theta = 360° - BAC - ACB - BCD = 176° 15'.0$,

or $= ABD + BDA + ADC = 176° 14'.9$.

12. In measuring the line from A to B, whose direction was known, it was necessary to pass an obstacle at F (Fig. 79). A distance $CD = 144.31$ feet was measured, making an angle $\gamma = 19° 53'.4$ with AB, and the angle $\delta = 140° 10'.3$

was laid off with the transit. It is required to find the distance DE to the line, the distance CE, and the angle ϵ, in order that the line AC may be prolonged.

Ans. $CE = 271.06$ feet; $DE = 143.98$ feet; $\epsilon = 160° 3'.7$.

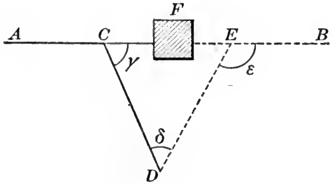


Fig. 79.

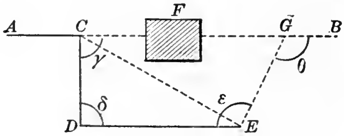


Fig. 80.

13. In passing an obstacle at F it was necessary to use the broken line $CDEG$ (Fig. 80). The distances CD and DE and the angles γ, δ, and ϵ were measured. It is required to find the distance EG to the line AB, the distance CG, and the angle θ, when $CD = 100.37$ feet, $DE = 94.367$ feet, $\gamma = 80°$, $\delta = 101° 19'.8$, and $\epsilon = 110°$.

$\therefore DCE = 37° 53'.3$; $DEC = 40° 46'.9$;

$CE = 150.67$ feet; $EG = 108.46$ feet;

$CG = 151.22$ feet; $\theta = 111° 19'.8$.

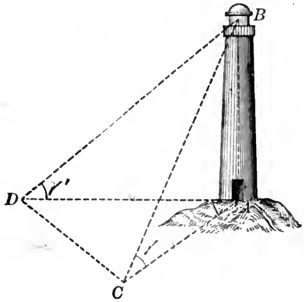


Fig. 81.

14. From the top of a lighthouse AB, 200 feet above the sea level, the angle of depression to a ship was $\gamma = 10° 14'.3$; an hour later it was $\gamma' = 11° 10'.6$; the horizontal angle between the directions of the ship at the two instants was $\alpha = 127° 14'.4$. Find the distance sailed by the ship.

$\therefore AC = 1107.3$ feet; $AD = 1012.2$ feet;

$CD = 1899.3$ feet.

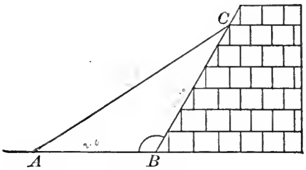


Fig. 82.

15. A ladder 52 feet long is set 20 feet in front of an inclined buttress, and reaches 46 feet up its face. Find the inclination of the face of the buttress.

Ans. $ABC = 95° 51'.8$, or $95° 51'.9$.

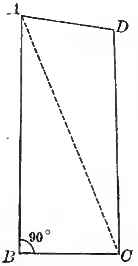


Fig. 83.

16. The sides of a city block measured $AB = 423.24$, $BC = 162.36$, $CD = 420.81$, and $DA = 160.62$ feet, the first two sides being perpendicular to each other. Find the angles between the other sides.

$\therefore AC = 453.31$ feet; $BCA = 69° 0'.8$;

$BAC = 20° 59'.2$; $ACD = 20° 45'.0$;

$CAD = 68° 8'.8$; $CDA = 91° 6'.4$;

$BCD = 89° 45'.8$; $BAD = 89° 8'.0$.

17. A ship B is 12 miles S. 45° W. of a lighthouse A, and sails S. 50° E. to C, a distance of 15 miles. Find its distance from the lighthouse.

Ans. $AC = 18.374$, or $18.37\bar{5}$ miles.

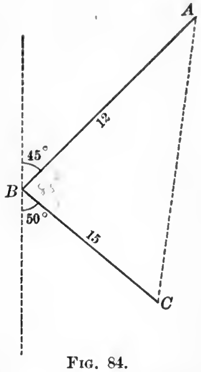


FIG. 84.

18. In surveying a field a thick wood prevents the measurement of the angle ABD and of the distance BD. The angle $ABC = 70° 14'.6$ is measured, a line BC is run 743.86 feet, the angle BCD is found to be 62° 14'.4, and the distance CD to be 912.82 feet.

$\therefore\ CBD = 68° 28'.1$; $CDB = 49° 17'.5$;

$BD = 868.34$, 868.36, or 868.38 feet;

$ABD = 138° 42'.7$.

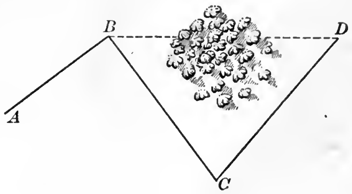


FIG. 85.

19. The distance OE and its bearing $E'OE$ are required, the engineer having measured the distances a, b, c, d, and e and their respective bearings, N. 30° W., S. 60° E., N. 20° E., N. 40° W., and N. 50° E.

$$OE' = OA' - B'A' + B'C' + C'D' + D'E'$$
$$= a \cos 30° - b \cos 60° + c \cos 20° + d \cos 40° + e \cos 50°.$$

$$E'E = -AA' + B''B + C''C - DD'' + E''E$$
$$= -a \sin 30° + b \sin 60° + c \sin 20° - d \sin 40° + e \sin 50°.$$

Then $\left.\begin{array}{l} OE \cos E'OE = OE', \\ OE \sin E'OE = E'E; \end{array}\right\}$

whence OE and $E'OE$ can be found. Then the quadrant of $E'OE$ fixes the direction of the line OE; thus, if $E'OE = 40°$, the bearing is N. 40° E.; if $E'OE = 110°$, the bearing is S. 70° E.; if $E'OE = 230°$, the bearing is S. 50° W.; if $E'OE = 310°$, the bearing is N. 50° W.

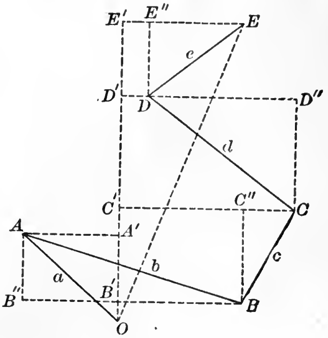


FIG. 86.

20. At a certain point the angles of elevation of the base of a vertical tower and of its top are α and β respectively, the height of the tower being h feet. Prove that the horizontal distance from the point to the tower is

$h \cos \alpha \cos \beta \operatorname{cosec}(\beta - \alpha)$, and that the elevation of its top above the point is $h \cos \alpha \sin \beta \operatorname{cosec}(\beta - \alpha)$.

21. At the top of a vertical tower whose height is h, the angles of depression to two points M and N in the same vertical plane with the tower were α and β respectively ($\beta > \alpha$), the points being in the same horizontal plane with the base of the tower. Prove that the distance MN is $h \sin(\beta - \alpha) \operatorname{cosec} \alpha \operatorname{cosec} \beta$.

22. Two points M and N in a horizontal plane are in the same vertical plane with a tower. The angle of elevation of the top of the tower from M is α, and from N it is β, β being greater than α. Prove that the horizontal distance of the tower from N is $MN \sin \alpha \cos \beta \operatorname{cosec}(\beta - \alpha)$.

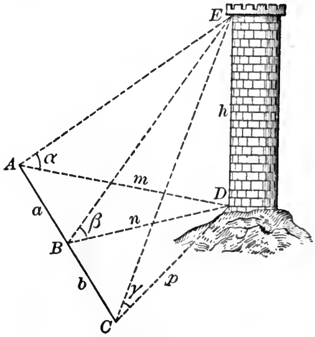


FIG. 87.

23. Three points, A, B, and C, are in the same horizontal line, the distances AB and BC being a and b feet respectively (Fig. 87). The angles of elevation of the top of a tower measured at A, B, and C were α, β, and γ respectively. Find the elevation of the top of the tower above the horizontal plane through the points, and the horizontal distances of the tower from the three points.

$$m = h \cot \alpha; \quad n = h \cot \beta; \quad p = h \cot \gamma;$$

$$m^2 = a^2 + n^2 - 2\,an \cos ABD;$$

$$p^2 = b^2 + n^2 + 2\,bn \cos ABD;$$

$$\therefore \frac{a^2 + n^2 - m^2}{2\,an} = \frac{p^2 - b^2 - n^2}{2\,bn};$$

$$\therefore h^2 = \frac{ab(a+b)}{a(\cot^2 \gamma - \cot^2 \beta) + b(\cot^2 \alpha - \cot^2 \beta)}.$$

24. In Fig. 88 the distances a and b and the angles α, β, and γ are known, and the distance $BC = x$ is required, $ABCD$ being an inaccessible straight line.

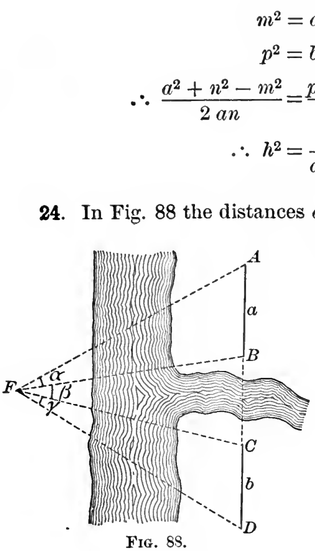


FIG. 88.

$$\frac{a}{\sin \alpha} = \frac{FB}{\sin A}; \quad \frac{a+x}{\sin(\alpha + \beta)} = \frac{FC}{\sin A};$$

$$\therefore \frac{FB}{FC} = \frac{a}{a+x} \cdot \frac{\sin(\alpha+\beta)}{\sin \alpha}. \qquad (1)$$

$$\frac{b}{\sin \gamma} = \frac{FC}{\sin D}; \quad \frac{b+x}{\sin(\beta+\gamma)} = \frac{FB}{\sin D};$$

$$\therefore \frac{FC}{FB} = \frac{b}{b+x} \cdot \frac{\sin(\beta+\gamma)}{\sin \gamma}. \qquad (2)$$

Multiplying (1) and (2), we have

$$(a+x)(b+x) \sin \alpha \sin \gamma = ab \sin(\alpha+\beta) \sin(\beta+\gamma),$$

from which x may be found, since the equation is a quadratic in x.

25. Two points A and B in the same vertical plane with the top of a tower are on a sidehill whose angle of inclination to a horizontal plane is δ, the inclined distance AB being a feet. The angles of elevation of the top of the tower were measured at A and B, and found to be α and β. Prove that the horizontal distance of the top of the tower from B is

$$a(\cos\delta\tan\alpha - \sin\delta)\cos\alpha\cos\beta\operatorname{cosec}(\beta-\alpha),$$

and that the elevation of the top above B is

$$a(\cos\delta\tan\alpha - \sin\delta)\cos\alpha\sin\beta\operatorname{cosec}(\beta-\alpha).$$

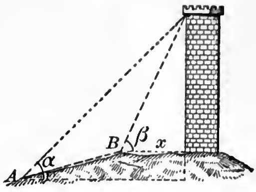


FIG. 89.

26. In a hydrographical survey, the distances between three points, A, B, and C, on the shore having been determined, the observer in the boat P measures the angles δ and ϵ subtended by AB and BC. It is required to find the distances of the boat from the three points.

(1) GRAPHICAL SOLUTION. — Construct on AB the segment of a circle APB that shall contain the measured angle δ, and on BC the segment of a circle BPC that shall contain the angle ϵ. Their point of intersection P will be the position of the boat. There are four possible solutions, only one being shown in the figure.

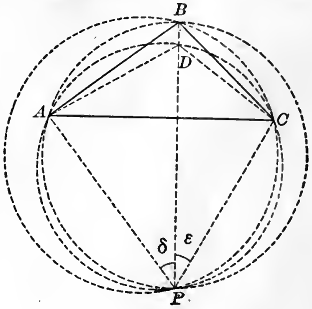


FIG. 90.

(2) ANALYTICAL SOLUTION.—Let $ADCP$ be the circle through A, C, and P. Then $DAC=\epsilon$, and $DCA=\delta$. Hence in the triangle ADC we know one side AC and the three angles; find AD and CD. In the triangle ABC we know the three sides; find the three angles. In the triangle DAB we know two sides and the angle $DAB = CAB - CAD$; find ABD. Then in the triangle ABP we know one side and the three angles; find AP and BP. Also, compute DBC from the triangle DCB, and then BP and CP from the triangle BPC. The values of BP should agree.

In the following examples find the last three elements, the first three being given:

27. $a = 1.0431$, $\beta = 4°\,4'.4$, $\gamma = 22°\,3'.6$.
$\therefore\ \alpha = 153°\,52'.0$; $b = 0.16822$; $c = 0.88942$.

28. $a = 103.37$, $\alpha = 10°\,11'.3$, $\beta = 83°\,43'.6$.
$\therefore\ \gamma = 86°\,5'.1$; $b = 580.89$; $c = 583.02$.

29. $c = 74.344$, $\alpha = 105°\,6'.7$, $\beta = 60°\,14'.4$.
$\therefore\ \gamma = 14°\,38'.9$; $a = 283.82$; $b = 255.21$.

30. $c = 0.047365$, $\beta = 40°\,7'.7$ $\gamma = 39°\,41'.9$.
$\therefore\ \alpha = 100°\,10'.4$; $a = 0.072990$; $b = 0.047792$.

31. $c = 4.4479$, $\alpha = 11°\,11'.3$, $\gamma = 57°\,37'.4$.
$\therefore\ \beta = 111°\,11'.3$; $a = 1.0219$; $b = 4.9106$.

32. $b = 143.97, \quad \beta = 30° \, 36'.8, \quad \gamma = 107° \, 15'.5.$

$\therefore \; \alpha = 42° \, 7'.7; \quad a = 189.64; \quad c = 269.98.$

33. $b = 10.467, \quad c = 1.4321, \quad \beta = 114° \, 10'.3.$

$\therefore \; \gamma = 7° \, 10'.2; \quad \alpha = 58° \, 39'.5; \quad a = 9.79875.$

34. $a = 0.67375, \quad b = 0.43213, \quad \alpha = 147° \, 11'.3.$

$\therefore \; \beta = 20° \, 20'.2; \quad \gamma = 12° \, 28'.5; \quad c = 0.26858.$

35. $a = 1.4742, \quad c = 0.97674, \quad \alpha = 25° \, 19'.9.$

$\therefore \; \gamma = 16° \, 28'.1; \quad \beta = 138° \, 12'.0; \quad b = 2.2966.$

36. $a = 943.42, \quad b = 647.15, \quad \alpha = 104° \, 6'.9.$

$\therefore \; \beta = 41° \, 42'.0; \quad \gamma = 34° \, 11'.1; \quad c = 546.59.$

37. $a = 0.10321, \quad c = 0.047323, \quad \alpha = 45° \, 9'.7.$

$\therefore \; \gamma = 18° \, 58'.4; \quad \beta = 115° \, 51'.9; \quad b = 0.13097.$

38. $a = 4.4321, \quad c = 5.4763, \quad \gamma = 100° \, 11'.9.$

$\therefore \; \alpha = 52° \, 48'.1; \quad \beta = 27° \, 0'.0; \quad b = 2.5261.$

39. $c = 23.111, \quad b = 19.476, \quad \gamma = 47° \, 16'.7.$

$\therefore \; \beta = 38° \, 15'.0; \quad \alpha = 94° \, 28'.3; \quad a = 31.363.$

40. $a = 0.11111, \quad c = 0.12767, \quad \alpha = 23° \, 15'.6.$

$\therefore \; \gamma = 26° \, 59'.1; \quad \beta = 129° \, 45'.3; \quad b = 0.21630;$

$\gamma' = 153° \, 0'.9; \quad \beta' = 3° \, 43'.5; \quad b' = 0.018279.$

41. $b = 1.4326, \quad c = 1.3671, \quad \gamma = 44° \, 17'.3.$

$\therefore \; \beta = 47° \, 1'.9; \quad \alpha = 88° \, 40'.8; \quad a = 1.9574;$

$\beta' = 132° \, 58'.1; \quad \alpha' = 2° \, 44'.6; \quad a' = 0.093706.$

42. $a = 46.703, \quad b = 57.147, \quad \alpha = 19° \, 17'.7.$

$\therefore \; \beta = 23° \, 50'.9; \quad \gamma = 136° \, 51'.4; \quad c = 96.652;$

$\beta' = 156° \, 9'.1; \quad \gamma' = 4° \, 33'.2; \quad c' = 11.221.$

43. $a = 9.4327, \quad c = 10.4751, \quad \alpha = 63° \, 17'.3.$

$\therefore \; \gamma = 82° \, 45'.0; \quad \beta = 33° \, 57'.7; \quad b = 5.8990;$

$\gamma' = 97° \, 15'.0; \quad \beta' = 19° \, 27'.7; \quad b' = 3.5182.$

44. $a = 0.034337, \quad c = 0.062774, \quad \alpha = 9° \, 6'.7.$

$\therefore \; \gamma = 16° \, 49'.7; \quad \beta = 154° \, 3'.6; \quad b = 0.094846;$

$\gamma' = 163° \, 10'.3; \quad \beta' = 7° \, 43'.0; \quad b' = 0.029115.$

45. $a = 0.79797, \quad b = 0.46731, \quad \beta = 23° \, 19'.6.$

$\therefore \; \alpha = 42° \, 32'.5; \quad \gamma = 114° \, 7'.9; \quad c = 1.07705;$

$\alpha' = 137° \, 27'.5; \quad \gamma' = 19° \, 12'.9; \quad c' = 0.38841.$

46. $a = 37.456, \quad b = 43.987, \quad c = 13.498.$

$\therefore \; \frac{1}{2}\alpha = 26° \, 31'.0; \quad \frac{1}{2}\beta = 55° \, 7'.0; \quad \frac{1}{2}\gamma = 8° \, 22'.0.$

47. $a = 2.4568, \quad b = 2.4743, \quad c = 1.0047.$

$\therefore \; \frac{1}{2}\alpha = 38° \, 38'.0; \quad \frac{1}{2}\beta = 39° \, 36'.7; \quad \frac{1}{2}\gamma = 11° \, 45'.3.$

48. $a = 47.474, \quad b = 100.980, \quad c = 93.929.$

$\therefore \; \frac{1}{2}\alpha = 13° \, 56'.8; \quad \frac{1}{2}\beta = 42° \, 10'.2; \quad \frac{1}{2}\gamma = 33° \, 53'.0.$

49. $a = 14.567, \quad b = 9.4769, \quad c = 11.113.$

$\therefore \; \frac{1}{2}\alpha = 44° \, 50'.9; \quad \frac{1}{2}\beta = 20° \, 17'.5; \quad \frac{1}{2}\gamma = 24° \, 51'.5.$

50. $a = 2.1476$, $b = 1.9397$, $c = 3.4345$.
$\therefore\ \frac{1}{2}\alpha = 17° 22'.8$; $\frac{1}{2}\beta = 15° 29'.8$; $\frac{1}{2}\gamma = 57° 7'.3$.

51. $a = 115.03$, $b = 129.15$, $c = 112.06$.
$\therefore\ \frac{1}{2}\alpha = 28° 12'.9$; $\frac{1}{2}\beta = 34° 39'.2$; $\frac{1}{2}\gamma = 27° 7'.9$.

52. $b = 113.47$, $c = 227.79$, $\alpha = 19° 43'.4$.
$\therefore\ \beta = 17° 33'.8$; $\gamma = 142° 42'.8$; $a = 126.90$;
or $\log\tan x = 9.68278$; $a = 126.89$.

53. $a = 99.416$, $c = 90.432$, $\beta = 11° 7'.8$.
$\therefore\ \alpha = 110° 20'.4$; $\gamma = 58° 31'.8$; $b = 20.467$;
or $\log\tan x = 0.31110$; $b = 20.467$.

54. $a = 1.4342$, $b = 9.7672$; $\gamma = 109° 19'.6$.
$\therefore\ \alpha = 7° 31'.7$; $\beta = 63° 8'.7$; $c = 10.330$, or 10.331;
or $\log\tan x = 9.86498$; $c = 10.331$.

55. $a = 1003.7$, $b = 943.67$, $\gamma = 101° 19'.8$.
$\therefore\ \alpha = 40° 46'.9$; $\beta = 37° 53'.3$; $c = 1506.7$;
or $\log\tan x = 1.39930$; $c = 1506.7$.

56. $a = 222.76$, $b = 444.38$, $\gamma = 17° 17'.6$.
$\therefore\ \alpha = 15° 57'.0$; $\beta = 146° 45'.4$; $c = 240.97$;
or $\log\tan x = 9.63029$; $c = 240.97$.

57. $a = 363.24$, $b = 146.18$, $\gamma = 68° 14'.4$.
$\therefore\ \alpha = 88° 2'.6$; $\beta = 23° 43'.0$; $c = 337.55$, or 337.56;
or $\log\tan x = 0.07590$; $c = 337.55$.

PART TWO.

SPHERICAL TRIGONOMETRY.

CHAPTER VIII.

DEFINITIONS AND CONSTRUCTIONS.

112. Spherical Trigonometry treats of the relations between the face angles and the edge angles of a trihedral angle.

An *edge angle* is the angle between two of the three planes forming the trihedral angle; it is measured by the angle between the lines cut from the two planes by a plane perpendicular to the edge in which the two planes intersect.

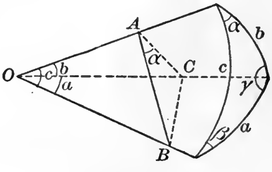


FIG. 91.

A *face angle* is the angle between two of the edges.

113. Representation of Trihedral Angles. — The relations between the elements of a trihedral angle are discussed by means of the spherical triangle formed by the intersections of the faces with a sphere described with any radius about the vertex as a center. The faces will cut arcs of great circles from the surface of the sphere, their angular measures being the same as those of the face angles; and the angles of the spherical triangle will correspond to the edge angles, each being measured by the angle between two lines lying in the planes of the faces and perpendicular to the line of intersection of the faces.

Hence, in the spherical triangle the sides correspond to the face angles, and the angles to the edge angles of the trihedral angle.

The lengths of the sides in linear measure will depend upon the radius of the sphere, and are computed, when the radius is known, by the proportion

$$360° : \alpha = 2\pi r : l, \tag{1}$$

where α is the number of degrees in the arc, and l is its length.

114. Limitation of Values. — We shall consider only those triangles in which each element is less than 180°. In the general spherical triangle the sides and angles may have values greater than 180°, but in such a case it is always possible to substitute for the triangle, in the computations, another in which each element shall be less than 180°.

115. Definitions and Relations. — A *great circle* is cut from the surface of a sphere by a plane passing through its center; its radius is equal to the radius of the sphere.

A *small circle* is cut from the surface by a plane not passing through the center; its radius is always less than the radius of the sphere.

Two planes passing through the center will intersect in a diameter of the sphere, and the two corresponding great circles will intersect at the ends of this diameter. Hence any two great circles will intersect at two points 180° apart.

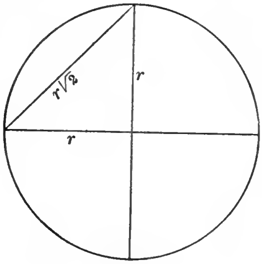


Fig. 92.

To describe a great circle on a sphere, separate the points of a pair of compasses by a distance equal to the chord of 90°, or $r\sqrt{2}$, and describe an arc about any point. If any other distance is used, a small circle will be described. The point used as the center is called the *pole* of the great circle; its distance from all points on the great circle is evidently 90°.

Any great circle passing through the pole of another great circle will be perpendicular to that great circle

Any two great circles drawn perpendicular to a third great circle will intersect in its pole.

A great circle perpendicular to two great circles will pass through the poles of both, and its plane will be perpendicular to the diameter joining the points of intersection of the two great circles.

The angle between two arcs of great circles is measured by the arc of a great circle described about the vertex as a pole, and limited by the sides, produced if necessary.

The shortest distance between two points on a sphere is the arc of the great circle passing through the points.

116. Constructions. — To find the pole of a given great circle: from any two points on the circle as poles, describe arcs of great circles, and their intersection will be the point required.

To draw a great circle through two points: find the pole as before, and describe the great circle.

To draw a great circle through a given point perpendicular to a given great circle: from the point as a pole describe an arc of a great circle; its point of intersection with the given circle will be the pole of the required circle. Or, find the pole of the given circle, and then draw the great circle through this pole and the given point.

To cut from a great circle an arc $n°$ long: separate the points of the compasses by a distance equal to the chord of $n°$, or $2\,r \sin \frac{1}{2}\, n°$, place the points on the great circle, and the arc intercepted will be the one required.

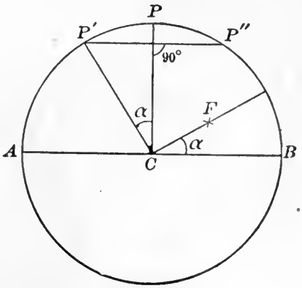


Fig. 93.

To construct a great circle passing through a given point and making a given angle with a given great circle: in Fig. 93, let ACB be the given great circle,* P its pole, F the given point, and α the given angle. With P as a pole, draw the small circle $P'P''$ such that the angular

* The planes of the great circles ACB and CF, and of the small circle $P'P''$, are perpendicular to the paper.

distance $PP' = a$; then the pole of the required great circle must be on this small circle. With F as a pole, describe an arc of a great circle cutting the small circle $P'P''$ in two points; these points will be the poles of two great circles through F, both of which satisfy the given conditions. Only the great circle CF, whose pole is P', is shown in the figure.

To construct a great circle making a given angle with a given great circle, the point of intersection being given: from the given point as a pole describe a great circle, lay off on it from the given circle a distance equal in angular measure to the given angle, and pass a great circle through the point thus found and the given point of intersection.

117. Definitions. — A right spherical triangle is one which has one angle equal to 90°; a birectangular triangle has two angles each equal to 90°; a trirectangular triangle has three angles each equal to 90°.

A quadrantal triangle has one side equal to a quadrant, or 90°; a biquadrantal triangle has two sides each equal to a quadrant; a triquadrantal triangle has three sides each equal to a quadrant.

A birectangular triangle is also biquadrantal, and a trirectangular triangle is also triquadrantal; and *vice versâ*.

118. The Polar Triangle of any triangle is constructed by describing arcs of great circles about the vertices of the original triangle as poles. Thus, about A, B, and C as poles, describe the arcs $B'C'$, $A'C'$, and $A'B'$, respectively; that triangle is called the polar in which the vertices A and A', B and B', C and C' are on the same side of BC, AC, and AB, respectively.

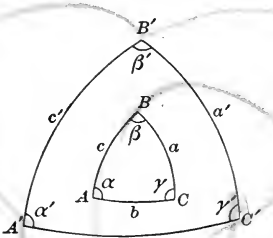


FIG. 94.

The vertices of the polar triangle will be the poles of the sides of the original triangle, so that either triangle will be the polar of the other.

The sides of a triangle are the supplements of the opposite angles of the polar, and the angles are the supplements of the opposite sides of the polar; $a' = 180° - \alpha$, $\alpha' = 180° - a$. Thus, if the angles of a triangle be 120°, 80°, and 60°, the opposite sides of the polar will be 60°, 100°, and 120°.

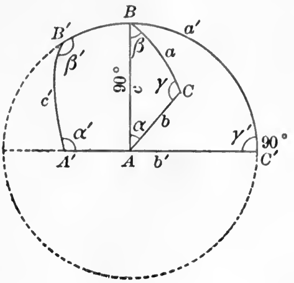


Fig. 95.

The polar of a quadrantal triangle is a right triangle, the angle in the polar opposite the quadrant being equal to the supplement of 90°; the polar of a biquadrantal triangle is birectangular; the polar of a triquadrantal triangle is trirectangular; and *vice versâ*.

The triquadrantal triangle is its own polar, each vertex being the pole of the opposite side.

119. In Any Spherical Triangle:

(1) Each side must be an arc of a great circle.

(2) Each side must be less than the sum of the other two.

(3) The greater side is opposite the greater angle, and conversely. Equal sides are opposite equal angles.

(4) The sum of the sides must be less than 360°.

(5) The sum of the angles must be greater than 180° and less than 540°.

120. Construction of Triangles. — (1) Given the three sides, a, b, c. — Draw an arc of a great circle and lay off on it an arc equal to one of the sides, as a. From the extremities of this arc as poles, with radii equal to the chords of b and c respectively, describe arcs of small circles with the compasses, and find their point of intersection. Join this point and the extremities of a by arcs of great circles, and the triangle will be constructed.

(2) Given the three angles, α, β, γ. — Find the sides of the polar triangle, construct it, and then construct the given triangle by using the vertices of the polar as poles.

(3) Given two sides and the included angle, a, b, γ. — Draw an arc of a great circle, and lay off on it an arc equal to one of the sides, as a. Pass an arc of a great circle through one extremity of a, making the angle γ with a, and lay off on it an arc equal to b. Join the extremities of a and b by an arc of a great circle, and the triangle will be constructed.

(4) Given two angles and their included side, α, β, c. — In the polar we know two sides and the included angle, and hence we can construct it by the method just given. Having the polar, we can then construct the required triangle.

Or, draw a great circle and lay off on it an arc equal to c; at the extremities of this arc, construct arcs of great circles making the angles α and β with c; their point of intersection will be the third vertex.

(5) Given two sides and the angle opposite one of them, a, b, α. — Draw any great circle ADA', and through any point on it, as A, draw a great circle making the angle $DAC = \alpha$ with it. On this circle lay off from A the distance $AC = b$. With C as a pole, describe a small circle whose radius is equal to the chord of a, using the compasses; pass arcs of great circles through C, and the points B and B' where this small circle intersects the first great circle ADA', and the triangle will be constructed.

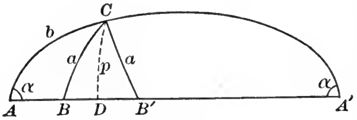


Fig. 96.

There will be, in general, two points of intersection, and there may therefore be two triangles that will satisfy the conditions of the problem. Only those triangles can be taken in which each side is less than 180°, *i.e.* both B and B' must lie on the arc ADA' between A and A', these points being 180° apart.

If α is acute, as in Fig. 96, a must be greater than p and less than the shorter of the two distances CA and CA' (b and $180° - b$) in order that there may be two solutions.

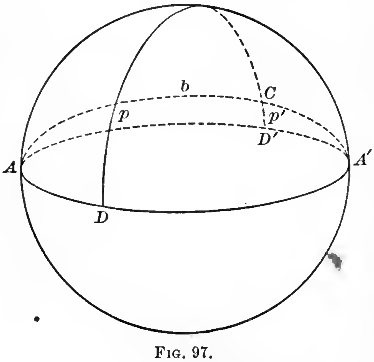


Fig. 97.

If α is obtuse, as in Fig. 97, CD' is the least and CD the greatest distance of C from $ADA'D'$, DCD' being perpendicular to $ADA'D'$. Therefore a must be less than p, in order that the small circle may cut $ADA'D'$; a must also be greater than the longer of the two distances CA and CA' (b and $180° - b$) in order that the two points of intersection may fall on the arc ADA'.

The conditions, therefore, for two solutions are :

$$\alpha \text{ acute}: \quad a > p, \quad a < b, \quad a < 180° - b.$$

$$\alpha \text{ obtuse}: \quad a < p, \quad a > b, \quad a > 180° - b.$$

Or, a must be intermediate in value between p and both b and $180° - b$.

If a is intermediate in value only between p and either b or $180° - b$, there will be one solution.

If a is not intermediate in value between p and either b or $180° - b$, no solution will be possible, but if $p = a$, there will be one solution — a right triangle.

(6) Given two angles and the side opposite one of them, α, β, a. — In the polar triangle we know two sides and the angle opposite one of them, and we can construct it; having the polar we can construct the required triangle.

As the polar triangle may admit of two solutions, there may be two solutions of the problem.

CHAPTER IX.

GENERAL FORMULAS.

121. The Cosine of Any Side of a Spherical Triangle *is equal to the product of the cosines of the other two sides, increased by the product of the sines of these two sides multiplied by the cosine of their included angle.* — Let the plane BAC be perpendicular to OA at any point A, and let AC, BC, and BA be its intersections with the faces of the trihedral angle.

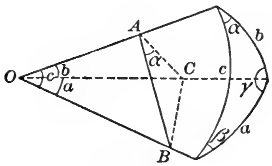


FIG. 98.

Then $BAC = \alpha$, and AB and AC are perpendicular to OA, *i.e.* OAB and OAC are triangles right-angled at A.

In the triangle BAC we have

$$BC^2 = AB^2 + AC^2 - 2\,AB \cdot AC \cos\alpha.$$

In the triangle BOC,

$$BC^2 = OB^2 + OC^2 - 2\,OB \cdot OC \cos a.$$

Equating the values of BC^2, and transposing,

$$2\,OB \cdot OC \cos a = OB^2 - AB^2 + OC^2 - AC^2 + 2\,AB \cdot AC \cos\alpha.$$

In the right triangles OAB and OAC,

$$OB^2 - AB^2 = OA^2, \text{ and } OC^2 - AC^2 = OA^2.$$

$$\therefore\ 2\,OB \cdot OC \cos a = OA^2 + OA^2 + 2\,AB \cdot AC \cos\alpha;$$

or $$OB \cdot OC \cos a = OA^2 + AB \cdot AC \cos\alpha.$$

$$\therefore\ \cos a = \frac{OA}{OC} \cdot \frac{OA}{OB} + \frac{AC}{OC} \cdot \frac{AB}{OB} \cos\alpha;$$

or $$\mathbf{\cos a = \cos b \cos c + \sin b \sin c \cos \alpha.} \qquad (1)$$

In this proof b and c are assumed to be less than 90°, while a and α may have any values less than 180°. The formula is true, however, when either b or c, or both b and c, exceed 90°.

If, in the triangle represented by the full lines (Fig. 99), b is greater than 90°, then in the dotted triangle formed by completing the arcs of great circles, the two sides are $180° - b$, and c, both less than 90°, and the other side and its opposite angle are $180° - a$, and $180° - \alpha$. Hence we can apply (1) to the dotted triangle, giving

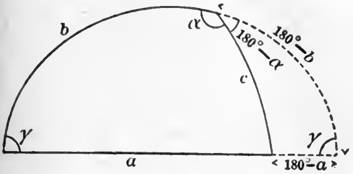


FIG. 99.

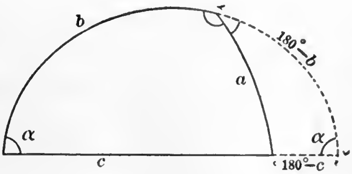


FIG. 100.

$$\cos(180° - a) = \cos(180° - b)\cos c + \sin(180° - b)\sin c\cos(180° - \alpha).$$

$$\therefore\ -\cos a = -\cos b\cos c - \sin b\sin c\cos\alpha.$$

$$\therefore\ \cos a = \cos b\cos c + \sin b\sin c\cos\alpha. \qquad \text{Q.E.D.}$$

If both b and c are greater than 90°, as in Fig. 100, then in the dotted triangle the two sides are $180° - b$, and $180° - c$, and the other side and its opposite angle are a and α.

$$\therefore\ \cos a = \cos(180° - b)\cos(180° - c) + \sin(180° - b)\sin(180° - c)\cos\alpha$$
$$= (-\cos b)(-\cos c) + \sin b\sin c\cos\alpha.$$

$$\therefore\ \cos a = \cos b\cos c + \sin b\sin c\cos\alpha. \qquad \text{Q.E.D.}$$

Therefore the formula is always true when each of the elements of the triangle is less than 180°.

No assumption, then, has been made concerning any element that is not true for all the others. We may therefore change any angle to another, as α to β, if at the same time we change the sides opposite, as a to b, making also the reverse changes, b to a and β to α, in the formula; for this is equivalent to changing the *names* arbitrarily assigned to the sides and angles. Thus, to permute (1) to find $\cos c$, we change a to c, α to γ, c to a, and γ to α, if they occur in the formula, while b and β will not be affected.

$$\therefore\ \cos c = \cos b\cos a + \sin b\sin a\cos\gamma.$$

If we assume that our triangle is right-angled, γ being equal to 90°, we can permute between a and b, and α and β, since no assumption is made concerning a and α that is not equally true concerning b and β. But we cannot permute between α and γ, because γ is assumed to be equal to 90°, while no such assumption is made concerning α.

Permuting (1) in the oblique-angled triangle, we have

$$\left.\begin{aligned}\cos a &= \cos b \cos c + \sin b \sin c \cos \alpha,\\ \cos b &= \cos a \cos c + \sin a \sin c \cos \beta,\\ \cos c &= \cos a \cos b + \sin a \sin b \cos \gamma.\end{aligned}\right\} \qquad (2)$$

Eq. (1) is called the fundamental equation of spherical trigonometry, since all the other formulas may be derived from it.

122. The Cosine of Any Angle of a Spherical Triangle *is equal to the product of the sines of the other two angles multiplied by the cosine of their included side, diminished by the product of the cosines of the other two angles.* — We have

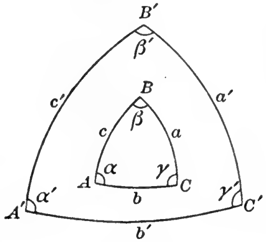


Fig. 101.

$$\cos a = \cos b \cos c + \sin b \sin c \cos \alpha. \qquad (1)$$

Since the angles of the polar triangle are the supplements of the sides opposite in the original triangle, and *vice versâ*, we have

$$a = 180° - \alpha',\ b = 180° - \beta',\ c = 180° - \gamma',\ \alpha = 180° - a'.$$

Substituting in (1),

$$\cos(180° - \alpha') = \cos(180° - \beta')\cos(180° - \gamma') + \sin(180° - \beta')\sin(180° - \gamma')\cos(180° - a'),$$

or $$-\cos\alpha' = (-\cos\beta')(-\cos\gamma') + \sin\beta'\sin\gamma'(-\cos a').$$

$$\therefore\ \cos\alpha' = -\cos\beta'\cos\gamma' + \sin\beta'\sin\gamma'\cos a'.$$

This formula expresses a relation between the elements of the polar triangle; but, since the polar may be *any* spherical triangle, it expresses the value of the cosine of an angle of *any* spherical triangle. Dropping the primes and permuting,

$$\left.\begin{aligned}\cos\alpha &= -\cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a,\\ \cos\beta &= -\cos\alpha\cos\gamma + \sin\alpha\sin\gamma\cos b,\\ \cos\gamma &= -\cos\alpha\cos\beta + \sin\alpha\sin\beta\cos c.\end{aligned}\right\} \qquad (2)$$

123. The Sine Proportion. — *The sines of the sides of a spherical triangle are to each other as the sines of the opposite angles.*

*First Proof.** — From (1), Art. 121,

$$\sin b \sin c \cos \alpha = \cos a - \cos b \cos c.$$

$$\therefore \sin^2 b \sin^2 c \cos^2 \alpha = \cos^2 a + \cos^2 b \cos^2 c - 2 \cos a \cos b \cos c.$$

$$\therefore \sin^2 b \sin^2 c (1 - \sin^2 \alpha) = \cos^2 a + \cos^2 b \cos^2 c - 2 \cos a \cos b \cos c.$$

$$\therefore \sin^2 b \sin^2 c \sin^2 \alpha = \sin^2 b \sin^2 c - \cos^2 a - \cos^2 b \cos^2 c + 2 \cos a \cos b \cos c$$

$$= (1 - \cos^2 b)(1 - \cos^2 c) - \cos^2 a - \cos^2 b \cos^2 c + 2 \cos a \cos b \cos c$$

$$= 1 - \cos^2 b - \cos^2 c - \cos^2 a + 2 \cos a \cos b \cos c.$$

Dividing both sides by $\sin^2 a \sin^2 b \sin^2 c$,

$$\frac{\sin^2 \alpha}{\sin^2 a} = \frac{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}{\sin^2 a \sin^2 b \sin^2 c}.$$

Permuting,

$$\frac{\sin^2 \beta}{\sin^2 b} = \frac{1 - \cos^2 b - \cos^2 a - \cos^2 c + 2 \cos b \cos a \cos c}{\sin^2 b \sin^2 a \sin^2 c},$$

$$\frac{\sin^2 \gamma}{\sin^2 c} = \frac{1 - \cos^2 c - \cos^2 b - \cos^2 a + 2 \cos c \cos b \cos a}{\sin^2 c \sin^2 b \sin^2 a}.$$

The second members of the three equations are identical.

$$\therefore \frac{\sin^2 \alpha}{\sin^2 a} = \frac{\sin^2 \beta}{\sin^2 b} = \frac{\sin^2 \gamma}{\sin^2 c},$$

or

$$\frac{\sin \alpha}{\sin a} = \frac{\sin \beta}{\sin b} = \frac{\sin \gamma}{\sin c}. \qquad (1)$$

Second Proof.—From any point A on OA pass the planes ADC and ADB perpendicular to OC and OB, respectively, and let AD be their line of intersection. Then AD will be perpendicular to the plane BOC, being the intersection of two planes perpendicular to BOC, and DB and DC will be perpendicular to OB and OC, respectively. The triangles ADC and ADB will be right-angled at D, and

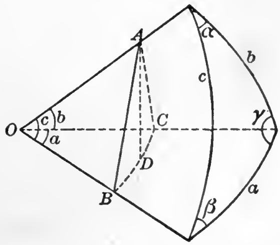


Fig. 102.

* Or find cos α from (1), Art. 121; then $\sin^2 \alpha = 1 - \cos^2 \alpha$, etc.

the triangles ACO and ABO will be right-angled at C and B. Also $ABD = \beta$ and $ACD = \gamma$. Then

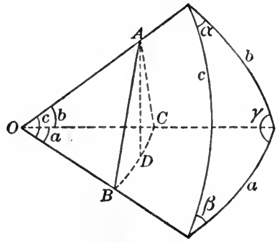


FIG. 103.

$$AD = AB \sin ABD = AB \sin \beta,$$

and $AD = AC \sin ACD = AC \sin \gamma$.

$$\therefore \quad AB \sin \beta = AC \sin \gamma$$

But $\qquad AB = OA \sin c,$

and $\qquad AC = OA \sin b.$

$$\therefore \quad OA \sin c \sin \beta = OA \sin b \sin \gamma.$$

$$\therefore \quad \frac{\sin \beta}{\sin b} = \frac{\sin \gamma}{\sin c}.$$

Permuting, $$\frac{\sin \alpha}{\sin a} = \frac{\sin \beta}{\sin b} = \frac{\sin \gamma}{\sin c}. \qquad (1)$$

124. Additional Formulas. — We have

$$\cos b = \cos a \cos c + \sin a \sin c \cos \beta.$$

$$\therefore \sin a \sin c \cos \beta = \cos b - (\cos b \cos c + \sin b \sin c \cos a) \cos c$$
$$= \cos b - \cos b \cos^2 c - \sin b \sin c \cos c \cos \alpha$$
$$= \cos b \sin^2 c - \sin b \sin c \cos c \cos \alpha.$$

$$\therefore \mathbf{\sin a \cos \beta = \cos b \sin c - \sin b \cos c \cos \alpha.} \qquad (1)$$

Applying (1) to the polar triangle and dropping the primes,

$$\mathbf{\sin \alpha \cos b = \cos \beta \sin \gamma + \sin \beta \cos \gamma \cos a.} \qquad (2)$$

Dividing (1), member for member, by the equation

$$\sin a \sin \beta = \sin b \sin \alpha,$$

we have $$\cot \beta = \frac{\cot b \sin c - \cos c \cos \alpha}{\sin \alpha};$$

$$\therefore \mathbf{\sin \alpha \cot \beta = \cot b \sin c - \cos c \cos \alpha.} \qquad (3)$$

Transposing,

$$\sin c \cot b = \sin \alpha \cot \beta + \cos c \cos \alpha.$$

Permuting,

$$\mathbf{\sin a \cot b = \cot \beta \sin \gamma + \cos a \cos \gamma.} \qquad (4)$$

Other formulas may be found by permuting (1), (2), and (3). Among these are the following:

from (1), $\quad \sin a \cos \gamma = \cos c \sin b - \sin c \cos b \cos \alpha;$ (5)

from (3), $\quad \sin \alpha \cot \gamma = \cot c \sin b - \cos b \cos \alpha,$ (6)

and $\quad \sin \gamma \cot \alpha = \cot a \sin b - \cos b \cos \gamma.$ (7)

CHAPTER X.

RIGHT SPHERICAL TRIANGLES.

125. Formulas for Right Spherical Triangles. — The following equations have been shown in Chap. IX to be true for all spherical triangles:

$$\cos c = \cos a \cos b + \sin a \sin b \cos \gamma, \tag{a}$$

$$\cos \alpha = -\cos \beta \cos \gamma + \sin \beta \sin \gamma \cos a, \tag{b}$$

$$\cos \gamma = -\cos \alpha \cos \beta + \sin \alpha \sin \beta \cos c, \tag{c}$$

$$\frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma}, \tag{d}$$

$$\sin a \cos \gamma = \cos c \sin b - \sin c \cos b \cos \alpha, \tag{e}$$

$$\sin \gamma \cot \alpha = \cot a \sin b - \cos b \cos \gamma. \tag{f}$$

By making $\gamma = 90°$ we get seven formulas applicable to right triangles, and by permuting these three others are found.

From (*a*), $\cos c = \cos a \cos b.$ (1)

From (*c*), $\cos c = \cot \alpha \cot \beta.$ (2)

From (*b*), $\cos \alpha = \sin \beta \cos a.$
Permuting, $\cos \beta = \sin \alpha \cos b.$ (3)

From (*e*), $\cos \alpha = \tan b \cot c.$
Permuting, $\cos \beta = \tan a \cot c.$ (4)

From (*d*), $\sin a = \sin c \sin \alpha.$
From (*d*), $\sin b = \sin c \sin \beta.$ (5)

From (*f*), $\sin b = \tan a \cot \alpha.$
Permuting, $\sin a = \tan b \cot \beta.$ (6)

126. Formulas for Right Spherical Triangles. Geometrical Proof. — Let OB be unity. From B pass the plane BAC perpendicular to OA. Then AB and AC are perpendicular to OA, and CB is perpendicular to OC.

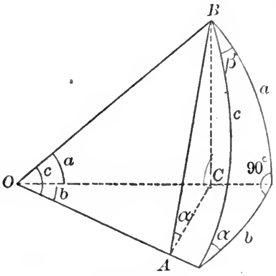


Fig. 104.

$\therefore$ $CB = \sin a$, $OC = \cos a$, $AB = \sin c$,

$OA = \cos c$, $CAB = \alpha$.

$$\therefore\ AC = CB \cot \alpha = \sin a \cot \alpha, \qquad (a)$$

$$AC = AB \cos \alpha = \sin c \cos \alpha, \qquad (b)$$

$$AC = OC \sin b = \cos a \sin b, \qquad (c)$$

$$AC = OA \tan b = \cos c \tan b. \qquad (d)$$

Equating these values of AC, we obtain the following formulas:

From (a) and (b), $\sin a = \sin c \sin \alpha.$
Permuting, $\sin b = \sin c \sin \beta.$ $\Big\}$ (5)

From (a) and (c), $\sin b = \tan a \cot \alpha.$
Permuting, $\sin a = \tan b \cot \beta.$ $\Big\}$ (6)

From (a) and (d), $\cos c = \dfrac{\sin a}{\tan b} \cot \alpha$;

$\therefore$ from (6), $\cos c = \cot \alpha \cot \beta.$ (2)

From (b) and (c), $\cos \alpha = \cos a \dfrac{\sin b}{\sin c}$;

$\therefore$ from the sine proportion or from (5)

$\cos \alpha = \cos a \sin \beta.$
Permuting, $\cos \beta = \cos b \sin \alpha.$ $\Big\}$ (3)

From (b) and (d), $\cos \alpha = \tan b \cot c.$
Permuting, $\cos \beta = \tan a \cot c.$ $\Big\}$ (4)

From (c) and (d), $\cos c = \cos a \cos b.$ (1)

127. Napier's Rules. — Napier, the celebrated Scotch mathematician, devised two rules by which the ten formulas connecting the elements of a right spherical triangle may be easily written.

He called the sides a and b about the right angle, and the complements of the two oblique angles and of the hypotenuse, the *parts* of the triangle, not considering the right angle as a part; the parts, then, are a, b, $90° - c$, $90° - \alpha$, $90° - \beta$, which we shall call a, b, c', α', β'. By reference to the circular figure, in which the parts are arranged in their order in going around the triangle, it will be seen that if any three parts are considered, either one will lie between the two others, being adjacent to both, or one will be separated from the other two by intermediate parts. Thus b lies between α' and a, being adjacent to both, and β' is separated from both α' and b. The part which lies between two others adjacent to it or is separated from both the others by intervening parts, is called the *middle part;* the two others, if adjacent to it, are called *adjacent parts*, and if separated, *opposite parts*. Thus, if c', α', and b are considered, α' is the middle part, and c' and b are the adjacent parts; if c', β', and b are considered, b is the middle part, and c' and β' are the opposite parts.

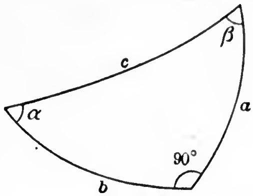


FIG. 105.

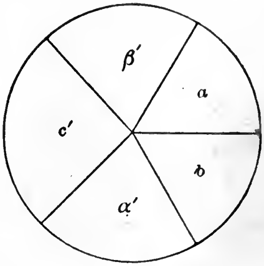


FIG. 106.

Napier's rules are:

1. *The sine of the middle part is equal to the product of the tangents of the adjacent parts.*

2. *The sine of the middle part is equal to the product of the cosines of the opposite parts.*

The rules may be easily remembered by the ***a*** in the words t***a***ngent and ***a***djacent and the ***o*** in c***o***sine and ***o***pposite.

If, in a right triangle, any two elements besides the right angle are given, the other elements may always be expressed in terms of these two by Napier's rules. Thus, let the given elements be a and c.

(1) To find α; of the three parts a, c', and α', a is the middle part, and c' and α' are the opposite parts.

$$\therefore \sin a = \cos c' \cos \alpha' = \cos(90° - c)\cos(90° - \alpha) = \sin c \sin \alpha.$$

$$\therefore \sin \alpha = \frac{\sin a}{\sin c}.$$

(2) To find β; of the three parts a, c', and β', β' is the middle part, and a and c' are the adjacent parts.

$$\therefore \sin \beta' = \tan a \tan c'.$$

$$\therefore \sin(90° - \beta) = \tan a \tan(90° - c).$$

$$\therefore \cos \beta = \tan a \cot c.$$

(3) To find b; of the three parts a, c', and b, c' is the middle part, and a and b are the opposite parts.

$$\therefore \sin c' = \cos a \cos b.$$

$$\therefore \sin(90° - c) = \cos a \cos b.$$

$$\therefore \cos c = \cos a \cos b.$$

$$\therefore \cos b = \frac{\cos c}{\cos a}.$$

128. Species.—Two angular quantities are said to be of the same species when both are less or both greater than 90°, *i.e.* when they are in the same quadrant; and of different species when they are in different quadrants.

129. Rules for Species in Right Spherical Triangles.

(1) *An oblique angle and its opposite side are always of the same species.* From Napier's rules,

$$\sin b = \tan a \cot \alpha.$$

But $\sin b$ is always positive, and therefore $\tan a$ and $\cot \alpha$ must have the same sign; if they are both positive a and α will be in the first quadrant, and if both are negative a and α will be in the second quadrant.

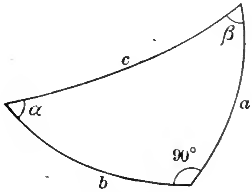


Fig. 107.

(2) *If the hypotenuse is less than 90°, the two oblique angles (and therefore the two sides) are of the same species; if it is greater than 90°, the two angles (and therefore the two sides) are of different species.* From Napier's rules,

$$\cos c = \cot \alpha \cot \beta = \cos a \cos b.$$

If c is less than 90° its cosine will be positive; $\cot \alpha$ and $\cot \beta$ must therefore have the same sign, and hence α and β must be in the same quadrant. If c is greater than 90° its cosine will be negative; $\cot \alpha$ and $\cot \beta$ must therefore have different signs, and hence α and β must be in different quadrants.

Thus, if $a = 40°$ and $\beta = 60°$, α and b must be, from the first rule, in the first quadrant; and, since α and β (or a and b) are in the same quadrant, c must be, from the second rule, in the first quadrant. If $\alpha = 70°$ and $c = 110°$, from the second rule we see that β must be in the second quadrant, and from the first rule that a is in the first and b in the second quadrant.

130. Solution of Right Spherical Triangles. — There are six possible cases, all of which may be solved by Napier's rules:

I. Given the hypotenuse and an angle.
II. Given the hypotenuse and a side.
III. Given the two angles.
IV. Given the two sides.
V. Given an angle and the adjacent side.
VI. Given an angle and the opposite side.

The required elements should always be determined directly from the given elements.

First write the three formulas, each containing the two given elements and one required element; arrange the three formulas for logarithmic computation; and then write the values of the functions in their proper places, being very careful about writing n after the logarithms of the negative functions. If the number of negative factors is even, the result will be positive; if it is odd, the result will be negative and n should be written after the resulting logarithm.

If the sine of a quantity is found by the computation to be positive, the quantity may be either in the first or in the second quadrant, the proper quadrant being determined by the rules for species; if the sine is negative the triangle is impossible, since the elements of the triangle are each less than 180°. If the cosine, tangent, or cotangent is found to be positive, the quantity lies in the first quadrant; if negative, the quantity lies in the second quadrant.

A check formula in each case is found by applying Napier's rules to the three unknown elements; thus, if a and b are given, α, β, and c will be computed, and the check formula is

$$\cos c = \cot \alpha \cot \beta.$$

131. Case I. Given the Hypotenuse and an Angle.

1. $c = 129°\,14'.6, \quad \alpha = 43°\,15'.7.$

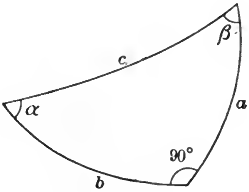


Fig. 108.

To find β; $\cos c = \cot \alpha \cot \beta$;

$$\therefore \cot \beta = \frac{\cos c}{\cot \alpha}.$$

To find b; $\cos \alpha = \tan b \cot c$;

$$\therefore \tan b = \frac{\cos \alpha}{\cot c}.$$

To find a; $\sin a = \sin \alpha \sin c$.

Check: $\sin a = \tan b \cot \beta$.

$\log \cos c = 9.80114\,n$	$\log \cos \alpha = 9.86227$	$\log \sin \alpha = 9.83590$
$-\log \cot \alpha = 0.02637$	$-\log \cot c = 9.91214\,n$	$+\log \sin c = 9.88900$
$\log \cot \beta = 9.77477\,n$	$\log \tan b = 9.95013\,n$	$\log \sin a = 9.72490$
$\therefore \beta = 120°\,46'.03$	$b = 138°\,16'.96$	$a = 32°\,3'.4$

By the rules for species β must be in the second, a in the first, and b in the second quadrant.

2. $c = 110°, \quad \beta = 48°\,28'.6.$

$$\therefore a = 118°\,46'.1; \; b = 44°\,42'.7; \; \alpha = 111°\,7'.2.$$

132. Case II. Given the Hypotenuse and a Side.

1. $c = 75°\,0'.4$, $a = 32°\,56'$. $\therefore b = 72°\,2'.8$; $\alpha = 34°\,15'.0$; $\beta = 80°\,0'.6$.
2. $c = 100°\,12'$, $b = 40°\,30'.3$. $\therefore a = 103°\,28'.1$; $\alpha = 98°\,50'.5$; $\beta = 41°\,17'.7$.

133. Case III. Given the Two Angles.

1. $\alpha = 30°\,51'.2$, $\beta = 71°\,36'$. $\therefore a = 25°\,12'.8$; $b = 52°\,0'.75$; $c = 56°\,9'.6$.
2. $\alpha = 130°\,20'$, $\beta = 100°\,10'.9$. $\therefore a = 131°\,7'.0$; $b = 103°\,24'.5$; $c = 81°\,13'.7$.

134. Case IV. Given the Two Sides.

1. $a = 43°\,20'$, $b = 74°\,13'$. $\therefore c = 78°\,35'.3$; $\alpha = 44°\,26'.0$; $\beta = 79°\,1'.4$.
2. $a = 100°$, $b = 98°\,20'$. $\therefore c = 88°\,33'.5$; $\alpha = 99°\,53'.8$; $\beta = 98°\,12'.5$.

135. Case V. Given One Side and the Adjacent Angle.

1. $b = 66°\,29'$, $\alpha = 50°\,17'$. $\therefore a = 47°\,49'.5$; $c = 74°\,27'.6$; $\beta = 72°\,7'.5$.
2. $a = 24°\,41'$, $\beta = 140°\,34'.7$. $\therefore b = 161°\,3'.2$; $c = 149°\,15'.0$; $\alpha = 54°\,45'.6$.

136. Case VI. Given One Side and the Angle Opposite. — Let a and α be the given elements.

To find b; $\sin b = \tan a \cot \alpha$.

To find c; $\sin a = \sin c \sin \alpha$; $\therefore \sin c = \dfrac{\sin a}{\sin \alpha}$.

To find β; $\cos \alpha = \cos a \sin \beta$; $\therefore \sin \beta = \dfrac{\cos \alpha}{\cos a}$.

All three quantities are found by their sines; each of the three quantities may therefore be in either the first or the second quadrant, and there will be two solutions.*

If P is the pole of AC, PC will be perpendicular to AC, and the distances BC will be equal if m is equal to n. Either of the triangles ABC will satisfy the conditions of the problem, being right-angled at C and having $A = \alpha$ and $BC = a$.

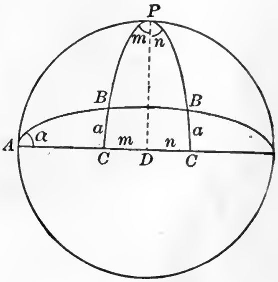


FIG. 109.

The rules for species must be applied in determining which values belong to each solution. Thus, if α is greater than 90°, and β in the first solution is less than 90°, c must be greater than 90° in that triangle; in the second triangle, where β is greater than 90°, c must be less than 90°. Each side, of course, is of the same species as its opposite angle. These results may be written:

First solution: $\alpha > 90°$; $\beta < 90°$; $c > 90°$; $a > 90°$; $b < 90°$.

Second solution: $\alpha > 90°$; $\beta > 90°$; $c < 90°$; $a > 90°$; $b > 90°$.

1. $a = 160° 12'.2$, $\alpha = 150° 37'$.

$$\log \tan a = 9.55625\,n \qquad \log \sin a = 9.52979 \qquad \log \cos \alpha = 9.94020\,n$$
$$+ \log \cot \alpha = 0.24942\,n \qquad - \log \sin \alpha = 9.69077 \qquad - \log \cos a = 9.97354\,n$$
$$\log \sin b = 9.80567 \qquad \log \sin c = 9.83902 \qquad \log \sin \beta = 9.96666$$
$$b = 39° 44'.1 \qquad c' = 43° 39'.1 \qquad \beta = 67° 50'.2$$
$$b' = 140° 15'.9 \qquad c = 136° 20'.9 \qquad \beta' = 112° \ 9'.8$$

2. $b = 40° 50'$, $\beta = 62° 14'$. $\therefore a = 27° 3'.9$; $\alpha = 38° 0'.4$; $c = 47° 38'.6$;
or $a' = 152° 56'.1$; $\alpha' = 141° 59'.6$; $c' = 132° 21'.4$.

* The triangle is supposed to be possible. The two solutions are identical when $a = \alpha$.

137. Special Cases.

1. $c = 90°,\ \alpha = 90°.$ $\therefore\ a = 90°$; b and β indeterminate.
2. $c = 90°,\ a = 90°.$ $\therefore\ \alpha = 90°$; b and β indeterminate.
3. $\alpha = 90°,\ \beta = 90°.$ $\therefore\ a = 90°;\ b = 90°;\ c = 90°.$
4. $a = 90°,\ b = 90°.$ $\therefore\ c = 90°;\ \alpha = 90°;\ \beta = 90°.$
5. $a = 90°,\ \beta = 90°.$ $\therefore\ c = 90°;\ b = 90°;\ \alpha = 90°.$
6. $a = 20°,\ \alpha = 20°.$ $\therefore\ c = 90°;\ b = 90°;\ \beta = 90°.$

138. Additional Examples.

1. $a = 40°\,42'.4,\ c = 63°\,20'.$
$\therefore\ b = 53°\,41'.9;\ \alpha = 46°\,52'.25;\ \beta = 64°\,24'.0.$

2. $a = 70°\,15'.5,\ \alpha = 81°\,42'.7.$
$\therefore\ b = 23°\,57'.0;\ \beta = 25°\,15'.7;\ c = 72°\ 1'.25;$
or $b' = 156°\,3'.0;\ \beta' = 154°\,44'.3;\ c' = 107°\,58'.75.$

3. $b = 30°\,32'.4,\ \alpha = 36°\,44'.$
$\therefore\ a = 20°\,46'.0;\ c = 36°\,21'.6;\ \beta = 58°\,59'.7.$

4. $c = 72°\,10',\ \alpha = 30°\,43'.$
$\therefore\ a = 29°\,5'.6;\ b = 69°\,29'.0;\ \beta = 79°\,41'.25.$

5. $\alpha = 106°\,34'.2,\ \beta = 33°\,11'.7.$
$\therefore\ a = 121°\,23'.6;\ b = 29°\,11'.0;\ c = 117°\,3'.0.$

6. $a = 28°\,47',\ b = 110°\,27'.3.$
$\therefore\ c = 107°\,50'.2;\ \alpha = 30°\,23'.1;\ \beta = 100°\,10'.9.$

7. $c = 54°\,12'.2,\ \beta = 164°\,50'.4.$
$\therefore\ \alpha = 99°\,0'.3;\ b = 167°\,45'.2;\ a = 126°\,45'.9.$

8. $a = 40°\,8',\ \beta = 74°\,30'.2.$
$\therefore\ b = 66°\,43'.5;\ c = 72°\,25'.0;\ \alpha = 42°\,32'.7.$

9. $c = 102°\,36',\ \alpha = 125°\,13'.4.$
$\therefore\ a = 127°\,8'.1;\ b = 68°\,49'.0;\ \beta = 72°\,49'.8.$

10. $\alpha = 40°\,42'.4,\ \beta = 67°\,51'.6.$
$\therefore\ a = 35°\,4'.4;\ b = 54°\,42'.0;\ c = 61°\,46'.6.$

11. $b = 163°\,14'.2,\ c = 112°\,41'.8.$
$\therefore\ a = 66°\,14'.1;\ \alpha = 82°\,45'.75;\ \beta = 161°\,46'.9.$

12. $a = 120°\,30'.2,\ b = 140°\,12'.$
$\therefore\ c = 67°\,2'.8;\ \alpha = 110°\,39'.7;\ \beta = 135°\,57'.7.$

13. $c = 50°\,20'.2,\ \beta = 101°\,29'.4.$
$\therefore\ a = 166°\,29'.5;\ b = 131°\,1'.7;\ \alpha = 162°\,20'.1.$

14. $\alpha = 82°\,4'.4,\ \beta = 8°\,22'.3.$
$\therefore\ a = 18°\,42'.2;\ c = 18°\,53'.25;\ b = 2°\,43'$ or $2°\,44'.$

15. $a = 130° 40'.7$, $c = 75° 31'.5$.

$\therefore\ b = 112° 33'.0$; $\alpha = 128° 26'.6$; $\beta = 107° 28'.75$.

16. $b = 10° 10'.2$, $\beta = 15° 40'.6$.

$\therefore\ a = 39° 43'.9$; $c = 40° 48'.1$; $\alpha = 78° 0'.7$;

or $a' = 140° 16'.1$; $c' = 139° 11'.9$; $\alpha' = 101° 59'.3$.

17. $b = 57° 8'.3$, $\alpha = 104° 16'.2$.

$\therefore\ a = 106° 50'.8$; $c = 99° 2'.8$; $\beta = 58° 16'.4$.

18. $a = 20° 54'$, $b = 64° 26'.7$.

$\therefore\ c = 66° 14'.1$; $\alpha = 22° 56'.5$; $\beta = 80° 19'.2$.

139. Isosceles Triangles. — If an arc of a great circle be drawn from the vertex perpendicular to the base, it will bisect both the base and the angle at the vertex, dividing the triangle into two equal right triangles that may be solved by Napier's rules.

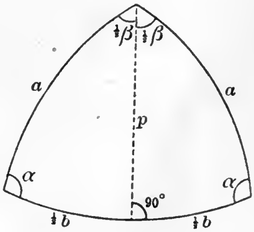


Fig. 110.

1. $a = 110° 47'.3$, $\beta = 92° 14'.6$.

$$\therefore\ \tfrac{1}{2}\beta = 46° 7'.3.$$

To find α: $\cos a = \cot \alpha \cot \tfrac{1}{2}\beta$;

$$\therefore\ \cot \alpha = \frac{\cos a}{\cot \tfrac{1}{2}\beta}.$$

To find $\tfrac{1}{2}b$: $\sin \tfrac{1}{2}b = \sin a \sin \tfrac{1}{2}\beta$.

To find p: $\cos \tfrac{1}{2}\beta = \tan p \cot a$; $\therefore\ \tan p = \dfrac{\cos \tfrac{1}{2}\beta}{\cot a}$.

Check: $\sin \tfrac{1}{2}b = \tan p \cot \alpha$.

$\log \cos a = 9.55013\,n$	$\log \sin a = 9.97076$	$\log \cos \tfrac{1}{2}\beta = 9.84081$
$-\log \cot \tfrac{1}{2}\beta = 9.98299$	$+\log \sin \tfrac{1}{2}\beta = 9.85783$	$-\log \cot a = 9.57936\,n$
$\log \cot \alpha = 9.56714\,n$	$\log \sin \tfrac{1}{2}b = 9.82859$	$\log \tan p = 0.26145\,n$
$110° 15'.54$	$\tfrac{1}{2}b = 42° 22'.1$	$p = 118° 42'.6$
	$b = 84° 44'.2$	

2. $\alpha = 82° 26'$, $\beta = 64° 42'$. $\therefore\ a = 77° 53'.6$; $\tfrac{1}{2}b = 31° 32'.75$.

3. $b = 56° 41'$, $\beta = 112° 44'.6$. $\therefore\ \alpha = 38° 59'.6$; $a = 34° 45'.6$;

or $\alpha' = 141° 0'.4$; $a' = 145° 14'.4$.

140. Quadrantal Triangles. — The polar of a quadrantal triangle is a right triangle whose angles are the supplements of the sides, and whose sides are the supplements of the angles, of the original triangle. We may therefore solve the polar by Napier's rules, and then find the elements of the original triangle by taking the supplements of the elements of the polar.

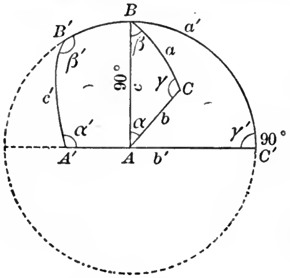


FIG. 111.

1. $c = 90°$, $a = 23° \; 14'.7$, $b = 27° \; 14'.6$.

$\therefore \; \gamma' = 90°$, $a' = 156° \; 45'.3$, $\beta' = 152° \; 45'.4$

are the elements of the polar triangle.*

To find c': $\cos \beta' = \tan a' \cot c'$; $\therefore \; \cot c' = \dfrac{\cos \beta'}{\tan a'}$.

To find b': $\sin a' = \tan b' \cot \beta'$; $\therefore \; \tan b' = \dfrac{\sin a'}{\cot \beta'}$.

To find α': $\cos \alpha' = \cos a' \sin \beta'$.

Check: $\cos \alpha' = \cot c' \tan b'$.

$\log \cos \beta' = 9.94894 \, n$	$\log \sin a' = 9.59623$	$\log \cos a' = 9.96324 \, n$
$- \log \tan a' = 9.63300 \, n$	$- \log \cot \beta' = 0.28828 \, n$	$+ \log \sin \beta' = 9.66065$
$\log \cot c' = 0.31594$	$\log \tan b' = 9.30795 \, n$	$\log \cos \alpha' = 9.62389 \, n$
$c' = 64° \; 12'.8$	$b' = 168° \; 30'.8$	$\alpha' = 114° \; 52'.4$
$\therefore \; \gamma = 115° \; 47'.2$	$\therefore \; \beta = 11° \; 29'.2$	$\therefore \; \alpha = 65° \; 7'.6$

2. $c = 90°$, $\gamma = 98° \; 22'.7$, $\alpha = 150° \; 47'$.
$\therefore \; a = 150° \; 26'.2$; $b = 94° \; 43'.5$; $\beta = 99° \; 36'.6$.

3. $c = 90°$, $a = 121° \; 30'$, $\beta = 112° \; 16'.2$.
$\therefore \; b = 108° \; 51'.1$; $\alpha = 123° \; 30'.75$; $\gamma = 102° \; 4'.7$.

4. $c = 90°$, $a = 138° \; 47'.8$, $b = 107° \; 54'.9$.
$\therefore \; \alpha = 142° \; 15'.2$; $\beta = 117° \; 50'.25$; $\gamma = 111° \; 40'.1$.

5. $c = 90°$, $a = 112° \; 6'.5$, $\gamma = 74° \; 30'$.
$\therefore \; b = 56° \; 39'.6$; $\alpha = 116° \; 46'.4$; $\beta = 53° \; 36'.9$.

6. $c = 90°$, $\alpha = 83° \; 20'.6$, $\beta = 77° \; 14'.3$.
$\therefore \; a = 83° \; 30'.3$; $b = 77° \; 19'.3$; $\gamma = 91° \; 28'.0$.

7. $c = 90°$, $a = 94° \; 22'.2$, $\alpha = 108° \; 13'.3$.
$\therefore \; b = 14° \; 6'.2$; $\beta = 13° \; 25'.3$; $\gamma = 72° \; 17'.5$;
or $b' = 165° \; 53'.8$; $\beta' = 166° \; 34'.7$; $\gamma' = 107° \; 42'.5$.

* Note that α', β', and c' are not the *parts* of the right triangle, but their complements.

141. Quadrantal Triangles may also be solved by the use of Fig. 112, in which B, one of the vertices adjacent to the quadrantal side, is the pole of the great circle $MDGN$.

If the triangle has one side less than 90°, as BC in the triangle ABC, produce that side to D. In the triangle ACD, $ADC = 90°$, $DAC = 90° - \alpha$, $ACD = 180° - \gamma$, $AC = b$, $CD = 90° - a$, and $AD = \beta$ since $AD = ABD$. Therefore, if any two elements of ACB besides the quadrant are given, we know two elements of the right triangle ACD in addition to the right angle. Hence we could solve it by Napier's rules, thence obtaining the elements of ABC.

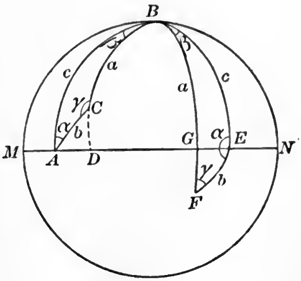


FIG. 112.

If one side of the triangle is greater than 90°, as in BEF, then in the triangle GEF we have $GF = a - 90°$, $GE = \beta$, $EF = b$, $FGE = 90°$, $GEF = \alpha - 90°$, and $GFE = \gamma$. If any two elements of BFE besides the quadrantal side are given, we then know two elements of the triangle GFE in addition to the right angle. Hence we could solve it by Napier's rules, thence finding the elements of BFE.

CHAPTER XI.

OBLIQUE SPHERICAL TRIANGLES.

142. To find an Angle, having given the Three Sides.

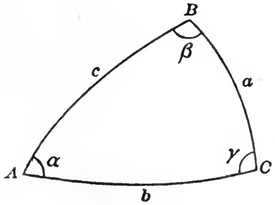


FIG. 113.

(*a*) $\cos a = \cos b \cos c + \sin b \sin c \cos \alpha;$

$$\therefore \cos \alpha = \frac{\cos a - \cos b \cos c}{\sin b \sin c}, \qquad (1)$$

which may be solved by the use of the natural functions.

(*b*)* To adapt (1) to logarithmic computation, subtract each member from unity.

$$\therefore 1 - \cos \alpha = 1 - \frac{\cos a - \cos b \cos c}{\sin b \sin c} = \frac{\sin b \sin c - \cos a + \cos b \cos c}{\sin b \sin c}.$$

$$\therefore 2 \sin^2 \tfrac{1}{2} \alpha = \frac{\cos (b - c) - \cos a}{\sin b \sin c}.$$

Applying (4) of Art. 73,

$$\cos u - \cos v = -2 \sin \tfrac{1}{2}(u + v) \sin \tfrac{1}{2}(u - v),$$

we have $\cos (b - c) - \cos a = -2 \sin \tfrac{1}{2}(b - c + a) \sin \tfrac{1}{2}(b - c - a)$

$$= +2 \sin \tfrac{1}{2}(a + b - c) \sin \tfrac{1}{2}(a - b + c),$$

since $\sin(-x) = -\sin x$.

Let $a + b + c = 2s$; $\therefore a + b - c = 2s - 2c = 2(s - c)$;

$$a - b + c = 2s - 2b = 2(s - b).$$

$$\therefore \cos (b - c) - \cos a = 2 \sin (s - c) \sin (s - b).$$

Permuting,

$$\left.\begin{aligned} \therefore \sin^2 \tfrac{1}{2} \alpha &= \frac{\sin (s - b) \sin (s - c)}{\sin b \sin c}. \\ \sin^2 \tfrac{1}{2} \beta &= \frac{\sin (s - a) \sin (s - c)}{\sin a \sin c}, \\ \sin^2 \tfrac{1}{2} \gamma &= \frac{\sin (s - a) \sin (s - b)}{\sin a \sin b}. \end{aligned}\right\} \qquad (2)$$

* Compare with Art. 99.

(*c*)* Add each member of (1) to unity.

$$\therefore\ 1+\cos\alpha = 1+\frac{\cos a-\cos b\cos c}{\sin b\sin c}$$

$$=\frac{\cos a-(\cos b\cos c-\sin b\sin c)}{\sin b\sin c}.$$

$$\therefore\ 2\cos^2\tfrac{1}{2}\alpha=\frac{\cos a-\cos(b+c)}{\sin b\sin c}.$$

Applying (4) of Art. 73, we have

$$\cos a-\cos(b+c)=-2\sin\tfrac{1}{2}(a+b+c)\sin\tfrac{1}{2}(a-b-c)$$
$$=+2\sin\tfrac{1}{2}(a+b+c)\sin\tfrac{1}{2}(b+c-a).$$

Let $a+b+c=2s$; $\therefore\ b+c-a=2s-2a=2(s-a)$.

$\therefore\ \cos a-\cos(b+c)=2\sin s\sin(s-a)$.

$$\left.\begin{aligned}\therefore\ \cos^2\tfrac{1}{2}\alpha&=\frac{\sin s\sin(s-a)}{\sin b\sin c}.\\ \cos^2\tfrac{1}{2}\beta&=\frac{\sin s\sin(s-b)}{\sin a\sin c},\\ \cos^2\tfrac{1}{2}\gamma&=\frac{\sin s\sin(s-c)}{\sin a\sin b}.\end{aligned}\right\}\qquad(3)$$

Permuting,

(*d*) Dividing $\sin^2\frac{1}{2}\alpha$ by $\cos^2\frac{1}{2}\alpha$, we have

$$\left.\begin{aligned}\tan^2\tfrac{1}{2}\alpha&=\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}.\\ \tan^2\tfrac{1}{2}\beta&=\frac{\sin(s-a)\sin(s-c)}{\sin s\sin(s-b)},\\ \tan^2\tfrac{1}{2}\gamma&=\frac{\sin(s-a)\sin(s-b)}{\sin s\sin(s-c)}.\end{aligned}\right\}\qquad(4)$$

Permuting,

We may write

$$\tan\tfrac{1}{2}\alpha=\frac{1}{\sin(s-a)}\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}.$$

Let $$r=\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}.\qquad(5)$$

$$\left.\begin{aligned}\therefore\ \tan\tfrac{1}{2}\alpha&=\frac{r}{\sin(s-a)}.\\ \tan\tfrac{1}{2}\beta&=\frac{r}{\sin(s-b)},\\ \tan\tfrac{1}{2}\gamma&=\frac{r}{\sin(s-c)}.\end{aligned}\right\}\qquad(6)$$

Permuting,

* Compare with Art. 99.

NOTE. — The center of the inscribed circle of a spherical triangle is the point of intersection of the arcs of great circles bisecting the angles of the triangle. From this point O draw the arcs OL, OM, and ON perpendicular to the sides of the triangle.

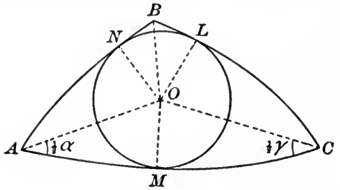


FIG. 114.

$$\therefore\ MA = AN;\ NB = BL;\ CM = LC.$$

$$\therefore\ MA + NB + CM = s.$$

$$\therefore\ b + NB = s,\ \text{since } MA + CM = b.$$

$$\therefore\ NB = s - b.$$

In the right triangle OBN, by Napier's rules,

$$\sin NB = \tan ON \cot NBO.$$

$$\therefore\ \tan ON = \frac{\sin NB}{\cot NBO} = \sin(s-b)\tan\tfrac{1}{2}\beta$$

$$= \sin(s-b)\sqrt{\frac{\sin(s-a)\sin(s-c)}{\sin s \sin(s-b)}}$$

$$= \sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}.$$

$$\therefore\ \tan ON = r.$$

Hence r is the tangent of the radius of the inscribed circle.

143. To find a Side, having given the Three Angles.

$$(a)\qquad \cos\alpha = -\cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a.$$

$$\therefore\ \cos a = \frac{\cos\alpha + \cos\beta\cos\gamma}{\sin\beta\sin\gamma}, \qquad (1)$$

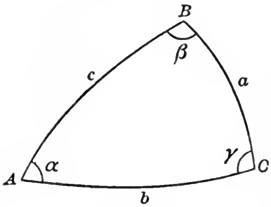


FIG. 115.

which may be solved by the use of the natural functions.

(b)* To adapt (1) to logarithmic computation, subtract each member from unity.

$$\therefore\ 1 - \cos a = 1 - \frac{\cos\alpha + \cos\beta\cos\gamma}{\sin\beta\sin\gamma}$$

$$= -\frac{\cos\beta\cos\gamma - \sin\beta\sin\gamma + \cos\alpha}{\sin\beta\sin\gamma}.$$

$$\therefore\ 2\sin^2\tfrac{1}{2}a = -\frac{\cos(\beta+\gamma) + \cos\alpha}{\sin\beta\sin\gamma}.$$

Applying the equation (Art. 73)

$$\cos u + \cos v = 2\cos\tfrac{1}{2}(u+v)\cos\tfrac{1}{2}(u-v),$$

we have $\cos(\beta+\gamma) + \cos\alpha = 2\cos\frac{1}{2}(\alpha+\beta+\gamma)\cos\frac{1}{2}(\beta+\gamma-\alpha)$

* Compare with Art. 99.

Let $\alpha + \beta + \gamma = 2S$; $\therefore\ \beta + \gamma - \alpha = 2S - 2\alpha = 2(S - \alpha)$.

$$\therefore\ \cos(\beta + \gamma) + \cos\alpha = 2\cos S \cos(S - \alpha).$$

Permuting,

$$\left.\begin{aligned} \therefore\ \sin^2 \tfrac{1}{2} a &= \frac{-\cos S \cos(S - \alpha)}{\sin\beta \sin\gamma}. \\ \sin^2 \tfrac{1}{2} b &= \frac{-\cos S \cos(S - \beta)}{\sin\alpha \sin\gamma}, \\ \sin^2 \tfrac{1}{2} c &= \frac{-\cos S \cos(S - \gamma)}{\sin\alpha \sin\beta}. \end{aligned}\right\} \qquad (2)$$

(c)* Add each member of (1) to unity.

$$\therefore\ 1 + \cos a = 1 + \frac{\cos\alpha + \cos\beta\cos\gamma}{\sin\beta\sin\gamma}$$

$$= \frac{\cos\alpha + \cos\beta\cos\gamma + \sin\beta\sin\gamma}{\sin\beta\sin\gamma}.$$

$$\therefore\ 2\cos^2 \tfrac{1}{2} a = \frac{\cos\alpha + \cos(\beta - \gamma)}{\sin\beta\sin\gamma}.$$

Applying the equation (Art. 73)

$$\cos u + \cos v = 2\cos\tfrac{1}{2}(u + v)\cos\tfrac{1}{2}(u - v),$$

we have $\cos\alpha + \cos(\beta - \gamma) = 2\cos\tfrac{1}{2}(\alpha + \beta - \gamma)\cos\tfrac{1}{2}(\alpha - \beta + \gamma)$.

Let $\alpha + \beta + \gamma = 2S$; $\therefore\ \alpha + \beta - \gamma = 2S - 2\gamma = 2(S - \gamma)$;

$$\alpha - \beta + \gamma = 2S - 2\beta = 2(S - \beta).$$

$$\therefore\ \cos\alpha + \cos(\beta - \gamma) = 2\cos(S - \beta)\cos(S - \gamma).$$

Permuting,

$$\left.\begin{aligned} \therefore\ \cos^2 \tfrac{1}{2} a &= \frac{\cos(S - \beta)\cos(S - \gamma)}{\sin\beta \sin\gamma}. \\ \cos^2 \tfrac{1}{2} b &= \frac{\cos(S - \alpha)\cos(S - \gamma)}{\sin\alpha \sin\gamma}, \\ \cos^2 \tfrac{1}{2} c &= \frac{\cos(S - \alpha)\cos(S - \beta)}{\sin\alpha \sin\beta}. \end{aligned}\right\} \qquad (3)$$

(d) Dividing $\sin^2 \tfrac{1}{2} a$ by $\cos^2 \tfrac{1}{2} a$, we have

Permuting,

$$\left.\begin{aligned} \tan^2 \tfrac{1}{2} a &= \frac{-\cos S \cos(S - \alpha)}{\cos(S - \beta)\cos(S - \gamma)}. \\ \tan^2 \tfrac{1}{2} b &= \frac{-\cos S \cos(S - \beta)}{\cos(S - \alpha)\cos(S - \gamma)}, \\ \tan^2 \tfrac{1}{2} c &= \frac{-\cos S \cos(S - \gamma)}{\cos(S - \alpha)\cos(S - \beta)}. \end{aligned}\right\} \qquad (4)$$

* Compare with Art. 99.

We may write

$$\tan \tfrac{1}{2} a = \cos (S - \alpha) \sqrt{\frac{-\cos S}{\cos (S - \alpha) \cos (S - \beta) \cos (S - \gamma)}}.$$

Let $$R = \sqrt{\frac{-\cos S}{\cos (S - \alpha) \cos (S - \beta) \cos (S - \gamma)}}. \qquad (5)$$

Permuting,
$$\left.\begin{aligned} \therefore \tan \tfrac{1}{2} a &= R \cos (S - \alpha). \\ \tan \tfrac{1}{2} b &= R \cos (S - \beta), \\ \tan \tfrac{1}{2} c &= R \cos (S - \gamma). \end{aligned}\right\} \qquad (6)$$

NOTE. — Since the sum of the angles $2S$ must be between 180° and 540°, S must be between 90° and 270°, so that $\cos S$ is always negative and hence $-\cos S$ is always positive.

NOTE. — The center of the circumscribed circle of a spherical triangle is the point of intersection of the arcs of great circles perpendicular to the sides of the triangle at their middle points.

$$\therefore AN = NB; \; BL = LC; \; CM = MA.$$

$$\therefore OAM = OCM; \; OAN = OBN; \; OCL = OBL.$$

$$\therefore OAM + OAN + OCL = S,$$

$$\therefore OCL = S - (OAM + OAN) = S - \alpha.$$

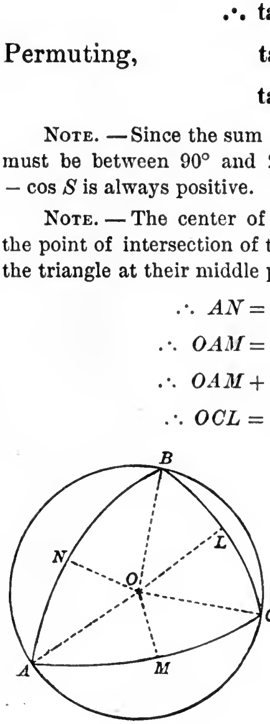


FIG. 116.

In the right triangle OCL, by Napier's rules,

$$\cos OCL = \tan LC \cot OC.$$

$$\therefore \tan OC = \frac{\tan LC}{\cos OCL} = \frac{\tan \frac{1}{2} a}{\cos (S - \alpha)}$$

$$= \frac{1}{\cos (S - \alpha)} \sqrt{\frac{-\cos S \cos (S - \alpha)}{\cos (S - \beta) \cos (S - \gamma)}}$$

$$= \sqrt{\frac{-\cos S}{\cos (S - \alpha) \cos (S - \beta) \cos (S - \gamma)}}.$$

$$\therefore \tan OC = R.$$

Hence R is the tangent of the radius of the circumscribed circle.

144. Napier's Analogies.

(1) From (4) or (6), Art. 142,

$$\frac{\tan \frac{1}{2} \alpha}{\tan \frac{1}{2} \beta} = \frac{\sin (s - b)}{\sin (s - a)}.$$

By division and composition,

$$\frac{\tan \frac{1}{2} \alpha - \tan \frac{1}{2} \beta}{\tan \frac{1}{2} \alpha + \tan \frac{1}{2} \beta} = \frac{\sin (s - b) - \sin (s - a)}{\sin (s - b) + \sin (s - a)}. \qquad (a)$$

But

$$\frac{\tan\frac{1}{2}\alpha - \tan\frac{1}{2}\beta}{\tan\frac{1}{2}\alpha + \tan\frac{1}{2}\beta} = \frac{\sin\frac{1}{2}\alpha\cos\frac{1}{2}\beta - \cos\frac{1}{2}\alpha\sin\frac{1}{2}\beta}{\sin\frac{1}{2}\alpha\cos\frac{1}{2}\beta + \cos\frac{1}{2}\alpha\sin\frac{1}{2}\beta} = \frac{\sin\frac{1}{2}(\alpha-\beta)}{\sin\frac{1}{2}(\alpha+\beta)}.$$

Also, from (1) and (2), Art. 73,

$$\frac{\sin(s-b)-\sin(s-a)}{\sin(s-b)+\sin(s-a)} = \frac{2\cos\frac{1}{2}(2s-a-b)\sin\frac{1}{2}(a-b)}{2\sin\frac{1}{2}(2s-a-b)\cos\frac{1}{2}(a-b)}$$

$$= \frac{\tan\frac{1}{2}(a-b)}{\tan\frac{1}{2}c}.$$

Substituting these values in (a), and reversing the order,

$$\frac{\tan\frac{1}{2}(a-b)}{\tan\frac{1}{2}c} = \frac{\sin\frac{1}{2}(\alpha-\beta)}{\sin\frac{1}{2}(\alpha+\beta)}. \qquad (1)$$

(2) Substituting the values of a, b, c, α, and β in terms of the elements of the polar triangle, (1) becomes

$$\frac{\tan\frac{1}{2}(180° - \alpha' - 180° + \beta')}{\tan\frac{1}{2}(180° - \gamma')} = \frac{\sin\frac{1}{2}(180° - a' - 180° + b')}{\sin\frac{1}{2}(180° - a' + 180° - b')}.$$

$$\therefore \frac{\tan\frac{1}{2}(\beta' - \alpha')}{\cot\frac{1}{2}\gamma'} = \frac{\sin\frac{1}{2}(b' - a')}{\sin\frac{1}{2}(a' + b')}.$$

$$\therefore \frac{-\tan\frac{1}{2}(\alpha' - \beta')}{\cot\frac{1}{2}\gamma'} = \frac{-\sin\frac{1}{2}(a' - b')}{\sin\frac{1}{2}(a' + b')}.$$

Changing the signs and dropping the primes,

$$\frac{\tan\frac{1}{2}(\alpha-\beta)}{\cot\frac{1}{2}\gamma} = \frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}. \qquad (2)$$

(3) From (4), Art. 142,

$$\tan\tfrac{1}{2}\alpha\tan\tfrac{1}{2}\beta = \frac{\sin(s-c)}{\sin s}.$$

$$\therefore \frac{1+\tan\frac{1}{2}\alpha\tan\frac{1}{2}\beta}{1-\tan\frac{1}{2}\alpha\tan\frac{1}{2}\beta} = \frac{\sin s + \sin(s-c)}{\sin s - \sin(s-c)} \qquad (b)$$

But $$\frac{1+\tan\frac{1}{2}\alpha\tan\frac{1}{2}\beta}{1-\tan\frac{1}{2}\alpha\tan\frac{1}{2}\beta} = \frac{\cos\frac{1}{2}\alpha\cos\frac{1}{2}\beta + \sin\frac{1}{2}\alpha\sin\frac{1}{2}\beta}{\cos\frac{1}{2}\alpha\cos\frac{1}{2}\beta - \sin\frac{1}{2}\alpha\sin\frac{1}{2}\beta}$$

$$= \frac{\cos\frac{1}{2}(\alpha-\beta)}{\cos\frac{1}{2}(\alpha+\beta)}.$$

Also, from (1) and (2), Art. 73,

$$\frac{\sin s + \sin(s-c)}{\sin s - \sin(s-c)} = \frac{2\sin\frac{1}{2}(2s-c)\cos\frac{1}{2}c}{2\cos\frac{1}{2}(2s-c)\sin\frac{1}{2}c} = \frac{\tan\frac{1}{2}(a+b)}{\tan\frac{1}{2}c}.$$

Substituting these values in (b), and reversing the order,

$$\frac{\tan\frac{1}{2}(a+b)}{\tan\frac{1}{2}c}=\frac{\cos\frac{1}{2}(\alpha-\beta)}{\cos\frac{1}{2}(\alpha+\beta)}. \tag{3}$$

(4) Passing to the polar triangle, (3) becomes

$$\frac{\tan\frac{1}{2}(180^\circ-\alpha'+180^\circ-\beta')}{\tan\frac{1}{2}(180^\circ-\gamma')}=\frac{\cos\frac{1}{2}(180^\circ-a'-180^\circ+b')}{\cos\frac{1}{2}(180^\circ-a'+180^\circ-b')}.$$

$$\therefore\ \frac{\tan[180^\circ-\frac{1}{2}(\alpha'+\beta')]}{\tan(90^\circ-\frac{1}{2}\gamma')}=\frac{\cos\frac{1}{2}(b'-a')}{\cos[180^\circ-\frac{1}{2}(a'+b')]}.$$

$$\therefore\ \frac{-\tan\frac{1}{2}(\alpha'+\beta')}{\cot\frac{1}{2}\gamma'}=\frac{\cos\frac{1}{2}(a'-b')}{-\cos\frac{1}{2}(a'+b')}.$$

Changing the signs and dropping the primes,

$$\frac{\tan\frac{1}{2}(\alpha+\beta)}{\cot\frac{1}{2}\gamma}=\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}. \tag{4}$$

Eqs. (1), (2), (3), and (4) are called *Napier's Analogies.*

145. Gauss's Equations. — From (2) and (3), Art. 142, we have

$$\sin\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta=\frac{\sin(s-b)}{\sin c}\sqrt{\frac{\sin s\sin(s-c)}{\sin a\sin b}}=\frac{\sin(s-b)}{\sin c}\cos\tfrac{1}{2}\gamma;$$

$$\cos\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\beta=\frac{\sin(s-a)}{\sin c}\sqrt{\frac{\sin s\sin(s-c)}{\sin a\sin b}}=\frac{\sin(s-a)}{\sin c}\cos\tfrac{1}{2}\gamma;$$

$$\cos\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta=\frac{\sin s}{\sin c}\sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a\sin b}}=\frac{\sin s}{\sin c}\sin\tfrac{1}{2}\gamma;$$

$$\sin\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\beta=\frac{\sin(s-c)}{\sin c}\sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a\sin b}}$$

$$=\frac{\sin(s-c)}{\sin c}\sin\tfrac{1}{2}\gamma.$$

(1) $$\sin\tfrac{1}{2}(\alpha+\beta)=\frac{\cos\frac{1}{2}\gamma}{\sin c}[\sin(s-b)+\sin(s-a)]$$

$$=\frac{\cos\frac{1}{2}\gamma\sin[s-\frac{1}{2}(a+b)]\cos\frac{1}{2}(a-b)}{\sin\frac{1}{2}c\cos\frac{1}{2}c}$$

$$=\frac{\cos\frac{1}{2}\gamma\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}c}.$$

$$\therefore\ \cos\tfrac{1}{2}c\sin\tfrac{1}{2}(\alpha+\beta)=\cos\tfrac{1}{2}\gamma\cos\tfrac{1}{2}(a-b). \tag{1}$$

(2) $$\cos\tfrac{1}{2}(\alpha+\beta)=\frac{\sin\frac{1}{2}\gamma}{\sin c}[\sin s-\sin(s-c)]$$

$$=\frac{\sin\frac{1}{2}\gamma\cos(s-\frac{1}{2}c)\sin\frac{1}{2}c}{\sin\frac{1}{2}c\cos\frac{1}{2}c}$$

$$=\frac{\sin\frac{1}{2}\gamma\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}c}.$$

$$\therefore\ \cos\tfrac{1}{2}c\cos\tfrac{1}{2}(\alpha+\beta)=\sin\tfrac{1}{2}\gamma\cos\tfrac{1}{2}(a+b). \qquad (2)$$

(3) $$\sin\tfrac{1}{2}(\alpha-\beta)=\frac{\cos\frac{1}{2}\gamma}{\sin c}[\sin(s-b)-\sin(s-a)]$$

$$=\frac{\cos\frac{1}{2}\gamma\cos[s-\frac{1}{2}(a+b)]\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}c\cos\frac{1}{2}c}$$

$$=\frac{\cos\frac{1}{2}\gamma\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}c}.$$

$$\therefore\ \sin\tfrac{1}{2}c\sin\tfrac{1}{2}(\alpha-\beta)=\cos\tfrac{1}{2}\gamma\sin\tfrac{1}{2}(a-b). \qquad (3)$$

(4) $$\cos\tfrac{1}{2}(\alpha-\beta)=\frac{\sin\frac{1}{2}\gamma}{\sin c}[\sin s+\sin(s-c)]$$

$$=\frac{\sin\frac{1}{2}\gamma\sin(s-\frac{1}{2}c)\cos\frac{1}{2}c}{\sin\frac{1}{2}c\cos\frac{1}{2}c}$$

$$=\frac{\sin\frac{1}{2}\gamma\sin\frac{1}{2}(a+b)}{\sin\frac{1}{2}c}.$$

$$\therefore\ \sin\tfrac{1}{2}c\cos\tfrac{1}{2}(\alpha-\beta)=\sin\tfrac{1}{2}\gamma\sin\tfrac{1}{2}(a+b). \qquad (4)$$

Eqs. (1), (2), (3), and (4) are known as *Gauss's Equations*, or *Delambre's Analogies.*

146. Rules for Species in Oblique Spherical Triangles.

(1) *If a side (or angle) differs more than another side (or angle) from 90°, it is of the same species as its opposite angle (or side).*

We wish to show that $\cos a$ and $\cos\alpha$ will have the same sign when the difference between a and 90° is numerically greater than the difference between b and 90°. In the formula

$$\cos\alpha=\frac{\cos a-\cos b\cos c}{\sin b\sin c} \qquad (1)$$

the denominator is always positive, so that the sign of the fraction, and hence that of $\cos\alpha$, is the same as that of the numerator. But if a differs more than b from 90°, $\cos a$ is numerically greater than $\cos b$, and hence greater than $\cos b \cos c$, since $\cos c$ cannot exceed unity. Therefore the numerator has the same sign as $\cos a$; *i.e.* $\cos a$ and $\cos \alpha$ have the same sign, so that a and α are in the same quadrant.

By a similar process, using the formula

$$\cos a = \frac{\cos \alpha + \cos \beta \cos \gamma}{\sin \beta \sin \gamma}, \tag{2}$$

we can show that, when α differs more than β or γ from 90°, a and α are of the same species.

Since two sides will in general differ more than the third from 90°, two angles will in general be of the same species as their opposite sides. Thus, if $a = 140°$, $b = 50°$, and $c = 110°$, we see that a and b differ more from 90° than c does; therefore α will lie in the second, and β in the first quadrant, while the quadrant of γ is not determined by this rule.

2. *Half the sum of two sides must be of the same species as half the sum of the two opposite angles.* — From (3), Art. 144,

$$\tan \tfrac{1}{2}(a + b) = \tan \tfrac{1}{2} c \, \frac{\cos \frac{1}{2}(\alpha - \beta)}{\cos \frac{1}{2}(\alpha + \beta)}. \tag{3}$$

But c must be less than 180°; hence $\frac{1}{2} c$ must be less than 90°, so that $\tan \frac{1}{2} c$ is positive. Also, $\alpha - \beta$ must be numerically less than 180°; hence $\frac{1}{2}(\alpha - \beta)$ must be numerically less than 90°, so that $\cos \frac{1}{2}(\alpha - \beta)$ is always positive. Hence $\tan \frac{1}{2}(a + b)$ and $\cos \frac{1}{2}(\alpha + \beta)$ must have the same sign. But $a + b$ and $\alpha + \beta$ must each be less than 360°; hence $\frac{1}{2}(a + b)$ and $\frac{1}{2}(\alpha + \beta)$ must each be less than 180°, so that they must be in the same quadrant in order that $\tan \frac{1}{2}(a + b)$ and $\cos \frac{1}{2}(\alpha + \beta)$ may have the same sign. Thus, if $\frac{1}{2}(\alpha + \beta)$ is in the first quadrant, its cosine will be positive; the second member of (2) will be positive, and therefore $\frac{1}{2}(a + b)$ must be in the first quadrant. If $\cos \frac{1}{2}(\alpha + \beta)$ is negative, $\tan \frac{1}{2}(a + b)$ will be negative; therefore $\frac{1}{2}(\alpha + \beta)$ and $\frac{1}{2}(a + b)$ must both be in the second quadrant.

In the example under the first rule, after α and β have been computed, the quadrant in which γ will lie may be determined by the second rule.

147. Solution of Oblique Spherical Triangles. — Any spherical triangle may be solved by the use of the following formulas:

$$\frac{\sin a}{\sin \alpha}=\frac{\sin b}{\sin \beta}=\frac{\sin c}{\sin \gamma}. \qquad (1)$$

$$\tan\tfrac{1}{2}\alpha=\frac{r}{\sin(s-a)}; \qquad r=\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}. \qquad (2)$$

$$\tan\tfrac{1}{2}a=R\cos(S-\alpha); \quad R=\sqrt{\frac{-\cos S}{\cos(S-\alpha)\cos(S-\beta)\cos(S-\gamma)}}. \qquad (3)$$

$$\frac{\tan\frac{1}{2}(a-b)}{\tan\frac{1}{2}c}=\frac{\sin\frac{1}{2}(\alpha-\beta)}{\sin\frac{1}{2}(\alpha+\beta)}. \qquad (4)$$

$$\frac{\tan\frac{1}{2}(a+b)}{\tan\frac{1}{2}c}=\frac{\cos\frac{1}{2}(\alpha-\beta)}{\cos\frac{1}{2}(\alpha+\beta)}. \qquad (5)$$

$$\frac{\tan\frac{1}{2}(\alpha-\beta)}{\cot\frac{1}{2}\gamma}=\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}. \qquad (6)$$

$$\frac{\tan\frac{1}{2}(\alpha+\beta)}{\cot\frac{1}{2}\gamma}=\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}. \qquad (7)$$

There are six possible cases:

I. Given the three sides.
II. Given the three angles.
III. Given two sides and the included angle.
IV. Given two angles and the included side.
V. Given two sides and the angle opposite one of them.
VI. Given two angles and the side opposite one of them.

148. Case I. Given the Three Sides (a, b, c). — Find the angles by the formulas

$$r=\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}},$$

$$\tan\tfrac{1}{2}\alpha=\frac{r}{\sin(s-a)}.$$

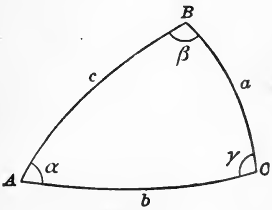


FIG. 117.

Check by the sine proportion.

1. Solve the triangle when $a = 114^\circ\, 43'.3$, $b = 136^\circ\, 19'.6$, $c = 43^\circ\, 18'.5$.

$s = 147^\circ\, 10'.7$	$\text{col}\sin s = 0.26598$	$\therefore\ \log\tan\frac{1}{2}\alpha = 9.89910$
$s - a = 32^\circ\, 27'.4$	$\log\sin(s-a) = 9.72970$	$\frac{1}{2}\alpha = 38^\circ\, 24'.2$
$s - b = 10^\circ\, 51'.1$	$\log\sin(s-b) = 9.27478$	$\log\tan\frac{1}{2}\beta = 0.35402$
$s - c = 103^\circ\, 52'.2$	$\log\sin(s-c) = 9.98714$	$\frac{1}{2}\beta = 66^\circ\, 7'.6$
$2s = 294^\circ\, 21'.4$	$\log r^2 = 29.25760 - 30$	$\log\tan\frac{1}{2}\gamma = 9.64166$
a check.	$\log r = 9.62880$	$\frac{1}{2}\gamma = 23^\circ\, 39'.7$

In finding $\log\tan\frac{1}{2}\alpha$ write $\log r$ on the margin of a slip of paper, place it above $\log\sin(s-a)$, and write the difference opposite $\log\tan\frac{1}{2}\alpha$; then find $\log\tan\frac{1}{2}\beta$ and $\log\tan\frac{1}{2}\gamma$ in the same manner.

2. $a = 76^\circ\, 40'.4$, $b = 54^\circ\, 21'.3$, $c = 36^\circ\, 8'.7$.
$\therefore\ \frac{1}{2}\alpha = 60^\circ\, 1'.8$; $\frac{1}{2}\beta = 23^\circ\, 8'.6$; $\frac{1}{2}\gamma = 15^\circ\, 49'.3$.

3. $a = 124^\circ\, 34'.9$, $b = 66^\circ\, 7'.2$, $c = 109^\circ\, 43'.5$.
$\therefore\ \frac{1}{2}\alpha = 60^\circ\, 1'.3$; $\frac{1}{2}\beta = 37^\circ\, 0'.8$; $\frac{1}{2}\gamma = 49^\circ\, 6'.8$.

4. $a = 30^\circ\, 17'.6$, $b = 22^\circ\, 14'.4$, $c = 18^\circ\, 51'.8$.
$\therefore\ \frac{1}{2}\alpha = 47^\circ\, 55'.0$; $\frac{1}{2}\beta = 24^\circ\, 8'.5$; $\frac{1}{2}\gamma = 19^\circ\, 48'.45$.

5. $a = 130^\circ\, 46'.0$, $b = 113^\circ\, 21'.4$, $c = 102^\circ\, 16'.2$.
$\therefore\ \frac{1}{2}\alpha = 72^\circ\, 38'.0$; $\frac{1}{2}\beta = 68^\circ\, 9'.6$; $\frac{1}{2}\gamma = 66^\circ\, 20'.5$.

149. Case II. Given the Three Angles (α, β, γ). — Find the sides by the formulas

$$\left.\begin{aligned} R &= \sqrt{\frac{-\cos S}{\cos(S-\alpha)\cos(S-\beta)\cos(S-\gamma)}},\\ \tan\tfrac{1}{2}a &= R\cos(S-\alpha). \end{aligned}\right\}$$

Check by the sine proportion.

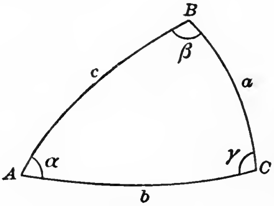


FIG. 118.

1. Solve the triangle when $\alpha = 116^\circ\, 19'.4$, $\beta = 83^\circ\, 19'.2$, $\gamma = 106^\circ\, 10'.6$.

$S = 152^\circ\, 54'.6$	$\text{col}(-\cos S) = 0.05047$	$\therefore\ \log\tan\frac{1}{2}a = 0.23789$
$S - \alpha = 36^\circ\, 35'.2$	$\log\cos(S-\alpha) = 9.90469$	$\frac{1}{2}a = 59^\circ\, 57'.7$
$S - \beta = 69^\circ\, 35'.4$	$\log\cos(S-\beta) = 9.54249$	$\log\tan\frac{1}{2}b = 9.87569$
$S - \gamma = 46^\circ\, 44'.0$	$\log\cos(S-\gamma) = 9.83594$	$\frac{1}{2}b = 36^\circ\, 54'.6$
$2S = 305^\circ\, 49'.2$	$\text{col}\, R^2 = 9.33359$	$\log\tan\frac{1}{2}c = 0.16914$
a check.	$\log R^2 = 0.66641$	$\frac{1}{2}c = 55^\circ\, 53'.1$
	$\log R = 0.33320$	

In finding log tan $\frac{1}{2}a$, write log R on the margin of a slip of paper, place it above log cos $(S-\alpha)$, and write the sum opposite log tan $\frac{1}{2}a$; then find log tan $\frac{1}{2}b$ and log tan $\frac{1}{2}c$ in a similar manner.

2. $\alpha = 110° 36'.4$, $\beta = 122° 8'.7$, $\gamma = 140° 20'.3$.

$\therefore\ \frac{1}{2}a = 41° 56'.3$; $\frac{1}{2}b = 57° 57'.5$; $\frac{1}{2}c = 68° 39'.4$.

3. $\alpha = 120° 50'.6$, $\beta = 78° 6'.1$, $\gamma = 81° 12'.3$.

$\therefore\ \frac{1}{2}a = 59° 55'.2$; $\frac{1}{2}b = 40° 40'.1$; $\frac{1}{2}c = 43° 23'.4$.

4. $\alpha = 80° 20'.2$, $\beta = 73° 46'.7$, $\gamma = 54° 8'.5$.

$\therefore\ \frac{1}{2}a = 32° 23'.6$; $\frac{1}{2}b = 30° 53'.7$; $\frac{1}{2}c = 24° 1'.7$.

5. $\alpha = 100° 51'.3$, $\beta = 80° 47'.6$, $\gamma = 74° 3'.3$.

$\therefore\ \frac{1}{2}a = 49° 22'.4$; $\frac{1}{2}b = 41° 42'.5$; $\frac{1}{2}c = 37° 41'.6$.

150. Case III. Given Two Sides and the Included Angle (b, c, α). — By permuting (6) and (7), Art. 147, we have

$$\tan\tfrac{1}{2}(\beta-\gamma) = \cot\tfrac{1}{2}\alpha\,\frac{\sin\frac{1}{2}(b-c)}{\sin\frac{1}{2}(b+c)}, \tag{1}$$

$$\tan\tfrac{1}{2}(\beta+\gamma) = \cot\tfrac{1}{2}\alpha\,\frac{\cos\frac{1}{2}(b-c)}{\cos\frac{1}{2}(b+c)}. \tag{2}$$

Then
$$\beta = \tfrac{1}{2}(\beta+\gamma) + \tfrac{1}{2}(\beta-\gamma),$$
$$\gamma = \tfrac{1}{2}(\beta+\gamma) - \tfrac{1}{2}(\beta-\gamma).$$

Note that the larger angle must be opposite the larger side.

To obtain a, we permute (4) and (5), Art. 147 :

$$\tan\tfrac{1}{2}a = \tan\tfrac{1}{2}(b-c)\,\frac{\sin\frac{1}{2}(\beta+\gamma)}{\sin\frac{1}{2}(\beta-\gamma)}, \tag{3}$$

$$\tan\tfrac{1}{2}a = \tan\tfrac{1}{2}(b+c)\,\frac{\cos\frac{1}{2}(\beta+\gamma)}{\cos\frac{1}{2}(\beta-\gamma)}. \tag{4}$$

The agreement of the values of $\frac{1}{2}a$ found from (3) and (4) is a check upon the computation. The sine proportion may also be used as a check.

NOTE. — In using these formulas, the larger side and the larger angle should be written first in the expressions $b-c$ and $\beta-\gamma$. Thus for $c > b$, (1) would be written

$$\tan\tfrac{1}{2}(\gamma-\beta) = \cot\tfrac{1}{2}\alpha\,\frac{\sin\frac{1}{2}(c-b)}{\sin\frac{1}{2}(c+b)}.$$

Eq. (1) may be read: "The tangent of half the difference of the required angles is equal to the cotangent of half the given angle, multiplied by the sine of half the difference of the given sides, and divided by the sine of half their sum."

1. Solve the triangle when $b = 105° 14'.8$, $c = 43° 17'.2$, $\alpha = 112° 47'.4$.

$b = 105° 14'.8$	$\log\cot\frac{1}{2}\alpha = 9.82251$	$\log\cot\frac{1}{2}\alpha = 9.82251$
$c = 43° 17'.2$	$\log\sin\frac{1}{2}(b-c) = 9.71159$	$\log\cos\frac{1}{2}(b-c) = 9.93316$
$\frac{1}{2}(b+c) = 74° 16'.0$	$\text{col}\sin\frac{1}{2}(b+c) = 0.01658$	$\text{col}\cos\frac{1}{2}(b+c) = 0.56677$
$\frac{1}{2}(b-c) = 30° 58'.8$	$\log\tan\frac{1}{2}(\beta-\gamma) = 9.55068$	$\log\tan\frac{1}{2}(\beta+\gamma) = 0.32244$
$\frac{1}{2}\alpha = 56° 23'.7$	$\frac{1}{2}(\beta-\gamma) = 19° 33'.8$	$\frac{1}{2}(\beta+\gamma) = 64° 32'.9$
		$\frac{1}{2}(\beta-\gamma) = 19° 33'.8$
		$\beta = 84° 6'.7$
		$\gamma = 44° 59'.1$

$\log\tan\frac{1}{2}(b-c) = 9.77843$	$\log\tan\frac{1}{2}(b+c) = 0.55019$
$\log\sin\frac{1}{2}(\beta+\gamma) = 9.95566$*	$\log\cos\frac{1}{2}(\beta+\gamma) = 9.63322$*
$\text{col}\sin\frac{1}{2}(\beta-\gamma) = 0.47514$*	$\text{col}\cos\frac{1}{2}(\beta-\gamma) = 0.02582$*
$\log\tan\frac{1}{2}a = 0.20923$	$\log\tan\frac{1}{2}a = 0.20923$
$\frac{1}{2}a = 58° 17'.8$	$\frac{1}{2}a = 58° 17'.8$

2. $a = 103° 44'.7$, $b = 64° 12'.3$, $\gamma = 98° 33'.8$.

$\therefore\ \frac{1}{2}(\alpha+\beta) = 82° 37'.0$; $\frac{1}{2}(\alpha-\beta) = 16° 19'.0$; $\alpha = 98° 56'.0$;
$\beta = 66° 18'.0$; $\frac{1}{2}c = 51° 45'.3$.

3. $a = 156° 12'.2$, $b = 112° 48'.6$, $\gamma = 76° 32'.4$.

$\therefore\ \frac{1}{2}(\alpha+\beta) = 120° 45'.6$; $\frac{1}{2}(\alpha-\beta) = 33° 18'.5$; $\alpha = 154° 4'.1$;
$\beta = 87° 27'.1$; $\frac{1}{2}c = 31° 54'.4$.

151. Case III. Second Method. Given *b*, *c*, *α*, to find One Element only.

(1) *To find a only.*

$$\cos a = \cos b\cos c + \sin b\sin c\cos\alpha.$$

Let
$$\left.\begin{aligned} m\sin M &= \sin c\cos\alpha,\\ m\cos M &= \cos c.\end{aligned}\right\}\qquad(1)$$

$$\therefore\ \cos a = m(\cos b\cos M + \sin b\sin M).$$

$$\therefore\ \cos a = m\cos(b-M).\qquad(2)$$

(2) *To find one angle only*, β or γ. — From (6), Art. 124,

$$\sin\alpha\cot\gamma = \cot c\sin b - \cos b\cos\alpha.$$

$$\therefore\ \cot\gamma = \frac{\cot c\sin b - \cos b\cos\alpha}{\sin\alpha} = \frac{\cos c\sin b - \cos b\sin c\cos\alpha}{\sin\alpha\sin c}.$$

* The functions of $\frac{1}{2}(\beta-\gamma)$ and of $\frac{1}{2}(\beta+\gamma)$ should be found by using the fraction from which the decimal of a minute is found. Thus,

$$\log\sin\tfrac{1}{2}(\beta+\gamma) = 9.95561 + \tfrac{29}{33}\times 6 = 9.95561 + 5 = 9.95566.$$

Let

$$\left.\begin{aligned} m \sin M &= \sin c \cos \alpha, \\ m \cos M &= \cos c. \end{aligned}\right\} \tag{3}$$

$$\therefore\ \cot \gamma = \frac{m \sin (b - M)}{\sin \alpha \sin c}. \tag{4}$$

The formula for $\cot \beta$ may be found by permuting b and c in (3) and (4).

1. $b = 105^\circ\, 14'.8,\ c = 43^\circ\, 17'.2,\ \alpha = 112^\circ\, 47'.4.$

To find a.		To find γ.		To find β.
$\log \sin c = 9.83611$	(1)	$\log \sin c =$	(1)	$\log \sin b = 9.98444$
$\log \cos \alpha = 9.58811\, n$	(3)	$\log \cos \alpha =$	(3)	$\log \cos \alpha = 9.58811\, n$
$\log (m \sin M) = 9.42422\, n$	(4)	$\log (m \sin M) =$	(5)	$\log (m \sin M) = 9.57255\, n$
$\log \sin M = 9.53499\, n$	(7)	$\log \sin M =$	(8)	$\log \sin M = 9.91266\, n$
$\log \cos M = 9.97286$	(8)	$\log \cos M =$	(9)	$\log \cos M = 9.76002\, n$
$\log \cos c = 9.86209$	(2)	$\log \cos c =$	(2)	$\log \cos b = 9.41992\, n$
$\log \tan M = 9.56213\, n$	(5)	$\log \tan M = 9.56213\, n$	(6)	$\log \tan M = 0.15263$
$M = -20^\circ\, 2'.7$	(6)	$M = -20^\circ\, 2'.7$	(7)	$M = 234^\circ\, 52'.0$
$b = 105^\circ\, 14'.8$	(10)	$b = 105^\circ\, 14'.8$	(11)	$c = 43^\circ\, 17'.2$
$b - M = 125^\circ\, 17'.5$	(11)	$b - M = 125^\circ\, 17'.5$	(12)	$c - M = -191^\circ\, 34'.8$
$\log \cos (b - M) = 9.76173\, n$	(12)	$\log \sin (b - M) = 9.91180$	(13)	$\log \sin (c - M) = 9.30263$
$\log m = 9.88923$	(9)	$\log m = 9.88923$	(10)	$\log m = 9.65989$
$\log \cos a = 9.65096\, n$	(13)	$\text{col} \sin \alpha = 0.03530$	(4)	$\text{col} \sin \alpha = 0.03530$
$a = 116^\circ\, 35'.7$	(14)	$\text{col} \sin c = 0.16389$	(14)	$\text{col} \sin b = 0.01556$
		$\log \cot \gamma = 0.00022$	(15)	$\log \cot \beta = 9.01338$
		$\gamma = 44^\circ\, 59'.1$	(16)	$\beta = 84^\circ\, 6'.7$

2. $a = 103^\circ\, 44'.7,\ b = 64^\circ\, 12'.3,\ \gamma = 98^\circ\, 33'.8.$

$\therefore\ M = 211^\circ\, 19'.8,\ c = 103^\circ\, 30'.6,\ \alpha = 98^\circ\, 56'.0;\ M = -17^\circ\, 7'.4,\ \beta = 66^\circ\, 18'.0.$

3. $a = 156^\circ\, 12'.2,\ b = 112^\circ\, 48'.6,\ \gamma = 76^\circ\, 32'.4.$

$\therefore\ M = 174^\circ\, 8'.4,\ c = 63^\circ\, 48'.9,\ \alpha = 154^\circ\, 4'.2;\ M = 151^\circ\, 2'.3,\ \beta = 87^\circ\, 27'.1.$

152. Case IV. Given Two Angles and the Included Side (α, β, c).—From (4) and (5), Art. 147, we have

$$\tan \tfrac{1}{2}(a - b) = \tan \tfrac{1}{2} c\, \frac{\sin \frac{1}{2}(\alpha - \beta)}{\sin \frac{1}{2}(\alpha + \beta)}, \tag{1}$$

$$\tan \tfrac{1}{2}(a + b) = \tan \tfrac{1}{2} c\, \frac{\cos \frac{1}{2}(\alpha - \beta)}{\cos \frac{1}{2}(\alpha + \beta)}. \tag{2}$$

Then

$$a = \tfrac{1}{2}(a + b) + \tfrac{1}{2}(a - b),$$

$$b = \tfrac{1}{2}(a + b) - \tfrac{1}{2}(a + b).$$

To obtain γ, use (6) and (7), Art. 147:

$$\cot \tfrac{1}{2}\gamma = \tan \tfrac{1}{2}(\alpha - \beta)\frac{\sin \tfrac{1}{2}(a+b)}{\sin \tfrac{1}{2}(a-b)}, \quad (3)$$

$$\cot \tfrac{1}{2}\gamma = \tan \tfrac{1}{2}(\alpha + \beta)\frac{\cos \tfrac{1}{2}(a+b)}{\cos \tfrac{1}{2}(a-b)}. \quad (4)$$

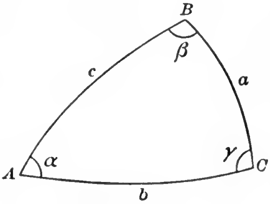


Fig. 119.

The agreement of the values of $\tfrac{1}{2}\gamma$ found from (3) and (4) is a check upon the computation. The sine proportion may also be used as a check. See the note, Art. 150.

1. Solve the triangle when $\alpha = 104° 30'.7$, $\beta = 62° 52'.1$, $c = 56° 6'.4$.

$$\begin{aligned} \alpha &= 104° 30'.7 \\ \beta &= 62° 52'.1 \\ \tfrac{1}{2}(\alpha+\beta) &= 83° 41'.4 \\ \tfrac{1}{2}(\alpha-\beta) &= 20° 49'.3 \\ \tfrac{1}{2}c &= 28° 3'.2 \end{aligned}$$

$$\begin{aligned} \log\tan\tfrac{1}{2}c &= 9.72665 \\ \log\sin\tfrac{1}{2}(\alpha-\beta) &= 9.55079 \\ \text{col}\sin\tfrac{1}{2}(\alpha+\beta) &= 0.00264 \\ \log\tan\tfrac{1}{2}(a-b) &= 9.28008 \\ \tfrac{1}{2}(a-b) &= 10° 47'.4 \end{aligned}$$

$$\begin{aligned} \log\tan\tfrac{1}{2}c &= 9.72665 \\ \log\cos\tfrac{1}{2}(\alpha-\beta) &= 9.97066 \\ \text{col}\cos\tfrac{1}{2}(\alpha+\beta) &= 0.95897 \\ \log\tan\tfrac{1}{2}(a+b) &= 0.65628 \\ \tfrac{1}{2}(a+b) &= 77° 33'.4 \\ \tfrac{1}{2}(a-b) &= 10° 47'.4 \\ a &= 88° 20'.8 \\ b &= 66° 46'.0 \end{aligned}$$

$$\begin{aligned} \log\tan\tfrac{1}{2}(\alpha-\beta) &= 9.58012 \\ \log\sin\tfrac{1}{2}(a+b) &= 9.98968 \\ \text{col}\sin\tfrac{1}{2}(a-b) &= 0.72768 \\ \log\cot\tfrac{1}{2}\gamma &= 0.29748 \\ \tfrac{1}{2}\gamma &= 26° 45'.2 \end{aligned}$$

$$\begin{aligned} \log\tan\tfrac{1}{2}(\alpha+\beta) &= 0.95633 \\ \log\cos\tfrac{1}{2}(a+b) &= 9.33339 \\ \text{col}\cos\tfrac{1}{2}(a-b) &= 0.00775 \\ \log\cot\tfrac{1}{2}\gamma &= 0.29747 \\ \tfrac{1}{2}\gamma &= 26° 45'.2 \end{aligned}$$

2. $\alpha = 140° 43'.2$, $\beta = 100° 4'.6$, $c = 60° 43'.6$.

$\therefore\ \tfrac{1}{2}(a+b) = 132° 38'.88$; $\tfrac{1}{2}(a-b) = 13° 16'.32$; $a = 145° 55'.20$;
$b = 119° 22'.56$; $\tfrac{1}{2}\gamma = 40° 7'.42$.

3. $\alpha = 140° 24'.6$, $\beta = 12° 18'.6$, $c = 28° 7'.7$.

$\therefore\ \tfrac{1}{2}(a+b) = 24° 55'.9$; $\tfrac{1}{2}(a-b) = 13° 3'.0$; $a = 37° 58'.9$;
$b = 11° 52'.9$; $\tfrac{1}{2}\gamma = 14° 36'.7$.

153. Case IV. Second Method. Given β, γ, a, to find One Element only.

(1) *To find α only.*

$$\cos\alpha = -\cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a.$$

Let

$$\left.\begin{aligned} m\sin M &= \sin\gamma\cos a, \\ m\cos M &= \cos\gamma. \end{aligned}\right\} \quad (1)$$

$$\therefore\ \cos\alpha = -m(\cos\beta\cos M - \sin\beta\sin M).$$

$$\therefore\ \cos\alpha = -m\cos(M+\beta). \quad (2)$$

(2) *To find one side only, b or c.* — Permuting (6), Art. 124,

$$\sin\beta\cot\gamma = \cot c\sin a - \cos a\cos\beta.$$

$$\therefore\ \cot c = \frac{\sin\beta\cot\gamma + \cos a\cos\beta}{\sin a} = \frac{\sin\beta\cos\gamma + \cos a\cos\beta\sin\gamma}{\sin a\sin\gamma}.$$

Let

$$\left.\begin{aligned} m\sin M &= \sin\gamma\cos a, \\ m\cos M &= \cos\gamma. \end{aligned}\right\} \qquad (3)$$

$$\therefore\ \cot c = \frac{m\sin(M+\beta)}{\sin a\sin\gamma}. \qquad (4)$$

The formula for $\cot b$ may be found by permuting b and c in (3) and (4).

1. $\beta = 140° 43'.2,\ \gamma = 100° 4'.6,\ a = 60° 43'.6.$

To find α.

$\log\sin\gamma$	$=9.99325$	(1)
$\log\cos a$	$=9.68929$	(3)
$\log(m\sin M)$	$=9.68254$	(4)
$\log\sin M$	$=9.97306$	(7)
$\log\cos M$	$=9.53348\,n$	(8)
$\log\cos\gamma$	$=9.24296\,n$	(2)
$\log\tan M$	$=0.43958\,n$	(5)
M	$=109° 58'.4$	(6)
β	$=140° 43'.2$	(10)
$M+\beta$	$=250° 41'.6$	(11)
$\log\cos(M+\beta)$	$=9.51933\,n$	(12)
$\log(-m)$	$=9.70948\,n$	(9)
$\log\cos\alpha$	$=9.22881$	(13)
α	$=80° 15'.0$	(14)

To find c.

$\log\sin\gamma$	$=$	(1)
$\log\cos a$	$=$	(3)
$\log(m\sin M)$	$=$	(5)
$\log\sin M$	$=$	(8)
$\log\cos M$	$=$	(9)
$\log\cos\gamma$	$=$	(2)
$\log\tan M$	$=0.43958\,n$	(6)
M	$=109° 58'.4$	(7)
β	$=140° 43'.2$	(11)
$M+\beta$	$=250° 41'.6$	(12)
$\log\sin(M+\beta)$	$=9.97486\,n$	(13)
$\log m$	$=9.70948$	(10)
col $\sin a$	$=0.05934$	(4)
col $\sin\gamma$	$=0.00675$	(14)
$\log\cot c$	$=9.75043\,n$	(15)
c	$=119° 22'.5$	(16)

To find b.

$\log\sin\beta$	$=9.80148$
$\log\cos a$	$=9.68929$
$\log(m\sin M)$	$=9.49077$
$\log\sin M$	$=9.56977$
$\log\cos M$	$=9.96778\,n$
$\log\cos\beta$	$=9.88877\,n$
$\log\tan M$	$=9.60200\,n$
M	$=158° 12'.1$
γ	$=100° 4'.6$
$M+\gamma$	$=258° 16'.7$
$\log\sin(M+\gamma)$	$=9.99085\,n$
$\log m$	$=9.92099$
col $\sin a$	$=0.05934$
col $\sin\beta$	$=0.19852$
$\log\cot b$	$=0.16970\,n$
b	$=145° 55'.2$

2. $\alpha = 104° 30'.7,\ \beta = 62° 52'.1,\ c = 56° 6'.4.$

$\therefore\ M = 114° 53'.9,\ \gamma = 53° 30'.5,\ a = 88° 20'.8;\ M = 47° 25'.2,\ b = 66° 46'.1.$

3. $\alpha = 140° 24'.6,\ \beta = 12° 18'.6,\ c = 28° 7'.7.$

$\therefore\ M = 143° 53'.7,\ \gamma = 29° 13'.3,\ a = 37° 58'.8;\ M = 10° 53'.6,\ b = 11° 52'.9.$

4. $\alpha = 109° 23'.5,\ \beta = 76° 47'.4,\ c = 121° 32'.8.$

$\therefore\ M = 236° 4'.1,\ \gamma = 113° 51'.9,\ a = 118° 28'.5;\ M = 294° 9'.8,\ b = 65° 7'.5.$

154. Case V. Given Two Sides and the Angle Opposite One of them (a, b, α). — Find β by the sine proportion,

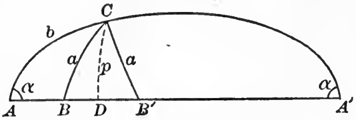


FIG. 120.

$$\sin\beta = \sin b\,\frac{\sin\alpha}{\sin a}. \tag{1}$$

Find c by (4) and (5), Art. 147,

$$\tan\tfrac{1}{2}c = \tan\tfrac{1}{2}(a-b)\,\frac{\sin\frac{1}{2}(\alpha+\beta)}{\sin\frac{1}{2}(\alpha-\beta)}, \tag{2}$$

$$\tan\tfrac{1}{2}c = \tan\tfrac{1}{2}(a+b)\,\frac{\cos\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)}. \tag{3}$$

Find γ by (6) and (7), Art. 147,

$$\cot\tfrac{1}{2}\gamma = \tan\tfrac{1}{2}(\alpha-\beta)\,\frac{\sin\frac{1}{2}(a+b)}{\sin\frac{1}{2}(a-b)}, \tag{4}$$

$$\cot\tfrac{1}{2}\gamma = \tan\tfrac{1}{2}(\alpha+\beta)\,\frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}(a-b)}. \tag{5}$$

The agreement of the values of $\frac{1}{2}c$ and of $\frac{1}{2}\gamma$ is a check upon the computation.

Since β is found by means of its sine, it may be either in the first or in the second quadrant; hence there may be two solutions. If b differs more than a from 90°, β must be of the same species as b, and the quadrant in which β lies is fixed. But if b does not differ more than a from 90°, we cannot determine by the first rule for species the quadrant in which β must lie, and both values of β may be admissible. Hence, *inspect for two solutions when the side opposite the required angle differs less from* 90° *than the side opposite the given angle.* After finding β, the second rule of Art. 146 will show whether both values are admissible.

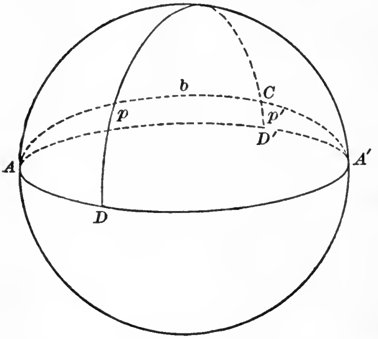


FIG. 121.

1. Solve the triangle when $a = 148°\,34'.4$, $b = 142°\,11'.6$, $\alpha = 153°\,17'.6$. Since b differs less than a from 90°, there may be two solutions.

$\log \sin b = 9.78746$	$\frac{1}{2}(a+b) = 145°\,23'.0$	$\frac{1}{2}(a-b) = 3°\,11'.4$
$\log \sin \alpha = 9.65265$	$\frac{1}{2}(\alpha+\beta) = 92°\,35'.65$	$\frac{1}{2}(\alpha+\beta') = 150°\,41'.95$
$\text{col} \sin a = \underline{0.28282}$	$\frac{1}{2}(\alpha-\beta) = 60°\,41'.95$	$\frac{1}{2}(\alpha-\beta') = 2°\,35'.65$
$\log \sin \beta = 9.72293$		
$\beta = 31°\,53'.7$		
$\beta' = \underline{148°\;\;6'.3}$		

The second rule for species is satisfied for both β and β'; hence there are two solutions.

	First.		Second.
$\log \tan \frac{1}{2}(a-b) =$	8.74612	or	8.74612
$\log \sin \frac{1}{2}(\alpha+\beta) =$	9.99955		9.68966
$\text{col} \sin \frac{1}{2}(\alpha-\beta) =$	0.05945		1.34427
$\log \tan \frac{1}{2} c =$	8.80512		9.78005
$\frac{1}{2} c =$	$3°\,39'.18$		$31°\,4'.46$
$c =$	$7°\,18'.36$	$c' =$	$62°\,8'.92$
$\log \tan \frac{1}{2}(a+b) =$	$9.83903\,n$	or	$9.83903\,n$
$\log \cos \frac{1}{2}(\alpha+\beta) =$	$8.65573\,n$		$9.94055\,n$
$\text{col} \cos \frac{1}{2}(\alpha-\beta) =$	0.31034		0.00045
$\log \tan \frac{1}{2} c =$	8.80510		9.78003
$\frac{1}{2} c =$	$3°\,39'.17$		$31°\,4'.39$
$c =$	$7°\,18'.34$	$c' =$	$62°\,8'.78$
$\log \tan \frac{1}{2}(\alpha-\beta) =$	0.25089	or	8.65617
$\log \sin \frac{1}{2}(a+b) =$	9.75441		9.75441
$\text{col} \sin \frac{1}{2}(a-b) =$	1.25456		1.25456
$\log \cot \frac{1}{2}\gamma =$	1.25986		9.66514
$\frac{1}{2}\gamma =$	$3°\;\;8'.79$		$65°\,10'.68$
$\gamma =$	$6°\,17'.58$	$\gamma' =$	$130°\,21'.36$
$\log \tan \frac{1}{2}(\alpha+\beta) =$	$1.34383\,n$	or	$9.74911\,n$
$\log \cos \frac{1}{2}(a+b) =$	$9.91538\,n$		$9.91538\,n$
$\text{col} \cos \frac{1}{2}(a-b) =$	0.00067		0.00067
$\log \cot \frac{1}{2}\gamma =$	1.25988		9.66516
$\frac{1}{2}\gamma =$	$3°\;\;8'.78$		$65°\,10'.62$
$\gamma =$	$6°\,17'.56$	$\gamma' =$	$130°\,21'.24$

2. $a = 40°\,20'.4,\;\; b = 20°\,18'.2,\;\; \alpha = 60°\,44'.4.$

$\therefore\;\; \beta = 27°\,52'.9\,;\;\; \frac{1}{2}c = 23°\,34'.34\,;\;\; \frac{1}{2}\gamma = 49°\,26'.7.$

3. $a = 98°\,16',\;\; b = 74°\,38',\;\; \alpha = 78°\,40'.$

$\therefore\;\; \beta = 72°\,49'.25\,;\;\; \frac{1}{2}c = 75°\,53'.0$ or $75°\,52'.6$*; $\frac{1}{2}\gamma = 76°\,1'.5$ or $76°\,1'.1$.*

155. Case V. Second Method. Given a, b, α, to find One Element only.

(1) *To find β only.*

$$\sin \beta = \frac{\sin b}{\sin a} \sin \alpha. \qquad (1)$$

* These values would be taken, since a small error in β will affect them less than if they had been computed from the other formulas.

(2) *To find c only.*

$$\cos b \cos c + \sin b \sin c \cos \alpha = \cos a.$$

Let
$$\left.\begin{aligned} m \sin M &= \sin b \cos \alpha, \\ m \cos M &= \cos b \end{aligned}\right\} \qquad (2)$$

$$\therefore\ m \cos (M - c) = \cos a.$$

$$\therefore\ \cos (M - c) = \frac{\cos a}{m}. \qquad (3)$$

$M - c$ may be either in the first and fourth quadrants or in the second and third; if there are two solutions both values of $M - c$ will give $c < 180°$.

(3) *To find γ only.* — From (7), Art. 124,

$$\cos b \cos \gamma + \sin \gamma \cot \alpha = \cot a \sin b.$$

$$\therefore\ \cos b \sin \alpha \cos \gamma + \sin \gamma \cos \alpha = \cot a \sin b \sin \alpha.$$

Let
$$\left.\begin{aligned} m \sin M &= \cos b \sin \alpha, \\ m \cos M &= \cos \alpha. \end{aligned}\right\} \qquad (4)$$

$$\therefore\ m \sin (M + \gamma) = \cot a \sin b \sin \alpha.$$

$$\therefore\ \sin (M + \gamma) = \frac{\cot a \sin b \sin \alpha}{m}. \qquad (5)$$

$M + \gamma$ may be either in the first and second quadrants or in the third and fourth; if there are two solutions, both values of $M + \gamma$ will give $\gamma < 180°$.

1. $a = 148° 34'.4,\ b = 142° 11'.6,\ \alpha = 153° 17'.6.$

To find c.

$\log \sin b =$	9.78746	(1)
$\log \cos \alpha =$	$9.95101\ n$	(3)
$\log (m \sin M) =$	$9.73847\ n$	(4)
$\log \sin M =$	$9.75561\ n$	(7)
$\log \cos M =$	$9.91481\ n$	(8)
$\log \cos b =$	$9.89767\ n$	(2)
$\log \tan M =$	9.84080	(5)
$M =$	$214° 43'.6$	(6)
$\text{colog}\, m =$	0.01714	(9)
$\log \cos a =$	$9.93111\ n$	(10)
$\log \cos (M - c) =$	$9.94825\ n$	(11)
$M - c =$	$152° 34'.8$	(12)
$M =$	$214° 43'.6$	(14)
$M - c' =$	$207° 25'.2$	(13)
$\therefore\ c =$	$62° 8'.8$	(15)
and $c' =$	$7° 18'.4$	(16)

Two values.

To find γ.

$\log \cos b =$	$9.89767\ n$	(1)
$\log \sin \alpha =$	9.65265	(3)
$\log (m \sin M) =$	$9.55032\ n$	(5)
$\log \sin M =$	$9.56746\ n$	(8)
$\log \cos M =$	$9.96815\ n$	(9)
$\log \cos \alpha =$	$9.95101\ n$	(4)
$\log \tan M =$	9.59931	(6)
$M =$	$201° 40'.6$	(7)
$\text{colog}\, m =$	0.01714	(10)
$\log \cot a =$	$0.21393\ n$	(11)
$\log \sin b =$	9.78746	(2)
$\log \sin \alpha =$	9.65265	(3)
$\log \sin (M + \gamma) =$	$9.67118\ n$	(12)
$M + \gamma =$	$207° 58'.2$	(13)
$M =$	$201° 40'.6$	(15)
$M + \gamma' =$	$332° 1'.8$	(14)
$\therefore\ \gamma =$	$6° 17'.6$	(16)
and $\gamma' =$	$130° 21'.2$	(17)

Two values.

2. $a = 40° 20'.4$, $b = 20° 18'.2$, $\alpha = 60° 44'.4$.

$\therefore M = 10° 15'.0$, $c = 47° 8'.7$; $M = 59° 8'.8$, $\gamma = 98° 53'.5$.

3. $a = 98° 16'$, $b = 74° 38'$, $\alpha = 78° 40'$.

$\therefore M = 35° 34'.0$, $c = 151° 45'.4$; $M = 52° 53'.9$, $\gamma = 152° 2'.5$.

156. Case VI. Given Two Angles and the Side Opposite One of them (α, β, *a*).—Find b by the sine proportion,

$$\sin b = \sin\beta \frac{\sin a}{\sin\alpha}. \tag{1}$$

Find c by (4) and (5), Art. 147,

$$\tan\tfrac{1}{2}c = \tan\tfrac{1}{2}(a-b)\frac{\sin\frac{1}{2}(\alpha+\beta)}{\sin\frac{1}{2}(\alpha-\beta)}, \tag{2}$$

$$\tan\tfrac{1}{2}c = \tan\tfrac{1}{2}(a+b)\frac{\cos\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)}. \tag{3}$$

Find γ by (6) and (7), Art. 147,

$$\cot\tfrac{1}{2}\gamma = \tan\tfrac{1}{2}(\alpha-\beta)\frac{\sin\frac{1}{2}(a+b)}{\sin\frac{1}{2}(a-b)}, \tag{4}$$

$$\cot\tfrac{1}{2}\gamma = \tan\tfrac{1}{2}(\alpha+\beta)\frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}(a-b)}. \tag{5}$$

The agreement of the values of $\frac{1}{2}c$ and of $\frac{1}{2}\gamma$ is a check upon the computation.

Since b is found by means of its sine, it may be either in the first or in the second quadrant; hence there may be two solutions. If β differs more than α from 90°, β and b must be of the same species, and the quadrant in which b lies is fixed. But if β does not differ more than α from 90°, we cannot determine by the first rule for species the quadrant in which b must lie, and both values of b may be admissible. Hence, *inspect for two solutions when the angle opposite the required side differs less from* 90° *than the angle opposite the given side.* After finding b, the second rule of Art. 146 will show whether both values are admissible.

1. Solve the triangle when $\alpha = 143° 17'.4$, $\beta = 70° 18'.4$, $a = 160° 40'.6$. Since β differs less than α from 90°, there may be two solutions.

$\log\sin a = 9.51969$	$\frac{1}{2}(\alpha+\beta) = 106° 47'.9$	$\frac{1}{2}(\alpha-\beta) = 36° 29'.5$
$\log\sin\beta = 9.97383$	$\frac{1}{2}(a+b) = 96° 2'.65$	$\frac{1}{2}(a+b') = 154° 37'.95$
$\text{col}\sin\alpha = 0.22347$	$\frac{1}{2}(a-b) = 64° 37'.95$	$\frac{1}{2}(a-b') = 6° 2'.65$
$\log\sin b = 9.71699$		
$b = 31° 24'.7$		
$b' = 148° 35'.3$		

The second rule for species is satisfied for both b and b'; hence there are two solutions.

	First.		Second.
$\log\tan\frac{1}{2}(a-b)=$	0.32409	or	9.02483
$\log\sin\frac{1}{2}(\alpha+\beta)=$	9.98106		9.98106
$\text{col}\sin\frac{1}{2}(\alpha-\beta)=$	0.22570		0.22570
$\log\tan\frac{1}{2}c=$	0.53085		9.23159
$\frac{1}{2}c=$	$73^\circ 35'.28$		$9^\circ 40'.38$
$c=$	$147^\circ 10'.56$	$c'=$	$19^\circ 20'.76$
$\log\tan\frac{1}{2}(a+b)=$	$0.97517\,n$	or	$9.67591\,n$
$\log\cos\frac{1}{2}(\alpha+\beta)=$	$9.46091\,n$		$9.46091\,n$
$\text{col}\cos\frac{1}{2}(\alpha-\beta)=$	0.09478		0.09478
$\log\tan\frac{1}{2}c=$	0.53086		9.23160
$\frac{1}{2}c=$	$73^\circ 35'.30$		$9^\circ 40'.39$
$c=$	$147^\circ 10'.60$	$c'=$	$19^\circ 20'.78$
$\log\tan\frac{1}{2}(\alpha-\beta)=$	9.86908	or	9.86908
$\log\sin\frac{1}{2}(a+b)=$	9.99758		9.63187
$\text{col}\sin\frac{1}{2}(a-b)=$	0.04403		0.97759
$\log\cot\frac{1}{2}\gamma=$	9.91069		0.47854
$\frac{1}{2}\gamma=$	$50^\circ 51'.00$		$18^\circ 22'.74$
$\gamma=$	$101^\circ 42'.00$	$\gamma'=$	$36^\circ 45'.48$
$\log\tan\frac{1}{2}(\alpha+\beta)=$	$0.52016\,n$	or	$0.52016\,n$
$\log\cos\frac{1}{2}(a+b)=$	$9.02241\,n$		$9.95597\,n$
$\text{col}\cos\frac{1}{2}(a-b)=$	0.36813		0.00242
$\log\cot\frac{1}{2}\gamma=$	9.91070		0.47855
$\frac{1}{2}\gamma=$	$50^\circ 50'.96$		$18^\circ 22'.71$
$\gamma=$	$101^\circ 41'.92$	$\gamma'=$	$36^\circ 45'.42$

2. $\alpha=117^\circ 54'.4,\ \beta=45^\circ 8'.6,\ a=76^\circ 37'.5.$

$\therefore\ b=51^\circ 17'.9;\ \frac{1}{2}c=20^\circ 32'.3$ or $20^\circ 32'.4;\ \frac{1}{2}\gamma=18^\circ 19'.4.$

3. $\alpha=104^\circ 40'.0,\ \beta=80^\circ 13'.6,\ a=126^\circ 50'.4.$

$\therefore\ b=54^\circ 36'.8;\ \frac{1}{2}c=73^\circ 48'.4$ or $73^\circ 48'.5;\ \frac{1}{2}\gamma=69^\circ 49'.5$ or $69^\circ 49'.6;$

and $b'=125^\circ 23'.2;\ \frac{1}{2}c'=3^\circ 25'.6$ or $3^\circ 25'.5;\ \frac{1}{2}\gamma'=4^\circ\ 8'.8.$

157. Case VI. Second Method. Given α, β, a, to find One Element only.

(1) *To find b only.*

$$\sin b=\frac{\sin\beta}{\sin\alpha}\sin a. \qquad (1)$$

(2) *To find c only.*—Permuting (3), Art. 124,

$$\cot a\sin c-\cos c\cos\beta=\sin\beta\cot\alpha.$$

$$\therefore\ \cos a\sin c-\sin a\cos c\cos\beta=\sin a\sin\beta\cot\alpha.$$

Let
$$\left.\begin{aligned} m\sin M &= \sin a\cos\beta, \\ m\cos M &= \cos a. \end{aligned}\right\} \qquad (2)$$

$$\therefore\ m\sin(c-M) = \sin a\sin\beta\cot\alpha.$$

$$\therefore\ \sin(c-M) = \frac{\sin a\sin\beta\cot\alpha}{m}. \qquad (3)$$

$c-M$ may be either in the first and second quadrants, or in the third and fourth; if there are two solutions, both values of $c-M$ will give $c<180°$.

(3) *To find* γ *only.*

$$-\cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a = \cos\alpha.$$

Let
$$\left.\begin{aligned} m\sin M &= \cos a\sin\beta, \\ m\cos M &= \cos\beta. \end{aligned}\right\} \qquad (4)$$

$$\therefore\ m\cos(M+\gamma) = -\cos\alpha.$$

$$\therefore\ \cos(M+\gamma) = -\frac{\cos\alpha}{m}. \qquad (5)$$

$M+\gamma$ may be either in the first and fourth quadrants, or in the second and third; if there are two solutions, both values of $M+\gamma$ will give $\gamma<180°$.

1. $\alpha = 143°\,17'.4,\ \beta = 70°\,18'.4,\ a = 160°\,40'.6.$

To find c.

$\log\sin a$	$= 9.51969$	(1)
$\log\cos\beta$	$= 9.52761$	(3)
$\log(m\sin M)$	$= 9.04730$	(5)
$\log\sin M$	$= 9.06947$	(8)
$\log\cos M$	$= 9.99699\,n$	(9)
$\log\cos a$	$= 9.97482\,n$	(2)
$\log\tan M$	$= 9.07248\,n$	(6)
M	$= 173°\,15'.7$	(7)
$\operatorname{colog} m$	$= 0.02217$	(10)
$\log\sin a$	$= 9.51969$	(1)
$\log\sin\beta$	$= 9.97383$	(4)
$\log\cot\alpha$	$= 0.12746\,n$	(11)
$\log\sin(c-M)$	$= 9.64315\,n$	(12)
$c-M$	$= 206°\,5'.1$	(13)
M	$= 173°\,15'.7$	(15)
$c'-M$	$= 333°\,54'.9$	(14)
c	$= 19°\,20'.8$	(16)
c'	$= 147°\,10'.6$	(17)

Two values.

To find γ.

$\log\cos a$	$= 9.97482\,n$	(1)
$\log\sin\beta$	$= 9.97383$	(2)
$\log(m\sin M)$	$= 9.94865\,n$	(4)
$\log\sin M$	$= 9.97082\,n$	(7)
$\log\cos M$	$= 9.54977$	(8)
$\log\cos\beta$	$= 9.52761$	(3)
$\log\tan M$	$= 0.42104\,n$	(5)
M	$= 290°\,46'.2$	(6)
$\operatorname{colog}(-m)$	$= 0.02217\,n$	(9)
$\log\cos\alpha$	$= 9.90400\,n$	(10)
$\log\cos(M+\gamma)$	$= 9.92617$	(11)
$M+\gamma$	$= 32°\,28'.2$	(12)
M	$= 290°\,46'.2$	(14)
$M+\gamma'$	$= 327°\,31'.8$	(13)
γ	$= 101°\,42'.0$	(15)
γ'	$= 36°\,45'.6$	(16)

Two values.

2. $\alpha = 117° 54'.4,\ \beta = 45° 8'.6,\ a = 76° 37'.5.$

$\therefore\ M = 71° 22'.3,\ c = 41° 4'.9;\ M = 13° 5'.3,\ \gamma = 36° 38'.8.$

3. $\alpha = 104° 40'.0,\ \beta = 80° 13'.6,\ a = 126° 50'.4.$

$\therefore\ M = 167° 14'.0,\ c = 147° 36'.9$ or $6° 51'.1$;
$M = -73° 58'.3,\ \gamma = 139° 39'.0$ or $8° 17'.6.$

OBLIQUE TRIANGLES SOLVED BY RIGHT TRIANGLES.

158. General Method. — From any vertex C of the triangle draw an arc p of a great circle perpendicular to the opposite

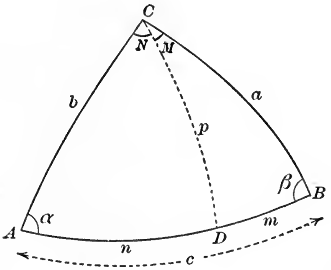


Fig. 122.

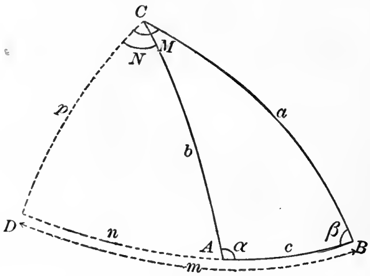


Fig. 123.

side, dividing the triangle into two right triangles. Denote the segments of the side by m and n, and the corresponding segments of the angle by M and N.

The opposite side must in some cases be produced to meet the perpendicular arc, as in Fig. 123. The segments of the side are AD and DB, and their signs are so taken that their algebraic sum shall be equal to the side; that is, *if a segment is entirely exterior to the triangle, it is negative.*

The perpendicular p may have either of two supplemental values; we shall always place it in the same quadrant as its opposite angle in the triangle first used in the solution, in accordance with the rule for species.

159. Special Formula. — To prove

$$\tan \tfrac{1}{2}(m + n) \tan \tfrac{1}{2}(m - n) = \tan \tfrac{1}{2}(a + b) \tan \tfrac{1}{2}(a - b). \qquad (1)$$

In both Fig. 122 and Fig. 123, by Napier's rules,

$$\cos a = \cos m \cos p, \quad \text{and} \quad \cos b = \cos n \cos p.$$

$$\therefore \cos p = \frac{\cos a}{\cos m} = \frac{\cos b}{\cos n}.$$

$$\therefore \frac{\cos m}{\cos n} = \frac{\cos a}{\cos b}.$$

$$\therefore \frac{\cos m - \cos n}{\cos m + \cos n} = \frac{\cos a - \cos b}{\cos a + \cos b},$$

which becomes, from (4) and (3), Art. 73,

$$\tan \tfrac{1}{2}(m+n) \tan \tfrac{1}{2}(m-n) = \tan \tfrac{1}{2}(a+b) \tan \tfrac{1}{2}(a-b). \quad \text{Q.E.D.}$$

160. Case I. Given a, b, c. — From (1), Art. 159, we have

$$\tan \tfrac{1}{2}(m-n) = \tan \tfrac{1}{2}(a+b) \tan \tfrac{1}{2}(a-b) \cot \tfrac{1}{2} c, \qquad (1)$$

since $m + n = c$. We shall consider $\frac{1}{2}(m-n)$ as being numerically less than 90°, so that it will be a negative angle when its tangent is negative. After $\frac{1}{2}(m-n)$ has been found, we have

$$\left.\begin{aligned} m &= \tfrac{1}{2}c + \tfrac{1}{2}(m-n), \\ n &= \tfrac{1}{2}c - \tfrac{1}{2}(m-n). \end{aligned}\right\} \qquad (2)$$

A negative value of m or of n indicates that the segment, and hence the corresponding triangle, is exterior to the given triangle. Note that m is always measured from the side that is called a, and n from b.

In the triangles ACD and DCB we now know the two sides, so that the other elements can be found by Napier's rules. The example shows the method of finding the elements of the original triangle from the results of the computation.

1. $a = 114° 43'.3$, $b = 136° 19'.6$, $c = 43° 18'.5$.

From (1), $\frac{1}{2}(m-n) = 33° 56'.81$, whence $m = 55° 36'.06$, $n = -12° 17'.56$. The negative value of n shows that ACD is exterior to the triangle.

From BCD we find $DBC = \beta = 132° 15'.3$, $DCB = M = 65° 17'.0$.

From ACD we find $DAC = 180° - \alpha = 103° 11'.6$, $ACD = N = -17° 57'.5$, giving N the negative sign since it is exterior to the triangle. Hence

$$\alpha = 76° 48'.4; \quad \gamma = M + N = 47° 19'.5.$$

2. $a = 76° 40'.4$, $b = 54° 21'.3$, $c = 36° 8'.7$.

$\therefore \frac{1}{2}(m-n) = 53° 0'.38$; $m = 71° 4'.73$; $n = -34° 56'.03$; $\beta = 46° 17'.3$; $M = 76° 27'.0$; $N = -44° 48'.2$; $\gamma = 31° 38'.8$; $\alpha = 120° 3'.6$.

3. $a = 124° 34'.9,\ b = 66° 7'.2,\ c = 109° 43'.5.$

$\therefore\ \frac{1}{2}(m-n) = -76° 37'.32;\ m = -21° 45'.57;\ n = +131° 29'.07;\ \beta = 74° 1'.7;$
$M = -26° 45'.6;\ \alpha = 120° 2'.7;\ N = +124° 59'.2;\ \gamma = 98° 13'.6.$

4. $a = 30° 17'.6,\ b = 22° 14'.4,\ c = 18° 51'.8.$

$\therefore\ m = 21° 14'.6;\ n = -2° 22'.8;\ \beta = 48° 17'.1;\ M = 45° 54'.8;$
$\alpha = 95° 50'.0;\ N = -6° 17'.9;\ \gamma = 39° 36'.9.$

5. $a = 130° 46'.0,\ b = 113° 21'.4,\ c = 102° 16'.2.$

$\therefore\ \frac{1}{2}(m-n) = -11° 8'.6;\ m = 39° 59'.5;\ n = 62° 16'.7;\ \beta = 136° 19'.25;$
$M = 58° 3'.4;\ \alpha = 145° 15'.9;\ N = 74° 37'.75;\ \gamma = 132° 41'.2.$

161. Case II. Given α, β, γ. — Apply the method of Case I to the polar triangle, and thence find the elements of the original triangle.

1. $\alpha = 116° 19'.4,\ \beta = 83° 19'.2,\ \gamma = 106° 10'.6.$

In the polar triangle,

$$a' = 63° 40'.6,\ b' = 96° 40'.8,\ c' = 73° 49'.4.$$

$$\therefore\ \tfrac{1}{2}(m'-n') = -66° 18'.1,\ m' = -29° 23'.4,\ n' = +103° 12'.8.$$

The negative value of m' shows that $B'C'D'$ is exterior to the triangle.

From $B'C'D'$ we find

$$D'B'C' = 180° - \beta' = 73° 49'.2,\ D'C'B' = M' = -33° 11'.8,$$

giving M' the negative sign since it is exterior to the triangle.

From $A'C'D'$ we find

$$D'A'C' = \alpha' = 60° 4'.7,\ N' = +101° 25'.5.$$

$$\therefore\ \beta' = 106° 10'.8,\ \gamma' = M' + N' = 68° 13'.7.$$

Passing from the polar to the original triangle,

$$a = 119° 55'.3;\ b = 73° 49'.2;\ c = 111° 46'.3.$$

2. $\alpha = 110° 36'.4,\ \beta = 122° 8'.7,\ \gamma = 140° 20'.3.$

$\therefore\ \frac{1}{2}(m'-n') = 29° 27'.90;\ m' = 49° 17'.75;\ n' = -9° 38'.05;$
$\beta' = 64° 4'.9;\ M' = 54° 5'.4;\ \alpha' = 96° 7'.4;\ N' = -11° 24'.0;\ \gamma' = 42° 41'.4;$
$\therefore\ a = 83° 52'.6,\ b = 115° 55'.1,\ c = 137° 18'.6.$

3. $\alpha = 120° 50'.6,\ \beta = 78° 6'.1,\ \gamma = 81° 12'.3.$

$\therefore\ \frac{1}{2}(m'-n') = -63° 33'.19;\ m' = -14° 9'.34;\ n' = 112° 57'.04;$
$\beta' = 98° 39'.7;\ M' = -16° 33'.0;\ \alpha' = 60° 9'.6;$
$N' = 109° 46'.0;\ \gamma' = 93° 13'.0;$
$\therefore\ a = 119° 50'.4,\ b = 81° 20'.3,\ c = 86° 47'.0.$

4. $\alpha = 80° 20'.2,\ \beta = 73° 46'.7,\ \gamma = 54° 8'.5.$

$\therefore\ \frac{1}{2}(m'-n') = 7° 15'.69;\ m' = 70° 11'.44;\ n' = 55° 40'.06;$
$\beta' = 118° 12'.7;\ M' = 72° 37'.5;\ \alpha' = 115° 12'.8;$
$N' = 59° 19'.1;\ \gamma' = 131° 56'.6;$
$\therefore\ a = 64° 47'.2,\ b = 61° 47'.3,\ c = 48° 3'.4.$

5. $\alpha = 100° 51'.3$, $\beta = 80° 47'.6$, $\gamma = 74° 3'.3$.

$\therefore \frac{1}{2}(m' - n') = -83° 50'.76$; $m' = -30° 52'.41$; $n' = 136° 49'.11$;

$\beta' = 96° 35'.0$; $M' = -31° 30'.0$; $\alpha' = 81° 15'.1$;

$N' = 136° 6'.8$; $\gamma' = 104° 36'.8$;

$\therefore a = 98° 44'.9$, $b = 83° 25'.0$, $c = 75° 23'.2$.

162. Case III. Given a, b, γ. — From the end of one of the sides, as b, let fall an arc of a great circle perpendicular to the other side. In the triangle DAC we know b and γ; hence we find n, N, and p by Napier's rules, considering p as of the same species as γ. Then $m = a - n$, being negative when $n > a$, showing that the triangle BAD is then exterior to the triangle BAC.

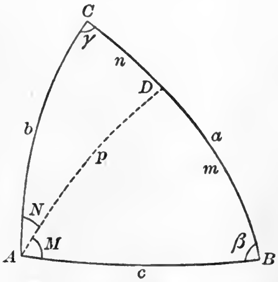


FIG. 124.

Now in the triangle BAD we know DB and AD, and we find c, M, and ABD by Napier's rules.

1. $a = 105° 14'.8$, $b = 43° 17'.2$, $\gamma = 112° 47'.4$.

$\therefore n = 159° 57'.3$, $N = 150° 0'.4$, $p = 140° 47'.53$.

$\therefore m = -54° 42'.5$, showing that BAD is exterior to BAC.

In the triangle BAD we find

$ABD = 180° - \beta = 135° 0'.8$, $c = 116° 35'.6$, $M = -65° 53'.7$,

giving M the negative sign since it is exterior to the triangle.

Hence $\beta = 44° 59'.2$, $\alpha = N + M = 84° 6'.7$.

2. $a = 103° 44'.7$, $b = 64° 12'.3$, $\gamma = 98° 33'.8$.

$\therefore N = 160° 54'.7$; $n = 162° 52'.6$; $p = 117° 5'.1$; $m = -59° 7'.9$;

$c = 103° 30'.6$; $\beta = 66° 18'.0$; $M = -61° 58'.7$; $\alpha = 98° 56'.0$.

3. $a = 156° 12'.2$, $b = 112° 48'.6$, $\gamma = 76° 32'.4$.

$\therefore N = 148° 18'.6$; $n = 151° 2'.3$; $p = 63° 41'.8$; $m = 5° 9'.9$;

$c = 63° 48'.8$; $\beta = 87° 27'.1$; $M = 5° 45'.5$; $\alpha = 154° 4'.1$.

163. Case IV. Given α, β, c. — Let fall from the vertex of one of the angles, as $\alpha = BAC$ (Fig. 124), an arc of a great circle perpendicular to the opposite side. In the triangle ABD we know c and β, and we find m, M, and p by Napier's rules, considering p as of the same species as β. Then $N = \alpha - M$, a negative value of N showing that the point D lies on BC produced, the triangle ACD being then exterior to the given triangle.

In the triangle ACD we now know p and CAD, and we find b, γ, and n by Napier's rules.

1. $\alpha = 140° \, 43'.2, \ \beta = 100° \, 4'.6, \ c = 60° \, 43'.6.$

$$\therefore \ m = 162° \, 39'.9, \ p = 120° \, 48'.86, \ M = 160° \, 1'.7.$$

Then $N = \alpha - M = -19° \, 18'.5.$

$$\therefore \ b = 119° \, 22'.5, \ ACD = 180° - \gamma = 99° \, 45'.1, \ n = -16° \, 44'.8,$$

giving n the negative sign since it is exterior to the triangle.

$$\therefore \ \gamma = 80° \, 14'.9, \ a = m + n = 145° \, 55'.1.$$

2. $\alpha = 104° \, 30'.7, \ \beta = 62° \, 52'.1, \ c = 56° \, 6'.4.$

$$\therefore \ M = 42° \, 34'.8 \, ; \ N = 61° \, 55'.9 \, ; \ m = 34° \, 10'.2 \, ; \ p = 47° \, 37'.5 \, ;$$

$$b = 66° \, 46'.0 \, ; \ \gamma = 53° \, 30'.4 \, ; \ n = 54° \, 10'.7 \, ; \ a = 88° \, 20'.9.$$

3. $\alpha = 140° \, 24'.6, \ \beta = 12° \, 18'.6, \ c = 28° \, 7'.7.$

$$\therefore \ M = 79° \, 6'.4 \, ; \ N = 61° \, 18'.2 \, ; \ m = 27° \, 34'.7 \, ; \ p = 5° \, 46'.1 \, ;$$

$$b = 11° \, 52'.9 \, ; \ \gamma = 29° \, 13'.3 \, ; \ n = 10° \, 24'.3 \, ; \ a = 37° \, 59'.0.$$

164. Case V. Given *a, b,* α. — Let fall an arc of a great circle from the intersection of a and b, perpendicular to c. In this case there will be two solutions if a is intermediate in value between p and both b and $180° - b$ (Art. 120).

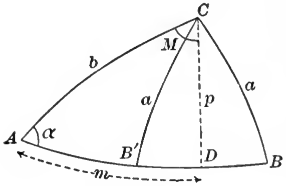


FIG. 125.

In the triangle ACD, knowing b and α, find m, M, and p by Napier's rules. Then in the triangle DCB, knowing p and a, find DB, DCB, and DBC. Then in the triangle ACB we have

$$c = AB = m + DB, \ \gamma = ACB = M + DCB, \ \beta = DBC;$$

and in the triangle ACB',

$$c' = AB' = m - DB, \ \gamma' = ACB' = M - DCB, \ \beta' = 180° - DBC.$$

1. $a = 148° \, 34'.4, \ b = 142° \, 11'.6, \ \alpha = 153° \, 17'.6.$

$\therefore \ p = 164° \, 0'.52$, and there are two solutions.

$$m = 34° \, 43'.5, \ M = 68° \, 19'.4.$$

Also, $DB = 27° \, 25'.1, \ DBC = 148° \, 6'.3, \ DCB = 62° \, 1'.8.$

$$\therefore \ c = 62° \, 8'.6, \ \gamma = 130° \, 21'.2, \ \beta = 148° \, 6'.3,$$

and $c' = 7° \, 18'.4, \ \gamma' = 6° \, 17'.6, \ \beta' = 31° \, 53'.7.$

2. $a = 40° \, 20'.4$, $b = 20° \, 18'.2$, $\alpha = 60° \, 44'.4$.

$\therefore \; p = 17° \, 37'.3$; $m = 10° \, 15'.0$; $M = 30° \, 51'.2$; $\beta = 27° \, 52'.9$;

$DB = 36° \, 53 .7$; $DCB = 68° \, 2'.3$; $c = 47° \, 8'.7$; $\gamma = 98° \, 53'.5$.

3. $a = 98° \, 16'$, $b = 74° \, 38'$, $\alpha = 78° \, 40'$.

$\therefore \; p = 70° \, 59'.25$; $m = 35° \, 34'.0$; $M = 37° \, 6'.1$; $\beta = 72° \, 49'.25$;

$DB = 116° \, 11'.4$; $DCB = 114° \, 56'.4$; $c = 151° \, 45'.4$; $\gamma = 152° \, 2'.5$.

165. Case VI. Given α, β, *a*.—Pass to the polar triangle, in which we shall know a', b', and a', and solve by the method of Art. 164. There may be two solutions of the polar triangle, and therefore of the triangle itself.

1. $\alpha = 143° \, 17'.4$, $\beta = 70° \, 18'.4$, $a = 160° \, 40'.6$.

$\therefore \; a' = 36° \, 42'.6$, $b' = 109° \, 41'.6$, $\alpha' = 19° \, 19'.4$.

$\therefore \; p' = 18° \, 9'.13$, and there will be two solutions.

$M' = 96° \, 44'.3$, $m' = 110° \, 46'.3$.

Also $D'B' = 32° \, 28'.25$, $D'C'B' = 63° \, 54'.9$, $D'B'C' = 31° \, 24'.7$.

$\therefore \; c_1' = 143° \, 14'.55$, $\gamma_1' = 160° \, 39'.2$, $\beta_1' = 31° \, 24'.7$,

and $c_1'' = 78° \, 18'.05$, $\gamma_1'' = 32° \, 49'.4$, $\beta_1'' = 148° \, 35'.3$.

Taking the supplements to obtain the elements of the original triangle,

$\gamma = 36° \, 45'.45$, $c = 19° \, 20'.8$, $b = 148° \, 35'.3$,

and $\gamma' = 101° \, 41'.95$, $c' = 147° \, 10'.6$, $b' = 31° \, 24'.7$.

2. $\alpha = 117° \, 54'.4$, $\beta = 45° \, 8'.6$, $a = 76° \, 37'.5$.

$\therefore \; p' = 136° \, 23'.8$; $M' = 18° \, 37'.7$; $m' = 13° \, 5'.3$;

$D'C'B' = 120° \, 17'.5$; $D'B'C' = 128° \, 42'.1$; $D'B' = 130° \, 15'.9$;

$\gamma' = 138° \, 55'.2$; $c' = 143° \, 21'.2$;

$\therefore \; b = 51° \, 17'.9$; $c = 41° \, 4'.8$; $\gamma = 36° \, 38'.8$.

3. $\alpha = 104° \, 40'.0$, $\beta = 80° \, 13'.6$, $a = 126° \, 50'.4$.

$\therefore \; p' = 52° \, 3'.8$; $M' = 102° \, 46'.0$; $m' = 106° \, 1'.7$;

$D'C'B' = 70° \, 22'.9$; $D'B'C' = 54° \, 36'.8$; $D'B' = 65° \, 40'.7$;

$\gamma_1' = 172° \, 8'.9$; $\gamma_1'' = 32° \, 23'.1$; $c_1' = 171° \, 42'.4$;

$c_1'' = 40° \, 21'.0$; $\beta_1' = 54° \, 36'.8$; $\beta_1'' = 125° \, 23'.2$.

$\therefore \; c = 7° \, 51'.1$; $\gamma = 8° \, 17'.6$; $b = 125° \, 23'.2$;

and $c' = 147° \, 36'.9$; $\gamma' = 139° \, 39'.0$; $b' = 54° \, 36'.8$.

CHAPTER XII.

APPLICATIONS OF SPHERICAL TRIGONOMETRY.

166. To find the Shortest Distance between Two Points on the Surface of the Earth,* the earth being treated as a sphere. — North latitudes and west longitudes are considered positive. Let QQ' be the equator, P the north pole, A and B the two points, and PM the meridian from which the longitudes are measured. The longitude of A is MPC and that of B is MPD, both being positive since they are measured westward. The latitudes are CA and DB, the former being negative since it is measured southward.

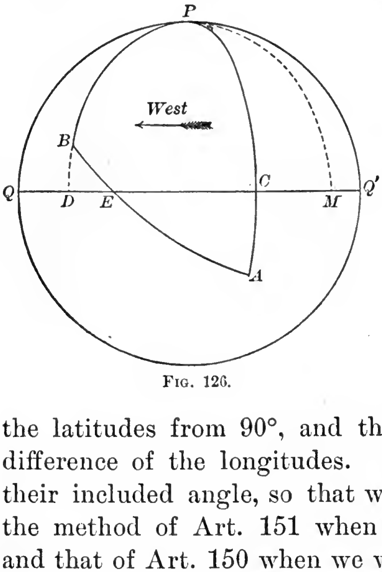


FIG. 126.

In the triangle APB the sides AP and BP are found by algebraically subtracting the latitudes from 90°, and the angle APB is the algebraic difference of the longitudes. Hence we know two sides and their included angle, so that we can solve the triangle, using the method of Art. 151 when the distance only is required, and that of Art. 150 when we wish to find all the elements.

1. Find the shortest distance between New York, 40° 45′.4 N., 73° 58′.4 W., and Rio Janeiro, 22° 54′.4 S., 43° 10′.4 W.

∴ $BP = 49° 14'.6$, $AP = 112° 54'.4$, $APB = 30° 48'.0$. *Ans.* $AB = 69° 48'.2$.

2. Find the shortest distance between New York, 40° 45′.4 N., 73° 58′.4 W., and Paris, 48° 50′.2 N., 2° 20′.2 E. *Ans.* $AB = 52° 26'.8$.

* The shortest distance between two points on a sphere is the arc of the great circle passing through the points.

If the bearings of the great circle AB at A and B are required, it will be necessary to find the angles PAB and PBA.

3. A ship sailed from Calcutta, 22° 34′.8 N., 88° 27′.3 E., on an arc of a great circle to Melbourne, 37° 48′.0 S., 144° 58′.0 E. Find the distance sailed and the bearings* at both points.

Ans. At Calcutta, S. 41° 56′.61 E.; at Melbourne, S. 51° 21′.47 E.; distance, 80° 22′.4 or 80° 22′.6.

4. A ship sailed from the Cape of Good Hope, 34° 22′ S., 18° 29′ E., on an arc of a great circle to Cape St. Roque, 5° 28′ S., 35° 16′ W. Find the distance sailed and the bearings* at both points.

Ans. At G. H., N. 72° 28′.0 W.; at S. R., N. 52° 15′.0 W.; distance, 57° 20′.4.

5. A ship sailed from Bombay, 18° 56′ N., 72° 53′ E., on an arc of a great circle to the Cape of Good Hope, 34° 22′ S., 18° 29′ E. Find the distance sailed and the bearings* at both points.

Ans. At Bombay, S. 44° 12′.8 W.; at G. H., S. 53° 2′.6 W.; distance, 74° 15′.2 or 74° 15′.4.

6. A ship sailed from Bombay, 18° 56′ N., 72° 53′ E., on an arc of a great circle for the Cape of Good Hope, 34° 22′ S., 18° 29′ E. Find the distance to the equator and the bearing* and longitude at the equator. [Use the triangle BDE; the angle $PBA = 135° 47'.2$ was found in Ex. 5.]

Ans. S. 41° 16′.1 W.; distance, 25° 34′.5; longitude, 55° 21′.8 E.

7. From a point whose latitude is 17° N. and longitude 130° W. a ship sailed an arc of a great circle over a distance of 4150 miles, starting S. 54° 20′ W. Find its latitude and longitude if the length of 1° is $69\frac{1}{6}$ miles.

Ans. Lat., 19° 40′.52 or 19° 40′.60 S.; Long., 178° 20′.9 W.

167. Given the Lengths of the Three Edges of a Parallelopiped that meet in a Point, and the Angles between them, to find the Surface and the Volume of the Parallelopiped. — Let OG be the solid, AD the perpendicular from A to BOC, and hence AOD a plane perpendicular to BOC. Let the angles and edges be

$$BOC = a, \quad AOC = b,$$
$$AOB = c, \quad OA = l,$$
$$OB = m, \quad OC = n.$$

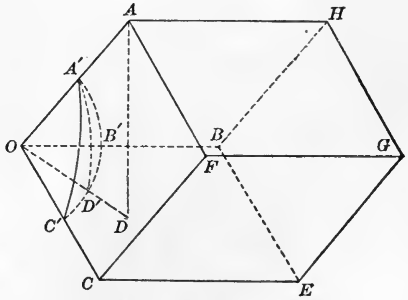


Fig. 127.

Describe a sphere with a radius of unity about O as a center, its intersections with the planes forming the figure marked by the primed letters.

* The *course* of the ship.

Then the surface is

$$S = 2\,OBEC + 2\,OAFC + 2\,OBHA$$
$$= 2\,(mn \sin a + ln \sin b + lm \sin c). \qquad (1)$$

In the triangle $A'D'B'$, right-angled at D', we have

$$\sin D'A' = \sin B'A' \sin A'B'D'\,;$$
$$\therefore\ \sin D'A' = \sin c \sin A'B'D'.$$

But in the triangle $A'B'C'$ we know the three sides a, b, c; hence

$$\sin A'B'D' = 2 \sin \tfrac{1}{2} A'B'D' \cos \tfrac{1}{2} A'B'D'$$
$$= \frac{2}{\sin a \sin c}\sqrt{\sin s \sin (s-a) \sin (s-b) \sin (s-c)}.$$

$$\therefore\ \sin D'A' = \sin DOA$$
$$= \frac{2}{\sin a}\sqrt{\sin s \sin (s-a) \sin (s-b) \sin (s-c)}.$$

$$\therefore\ DA = OA \sin DOA$$
$$= \frac{2\,l}{\sin a}\sqrt{\sin s \sin (s-a) \sin (s-b) \sin (s-c)}.$$

Hence the volume is

$$V = OBEC \times DA$$
$$= 2\,lmn\sqrt{\sin s \sin (s-a) \sin (s-b) \sin (s-c)}. \qquad (2)$$

168. To find the Volume of a Regular Polyhedron. — Let AB be the edge in which two adjacent faces intersect, D its middle point, C and E the centers of the polygonal faces, and O the center of the sphere inscribed in the polyhedron, the faces being tangent to the sphere at C and E. Then

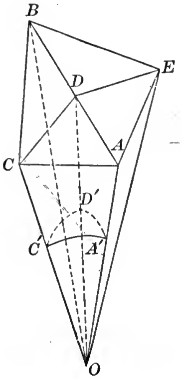


FIG. 128.

$$DC = DE\,;\ \ DA = DB\,;$$
$$CDA = CDB = EDA = EDB = 90°;$$
$$DCO = 90°;\ \ DEO = 90°.$$

Let a = length of an edge AB,
s = number of sides of each polygonal face,
n = number of faces meeting at a vertex of the polyhedron,

$N =$ number of faces of the polyhedron,

$E =$ edge angle CDE of the polyhedron.

Then $\quad CD = AD \cot ACD = \frac{1}{2} a \cot \frac{180°}{s}.$

$$CO = CD \tan CDO = CD \tan \tfrac{1}{2} E.$$

$$\therefore \quad CO = \tfrac{1}{2} a \cot \frac{180°}{s} \tan \tfrac{1}{2} E. \qquad (1)$$

About O as a center, with a unit radius, describe a sphere, and let its intersections with the three planes form the triangle $A'C'D'$. Then

$$A'C'D' = ACD = \frac{180°}{s}; \quad A'D'C' = 90°; \quad C'A'D' = \tfrac{1}{2} \frac{360°}{n}.$$

By Napier's rules,

$$\cos C'A'D' = \cos C'D' \sin A'C'D',$$

or
$$\cos \frac{180°}{n} = \cos C'D' \sin \frac{180°}{s}.$$

But

$$\cos C'D' = \cos COD = \cos (90° - CDO) = \sin CDO = \sin \tfrac{1}{2} E.$$

$$\therefore \quad \cos \frac{180°}{n} = \sin \tfrac{1}{2} E \sin \frac{180°}{s}.$$

$$\therefore \quad \sin \tfrac{1}{2} E = \cos \frac{180°}{n} \operatorname{cosec} \frac{180°}{s}. \qquad (2)$$

Then, if A is the area of a face, the volume is

$$V = \tfrac{1}{3}\, CO \times A \times N = \tfrac{1}{24} N s a^3 \cot^2 \frac{180°}{s} \tan \tfrac{1}{2} E. \qquad (3)$$

Find $\frac{1}{2} E$ from (2) and then V from (3).

1. Dodecahedron, formed by 12 regular pentagons, 3 meeting at a vertex.

$\therefore\ s = 5,\ n = 3,\ N = 12.$

$$\log \cos 60° = 9.69897$$
$$\log \operatorname{cosec} 36° = 0.23078$$
$$\log \sin \tfrac{1}{2} E = 9.92975$$

$$\log \frac{Ns}{24} = 0.39794$$
$$\log \cot^2 36° = 0.27748$$
$$\log \tan \tfrac{1}{2} E = 0.20896$$
$$0.88438$$

$$\therefore \quad V = 7.663\, a^3.$$

2. Tetrahedron, formed by 4 equilateral triangles, 3 meeting at a vertex.

$\therefore\ s = 3,\ n = 3,\ N = 4.$ *Ans.* $V = 0.1179\, a^3.$

3. Cube, formed by 6 squares, 3 meeting at a vertex.

$\therefore\ s = 4,\ n = 3,\ N = 6.$ *Ans.* $V = a^3.$

4. Octahedron, formed by 8 equilateral triangles, 4 meeting at a vertex.

$\therefore\ s = 3,\ n = 4,\ N = 8.$ *Ans.* $V = 0.4714\,a^3.$

5. Icosahedron, formed by 20 equilateral triangles, 5 meeting at a vertex.

$\therefore\ s = 3,\ n = 5,\ N = 20.$ *Ans.* $V = 2.182\,a^3.$

169. If from Any Point in a Trirectangular Triangle Arcs of Great Circles are drawn to the Vertices,

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1,$$

where α, β, and γ are the arcs. — In Fig. 129, produce YP and ZP to D and E. In the right triangle PDX,

$$\sin PD = \sin\alpha \sin PXD\,;\quad \therefore\ \cos\beta = \sin\alpha\sin PXD. \quad (1)$$

In the right triangle PEX,

$$\sin PE = \sin\alpha \sin PXE\,;\quad \therefore\ \cos\gamma = \sin\alpha\cos PXD. \quad (2)$$

Squaring (1) and (2), and adding, we have

$$\cos^2\beta + \cos^2\gamma = \sin^2\alpha.$$

$$\therefore\ \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1. \qquad \text{Q.E.D.}$$

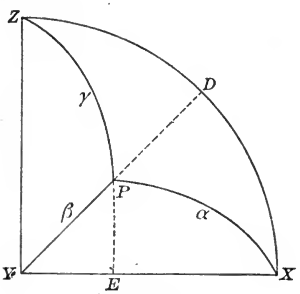


FIG. 129.

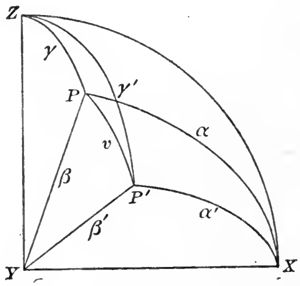


FIG. 130.

170. If from Any Two Points P and P' in a Trirectangular Triangle Arcs of Great Circles are drawn to the Three Vertices, and if v is the Length of the Arc PP', prove that

$$\cos v = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma'.$$

In the triangle PYP' (Fig. 130),

$$\cos v = \cos\beta\cos\beta' + \sin\beta\sin\beta'\cos PYP'. \quad (1)$$

But

$$\cos PYP' = \cos(ZYP' - ZYP).$$

$$\therefore\ \cos PYP' = \cos ZYP'\cos ZYP + \sin ZYP'\sin ZYP. \quad (2)$$

$$\left.\begin{aligned}
&\text{In } ZYP, && \cos\gamma = \sin\beta\cos ZYP.^* \\
&\text{In } ZYP', && \cos\gamma' = \sin\beta'\cos ZYP'.^* \\
&\text{In } XYP, && \cos\alpha = \sin\beta\cos XYP^* = \sin\beta\sin ZYP. \\
&\text{In } XYP', && \cos\alpha' = \sin\beta'\cos XYP'^* = \sin\beta'\sin ZYP'.
\end{aligned}\right\} \quad (3)$$

Substituting in (1) the values found from (2) and (3),

$$\cos v = \cos\beta\cos\beta' + \cos\gamma\cos\gamma' + \cos\alpha\cos\alpha'. \qquad \text{Q.E.D.}$$

This is the formula for the cosine of the angle between two lines in space, the angles made by them with three lines at right angles to each other being α, β, γ, and α', β', γ', respectively.

171. To find the Angle α' between the Chords of Two Sides of a Spherical Triangle, having given the Two Sides b and c, and the Angle α between them. — Let $AB = c$, $AC = b$, the spherical angle $BAC = \alpha$, and the plane angle $BAC = \alpha'$, O being the center of the sphere. About A as a center describe a sphere, and let its intersections with the planes OAB, OAC, and BAC form the triangle DEF. Then

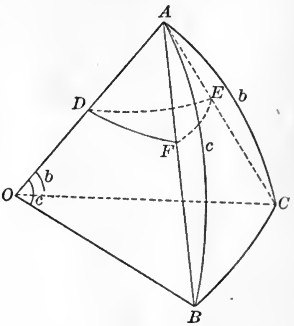


FIG. 131.

$$DF = OAB = 90^\circ - \tfrac{1}{2}c;$$
$$DE = OAC = 90^\circ - \tfrac{1}{2}b;$$
$$FDE = \alpha; \quad FE = BAC = \alpha'.$$

$$\therefore \cos FE = \cos DE\cos DF + \sin DE\sin DF\cos FDE.$$

$$\therefore \cos\alpha' = \sin\tfrac{1}{2}b\sin\tfrac{1}{2}c + \cos\tfrac{1}{2}b\cos\tfrac{1}{2}c\cos\alpha. \qquad (1)$$

This formula is true for all values of b, c, and α. When b and c are small, the *correction* that must be applied to α to obtain α' may be found from (1) as follows:

Let $p = b + c$, and $q = b - c$. Then, from Art. 72,

$$\begin{aligned}
\cos\alpha' &= \tfrac{1}{2}\cos\tfrac{1}{2}q - \tfrac{1}{2}\cos\tfrac{1}{2}p + \tfrac{1}{2}(\cos\tfrac{1}{2}p + \cos\tfrac{1}{2}q)\cos\alpha \\
&= -\sin^2\tfrac{1}{4}q + \sin^2\tfrac{1}{4}p + (1 - \sin^2\tfrac{1}{4}p - \sin^2\tfrac{1}{4}q)\cos\alpha \\
&= (\sin^2\tfrac{1}{4}p - \sin^2\tfrac{1}{4}q)(\sin^2\tfrac{1}{2}\alpha + \cos^2\tfrac{1}{2}\alpha) + \cos\alpha \\
&\qquad - (\sin^2\tfrac{1}{4}p + \sin^2\tfrac{1}{4}q)(\cos^2\tfrac{1}{2}\alpha - \sin^2\tfrac{1}{2}\alpha).
\end{aligned}$$

$$\therefore \cos\alpha' = \cos\alpha - 2\sin^2\tfrac{1}{4}q\cos^2\tfrac{1}{2}\alpha + 2\sin^2\tfrac{1}{4}p\sin^2\tfrac{1}{2}\alpha. \qquad (2)$$

* Eq. (2), Art. 121.

Let $a' = a + \theta$, where θ is so small that we may place

$$\sin\theta = \theta, \text{ and } \cos\theta = 1.$$

$$\because \cos a' = \cos a \cos\theta - \sin a \sin\theta.$$

$$\therefore \cos a' = \cos a - \theta \sin a. \tag{3}$$

Comparing (2) and (3),

$$2\,\theta \sin\tfrac{1}{2}a \cos\tfrac{1}{2}a = 2\sin^2\tfrac{1}{4}q\cos^2\tfrac{1}{2}a - 2\sin^2\tfrac{1}{4}p\sin^2\tfrac{1}{2}a.$$

$$\therefore \theta = \sin^2\tfrac{1}{4}q\cot\tfrac{1}{2}a - \sin^2\tfrac{1}{4}p\tan\tfrac{1}{2}a.$$

$$\therefore \theta'' = \frac{1}{\sin 1''}\sin^2\tfrac{1}{4}q\cot\tfrac{1}{2}a - \frac{1}{\sin 1''}\sin^2\tfrac{1}{4}p\tan\tfrac{1}{2}a, \tag{4}$$

since $\theta = \theta'' \sin 1''$ (Art. 81).

172. The Angles of Elevation of Two Points, in the Directions *OA* and *OB*, above a Horizontal Plane, and the Inclined Angle *AOB*, were measured with a Sextant. Find the Horizontal Angle between the Points, as seen from *O*. — Let OZ be the vertical line, Oab the horizontal plane; $aOA = h$, and $bOB = k$ the measured altitudes; and $AOB = c$ the inclined angle. Describe a sphere about O as a center. Then in the triangle AZB,

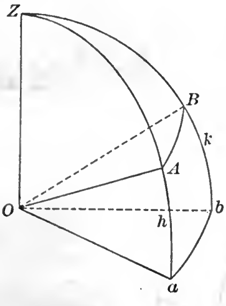


FIG. 132.

$$AZ = 90° - h,\ BZ = 90° - k,\ AB = c,$$

and hence the required angle $aOb = AZB$ may be computed, since we know the three sides of the triangle.

When h and k are small, the correction to be applied to the measured value c to obtain aOb may be found as follows:*

From (2), Art. 121,

$$\cos AZB = \frac{\cos c - \sin h \sin k}{\cos h \cos k} = \frac{\cos c - hk}{(1 - \frac{1}{2}h^2)(1 - \frac{1}{2}k^2)} \quad \text{(Art. 78)}$$

$$= \frac{\cos c - hk}{1 - \frac{1}{2}(h^2 + k^2)} = (\cos c - hk)[1 + \tfrac{1}{2}(h^2 + k^2)].$$

$$\therefore \cos AZB = \cos c + \tfrac{1}{2}(h^2 + k^2)\cos c - hk. \tag{1}$$

* Neglecting powers of h and k above the second.

Let θ be the correction to c so that $AZB = c + \theta$.

$$\therefore \quad \cos AZB = \cos c \cos \theta - \sin c \sin \theta.$$

$$\therefore \quad \cos AZB = \cos c - \theta \sin c. \tag{2}$$

Comparing (1) and (2),

$$\theta = -\frac{(h^2 + k^2)(\cos^2 \tfrac{1}{2} c - \sin^2 \tfrac{1}{2} c) - 2\,hk\,(\cos^2 \tfrac{1}{2} c + \sin^2 \tfrac{1}{2} c)}{4 \sin \tfrac{1}{2} c \cos \tfrac{1}{2} c}.$$

$$\therefore \quad \theta = \tfrac{1}{4}(h + k)^2 \tan \tfrac{1}{2} c - \tfrac{1}{4}(h - k)^2 \cot \tfrac{1}{2} c, \tag{3}$$

where θ, h, and k are expressed in circular measure. To find θ in seconds, let $\theta = \theta'' \sin 1''$, $h = h'' \sin 1''$, $k = k'' \sin 1''$.

$$\therefore \quad \theta'' = \tfrac{1}{4}(h'' + k'')^2 \sin 1'' \tan \tfrac{1}{2} c - \tfrac{1}{4}(h'' - k'')^2 \sin 1'' \cot \tfrac{1}{2} c. \tag{4}$$

SPHERICAL EXCESS.

173. Area of a Spherical Triangle. — From geometry we know that the areas of any two triangles are to each other as their spherical excesses, the spherical excess being the amount by which the sum of the three angles exceeds 180°. We also know that the area of the trirectangular triangle is $\frac{1}{2}\pi r^2$, and that its spherical excess is 90°. If A is the area of any triangle, and E its spherical excess expressed in degrees, we have

$$A : \tfrac{1}{2}\pi r^2 = E : 90°. \tag{1}$$

$$\therefore \quad A = E \frac{\pi r^2}{180°}, \tag{2}$$

and

$$E = A \frac{180°}{\pi r^2}. \tag{3}$$

174. Lhuillier's Theorem. — We have

$$\tan \tfrac{1}{4} E = \frac{\sin \frac{1}{4}(\alpha + \beta + \gamma - \pi)}{\cos \frac{1}{4}(\alpha + \beta + \gamma - \pi)} \cdot \frac{2 \cos \frac{1}{4}(\alpha + \beta + \pi - \gamma)}{2 \cos \frac{1}{4}(\alpha + \beta + \pi - \gamma)}$$

$$= \frac{\sin \frac{1}{2}(\alpha + \beta) - \sin \frac{1}{2}(\pi - \gamma)}{\cos \frac{1}{2}(\alpha + \beta) + \cos \frac{1}{2}(\pi - \gamma)},$$

from (6) and (7), Art. 72.

$$\therefore \quad \tan \tfrac{1}{4} E = \frac{\sin \frac{1}{2}(\alpha + \beta) - \cos \frac{1}{2}\gamma}{\cos \frac{1}{2}(\alpha + \beta) + \sin \frac{1}{2}\gamma}.$$

Hence, from (1) and (2), Art. 145, substituting for $\sin\frac{1}{2}(\alpha+\beta)$ and $\cos\frac{1}{2}(\alpha+\beta)$, we have

$$\tan\tfrac{1}{4}E = \frac{\cos\frac{1}{2}(a-b)-\cos\frac{1}{2}c}{\cos\frac{1}{2}(a+b)+\cos\frac{1}{2}c}\cdot\frac{\cos\frac{1}{2}\gamma}{\sin\frac{1}{2}\gamma}$$
$$= \frac{\sin\frac{1}{4}(a-b+c)\sin\frac{1}{4}(b+c-a)}{\cos\frac{1}{4}(a+b+c)\cos\frac{1}{4}(a+b-c)}\cot\tfrac{1}{2}\gamma,$$

from (4) and (3), Art. 73.

$$\therefore\ \tan\tfrac{1}{4}E = \frac{\sin\frac{1}{2}(s-b)\sin\frac{1}{2}(s-a)}{\cos\frac{1}{2}s\cos\frac{1}{2}(s-c)}\sqrt{\frac{\sin s\sin(s-c)}{\sin(s-a)\sin(s-b)}}$$

$$=\sqrt{\left[\frac{\sin^2\frac{1}{2}(s-b)\sin^2\frac{1}{2}(s-a)}{\cos^2\frac{1}{2}s\cos^2\frac{1}{2}(s-c)}\times\frac{\sin\frac{1}{2}s\cos\frac{1}{2}s\sin\frac{1}{2}(s-c)\cos\frac{1}{2}(s-c)}{\sin\frac{1}{2}(s-a)\cos\frac{1}{2}(s-a)\sin\frac{1}{2}(s-b)\cos\frac{1}{2}(s-b)}\right]}.$$

$$\therefore\ \tan\tfrac{1}{4}E = \sqrt{\tan\tfrac{1}{2}s\tan\tfrac{1}{2}(s-a)\tan\tfrac{1}{2}(s-b)\tan\tfrac{1}{2}(s-c)}. \quad \text{Q.E.I.}$$

175. Spherical Excess in Terms of Two Sides and their Included Angle.

$$\tan\tfrac{1}{2}E = \frac{\sin\frac{1}{2}(\alpha+\beta+\gamma-\pi)}{\cos\frac{1}{2}(\alpha+\beta+\gamma-\pi)} = \frac{-\cos\frac{1}{2}(\alpha+\beta+\gamma)}{\sin\frac{1}{2}(\alpha+\beta+\gamma)}$$
$$= \frac{\sin\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}\gamma-\cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}\gamma}{\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}\gamma+\cos\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}\gamma}.$$

Substituting for $\sin\frac{1}{2}(\alpha+\beta)$ and $\cos\frac{1}{2}(\alpha+\beta)$ from (1) and (2), Art. 145,

$$\tan\tfrac{1}{2}E = \frac{\sin\frac{1}{2}\gamma\cos\frac{1}{2}\gamma[\cos\frac{1}{2}(a-b)-\cos\frac{1}{2}(a+b)]}{\cos\frac{1}{2}(a-b)\cos^2\frac{1}{2}\gamma+\cos\frac{1}{2}(a+b)\sin^2\frac{1}{2}\gamma}$$

$$= \frac{\sin\frac{1}{2}\gamma\cos\frac{1}{2}\gamma[+2\sin\frac{1}{2}a\sin\frac{1}{2}b]}{\frac{1}{2}[\cos\frac{1}{2}(a-b)+\cos\frac{1}{2}(a+b)]+\frac{1}{2}[\cos\frac{1}{2}(a-b)-\cos\frac{1}{2}(a+b)]\cos\gamma}$$

$$= \frac{\sin\frac{1}{2}a\sin\frac{1}{2}b\sin\gamma}{\cos\frac{1}{2}a\cos\frac{1}{2}b+\sin\frac{1}{2}a\sin\frac{1}{2}b\cos\gamma}.$$

$$\therefore\ \tan\tfrac{1}{2}E = \frac{\tan\frac{1}{2}a\tan\frac{1}{2}b\sin\gamma}{1+\tan\frac{1}{2}a\tan\frac{1}{2}b\cos\gamma}. \quad \text{Q.E.I.}$$

176. Approximate Value of the Spherical Excess, neglecting Powers above the Second. — Let the sides of the triangle be so small that the powers of their circular measures higher than the second may be neglected. We have, from Art. 78,

$$\tan x = x + \tfrac{1}{3}x^3 + \cdots, \tag{1}$$

where x is expressed in circular measure.

Let the lengths of the sides be a, b, and c when expressed in circular measure, and a', b', and c' in linear measure, r being the radius of the sphere. Then

$$a = \frac{a'}{r}, \quad b = \frac{b'}{r}, \quad c = \frac{c'}{r}. \tag{2}$$

Placing these values of a, b, and c for x in (1), and substituting in Lhuillier's theorem, we have, neglecting powers above the second,

$$\tan \tfrac{1}{4}E = \sqrt{\frac{s'}{2r} \cdot \frac{s'-a'}{2r} \cdot \frac{s'-b'}{2r} \cdot \frac{s'-c'}{2r}}, \tag{3}$$

where $$s' = \tfrac{1}{2}(a' + b' + c'). \tag{4}$$

$$\therefore \tan \tfrac{1}{4}E = \frac{1}{4r^2}\sqrt{s'(s'-a')(s'-b')(s'-c')}. \tag{5}$$

Since $\frac{1}{4}E$ is small, we place its tangent equal to its arc.

$$\therefore \tfrac{1}{4}E = \frac{1}{4r^2}\sqrt{s'(s'-a')(s'-b')(s'-c')} \tag{6}$$

or $$E = \frac{1}{r^2}A, \tag{7}$$

where A is the area of the plane triangle whose sides are a', b', and c', E being expressed in circular measure.

To find the value of E in seconds of arc, divide both sides by $\sin 1''$.

$$\therefore \frac{E}{\sin 1''} = E'' = \frac{A}{r^2 \sin 1''}. \tag{8}$$

Hence, whenever the third powers of the circular measures of the sides can be neglected, the spherical excess is found by computing the area of the triangle, considering it as a plane triangle, and dividing the area by $r^2 \sin 1''$.

177. Approximate Value of the Spherical Excess, neglecting Powers above the Fourth. — From Lhuillier's theorem,

$$\tan^2 \tfrac{1}{4} E = \left[\frac{s'}{2r} + \frac{s'^3}{24r^3}\right]\left[\frac{s'-a'}{2r} + \frac{(s'-a')^3}{24r^3}\right]$$

$$\left[\frac{s'-b'}{2r} + \frac{(s'-b')^3}{24r^3}\right]\left[\frac{s'-c'}{2r} + \frac{(s'-c')^3}{24r^3}\right]$$

$$= \frac{A^2}{16r^4} + \frac{A^2}{192r^6}[(s'-c')^2 + (s'-b')^2 + (s'-a')^2 + s'^2],$$

where $$A^2 = s'(s'-a')(s'-b')(s'-c').$$

$$\therefore \tan^2 \tfrac{1}{4} E = \frac{A^2}{16r^4} + \frac{A^2}{192r^6}(a'^2 + b'^2 + c'^2).$$

$$\therefore \tan \tfrac{1}{4} E = \frac{A}{4r^2}\left(1 + \frac{a'^2 + b'^2 + c'^2}{12r^2}\right)^{\frac{1}{2}}$$

$$= \frac{A}{4r^2}\left(1 + \frac{a'^2 + b'^2 + c'^2}{24r^2}\right).$$

$$\therefore \tfrac{1}{4} E'' \sin 1'' = \frac{A}{4r^2}\left(1 + \frac{a'^2 + b'^2 + c'^2}{24r^2}\right).$$

$$\therefore E'' = \frac{A}{r^2 \sin 1''}\left(1 + \frac{a'^2 + b'^2 + c'^2}{24r^2}\right). \qquad \text{Q.E.I.}$$

This value exceeds that found in Art. 176 by

$$\frac{A}{r^2 \sin 1''} \cdot \frac{a'^2 + b'^2 + c'^2}{24r^2}.$$

If $a' = b' = c' = 100$ miles, and $r = 3963.3$ miles, we obtain

$$\frac{A}{r^2 \sin 1''} = 56''.863\,; \qquad \frac{a'^2 + b'^2 + c'^2}{24r^2} = 0.00008;$$

so that the correction to the value of E'' given by (8), Art. 176, is only

$$56''.863 \times 0.00008 = 0''.005.$$

178. Legendre's Theorem. — *If the sides of a spherical triangle are very small compared with the radius of the sphere, the angles of the plane triangle whose sides are of the same length as* *those of the spherical triangle, are equal to the corresponding angles of the spherical triangle diminished by one third of the spherical excess.* — Let a', b', and c' be the lengths of the sides of the spherical triangle expressed in linear measure, and a, b, and c the lengths in circular measure.

$$\therefore\ a = \frac{a'}{r}, \quad b = \frac{b'}{r}, \quad c = \frac{c'}{r}. \tag{1}$$

Let α be an angle of the spherical triangle and α' the corresponding angle of the plane triangle. We have

$$\cos\alpha = \frac{\cos a - \cos b \cos c}{\sin b \sin c}. \tag{2}$$

From Art. 78,

$$\cos a = 1 - \tfrac{1}{2}a^2 + \tfrac{1}{24}a^4 - \cdots \qquad \sin b = b - \tfrac{1}{6}b^3 + \cdots$$
$$\cos b = 1 - \tfrac{1}{2}b^2 + \tfrac{1}{24}b^4 - \cdots \qquad \sin c = c - \tfrac{1}{6}c^3 + \cdots$$
$$\cos c = 1 - \tfrac{1}{2}c^2 + \tfrac{1}{24}c^4 - \cdots$$

$$\therefore\ \cos\alpha = \frac{\tfrac{1}{2}(b^2 + c^2 - a^2) + \tfrac{1}{24}(a^4 - b^4 - c^4 - 6b^2c^2)}{bc\left[1 - \tfrac{1}{6}(b^2 + c^2)\right]}, \tag{3}$$

the terms of orders higher than the fourth being neglected.

$$\therefore\ \cos\alpha = \frac{1}{2bc}\left[(b^2 + c^2 - a^2) + \tfrac{1}{12}(a^4 - b^4 - c^4 - 6b^2c^2)\right]\left[1 - \tfrac{1}{6}(b^2 + c^2)\right]^{-1}$$
$$= \frac{1}{2bc}\left[(b^2 + c^2 - a^2) + \tfrac{1}{12}(a^4 - b^4 - c^4 - 6b^2c^2)\right]\left[1 + \tfrac{1}{6}(b^2 + c^2) + \cdots\right].$$

$$\therefore\ \cos\alpha = \frac{b^2 + c^2 - a^2}{2bc} + \frac{a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2}{24bc}, \tag{4}$$

the terms of orders higher than the fourth being neglected, as before.

In the plane triangle,

$$\cos\alpha' = \frac{b'^2 + c'^2 - a'^2}{2b'c'} = \frac{b^2 + c^2 - a^2}{2bc}, \tag{5}$$

from (1)

$$\therefore\ \cos\alpha = \cos\alpha' + \frac{a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2}{24bc}. \tag{6}$$

$$\therefore\ \cos\alpha = \cos\alpha' + \frac{1}{r^2}\cdot\frac{a'^4 + b'^4 + c'^4 - 2a'^2b'^2 - 2a'^2c'^2 - 2b'^2c'^2}{24b'c'}. \tag{7}$$

Let $s' = \frac{1}{2}(a' + b' + c')$; then

$$s'(s' - a')(s' - b')(s' - c')$$
$$= -\tfrac{1}{16}(a'^4 + b'^4 + c'^4 - 2a'^2b'^2 - 2a'^2c'^2 - 2b'^2c'^2). \quad (8)$$

But the area of the plane triangle is

$$\sqrt{s'(s' - a')(s' - b')(s' - c')} = \tfrac{1}{2}b'c' \sin \alpha'. \quad (9)$$

Hence (7) becomes, from (8) and (9),

$$\cos \alpha = \cos \alpha' - \frac{1}{6r^2} b'c' \sin^2 \alpha'. \quad (10)$$

Let $$\alpha = \alpha' + \theta. \quad (11)$$

$$\therefore \cos \alpha = \cos \alpha' \cos \theta - \sin \alpha' \sin \theta. \quad (12)$$

Since θ is small, we may place $\cos \theta = 1$, and $\sin \theta = \theta$.

$$\therefore \cos \alpha = \cos \alpha' - \theta \sin \alpha'. \quad (13)$$

Comparing (10) and (13),

$$\theta = \frac{1}{6r^2} b'c' \sin \alpha' = \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{b'c' \sin \alpha'}{r^2}. \quad (14)$$

Hence, from (7), Art. 176,

$$\theta = \tfrac{1}{3}E,$$

and, from (11), $$\alpha' = \alpha - \tfrac{1}{3}E.$$ Q.E.D.

179. Application of Legendre's Theorem. — In the New York State Survey the angles of the spherical triangle, whose vertices were at Howlett, Gilbertsville, and Eagle, were measured, the distance from Howlett to Gilbertsville having been already computed. The measured values were

At Howlett,	$\alpha =$	$85° \; 18' \; 57''.71$	$\log b = 4.54227\ 32$
At Eagle,	$\beta =$	$51° \; 35' \; 41''.61$	$\log r = 6.80459\ 32$
At Gilbertsville,	$\gamma =$	$43° \; \; 5' \; 24''.24$	
	$\therefore \alpha + \beta + \gamma =$	$180° \; \; 0' \; \; 3''.56$	

The formula for the spherical excess is (Art. 176)

$$E'' = \frac{A}{r^2 \sin 1''} = \frac{1}{2} \cdot \frac{b^2 \sin \alpha \sin \gamma}{\sin \beta} \cdot \frac{1}{r^2 \sin 1''}.$$

$$
\begin{aligned}
\log b^2 &= 9.08455 \\
\text{colog } 2 &= 9.69897 - 10 \\
\log \sin \alpha &= 9.99855 - 10 \\
\log \sin \gamma &= 9.83451 - 10 \\
\text{col sin } \beta &= 0.10588 \\
\text{colog } r^2 &= 6.39081 - 20 \\
\text{col sin } 1'' &= 5.31443 \\
\hline
\log E'' &= 0.42770
\end{aligned}
$$

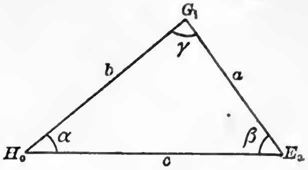


FIG. 133.

$$\therefore\ E'' = 2''.6773;$$
$$\therefore\ \tfrac{1}{3}E'' = 0''.8924.$$

The errors due to observation therefore amounted to 3''.56 − 2''.677 = 0''.883. This discrepancy was distributed among the three angles according to the method of least squares,* giving the following results:

	Observed Angles.	Correction.	Spherical Angles.	$\frac{1}{3}E''$.	Plane Angles.
$\alpha =$ 85° 18′	57″.71	− 0″.747	56″.963	0″.893	56″.070 = α'
$\beta =$ 51° 35′	41″.61	+ 1″.355	42″.965	0″.892	42″.073 = β'
$\gamma =$ 43° 5′	24″.24	− 1″.491	22″.749	0″.892	21″.857 = γ'
Sum = 180° 0′	3″.56	− 0″.883	2″.677	2″.677	0″.000

Using the plane triangle, we find by the sine proportion:

$$
\begin{aligned}
\log b' &= 4.542\ 2732 \\
\text{col sin } \beta' &= 0.105\ 8837 \\
\log \sin \alpha' &= 9.998\ 5468 - 10 \\
\hline
\log a' &= 4.646\ 7037 \\
a' &= 44330.61 \text{ meters.}
\end{aligned}
\qquad
\begin{aligned}
\log b' &= 4.542\ 2732 \\
\text{col sin } \beta' &= 0.105\ 8837 \\
\log \sin \gamma' &= 9.834\ 5089 - 10 \\
\hline
\log c' &= 4.482\ 6658 \\
c' &= 30385.46 \text{ meters.}
\end{aligned}
$$

These are the distances between the points measured on the great circles joining them.

ASTRONOMICAL APPLICATIONS.

180. Definitions. — Let us consider the earth as a point O (Fig. 134), and let a sphere be described about O as a center, with a radius indefinitely great, so that all the stars shall be within the sphere. The figure represents the sphere as seen from the outside.

* Eleven angles were involved in the adjustment.

The *zenith* Z is the point where a vertical line — the plumb line — pierces the sphere.

The *horizon* $HWNE$ is the great circle cut from the sphere by a plane through O perpendicular to the plumb line. N, E, H, and W are the north, east, south, and west points of the horizon.

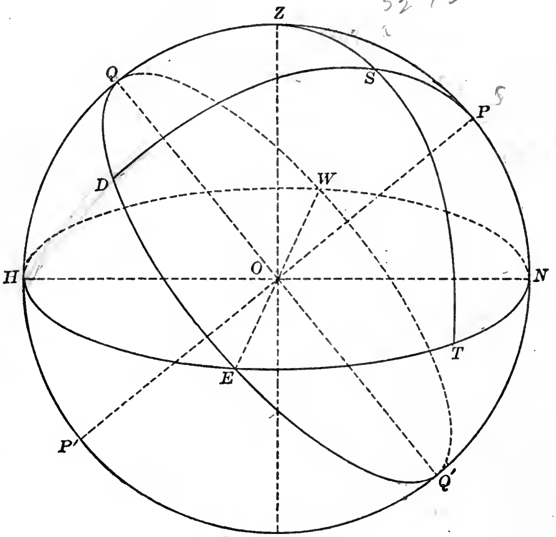


FIG. 134.

Vertical circles are great circles whose planes pass through the plumb line OZ, as ZST in the plane OZT.

The *meridian* HZN is the vertical circle passing through the north and south points of the horizon.

The *altitude* TS of a star or point is its angular distance above the horizon, measured on a vertical circle.

The *zenith distance* ZS is the complement of the altitude.

The *azimuth* of a star or point is the arc NT or the angle NZT between the meridian and the vertical circle through the star * or point. It is usually measured from the south point of the horizon through the west.

The *poles* P and P' are the intersections of the axis of the earth with the sphere. P is here the north pole. In consequence of the earth's rotation about its axis the stars appear to

* That is, whose plane passes through the star.

describe small circles about P as the pole, apparently moving in the direction $EQWQ'$.

The *equator* $EQWQ'$ is the great circle cut from the sphere by a plane through O perpendicular to the axis of the earth.

The *latitude* of the observer is the angular distance QZ from the equator to the zenith. Since $PQ = 90°$ and $ZN = 90°$, we have $NP = QZ$, *i.e. the elevation of the pole above the horizon is equal to the latitude of the place.*

The *hour circle* of a star is the great circle PSD through the star * and the pole. All the hour circles are perpendicular to the equator.

The *hour angle* of a star is the angle at the pole between the meridian and the hour circle of the star, measured from the meridian to the west. Thus the hour angle of S is $-ZPS$, negative since it is measured to the east. It is so named because, if the angle ZPS is 15°, one hour will elapse before PS coincides with PZ; for $15° = 360° \div 24$, and the star appears to make a complete revolution about P in 24 hours of sidereal (*i.e.* star) time.

The *declination* DS of a star is its angular distance from the equator, measured on its hour circle, and positive when the star is north of the equator.

The *right ascension* of a star is the angular distance along the equator from a certain point on the equator, called the *vernal equinox*, to the foot of the hour circle through the star, measured towards the east; or it is the angle at the pole between the hour circle of the vernal equinox and that of the star.

Hence the angle between the hour circles of two stars is equal to the difference between their right ascensions.

181. At a Place in Latitude 42° N. the Altitude of a Star, whose Declination is + 60°, was measured and found to be 50°, the Star being East of the Meridian. At what Time did the Star reach the Meridian? — In the triangle ZPS, $ZP = 48°$, $ZS = 40°$, $PS = 30°$; $\therefore$ by Art. 148, $\frac{1}{2} ZPS = 29° \ 55'.9$; $\therefore$ $ZPS = 59° \ 51'.8$ or $3^h \ 59^m.5$. Hence the star reached the meridian $3^h \ 59^m.5$ after the observation was made.†

* That is, whose plane passes through the star. † Sidereal time.

182. The Latitude of the Place being 42° N., find the Interval of Time between the Rising of a Star above the Horizon and its Passage across the Meridian, its Declination being + 10°. — In the triangle ZPS, S will be on the horizon NEH at the instant of rising, so that $ZS = 90°$.

$$\therefore \cos ZS = 0 = \cos ZP \cos SP + \sin ZP \sin SP \cos ZPS.$$

$$\therefore \cos ZPS = -\cot ZP \cot SP = -\cot 48° \cot 80°.$$

$$\therefore ZPS = 99° \; 8'.1 \text{ or } 6^h \; 36^m.5.*$$

Hence the star will be above the horizon $13^h \; 13^m.0$.*

183. The Latitude of the Place being 42° N., and the Declination of the Star + 20°, find the Interval between the Instant when it is due East and that when it is due West. — In the triangle ZPS, $PZS = 90°$.

$$\therefore \cos ZPS = \tan ZP \cot SP = \tan 48° \cot 70°.$$

$$\therefore ZPS = 66° \; 9'.4. \quad \therefore 2\,ZPS = 132° \; 18'.8 = 8^h \; 49^m.3.$$

Hence the interval required is $8^h \; 49^m.3$.*

184. The Latitude being 42° N. and the Declination of the Star + 80°, find the Azimuth of the Star when it is at its Greatest Western Elongation; that is, when the Star has reached its Farthest Distance towards the West, afterwards moving East. — In the figure the ZPS triangle is projected upon the plane of the horizon, so that Z is the zenith, P the pole, S the star, MSM' the apparent diurnal path of the star about the pole, ZP the meridian, ZS the vertical circle of the star, and PZS the angle required, the angle ZSP being a right angle.

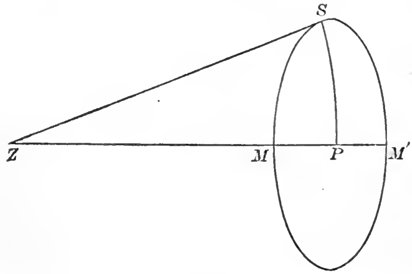


Fig. 135.

* Sidereal time.

$\therefore \sin SP = \sin ZP \sin PZS.$ $\quad\therefore \sin PZS = \sin 10° \operatorname{cosec} 48°;$

$\therefore PZS = 13° 30'.8.$

NOTE. — This is the method ordinarily used by the engineer to determine the north and south line.

185. The Right Ascensions of Two Stars are α and α', and their Declinations δ and δ'; find the Angular Distance between the Two Stars.

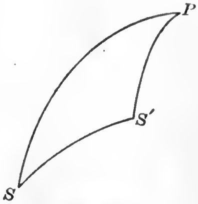


FIG. 136.

$SP = 90° - \delta$, $S'P = 90° - \delta'$, $SPS' = \alpha' - \alpha.$

Hence we know two sides and the included angle, and we find the third side SS' by Art. 151 or by Art. 150.

186. If α' and α'' are the Right Ascensions, and δ' and δ'' the Declinations of Two Stars, find the Inclination to the Equator of the Great Circle passing through the Stars, and also the Right Ascension of the Point where it cuts the Equator. — Let B and D be the two stars, EQ the equator, V the vernal equinox, E the intersection of the great circle BD with the equator, $VE = \alpha_1$. In the right triangle EAB,

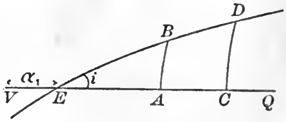


FIG. 137.

$$\sin EA = \tan AB \cot AEB. \quad \therefore \cot i = \sin(\alpha' - \alpha_1) \cot \delta'. \quad (1)$$

In the right triangle ECD,

$$\sin EC = \tan CD \cot CED. \quad \therefore \cot i = \sin(\alpha'' - \alpha_1) \cot \delta''. \quad (2)$$

$$\therefore \frac{\sin(\alpha'' - \alpha_1)}{\sin(\alpha' - \alpha_1)} = \frac{\cot \delta'}{\cot \delta''}.$$

$$\therefore \frac{\sin(\alpha'' - \alpha_1) + \sin(\alpha' - \alpha_1)}{\sin(\alpha'' - \alpha_1) - \sin(\alpha' - \alpha_1)} = \frac{\cot \delta' + \cot \delta''}{\cot \delta' - \cot \delta''}.$$

$$\therefore \frac{\tan \frac{1}{2}(\alpha'' + \alpha' - 2\alpha_1)}{\tan \frac{1}{2}(\alpha'' - \alpha')} = \frac{\sin(\delta'' + \delta')}{\sin(\delta'' - \delta')}.$$

$$\therefore \tan \tfrac{1}{2}(\alpha'' + \alpha' - 2\alpha_1) = \frac{\sin(\delta'' + \delta')}{\sin(\delta'' - \delta')} \tan \tfrac{1}{2}(\alpha'' - \alpha'). \quad (3)$$

From (3) find $\frac{1}{2}(\alpha'' + \alpha' - 2\alpha_1)$, thence finding α_1; then i may be found from either (1) or (2).

187. The Right Ascension and Declination of a Star are α and δ, and those of Another Star are α' and δ'; find the Hour Angle of the First Star and their Common Azimuth when the Stars are in the Same Vertical Circle, the Latitude of the Place being ϕ. — There are two positions, one when both stars are west, and the other when they are both east, of the meridian.

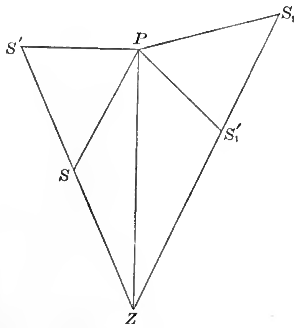


FIG. 138.

(1) $S'P=90°-\delta'$; $SP=90°-\delta$; $SPS'=\alpha-\alpha'$; $ZP=90°-\phi$. In the triangle SPS', find $PS'S$ and PSS'. Then in the triangle $S'PZ$ we know $S'P$, ZP, and $PS'Z$, and we find PZS' and ZPS'. In the triangle SPZ we know SP, ZP, and $PSZ = 180° - PSS'$, and we find PZS and ZPS.

The checks are $PZS'=PZS$, and $S'PZ-SPZ=\alpha-\alpha'$.

(2) $S_1P = 90° - \delta$, $S_1'P = 90° - \delta'$, $S_1PS_1' = \alpha - \alpha'$; find PS_1S_1' and $PS_1'S_1$, these angles being the same as those at S and S' in the first case. Then from the two triangles PS_1Z and $PS_1'Z$ we find the angles PZS_1 and PZS_1', which should be identical, and also the angles S_1PZ and $S_1'PZ$, whose difference should be $\alpha - \alpha'$.

LOGARITHMIC AND TRIGONOMETRIC

TABLES

FIVE DECIMAL PLACES

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CONTENTS.

NOTE. — The well-known tables of Gauss, Becker, and Albrecht have been taken as the standards, the proof sheets have been read with great care, and it is believed that the number of errors cannot be large. The arrangement of the figures on the page is in accordance with that adopted in the standard six and seven place tables.

The natural tables were reduced from seven-place tables and compared with published five-place tables.

For convenience in using the tables, the explanation has been placed after them instead of before them.

W. P. 3

I.

COMMON

LOGARITHMS OF NUMBERS

FROM 1 TO 11000.

N.	Log.	N.	Log.	N.	Log.	N.	Log.	N.	Log.
0	—	**20**	1.30 103	**40**	1.60 206	**60**	1.77 815	**80**	1.90 309
1	0.00 000	21	1.32 222	41	1.61 278	61	1.78 533	81	1.90 849
2	0.30 103	22	1.34 242	42	1.62 32$\bar{5}$	62	1.79 239	82	1.91 381
3	0.47 712	23	1.36 173	43	1.63 347	63	1.79 934	83	1.91 908
4	0.60 206	24	1.38 021	44	1.64 345	64	1.80 618	84	1.92 428
5	0.69 897	25	1.39 794	45	1.65 321	65	1.81 291	85	1.92 942
6	0.77 815	26	1.41 497	46	1.66 276	66	1.81 954	86	1.93 4$\bar{5}$0
7	0.84 510	27	1.43 136	47	1.67 210	67	1.82 607	87	1.93 952
8	0.90 309	28	1.44 716	48	1.68 124	68	1.83 251	88	1.94 448
9	0.95 424	29	1.46 240	49	1.69 020	69	1.83 88$\bar{5}$	89	1.94 939
10	1.00 000	**30**	1.47 712	**50**	1.69 897	**70**	1.84 510	**90**	1.95 424
11	1.04 139	31	1.49 [illegible]	51	1.70 757	71	1.85 126	91	1.95 904
12	1.07 918	32	1.50 5[illegible]	52	1.71 600	72	1.85 733	92	1.96 379
13	1.11 394	33	1.51 851	53	1.72 428	73	1.86 332	93	1.96 848
14	1.14 613	34	1.53 148	54	1.73 239	74	1.86 923	94	1.97 313
15	1.17 609	35	1.54 407	55	1.74 036	75	1.87 506	95	1.97 772
16	1.20 412	36	1.55 630	56	1.74 819	76	1.88 081	96	1.98 227
17	1.23 04$\bar{5}$	37	1.56 820	57	1.75 587	77	1.88 649	97	1.98 677
18	1.25 527	38	1.57 978	58	1.76 343	78	1.89 209	98	1.99 123
19	1.27 875	39	1.59 106	59	1.77 085	79	1.89 763	99	1.99 564
20	1.30 103	**40**	1.60 206	**60**	1.77 815	**80**	1.90 309	**100**	2.00 000

	S′.	T′.
0′	6.46 373	373
1	373	373

	S″.	T″.
0° 0′ = 0″	4.68 557	557
0 1 = 60	557	557
0 2 = 120	557	557

S′. (6.46)	T′.	N.	L. 0	1	2	3	4	5	6	7	8	9
366	[illegible]	**100**	00 000	043	087	130	173	217	260	303	346	389
366	385	101	432	475	518	561	604	647	689	732	775	817
366	385	102	860	903	945	988	*030	*072	*115	*157	*199	*242
366	386	103	01 284	326	368	410	452	494	536	578	620	662
366	386	104	703	745	787	828	870	912	953	995	*036	*078
366	386	105	02 119	160	202	243	284	325	366	407	449	490
366	386	106	531	572	612	653	694	735	776	816	857	898
366	387	107	938	979	*019	*060	*100	*141	*181	*222	*262	*302
365	387	108	03 342	383	423	463	503	543	583	623	663	703
365	387	109	743	782	822	862	902	941	981	*021	*060	*100
365	387	**110**	04 139	179	218	258	297	336	376	415	454	493
365	388	111	532	571	610	650	689	727	766	805	844	883
365	388	112	922	961	999	*038	*077	*115	*154	*192	*231	*269
365	388	113	05 308	346	385	423	461	500	538	576	614	652
365	389	114	690	729	767	805	843	881	918	956	994	*032
365	389	115	06 070	108	145	183	221	258	296	333	371	408
364	389	116	446	483	521	558	595	633	670	707	744	781
364	389	117	819	856	893	930	967	*004	*041	*078	*115	*151
364	390	118	07 188	225	262	298	335	372	408	445	482	518
364	390	119	555	591	628	664	700	737	773	809	846	882
364	390	**120**	918	954	990	*027	*063	*099	*135	*171	*207	*243
364	391	121	08 279	314	350	386	422	458	493	529	565	600
363	391	122	636	672	707	743	778	814	849	884	920	955
363	391	123	991	*026	*061	*096	*132	*167	*202	*237	*272	*307
363	391	124	09 342	377	412	447	482	517	552	587	621	656
363	392	125	691	726	760	795	830	864	899	934	968	*003
363	392	126	10 037	072	106	140	175	209	243	278	312	346
363	392	127	380	415	449	483	517	551	585	619	653	687
363	393	128	721	755	789	823	857	890	924	958	992	*025
362	393	129	11 059	093	126	160	193	227	261	294	327	361
362	393	**130**	394	428	461	494	528	561	594	628	661	694
362	394	131	727	760	793	826	860	893	926	959	992	*024
362	394	132	12 057	090	123	156	189	222	254	287	320	352
362	394	133	385	418	450	483	516	548	581	613	646	678
362	395	134	710	743	775	808	840	872	905	937	969	*001
361	395	135	13 033	066	098	130	162	194	226	258	290	322
361	395	136	354	386	418	450	481	513	545	577	609	640
361	396	137	672	704	735	767	799	830	862	893	925	956
361	396	138	988	*019	*051	*082	*114	*145	*176	*208	*239	*270
361	396	139	14 301	333	364	395	426	457	489	520	551	582
361	397	**140**	613	644	675	706	737	768	799	829	860	891
360	397	141	922	953	983	*014	*045	*076	*106	*137	*168	*198
360	397	142	15 229	259	290	320	351	381	412	442	473	503
360	398	143	534	564	594	625	655	685	715	746	776	806
360	398	144	836	866	897	927	957	987	*017	*047	*077	*107
360	398	145	16 137	167	197	227	256	286	316	346	376	406
360	399	146	435	465	495	524	554	584	613	643	673	702
359	399	147	732	761	791	820	850	879	909	938	967	997
359	399	148	17 026	056	085	114	143	173	202	231	260	289
359	400	149	319	348	377	406	435	464	493	522	551	580
359	400	**150**	609	638	667	696	725	754	782	811	840	869
		N.	L. 0	1	2	3	4	5	6	7	8	9

P. P.

	44	43	42	41	40	39	38	37	36	35	34	33	32	31	30
1	4.4	4.3	4.2	4.1	4.0	3.9	3.8	3.7	3.6	3.5	3.4	3.3	3.2	3.1	3.0
2	8.8	8.6	8.4	8.2	8.0	7.8	7.6	7.4	7.2	7.0	6.8	6.6	6.4	6.2	6.0
3	13.2	12.9	12.6	12.3	12.0	11.7	11.4	11.1	10.8	10.5	10.2	9.9	9.6	9.3	9.0
4	17.6	17.2	16.8	16.4	16.0	15.6	15.2	14.8	14.4	14.0	13.6	13.2	12.8	12.4	12.0
5	22.0	21.5	21.0	20.5	20.0	19.5	19.0	18.5	18.0	17.5	17.0	16.5	16.0	15.5	15.0
6	26.4	25.8	25.2	24.6	24.0	23.4	22.8	22.2	21.6	21.0	20.4	19.8	19.2	18.6	18.0
7	30.8	30.1	29.4	28.7	28.0	27.3	26.6	25.9	25.2	24.5	23.8	23.1	22.4	21.7	21.0
8	35.2	34.4	33.6	32.8	32.0	31.2	30.4	29.6	28.8	28.0	27.2	26.4	25.6	24.8	24.0
9	39.6	38.7	37.8	36.9	36.0	35.1	34.2	33.3	32.4	31.5	30.6	29.7	28.8	27.9	27.0

		S.′	T.′
1′	6.46	373	373
2		373	373
		373	373
		373	373
		372	373
		372	373

			S.″	T.″
0°	1′ = 60″	4.68	557	557
0	2 = 120		557	557
0	3 = 180		557	557
0	16 = 960		557	558
0	17 = 1020		557	558
0	18 = 1080		557	558
0	19 = 1140		557	558

			S.″	T.″
0°	19′ = 1140″	4.68	557	558
0	20 = 1200		557	558
0	21 = 1260		557	558
0	22 = 1320		557	558
0	23 = 1380		557	558
0	24 = 1440		557	558
0	25 = 1500		557	558

S′.	T′. 6.46	N.	L. 0	1	2	3	4	5	6	7	8	9
359	400	**150**	17 609	[illegible]	667	696	725	754	782	811	840	869
359	401	151	898	926	955	984	*013	*041	*070	*099	*127	*156
358	401	152	18 184	213	241	270	298	327	355	384	412	441
358	401	153	469	498	526	554	583	611	639	667	696	724
358	402	154	752	780	808	837	865	893	921	949	977	*005
358	402	155	19 033	061	089	117	145	173	201	229	257	285
358	402	156	312	340	368	396	424	451	479	507	535	562
358	403	157	590	618	645	673	700	728	756	783	811	838
357	403	158	866	893	921	948	976	*003	*030	*058	*085	*112
357	404	159	20 140	167	194	222	249	276	303	330	358	385
357	404	**160**	412	439	466	493	520	548	575	602	629	656
357	404	161	683	710	737	763	790	817	844	871	898	925
357	405	162	952	978	*005	*032	*059	*085	*112	*139	*165	*192
356	405	163	21 219	245	272	299	325	352	378	405	431	458
356	406	164	484	511	537	564	590	617	643	669	696	722
356	406	165	748	775	801	827	854	880	906	932	958	985
356	406	166	22 011	037	063	089	115	141	167	194	220	246
356	407	167	272	298	324	350	376	401	427	453	479	505
355	407	168	531	557	583	608	634	660	686	712	737	763
355	408	169	789	814	840	866	891	917	943	968	994	*019
355	408	**170**	23 045	070	096	121	147	172	198	223	249	274
355	408	171	300	325	350	376	401	426	452	477	502	528
354	409	172	553	578	603	629	654	679	704	729	754	779
354	409	173	805	830	855	880	905	930	955	980	*005	*030
354	410	174	24 055	080	105	130	155	180	204	229	254	279
354	410	175	304	329	353	378	403	428	452	477	502	527
354	411	176	551	576	601	625	650	674	699	724	748	773
353	411	177	797	822	846	[illegible]	895	920	944	969	993	*018
353	411	178	25 042	066	091	115	139	164	188	212	237	261
353	412	179	285	310	334	358	382	406	431	455	479	503
353	412	**180**	527	551	575	600	624	648	672	696	720	744
353	413	181	768	792	816	840	864	888	912	935	959	*983
352	413	182	26 007	031	055	079	102	126	150	174	198	221
352	414	183	245	269	293	316	340	364	387	411	435	458
352	414	184	482	505	529	553	576	600	623	647	670	694
352	415	185	717	741	764	788	811	834	858	881	905	928
351	415	186	951	975	998	*021	*045	*068	*091	*114	*138	*161
351	415	187	27 184	207	231	254	277	300	323	346	370	393
351	416	188	416	439	462	485	508	531	554	577	600	623
351	416	189	646	669	692	715	738	761	784	807	830	852
350	417	**190**	875	898	921	944	967	989	*012	*035	*058	*081
350	417	191	28 103	126	149	171	194	217	240	262	285	307
350	418	192	330	353	375	398	421	443	466	488	511	533
350	418	193	556	578	601	623	646	668	691	713	735	758
350	419	194	780	803	825	847	870	892	914	937	959	981
349	419	195	29 003	026	048	070	092	115	137	159	181	203
349	420	196	226	248	270	292	314	336	358	380	403	425
[illegible]	420	197	447	469	491	513	535	557	579	601	623	645
[illegible]	421	198	667	688	710	732	754	776	798	820	842	863
[illegible]	421	199	885	907	929	951	973	994	*016	*038	*060	*081
[illegible]	422	**200**	30 103	125	146	168	190	211	233	255	276	298
		N.	L. 0	1	2	3	4	5	6	7	8	9

P. P.

	29	28
1	2.9	2.8
2	5.8	5.6
3	8.7	8.4
4	11.6	11.2
5	14.5	14.0
6	17.4	16.8
7	20.3	19.6
8	23.2	22.4
9	26.1	25.2

	27	26
1	2.7	2.6
2	5.4	5.2
3	8.1	7.8
4	10.8	10.4
5	13.5	13.0
6	16.2	15.6
7	18.9	18.2
8	21.6	20.8
9	24.3	23.4

	25
1	2.5
2	5.0
3	7.5
4	10.0
5	12.5
6	15.0
7	17.5
8	20.0
9	22.5

	24	23
1	2.4	2.3
2	4.8	4.6
3	7.2	6.9
4	9.6	[illegible]
5	12.0	11.5
6	14.4	13.8
7	16.8	16.1
8	19.2	18.4
9	21.6	20.7

	22	21
1	2.2	2.1
2	4.4	4.2
3	6.6	6.3
4	8.8	8.4
5	11.0	10.5
6	13.2	12.6
7	15.4	14.7
8	17.6	16.8
9	19.8	18.9

	S.′	T.′
6.46	373	373
	373	373
	372	373
	372	373

		S.″	T.″
0° 2′ = 120″	4.68	557	557
0 3 = 180		557	557
0 4 = 240		557	558
0 25 = 1500		557	558
0 26 = 1560		557	558
0 27 = 1620		557	558
0 28 = 1680		557	558

		S.″	T.″
0° 28′ = 1680″	4.68	55[illegible]	[illegible]
0 29 = 1740		5[illegible]	[illegible]
0 30 = 1800		5[illegible]	[illegible]
0 31 = 1860		5[illegible]	[illegible]
0 32 = 1920		5[illegible]	[illegible]
0 33 = 1980		5[illegible]	[illegible]
0 34 = 2040		5[illegible]	[illegible]

N.	L. 0	1	2	3	4	5	6	7	8	9
200	30 103	125	146	168	190	211	233	255	276	298
201	320	341	363	384	406	428	449	471	492	514
202	535	557	578	600	621	643	664	685	707	728
203	750	771	792	814	835	856	878	899	920	942
204	963	984	*006	*027	*048	*069	*091	*112	*133	*154
205	31 175	197	218	239	260	281	302	323	345	366
206	387	408	429	450	471	492	513	534	555	576
207	597	618	639	660	681	702	723	744	765	785
208	806	827	848	869	890	911	931	952	973	994
209	32 015	035	056	077	098	118	139	160	181	201
210	222	243	263	284	305	325	346	366	387	408
211	428	449	469	490	510	531	552	572	593	613
212	634	654	675	695	715	736	756	777	797	818
213	838	858	879	899	919	940	960	980	*001	*021
214	33 041	062	082	102	122	143	163	183	203	224
215	244	264	284	304	325	345	365	385	405	425
216	445	465	486	506	526	546	566	586	606	626
217	646	666	686	706	726	746	766	786	806	826
218	846	866	885	905	925	945	965	985	*005	*025
219	34 044	064	084	104	124	143	163	183	203	223
220	242	262	282	301	321	341	361	380	400	420
221	439	459	479	498	518	537	557	577	596	616
222	635	655	674	694	713	733	753	772	792	811
223	830	850	869	889	908	928	947	967	986	*005
224	35 025	044	064	083	102	122	141	160	180	199
225	218	238	257	276	295	315	334	353	372	392
226	411	430	449	468	488	507	526	545	564	583
227	603	622	641	660	679	698	717	736	755	774
228	793	813	832	851	870	889	908	927	946	965
229	984	*003	*021	*040	*059	*078	*097	*116	*135	*154
230	36 173	192	211	229	248	267	286	305	324	342
231	361	380	399	418	436	455	474	493	511	530
232	549	568	586	605	624	642	661	680	698	717
233	736	754	773	791	810	829	847	866	884	903
234	922	940	959	977	996	*014	*033	*051	*070	*088
235	37 107	125	144	162	181	199	218	236	254	273
236	291	310	328	346	365	383	401	420	438	457
237	475	493	511	530	548	566	585	603	621	639
238	658	676	694	712	731	749	767	785	803	822
239	840	858	876	894	912	931	949	967	985	*003
240	38 021	039	057	075	093	112	130	148	166	184
241	202	220	238	256	274	292	310	328	346	364
242	382	399	417	435	453	471	489	507	525	543
243	561	578	596	614	632	650	668	686	703	721
244	739	757	775	792	810	828	846	863	881	899
245	917	934	952	970	987	*005	*023	*041	*058	*076
246	39 094	111	129	146	164	182	199	217	235	252
247	270	287	305	322	340	358	375	393	410	428
248	445	463	480	498	515	533	550	568	585	602
249	620	637	655	672	690	707	724	742	759	777
250	794	811	829	846	863	881	898	915	933	950
N.	L. 0	1	2	3	4	5	6	7	8	9

P. P.

	22	21
1	2.2	2.1
2	4.4	4.2
3	6.6	6.3
4	8.8	8.4
5	11.0	10.5
6	13.2	12.6
7	15.4	14.7
8	17.6	16.8
9	19.8	18.9

	20	19	18	17
1	2.0	1.9	1.8	1.7
2	4.0	3.8	3.6	3.4
3	6.0	5.7	5.4	5.1
4	8.0	7.6	7.2	6.8
5	10.0	9.5	9.0	8.5
6	12.0	11.4	10.8	10.2
7	14.0	13.3	12.6	11.9
8	16.0	15.2	14.4	13.6
9	18.0	17.1	16.2	15.3

		S.′	T.′
2′	6.46	373	373
3		373	373
20		372	373
25		372	373

		S.″	T.″
0° 3′ = 180″	4.68	557	557
0 4 = 240		557	558
0 5 = 300		557	558
0 33 = 1980		557	[illegible]
0 34 = 2040		557	[illegible]
0 35 = 2100		557	[illegible]
0 36 = 2160		557	[illegible]

		S.″	T.″
0° 36′ = 2160″	4.68	557	559
0 37 = 2220		557	559
0 38 = 2280		557	559
0 39 = 2340		557	559
0 40 = 2400		557	559
0 41 = 2460		556	560
0 42 = 2520		556	560

N.	L. 0	1	2	3	4	5	6	7	8	9
250	39 794	811	829	846	863	881	898	915	933	950
251	967	985	*002	*019	*037	*054	*071	*088	*106	*123
252	40 140	157	175	192	209	226	243	261	278	295
253	312	329	346	364	381	398	415	432	449	466
254	483	500	518	535	552	569	586	603	620	637
255	654	671	688	705	722	739	756	773	790	807
256	824	841	858	875	892	909	926	943	960	977
257	993	*010	*027	*044	*061	*078	*095	*111	*128	*145
258	41 162	179	196	212	229	246	263	280	296	313
259	330	347	363	380	397	414	430	447	464	481
260	497	514	531	547	564	581	597	614	631	647
261	664	681	697	714	731	747	764	780	797	814
262	830	847	863	880	896	913	929	946	963	979
263	996	*012	*029	*045	*062	*078	*095	*111	*127	*144
264	42 160	177	193	210	226	243	259	275	292	308
265	325	341	357	374	390	406	423	439	455	472
266	488	504	521	537	553	570	586	602	619	635
267	651	667	684	700	716	732	749	765	781	797
268	813	830	846	862	878	894	911	927	943	959
269	975	991	*008	*024	*040	*056	*072	*088	*104	*120
270	43 136	152	169	185	201	217	233	249	265	281
271	297	313	329	345	361	377	393	409	425	441
272	457	473	489	505	521	537	553	569	584	600
273	616	632	648	664	680	696	712	727	743	759
274	775	791	807	823	838	854	870	886	902	917
275	933	949	965	981	996	*012	*028	*044	*059	*075
276	44 091	107	122	138	154	170	185	201	217	232
277	248	264	279	295	311	326	342	358	373	389
278	404	420	436	451	467	483	498	514	529	545
279	560	576	592	607	623	638	654	669	685	700
280	716	731	747	762	778	793	809	824	840	855
281	871	886	902	917	932	948	963	979	994	*010
282	45 025	040	056	071	086	102	117	133	148	163
283	179	194	209	225	240	255	271	286	301	317
284	332	347	362	378	393	408	423	439	454	469
285	484	500	515	530	545	561	576	591	606	621
286	637	652	667	682	697	712	728	743	758	773
287	788	803	818	834	849	864	879	894	909	924
288	939	954	969	984	*000	*015	*030	*045	*060	*075
289	46 090	105	120	135	150	165	180	195	210	225
290	240	255	270	285	300	315	330	345	359	374
291	389	404	419	434	449	464	479	494	509	523
292	538	553	568	583	598	613	627	642	657	672
293	687	702	716	731	746	761	776	790	805	820
294	835	850	864	879	894	909	923	938	953	967
295	982	997	*012	*026	*041	*056	*070	*085	*100	*114
296	47 129	144	159	173	188	202	217	232	246	261
297	276	290	305	319	334	349	363	378	392	407
298	422	436	451	465	480	494	509	524	538	553
299	567	582	596	611	625	640	654	669	683	698
300	712	727	741	756	770	784	799	813	828	8[illegible]
N.	L. 0	1	2	3	4	5	6	7	8	9

P. P.

	18	17	16	15	14
1	1.8	1.7	1.6	1.5	1.4
2	3.6	3.4	3.2	3.0	2.8
3	5.4	5.1	4.8	4.5	4.2
4	7.2	6.8	6.4	6.0	5.6
5	9.0	8.5	8.0	7.5	7.0
6	10.8	10.2	9.6	9.0	8.4
7	12.6	11.9	11.2	10.5	9.8
8	14.4	13.6	12.8	12.0	11.2
9	16.2	15.3	14.4	13.5	12.6

		S.′	T.′
2′	6.46	373	373
3		373	373
25		372	373
26		372	373
27		372	374
30		37[illegible]	374

		S.″	T.″
0° 4′ = 240″	4.68	557	558
0 5 = 300		557	558
0 41 = 2460		556	560
0 42 = 2520		556	560
0 43 = 2580		556	560
0 44 = 2640		556	560
0 45 = 2700		556	560

		S.″	T.″
0° 45′ = 2700″	4.68	556	560
0 46 = 2760		556	560
0 47 = 2820		556	560
0 48 = 2880		556	560
0 49 = 2940		556	560
0 50 = 3000		556	561

Log π = 0.49715

N.	L. 0	1	2	3	4	5	6	7	8	9
300	47 712	727	741	756	770	784	799	813	828	842
301	857	871	885	900	914	929	943	958	972	986
302	48 001	015	029	044	058	073	087	101	116	130
303	144	159	173	187	202	216	230	244	259	273
304	287	302	316	330	344	359	373	387	401	416
305	430	444	458	473	487	501	515	530	544	558
306	572	586	601	615	629	643	657	671	686	700
307	714	728	742	756	770	785	799	813	827	841
308	855	869	883	897	911	926	940	954	968	982
309	996	*010	*024	*038	*052	*066	*080	*094	*108	*122
310	49 136	150	164	178	192	206	220	234	248	262
311	276	290	304	318	332	346	360	374	388	402
312	415	429	443	457	471	485	499	513	527	541
313	554	568	582	596	610	624	638	651	665	679
314	693	707	721	734	748	762	776	790	803	817
315	831	845	859	872	886	900	914	927	941	955
316	969	982	996	*010	*024	*037	*051	*065	*079	*092
317	50 106	120	133	147	161	174	188	202	215	229
318	243	256	270	284	297	311	325	338	352	365
319	379	393	406	420	433	447	461	474	488	501
320	515	529	542	556	569	583	596	610	623	637
321	651	664	678	691	705	718	732	745	759	772
322	786	799	813	826	840	853	866	880	893	907
323	920	934	947	961	974	987	*001	*014	*028	*041
324	51 055	068	081	095	108	121	135	148	162	175
325	188	202	215	228	242	255	268	282	295	308
326	322	335	348	362	375	388	402	415	428	441
327	455	468	481	495	508	521	534	548	561	574
328	587	601	614	627	640	654	667	680	693	706
329	720	733	746	759	772	786	799	812	825	838
330	851	865	878	891	904	917	930	943	957	970
331	983	996	*009	*022	*035	*048	*061	*075	*088	*101
332	52 114	127	140	153	166	179	192	205	218	231
333	244	257	270	284	297	310	323	336	349	362
334	375	388	401	414	427	440	453	466	479	492
335	504	517	530	543	556	569	582	595	608	621
336	634	647	660	673	686	699	711	724	737	750
337	763	776	789	802	815	827	840	853	866	879
338	892	905	917	930	943	956	969	982	994	*007
339	53 020	033	046	058	071	084	097	110	122	135
340	148	161	173	186	199	212	224	237	250	263
341	275	288	301	314	326	339	352	364	377	390
342	403	415	428	441	453	466	479	491	504	517
343	529	542	555	567	580	593	605	618	631	643
344	656	668	681	694	706	719	732	744	757	769
345	782	794	807	820	832	845	857	870	882	895
346	908	920	933	945	958	970	983	995	*008	*020
347	54 033	045	058	070	083	095	108	120	133	145
348	158	170	183	195	208	220	233	245	258	270
349	283	295	307	320	332	345	357	370	382	394
350	407	419	432	444	456	469	481	494	506	518
N.	L. 0	1	2	3	4	5	6	7	8	9

P. P.

	15	14	13	12
1	1.5	1.4	1.3	1.2
2	3.0	2.8	2.6	2.4
3	4.5	4.2	3.9	3.6
4	6.0	5.6	5.2	4.8
5	7.5	7.0	6.5	6.0
6	9.0	8.4	7.8	7.2
7	10.5	9.8	9.1	8.4
8	12.0	11.2	10.4	9.6
9	13.5	12.6	11.7	10.8

		S.′	T.′
3′	6.46	373	373
4		373	373
30		372	374
35		372	374

		S.″	T.″
0° 5′ = 300″	4.68	557	558
0 6 = 360		557	558
0 50 = 3000		556	561
0 51 = 3060		556	561
0 52 = 3120		556	561
0 53 = 3180		556	561
0 54 = 3240		556	561

		S.″	T.″
0° 54′ = 3240″	4.68	556	561
0 55 = 3300		556	561
0 56 = 3360		556	561
0 57 = 3420		555	561
0 58 = 3480		555	562
0 59 = 3540		555	562

N.	L. 0	1	2	3	4	5	6	7	8	9
350	54 407	419	432	444	456	469	481	494	506	518
351	531	543	555	568	580	593	605	617	630	642
352	654	667	679	691	704	716	728	741	753	765
353	777	790	802	814	827	839	851	864	876	888
354	900	913	925	937	949	962	974	986	998	*011
355	55 023	035	047	060	072	084	096	108	121	133
356	145	157	169	182	194	206	218	230	242	255
357	267	279	291	303	315	328	340	352	364	376
358	388	400	413	425	437	449	461	473	485	497
359	509	522	534	546	558	570	582	594	606	618
360	630	642	654	666	678	691	703	715	727	739
361	751	763	775	787	799	811	823	835	847	859
362	871	883	895	907	919	931	943	955	967	979
363	991	*003	*015	*027	*038	*050	*062	*074	*086	*098
364	56 110	122	134	146	158	170	182	194	205	217
365	229	241	253	265	277	289	301	312	324	336
366	348	360	372	384	396	407	419	431	443	455
367	467	478	490	502	514	526	538	549	561	573
368	585	597	608	620	632	644	656	667	679	691
369	703	714	726	738	750	761	773	785	797	808
370	820	832	844	855	867	879	891	902	914	926
371	937	949	961	972	984	996	*008	*019	*031	*043
372	57 054	066	078	089	101	113	124	136	148	159
373	171	183	194	206	217	229	241	252	264	276
374	287	299	310	322	334	345	357	368	380	392
375	403	415	426	438	449	461	473	484	496	507
376	519	530	542	553	565	576	588	600	611	6[illegible]
377	634	646	657	669	680	692	703	715	726	[illegible]
378	749	761	772	784	795	807	818	830	841	[illegible]52
379	864	875	887	898	910	921	933	944	955	967
380	978	990	*001	*013	*024	*035	*047	*058	*070	*081
381	58 092	104	115	127	138	149	161	172	184	195
382	206	218	229	240	252	263	274	286	297	309
383	320	331	343	354	365	377	388	399	410	422
384	433	444	456	467	478	490	501	512	524	535
385	546	557	569	580	591	602	614	625	636	647
386	659	670	681	692	704	715	726	737	749	760
387	771	782	794	805	816	827	838	850	861	872
388	883	894	906	917	928	939	950	961	973	984
389	995	*006	*017	*028	*040	*051	*062	*073	*084	*095
390	59 106	118	129	140	151	162	173	184	195	207
391	218	229	240	251	262	273	284	295	306	318
392	329	340	351	362	373	384	395	406	417	428
393	439	450	461	472	483	494	506	517	528	539
394	550	561	572	583	594	605	616	627	638	649
395	660	671	682	693	704	715	726	737	748	759
396	770	780	791	802	813	824	835	846	857	868
397	879	890	901	912	923	934	945	956	966	977
398	988	999	*010	*021	*032	*043	*054	*065	*076	*086
399	60 097	108	119	130	141	152	163	173	184	195
400	206	217	228	239	249	260	271	282	293	304
N.	L. 0	1	2	3	4	5	6	7	8	9

P. P.

	13
1	1.3
2	2.6
3	3.9
4	5.2
5	6.5
6	7.8
7	9.1
8	10.4
9	11.7

	12
1	1.2
2	2.4
3	3.6
4	4.8
5	6.0
6	7.2
7	8.4
8	9.6
9	10.8

	11
1	1.1
2	2.2
3	3.3
4	4.4
5	5.5
6	6.6
7	7.7
8	8.8
9	9.9

	10
1	1.0
2	2.0
3	3.0
4	4.0
5	5.0
6	6.0
7	7.0
8	8.0
9	9.0

	S.′	T.′
[illegible]′	6.46 373	373
4	[illegible]	3[illegible]
35[illegible]	3[illegible]	374
39	372	37[illegible]
40	37[illegible]	37[illegible]

		S.″	T.″
0° 5′ = 300″	4.68	557	558
[illegible]		557	558
[illegible]		557	558
[illegible] 58 = [illegible]		555	562
[illegible] 59 = [illegible]		555	562
1 0 = [illegible]		555	562
1 1 = 36[illegible]		555	562

		S.″	T.″
1° 1′ = 3660″	4.68	555	562
1 2 = 3720		[illegible]	[illegible]
1 3 = 3780		[illegible]	[illegible]
1 4 = 3840		[illegible]	[illegible]
1 5 = 3900		[illegible]	[illegible]
1 6 = 3960		[illegible]	[illegible]
1 7 = 4020		[illegible]	[illegible]

N.	L. 0	1	2	3	4	5	6	7	8	9
400	60 206	217	228	239	249	260	271	282	293	304
401	314	325	336	347	358	369	379	390	401	412
402	423	433	444	455	466	477	487	498	509	520
403	531	541	552	563	574	584	595	606	617	627
404	638	649	660	670	681	692	703	713	724	735
405	746	756	767	778	788	799	810	821	831	842
406	853	863	874	885	895	906	917	927	938	949
407	959	970	981	991	*002	*013	*023	*034	*045	*055
408	61 066	077	087	098	109	119	130	140	151	162
409	172	183	194	204	215	225	236	247	257	268
410	278	289	300	310	321	331	342	352	363	374
411	384	395	405	416	426	437	448	458	469	479
412	490	500	511	521	532	542	553	563	574	584
413	595	606	616	627	637	648	658	669	679	690
414	700	711	721	731	742	752	763	773	784	794
415	805	815	826	836	847	857	868	878	888	899
416	909	920	930	941	951	962	972	982	993	*003
417	62 014	024	034	045	055	066	076	086	097	107
418	118	128	138	149	159	170	180	190	201	211
419	221	232	242	252	263	273	284	294	304	315
420	325	335	346	356	366	377	387	397	408	418
421	428	439	449	459	469	480	490	500	511	521
422	531	542	552	562	572	583	593	603	613	624
423	634	644	655	665	675	685	696	706	716	726
424	737	747	757	767	778	788	798	808	818	829
425	839	849	859	870	880	890	900	910	921	931
426	941	951	961	972	982	992	*002	*012	*022	*033
427	63 043	053	063	073	083	094	104	114	124	134
428	144	155	165	175	185	195	205	215	225	236
429	246	256	266	276	286	296	306	317	327	337
430	347	357	367	377	387	397	407	417	428	438
431	448	458	468	478	488	498	508	518	528	538
432	548	558	568	579	589	599	609	619	629	639
433	649	659	669	679	689	699	709	719	729	739
434	749	759	769	779	789	799	809	819	829	839
435	849	859	869	879	889	899	909	919	929	939
436	949	959	969	979	988	998	*008	*018	*028	*038
437	64 048	058	068	078	088	098	108	118	128	137
438	147	157	167	177	187	197	207	217	227	237
439	246	256	266	276	286	296	306	316	326	335
440	345	355	365	375	385	395	404	414	424	434
441	444	454	464	473	483	493	503	513	523	532
442	542	552	562	572	582	591	601	611	621	631
443	640	650	660	670	680	689	699	709	719	729
444	738	748	758	768	777	787	797	807	816	826
445	836	846	856	865	875	885	895	904	914	924
446	933	943	953	963	972	982	992	*002	*011	*021
447	65 031	040	050	060	070	079	089	099	108	118
448	128	137	147	157	167	176	186	196	205	215
449	225	234	244	254	263	273	283	292	302	312
450	321	331	341	350	360	369	379	389	398	408
N.	L. 0	1	2	3	4	5	6	7	8	9

P. P.

	11
1	1.1
2	2.2
3	3.3
4	4.4
5	5.5
6	6.6
7	7.7
8	8.8
9	9.9

	10
1	1.0
2	2.0
3	3.0
4	4.0
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7	7.0
8	8.0
9	9.0

	9
1	0.9
2	1.8
3	2.7
4	3.6
5	4.5
6	5.4
7	6.3
8	7.2
9	8.1

		S.′	T.′
4′	6.46	373	373
5		373	373
40		372	375
42		372	375
43		371	375
44		371	375
45		371	375

		S.″	T.″
0° 6′ = 360″	4.68	557	558
0 7 = 420		557	558
0 8 = 480		557	558
1 6 = 3960		555	563
1 7 = 4020		555	563
1 8 = 4080		555	563
1 9 = 4140		555	563

		S.″	T.″
1° 9′ = 4140″	4.68	555	563
1 10 = 4200		554	563
1 11 = 4260		554	564
1 12 = 4320		554	564
1 13 = 4380		554	564
1 14 = 4440		554	564
1 15 = 4500		554	564

N.	L. 0	1	2	3	4	5	6	7	8	9
450	65 321	331	341	350	360	369	379	389	398	408
451	418	427	437	447	456	466	475	485	495	504
452	514	523	533	543	552	562	571	581	591	600
453	610	619	629	639	648	658	667	677	686	696
454	706	715	725̄	734	744	753	763	772	782	792
455	801	811	820	830	839	849	858	868	877	887
456	896	906	916	925	935̄	944	954	963	973	982
457	992	*001	*011	*020	*030	*039	*049	*058	*068	*077
458	66 087	096	106	115̄	124	134	143	153	162	172
459	181	191	200	210	219	229	238	247	257	266
460	276	285	295̄	304	314	323	332	342	351	361
461	370	380	389	398	408	417	427	436	445	455̄
462	464	474	483	492	502	511	521	530	539	549
463	558	567	577	586	596	605̄	614	624	633	642
464	652	661	671	680	689	699	708	717	727	736
465	745	755̄	764	773	783	792	801	811	820	829
466	839	848	857	867	876	885	894	904	913	922
467	932	941	950	960	969	978	987	997	*006	*015
468	67 025̄	034	043	052	062	071	080	089	099	108
469	117	127	136	145	154	164	173	182	191	201
470	210	219	228	237	247	256	265	274	284	293
471	302	311	321	330	339	348	357	367	376	385̄
472	394	403	413	422	431	440	449	459	468	477
473	486	495	504	514	523	532	541	550	560	569
474	578	587	596	605	614	624	633	642	651	660
475	669	679	688	697	706	715	724	733	742	752
476	761	770	779	788	797	806	815	825̄	834	843
477	852	861	870	879	888	897	906	916	925̄	934
478	943	952	961	970	979	988	997	*006	*015	*024
479	68 034	043	052	061	070	079	088	097	106	115
480	124	133	142	151	160	169	178	187	196	205
481	215̄	224	233	242	251	260	269	278	287	296
482	305̄	314	323	332	341	350	359	368	377	386
483	395̄	404	413	422	431	440	449	458	467	476
484	485̄	494	502	511	520	529	538	547	556	565
485	574	583	592	601	610	619	628	637	646	655̄
486	664	673	681	690	699	708	717	726	735	744
487	753	762	771	780	789	797	806	815	824	833
488	842	851	860	869	878	886	895	904	913	922
489	931	940	949	958	966	975	984	993	*002	*011
490	69 020	028	037	046	055	064	073	082	090	099
491	108	117	126	135̄	144	152	161	170	179	188
492	197	205	214	223	232	241	249	258	267	276
493	285̄	294	302	311	320	329	338	346	355	364
494	373	381	390	399	408	417	425	434	443	452
495	461	469	478	487	496	504	513	522	531	539
496	548	557	566	574	583	592	601	609	618	627
497	636	644	653	662	671	679	688	697	705	714
498	723	732	740	749	758	767	775	784	793	801
499	810	819	827	836	845̄	854	862	871	880	888
500	897	906	914	923	932	940	949	958	966	975
N.	L. 0	1	2	3	4	5	6	7	8	9

P. P.

	10
1	1.0
2	2.0
3	3.0
4	4.0
5	5.0
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7	7.0
8	8.0
9	9.0

	9
1	0.9
2	1.8
3	2.7
4	3.6
5	4.5
6	5.4
7	6.3
8	7.2
9	8.1

	8
1	0.8
2	1.6
3	2.4
4	3.2
5	4.0
6	4.8
7	5.6
8	6.4
9	7.2

		S.′	T.′
4′	6.46	373	373
5		373	373
45		371	375
48		371	375
49		371	376
50		371	376

		S.″	T.″
0° 7′ = 420″	4.68	557	558
0 8 = 480		557	558
0 9 = 540		557	558
1 15 = 4500		554	564
1 16 = 4560		554	565̄
1 17 = 4620		554	565̄
1 18 = 4680		554	565̄

		S.′	T.″
1° 18′ = 4680″	4.68	554	565
1 19 = 4740		554	565
1 20 = 4800		554	565
1 21 = 4860		553	566
1 22 = 4920		553	566
1 23 = 4980		553	566
1 24 = 5040		553	566

N.	L. 0	1	2	3	4	5	6	7	8	9
500	69 897	906	914	923	932	940	949	958	966	975
501	984	992	*001	*010	*018	*027	*036	*044	*053	*062
502	70 070	079	088	096	105	114	122	131	140	148
503	157	165	174	183	191	200	209	217	226	234
504	243	252	260	269	278	286	295	303	312	321
505	329	338	346	355	364	372	381	389	398	406
506	415	424	432	441	449	458	467	475	484	492
507	501	509	518	526	535	544	552	561	569	578
508	586	595	603	612	621	629	638	646	655	663
509	672	680	689	697	706	714	723	731	740	749
510	757	766	774	783	791	800	808	817	825	834
511	842	851	859	868	876	885	893	902	910	919
512	927	935	944	952	961	969	978	986	995	*003
513	71 012	020	029	037	046	054	063	071	079	088
514	096	105	113	122	130	139	147	155	164	172
515	181	189	198	206	214	223	231	240	248	257
516	265	273	282	290	299	307	315	324	332	341
517	349	357	366	374	383	391	399	408	416	425
518	433	441	450	458	466	475	483	492	500	508
519	517	525	533	542	550	559	567	575	584	592
520	600	609	617	625	634	642	650	659	667	675
521	684	692	700	709	717	725	734	742	750	759
522	767	775	784	792	800	809	817	825	834	842
523	850	858	867	875	883	892	900	908	917	925
524	933	941	950	958	966	975	983	991	999	*008
525	72 016	024	032	041	049	057	066	074	082	090
526	099	107	115	123	132	140	148	156	165	173
527	181	189	198	206	214	222	230	239	247	255
528	263	272	280	288	296	304	313	321	329	337
529	346	354	362	370	378	387	395	403	411	419
530	428	436	444	452	460	469	477	485	493	501
531	509	518	526	534	542	550	558	567	575	583
532	591	599	607	616	624	632	640	648	656	665
533	673	681	689	697	705	713	722	730	738	746
534	754	762	770	779	787	795	803	811	819	827
535	835	843	852	860	868	876	884	892	900	908
536	916	925	933	941	949	957	965	973	981	989
537	997	*006	*014	*022	*030	*038	*046	*054	*062	*070
538	73 078	086	094	102	111	119	127	135	143	151
539	159	167	175	183	191	199	207	215	223	231
540	239	247	255	263	272	280	288	296	304	312
541	320	328	336	344	352	360	368	376	384	392
542	400	408	416	424	432	440	448	456	464	472
543	480	488	496	504	512	520	528	536	544	552
544	560	568	576	584	592	600	608	616	624	632
545	640	648	656	664	672	679	687	695	703	711
546	719	727	735	743	751	759	767	775	783	791
547	799	807	815	823	830	838	846	854	862	870
548	878	886	894	902	910	918	926	933	941	949
549	957	965	973	981	989	997	*005	*013	*020	*028
550	74 036	044	052	060	068	076	084	092	099	107
N.	L. 0	1	2	3	4	5	6	7	8	9

P. P.

	9
1	0.9
2	1.8
3	2.7
4	3.6
5	4.5
6	5.4
7	6.3
8	7.2
9	8.1

	8
1	0.8
2	1.6
3	2.4
4	3.2
5	4.0
6	4.8
7	5.6
8	6.4
9	7.2

1	[illegible]
2	[illegible]
3	[illegible]
4	[illegible]
5	[illegible]
6	[illegible]
7	[illegible]
8	[illegible]
9	[illegible]

		S.′	T.′
5′	6.46	373	373
6		373	373
50		371	376
55		371	376

		S.″	T.″
0° 8′ = 480″	4.68	557	558
0 9 = 540		557	558
0 10 = 600		557	558
1 23 = 4980		553	566
1 24 = 5040		553	566
1 25 = 5100		553	566
1 26 = 5160		553	567

		S.″	T.″
1° 26′ = 5160″	4.68	553	567
1 27 = 5220		553	[illegible]
1 28 = 5280		553	[illegible]
1 29 = 5340		553	[illegible]
1 30 = 5400		553	[illegible]
1 31 = 5460		552	[illegible]
1 32 = 5520		552	568

N.	L. 0	1	2	3	4	5	6	7	8	9	P. P.
550	74 036	044	052	060	068	076	084	092	099	107	
551	115	123	131	139	147	155	162	170	178	186	
552	194	202	210	218	225	233	241	249	257	265	
553	273	280	288	296	304	312	320	327	335	343	
554	351	359	367	374	382	390	398	406	414	421	
555	429	437	445	453	461	468	476	484	492	500	
556	507	515	523	531	539	547	554	562	570	578	
557	586	593	601	609	617	624	632	640	648	656	
558	663	671	679	687	695	702	710	718	726	733	
559	741	749	757	764	772	780	788	796	803	811	
560	819	827	834	842	850	858	865	873	881	889	8
561	896	904	912	920	927	935	943	950	958	966	1 0.8
562	974	981	989	997	*005	*012	*020	*028	*035	*043	2 1.6
563	75 051	059	066	074	082	089	097	105	113	120	3 2.4
564	128	136	143	151	159	166	174	182	189	197	4 3.2
565	205	213	220	228	236	243	251	259	266	274	5 4.0
566	282	289	297	305	312	320	328	335	343	351	6 4.8
567	358	366	374	381	389	397	404	412	420	427	7 5.6
568	435	442	450	458	465	473	481	488	496	504	8 6.4
569	511	519	526	534	542	549	557	565	572	580	9 7.2
570	587	595	603	610	618	626	633	641	648	656	
571	664	671	679	686	694	702	709	717	724	732	
572	740	747	755	762	770	778	785	793	800	808	
573	815	823	831	838	846	853	861	868	876	884	
574	891	899	906	914	921	929	937	944	952	959	
575	967	974	982	989	997	*005	*012	*020	*027	*035	
576	76 042	050	057	065	072	080	087	095	103	110	
577	118	125	133	140	148	155	163	170	178	185	
578	193	200	208	215	223	230	238	245	253	260	
579	268	275	283	290	298	305	313	320	328	335	
580	343	350	358	365	373	380	388	395	403	410	7
581	418	425	433	440	448	455	462	470	477	485	1 0.7
582	492	500	507	515	522	530	537	545	552	559	2 1.4
583	567	574	582	589	597	604	612	619	626	634	3 2.1
584	641	649	656	664	671	678	686	693	701	708	4 2.8
585	716	723	730	738	745	753	760	768	775	782	5 3.5
586	790	797	805	812	819	827	834	842	849	856	6 4.2
587	864	871	879	886	893	901	908	916	923	930	7 4.9
588	938	945	953	960	967	975	982	989	997	*004	8 5.6
589	77 012	019	026	034	041	048	056	063	070	078	9 6.3
590	085	093	100	107	115	122	129	137	144	151	
591	159	166	173	181	188	195	203	210	217	225	
592	232	240	247	254	262	269	276	283	291	298	
593	305	313	320	327	335	342	349	357	364	371	
594	379	386	393	401	408	415	422	430	437	444	
595	452	459	466	474	481	488	495	503	510	517	
596	525	532	539	546	554	561	568	576	583	590	
597	597	605	612	619	627	634	641	648	656	663	
598	670	677	685	692	699	706	714	721	728	735	
599	743	750	757	764	772	779	786	793	801	808	
600	815	822	830	837	844	851	859	866	873	880	
N.	L. 0	1	2	3	4	5	6	7	8	9	P. P.

		S.′	T.′
6′	6.46	373	373
55		371	376
56		371	376
57		371	377
58		371	377
59		370	377
60		370	377

		S.″	T.″
0° 9′ = 540″	4.68	557	558
0 10 = 600		557	558
1 31 = 5460		552	568
1 32 = 5520		552	568
1 33 = 5580		552	568
1 34 = 5640		552	568
1 35 = 5700		552	569

1° 35′ = 5700″	4.(
1 36 = 5760	
1 37 = 5820	
1 38 = 5880	
1 39 = 5940	
1 40 = 6000	

N.	L. 0	1	2	3	4	5	6	7	8	9
600	77 815	822	830	837	844	851	859	866	873	880
601	887	895̅	902	909	916	924	931	938	945	952
602	960	967	974	981	988	996	*003	*010	*017	*025̅
603	78 032	039	046	053	061	068	075̅	082	089	097
604	104	111	118	125	132	140	147	154	161	168
605	176	183	190	197	204	211	219	226	233	240
606	247	254	262	269	276	283	290	297	305̅	312
607	319	326	333	340	347	355̅	362	369	376	383
608	390	398	405̅	412	419	426	433	440	447	455̅
609	462	469	476	483	490	497	504	512	519	526
610	533	540	547	554	561	569	576	583	590	597
611	604	611	618	625	633	640	647	654	661	668
612	675	682	689	696	704	711	718	725̅	732	739
613	746	753	760	767	774	781	789	796	803	810
614	817	824	831	838	845	852	859	866	873	880
615	888	895̅	902	909	916	923	930	937	944	951
616	958	965	972	979	986	993	*000	*007	*014	*021
617	79 029	036	043	050	057	064	071	078	085̅	092
618	099	106	113	120	127	134	141	148	155	162
619	169	176	183	190	197	204	211	218	225	232
620	239	246	253	260	267	274	281	288	295	302
621	309	316	323	330	337	344	351	358	365	372
622	379	386	393	400	407	414	421	428	435̅	442
623	449	456	463	470	477	484	491	498	505̅	511
624	518	525	532	539	546	553	560	567	574	581
625	588	595̅	602	609	616	623	630	637	644	650
626	657	664	671	678	685	692	699	706	713	720
627	727	734	741	748	754	761	768	775	782	789
628	796	803	810	817	824	831	837	844	851	858
629	865	872	879	886	893	900	906	913	920	927
630	934	941	948	955̅	962	969	975	982	989	996
631	80 003	010	017	024	030	037	044	051	058	065̅
632	072	079	085	092	099	106	113	120	127	134
633	140	147	154	161	168	175̅	182	188	195	202
634	209	216	223	229	236	243	250	257	264	271
635	277	284	291	298	305̅	312	318	325	332	339
636	346	353	359	366	373	380	387	393	400	407
637	414	421	428	434	441	448	455̅	462	468	475
638	482	489	496	502	509	516	523	530	536	543
639	550	557	564	570	577	584	591	598	604	611
640	618	625	632	638	645	652	659	665	672	679
641	686	693	699	706	713	720	726	733	740	747
642	754	760	767	774	781	787	794	801	808	814
643	[illegible]	[illegible]	35̅	841	848	855̅	862	868	875	882
644	889	[illegible]	02	909	916	922	929	936	943	949
645	956	963	69	976	983	990	996	*003	*010	*017
646	81 023	030	37	043	050	057	064	070	077	084
647	090	097	104	111	117	124	131	137	144	151
648	158	164	171	178	184	191	198	204	211	218
649	224	231	238	245̅	251	258	265̅	271	278	285̅
	291	298	305̅	311	318	325̅	331	338	345̅	351
	0	1	2	3	4	5	6	7	8	9

P. P.

8	
1	0.8
2	1.6
3	2.4
4	3.2
5	4.0
6	4.8
7	5.6
8	6.4
9	7.2

7	
1	0.7
2	1.4
3	2.1
4	2.8
5	3.5
6	4.2
7	4.9
8	5.6
9	6.3

6	
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

T.′
73
73
7
7
3
3

		S.″	T.″
0° 10′ = 600″	4.68	557	558
0 11 = 660		557	558
1 40 = 6000		551	570
1 41 = 6060		551	570
1 42 = 6120		551	570
1 43 = 6180		551	570
1 44 = 6240		551	571

		S.″	T.″
1° 44′ = 6240″	4.68	551	571
1 45 = 6300		551	571
1 46 = 6360		551	571
1 47 = 6420		550	572
1 48 = 6480		550	572
1 49 = 6540		550	572

N.	L. 0	1	2	3	4	5	6	7	8	9	P. P.
750	87 506	512	518	523	529	535	541	547	552	558	
751	564	570	576	581	587	593	599	604	610	616	
752	622	628	633	639	645	651	656	662	668	674	
753	679	685	691	697	703	708	714	720	726	731	
754	737	743	749	754	760	766	772	777	783	789	
755	795	800	806	812	818	823	829	835	841	846	
756	852	858	864	869	875	881	887	892	898	904	
757	910	915	921	927	933	938	944	950	955	961	
758	967	973	978	984	990	996	*001	*007	*013	*018	
759	88 024	030	036	041	047	053	058	064	070	076	
760	081	087	093	098	104	110	116	121	127	133	6
761	138	144	150	156	161	167	173	178	184	190	1 0.6
762	195	201	207	213	218	224	230	235	241	247	2 1.2
763	252	258	264	270	275	281	287	292	298	304	3 1.8
764	309	315	321	326	332	338	343	349	355	360	4 2.4
765	366	372	377	383	389	395	400	406	412	417	5 3.0
766	423	429	434	440	446	451	457	463	468	474	6 3.6
767	480	485	491	497	502	508	513	519	525	530	7 4.2
768	536	542	547	553	559	564	570	576	581	587	8 4.8
769	593	598	604	610	615	621	627	632	638	643	9 5.4
770	649	655	660	666	672	677	683	689	694	700	
771	705	711	717	722	728	734	739	745	750	756	
772	762	767	773	779	784	790	795	801	807	812	
773	818	824	829	835	840	846	852	857	863	868	
774	874	880	885	891	897	902	908	913	919	925	
775	930	936	941	947	953	958	964	969	975	981	
776	986	992	997	*003	*009	*014	*020	*025	*031	*037	
777	89 042	048	053	059	064	070	076	081	087	092	
778	098	104	109	115	120	126	131	137	143	148	
779	154	159	165	170	176	182	187	193	198	204	
780	209	215	221	226	232	237	243	248	254	260	5
781	265	271	276	282	287	293	298	304	310	315	1 0.5
782	321	326	332	337	343	348	354	360	365	371	2 1.0
783	376	382	387	393	398	404	409	415	421	426	3 1.5
784	432	437	443	448	454	459	465	470	476	481	4 2.0
785	487	492	498	504	509	515	520	526	531	537	5 2.5
786	542	548	553	559	564	570	575	581	586	592	6 3.0
787	597	603	609	614	620	625	631	636	642	647	7 3.5
788	653	658	664	669	675	680	686	691	697	702	8 4.0
789	708	713	719	724	730	735	741	746	752	757	9 4.5
790	763	768	774	779	785	790	796	801	807	812	
791	818	823	829	834	840	845	851	856	862	867	
792	873	878	883	889	894	900	905	911	916	922	
793	927	933	938	944	949	955	960	966	971	977	
794	982	988	993	998	*004	*009	*015	*020	*026	*031	
795	90 037	042	048	053	059	064	069	075	080	086	
796	091	097	102	108	113	119	124	129	135	140	
797	146	151	157	162	168	173	179	184	189	195	
798	200	206	211	217	222	227	233	238	244	249	
799	255	260	266	271	276	282	287	293	298	304	
800	309	314	320	325	331	336	342	347	352	358	
N.	L. 0	1	2	3	4	5	6	7	8	9	P. P.

		S.′	T.′
7′	6.46	373	373
8		373	373
75		369	380
80		369	380

		S.″	T.″
0° 12′ = 720″	4.68	557	558
0 13 = 780		557	558
0 14 = 840		557	558
2 5 = 7500		548	577
2 6 = 7560		548	577
2 7 = 7620		548	577
2 8 = 7680		547	578

		S.″	T.″
2° 8′ = 7680″	4.68	547	578
2 9 = 7740		547	578
2 10 = 7800		547	578
2 11 = 7860		547	579
2 12 = 7920		547	579
2 13 = 7980		547	579
2 14 = 8040		546	579

N.	L. 0	1	2	3	4	5	6	7	8	9	P. P.
800	90 309	314	320	325	331	336	342	347	352	358	
801	363	369	374	380	385	[illegible]	396	401	407	412	
802	417	423	428	434	439	445	450	455	461	466	
803	472	477	482	488	493	499	504	509	515	520	
804	526	531	536	542	547	553	558	563	569	574	
805	580	585	590	596	601	607	612	617	623	628	
806	634	639	644	650	655	660	666	671	677	682	
807	687	693	698	703	709	714	720	725	730	736	
808	741	747	752	757	763	768	773	779	784	789	
809	795	800	806	811	816	822	827	832	838	843	
810	849	854	859	865	870	875	881	886	891	897	
811	902	907	913	918	924	929	934	940	945	950	6
812	956	961	966	972	977	982	988	993	998	*004	1 0.6
813	91 009	014	020	025	030	036	041	046	052	057	2 1.2
814	062	068	073	078	084	089	094	100	105	110	3 1.8
815	116	121	126	132	137	142	148	153	158	164	4 2.4
816	169	174	180	185	190	196	201	206	212	217	5 3.0
817	222	228	233	238	243	249	254	259	265	270	6 3.6
818	275	281	286	291	297	302	307	312	318	323	7 4.2
819	328	334	339	344	350	355	360	365	371	376	8 4.8
820	381	387	392	397	403	408	413	418	424	429	9 5.4
821	434	440	445	450	455	461	466	471	477	482	
822	487	492	498	503	508	514	519	524	529	535	
823	540	545	551	556	561	566	572	577	582	587	
824	593	598	603	609	614	619	624	630	635	640	
825	645	651	656	661	666	672	677	682	687	693	
826	698	703	709	714	719	724	730	735	740	745	
827	751	756	761	766	772	777	782	787	793	798	
828	803	808	814	819	824	829	834	840	845	850	
829	855	861	866	871	876	882	887	892	897	903	
830	908	913	918	924	929	934	939	944	950	955	5
831	960	965	971	976	981	986	991	997	*002	*007	1 0.5
832	92 012	018	023	028	033	038	044	049	054	059	2 1.0
833	065	070	075	080	085	091	096	101	106	111	3 1.5
834	117	122	127	132	137	143	148	153	158	163	4 2.0
835	169	174	179	184	189	195	200	205	210	215	5 2.5
836	221	226	231	236	241	247	252	257	262	267	6 3.0
837	273	278	283	288	293	298	304	309	314	319	7 3.5
838	324	330	335	340	345	350	355	361	366	371	8 4.0
839	376	381	387	392	397	402	407	412	418	423	9 4.5
840	428	433	438	443	449	454	459	464	469	474	
841	480	485	490	495	500	505	511	516	521	526	
842	531	536	542	547	552	557	562	567	572	578	
843	583	588	593	598	603	609	614	619	624	629	
844	634	639	645	650	655	660	665	670	675	681	
845	686	691	696	701	706	711	716	722	727	732	
846	737	742	747	752	758	763	768	773	778	783	
847	788	793	799	804	809	814	819	824	829	834	
848	840	845	850	855	860	865	870	875	881	886	
849	891	896	901	906	911	916	921	927	932	937	
850	942	947	952	957	962	967	973	978	983	988	
N.	L. 0	1	2	3	4	5	6	7	8	9	P. P.

		S.′	T.′
8′	6.46	373	373
9		373	373
80		369	380
81		369	381
82		368	381
85		368	381

		S.″	T.″
0° 13′ = 780″	4.68	557	558
0 14 = 840		557	558
0 15 = 900		557	558
2 13 = 7980		547	579
2 14 = 8040		546	579
2 15 = 8100		546	580
2 16 = 8160		546	580

		S.″	T.″
2° 16′ = 8160″	4.68	546	580
2 17 = 8220		546	580
2 18 = 8280		546	581
2 19 = 8340		546	581
2 20 = 8400		545	582
2 21 = 8460		545	582
2 22 = 8520		545	582

N.	L. 0	1	2	3	4	5	6	7	8	9
850	92 942	947	952	957	962	967	973	978	983	988
851	993	998	*003	*008	*013	*018	*024	*029	*034	*039
852	93 044	049	054	059	064	069	075	080	085	090
853	095	100	105	110	115	120	125	131	136	141
854	146	151	156	161	166	171	176	181	186	192
855	197	202	207	212	217	222	227	232	237	242
856	247	252	258	263	268	273	278	283	288	293
857	298	303	308	313	318	323	328	334	339	344
858	349	354	359	364	369	374	379	384	389	394
859	399	404	409	414	420	425	430	435	440	445
860	450	455	460	465	470	475	480	485	490	495
861	500	505	510	515	520	526	531	536	541	546
862	551	556	561	566	571	576	581	586	591	596
863	601	606	611	616	621	626	631	636	641	646
864	651	656	661	666	671	676	682	687	692	697
865	702	707	712	717	722	727	732	737	742	747
866	752	757	762	767	772	777	782	787	792	797
867	802	807	812	817	822	827	832	837	842	847
868	852	857	862	867	872	877	882	887	892	897
869	902	907	912	917	922	927	932	937	942	947
870	952	957	962	967	972	977	982	987	992	997
871	94 002	007	012	017	022	027	032	037	042	047
872	052	057	062	067	072	077	082	086	091	096
873	101	106	111	116	121	126	131	136	141	146
874	151	156	161	166	171	176	181	186	191	196
875	201	206	211	216	221	226	231	236	240	245
876	250	255	260	265	270	275	280	285	290	295
877	300	305	310	315	320	325	330	335	340	345
878	349	354	359	364	369	374	379	384	389	394
879	399	404	409	414	419	424	429	433	438	443
880	448	453	458	463	468	473	478	483	488	493
881	498	503	507	512	517	522	527	532	537	542
882	547	552	557	562	567	571	576	581	586	591
883	596	601	606	611	616	621	626	630	635	640
884	645	650	655	660	665	670	675	680	685	689
885	694	699	704	709	714	719	724	729	734	738
886	743	748	753	758	763	768	773	778	783	787
887	792	797	802	807	812	817	822	827	832	836
888	841	846	851	856	861	866	871	876	880	885
889	890	895	900	905	910	915	919	924	929	934
890	939	944	949	954	959	963	968	973	978	983
891	988	993	998	*002	*007	*012	*017	*022	*027	*032
892	95 036	041	046	051	056	061	066	071	075	080
893	085	090	095	100	105	109	114	119	124	129
894	134	139	143	148	153	158	163	168	173	177
895	182	187	192	197	202	207	211	216	221	226
896	231	236	240	245	250	255	260	265	270	274
897	279	284	289	294	299	303	308	313	318	323
898	328	332	337	342	347	352	357	361	366	371
899	376	381	386	390	395	400	405	410	415	419
900	424	429	434	439	444	448	453	458	463	468

| N. | L. 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | P. P. |

P. P.

	6	5	4
1	0.6	0.5	0.4
2	1.2	1.0	0.8
3	1.8	1.5	1.2
4	2.4	2.0	1.6
5	3.0	2.5	2.0
6	3.6	3.0	2.4
7	4.2	3.5	2.8
8	4.8	4.0	3.2
9	5.4	4.5	3.6

		S.′	T.′
8′	6.46	373	373
9		373	373
85		368	381
86		368	382
89		368	382
90		368	383

		S.″	T.″
0° 14′ = 840″	4.68	557	558
0 15 = 900		557	558
2 21 = 8460		545	582
2 22 = 8520		545	582
2 23 = 8580		545	583
2 24 = 8640		545	583
2 25 = 8700		545	583

		S.″	T.″
2° 25′ = 8700″	4.68	545	583
2 26 = 8760		544	584
2 27 = 8820		544	584
2 28 = 8880		544	584
2 29 = 8940		544	585
2 30 = 9000		544	585

N.	L. 0	1	2	3	4	5	6	7	8	9	P. P.
900	95 424	429	434	439	444	448	453	458	463	468	
901	472	477	482	487	492	497	501	506	511	516	
902	521	525	530	535	540	545	550	554	559	564	
903	569	574	578	583	588	593	598	602	607	612	
904	617	622	626	631	636	641	646	650	655	660	
905	665	670	674	679	684	689	694	698	703	708	
906	713	718	722	727	732	737	742	746	751	756	
907	761	766	770	775	780	785	789	794	799	804	
908	809	813	818	823	828	832	837	842	847	852	
909	856	861	866	871	875	880	885	890	895	899	
910	904	909	914	918	923	928	933	938	942	947	5
911	952	957	961	966	971	976	980	985	990	995	1 0.5
912	999	*004	*009	*014	*019	*023	*028	*033	*038	*042	2 1.0
913	96 047	052	057	061	066	071	076	080	085	090	3 1.5
914	095	099	104	109	114	118	123	128	133	137	4 2.0
915	142	147	152	156	161	166	171	175	180	185	5 2.5
916	190	194	199	204	209	213	218	223	227	232	6 3.0
917	237	242	246	251	256	261	265	270	275	280	7 3.5
918	284	289	294	298	303	308	313	317	322	327	8 4.0
919	332	336	341	346	350	355	360	365	369	374	9 4.5
920	379	384	388	393	398	402	407	412	417	421	
921	426	431	435	440	445	450	454	459	464	468	
922	473	478	483	487	492	497	501	506	511	515	
923	520	525	530	534	539	544	548	553	558	562	
924	567	572	577	581	586	591	595	600	605	609	
925	614	619	624	628	633	638	642	647	652	656	
926	661	666	670	675	680	685	689	694	699	703	
927	708	713	717	722	727	731	736	741	745	750	
928	755	759	764	769	774	778	783	788	792	797	
929	802	806	811	816	820	825	830	834	839	844	
930	848	453	858	862	867	872	876	881	886	890	4
931	895	900	904	909	914	918	923	928	932	937	1 0.4
932	942	946	951	956	960	965	970	974	979	984	2 0.8
933	988	993	997	*002	*007	*011	*016	*021	*025	*030	3 1.2
934	97 035	039	044	049	053	058	063	067	072	077	4 1.6
935	081	086	090	095	100	104	109	114	118	123	5 2.0
936	128	132	137	142	146	151	155	160	165	169	6 2.4
937	174	179	183	188	192	197	202	206	211	216	7 2.8
938	220	225	230	234	239	243	248	253	257	262	8 3.2
939	267	271	276	280	285	290	294	299	304	308	9 3.6
940	313	317	322	327	331	336	340	345	350	354	
941	359	364	368	373	377	382	387	391	396	400	
942	405	410	414	419	424	428	433	437	442	447	
943	451	456	460	465	470	474	479	483	488	493	
944	497	502	506	511	516	520	525	529	534	539	
945	543	548	552	557	562	566	571	575	580	585	
946	589	594	598	603	607	612	617	621	626	630	
947	635	640	644	649	653	658	663	667	672	676	
948	681	685	690	695	699	704	708	713	717	722	
949	727	731	736	740	745	749	754	759	763	768	
950	772	777	782	786	791	795	800	804	809	813	
N.	L. 0	1	2	3	4	5	6	7	8	9	P. P.

		S.′	T.′
9′	6.46	373	373
10		373	373
90		368	383
91		368	383
92		367	383
94		367	383
95		367	384

		S.″	T.″
0° 15′ = 900″	4.68	557	558
0 16 = 960		557	558
2 30 = 9000		544	585
2 31 = 9060		544	585
2 32 = 9120		543	586
2 33 = 9180		543	586
2 34 = 9240		543	587

		S.″	T.″
2° 34′ = 9240″	4.68	543	587
2 35 = 9300		543	587
2 36 = 9360		543	587
2 37 = 9420		542	588
2 38 = 9480		542	588
2 39 = 9540		542	588

N.	L. 0	1	2	3	4	5	6	7	8	9
950	97 772	777	782	786	791	795	800	804	809	813
951	818	823	827	832	836	841	845	850	855	859
952	864	868	873	877	882	886	891	896	900	905
953	909	914	918	923	928	932	937	941	946	950
954	955	959	964	968	973	978	982	987	991	996
955	98 000	005	009	014	019	023	028	032	037	041
956	046	050	055	059	064	068	073	078	082	087
957	091	096	100	105	109	114	118	123	127	132
958	137	141	146	150	155	159	164	168	173	177
959	182	186	191	195	200	204	209	214	218	223
960	227	232	236	241	245	250	254	259	263	268
961	272	277	281	286	290	295	299	304	308	313
962	318	322	327	331	336	340	345	349	354	358
963	363	367	372	376	381	385	390	394	399	403
964	408	412	417	421	426	430	435	439	444	448
965	453	457	462	466	471	475	480	484	489	493
966	498	502	507	511	516	520	525	529	534	538
967	543	547	552	556	561	565	570	574	579	583
968	588	592	597	601	605	610	614	619	623	628
969	632	637	641	646	650	655	659	664	668	673
970	677	682	686	691	695	700	704	709	713	717
971	722	726	731	735	740	744	749	753	758	762
972	767	771	776	780	784	789	793	798	802	807
973	811	816	820	825	829	834	838	843	847	851
974	856	860	865	869	874	878	883	887	892	896
975	900	905	909	914	918	923	927	932	936	941
976	945	949	954	958	963	967	972	976	981	985
977	989	994	998	*003	*007	*012	*016	*021	*025	*029
978	99 034	038	043	047	052	056	061	065	069	074
979	078	083	087	092	096	100	105	109	114	118
980	123	127	131	136	140	145	149	154	158	162
981	167	171	176	180	185	189	193	198	202	207
982	211	216	220	224	229	233	238	242	247	251
983	255	260	264	269	273	277	282	286	291	295
984	300	304	308	313	317	322	326	330	335	339
985	344	348	352	357	361	366	370	374	379	383
986	388	392	396	401	405	410	414	419	423	427
987	432	436	441	445	449	454	458	463	467	471
988	476	480	484	489	493	498	502	506	511	515
989	520	524	528	533	537	542	546	550	555	559
990	564	568	572	577	581	585	590	594	599	603
991	607	612	616	621	625	629	634	638	642	647
992	651	656	660	664	669	673	677	682	686	691
993	695	699	704	708	712	717	721	726	730	734
994	739	743	747	752	756	760	765	769	774	778
995	782	787	791	795	800	804	808	813	817	822
996	826	830	835	839	843	848	852	856	861	865
997	870	874	878	883	887	891	896	900	904	909
998	913	917	922	926	930	935	939	944	948	952
999	957	961	965	970	974	978	983	987	991	996
1000	00 000	004	009	013	017	022	026	030	035	039
N.	L. 0	1	2	3	4	5	6	7	8	9

P. P.

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

	4
1	0.4
2	0.8
3	1.2
4	1.6
5	2.0
6	2.4
7	2.8
8	3.2
9	3.6

		S.′	T.′
9′	6.46	373	373
10		373	373
95		367	384
98		367	384
99		367	385
100		366	385

		S.″	T.″
0° 15′ = 900″	4.68	557	558
0 16 = 960		557	558
0 17 = 1020		557	558
2 38 = 9480		542	588
2 39 = 9540		542	588
2 40 = 9600		542	589
2 41 = 9660		542	589

		S.″	T.″
2° 41′ = 9660″	4.68	542	589
2 42 = 9720		541	590
2 43 = 9780		541	590
2 44 = 9840		541	590
2 45 = 9900		541	591
2 46 = 9960		541	591
2 47 = 10020		540	592

N.	L. 0	1	2	3	4	5	6	7	8	9
1000	000 0000	0434	0869	1303	1737	2171	2605	3039	3473	3907
1001	4341	4775	5208	5642	6076	6510	6943	7377	7810	8244
1002	8677	9111	9544	9977	*0411	*0844	*1277	*1710	*2143	*2576
1003	001 3009	3442	3875	4308	4741	5174	5607	6039	6472	6905
1004	7337	7770	8202	8635	9067	9499	9932	*0364	*0796	*1228
1005	002 1661	2093	2525	2957	3389	3821	4253	4685	5116	5548
1006	5980	6411	6843	7275	7706	8138	8569	9001	9432	9863
1007	003 0295	0726	1157	1588	2019	2451	2882	3313	3744	4174
1008	4605	5036	5467	5898	6328	6759	7190	7620	8051	8481
1009	8912	9342	9772	*0203	*0633	*1063	*1493	*1924	*2354	*2784
1010	004 3214	3644	4074	4504	4933	5363	5793	6223	6652	7082
1011	7512	7941	8371	8800	9229	9659	*0088	*0517	*0947	*1376
1012	005 1805	2234	2663	3092	3521	3950	4379	4808	5237	5666
1013	6094	6523	6952	7380	7809	8238	8666	9094	9523	9951
1014	006 0380	0808	1236	1664	2092	2521	2949	3377	3805	4233
1015	4660	5088	5516	5944	6372	6799	7227	7655	8082	8510
1016	8937	9365	9792	*0219	*0647	*1074	*1501	*1928	*2355	*2782
1017	007 3210	3637	4064	4490	4917	5344	5771	6198	6624	7051
1018	7478	7904	8331	8757	9184	9610	*0037	*0463	*0889	*1316
1019	008 1742	2168	2594	3020	3446	3872	4298	4724	5150	5576
1020	6002	6427	6853	7279	7704	8130	8556	8981	9407	9832
1021	009 0257	0683	1108	1533	1959	2384	2809	3234	3659	4084
1022	4509	4934	5359	5784	6208	6633	7058	7483	7907	8332
1023	8756	9181	9605	*0030	*0454	*0878	*1303	*1727	*2151	*2575
1024	010 3000	3424	3848	4272	4696	5120	5544	5967	6391	6815
1025	7239	7662	8086	8510	8933	9357	9780	*0204	*0627	*1050
1026	011 1474	1897	2320	2743	3166	3590	4013	4436	4859	5282
1027	5704	6127	6550	6973	7396	7818	8241	8664	9086	9509
1028	9931	*0354	*0776	*1198	*1621	*2043	*2465	*2887	*3310	*3732
1029	012 4154	4576	4998	5420	5842	6264	6685	7107	7529	7951
1030	8372	8794	9215	9637	*0059	*0480	*0901	*1323	*1744	*2165
1031	013 2587	3008	3429	3850	4271	4692	5113	5534	5955	6376
1032	6797	7218	7639	8059	8480	8901	9321	9742	*0162	*0583
1033	014 1003	1424	1844	2264	2685	3105	3525	3945	4365	4785
1034	5205	5625	6045	6465	6885	7305	7725	8144	8564	8984
1035	9403	9823	*0243	*0662	*1082	*1501	*1920	*2340	*2759	*3178
1036	015 3598	4017	4436	4855	5274	5693	6112	6531	6950	7369
1037	7788	8206	8625	9044	9462	9881	*0300	*0718	*1137	*1555
1038	016 1974	2392	2810	3229	3647	4065	4483	4901	5319	5737
1039	6155	6573	6991	7409	7827	8245	8663	9080	9498	9916
1040	017 0333	0751	1168	1586	2003	2421	2838	3256	3673	4090
1041	4507	4924	5342	5759	6176	6593	7010	7427	7844	8260
1042	8677	9094	9511	9927	*0344	*0761	*1177	*1594	*2010	*2427
1043	018 2843	3259	3676	4092	4508	4925	5341	5757	6173	6589
1044	7005	7421	7837	8253	8669	9084	9500	9916	*0332	*0747
1045	019 1163	1578	1994	2410	2825	3240	3656	4071	4486	4902
1046	5317	5732	6147	6562	6977	7392	7807	8222	8637	9052
1047	9467	9882	*0296	*0711	*1126	*1540	*1955	*2369	*2784	*3198
1048	020 3613	4027	4442	4856	5270	5684	6099	6513	6927	7341
1049	7755	8169	8583	8997	9411	9824	*0238	*0652	*1066	*1479
1050	021 1893	2307	2720	3134	3547	3961	4374	4787	5201	5614
N.	L. 0	1	2	3	4	5	6	7	8	9

		S.″	T.″			S.″	T.″
2° 46′ = 9960′	4.68	541	591	2° 51′ = 10260″	4.68	540	593
2 47 = 10020		540	592	2 52 = 10320		539	594
2 48 = 10080		540	592	2 53 = 10380		539	594
2 49 = 10140		540	592	2 54 = 10440		539	595
2 50 = 10200		540	593	2 55 = 10500		539	595

N.	L. 0	1	2	3	4	5	6	7	8	9
1050	021 1893	2307	2720	3134	3547	3961	4374	4787	5201	5614
1051	6027	6440	6854	7267	7680	8093	8506	8919	9332	9745
1052	022 0157	0570	0983	1396	1808	2221	2634	3046	3459	3871
1053	4284	4696	5109	5521	5933	6345	6758	7170	7582	7994
1054	8406	8818	9230	9642	*0054	*0466	*0878	*1289	*1701	*2113
1055	023 2525	2936	3348	3759	4171	4582	4994	5405	5817	6228
1056	6639	7050	7462	7873	8284	8695	9106	9517	9928	*0339
1057	024 0750	1161	1572	1982	2393	2804	3214	3625	4036	4446
1058	4857	5267	5678	6088	6498	6909	7319	7729	8139	8549
1059	8960	9370	9780	*0190	*0600	*1010	*1419	*1829	*2239	*2649
1060	025 3059	3468	3878	4288	4697	5107	5516	5926	6335	6744
1061	7154	7563	7972	8382	8791	9200	9609	*0018	*0427	*0836
1062	026 1245	1654	2063	2472	2881	3289	3698	4107	4515	4924
1063	5333	5741	6150	6558	6967	7375	7783	8192	8600	9008
1064	9416	9824	*0233	*0641	*1049	*1457	*1865	*2273	*2680	*3088
1065	027 3496	3904	4312	4719	5127	5535	5942	6350	6757	7165
1066	7572	7979	8387	8794	9201	9609	*0016	*0423	*0830	*1237
1067	028 1644	2051	2458	2865	3272	3679	4086	4492	4899	5306
1068	5713	6119	6526	6932	7339	7745	8152	8558	8964	9371
1069	9777	*0183	*0590	*0996	*1402	*1808	*2214	*2620	*3026	*3432
1070	029 3838	4244	4649	5055	5461	5867	6272	6678	7084	7489
1071	7895	8300	8706	9111	9516	9922	*0327	*0732	*1138	*1543
1072	030 1948	2353	2758	3163	3568	3973	4378	4783	5188	5592
1073	5997	6402	6807	7211	7616	8020	8425	8830	9234	9638
1074	031 0043	0447	0851	1256	1660	2064	2468	2872	3277	3681
1075	4085	4489	4893	5296	5700	6104	6508	6912	7315	7719
1076	8123	8526	8930	9333	9737	*0140	*0544	*0947	*1350	*1754
1077	032 2157	2560	2963	3367	3770	4173	4576	4979	5382	5785
1078	6188	6590	6993	7396	7799	8201	8604	9007	9409	9812
1079	033 0214	0617	1019	1422	1824	2226	2629	3031	3433	3835
1080	4238	4640	5042	5444	5846	6248	6650	7052	7453	7855
1081	8257	8659	9060	9462	9864	*0265	*0667	*1068	*1470	*1871
1082	034 2273	2674	3075	3477	3878	4279	4680	5081	5482	5884
1083	6285	6686	7087	7487	7888	8289	8690	9091	9491	9892
1084	035 0293	0693	1094	1495	1895	2296	2696	3096	3497	3897
1085	4297	4698	5098	5498	5898	6298	6698	7098	7498	7898
1086	8298	8698	9098	9498	9898	*0297	*0697	*1097	*1496	*1896
1087	036 2295	2695	3094	3494	3893	4293	4692	5091	5491	5890
1088	6289	6688	7087	7486	7885	8284	8683	9082	9481	9880
1089	037 0279	0678	1076	1475	1874	2272	2671	3070	3468	3867
1090	4265	4663	5062	5460	5858	6257	6655	7053	7451	7849
1091	8248	8646	9044	9442	9839	*0237	*0635	*1033	*1431	*1829
1092	038 2226	2624	3022	3419	3817	4214	4612	5009	5407	5804
1093	6202	6599	6996	7393	7791	8188	8585	8982	9379	9776
1094	039 0173	0570	0967	1364	1761	2158	2554	2951	3348	3745
1095	4141	4538	4934	5331	5727	6124	6520	6917	7313	7709
1096	8106	8502	8898	9294	9690	*0086	*0482	*0878	*1274	*1670
1097	040 2066	2462	2858	3254	3650	4045	4441	4837	5232	5628
1098	6023	6419	6814	7210	7605	8001	8396	8791	9187	9582
1099	9977	*0372	*0767	*1162	*1557	*1952	*2347	*2742	*3137	*3532
1100	041 3927	4322	4716	5111	5506	5900	6295	6690	7084	7479

| N. | L. 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

		S.″	T.″			S.″	T.″
2° 55′ = 10500″	4.68	539	595	3° 0′ = 10800″	4.68	538	597
2 56 = 10560		539	595	3 1 = 10860		537	598
2 57 = 10620		538	596	3 2 = 10920		537	598
2 58 = 10680		538	596	3 3 = 10980		537	599
2 59 = 10740		538	597	3 4 = 11040		537	599

3°

′	M.	S′.	T′.	Sec.	S″.	T″.
		6.46			4.68	
0	180	353	412	10800	538	597
1	181	353	413	10860	537	598
2	182	352	413	10920	537	598
3	183	352	414	10980	537	599
4	184	352	414	11040	537	599
5	185	352	415	11100	537	599
6	186	351	415	11160	536	600
7	187	351	415	11220	536	600
8	188	351	416	11280	536	601
9	189	351	416	11340	536	601
10	190	350	417	11400	535	602
11	191	350	417	11460	535	602
12	192	350	418	11520	535	603
13	193	350	418	11580	535	603
14	194	350	419	11640	534	604
15	195	349	419	11700	534	604
16	196	349	420	11760	534	605
17	197	349	420	11820	534	605
18	198	349	421	11880	533	606
19	199	348	421	11940	533	606
20	200	348	422	12000	533	607
21	201	348	422	12060	533	607
22	202	348	423	12120	532	608
23	203	347	423	12180	532	608
24	204	347	424	12240	532	609
25	205	347	424	12300	532	609
26	206	347	425	12360	531	610
27	207	346	425	12420	531	610
28	208	346	426	12480	531	611
29	209	346	426	12540	531	611
30	210	346	427	12600	530	612
31	211	345	427	12660	530	612
32	212	345	428	12720	530	613
33	213	345	428	12780	530	613
34	214	345	429	12840	529	614
35	215	344	429	12900	529	614
36	216	344	430	12960	529	615
37	217	344	430	13020	529	615
38	218	344	431	13080	528	616
39	219	343	431	13140	528	616
40	220	343	432	13200	528	617
41	221	343	432	13260	528	617
42	222	342	433	13320	527	618
43	223	342	434	13380	527	618
44	224	342	434	13440	527	619
45	225	342	435	13500	526	620
46	226	341	435	13560	526	620
47	227	341	436	13620	526	621
48	228	341	436	13680	526	621
49	229	340	437	13740	525	622
50	230	340	437	13800	525	622
51	231	340	438	13860	525	623
52	232	340	439	13920	525	623
53	233	339	439	13980	524	624
54	234	339	440	14040	524	625
55	235	339	440	14100	524	625
56	236	338	441	14160	523	626
57	237	338	441	14220	523	626
58	238	338	442	14280	523	627
59	239	338	443	14340	522	628
60	240	337	443	14400	522	628

4°

′	M.	S′.	T′.	Sec.	S″.	T″.
		6.46			4.68	
0	240	337	443	14400	522	628
1	241	337	444	14460	522	629
2	242	337	444	14520	522	629
3	243	336	445	14580	521	630
4	244	336	446	14640	521	631
5	245	336	446	14700	521	631
6	246	336	447	14760	520	632
7	247	335	447	14820	520	632
8	248	335	448	14880	520	633
9	249	335	449	14940	520	634
10	250	334	449	15000	519	634
11	251	334	450	15060	519	635
12	252	334	450	15120	519	635
13	253	333	451	15180	518	636
14	254	333	452	15240	518	637
15	255	333	452	15300	518	637
16	256	332	453	15360	517	638
17	257	332	454	15420	517	638
18	258	332	454	15480	517	639
19	259	332	455	15540	516	640
20	260	331	456	15600	516	640
21	261	331	456	15660	516	641
22	262	331	457	15720	515	642
23	263	330	457	15780	515	642
24	264	330	458	15840	515	643
25	265	330	459	15900	514	644
26	266	329	459	15960	514	644
27	267	329	460	16020	514	645
28	268	329	461	16080	513	646
29	269	328	461	16140	513	646
30	270	328	462	16200	513	647
31	271	328	463	16260	512	648
32	272	327	463	16320	512	648
33	273	327	464	16380	512	649
34	274	327	465	16440	511	650
35	275	326	465	16500	511	650
36	276	326	466	16560	511	651
37	277	326	467	16620	510	652
38	278	325	467	16680	510	652
39	279	325	468	16740	510	653
40	280	325	469	16800	509	654
41	281	324	469	16860	509	654
42	282	324	470	16920	509	655
43	283	324	471	16980	508	656
44	284	323	472	17040	508	656
45	285	323	472	17100	508	657
46	286	323	473	17160	507	658
47	287	322	474	17220	507	659
48	288	322	474	17280	507	659
49	289	321	475	17340	506	660
50	290	321	476	17400	506	661
51	291	321	477	17460	506	661
52	292	320	477	17520	505	662
53	293	320	478	17580	505	663
54	294	320	479	17640	505	664
55	295	319	479	17700	504	664
56	296	319	480	17760	504	665
57	297	319	481	17820	503	666
58	298	318	482	17880	503	666
59	299	318	482	17940	503	667
60	300	317	483	18000	502	668

II.

THE LOGARITHMS

OF THE

TRIGONOMETRIC FUNCTIONS

FOR EACH MINUTE.

Formulas for the Use of the Auxiliaries *S* and *T*.

1. When α is in the first five degrees of the quadrant:

$$\log\sin\alpha = \log\alpha' + S.' \qquad \log\alpha' = \log\sin\alpha + \text{cpl } S.'$$
$$\log\tan\alpha = \log\alpha' + T.' \qquad = \log\tan\alpha + \text{cpl } T.'$$
$$\log\cot\alpha = \text{cpl}\log\tan\alpha. \qquad = \text{cpl}\log\cot\alpha + \text{cpl } T.'$$

$$\log\sin\alpha = \log\alpha'' + S.'' \qquad \log\alpha'' = \log\sin\alpha + \text{cpl } S.''$$
$$\log\tan\alpha = \log\alpha'' + T.'' \qquad = \log\tan\alpha + \text{cpl } T.''$$
$$\log\cot\alpha = \text{cpl}\log\tan\alpha. \qquad = \text{cpl}\log\cot\alpha + \text{cpl } T.''$$

2. When α is in the last five degrees of the quadrant:

$$\log\cos\alpha = \log(90^\circ - \alpha)' + S.' \qquad \log(90^\circ - \alpha)' = \log\cos\alpha + \text{cpl } S.'$$
$$\log\cot\alpha = \log(90^\circ - \alpha)' + T.' \qquad = \log\cot\alpha + \text{cpl } T.'$$
$$\log\tan\alpha = \text{cpl}\log\cot\alpha. \qquad = \text{cpl}\log\tan\alpha + \text{cpl } T.'$$

$$\log\cos\alpha = \log(90^\circ - \alpha)'' + S.'' \qquad \log(90^\circ - \alpha)'' = \log\cos\alpha + \text{cpl } S.''$$
$$\log\cot\alpha = \log(90^\circ - \alpha)'' + T.'' \qquad = \log\cot\alpha + \text{cpl } T.''$$
$$\log\tan\alpha = \text{cpl}\log\cot\alpha. \qquad = \text{cpl}\log\tan\alpha + \text{cpl } T.''$$

$$\alpha = 90^\circ - (90^\circ - \alpha).$$

0°

″	′	L. Sin.	d.	Cpl. S′.	Cpl. T′.	L. Tan.	c. d.	L. Cot.	L. Cos.	
0	**0**	—		—	—	—		—	0.00 000	**60**
60	1	6.46 373	30103	3.53 627	3.53 627	6.46 373	30103	3.53 627	0.00 000	59
120	2	6.76 476	17609	3.53 627	3.53 627	6.76 476	17609	3.23 524	0.00 000	58
180	3	6.94 085̄	12494	3.53 627	3.53 627	6.94 085̄	12494	3.05 915	0.00 000	57
240	4	7.06 579	9691	3.53 627	3.53 627	7.06 579	9691	2.93 421	0.00 000	56
300	5	7.16 270	7918	3.53 627	3.53 627	7.16 270	7918	2.83 730	0.00 000	55
360	6	7.24 188	6694	3.53 627	3.53 627	7.24 188	6694	2.75 812	0.00 000	54
420	7	7.30 882	5800	3.53 627	3.53 627	7.30 882	5800	2.69 118	0.00 000	53
480	8	7.36 682	5115	3.53 627	3.53 627	7.36 682	5115	2.63 318	0.00 000	52
540	9	7.41 797	4576	3.53 627	3.53 627	7.41 797	4576	2.58 203	0.00 000	51
600	**10**	7.46 373	4139	3.53 627	3.53 627	7.46 373	4139	2.53 627	0.00 000	**50**
660	11	7.50 512	3779	3.53 627	3.53 627	7.50 512	3779	2.49 488	0.00 000	49
720	12	7.54 291	3476	3.53 627	3.53 627	7.54 291	3476	2.45 709	0.00 000	48
780	13	7.57 767	3218	3.53 627	3.53 627	7.57 767	3219	2.42 233	0.00 000	47
840	14	7.60 985	2997	3.53 628	3.53 627	7.60 986	2996	2.39 014	0.00 000	46
900	15	7.63 982	2802	3.53 628	3.53 627	7.63 982	2803	2.36 018	0.00 000	45
960	16	7.66 784	2633	3.53 628	3.53 627	7.66 785̄	2633	2.33 215	0.00 000	44
1020	17	7.69 417	2483	3.53 628	3.53 627	7.69 418	2482	2.30 582	9.99 999	43
1080	18	7.71 900	2348	3.53 628	3.53 627	7.71 900	2348	2.28 100	9.99 999	42
1140	19	7.74 248	2227	3.53 628	3.53 627	7.74 248	2228	2.25 752	9.99 999	41
1200	**20**	7.76 475	2119	3.53 628	3.53 627	7.76 476	2119	2.23 524	9.99 999	**40**
1260	21	7.78 594	2021	3.53 628	3.53 627	7.78 595	2020	2.21 405̄	9.99 999	39
1320	22	7.80 615̄	1930	3.53 628	3.53 627	7.80 615	1931	2.19 385̄	9.99 999	38
1380	23	7.82 545	1848	3.53 628	3.53 627	7.82 546	1848	2.17 454	9.99 999	37
1440	24	7.84 393	1773	3.53 628	3.53 627	7.84 394	1773	2.15 606	9.99 999	36
1500	25	7.86 166	1704	3.53 628	3.53 627	7.86 167	1704	2.13 833	9.99 999	35
1560	26	7.87 870	1639	3.53 628	3.53 627	7.87 871	1639	2.12 129	9.99 999	34
1620	27	7.89 509	1579	3.53 628	3.53 626	7.89 510	1579	2.10 490	9.99 999	33
1680	28	7.91 088	1524	3.53 628	3.53 626	7.91 089	1524	2.08 911	9.99 999	32
1740	29	7.92 612	1472	3.53 628	3.53 626	7.92 613	1473	2 07 387	9.99 998	31
1800	**30**	7.94 084	1424	3.53 628	3.53 626	7.94 086	1424	2.05 914	9.99 998	**30**
1860	31	7.95 508	1379	3.53 628	3.53 626	7.95 510	1379	2.04 490	9.99 998	29
1920	32	7.96 887	1336	3.53 628	3.53 626	7.96 889	1336	2.03 111	9.99 998	28
1980	33	7.98 223	1297	3.53 628	3.53 626	7.98 225	1297	2.01 775̄	9.99 998	27
2040	34	7.99 520	1259	3.53 628	3.53 626	7.99 522	1259	2.00 478	9.99 998	26
2100	35	8.00 779	1223	3.53 628	3.53 626	8.00 781	1223	1.99 219	9.99 998	25
2160	36	8.02 002	1190	3.53 628	3.53 626	8.02 004	1190	1.97 996	9.99 998	24
2220	37	8.03 192	1158	3.53 628	3.53 626	8.03 194	1159	1.96 806	9.99 997	23
2280	38	8.04 350	1128	3.53 628	3.53 626	8.04 353	1128	1.95 647	9.99 997	22
2340	39	8.05 478	1100	3.53 628	3.53 626	8.05 481	1100	1.94 519	9.99 997	21
2400	**40**	8.06 578	1072	3.53 628	3.53 625	8.06 581	1072	1.93 419	9.99 997	**20**
2460	41	8.07 650	1046	3.53 628	3.53 625	8.07 653	1047	1.92 347	9.99 997	19
2520	42	8.08 696	1022	3.53 628	3.53 625	8.08 700	1022	1.91 300	9.99 997	18
2580	43	8.09 718	999	3.53 629	3.53 625	8.09 722	998	1.90 278	9.99 997	17
2640	44	8.10 717	976	3.53 629	3.53 625	8.10 720	976	1.89 280	9.99 996	16
2700	45	8.11 693	954	3.53 629	3.53 625̄	8.11 696	955	1.88 304	9.99 996	15
2760	46	8.12 647	934	3.53 629	3.53 625̄	8.12 651	934	1.87 349	9.99 996	14
2820	47	8.13 581	914	3.53 629	3.53 625̄	8.13 585	915	1.86 415̄	9.99 996	13
2880	48	8.14 495	896	3.53 629	3.53 625̄	8.14 5̄00	895	1.85 500	9.99 996	12
2940	49	8.15 391	877	3.53 629	3.53 624	8.15 395	878	1.84 605̄	9.99 996	11
3000	**50**	8.16 268	860	3.53 629	3.53 624	8.16 273	860	1.83 727	9.99 995	**10**
3060	51	8.17 128	843	3.53 629	3.53 624	8.17 133	843	1.82 867	9.99 995	9
3120	52	8.17 971	827	3.53 629	3.53 624	8.17 976	828	1.82 024	9.99 995	8
3180	53	8.18 798	812	3.53 629	3.53 624	8.18 804	812	1.81 196	9.99 995̄	7
3240	54	8.19 610	797	3.53 629	3.53 624	8.19 616	797	1.80 384	9.99 995̄	6
3300	55	8.20 407	782	3.53 629	3.53 624	8.20 413	782	1.79 587	9.99 994	5
3360	56	8.21 189	769	3.53 629	3.53 624	8.21 195	769	1.78 805̄	9.99 994	4
3420	57	8.21 958	755	3.53 629	3.53 623	8.21 964	756	1.78 036	9.99 994	3
3480	58	8.22 713	743	3.53 629	3.53 623	8.22 720	742	1.77 280	9.99 994	2
3540	59	8.23 456	730	3.53 630	3.53 623	8.23 462	730	1.76 538	9.99 994	1
3600	**60**	8.24 186		3.53 630	3.53 623	8.24 192		1.75 808	9.99 993	**0**
		L. Cos.	d.			L. Cot.	c. d.	L. Tan.	L. Sin.	′

89°

″	′	L. Sin.	d.	Cpl. S′.	Cpl. T′.	L. Tan.	c. d.	L. Cot.	L. Cos.	
3600	0	8.24 186	717	3.53 630	3.53 623	8.24 192	718	1.75 808	9.99 993	60
3660	1	8.24 903	706	3.53 630	3.53 623	8.24 910	706	1.75 090	9.99 993	59
3720	2	8.25 609	695	3.53 630	3.53 623	8.25 616	696	1.74 384	9.99 993	58
3780	3	8.26 304	684	3.53 630	3.53 623	8.26 312	684	1.73 688	9.99 993	57
3840	4	8.26 988	673	3.53 630	3.53 622	8.26 996	673	1.73 004	9.99 992	56
3900	5	8.27 661	663	3.53 630	3.53 622	8.27 669	663	1.72 331	9.99 992	55
3960	6	8.28 324	653	3.53 630	3.53 622	8.28 332	654	1.71 668	9.99 992	54
4020	7	8.28 977	644	3.53 630	3.53 622	8.28 986	643	1.71 014	9.99 992	53
4080	8	8.29 621	634	3.53 630	3.53 622	8.29 629	634	1.70 371	9.99 992	52
4140	9	8.30 255	624	3.53 630	3.53 622	8.30 263	625	1.69 737	9.99 991	51
4200	10	8.30 879	616	3.53 630	3.53 621	8.30 888	617	1.69 112	9.99 991	50
4260	11	8.31 495	608	3.53 630	3.53 621	8.31 505	607	1.68 495	9.99 991	49
4320	12	8.32 103	599	3.53 631	3.53 621	8.32 112	599	1.67 888	9.99 990	48
4380	13	8.32 702	590	3.53 631	3.53 621	8.32 711	591	1.67 289	9.99 990	47
4440	14	8.33 292	583	3.53 631	3.53 621	8.33 302	584	1.66 698	9.99 990	46
4500	15	8.33 875	575	3.53 631	3.53 620	8.33 886	575	1.66 114	9.99 990	45
4560	16	8.34 450	568	3.53 631	3.53 620	8.34 461	568	1.65 539	9.99 989	44
4620	17	8.35 018	560	3.53 631	3.53 620	8.35 029	561	1.64 971	9.99 989	43
4680	18	8.35 578	553	3.53 631	3.53 620	8.35 590	553	1.64 410	9.99 989	42
4740	19	8.36 131	547	3.53 631	3.53 620	8.36 143	546	1.63 857	9.99 989	41
4800	20	8.36 678	539	3.53 631	3.53 620	8.36 689	540	1.63 311	9.99 988	40
4860	21	8.37 217	533	3.53 631	3.53 619	8.37 229	533	1.62 771	9.99 988	39
4920	22	8.37 750	526	3.53 632	3.53 619	8.37 762	527	1.62 238	9.99 988	38
4980	23	8.38 276	520	3.53 632	3.53 619	8.38 289	520	1.61 711	9.99 987	37
5040	24	8.38 796	514	3.53 632	3.53 619	8.38 809	514	1.61 191	9.99 987	36
5100	25	8.39 310	508	3.53 632	3.53 619	8.39 323	509	1.60 677	9.99 987	35
5160	26	8.39 818	502	3.53 632	3.53 618	8.39 832	502	1.60 168	9.99 986	34
5220	27	8.40 320	496	3.53 632	3.53 618	8.40 334	496	1.59 666	9.99 986	33
5280	28	8.40 816	491	3.53 632	3.53 618	8.40 830	491	1.59 170	9.99 986	32
5340	29	8.41 307	485	3.53 632	3.53 618	8.41 321	486	1.58 679	9.99 985	31
5400	30	8.41 792	480	3.53 632	3.53 617	8.41 807	480	1.58 193	9.99 985	30
5460	31	8.42 272	474	3.53 632	3.53 617	8.42 287	475	1.57 713	9.99 985	29
5520	32	8.42 746	470	3.53 633	3.53 617	8.42 762	470	1.57 238	9.99 984	28
5580	33	8.43 216	464	3.53 633	3.53 617	8.43 232	464	1.56 768	9.99 984	27
5640	34	8.43 680	459	3.53 633	3.53 617	8.43 696	460	1.56 304	9.99 984	26
5700	35	8.44 139	455	3.53 633	3.53 616	8.44 156	455	1.55 844	9.99 983	25
5760	36	8.44 594	450	3.53 633	3.53 616	8.44 611	450	1.55 389	9.99 983	24
5820	37	8.45 044	445	3.53 633	3.53 616	8.45 061	446	1.54 939	9.99 983	23
5880	38	8.45 489	441	3.53 633	3.53 616	8.45 507	441	1.54 493	9.99 982	22
5940	39	8.45 930	436	3.53 633	3.53 615	8.45 948	437	1.54 052	9.99 982	21
6000	40	8.46 366	433	3.53 634	3.53 615	8.46 385	432	1.53 615	9.99 982	20
6060	41	8.46 799	427	3.53 634	3.53 615	8.46 817	428	1.53 183	9.99 981	19
6120	42	8.47 226	424	3.53 634	3.53 615	8.47 245	424	1.52 755	9.99 981	18
6180	43	8.47 650	419	3.53 634	3.53 614	8.47 669	420	1.52 331	9.99 981	17
6240	44	8.48 069	416	3.53 634	3.53 614	8.48 089	416	1.51 911	9.99 980	16
6300	45	8.48 485	411	3.53 634	3.53 614	8.48 505	412	1.51 495	9.99 980	15
6360	46	8.48 896	408	3.53 634	3.53 614	8.48 917	408	1.51 083	9.99 979	14
6420	47	8.49 304	404	3.53 634	3.53 613	8.49 325	404	1.50 675	9.99 979	13
6480	48	8.49 708	400	3.53 635	3.53 613	8.49 729	401	1.50 271	9.99 979	12
6540	49	8.50 108	396	3.53 635	3.53 613	8.50 130	397	1.49 870	9.99 978	11
6600	50	8.50 504	393	3.53 635	3.53 613	8.50 527	393	1.49 473	9.99 978	10
6660	51	8.50 897	390	3.53 635	3.53 612	8.50 920	390	1.49 080	9.99 977	9
6720	52	8.51 287	386	3.53 635	3.53 612	8.51 310	386	1.48 690	9.99 977	8
6780	53	8.51 673	382	3.53 635	3.53 612	8.51 696	383	1.48 304	9.99 977	7
6840	54	8.52 055	379	3.53 635	3.53 611	8.52 079	380	1.47 921	9.99 976	6
6900	55	8.52 434	376	3.53 635	3.53 611	8.52 459	376	1.47 541	9.99 976	5
6960	56	8.52 810	373	3.53 636	3.53 611	8.52 835	373	1.47 165	9.99 975	4
7020	57	8.53 183	369	3.53 636	3.53 611	8.53 208	370	1.46 792	9.99 975	3
7080	58	8.53 552	367	3.53 636	3.53 610	8.53 578	367	1.46 422	9.99 974	2
7140	59	8.53 919	363	3.53 636	3.53 610	8.53 945	363	1.46 055	9.99 974	1
7200	60	8.54 282		3.53 636	3.53 610	8.54 308		1.45 692	9.99 974	0
		L. Cos.	d.			L. Cot.	c. d.	L. Tan.	L. Sin.	′

2°

″	′	L. Sin.	d.	Cpl. S′.	Cpl. T′.	L. Tan.	c. d.	L. Cot.	L. Cos.	
7200	**0**	8.54 282	360	3.53 636	3.53 610	8.54 308	361	1.45 692	9.99 974	**60**
7260	1	8.54 642	357	3.53 636	3.53 609	8.54 669	358	1.45 331	9.99 973	59
7320	2	8.54 999	355	3.53 637	3.53 609	8.55 027	355	1.44 973	9.99 973	58
7380	3	8.55 354	351	3.53 637	3.53 609	8.55 382	352	1.44 618	9.99 972	57
7440	4	8.55 705	349	3.53 637	3.53 609	8.55 734	349	1.44 266	9.99 972	56
7500	5	8.56 054	346	3.53 637	3.53 608	8.56 083	346	1.43 917	9.99 971	55
7560	6	8.56 400	343	3.53 637	3.53 608	8.56 429	344	1.43 571	9.99 971	54
7620	7	8.56 743	341	3.53 637	3.53 608	8.56 773	341	1.43 227	9.99 970	53
7680	8	8.57 084	337	3.53 637	3.53 607	8.57 114	338	1.42 886	9.99 970	52
7740	9	8.57 421	336	3.53 638	3.53 607	8.57 452	336	1.42 548	9.99 969	51
7800	**10**	8.57 757	332	3.53 638	3.53 607	8.57 788	333	1.42 212	9.99 969	**50**
7860	11	8.58 089	330	3.53 638	3.53 606	8.58 121	330	1.41 879	9.99 968	49
7920	12	8.58 419	328	3.53 638	3.53 606	8.58 451	328	1.41 549	9.99 968	48
7980	13	8.58 747	325	3.53 638	3.53 606	8.58 779	326	1.41 221	9.99 967	47
8040	14	8.59 072	323	3.53 638	3.53 605	8.59 105	323	1.40 895̄	9.99 967	46
8100	15	8.59 395̄	320	3.53 639	3.53 605	8.59 428	321	1.40 572	9.99 967	45
8160	16	8.59 715	318	3.53 639	3.53 605̄	8.59 749	319	1.40 251	9.99 966	44
8220	17	8.60 033	316	3.53 639	3.53 604	8.60 068	316	1.39 932	9.99 966	43
8280	18	8.60 349	313	3.53 639	3.53 604	8.60 384	314	1.39 616	9.99 965̄	42
8340	19	8.60 662	311	3.53 639	3.53 604	8.60 698	311	1.39 302	9.99 964	41
8400	**20**	8.60 973	309	3.53 639	3.53 603	8.61 009	310	1.38 991	9.99 964	**40**
8460	21	8.61 282	307	3.53 640	3.53 603	8.61 319	307	1.38 681	9.99 963	39
8520	22	8.61 589	305	3.53 640	3.53 603	8.61 626	305	1.38 374	9.99 963	38
8580	23	8.61 894	302	3.53 640	3.53 602	8.61 931	303	1.38 069	9.99 962	37
8640	24	8.62 196	301	3.53 640	3.53 602	8.62 234	301	1.37 766	9.99 962	36
8700	25	8.62 497	298	3.53 640	3.53 602	8.62 535	299	1.37 465̄	9.99 961	35
8760	26	8.62 795̄	296	3.53 640	3.53 601	8.62 834	297	1.37 166	9.99 961	34
8820	27	8.63 091	294	3.53 641	3.53 601	8.63 131	295	1.36 869	9.99 960	33
8880	28	8.63 385	293	3.53 641	3.53 601	8.63 426	292	1.36 574	9.99 960	32
8940	29	8.63 678	290	3.53 641	3.53 600	8.63 718	291	1.36 282	9.99 959	31
9000	**30**	8.63 968	288	3.53 641	3.53 600	8.64 009	289	1.35 991	9.99 959	**30**
9060	31	8.64 256	287	3.53 641	3.53 599	8.64 298	287	1.35 702	9.99 958	29
9120	32	8.64 543	284	3.53 642	3.53 599	8.64 585	285	1.35 415̄	9.99 958	28
9180	33	8.64 827	283	3.53 642	3.53 599	8.64 870	284	1.35 130	9.99 957	27
9240	34	8.65 110	281	3.53 642	3.53 598	8.65 154	281	1.34 846	9.99 956	26
9300	35	8.65 391	279	3.53 642	3.53 598	8.65 435	280	1.34 565̄	9.99 956	25
9360	36	8.65 670	277	3.53 642	3.53 598	8.65 715̄	278	1.34 285	9.99 955	24
9420	37	8.65 947	276	3.53 642	3.53 597	8.65 993	276	1.34 007	9.99 955̄	23
9480	38	8.66 223	274	3.53 643	3.53 597	8.66 269	274	1.33 731	9.99 954	22
9540	39	8.66 497	272	3.53 643	3.53 596	8.66 543	273	1.33 457	9.99 954	21
9600	**40**	8.66 769	270	3.53 643	3.53 596	8.66 816	271	1.33 184	9.99 953	**20**
9660	41	8.67 039	269	3.53 643	3.53 596	8.67 087	269	1.32 913	9.99 952	19
9720	42	8.67 308	267	3.53 643	3.53 595	8.67 356	268	1.32 644	9.99 952	18
9780	43	8.67 575	266	3.53 644	3.53 595̄	8.67 624	266	1.32 376	9.99 951	17
9840	44	8.67 841	263	3.53 644	3.53 594	8.67 890	264	1.32 110	9.99 951	16
9900	45	8.68 104	263	3.53 644	3.53 594	8.68 154	263	1.31 846	9.99 950	15
9960	46	8.68 367	260	3.53 644	3.53 594	8.68 417	261	1.31 583	9 99 949	14
10020	47	8.68 627	259	3.53 644	3.53 593	8.68 678	260	1.31 322	9.99 949	13
10080	48	8.68 886	258	3.53 645̄	3.53 593	8.68 938	258	1.31 062	9.99 948	12
10140	49	8.69 144	256	3.53 645̄	3.53 592	8.69 196	257	1.30 804	9.99 948	11
10200	**50**	8.69 400	254	3.53 645	3.53 592	8.69 453	255	1.30 547	9.99 947	**10**
10260	51	8.69 654	253	3.53 645	3.53 592	8.69 708	254	1.30 292	9.99 946	9
10320	52	8.69 907	252	3.53 646	3.53 591	8.69 962	252	1.30 038	9.99 946	8
10380	53	8.70 159	250	3.53 646	3.53 591	8.70 214	251	1.29 786	9.99 945̄	7
10440	54	8.70 409	249	3.53 646	3.53 590	8.70 465̄	249	1.29 535	9.99 944	6
10500	55	8.70 658	247	3.53 646	3.53 590	8.70 714	248	1.29 286	9.99 944	5
10560	56	8.70 905̄	246	3.53 646	3.53 589	8.70 962	246	1.29 038	9.99 943	4
10620	57	8.71 151	244	3.53 647	3.53 589	8.71 208	245	1.28 792	9.99 942	3
10680	58	8.71 395	243	3.53 647	3.53 589	8.71 453	244	1.28 547	9.99 942	2
10740	59	8.71 638	242	3.53 647	3.53 588	8.71 697	243	1.28 303	9.99 941	1
10800	**60**	8.71 880		3.53 647	3.53 588	8.71 940		1.28 060	9.99 940	**0**
		L. Cos.	d.			L. Cot.	c. d.	L. Tan.	L. Sin.	′

87°

′	L. Sin.	d.	L. Tan.	c.d.	L. Cot.	L. Cos.	
0	8.71 880	240	8.71 940	241	1.28 060	9.99 940	60
1	8.72 120	239	8.72 181	239	1.27 819	9.99 940	59
2	8.72 359	238	8.72 420	239	1.27 580	9.99 939	58
3	8.72 597	237	8.72 659	237	1.27 341	9.99 938	57
4	8.72 834	235	8.72 896	236	1.27 104	9.99 938	56
5	8.73 069	234	8.73 132	234	1.26 868	9.99 937	55
6	8.73 303	232	8.73 366	234	1.26 634	9.99 936	54
7	8.73 535	232	8.73 600	232	1.26 400	9.99 936	53
8	8.73 767	230	8.73 832	231	1.26 168	9.99 935	52
9	8.73 997	229	8.74 063	229	1.25 937	9.99 934	51
10	8.74 226	228	8.74 292	229	1.25 708	9.99 934	50
11	8.74 454	226	8.74 521	227	1.25 479	9.99 933	49
12	8.74 680	226	8.74 748	226	1.25 252	9.99 932	48
13	8.74 906	224	8.74 974	225	1.25 026	9.99 932	47
14	8.75 130	223	8.75 199	224	1.24 801	9.99 931	46
15	8.75 353	222	8.75 423	222	1.24 577	9.99 930	45
16	8.75 575̄	220	8.75 645	222	1.24 355̄	9.99 929	44
17	8.75 795	220	8.75 867	220	1.24 133	9.99 929	43
18	8.76 015	219	8.76 087	219	1.23 913	9.99 928	42
19	8.76 234	217	8.76 306	219	1.23 694	9.99 927	41
20	8.76 451	216	8.76 525̄	217	1.23 475	9.99 926	40
21	8.76 667	216	8.76 742	216	1.23 258	9.99 926	39
22	8.76 883	214	8.76 958	215	1.23 042	9.99 925̄	38
23	8.77 097	213	8.77 173	214	1.22 827	9.99 924	37
24	8.77 310	212	8.77 387	213	1.22 613	9.99 923	36
25	8.77 522	211	8.77 600	211	1.22 400	9.99 923	35
26	8.77 733	210	8.77 811	211	1.22 189	9.99 922	34
27	8.77 943	209	8.78 022	210	1.21 978	9.99 921	33
28	8.78 152	208	8.78 232	209	1.21 768	9.99 920	32
29	8.78 360	208	8.78 441	208	1.21 559	9.99 920	31
30	8.78 568	206	8.78 649	206	1.21 351	9.99 919	30
31	8.78 774	205	8.78 855	206	1.21 145̄	9.99 918	29
32	8.78 979	204	8.79 061	205	1.20 939	9.99 917	28
33	8.79 183	203	8.79 266	204	1.20 734	9.99 917	27
34	8.79 386	202	8.79 470	203	1.20 530	9.99 916	26
35	8.79 588	201	8.79 673	202	1.20 327	9.99 915	25
36	8.79 789	201	8.79 875	201	1.20 125̄	9.99 914	24
37	8.79 990	199	8.80 076	201	1.19 924	9.99 913	23
38	8.80 189	199	8.80 277	199	1.19 723	9.99 913	22
39	8.80 388	197	8.80 476	198	1.19 524	9.99 912	21
40	8.80 585	197	8.80 674	198	1.19 326	9.99 911	20
41	8.80 782	196	8.80 872	196	1.19 128	9.99 910	19
42	8.80 978	195	8.81 068	196	1.18 932	9.99 909	18
43	8.81 173	194	8.81 264	195	1.18 736	9.99 909	17
44	8.81 367	193	8.81 459	194	1.18 541	9.99 908	16
45	8.81 560	192	8.81 653	193	1.18 347	9.99 907	15
46	8.81 752	192	8.81 846	192	1.18 154	9.99 906	14
47	8.81 944	190	8.82 038	192	1.17 962	9.99 905	13
48	8.82 134	190	8.82 230	190	1.17 770	9.99 904	12
49	8.82 324	189	8.82 420	190	1.17 580	9.99 904	11
50	8.82 513	188	8.82 610	189	1.17 390	9.99 903	10
51	8.82 701	187	8.82 799	188	1.17 201	9.99 902	9
52	8.82 888	187	8.82 987	188	1.17 013	9.99 901	8
53	8.83 075̄	186	8.83 175̄	186	1.16 825	9.99 900	7
54	8.83 261	185	8.83 361	186	1.16 639	9.99 899	6
55	8.83 446	184	8.83 547	185	1.16 453	9.99 898	5
56	8.83 630	183	8.83 732	184	1.16 268	9.99 898	4
57	8.83 813	183	8.83 916	184	1.16 084	9.99 897	3
58	8.83 996	181	8.84 100	182	1.15 900	9.99 896	2
59	8.84 177	181	8.84 282	182	1.15 718	9.99 895̄	1
60	8.84 358		8.84 464		1.15 536	9.99 894	0
	L. Cos.	d.	L. Cot.	c.d.	L. Tan.	L. Sin.	′

P. P.

	241	239	237	236	234
1	24.1	23.9	23.7	23.6	23.4
2	48.2	47.8	47.4	47.2	46.8
3	72.3	71.7	71.1	70.8	70.2
4	96.4	95.6	94.8	94.4	93.6
5	120.5	119.5	118.5	118.0	117.0
6	144.6	143.4	142.2	141.6	140.4
7	168.7	167.3	165.9	165.2	163.8
8	192.8	191.2	189.6	188.8	187.2
9	216.9	215.1	213.3	212.4	210.6

	232	231	229	227	226
1	23.2	23.1	22.9	22.7	22.6
2	46.4	46.2	45.8	45.4	45.2
3	69.6	69.3	68.7	68.1	67.8
4	92.8	92.4	91.6	90.8	90.4
5	116.0	115.5	114.5	113.5	113.0
6	139.2	138.6	137.4	136.2	135.6
7	162.4	161.7	160.3	158.9	158.2
8	185.6	184.8	183.2	181.6	180.8
9	208.8	207.9	206.1	204.3	203.4

	224	222	220	219	217
1	22.4	22.2	22.0	21.9	21.7
2	44.8	44.4	44.0	43.8	43.4
3	67 2	66.6	66.0	65.7	65.1
4	89.6	88.8	88.0	87.6	86.8
5	112.0	111.0	110.0	109.5	108.5
6	134.4	133.2	132.0	131.4	130.2
7	156.8	155.4	154.0	153.3	151.9
8	179.2	177.6	176.0	175.2	173.6
9	201.6	199.8	198.0	197.1	195.3

	216	214	213	211	209
1	21.6	21.4	21.3	21 1	20.9
2	43.2	42.8	42.6	42.2	41.8
3	64.8	64.2	63.9	63.3	62.7
4	86.4	85.6	85.2	84.4	83.6
5	108.0	107.0	106.5	105.5	104.5
6	129.6	128.4	127.8	126.6	125.4
7	151.2	149.8	149.1	147.7	146.3
8	172.8	171.2	170.4	168.8	167.2
9	194.4	192.6	191.7	189.9	188.1

	208	206	203	201	199
1	20.8	20.6	20.3	20.1	19.9
2	41.6	41.2	40.6	40.2	39.8
3	62.4	61.8	60.9	60.3	59.7
4	83.2	82.4	81.2	80.4	79.6
5	104.0	103.0	101.5	100.5	99.5
6	124.8	123.6	121.8	120.6	119.4
7	145.6	144.2	142 1	140.7	139.3
8	166.4	164.8	162.4	160.8	159.2
9	187.2	185.4	182.7	180.9	179.1

	198	196	194	192	190
1	19.8	19.6	19.4	19.2	19.0
2	39.6	39.2	38.8	38.4	38.0
3	59.4	58.8	58.2	57.6	57.0
4	79.2	78.4	77.6	76.8	76.0
5	99.0	98.0	97.0	96.0	95 0
6	118.8	117.6	116.4	115.2	114.0
7	138.6	137.2	135.8	134.4	133.0
8	158.4	156.8	155.2	153.6	152.0
9	178.2	176.4	174.6	172.8	171.0

	188	186	184	182	181
1	18.8	18.6	18.4	18 2	18.1
2	37.6	37.2	36.8	36.4	36.2
3	56.4	55.8	55 2	54.6	54.3
4	75.2	74.4	73 6	72.8	72.4
5	94.0	93.0	92.0	91.0	90.5
6	112.8	111.6	110.4	109.2	108.6
7	131.6	130.2	128.8	127.4	126.7
8	150.4	148.8	147.2	145.6	144.8
9	169.2	167.4	165.6	163 8	162.9

P. P.

4°

′	L. Sin.	d.	L. Tan.	c.d.	L. Cot.	L. Cos.	
0	8.84 358	181	8.84 464	182	1.15 536	9.99 894	60
1	8.84 539	179	8.84 646	180	1.15 354	9.99 893	59
2	8.84 718	179	8.84 826	180	1.15 174	9.99 892	58
3	8.84 897	178	8.85 006	179	1.14 994	9.99 891	57
4	8.85 075	177	8.85 185	178	1.14 815	9.99 891	56
5	8.85 252	177	8.85 363	177	1.14 637	9.99 890	55
6	8.85 429	176	8.85 540	177	1.14 460	9.99 889	54
7	8.85 605	175	8.85 717	176	1.14 283	9.99 888	53
8	8.85 780	175	8.85 893	176	1.14 107	9.99 887	52
9	8.85 955	173	8.86 069	174	1.13 931	9.99 886	51
10	8.86 128	173	8.86 243	174	1.13 757	9.99 885	50
11	8.86 301	173	8.86 417	174	1.13 583	9.99 884	49
12	8.86 474	171	8.86 591	172	1.13 409	9.99 883	48
13	8.86 645	171	8.86 763	172	1.13 237	9.99 882	47
14	8.86 816	171	8.86 935	171	1.13 065	9.99 881	46
15	8.86 987	169	8.87 106	171	1.12 894	9.99 880	45
16	8.87 156	169	8.87 277	170	1.12 723	9.99 879	44
17	8.87 325	169	8.87 447	169	1.12 553	9.99 879	43
18	8.87 494	167	8.87 616	169	1.12 384	9.99 878	42
19	8.87 661	168	8.87 785	168	1.12 215	9.99 877	41
20	8.87 829	166	8.87 953	167	1.12 047	9.99 876	40
21	8.87 995	166	8.88 120	167	1.11 880	9.99 875	39
22	8.88 161	165	8.88 287	166	1.11 713	9.99 874	38
23	8.88 326	164	8.88 453	165	1.11 547	9.99 873	37
24	8.88 490	164	8.88 618	165	1.11 382	9.99 872	36
25	8.88 654	163	8.88 783	165	1.11 217	9.99 871	35
26	8.88 817	163	8.88 948	163	1.11 052	9.99 870	34
27	8.88 980	162	8.89 111	163	1.10 889	9.99 869	33
28	8.89 142	162	8.89 274	163	1.10 726	9.99 868	32
29	8.89 304	160	8.89 437	161	1.10 563	9.99 867	31
30	8.89 464	161	8.89 598	162	1.10 402	9.99 866	30
31	8.89 625	159	8.89 760	160	1.10 240	9.99 865	29
32	8.89 784	159	8.89 920	160	1.10 080	9.99 864	28
33	8.89 943	159	8.90 080	160	1.09 920	9.99 863	27
34	8.90 102	158	8.90 240	159	1.09 760	9.99 862	26
35	8.90 260	157	8.90 399	158	1.09 601	9.99 861	25
36	8.90 417	157	8.90 557	158	1.09 443	9.99 860	24
37	8.90 574	156	8.90 715	157	1.09 285	9.99 859	23
38	8.90 730	155	8.90 872	157	1.09 128	9.99 858	22
39	8.90 885	155	8.91 029	156	1.08 971	9.99 857	21
40	8.91 040	155	8.91 185	155	1.08 815	9.99 856	20
41	8.91 195	154	8.91 340	155	1.08 660	9.99 855	19
42	8.91 349	153	8.91 495	155	1.08 505	9.99 854	18
43	8.91 502	153	8.91 650	153	1.08 350	9.99 853	17
44	8.91 655	152	8.91 803	154	1.08 197	9.99 852	16
45	8.91 807	152	8.91 957	153	1.08 043	9.99 851	15
46	8.91 959	151	8.92 110	152	1.07 890	9.99 850	14
47	8.92 110	151	8.92 262	152	1.07 738	9.99 848	13
48	8.92 261	150	8.92 414	151	1.07 586	9.99 847	12
49	8.92 411	150	8.92 565	151	1.07 435	9.99 846	11
50	8.92 561	149	8.92 716	150	1.07 284	9.99 845	10
51	8.92 710	149	8.92 866	150	1.07 134	9.99 844	9
52	8.92 859	148	8.93 016	149	1.06 984	9.99 843	8
53	8.93 007	147	8.93 165	148	1.06 835	9.99 842	7
54	8.93 154	147	8.93 313	149	1.06 687	9.99 841	6
55	8.93 301	147	8.93 462	147	1.06 538	9.99 840	5
56	8.93 448	146	8.93 609	147	1.06 391	9.99 839	4
57	8.93 594	146	8.93 756	147	1.06 244	9.99 838	3
58	8.93 740	145	8.93 903	146	1.06 097	9.99 837	2
59	8.93 885	145	8.94 049	146	1.05 951	9.99 836	1
60	8.94 030		8.94 195		1.05 805	9.99 834	0
	L. Cos.	d.	L. Cot.	c.d.	L. Tan.	L. Sin.	′

P. P.

	182	181	180	179	178
1	18.2	18.1	18.0	17.9	17.8
2	36.4	36.2	36.0	35.8	35.6
3	54.6	54.3	54.0	53.7	53.4
4	72.8	72.4	72.0	71.6	71.2
5	91.0	90.5	90.0	89.5	89.0
6	109.2	108.6	108.0	107.4	106.8
7	127.4	126.7	126.0	125.3	124.6
8	145.6	144.8	144.0	143.2	142.4
9	163.8	162.9	162.0	161.1	160.2

	177	176	175	174	173
1	17.7	17.6	17.5	17.4	17.3
2	35.4	35.2	35.0	34.8	34.6
3	53.1	52.8	52.5	52.2	51.9
4	70.8	70.4	70.0	69.6	69.2
5	88.5	88.0	87.5	87.0	86.5
6	106.2	105.6	105.0	104.4	103.8
7	123.9	123.2	122.5	121.8	121.1
8	141.6	140.8	140.0	139.2	138.4
9	159.3	158.4	157.5	156.6	155.7

	172	171	170	169	168
1	17.2	17.1	17.0	16.9	16.8
2	34.4	34.2	34.0	33.8	33.6
3	51.6	51.3	51.0	50.7	50.4
4	68.8	68.4	68.0	67.6	67.2
5	86.0	85.5	85.0	84.5	84.0
6	103.2	102.6	102.0	101.4	100.8
7	120.4	119.7	119.0	118.3	117.6
8	137.6	136.8	136.0	135.2	134.4
9	154.8	153.9	153.0	152.1	151.2

	167	166	165	164	163
1	16.7	16.6	16.5	16.4	16.3
2	33.4	33.2	33.0	32.8	32.6
3	50.1	49.8	49.5	49.2	48.9
4	66.8	66.4	66.0	65.6	65.2
5	83.5	83.0	82.5	82.0	81.5
6	100.2	99.6	99.0	98.4	97.8
7	116.9	116.2	115.5	114.8	114.1
8	133.6	132.8	132.0	131.2	130.4
9	150.3	149.4	148.5	147.6	146.7

	162	161	160	159	158
1	16.2	16.1	16.0	15.9	15.8
2	32.4	32.2	32.0	31.8	31.6
3	48.6	48.3	48.0	47.7	47.4
4	64.8	64.4	64.0	63.6	63.2
5	81.0	80.5	80.0	79.5	79.0
6	97.2	96.6	96.0	95.4	94.8
7	113.4	112.7	112.0	111.3	110.6
8	129.6	128.8	128.0	127.2	126.4
9	145.8	144.9	144.0	143.1	142.2

	157	156	155	154	153
1	15.7	15.6	15.5	15.4	15.3
2	31.4	31.2	31.0	30.8	30.6
3	47.1	46.8	46.5	46.2	45.9
4	62.8	62.4	62.0	61.6	61.2
5	78.5	78.0	77.5	77.0	76.5
6	94.2	93.6	93.0	92.4	91.8
7	109.9	109.2	108.5	107.8	107.1
8	125.6	124.8	124.0	123.2	122.4
9	141.3	140.4	139.5	138.6	137.7

	152	151	150	149	148
1	15.2	15.1	15.0	14.9	14.8
2	30.4	30.2	30.0	29.8	29.6
3	45.6	45.3	45.0	44.7	44.4
4	60.8	60.4	60.0	59.6	59.2
5	76.0	75.5	75.0	74.5	74.0
6	91.2	90.6	90.0	89.4	88.8
7	106.4	105.7	105.0	104.3	103.6
8	121.6	120.8	120.0	119.2	118.4
9	136.8	135.9	135.0	134.1	133.2

P. P.

85°

′	L. Sin.	d.	L. Tan.	c.d.	L. Cot.	L. Cos.	
0	8.94 030	144	8.94 195	145	1.05 805	9.99 834	60
1	8.94 174	143	8.94 340	145	1.05 660	9.99 833	59
2	8.94 317	144	8.94 485	145	1.05 515	9.99 832	58
3	8.94 461	142	8.94 630	143	1.05 370	9.99 831	57
4	8.94 603	143	8.94 773	144	1.05 227	9.99 830	56
5	8.94 746	141	8.94 917	143	1.05 083	9.99 829	55
6	8.94 887	142	8.95 060	142	1.04 940	9.99 828	54
7	8.95 029	141	8.95 202	142	1.04 798	9.99 827	53
8	8.95 170	140	8.95 344	142	1.04 656	9.99 825	52
9	8.95 310	140	8.95 486	141	1.04 514	9.99 824	51
10	8.95 450	139	8.95 627	140	1.04 373	9.99 823	50
11	8.95 589	139	8.95 767	141	1.04 233	9.99 822	49
12	8.95 728	139	8.95 908	139	1.04 092	9.99 821	48
13	8.95 867	138	8.96 047	140	1.03 953	9.99 820	47
14	8.96 005	138	8.96 187	138	1.03 813	9.99 819	46
15	8.96 143	137	8.96 325	139	1.03 675	9.99 817	45
16	8.96 280	137	8.96 464	138	1.03 536	9.99 816	44
17	8.96 417	136	8.96 602	137	1.03 398	9.99 815	43
18	8.96 553	136	8.96 739	138	1.03 261	9.99 814	42
19	8.96 689	136	8.96 877	136	1.03 123	9.99 813	41
20	8.96 825	135	8.97 013	137	1.02 987	9.99 812	40
21	8.96 960	135	8.97 150	135	1.02 850	9.99 810	39
22	8.97 095	134	8.97 285	136	1.02 715	9.99 809	38
23	8.97 229	134	8.97 421	135	1.02 579	9.99 808	37
24	8.97 363	133	8.97 556	135	1.02 444	9.99 807	36
25	8.97 496	133	8.97 691	134	1.02 309	9.99 806	35
26	8.97 629	133	8.97 825	134	1.02 175	9.99 804	34
27	8.97 762	132	8.97 959	133	1.02 041	9.99 803	33
28	8.97 894	132	8.98 092	133	1.01 908	9.99 802	32
29	8.98 026	131	8.98 225	133	1.01 775	9.99 801	31
30	8.98 157	131	8.98 358	132	1.01 642	9.99 800	30
31	8.98 288	131	8.98 490	132	1.01 510	9.99 798	29
32	8.98 419	130	8.98 622	131	1.01 378	9.99 797	28
33	8.98 549	130	8.98 753	131	1.01 247	9.99 796	27
34	8.98 679	129	8.98 884	131	1.01 116	9.99 795	26
35	8.98 808	129	8.99 015	130	1.00 985	9.99 793	25
36	8.98 937	129	8.99 145	130	1.00 855	9.99 792	24
37	8.99 066	128	8.99 275	130	1.00 725	9.99 791	23
38	8.99 194	128	8.99 405	129	1.00 595	9.99 790	22
39	8.99 322	128	8.99 534	128	1.00 466	9.99 788	21
40	8.99 450	127	8.99 662	129	1.00 338	9.99 787	20
41	8.99 577	127	8.99 791	128	1.00 209	9.99 786	19
42	8.99 704	126	8.99 919	127	1.00 081	9.99 785	18
43	8.99 830	126	9.00 046	128	0.99 954	9.99 783	17
44	8.99 956	126	9.00 174	127	0.99 826	9.99 782	16
45	9.00 082	125	9.00 301	126	0.99 699	9.99 781	15
46	9.00 207	125	9.00 427	126	0.99 573	9.99 780	14
47	9.00 332	124	9.00 553	126	0.99 447	9.99 778	13
48	9.00 456	125	9.00 679	126	0.99 321	9.99 777	12
49	9.00 581	123	9.00 805	125	0.99 195	9.99 776	11
50	9.00 704	124	9.00 930	125	0.99 070	9.99 775	10
51	9.00 828	123	9.01 055	124	0.98 945	9.99 773	9
52	9.00 951	123	9.01 179	124	0.98 821	9.99 772	8
53	9.01 074	122	9.01 303	124	0.98 697	9.99 771	7
54	9.01 196	122	9.01 427	123	0.98 573	9.99 769	6
55	9.01 318	122	9.01 550	123	0.98 450	9.99 768	5
56	9.01 440	121	9.01 673	123	0.98 327	9.99 767	4
57	9.01 561	121	9.01 796	122	0.98 204	9.99 765	3
58	9.01 682	121	9.01 918	122	0.98 082	9.99 764	2
59	9.01 803	120	9.02 040	122	0.97 960	9.99 763	1
60	9.01 923		9.02 162		0.97 838	9.99 761	0
	L. Cos.	d.	L. Cot.	c.d.	L. Tan.	L. Sin.	′

P. P.

	147	146	145	144
1	14.7	14.6	14.5	14.4
2	29.4	29.2	29.0	28.8
3	44.1	43.8	43.5	43.2
4	58.8	58.4	58.0	57.6
5	73.5	73.0	72.5	72.0
6	88.2	87.6	87.0	86.4
7	102.9	102.2	101.5	100.8
8	117.6	116.8	116.0	115.2
9	132.3	131.4	130.5	129.6

	143	142	141	140
1	14.3	14.2	14.1	14.0
2	28.6	28.4	28.2	28.0
3	42.9	42.6	42.3	42.0
4	57.2	56.8	56.4	56.0
5	71.5	71.0	70.5	70.0
6	85.8	85.2	84.6	84.0
7	100.1	99.4	98.7	98.0
8	114.4	113.6	112.8	112.0
9	128.7	127.8	126.9	126.0

	139	138	137	136
1	13.9	13.8	13.7	13.6
2	27.8	27.6	27.4	27.2
3	41.7	41.4	41.1	40.8
4	55.6	55.2	54.8	54.4
5	69.5	69.0	68.5	68.0
6	83.4	82.8	82.2	81.6
7	97.3	96.6	95.9	95.2
8	111.2	110.4	109.6	108.8
9	125.1	124.2	123.3	122.4

	135	134	133	132
1	13.5	13.4	13.3	13.2
2	27.0	26.8	26.6	26.4
3	40.5	40.2	39.9	39.6
4	54.0	53.6	53.2	52.8
5	67.5	67.0	66.5	66.0
6	81.0	80.4	79.8	79.2
7	94.5	93.8	93.1	92.4
8	108.0	107.2	106.4	105.6
9	121.5	120.6	119.7	118.8

	131	130	129	128
1	13.1	13.0	12.9	12.8
2	26.2	26.0	25.8	25.6
3	39.3	39.0	38.7	38.4
4	52.4	52.0	51.6	51.2
5	65.5	65.0	64.5	64.0
6	78.6	78.0	77.4	76.8
7	91.7	91.0	90.3	89.6
8	104.8	104.0	103.2	102.4
9	117.9	117.0	116.1	115.2

	127	126	125	124
1	12.7	12.6	12.5	12.4
2	25.4	25.2	25.0	24.8
3	38.1	37.8	37.5	37.2
4	50.8	50.4	50.0	49.6
5	63.5	63.0	62.5	62.0
6	76.2	75.6	75.0	74.4
7	88.9	88.2	87.5	86.8
8	101.6	100.8	100.0	99.2
9	114.3	113.4	112.5	111.6

	123	122	121	120
1	12.3	12.2	12.1	12.0
2	24.6	24.4	24.2	24.0
3	36.9	36.6	36.3	36.0
4	49.2	48.8	48.4	48.0
5	61.5	61.0	60.5	60.0
6	73.8	73.2	72.6	72.0
7	86.1	85.4	84.7	84.0
8	98.4	97.6	96.8	96.0
9	110.7	109.8	108.9	108.0

P. P.

6°

′	L. Sin.	d.	L. Tan.	c.d.	L. Cot.	L. Cos.	
0	9.01 923	120	9.02 162	121	0.97 838	9.99 761	60
1	9.02 043	120	9.02 283	121	0.97 717	9.99 760	59
2	9.02 163	120	9.02 404	121	0.97 596	9.99 759	58
3	9.02 283	119	9.02 525	120	0.97 475	9.99 757	57
4	9.02 402	118	9.02 645	121	0.97 355	9.99 756	56
5	9.02 520	119	9.02 766	119	0.97 234	9.99 755	55
6	9.02 639	118	9.02 885	120	0.97 115	9.99 753	54
7	9.02 757	117	9.03 005	119	0.96 995	9.99 752	53
8	9.02 874	118	9.03 124	118	0.96 876	9.99 751	52
9	9.02 992	117	9.03 242	119	0.96 758	9.99 749	51
10	9.03 109	117	9.03 361	118	0.96 639	9.99 748	50
11	9.03 226	116	9.03 479	118	0.96 521	9.99 747	49
12	9.03 342	116	9.03 597	117	0.96 403	9.99 745	48
13	9.03 458	116	9.03 714	118	0.96 286	9.99 744	47
14	9.03 574	116	9.03 832	116	0.96 168	9.99 742	46
15	9.03 690	115	9.03 948	117	0.96 052	9.99 741	45
16	9.03 805	115	9.04 065	116	0.95 935	9.99 740	44
17	9.03 920	114	9.04 181	116	0.95 819	9.99 738	43
18	9.04 034	115	9.04 297	116	0.95 703	9.99 737	42
19	9.04 149	113	9.04 413	115	0.95 587	9.99 736	41
20	9.04 262	114	9.04 528	115	0.95 472	9.99 734	40
21	9.04 376	114	9.04 643	115	0.95 357	9.99 733	39
22	9.04 490	113	9.04 758	115	0.95 242	9.99 731	38
23	9.04 603	112	9.04 873	114	0.95 127	9.99 730	37
24	9.04 715	113	9.04 987	114	0.95 013	9.99 728	36
25	9.04 828	112	9.05 101	113	0.94 899	9.99 727	35
26	9.04 940	112	9.05 214	114	0.94 786	9.99 726	34
27	9.05 052	112	9.05 328	113	0.94 672	9.99 724	33
28	9.05 164	111	9.05 441	112	0.94 559	9.99 723	32
29	9.05 275	111	9.05 553	113	0.94 447	9.99 721	31
30	9.05 386	111	9.05 666	112	0.94 334	9.99 720	30
31	9.05 497	110	9.05 778	112	0.94 222	9.99 718	29
32	9.05 607	110	9.05 890	112	0.94 110	9.99 717	28
33	9.05 717	110	9.06 002	111	0.93 998	9.99 716	27
34	9.05 827	110	9.06 113	111	0.93 887	9.99 714	26
35	9.05 937	109	9.06 224	111	0.93 776	9.99 713	25
36	9.06 046	109	9.06 335	110	0.93 665	9.99 711	24
37	9.06 155	109	9.06 445	111	0.93 555	9.99 710	23
38	9.06 264	108	9.06 556	110	0.93 444	9.99 708	22
39	9.06 372	109	9.06 666	109	0.93 334	9.99 707	21
40	9.06 481	108	9.06 775	110	0.93 225	9.99 705	20
41	9.06 589	107	9.06 885	109	0.93 115	9.99 704	19
42	9.06 696	108	9.06 994	109	0.93 006	9.99 702	18
43	9.06 804	107	9.07 103	108	0.92 897	9.99 701	17
44	9.06 911	107	9.07 211	109	0.92 789	9.99 699	16
45	9.07 018	106	9.07 320	108	0.92 680	9.99 698	15
46	9.07 124	107	9.07 428	108	0.92 572	9.99 696	14
47	9.07 231	106	9.07 536	107	0.92 464	9.99 695	13
48	9.07 337	105	9.07 643	108	0.92 357	9.99 693	12
49	9.07 442	106	9.07 751	107	0.92 249	9.99 692	11
50	9.07 548	105	9.07 858	106	0.92 142	9.99 690	10
51	9.07 653	105	9.07 964	107	0.92 036	9.99 689	9
52	9.07 758	105	9.08 071	106	0.91 929	9.99 687	8
53	9.07 863	105	9.08 177	106	0.91 823	9.99 686	7
54	9.07 968	104	9.08 283	106	0.91 717	9.99 684	6
55	9.08 072	104	9.08 389	106	0.91 611	9.99 683	5
56	9.08 176	104	9.08 495	105	0.91 505	9.99 681	4
57	9.08 280	103	9.08 600	105	0.91 400	9.99 680	3
58	9.08 383	103	9.08 705	105	0.91 295	9.99 678	2
59	9.08 486	103	9.08 810	104	0.91 190	9.99 677	1
60	9.08 589		9.08 914		0.91 086	9.99 675	0
	L. Cos.	d.	L. Cot.	c.d.	L. Tan.	L. Sin.	′

P. P.

	121	120	119	118
1	12.1	12.0	11.9	11.8
2	24.2	24.0	23.8	23.6
3	36.3	36.0	35.7	35.4
4	48.4	48.0	47.6	47.2
5	60.5	60.0	59.5	59.0
6	72.6	72.0	71.4	70.8
7	84.7	84.0	83.3	82.6
8	96.8	96.0	95.2	94.4
9	108.9	108.0	107.1	106.2

	117	116	115	114
1	11.7	11.6	11.5	11.4
2	23.4	23.2	23.0	22.8
3	35.1	34.8	34.5	34.2
4	46.8	46.4	46.0	45.6
5	58.5	58.0	57.5	57.0
6	70.2	69.6	69.0	68.4
7	81.9	81.2	80.5	79.8
8	93.6	92.8	92.0	91.2
9	105.3	104.4	103.5	102.6

	113	112	111	110
1	11.3	11.2	11.1	11.0
2	22.6	22.4	22.2	22.0
3	33.9	33.6	33.3	33.0
4	45.2	44.8	44.4	44.0
5	56.5	56.0	55.5	55.0
6	67.8	67.2	66.6	66.0
7	79.1	78.4	77.7	77.0
8	90.4	89.6	88.8	88.0
9	101.7	100.8	99.9	99.0

	109	108	107	106
1	10.9	10.8	10.7	10.6
2	21.8	21.6	21.4	21.2
3	32.7	32.4	32.1	31.8
4	43.6	43.2	42.8	42.4
5	54.5	54.0	53.5	53.0
6	65.4	64.8	64.2	63.6
7	76.3	75.6	74.9	74.2
8	87.2	86.4	85.6	84.8
9	98.1	97.2	96.3	95.4

	105	104	103
1	10.5	10.4	10.3
2	21.0	20.8	20.6
3	31.5	31.2	30.9
4	42.0	41.6	41.2
5	52.5	52.0	51.5
6	63.0	62.4	61.8
7	73.5	72.8	72.1
8	84.0	83.2	82.4
9	94.5	93.6	92.7

P. P.

83°

′	L. Sin.	d.	L. Tan.	c.d.	L. Cot.	L. Cos.	′
0	9.08 589	103	9.08 914	105	0.91 086	9.99 675	60
1	9.08 692	103	9.09 019	104	0.90 981	9.99 674	59
2	9.08 795	102	9.09 123	104	0.90 877	9.99 672	58
3	9.08 897	102	9.09 227	103	0.90 773	9.99 670	57
4	9.08 999	102	9.09 330	104	0.90 670	9.99 669	56
5	9.09 101	101	9.09 434	103	0.90 566	9.99 667	55
6	9.09 202	102	9.09 537	103	0.90 463	9.99 666	54
7	9.09 304	101	9.09 640	102	0.90 360	9.99 664	53
8	9.09 405	101	9.09 742	103	0.90 258	9.99 663	52
9	9.09 506	100	9.09 845	102	0.90 155	9.99 661	51
10	9.09 606	101	9.09 947	102	0.90 053	9.99 659	50
11	9.09 707	100	9.10 049	101	0.89 951	9.99 658	49
12	9.09 807	100	9.10 150	102	0.89 850	9.99 656	48
13	9.09 907	99	9.10 252	101	0.89 748	9.99 655	47
14	9.10 006	100	9.10 353	101	0.89 647	9.99 653	46
15	9.10 106	99	9.10 454	101	0.89 546	9.99 651	45
16	9.10 205	99	9.10 555	101	0.89 445	9.99 650	44
17	9.10 304	98	9.10 656	100	0.89 344	9.99 648	43
18	9.10 402	99	9.10 756	100	0.89 244	9.99 647	42
19	9.10 501	98	9.10 856	100	0.89 144	9.99 645	41
20	9.10 599	98	9.10 956	100	0.89 044	9.99 643	40
21	9.10 697	98	9.11 056	99	0.88 944	9.99 642	39
22	9.10 795	98	9.11 155	99	0.88 845	9.99 640	38
23	9.10 893	97	9.11 254	99	0.88 746	9.99 638	37
24	9.10 990	97	9.11 353	99	0.88 647	9.99 637	36
25	9.11 087	97	9.11 452	99	0.88 548	9.99 635	35
26	9.11 184	97	9.11 551	98	0.88 449	9.99 633	34
27	9.11 281	96	9.11 649	98	0.88 351	9.99 632	33
28	9.11 377	97	9.11 747	98	0.88 253	9.99 630	32
29	9.11 474	96	9.11 845	98	0.88 155	9.99 629	31
30	9.11 570	96	9.11 943	97	0.88 057	9.99 627	30
31	9.11 666	95	9.12 040	98	0.87 960	9.99 625	29
32	9.11 761	96	9.12 138	97	0.87 862	9.99 624	28
33	9.11 857	95	9.12 235	97	0.87 765	9.99 622	27
34	9.11 952	95	9.12 332	96	0.87 668	9.99 620	26
35	9.12 047	95	9.12 428	97	0.87 572	9.99 618	25
36	9.12 142	94	9.12 525	96	0.87 475	9.99 617	24
37	9.12 236	95	9.12 621	96	0.87 379	9.99 615	23
38	9.12 331	94	9.12 717	96	0.87 283	9.99 613	22
39	9.12 425	94	9.12 813	96	0.87 187	9.99 612	21
40	9.12 519	93	9.12 909	95	0.87 091	9.99 610	20
41	9.12 612	94	9.13 004	95	0.86 996	9.99 608	19
42	9.12 706	93	9.13 099	95	0.86 901	9.99 607	18
43	9.12 799	93	9.13 194	95	0.86 806	9.99 605	17
44	9.12 892	93	9.13 289	95	0.86 711	9.99 603	16
45	9.12 985	93	9.13 384	94	0.86 616	9.99 601	15
46	9.13 078	93	9.13 478	95	0.86 522	9.99 600	14
47	9.13 171	92	9.13 573	94	0.86 427	9.99 598	13
48	9.13 263	92	9.13 667	94	0.86 333	9.99 596	12
49	9.13 355	92	9.13 761	93	0.86 239	9.99 595	11
50	9.13 447	92	9.13 854	94	0.86 146	9.99 593	10
51	9.13 539	91	9.13 948	93	0.86 052	9.99 591	9
52	9.13 630	92	9.14 041	93	0.85 959	9.99 589	8
53	9.13 722	91	9.14 134	93	0.85 866	9.99 588	7
54	9.13 813	91	9.14 227	93	0.85 773	9.99 586	6
55	9.13 904	90	9.14 320	92	0.85 680	9.99 584	5
56	9.13 994	91	9.14 412	92	0.85 588	9.99 582	4
57	9.14 085	90	9.14 504	93	0.85 496	9.99 581	3
58	9.14 175	91	9.14 597	91	0.85 403	9.99 579	2
59	9.14 266	90	9.14 688	92	0.85 312	9.99 577	1
60	9.14 356		9.14 780		0.85 220	9.99 575	0
	L. Cos.	d.	L. Cot.	c.d.	L. Tan.	L. Sin.	′

P. P.

	105	104	103
1	10.5	10.4	10.3
2	21.0	20.8	20.6
3	31.5	31.2	30.9
4	42.0	41.6	41.2
5	52.5	52.0	51.5
6	63.0	62.4	61.8
7	73.5	72.8	72.1
8	84.0	83.2	82.4
9	94.5	93.6	92.7

	102	101	99
1	10.2	10.1	9.9
2	20.4	20.2	19.8
3	30.6	30.3	29.7
4	40.8	40.4	39.6
5	51.0	50.5	49.5
6	61.2	60.6	59.4
7	71.4	70.7	69.3
8	81.6	80.8	79.2
9	91.8	90.9	89.1

	98	97	96
1	9.8	9.7	9.6
2	19.6	19.4	19.2
3	29.4	29.1	28.8
4	39.2	38.8	38.4
5	49.0	48.5	48.0
6	58.8	58.2	57.6
7	68.6	67.9	67.2
8	78.4	77.6	76.8
9	88.2	87.3	86.4

	95	94	93
1	9.5	9.4	9.3
2	19.0	18.8	18.6
3	28.5	28.2	27.9
4	38.0	37.6	37.2
5	47.5	47.0	46.5
6	57.0	56.4	55.8
7	66.5	65.8	65.1
8	76.0	75.2	74.4
9	85.5	84.6	83.7

	92	91	90
1	9.2	9.1	9.0
2	18.4	18.2	18.0
3	27.6	27.3	27.0
4	36.8	36.4	36.0
5	46.0	45.5	45.0
6	55.2	54.6	54.0
7	64.4	63.7	63.0
8	73.6	72.8	72.0
9	82.8	81.9	81.0

P. P.

8°

′	L. Sin.	d.	L. Tan.	c.d.	L. Cot.	L. Cos.	
0	9.14 356	89	9.14 780	92	0.85 220	9.99 575	60
1	9.14 445	90	9.14 872	91	0.85 128	9.99 574	59
2	9.14 535	89	9.14 963	91	0.85 037	9.99 572	58
3	9.14 624	90	9.15 054	91	0.84 946	9.99 570	57
4	9.14 714	89	9.15 145	91	0.84 855	9.99 568	56
5	9.14 803	88	9.15 236	91	0.84 764	9.99 566	55
6	9.14 891	89	9.15 327	90	0.84 673	9.99 565	54
7	9.14 980	89	9.15 417	91	0.84 583	9.99 563	53
8	9.15 069	88	9.15 508	90	0.84 492	9.99 561	52
9	9.15 157	88	9.15 598	90	0.84 402	9.99 559	51
10	9.15 245	88	9.15 688	89	0.84 312	9.99 557	50
11	9.15 333	88	9.15 777	90	0.84 223	9.99 556	49
12	9.15 421	87	9.15 867	89	0.84 133	9.99 554	48
13	9.15 508	88	9.15 956	90	0.84 044	9.99 552	47
14	9.15 596	87	9.16 046	89	0.83 954	9.99 550	46
15	9.15 683	87	9.16 135	89	0.83 865	9.99 548	45
16	9.15 770	87	9.16 224	88	0.83 776	9.99 546	44
17	9.15 857	87	9.16 312	89	0.83 688	9.99 545	43
18	9.15 944	86	9.16 401	88	0.83 599	9.99 543	42
19	9.16 030	86	9.16 489	88	0.83 511	9.99 541	41
20	9.16 116	87	9.16 577	88	0.83 423	9.99 539	40
21	9.16 203	86	9.16 665	88	0.83 335	9.99 537	39
22	9.16 289	85	9.16 753	88	0.83 247	9.99 535	38
23	9.16 374	86	9.16 841	87	0.83 159	9.99 533	37
24	9.16 460	85	9.16 928	88	0.83 072	9.99 532	36
25	9.16 545	86	9.17 016	87	0.82 984	9.99 530	35
26	9.16 631	85	9.17 103	87	0.82 897	9.99 528	34
27	9.16 716	85	9.17 190	87	0.82 810	9.99 526	33
28	9.16 801	85	9.17 277	86	0.82 723	9.99 524	32
29	9.16 886	84	9.17 363	87	0.82 637	9.99 522	31
30	9.16 970	85	9.17 450	86	0.82 550	9.99 520	30
31	9.17 055	84	9.17 536	86	0.82 464	9.99 518	29
32	9.17 139	84	9.17 622	86	0.82 378	9.99 517	28
33	9.17 223	84	9.17 708	86	0.82 292	9.99 515	27
34	9.17 307	84	9.17 794	86	0.82 206	9.99 513	26
35	9.17 391	83	9.17 880	85	0.82 120	9.99 511	25
36	9.17 474	84	9.17 965	86	0.82 035	9.99 509	24
37	9.17 558	83	9.18 051	85	0.81 949	9.99 507	23
38	9.17 641	83	9.18 136	85	0.81 864	9.99 505	22
39	9.17 724	83	9.18 221	85	0.81 779	9.99 503	21
40	9.17 807	83	9.18 306	85	0.81 694	9.99 501	20
41	9.17 890	83	9.18 391	84	0.81 609	9.99 499	19
42	9.17 973	82	9.18 475	85	0.81 525	9.99 497	18
43	9.18 055	82	9.18 560	84	0.81 440	9.99 495	17
44	9.18 137	83	9.18 644	84	0.81 356	9.99 494	16
45	9.18 220	82	9.18 728	84	0.81 272	9.99 492	15
46	9.18 302	81	9.18 812	84	0.81 188	9.99 490	14
47	9.18 383	82	9.18 896	83	0.81 104	9.99 488	13
48	9.18 465	82	9.18 979	84	0.81 021	9.99 486	12
49	9.18 547	81	9.19 063	83	0.80 937	9.99 484	11
50	9.18 628	81	9.19 146	83	0.80 854	9.99 482	10
51	9.18 709	81	9.19 229	83	0.80 771	9.99 480	9
52	9.18 790	81	9.19 312	83	0.80 688	9.99 478	8
53	9.18 871	81	9.19 395	83	0.80 605	9.99 476	7
54	9.18 952	81	9.19 478	83	0.80 522	9.99 474	6
55	9.19 033	80	9.19 561	82	0.80 439	9.99 472	5
56	9.19 113	80	9.19 643	82	0.80 357	9.99 470	4
57	9.19 193	80	9.19 725	82	0.80 275	9.99 468	3
58	9.19 273	80	9.19 807	82	0.80 193	9.99 466	2
59	9.19 353	80	9.19 889	82	0.80 111	9.99 464	1
60	9.19 433		9.19 971		0.80 029	9.99 462	0
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	′

P. P.

	92	91	90
1	9.2	9.1	9.0
2	18.4	18.2	18.0
3	27.6	27.3	27.0
4	36.8	36.4	36.0
5	46.0	45.5	45.0
6	55.2	54.6	54.0
7	64.4	63.7	63.0
8	73.6	72.8	72.0
9	82.8	81.9	81.0

	89	88
1	8.9	8.8
2	17.8	17.6
3	26.7	26.4
4	35.6	35.2
5	44.5	44.0
6	53.4	52.8
7	62.3	61.6
8	71.2	70.4
9	80.1	79.2

	87	86	85
1	8.7	8.6	8.5
2	17.4	17.2	17.0
3	26.1	25.8	25.5
4	34.8	34.4	34.0
5	43.5	43.0	42.5
6	52.2	51.6	51.0
7	60.9	60.2	59.5
8	69.6	68.8	68.0
9	78.3	77.4	76.5

	84	83
1	8.4	8.3
2	16.8	16.6
3	25.2	24.9
4	33.6	33.2
5	42.0	41.5
6	50.4	49.8
7	58.8	58.1
8	67.2	66.4
9	75.6	74.7

	82	81	80
1	8.2	8.1	8.0
2	16.4	16.2	16.0
3	24.6	24.3	24.0
4	32.8	32.4	32.0
5	41.0	40.5	40.0
6	49.2	48.6	48.0
7	57.4	56.7	56.0
8	65.6	64.8	64.0
9	73.8	72.9	72.0

P. P.

81°

′	L. Sin.	d.	L. Tan.	c.d.	L. Cot.	L. Cos.	
0	9.19 433	80	9.19 971	82	0.80 029	9.99 462	60
1	9.19 513	79	9.20 053	81	0.79 947	9.99 460	59
2	9.19 592	80	9.20 134	82	0.79 866	9.99 458	58
3	9.19 672	79	9.20 216	81	0.79 784	9.99 456	57
4	9.19 751	79	9.20 297	81	0.79 703	9.99 454	56
5	9.19 830	79	9.20 378	81	0.79 622	9.99 452	55
6	9.19 909	79	9.20 459	81	0.79 541	9.99 450	54
7	9.19 988	79	9.20 540	81	0.79 460	9.99 448	53
8	9.20 067	78	9.20 621	80	0.79 379	9.99 446	52
9	9.20 145	78	9.20 701	81	0.79 299	9.99 444	51
10	9.20 223	79	9.20 782	80	0.79 218	9.99 442	50
11	9.20 302	78	9.20 862	80	0.79 138	9.99 440	49
12	9.20 380	78	9.20 942	80	0.79 058	9.99 438	48
13	9.20 458	77	9.21 022	80	0.78 978	9.99 436	47
14	9.20 535	78	9.21 102	80	0.78 898	9.99 434	46
15	9.20 613	78	9.21 182	79	0.78 818	9.99 432	45
16	9.20 691	77	9.21 261	80	0.78 739	9.99 429	44
17	9.20 768	77	9.21 341	79	0.78 659	9.99 427	43
18	9.20 845	77	9.21 420	79	0.78 580	9.99 425	42
19	9.20 922	77	9.21 499	79	0.78 501	9.99 423	41
20	9.20 999	77	9.21 578	79	0.78 422	9.99 421	40
21	9.21 076	77	9.21 657	79	0.78 343	9.99 419	39
22	9.21 153	76	9.21 736	78	0.78 264	9.99 417	38
23	9.21 229	77	9.21 814	79	0.78 186	9.99 415	37
24	9.21 306	76	9.21 893	78	0.78 107	9.99 413	36
25	9.21 382	76	9.21 971	78	0.78 029	9.99 411	35
26	9.21 458	76	9.22 049	78	0.77 951	9.99 409	34
27	9.21 534	76	9.22 127	78	0.77 873	9.99 407	33
28	9.21 610	75	9.22 205	78	0.77 795	9.99 404	32
29	9.21 685	76	9.22 283	78	0.77 717	9.99 402	31
30	9.21 761	75	9.22 361	77	0.77 639	9.99 400	30
31	9.21 836	76	9.22 438	78	0.77 562	9.99 398	29
32	9.21 912	75	9.22 516	77	0.77 484	9.99 396	28
33	9.21 987	75	9.22 593	77	0.77 407	9.99 394	27
34	9.22 062	75	9.22 670	77	0.77 330	9.99 392	26
35	9.22 137	74	9.22 747	77	0.77 253	9.99 390	25
36	9.22 211	75	9.22 824	77	0.77 176	9.99 388	24
37	9.22 286	75	9.22 901	76	0.77 099	9.99 385	23
38	9.22 361	74	9.22 977	77	0.77 023	9.99 383	22
39	9.22 435	74	9.23 054	76	0.76 946	9.99 381	21
40	9.22 509	74	9.23 130	76	0.76 870	9.99 379	20
41	9.22 583	74	9.23 206	77	0.76 794	9.99 377	19
42	9.22 657	74	9.23 283	76	0.76 717	9.99 375	18
43	9.22 731	74	9.23 359	76	0.76 641	9.99 372	17
44	9.22 805	73	9.23 435	75	0.76 565	9.99 370	16
45	9.22 878	74	9.23 510	76	0.76 490	9.99 368	15
46	9.22 952	73	9.23 586	75	0.76 414	9.99 366	14
47	9.23 025	73	9.23 661	76	0.76 339	9.99 364	13
48	9.23 098	73	9.23 737	75	0.76 263	9.99 362	12
49	9.23 171	73	9.23 812	75	0.76 188	9.99 359	11
50	9.23 244	73	9.23 887	75	0.76 113	9.99 357	10
51	9.23 317	73	9.23 962	75	0.76 038	9.99 355	9
52	9.23 390	72	9.24 037	75	0.75 963	9.99 353	8
53	9.23 462	73	9.24 112	74	0.75 888	9.99 351	7
54	9.23 535	72	9.24 186	75	0.75 814	9.99 348	6
55	9.23 607	72	9.24 261	74	0.75 739	9.99 346	5
56	9.23 679	73	9.24 335	75	0.75 665	9.99 344	4
57	9.23 752	71	9.24 410	74	0.75 590	9.99 342	3
58	9.23 823	72	9.24 484	74	0.75 516	9.99 340	2
59	9.23 895	72	9.24 558	74	0.75 442	9.99 337	1
60	9.23 967		9.24 632		0.75 368	9.99 335	0
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	′

P. P.

	82	81	80
1	8.2	8.1	8.0
2	16.4	16.2	16.0
3	24.6	24.3	24.0
4	32.8	32.4	32.0
5	41.0	40.5	40.0
6	49.2	48.6	48.0
7	57.4	56.7	56.0
8	65.6	64.8	64.0
9	73.8	72.9	72.0

	79	78	77
1	7.9	7.8	7.7
2	15.8	15.6	15.4
3	23.7	23.4	23.1
4	31.6	31.2	30.8
5	39.5	39.0	38.5
6	47.4	46.8	46.2
7	55.3	54.6	53.9
8	63.2	62.4	61.6
9	71.1	70.2	69.3

	76	75	74
1	7.6	7.5	7.4
2	15.2	15.0	14.8
3	22.8	22.5	22.2
4	30.4	30.0	29.6
5	38.0	37.5	37.0
6	45.6	45.0	44.4
7	53.2	52.5	51.8
8	60.8	60.0	59.2
9	68.4	67.5	66.6

	73	72	71
1	7.3	7.2	7.1
2	14.6	14.4	14.2
3	21.9	21.6	21.3
4	29.2	28.8	28.4
5	36.5	36.0	35.5
6	43.8	43.2	42.6
7	51.1	50.4	49.7
8	58.4	57.6	56.8
9	65.7	64.8	63.9

	3/79	3/78	3/77
0	13.2	13.0	12.8
1	39.5	39.0	38.5
2	65.8	65.0	64.2
3			

	3/76	3/75	3/74
0	12.7	12.5	12.3
1	38.0	37.5	37.0
2	63.3	62.5	61.7
3			

P. P.

10°

′	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.	
0	9.23 967	72	9.24 632	74	0.75 368	9.99 335	2	60
1	9.24 039	71	9.24 706	73	0.75 294	9.99 333	2	59
2	9.24 110	71	9.24 779	74	0.75 221	9.99 331	3	58
3	9.24 181	72	9.24 853	73	0.75 147	9.99 328	2	57
4	9.24 253	71	9.24 926	74	0.75 074	9.99 326	2	56
5	9.24 324	71	9.25 000	73	0.75 000	9.99 324	2	55
6	9.24 395	71	9.25 073	73	0.74 927	9.99 322	3	54
7	9.24 466	70	9.25 146	73	0.74 854	9.99 319	2	53
8	9.24 536	71	9.25 219	73	0.74 781	9.99 317	2	52
9	9.24 607	70	9.25 292	73	0.74 708	9.99 315	2	51
10	9.24 677	71	9.25 365	72	0.74 635	9.99 313	3	50
11	9.24 748	70	9.25 437	73	0.74 563	9.99 310	2	49
12	9.24 818	70	9.25 510	72	0.74 490	9.99 308	2	48
13	9.24 888	70	9.25 582	73	0.74 418	9.99 306	2	47
14	9.24 958	70	9.25 655	72	0.74 345	9.99 304	3	46
15	9.25 028	70	9.25 727	72	0.74 273	9.99 301	2	45
16	9.25 098	70	9.25 799	72	0.74 201	9.99 299	2	44
17	9.25 168	69	9.25 871	72	0.74 129	9.99 297	3	43
18	9.25 237	70	9.25 943	72	0.74 057	9.99 294	2	42
19	9.25 307	69	9.26 015	71	0.73 985	9.99 292	2	41
20	9.25 376	69	9.26 086	72	0.73 914	9.99 290	2	40
21	9.25 445	69	9.26 158	71	0.73 842	9.99 288	3	39
22	9.25 514	69	9.26 229	72	0.73 771	9.99 285	2	38
23	9.25 583	69	9.26 301	71	0.73 699	9.99 283	2	37
24	9.25 652	69	9.26 372	71	0.73 628	9.99 281	3	36
25	9.25 721	69	9.26 443	71	0.73 557	9.99 278	2	35
26	9.25 790	68	9.26 514	71	0.73 486	9.99 276	2	34
27	9.25 858	69	9.26 585	70	0.73 415	9.99 274	3	33
28	9.25 927	68	9.26 655	71	0.73 345	9.99 271	2	32
29	9.25 995	68	9.26 726	71	0.73 274	9.99 269	2	31
30	9.26 063	68	9.26 797	70	0.73 203	9.99 267	3	30
31	9.26 131	68	9.26 867	70	0.73 133	9.99 264	2	29
32	9.26 199	68	9.26 937	71	0.73 063	9.99 262	2	28
33	9.26 267	68	9.27 008	70	0.72 992	9.99 260	3	27
34	9.26 335	68	9.27 078	70	0.72 922	9.99 257	2	26
35	9.26 403	67	9.27 148	70	0.72 852	9.99 255	3	25
36	9.26 470	68	9.27 218	70	0.72 782	9.99 252	2	24
37	9.26 538	67	9.27 288	69	0.72 712	9.99 250	2	23
38	9.26 605	67	9.27 357	70	0.72 643	9.99 248	3	22
39	9.26 672	67	9.27 427	69	0.72 573	9.99 245	2	21
40	9.26 739	67	9.27 496	70	0.72 504	9.99 243	2	20
41	9.26 806	67	9.27 566	69	0.72 434	9.99 241	3	19
42	9.26 873	67	9.27 635	69	0.72 365	9.99 238	2	18
43	9.26 940	67	9.27 704	69	0.72 296	9.99 236	3	17
44	9.27 007	66	9.27 773	69	0.72 227	9.99 233	2	16
45	9.27 073	67	9.27 842	69	0.72 158	9.99 231	2	15
46	9.27 140	66	9.27 911	69	0.72 089	9.99 229	3	14
47	9.27 206	67	9.27 980	69	0.72 020	9.99 226	2	13
48	9.27 273	66	9.28 049	68	0.71 951	9.99 224	3	12
49	9.27 339	66	9.28 117	69	0.71 883	9.99 221	2	11
50	9.27 405	66	9.28 186	68	0.71 814	9.99 219	2	10
51	9.27 471	66	9.28 254	69	0.71 746	9.99 217	3	9
52	9.27 537	65	9.28 323	68	0.71 677	9.99 214	2	8
53	9.27 602	66	9.28 391	68	0.71 609	9.99 212	3	7
54	9.27 668	66	9.28 459	68	0.71 541	9.99 209	2	6
55	9.27 734	65	9.28 527	68	0.71 473	9.99 207	3	5
56	9.27 799	65	9.28 595	67	0.71 405	9.99 204	2	4
57	9.27 864	66	9.28 662	68	0.71 338	9.99 202	2	3
58	9.27 930	65	9.28 730	68	0.71 270	9.99 200	3	2
59	9.27 995	65	9.28 798	67	0.71 202	9.99 197	2	1
60	9.28 060		9.28 865		0.71 135	9.99 195		0
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.	′

P. P.

	74	73	72
1	7.4	7.3	7.2
2	14.8	14.6	14.4
3	22.2	21.9	21.6
4	29.6	29.2	28.8
5	37.0	36.5	36.0
6	44.4	43.8	43.2
7	51.8	51.1	50.4
8	59.2	58.4	57.6
9	66.6	65.7	64.8

	71	70	69
1	7.1	7.0	6.9
2	14.2	14.0	13.8
3	21.3	21.0	20.7
4	28.4	28.0	27.6
5	35.5	35.0	34.5
6	42.6	42.0	41.4
7	49.7	49.0	48.3
8	56.8	56.0	55.2
9	63.9	63.0	62.1

	68	67	66
1	6.8	6.7	6.6
2	13.6	13.4	13.2
3	20.4	20.1	19.8
4	27.2	26.8	26.4
5	34.0	33.5	33.0
6	40.8	40.2	39.6
7	47.6	46.9	46.2
8	54.4	53.6	52.8
9	61.2	60.3	59.4

	65	3
1	6.5	0.3
2	13.0	0.6
3	19.5	0.9
4	26.0	1.2
5	32.5	1.5
6	39.0	1.8
7	45.5	2.1
8	52.0	2.4
9	58.5	2.7

	3/74	3/73	3/72
0			
1	12.3	12.2	12.0
2	37.0	36.5	36.0
3	61.7	60.8	60.0

	3/71	3/70	3/69	3/68
0				
1	11.8	11.7	11.5	11.3
2	35.5	35.0	34.5	34.0
3	59.2	58.3	57.5	56.7

79°

′	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.	′
0	9.28 060	65	9.28 865	68	0.71 135	9.99 195	3	60
1	9.28 125	65	9.28 933	67	0.71 067	9.99 192	2	59
2	9.28 190	64	9.29 000	67	0.71 000	9.99 190	3	58
3	9.28 254	65	9.29 067	67	0.70 933	9.99 187	2	57
4	9.28 319	65	9.29 134	67	0.70 866	9.99 185	3	56
5	9.28 384	64	9.29 201	67	0.70 799	9.99 182	2	55
6	9.28 448	64	9.29 268	67	0.70 732	9.99 180	3	54
7	9.28 512	65	9.29 335	67	0.70 665	9.99 177	2	53
8	9.28 577	64	9.29 402	66	0.70 598	9.99 175	3	52
9	9.28 641	64	9.29 468	67	0.70 532	9.99 172	2	51
10	9.28 705	64	9.29 535	66	0.70 465	9.99 170	3	50
11	9.28 769	64	9.29 601	67	0.70 399	9.99 167	2	49
12	9.28 833	63	9.29 668	66	0.70 332	9.99 165	3	48
13	9.28 896	64	9.29 734	66	0.70 266	9.99 162	2	47
14	9.28 960	64	9.29 800	66	0.70 200	9.99 160	3	46
15	9.29 024	63	9.29 866	66	0.70 134	9.99 157	2	45
16	9.29 087	63	9.29 932	66	0.70 068	9.99 155	3	44
17	9.29 150	64	9.29 998	66	0.70 002	9.99 152	2	43
18	9.29 214	63	9.30 064	66	0.69 936	9.99 150	3	42
19	9.29 277	63	9.30 130	65	0.69 870	9.99 147	2	41
20	9.29 340	63	9.30 195	66	0.69 805	9.99 145	3	40
21	9.29 403	63	9.30 261	65	0.69 739	9.99 142	2	39
22	9.29 466	63	9.30 326	65	0.69 674	9.99 140	3	38
23	9.29 529	62	9.30 391	66	0.69 609	9.99 137	2	37
24	9.29 591	63	9.30 457	65	0.69 543	9.99 135	3	36
25	9.29 654	62	9.30 522	65	0.69 478	9.99 132	2	35
26	9.29 716	63	9.30 587	65	0.69 413	9.99 130	3	34
27	9.29 779	62	9.30 652	65	0.69 348	9.99 127	3	33
28	9.29 841	62	9.30 717	65	0.69 283	9.99 124	2	32
29	9.29 903	63	9.30 782	64	0.69 218	9.99 122	3	31
30	9.29 966	62	9.30 846	65	0.69 154	9.99 119	2	30
31	9.30 028	62	9.30 911	64	0.69 089	9.99 117	3	29
32	9.30 090	61	9.30 975	65	0.69 025	9.99 114	2	28
33	9.30 151	62	9.31 040	64	0.68 960	9.99 112	3	27
34	9.30 213	62	9.31 104	64	0.68 896	9.99 109	3	26
35	9.30 275	61	9.31 168	65	0.68 832	9.99 106	2	25
36	9.30 336	62	9.31 233	64	0.68 767	9.99 104	3	24
37	9.30 398	61	9.31 297	64	0.68 703	9.99 101	2	23
38	9.30 459	62	9.31 361	64	0.68 639	9.99 099	3	22
39	9.30 521	61	9.31 425	64	0.68 575	9.99 096	3	21
40	9.30 582	61	9.31 489	63	0.68 511	9.99 093	2	20
41	9.30 643	61	9.31 552	64	0.68 448	9.99 091	3	19
42	9.30 704	61	9.31 616	63	0.68 384	9.99 088	2	18
43	9.30 765	61	9.31 679	64	0.68 321	9.99 086	3	17
44	9.30 826	61	9.31 743	63	0.68 257	9.99 083	3	16
45	9.30 887	60	9.31 806	64	0.68 194	9.99 080	2	15
46	9.30 947	61	9.31 870	63	0.68 130	9.99 078	3	14
47	9.31 008	60	9.31 933	63	0.68 067	9.99 075	3	13
48	9.31 068	61	9.31 996	63	0.68 004	9.99 072	2	12
49	9.31 129	60	9.32 059	63	0.67 941	9.99 070	3	11
50	9.31 189	61	9.32 122	63	0.67 878	9.99 067	3	10
51	9.31 250	60	9.32 185	63	0.67 815	9.99 064	2	9
52	9.31 310	60	9.32 248	63	0.67 752	9.99 062	3	8
53	9.31 370	60	9.32 311	62	0.67 689	9.99 059	3	7
54	9.31 430	60	9.32 373	63	0.67 627	9.99 056	2	6
55	9.31 490	59	9.32 436	62	0.67 564	9.99 054	3	5
56	9.31 549	60	9.32 498	63	0.67 502	9.99 051	3	4
57	9.31 609	60	9.32 561	62	0.67 439	9.99 048	2	3
58	9.31 669	59	9.32 623	62	0.67 377	9.99 046	3	2
59	9.31 728	60	9.32 685	62	0.67 315	9.99 043	3	1
60	9.31 788		9.32 747		0.67 253	9.99 040		0
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.	′

P. P.

	68	67	66
1	6.8	6.7	6.6
2	13.6	13.4	13.2
3	20.4	20.1	19.8
4	27.2	26.8	26.4
5	34.0	33.5	33.0
6	40.8	40.2	39.6
7	47.6	46.9	46.2
8	54.4	53.6	52.8
9	61.2	60.3	59.4

	65	64	63
1	6.5	6.4	6.3
2	13.0	12.8	12.6
3	19.5	19.2	18.9
4	26.0	25.6	25.2
5	32.5	32.0	31.5
6	39.0	38.4	37.8
7	45.5	44.8	44.1
8	52.0	51.2	50.4
9	58.5	57.6	56.7

	62	61	60
1	6.2	6.1	6.0
2	12.4	12.2	12.0
3	18.6	18.3	18.0
4	24.8	24.4	24.0
5	31.0	30.5	30.0
6	37.2	36.6	36.0
7	43.4	42.7	42.0
8	49.6	48.8	48.0
9	55.8	54.9	54.0

	59	3
1	5.9	0.3
2	11.8	0.6
3	17.7	0.9
4	23.6	1.2
5	29.5	1.5
6	35.4	1.8
7	41.3	2.1
8	47.2	2.4
9	53.1	2.7

	3/67	3/66	3/65
0			
	11.2	11.0	10.8
1			
	33.5	33.0	32.5
2			
	55.8	55.0	54.2
3			

	3/64	3/63	3/62
0			
	10.7	10.5	10.3
1			
	32.0	31.5	31.0
2			
	53.3	52.5	51.7
3			

P. P.

′	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.	
0	9.31 788	59	9.32 747	63	0.67 253	9.99 040	2	**60**
1	9.31 847	60	9.32 810	62	0.67 190	9.99 038	3	59
2	9.31 907	59	9.32 872	61	0.67 128	9.99 035	3	58
3	9.31 966	59	9.32 933	62	0.67 067	9.99 032	2	57
4	9.32 025	59	9.32 995	62	0.67 005	9.99 030	3	56
5	9.32 084	59	9.33 057	62	0.66 943	9.99 027	3	55
6	9.32 143	59	9.33 119	61	0.66 881	9.99 024	2	54
7	9.32 202	59	9.33 180	62	0.66 820	9.99 022	3	53
8	9.32 261	58	9.33 242	61	0.66 758	9.99 019	3	52
9	9.32 319	59	9.33 303	62	0.66 697	9.99 016	3	51
10	9.32 378	59	9.33 365	61	0.66 635	9.99 013	2	**50**
11	9.32 437	58	9.33 426	61	0.66 574	9.99 011	3	49
12	9.32 495	58	9.33 487	61	0.66 513	9.99 008	3	48
13	9.32 553	59	9.33 548	61	0.66 452	9.99 005	3	47
14	9.32 612	58	9.33 609	61	0.66 391	9.99 002	2	46
15	9.32 670	58	9.33 670	61	0.66 330	9.99 000	3	45
16	9.32 728	58	9.33 731	61	0.66 269	9.98 997	3	44
17	9.32 786	58	9.33 792	61	0.66 208	9.98 994	3	43
18	9.32 844	58	9.33 853	60	0.66 147	9.98 991	2	42
19	9.32 902	58	9.33 913	61	0.66 087	9.98 989	3	41
20	9.32 960	58	9.33 974	60	0.66 026	9.98 986	3	**40**
21	9.33 018	57	9.34 034	61	0.65 966	9.98 983	3	39
22	9.33 075	58	9.34 095	60	0.65 905	9.98 980	2	38
23	9.33 133	57	9.34 155	60	0.65 845	9.98 978	3	37
24	9.33 190	58	9.34 215	61	0.65 785	9.98 975	3	36
25	9.33 248	57	9.34 276	60	0.65 724	9.98 972	3	35
26	9.33 305	57	9.34 336	60	0.65 664	9.98 969	2	34
27	9.33 362	58	9.34 396	60	0.65 604	9.98 967	3	33
28	9.33 420	57	9.34 456	60	0.65 544	9.98 964	3	32
29	9.33 477	57	9.34 516	60	0.65 484	9.98 961	3	31
30	9.33 534	57	9.34 576	59	0.65 424	9.98 958	3	**30**
31	9.33 591	56	9.34 635	60	0.65 365	9.98 955	2	29
32	9.33 647	57	9.34 695	60	0.65 305	9.98 953	3	28
33	9.33 704	57	9.34 755	59	0.65 245	9.98 950	3	27
34	9.33 761	57	9.34 814	60	0.65 186	9.98 947	3	26
35	9.33 818	56	9.34 874	59	0.65 126	9.98 944	3	25
36	9.33 874	57	9.34 933	59	0.65 067	9.98 941	3	24
37	9.33 931	56	9.34 992	59	0.65 008	9.98 938	2	23
38	9.33 987	56	9.35 051	60	0.64 949	9.98 936	3	22
39	9.34 043	57	9.35 111	59	0.64 889	9.98 933	3	21
40	9.34 100	56	9.35 170	59	0.64 830	9.98 930	3	**20**
41	9.34 156	56	9.35 229	59	0.64 771	9.98 927	3	19
42	9.34 212	56	9.35 288	59	0.64 712	9.98 924	3	18
43	9.34 268	56	9.35 347	58	0.64 653	9.98 921	2	17
44	9.34 324	56	9.35 405	59	0.64 595	9.98 919	3	16
45	9.34 380	56	9.35 464	59	0.64 536	9.98 916	3	15
46	9.34 436	55	9.35 523	58	0.64 477	9.98 913	3	14
47	9.34 491	56	9.35 581	59	0.64 419	9.98 910	3	13
48	9.34 547	55	9.35 640	58	0.64 360	9.98 907	3	12
49	9.34 602	56	9.35 698	59	0.64 302	9.98 904	3	11
50	9.34 658	55	9.35 757	58	0.64 243	9.98 901	3	**10**
51	9.34 713	56	9.35 815	58	0.64 185	9.98 898	2	9
52	9.34 769	55	9.35 873	58	0.64 127	9.98 896	3	8
53	9.34 824	55	9.35 931	58	0.64 069	9.98 893	3	7
54	9.34 879	55	9.35 989	58	0.64 011	9.98 890	3	6
55	9.34 934	55	9.36 047	58	0.63 953	9.98 887	3	5
56	9.34 989	55	9.36 105	58	0.63 895	9.98 884	3	4
57	9.35 044	55	9.36 163	58	0.63 837	9.98 881	3	3
58	9.35 099	55	9.36 221	58	0.63 779	9.98 878	3	2
59	9.35 154	55	9.36 279	57	0.63 721	9.98 875	3	1
60	9.35 209		9.36 336		0.63 664	9.98 872		**0**
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.	′

P. P.

	63	62	61
1	6.3	6.2	6.1
2	12.6	12.4	12.2
3	18.9	18.6	18.3
4	25.2	24.8	24.4
5	31.5	31.0	30.5
6	37.8	37.2	36.6
7	44.1	43.4	42.7
8	50.4	49.6	48.8
9	56.7	55.8	54.9

	60	59
1	6.0	5.9
2	12.0	11.8
3	18.0	17.7
4	24.0	23.6
5	30.0	29.5
6	36.0	35.4
7	42.0	41.3
8	48.0	47.2
9	54.0	53.1

	58	57
1	5.8	5.7
2	11.6	11.4
3	17.4	17.1
4	23.2	22.8
5	29.0	28.5
6	34.8	34.2
7	40.6	39.9
8	46.4	45.6
9	52.2	51.3

	56	55	3
1	5.6	5.5	0.3
2	11.2	11.0	0.6
3	16.8	16.5	0.9
4	22.4	22.0	1.2
5	28.0	27.5	1.5
6	33.6	33.0	1.8
7	39.2	38.5	2.1
8	44.8	44.0	2.4
9	50.4	49.5	2.7

	3/62	3/61	3/60
0			
1	10.3	10.2	10.0
2	31.0	30.5	30.0
3	51.7	50.8	50.0

	3/59	3/58	3/57
0			
1	9.8	9.7	9.5
2	29.5	29.0	28.5
3	49.2	48.3	47.5

′	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.	′
0	9.35 209	54	9.36 336	58	0.63 664	9.98 872	3	60
1	9.35 263	55	9.36 394	58	0.63 606	9.98 869	2	59
2	9.35 318	55	9.36 452	57	0.63 548	9.98 867	3	58
3	9.35 373	54	9.36 509	57	0.63 491	9.98 864	3	57
4	9.35 427	54	9.36 566	58	0.63 434	9.98 861	3	56
5	9.35 481	55	9.36 624	57	0.63 376	9.98 858	3	55
6	9.35 536	54	9.36 681	57	0.63 319	9.98 855	3	54
7	9.35 590	54	9.36 738	57	0.63 262	9.98 852	3	53
8	9.35 644	54	9.36 795	57	0.63 205	9.98 849	3	52
9	9.35 698	54	9.36 852	57	0.63 148	9.98 846	3	51
10	9.35 752	54	9.36 909	57	0.63 091	9.98 843	3	50
11	9.35 806	54	9.36 966	57	0.63 034	9.98 840	3	49
12	9.35 860	54	9.37 023	57	0.62 977	9.98 837	3	48
13	9.35 914	54	9.37 080	57	0.62 920	9.98 834	3	47
14	9.35 968	54	9.37 137	56	0.62 863	9.98 831	3	46
15	9.36 022	53	9.37 193	57	0.62 807	9.98 828	3	45
16	9.36 075	54	9.37 250	56	0.62 750	9.98 825	3	44
17	9.36 129	53	9.37 306	57	0.62 694	9.98 822	3	43
18	9.36 182	54	9.37 363	56	0.62 637	9.98 819	3	42
19	9.36 236	53	9.37 419	57	0.62 581	9.98 816	3	41
20	9.36 289	53	9.37 476	56	0.62 524	9.98 813	3	40
21	9.36 342	53	9.37 532	56	0.62 468	9.98 810	3	39
22	9.36 395	54	9.37 588	56	0.62 412	9.98 807	3	38
23	9.36 449	53	9.37 644	56	0.62 356	9.98 804	3	37
24	9.36 502	53	9.37 700	56	0.62 300	9.98 801	3	36
25	9.36 555	53	9.37 756	56	0.62 244	9.98 798	3	35
26	9.36 608	52	9.37 812	56	0.62 188	9.98 795	3	34
27	9.36 660	53	9.37 868	56	0.62 132	9.98 792	3	33
28	9.36 713	53	9.37 924	56	0.62 076	9.98 789	3	32
29	9.36 766	53	9.37 980	55	0.62 020	9.98 786	3	31
30	9.36 819	52	9.38 035	56	0.61 965	9.98 783	3	30
31	9.36 871	53	9.38 091	56	0.61 909	9.98 780	3	29
32	9.36 924	52	9.38 147	55	0.61 853	9.98 777	3	28
33	9.36 976	52	9.38 202	55	0.61 798	9.98 774	3	27
34	9.37 028	53	9.38 257	56	0.61 743	9.98 771	3	26
35	9.37 081	52	9.38 313	55	0.61 687	9.98 768	3	25
36	9.37 133	52	9.38 368	55	0.61 632	9.98 765	3	24
37	9.37 185	52	9.38 423	56	0.61 577	9.98 762	3	23
38	9.37 237	52	9.38 479	55	0.61 521	9.98 759	3	22
39	9.37 289	52	9.38 534	55	0.61 466	9.98 756	3	21
40	9.37 341	52	9.38 589	55	0.61 411	9.98 753	3	20
41	9.37 393	52	9.38 644	55	0.61 356	9.98 750	4	19
42	9.37 445	52	9.38 699	55	0.61 301	9.98 746	3	18
43	9.37 497	52	9.38 754	54	0.61 246	9.98 743	3	17
44	9.37 549	51	9.38 808	55	0.61 192	9.98 740	3	16
45	9.37 600	52	9.38 863	55	0.61 137	9.98 737	3	15
46	9.37 652	51	9.38 918	54	0.61 082	9.98 734	3	14
47	9.37 703	52	9.38 972	55	0.61 028	9.98 731	3	13
48	9.37 755	51	9.39 027	55	0.60 973	9.98 728	3	12
49	9.37 806	52	9.39 082	54	0.60 918	9.98 725	3	11
50	9.37 858	51	9.39 136	54	0.60 864	9.98 722	3	10
51	9.37 909	51	9.39 190	55	0.60 810	9.98 719	4	9
52	9.37 960	51	9.39 245	54	0.60 755	9.98 715	3	8
53	9.38 011	51	9.39 299	54	0.60 701	9.98 712	3	7
54	9.38 062	51	9.39 353	54	0.60 647	9.98 709	3	6
55	9.38 113	51	9.39 407	54	0.60 593	9.98 706	3	5
56	9.38 164	51	9.39 461	54	0.60 539	9.98 703	3	4
57	9.38 215	51	9.39 515	54	0.60 485	9.98 700	3	3
58	9.38 266	51	9.39 569	54	0.60 431	9.98 697	3	2
59	9.38 317	51	9.39 623	54	0.60 377	9.98 694	4	1
60	9.38 368		9.39 677		0.60 323	9.98 690		0
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.	′

P. P.

	58	57	56
1	5.8	5.7	5.6
2	11.6	11.4	11.2
3	17.4	17.1	16.8
4	23.2	22.8	22.4
5	29.0	28.5	28.0
6	34.8	34.2	33.6
7	40.6	39.9	39.2
8	46.4	45.6	44.8
9	52.2	51.3	50.4

	55	54	53
1	5.5	5.4	5.3
2	11.0	10.8	10.6
3	16.5	16.2	15.9
4	22.0	21.6	21.2
5	27.5	27.0	26.5
6	33.0	32.4	31.8
7	38.5	37.8	37.1
8	44.0	43.2	42.4
9	49.5	48.6	47.7

	52	51
1	5.2	5.1
2	10.4	10.2
3	15.6	15.3
4	20.8	20.4
5	26.0	25.5
6	31.2	30.6
7	36.4	35.7
8	41.6	40.8
9	46.8	45.9

	4	3
1	0.4	0.3
2	0.8	0.6
3	1.2	0.9
4	1.6	1.2
5	2.0	1.5
6	2.4	1.8
7	2.8	2.1
8	3.2	2.4
9	3.6	2.7

	4/55	4/54	3/58	3/57
0	6.9	6.8	9.7	9.5
1	20.6	20.2	29.0	28.5
2	34.4	33.8	48.3	47.5
3	48.1	47.2	—	—
4				

	3/56	3/55	3/54
0	9.3	9.2	9.0
1	28.0	27.5	27.0
2	46.7	45.8	45.0
3			

P. P.

14°

′	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.	
0	9.38 368	50	9.39 677	54	0.60 323	9.98 690	3	60
1	9.38 418	51	9.39 731	54	0.60 269	9.98 687	3	59
2	9.38 469	50	9.39 785	53	0.60 215	9.98 684	3	58
3	9.38 519	51	9.39 838	54	0.60 162	9.98 681	3	57
4	9.38 570	50	9.39 892	53	0.60 108	9.98 678	3	56
5	9.38 620	50	9.39 945	54	0.60 055	9.98 675	4	55
6	9.38 670	51	9.39 999	53	0.60 001	9.98 671	3	54
7	9.38 721	50	9.40 052	54	0.59 948	9.98 668	3	53
8	9.38 771	50	9.40 106	53	0.59 894	9.98 665	3	52
9	9.38 821	50	9.40 159	53	0.59 841	9.98 662	3	51
10	9.38 871	50	9.40 212	54	0.59 788	9.98 659	3	50
11	9.38 921	50	9.40 266	53	0.59 734	9.98 656	4	49
12	9.38 971	50	9.40 319	53	0.59 681	9.98 652	3	48
13	9.39 021	50	9.40 372	53	0.59 628	9.98 649	3	47
14	9.39 071	50	9.40 425	53	0.59 575	9.98 646	3	46
15	9.39 121	49	9.40 478	53	0.59 522	9.98 643	3	45
16	9.39 170	50	9.40 531	53	0.59 469	9.98 640	4	44
17	9.39 220	50	9.40 584	52	0.59 416	9.98 636	3	43
18	9.39 270	49	9.40 636	53	0.59 364	9.98 633	3	42
19	9.39 319	50	9.40 689	53	0.59 311	9.98 630	3	41
20	9.39 369	49	9.40 742	53	0.59 258	9.98 627	4	40
21	9.39 418	49	9.40 795	52	0.59 205	9.98 623	3	39
22	9.39 467	50	9.40 847	53	0.59 153	9.98 620	3	38
23	9.39 517	49	9.40 900	52	0.59 100	9.98 617	3	37
24	9.39 566	49	9.40 952	53	0.59 048	9.98 614	4	36
25	9.39 615	49	9.41 005	52	0.58 995	9.98 610	3	35
26	9.39 664	49	9.41 057	52	0.58 943	9.98 607	3	34
27	9.39 713	49	9.41 109	52	0.58 891	9.98 604	3	33
28	9.39 762	49	9.41 161	53	0.58 839	9.98 601	4	32
29	9.39 811	49	9.41 214	52	0.58 786	9.98 597	3	31
30	9.39 860	49	9.41 266	52	0.58 734	9.98 594	3	30
31	9.39 909	49	9.41 318	52	0.58 682	9.98 591	3	29
32	9.39 958	48	9.41 370	52	0.58 630	9.98 588	4	28
33	9.40 006	49	9.41 422	52	0.58 578	9.98 584	3	27
34	9.40 055	48	9.41 474	52	0.58 526	9.98 581	3	26
35	9.40 103	49	9.41 526	52	0.58 474	9.98 578	4	25
36	9.40 152	48	9.41 578	51	0.58 422	9.98 574	3	24
37	9.40 200	49	9.41 629	52	0.58 371	9.98 571	3	23
38	9.40 249	48	9.41 681	52	0.58 319	9.98 568	3	22
39	9.40 297	49	9.41 733	51	0.58 267	9.98 565	4	21
40	9.40 346	48	9.41 784	52	0.58 216	9.98 561	3	20
41	9.40 394	48	9.41 836	51	0.58 164	9.98 558	3	19
42	9.40 442	48	9.41 887	52	0.58 113	9.98 555	4	18
43	9.40 490	48	9.41 939	51	0.58 061	9.98 551	3	17
44	9.40 538	48	9.41 990	51	0.58 010	9.98 548	3	16
45	9.40 586	48	9.42 041	52	0.57 959	9.98 545	4	15
46	9.40 634	48	9.42 093	51	0.57 907	9.98 541	3	14
47	9.40 682	48	9.42 144	51	0.57 856	9.98 538	3	13
48	9.40 730	48	9.42 195	51	0.57 805	9.98 535	4	12
49	9.40 778	47	9.42 246	51	0.57 754	9.98 531	3	11
50	9.40 825	48	9.42 297	51	0.57 703	9.98 528	3	10
51	9.40 873	48	9.42 348	51	0.57 652	9.98 525	4	9
52	9.40 921	47	9.42 399	51	0.57 601	9.98 521	3	8
53	9.40 968	48	9.42 450	51	0.57 550	9.98 518	3	7
54	9.41 016	47	9.42 501	51	0.57 499	9.98 515	4	6
55	9.41 063	48	9.42 552	51	0.57 448	9.98 511	3	5
56	9.41 111	47	9.42 603	50	0.57 397	9.98 508	3	4
57	9.41 158	47	9.42 653	51	0.57 347	9.98 505	4	3
58	9.41 205	47	9.42 704	51	0.57 296	9.98 501	3	2
59	9.41 252	48	9.42 755	50	0.57 245	9.98 498	4	1
60	9.41 300		9.42 805		0.57 195	9.98 494		0
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.	′

P. P.

	54	53
1	5.4	5.3
2	10.8	10.6
3	16.2	15.9
4	21.6	21.2
5	27.0	26.5
6	32.4	31.8
7	37.8	37.1
8	43.2	42.4
9	48.6	47.7

	52	51	50
1	5.2	5.1	5.0
2	10.4	10.2	10.0
3	15.6	15.3	15.0
4	20.8	20.4	20.0
5	26.0	25.5	25.0
6	31.2	30.6	30.0
7	36.4	35.7	35.0
8	41.6	40.8	40.0
9	46.8	45.9	45.0

	49	48	47
1	4.9	4.8	4.7
2	9.8	9.6	9.4
3	14.7	14.4	14.1
4	19.6	19.2	18.8
5	24.5	24.0	23.5
6	29.4	28.8	28.2
7	34.3	33.6	32.9
8	39.2	38.4	37.6
9	44.1	43.2	42.3

	4	3
1	0.4	0.3
2	0.8	0.6
3	1.2	0.9
4	1.6	1.2
5	2.0	1.5
6	2.4	1.8
7	2.8	2.1
8	3.2	2.4
9	3.6	2.7

	4/54	4/53	4/52	4/51
0	6.8	6.6	6.5	6.4
1	20.2	19.9	19.5	19.1
2	33.8	33.1	32.5	31.9
3	47.2	46.4	45.5	44.6
4				

	3/54	3/53	3/52	3/51
0	9.0	8.8	8.7	8.5
1	27.0	26.5	26.0	25.5
2	45.0	44.2	43.3	42.5
3				

'	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.	
0	9.41 300	47	9.42 805	51	0.57 195	9.98 494	3	60
1	9.41 347	47	9.42 856	50	0.57 144	9.98 491	3	59
2	9.41 394	47	9.42 906	51	0.57 094	9.98 488	4	58
3	9.41 441	47	9.42 957	50	0.57 043	9.98 484	3	57
4	9.41 488	47	9.43 007	50	0.56 993	9.98 481	4	56
5	9.41 535	47	9.43 057	51	0.56 943	9.98 477	3	55
6	9.41 582	46	9.43 108	50	0.56 892	9.98 474	3	54
7	9.41 628	47	9.43 158	50	0.56 842	9.98 471	4	53
8	9.41 675	47	9.43 208	50	0.56 792	9.98 467	3	52
9	9.41 722	46	9.43 258	50	0.56 742	9.98 464	4	51
10	9.41 768	47	9.43 308	50	0.56 692	9.98 460	3	50
11	9.41 815	46	9.43 358	50	0.56 642	9.98 457	4	49
12	9.41 861	47	9.43 408	50	0.56 592	9.98 453	3	48
13	9.41 908	46	9.43 458	50	0.56 542	9.98 450	3	47
14	9.41 954	47	9.43 508	50	0.56 492	9.98 447	4	46
15	9.42 001	46	9.43 558	49	0.56 442	9.98 443	3	45
16	9.42 047	46	9.43 607	50	0.56 393	9.98 440	4	44
17	9.42 093	47	9.43 657	50	0.56 343	9.98 436	3	43
18	9.42 140	46	9.43 707	49	0.56 293	9.98 433	4	42
19	9.42 186	46	9.43 756	50	0.56 244	9.98 429	3	41
20	9.42 232	46	9.43 806	49	0.56 194	9.98 426	4	40
21	9.42 278	46	9.43 855	50	0.56 145	9.98 422	3	39
22	9.42 324	46	9.43 905	49	0.56 095	9.98 419	4	38
23	9.42 370	46	9.43 954	50	0.56 046	9.98 415	3	37
24	9.42 416	45	9.44 004	49	0.55 996	9.98 412	3	36
25	9.42 461	46	9.44 053	49	0.55 947	9.98 409	4	35
26	9.42 507	46	9.44 102	49	0.55 898	9.98 405	3	34
27	9.42 553	46	9.44 151	50	0.55 849	9.98 402	4	33
28	9.42 599	45	9.44 201	49	0.55 799	9.98 398	3	32
29	9.42 644	46	9.44 250	49	0.55 750	9.98 395	4	31
30	9.42 690	45	9.44 299	49	0.55 701	9.98 391	3	30
31	9.42 735	46	9.44 348	49	0.55 652	9.98 388	4	29
32	9.42 781	45	9.44 397	49	0.55 603	9.98 384	3	28
33	9.42 826	46	9.44 446	49	0.55 554	9.98 381	4	27
34	9.42 872	45	9.44 495	49	0.55 505	9.98 377	4	26
35	9.42 917	45	9.44 544	48	0.55 456	9.98 373	3	25
36	9.42 962	46	9.44 592	49	0.55 408	9.98 370	4	24
37	9.43 008	45	9.44 641	49	0.55 359	9.98 366	3	23
38	9.43 053	45	9.44 690	48	0.55 310	9.98 363	4	22
39	9.43 098	45	9.44 738	49	0.55 262	9.98 359	3	21
40	9.43 143	45	9.44 787	49	0.55 213	9.98 356	4	20
41	9.43 188	45	9.44 836	48	0.55 164	9.98 352	3	19
42	9.43 233	45	9.44 884	49	0.55 116	9.98 349	4	18
43	9.43 278	45	9.44 933	48	0.55 067	9.98 345	3	17
44	9.43 323	44	9.44 981	48	0.55 019	9.98 342	4	16
45	9.43 367	45	9.45 029	49	0.54 971	9.98 338	4	15
46	9.43 412	45	9.45 078	48	0.54 922	9.98 334	3	14
47	9.43 457	45	9.45 126	48	0.54 874	9.98 331	4	13
48	9.43 502	44	9.45 174	48	0.54 826	9.98 327	3	12
49	9.43 546	45	9.45 222	49	0.54 778	9.98 324	4	11
50	9.43 591	44	9.45 271	48	0.54 729	9.98 320	3	10
51	9.43 635	45	9.45 319	48	0.54 681	9.98 317	4	9
52	9.43 680	44	9.45 367	48	0.54 633	9.98 313	4	8
53	9.43 724	45	9.45 415	48	0.54 585	9.98 309	3	7
54	9.43 769	44	9.45 463	48	0.54 537	9.98 306	4	6
55	9.43 813	44	9.45 511	48	0.54 489	9.98 302	3	5
56	9.43 857	44	9.45 559	47	0.54 441	9.98 299	4	4
57	9.43 901	45	9.45 606	48	0.54 394	9.98 295	4	3
58	9.43 946	44	9.45 654	48	0.54 346	9.98 291	3	2
59	9.43 990	44	9.45 702	48	0.54 298	9.98 288	4	1
60	9.44 034		9.45 750		0.54 250	9.98 284		0
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.	'

P. P.

	51	50	49
1	5.1	5.0	4.9
2	10.2	10.0	9.8
3	15.3	15.0	14.7
4	20.4	20.0	19.6
5	25.5	25.0	24.5
6	30.6	30.0	29.4
7	35.7	35.0	34.3
8	40.8	40.0	39.2
9	45.9	45.0	44.1

	48	47	46
1	4.8	4.7	4.6
2	9.6	9.4	9.2
3	14.4	14.1	13.8
4	19.2	18.8	18.4
5	24.0	23.5	23.0
6	28.8	28.2	27.6
7	33.6	32.9	32.2
8	38.4	37.6	36.8
9	43.2	42.3	41.4

	45	44
1	4.5	4.4
2	9.0	8.8
3	13.5	13.2
4	18.0	17.6
5	22.5	22.0
6	27.0	26.4
7	31.5	30.8
8	36.0	35.2
9	40.5	39.6

	4	3
1	0.4	0.3
2	0.8	0.6
3	1.2	0.9
4	1.6	1.2
5	2.0	1.5
6	2.4	1.8
7	2.8	2.1
8	3.2	2.4
9	3.6	2.7

	4/50	4/49	4/48	4/47
0	6.2	6.1	6.0	5.9
1	18.8	18.4	18.0	17.6
2	31.2	30.6	30.0	29.4
3	43.8	42.9	42.0	41.1
4				

	3/51	3/50	3/49	3/48
0	8.5	8.3	8.2	8.0
1	25.5	25.0	24.5	24.0
2	42.5	41.7	40.8	40.0
3				

16°

′	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.	
0	9.44 034	44	9.45 750	47	0.54 250	9.98 284	3	60
1	9.44 078	44	9.45 797	48	0.54 203	9.98 281	4	59
2	9.44 122	44	9.45 845	47	0.54 155	9.98 277	4	58
3	9.44 166	44	9.45 892	48	0.54 108	9.98 273	3	57
4	9.44 210	43	9.45 940	47	0.54 060	9.98 270	4	56
5	9.44 253	44	9.45 987	48	0.54 013	9.98 266	4	55
6	9.44 297	44	9.46 035	47	0.53 965	9.98 262	3	54
7	9.44 341	44	9.46 082	48	0.53 918	9.98 259	4	53
8	9.44 385	43	9.46 130	47	0.53 870	9.98 255	4	52
9	9.44 428	44	9.46 177	47	0.53 823	9.98 251	3	51
10	9.44 472	44	9.46 224	47	0.53 776	9.98 248	4	50
11	9.44 516	43	9.46 271	48	0.53 729	9.98 244	4	49
12	9.44 559	43	9.46 319	47	0.53 681	9.98 240	3	48
13	9.44 602	44	9.46 366	47	0.53 634	9.98 237	4	47
14	9.44 646	43	9.46 413	47	0.53 587	9.98 233	4	46
15	9.44 689	44	9.46 460	47	0.53 540	9.98 229	3	45
16	9.44 733	43	9.46 507	47	0.53 493	9.98 226	4	44
17	9.44 776	43	9.46 554	47	0.53 446	9.98 222	4	43
18	9.44 819	43	9.46 601	47	0.53 399	9.98 218	3	42
19	9.44 862	43	9.46 648	46	0.53 352	9.98 215	4	41
20	9.44 905	43	9.46 694	47	0.53 306	9.98 211	4	40
21	9.44 948	44	9.46 741	47	0.53 259	9.98 207	3	39
22	9.44 992	43	9.46 788	47	0.53 212	9.98 204	4	38
23	9.45 035	42	9.46 835	46	0.53 165	9.98 200	4	37
24	9.45 077	43	9.46 881	47	0.53 119	9.98 196	4	36
25	9.45 120	43	9.46 928	47	0.53 072	9.98 192	3	35
26	9.45 163	43	9.46 975	46	0.53 025	9.98 189	4	34
27	9.45 206	43	9.47 021	47	0.52 979	9.98 185	4	33
28	9.45 249	43	9.47 068	46	0.52 932	9.98 181	4	32
29	9.45 292	42	9.47 114	46	0.52 886	9.98 177	3	31
30	9.45 334	43	9.47 160	47	0.52 840	9.98 174	4	30
31	9.45 377	42	9.47 207	46	0.52 793	9.98 170	4	29
32	9.45 419	43	9.47 253	46	0.52 747	9.98 166	4	28
33	9.45 462	42	9.47 299	47	0.52 701	9.98 162	3	27
34	9.45 504	43	9.47 346	46	0.52 654	9.98 159	4	26
35	9.45 547	42	9.47 392	46	0.52 608	9.98 155	4	25
36	9.45 589	43	9.47 438	46	0.52 562	9.98 151	4	24
37	9.45 632	42	9.47 484	46	0.52 516	9.98 147	3	23
38	9.45 674	42	9.47 530	46	0.52 470	9.98 144	4	22
39	9.45 716	42	9.47 576	46	0.52 424	9.98 140	4	21
40	9.45 758	43	9.47 622	46	0.52 378	9.98 136	4	20
41	9.45 801	42	9.47 668	46	0.52 332	9.98 132	3	19
42	9.45 843	42	9.47 714	46	0.52 286	9.98 129	4	18
43	9.45 885	42	9.47 760	46	0.52 240	9.98 125	4	17
44	9.45 927	42	9.47 806	46	0.52 194	9.98 121	4	16
45	9.45 969	42	9.47 852	45	0.52 148	9.98 117	4	15
46	9.46 011	42	9.47 897	46	0.52 103	9.98 113	3	14
47	9.46 053	42	9.47 943	46	0.52 057	9.98 110	4	13
48	9.46 095	41	9.47 989	46	0.52 011	9.98 106	4	12
49	9.46 136	42	9.48 035	45	0.51 965	9.98 102	4	11
50	9.46 178	42	9.48 080	46	0.51 920	9.98 098	4	10
51	9.46 220	42	9.48 126	45	0.51 874	9.98 094	4	9
52	9.46 262	41	9.48 171	46	0.51 829	9.98 090	3	8
53	9.46 303	42	9.48 217	45	0.51 783	9.98 087	4	7
54	9.46 345	41	9.48 262	45	0.51 738	9.98 083	4	6
55	9.46 386	42	9.48 307	46	0.51 693	9.98 079	4	5
56	9.46 428	41	9.48 353	45	0.51 647	9.98 075	4	4
57	9.46 469	42	9.48 398	45	0.51 602	9.98 071	4	3
58	9.46 511	41	9.48 443	46	0.51 557	9.98 067	4	2
59	9.46 552	42	9.48 489	45	0.51 511	9.98 063	3	1
60	9.46 594		9.48 534		0.51 466	9.98 060		0
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.	′

P. P.

	48	47	46
1	4.8	4.7	4.6
2	9.6	9.4	9.2
3	14.4	14.1	13.8
4	19.2	18.8	18.4
5	24.0	23.5	23.0
6	28.8	28.2	27.6
7	33.6	32.9	32.2
8	38.4	37.6	36.8
9	43.2	42.3	41.4

	45	44	43
1	4.5	4.4	4.3
2	9.0	8.8	8.6
3	13.5	13.2	12.9
4	18.0	17.6	17.2
5	22.5	22.0	21.5
6	27.0	26.4	25.8
7	31.5	30.8	30.1
8	36.0	35.2	34.4
9	40.5	39.6	38.7

	42	41
1	4.2	4.1
2	8.4	8.2
3	12.6	12.3
4	16.8	16.4
5	21.0	20.5
6	25.2	24.6
7	29.4	28.7
8	33.6	32.8
9	37.8	36.9

	4	3
1	0.4	0.3
2	0.8	0.6
3	1.2	0.9
4	1.6	1.2
5	2.0	1.5
6	2.4	1.8
7	2.8	2.1
8	3.2	2.4
9	3.6	2.7

	4/48	4/47	4/46	4/45
0	6.0	5.9	5.8	5.6
1	18.0	17.6	17.2	16.9
2	30.0	29.4	28.8	28.1
3	42.0	41.1	40.2	39.4
4				

	3/48	3/47	3/46	3/45
0	8.0	7.8	7.7	7.5
1	24.0	23.5	23.0	22.5
2	40.0	39.2	38.3	37.5
3				

17°

′	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.	
0	9.46 594	41	9.48 534	45	0.51 466	9.98 060	4	60
1	9.46 635	41	9.48 579	45	0.51 421	9.98 056	4	59
2	9.46 676	41	9.48 624	45	0.51 376	9.98 052	4	58
3	9.46 717	41	9.48 669	45	0.51 331	9.98 048	4	57
4	9.46 758	42	9.48 714	45	0.51 286	9.98 044	4	56
5	9.46 800	41	9.48 759	45	0.51 241	9.98 040	4	55
6	9.46 841	41	9.48 804	45	0.51 196	9.98 036	4	54
7	9.46 882	41	9.48 849	45	0.51 151	9.98 032	3	53
8	9.46 923	41	9.48 894	45	0.51 106	9.98 029	4	52
9	9.46 964	41	9.48 939	45	0.51 061	9.98 025	4	51
10	9.47 005	40	9.48 984	45	0.51 016	9.98 021	4	50
11	9.47 045	41	9.49 029	44	0.50 971	9.98 017	4	49
12	9.47 086	41	9.49 073	45	0.50 927	9.98 013	4	48
13	9.47 127	41	9.49 118	45	0.50 882	9.98 009	4	47
14	9.47 168	41	9.49 163	44	0.50 837	9.98 005	4	46
15	9.47 209	40	9.49 207	45	0.50 793	9.98 001	4	45
16	9.47 249	41	9.49 252	44	0.50 748	9.97 997	4	44
17	9.47 290	40	9.49 296	45	0.50 704	9.97 993	4	43
18	9.47 330	41	9.49 341	44	0.50 659	9.97 989	3	42
19	9.47 371	40	9.49 385	45	0.50 615	9.97 986	4	41
20	9.47 411	41	9.49 430	44	0.50 570	9.97 982	4	40
21	9.47 452	40	9.49 474	45	0.50 526	9.97 978	4	39
22	9.47 492	41	9.49 519	44	0.50 481	9.97 974	4	38
23	9.47 533	40	9.49 563	44	0.50 437	9.97 970	4	37
24	9.47 573	40	9.49 607	45	0.50 393	9.97 966	4	36
25	9.47 613	41	9.49 652	44	0.50 348	9.97 962	4	35
26	9.47 654	40	9.49 696	44	0.50 304	9.97 958	4	34
27	9.47 694	40	9.49 740	44	0.50 260	9.97 954	4	33
28	9.47 734	40	9.49 784	44	0.50 216	9.97 950	4	32
29	9.47 774	40	9.49 828	44	0.50 172	9.97 946	4	31
30	9.47 814	40	9.49 872	44	0.50 128	9.97 942	4	30
31	9.47 854	40	9.49 916	44	0.50 084	9.97 938	4	29
32	9.47 894	40	9.49 960	44	0.50 040	9.97 934	4	28
33	9.47 934	40	9.50 004	44	0.49 996	9.97 930	4	27
34	9.47 974	40	9.50 048	44	0.49 952	9.97 926	4	26
35	9.48 014	40	9.50 092	44	0.49 908	9.97 922	4	25
36	9.48 054	40	9.50 136	44	0.49 864	9.97 918	4	24
37	9.48 094	39	9.50 180	43	0.49 820	9.97 914	4	23
38	9.48 133	40	9.50 223	44	0.49 777	9.97 910	4	22
39	9.48 173	40	9.50 267	44	0.49 733	9.97 906	4	21
40	9.48 213	39	9.50 311	44	0.49 689	9.97 902	4	20
41	9.48 252	40	9.50 355	43	0.49 645	9.97 898	4	19
42	9.48 292	40	9.50 398	44	0.49 602	9.97 894	4	18
43	9.48 332	39	9.50 442	43	0.49 558	9.97 890	4	17
44	9.48 371	40	9.50 485	44	0.49 515	9.97 886	4	16
45	9.48 411	39	9.50 529	43	0.49 471	9.97 882	4	15
46	9.48 450	40	9.50 572	44	0.49 428	9.97 878	4	14
47	9.48 490	39	9.50 616	43	0.49 384	9.97 874	4	13
48	9.48 529	39	9.50 659	44	0.49 341	9.97 870	4	12
49	9.48 568	39	9.50 703	43	0.49 297	9.97 866	5	11
50	9.48 607	40	9.50 746	43	0.49 254	9.97 861	4	10
51	9.48 647	39	9.50 789	44	0.49 211	9.97 857	4	9
52	9.48 686	39	9.50 833	43	0.49 167	9.97 853	4	8
53	9.48 725	39	9.50 876	43	0.49 124	9.97 849	4	7
54	9.48 764	39	9.50 919	43	0.49 081	9.97 845	4	6
55	9.48 803	39	9.50 962	43	0.49 038	9.97 841	4	5
56	9.48 842	39	9.51 005	43	0.48 995	9.97 837	4	4
57	9.48 881	39	9.51 048	44	0.48 952	9.97 833	4	3
58	9.48 920	39	9.51 092	43	0.48 908	9.97 829	4	2
59	9.48 959	39	9.51 135	43	0.48 865	9.97 825	4	1
60	9.48 998		9.51 178		0.48 822	9.97 821		0
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.	′

P. P.

	45	44	43
1	4.5	4.4	4.3
2	9.0	8.8	8.6
3	13.5	13.2	12.9
4	18.0	17.6	17.2
5	22.5	22.0	21.5
6	27.0	26.4	25.8
7	31.5	30.8	30.1
8	36.0	35.2	34.4
9	40.5	39.6	38.7

	42	41	40
1	4.2	4.1	4.0
2	8.4	8.2	8.0
3	12.6	12.3	12.0
4	16.8	16.4	16.0
5	21.0	20.5	20.0
6	25.2	24.6	24.0
7	29.4	28.7	28.0
8	33.6	32.8	32.0
9	37.8	36.9	36.0

	39	5	4	3
1	3.9	0.5	0.4	0.3
2	7.8	1.0	0.8	0.6
3	11.7	1.5	1.2	0.9
4	15.6	2.0	1.6	1.2
5	19.5	2.5	2.0	1.5
6	23.4	3.0	2.4	1.8
7	27.3	3.5	2.8	2.1
8	31.2	4.0	3.2	2.4
9	35.1	4.5	3.6	2.7

	5/43	4/45	4/44
0	4.3	5.6	5.5
1	12.9	16.9	16.5
2	21.5	28.1	27.5
3	30.1	39.4	38.5
4	38.7	—	—
5			

	4/43	3/45	3/44
0	5.4	7.5	7.3
1	16.1	22.5	22.0
2	26.9	37.5	36.7
3	37.6	—	—
4			

72°

18°

′	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.	′
0	9.48 998	39	9.51 178	43	0.48 822	9.97 821	4	60
1	9.49 037	39	9.51 221	43	0.48 779	9.97 817	5	59
2	9.49 076	39	9.51 264	42	0.48 736	9.97 812	4	58
3	9.49 115	38	9.51 306	43	0.48 694	9.97 808	4	57
4	9.49 153	39	9.51 349	43	0.48 651	9.97 804	4	56
5	9.49 192	39	9.51 392	43	0.48 608	9.97 800	4	55
6	9.49 231	38	9.51 435	43	0.48 565	9.97 796	4	54
7	9.49 269	39	9.51 478	42	0.48 522	9.97 792	4	53
8	9.49 308	39	9.51 520	43	0.48 480	9.97 788	4	52
9	9.49 347	38	9.51 563	43	0.48 437	9.97 784	5	51
10	9.49 385	39	9.51 606	42	0.48 394	9.97 779	4	50
11	9.49 424	38	9.51 648	43	0.48 352	9.97 775	4	49
12	9.49 462	38	9.51 691	43	0.48 309	9.97 771	4	48
13	9.49 500	39	9.51 734	42	0.48 266	9.97 767	4	47
14	9.49 539	38	9.51 776	43	0.48 224	9.97 763	4	46
15	9.49 577	38	9.51 819	42	0.48 181	9.97 759	5	45
16	9.49 615	39	9.51 861	42	0.48 139	9.97 754	4	44
17	9.49 654	38	9.51 903	43	0.48 097	9.97 750	4	43
18	9.49 692	38	9.51 946	42	0.48 054	9.97 746	4	42
19	9.49 730	38	9.51 988	43	0.48 012	9.97 742	4	41
20	9.49 708	38	9.52 031	42	0.47 969	9.97 738	4	40
21	9.49 806	38	9.52 073	42	0.47 927	9.97 734	5	39
22	9.49 844	38	9.52 115	42	0.47 885	9.97 729	4	38
23	9.49 882	38	9.52 157	43	0.47 843	9.97 725	4	37
24	9.49 920	38	9.52 200	42	0.47 800	9.97 721	4	36
25	9.49 958	38	9.52 242	42	0.47 758	9.97 717	4	35
26	9.49 996	38	9.52 284	42	0.47 716	9.97 713	5	34
27	9.50 034	38	9.52 326	42	0.47 674	9.97 708	4	33
28	9.50 072	38	9.52 368	42	0.47 632	9.97 704	4	32
29	9.50 110	38	9.52 410	42	0.47 590	9.97 700	4	31
30	9.50 148	37	9.52 452	42	0.47 548	9.97 696	5	30
31	9.50 185	38	9.52 494	42	0.47 506	9.97 691	4	29
32	9.50 223	38	9.52 536	42	0.47 464	9.97 687	4	28
33	9.50 261	37	9.52 578	42	0.47 422	9.97 683	4	27
34	9.50 298	38	9.52 620	41	0.47 380	9.97 679	5	26
35	9.50 336	38	9.52 661	42	0.47 339	9.97 674	4	25
36	9.50 374	37	9.52 703	42	0.47 297	9.97 670	4	24
37	9.50 411	38	9.52 745	42	0.47 255	9.97 666	4	23
38	9.50 449	37	9.52 787	42	0.47 213	9.97 662	5	22
39	9.50 486	37	9.52 829	41	0.47 171	9.97 657	4	21
40	9.50 523	38	9.52 870	42	0.47 130	9.97 653	4	20
41	9.50 561	37	9.52 912	41	0.47 088	9.97 649	4	19
42	9.50 598	37	9.52 953	42	0.47 047	9.97 645	5	18
43	9.50 635	38	9.52 995	42	0.47 005	9.97 640	4	17
44	9.50 673	37	9.53 037	41	0.46 963	9.97 636	4	16
45	9.50 710	37	9.53 078	42	0.46 922	9.97 632	4	15
46	9.50 747	37	9.53 120	41	0.46 880	9.97 628	5	14
47	9.50 784	37	9.53 161	41	0.46 839	9.97 623	4	13
48	9.50 821	37	9.53 202	42	0.46 798	9.97 619	4	12
49	9.50 858	38	9.53 244	41	0.46 756	9.97 615	5	11
50	9.50 896	37	9.53 285	42	0.46 715	9.97 610	4	10
51	9.50 933	37	9.53 327	41	0.46 673	9.97 606	4	9
52	9.50 970	37	9.53 368	41	0.46 632	9.97 602	5	8
53	9.51 007	36	9.53 409	41	0.46 591	9.97 597	4	7
54	9.51 043	37	9.53 450	42	0.46 550	9.97 593	4	6
55	9.51 080	37	9.53 492	41	0.46 508	9.97 589	5	5
56	9.51 117	37	9.53 533	41	0.46 467	9.97 584	4	4
57	9.51 154	37	9.53 574	41	0.46 426	9.97 580	4	3
58	9.51 191	36	9.53 615	41	0.46 385	9.97 576	5	2
59	9.51 227	37	9.53 656	41	0.46 344	9.97 571	4	1
60	9.51 264		9.53 697		0.46 303	9.97 567		0
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.	′

P. P.

	43	42	41
1	4.3	4.2	4.1
2	8.6	8.4	8.2
3	12.9	12.6	12.3
4	17.2	16.8	16.4
5	21.5	21.0	20.5
6	25.8	25.2	24.6
7	30.1	29.4	28.7
8	34.4	33.6	32.8
9	38.7	37.8	36.9

	39	38	37
1	3.9	3.8	3.7
2	7.8	7.6	7.4
3	11.7	11.4	11.1
4	15.6	15.2	14.8
5	19.5	19.0	18.5
6	23.4	22.8	22.2
7	27.3	26.6	25.9
8	31.2	30.4	29.6
9	35.1	34.2	33.3

	36	5	4
1	3.6	0.5	0.4
2	7.2	1.0	0.8
3	10.8	1.5	1.2
4	14.4	2.0	1.6
5	18.0	2.5	2.0
6	21.6	3.0	2.4
7	25.2	3.5	2.8
8	28.8	4.0	3.2
9	32.4	4.5	3.6

	5/43	5/42	5/41
0			
1	4.3	4.2	4.1
2	12.9	12.6	12.3
3	21.5	21.0	20.5
4	30.1	29.4	28.7
5	38.7	37.8	36.9

	4/43	4/42	4/41
0			
1	5.4	5.2	5.1
2	16.1	15.8	15.4
3	26.9	26.2	25.6
4	37.6	36.8	35.9

P. P.

71°

′	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.	
0	9.51 264	37	9.53 697	41	0.46 303	9.97 567	4	**60**
1	9.51 301	37	9.53 738	41	0.46 262	9.97 563	5	59
2	9.51 338	36	9.53 779	41	0.46 221	9.97 558	4	58
3	9.51 374	37	9.53 820	41	0.46 180	9.97 554	4	57
4	9.51 411	36	9.53 861	41	0.46 139	9.97 550	5	56
5	9.51 447	37	9.53 902	41	0.46 098	9.97 545	4	55
6	9.51 484	36	9.53 943	41	0.46 057	9.97 541	5	54
7	9.51 520	37	9.53 984	41	0.46 016	9.97 536	4	53
8	9.51 557	36	9.54 025	40	0.45 975	9.97 532	4	52
9	9.51 593	36	9.54 065	41	0.45 935	9.97 528	5	51
10	9.51 629	37	9.54 106	41	0.45 894	9.97 523	4	**50**
11	9.51 666	36	9.54 147	40	0.45 853	9.97 519	4	49
12	9.51 702	36	9.54 187	41	0.45 813	9.97 515	5	48
13	9.51 738	36	9.54 228	41	0.45 772	9.97 510	4	47
14	9.51 774	37	9.54 269	40	0.45 731	9.97 506	5	46
15	9.51 811	36	9.54 309	41	0.45 691	9.97 501	4	45
16	9.51 847	36	9.54 350	40	0.45 650	9.97 497	5	44
17	9.51 883	36	9.54 390	41	0.45 610	9.97 492	4	43
18	9.51 919	36	9.54 431	40	0.45 569	9.97 488	4	42
19	9.51 955	36	9.54 471	41	0.45 529	9.97 484	5	41
20	9.51 991	36	9.54 512	40	0.45 488	9.97 479	4	**40**
21	9.52 027	36	9.54 552	41	0.45 448	9.97 475	5	39
22	9.52 063	36	9.54 593	40	0.45 407	9.97 470	4	38
23	9.52 099	36	9.54 633	40	0.45 367	9.97 466	5	37
24	9.52 135	36	9.54 673	41	0.45 327	9.97 461	4	36
25	9.52 171	36	9.54 714	40	0.45 286	9.97 457	4	35
26	9.52 207	35	9.54 754	40	0.45 246	9.97 453	5	34
27	9.52 242	36	9.54 794	41	0.45 206	9.97 448	4	33
28	9.52 278	36	9.54 835	40	0.45 165	9.97 444	5	32
29	9.52 314	36	9.54 875	40	0.45 125	9.97 439	4	31
30	9.52 350	35	9.54 915	40	0.45 085	9.97 435	5	**30**
31	9.52 385	36	9.54 955	40	0.45 045	9.97 430	4	29
32	9.52 421	35	9.54 995	40	0.45 005	9.97 426	5	28
33	9.52 456	36	9.55 035	40	0.44 965	9.97 421	4	27
34	9.52 492	35	9.55 075	40	0.44 925	9.97 417	5	26
35	9.52 527	36	9.55 115	40	0.44 885	9.97 412	4	25
36	9.52 563	35	9.55 155	40	0.44 845	9.97 408	5	24
37	9.52 598	36	9.55 195	40	0.44 805	9.97 403	4	23
38	9.52 634	35	9.55 235	40	0.44 765	9.97 399	5	22
39	9.52 669	36	9.55 275	40	0.44 725	9.97 394	4	21
40	9.52 705	35	9.55 315	40	0.44 685	9.97 390	5	**20**
41	9.52 740	35	9.55 355	40	0.44 645	9.97 385	4	19
42	9.52 775	36	9.55 395	39	0.44 605	9.97 381	5	18
43	9.52 811	35	9.55 434	40	0.44 566	9.97 376	4	17
44	9.52 846	35	9.55 474	40	0.44 526	9.97 372	5	16
45	9.52 881	35	9.55 514	40	0.44 486	9.97 367	4	15
46	9.52 916	35	9.55 554	39	0.44 446	9.97 363	5	14
47	9.52 951	35	9.55 593	40	0.44 407	9.97 358	5	13
48	9.52 986	35	9.55 633	40	0.44 367	9.97 353	4	12
49	9.53 021	35	9.55 673	39	0.44 327	9.97 349	5	11
50	9.53 056	36	9.55 712	40	0.44 288	9.97 344	4	**10**
51	9.53 092	34	9.55 752	39	0.44 248	9.97 340	5	9
52	9.53 126	35	9.55 791	40	0.44 209	9.97 335	4	8
53	9.53 161	35	9.55 831	39	0.44 169	9.97 331	5	7
54	9.53 196	35	9.55 870	40	0.44 130	9.97 326	4	6
55	9.53 231	35	9.55 910	39	0.44 090	9.97 322	5	5
56	9.53 266	35	9.55 949	40	0.44 051	9.97 317	5	4
57	9.53 301	35	9.55 989	39	0.44 011	9.97 312	4	3
58	9.53 336	34	9.56 028	39	0.43 972	9.97 308	5	2
59	9.53 370	35	9.56 067	40	0.43 933	9.97 303	4	1
60	9.53 405		9.56 107		0.43 893	9.97 299		**0**
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.	′

P. P.

	41	40	39
1	4.1	4.0	3.9
2	8.2	8.0	7.8
3	12.3	12.0	11.7
4	16.4	16.0	15.6
5	20.5	20.0	19.5
6	24.6	24.0	23.4
7	28.7	28.0	27.3
8	32.8	32.0	31.2
9	36.9	36.0	35.1

	37	36	35
1	3.7	3.6	3.5
2	7.4	7.2	7.0
3	11.1	10.8	10.5
4	14.8	14.4	14.0
5	18.5	18.0	17.5
6	22.2	21.6	21.0
7	25.9	25.2	24.5
8	29.6	28.8	28.0
9	33.3	32.4	31.5

	34	5	4
1	3.4	0.5	0.4
2	6.8	1.0	0.8
3	10.2	1.5	1.2
4	13.6	2.0	1.6
5	17.0	2.5	2.0
6	20.4	3.0	2.4
7	23.8	3.5	2.8
8	27.2	4.0	3.2
9	30.6	4.5	3.6

	5/41	5/40	5/39
0	4.1	4.0	3.9
1	12.3	12.0	11.7
2	20.5	20.0	19.5
3	28.7	28.0	27.3
4	36.9	36.0	35.1
5			

	4/41	4/40	4/39
0	5.1	5.0	4.9
1	15.4	15.0	14.6
2	25.6	25.0	24.4
3	35.9	35.0	34.1
4			

P. P.

20°

′	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.	
0	9.53 405	35	9.56 107	39	0.43 893	9.97 299	5	60
1	9.53 440	35	9.56 146	39	0.43 854	9.97 294	5	59
2	9.53 475	34	9.56 185	39	0.43 815	9.97 289	4	58
3	9.53 509	35	9.56 224	40	0.43 776	9.97 285	5	57
4	9.53 544	34	9.56 264	39	0.43 736	9.97 280	4	56
5	9.53 578	35	9.56 303	39	0.43 697	9.97 276	5	55
6	9.53 613	34	9.56 342	39	0.43 658	9.97 271	5	54
7	9.53 647	35	9.56 381	39	0.43 619	9.97 266	4	53
8	9.53 682	34	9.56 420	39	0.43 580	9.97 262	5	52
9	9.53 716	35	9.56 459	39	0.43 541	9.97 257	5	51
10	9.53 751	34	9.56 498	39	0.43 502	9.97 252	4	50
11	9.53 785	34	9.56 537	39	0.43 463	9.97 248	5	49
12	9.53 819	35	9.56 576	39	0.43 424	9.97 243	5	48
13	9.53 854	34	9.56 615	39	0.43 385	9.97 238	4	47
14	9.53 888	34	9.56 654	39	0.43 346	9.97 234	5	46
15	9.53 922	35	9.56 693	39	0.43 307	9.97 229	5	45
16	9.53 957	34	9.56 732	39	0.43 268	9.97 224	4	44
17	9.53 991	34	9.56 771	39	0.43 229	9.97 220	5	43
18	9.54 025	34	9.56 810	39	0.43 190	9.97 215	5	42
19	9.54 059	34	9.56 849	38	0.43 151	9.97 210	4	41
20	9.54 093	34	9.56 887	39	0.43 113	9.97 206	5	40
21	9.54 127	34	9.56 926	39	0.43 074	9.97 201	5	39
22	9.54 161	34	9.56 965	39	0.43 035	9.97 196	4	38
23	9.54 195	34	9.57 004	38	0.42 996	9.97 192	5	37
24	9.54 229	34	9.57 042	39	0.42 958	9.97 187	5	36
25	9.54 263	34	9.57 081	39	0.42 919	9.97 182	4	35
26	9.54 297	34	9.57 120	38	0.42 880	9.97 178	5	34
27	9.54 331	34	9.57 158	39	0.42 842	9.97 173	5	33
28	9.54 365	34	9.57 197	38	0.42 803	9.97 168	5	32
29	9.54 399	34	9.57 235	39	0.42 765	9.97 163	4	31
30	9.54 433	33	9.57 274	38	0.42 726	9.97 159	5	30
31	9.54 466	34	9.57 312	39	0.42 688	9.97 154	5	29
32	9.54 500	34	9.57 351	38	0.42 649	9.97 149	4	28
33	9.54 534	33	9.57 389	39	0.42 611	9.97 145	5	27
34	9.54 567	34	9.57 428	38	0.42 572	9.97 140	5	26
35	9.54 601	34	9.57 466	38	0.42 534	9.97 135	5	25
36	9.54 635	33	9.57 504	39	0.42 496	9.97 130	4	24
37	9.54 668	34	9.57 543	38	0.42 457	9.97 126	5	23
38	9.54 702	33	9.57 581	38	0.42 419	9.97 121	5	22
39	9.54 735	34	9.57 619	39	0.42 381	9.97 116	5	21
40	9.54 769	33	9.57 658	38	0.42 342	9.97 111	4	20
41	9.54 802	34	9.57 696	38	0.42 304	9.97 107	5	19
42	9.54 836	33	9.57 734	38	0.42 266	9.97 102	5	18
43	9.54 869	34	9.57 772	38	0.42 228	9.97 097	5	17
44	9.54 903	33	9.57 810	39	0.42 190	9.97 092	5	16
45	9.54 936	33	9.57 849	38	0.42 151	9.97 087	4	15
46	9.54 969	34	9.57 887	38	0.42 113	9.97 083	5	14
47	9.55 003	33	9.57 925	38	0.42 075	9.97 078	5	13
48	9.55 036	33	9.57 963	38	0.42 037	9.97 073	5	12
49	9.55 069	33	9.58 001	38	0.41 999	9.97 068	5	11
50	9.55 102	34	9.58 039	38	0.41 961	9.97 063	4	10
51	9.55 136	33	9.58 077	38	0.41 923	9.97 059	5	9
52	9.55 169	33	9.58 115	38	0.41 885	9.97 054	5	8
53	9.55 202	33	9.58 153	38	0.41 847	9.97 049	5	7
54	9.55 235	33	9.58 191	38	0.41 809	9.97 044	5	6
55	9.55 268	33	9.58 229	38	0.41 771	9.97 039	4	5
56	9.55 301	33	9.58 267	37	0.41 733	9.97 035	5	4
57	9.55 334	33	9.58 304	38	0.41 696	9.97 030	5	3
58	9.55 367	33	9.58 342	38	0.41 658	9.97 025	5	2
59	9.55 400	33	9.58 380	38	0.41 620	9.97 020	5	1
60	9.55 433		9.58 418		0.41 582	9.97 015		0
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.	′

P. P.

	40	39	38
1	4.0	3.9	3.8
2	8.0	7.8	7.6
3	12.0	11.7	11.4
4	16.0	15.6	15.2
5	20.0	19.5	19.0
6	24.0	23.4	22.8
7	28.0	27.3	26.6
8	32.0	31.2	30.4
9	36.0	35.1	34.2

	37	35	34
1	3.7	3.5	3.4
2	7.4	7.0	6.8
3	11.1	10.5	10.2
4	14.8	14.0	13.6
5	18.5	17.5	17.0
6	22.2	21.0	20.4
7	25.9	24.5	23.8
8	29.6	28.0	27.2
9	33.3	31.5	30.6

	33	5	4
1	3.3	0.5	0.4
2	6.6	1.0	0.8
3	9.9	1.5	1.2
4	13.2	2.0	1.6
5	16.5	2.5	2.0
6	19.8	3.0	2.4
7	23.1	3.5	2.8
8	26.4	4.0	3.2
9	29.7	4.5	3.6

	5/40	5/39	5/38
0			
1	4.0	3.9	3.8
2	12.0	11.7	11.4
3	20.0	19.5	19.0
4	28.0	27.3	26.6
5	36.0	35.1	34.2

	5/37	4/39	4/38
0			
1	3.7	4.9	4.8
2	11.1	14.6	14.2
3	18.5	24.4	23.8
4	25.9	34.1	33.2
5	33.3	—	—

P. P.

′	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.	
0	9.55 433	33	9.58 418	37	0.41 582	9.97 015	5	60
1	9.55 466	33	9.58 455	38	0.41 545	9.97 010	5	59
2	9.55 499	33	9.58 493	38	0.41 507	9.97 005	4	58
3	9.55 532	32	9.58 531	38	0.41 469	9.97 001	5	57
4	9.55 564	33	9.58 569	37	0.41 431	9.96 996	5	56
5	9.55 597	33	9.58 606	38	0.41 394	9.96 991	5	55
6	9.55 630	33	9.58 644	37	0.41 356	9.96 986	5	54
7	9.55 663	32	9.58 681	38	0.41 319	9.96 981	5	53
8	9.55 695	33	9.58 719	38	0.41 281	9.96 976	5	52
9	9.55 728	33	9.58 757	37	0.41 243	9.96 971	5	51
10	9.55 761	32	9.58 794	38	0.41 206	9.96 966	4	50
11	9.55 793	33	9.58 832	37	0.41 168	9.96 962	5	49
12	9.55 826	32	9.58 869	38	0.41 131	9.96 957	5	48
13	9.55 858	33	9.58 907	37	0.41 093	9.96 952	5	47
14	9.55 891	32	9.58 944	37	0.41 056	9.96 947	5	46
15	9.55 923	33	9.58 981	38	0.41 019	9.96 942	5	45
16	9.55 956	32	9.59 019	37	0.40 981	9.96 937	5	44
17	9.55 988	33	9.59 056	38	0.40 944	9.96 932	5	43
18	9.56 021	32	9.59 094	37	0.40 906	9.96 927	5	42
19	9.56 053	32	9.59 131	37	0.40 869	9.96 922	5	41
20	9.56 085	33	9.59 168	37	0.40 832	9.96 917	5	40
21	9.56 118	32	9.59 205	38	0.40 795	9.96 912	5	39
22	9.56 150	32	9.59 243	37	0.40 757	9.96 907	4	38
23	9.56 182	33	9.59 280	37	0.40 720	9.96 903	5	37
24	9.56 215	32	9.59 317	37	0.40 683	9.96 898	5	36
25	9.56 247	32	9.59 354	37	0.40 646	9.96 893	5	35
26	9.56 279	32	9.59 391	38	0.40 609	9.96 888	5	34
27	9.56 311	32	9.59 429	37	0.40 571	9.96 883	5	33
28	9.56 343	32	9.59 466	37	0.40 534	9.96 878	5	32
29	9.56 375	33	9.59 503	37	0.40 497	9.96 873	5	31
30	9.56 408	32	9.59 540	37	0.40 460	9.96 868	5	30
31	9.56 440	32	9.59 577	37	0.40 423	9.96 863	5	29
32	9.56 472	32	9.59 614	37	0.40 386	9.96 858	5	28
33	9.56 504	32	9.59 651	37	0.40 349	9.96 853	5	27
34	9.56 536	32	9.59 688	37	0.40 312	9.96 848	5	26
35	9.56 568	31	9.59 725	37	0.40 275	9.96 843	5	25
36	9.56 599	32	9.59 762	37	0.40 238	9.96 838	5	24
37	9.56 631	32	9.59 799	36	0.40 201	9.96 833	5	23
38	9.56 663	32	9.59 835	37	0.40 165	9.96 828	5	22
39	9.56 695	32	9.59 872	37	0.40 128	9.96 823	5	21
40	9.56 727	32	9.59 909	37	0.40 091	9.96 818	5	20
41	9.56 759	31	9.59 946	37	0.40 054	9.96 813	5	19
42	9.56 790	32	9.59 983	36	0.40 017	9.96 808	5	18
43	9.56 822	32	9.60 019	37	0.39 981	9.96 803	5	17
44	9.56 854	32	9.60 056	37	0.39 944	9.96 798	5	16
45	9.56 886	31	9.60 093	37	0.39 907	9.96 793	5	15
46	9.56 917	32	9.60 130	36	0.39 870	9.96 788	5	14
47	9.56 949	31	9.60 166	37	0.39 834	9.96 783	5	13
48	9.56 980	32	9.60 203	37	0.39 797	9.96 778	6	12
49	9.57 012	32	9.60 240	36	0.39 760	9.96 772	5	11
50	9.57 044	31	9.60 276	37	0.39 724	9.96 767	5	10
51	9.57 075	32	9.60 313	36	0.39 687	9.96 762	5	9
52	9.57 107	31	9.60 349	37	0.39 651	9.96 757	5	8
53	9.57 138	31	9.60 386	36	0.39 614	9.96 752	5	7
54	9.57 169	32	9.60 422	37	0.39 578	9.96 747	5	6
55	9.57 201	31	9.60 459	36	0.39 541	9.96 742	5	5
56	9.57 232	32	9.60 495	37	0.39 505	9.96 737	5	4
57	9.57 264	31	9.60 532	36	0.39 468	9.96 732	5	3
58	9.57 295	31	9.60 568	37	0.39 432	9.96 727	5	2
59	9.57 326	32	9.60 605	36	0.39 395	9.96 722	5	1
60	9.57 358		9.60 641		0.39 359	9.96 717		0
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.	′

P. P.

	38	37	36
1	3.8	3.7	3.6
2	7.6	7.4	7.2
3	11.4	11.1	10.8
4	15.2	14.8	14.4
5	19.0	18.5	18.0
6	22.8	22.2	21.6
7	26.6	25.9	25.2
8	30.4	29.6	28.8
9	34.2	33.3	32.4

	33	32	31
1	3.3	3.2	3.1
2	6.6	6.4	6.2
3	9.9	9.6	9.3
4	13.2	12.8	12.4
5	16.5	16.0	15.5
6	19.8	19.2	18.6
7	23.1	22.4	21.7
8	26.4	25.6	24.8
9	29.7	28.8	27.9

	6	5	4
1	0.6	0.5	0.4
2	1.2	1.0	0.8
3	1.8	1.5	1.2
4	2.4	2.0	1.6
5	3.0	2.5	2.0
6	3.6	3.0	2.4
7	4.2	3.5	2.8
8	4.8	4.0	3.2
9	5.4	4.5	3.6

	6/37	5/38	5/37
0	3.1	3.8	3.7
1	9.2	11.4	11.1
2	15.4	19.0	18.5
3	21.6	26.6	25.9
4	27.8	34.2	33.3
5	33.9	—	—
6			

	5/36	4/38	4/37
0	3.6	4.8	4.6
1	10.8	14.2	13.9
2	18.0	23.8	23.1
3	25.2	33.2	32.4
4	32.4	—	—
5			

′	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.	
0	9.57 358	31	9.60 641	36	0.39 359	9.96 717	6	60
1	9.57 389	31	9.60 677	37	0.39 323	9.96 711	5	59
2	9.57 420	31	9.60 714	36	0.39 286	9.96 706	5	58
3	9.57 451	31	9.60 750	36	0.39 250	9.96 701	5	57
4	9.57 482	32	9.60 786	37	0.39 214	9.96 696	5	56
5	9.57 514	31	9.60 823	36	0.39 177	9.96 691	5	55
6	9.57 545	31	9.60 859	36	0.39 141	9.96 686	5	54
7	9.57 576	31	9.60 895	36	0.39 105	9.96 681	5	53
8	9.57 607	31	9.60 931	36	0.39 069	9.96 676	6	52
9	9.57 638	31	9.60 967	37	0.39 033	9.96 670	5	51
10	9.57 669	31	9.61 004	36	0.38 996	9.96 665	5	50
11	9.57 700	31	9.61 040	36	0.38 960	9.96 660	5	49
12	9.57 731	31	9.61 076	36	0.38 924	9.96 655	5	48
13	9.57 762	31	9.61 112	36	0.38 888	9.96 650	5	47
14	9.57 793	31	9.61 148	36	0.38 852	9.96 645	5	46
15	9.57 824	31	9.61 184	36	0.38 816	9.96 640	6	45
16	9.57 855	30	9.61 220	36	0.38 780	9.96 634	5	44
17	9.57 885	31	9.61 256	36	0.38 744	9.96 629	5	43
18	9.57 916	31	9.61 292	36	0.38 708	9.96 624	5	42
19	9.57 947	31	9.61 328	36	0.38 672	9.96 619	5	41
20	9.57 978	30	9.61 364	36	0.38 636	9.96 614	6	40
21	9.58 008	31	9.61 400	36	0.38 600	9.96 608	5	39
22	9.58 039	31	9.61 436	36	0.38 564	9.96 603	5	38
23	9.58 070	31	9.61 472	36	0.38 528	9.96 598	5	37
24	9.58 101	30	9.61 508	36	0.38 492	9.96 593	5	36
25	9.58 131	31	9.61 544	35	0.38 456	9.96 588	6	35
26	9.58 162	30	9.61 579	36	0.38 421	9.96 582	5	34
27	9.58 192	31	9.61 615	36	0.38 385	9.96 577	5	33
28	9.58 223	30	9.61 651	36	0.38 349	9.96 572	5	32
29	9.58 253	31	9.61 687	35	0.38 313	9.96 567	5	31
30	9.58 284	30	9.61 722	36	0.38 278	9.96 562	6	30
31	9.58 314	31	9.61 758	36	0.38 242	9.96 556	5	29
32	9.58 345	30	9.61 794	36	0.38 206	9.96 551	5	28
33	9.58 375	31	9.61 830	35	0.38 170	9.96 546	5	27
34	9.58 406	30	9.61 865	36	0.38 135	9.96 541	6	26
35	9.58 436	31	9.61 901	35	0.38 099	9.96 535	5	25
36	9.58 467	30	9.61 936	36	0.38 064	9.96 530	5	24
37	9.58 497	30	9.61 972	36	0.38 028	9.96 525	5	23
38	9.58 527	30	9.62 008	35	0.37 992	9.96 520	6	22
39	9.58 557	31	9.62 043	36	0.37 957	9.96 514	5	21
40	9.58 588	30	9.62 079	35	0.37 921	9.96 509	5	20
41	9.58 618	30	9.62 114	36	0.37 886	9.96 504	6	19
42	9.58 648	30	9.62 150	35	0.37 850	9.96 498	5	18
43	9.58 678	31	9.62 185	36	0.37 815	9.96 493	5	17
44	9.58 709	30	9.62 221	35	0.37 779	9.96 488	5	16
45	9.58 739	30	9.62 256	36	0.37 744	9.96 483	6	15
46	9.58 769	30	9.62 292	35	0.37 708	9.96 477	5	14
47	9.58 799	30	9.62 327	35	0.37 673	9.96 472	5	13
48	9.58 829	30	9.62 362	36	0.37 638	9.96 467	6	12
49	9.58 859	30	9.62 398	35	0.37 602	9.96 461	5	11
50	9.58 889	30	9.62 433	35	0.37 567	9.96 456	5	10
51	9.58 919	30	9.62 468	36	0.37 532	9.96 451	6	9
52	9.58 949	30	9.62 504	35	0.37 496	9.96 445	5	8
53	9.58 979	30	9.62 539	35	0.37 461	9.96 440	5	7
54	9.59 009	30	9.62 574	35	0.37 426	9.96 435	6	6
55	9.59 039	30	9.62 609	36	0.37 391	9.96 429	5	5
56	9.59 069	29	9.62 645	35	0.37 355	9.96 424	5	4
57	9.59 098	30	9.62 680	35	0.37 320	9.96 419	6	3
58	9.59 128	30	9.62 715	35	0.37 285	9.96 413	5	2
59	9.59 158	30	9.62 750	35	0.37 250	9.96 408	5	1
60	9.59 188		9.62 785		0.37 215	9.96 403		0
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.	′

P. P.

	37	36	35
1	3.7	3.6	3.5
2	7.4	7.2	7.0
3	11.1	10.8	10.5
4	14.8	14.4	14.0
5	18.5	18.0	17.5
6	22.2	21.6	21.0
7	25.9	25.2	24.5
8	29.6	28.8	28.0
9	33.3	32.4	31.5

	32	31	30
1	3.2	3.1	3.0
2	6.4	6.2	6.0
3	9.6	9.3	9.0
4	12.8	12.4	12.0
5	16.0	15.5	15.0
6	19.2	18.6	18.0
7	22.4	21.7	21.0
8	25.6	24.8	24.0
9	28.8	27.9	27.0

	29	6	5
1	2.9	0.6	0.5
2	5.8	1.2	1.0
3	8.7	1.8	1.5
4	11.6	2.4	2.0
5	14.5	3.0	2.5
6	17.4	3.6	3.0
7	20.3	4.2	3.5
8	23.2	4.8	4.0
9	26.1	5.4	4.5

	6/36	6/35
0	3.0	2.9
1	9.0	8.8
2	15.0	14.6
3	21.0	20.4
4	27.0	26.2
5	33.0	32.1
6		

	5/37	5/36	5/35
0	3.7	3.6	3.5
1	11.1	10.8	10.5
2	18.5	18.0	17.5
3	25.9	25.2	24.5
4	33.3	32.4	31.5
5			

′	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.	
0	9.59 188	30	9.62 785	35	0.37 215̄	9.96 403	6	**60**
1	9.59 218	29	9.62 820	35	0.37 180	9.96 397	5	59
2	9.59 247	30	9.62 855	35	0.37 145̄	9.96 392	5	58
3	9.59 277	30	9.62 890	36	0.37 110	9.96 387	6	57
4	9.59 307	29	9.62 926	35	0.37 074	9.96 381	5	56
5	9.59 336	30	9.62 961	35	0.37 039	9.96 376	6	55
6	9.59 366	30	9.62 996	35	0.37 004	9.96 370	5	54
7	9.59 396	29	9.63 031	35	0.36 969	9.96 365̄	5	53
8	9.59 425	30	9.63 066	35	0.36 934	9.96 360	6	52
9	9.59 455̄	29	9.63 101	34	0.36 899	9.96 354	5	51
10	9.59 484	30	9.63 135	35	0.36 865̄	9.96 349	6	**50**
11	9.59 514	29	9.63 170	35	0.36 830	9.96 343	5	49
12	9.59 543	30	9.63 205	35	0.36 795̄	9.96 338	5	48
13	9.59 573	29	9.63 240	35	0.36 760	9.96 333	6	47
14	9.59 602	30	9.63 275	35	0.36 725̄	9.96 327	5	46
15	9.59 632	29	9.63 310	35	0.36 690	9.96 322	6	45
16	9.59 661	29	9.63 345̄	34	0.36 655	9.96 316	5	44
17	9.59 690	30	9.63 379	35	0.36 621	9.96 311	6	43
18	9.59 720	29	9.63 414	35	0.36 586	9.96 305	5	42
19	9.59 749	29	9.63 449	35	0.36 551	9.96 300	6	41
20	9.59 778	30	9.63 484	35	0.36 516	9.96 294	5	**40**
21	9.59 808	29	9.63 519	34	0.36 481	9.96 289	5	39
22	9.59 837	29	9.63 553	35	0.36 447	9.96 284	6	38
23	9.59 866	29	9.63 588	35	0.36 412	9.96 278	5	37
24	9.59 895	29	9.63 623	34	0.36 377	9.96 273	6	36
25	9.59 924	30	9.63 657	35	0.36 343	9.96 267	5	35
26	9.59 954	29	9.63 692	34	0.36 308	9.96 262	6	34
27	9.59 983	29	9.63 726	35	0.36 274	9.96 256	5	33
28	9.60 012	29	9.63 761	35	0.36 239	9.96 251	6	32
29	9.60 041	29	9.63 796	34	0.36 204	9.96 245	5	31
30	9.60 070	29	9.63 830	35	0.36 170	9.96 240	6	**30**
31	9.60 099	29	9.63 865̄	34	0.36 135	9.96 234	5	29
32	9.60 128	29	9.63 899	35	0.36 101	9.96 229	6	28
33	9.60 157	29	9.63 934	34	0.36 066	9.96 223	5	27
34	9.60 186	29	9.63 968	35	0.36 032	9.96 218	6	26
35	9.60 215̄	29	9.64 003	34	0.35 997	9.96 212	5	25
36	9.60 244	29	9.64 037	35	0.35 963	9.96 207	6	24
37	9.60 273	29	9.64 072	34	0.35 928	9.96 201	5	23
38	9.60 302	29	9.64 106	34	0.35 894	9.96 196	6	22
39	9.60 331	28	9.64 140	35	0.35 860	9.96 190	5	21
40	9.60 359	29	9.64 175̄	34	0.35 825	9.96 185̄	6	**20**
41	9.60 388	29	9.64 209	34	0.35 791	9.96 179	5	19
42	9.60 417	29	9.64 243	35	0.35 757	9.96 174	6	18
43	9.60 446	28	9.64 278	34	0.35 722	9.96 168	6	17
44	9.60 474	29	9.64 312	34	0.35 688	9.96 162	5	16
45	9.60 503	29	9.64 346	35	0.35 654	9.96 157	6	15
46	9.60 532	29	9.64 381	34	0.35 619	9.96 151	5	14
47	9.60 561	28	9.64 415̄	34	0.35 585	9.96 146	6	13
48	9.60 589	29	9.64 449	34	0.35 551	9.96 140	5	12
49	9.60 618	28	9.64 483	34	0.35 517	9.96 135̄	6	11
50	9.60 646	29	9.64 517	35	0.35 483	9.96 129	6	**10**
51	9.60 675	29	9.64 552	34	0.35 448	9.96 123	5	9
52	9.60 704	28	9.64 586	34	0.35 414	9.96 118	6	8
53	9.60 732	29	9.64 620	34	0.35 380	9.96 112	5	7
54	9.60 761	28	9.64 654	34	0.35 346	9.96 107	6	6
55	9.60 789	29	9.64 688	34	0.35 312	9.96 101	6	5
56	9.60 818	28	9.64 722	34	0.35 278	9.96 095	5	4
57	9.60 846	29	9.64 756	34	0.35 244	9.96 090	6	3
58	9.60 875̄	28	9.64 790	34	0.35 210	9.96 084	5	2
59	9.60 903	28	9.64 824	34	0.35 176	9.96 079	6	1
60	9.60 931		9.64 858		0.35 142	9.96 073		**0**
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.	′

P. P.

	36	35	34
1	3.6	3.5	3.4
2	7.2	7.0	6.8
3	10.8	10.5	10.2
4	14.4	14.0	13.6
5	18.0	17.5	17.0
6	21.6	21.0	20.4
7	25.2	24.5	23.8
8	28.8	28.0	27.2
9	32.4	31.5	30.6

	30	29	28
1	3.0	2.9	2.8
2	6.0	5.8	5.6
3	9.0	8.7	8.4
4	12.0	11.6	11.2
5	15.0	14.5	14.0
6	18.0	17.4	16.8
7	21.0	20.3	19.6
8	24.0	23.2	22.4
9	27.0	26.1	25.2

	6	5
1	0.6	0.5
2	1.2	1.0
3	1.8	1.5
4	2.4	2.0
5	3.0	2.5
6	3.6	3.0
7	4.2	3.5
8	4.8	4.0
9	5.4	4.5

	6/36	6/35	6/34
0	3.0	2.9	2.8
1	9.0	8.8	8.5
2	15.0	14.6	14.2
3	21.0	20.4	19.8
4	27.0	26.2	25.5
5	33.0	32.1	31.2
6			

	5/35	5/34
0	3.5	3.4
1	10.5	10.2
2	17.5	17.0
3	24.5	23.8
4	31.5	30.6
5		

P. P.

24°

′	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.	′
0	9.60 931	29	9.64 858	34	0.35 142	9.96 073	6	**60**
1	9.60 960	28	9.64 892	34	0.35 108	9.96 067	5	59
2	9.60 988	28	9.64 926	34	0.35 074	9.96 062	6	58
3	9.61 016	29	9.64 960	34	0.35 040	9.96 056	6	57
4	9.61 045	28	9.64 994	34	0.35 006	9.96 050	5	56
5	9.61 073	28	9.65 028	34	0.34 972	9.96 045	6	55
6	9.61 101	28	9.65 062	34	0.34 938	9.96 039	5	54
7	9.61 129	29	9.65 096	34	0.34 904	9.96 034	6	53
8	9.61 158	28	9.65 130	34	0.34 870	9.96 028	6	52
9	9.61 186	28	9.65 164	33	0.34 836	9.96 022	5	51
10	9.61 214	28	9.65 197	34	0.34 803	9.96 017	6	**50**
11	9.61 242	28	9.65 231	34	0.34 769	9.96 011	6	49
12	9.61 270	28	9.65 265	34	0.34 735	9.96 005	5	48
13	9.61 298	28	9.65 299	34	0.34 701	9.96 000	6	47
14	9.61 326	28	9.65 333	33	0.34 667	9.95 994	6	46
15	9.61 354	28	9.65 366	34	0.34 634	9.95 988	6	45
16	9.61 382	29	9.65 400	34	0.34 600	9.95 982	5	44
17	9.61 411	27	9.65 434	33	0.34 566	9.95 977	6	43
18	9.61 438	28	9.65 467	34	0.34 533	9.95 971	6	42
19	9.61 466	28	9.65 501	34	0.34 499	9.95 965	5	41
20	9.61 494	28	9.65 535	33	0.34 465	9.95 960	6	**40**
21	9.61 522	28	9.65 568	34	0.34 432	9.95 954	6	39
22	9.61 550	28	9.65 602	34	0.34 398	9.95 948	6	38
23	9.61 578	28	9.65 636	33	0.34 364	9.95 942	5	37
24	9.61 606	28	9.65 669	34	0.34 331	9.95 937	6	36
25	9.61 634	28	9.65 703	33	0.34 297	9.95 931	6	35
26	9.61 662	27	9.65 736	34	0.34 264	9.95 925	5	34
27	9.61 689	28	9.65 770	33	0.34 230	9.95 920	6	33
28	9.61 717	28	9.65 803	34	0.34 197	9.95 914	6	32
29	9.61 745	28	9.65 837	33	0.34 163	9.95 908	6	31
30	9.61 773	27	9.65 870	34	0.34 130	9.95 902	5	**30**
31	9.61 800	28	9.65 904	33	0.34 096	9.95 897	6	29
32	9.61 828	28	9.65 937	34	0.34 063	9.95 891	6	28
33	9.61 856	27	9.65 971	33	0.34 029	9.95 885	6	27
34	9.61 883	28	9.66 004	34	0.33 996	9.95 879	6	26
35	9.61 911	28	9.66 038	33	0.33 962	9.95 873	5	25
36	9.61 939	27	9.66 071	33	0.33 929	9.95 868	6	24
37	9.61 966	28	9.66 104	34	0.33 896	9.95 862	6	23
38	9.61 994	27	9.66 138	33	0.33 862	9.95 856	6	22
39	9.62 021	28	9.66 171	33	0.33 829	9.95 850	6	21
40	9.62 049	27	9.66 204	34	0.33 796	9.95 844	5	**20**
41	9.62 076	28	9.66 238	33	0.33 762	9.95 839	6	19
42	9.62 104	27	9.66 271	33	0.33 729	9.95 833	6	18
43	9.62 131	28	9.66 304	33	0.33 696	9.95 827	6	17
44	9.62 159	27	9.66 337	34	0.33 663	9.95 821	6	16
45	9.62 186	28	9.66 371	33	0.33 629	9.95 815	5	15
46	9.62 214	27	9.66 404	33	0.33 596	9.95 810	6	14
47	9.62 241	27	9.66 437	33	0.33 563	9.95 804	6	13
48	9.62 268	28	9.66 470	33	0.33 530	9.95 798	6	12
49	9.62 296	27	9.66 503	34	0.33 497	9.95 792	6	11
50	9.62 323	27	9.66 537	33	0.33 463	9.95 786	6	**10**
51	9.62 350	27	9.66 570	33	0.33 430	9.95 780	5	9
52	9.62 377	28	9.66 603	33	0.33 397	9.95 775	6	8
53	9.62 405	27	9.66 636	33	0.33 364	9.95 769	6	7
54	9.62 432	27	9.66 669	33	0.33 331	9.95 763	6	6
55	9.62 459	27	9.66 702	33	0.33 298	9.95 757	6	5
56	9.62 486	27	9.66 735	33	0.33 265	9.95 751	6	4
57	9.62 513	28	9.66 768	33	0.33 232	9.95 745	6	3
58	9.62 541	27	9.66 801	33	0.33 199	9.95 739	6	2
59	9.62 568	27	9.66 834	33	0.33 166	9.95 733	5	1
60	9.62 595		9.66 867		0.33 133	9.95 728		**0**
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.	′

P. P.

	34	33
1	3.4	3.3
2	6.8	6.6
3	10.2	9.9
4	13.6	13.2
5	17.0	16.5
6	20.4	19.8
7	23.8	23.1
8	27.2	26.4
9	30.6	29.7

	29	28	27
1	2.9	2.8	2.7
2	5.8	5.6	5.4
3	8.7	8.4	8.1
4	11.6	11.2	10.8
5	14.5	14.0	13.5
6	17.4	16.8	16.2
7	20.3	19.6	18.9
8	23.2	22.4	21.6
9	26.1	25.2	24.3

	6	5
1	0.6	0.5
2	1.2	1.0
3	1.8	1.5
4	2.4	2.0
5	3.0	2.5
6	3.6	3.0
7	4.2	3.5
8	4.8	4.0
9	5.4	4.5

	6/34	6/33	5/34
0	2.8	2.8	3.4
1	8.5	8.2	10.2
2	14.2	13.8	17.0
3	19.8	19.2	23.8
4	25.5	24.8	30.6
5	31.2	30.2	—

′	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.	
0	9.62 595	27	9.66 867	33	0.33 133	9.95 728	6	60
1	9.62 622	27	9.66 900	33	0.33 100	9.95 722	6	59
2	9.62 649	27	9.66 933	33	0.33 067	9.95 716	6	58
3	9.62 676	27	9.66 966	33	0.33 034	9.95 710	6	57
4	9.62 703	27	9.66 999	33	0.33 001	9.95 704	6	56
5	9.62 730	27	9.67 032	33	0.32 968	9.95 698	6	55
6	9.62 757	27	9.67 065	33	0.32 935	9.95 692	6	54
7	9.62 784	27	9.67 098	33	0.32 902	9.95 686	6	53
8	9.62 811	27	9.67 131	32	0.32 869	9.95 680	6	52
9	9.62 838	27	9.67 163	33	0.32 837	9.95 674	6	51
10	9.62 865	27	9.67 196	33	0.32 804	9.95 668	5	50
11	9.62 892	26	9.67 229	33	0.32 771	9.95 663	6	49
12	9.62 918	27	9.67 262	33	0.32 738	9.95 657	6	48
13	9.62 945	27	9.67 295	32	0.32 705	9.95 651	6	47
14	9.62 972	27	9.67 327	33	0.32 673	9.95 645	6	46
15	9.62 999	27	9.67 360	33	0.32 640	9.95 639	6	45
16	9.63 026	26	9.67 393	33	0.32 607	9.95 633	6	44
17	9.63 052	27	9.67 426	32	0.32 574	9.95 627	6	43
18	9.63 079	27	9.67 458	33	0.32 542	9.95 621	6	42
19	9.63 106	27	9.67 491	33	0.32 509	9.95 615	6	41
20	9.63 133	26	9.67 524	32	0.32 476	9.95 609	6	40
21	9.63 159	27	9.67 556	33	0.32 444	9.95 603	6	39
22	9.63 186	27	9.67 589	33	0.32 411	9.95 597	6	38
23	9.63 213	26	9.67 622	32	0.32 378	9.95 591	6	37
24	9.63 239	27	9.67 654	33	0.32 346	9.95 585	6	36
25	9.63 266	26	9.67 687	32	0.32 313	9.95 579	6	35
26	9.63 292	27	9.67 719	33	0.32 281	9.95 573	6	34
27	9.63 319	26	9.67 752	33	0.32 248	9.95 567	6	33
28	9.63 345	27	9.67 785	32	0.32 215	9.95 561	6	32
29	9.63 372	26	9.67 817	33	0.32 183	9.95 555	6	31
30	9.63 398	27	9.67 850	32	0.32 150	9.95 549	6	30
31	9.63 425	26	9.67 882	33	0.32 118	9.95 543	6	29
32	9.63 451	27	9.67 915	32	0.32 085	9.95 537	6	28
33	9.63 478	26	9.67 947	33	0.32 053	9.95 531	6	27
34	9.63 504	27	9.67 980	32	0.32 020	9.95 525	6	26
35	9.63 531	26	9.68 012	32	0.31 988	9.95 519	6	25
36	9.63 557	26	9.68 044	33	0.31 956	9.95 513	6	24
37	9.63 583	27	9.68 077	32	0.31 923	9.95 507	7	23
38	9.63 610	26	9.68 109	33	0.31 891	9.95 500	6	22
39	9.63 636	26	9.68 142	32	0.31 858	9.95 494	6	21
40	9.63 662	27	9.68 174	32	0.31 826	9.95 488	6	20
41	9.63 689	26	9.68 206	33	0.31 794	9.95 482	6	19
42	9.63 715	26	9.68 239	32	0.31 761	9.95 476	6	18
43	9.63 741	26	9.68 271	32	0.31 729	9.95 470	6	17
44	9.63 767	27	9.68 303	33	0.31 697	9.95 464	6	16
45	9.63 794	26	9.68 336	32	0.31 664	9.95 458	6	15
46	9.63 820	26	9.68 368	32	0.31 632	9.95 452	6	14
47	9.63 846	26	9.68 400	32	0.31 600	9.95 446	6	13
48	9.63 872	26	9.68 432	33	0.31 568	9.95 440	6	12
49	9.63 898	26	9.68 465	32	0.31 535	9.95 434	7	11
50	9.63 924	26	9.68 497	32	0.31 503	9.95 427	6	10
51	9.63 950	26	9.68 529	32	0.31 471	9.95 421	6	9
52	9.63 976	26	9.68 561	32	0.31 439	9.95 415	6	8
53	9.64 002	26	9.68 593	33	0.31 407	9.95 409	6	7
54	9.64 028	26	9.68 626	32	0.31 374	9.95 403	6	6
55	9.64 054	26	9.68 658	32	0.31 342	9.95 397	6	5
56	9.64 080	26	9.68 690	32	0.31 310	9.95 391	7	4
57	9.64 106	26	9.68 722	32	0.31 278	9.95 384	6	3
58	9.64 132	26	9.68 754	32	0.31 246	9.95 378	6	2
59	9.64 158	26	9.68 786	32	0.31 214	9.95 372	6	1
60	9.64 184		9.68 818		0.31 182	9.95 366		0
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.	′

P. P.

	33	32
1	3.3	3.2
2	6.6	6.4
3	9.9	9.6
4	13.2	12.8
5	16.5	16.0
6	19.8	19.2
7	23.1	22.4
8	26.4	25.6
9	29.7	28.8

	27	26
1	2.7	2.6
2	5.4	5.2
3	8.1	7.8
4	10.8	10.4
5	13.5	13.0
6	16.2	15.6
7	18.9	18.2
8	21.6	20.8
9	24.3	23.4

	7	6	5
1	0.7	0.6	0.5
2	1.4	1.2	1.0
3	2.1	1.8	1.5
4	2.8	2.4	2.0
5	3.5	3.0	2.5
6	4.2	3.6	3.0
7	4.9	4.2	3.5
8	5.6	4.8	4.0
9	6.3	5.4	4.5

	7/32	6/32	5/33
0	2.3	2.7	3.3
1	6.9	8.0	9.9
2	11.4	13.3	16.5
3	16.0	18.7	23.1
4	20.6	24.0	29.7
5	25.1	29.3	—
6	29.7	—	—
7			

′	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.	
0	9.64 184	26	9.68 818	32	0.31 182	9.95 366	6	**60**
1	9.64 210	26	9.68 850	32	0.31 150	9.95 360	6	59
2	9.64 236	26	9.68 882	32	0.31 118	9.95 354	6	58
3	9.64 262	26	9.68 914	32	0.31 086	9.95 348	7	57
4	9.64 288	25	9.68 946	32	0.31 054	9.95 341	6	56
5	9.64 313	26	9.68 978	32	0.31 022	9.95 335	6	55
6	9.64 339	26	9.69 010	32	0.30 990	9.95 329	6	54
7	9.64 365	26	9.69 042	32	0.30 958	9.95 323	6	53
8	9.64 391	26	9.69 074	32	0.30 926	9.95 317	7	52
9	9.64 417	25	9.69 106	32	0.30 894	9.95 310	6	51
10	9.64 442	26	9.69 138	32	0.30 862	9.95 304	6	**50**
11	9.64 468	26	9.69 170	32	0.30 830	9.95 298	6	49
12	9.64 494	25	9.69 202	32	0.30 798	9.95 292	6	48
13	9.64 519	26	9.69 234	32	0.30 766	9.95 286	7	47
14	9.64 545	26	9.69 266	32	0.30 734	9.95 279	6	46
15	9.64 571	25	9.69 298	31	0.30 702	9.95 273	6	45
16	9.64 596	26	9.69 329	32	0.30 671	9.95 267	6	44
17	9.64 622	25	9.69 361	32	0.30 639	9.95 261	7	43
18	9.64 647	26	9.69 393	32	0.30 607	9.95 254	6	42
19	9.64 673	25	9.69 425	32	0.30 575	9.95 248	6	41
20	9.64 698	26	9.69 457	31	0.30 543	9.95 242	6	**40**
21	9.64 724	25	9.69 488	32	0.30 512	9.95 236	7	39
22	9.64 749	26	9.69 520	32	0.30 480	9.95 229	6	38
23	9.64 775	25	9.69 552	32	0.30 448	9.95 223	6	37
24	9.64 800	26	9.69 584	31	0.30 416	9.95 217	6	36
25	9.64 826	25	9.69 615	32	0.30 385	9.95 211	7	35
26	9.64 851	26	9.69 647	32	0.30 353	9.95 204	6	34
27	9.64 877	25	9.69 679	31	0.30 321	9.95 198	6	33
28	9.64 902	25	9.69 710	32	0.30 290	9.95 192	7	32
29	9.64 927	26	9.69 742	32	0.30 258	9.95 185	6	31
30	9.64 953	25	9.69 774	31	0.30 226	9.95 179	6	**30**
31	9.64 978	25	9.69 805	32	0.30 195	9.95 173	6	29
32	9.65 003	26	9.69 837	31	0.30 163	9.95 167	7	28
33	9.65 029	25	9.69 868	32	0.30 132	9.95 160	6	27
34	9.65 054	25	9.69 900	32	0.30 100	9.95 154	6	26
35	9.65 079	25	9.69 932	31	0.30 068	9.95 148	7	25
36	9.65 104	26	9.69 963	32	0.30 037	9.95 141	6	24
37	9.65 130	25	9.69 995	31	0.30 005	9.95 135	6	23
38	9.65 155	25	9.70 026	32	0.29 974	9.95 129	7	22
39	9.65 180	25	9.70 058	31	0.29 942	9.95 122	6	21
40	9.65 205	25	9.70 089	32	0.29 911	9.95 116	6	**20**
41	9.65 230	25	9.70 121	31	0.29 879	9.95 110	7	19
42	9.65 255	26	9.70 152	32	0.29 848	9.95 103	6	18
43	9.65 281	25	9.70 184	31	0.29 816	9.95 097	7	17
44	9.65 306	25	9.70 215	32	0.29 785	9.95 090	6	16
45	9.65 331	25	9.70 247	31	0.29 753	9.95 084	6	15
46	9.65 356	25	9.70 278	31	0.29 722	9.95 078	7	14
47	9.65 381	25	9.70 309	32	0.29 691	9.95 071	6	13
48	9.65 406	25	9.70 341	31	0.29 659	9.95 065	6	12
49	9.65 431	25	9.70 372	32	0.29 628	9.95 059	7	11
50	9.65 456	25	9.70 404	31	0.29 596	9.95 052	6	**10**
51	9.65 481	25	9.70 435	31	0.29 565	9.95 046	7	9
52	9.65 506	25	9.70 466	32	0.29 534	9.95 039	6	8
53	9.65 531	25	9.70 498	31	0.29 502	9.95 033	6	7
54	9.65 556	24	9.70 529	31	0.29 471	9.95 027	7	6
55	9.65 580	25	9.70 560	32	0.29 440	9.95 020	6	5
56	9.65 605	25	9.70 592	31	0.29 408	9.95 014	7	4
57	9.65 630	25	9.70 623	31	0.29 377	9.95 007	6	3
58	9.65 655	25	9.70 654	31	0.29 346	9.95 001	6	2
59	9.65 680	25	9.70 685	32	0.29 315	9.94 995	7	1
60	9.65 705		9.70 717		0.29 283	9.94 988		**0**
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.	′

P. P.

	32	31
1	3.2	3.1
2	6.4	6.2
3	9.6	9.3
4	12.8	12.4
5	16.0	15.5
6	19.2	18.6
7	22.4	21.7
8	25.6	24.8
9	28.8	27.9

	26	25	24
1	2.6	2.5	2.4
2	5.2	5.0	4.8
3	7.8	7.5	7.2
4	10.4	10.0	9.6
5	13.0	12.5	12.0
6	15.6	15.0	14.4
7	18.2	17.5	16.8
8	20.8	20.0	19.2
9	23.4	22.5	21.6

	7	6
1	0.7	0.6
2	1.4	1.2
3	2.1	1.8
4	2.8	2.4
5	3.5	3.0
6	4.2	3.6
7	4.9	4.2
8	5.6	4.8
9	6.3	5.4

	7/32	7/31	6/32
0			
1	2.3	2.2	2.7
2	6.9	6.6	8.0
3	11.4	11.1	13.3
4	16.0	15.5	18.7
5	20.6	19.9	24.0
6	25.1	24.4	29.3
7	29.7	28.8	—

′	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.	′
0	9.65 705	24	9.70 717	31	0.29 283	9.94 988	6	60
1	9.65 729	25	9.70 748	31	0.29 252	9.94 982	7	59
2	9.65 754	25	9.70 779	31	0.29 221	9.94 975	6	58
3	9.65 779	25	9.70 810	31	0.29 190	9.94 969	7	57
4	9.65 804	24	9.70 841	32	0.29 159	9.94 962	6	56
5	9.65 828	25	9.70 873	31	0.29 127	9.94 956	7	55
6	9.65 853	25	9.70 904	31	0.29 096	9.94 949	6	54
7	9.65 878	24	9.70 935	31	0.29 065	9.94 943	7	53
8	9.65 902	25	9.70 966	31	0.29 034	9.94 936	6	52
9	9.65 927	25	9.70 997	31	0.29 003	9.94 930	7	51
10	9.65 952	24	9.71 028	31	0.28 972	9.94 923	6	50
11	9.65 976	25	9.71 059	31	0.28 941	9.94 917	6	49
12	9.66 001	24	9.71 090	31	0.28 910	9.94 911	7	48
13	9.66 025	25	9.71 121	32	0.28 879	9.94 904	6	47
14	9.66 050	25	9.71 153	31	0.28 847	9.94 898	7	46
15	9.66 075	24	9.71 184	31	0.28 816	9.94 891	6	45
16	9.66 099	25	9.71 215	31	0.28 785	9.94 885	7	44
17	9.66 124	24	9.71 246	31	0.28 754	9.94 878	7	43
18	9.66 148	25	9.71 277	31	0.28 723	9.94 871	6	42
19	9.66 173	24	9.71 308	31	0.28 692	9.94 865	7	41
20	9.66 197	24	9.71 339	31	0.28 661	9.94 858	6	40
21	9.66 221	25	9.71 370	31	0.28 630	9.94 852	7	39
22	9.66 246	24	9.71 401	30	0.28 599	9.94 845	6	38
23	9.66 270	25	9.71 431	31	0.28 569	9.94 839	7	37
24	9.66 295	24	9.71 462	31	0.28 538	9.94 832	6	36
25	9.66 319	24	9.71 493	31	0.28 507	9.94 826	7	35
26	9.66 343	25	9.71 524	31	0.28 476	9.94 819	6	34
27	9.66 368	24	9.71 555	31	0.28 445	9.94 813	7	33
28	9.66 392	24	9.71 586	31	0.28 414	9.94 806	7	32
29	9.66 416	25	9.71 617	31	0.28 383	9.94 799	6	31
30	9.66 441	24	9.71 648	31	0.28 352	9.94 793	7	30
31	9.66 465	24	9.71 679	30	0.28 321	9.94 786	6	29
32	9.66 489	24	9.71 709	31	0.28 291	9.94 780	7	28
33	9.66 513	24	9.71 740	31	0.28 260	9.94 773	6	27
34	9.66 537	25	9.71 771	31	0.28 229	9.94 767	7	26
35	9.66 562	24	9.71 802	31	0.28 198	9.94 760	7	25
36	9.66 586	24	9.71 833	30	0.28 167	9.94 753	6	24
37	9.66 610	24	9.71 863	31	0.28 137	9.94 747	7	23
38	9.66 634	24	9.71 894	31	0.28 106	9.94 740	6	22
39	9.66 658	24	9.71 925	30	0.28 075	9.94 734	7	21
40	9.66 682	24	9.71 955	31	0.28 045	9.94 727	7	20
41	9.66 706	25	9.71 986	31	0.28 014	9.94 720	6	19
42	9.66 731	24	9.72 017	31	0.27 983	9.94 714	7	18
43	9.66 755	24	9.72 048	30	0.27 952	9.94 707	7	17
44	9.66 779	24	9.72 078	31	0.27 922	9.94 700	6	16
45	9.66 803	24	9.72 109	31	0.27 891	9.94 694	7	15
46	9.66 827	24	9.72 140	30	0.27 860	9.94 687	7	14
47	9.66 851	24	9.72 170	31	0.27 830	9.94 680	6	13
48	9.66 875	24	9.72 201	30	0.27 799	9.94 674	7	12
49	9.66 899	23	9.72 231	31	0.27 769	9.94 667	7	11
50	9.66 922	24	9.72 262	31	0.27 738	9.94 660	6	10
51	9.66 946	24	9.72 293	30	0.27 707	9.94 654	7	9
52	9.66 970	24	9.72 323	31	0.27 677	9.94 647	7	8
53	9.66 994	24	9.72 354	30	0.27 646	9.94 640	6	7
54	9.67 018	24	9.72 384	31	0.27 616	9.94 634	7	6
55	9.67 042	24	9.72 415	30	0.27 585	9.94 627	7	5
56	9.67 066	24	9.72 445	31	0.27 555	9.94 620	6	4
57	9.67 090	23	9.72 476	30	0.27 524	9.94 614	7	3
58	9.67 113	24	9.72 506	31	0.27 494	9.94 607	7	2
59	9.67 137	24	9.72 537	30	0.27 463	9.94 600	7	1
60	9.67 161		9.72 567		0.27 433	9.94 593		0
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.	′

P. P.

	32	31	30
1	3.2	3.1	3.0
2	6.4	6.2	6.0
3	9.6	9.3	9.0
4	12.8	12.4	12.0
5	16.0	15.5	15.0
6	19.2	18.6	18.0
7	22.4	21.7	21.0
8	25.6	24.8	24.0
9	28.8	27.9	27.0

	25	24	23
1	2.5	2.4	2.3
2	5.0	4.8	4.6
3	7.5	7.2	6.9
4	10.0	9.6	9.2
5	12.5	12.0	11.5
6	15.0	14.4	13.8
7	17.5	16.8	16.1
8	20.0	19.2	18.4
9	22.5	21.6	20.7

	7	6
1	0.7	0.6
2	1.4	1.2
3	2.1	1.8
4	2.8	2.4
5	3.5	3.0
6	4.2	3.6
7	4.9	4.2
8	5.6	4.8
9	6.3	5.4

	7/30	6/31	6/30
0	2.1	2.6	2.5
1	6.4	7.8	7.5
2	10.7	12.9	12.5
3	15.0	18.1	17.5
4	19.3	23.2	22.5
5	23.6	28.4	27.5
6	27.9	—	—
7			

P. P.

′	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.	′
0	9.67 161	24	9.72 567	31	0.27 433	9.94 593	6	60
1	9.67 185	23	9.72 598	30	0.27 402	9.94 587	7	59
2	9.67 208	24	9.72 628	31	0.27 372	9.94 580	7	58
3	9.67 232	24	9.72 659	30	0.27 341	9.94 573	6	57
4	9.67 256	24	9.72 689	31	0.27 311	9.94 567	7	56
5	9.67 280	23	9.72 720	30	0.27 280	9.94 560	7	55
6	9.67 303	24	9.72 750	30	0.27 250	9.94 553	7	54
7	9.67 327	23	9.72 780	31	0.27 220	9.94 546	6	53
8	9.67 350	24	9.72 811	30	0.27 189	9.94 540	7	52
9	9.67 374	24	9.72 841	31	0.27 159	9.94 533	7	51
10	9.67 398	23	9.72 872	30	0.27 128	9.94 526	7	50
11	9.67 421	24	9.72 902	30	0.27 098	9.94 519	6	49
12	9.67 445	23	9.72 932	31	0.27 068	9.94 513	7	48
13	9.67 468	24	9.72 963	30	0.27 037	9.94 506	7	47
14	9.67 492	23	9.72 993	30	0.27 007	9.94 499	7	46
15	9.67 515	24	9.73 023	31	0.26 977	9.94 492	7	45
16	9.67 539	23	9.73 054	30	0.26 946	9.94 485	6	44
17	9.67 562	24	9.73 084	30	0.26 916	9.94 479	7	43
18	9.67 586	23	9.73 114	30	0.26 886	9.94 472	7	42
19	9.67 609	24	9.73 144	31	0.26 856	9.94 465	7	41
20	9.67 633	23	9.73 175	30	0.26 825	9.94 458	7	40
21	9.67 656	24	9.73 205	30	0.26 795	9.94 451	6	39
22	9.67 680	23	9.73 235	30	0.26 765	9.94 445	7	38
23	9.67 703	23	9.73 265	30	0.26 735	9.94 438	7	37
24	9.67 726	24	9.73 295	31	0.26 705	9.94 431	7	36
25	9.67 750	23	9.73 326	30	0.26 674	9.94 424	7	35
26	9.67 773	23	9.73 356	30	0.26 644	9.94 417	7	34
27	9.67 796	24	9.73 386	30	0.26 614	9.94 410	6	33
28	9.67 820	23	9.73 416	30	0.26 584	9.94 404	7	32
29	9.67 843	23	9.73 446	30	0.26 554	9.94 397	7	31
30	9.67 866	24	9.73 476	31	0.26 524	9.94 390	7	30
31	9.67 890	23	9.73 507	30	0.26 493	9.94 383	7	29
32	9.67 913	23	9.73 537	30	0.26 463	9.94 376	7	28
33	9.67 936	23	9.73 567	30	0.26 433	9.94 369	7	27
34	9.67 959	23	9.73 597	30	0.26 403	9.94 362	7	26
35	9.67 982	24	9.73 627	30	0.26 373	9.94 355	6	25
36	9.68 006	23	9.73 657	30	0.26 343	9.94 349	7	24
37	9.68 029	23	9.73 687	30	0.26 313	9.94 342	7	23
38	9.68 052	23	9.73 717	30	0.26 283	9.94 335	7	22
39	9.68 075	23	9.73 747	30	0.26 253	9.94 328	7	21
40	9.68 098	23	9.73 777	30	0.26 223	9.94 321	7	20
41	9.68 121	23	9.73 807	30	0.26 193	9.94 314	7	19
42	9.68 144	23	9.73 837	30	0.26 163	9.94 307	7	18
43	9.68 167	23	9.73 867	30	0.26 133	9.94 300	7	17
44	9.68 190	23	9.73 897	30	0.26 103	9.94 293	7	16
45	9.68 213	24	9.73 927	30	0.26 073	9.94 286	7	15
46	9.68 237	23	9.73 957	30	0.26 043	9.94 279	6	14
47	9.68 260	23	9.73 987	30	0.26 013	9.94 273	7	13
48	9.68 283	22	9.74 017	30	0.25 983	9.94 266	7	12
49	9.68 305	23	9.74 047	30	0.25 953	9.94 259	7	11
50	9.68 328	23	9.74 077	30	0.25 923	9.94 252	7	10
51	9.68 351	23	9.74 107	30	0.25 893	9.94 245	7	9
52	9.68 374	23	9.74 137	29	0.25 863	9.94 238	7	8
53	9.68 397	23	9.74 166	30	0.25 834	9.94 231	7	7
54	9.68 420	23	9.74 196	30	0.25 804	9.94 224	7	6
55	9.68 443	23	9.74 226	30	0.25 774	9.94 217	7	5
56	9.68 466	23	9.74 256	30	0.25 744	9.94 210	7	4
57	9.68 489	23	9.74 286	30	0.25 714	9.94 203	7	3
58	9.68 512	22	9.74 316	29	0.25 684	9.94 196	7	2
59	9.68 534	23	9.74 345	30	0.25 655	9.94 189	7	1
60	9.68 557		9.74 375		0.25 625	9.94 182		0
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.	′

P. P.

	31	30	29
1	3.1	3.0	2.9
2	6.2	6.0	5.8
3	9.3	9.0	8.7
4	12.4	12.0	11.6
5	15.5	15.0	14.5
6	18.6	18.0	17.4
7	21.7	21.0	20.3
8	24.8	24.0	23.2
9	27.9	27.0	26.1

	24	23	22
1	2.4	2.3	2.2
2	4.8	4.6	4.4
3	7.2	6.9	6.6
4	9.6	9.2	8.8
5	12.0	11.5	11.0
6	14.4	13.8	13.2
7	16.8	16.1	15.4
8	19.2	18.4	17.6
9	21.6	20.7	19.8

	7	6
1	0.7	0.6
2	1.4	1.2
3	2.1	1.8
4	2.8	2.4
5	3.5	3.0
6	4.2	3.6
7	4.9	4.2
8	5.6	4.8
9	6.3	5.4

	7/31	6/31	6/30
0	2.2	2.6	2.5
1	6.6	7.8	7.5
2	11.1	12.9	12.5
3	15.5	18.1	17.5
4	19.9	23.2	22.5
5	24.4	28.4	27.5
6	28.8	—	—
7			

′	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.	
0	9.68 557	23	9.74 375	30	0.25 625	9.94 182	7	**60**
1	9.68 580	23	9.74 405	30	0.25 595	9.94 175	7	59
2	9.68 603	22	9.74 435	30	0.25 565	9.94 168	7	58
3	9.68 625	23	9.74 465	29	0.25 535	9.94 161	7	57
4	9.68 648	23	9.74 494	30	0.25 506	9.94 154	7	56
5	9.68 671	23	9.74 524	30	0.25 476	9.94 147	7	55
6	9.68 694	22	9.74 554	29	0.25 446	9.94 140	7	54
7	9.68 716	23	9.74 583	30	0.25 417	9.94 133	7	53
8	9.68 739	23	9.74 613	30	0.25 387	9.94 126	7	52
9	9.68 762	22	9.74 643	30	0.25 357	9.94 119	7	51
10	9.68 784	23	9.74 673	29	0.25 327	9.94 112	7	**50**
11	9.68 807	22	9.74 702	30	0.25 298	9.94 105	7	49
12	9.68 829	23	9.74 732	30	0.25 268	9.94 098	8	48
13	9.68 852	23	9.74 762	29	0.25 238	9.94 090	7	47
14	9.68 875	22	9.74 791	30	0.25 209	9.94 083	7	46
15	9.68 897	23	9.74 821	30	0.25 179	9.94 076	7	45
16	9.68 920	22	9.74 851	29	0.25 149	9.94 069	7	44
17	9.68 942	23	9.74 880	30	0.25 120	9.94 062	7	43
18	9.68 965	22	9.74 910	29	0.25 090	9.94 055	7	42
19	9.68 987	23	9.74 939	30	0.25 061	9.94 048	7	41
20	9.69 010	22	9.74 969	29	0.25 031	9.94 041	7	**40**
21	9.69 032	23	9.74 998	30	0.25 002	9.94 034	7	39
22	9.69 055	22	9.75 028	30	0.24 972	9.94 027	7	38
23	9.69 077	23	9.75 058	29	0.24 942	9.94 020	8	37
24	9.69 100	22	9.75 087	30	0.24 913	9.94 012	7	36
25	9.69 122	22	9.75 117	29	0.24 883	9.94 005	7	35
26	9.69 144	23	9.75 146	30	0.24 854	9.93 998	7	34
27	9.69 167	22	9.75 176	29	0.24 824	9.93 991	7	33
28	9.69 189	23	9.75 205	30	0.24 795	9.93 984	7	32
29	9.69 212	22	9.75 235	29	0.24 765	9.93 977	7	31
30	9.69 234	22	9.75 264	30	0.24 736	9.93 970	7	**30**
31	9.69 256	23	9.75 294	29	0.24 706	9.93 963	8	29
32	9.69 279	22	9.75 323	30	0.24 677	9.93 955	7	28
33	9.69 301	22	9.75 353	29	0.24 647	9.93 948	7	27
34	9.69 323	22	9.75 382	29	0.24 618	9.93 941	7	26
35	9.69 345	23	9.75 411	30	0.24 589	9.93 934	7	25
36	9.69 368	22	9.75 441	29	0.24 559	9.93 927	7	24
37	9.69 390	22	9.75 470	30	0.24 530	9.93 920	8	23
38	9.69 412	22	9.75 500	29	0.24 500	9.93 912	7	22
39	9.69 434	22	9.75 529	29	0.24 471	9.93 905	7	21
40	9.69 456	23	9.75 558	30	0.24 442	9.93 898	7	**20**
41	9.69 479	22	9.75 588	29	0.24 412	9.93 891	7	19
42	9.69 501	22	9.75 617	30	0.24 383	9.93 884	8	18
43	9.69 523	22	9.75 647	29	0.24 353	9.93 876	7	17
44	9.69 545	22	9.75 676	29	0.24 324	9.93 869	7	16
45	9.69 567	22	9.75 705	30	0.24 295	9.93 862	7	15
46	9.69 589	22	9.75 735	29	0.24 265	9.93 855	8	14
47	9.69 611	22	9.75 764	29	0.24 236	9.93 847	7	13
48	9.69 633	22	9.75 793	29	0.24 207	9.93 840	7	12
49	9.69 655	22	9.75 822	30	0.24 178	9.93 833	7	11
50	9.69 677	22	9.75 852	29	0.24 148	9.93 826	7	**10**
51	9.69 699	22	9.75 881	29	0.24 119	9.93 819	8	9
52	9.69 721	22	9.75 910	29	0.24 090	9.93 811	7	8
53	9.69 743	22	9.75 939	30	0.24 061	9.93 804	7	7
54	9.69 765	22	9.75 969	29	0.24 031	9.93 797	8	6
55	9.69 787	22	9.75 998	29	0.24 002	9.93 789	7	5
56	9.69 809	22	9.76 027	29	0.23 973	9.93 782	7	4
57	9.69 831	22	9.76 056	30	0.23 944	9.93 775	7	3
58	9.69 853	22	9.76 086	29	0.23 914	9.93 768	8	2
59	9.69 875	22	9.76 115	29	0.23 885	9.93 760	7	1
60	9.69 897		9.76 144		0.23 856	9.93 753		**0**
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.	′

P. P.

	30	29	23
1	3.0	2.9	2.3
2	6.0	5.8	4.6
3	9.0	8.7	6.9
4	12.0	11.6	9.2
5	15.0	14.5	11.5
6	18.0	17.4	13.8
7	21.0	20.3	16.1
8	24.0	23.2	18.4
9	27.0	26.1	20.7

	22	8	7
1	2.2	0.8	0.7
2	4.4	1.6	1.4
3	6.6	2.4	2.1
4	8.8	3.2	2.8
5	11.0	4.0	3.5
6	13.2	4.8	4.2
7	15.4	5.6	4.9
8	17.6	6.4	5.6
9	19.8	7.2	6.3

	8/30	8/29
0	1.9	1.8
1	5.6	5.4
2	9.4	9.1
3	13.1	12.7
4	16.9	16.3
5	20.6	19.9
6	24.4	23.6
7	28.1	27.2
8		

	7/30	7/29
0	2.1	2.1
1	6.4	6.2
2	10.7	10.4
3	15.0	14.5
4	19.3	18.6
5	23.6	22.8
6	27.9	26.9
7		

P. P.

′	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.	′
0	9.69 897	22	9.76 144	29	0.23 856	9.93 753	7	60
1	9.69 919	22	9.76 173	29	0.23 827	9.93 746	8	59
2	9.69 941	22	9.76 202	29	0.23 798	9.93 738	7	58
3	9.69 963	21	9.76 231	30	0.23 769	9.93 731	7	57
4	9.69 984	22	9.76 261	29	0.23 739	9.93 724	7	56
5	9.70 006	22	9.76 290	29	0.23 710	9.93 717	8	55
6	9.70 028	22	9.76 319	29	0.23 681	9.93 709	7	54
7	9.70 050	22	9.76 348	29	0.23 652	9.93 702	7	53
8	9.70 072	21	9.76 377	29	0.23 623	9.93 695	8	52
9	9.70 093	22	9.76 406	29	0.23 594	9.93 687	7	51
10	9.70 115	22	9.76 435	29	0.23 565	9.93 680	7	50
11	9.70 137	22	9.76 464	29	0.23 536	9.93 673	8	49
12	9.70 159	21	9.76 493	29	0.23 507	9.93 665	7	48
13	9.70 180	22	9.76 522	29	0.23 478	9.93 658	8	47
14	9.70 202	22	9.76 551	29	0.23 449	9.93 650	7	46
15	9.70 224	21	9.76 580	29	0.23 420	9.93 643	7	45
16	9.70 245	22	9.76 609	30	0.23 391	9.93 636	8	44
17	9.70 267	21	9.76 639	29	0.23 361	9.93 628	7	43
18	9.70 288	22	9.76 668	29	0.23 332	9.93 621	7	42
19	9.70 310	22	9.76 697	28	0.23 303	9.93 614	8	41
20	9.70 332	21	9.76 725	29	0.23 275	9.93 606	7	40
21	9.70 353	22	9.76 754	29	0.23 246	9.93 599	8	39
22	9.70 375	21	9.76 783	29	0.23 217	9.93 591	7	38
23	9.70 396	22	9.76 812	29	0.23 188	9.93 584	7	37
24	9.70 418	21	9.76 841	29	0.23 159	9.93 577	8	36
25	9.70 439	22	9.76 870	29	0.23 130	9.93 569	7	35
26	9.70 461	21	9.76 899	29	0.23 101	9.93 562	8	34
27	9.70 482	22	9.76 928	29	0.23 072	9.93 554	7	33
28	9.70 504	21	9.76 957	29	0.23 043	9.93 547	8	32
29	9.70 525	22	9.76 986	29	0.23 014	9.93 539	7	31
30	9.70 547	21	9.77 015	29	0.22 985	9.93 532	7	30
31	9.70 568	22	9.77 044	29	0.22 956	9.93 525	8	29
32	9.70 590	21	9.77 073	28	0.22 927	9.93 517	7	28
33	9.70 611	22	9.77 101	29	0.22 899	9.93 510	8	27
34	9.70 633	21	9.77 130	29	0.22 870	9.93 502	7	26
35	9.70 654	21	9.77 159	29	0.22 841	9.93 495	8	25
36	9.70 675	22	9.77 188	29	0.22 812	9.93 487	7	24
37	9.70 697	21	9.77 217	29	0.22 783	9.93 480	8	23
38	9.70 718	21	9.77 246	28	0.22 754	9.93 472	7	22
39	9.70 739	22	9.77 274	29	0.22 726	9.93 465	8	21
40	9.70 761	21	9.77 303	29	0.22 697	9.93 457	7	20
41	9.70 782	21	9.77 332	29	0.22 668	9.93 450	8	19
42	9.70 803	21	9.77 361	29	0.22 639	9.93 442	7	18
43	9.70 824	22	9.77 390	28	0.22 610	9.93 435	8	17
44	9.70 846	21	9.77 418	29	0.22 582	9.93 427	7	16
45	9.70 867	21	9.77 447	29	0.22 553	9.93 420	8	15
46	9.70 888	21	9.77 476	29	0.22 524	9.93 412	7	14
47	9.70 909	22	9.77 505	28	0.22 495	9.93 405	8	13
48	9.70 931	21	9.77 533	29	0.22 467	9.93 397	7	12
49	9.70 952	21	9.77 562	29	0.22 438	9.93 390	8	11
50	9.70 973	21	9.77 591	28	0.22 409	9.93 382	7	10
51	9.70 994	21	9.77 619	29	0.22 381	9.93 375	8	9
52	9.71 015	21	9.77 648	29	0.22 352	9.93 367	7	8
53	9.71 036	22	9.77 677	29	0.22 323	9.93 360	8	7
54	9.71 058	21	9.77 706	28	0.22 294	9.93 352	8	6
55	9.71 079	21	9.77 734	29	0.22 266	9.93 344	7	5
56	9.71 100	21	9.77 763	28	0.22 237	9.93 337	8	4
57	9.71 121	21	9.77 791	29	0.22 209	9.93 329	7	3
58	9.71 142	21	9.77 820	29	0.22 180	9.93 322	8	2
59	9.71 163	21	9.77 849	28	0.22 151	9.93 314	7	1
60	9.71 184		9.77 877		0.22 123	9.93 307		0
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.	′

P. P.

	30	29	28
1	3.0	2.9	2.8
2	6.0	5.8	5.6
3	9.0	8.7	8.4
4	12.0	11.6	11.2
5	15.0	14.5	14.0
6	18.0	17.4	16.8
7	21.0	20.3	19.6
8	24.0	23.2	22.4
9	27.0	26.1	25.2

	22	21
1	2.2	2.1
2	4.4	4.2
3	6.6	6.3
4	8.8	8.4
5	11.0	10.5
6	13.2	12.6
7	15.4	14.7
8	17.6	16.8
9	19.8	18.9

	8	7
1	0.8	0.7
2	1.6	1.4
3	2.4	2.1
4	3.2	2.8
5	4.0	3.5
6	4.8	4.2
7	5.6	4.9
8	6.4	5.6
9	7.2	6.3

	7/30	7/29	7/28
0	2.1	2.1	2.0
1	6.4	6.2	6.0
2	10.7	10.4	10.0
3	15.0	14.5	14.0
4	19.3	18.6	18.0
5	23.6	22.8	22.0
6	27.9	26.9	26.0
7			

P. P.

31°

′	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.	′
0	9.71 184	21	9.77 877	29	0.22 123	9.93 307	8	**60**
1	9.71 205	21	9.77 906	29	0.22 094	9.93 299	8	59
2	9.71 226	21	9.77 935	28	0.22 065	9.93 291	7	58
3	9.71 247	21	9.77 963	29	0.22 037	9.93 284	8	57
4	9.71 268	21	9.77 992	28	0.22 008	9.93 276	7	56
5	9.71 289	21	9.78 020	29	0.21 980	9.93 269	8	55
6	9.71 310	21	9.78 049	28	0.21 951	9.93 261	8	54
7	9.71 331	21	9.78 077	29	0.21 923	9.93 253	7	53
8	9.71 352	21	9.78 106	29	0.21 894	9.93 246	8	52
9	9.71 373	20	9.78 135	28	0.21 865	9.93 238	8	51
10	9.71 393	21	9.78 163	29	0.21 837	9.93 230	7	**50**
11	9.71 414	21	9.78 192	28	0.21 808	9.93 223	8	49
12	9.71 435	21	9.78 220	29	0.21 780	9.93 215	8	48
13	9.71 456	21	9.78 249	28	0.21 751	9.93 207	7	47
14	9.71 477	21	9.78 277	29	0.21 723	9.93 200	8	46
15	9.71 498	21	9.78 306	28	0.21 694	9.93 192	8	45
16	9.71 519	20	9.78 334	29	0.21 666	9.93 184	7	44
17	9.71 539	21	9.78 363	28	0.21 637	9.93 177	8	43
18	9.71 560	21	9.78 391	28	0.21 609	9.93 169	8	42
19	9.71 581	21	9.78 419	29	0.21 581	9.93 161	7	41
20	9.71 602	20	9.78 448	28	0.21 552	9.93 154	8	**40**
21	9.71 622	21	9.78 476	29	0.21 524	9.93 146	8	39
22	9.71 643	21	9.78 505	28	0.21 495	9.93 138	7	38
23	9.71 664	21	9.78 533	29	0.21 467	9.93 131	8	37
24	9.71 685	20	9.78 562	28	0.21 438	9.93 123	8	36
25	9.71 705	21	9.78 590	28	0.21 410	9.93 115	7	35
26	9.71 726	21	9.78 618	29	0.21 382	9.93 108	8	34
27	9.71 747	20	9.78 647	28	0.21 353	9.93 100	8	33
28	9.71 767	21	9.78 675	29	0.21 325	9.93 092	8	32
29	9.71 788	21	9.78 704	28	0.21 296	9.93 084	7	31
30	9.71 809	20	9.78 732	28	0.21 268	9.93 077	8	**30**
31	9.71 829	21	9.78 760	29	0.21 240	9.93 069	8	29
32	9.71 850	20	9.78 789	28	0.21 211	9.93 061	8	28
33	9.71 870	21	9.78 817	28	0.21 183	9.93 053	7	27
34	9.71 891	20	9.78 845	29	0.21 155	9.93 046	8	26
35	9.71 911	21	9.78 874	28	0.21 126	9.93 038	8	25
36	9.71 932	20	9.78 902	28	0.21 098	9.93 030	8	24
37	9.71 952	21	9.78 930	29	0.21 070	9.93 022	8	23
38	9.71 973	21	9.78 959	28	0.21 041	9.93 014	7	22
39	9.71 994	20	9.78 987	28	0.21 013	9.93 007	8	21
40	9.72 014	20	9.79 015	28	0.20 985	9.92 999	8	**20**
41	9.72 034	21	9.79 043	29	0.20 957	9.92 991	8	19
42	9.72 055	20	9.79 072	28	0.20 928	9.92 983	7	18
43	9.72 075	21	9.79 100	28	0.20 900	9.92 976	8	17
44	9.72 096	20	9.79 128	28	0.20 872	9.92 968	8	16
45	9.72 116	21	9.79 156	29	0.20 844	9.92 960	8	15
46	9.72 137	20	9.79 185	28	0.20 815	9.92 952	8	14
47	9.72 157	20	9.79 213	28	0.20 787	9.92 944	8	13
48	9.72 177	21	9.79 241	28	0.20 759	9.92 936	7	12
49	9.72 198	20	9.79 269	28	0.20 731	9.92 929	8	11
50	9.72 218	20	9.79 297	29	0.20 703	9.92 921	8	**10**
51	9.72 238	21	9.79 326	28	0.20 674	9.92 913	8	9
52	9.72 259	20	9.79 354	28	0.20 646	9.92 905	8	8
53	9.72 279	20	9.79 382	28	0.20 618	9.92 897	8	7
54	9.72 299	21	9.79 410	28	0.20 590	9.92 889	8	6
55	9.72 320	20	9.79 438	28	0.20 562	9.92 881	7	5
56	9.72 340	20	9.79 466	29	0.20 534	9.92 874	8	4
57	9.72 360	21	9.79 495	28	0.20 505	9.92 866	8	3
58	9.72 381	20	9.79 523	28	0.20 477	9.92 858	8	2
59	9.72 401	20	9.79 551	28	0.20 449	9.92 850	8	1
60	9.72 421		9.79 579		0.20 421	9.92 842		**0**
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.	′

P. P.

	29	28
1	2.9	2.8
2	5.8	5.6
3	8.7	8.4
4	11.6	11.2
5	14.5	14.0
6	17.4	16.8
7	20.3	19.6
8	23.2	22.4
9	26.1	25.2

	21	20
1	2.1	2.0
2	4.2	4.0
3	6.3	6.0
4	8.4	8.0
5	10.5	10.0
6	12.6	12.0
7	14.7	14.0
8	16.8	16.0
9	18.9	18.0

	8	7
1	0.8	0.7
2	1.6	1.4
3	2.4	2.1
4	3.2	2.8
5	4.0	3.5
6	4.8	4.2
7	5.6	4.9
8	6.4	5.6
9	7.2	6.3

	8/30	8/29	8/28
0			
	1.9	1.8	1.8
1			
	5.6	5.4	5.2
2			
	9.4	9.1	8.8
3			
	13.1	12.7	12.2
4			
	16.9	16.3	15.8
5			
	20.6	19.9	19.2
6			
	24.4	23.6	22.8
7			
	28.1	27.2	26.2
8			

′	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.	
0	9.72 421	20	9.79 579	28	0.20 421	9.92 842	8	60
1	9.72 441	20	9.79 607	28	0.20 393	9.92 834	8	59
2	9.72 461	21	9.79 635	28	0.20 365	9.92 826	8	58
3	9.72 482	20	9.79 663	28	0.20 337	9.92 818	8	57
4	9.72 502	20	9.79 691	28	0.20 309	9.92 810	7	56
5	9.72 522	20	9.79 719	28	0.20 281	9.92 803	8	55
6	9.72 542	20	9.79 747	29	0.20 253	9.92 795	8	54
7	9.72 562	20	9.79 776	28	0.20 224	9.92 787	8	53
8	9.72 582	20	9.79 804	28	0.20 196	9.92 779	8	52
9	9.72 602	20	9.79 832	28	0.20 168	9.92 771	8	51
10	9.72 622	21	9.79 860	28	0.20 140	9.92 763	8	50
11	9.72 643	20	9.79 888	28	0.20 112	9.92 755	8	49
12	9.72 663	20	9.79 916	28	0.20 084	9.92 747	8	48
13	9.72 683	20	9.79 944	28	0.20 056	9.92 739	8	47
14	9.72 703	20	9.79 972	28	0.20 028	9.92 731	8	46
15	9.72 723	20	9.80 000	28	0.20 000	9.92 723	8	45
16	9.72 743	20	9.80 028	28	0.19 972	9.92 715	8	44
17	9.72 763	20	9.80 056	28	0.19 944	9.92 707	8	43
18	9.72 783	20	9.80 084	28	0.19 916	9.92 699	8	42
19	9.72 803	20	9.80 112	28	0.19 888	9.92 691	8	41
20	9.72 823	20	9.80 140	28	0.19 860	9.92 683	8	40
21	9.72 843	20	9.80 168	27	0.19 832	9.92 675	8	39
22	9.72 863	20	9.80 195	28	0.19 805	9.92 667	8	38
23	9.72 883	19	9.80 223	28	0.19 777	9.92 659	8	37
24	9.72 902	20	9.80 251	28	0.19 749	9.92 651	8	36
25	9.72 922	20	9.80 279	28	0.19 721	9.92 643	8	35
26	9.72 942	20	9.80 307	28	0.19 693	9.92 635	8	34
27	9.72 962	20	9.80 335	28	0.19 665	9.92 627	8	33
28	9.72 982	20	9.80 363	28	0.19 637	9.92 619	8	32
29	9.73 002	20	9.80 391	28	0.19 609	9.92 611	8	31
30	9.73 022	19	9.80 419	28	0.19 581	9.92 603	8	30
31	9.73 041	20	9.80 447	27	0.19 553	9.92 595	8	29
32	9.73 061	20	9.80 474	28	0.19 526	9.92 587	8	28
33	9.73 081	20	9.80 502	28	0.19 498	9.92 579	8	27
34	9.73 101	20	9.80 530	28	0.19 470	9.92 571	8	26
35	9.73 121	19	9.80 558	28	0.19 442	9.92 563	8	25
36	9.73 140	20	9.80 586	28	0.19 414	9.92 555	9	24
37	9.73 160	20	9.80 614	28	0.19 386	9.92 546	8	23
38	9.73 180	20	9.80 642	27	0.19 358	9.92 538	8	22
39	9.73 200	19	9.80 669	28	0.19 331	9.92 530	8	21
40	9.73 219	20	9.80 697	28	0.19 303	9.92 522	8	20
41	9.73 239	20	9.80 725	28	0.19 275	9.92 514	8	19
42	9.73 259	19	9.80 753	28	0.19 247	9.92 506	8	18
43	9.73 278	20	9.80 781	27	0.19 219	9.92 498	8	17
44	9.73 298	20	9.80 808	28	0.19 192	9.92 490	8	16
45	9.73 318	19	9.80 836	28	0.19 164	9.92 482	9	15
46	9.73 337	20	9.80 864	28	0.19 136	9.92 473	8	14
47	9.73 357	20	9.80 892	27	0.19 108	9.92 465	8	13
48	9.73 377	19	9.80 919	28	0.19 081	9.92 457	8	12
49	9.73 396	20	9.80 947	28	0.19 053	9.92 449	8	11
50	9.73 416	19	9.80 975	28	0.19 025	9.92 441	8	10
51	9.73 435	20	9.81 003	27	0.18 997	9.92 433	8	9
52	9.73 455	19	9.81 030	28	0.18 970	9.92 425	9	8
53	9.73 474	20	9.81 058	28	0.18 942	9.92 416	8	7
54	9.73 494	19	9.81 086	27	0.18 914	9.92 408	8	6
55	9.73 513	20	9.81 113	28	0.18 887	9.92 400	8	5
56	9.73 533	19	9.81 141	28	0.18 859	9.92 392	8	4
57	9.73 552	20	9.81 169	27	0.18 831	9.92 384	8	3
58	9.73 572	19	9.81 196	28	0.18 804	9.92 376	9	2
59	9.73 591	20	9.81 224	28	0.18 776	9.92 367	8	1
60	9.73 611		9.81 252		0.18 748	9.92 359		0
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.	′

P. P.

	29	28	27
1	2.9	2.8	2.7
2	5.8	5.6	5.4
3	8.7	8.4	8.1
4	11.6	11.2	10.8
5	14.5	14.0	13.5
6	17.4	16.8	16.2
7	20.3	19.6	18.9
8	23.2	22.4	21.6
9	26.1	25.2	24.3

	21	20	19
1	2.1	2.0	1.9
2	4.2	4.0	3.8
3	6.3	6.0	5.7
4	8.4	8.0	7.6
5	10.5	10.0	9.5
6	12.6	12.0	11.4
7	14.7	14.0	13.3
8	16.8	16.0	15.2
9	18.9	18.0	17.1

	9	8	7
1	0.9	0.8	0.7
2	1.8	1.6	1.4
3	2.7	2.4	2.1
4	3.6	3.2	2.8
5	4.5	4.0	3.5
6	5.4	4.8	4.2
7	6.3	5.6	4.9
8	7.2	6.4	5.6
9	8.1	7.2	6.3

	8/29	8/28	7/28
0	1.8	1.8	2.0
1	5.4	5.2	6.0
2	9.1	8.8	10.0
3	12.7	12.2	14.0
4	16.3	15.8	18.0
5	19.9	19.2	22.0
6	23.6	22.8	26.0
7	27.2	26.2	—
8			

′	P. P.

′	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.	
0	9.73 611	19	9.81 252	27	0.18 748	9.92 359	8	**60**
1	9.73 630	20	9.81 279	28	0.18 721	9.92 351	8	59
2	9.73 650	19	9.81 307	28	0.18 693	9.92 343	8	58
3	9.73 669	20	9.81 335	27	0.18 665	9.92 335	9	57
4	9.73 689	19	9.81 362	28	0.18 638	9.92 326	8	56
5	9.73 708	19	9.81 390	28	0.18 610	9.92 318	8	55
6	9.73 727	20	9.81 418	27	0.18 582	9.92 310	8	54
7	9.73 747	19	9.81 445	28	0.18 555	9.92 302	9	53
8	9.73 766	19	9.81 473	27	0.18 527	9.92 293	8	52
9	9.73 785	20	9.81 500	28	0.18 500	9.92 285	8	51
10	9.73 805	19	9.81 528	28	0.18 472	9.92 277	8	**50**
11	9.73 824	19	9.81 556	27	0.18 444	9.92 269	9	49
12	9.73 843	20	9.81 583	28	0.18 417	9.92 260	8	48
13	9.73 863	19	9.81 611	27	0.18 389	9.92 252	8	47
14	9.73 882	19	9.81 638	28	0.18 362	9.92 244	9	46
15	9.73 901	20	9.81 666	27	0.18 334	9.92 235	8	45
16	9.73 921	19	9.81 693	28	0.18 307	9.92 227	8	44
17	9.73 940	19	9.81 721	27	0.18 279	9.92 219	8	43
18	9.73 959	19	9.81 748	28	0.18 252	9.92 211	9	42
19	9.73 978	19	9.81 776	27	0.18 224	9.92 202	8	41
20	9.73 997	20	9.81 803	28	0.18 197	9.92 194	8	**40**
21	9.74 017	19	9.81 831	27	0.18 169	9.92 186	9	39
22	9.74 036	19	9.81 858	28	0.18 142	9.92 177	8	38
23	9.74 055	19	9.81 886	27	0.18 114	9.92 169	8	37
24	9.74 074	19	9.81 913	28	0.18 087	9.92 161	9	36
25	9.74 093	20	9.81 941	27	0.18 059	9.92 152	8	35
26	9.74 113	19	9.81 968	28	0.18 032	9.92 144	8	34
27	9.74 132	19	9.81 996	27	0.18 004	9.92 136	9	33
28	9.74 151	19	9.82 023	28	0.17 977	9.92 127	8	32
29	9.74 170	19	9.82 051	27	0.17 949	9.92 119	8	31
30	9.74 189	19	9.82 078	28	0.17 922	9.92 111	9	**30**
31	9.74 208	19	9.82 106	27	0.17 894	9.92 102	8	29
32	9.74 227	19	9.82 133	28	0.17 867	9.92 094	8	28
33	9.74 246	19	9.82 161	27	0.17 839	9.92 086	9	27
34	9.74 265	19	9.82 188	27	0.17 812	9.92 077	8	26
35	9.74 284	19	9.82 215	28	0.17 785	9.92 069	9	25
36	9.74 303	19	9.82 243	27	0.17 757	9.92 060	8	24
37	9.74 322	19	9.82 270	28	0.17 730	9.92 052	8	23
38	9.74 341	19	9.82 298	27	0.17 702	9.92 044	9	22
39	9.74 360	19	9.82 325	27	0.17 675	9.92 035	8	21
40	9.74 379	19	9.82 352	28	0.17 648	9.92 027	9	**20**
41	9.74 398	19	9.82 380	27	0.17 620	9.92 018	8	19
42	9.74 417	19	9.82 407	28	0.17 593	9.92 010	8	18
43	9.74 436	19	9.82 435	27	0.17 565	9.92 002	9	17
44	9.74 455	19	9.82 462	27	0.17 538	9.91 993	8	16
45	9.74 474	19	9.82 489	28	0.17 511	9.91 985	9	15
46	9.74 493	19	9.82 517	27	0.17 483	9.91 976	8	14
47	9.74 512	19	9.82 544	27	0.17 456	9.91 968	9	13
48	9.74 531	18	9.82 571	28	0.17 429	9.91 959	8	12
49	9.74 549	19	9.82 599	27	0.17 401	9.91 951	9	11
50	9.74 568	19	9.82 626	27	0.17 374	9.91 942	8	**10**
51	9.74 587	19	9.82 653	28	0.17 347	9.91 934	9	9
52	9.74 606	19	9.82 681	27	0.17 319	9.91 925	8	8
53	9.74 625	19	9.82 708	27	0.17 292	9.91 917	9	7
54	9.74 644	18	9.82 735	27	0.17 265	9.91 908	8	6
55	9.74 662	19	9.82 762	28	0.17 238	9.91 900	9	5
56	9.74 681	19	9.82 790	27	0.17 210	9.91 891	8	4
57	9.74 700	19	9.82 817	27	0.17 183	9.91 883	9	3
58	9.74 719	18	9.82 844	27	0.17 156	9.91 874	8	2
59	9.74 737	19	9.82 871	28	0.17 129	9.91 866	9	1
60	9.74 756		9.82 899		0.17 101	9.91 857		**0**
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.	′

P. P.

	28	27
1	2.8	2.7
2	5.6	5.4
3	8.4	8.1
4	11.2	10.8
5	14.0	13.5
6	16.8	16.2
7	19.6	18.9
8	22.4	21.6
9	25.2	24.3

	20	19	18
1	2.0	1.9	1.8
2	4.0	3.8	3.6
3	6.0	5.7	5.4
4	8.0	7.6	7.2
5	10.0	9.5	9.0
6	12.0	11.4	10.8
7	14.0	13.3	12.6
8	16.0	15.2	14.4
9	18.0	17.1	16.2

	9	8
1	0.9	0.8
2	1.8	1.6
3	2.7	2.4
4	3.6	3.2
5	4.5	4.0
6	5.4	4.8
7	6.3	5.6
8	7.2	6.4
9	8.1	7.2

	9/28	9/27	8/27
0	1.6	1.5	1.7
1	4.7	4.5	5.1
2	7.8	7.5	8.4
3	10.9	10.5	11.8
4	14.0	13.5	15.2
5	17.1	16.5	18.6
6	20.2	19.5	21.9
7	23.3	22.5	25.3
8	26.4	25.5	—
9			

34°

′	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.	
0	9.74 756	19	9.82 899	27	0.17 101	9.91 857	8	**60**
1	9.74 775	19	9.82 926	27	0.17 074	9.91 849	9	59
2	9.74 794	18	9.82 953	27	0.17 047	9.91 840	8	58
3	9.74 812	19	9.82 980	28	0.17 020	9.91 832	9	57
4	9.74 831	19	9.83 008	27	0.16 992	9.91 823	8	56
5	9.74 850	18	9.83 035	27	0.16 965	9.91 815	9	55
6	9.74 868	19	9.83 062	27	0.16 938	9.91 806	8	54
7	9.74 887	19	9.83 089	28	0.16 911	9.91 798	9	53
8	9.74 906	18	9.83 117	27	0.16 883	9.91 789	8	52
9	9.74 924	19	9.83 144	27	0.16 856	9.91 781	9	51
10	9.74 943	18	9.83 171	27	0.16 829	9.91 772	9	**50**
11	9.74 961	19	9.83 198	27	0.16 802	9.91 763	8	49
12	9.74 980	19	9.83 225	27	0.16 775	9.91 755	9	48
13	9.74 999	18	9.83 252	28	0.16 748	9.91 746	8	47
14	9.75 017	19	9.83 280	27	0.16 720	9.91 738	9	46
15	9.75 036	18	9.83 307	27	0.16 693	9.91 729	9	45
16	9.75 054	19	9.83 334	27	0.16 666	9.91 720	8	44
17	9.75 073	18	9.83 361	27	0.16 639	9.91 712	9	43
18	9.75 091	19	9.83 388	27	0.16 612	9.91 703	8	42
19	9.75 110	18	9.83 415	27	0.16 585	9.91 695	9	41
20	9.75 128	19	9.83 442	28	0.16 558	9.91 686	9	**40**
21	9.75 147	18	9.83 470	27	0.16 530	9.91 677	8	39
22	9.75 165	19	9.83 497	27	0.16 503	9.91 669	9	38
23	9.75 184	18	9.83 524	27	0.16 476	9.91 660	9	37
24	9.75 202	19	9.83 551	27	0.16 449	9.91 651	8	36
25	9.75 221	18	9.83 578	27	0.16 422	9.91 643	9	35
26	9.75 239	19	9.83 605	27	0.16 395	9.91 634	9	34
27	9.75 258	18	9.83 632	27	0.16 368	9.91 625	8	33
28	9.75 276	18	9.83 659	27	0.16 341	9.91 617	9	32
29	9.75 294	19	9.83 686	27	0.16 314	9.91 608	9	31
30	9.75 313	18	9.83 713	27	0.16 287	9.91 599	8	**30**
31	9.75 331	19	9.83 740	28	0.16 260	9.91 591	9	29
32	9.75 350	18	9.83 768	27	0.16 232	9.91 582	9	28
33	9.75 368	18	9.83 795	27	0.16 205	9.91 573	8	27
34	9.75 386	19	9.83 822	27	0.16 178	9.91 565	9	26
35	9.75 405	18	9.83 849	27	0.16 151	9.91 556	9	25
36	9.75 423	18	9.83 876	27	0.16 124	9.91 547	9	24
37	9.75 441	18	9.83 903	27	0.16 097	9.91 538	8	23
38	9.75 459	19	9.83 930	27	0.16 070	9.91 530	9	22
39	9.75 478	18	9.83 957	27	0.16 043	9.91 521	9	21
40	9.75 496	18	9.83 984	27	0.16 016	9.91 512	8	**20**
41	9.75 514	19	9.84 011	27	0.15 989	9.91 504	9	19
42	9.75 533	18	9.84 038	27	0.15 962	9.91 495	9	18
43	9.75 551	18	9.84 065	27	0.15 935	9.91 486	9	17
44	9.75 569	18	9.84 092	27	0.15 908	9.91 477	8	16
45	9.75 587	18	9.84 119	27	0.15 881	9.91 469	9	15
46	9.75 605	19	9.84 146	27	0.15 854	9.91 460	9	14
47	9.75 624	18	9.84 173	27	0.15 827	9.91 451	9	13
48	9.75 642	18	9.84 200	27	0.15 800	9.91 442	9	12
49	9.75 660	18	9.84 227	27	0.15 773	9.91 433	8	11
50	9.75 678	18	9.84 254	26	0.15 746	9.91 425	9	**10**
51	9.75 696	18	9.84 280	27	0.15 720	9.91 416	9	9
52	9.75 714	19	9.84 307	27	0.15 693	9.91 407	9	8
53	9.75 733	18	9.84 334	27	0.15 666	9.91 398	9	7
54	9.75 751	18	9.84 361	27	0.15 639	9.91 389	8	6
55	9.75 769	18	9.84 388	27	0.15 612	9.91 381	9	5
56	9.75 787	18	9.84 415	27	0.15 585	9.91 372	9	4
57	9.75 805	18	9.84 442	27	0.15 558	9.91 363	9	3
58	9.75 823	18	9.84 469	27	0.15 531	9.91 354	9	2
59	9.75 841	18	9.84 496	27	0.15 504	9.91 345	9	1
60	9.75 859		9.84 523		0.15 477	9.91 336		**0**
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.	′

P. P.

	28	27	26
1	2.8	2.7	2.6
2	5.6	5.4	5.2
3	8.4	8.1	7.8
4	11.2	10.8	10.4
5	14.0	13.5	13.0
6	16.8	16.2	15.6
7	19.6	18.9	18.2
8	22.4	21.6	20.8
9	25.2	24.3	23.4

	19	18
1	1.9	1.8
2	3.8	3.6
3	5.7	5.4
4	7.6	7.2
5	9.5	9.0
6	11.4	10.8
7	13.3	12.6
8	15.2	14.4
9	17.1	16.2

	9	8
1	0.9	0.8
2	1.8	1.6
3	2.7	2.4
4	3.6	3.2
5	4.5	4.0
6	5.4	4.8
7	6.3	5.6
8	7.2	6.4
9	8.1	7.2

	9/28	8/28	8/27
0	1.6	1.8	1.7
1	4.7	5.2	5.1
2	7.8	8.8	8.4
3	10.9	12.2	11.8
4	14.0	15.8	15.2
5	17.1	19.2	18.6
6	20.2	22.8	21.9
7	23.3	26.2	25.3
8	26.4	—	—
9			

P. P.

55°

′	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.	′
0	9.75 859	18	9.84 523	27	0.15 477	9.91 336	8	60
1	9.75 877	18	9.84 550	26	0.15 450	9.91 328	9	59
2	9.75 895	18	9.84 576	27	0.15 424	9.91 319	9	58
3	9.75 913	18	9.84 603	27	0.15 397	9.91 310	9	57
4	9.75 931	18	9.84 630	27	0.15 370	9.91 301	9	56
5	9.75 949	18	9.84 657	27	0.15 343	9.91 292	9	55
6	9.75 967	18	9.84 684	27	0.15 316	9.91 283	9	54
7	9.75 985	18	9.84 711	27	0.15 289	9.91 274	8	53
8	9.76 003	18	9.84 738	26	0.15 262	9.91 266	9	52
9	9.76 021	18	9.84 764	27	0.15 236	9.91 257	9	51
10	9.76 039	18	9.84 791	27	0.15 209	9.91 248	9	50
11	9.76 057	18	9.84 818	27	0.15 182	9.91 239	9	49
12	9.76 075̄	18	9.84 845̄	27	0.15 155	9.91 230	9	48
13	9.76 093	18	9.84 872	27	0.15 128	9.91 221	9	47
14	9.76 111	18	9.84 899	26	0.15 101	9.91 212	9	46
15	9.76 129	17	9.84 925	27	0.15 075̄	9.91 203	9	45
16	9.76 146	18	9.84 952	27	0.15 048	9.91 194	9	44
17	9.76 164	18	9.84 979	27	0.15 021	9.91 185	9	43
18	9.76 182	18	9.85 006	27	0.14 994	9.91 176	9	42
19	9.76 200	18	9.85 033	26	0.14 967	9.91 167	9	41
20	9.76 218	18	9.85 059	27	0.14 941	9.91 158	9	40
21	9.76 236	17	9.85 086	27	0.14 914	9.91 149	8	39
22	9.76 253	18	9.85 113	27	0.14 887	9.91 141	9	38
23	9.76 271	18	9.85 140	26	0.14 860	9.91 132	9	37
24	9.76 289	18	9.85 166	27	0.14 834	9.91 123	9	36
25	9.76 307	17	9.85 193	27	0.14 807	9.91 114	9	35
26	9.76 324	18	9.85 220	27	0.14 780	9.91 105̄	9	34
27	9.76 342	18	9.85 247	26	0.14 753	9.91 096	9	33
28	9.76 360	18	9.85 273	27	0.14 727	9.91 087	9	32
29	9.76 378	17	9.85 300	27	0.14 700	9.91 078	9	31
30	9.76 395	18	9.85 327	27	0.14 673	9.91 069	9	30
31	9.76 413	18	9.85 354	26	0.14 646	9.91 060	9	29
32	9.76 431	17	9.85 380	27	0.14 620	9.91 051	9	28
33	9.76 448	18	9.85 407	27	0.14 593	9.91 042	9	27
34	9.76 466	18	9.85 434	26	0.14 566	9.91 033	10	26
35	9.76 484	17	9.85 460	27	0.14 540	9.91 023	9	25
36	9.76 501	18	9.85 487	27	0.14 513	9.91 014	9	24
37	9.76 519	18	9.85 514	26	0.14 486	9.91 005	9	23
38	9.76 537	17	9.85 540	27	0.14 460	9.90 996	9	22
39	9.76 554	18	9.85 567	27	0.14 433	9.90 987	9	21
40	9.76 572	18	9.85 594	26	0.14 406	9.90 978	9	20
41	9.76 590	17	9.85 620	27	0.14 380	9.90 969	9	19
42	9.76 607	18	9.85 647	27	0.14 353	9.90 960	9	18
43	9.76 625̄	17	9.85 674	26	0.14 326	9.90 951	9	17
44	9.76 642	18	9.85 700	27	0.14 300	9.90 942	9	16
45	9.76 660	17	9.85 727	27	0.14 273	9.90 933	9	15
46	9.76 677	18	9.85 754	26	0.14 246	9.90 924	9	14
47	9.76 695̄	17	9.85 780	27	0.14 220	9.90 915̄	9	13
48	9.76 712	18	9.85 807	27	0.14 193	9.90 906	10	12
49	9.76 730	17	9.85 834	26	0.14 166	9.90 896	9	11
50	9.76 747	18	9.85 860	27	0.14 140	9.90 887	9	10
51	9.76 765̄	17	9.85 887	26	0.14 113	9.90 878	9	9
52	9.76 782	18	9.85 913	27	0.14 087	9.90 869	9	8
53	9.76 800	17	9.85 940	27	0.14 060	9.90 860	9	7
54	9.76 817	18	9.85 967	26	0.14 033	9.90 851	9	6
55	9.76 835̄	17	9.85 993	27	0.14 007	9.90 842	10	5
56	9.76 852	18	9.86 020	26	0.13 980	9.90 832	9	4
57	9.76 870	17	9.86 046	27	0.13 954	9.90 823	9	3
58	9.76 887	17	9.86 073	27	0.13 927	9.90 814	9	2
59	9.76 904	18	9.86 100	26	0.13 900	9.90 805̄	9	1
60	9.76 922		9.86 126		0.13 874	9.90 796		0
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.	′

P. P.

	27	26
1	2.7	2.6
2	5.4	5.2
3	8.1	7.8
4	10.8	10.4
5	13.5	13.0
6	16.2	15.6
7	18.9	18.2
8	21.6	20.8
9	24.3	23.4

	18	17
1	1.8	1.7
2	3.6	3.4
3	5.4	5.1
4	7.2	6.8
5	9.0	8.5
6	10.8	10.2
7	12.6	11.9
8	14.4	13.6
9	16.2	15.3

	10	9	8
1	1.0	0.9	0.8
2	2.0	1.8	1.6
3	3.0	2.7	2.4
4	4.0	3.6	3.2
5	5.0	4.5	4.0
6	6.0	5.4	4.8
7	7.0	6.3	5.6
8	8.0	7.2	6.4
9	9.0	8.1	7.2

	10/27	10/26
0	1.4	1.3
1	4.0	3.9
2	6.8	6.5
3	9.4	9.1
4	12.2	11.7
5	14.8	14.3
6	17.6	16.9
7	20.2	19.5
8	23.0	22.1
9	25.6	24.7
10		

	9/27	9/26
0	1.5	1.4
1	4.5	4.3
2	7.5	7.2
3	10.5	10.1
4	13.5	13.0
5	16.5	15.9
6	19.5	18.8
7	22.5	21.7
8	25.5	24.6
9		

P. P.

36°

′	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.	′
0	9.76 922	17	9.86 126	27	0.13 874	9.90 796	9	60
1	9.76 939	18	9.86 153	26	0.13 847	9.90 787	10	59
2	9.76 957	17	9.86 179	27	0.13 821	9.90 777	9	58
3	9.76 974	17	9.86 206	26	0.13 794	9.90 768	9	57
4	9.76 991	18	9.86 232	27	0.13 768	9.90 759	9	56
5	9.77 009	17	9.86 259	26	0.13 741	9.90 750	9	55
6	9.77 026	17	9.86 285	27	0.13 715	9.90 741	10	54
7	9.77 043	18	9.86 312	26	0.13 688	9.90 731	9	53
8	9.77 061	17	9.86 338	27	0.13 662	9.90 722	9	52
9	9.77 078	17	9.86 365	27	0.13 635	9.90 713	9	51
10	9.77 095	17	9.86 392	26	0.13 608	9.90 704	10	50
11	9.77 112	18	9.86 418	27	0.13 582	9.90 694	9	49
12	9.77 130	17	9.86 445	26	0.13 555	9.90 685	9	48
13	9.77 147	17	9.86 471	27	0.13 529	9.90 676	9	47
14	9.77 164	17	9.86 498	26	0.13 502	9.90 667	10	46
15	9.77 181	18	9.86 524	27	0.13 476	9.90 657	9	45
16	9.77 199	17	9.86 551	26	0.13 449	9.90 648	9	44
17	9.77 216	17	9.86 577	26	0.13 423	9.90 639	9	43
18	9.77 233	17	9.86 603	27	0.13 397	9.90 630	10	42
19	9.77 250	18	9.86 630	26	0.13 370	9.90 620	9	41
20	9.77 268	17	9.86 656	27	0.13 344	9.90 611	9	40
21	9.77 285	17	9.86 683	26	0.13 317	9.90 602	10	39
22	9.77 302	17	9.86 709	27	0.13 291	9.90 592	9	38
23	9.77 319	17	9.86 736	26	0.13 264	9.90 583	9	37
24	9.77 336	17	9.86 762	27	0.13 238	9.90 574	9	36
25	9.77 353	17	9.86 789	26	0.13 211	9.90 565	10	35
26	9.77 370	17	9.86 815	27	0.13 185	9.90 555	9	34
27	9.77 387	18	9.86 842	26	0.13 158	9.90 546	9	33
28	9.77 405	17	9.86 868	26	0.13 132	9.90 537	10	32
29	9.77 422	17	9.86 894	27	0.13 106	9.90 527	9	31
30	9.77 439	17	9.86 921	26	0.13 079	9.90 518	9	30
31	9.77 456	17	9.86 947	27	0.13 053	9.90 509	10	29
32	9.77 473	17	9.86 974	26	0.13 026	9.90 499	9	28
33	9.77 490	17	9.87 000	27	0.13 000	9.90 490	10	27
34	9.77 507	17	9.87 027	26	0.12 973	9.90 480	9	26
35	9.77 524	17	9.87 053	26	0.12 947	9.90 471	9	25
36	9.77 541	17	9.87 079	27	0.12 921	9.90 462	10	24
37	9.77 558	17	9.87 106	26	0.12 894	9.90 452	9	23
38	9.77 575	17	9.87 132	26	0.12 868	9.90 443	9	22
39	9.77 592	17	9.87 158	27	0.12 842	9.90 434	10	21
40	9.77 609	17	9.87 185	26	0.12 815	9.90 424	9	20
41	9.77 626	17	9.87 211	27	0.12 789	9.90 415	10	19
42	9.77 643	17	9.87 238	26	0.12 762	9.90 405	9	18
43	9.77 660	17	9.87 264	26	0.12 736	9.90 396	10	17
44	9.77 677	17	9.87 290	27	0.12 710	9.90 386	9	16
45	9.77 694	17	9.87 317	26	0.12 683	9.90 377	9	15
46	9.77 711	17	9.87 343	26	0.12 657	9.90 368	10	14
47	9.77 728	16	9.87 369	27	0.12 631	9.90 358	9	13
48	9.77 744	17	9.87 396	26	0.12 604	9.90 349	10	12
49	9.77 761	17	9.87 422	26	0.12 578	9.90 339	9	11
50	9.77 778	17	9.87 448	27	0.12 552	9.90 330	10	10
51	9.77 795	17	9.87 475	26	0.12 525	9.90 320	9	9
52	9.77 812	17	9.87 501	26	0.12 499	9.90 311	10	8
53	9.77 829	17	9.87 527	27	0.12 473	9.90 301	9	7
54	9.77 846	16	9.87 554	26	0.12 446	9.90 292	10	6
55	9.77 862	17	9.87 580	26	0.12 420	9.90 282	9	5
56	9.77 879	17	9.87 606	27	0.12 394	9.90 273	10	4
57	9.77 896	17	9.87 633	26	0.12 367	9.90 263	9	3
58	9.77 913	17	9.87 659	26	0.12 341	9.90 254	10	2
59	9.77 930	16	9.87 685	26	0.12 315	9.90 244	9	1
60	9.77 946		9.87 711		0.12 289	9.90 235		0
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.	′

P. P.

	27	26
1	2.7	2.6
2	5.4	5.2
3	8.1	7.8
4	10.8	10.4
5	13.5	13.0
6	16.2	15.6
7	18.9	18.2
8	21.6	20.8
9	24.3	23.4

	18	17	16
1	1.8	1.7	1.6
2	3.6	3.4	3.2
3	5.4	5.1	4.8
4	7.2	6.8	6.4
5	9.0	8.5	8.0
6	10.8	10.2	9.6
7	12.6	11.9	11.2
8	14.4	13.6	12.8
9	16.2	15.3	14.4

	10	9
1	1.0	0.9
2	2.0	1.8
3	3.0	2.7
4	4.0	3.6
5	5.0	4.5
6	6.0	5.4
7	7.0	6.3
8	8.0	7.2
9	9.0	8.1

	9/27	9/26
0	1.5	1.4
1	4.5	4.3
2	7.5	7.2
3	10.5	10.1
4	13.5	13.0
5	16.5	15.9
6	19.5	18.8
7	22.5	21.7
8	25.5	24.6
9		

P. P.

53°

′	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.	
0	9.77 946	17	9.87 711	27	0.12 289	9.90 235	10	60
1	9.77 963	17	9.87 738	26	0.12 262	9.90 225	9	59
2	9.77 980	17	9.87 764	26	0.12 236	9.90 216	10	58
3	9.77 997	16	9.87 790	27	0.12 210	9.90 206	9	57
4	9.78 013	17	9.87 817	26	0.12 183	9.90 197	10	56
5	9.78 030	17	9.87 843	26	0.12 157	9.90 187	9	55
6	9.78 047	16	9.87 869	26	0.12 131	9.90 178	10	54
7	9.78 063	17	9.87 895	27	0.12 105	9.90 168	9	53
8	9.78 080	17	9.87 922	26	0.12 078	9.90 159	10	52
9	9.78 097	16	9.87 948	26	0.12 052	9.90 149	10	51
10	9.78 113	17	9.87 974	26	0.12 026	9.90 139	9	50
11	9.78 130	17	9.88 000	27	0.12 000	9.90 130	10	49
12	9.78 147	16	9.88 027	26	0.11 973	9.90 120	9	48
13	9.78 163	17	9.88 053	26	0.11 947	9.90 111	10	47
14	9.78 180	17	9.88 079	26	0.11 921	9.90 101	10	46
15	9.78 197	16	9.88 105	26	0.11 895	9.90 091	9	45
16	9.78 213	17	9.88 131	27	0.11 869	9.90 082	10	44
17	9.78 230	16	9.88 158	26	0.11 842	9.90 072	9	43
18	9.78 246	17	9.88 184	26	0.11 816	9.90 063	10	42
19	9.78 263	17	9.88 210	26	0.11 790	9.90 053	10	41
20	9.78 280	16	9.88 236	26	0.11 764	9.90 043	9	40
21	9.78 296	17	9.88 262	27	0.11 738	9.90 034	10	39
22	9.78 313	16	9.88 289	26	0.11 711	9.90 024	10	38
23	9.78 329	17	9.88 315	26	0.11 685	9.90 014	9	37
24	9.78 346	16	9.88 341	26	0.11 659	9.90 005	10	36
25	9.78 362	17	9.88 367	26	0.11 633	9.89 995	10	35
26	9.78 379	16	9.88 393	27	0.11 607	9.89 985	9	34
27	9.78 395	17	9.88 420	26	0.11 580	9.89 976	10	33
28	9.78 412	16	9.88 446	26	0.11 554	9.89 966	10	32
29	9.78 428	17	9.88 472	26	0.11 528	9.89 956	9	31
30	9.78 445	16	9.88 498	26	0.11 502	9.89 947	10	30
31	9.78 461	17	9.88 524	26	0.11 476	9.89 937	10	29
32	9.78 478	16	9.88 550	27	0.11 450	9.89 927	9	28
33	9.78 494	16	9.88 577	26	0.11 423	9.89 918	10	27
34	9.78 510	17	9.88 603	26	0.11 397	9.89 908	10	26
35	9.78 527	16	9.88 629	26	0.11 371	9.89 898	10	25
36	9.78 543	17	9.88 655	26	0.11 345	9.89 888	9	24
37	9.78 560	16	9.88 681	26	0.11 319	9.89 879	10	23
38	9.78 576	16	9.88 707	26	0.11 293	9.89 869	10	22
39	9.78 592	17	9.88 733	26	0.11 267	9.89 859	10	21
40	9.78 609	16	9.88 759	27	0.11 241	9.89 849	9	20
41	9.78 625	17	9.88 786	26	0.11 214	9.89 840	10	19
42	9.78 642	16	9.88 812	26	0.11 188	9.89 830	10	18
43	9.78 658	16	9.88 838	26	0.11 162	9.89 820	10	17
44	9.78 674	17	9.88 864	26	0.11 136	9.89 810	9	16
45	9.78 691	16	9.88 890	26	0.11 110	9.89 801	10	15
46	9.78 707	16	9.88 916	26	0.11 084	9.89 791	10	14
47	9.78 723	16	9.88 942	26	0.11 058	9.89 781	10	13
48	9.78 739	17	9.88 968	26	0.11 032	9.89 771	10	12
49	9.78 756	16	9.88 994	26	0.11 006	9.89 761	9	11
50	9.78 772	16	9.89 020	26	0.10 980	9.89 752	10	10
51	9.78 788	17	9.89 046	27	0.10 954	9.89 742	10	9
52	9.78 805	16	9.89 073	26	0.10 927	9.89 732	10	8
53	9.78 821	16	9.89 099	26	0.10 901	9.89 722	10	7
54	9.78 837	16	9.89 125	26	0.10 875	9.89 712	10	6
55	9.78 853	16	9.89 151	26	0.10 849	9.89 702	9	5
56	9.78 869	17	9.89 177	26	0.10 823	9.89 693	10	4
57	9.78 886	16	9.89 203	26	0.10 797	9.89 683	10	3
58	9.78 902	16	9.89 229	26	0.10 771	9.89 673	10	2
59	9.78 918	16	9.89 255	26	0.10 745	9.89 663	10	1
60	9.78 934		9.89 281		0.10 719	9.89 653		0
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.	′

P. P.

	27	26
1	2.7	2.6
2	5.4	5.2
3	8.1	7.8
4	10.8	10.4
5	13.5	13.0
6	16.2	15.6
7	18.9	18.2
8	21.6	20.8
9	24.3	23.4

	17	16
1	1.7	1.6
2	3.4	3.2
3	5.1	4.8
4	6.8	6.4
5	8.5	8.0
6	10.2	9.6
7	11.9	11.2
8	13.6	12.8
9	15.3	14.4

	10	9
1	1.0	0.9
2	2.0	1.8
3	3.0	2.7
4	4.0	3.6
5	5.0	4.5
6	6.0	5.4
7	7.0	6.3
8	8.0	7.2
9	9.0	8.1

	10/27	10/26
0	1.4	1.3
1	4.0	3.9
2	6.8	6.5
3	9.4	9.1
4	12.2	11.7
5	14.8	14.3
6	17.6	16.9
7	20.2	19.5
8	23.0	22.1
9	25.6	24.7
10		

P. P.

38°

′	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.	′
0	9.78 934	16	9.89 281	26	0.10 719	9.89 653	10	**60**
1	9.78 950	17	9.89 307	26	0.10 693	9.89 643	10	59
2	9.78 967	16	9.89 333	26	0.10 667	9.89 633	9	58
3	9.78 983	16	9.89 359	26	0.10 641	9.89 624	10	57
4	9.78 999	16	9.89 385	26	0.10 615	9.89 614	10	56
5	9.79 015	16	9.89 411	26	0.10 589	9.89 604	10	55
6	9.79 031	16	9.89 437	26	0.10 563	9.89 594	10	54
7	9.79 047	16	9.89 463	26	0.10 537	9.89 584	10	53
8	9.79 063	16	9.89 489	26	0.10 511	9.89 574	10	52
9	9.79 079	16	9.89 515	26	0.10 485	9.89 564	10	51
10	9.79 095	16	9.89 541	26	0.10 459	9.89 554	10	**50**
11	9.79 111	17	9.89 567	26	0.10 433	9.89 544	10	49
12	9.79 128	16	9.89 593	26	0.10 407	9.89 534	10	48
13	9.79 144	16	9.89 619	26	0.10 381	9.89 524	10	47
14	9.79 160	16	9.89 645	26	0.10 355	9.89 514	10	46
15	9.79 176	16	9.89 671	26	0.10 329	9.89 504	9	45
16	9.79 192	16	9.89 697	26	0.10 303	9.89 495	10	44
17	9.79 208	16	9.89 723	26	0.10 277	9.89 485	10	43
18	9.79 224	16	9.89 749	26	0.10 251	9.89 475	10	42
19	9.79 240	16	9.89 775	26	0.10 225	9.89 465	10	41
20	9.79 256	16	9.89 801	26	0.10 199	9.89 455	10	**40**
21	9.79 272	16	9.89 827	26	0.10 173	9.89 445	10	39
22	9.79 288	16	9.89 853	26	0.10 147	9.89 435	10	38
23	9.79 304	15	9.89 879	26	0.10 121	9.89 425	10	37
24	9.79 319	16	9.89 905	26	0.10 095	9.89 415	10	36
25	9.79 335	16	9.89 931	26	0.10 069	9.89 405	10	35
26	9.79 351	16	9.89 957	26	0.10 043	9.89 395	10	34
27	9.79 367	16	9.89 983	26	0.10 017	9.89 385	10	33
28	9.79 383	16	9.90 009	26	0.09 991	9.89 375	11	32
29	9.79 399	16	9.90 035	26	0.09 965	9.89 364	10	31
30	9.79 415	16	9.90 061	25	0.09 939	9.89 354	10	**30**
31	9.79 431	16	9.90 086	26	0.09 914	9.89 344	10	29
32	9.79 447	16	9.90 112	26	0.09 888	9.89 334	10	28
33	9.79 463	15	9.90 138	26	0.09 862	9.89 324	10	27
34	9.79 478	16	9.90 164	26	0.09 836	9.89 314	10	26
35	9.79 494	16	9.90 190	26	0.09 810	9.89 304	10	25
36	9.79 510	16	9.90 216	26	0.09 784	9.89 294	10	24
37	9.79 526	16	9.90 242	26	0.09 758	9.89 284	10	23
38	9.79 542	16	9.90 268	26	0.09 732	9.89 274	10	22
39	9.79 558	15	9.90 294	26	0.09 706	9.89 264	10	21
40	9.79 573	16	9.90 320	26	0.09 680	9.89 254	10	**20**
41	9.79 589	16	9.90 346	25	0.09 654	9.89 244	11	19
42	9.79 605	16	9.90 371	26	0.09 629	9.89 233	10	18
43	9.79 621	15	9.90 397	26	0.09 603	9.89 223	10	17
44	9.79 636	16	9.90 423	26	0.09 577	9.89 213	10	16
45	9.79 652	16	9.90 449	26	0.09 551	9.89 203	10	15
46	9.79 668	16	9.90 475	26	0.09 525	9.89 193	10	14
47	9.79 684	15	9.90 501	26	0.09 499	9.89 183	10	13
48	9.79 699	16	9.90 527	26	0.09 473	9.89 173	11	12
49	9.79 715	16	9.90 553	25	0.09 447	9.89 162	10	11
50	9.79 731	15	9.90 578	26	0.09 422	9.89 152	10	**10**
51	9.79 746	16	9.90 604	26	0.09 396	9.89 142	10	9
52	9.79 762	16	9.90 630	26	0.09 370	9.89 132	10	8
53	9.79 778	15	9.90 656	26	0.09 344	9.89 122	10	7
54	9.79 793	16	9.90 682	26	0.09 318	9.89 112	11	6
55	9.79 809	16	9.90 708	26	0.09 292	9.89 101	10	5
56	9.79 825	15	9.90 734	25	0.09 266	9.89 091	10	4
57	9.79 840	16	9.90 759	26	0.09 241	9.89 081	10	3
58	9.79 856	16	9.90 785	26	0.09 215	9.89 071	11	2
59	9.79 872	15	9.90 811	26	0.09 189	9.89 060	10	1
60	9.79 887		9.90 837		0.09 163	9.89 050		**0**
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.	′

P. P.

	26	25
1	2.6	2.5
2	5.2	5.0
3	7.8	7.5
4	10.4	10.0
5	13.0	12.5
6	15.6	15.0
7	18.2	17.5
8	20.8	20.0
9	23.4	22.5

	17	16	15
1	1.7	1.6	1.5
2	3.4	3.2	3.0
3	5.1	4.8	4.5
4	6.8	6.4	6.0
5	8.5	8.0	7.5
6	10.2	9.6	9.0
7	11.9	11.2	10.5
8	13.6	12.8	12.0
9	15.3	14.4	13.5

	11	10	9
1	1.1	1.0	0.9
2	2.2	2.0	1.8
3	3.3	3.0	2.7
4	4.4	4.0	3.6
5	5.5	5.0	4.5
6	6.6	6.0	5.4
7	7.7	7.0	6.3
8	8.8	8.0	7.2
9	9.9	9.0	8.1

	10/26	10/25	9/26
0			
	1.3	1.2	1.4
1			
	3.9	3.8	4.3
2			
	6.5	6.2	7.2
3			
	9.1	8.8	10.1
4			
	11.7	11.2	13.0
5			
	14.3	13.8	15.9
6			
	16.9	16.2	18.8
7			
	19.5	18.8	21.7
8			
	22.1	21.2	24.6
9			
	24.7	23.8	—
10			

P. P.

′	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.	′
0	9.79 887	16	9.90 837	26	0.09 163	9.89 050	10	60
1	9.79 903	15	9.90 863	26	0.09 137	9.89 040	10	59
2	9.79 918	16	9.90 889	25	0.09 111	9.89 030	10	58
3	9.79 934	16	9.90 914	26	0.09 086	9.89 020	11	57
4	9.79 950	15	9.90 940	26	0.09 060	9.89 009	10	56
5	9.79 965	16	9.90 966	26	0.09 034	9.88 999	10	55
6	9.79 981	15	9.90 992	26	0.09 008	9.88 989	11	54
7	9.79 996	16	9.91 018	25	0.08 982	9.88 978	10	53
8	9.80 012	15	9.91 043	26	0.08 957	9.88 968	10	52
9	9.80 027	16	9.91 069	26	0.08 931	9.88 958	10	51
10	9.80 043	15	9.91 095	26	0.08 905	9.88 948	11	50
11	9.80 058	16	9.91 121	26	0.08 879	9.88 937	10	49
12	9.80 074	15	9.91 147	25	0.08 853	9.88 927	10	48
13	9.80 089	16	9.91 172	26	0.08 828	9.88 917	11	47
14	9.80 105	15	9.91 198	26	0.08 802	9.88 906	10	46
15	9.80 120	16	9.91 224	26	0.08 776	9.88 896	10	45
16	9.80 136	15	9.91 250	26	0.08 750	9.88 886	11	44
17	9.80 151	15	9.91 276	25	0.08 724	9.88 875	10	43
18	9.80 166	16	9.91 301	26	0.08 699	9.88 865	10	42
19	9.80 182	15	9.91 327	26	0.08 673	9.88 855	11	41
20	9.80 197	16	9.91 353	26	0.08 647	9.88 844	10	40
21	9.80 213	15	9.91 379	25	0.08 621	9.88 834	10	39
22	9.80 228	16	9.91 404	26	0.08 596	9.88 824	11	38
23	9.80 244	15	9.91 430	26	0.08 570	9.88 813	10	37
24	9.80 259	15	9.91 456	26	0.08 544	9.88 803	10	36
25	9.80 274	16	9.91 482	25	0.08 518	9.88 793	11	35
26	9.80 290	15	9.91 507	26	0.08 493	9.88 782	10	34
27	9.80 305	15	9.91 533	26	0.08 467	9.88 772	11	33
28	9.80 320	16	9.91 559	26	0.08 441	9.88 761	10	32
29	9.80 336	15	9.91 585	25	0.08 415	9.88 751	10	31
30	9.80 351	15	9.91 610	26	0.08 390	9.88 741	11	30
31	9.80 366	16	9.91 636	26	0.08 364	9.88 730	10	29
32	9.80 382	15	9.91 662	26	0.08 338	9.88 720	11	28
33	9.80 397	15	9.91 688	25	0.08 312	9.88 709	10	27
34	9.80 412	16	9.91 713	26	0.08 287	9.88 699	11	26
35	9.80 428	15	9.91 739	26	0.08 261	9.88 688	10	25
36	9.80 443	15	9.91 765	26	0.08 235	9.88 678	10	24
37	9.80 458	15	9.91 791	25	0.08 209	9.88 668	11	23
38	9.80 473	16	9.91 816	26	0.08 184	9.88 657	10	22
39	9.80 489	15	9.91 842	26	0.08 158	9.88 647	11	21
40	9.80 504	15	9.91 868	25	0.08 132	9.88 636	10	20
41	9.80 519	15	9.91 893	26	0.08 107	9.88 626	11	19
42	9.80 534	16	9.91 919	26	0.08 081	9.88 615	10	18
43	9.80 550	15	9.91 945	26	0.08 055	9.88 605	11	17
44	9.80 565	15	9.91 971	25	0.08 029	9.88 594	10	16
45	9.80 580	15	9.91 996	26	0.08 004	9.88 584	11	15
46	9.80 595	15	9.92 022	26	0.07 978	9.88 573	10	14
47	9.80 610	15	9.92 048	25	0.07 952	9.88 563	11	13
48	9.80 625	16	9.92 073	26	0.07 927	9.88 552	10	12
49	9.80 641	15	9.92 099	26	0.07 901	9.88 542	11	11
50	9.80 656	15	9.92 125	25	0.07 875	9.88 531	10	10
51	9.80 671	15	9.92 150	26	0.07 850	9.88 521	11	9
52	9.80 686	15	9.92 176	26	0.07 824	9.88 510	11	8
53	9.80 701	15	9.92 202	25	0.07 798	9.88 499	10	7
54	9.80 716	15	9.92 227	26	0.07 773	9.88 489	11	6
55	9.80 731	15	9.92 253	26	0.07 747	9.88 478	10	5
56	9.80 746	16	9.92 279	25	0.07 721	9.88 468	11	4
57	9.80 762	15	9.92 304	26	0.07 696	9.88 457	10	3
58	9.80 777	15	9.92 330	26	0.07 670	9.88 447	11	2
59	9.80 792	15	9.92 356	25	0.07 644	9.88 436	11	1
60	9.80 807		9.92 381		0.07 619	9.88 425		0
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.	′

P. P.

	26	25	16	15	11	10
1	2.6	2.5	1.6	1.5	1.1	1.0
2	5.2	5.0	3.2	3.0	2.2	2.0
3	7.8	7.5	4.8	4.5	3.3	3.0
4	10.4	10.0	6.4	6.0	4.4	4.0
5	13.0	12.5	8.0	7.5	5.5	5.0
6	15.6	15.0	9.6	9.0	6.6	6.0
7	18.2	17.5	11.2	10.5	7.7	7.0
8	20.8	20.0	12.8	12.0	8.8	8.0
9	23.4	22.5	14.4	13.5	9.9	9.0

	11/26	11/25
0	1.2	1.1
1	3.5	3.4
2	5.9	5.7
3	8.3	8.0
4	10.6	10.2
5	13.0	12.5
6	15.4	14.8
7	17.7	17.0
8	20.1	19.3
9	22.5	21.6
10	24.8	23.9
11		

′	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.	
0	9.80 807	15	9.92 381	26	0.07 619	9.88 425	10	60
1	9.80 822	15	9.92 407	26	0.07 593	9.88 415	11	59
2	9.80 837	15	9.92 433	25	0.07 567	9.88 404	10	58
3	9.80 852	15	9.92 458	26	0.07 542	9.88 394	11	57
4	9.80 867	15	9.92 484	26	0.07 516	9.88 383	11	56
5	9.80 882	15	9.92 510	25	0.07 490	9.88 372	10	55
6	9.80 897	15	9.92 535	26	0.07 465	9.88 362	11	54
7	9.80 912	15	9.92 561	26	0.07 439	9.88 351	11	53
8	9.80 927	15	9.92 587	25	0.07 413	9.88 340	10	52
9	9.80 942	15	9.92 612	26	0.07 388	9.88 330	11	51
10	9.80 957	15	9.92 638	25	0.07 362	9.88 319	11	50
11	9.80 972	15	9.92 663	26	0.07 337	9.88 308	10	49
12	9.80 987	15	9.92 689	26	0.07 311	9.88 298	11	48
13	9.81 002	15	9.92 715	25	0.07 285	9.88 287	11	47
14	9.81 017	15	9.92 740	26	0.07 260	9.88 276	10	46
15	9.81 032	15	9.92 766	26	0.07 234	9.88 266	11	45
16	9.81 047	14	9.92 792	25	0.07 208	9.88 255	11	44
17	9.81 061	15	9.92 817	26	0.07 183	9.88 244	10	43
18	9.81 076	15	9.92 843	25	0.07 157	9.88 234	11	42
19	9.81 091	15	9.92 868	26	0.07 132	9.88 223	11	41
20	9.81 106	15	9.92 894	26	0.07 106	9.88 212	11	40
21	9.81 121	15	9.92 920	25	0.07 080	9.88 201	10	39
22	9.81 136	15	9.92 945	26	0.07 055	9.88 191	11	38
23	9.81 151	15	9.92 971	25	0.07 029	9.88 180	11	37
24	9.81 166	14	9.92 996	26	0.07 004	9.88 169	11	36
25	9.81 180	15	9.93 022	26	0.06 978	9.88 158	10	35
26	9.81 195	15	9.93 048	25	0.06 952	9.88 148	11	34
27	9.81 210	15	9.93 073	26	0.06 927	9.88 137	11	33
28	9.81 225	15	9.93 099	25	0.06 901	9.88 126	11	32
29	9.81 240	14	9.93 124	26	0.06 876	9.88 115	10	31
30	9.81 254	15	9.93 150	25	0.06 850	9.88 105	11	30
31	9.81 269	15	9.93 175	26	0.06 825	9.88 094	11	29
32	9.81 284	15	9.93 201	26	0.06 799	9.88 083	11	28
33	9.81 299	15	9.93 227	25	0.06 773	9.88 072	11	27
34	9.81 314	14	9.93 252	26	0.06 748	9.88 061	10	26
35	9.81 328	15	9.93 278	25	0.06 722	9.88 051	11	25
36	9.81 343	15	9.93 303	26	0.06 697	9.88 040	11	24
37	9.81 358	14	9.93 329	25	0.06 671	9.88 029	11	23
38	9.81 372	15	9.93 354	26	0.06 646	9.88 018	11	22
39	9.81 387	15	9.93 380	26	0.06 620	9.88 007	11	21
40	9.81 402	15	9.93 406	25	0.06 594	9.87 996	11	20
41	9.81 417	14	9.93 431	26	0.06 569	9.87 985	10	19
42	9.81 431	15	9.93 457	25	0.06 543	9.87 975	11	18
43	9.81 446	15	9.93 482	26	0.06 518	9.87 964	11	17
44	9.81 461	14	9.93 508	25	0.06 492	9.87 953	11	16
45	9.81 475	15	9.93 533	26	0.06 467	9.87 942	11	15
46	9.81 490	15	9.93 559	25	0.06 441	9.87 931	11	14
47	9.81 505	14	9.93 584	26	0.06 416	9.87 920	11	13
48	9.81 519	15	9.93 610	26	0.06 390	9.87 909	11	12
49	9.81 534	15	9.93 636	25	0.06 364	9.87 898	11	11
50	9.81 549	14	9.93 661	26	0.06 339	9.87 887	10	10
51	9.81 563	15	9.93 687	25	0.06 313	9.87 877	11	9
52	9.81 578	14	9.93 712	26	0.06 288	9.87 866	11	8
53	9.81 592	15	9.93 738	25	0 06 262	9.87 855	11	7
54	9.81 607	15	9.93 763	26	0.06 237	9.87 844	11	6
55	9.81 622	14	9.93 789	25	0.06 211	9.87 833	11	5
56	9.81 636	15	9.93 814	26	0.06 186	9.87 822	11	4
57	9.81 651	14	9.93 840	25	0.06 160	9.87 811	11	3
58	9.81 665	15	9.93 865	26	0.06 135	9.87 800	11	2
59	9.81 680	14	9.93 891	25	0.06 109	9.87 789	11	1
60	9.81 694		9.93 916		0.06 084	9.87 778		0
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.	′

P. P.

	26	25
1	2.6	2.5
2	5.2	5.0
3	7.8	7.5
4	10.4	10.0
5	13.0	12.5
6	15.6	15.0
7	18.2	17.5
8	20.8	20.0
9	23.4	22.5

	15	14
1	1.5	1.4
2	3.0	2.8
3	4.5	4.2
4	6.0	5.6
5	7.5	7.0
6	9.0	8.4
7	10.5	9.8
8	12.0	11.2
9	13.5	12.6

	11	10
1	1.1	1.0
2	2.2	2.0
3	3.3	3.0
4	4.4	4.0
5	5.5	5.0
6	6.6	6.0
7	7.7	7.0
8	8.8	8.0
9	9.9	9.0

	11/26	10/26	10/25
0	1.2	1.3	1.2
1	3.5	3.9	3.8
2	5.9	6.5	6.2
3	8.3	9.1	8.8
4	10.6	11.7	11.2
5	13.0	14.3	13.8
6	15.4	16.9	16.2
7	17.7	19.5	18.8
8	20.1	22.1	21.2
9	22.5	24.7	23.8
10	24.8	—	—
11			

′	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.	
0	9.81 694	15	9.93 916	26	0.06 084	9.87 778	11	60
1	9.81 709	14	9.93 942	25	0.06 058	9.87 767	11	59
2	9.81 723	15	9.93 967	26	0.06 033	9.87 756	11	58
3	9.81 738	14	9.93 993	25	0.06 007	9.87 745	11	57
4	9.81 752	15	9.94 018	26	0.05 982	9.87 734	11	56
5	9.81 767	14	9.94 044	25	0.05 956	9.87 723	11	55
6	9.81 781	15	9.94 069	26	0.05 931	9.87 712	11	54
7	9.81 796	14	9.94 095	25	0.05 905	9.87 701	11	53
8	9.81 810	15	9.94 120	26	0.05 880	9.87 690	11	52
9	9.81 825	14	9.94 146	25	0.05 854	9.87 679	11	51
10	9.81 839	15	9.94 171	26	0.05 829	9.87 668	11	50
11	9.81 854	14	9.94 197	25	0.05 803	9.87 657	11	49
12	9.81 868	14	9.94 222	26	0.05 778	9.87 646	11	48
13	9.81 882	15	9.94 248	25	0.05 752	9.87 635	11	47
14	9.81 897	14	9.94 273	26	0.05 727	9.87 624	11	46
15	9.81 911	15	9.94 299	25	0.05 701	9.87 613	12	45
16	9.81 926	14	9.94 324	26	0.05 676	9.87 601	11	44
17	9.81 940	15	9.94 350	25	0.05 650	9.87 590	11	43
18	9.81 955	14	9.94 375	26	0.05 625	9.87 579	11	42
19	9.81 969	14	9.94 401	25	0.05 599	9.87 568	11	41
20	9.81 983	15	9.94 426	26	0.05 574	9.87 557	11	40
21	9.81 998	14	9.94 452	25	0.05 548	9.87 546	11	39
22	9.82 012	14	9.94 477	26	0.05 523	9.87 535	11	38
23	9.82 026	15	9.94 503	25	0.05 497	9.87 524	11	37
24	9.82 041	14	9.94 528	26	0.05 472	9.87 513	12	36
25	9.82 055	14	9.94 554	25	0.05 446	9.87 501	11	35
26	9.82 069	15	9.94 579	25	0.05 421	9.87 490	11	34
27	9.82 084	14	9.94 604	26	0.05 396	9.87 479	11	33
28	9.82 098	14	9.94 630	25	0.05 370	9.87 468	11	32
29	9.82 112	14	9.94 655	26	0.05 345	9.87 457	11	31
30	9.82 126	15	9.94 681	25	0.05 319	9.87 446	12	30
31	9.82 141	14	9.94 706	26	0.05 294	9.87 434	11	29
32	9.82 155	14	9.94 732	25	0.05 268	9.87 423	11	28
33	9.82 169	15	9.94 757	26	0.05 243	9.87 412	11	27
34	9.82 184	14	9.94 783	25	0.05 217	9.87 401	11	26
35	9.82 198	14	9.94 808	26	0.05 192	9.87 390	12	25
36	9.82 212	14	9.94 834	25	0.05 166	9.87 378	11	24
37	9.82 226	14	9.94 859	25	0.05 141	9.87 367	11	23
38	9.82 240	15	9.94 884	26	0.05 116	9.87 356	11	22
39	9.82 255	14	9.94 910	25	0.05 090	9.87 345	11	21
40	9.82 269	14	9.94 935	26	0.05 065	9.87 334	12	20
41	9.82 283	14	9.94 961	25	0.05 039	9.87 322	11	19
42	9.82 297	14	9.94 986	26	0.05 014	9.87 311	11	18
43	9.82 311	15	9.95 012	25	0.04 988	9.87 300	12	17
44	9.82 326	14	9.95 037	25	0.04 963	9.87 288	11	16
45	9.82 340	14	9.95 062	26	0.04 938	9.87 277	11	15
46	9.82 354	14	9.95 088	25	0.04 912	9.87 266	11	14
47	9.82 368	14	9.95 113	26	0.04 887	9.87 255	12	13
48	9.82 382	14	9.95 139	25	0.04 861	9.87 243	11	12
49	9.82 396	14	9.95 164	26	0.04 836	9.87 232	11	11
50	9.82 410	14	9.95 190	25	0.04 810	9.87 221	12	10
51	9.82 424	15	9.95 215	25	0.04 785	9.87 209	11	9
52	9.82 439	14	9.95 240	26	0.04 760	9.87 198	11	8
53	9.82 453	14	9.95 266	25	0.04 734	9.87 187	12	7
54	9.82 467	14	9.95 291	26	0.04 709	9.87 175	11	6
55	9.82 481	14	9.95 317	25	0.04 683	9.87 164	11	5
56	9.82 495	14	9.95 342	26	0.04 658	9.87 153	12	4
57	9.82 509	14	9.95 368	25	0.04 632	9.87 141	11	3
58	9.82 523	14	9.95 393	25	0.04 607	9.87 130	11	2
59	9.82 537	14	9.95 418	26	0.04 582	9.87 119	12	1
60	9.82 551		9.95 444		0.04 556	9.87 107		0
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.	′

P. P.

	26	25
1	2.6	2.5
2	5.2	5.0
3	7.8	7.5
4	10.4	10.0
5	13.0	12.5
6	15.6	15.0
7	18.2	17.5
8	20.8	20.0
9	23.4	22.5

	15	14
1	1.5	1.4
2	3.0	2.8
3	4.5	4.2
4	6.0	5.6
5	7.5	7.0
6	9.0	8.4
7	10.5	9.8
8	12.0	11.2
9	13.5	12.6

	12	11
1	1.2	1.1
2	2.4	2.2
3	3.6	3.3
4	4.8	4.4
5	6.0	5.5
6	7.2	6.6
7	8.4	7.7
8	9.6	8.8
9	10.8	9.9

	12/26	12/25	11/25
0	1.1	1.0	1.1
1	3.2	3.1	3.4
2	5.4	5.2	5.7
3	7.6	7.3	8.0
4	9.8	9.4	10.2
5	11.9	11.5	12.5
6	14.1	13.5	14.8
7	16.2	15.6	17.0
8	18.4	17.7	19.3
9	20.6	19.8	21.6
10	22.8	21.9	23.9
11	24.9	24.0	—
12			

P. P.

42°

′	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.	
0	9.82 551	14	9.95 444	25	0.04 556	9.87 107	11	60
1	9.82 565	14	9.95 469	26	0.04 531	9.87 096	11	59
2	9.82 579	14	9.95 495	25	0.04 505	9.87 085	12	58
3	9.82 593	14	9.95 520	25	0.04 480	9.87 073	11	57
4	9.82 607	14	9.95 545	26	0.04 455	9.87 062	12	56
5	9.82 621	14	9.95 571	25	0.04 429	9.87 050	11	55
6	9.82 635	14	9.95 596	26	0.04 404	9.87 039	11	54
7	9.82 649	14	9.95 622	25	0.04 378	9.87 028	12	53
8	9.82 663	14	9.95 647	25	0.04 353	9.87 016	11	52
9	9.82 677	14	9.95 672	26	0.04 328	9.87 005	12	51
10	9.82 691	14	9.95 698	25	0.04 302	9.86 993	11	50
11	9.82 705	14	9.95 723	25	0.04 277	9.86 982	12	49
12	9.82 719	14	9.95 748	26	0.04 252	9.86 970	11	48
13	9.82 733	14	9.95 774	25	0.04 226	9.86 959	12	47
14	9.82 747	14	9.95 799	26	0.04 201	9.86 947	11	46
15	9.82 761	14	9.95 825	25	0.04 175	9.86 936	12	45
16	9.82 775	13	9.95 850	25	0.04 150	9.86 924	11	44
17	9.82 788	14	9.95 875	26	0.04 125	9.86 913	11	43
18	9.82 802	14	9.95 901	25	0.04 099	9.86 902	12	42
19	9.82 816	14	9.95 926	26	0.04 074	9.86 890	11	41
20	9.82 830	14	9.95 952	25	0.04 048	9.86 879	12	40
21	9.82 844	14	9.95 977	25	0.04 023	9.86 867	12	39
22	9.82 858	14	9.96 002	26	0.03 998	9.86 855	11	38
23	9.82 872	13	9.96 028	25	0.03 972	9.86 844	12	37
24	9.82 885	14	9.96 053	25	0.03 947	9.86 832	11	36
25	9.82 899	14	9.96 078	26	0.03 922	9.86 821	12	35
26	9.82 913	14	9.96 104	25	0.03 896	9.86 809	11	34
27	9.82 927	14	9.96 129	26	0.03 871	9.86 798	12	33
28	9.82 941	14	9.96 155	25	0.03 845	9.86 786	11	32
29	9.82 955	13	9.96 180	25	0.03 820	9.86 775	12	31
30	9.82 968	14	9.96 205	26	0.03 795	9.86 763	11	30
31	9.82 982	14	9.96 231	25	0.03 769	9.86 752	12	29
32	9.82 996	14	9.96 256	25	0.03 744	9.86 740	12	28
33	9.83 010	13	9.96 281	26	0.03 719	9.86 728	11	27
34	9.83 023	14	9.96 307	25	0.03 693	9.86 717	12	26
35	9.83 037	14	9.96 332	25	0.03 668	9.86 705	11	25
36	9.83 051	14	9.96 357	26	0.03 643	9.86 694	12	24
37	9.83 065	13	9.96 383	25	0.03 617	9.86 682	12	23
38	9.83 078	14	9.96 408	25	0.03 592	9.86 670	11	22
39	9.83 092	14	9.96 433	26	0.03 567	9.86 659	12	21
40	9.83 106	14	9.96 459	25	0.03 541	9.86 647	12	20
41	9.83 120	13	9.96 484	26	0.03 516	9.86 635	11	19
42	9.83 133	14	9.96 510	25	0.03 490	9.86 624	12	18
43	9.83 147	14	9.96 535	25	0.03 465	9.86 612	12	17
44	9.83 161	13	9.96 560	26	0.03 440	9.86 600	11	16
45	9.83 174	14	9.96 586	25	0.03 414	9.86 589	12	15
46	9.83 188	14	9.96 611	25	0.03 389	9.86 577	12	14
47	9.83 202	13	9.96 636	26	0.03 364	9.86 565	11	13
48	9.83 215	14	9.96 662	25	0.03 338	9.86 554	12	12
49	9.83 229	13	9.96 687	25	0.03 313	9.86 542	12	11
50	9.83 242	14	9.96 712	26	0.03 288	9.86 530	12	10
51	9.83 256	14	9.96 738	25	0.03 262	9.86 518	11	9
52	9.83 270	13	9.96 763	25	0.03 237	9.86 507	12	8
53	9.83 283	14	9.96 788	26	0.03 212	9.86 495	12	7
54	9.83 297	13	9.96 814	25	0.03 186	9.86 483	11	6
55	9.83 310	14	9.96 839	25	0.03 161	9.86 472	12	5
56	9.83 324	14	9.96 864	26	0.03 136	9.86 460	12	4
57	9.83 338	13	9.96 890	25	0.03 110	9.86 448	12	3
58	9.83 351	14	9.96 915	25	0.03 085	9.86 436	11	2
59	9.83 365	13	9.96 940	26	0.03 060	9.86 425	12	1
60	9.83 378		9.96 966		0.03 034	9.86 413		0
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.	′

P. P.

	26	25
1	2.6	2.5
2	5.2	5.0
3	7.8	7.5
4	10.4	10.0
5	13.0	12.5
6	15.6	15.0
7	18.2	17.5
8	20.8	20.0
9	23.4	22.5

	14	13
1	1.4	1.3
2	2.8	2.6
3	4.2	3.9
4	5.6	5.2
5	7.0	6.5
6	8.4	7.8
7	9.8	9.1
8	11.2	10.4
9	12.6	11.7

	12	11
1	1.2	1.1
2	2.4	2.2
3	3.6	3.3
4	4.8	4.4
5	6.0	5.5
6	7.2	6.6
7	8.4	7.7
8	9.6	8.8
9	10.8	9.9

	12/26	11/26	11/25
0	1.1	1.2	1.1
1	3.2	3.5	3.4
2	5.4	5.9	5.7
3	7.6	8.3	8.0
4	9.8	10.6	10.2
5	11.9	13.0	12.5
6	14.1	15.4	14.8
7	16.2	17.7	17.0
8	18.4	20.1	19.3
9	20.6	22.5	21.6
10	22.8	24.8	23.9
11	24.9	—	—
12			

′	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.	′
0	9.83 378	14	9.96 966	25	0.03 034	9.86 413	12	60
1	9.83 392	13	9.96 991	25	0.03 009	9.86 401	12	59
2	9.83 405	14	9.97 016	26	0.02 984	9.86 389	12	58
3	9.83 419	13	9.97 042	25	0.02 958	9.86 377	11	57
4	9.83 432	14	9.97 067	25	0.02 933	9.86 366	12	56
5	9.83 446	13	9.97 092	26	0.02 908	9.86 354	12	55
6	9.83 459	14	9.97 118	25	0.02 882	9.86 342	12	54
7	9.83 473	13	9.97 143	25	0.02 857	9.86 330	12	53
8	9.83 486	14	9.97 168	25	0.02 832	9.86 318	12	52
9	9.83 500	13	9.97 193	26	0.02 807	9.86 306	11	51
10	9.83 513	14	9.97 219	25	0.02 781	9.86 295	12	50
11	9.83 527	13	9.97 244	25	0.02 756	9.86 283	12	49
12	9.83 540	14	9.97 269	26	0.02 731	9.86 271	12	48
13	9.83 554	13	9.97 295	25	0.02 705	9.86 259	12	47
14	9.83 567	14	9.97 320	25	0.02 680	9.86 247	12	46
15	9.83 581	13	9.97 345	26	0.02 655	9.86 235	12	45
16	9.83 594	14	9.97 371	25	0.02 629	9.86 223	12	44
17	9.83 608	13	9.97 396	25	0.02 604	9.86 211	11	43
18	9.83 621	13	9.97 421	26	0.02 579	9.86 200	12	42
19	9.83 634	14	9.97 447	25	0.02 553	9.86 188	12	41
20	9.83 648	13	9.97 472	25	0.02 528	9.86 176	12	40
21	9.83 661	13	9.97 497	26	0.02 503	9.86 164	12	39
22	9.83 674	14	9.97 523	25	0.02 477	9.86 152	12	38
23	9.83 688	13	9.97 548	25	0.02 452	9.86 140	12	37
24	9.83 701	14	9.97 573	25	0.02 427	9.86 128	12	36
25	9.83 715	13	9.97 598	26	0.02 402	9.86 116	12	35
26	9.83 728	13	9.97 624	25	0.02 376	9.86 104	12	34
27	9.83 741	14	9.97 649	25	0.02 351	9.86 092	12	33
28	9.83 755	13	9.97 674	26	0.02 326	9.86 080	12	32
29	9.83 768	13	9.97 700	25	0.02 300	9.86 068	12	31
30	9.83 781	14	9.97 725	25	0.02 275	9.86 056	12	30
31	9.83 795	13	9.97 750	26	0.02 250	9.86 044	12	29
32	9.83 808	13	9.97 776	25	0.02 224	9.86 032	12	28
33	9.83 821	13	9.97 801	25	0.02 199	9.86 020	12	27
34	9.83 834	14	9.97 826	25	0.02 174	9.86 008	12	26
35	9.83 848	13	9.97 851	26	0.02 149	9.85 996	12	25
36	9.83 861	13	9.97 877	25	0.02 123	9.85 984	12	24
37	9.83 874	13	9.97 902	25	0.02 098	9.85 972	12	23
38	9.83 887	14	9.97 927	26	0.02 073	9.85 960	12	22
39	9.83 901	13	9.97 953	25	0.02 047	9.85 948	12	21
40	9.83 914	13	9.97 978	25	0.02 022	9.85 936	12	20
41	9.83 927	13	9.98 003	26	0.01 997	9.85 924	12	19
42	9.83 940	14	9.98 029	25	0.01 971	9.85 912	12	18
43	9.83 954	13	9.98 054	25	0.01 946	9.85 900	12	17
44	9.83 967	13	9.98 079	25	0.01 921	9.85 888	12	16
45	9.83 980	13	9.98 104	26	0.01 896	9.85 876	12	15
46	9.83 993	13	9.98 130	25	0.01 870	9.85 864	13	14
47	9.84 006	14	9.98 155	25	0.01 845	9.85 851	12	13
48	9.84 020	13	9.98 180	26	0.01 820	9.85 839	12	12
49	9.84 033	13	9.98 206	25	0.01 794	9.85 827	12	11
50	9.84 046	13	9.98 231	25	0.01 769	9.85 815	12	10
51	9.84 059	13	9.98 256	25	0.01 744	9.85 803	12	9
52	9.84 072	13	9.98 281	26	0.01 719	9.85 791	12	8
53	9.84 085	13	9.98 307	25	0.01 693	9.85 779	13	7
54	9.84 098	14	9.98 332	25	0.01 668	9.85 766	12	6
55	9.84 112	13	9.98 357	26	0.01 643	9.85 754	12	5
56	9.84 125	13	9.98 383	25	0.01 617	9.85 742	12	4
57	9.84 138	13	9.98 408	25	0.01 592	9.85 730	12	3
58	9.84 151	13	9.98 433	25	0.01 567	9.85 718	12	2
59	9.84 164	13	9.98 458	26	0.01 542	9.85 706	13	1
60	9.84 177		9.98 484		0.01 516	9.85 693		0
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.	′

P. P.

	26	25
1	2.6	2.5
2	5.2	5.0
3	7.8	7.5
4	10.4	10.0
5	13.0	12.5
6	15.6	15.0
7	18.2	17.5
8	20.8	20.0
9	23.4	22.5

	14	13
1	1.4	1.3
2	2.8	2.6
3	4.2	3.9
4	5.6	5.2
5	7.0	6.5
6	8.4	7.8
7	9.8	9.1
8	11.2	10.4
9	12.6	11.7

	12	11
1	1.2	1.1
2	2.4	2.2
3	3.6	3.3
4	4.8	4.4
5	6.0	5.5
6	7.2	6.6
7	8.4	7.7
8	9.6	8.8
9	10.8	9.9

	13/26	13/25	12/25
0	1.0	1.0	1.0
1	3.0	2.9	3.1
2	5.0	4.8	5.2
3	7.0	6.7	7.3
4	9.0	8.7	9.4
5	11.0	10.6	11.5
6	13.0	12.5	13.5
7	15.0	14.4	15.6
8	17.0	16.3	17.7
9	19.0	18.3	19.8
10	21.0	20.2	21.9
11	23.0	22.1	24.0
12	25.0	24.0	—

′	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.	′
0	9.84 177	13	9.98 484	25	0.01 516	9.85 693	12	60
1	9.84 190	13	9.98 509	25	0.01 491	9.85 681	12	59
2	9.84 203	13	9.98 534	26	0.01 466	9.85 669	12	58
3	9.84 216	13	9.98 560	25	0.01 440	9.85 657	12	57
4	9.84 229	13	9.98 585	25	0.01 415	9.85 645	13	56
5	9.84 242	13	9.98 610	25	0.01 390	9.85 632	12	55
6	9.84 255	14	9.98 635	26	0.01 365	9.85 620	12	54
7	9.84 269	13	9.98 661	25	0.01 339	9.85 608	12	53
8	9.84 282	13	9.98 686	25	0.01 314	9.85 596	13	52
9	9.84 295	13	9.98 711	26	0.01 289	9.85 583	12	51
10	9.84 308	13	9.98 737	25	0.01 263	9.85 571	12	50
11	9.84 321	13	9.98 762	25	0.01 238	9.85 559	12	49
12	9.84 334	13	9.98 787	25	0.01 213	9.85 547	13	48
13	9.84 347	13	9.98 812	26	0.01 188	9.85 534	12	47
14	9.84 360	13	9.98 838	25	0.01 162	9.85 522	12	46
15	9.84 373	12	9.98 863	25	0.01 137	9.85 510	13	45
16	9.84 385	13	9.98 888	25	0.01 112	9.85 497	12	44
17	9.84 398	13	9.98 913	26	0.01 087	9.85 485	12	43
18	9.84 411	13	9.98 939	25	0.01 061	9.85 473	13	42
19	9.84 424	13	9.98 964	25	0.01 036	9.85 460	12	41
20	9.84 437	13	9.98 989	26	0.01 011	9.85 448	12	40
21	9.84 450	13	9.99 015	25	0.00 985	9.85 436	13	39
22	9.84 463	13	9.99 040	25	0.00 960	9.85 423	12	38
23	9.84 476	13	9.99 065	25	0.00 935	9.85 411	12	37
24	9.84 489	13	9.99 090	26	0.00 910	9.85 399	13	36
25	9.84 502	13	9.99 116	25	0.00 884	9.85 386	12	35
26	9.84 515	13	9.99 141	25	0.00 859	9.85 374	13	34
27	9.84 528	12	9.99 166	25	0.00 834	9.85 361	12	33
28	9.84 540	13	9.99 191	26	0.00 809	9.85 349	12	32
29	9.84 553	13	9.99 217	25	0.00 783	9.85 337	13	31
30	9.84 566	13	9.99 242	25	0.00 758	9.85 324	12	30
31	9.84 579	13	9.99 267	26	0.00 733	9.85 312	13	29
32	9.84 592	13	9.99 293	25	0.00 707	9.85 299	12	28
33	9.84 605	13	9.99 318	25	0.00 682	9.85 287	13	27
34	9.84 618	12	9.99 343	25	0.00 657	9.85 274	12	26
35	9.84 630	13	9.99 368	26	0.00 632	9.85 262	12	25
36	9.84 643	13	9.99 394	25	0.00 606	9.85 250	13	24
37	9.84 656	13	9.99 419	25	0.00 581	9.85 237	12	23
38	9.84 669	13	9.99 444	25	0.00 556	9.85 225	13	22
39	9.84 682	12	9.99 469	26	0.00 531	9.85 212	12	21
40	9.84 694	13	9.99 495	25	0.00 505	9.85 200	13	20
41	9.84 707	13	9.99 520	25	0.00 480	9.85 187	12	19
42	9.84 720	13	9.99 545	25	0.00 455	9.85 175	13	18
43	9.84 733	12	9.99 570	26	0.00 430	9.85 162	12	17
44	9.84 745	13	9.99 596	25	0.00 404	9.85 150	13	16
45	9.84 758	13	9.99 621	25	0.00 379	9.85 137	12	15
46	9.84 771	13	9.99 646	26	0.00 354	9.85 125	13	14
47	9.84 784	12	9.99 672	25	0.00 328	9.85 112	12	13
48	9.84 796	13	9.99 697	25	0.00 303	9.85 100	13	12
49	9.84 809	13	9.99 722	25	0.00 278	9.85 087	13	11
50	9.84 822	13	9.99 747	26	0.00 253	9.85 074	12	10
51	9.84 835	12	9.99 773	25	0.00 227	9.85 062	13	9
52	9.84 847	13	9.99 798	25	0.00 202	9.85 049	12	8
53	9.84 860	13	9.99 823	25	0.00 177	9.85 037	13	7
54	9.84 873	12	9.99 848	26	0.00 152	9.85 024	12	6
55	9.84 885	13	9.99 874	25	0.00 126	9.85 012	13	5
56	9.84 898	13	9.99 899	25	0.00 101	9.84 999	13	4
57	9.84 911	12	9.99 924	25	0.00 076	9.84 986	12	3
58	9.84 923	13	9.99 949	26	0.00 051	9.84 974	13	2
59	9.84 936	13	9.99 975	25	0.00 025	9.84 961	12	1
60	9.84 949		0.00 000		0.00 000	9.84 949		0
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.	′

P. P.

	26	25	14
1	2.6	2.5	1.4
2	5.2	5.0	2.8
3	7.8	7.5	4.2
4	10.4	10.0	5.6
5	13.0	12.5	7.0
6	15.6	15.0	8.4
7	18.2	17.5	9.8
8	20.8	20.0	11.2
9	23.4	22.5	12.6

	13	12
1	1.3	1.2
2	2.6	2.4
3	3.9	3.6
4	5.2	4.8
5	6.5	6.0
6	7.8	7.2
7	9.1	8.4
8	10.4	9.6
9	11.7	10.8

	13/26	13/25
0	1.0	1.0
1	3.0	2.9
2	5.0	4.8
3	7.0	6.7
4	9.0	8.7
5	11.0	10.6
6	13.0	12.5
7	15.0	14.4
8	17.0	16.3
9	19.0	18.3
10	21.0	20.2
11	23.0	22.1
12	25.0	24.0
13		

	12/26	12/25
0	1.1	1.0
1	3.2	3.1
2	5.4	5.2
3	7.6	7.3
4	9.8	9.4
5	11.9	11.5
6	14.1	13.5
7	16.2	15.6
8	18.4	17.7
9	20.6	19.8
10	22.8	21.9
11	24.9	24.0
12		

P. P.

III.

NATURAL
TRIGONOMETRIC FUNCTIONS

FOR EACH MINUTE.

0°

′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.00000	.00000	∞	1.0000	60
1	029	029	3437.7	000	59
2	058	058	1718.9	000	58
3	087	087	1145.9	000	57
4	116	116	859.44	000	56
5	.00145	.00145	687.55	1.0000	55
6	175	175	572.96	000	54
7	204	204	491.11	000	53
8	233	233	429.72	000	52
9	262	262	381.97	000	51
10	.00291	.00291	343.77	1.0000	50
11	320	320	312.52	.99999	49
12	349	349	286.48	999	48
13	378	378	264.44	999	47
14	407	407	245.55	999	46
15	.00436	.00436	229.18	.99999	45
16	465	465	214.86	999	44
17	495	495	202.22	999	43
18	524	524	190.98	999	42
19	553	553	180.93	998	41
20	.00582	.00582	171.89	.99998	40
21	611	611	163.70	998	39
22	640	640	156.26	998	38
23	669	669	149.47	998	37
24	698	698	143.24	998	36
25	.00727	.00727	137.51	.99997	35
26	756	756	132.22	997	34
27	785	785	127.32	997	33
28	814	815	122.77	997	32
29	844	844	118.54	996	31
30	.00873	.00873	114.59	.99996	30
31	902	902	110.89	996	29
32	931	931	107.43	996	28
33	960	960	104.17	995	27
34	.00989	.00989	101.11	995	26
35	.01018	.01018	98.218	.99995	25
36	047	047	95.489	995	24
37	076	076	92.908	994	23
38	105	105	90.463	994	22
39	134	135	88.144	994	21
40	.01164	.01164	85.940	.99993	20
41	193	193	83.844	993	19
42	222	222	81.847	993	18
43	251	251	79.943	992	17
44	280	280	78.126	992	16
45	.01309	.01309	76.390	.99991	15
46	338	338	74.729	991	14
47	367	367	73.139	991	13
48	396	396	71.615	990	12
49	425	425	70.153	990	11
50	.01454	.01455	68.750	.99989	10
51	483	484	67.402	989	9
52	513	513	66.105	989	8
53	542	542	64.858	988	7
54	571	571	63.657	988	6
55	.01600	.01600	62.499	.99987	5
56	629	629	61.383	987	4
57	658	658	60.306	986	3
58	687	687	59.266	986	2
59	716	716	58.261	985	1
60	.01745	.01746	57.290	.99985	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	′

89°

1°

′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.01745	.01746	57.290	.99985	60
1	774	775	56.351	984	59
2	803	804	55.442	984	58
3	832	833	54.561	983	57
4	862	862	53.709	983	56
5	.01891	.01891	52.882	.99982	55
6	920	920	52.081	982	54
7	949	949	51.303	981	53
8	.01978	.01978	50.549	980	52
9	.02007	.02007	49.816	980	51
10	.02036	.02036	49.104	.99979	50
11	065	066	48.412	979	49
12	094	095	47.740	978	48
13	123	124	47.085	977	47
14	152	153	46.449	977	46
15	.02181	.02182	45.829	.99976	45
16	211	211	45.226	976	44
17	240	240	44.639	975	43
18	269	269	44.066	974	42
19	298	298	43.508	974	41
20	.02327	.02328	42.964	.99973	40
21	356	357	42.433	972	39
22	385	386	41.916	972	38
23	414	415	41.411	971	37
24	443	444	40.917	970	36
25	.02472	.02473	40.436	.99969	35
26	501	502	39.965	969	34
27	530	531	39.506	968	33
28	560	560	39.057	967	32
29	589	589	38.618	966	31
30	.02618	.02619	38.188	.99966	30
31	647	648	37.769	965	29
32	676	677	37.358	964	28
33	705	706	36.956	963	27
34	734	735	36.563	963	26
35	.02763	.02764	36.178	.99962	25
36	792	793	35.801	961	24
37	821	822	35.431	960	23
38	850	851	35.070	959	22
39	879	881	34.715	959	21
40	.02908	.02910	34.368	.99958	20
41	938	939	34.027	957	19
42	967	968	33.694	956	18
43	.02996	.02997	33.366	955	17
44	.03025	.03026	33.045	954	16
45	.03054	.03055	32.730	.99953	15
46	083	084	32.421	952	14
47	112	114	32.118	952	13
48	141	143	31.821	951	12
49	170	172	31.528	950	11
50	.03199	.03201	31.242	.99949	10
51	228	230	30.960	948	9
52	257	259	30.683	947	8
53	286	288	30.412	946	7
54	316	317	30.145	945	6
55	.03345	.03346	29.882	.99944	5
56	374	376	29.624	943	4
57	403	405	29.371	942	3
58	432	434	29.122	941	2
59	461	463	28.877	940	1
60	.03490	.03492	28.636	.99939	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	′

88°

2°

′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.03490	.03492	28.636	.99939	60
1	519	521	.399	938	59
2	548	550	28.166	937	58
3	577	579	27.937	936	57
4	606	609	.712	935	56
5	.03635	.03638	27.490	.99934	55
6	664	667	.271	933	54
7	693	696	27.057	932	53
8	723	725	26.845	931	52
9	752	754	.637	930	51
10	.03781	.03783	26.432	.99929	50
11	810	812	.230	927	49
12	839	842	26.031	926	48
13	868	871	25.835	925	47
14	897	900	.642	924	46
15	.03926	.03929	25.452	.99923	45
16	955	958	.264	922	44
17	.03984	.03987	25.080	921	43
18	.04013	.04016	24.898	919	42
19	042	046	.719	918	41
20	.04071	.04075	24.542	.99917	40
21	100	104	.368	916	39
22	129	133	.196	915	38
23	159	162	24.026	913	37
24	188	191	23.859	912	36
25	.04217	.04220	23.695	.99911	35
26	246	250	.532	910	34
27	275	279	.372	909	33
28	304	308	.214	907	32
29	333	337	23.058	906	31
30	.04362	.04366	22.904	.99905	30
31	391	395	.752	904	29
32	420	424	.602	902	28
33	449	454	.454	901	27
34	478	483	.308	900	26
35	.04507	.04512	22.164	.99898	25
36	536	541	22.022	897	24
37	565	570	21.881	896	23
38	594	599	.743	894	22
39	623	628	.606	893	21
40	.04653	.04658	21.470	.99892	20
41	682	687	.337	890	19
42	711	716	.205	889	18
43	740	745	21.075	888	17
44	769	774	20.946	886	16
45	.04798	.04803	20.819	.99885	15
46	827	833	.693	883	14
47	856	862	.569	882	13
48	885	891	.446	881	12
49	914	920	.325	879	11
50	.04943	.04949	20.206	.99878	10
51	.04972	.04978	20.087	876	9
52	.05001	.05007	19.970	875	8
53	030	037	.855	873	7
54	059	066	.740	872	6
55	.05088	.05095	19.627	.99870	5
56	117	124	.516	869	4
57	146	153	.405	867	3
58	175	182	.296	866	2
59	205	212	.188	864	1
60	.05234	.05241	19.081	.99863	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	′

87°

3°

′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.05234	.05241	19.081	.99863	60
1	263	270	18.976	861	59
2	292	299	.871	860	58
3	321	328	.768	858	57
4	350	357	.666	857	56
5	.05379	.05387	18.564	.99855	55
6	408	416	.464	854	54
7	437	445	.366	852	53
8	466	474	.268	851	52
9	495	503	.171	849	51
10	.05524	.05533	18.075	.99847	50
11	553	562	17.980	846	49
12	582	591	.886	844	48
13	611	620	.793	842	47
14	640	649	.702	841	46
15	.05669	.05678	17.611	.99839	45
16	698	708	.521	838	44
17	727	737	.431	836	43
18	756	766	.343	834	42
19	785	795	.256	833	41
20	.05814	.05824	17.169	.99831	40
21	844	854	17.084	829	39
22	873	883	16.999	827	38
23	902	912	.915	826	37
24	931	941	.832	824	36
25	.05960	.05970	16.750	.99822	35
26	.05989	.05999	.668	821	34
27	.06018	.06029	.587	819	33
28	047	058	.507	817	32
29	076	087	.428	815	31
30	.06105	.06116	16.350	.99813	30
31	134	145	.272	812	29
32	163	175	.195	810	28
33	192	204	.119	808	27
34	221	233	16.043	806	26
35	.06250	.06262	15.969	.99804	25
36	279	291	.895	803	24
37	308	321	.821	801	23
38	337	350	.748	799	22
39	366	379	.676	797	21
40	.06395	.06408	15.605	.99795	20
41	424	438	.534	793	19
42	453	467	.464	792	18
43	482	496	.394	790	17
44	511	525	.325	788	16
45	.06540	.06554	15.257	.99786	15
46	569	584	.189	784	14
47	598	613	.122	782	13
48	627	642	15.056	780	12
49	656	671	14.990	778	11
50	.06685	.06700	14.924	.99776	10
51	714	730	.860	774	9
52	743	759	.795	772	8
53	773	788	.732	770	7
54	802	817	.669	768	6
55	.06831	.06847	14.606	.99766	5
56	860	876	.544	764	4
57	889	905	.482	762	3
58	918	934	.421	760	2
59	947	963	.361	758	1
60	.06976	.06993	14.301	.99756	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	′

86°

4°

′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.06976	.06993	14.301	.99756	60
1	.07005	.07022	.241	754	59
2	034	051	.182	752	58
3	063	080	.124	750	57
4	092	110	.065	748	56
5	.07121	.07139	14.008	.99746	55
6	150	168	13.951	744	54
7	179	197	.894	742	53
8	208	227	.838	740	52
9	237	256	.782	738	51
10	.07266	.07285	13.727	.99736	50
11	295	314	.672	734	49
12	324	344	.617	731	48
13	353	373	.563	729	47
14	382	402	.510	727	46
15	.07411	.07431	13.457	.99725	45
16	440	461	.404	723	44
17	469	490	.352	721	43
18	498	519	.300	719	42
19	527	548	.248	716	41
20	.07556	.07578	13.197	.99714	40
21	585	607	.146	712	39
22	614	636	.096	710	38
23	643	665	13.046	708	37
24	672	695	12.996	705	36
25	.07701	.07724	12.947	.99703	35
26	730	753	.898	701	34
27	759	782	.850	699	33
28	788	812	.801	696	32
29	817	841	.754	694	31
30	.07846	.07870	12.706	.99692	30
31	875	899	.659	689	29
32	904	929	.612	687	28
33	933	958	.566	685	27
34	962	.07987	.520	683	26
35	.07991	.08017	12.474	.99680	25
36	.08020	046	.429	678	24
37	049	075	.384	676	23
38	078	104	.339	673	22
39	107	134	.295	671	21
40	.08136	.08163	12.251	.99668	20
41	165	192	.207	666	19
42	194	221	.163	664	18
43	223	251	.120	661	17
44	252	280	.077	659	16
45	.08281	.08309	12.035	.99657	15
46	310	339	11.992	654	14
47	339	368	.950	652	13
48	368	397	.909	649	12
49	397	427	.867	647	11
50	.08426	.08456	11.826	.99644	10
51	455	485	.785	642	9
52	484	514	.745	639	8
53	513	544	.705	637	7
54	542	573	.664	635	6
55	.08571	.08602	11.625	.99632	5
56	600	632	.585	630	4
57	629	661	.546	627	3
58	658	690	.507	625	2
59	687	720	.468	622	1
60	.08716	.08749	11.430	.99619	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	′

85°

5°

′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.08716	.08749	11.430	.99619	60
1	745	778	.392	617	59
2	774	807	.354	614	58
3	803	837	.316	612	57
4	831	866	.279	609	56
5	.08860	.08895	11.242	.99607	55
6	889	925	.205	604	54
7	918	954	.168	602	53
8	947	.08983	.132	599	52
9	.08976	.09013	.095	596	51
10	.09005	.09042	11.059	.99594	50
11	034	071	11.024	591	49
12	063	101	10.988	588	48
13	092	130	.953	586	47
14	121	159	.918	583	46
15	.09150	.09189	10.883	.99580	45
16	179	218	.848	578	44
17	208	247	.814	575	43
18	237	277	.780	572	42
19	266	306	.746	570	41
20	.09295	.09335	10.712	.99567	40
21	324	365	.678	564	39
22	353	394	.645	562	38
23	382	423	.612	559	37
24	411	453	.579	556	36
25	.09440	.09482	10.546	.99553	35
26	469	511	.514	551	34
27	498	541	.481	548	33
28	527	570	.449	545	32
29	556	600	.417	542	31
30	.09585	.09629	10.385	.99540	30
31	614	658	.354	537	29
32	642	688	.322	534	28
33	671	717	.291	531	27
34	700	746	.260	528	26
35	.09729	.09776	10.229	.99526	25
36	758	805	.199	523	24
37	787	834	.168	520	23
38	816	864	.138	517	22
39	845	893	.108	514	21
40	.09874	.09923	10.078	.99511	20
41	903	952	.048	508	19
42	932	.09981	10.019	506	18
43	961	.10011	9.9893	503	17
44	.09990	040	.9601	500	16
45	.10019	.10069	9.9310	.99497	15
46	048	099	.9021	494	14
47	077	128	.8734	491	13
48	106	158	.8448	488	12
49	135	187	.8164	485	11
50	.10164	.10216	9.7882	.99482	10
51	192	246	.7601	479	9
52	221	275	.7322	476	8
53	250	305	.7044	473	7
54	279	334	.6768	470	6
55	.10308	.10363	9.6493	.99467	5
56	337	393	.6220	464	4
57	366	422	.5949	461	3
58	395	452	.5679	458	2
59	424	481	.5411	455	1
60	.10453	.10510	9.5144	.99452	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	′

84°

6°

′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.10453	.10510	9.5144	.99452	**60**
1	482	540	.4878	449	59
2	511	569	.4614	446	58
3	540	599	.4352	443	57
4	569	628	.4090	440	56
5	.10597	.10657	9.3831	.99437	55
6	626	687	.3572	434	54
7	655	716	.3315	431	53
8	684	746	.3060	428	52
9	713	775	.2806	424	51
10	.10742	.10805	9.2553	.99421	**50**
11	771	834	.2302	418	49
12	800	863	.2052	415	48
13	829	893	.1803	412	47
14	858	922	.1555	409	46
15	.10887	.10952	9.1309	.99406	45
16	916	.10981	.1065	402	44
17	945	.11011	.0821	399	43
18	.10973	040	.0579	396	42
19	.11002	070	.0338	393	41
20	.11031	.11099	9.0098	.99390	**40**
21	060	128	8.9860	386	39
22	089	158	.9623	383	38
23	118	187	.9387	380	37
24	147	217	.9152	377	36
25	.11176	.11246	8.8919	.99374	35
26	205	276	.8686	370	34
27	234	305	.8455	367	33
28	263	335	.8225	364	32
29	291	364	.7996	360	31
30	.11320	.11394	8.7769	.99357	**30**
31	349	423	.7542	354	29
32	378	452	.7317	351	28
33	407	482	.7093	347	27
34	436	511	.6870	344	26
35	.11465	.11541	8.6648	.99341	25
36	494	570	.6427	337	24
37	523	600	.6208	334	23
38	552	629	.5989	331	22
39	580	659	.5772	327	21
40	.11609	.11688	8.5555	.99324	**20**
41	638	718	.5340	320	19
42	667	747	.5126	317	18
43	696	777	.4913	314	17
44	725	806	.4701	310	16
45	.11754	.11836	8.4490	.99307	15
46	783	865	.4280	303	14
47	812	895	.4071	300	13
48	840	924	.3863	297	12
49	869	954	.3656	293	11
50	.11898	.11983	8.3450	.99290	**10**
51	927	.12013	.3245	286	9
52	956	042	.3041	283	8
53	.11985	072	.2838	279	7
54	.12014	101	.2636	276	6
55	.12043	.12131	8.2434	.99272	5
56	071	160	.2234	269	4
57	100	190	.2035	265	3
58	129	219	.1837	262	2
59	158	249	.1640	258	1
60	.12187	.12278	8.1443	.99255	**0**
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	′

83°

7°

′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.12187	.12278	8.1443	.99255	**60**
1	216	308	.1248	251	59
2	245	338	.1054	248	58
3	274	367	.0860	244	57
4	302	397	.0667	240	56
5	.12331	.12426	8.0476	.99237	55
6	360	456	.0285	233	54
7	389	485	8.0095	230	53
8	418	515	7.9906	226	52
9	447	544	.9718	222	51
10	.12476	.12574	7.9530	.99219	**50**
11	504	603	.9344	215	49
12	533	633	.9158	211	48
13	562	662	.8973	208	47
14	591	692	.8789	204	46
15	.12620	.12722	7.8606	.99200	45
16	649	751	.8424	197	44
17	678	781	.8243	193	43
18	706	810	.8062	189	42
19	735	840	.7882	186	41
20	.12764	.12869	7.7704	.99182	**40**
21	793	899	.7525	178	39
22	822	929	.7348	175	38
23	851	958	.7171	171	37
24	880	.12988	.6996	167	36
25	.12908	.13017	7.6821	.99163	35
26	937	047	.6647	160	34
27	966	076	.6473	156	33
28	.12995	106	.6301	152	32
29	.13024	136	.6129	148	31
30	.13053	.13165	7.5958	.99144	**30**
31	081	195	.5787	141	29
32	110	224	.5618	137	28
33	139	254	.5449	133	27
34	168	284	.5281	129	26
35	.13197	.13313	7.5113	.99125	25
36	226	343	.4947	122	24
37	254	372	.4781	118	23
38	283	402	.4615	114	22
39	312	432	.4451	110	21
40	.13341	.13461	7.4287	.99106	**20**
41	370	491	.4124	102	19
42	399	521	.3962	098	18
43	427	550	.3800	094	17
44	456	580	.3639	091	16
45	.13485	.13609	7.3479	.99087	15
46	514	639	.3319	083	14
47	543	669	.3160	079	13
48	572	698	.3002	075	12
49	600	728	.2844	071	11
50	.13629	.13758	7.2687	.99067	**10**
51	658	787	.2531	063	9
52	687	817	.2375	059	8
53	716	846	.2220	055	7
54	744	876	.2066	051	6
55	.13773	.13906	7.1912	.99047	5
56	802	935	.1759	043	4
57	831	965	.1607	039	3
58	860	.13995	.1455	035	2
59	889	.14024	.1304	031	1
60	.13917	.14054	7.1154	.99027	**0**
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	′

82°

8°

′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.13917	.14054	7.1154	.99027	60
1	946	084	.1004	023	59
2	.13975	113	.0855	019	58
3	.14004	143	.0706	015	57
4	033	173	.0558	011	56
5	.14061	.14202	7.0410	.99006	55
6	090	232	.0264	.99002	54
7	119	262	7.0117	.98998	53
8	148	291	6.9972	994	52
9	177	321	.9827	990	51
10	.14205	.14351	6.9682	.98986	50
11	234	381	.9538	982	49
12	263	410	.9395	978	48
13	292	440	.9252	973	47
14	320	470	.9110	969	46
15	.14349	.14499	6.8969	.98965	45
16	378	529	.8828	961	44
17	407	559	.8687	957	43
18	436	588	.8548	953	42
19	464	618	.8408	948	41
20	.14493	.14648	6.8269	.98944	40
21	522	678	.8131	940	39
22	551	707	.7994	936	38
23	580	737	.7856	931	37
24	608	767	.7720	927	36
25	.14637	.14796	6.7584	.98923	35
26	666	826	.7448	919	34
27	695	856	.7313	914	33
28	723	886	.7179	910	32
29	752	915	.7045	906	31
30	.14781	.14945	6.6912	.98902	30
31	810	.14975	.6779	897	29
32	838	.15005	.6646	893	28
33	867	034	.6514	889	27
34	896	064	.6383	884	26
35	.14925	.15094	6.6252	.98880	25
36	954	124	.6122	876	24
37	.14982	153	.5992	871	23
38	.15011	183	.5863	867	22
39	040	213	.5734	863	21
40	.15069	.15243	6.5606	.98858	20
41	097	272	.5478	854	19
42	126	302	.5350	849	18
43	155	332	.5223	845	17
44	184	362	.5097	841	16
45	.15212	.15391	6.4971	.98836	15
46	241	421	.4846	832	14
47	270	451	.4721	827	13
48	299	481	.4596	823	12
49	327	511	.4472	818	11
50	.15356	.15540	6.4348	.98814	10
51	385	570	.4225	809	9
52	414	600	.4103	805	8
53	442	630	.3980	800	7
54	471	660	.3859	796	6
55	.15500	.15689	6.3737	.98791	5
56	529	719	.3617	787	4
57	557	749	.3496	782	3
58	586	779	.3376	778	2
59	615	809	.3257	773	1
60	.15643	.15838	6.3138	.98769	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	′

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9°

′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.15643	.15838	6.3138	.98769	60
1	672	868	.3019	764	59
2	701	898	.2901	760	58
3	730	928	.2783	755	57
4	758	958	.2666	751	56
5	.15787	.15988	6.2549	.98746	55
6	816	.16017	.2432	741	54
7	845	047	.2316	737	53
8	873	077	.2200	732	52
9	902	107	.2085	728	51
10	.15931	.16137	6.1970	.98723	50
11	959	167	.1856	718	49
12	.15988	196	.1742	714	48
13	.16017	226	.1628	709	47
14	046	256	.1515	704	46
15	.16074	.16286	6.1402	.98700	45
16	103	316	.1290	695	44
17	132	346	.1178	690	43
18	160	376	.1066	686	42
19	189	405	.0955	681	41
20	.16218	.16435	6.0844	.98676	40
21	246	465	.0734	671	39
22	275	495	.0624	667	38
23	304	525	.0514	662	37
24	333	555	.0405	657	36
25	.16361	.16585	6.0296	.98652	35
26	390	615	.0188	648	34
27	419	645	6.0080	643	33
28	447	674	5.9972	638	32
29	476	704	.9865	633	31
30	.16505	.16734	5.9758	.98629	30
31	533	764	.9651	624	29
32	562	794	.9545	619	28
33	591	824	.9439	614	27
34	620	854	.9333	609	26
35	.16648	.16884	5.9228	.98604	25
36	677	914	.9124	600	24
37	706	944	.9019	595	23
38	734	.16974	.8915	590	22
39	763	.17004	.8811	585	21
40	.16792	.17033	5.8708	.98580	20
41	820	063	.8605	575	19
42	849	093	.8502	570	18
43	878	123	.8400	565	17
44	906	153	.8298	561	16
45	.16935	.17183	5.8197	.98556	15
46	964	213	.8095	551	14
47	.16992	243	.7994	546	13
48	.17021	273	.7894	541	12
49	050	303	.7794	536	11
50	.17078	.17333	5.7694	.98531	10
51	107	363	.7594	526	9
52	136	393	.7495	521	8
53	164	423	.7396	516	7
54	193	453	.7297	511	6
55	.17222	.17483	5.7199	.98506	5
56	250	513	.7101	501	4
57	279	543	.7004	496	3
58	308	573	.6906	491	2
59	336	603	.6809	486	1
60	.17365	.17633	5.6713	.98481	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	′

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10°

′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.17365	.17633	5.6713	.98481	60
1	393	663	.6617	476	59
2	422	693	.6521	471	58
3	451	723	.6425	466	57
4	479	753	.6329	461	56
5	.17508	.17783	5.6234	.98455	55
6	537	813	.6140	450	54
7	565	843	.6045	445	53
8	594	873	.5951	440	52
9	623	903	.5857	435	51
10	.17651	.17933	5.5764	.98430	50
11	680	963	.5671	425	49
12	708	.17993	.5578	420	48
13	737	.18023	.5485	414	47
14	766	053	.5393	409	46
15	.17794	.18083	5.5301	.98404	45
16	823	113	.5209	399	44
17	852	143	.5118	394	43
18	880	173	.5026	389	42
19	909	203	.4936	383	41
20	.17937	.18233	5.4845	.98378	40
21	966	263	.4755	373	39
22	.17995	293	.4665	368	38
23	.18023	323	.4575	362	37
24	052	353	.4486	357	36
25	.18081	.18384	5.4397	.98352	35
26	109	414	.4308	347	34
27	138	444	.4219	341	33
28	166	474	.4131	336	32
29	195	504	.4043	331	31
30	.18224	.18534	5.3955	.98325	30
31	252	564	.3868	320	29
32	281	594	.3781	315	28
33	309	624	.3694	310	27
34	338	654	.3607	304	26
35	.18367	.18684	5.3521	.98299	25
36	395	714	.3435	294	24
37	424	745	.3349	288	23
38	452	775	.3263	283	22
39	481	805	.3178	277	21
40	.18509	.18835	5.3093	.98272	20
41	538	865	.3008	267	19
42	567	895	.2924	261	18
43	595	925	.2839	256	17
44	624	955	.2755	250	16
45	.18652	.18986	5.2672	.98245	15
46	681	.19016	.2588	240	14
47	710	046	.2505	234	13
48	738	076	.2422	229	12
49	767	106	.2339	223	11
50	.18795	.19136	5.2257	.98218	10
51	824	166	.2174	212	9
52	852	197	.2092	207	8
53	881	227	.2011	201	7
54	910	257	.1929	196	6
55	.18938	.19287	5.1848	.98190	5
56	967	317	.1767	185	4
57	.18995	347	.1686	179	3
58	.19024	378	.1606	174	2
59	052	408	.1526	168	1
60	.19081	.19438	5.1446	.98163	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	′

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11°

′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.19081	.19438	5.1446	.98163	60
1	109	468	.1366	157	59
2	138	498	.1286	152	58
3	167	529	.1207	146	57
4	195	559	.1128	140	56
5	.19224	.19589	5.1049	.98135	55
6	252	619	.0970	129	54
7	281	649	.0892	124	53
8	309	680	.0814	118	52
9	338	710	.0736	112	51
10	.19366	.19740	5.0658	.98107	50
11	395	770	.0581	101	49
12	423	801	.0504	096	48
13	452	831	.0427	090	47
14	481	861	.0350	084	46
15	.19509	.19891	5.0273	.98079	45
16	538	921	.0197	073	44
17	566	952	.0121	067	43
18	595	.19982	5.0045	061	42
19	623	.20012	4.9969	056	41
20	.19652	.20042	4.9894	.98050	40
21	680	073	.9819	044	39
22	709	103	.9744	039	38
23	737	133	.9669	033	37
24	766	164	.9594	027	36
25	.19794	.20194	4.9520	.98021	35
26	823	224	.9446	016	34
27	851	254	.9372	010	33
28	880	285	.9298	.98004	32
29	908	315	.9225	.97998	31
30	.19937	.20345	4.9152	.97992	30
31	965	376	.9078	987	29
32	.19994	406	.9006	981	28
33	.20022	436	.8933	975	27
34	051	466	.8860	969	26
35	.20079	.20497	4.8788	.97963	25
36	108	527	.8716	958	24
37	136	557	.8644	952	23
38	165	588	.8573	946	22
39	193	618	.8501	940	21
40	.20222	.20648	4.8430	.97934	20
41	250	679	.8359	928	19
42	279	709	.8288	922	18
43	307	739	.8218	916	17
44	336	770	.8147	910	16
45	.20364	.20800	4.8077	.97905	15
46	393	830	.8007	899	14
47	421	861	.7937	893	13
48	450	891	.7867	887	12
49	478	921	.7798	881	11
50	.20507	.20952	4.7729	.97875	10
51	535	.20982	.7659	869	9
52	563	.21013	.7591	863	8
53	592	043	.7522	857	7
54	620	073	.7453	851	6
55	.20649	.21104	4.7385	.97845	5
56	677	134	.7317	839	4
57	706	164	.7249	833	3
58	734	195	.7181	827	2
59	763	225	.7114	821	1
60	.20791	.21256	4.7046	.97815	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	′

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12°

′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.20791	.21256	4.7046	.97815	60
1	820	286	.6979	809	59
2	848	316	.6912	803	58
3	877	347	.6845	797	57
4	905	377	.6779	791	56
5	.20933	.21408	4.6712	.97784	55
6	962	438	.6646	778	54
7	.20990	469	.6580	772	53
8	.21019	499	.6514	766	52
9	047	529	.6448	760	51
10	.21076	.21560	4.6382	.97754	50
11	104	590	.6317	748	49
12	132	621	.6252	742	48
13	161	651	.6187	735	47
14	189	682	.6122	729	46
15	.21218	.21712	4.6057	.97723	45
16	246	743	.5993	717	44
17	275	773	.5928	711	43
18	303	804	.5864	705	42
19	331	834	.5800	698	41
20	.21360	.21864	4.5736	.97692	40
21	388	895	.5673	686	39
22	417	925	.5609	680	38
23	445	956	.5546	673	37
24	474	.21986	.5483	667	36
25	.21502	.22017	4.5420	.97661	35
26	530	047	.5357	655	34
27	559	078	.5294	648	33
28	587	108	.5232	642	32
29	616	139	.5169	636	31
30	.21644	.22169	4.5107	.97630	30
31	672	200	.5045	623	29
32	701	231	.4983	617	28
33	729	261	.4922	611	27
34	758	292	.4860	604	26
35	.21786	.22322	4.4799	.97598	25
36	814	353	.4737	592	24
37	843	383	.4676	585	23
38	871	414	.4615	579	22
39	899	444	.4555	573	21
40	.21928	.22475	4.4494	.97566	20
41	956	505	.4434	560	19
42	.21985	536	.4373	553	18
43	.22013	567	.4313	547	17
44	041	597	.4253	541	16
45	.22070	.22628	4.4194	.97534	15
46	098	658	.4134	528	14
47	126	689	.4075	521	13
48	155	719	.4015	515	12
49	183	750	.3956	508	11
50	.22212	.22781	4.3897	.97502	10
51	240	811	.3838	496	9
52	268	842	.3779	489	8
53	297	872	.3721	483	7
54	325	903	.3662	476	6
55	.22353	.22934	4.3604	.97470	5
56	382	964	.3546	463	4
57	410	.22995	.3488	457	3
58	438	.23026	.3430	450	2
59	467	056	.3372	444	1
60	22495	.23087	4.3315	.97437	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	′

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13°

′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.22495	.23087	4.3315	.97437	60
1	523	117	.3257	430	59
2	552	148	.3200	424	58
3	580	179	.3143	417	57
4	608	209	.3086	411	56
5	.22637	.23240	4.3029	.97404	55
6	665	271	.2972	398	54
7	693	301	.2916	391	53
8	722	332	.2859	384	52
9	750	363	.2803	378	51
10	.22778	.23393	4.2747	.97371	50
11	807	424	.2691	365	49
12	835	455	.2635	358	48
13	863	485	.2580	351	47
14	892	516	.2524	345	46
15	.22920	.23547	4.2468	.97338	45
16	948	578	.2413	331	44
17	.22977	608	.2358	325	43
18	.23005	639	.2303	318	42
19	033	670	.2248	311	41
20	.23062	.23700	4.2193	.97304	40
21	090	731	.2139	298	39
22	118	762	.2084	291	38
23	146	793	.2030	284	37
24	175	823	.1976	278	36
25	.23203	.23854	4.1922	.97271	35
26	231	885	.1868	264	34
27	260	916	.1814	257	33
28	288	946	.1760	251	32
29	316	.23977	.1706	244	31
30	.23345	.24008	4.1653	.97237	30
31	373	039	.1600	230	29
32	401	069	.1547	223	28
33	429	100	.1493	217	27
34	458	131	.1441	210	26
35	.23486	.24162	4.1388	.97203	25
36	514	193	.1335	196	24
37	542	223	.1282	189	23
38	571	254	.1230	182	22
39	599	285	.1178	176	21
40	.23627	.24316	4.1126	.97169	20
41	656	347	.1074	162	19
42	684	377	.1022	155	18
43	712	408	.0970	148	17
44	740	439	.0918	141	16
45	.23769	.24470	4.0867	.97134	15
46	797	501	.0815	127	14
47	825	532	.0764	120	13
48	853	562	.0713	113	12
49	882	593	.0662	106	11
50	.23910	.24624	4.0611	.97100	10
51	938	655	.0560	093	9
52	966	686	.0509	086	8
53	.23995	717	.0459	079	7
54	.24023	747	.0408	072	6
55	.24051	.24778	4.0358	.97065	5
56	079	809	.0308	058	4
57	108	840	.0257	051	3
58	136	871	.0207	044	2
59	164	902	.0158	037	1
60	.24192	.24933	4.0108	.97030	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	′

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14°

′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.24192	.24933	4.0108	.97030	**60**
1	220	964	.0058	023	59
2	249	.24995	4.0009	015	58
3	277	.25026	3.9959	008	57
4	305	056	.9910	.97001	56
5	.24333	.25087	3.9861	.96994	55
6	362	118	.9812	987	54
7	390	149	.9763	980	53
8	418	180	.9714	973	52
9	446	211	.9665	966	51
10	.24474	.25242	3.9617	.96959	**50**
11	503	273	.9568	952	49
12	531	304	.9520	945	48
13	559	335	.9471	937	47
14	587	366	.9423	930	46
15	.24615	.25397	3.9375	.96923	45
16	644	428	.9327	916	44
17	672	459	.9279	909	43
18	700	490	.9232	902	42
19	728	521	.9184	894	41
20	.24756	.25552	3.9136	.96887	**40**
21	784	583	.9089	880	39
22	813	614	.9042	873	38
23	841	645	.8995	866	37
24	869	676	.8947	858	36
25	.24897	.25707	3.8900	.96851	35
26	925	738	.8854	844	34
27	954	769	.8807	837	33
28	.24982	800	.8760	829	32
29	.25010	831	.8714	822	31
30	.25038	.25862	3.8667	.96815	**30**
31	066	893	.8621	807	29
32	094	924	.8575	800	28
33	122	955	.8528	793	27
34	151	.25986	.8482	786	26
35	.25179	.26017	3.8436	.96778	25
36	207	048	.8391	771	24
37	235	079	.8345	764	23
38	263	110	.8299	756	22
39	291	141	.8254	749	21
40	.25320	.26172	3.8208	.96742	**20**
41	348	203	.8163	734	19
42	376	235	.8118	727	18
43	404	266	.8073	719	17
44	432	297	.8028	712	16
45	.25460	.26328	3.7983	.96705	15
46	488	359	.7938	697	14
47	516	390	.7893	690	13
48	545	421	.7848	682	12
49	573	452	.7804	675	11
50	.25601	.26483	3.7760	.96667	**10**
51	629	515	.7715	660	9
52	657	546	.7671	653	8
53	685	577	.7627	645	7
54	713	608	.7583	638	6
55	.25741	.26639	3.7539	.96630	5
56	769	670	.7495	623	4
57	798	701	.7451	615	3
58	826	733	.7408	608	2
59	854	764	.7364	600	1
60	.25882	.26795	3.7321	.96593	**0**
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	′

75°

15°

′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.25882	.26795	3.7321	.96593	**60**
1	910	826	.7277	585	59
2	938	857	.7234	578	58
3	966	888	.7191	570	57
4	.25994	920	.7148	562	56
5	.26022	.26951	3.7105	.96555	55
6	050	.26982	.7062	547	54
7	079	.27013	.7019	540	53
8	107	044	.6976	532	52
9	135	076	.6933	524	51
10	.26163	.27107	3.6891	.96517	**50**
11	191	138	.6848	509	49
12	219	169	.6806	502	48
13	247	201	.6764	494	47
14	275	232	.6722	486	46
15	.26303	.27263	3.6680	.96479	45
16	331	294	.6638	471	44
17	359	326	.6596	463	43
18	387	357	.6554	456	42
19	415	388	.6512	448	41
20	.26443	.27419	3.6470	.96440	**40**
21	471	451	.6429	433	39
22	500	482	.6387	425	38
23	528	513	.6346	417	37
24	556	545	.6305	410	36
25	.26584	.27576	3.6264	.96402	35
26	612	607	.6222	394	34
27	640	638	.6181	386	33
28	668	670	.6140	379	32
29	696	701	.6100	371	31
30	.26724	.27732	3.6059	.96363	**30**
31	752	764	.6018	355	29
32	780	795	.5978	347	28
33	808	826	.5937	340	27
34	836	858	.5897	332	26
35	.26864	.27889	3.5856	.96324	25
36	892	921	.5816	316	24
37	920	952	.5776	308	23
38	948	.27983	.5736	301	22
39	.26976	.28015	.5696	293	21
40	.27004	.28046	3.5656	.96285	**20**
41	032	077	.5616	277	19
42	060	109	.5576	269	18
43	088	140	.5536	261	17
44	116	172	.5497	253	16
45	.27144	.28203	3.5457	.96246	15
46	172	234	.5418	238	14
47	200	266	.5379	230	13
48	228	297	.5339	222	12
49	256	329	.5300	214	11
50	.27284	.28360	3.5261	.96206	**10**
51	312	391	.5222	198	9
52	340	423	.5183	190	8
53	368	454	.5144	182	7
54	396	486	.5105	174	6
55	.27424	.28517	3.5067	.96166	5
56	452	549	.5028	158	4
57	480	580	.4989	150	3
58	508	612	.4951	142	2
59	536	643	.4912	134	1
60	.27564	.28675	3.4874	.96126	**0**
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	′

74°

16°

′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.27564	.28675	3.4874	.96126	60
1	592	706	.4836	118	59
2	620	738	.4798	110	58
3	648	769	.4760	102	57
4	676	801	.4722	094	56
5	.27704	.28832	3.4684	.96086	55
6	731	864	.4646	078	54
7	759	895	.4608	070	53
8	787	927	.4570	062	52
9	815	958	.4533	054	51
10	.27843	.28990	3.4495	.96046	50
11	871	.29021	.4458	037	49
12	899	053	.4420	029	48
13	927	084	.4383	021	47
14	955	116	.4346	013	46
15	.27983	.29147	3.4308	.96005	45
16	.28011	179	.4271	.95997	44
17	039	210	.4234	989	43
18	067	242	.4197	981	42
19	095	274	.4160	972	41
20	.28123	.29305	3.4124	.95964	40
21	150	337	.4087	956	39
22	178	368	.4050	948	38
23	206	400	.4014	940	37
24	234	432	.3977	931	36
25	.28262	.29463	3.3941	.95923	35
26	290	495	.3904	915	34
27	318	526	.3868	907	33
28	346	558	.3832	898	32
29	374	590	.3796	890	31
30	.28402	.29621	3.3759	.95882	30
31	429	653	.3723	874	29
32	457	685	.3687	865	28
33	485	716	.3652	857	27
34	513	748	.3616	849	26
35	.28541	.29780	3.3580	.95841	25
36	569	811	.3544	832	24
37	597	843	.3509	824	23
38	625	875	.3473	816	22
39	652	906	.3438	807	21
40	.28680	.29938	3.3402	.95799	20
41	708	.29970	.3367	791	19
42	736	.30001	.3332	782	18
43	764	033	.3297	774	17
44	792	065	.3261	766	16
45	.28820	.30097	3.3226	.95757	15
46	847	128	.3191	749	14
47	875	160	.3156	740	13
48	903	192	.3122	732	12
49	931	224	.3087	724	11
50	.28959	.30255	3.3052	.95715	10
51	.28987	287	.3017	707	9
52	.29015	319	.2983	698	8
53	042	351	.2948	690	7
54	070	382	.2914	681	6
55	.29098	.30414	3.2879	.95673	5
56	126	446	.2845	664	4
57	154	478	.2811	656	3
58	182	509	.2777	647	2
59	209	541	.2743	639	1
60	.29237	.30573	3.2709	.95630	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	′

73°

17°

′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.29237	.30573	3.2709	.95630	60
1	265	605	.2675	622	59
2	293	637	.2641	613	58
3	321	669	.2607	605	57
4	348	700	.2573	596	56
5	.29376	.30732	3.2539	.95588	55
6	404	764	.2506	579	54
7	432	796	.2472	571	53
8	460	828	.2438	562	52
9	487	860	.2405	554	51
10	.29515	.30891	3.2371	.95545	50
11	543	923	.2338	536	49
12	571	955	.2305	528	48
13	599	.30987	.2272	519	47
14	626	.31019	.2238	511	46
15	.29654	.31051	3.2205	.95502	45
16	682	083	.2172	493	44
17	710	115	.2139	485	43
18	737	147	.2106	476	42
19	765	178	.2073	467	41
20	.29793	.31210	3.2041	.95459	40
21	821	242	.2008	450	39
22	849	274	.1975	441	38
23	876	306	.1943	433	37
24	904	338	.1910	424	36
25	.29932	.31370	3.1878	.95415	35
26	960	402	.1845	407	34
27	.29987	434	.1813	398	33
28	.30015	466	.1780	389	32
29	043	498	.1748	380	31
30	.30071	.31530	3.1716	.95372	30
31	098	562	.1684	363	29
32	126	594	.1652	354	28
33	154	626	.1620	345	27
34	182	658	.1588	337	26
35	.30209	.31690	3.1556	.95328	25
36	237	722	.1524	319	24
37	265	754	.1492	310	23
38	292	786	.1460	301	22
39	320	818	.1429	293	21
40	.30348	.31850	3.1397	.95284	20
41	376	882	.1366	275	19
42	403	914	.1334	266	18
43	431	946	.1303	257	17
44	459	.31978	.1271	248	16
45	.30486	.32010	3.1240	.95240	15
46	514	042	.1209	231	14
47	542	074	.1178	222	13
48	570	106	.1146	213	12
49	597	139	.1115	204	11
50	.30625	.32171	3.1084	.95195	10
51	653	203	.1053	186	9
52	680	235	.1022	177	8
53	708	267	.0991	168	7
54	736	299	.0961	159	6
55	.30763	.32331	3.0930	.95150	5
56	791	363	.0899	142	4
57	819	396	.0868	133	3
58	846	428	.0838	124	2
59	874	460	.0807	115	1
60	.30902	.32492	3.0777	.95106	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	′

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18°

′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.30902	.32492	3.0777	.95106	60
1	929	524	.0746	097	59
2	957	556	.0716	088	58
3	.30985	588	.0686	079	57
4	.31012	621	.0655	070	56
5	.31040	.32653	3.0625	.95061	55
6	068	685	.0595	052	54
7	095	717	.0565	043	53
8	123	749	.0535	033	52
9	151	782	.0505	024	51
10	.31178	.32814	3.0475	.95015	50
11	206	846	.0445	.95006	49
12	233	878	.0415	.94997	48
13	261	911	.0385	988	47
14	289	943	.0356	979	46
15	.31316	.32975	3.0326	.94970	45
16	344	.33007	.0296	961	44
17	372	040	.0267	952	43
18	399	072	.0237	943	42
19	427	104	.0208	933	41
20	.31454	.33136	3.0178	.94924	40
21	482	169	.0149	915	39
22	510	201	.0120	906	38
23	537	233	.0090	897	37
24	565	266	.0061	888	36
25	.31593	.33298	3.0032	.94878	35
26	620	330	3.0003	869	34
27	648	363	2.9974	860	33
28	675	395	.9945	851	32
29	703	427	.9916	842	31
30	.31730	.33460	2.9887	.94832	30
31	758	492	.9858	823	29
32	786	524	.9829	814	28
33	813	557	.9800	805	27
34	841	589	.9772	795	26
35	.31868	.33621	2.9743	.94786	25
36	896	654	.9714	777	24
37	923	686	.9686	768	23
38	951	718	.9657	758	22
39	.31979	751	.9629	749	21
40	.32006	.33783	2.9600	.94740	20
41	034	816	.9572	730	19
42	061	848	.9544	721	18
43	089	881	.9515	712	17
44	116	913	.9487	702	16
45	.32144	.33945	2.9459	.94693	15
46	171	.33978	.9431	684	14
47	199	.34010	.9403	674	13
48	227	043	.9375	665	12
49	254	075	.9347	656	11
50	.32282	.34108	2.9319	.94646	10
51	309	140	.9291	637	9
52	337	173	.9263	627	8
53	364	205	.9235	618	7
54	392	238	.9208	609	6
55	.32419	.34270	2.9180	.94599	5
56	447	303	.9152	590	4
57	474	335	.9125	580	3
58	502	368	.9097	571	2
59	529	400	.9070	561	1
60	.32557	.34433	2.9042	.94552	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	′

71°

19°

′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.32557	.34433	2.9042	.94552	60
1	584	465	.9015	542	59
2	612	498	.8987	533	58
3	639	530	.8960	523	57
4	667	563	.8933	514	56
5	.32694	.34596	2.8905	.94504	55
6	722	628	.8878	495	54
7	749	661	.8851	485	53
8	777	693	.8824	476	52
9	804	726	.8797	466	51
10	.32832	.34758	2.8770	.94457	50
11	859	791	.8743	447	49
12	887	824	.8716	438	48
13	914	856	.8689	428	47
14	942	889	.8662	418	46
15	.32969	.34922	2.8636	.94409	45
16	.32997	954	.8609	399	44
17	.33024	.34987	.8582	390	43
18	051	.35020	.8556	380	42
19	079	052	.8529	370	41
20	.33106	.35085	2.8502	.94361	40
21	134	118	.8476	351	39
22	161	150	.8449	342	38
23	189	183	.8423	332	37
24	216	216	.8397	322	36
25	.33244	.35248	2.8370	.94313	35
26	271	281	.8344	303	34
27	298	314	.8318	293	33
28	326	346	.8291	284	32
29	353	379	.8265	274	31
30	.33381	.35412	2.8239	.94264	30
31	408	445	.8213	254	29
32	436	477	.8187	245	28
33	463	510	.8161	235	27
34	490	543	.8135	225	26
35	.33518	.35576	2.8109	.94215	25
36	545	608	.8083	206	24
37	573	641	.8057	196	23
38	600	674	.8032	186	22
39	627	707	.8006	176	21
40	.33655	.35740	2.7980	.94167	20
41	682	772	.7955	157	19
42	710	805	.7929	147	18
43	737	838	.7903	137	17
44	764	871	.7878	127	16
45	.33792	.35904	2.7852	.94118	15
46	819	937	.7827	108	14
47	846	.35969	.7801	098	13
48	874	.36002	.7776	088	12
49	901	035	.7751	078	11
50	.33929	.36068	2.7725	.94068	10
51	956	101	.7700	058	9
52	.33983	134	.7675	049	8
53	.34011	167	.7650	039	7
54	038	199	.7625	029	6
55	.34065	.36232	2.7600	.94019	5
56	093	265	.7575	.94009	4
57	120	298	.7550	.93999	3
58	147	331	.7525	989	2
59	175	364	.7500	979	1
60	.34202	.36397	2.7475	.93969	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	′

70°

20°

′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.34202	.36397	2.7475	.93969	60
1	229	430	.7450	959	59
2	257	463	.7425	949	58
3	284	496	.7400	939	57
4	311	529	.7376	929	56
5	.34339	.36562	2.7351	.93919	55
6	366	595	.7326	909	54
7	393	628	.7302	899	53
8	421	661	.7277	889	52
9	448	694	.7253	879	51
10	.34475	.36727	2.7228	.93869	50
11	503	760	.7204	859	49
12	530	793	.7179	849	48
13	557	826	.7155	839	47
14	584	859	.7130	829	46
15	.34612	.36892	2.7106	.93819	45
16	639	925	.7082	809	44
17	666	958	.7058	799	43
18	694	.36991	.7034	789	42
19	721	.37024	.7009	779	41
20	.34748	.37057	2.6985	.93769	40
21	775	090	.6961	759	39
22	803	123	.6937	748	38
23	830	157	.6913	738	37
24	857	190	.6889	728	36
25	.34884	.37223	2.6865	.93718	35
26	912	256	.6841	708	34
27	939	289	.6818	698	33
28	966	322	.6794	688	32
29	.34993	355	.6770	677	31
30	.35021	.37388	2.6746	.93667	30
31	048	422	.6723	657	29
32	075	455	.6699	647	28
33	102	488	.6675	637	27
34	130	521	.6652	626	26
35	.35157	.37554	2.6628	.93616	25
36	184	588	.6605	606	24
37	211	621	.6581	596	23
38	239	654	.6558	585	22
39	266	687	.6534	575	21
40	.35293	.37720	2.6511	.93565	20
41	320	754	.6488	555	19
42	347	787	.6464	544	18
43	375	820	.6441	534	17
44	402	853	.6418	524	16
45	.35429	.37887	2.6395	.93514	15
46	456	920	.6371	503	14
47	484	953	.6348	493	13
48	511	.37986	.6325	483	12
49	538	.38020	.6302	472	11
50	.35565	.38053	2.6279	.93462	10
51	592	086	.6256	452	9
52	619	120	.6233	441	8
53	647	153	.6210	431	7
54	674	186	.6187	420	6
55	.35701	.38220	2.6165	.93410	5
56	728	253	.6142	400	4
57	755	286	.6119	389	3
58	782	320	.6096	379	2
59	810	353	.6074	368	1
60	.35837	.38386	2.6051	.93358	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	′

69°

21°

′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.35837	.38386	2.6051	.93358	60
1	864	420	.6028	348	59
2	891	453	.6006	337	58
3	918	487	.5983	327	57
4	945	520	.5961	316	56
5	.35973	.38553	2.5938	.93306	55
6	.36000	587	.5916	295	54
7	027	620	.5893	285	53
8	054	654	.5871	274	52
9	081	687	.5848	264	51
10	.36108	.38721	2.5826	.93253	50
11	135	754	.5804	243	49
12	162	787	.5782	232	48
13	190	821	.5759	222	47
14	217	854	.5737	211	46
15	.36244	.38888	2.5715	.93201	45
16	271	921	.5693	190	44
17	298	955	.5671	180	43
18	325	.38988	.5649	169	42
19	352	.39022	.5627	159	41
20	.36379	.39055	2.5605	.93148	40
21	406	089	.5583	137	39
22	434	122	.5561	127	38
23	461	156	.5539	116	37
24	488	190	.5517	106	36
25	.36515	.39223	2.5495	.93095	35
26	542	257	.5473	084	34
27	569	290	.5452	074	33
28	596	324	.5430	063	32
29	623	357	.5408	052	31
30	.36650	.39391	2.5386	.93042	30
31	677	425	.5365	031	29
32	704	458	.5343	020	28
33	731	492	.5322	.93010	27
34	758	526	.5300	.92999	26
35	.36785	.39559	2.5279	.92988	25
36	812	593	.5257	978	24
37	839	626	.5236	967	23
38	867	660	.5214	956	22
39	894	694	.5193	945	21
40	.36921	.39727	2.5172	.92935	20
41	948	761	.5150	924	19
42	.36975	795	.5129	913	18
43	.37002	829	.5108	902	17
44	029	862	.5086	892	16
45	.37056	.39896	2.5065	.92881	15
46	083	930	.5044	870	14
47	110	963	.5023	859	13
48	137	.39997	.5002	849	12
49	164	.40031	.4981	838	11
50	.37191	.40065	2.4960	.92827	10
51	218	098	.4939	816	9
52	245	132	.4918	805	8
53	272	166	.4897	794	7
54	299	200	.4876	784	6
55	.37326	.40234	2.4855	.92773	5
56	353	267	.4834	762	4
57	380	301	.4813	751	3
58	407	335	.4792	740	2
59	434	369	.4772	729	1
60	.37461	.40403	2.4751	.92718	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	′

68°

22°

′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.37461	.40403	2.4751	.92718	60
1	488	436	.4730	707	59
2	515	470	.4709	697	58
3	542	504	.4689	686	57
4	569	538	.4668	675	56
5	.37595	.40572	2.4648	.92664	55
6	622	606	.4627	653	54
7	649	640	.4606	642	53
8	676	674	.4586	631	52
9	703	707	.4566	620	51
10	.37730	.40741	2.4545	.92609	50
11	757	775	.4525	598	49
12	784	809	.4504	587	48
13	811	843	.4484	576	47
14	838	877	.4464	565	46
15	.37865	.40911	2.4443	.92554	45
16	892	945	.4423	543	44
17	919	.40979	.4403	532	43
18	946	.41013	.4383	521	42
19	973	047	.4362	510	41
20	.37999	.41081	2.4342	.92499	40
21	.38026	115	.4322	488	39
22	053	149	.4302	477	38
23	080	183	.4282	466	37
24	107	217	.4262	455	36
25	.38134	.41251	2.4242	.92444	35
26	161	285	.4222	432	34
27	188	319	.4202	421	33
28	215	353	.4182	410	32
29	241	387	.4162	399	31
30	.38268	.41421	2.4142	.92388	30
31	295	455	.4122	377	29
32	322	490	.4102	366	28
33	349	524	.4083	355	27
34	376	558	.4063	343	26
35	.38403	.41592	2.4043	.92332	25
36	430	626	.4023	321	24
37	456	660	.4004	310	23
38	483	694	.3984	299	22
39	510	728	.3964	287	21
40	.38537	.41763	2.3945	.92276	20
41	564	797	.3925	265	19
42	591	831	.3906	254	18
43	617	865	.3886	243	17
44	644	899	.3867	231	16
45	.38671	.41933	2.3847	.92220	15
46	698	.41968	.3828	209	14
47	725	.42002	.3808	198	13
48	752	036	.3789	186	12
49	778	070	.3770	175	11
50	.38805	.42105	2.3750	.92164	10
51	832	139	.3731	152	9
52	859	173	.3712	141	8
53	886	207	.3693	130	7
54	912	242	.3673	119	6
55	.38939	.42276	2.3654	.92107	5
56	966	310	.3635	096	4
57	.38993	345	.3616	085	3
58	.39020	379	.3597	073	2
59	046	413	.3578	062	1
60	.39073	.42447	2.3559	.92050	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	′

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23°

′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.39073	.42447	2.3559	.92050	60
1	100	482	.3539	039	59
2	127	516	.3520	028	58
3	153	551	.3501	016	57
4	180	585	.3483	.92005	56
5	.39207	.42619	2.3464	.91994	55
6	234	654	.3445	982	54
7	260	688	.3426	971	53
8	287	722	.3407	959	52
9	314	757	.3388	948	51
10	.39341	.42791	2.3369	.91936	50
11	367	826	.3351	925	49
12	394	860	.3332	914	48
13	421	894	.3313	902	47
14	448	929	.3294	891	46
15	.39474	.42963	2.3276	.91879	45
16	501	.42998	.3257	868	44
17	528	.43032	.3238	856	43
18	555	067	.3220	845	42
19	581	101	.3201	833	41
20	.39608	.43136	2.3183	.91822	40
21	635	170	.3164	810	39
22	661	205	.3146	799	38
23	688	239	.3127	787	37
24	715	274	.3109	775	36
25	.39741	.43308	2.3090	.91764	35
26	768	343	.3072	752	34
27	795	378	.3053	741	33
28	822	412	.3035	729	32
29	848	447	.3017	718	31
30	.39875	.43481	2.2998	.91706	30
31	902	516	.2980	694	29
32	928	550	.2962	683	28
33	955	585	.2944	671	27
34	.39982	620	.2925	660	26
35	.40008	.43654	2.2907	.91648	25
36	035	689	.2889	636	24
37	062	724	.2871	625	23
38	088	758	.2853	613	22
39	115	793	.2835	601	21
40	.40141	.43828	2.2817	.91590	20
41	168	862	.2799	578	19
42	195	897	.2781	566	18
43	221	932	.2763	555	17
44	248	.43966	.2745	543	16
45	.40275	.44001	2.2727	.91531	15
46	301	036	.2709	519	14
47	328	071	.2691	508	13
48	355	105	.2673	496	12
49	381	140	.2655	484	11
50	.40408	.44175	2.2637	.91472	10
51	434	210	.2620	461	9
52	461	244	.2602	449	8
53	488	279	.2584	437	7
54	514	314	.2566	425	6
55	.40541	.44349	2.2549	.91414	5
56	567	384	.2531	402	4
57	594	418	.2513	390	3
58	621	453	.2496	378	2
59	647	488	.2478	366	1
60	.40674	.44523	2.2460	.91355	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	′

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24°

′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.40674	.44523	2.2460	.91355	60
1	700	558	.2443	343	59
2	727	593	.2425	331	58
3	753	627	.2408	319	57
4	780	662	.2390	307	56
5	.40806	.44697	2.2373	.91295	55
6	833	732	.2355	283	54
7	860	767	.2338	272	53
8	886	802	.2320	260	52
9	913	837	.2303	248	51
10	.40939	.44872	2.2286	.91236	50
11	966	907	.2268	224	49
12	.40992	942	.2251	212	48
13	.41019	.44977	.2234	200	47
14	045	.45012	.2216	188	46
15	.41072	.45047	2.2199	.91176	45
16	098	082	.2182	164	44
17	125	117	.2165	152	43
18	151	152	.2148	140	42
19	178	187	.2130	128	41
20	.41204	.45222	2.2113	.91116	40
21	231	257	.2096	104	39
22	257	292	.2079	092	38
23	284	327	.2062	080	37
24	310	362	.2045	068	36
25	.41337	.45397	2.2028	.91056	35
26	363	432	.2011	044	34
27	390	467	.1994	032	33
28	416	502	.1977	020	32
29	443	538	.1960	.91008	31
30	.41469	.45573	2.1943	.90996	30
31	496	608	.1926	984	29
32	522	643	.1909	972	28
33	549	678	.1892	960	27
34	575	713	.1876	948	26
35	.41602	.45748	2.1859	.90936	25
36	628	784	.1842	924	24
37	655	819	.1825	911	23
38	681	854	.1808	899	22
39	707	889	.1792	887	21
40	.41734	.45924	2.1775	.90875	20
41	760	960	.1758	863	19
42	787	.45995	.1742	851	18
43	813	.46030	.1725	839	17
44	840	065	.1708	826	16
45	.41866	.46101	2.1692	.90814	15
46	892	136	.1675	802	14
47	919	171	.1659	790	13
48	945	206	.1642	778	12
49	972	242	.1625	766	11
50	.41998	.46277	2.1609	.90753	10
51	.42024	312	.1592	741	9
52	051	348	.1576	729	8
53	077	383	.1560	717	7
54	104	418	.1543	704	6
55	.42130	.46454	2.1527	.90692	5
56	156	489	.1510	680	4
57	183	525	.1494	668	3
58	209	560	.1478	655	2
59	235	595	.1461	643	1
60	.42262	.46631	2.1445	.90631	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	′

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25°

′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.42262	.46631	2.1445	.90631	60
1	288	666	.1429	618	59
2	315	702	.1413	606	58
3	341	737	.1396	594	57
4	367	772	.1380	582	56
5	.42394	.46808	2.1364	.90569	55
6	420	843	.1348	557	54
7	446	879	.1332	545	53
8	473	914	.1315	532	52
9	499	950	.1299	520	51
10	.42525	.46985	2.1283	.90507	50
11	552	.47021	.1267	495	49
12	578	056	.1251	483	48
13	604	092	.1235	470	47
14	631	128	.1219	458	46
15	.42657	.47163	2.1203	.90446	45
16	683	199	.1187	433	44
17	709	234	.1171	421	43
18	736	270	.1155	408	42
19	762	305	.1139	396	41
20	.42788	.47341	2.1123	.90383	40
21	815	377	.1107	371	39
22	841	412	.1092	358	38
23	867	448	.1076	346	37
24	894	483	.1060	334	36
25	.42920	.47519	2.1044	.90321	35
26	946	555	.1028	309	34
27	972	590	.1013	296	33
28	.42999	626	.0997	284	32
29	.43025	662	.0981	271	31
30	.43051	.47698	2.0965	.90259	30
31	077	733	.0950	246	29
32	104	769	.0934	233	28
33	130	805	.0918	221	27
34	156	840	.0903	208	26
35	.43182	.47876	2.0887	.90196	25
36	209	912	.0872	183	24
37	235	948	.0856	171	23
38	261	.47984	.0840	158	22
39	287	.48019	.0825	146	21
40	.43313	.48055	2.0809	.90133	20
41	340	091	.0794	120	19
42	366	127	.0778	108	18
43	392	163	.0763	095	17
44	418	198	.0748	082	16
45	.43445	.48234	2.0732	.90070	15
46	471	270	.0717	057	14
47	497	306	.0701	045	13
48	523	342	.0686	032	12
49	549	378	.0671	019	11
50	.43575	.48414	2.0655	.90007	10
51	602	450	.0640	.89994	9
52	628	486	.0625	981	8
53	654	521	.0609	968	7
54	680	557	.0594	956	6
55	.43706	.48593	2.0579	.89943	5
56	733	629	.0564	930	4
57	759	665	.0549	918	3
58	785	701	.0533	905	2
59	811	737	.0518	892	1
60	.43837	.48773	2.0503	.89879	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	′

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26°

′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.43837	.48773	2.0503	.89879	60
1	863	809	.0488	867	59
2	889	845	.0473	854	58
3	916	881	.0458	841	57
4	942	917	.0443	828	56
5	.43968	.48953	2.0428	.89816	55
6	.43994	.48989	.0413	803	54
7	.44020	.49026	.0398	790	53
8	046	062	.0383	777	52
9	072	098	.0368	764	51
10	.44098	.49134	2.0353	.89752	50
11	124	170	.0338	739	49
12	151	206	.0323	726	48
13	177	242	.0308	713	47
14	203	278	.0293	700	46
15	.44229	.49315	2.0278	.89687	45
16	255	351	.0263	674	44
17	281	387	.0248	662	43
18	307	423	.0233	649	42
19	333	459	.0219	636	41
20	.44359	.49495	2.0204	.89623	40
21	385	532	.0189	610	39
22	411	568	.0174	597	38
23	437	604	.0160	584	37
24	464	640	.0145	571	36
25	.44490	.49677	2.0130	.89558	35
26	516	713	.0115	545	34
27	542	749	.0101	532	33
28	568	786	.0086	519	32
29	594	822	.0072	506	31
30	.44620	.49858	2.0057	.89493	30
31	646	894	.0042	480	29
32	672	931	.0028	467	28
33	698	.49967	2.0013	454	27
34	724	.50004	1.9999	441	26
35	.44750	.50040	1.9984	.89428	25
36	776	076	.9970	415	24
37	802	113	.9955	402	23
38	828	149	.9941	389	22
39	854	185	.9926	376	21
40	.44880	.50222	1.9912	.89363	20
41	906	258	.9897	350	19
42	932	295	.9883	337	18
43	958	331	.9868	324	17
44	.44984	368	.9854	311	16
45	.45010	.50404	1.9840	.89298	15
46	036	441	.9825	285	14
47	062	477	.9811	272	13
48	088	514	.9797	259	12
49	114	550	.9782	245	11
50	.45140	.50587	1.9768	.89232	10
51	166	623	.9754	219	9
52	192	660	.9740	206	8
53	218	696	.9725	193	7
54	243	733	.9711	180	6
55	.45269	.50769	1.9697	.89167	5
56	295	806	.9683	153	4
57	321	843	.9669	140	3
58	347	879	.9654	127	2
59	373	916	.9640	114	1
60	.45399	.50953	1.9626	.89101	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	′

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27°

′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.45399	.50953	1.9626	.89101	60
1	425	.50989	.9612	087	59
2	451	.51026	.9598	074	58
3	477	063	.9584	061	57
4	503	099	.9570	048	56
5	.45529	.51136	1.9556	.89035	55
6	554	173	.9542	021	54
7	580	209	.9528	.89008	53
8	606	246	.9514	.88995	52
9	632	283	.9500	981	51
10	.45658	.51319	1.9486	.88968	50
11	684	356	.9472	955	49
12	710	393	.9458	942	48
13	736	430	.9444	928	47
14	762	467	.9430	915	46
15	.45787	.51503	1.9416	.88902	45
16	813	540	.9402	888	44
17	839	577	.9388	875	43
18	865	614	.9375	862	42
19	891	651	.9361	848	41
20	.45917	.51688	1.9347	.88835	40
21	942	724	.9333	822	39
22	968	761	.9319	808	38
23	.45994	798	.9306	795	37
24	.46020	835	.9292	782	36
25	.46046	.51872	1.9278	.88768	35
26	072	909	.9265	755	34
27	097	946	.9251	741	33
28	123	.51983	.9237	728	32
29	149	.52020	.9223	715	31
30	.46175	.52057	1.9210	.88701	30
31	201	094	.9196	688	29
32	226	131	.9183	674	28
33	252	168	.9169	661	27
34	278	205	.9155	647	26
35	.46304	.52242	1.9142	.88634	25
36	330	279	.9128	620	24
37	355	316	.9115	607	23
38	381	353	.9101	593	22
39	407	390	.9088	580	21
40	.46433	.52427	1.9074	.88566	20
41	458	464	.9061	553	19
42	484	501	.9047	539	18
43	510	538	.9034	526	17
44	536	575	.9020	512	16
45	.46561	.52613	1.9007	.88499	15
46	587	650	.8993	485	14
47	613	687	.8980	472	13
48	639	724	.8967	458	12
49	664	761	.8953	445	11
50	.46690	.52798	1.8940	.88431	10
51	716	836	.8927	417	9
52	742	873	.8913	404	8
53	767	910	.8900	390	7
54	793	947	.8887	377	6
55	.46819	.52985	1.8873	.88363	5
56	844	.53022	.8860	349	4
57	870	059	.8847	336	3
58	896	096	.8834	322	2
59	921	134	.8820	308	1
60	.46947	.53171	1.8807	.88295	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	′

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28°

′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.46947	.53171	1.8807	.88295	60
1	973	208	.8794	281	59
2	.46999	246	.8781	267	58
3	.47024	283	.8768	254	57
4	050	320	.8755	240	56
5	.47076	.53358	1.8741	.88226	55
6	101	395	.8728	213	54
7	127	432	.8715	199	53
8	153	470	.8702	185	52
9	178	507	.8689	172	51
10	.47204	.53545	1.8676	.88158	50
11	229	582	.8663	144	49
12	255	620	.8650	130	48
13	281	657	.8637	117	47
14	306	694	.8624	103	46
15	.47332	.53732	1.8611	.88089	45
16	358	769	.8598	075	44
17	383	807	.8585	062	43
18	409	844	.8572	048	42
19	434	882	.8559	034	41
20	.47460	.53920	1.8546	.88020	40
21	486	957	.8533	.88006	39
22	511	.53995	.8520	.87993	38
23	537	.54032	.8507	979	37
24	562	070	.8495	965	36
25	.47588	.54107	1.8482	.87951	35
26	614	145	.8469	937	34
27	639	183	.8456	923	33
28	665	220	.8443	909	32
29	690	258	.8430	896	31
30	.47716	.54296	1.8418	.87882	30
31	741	333	.8405	868	29
32	767	371	.8392	854	28
33	793	409	.8379	840	27
34	818	446	.8367	826	26
35	.47844	.54484	1.8354	.87812	25
36	869	522	.8341	798	24
37	895	560	.8329	784	23
38	920	597	.8316	770	22
39	946	635	.8303	756	21
40	.47971	.54673	1.8291	.87743	20
41	.47997	711	.8278	729	19
42	.48022	748	.8265	715	18
43	048	786	.8253	701	17
44	073	824	.8240	687	16
45	.48099	.54862	1.8228	.87673	15
46	124	900	.8215	659	14
47	150	938	.8202	645	13
48	175	.54975	.8190	631	12
49	201	.55013	.8177	617	11
50	.48226	.55051	1.8165	.87603	10
51	252	089	.8152	589	9
52	277	127	.8140	575	8
53	303	165	.8127	561	7
54	328	203	.8115	546	6
55	.48354	.55241	1.8103	.87532	5
56	379	279	.8090	518	4
57	405	317	.8078	504	3
58	430	355	.8065	490	2
59	456	393	.8053	476	1
60	.48481	.55431	1.8040	.87462	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	′

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29°

′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.48481	.55431	1.8040	.87462	60
1	506	469	.8028	448	59
2	532	507	.8016	434	58
3	557	545	.8003	420	57
4	583	583	.7991	406	56
5	.48608	.55621	1.7979	.87391	55
6	634	659	.7966	377	54
7	659	697	.7954	363	53
8	684	736	.7942	349	52
9	710	774	.7930	335	51
10	.48735	.55812	1.7917	.87321	50
11	761	850	.7905	306	49
12	786	888	.7893	292	48
13	811	926	.7881	278	47
14	837	.55964	.7868	264	46
15	.48862	.56003	1.7856	.87250	45
16	888	041	.7844	235	44
17	913	079	.7832	221	43
18	938	117	.7820	207	42
19	964	156	.7808	193	41
20	.48989	.56194	1.7796	.87178	40
21	.49014	232	.7783	164	39
22	040	270	.7771	150	38
23	065	309	.7759	136	37
24	090	347	.7747	121	36
25	.49116	.56385	1.7735	.87107	35
26	141	424	.7723	093	34
27	166	462	.7711	079	33
28	192	501	.7699	064	32
29	217	539	.7687	050	31
30	.49242	.56577	1.7675	.87036	30
31	268	616	.7663	021	29
32	293	654	.7651	.87007	28
33	318	693	.7639	.86993	27
34	344	731	.7627	978	26
35	.49369	.56769	1.7615	.86964	25
36	394	808	.7603	949	24
37	419	846	.7591	935	23
38	445	885	.7579	921	22
39	470	923	.7567	906	21
40	.49495	.56962	1.7556	.86892	20
41	521	.57000	.7544	878	19
42	546	039	.7532	863	18
43	571	078	.7520	849	17
44	596	116	.7508	834	16
45	.49622	.57155	1.7496	.86820	15
46	647	193	.7485	805	14
47	672	232	.7473	791	13
48	697	271	.7461	777	12
49	723	309	.7449	762	11
50	.49748	.57348	1.7437	.86748	10
51	773	386	.7426	733	9
52	798	425	.7414	719	8
53	824	464	.7402	704	7
54	849	503	.7391	690	6
55	.49874	.57541	1.7379	.86675	5
56	899	580	.7367	661	4
57	924	619	.7355	646	3
58	950	657	.7344	632	2
59	.49975	696	.7332	617	1
60	.50000	.57735	1.7321	.86603	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	′

60°

30°

′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.50000	.57735	1.7321	.86603	60
1	025	774	.7309	588	59
2	050	813	.7297	573	58
3	076	851	.7286	559	57
4	101	890	.7274	544	56
5	.50126	.57929	1.7262	.86530	55
6	151	.57968	.7251	515	54
7	176	.58007	.7239	501	53
8	201	046	.7228	486	52
9	227	085	.7216	471	51
10	.50252	.58124	1.7205	.86457	50
11	277	162	.7193	442	49
12	302	201	.7182	427	48
13	327	240	.7170	413	47
14	352	279	.7159	398	46
15	.50377	.58318	1.7147	.86384	45
16	403	357	.7136	369	44
17	428	396	.7124	354	43
18	453	435	.7113	340	42
19	478	474	.7102	325	41
20	.50503	.58513	1.7090	.86310	40
21	528	552	.7079	295	39
22	553	591	.7067	281	38
23	578	631	.7056	266	37
24	603	670	.7045	251	36
25	.50628	.58709	1.7033	.86237	35
26	654	748	.7022	222	34
27	679	787	.7011	207	33
28	704	826	.6999	192	32
29	729	865	.6988	178	31
30	.50754	.58905	1.6977	.86163	30
31	779	944	.6965	148	29
32	804	.58983	.6954	133	28
33	829	.59022	.6943	119	27
34	854	061	.6932	104	26
35	.50879	.59101	1.6920	.86089	25
36	904	140	.6909	074	24
37	929	179	.6898	059	23
38	954	218	.6887	045	22
39	.50979	258	.6875	030	21
40	.51004	.59297	1.6864	.86015	20
41	029	336	.6853	.86000	19
42	054	376	.6842	.85985	18
43	079	415	.6831	970	17
44	104	454	.6820	956	16
45	.51129	.59494	1.6808	.85941	15
46	154	533	.6797	926	14
47	179	573	.6786	911	13
48	204	612	.6775	896	12
49	229	651	.6764	881	11
50	.51254	.59691	1.6753	.85866	10
51	279	730	.6742	851	9
52	304	770	.6731	836	8
53	329	809	.6720	821	7
54	354	849	.6709	806	6
55	.51379	.59888	1.6698	.85792	5
56	404	928	.6687	777	4
57	429	.59967	.6676	762	3
58	454	.60007	.6665	747	2
59	479	046	.6654	732	1
60	.51504	.60086	1.6643	.85717	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	′

59°

31°

′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.51504	.60086	1.6643	.85717	60
1	529	126	.6632	702	59
2	554	165	.6621	687	58
3	579	205	.6610	672	57
4	604	245	.6599	657	56
5	.51628	.60284	1.6588	.85642	55
6	653	324	.6577	627	54
7	678	364	.6566	612	53
8	703	403	.6555	597	52
9	728	443	.6545	582	51
10	.51753	.60483	1.6534	.85567	50
11	778	522	.6523	551	49
12	803	562	.6512	536	48
13	828	602	.6501	521	47
14	852	642	.6490	506	46
15	.51877	.60681	1.6479	.85491	45
16	902	721	.6469	476	44
17	927	761	.6458	461	43
18	952	801	.6447	446	42
19	.51977	841	.6436	431	41
20	.52002	.60881	1.6426	.85416	40
21	026	921	.6415	401	39
22	051	.60960	.6404	385	38
23	076	.61000	.6393	370	37
24	101	040	.6383	355	36
25	.52126	.61080	1.6372	.85340	35
26	151	120	.6361	325	34
27	175	160	.6351	310	33
28	200	200	.6340	294	32
29	225	240	.6329	279	31
30	.52250	.61280	1.6319	.85264	30
31	275	320	.6308	249	29
32	299	360	.6297	234	28
33	324	400	.6287	218	27
34	349	440	.6276	203	26
35	.52374	.61480	1.6265	.85188	25
36	399	520	.6255	173	24
37	423	561	.6244	157	23
38	448	601	.6234	142	22
39	473	641	.6223	127	21
40	.52498	.61681	1.6212	.85112	20
41	522	721	.6202	096	19
42	547	761	.6191	081	18
43	572	801	.6181	066	17
44	597	842	.6170	051	16
45	.52621	.61882	1.6160	.85035	15
46	646	922	.6149	020	14
47	671	.61962	.6139	.85005	13
48	696	.62003	.6128	.84989	12
49	720	043	.6118	974	11
50	.52745	.62083	1.6107	.84959	10
51	770	124	.6097	943	9
52	794	164	.6087	928	8
53	819	204	.6076	913	7
54	844	245	.6066	897	6
55	.52869	.62285	1.6055	.84882	5
56	893	325	.6045	866	4
57	918	366	.6034	851	3
58	943	406	.6024	836	2
59	967	446	.6014	820	1
60	.52992	.62487	1.6003	.84805	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	′

58°

32°

′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.52992	.62487	1.6003	.84805	60
1	.53017	527	.5993	789	59
2	041	568	.5983	774	58
3	066	608	.5972	759	57
4	091	649	.5962	743	56
5	.53115	.62689	1.5952	.84728	55
6	140	730	.5941	712	54
7	164	770	.5931	697	53
8	189	811	.5921	681	52
9	214	852	.5911	666	51
10	.53238	.62892	1.5900	.84650	50
11	263	933	.5890	635	49
12	288	.62973	.5880	619	48
13	312	.63014	.5869	604	47
14	337	055	.5859	588	46
15	.53361	.63095	1.5849	.84573	45
16	386	136	.5839	557	44
17	411	177	.5829	542	43
18	435	217	.5818	526	42
19	460	258	.5808	511	41
20	.53484	.63299	1.5798	.84495	40
21	509	340	.5788	480	39
22	534	380	.5778	464	38
23	558	421	.5768	448	37
24	583	462	.5757	433	36
25	.53607	.63503	1.5747	.84417	35
26	632	544	.5737	402	34
27	656	584	.5727	386	33
28	681	625	.5717	370	32
29	705	666	.5707	355	31
30	.53730	.63707	1.5697	.84339	30
31	754	748	.5687	324	29
32	779	789	.5677	308	28
33	804	830	.5667	292	27
34	828	871	.5657	277	26
35	.53853	.63912	1.5647	.84261	25
36	877	953	.5637	245	24
37	902	.63994	.5627	230	23
38	926	.64035	.5617	214	22
39	951	076	.5607	198	21
40	.53975	.64117	1.5597	.84182	20
41	.54000	158	.5587	167	19
42	024	199	.5577	151	18
43	049	240	.5567	135	17
44	073	281	.5557	120	16
45	.54097	.64322	1.5547	.84104	15
46	122	363	.5537	088	14
47	146	404	.5527	072	13
48	171	446	.5517	057	12
49	195	487	.5507	041	11
50	.54220	.64528	1.5497	.84025	10
51	244	569	.5487	.84009	9
52	269	610	.5477	.83994	8
53	293	652	.5468	978	7
54	317	693	.5458	962	6
55	.54342	.64734	1.5448	.83946	5
56	366	775	.5438	930	4
57	391	817	.5428	915	3
58	415	858	.5418	899	2
59	440	899	.5408	883	1
60	.54464	.64941	1.5399	.83867	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	′

57°

33°

′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.54464	.64941	1.5399	.83867	60
1	488	.64982	.5389	851	59
2	513	.65024	.5379	835	58
3	537	065	.5369	819	57
4	561	106	.5359	804	56
5	.54586	.65148	1.5350	.83788	55
6	610	189	.5340	772	54
7	635	231	.5330	756	53
8	659	272	.5320	740	52
9	683	314	.5311	724	51
10	.54708	.65355	1.5301	.83708	50
11	732	397	.5291	692	49
12	756	438	.5282	676	48
13	781	480	.5272	660	47
14	805	521	.5262	645	46
15	.54829	.65563	1.5253	.83629	45
16	854	604	.5243	613	44
17	878	646	.5233	597	43
18	902	688	.5224	581	42
19	927	729	.5214	565	41
20	.54951	.65771	1.5204	.83549	40
21	975	813	.5195	533	39
22	.54999	854	.5185	517	38
23	.55024	896	.5175	501	37
24	048	938	.5166	485	36
25	.55072	.65980	1.5156	.83469	35
26	097	.66021	.5147	453	34
27	121	063	.5137	437	33
28	145	105	.5127	421	32
29	169	147	.5118	405	31
30	.55194	.66189	1.5108	.83389	30
31	218	230	.5099	373	29
32	242	272	.5089	356	28
33	266	314	.5080	340	27
34	291	356	.5070	324	26
35	.55315	.66398	1.5061	.83308	25
36	339	440	.5051	292	24
37	363	482	.5042	276	23
38	388	524	.5032	260	22
39	412	566	.5023	244	21
40	.55436	.66608	1.5013	.83228	20
41	460	650	.5004	212	19
42	484	692	.4994	195	18
43	509	734	.4985	179	17
44	533	776	.4975	163	16
45	.55557	.66818	1.4966	.83147	15
46	581	860	.4957	131	14
47	605	902	.4947	115	13
48	630	944	.4938	098	12
49	654	.66986	.4928	082	11
50	.55678	.67028	1.4919	.83066	10
51	702	071	.4910	050	9
52	726	113	.4900	034	8
53	750	155	.4891	017	7
54	775	197	.4882	.83001	6
55	.55799	.67239	1.4872	.82985	5
56	823	282	.4863	969	4
57	847	324	.4854	953	3
58	871	366	.4844	936	2
59	895	409	.4835	920	1
60	.55919	.67451	1.4826	.82904	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	′

56°

34°

′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.55919	.67451	1.4826	.82904	60
1	943	493	.4816	887	59
2	968	536	.4807	871	58
3	.55992	578	.4798	855	57
4	.56016	620	.4788	839	56
5	.56040	.67663	1.4779	.82822	55
6	064	705	.4770	806	54
7	088	748	.4761	790	53
8	112	790	.4751	773	52
9	136	832	.4742	757	51
10	.56160	.67875	1.4733	.82741	50
11	184	917	.4724	724	49
12	208	.67960	.4715	708	48
13	232	.68002	.4705	692	47
14	256	045	.4696	675	46
15	.56280	.68088	1.4687	.82659	45
16	305	130	.4678	643	44
17	329	173	.4669	626	43
18	353	215	.4659	610	42
19	377	258	.4650	593	41
20	.56401	.68301	1.4641	.82577	40
21	425	343	.4632	561	39
22	449	386	.4623	544	38
23	473	429	.4614	528	37
24	497	471	.4605	511	36
25	.56521	.68514	1.4596	.82495	35
26	545	557	.4586	478	34
27	569	600	.4577	462	33
28	593	642	.4568	446	32
29	617	685	.4559	429	31
30	.56641	.68728	1.4550	.82413	30
31	665	771	.4541	396	29
32	689	814	.4532	380	28
33	713	857	.4523	363	27
34	736	900	.4514	347	26
35	.56760	.68942	1.4505	.82330	25
36	784	.68985	.4496	314	24
37	808	.69028	.4487	297	23
38	832	071	.4478	281	22
39	856	114	.4469	264	21
40	.56880	.69157	1.4460	.82248	20
41	904	200	.4451	231	19
42	928	243	.4442	214	18
43	952	286	.4433	198	17
44	.56976	329	.4424	181	16
45	.57000	.69372	1.4415	.82165	15
46	024	416	.4406	148	14
47	047	459	.4397	132	13
48	071	502	.4388	115	12
49	095	545	.4379	098	11
50	.57119	.69588	1.4370	.82082	10
51	143	631	.4361	065	9
52	167	675	.4352	048	8
53	191	718	.4344	032	7
54	215	761	.4335	.82015	6
55	.57238	.69804	1.4326	.81999	5
56	262	847	.4317	982	4
57	286	891	.4308	965	3
58	310	934	.4299	949	2
59	334	.69977	.4290	932	1
60	.57358	.70021	1.4281	.81915	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	′

55°

35°

′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.57358	.70021	1.4281	.81915	60
1	381	064	.4273	899	59
2	405	107	.4264	882	58
3	429	151	.4255	865	57
4	453	194	.4246	848	56
5	.57477	.70238	1.4237	.81832	55
6	501	281	.4229	815	54
7	524	325	.4220	798	53
8	548	368	.4211	782	52
9	572	412	.4202	765	51
10	.57596	.70455	1.4193	.81748	50
11	619	499	.4185	731	49
12	643	542	.4176	714	48
13	667	586	.4167	698	47
14	691	629	.4158	681	46
15	.57715	.70673	1.4150	.81664	45
16	738	717	.4141	647	44
17	762	760	.4132	631	43
18	786	804	.4124	614	42
19	810	848	.4115	597	41
20	.57833	.70891	1.4106	.81580	40
21	857	935	.4097	563	39
22	881	.70979	.4089	546	38
23	904	.71023	.4080	530	37
24	928	066	.4071	513	36
25	.57952	.71110	1.4063	.81496	35
26	976	154	.4054	479	34
27	.57999	198	.4045	462	33
28	.58023	242	.4037	445	32
29	047	285	.4028	428	31
30	.58070	.71329	1.4019	.81412	30
31	094	373	.4011	395	29
32	118	417	.4002	378	28
33	141	461	.3994	361	27
34	165	505	.3985	344	26
35	.58189	.71549	1.3976	.81327	25
36	212	593	.3968	310	24
37	236	637	.3959	293	23
38	260	681	.3951	276	22
39	283	725	.3942	259	21
40	.58307	.71769	1.3934	.81242	20
41	330	813	.3925	225	19
42	354	857	.3916	208	18
43	378	901	.3908	191	17
44	401	946	.3899	174	16
45	.58425	.71990	1.3891	.81157	15
46	449	.72034	.3882	140	14
47	472	078	.3874	123	13
48	496	122	.3865	106	12
49	519	167	.3857	089	11
50	.58543	.72211	1.3848	.81072	10
51	567	255	.3840	055	9
52	590	299	.3831	038	8
53	614	344	.3823	021	7
54	637	388	.3814	.81004	6
55	.58661	.72432	1.3806	.80987	5
56	684	477	.3798	970	4
57	708	521	.3789	953	3
58	731	565	.3781	936	2
59	755	610	.3772	919	1
60	.58779	.72654	1.3764	.80902	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	′

54°

36°

′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.58779	.72654	1.3764	.80902	60
1	802	699	.3755	885	59
2	826	743	.3747	867	58
3	849	788	.3739	850	57
4	873	832	.3730	833	56
5	.58896	.72877	1.3722	.80816	55
6	920	921	.3713	799	54
7	943	.72966	.3705	782	53
8	967	.73010	.3697	765	52
9	.58990	055	.3688	748	51
10	.59014	.73100	1.3680	.80730	50
11	037	144	.3672	713	49
12	061	189	.3663	696	48
13	084	234	.3655	679	47
14	108	278	.3647	662	46
15	.59131	.73323	1.3638	.80644	45
16	154	368	.3630	627	44
17	178	413	.3622	610	43
18	201	457	.3613	593	42
19	225	502	.3605	576	41
20	.59248	.73547	1.3597	.80558	40
21	272	592	.3588	541	39
22	295	637	.3580	524	38
23	318	681	.3572	507	37
24	342	726	.3564	489	36
25	.59365	.73771	1.3555	.80472	35
26	389	816	.3547	455	34
27	412	861	.3539	438	33
28	436	906	.3531	420	32
29	459	951	.3522	403	31
30	.59482	.73996	1.3514	.80386	30
31	506	.74041	.3506	368	29
32	529	086	.3498	351	28
33	552	131	.3490	334	27
34	576	176	.3481	316	26
35	.59599	.74221	1.3473	.80299	25
36	622	267	.3465	282	24
37	646	312	.3457	264	23
38	669	357	.3449	247	22
39	693	402	.3440	230	21
40	.59716	.74447	1.3432	.80212	20
41	739	492	.3424	195	19
42	763	538	.3416	178	18
43	786	583	.3408	160	17
44	809	628	.3400	143	16
45	.59832	.74674	1.3392	.80125	15
46	856	719	.3384	108	14
47	879	764	.3375	091	13
48	902	810	.3367	073	12
49	926	855	.3359	056	11
50	.59949	.74900	1.3351	.80038	10
51	972	946	.3343	021	9
52	.59995	.74991	.3335	.80003	8
53	.60019	.75037	.3327	.79986	7
54	042	082	.3319	968	6
55	.60065	.75128	1.3311	.79951	5
56	089	173	.3303	934	4
57	112	219	.3295	916	3
58	135	264	.3287	899	2
59	158	310	.3278	881	1
60	.60182	.75355	1.3270	.79864	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	′

53°

37°

′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.60182	.75355	1.3270	.79864	60
1	205	401	.3262	846	59
2	228	447	.3254	829	58
3	251	492	.3246	811	57
4	274	538	.3238	793	56
5	.60298	.75584	1.3230	.79776	55
6	321	629	.3222	758	54
7	344	675	.3214	741	53
8	367	721	.3206	723	52
9	390	767	.3198	706	51
10	.60414	.75812	1.3190	.79688	50
11	437	858	.3182	671	49
12	460	904	.3175	653	48
13	483	950	.3167	635	47
14	506	.75996	.3159	618	46
15	.60529	.76042	1.3151	.79600	45
16	553	088	.3143	583	44
17	576	134	.3135	565	43
18	599	180	.3127	547	42
19	622	226	.3119	530	41
20	.60645	.76272	1.3111	.79512	40
21	668	318	.3103	494	39
22	691	364	.3095	477	38
23	714	410	.3087	459	37
24	738	456	.3079	441	36
25	.60761	.76502	1.3072	.79424	35
26	784	548	.3064	406	34
27	807	594	.3056	388	33
28	830	640	.3048	371	32
29	853	686	.3040	353	31
30	.60876	.76733	1.3032	.79335	30
31	899	779	.3024	318	29
32	922	825	.3017	300	28
33	945	871	.3009	282	27
34	968	918	.3001	264	26
35	.60991	.76964	1.2993	.79247	25
36	.61015	.77010	.2985	229	24
37	038	057	.2977	211	23
38	061	103	.2970	193	22
39	084	149	.2962	176	21
40	.61107	.77196	1.2954	.79158	20
41	130	242	.2946	140	19
42	153	289	.2938	122	18
43	176	335	.2931	105	17
44	199	382	.2923	087	16
45	.61222	.77428	1.2915	.79069	15
46	245	475	.2907	051	14
47	268	521	.2900	033	13
48	291	568	.2892	.79016	12
49	314	615	.2884	.78998	11
50	.61337	.77661	1.2876	.78980	10
51	360	708	.2869	962	9
52	383	754	.2861	944	8
53	406	801	.2853	926	7
54	429	848	.2846	908	6
55	.61451	.77895	1.2838	.78891	5
56	474	941	.2830	873	4
57	497	.77988	.2822	855	3
58	520	.78035	.2815	837	2
59	543	082	.2807	819	1
60	.61566	.78129	1.2799	.78801	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	′

52°

38°

′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.61566	.78129	1.2799	.78801	60
1	589	175	.2792	783	59
2	612	222	.2784	765	58
3	635	269	.2776	747	57
4	658	316	.2769	729	56
5	.61681	.78363	1.2761	.78711	55
6	704	410	.2753	694	54
7	726	457	.2746	676	53
8	749	504	.2738	658	52
9	772	551	.2731	640	51
10	.61795	.78598	1.2723	.78622	50
11	818	645	.2715	604	49
12	841	692	.2708	586	48
13	864	739	.2700	568	47
14	887	786	.2693	550	46
15	.61909	.78834	1.2685	.78532	45
16	932	881	.2677	514	44
17	955	928	.2670	496	43
18	.61978	.78975	.2662	478	42
19	.62001	.79022	.2655	460	41
20	.62024	.79070	1.2647	.78442	40
21	046	117	.2640	424	39
22	069	164	.2632	405	38
23	092	212	.2624	387	37
24	115	259	.2617	369	36
25	.62138	.79306	1.2609	.78351	35
26	160	354	.2602	333	34
27	183	401	.2594	315	33
28	206	449	.2587	297	32
29	229	496	.2579	279	31
30	.62251	.79544	1.2572	.78261	30
31	274	591	.2564	243	29
32	297	639	.2557	225	28
33	320	686	.2549	206	27
34	342	734	.2542	188	26
35	.62365	.79781	1.2534	.78170	25
36	388	829	.2527	152	24
37	411	877	.2519	134	23
38	433	924	.2512	116	22
39	456	.79972	.2504	098	21
40	.62479	.80020	1.2497	.78079	20
41	502	067	.2489	061	19
42	524	115	.2482	043	18
43	547	163	.2475	025	17
44	570	211	.2467	.78007	16
45	.62592	.80258	1.2460	.77988	15
46	615	306	.2452	970	14
47	638	354	.2445	952	13
48	660	402	.2437	934	12
49	683	450	.2430	916	11
50	.62706	.80498	1.2423	.77897	10
51	728	546	.2415	879	9
52	751	594	.2408	861	8
53	774	642	.2401	843	7
54	796	690	.2393	824	6
55	.62819	.80738	1.2386	.77806	5
56	842	786	.2378	788	4
57	864	834	.2371	769	3
58	887	882	.2364	751	2
59	909	930	.2356	733	1
60	.62932	.80978	1.2349	.77715	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	′

51°

39°

′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.62932	.80978	1.2349	.77715	60
1	955	.81027	.2342	696	59
2	.62977	075	.2334	678	58
3	.63000	123	.2327	660	57
4	022	171	.2320	641	56
5	.63045	.81220	1.2312	.77623	55
6	068	268	.2305	605	54
7	090	316	.2298	586	53
8	113	364	.2290	568	52
9	135	413	.2283	550	51
10	.63158	.81461	1.2276	.77531	50
11	180	510	.2268	513	49
12	203	558	.2261	494	48
13	225	606	.2254	476	47
14	248	655	.2247	458	46
15	.63271	.81703	1.2239	.77439	45
16	293	752	.2232	421	44
17	316	800	.2225	402	43
18	338	849	.2218	384	42
19	361	898	.2210	366	41
20	.63383	.81946	1.2203	.77347	40
21	406	.81995	.2196	329	39
22	428	.82044	.2189	310	38
23	451	092	.2181	292	37
24	473	141	.2174	273	36
25	.63496	.82190	1.2167	.77255	35
26	518	238	.2160	236	34
27	540	287	.2153	218	33
28	563	336	.2145	199	32
29	585	385	.2138	181	31
30	.63608	.82434	1.2131	.77162	30
31	630	483	.2124	144	29
32	653	531	.2117	125	28
33	675	580	.2109	107	27
34	698	629	.2102	088	26
35	.63720	.82678	1.2095	.77070	25
36	742	727	.2088	051	24
37	765	776	.2081	033	23
38	787	825	.2074	.77014	22
39	810	874	.2066	.76996	21
40	.63832	.82923	1.2059	.76977	20
41	854	.82972	.2052	959	19
42	877	.83022	.2045	940	18
43	899	071	.2038	921	17
44	922	120	.2031	903	16
45	.63944	.83169	1.2024	.76884	15
46	966	218	.2017	866	14
47	.63989	268	.2009	847	13
48	.64011	317	.2002	828	12
49	033	366	.1995	810	11
50	.64056	.83415	1.1988	.76791	10
51	078	465	.1981	772	9
52	100	514	.1974	754	8
53	123	564	.1967	735	7
54	145	613	.1960	717	6
55	.64167	.83662	1.1953	.76698	5
56	190	712	.1946	679	4
57	212	761	.1939	661	3
58	234	811	.1932	642	2
59	256	860	.1925	623	1
60	.64279	.83910	1.1918	.76604	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	′

50°

40°

′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.64279	.83910	1.1918	.76604	60
1	301	.83960	.1910	586	59
2	323	.84009	.1903	567	58
3	346	059	.1896	548	57
4	368	108	.1889	530	56
5	.64390	.84158	1.1882	.76511	55
6	412	208	.1875	492	54
7	435	258	.1868	473	53
8	457	307	.1861	455	52
9	479	357	.1854	436	51
10	.64501	.84407	1.1847	.76417	50
11	524	457	.1840	398	49
12	546	507	.1833	380	48
13	568	556	.1826	361	47
14	590	606	.1819	342	46
15	.64612	.84656	1.1812	.76323	45
16	635	706	.1806	304	44
17	657	756	.1799	286	43
18	679	806	.1792	267	42
19	701	856	.1785	248	41
20	.64723	.84906	1.1778	.76229	40
21	746	.84956	.1771	210	39
22	768	.85006	.1764	192	38
23	790	057	.1757	173	37
24	812	107	.1750	154	36
25	.64834	.85157	1.1743	.76135	35
26	856	207	.1736	116	34
27	878	257	.1729	097	33
28	901	308	.1722	078	32
29	923	358	.1715	059	31
30	.64945	.85408	1.1708	.76041	30
31	967	458	.1702	022	29
32	.64989	509	.1695	.76003	28
33	.65011	559	.1688	.75984	27
34	033	609	.1681	965	26
35	.65055	.85660	1.1674	.75946	25
36	077	710	.1667	927	24
37	100	761	.1660	908	23
38	122	811	.1653	889	22
39	144	862	.1647	870	21
40	.65166	.85912	1.1640	.75851	20
41	188	.85963	.1633	832	19
42	210	.86014	.1626	813	18
43	232	064	.1619	794	17
44	254	115	.1612	775	16
45	.65276	.86166	1.1606	.75756	15
46	298	216	.1599	738	14
47	320	267	.1592	719	13
48	342	318	.1585	700	12
49	364	368	.1578	680	11
50	.65386	.86419	1.1571	.75661	10
51	408	470	.1565	642	9
52	430	521	.1558	623	8
53	452	572	.1551	604	7
54	474	623	.1544	585	6
55	.65496	.86674	1.1538	.75566	5
56	518	725	.1531	547	4
57	540	776	.1524	528	3
58	562	827	.1517	509	2
59	584	878	.1510	490	1
60	.65606	.86929	1.1504	.75471	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	′

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41°

′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.65606	.86929	1.1504	.75471	60
1	628	.86980	.1497	452	59
2	650	.87031	.1490	433	58
3	672	082	.1483	414	57
4	694	133	.1477	395	56
5	.65716	.87184	1.1470	.75375	55
6	738	236	.1463	356	54
7	759	287	.1456	337	53
8	781	338	.1450	318	52
9	803	389	.1443	299	51
10	.65825	.87441	1.1436	.75280	50
11	847	492	.1430	261	49
12	869	543	.1423	241	48
13	891	595	.1416	222	47
14	913	646	.1410	203	46
15	.65935	.87698	1.1403	.75184	45
16	956	749	.1396	165	44
17	.65978	801	.1389	146	43
18	.66000	852	.1383	126	42
19	022	904	.1376	107	41
20	.66044	.87955	1.1369	.75088	40
21	066	.88007	.1363	069	39
22	088	059	.1356	050	38
23	109	110	.1349	030	37
24	131	162	.1343	.75011	36
25	.66153	.88214	1.1336	.74992	35
26	175	265	.1329	973	34
27	197	317	.1323	953	33
28	218	369	.1316	934	32
29	240	421	.1310	915	31
30	.66262	.88473	1.1303	.74896	30
31	284	524	.1296	876	29
32	306	576	.1290	857	28
33	327	628	.1283	838	27
34	349	680	.1276	818	26
35	.66371	.88732	1.1270	.74799	25
36	393	784	.1263	780	24
37	414	836	.1257	760	23
38	436	888	.1250	741	22
39	458	940	.1243	722	21
40	.66480	.88992	1.1237	.74703	20
41	501	.89045	.1230	683	19
42	523	097	.1224	664	18
43	545	149	.1217	644	17
44	566	201	.1211	625	16
45	.66588	.89253	1.1204	.74606	15
46	610	306	.1197	586	14
47	632	358	.1191	567	13
48	653	410	.1184	548	12
49	675	463	.1178	528	11
50	.66697	.89515	1.1171	.74509	10
51	718	567	.1165	489	9
52	740	620	.1158	470	8
53	762	672	.1152	451	7
54	783	725	.1145	431	6
55	.66805	.89777	1.1139	.74412	5
56	827	830	.1132	392	4
57	848	883	.1126	373	3
58	870	935	.1119	353	2
59	891	.89988	.1113	334	1
60	.66913	.90040	1.1106	.74314	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	′

48°

42°

′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.66913	.90040	1.1106	.74314	60
1	935	093	.1100	295	59
2	956	146	.1093	276	58
3	978	199	.1087	256	57
4	.66999	251	.1080	237	56
5	.67021	.90304	1.1074	.74217	55
6	043	357	.1067	198	54
7	064	410	.1061	178	53
8	086	463	.1054	159	52
9	107	516	.1048	139	51
10	.67129	.90569	1.1041	.74120	50
11	151	621	.1035	100	49
12	172	674	.1028	080	48
13	194	727	.1022	061	47
14	215	781	.1016	041	46
15	.67237	.90834	1.1009	.74022	45
16	258	887	.1003	.74002	44
17	280	940	.0996	.73983	43
18	301	.90993	.0990	963	42
19	323	.91046	.0983	944	41
20	.67344	.91099	1.0977	.73924	40
21	366	153	.0971	904	39
22	387	206	.0964	885	38
23	409	259	.0958	865	37
24	430	313	.0951	846	36
25	.67452	.91366	1.0945	.73826	35
26	473	419	.0939	806	34
27	495	473	.0932	787	33
28	516	526	.0926	767	32
29	538	580	.0919	747	31
30	.67559	.91633	1.0913	.73728	30
31	580	687	.0907	708	29
32	602	740	.0900	688	28
33	623	794	.0894	669	27
34	645	847	.0888	649	26
35	.67666	.91901	1.0881	.73629	25
36	688	.91955	.0875	610	24
37	709	.92008	.0869	590	23
38	730	062	.0862	570	22
39	752	116	.0856	551	21
40	.67773	.92170	1.0850	.73531	20
41	795	224	.0843	511	19
42	816	277	.0837	491	18
43	837	331	.0831	472	17
44	859	385	.0824	452	16
45	.67880	.92439	1.0818	.73432	15
46	901	493	.0812	413	14
47	923	547	.0805	393	13
48	944	601	.0799	373	12
49	965	655	.0793	353	11
50	.67987	.92709	1.0786	.73333	10
51	.68008	763	.0780	314	9
52	029	817	.0774	294	8
53	051	872	.0768	274	7
54	072	926	.0761	254	6
55	.68093	.92980	1.0755	.73234	5
56	115	.93034	.0749	215	4
57	136	088	.0742	195	3
58	157	143	.0736	175	2
59	179	197	.0730	155	1
60	.68200	.93252	1.0724	.73135	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	′

47°

43°

′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.68200	.93252	1.0724	.73135	60
1	221	306	.0717	116	59
2	242	360	.0711	096	58
3	264	415	.0705	076	57
4	285	469	.0699	056	56
5	.68306	.93524	1.0692	.73036	55
6	327	578	.0686	.73016	54
7	349	633	.0680	.72996	53
8	370	688	.0674	976	52
9	391	742	.0668	957	51
10	.68412	.93797	1.0661	.72937	50
11	434	852	.0655	917	49
12	455	906	.0649	897	48
13	476	.93961	.0643	877	47
14	497	.94016	.0637	857	46
15	.68518	.94071	1.0630	.72837	45
16	539	125	.0624	817	44
17	561	180	.0618	797	43
18	582	235	.0612	777	42
19	603	290	.0606	757	41
20	.68624	.94345	1.0599	.72737	40
21	645	400	.0593	717	39
22	666	455	.0587	697	38
23	688	510	.0581	677	37
24	709	565	.0575	657	36
25	.68730	.94620	1.0569	.72637	35
26	751	676	.0562	617	34
27	772	731	.0556	597	33
28	793	786	.0550	577	32
29	814	841	.0544	557	31
30	.68835	.94896	1.0538	.72537	30
31	857	.94952	.0532	517	29
32	878	.95007	.0526	497	28
33	899	062	.0519	477	27
34	920	118	.0513	457	26
35	.68941	.95173	1.0507	.72437	25
36	962	229	.0501	417	24
37	.68983	284	.0495	397	23
38	.69004	340	.0489	377	22
39	025	395	.0483	357	21
40	.69046	.95451	1.0477	.72337	20
41	067	506	.0470	317	19
42	088	562	.0464	297	18
43	109	618	.0458	277	17
44	130	673	.0452	257	16
45	.69151	.95729	1.0446	.72236	15
46	172	785	.0440	216	14
47	193	841	.0434	196	13
48	214	897	.0428	176	12
49	235	.95952	.0422	156	11
50	.69256	.96008	1.0416	.72136	10
51	277	064	.0410	116	9
52	298	120	.0404	095	8
53	319	176	.0398	075	7
54	340	232	.0392	055	6
55	.69361	.96288	1.0385	.72035	5
56	382	344	.0379	.72015	4
57	403	400	.0373	.71995	3
58	424	457	.0367	974	2
59	445	513	.0361	954	1
60	.69466	.96569	1.0355	.71934	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	′

46°

44°

′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.69466	.96569	1.0355	.71934	60
1	487	625	.0349	914	59
2	508	681	.0343	894	58
3	529	738	.0337	873	57
4	549	794	.0331	853	56
5	.69570	.96850	1.0325	.71833	55
6	591	907	.0319	813	54
7	612	.96963	.0313	792	53
8	633	.97020	.0307	772	52
9	654	076	.0301	752	51
10	.69675	.97133	1.0295	.71732	50
11	696	189	.0289	711	49
12	717	246	.0283	691	48
13	737	302	.0277	671	47
14	758	359	.0271	650	46
15	.69779	.97416	1.0265	.71630	45
16	800	472	.0259	610	44
17	821	529	.0253	590	43
18	842	586	.0247	569	42
19	862	643	.0241	549	41
20	.69883	.97700	1.0235	.71529	40
21	904	756	.0230	508	39
22	925	813	.0224	488	38
23	946	870	.0218	468	37
24	966	927	.0212	447	36
25	.69987	.97984	1.0206	.71427	35
26	.70008	.98041	.0200	407	34
27	029	098	.0194	386	33
28	049	155	.0188	366	32
29	070	213	.0182	345	31
30	.70091	.98270	1.0176	.71325	30
31	112	327	.0170	305	29
32	132	384	.0164	284	28
33	153	441	.0158	264	27
34	174	499	.0152	243	26
35	.70195	.98556	1.0147	.71223	25
36	215	613	.0141	203	24
37	236	671	.0135	182	23
38	257	728	.0129	162	22
39	277	786	.0123	141	21
40	.70298	.98843	1.0117	.71121	20
41	319	901	.0111	100	19
42	339	.98958	.0105	080	18
43	360	.99016	.0099	059	17
44	381	073	.0094	039	16
45	.70401	.99131	1.0088	.71019	15
46	422	189	.0082	.70998	14
47	443	247	.0076	978	13
48	463	304	.0070	957	12
49	484	362	.0064	937	11
50	.70505	.99420	1.0058	.70916	10
51	525	478	.0052	896	9
52	546	536	.0047	875	8
53	567	594	.0041	855	7
54	587	652	.0035	834	6
55	.70608	.99710	1.0029	.70813	5
56	628	768	.0023	793	4
57	649	826	.0017	772	3
58	670	884	.0012	752	2
59	690	.99942	.0006	731	1
60	.70711	1.0000	1.0000	.70711	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	′

45°

DEGREES.						MINUTES.		SECONDS.	
0°	0.00000 00	60°	1.04719 76	120°	2.09439 51	0′	0.00000 00	0″	0.00000 00
1	0.01745 33	61	1.06465 08	121	2.11184 84	1	0.00029 09	1	0.00000 48
2	0.03490 66	62	1.08210 41	122	2.12930 17	2	0.00058 18	2	0.00000 97
3	0.05235 99	63	1.09955 74	123	2.14675 $\bar{5}$0	3	0.00087 27	3	0.00001 45
4	0.06981 32	64	1.11701 07	124	2.16420 83	4	0.00116 36	4	0.00001 94
5	0.08726 6$\bar{5}$	65	1.13446 40	125	2.18166 16	5	0.00145 44	5	0.00002 42
6	0.10471 98	66	1.15191 73	126	2.19911 49	6	0.00174 53	6	0.00002 91
7	0.12217 30	67	1.16937 06	127	2.21656 82	7	0.00203 62	7	0.00003 39
8	0.13962 63	68	1.18682 39	128	2.23402 14	8	0.00232 71	8	0.00003 88
9	0.15707 96	69	1.20427 72	129	2.25147 47	9	0.00261 80	9	0.00004 36
10	0.17453 29	**70**	1.22173 0$\bar{5}$	**130**	2.26892 80	**10**	0.00290 89	**10**	0.00004 8$\bar{5}$
11	0.19198 62	71	1.23918 38	131	2.28638 13	11	0.00319 98	11	0.00005 33
12	0.20943 95	72	1.25663 71	132	2.30383 46	12	0.00349 07	12	0.00005 82
13	0.22689 28	73	1.27409 04	133	2.32128 79	13	0.00378 15	13	0.00006 30
14	0.24434 61	74	1.29154 36	134	2.33874 12	14	0.00407 24	14	0.00006 79
15	0.26179 94	75	1.30899 69	135	2.35619 4$\bar{5}$	15	0.00436 33	15	0.00007 27
16	0.27925 27	76	1.32645 02	136	2.37364 78	16	0.00465 42	16	0.00007 76
17	0.29670 60	77	1.34390 35	137	2.39110 11	17	0.00494 51	17	0.00008 24
18	0.31415 93	78	1.36135 68	138	2.40855 44	18	0.00523 60	18	0.00008 73
19	0.33161 26	79	1.37881 01	139	2.42600 77	19	0.00552 69	19	0.00009 21
20	0.34906 59	**80**	1.39626 34	**140**	2.44346 10	**20**	0.00581 78	**20**	0.00009 70
21	0.36651 91	81	1.41371 67	141	2.46091 42	21	0.00610 87	21	0.00010 18
22	0.38397 24	82	1.43117 00	142	2.47836 75	22	0.00639 95	22	0.00010 67
23	0.40142 57	83	1.44862 33	143	2.49582 08	23	0.00669 04	23	0.00011 15
24	0.41887 90	84	1.46607 66	144	2.51327 41	24	0.00698 13	24	0.00011 64
25	0.43633 23	85	1.48352 99	145	2.53072 74	25	0.00727 22	25	0.00012 12
26	0.45378 56	86	1.50098 32	146	2.54818 07	26	0.00756 31	26	0.00012 61
27	0.47123 89	87	1.51843 64	147	2.56563 40	27	0.00785 40	27	0.00013 09
28	0.48869 22	88	1.53588 97	148	2.58308 73	28	0.00814 49	28	0.00013 57
29	0.50614 5$\bar{5}$	89	1.55334 30	149	2.60054 06	29	0.00843 58	29	0.00014 06
30	0.52359 88	**90**	1.57079 63	**150**	2.61799 39	**30**	0.00872 66	**30**	0.00014 54
31	0.54105 21	91	1.58824 96	151	2.63544 72	31	0.00901 75	31	0.00015 03
32	0.55850 54	92	1.60570 29	152	2.65290 0$\bar{5}$	32	0.00930 84	32	0.00015 51
33	0.57595 87	93	1.62315 62	153	2.67035 38	33	0.00959 93	33	0.00016 00
34	0.59341 19	94	1.64060 9$\bar{5}$	154	2.68780 70	34	0.00989 02	34	0.00016 48
35	0.61086 52	95	1.65806 28	155	2.70526 03	35	0.01018 11	35	0.00016 97
36	0.62831 85	96	1.67551 61	156	2.72271 36	36	0.01047 20	36	0.00017 45
37	0.64577 18	97	1.69296 94	157	2.74016 69	37	0.01076 29	37	0.00017 94
38	0.66322 51	98	1.71042 27	158	2.75762 02	38	0.01105 38	38	0.00018 42
39	0.68067 84	99	1.72787 60	159	2.77507 35	39	0.01134 46	39	0.00018 91
40	0.69813 17	**100**	1.74532 93	**160**	2.79252 68	**40**	0.01163 55	**40**	0.00019 39
41	0.71558 $\bar{5}$0	101	1.76278 25	161	2.80998 01	41	0.01192 64	41	0.00019 88
42	0.73303 83	102	1.78023 58	162	2.82743 34	42	0.01221 73	42	0.00020 36
43	0.75049 16	103	1.79768 91	163	2.84488 67	43	0.01250 82	43	0.00020 8$\bar{5}$
44	0.76794 49	104	1.81514 24	164	2.86234 00	44	0.01279 91	44	0.00021 33
45	0.78539 82	105	1.83259 57	165	2.87979 33	45	0.01309 00	45	0.00021 82
46	0.80285 1$\bar{5}$	106	1.85004 90	166	2.89724 66	46	0.01338 09	46	0.00022 30
47	0.82030 47	107	1.86750 23	167	2.91469 99	47	0.01367 17	47	0.00022 79
48	0.83775 80	108	1.88495 56	168	2.93215 31	48	0.01396 26	48	0.00023 27
49	0.85521 13	109	1.90240 89	169	2.94960 64	49	0.01425 35	49	0.00023 76
50	0.87266 46	**110**	1.91986 22	**170**	2.96705 97	**50**	0.01454 44	**50**	0.00024 24
51	0.89011 79	111	1.93731 5$\bar{5}$	171	2.98451 30	51	0.01483 53	51	0.00024 73
52	0.90757 12	112	1.95476 88	172	3.00196 63	52	0.01512 62	52	0.00025 21
53	0.92502 45	113	1.97222 21	173	3.01941 96	53	0.01541 71	53	0.00025 70
54	0.94247 78	114	1.98967 53	174	3.03687 29	54	0.01570 80	54	0.00026 18
55	0.95993 11	115	2.00712 86	175	3.05432 62	55	0.01599 89	55	0.00026 66
56	0.97738 44	116	2.02458 19	176	3.07177 9$\bar{5}$	56	0.01628 97	56	0.00027 1$\bar{5}$
57	0.99483 77	117	2.04203 52	177	3.08923 28	57	0.01658 06	57	0.00027 63
58	1.01229 10	118	2.05948 85	178	3.10668 61	58	0.01687 15	58	0.00028 12
59	1.02974 43	119	2.07694 18	179	3.12413 94	59	0.01716 24	59	0.00028 60
60	1.04719 76	**120**	2.09439 51	**180**	3.14159 27	**60**	0.01745 33	**60**	0.00029 09
DEGREES.						MINUTES.		SECONDS.	

Base of common logarithms $= 10.$

Base of Naperian logarithms (e) $= 2.71828\ 18284\ 59045\ 23536$

Com. Log. $e = M$ (Modulus of Com. Logs.) $= 0.43429\ 44819\ 03251\ 82765$

Nap. Log. $10 = \frac{1}{M} = 2.30258\ 50929\ 94045\ 68402$

$$\left.\begin{array}{l}\text{Com. Log. } N = M \times \text{Nap. Log. } N. \\ \text{Nap. Log. } N = \frac{1}{M} \times \text{Com. Log. } N.\end{array}\right\}\ \text{where } N \text{ denotes any number.}$$

Multiples of M.				Multiples of $\frac{1}{M}$.			
0	0.00000 000	**50**	21.71472 410	**0**	0.00000 000	**50**	115.12925 465
1	0.43429 448	51	22.14901 858	1	2.30258 509	51	117.43183 974
2	0.86858 896	52	22.58331 306	2	4.60517 019	52	119.73442 484
3	1.30288 345	53	23.01760 754	3	6.90775 528	53	122.03700 993
4	1.73717 793	54	23.45190 202	4	9.21034 037	54	124.33959 502
5	2.17147 241	55	23.88619 650	5	11.51292 546	55	126.64218 011
6	2.60576 689	56	24.32049 099	6	13.81551 056	56	128.94476 521
7	3.04006 137	57	24.75478 547	7	16.11809 565	57	131.24735 030
8	3.47435 586	58	25.18907 995	8	18.42068 074	58	133.54993 539
9	3.90865 034	59	25.62337 443	9	20.72326 584	59	135.85252 049
10	4.34294 482	**60**	26.05766 891	**10**	23.02585 093	**60**	138.15510 558
11	4.77723 930	61	26.49196 340	11	25.32843 602	61	140.45769 067
12	5.21153 378	62	26.92625 788	12	27.63102 112	62	142.76027 577
13	5.64582 826	63	27.36055 236	13	29.93360 621	63	145.06286 086
14	6.08012 275	64	27.79484 684	14	32.23619 130	64	147.36544 595
15	6.51441 723	65	28.22914 132	15	34.53877 639	65	149.66803 104
16	6.94871 171	66	28.66343 581	16	36.84136 149	66	151.97061 614
17	7.38300 619	67	29.09773 029	17	39.14394 658	67	154.27320 123
18	7.81730 067	68	29.53202 477	18	41.44653 167	68	156.57578 632
19	8.25159 516	69	29.96631 925	19	43.74911 677	69	158.87837 142
20	8.68588 964	**70**	30.40061 373	**20**	46.05170 186	**70**	161.18095 651
21	9.12018 412	71	30.83490 822	21	48.35428 695	71	163.48354 160
22	9.55447 860	72	31.26920 270	22	50.65687 205	72	165.78612 670
23	9.98877 308	73	31.70349 718	23	52.95945 714	73	168.08871 179
24	10.42306 757	74	32.13779 166	24	55.26204 223	74	170.39129 688
25	10.85736 205	75	32.57208 614	25	57.56462 732	75	172.69388 197
26	11.29165 653	76	33.00638 062	26	59.86721 242	76	174.99646 707
27	11.72595 101	77	33.44067 511	27	62.16979 751	77	177.29905 216
28	12.16024 549	78	33.87496 959	28	64.47238 260	78	179.60163 725
29	12.59453 998	79	34.30926 407	29	66.77496 770	79	181.90422 235
30	13.02883 446	**80**	34.74355 855	**30**	69.07755 279	**80**	184.20680 744
31	13.46312 894	81	35.17785 303	31	71.38013 788	81	186.50939 253
32	13.89742 342	82	35.61214 752	32	73.68272 298	82	188.81197 763
33	14.33171 790	83	36.04644 200	33	75.98530 807	83	191.11456 272
34	14.76601 238	84	36.48073 648	34	78.28789 316	84	193.41714 781
35	15.20030 687	85	36.91503 096	35	80.59047 825	85	195.71973 290
36	15.63460 135	86	37.34932 544	36	82.89306 335	86	198.02231 800
37	16.06889 583	87	37.78361 993	37	85.19564 844	87	200.32490 309
38	16.50319 031	88	38.21791 441	38	87.49823 353	88	202.62748 818
39	16.93748 479	89	38.65220 889	39	89.80081 863	89	204.93007 328
40	17.37177 928	**90**	39.08650 337	**40**	92.10340 372	**90**	207.23265 837
41	17.80607 376	91	39.52079 785	41	94.40598 881	91	209.53524 346
42	18.24036 824	92	39.95509 234	42	96.70857 391	92	211.83782 856
43	18.67466 272	93	40.38938 682	43	99.01115 900	93	214.14041 365
44	19.10895 720	94	40.82368 130	44	101.31374 409	94	216.44299 874
45	19.54325 169	95	41.25797 578	45	103.61632 918	95	218.74558 383
46	19.97754 617	96	41.69227 026	46	105.91891 428	96	221.04816 893
47	20.41184 065	97	42.12656 474	47	108.22149 937	97	223.35075 402
48	20.84613 513	98	42.56085 923	48	110.52408 446	98	225.65333 911
49	21.28042 961	99	42.99515 371	49	112.82666 956	99	227.95592 421
50	21.71472 410	**100**	43.42944 819	**50**	115.12925 465	**100**	230.25850 930

TRIGONOMETRIC FORMULAS.

$\sin^2 \alpha + \cos^2 \alpha = 1.$ $\qquad \tan \alpha = \dfrac{\sin \alpha}{\cos \alpha}.$ $\qquad \sec \alpha = \dfrac{1}{\cos \alpha}.$

$\sec^2 \alpha = 1 + \tan^2 \alpha.$

$\operatorname{cosec}^2 \alpha = 1 + \cot^2 \alpha.$ $\qquad \cot \alpha = \dfrac{\cos \alpha}{\sin \alpha}.$ $\qquad \operatorname{cosec} \alpha = \dfrac{1}{\sin \alpha}.$

$$\sin \alpha = \pm \frac{\tan \alpha}{\sqrt{1 + \tan^2 \alpha}}. \qquad \cos \alpha = \pm \frac{1}{\sqrt{1 + \tan^2 \alpha}}.$$

$$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta.$$

$$\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta.$$

$$\tan(\alpha \pm \beta) = \frac{\tan \alpha \pm \tan \beta}{1 \mp \tan \alpha \tan \beta}.$$

$$\sin \alpha + \sin \beta = 2 \sin \tfrac{1}{2}(\alpha + \beta) \cos \tfrac{1}{2}(\alpha - \beta).$$

$$\sin \alpha - \sin \beta = 2 \cos \tfrac{1}{2}(\alpha + \beta) \sin \tfrac{1}{2}(\alpha - \beta).$$

$$\cos \alpha + \cos \beta = 2 \cos \tfrac{1}{2}(\alpha + \beta) \cos \tfrac{1}{2}(\alpha - \beta).$$

$$\cos \alpha - \cos \beta = -2 \sin \tfrac{1}{2}(\alpha + \beta) \sin \tfrac{1}{2}(\alpha - \beta).$$

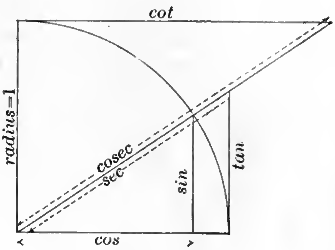

FIG. 1.

$$\sin \alpha \sin \beta = \tfrac{1}{2} \cos(\alpha - \beta) - \tfrac{1}{2} \cos(\alpha + \beta).$$

$$\cos \alpha \cos \beta = \tfrac{1}{2} \cos(\alpha - \beta) + \tfrac{1}{2} \cos(\alpha + \beta).$$

$$\sin \alpha \cos \beta = \tfrac{1}{2} \sin(\alpha + \beta) + \tfrac{1}{2} \sin(\alpha - \beta).$$

$$\sin^2 \alpha - \sin^2 \beta = \cos^2 \beta - \cos^2 \alpha = \sin(\alpha + \beta) \sin(\alpha - \beta).$$

$$\cos^2 \alpha - \sin^2 \beta = \cos^2 \beta - \sin^2 \alpha = \cos(\alpha + \beta) \cos(\alpha - \beta).$$

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha.$$

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha. \qquad \tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}.$$

$$2 \sin^2 \tfrac{1}{2} \alpha = 1 - \cos \alpha. \qquad 2 \cos^2 \tfrac{1}{2} \alpha = 1 + \cos \alpha.$$

$$\tan \tfrac{1}{2} \alpha = \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} = \frac{\sin \alpha}{1 + \cos \alpha} = \frac{1 - \cos \alpha}{\sin \alpha}.$$

$$\sin \alpha + \sin(\alpha + x) + \sin(\alpha + 2x) + \cdots + \sin(\alpha + nx) = \frac{\sin \tfrac{1}{2}(n+1)x \sin(\alpha + \tfrac{1}{2} nx)}{\sin \tfrac{1}{2} x}.$$

$$\cos \alpha + \cos(\alpha + x) + \cos(\alpha + 2x) + \cdots + \cos(\alpha + nx) = \frac{\sin \tfrac{1}{2}(n+1)x \cos(\alpha + \tfrac{1}{2} nx)}{\sin \tfrac{1}{2} x}.$$

$$i = \sqrt{-1}. \qquad e^{xi} = \cos x + i \sin x. \qquad e^{-xi} = \cos x - i \sin x.$$

$$\cos x = \frac{1}{2}(e^{xi} + e^{-xi}). \qquad \sin x = \frac{1}{2i}(e^{xi} - e^{-xi}).$$

$$e^{nxi} = (\cos x + i \sin x)^n = \cos nx + i \sin nx.$$

PLANE TRIANGLES.

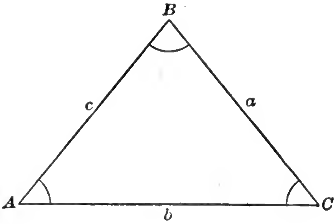

FIG. 2.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

$$a = b\cos C + c\cos B.$$

$$a^2 = b^2 + c^2 - 2bc\cos A.$$

$$a\sin\tfrac{1}{2}(B - C) = (b - c)\cos\tfrac{1}{2}A.$$

$$a\cos\tfrac{1}{2}(B - C) = (b + c)\sin\tfrac{1}{2}A.$$

$$\frac{a+b}{a-b} = \frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}. \qquad \tan A = \frac{a\sin B}{c - a\cos B}.$$

$$c = \frac{a-b}{\cos x}, \text{ if } \tan x = \frac{2\sin\frac{1}{2}C}{a-b}\sqrt{ab}.$$

If $s = \frac{1}{2}(a + b + c)$:

$$\sin\tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{bc}}. \qquad \cos\tfrac{1}{2}A = \sqrt{\frac{s(s-a)}{bc}}. \qquad \tan\tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}. \qquad \tan\tfrac{1}{2}A = \frac{r}{s-a}. \qquad \tan\tfrac{1}{2}B = \frac{r}{s-b}. \qquad \tan\tfrac{1}{2}C = \frac{r}{s-c}.$$

$$\text{Area} = \tfrac{1}{2}ab\sin C = \frac{c^2}{2}\cdot\frac{\sin A\sin B}{\sin C} = \sqrt{s(s-a)(s-b)(s-c)}.$$

Radius of inscribed circle $= r$.

Diameter of circumscribed circle $= \dfrac{a}{\sin A}$.

DIFFERENTIAL FORMULAS FOR PLANE TRIANGLES.

$$dA + dB + dC = 0.$$

$$\frac{da}{a} - \cot A\,dA = \frac{db}{b} - \cot B\,dB = \frac{dc}{c} - \cot C\,dC.$$

$$da = \cos C\,db + \cos B\,dc + b\sin C\,dA.$$

$$a\,dB = \sin C\,db - \sin B\,dc - b\cos C\,dA.$$

RIGHT SPHERICAL TRIANGLES ($C = 90°$).

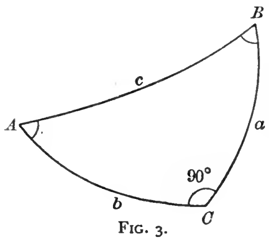

FIG. 3.

$\sin a = \sin A\sin c.$ $\qquad$ $\sin b = \sin B\sin c.$

$\sin a = \cot B\tan b.$ $\qquad$ $\sin b = \cot A\tan a.$

$\cos A = \sin B\cos a.$ $\qquad$ $\cos B = \sin A\cos b.$

$\cos A = \tan b\cot c.$ $\qquad$ $\cos B = \tan a\cot c.$

$$\cos c = \cos a\cos b = \cot A\cot B.$$

OBLIQUE SPHERICAL TRIANGLES.

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}.$$

$\cos a = \cos b \cos c + \sin b \sin c \cos A.$

$\cos A = - \cos B \cos C + \sin B \sin C \cos a.$

$\sin a \cos B = \cos b \sin c - \sin b \cos c \cos A.$

$\sin A \cos b = \cos B \sin C + \sin B \cos C \cos a.$

$\sin a \cos b = \sin c \cos B + \cos a \sin b \cos C.$

$\sin A \cos B = \cos b \sin C - \cos c \cos A \sin B.$

$\sin a \cot b = \cot B \sin C + \cos a \cos C.$

$\sin A \cot B = \cot b \sin c - \cos c \cos A.$

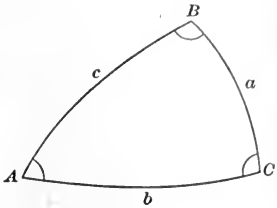

FIG. 4.

$s = \frac{1}{2}(a + b + c).$

$$\sin^2 \tfrac{1}{2} A = \frac{\sin (s - b) \sin (s - c)}{\sin b \sin c}.$$

$$\cos^2 \tfrac{1}{2} A = \frac{\sin s \sin (s - a)}{\sin b \sin c}.$$

$$\tan^2 \tfrac{1}{2} A = \frac{\sin (s - b) \sin (s - c)}{\sin s \sin (s - a)}.$$

$$r^2 = \frac{\sin (s - a) \sin (s - b) \sin (s - c)}{\sin s}.$$

$$\tan \tfrac{1}{2} A = \frac{r}{\sin (s - a)}.$$

$S = \frac{1}{2}(A + B + C).$

$$\sin^2 \tfrac{1}{2} a = \frac{- \cos S \cos (S - A)}{\sin B \sin C}.$$

$$\cos^2 \tfrac{1}{2} a = \frac{\cos (S - B) \cos (S - C)}{\sin B \sin C}.$$

$$\tan^2 \tfrac{1}{2} a = \frac{- \cos S \cos (S - A)}{\cos (S - B) \cos (S - C)}.$$

$$R^2 = \frac{- \cos S}{\cos (S - A) \cos (S - B) \cos (S - C)}.$$

$$\tan \tfrac{1}{2} a = R \cos (S - A).$$

$\sin \frac{1}{2} c \sin \frac{1}{2}(A - B) = \cos \frac{1}{2} C \sin \frac{1}{2}(a - b).$

$\sin \frac{1}{2} c \cos \frac{1}{2}(A - B) = \sin \frac{1}{2} C \sin \frac{1}{2}(a + b).$

$\cos \frac{1}{2} c \sin \frac{1}{2}(A + B) = \cos \frac{1}{2} C \cos \frac{1}{2}(a - b).$

$\cos \frac{1}{2} c \cos \frac{1}{2}(A + B) = \sin \frac{1}{2} C \cos \frac{1}{2}(a + b).$

$$\frac{\tan \frac{1}{2} c}{\tan \frac{1}{2}(a - b)} = \frac{\sin \frac{1}{2}(A + B)}{\sin \frac{1}{2}(A - B)}. \qquad \frac{\cot \frac{1}{2} C}{\tan \frac{1}{2}(A - B)} = \frac{\sin \frac{1}{2}(a + b)}{\sin \frac{1}{2}(a - b)}.$$

$$\frac{\tan \frac{1}{2} c}{\tan \frac{1}{2}(a + b)} = \frac{\cos \frac{1}{2}(A + B)}{\cos \frac{1}{2}(A - B)}. \qquad \frac{\cot \frac{1}{2} C}{\tan \frac{1}{2}(A + B)} = \frac{\cos \frac{1}{2}(a + b)}{\cos \frac{1}{2}(a - b)}.$$

r = tangent of the angular radius of the inscribed small circle.

R = tangent of the angular radius of the circumscribed small circle.

SPHERICAL EXCESS.

$E = A + B + C - 180°.$

$$\sin \tfrac{1}{2} E = \frac{\sin \frac{1}{2} a \sin \frac{1}{2} b}{\cos \frac{1}{2} c} \sin C. \qquad \tan \tfrac{1}{2} E = \frac{\tan \frac{1}{2} a \tan \frac{1}{2} b \sin C}{1 + \tan \frac{1}{2} a \tan \frac{1}{2} b \cos C}.$$

$\tan^2 \frac{1}{4} E = \tan \frac{1}{2} s \tan \frac{1}{2}(s - a) \tan \frac{1}{2}(s - b) \tan \frac{1}{2}(s - c).$

$E'' = \text{area} \div r^2 \sin 1''.$

DIFFERENTIAL FORMULAS FOR SPHERICAL TRIANGLES.

$$\cot a\,da - \cot A\,dA = \cot b\,db - \cot B\,dB = \cot c\,dc - \cot C\,dC.$$

$$da = \cos C\,db + \cos B\,dc + \sin c \sin B\,dA.$$

$$dA = \sin b \sin C\,da - \cos c\,dB - \cos b\,dC.$$

$$\sin c\,dB = -\cos c \sin B\,da + \sin A\,db - \sin b \cos A\,dC.$$

MACLAURIN'S THEOREM.*

$$f(x) = f(0) + f'(0)\frac{x}{1} + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \cdots.$$

TAYLOR'S THEOREM.*

$$f(x+h) = f(x) + f'(x)\frac{h}{1} + f''(x)\frac{h^2}{2!} + f'''(x)\frac{h^3}{3!} + \cdots.$$

$$f(x+h,\, y+k) = f(x, y) + \frac{du}{dx}\frac{h}{1} + \frac{du}{dy}\frac{k}{1} + \frac{d^2u}{dx^2}\frac{h^2}{2!} + \frac{d^2u}{dy^2}\frac{k^2}{2!} + \frac{d^2u}{dxdy}hk + \cdots,$$

where $u = f(x, y)$.

LAGRANGE'S THEOREM.*

$u = f(z)$, and $z = y + x\phi(z)$;

$$u = f(y) + \phi(y)\frac{df(y)}{dy}\frac{x}{1} + \frac{d\left[\overline{\phi(y)}^2 \frac{df(y)}{dy}\right]}{dy}\frac{x^2}{2!} + \frac{d^2\left[\overline{\phi(y)}^3 \frac{df(y)}{dy}\right]}{dy^2}\frac{x^3}{3!} + \cdots.$$

BINOMIAL THEOREM.

$$(a \pm b)^n = a^n \pm \frac{n}{1}a^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 \pm \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \cdots.$$

EXPONENTIAL THEOREM.*

$$a^x = 1 + \frac{\log a}{M}x + \left(\frac{\log a}{M}\right)^2\frac{x^2}{2!} + \left(\frac{\log a}{M}\right)^3\frac{x^3}{3!} + \left(\frac{\log a}{M}\right)^4\frac{x^4}{4!} + \cdots.$$

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \cdots.$$

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* $n!$ denotes "factorial n," or the product $1\cdot 2\cdot 3\cdot 4 \cdots n$.

LOGARITHMIC SERIES.*

$$\log(a+b)=\log a+M\left[\frac{1}{1}\left(\frac{b}{a}\right)-\frac{1}{2}\left(\frac{b}{a}\right)^2+\frac{1}{3}\left(\frac{b}{a}\right)^3-\frac{1}{4}\left(\frac{b}{a}\right)^4+\cdots\right].$$

$$\log(a+b)=\log a+2M\left[\frac{1}{1}\left(\frac{b}{2a+b}\right)+\frac{1}{3}\left(\frac{b}{2a+b}\right)^3+\frac{1}{5}\left(\frac{b}{2a+b}\right)^5+\cdots\right].$$

$$\log(1+x)=M\left(x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\frac{1}{5}x^5-\cdots\right).$$

$$\log(1-x)=-M\left(x+\frac{1}{2}x^2+\frac{1}{3}x^3+\frac{1}{4}x^4+\frac{1}{5}x^5+\cdots\right).$$

$$\log\frac{a}{b}=2M\left[\frac{1}{1}\left(\frac{a-b}{a+b}\right)+\frac{1}{3}\left(\frac{a-b}{a+b}\right)^3+\frac{1}{5}\left(\frac{a-b}{a+b}\right)^5+\cdots\right].$$

$$a=1+\frac{1}{1}\left(\frac{\log a}{M}\right)+\frac{1}{2!}\left(\frac{\log a}{M}\right)^2+\frac{1}{3!}\left(\frac{\log a}{M}\right)^3+\frac{1}{4!}\left(\frac{\log a}{M}\right)^4+\cdots.$$

TRIGONOMETRIC SERIES.* †

$$\sin x=\frac{x}{1}-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots.$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots.$$

$$\tan x=x+\frac{1}{3}x^3+\frac{2}{15}x^5+\frac{17}{315}x^7+\frac{62}{2835}x^9+\frac{1382}{155925}x^{11}+\cdots.$$

$$\cot x=\frac{1}{x}-\frac{1}{3}x-\frac{1}{45}x^3-\frac{2}{945}x^5-\frac{1}{4725}x^7-\frac{2}{93555}x^9-\cdots.$$

$$\sec x=1+\frac{1}{2}x^2+\frac{5}{24}x^4+\frac{61}{720}x^6+\frac{277}{8064}x^8+\cdots.$$

$$\operatorname{cosec} x=\frac{1}{x}+\frac{1}{6}x+\frac{7}{360}x^3+\frac{31}{15120}x^5+\frac{127}{604800}x^7+\cdots.$$

$$\sin^{-1}y=y+\frac{1}{6}y^3+\frac{3}{40}y^5+\frac{5}{112}y^7+\frac{35}{1152}y^9+\cdots.$$

$$\cos^{-1}y=\frac{\pi}{2}-y-\frac{1}{6}y^3-\frac{3}{40}y^5-\frac{5}{112}y^7-\frac{35}{1152}y^9-\cdots.$$

$$\tan^{-1}y=y-\frac{1}{3}y^3+\frac{1}{5}y^5-\frac{1}{7}y^7+\frac{1}{9}y^9-\cdots.$$

$$\cot^{-1}y=\frac{1}{y}-\frac{1}{3y^3}+\frac{1}{5y^5}-\frac{1}{7y^7}+\frac{1}{9y^9}-\cdots.$$

$$\log\sin x=\log x-M\left(\frac{1}{6}x^2+\frac{1}{180}x^4+\frac{1}{2835}x^6+\frac{1}{37800}x^8+\cdots\right).$$

$$\log\cos x=-M\left(\frac{1}{2}x^2+\frac{1}{12}x^4+\frac{1}{45}x^6+\frac{17}{2520}x^8+\cdots\right).$$

$$\log\tan x=\log x+M\left(\frac{1}{3}x^2+\frac{7}{90}x^4+\frac{62}{2835}x^6+\frac{127}{18900}x^8+\cdots\right).$$

$$\log\sin^{-1}y=\log y+M\left(\frac{1}{6}y^2+\frac{11}{180}y^4+\frac{191}{5670}y^6+\cdots\right).$$

$$\log\tan^{-1}y=\log y-M\left(\frac{1}{3}y^2-\frac{13}{90}y^4+\frac{251}{2835}y^6-\cdots\right).$$

$$\log\sin x=\log\tan x-M\left(\frac{1}{2}\tan^2 x-\frac{1}{4}\tan^4 x+\frac{1}{6}\tan^6 x-\frac{1}{8}\tan^8 x+\cdots\right).$$

$$\log\tan x=\log\sin x+M\left(\frac{1}{2}\sin^2 x+\frac{1}{4}\sin^4 x+\frac{1}{6}\sin^6 x+\frac{1}{8}\sin^8 x+\cdots\right).$$

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* $n!$ denotes "factorial n," or the product $1\cdot 2\cdot 3\cdot 4\cdots n$.

† The angles are expressed in circular measure.

DIFFERENTIATION.

$d(ax+b)=a\,dx.$ $\qquad d(u\pm v)=du\pm dv.$ $\qquad d(uv)=u\,dv+v\,du.$

$d\left(\frac{x}{y}\right)=\frac{y\,dx-x\,dy}{y^2}.$ $\qquad d(x^n)=nx^{n-1}\,dx.$ $\qquad d(\sqrt{x})=\frac{dx}{2\sqrt{x}}.$

$d(\log x)=M\frac{dx}{x}.$ $\qquad d(a^x)=\frac{1}{M}a^x\log a\,dx.$ $\qquad d(e^x)=e^x\,dx.$

$$d(x^y)=x^y\log_e x\,dy+yx^{y-1}\,dx.$$

$d(\sin x)=\cos x\,dx.$ $\qquad d(\cos x)=-\sin x\,dx.$

$d(\tan x)=\sec^2 x\,dx.$ $\qquad d(\cot x)=-\operatorname{cosec}^2 x\,dx.$

$d(\sec x)=\sec x\tan x\,dx.$ $\qquad d(\operatorname{cosec} x)=-\operatorname{cosec} x\cot x\,dx.$

$d(\sin^{-1}x)=\frac{dx}{\sqrt{1-x^2}}.$ $\qquad d(\tan^{-1}x)=\frac{dx}{1+x^2}.$ $\qquad d(\sec^{-1}x)=\frac{dx}{x\sqrt{x^2-1}}.$

$d(\cos^{-1}x)=-\frac{dx}{\sqrt{1-x^2}}.$ $\qquad d(\cot^{-1}x)=-\frac{dx}{1+x^2}.$ $\qquad d(\operatorname{cosec}^{-1}x)=-\frac{dx}{x\sqrt{x^2-1}}.$

$d(\operatorname{vers}^{-1}x)=\frac{dx}{\sqrt{2x-x^2}}.$ $\qquad d(\operatorname{covers}^{-1}x)=-\frac{dx}{\sqrt{2x-x^2}}.$

APPROXIMATE INTEGRATION.

Let Δx be the common distance between the ordinates $y_0, y_1, y_2, \cdots y_n$, where n is even.

1. $\int f(x)\,dx=\frac{P+2Q}{3}-\frac{y_n'''-y_0'''}{180}\Delta x^4+\frac{y_n^{\text{v}}-y_0^{\text{v}}}{1512}\Delta x^6-\cdots,$

where $\qquad P=\Delta x\,[y_0+y_n+2(y_2+y_4+\cdots+y_{n-2})],$

$\qquad Q=2\,\Delta x\,[y_1+y_3+\cdots+y_{n-1}],$

$\qquad Y_n'''=\frac{d^3}{dx^3}f(x)$ when $x=$ abscissa of y_n.

2. Simpson's rule:

$$A=\frac{\Delta x}{3}[y_0+y_n+2(y_2+y_4+\cdots+y_{n-2})+4(y_1+y_3+\cdots+y_{n-1})].$$

3. Weddle's rule (for seven ordinates):

$$A=\frac{3\,\Delta x}{10}[y_0+y_2+y_4+y_6+y_3+5(y_1+y_3+y_5)].$$

4. Prismoidal formula: $V=\frac{\Delta x}{3}[A+A'+4A_m]=\frac{h}{6}[A+A'+4A_m].$

Constants.		Logarithms.
Base of Naperian logs: $e =$	2.71828 183	0.43429 448
Modulus of common logs: $\log e = M =$	0.43429 448	9.63778 431 − 10
Degrees in arc = radius: $180° \div \pi =$	57°.29577 951	1.75812 263
Minutes in arc = radius:	3 437′.74677	3.53627 388
Seconds in arc = radius:	206 264″.806	5.31442 513
360° expressed in minutes of arc:	21 600′	4.33445 375
360° expressed in seconds of arc:	1 296 000″	6.11260 500
24 hours expressed in minutes of time:	1 440^m	3.15836 249
24 hours expressed in seconds of time:	86 400^s	4.93651 374
$\pi =$ 3.14159 26535 89793 23846		0.49714 987
$\log \pi$ = 0.49714 98726 94133 85435		
sin 1″ = 0.00000 48481 36811 07637		4.68557 487 − 10
arc 1″ = 0.00000 48481 36811 09536		4.68557 487 − 10

			Logarithms.
1 Eng. inch	0.02540 01	meters	8.40483 $\bar{5}$ − 10
1 Eng. foot	0.30480 1	meters	9.48401 6 − 10
1 Eng. yard	0.91440 2	meters	9.96113 7 − 10
1 Eng. statute mile	1.6093$\bar{5}$	kilometers	0.2066$\bar{5}$ 0
1 meter	39.3700	Eng. inches	1.59516 5
1 meter	3.28083	Eng. feet	0.51598 4
1 meter	1.09361 1	Eng. yards	0.03886 3
1 kilometer	0.62137 0	Eng. statute miles	9.79335 0 − 10
1 sq. foot	9.29034	sq. decimeters	0.96803 2
1 sq. inch	6.45163	sq. centimeters	0.80966 9
1 sq. meter	10.7639	sq. feet	1.03196 8
1 sq. centimeter	0.1$\bar{5}$500 0	sq. inches	9.19033 1 − 10
1 cubic foot	0.02831 70	cubic meters	8.45204 7 − 10
1 cubic inch	16.3872	cubic centimeters	1.21450 4
1 cubic meter	35.314$\bar{5}$	cubic feet	1.54795 3
1 cubic decimeter (liter)	61.0234	cubic inches	1.78549 6
1 avoirdupois pound	453.59242 77	grams	2.65666 6
1 avoirdupois ounce	28.34953	grams	1.45254 6
1 Troy ounce	31.10348	grams	1.49280 9
1 grain	64.79892	milligrams	1.81156 8
1 kilogram	2.20462	avdp. pounds	0.34333 4
1 kilogram	35.2740	avdp. ounces	1.54745 4
1 kilogram	32.1507	Troy ounces	1.50719 1
1 gram	15.43235 639	grains	1.18843 2
1 foot-pound	0.13825 5	kilogram-meters	9.14068 2 − 10
1 kilogram-meter	7.23300	foot-pounds	0.85931 8
1 pound per sq. in.	70.3067	grams per sq. cm.	1.84699 7
1 gram per sq. cm.	0.01422 34	lbs. per sq. in.	8.15300 3 − 10
1 pound per cu. ft.	0.01601 84	grams per cu. cm.	8.20461 8 − 10
1 grain per cu. in.	0.00395 42$\bar{5}$	grams per cu. cm.	7.59706 4 − 10
1 gram per cu. cm.	62.4283	lbs. per cu. ft.	1.79538 2
1 gram per cu. cm.	252.8925	grains per cu. in.	2.40293 6

			Logarithms.
Wt. of mass of 1 gram	100g	dynes (g in meters).	
Wt. of mass of 1 grain	6.47989 2g	dynes (g in meters)	0.81156 8
1 foot-pound	13825.5g	ergs (g in centimeters)	4.14068 2
1 kilogram-meter	100000g	ergs (g in centimeters).	
1 watt	10^7	ergs per sec.	
1 horse-power	76.0404g	watts (g in meters)	1.88104 4
1 horse-power	746	watts (approximately)	2.87273 9

$g = 32.086\,528 + 0.171\,293 \sin^2\phi - 0.000\,003\,h$. in feet (Harkness).
$\;\;= 9.779\,886 + 0.052\,210 \sin^2\phi - 0.000\,003\,h$. in meters (Harkness).
$l = 39.012\,540 + 0.208\,268 \sin^2\phi - 0.000\,000\,3\,h$. in inches (Harkness)
$\;\;= 0.990\,910 + 0.005\,290 \sin^2\phi - 0.000\,000\,3\,h$. in meters (Harkness).

EXPLANATION OF THE TABLES.

INTRODUCTORY.

1. When we have a number with six or more decimal places, and we wish to use only five:

(*a*) If the sixth and following figures of the decimal are less than 5 in the sixth place, they are dropped; thus, 0.46437 4999 is called 0.46437.

(*b*) If the sixth and following figures of the decimal are greater than 5 in the sixth place, the fifth place is increased by unity and the sixth and following places are dropped; thus, 0.46437 5001 is called 0.46438.

(*c*) If the sixth figure of the decimal is 5, and if it is followed only by zeros, make the fifth figure the nearest *even* figure; thus, 0.46437 500 is called 0.46438, while 0.46438 500 is also called 0.46438. The number is thus increased when the fifth figure is odd, and decreased when it is even, the two operations tending to neutralize each other in a series of computations, and hence to diminish the resultant error.

2. Hence any number obtained according to Art. 1 may be in error by half a unit in the fifth decimal place.

3. When the last figure of a number in these tables is 5, the number printed is too large, the 5 having been obtained according to Art. 1 (*b*); if the 5 is without the minus sign, the number printed is too small, the figures following the 5 having been dropped according to Art. 1 (*a*).

4. The marginal tables contain the products of the numbers at the top of the columns by 1, 2, 3, ⋯ 9 *tenths*, and may be used in multiplying and dividing in interpolation.

(*a*) To multiply 38 by .746:

	38
1	3.8
2	7.6
3	11.4
4	15.2
5	19.0
6	22.8
7	26.6
8	30.4
9	34.2

$$38 \times .7 = \qquad\qquad = 26.6$$
$$38 \times .4 = 15.2;\ \therefore\ 38 \times .04 = 1.52$$
$$38 \times .6 = 22.8;\ \therefore\ 38 \times .006 = .228$$
$$\therefore\ 38 \times .746 = 28.348$$

In multiplying by the second figure (hundredths), the decimal point in the table is moved *one* place to the left; in multiplying by the third (thousandths), *two* to the left; and so on.

(*b*) To divide 28 by 38:

	Dividend,	28		
	Next less,	26.6	corresponding to .7
	Remainder,	1 4	
	Next less,	1 1.4	corresponding to .03
	Remainder,	2 6	
	Nearest,	2 6.6	corresponding to .007
∴	Quotient,		.737

	38
1	3.8
2	7.6
3	11.4
4	15.2
5	19.0
6	22.8
7	26.6
8	30.4
9	34.2

to the nearest third decimal place. The decimal point is moved one place to the right in each remainder, since the next figure in the quotient will be one place farther to the right.

To divide 23 by 38:

	Dividend,	23	
		22.8	corresponding to .6
		2	
		0.0	corresponding to .00
		2 0.	
	Nearest,	1 9.0	corresponding to .005
∴	Quotient,		.605

The computer should use the marginal tables mentally.

LOGARITHMS.

5. The logarithm of a number is the exponent of the power to which a given number called the *base* must be raised to produce the first number. If $A = e^a$, a is called the logarithm of the number A to the base e, written $\log_e A = a$.

6. If $A = e^a$, and $B = e^b$, or (omitting subscripts) $\log A = a$, and $\log B = b$, we have

$$A \times B = e^{a+b}; \quad \therefore \log(A \times B) = a + b; \quad \therefore \log(A \times B) = \log A + \log B.$$

$$A \div B = e^{a-b}; \quad \therefore \log(A \div B) = a - b; \quad \therefore \log(A \div B) = \log A - \log B.$$

$$A^n = e^{na}; \quad \therefore \log(A^n) = na; \quad \therefore \log(A^n) = n \log A.$$

$$\sqrt[n]{A} = e^{\frac{1}{n}a}; \quad \therefore \log \sqrt[n]{A} = \frac{1}{n} a; \quad \therefore \log \sqrt[n]{A} = \frac{1}{n} \log A.$$

7. When the base is not specified, it is generally understood that logarithms to the base 10, or *common logarithms*, are meant. In this system, since

$$0.001 = \frac{1}{1000} = \frac{1}{10^3} = 10^{-3}, \quad \log 0.001 = -3;$$

$$0.01 = \frac{1}{100} = \frac{1}{10^2} = 10^{-2}, \quad \log 0.01 = -2;$$

$$0.1 = \frac{1}{10} = \frac{1}{10} = 10^{-1}, \quad \log 0.1 = -1;$$

$$1. = 10^0, \quad \log 1 = 0;$$

$$10. = 10^1, \quad \log 10 = +1;$$

$$100. = 10^2, \quad \log 100 = +2;$$

$$1000. = 10^3, \quad \log 1000 = +3.$$

8. The logarithm of a number between 100 and 1000 will be a number between 2 and 3, or $2 + m$ where m will be a decimal called the *mantissa*, the integral portion of the logarithm being the *characteristic*. The mantissa is always considered *positive;* thus log 0.002 will be a number between -2 and -3, that is, either $-3 + m$ or $-2 - m'$, the first form being used. We write $\log 0.002 = \bar{3}.30103$, the negative sign being placed over the characteristic to show that the characteristic alone is negative.

9. Since

$$\log (A \times 10^n) = \log A + \log 10^n = \log A + n \log 10 = \log A + n,$$

and $\log (A \div 10^n) = \log A - \log 10^n = \log A - n \log 10 = \log A - n,$

we have, if $\log 37.3 = 1.57171$,

$$\log 373. = 2.57171, \text{ and } \log 3.73 = 0.57171,$$

$$\log 3730 = 3.57171, \text{ and } \log 0.373 = \bar{1}.57171;$$

$$\log 37300 = 4.57171, \text{ and } \log 0.0373 = \bar{2}.57171.$$

Hence the position of the decimal point affects the characteristic alone, the mantissa being always the same for the same sequence of figures. For this reason the common system of logarithms is used in practice.

10. The characteristic is found as follows: *When the number is greater than* 1, *the characteristic is positive, and is one* less *than the number of digits to the left of the decimal point; when the number is less than* 1, *the characteristic is negative, and is one* more *than the number of zeros between the decimal point and the first significant figure.*

11. To avoid the use of negative characteristics we add 10 to the characteristic and write -10 after the mantissa, *i.e.* adding and subtracting the same quantity, 10. Thus $\log 0.2 = \bar{1}.30103$ would be written

9.30103 — 10. The — 10 is often omitted for brevity when there is no danger of confusion, but its existence must not be forgotten. Such logarithms are called *augmented* logarithms.

In this case the characteristic of the logarithm of a pure decimal is 9 *diminished by the number of ciphers to the left of the first significant figure.* Thus the characteristic of log 0.004 is 9 — 2, or 7, and that of log 0.94 is 9 — 0, or 9.

12. The arithmetical complement of the logarithm (written *colog*) of a number is the logarithm of its reciprocal, and is found by subtracting each figure of the logarithm from 9, commencing at the left, except the last significant figure on the right, which is subtracted from 10.

For $\log \frac{1}{x} = -\log x = 10 - \log x - 10$;

thus, if $\log x = 2.46403$, $\text{colog}\, x = 7.53597 - 10$;

if $\log x = 8.43000 - 10$, $\text{colog}\, x = 1.57000$.

TABLE I.

13. Page 3 contains the logarithms of numbers from 1 to 100, to five decimal places.

Pages 4–21 contain the mantissas of the logarithms of numbers from 1000 to 10009, to five decimal places.

Pages 22, 23, contain the mantissas of the logarithms of numbers from 10000 to 11009, to seven decimal places.

NOTE. — The mantissas of the logarithms of numbers, except those of the integral powers of 10, are incommensurable, the mantissas in the tables being found as shown in Art. 1.

To find the Logarithm of a Number.

14. The *characteristic* is found by the rules in Arts. 10 and 11, and the *mantissa* from the tables, as shown in Arts. 15, 16, 17, 18.

15. *When the number has four figures.* — Find on pages 4–21 the first three figures in the column marked *N*, and the fourth at the top of one of the other columns. The last three figures of the mantissa are found in this column on the horizontal line through the first three figures of the given number in column *N*. The first two figures of the mantissa are those under *L* in the same line with the number, or else those nearest above it, unless the last three figures of the mantissa as given in the tables are preceded by a *, when the first two figures are found under *L* in the first line *below* the number. Thus (page 4),

log 1136 = 3.05538; log 1137 = 3.05576; log 1138 = 3.05614;
log 1370 = 3.13672; log 1371 = 3.13704; log 1372 = 3.13735;
log 1380 = 3.13988; log 1381 = 3.14019; log 1382 = 3.14051.

16. *When the number has less than four figures*, annex ciphers on the right and proceed as in Art. 15. Thus,

log 1.13 = 0.05308 ; log 12.8 = 1.10721 ; log 130 = 2.11394 ;
log 15 = 1.17609 ; log 16 = 1.20412 ; log 17 = 1.23045.

17. *When the number has more than four figures*, as 11.4672. — Since the mantissa is independent of the position of the decimal point, point off the first four figures and find the mantissa of log 1146.72. This will be between the mantissas of log 1146 and log 1147. Hence find from the tables the mantissas corresponding to 1146 and 1147; multiply the difference between them (called the tabular difference) by .72, and add the product (called the correction) to log 11.46; the result will be the logarithm required.

Mantissa of log 1146 =	05918	log 11.46 =	1.05918
Mantissa of log 1147 =	05956	correction = 38 × .72 =	27.36
Tabular difference =	38	∴ log 11.4672 =	1.05945 36
		or =	1.05945

NOTE. — Since any mantissa in the tables may be in error by half a unit in the fifth decimal place (Art. 2), no advantage is gained by using the sixth place in the interpolated logarithm. Thus, according to Art. 1, we drop the .36, and call log 11.4672 = 1.05945.

NOTE. — The marginal tables should be used in multiplying the tabular difference to find the correction (Art. 4).

NOTE. — It is assumed that the change in the mantissa is proportional to that in the number, as the latter increases from 1146 to 1147. An increase of 1 in the number causes an increase of 38 in the mantissa; hence an increase of .72 in the number will cause an increase of 38 × .72 in the mantissa.

NOTE. — We could also find the mantissa of log 11.4672 by subtracting the product of the tabular difference by .28 (or 1.00 — .72) from the mantissa corresponding to 1147; that is, the required mantissa is 05956 — (38 × .28) = 05956 — 10.64 = 05945 as before.

18. *The general rule is: Find the mantissa corresponding to the first four figures of the number; multiply the tabular difference by the fifth and following figures treated as a decimal; and add the product to the mantissa just found.*

The tabular difference is the difference between the mantissas corresponding to the two numbers in the tables, between which the given number lies.

log 1.62163 = 0.20995 ; log 0.38024 = $\bar{1}$.58006 ; log 0.0852763 = $\bar{2}$.93083 ;
log 189.524 = 2.27767 ; log 0.38602 = $\bar{1}$.58661 ; log 0.0085938 = $\bar{3}$.93419 ;
log 19983.4 = 4.30067 ; log 3.98743 = 0.60070 ; log 0.090046 = $\bar{2}$.95446.

NOTE. — Page 3 is used when the number contains less than three figures, the number being found in the column *N*, and the logarithm in the column headed *Log.* The characteristic is given for whole numbers, and must be changed for decimals.

NOTE. — When a number is composed of three figures, find on pages 4–21 the number in the column *N*, and the mantissa corresponding in the column *L.* 0.

To find the Number corresponding to a Given Logarithm.

19. From the tables we find the sequence of figures corresponding to the given mantissa, as shown in Arts. 20, 21, and 22, the position of the decimal point being determined by the characteristic (Arts. 10, 11).

20. *When the given mantissa can be found in the tables.*— Find on pages 4–21 the first two figures of the mantissa under *L* in the column headed *L.* 0. The last three figures of the mantissa are then sought for in the columns headed 0, 1, 2, ⋯ 9, in the same line with the first two figures, or in one of the lines just below, or in the line next above (where they would be preceded by a *). The first three figures of the required number will be found in the column headed *N*, in the same horizontal line with the last three figures of the mantissa, and the fourth figure of the number at the top of the column in which the last three figures of the mantissa are found. Thus (page 4),

$0.06221 = \log 1.154$; $0.06558 = \log 1.163$; $0.06893 = \log 1.172$;
$0.07004 = \log 1.175$; $0.07188 = \log 1.180$; $0.08063 = \log 1.204$.

21. *When the given mantissa can not be found in the tables.* — If we wish to find the number whose logarithm is 2.16531, we enter the tables with 16531, and find that it lies between 16524 and 16554, so that the given mantissa corresponds to a number between 1463 and 1464. Also 16531 exceeds 16524 by 7, and this difference, divided by the tabular difference 30, gives .23⋯ as the amount by which the required number exceeds 1463. Hence $2.16531 = \log 146.323\cdots$, which we call 146.32, according to Art. 1, the incompleteness of the tables making the sixth figure uncertain.

NOTE. — The marginal tables should be used in dividing the difference between the given mantissa and the one next less in the tables by the tabular difference.

22. *The general rule is: Find the number corresponding to the mantissa in the tables next less than the given mantissa; divide the excess of the given mantissa over the one found in the tables by the tabular difference; and annex the quotient to the first four figures already found.*

The tabular difference is the difference between the two mantissas in the tables, between which the given mantissa lies.

$\bar{1}.16600 = \log 0.14656$; $0.18002 = \log 1.5136$; $2.18200 = \log 152.06$;
$1.19000 = \log 15.488$; $4.19680 = \log 15773$; $1.20020 = \log 15.856$.

23. For the use of the numbers S', T', S'', T'', see Arts. 35–38.

TABLE II.

24. This table (pages 26–70) contains the logarithms, to five decimal places, of the trigonometric sines, cosines, tangents, and cotangents of angles from 0° to 90°, for each minute. The logarithms in the columns headed *L. Sin*, *L. Tan*, and *L. Cos*, are augmented, and should be diminished by 10 (Art. 11), while those in the columns headed *L. Cot* are correctly given.

25. Since $\sec x = \frac{1}{\cos x}$, and $\operatorname{cosec} x = \frac{1}{\sin x}$, the logarithms of the secant and cosecant of an angle are the arithmetical complements of those of the cosine and sine respectively (Art. 12).

To find the Logarithmic Functions of an Angle Less than 90°.

26. *When the angle is less than* 45°, the degrees are found at the *top* of the page, and the minutes on the *left.* The numbers in the same horizontal line with the minutes of the angle are the logarithmic functions indicated by the notation at the *top* of the columns. Thus (page 34),

$$\log \sin 8^\circ 4' = 9.14714 - 10, \qquad \log \tan 8^\circ 4' = 9.15145 - 10,$$
$$\log \cot 8^\circ 4' = 0.84855, \qquad \log \cos 8^\circ 4' = 9.99568 - 10.$$

27. *When the angle is greater than* 45°, the degrees are found at the *bottom* of the page, and the minutes on the *right.* The numbers in the same horizontal line with the minutes of the angle are the logarithmic functions indicated by the notation at the *bottom* of the columns. Thus (page 34),

$$\log \sin 81^\circ 25' = 9.99511 - 10, \qquad \log \tan 81^\circ 25' = 0.82120,$$
$$\log \cot 81^\circ 25' = 9.17880 - 10, \qquad \log \cos 81^\circ 25' = 9.17391 - 10.$$

28. *When the angle is given to decimals of a minute.*—In finding $\log \sin 30^\circ 8'.48$, for example, we see that it will lie between the logarithmic sines of $30^\circ 8'$ and $30^\circ 9'$, that is, between 9.70072 and 9.70093, their difference 21 being the change in the logarithmic sine caused by a change of $1'$ in the angle. Hence, to find the correction to $\log \sin 30^\circ 8'$ that would give $\log \sin 30^\circ 8'.48$ we multiply 21 by .48 (Art. 4). The product 10.08 added to $\log \sin 30^\circ 8'$, since $\log \sin 30^\circ 9'$ is greater than $\log \sin 30^\circ 8'$, gives $\log \sin 30^\circ 8'.48 = 9.70082$ (Art. 1). Similarly, $\log \tan 30^\circ 8'.48 = 9.76391$, $\log \cot 30^\circ 8'.48 = 0.23609$, $\log \cos 30^\circ 8'.48 = 9.93691$, the correction being subtracted in the last two cases, since the cotangent and the cosine decrease as the angle increases.

29. *The general rule is: Find the function corresponding to the given degrees and minutes; multiply the tabular difference by the decimals of a minute; add the product to the function corresponding to the given degrees and minutes when finding the logarithmic sine or tangent, and subtract it when finding the logarithmic cosine or cotangent.*

The tabular differences are given in the columns headed *d.* and *c. d.*, the latter containing the common difference for the *L. Tan* and *L. Cot* columns. *The difference to be used is that between the functions corresponding to the two angles between which the given angle lies.*

For $30° 39'.38$: log sin = 9.70747; log cos = 9.93462;
log tan = 9.77285; log cot = 0.22715.

For $59° 43'.46$: log sin = 9.93632; log cos = 9.70257;
log tan = 0.23375; log cot = 9.76625.

30. *When the angle is given to seconds*, the correction may be found by multiplying the tabular difference by the number of seconds, and dividing the product by 60.

To find the Acute Angle corresponding to a Given Logarithmic Function.

31. The column headed *L. Sin* is marked *L. Cos* at the bottom, while that headed *L. Cos* is marked *L. Sin* at the bottom; hence, if a logarithmic sine or cosine were given, we should expect to find it in one of these two columns. Similarly, we should expect to find a given logarithmic tangent or cotangent in one of the two columns headed *L. Tan* and *L. Cot*.

32. *When the function can be found in the tables.*—If a logarithmic sine is given, find it in one of the two columns marked *L. Sin* and *L. Cos;* if found in the column headed *L. Sin*, the degrees are taken from the top, and the minutes from the left of the page; if in the column headed *L. Cos* but marked *L. Sin* at the bottom, the degrees are taken from the bottom, and the minutes from the right of the page. Thus,

9.70115 = log sin 30° 10′; 9.93457 = log sin 59° 20′;
9.93724 = log cos 30° 4′; 9.70590 = log cos 59° 28′;
9.76406 = log tan 30° 9′; 0.23130 = log tan 59° 35′;
0.23420 = log cot 30° 15′; 9.76870 = log cot 59° 35′.

33. *When the function can not be found in the tables.*—If we wish to find the angle whose logarithmic sine is 9.70170, we see on page 56 that the given logarithmic sine lies between 9.70159 and 9.70180, and hence the angle is between 30° 12′ and 30° 13′. The given logarithmic sine differs from log sin 30° 12′ by 11, and this difference, divided by the tabular difference 21, gives .52 + as the decimal of a minute by which the angle exceeds 30° 12′. Hence 9.70170 = log sin 30° 12′.52, which we call 30° 12′.5, since the incompleteness of the tables (Art. 1) makes the hundredths of a minute uncertain.

34. *The rule is: For a logarithmic sine or tangent find the degrees and minutes corresponding to the function in the tables next* less *than the given function; divide the difference between the given function and the one next less by the tabular difference; and the quotient will be the decimal of a minute to be added to the degrees and minutes already found. For a logarithmic cosine or cotangent find the degrees and minutes corresponding to the function next* greater *than the given function, since the cosine and cotangent decrease as the angle increases, and divide the difference between the given function and the one next greater by the tabular difference, to find the decimal of a minute.*

The tabular difference is the difference between the two functions in the tables, between which the given function lies.

9.70000 = log sin 30° 4′.7; 9.93500 = log sin 59° 25′.7;
9.93400 = log cos 30° 47′.6; 9.70500 = log cos 59° 32′.2;
9.77000 = log tan 30° 29′.5; 0.23200 = log tan 59° 37′.4;
0.23300 = log cot 30° 19′.1; 9.76400 = log cot 59° 51′.2.

Angles Near 0° *or* 90°.

35. The assumption that the variations in the functions are proportional to the variations in the angles if the latter are less than 1′ fails when the angle is small, shown by the rapid changes in the tabular differences on pages 26, 27, and 28.

36. The quantities S' and T' which are used in this case are defined by the equations

$$S' = \log \frac{\sin \alpha}{\alpha'},$$

$$T' = \log \frac{\tan \alpha}{\alpha'},$$

where α' is the number of minutes in the angle. Their values from 0° to 1° 40′ (= 100′) are given at the bottom of pages 3–21; from 1° 40′ to 3° 20′ at the left margin of pages 4 and 5, the first three figures being found at the top; and from 3° to 5° on page 24. Thus,

for 1′ = 1′ (page 5), $S' = 6.46\,373$, $T' = 6.46\,373$;
for 15′ = 15′ (page 5), $S' = 6.46\,372$, $T' = 6.46\,373$;
for 2° 40′ = 160′ (page 5), $S' = 6.46\,357$, $T' = 6.46\,404$;
for 4° 20′ = 260′ (page 24), $S' = 6.46\,331$, $T' = 6.46\,456$.

Each of these numbers should have −10 written after it (Art. 11).

NOTE. — The logarithmic cosine of a small angle is found by the ordinary method. The cotangent of an angle is the reciprocal of the tangent, and hence the logarithmic cotangent is the arithmetical complement of the logarithmic tangent. The formulas for finding the logarithmic cosine, tangent, and cotangent of angles near 90° are given on page 25.

37. *To find the logarithmic sine or tangent of a small angle.* — From Art. 36, we have

$$\log \sin \alpha = S' + \log \alpha',$$
$$\log \tan \alpha = T' + \log \alpha'.$$

Hence, to find the logarithmic sine or tangent of an angle less than 5°, find the value of the S' or T' corresponding to the angle, interpolating if necessary, and add it to the logarithm of the number of minutes in the angle.

Find $\log \sin 0° 42'.6$. Since the angle is nearer 43′ than 42′, we take

$$S' = 6.46\,371$$
$$\log 42.6 = 1.62\,941$$
$$\therefore \log \sin 0° 42'.6 = 8.09\,312$$

Find $\log \tan 1° 53'.2$. Since the angle is nearer 1° 53′ (= 113′) than 114′, we take

$$T' = 6.46\,388$$
$$\log 113.2 = 2.05\,385$$
$$\therefore \log \tan 1° 53'.2 = 8.51\,773$$

NOTE. — When the angle is given in seconds, either reduce the seconds to decimals of a minute, or use the values of S'' and T'' given at the bottom of pages 3–23 and on page 24. They are defined by the equations

$$S'' = \log \frac{\sin a}{a''}, \text{ and } T'' = \log \frac{\tan a}{a''},$$

where a'' is the number of seconds in the angle. Hence

$$\log \sin a = S'' + \log a'', \text{ and } \log \tan a = T'' + \log a''.$$

38. *To find the small angle corresponding to a given logarithmic sine or tangent.* — From Art. 36,

$$\left.\begin{aligned} \log \alpha' &= \log \sin \alpha - S', \\ \log \alpha' &= \log \tan \alpha - T', \end{aligned}\right\}$$

or

$$\left.\begin{aligned} \log \alpha' &= \log \sin \alpha + \text{cpl } S', \\ \log \alpha' &= \log \tan \alpha + \text{cpl } T'. \end{aligned}\right\}$$

When the angle is less than 3°, find on pages 26–28 the value of cpl S' (or cpl T') corresponding to the function, interpolating if necessary, and add it to $\log \sin \alpha$ (or $\log \tan \alpha$); the sum will be the logarithm of the number of minutes in the angle.

In finding the angle whose logarithmic sine is 8.09006, we see from the *L. Sin* column (page 26) that the angle is between 0° 42′ and 0° 43′, and that the value of cpl S' must be either 3.53628 or 3.53629. The given logarithmic sine is nearer that of 42′ than that of 43′; hence we take

$$\text{cpl } S' = 3.53628$$
$$\log \sin \alpha = \underline{8.09006}$$
$$\log \alpha' = 1.62634 \quad \therefore \ \alpha' = 42'.300.$$

When the angle is between 3° and 5°, we may find S' and T' from page 24 after finding the angle approximately from pages 29 and 30. Thus in finding the angle whose logarithmic tangent is 8.77237 we find from page 29 that the angle is between 3° 23′ and 3° 24′, being nearer 3° 23′. Then on page 24 we have

$$T' = 6.46423$$
$$\log \tan \alpha = \underline{8.77237}$$
$$\therefore \ \log \tan \alpha - T' = \log \alpha' = 2.30814 \quad \therefore \ \alpha' = 203'.30 = 3° \, 23'.30.$$

a.) $S = \frac{n}{2}(a+l)$ or $\frac{n}{2}(2a+(n-1)d)$

$l = a+(n-1)d$

Angles Greater than 90°.

$S = \frac{a(1-r^n)}{1-r}$

39. To find the logarithmic sine, cosine, tangent, or cotangent of an angle greater than 90°, subtract from the given angle the largest multiple of 90° contained therein. If this multiple is even, find from the tables the logarithmic sine, cosine, tangent, or cotangent of the remaining acute angle. If the multiple is odd, the logarithmic *cosine, sine, cotangent,* or *tangent,* respectively, of the remaining acute angle will be the function required; thus, sin 120° = sin (90° + 30°) = cos 30°.

$x =$	I. QUADRANT. α	II. QUADRANT. $90°+\alpha$	III. QUADRANT. $180°+\alpha$	IV. QUADRANT. $270°+\alpha$
$\sin x =$	$+\sin \alpha$	$+\cos \alpha$	$-\sin \alpha$	$-\cos \alpha$
$\cos x =$	$+\cos \alpha$	$-\sin \alpha$	$-\cos \alpha$	$+\sin \alpha$
$\tan x =$	$+\tan \alpha$	$-\cot \alpha$	$+\tan \alpha$	$-\cot \alpha$
$\cot x =$	$+\cot \alpha$	$-\tan \alpha$	$+\cot \alpha$	$-\tan \alpha$

Or we could find the difference between the angle and 180° or 360°, and find from the tables the same function of the remaining acute angle; thus, cos 300° = cos (360° − 60°) = cos 60°, etc.

$x =$	I. QUADRANT. α	II. QUADRANT. $180° - \alpha$	III. QUADRANT. $180°+\alpha$	IV. QUADRANT. $360°-\alpha$ or $-\alpha$
$\sin x =$	$+\sin \alpha$	$+\sin \alpha$	$-\sin \alpha$	$-\sin \alpha$
$\cos x =$	$+\cos \alpha$	$-\cos \alpha$	$-\cos \alpha$	$+\cos \alpha$
$\tan x =$	$+\tan \alpha$	$-\tan \alpha$	$+\tan \alpha$	$-\tan \alpha$
$\cot x =$	$+\cot \alpha$	$-\cot \alpha$	$+\cot \alpha$	$-\cot \alpha$

To indicate that the trigonometric function is negative, *n* is written after its logarithm.

40. To find the angle corresponding to a given function, find the acute angle α corresponding thereto, and the required angle will be α, $180° \pm \alpha$, or $360° - \alpha$, according to the quadrant in which the angle should be placed.

41. There are always two angles less than 360° corresponding to any given function. Hence there will be ambiguity in the result unless some condition is known that will fix the angle; thus, if the sine is positive, the angle may be in either of the first two quadrants, but if we also know that the cosine is negative, the angle must be in the second quadrant.

Given One Function of an Angle, to find Another without finding the Angle.

42. Suppose $\log \tan \alpha = 9.79361$, and $\log \cos \alpha$ is sought. On page 57 the tabular difference for $\log \tan \alpha$ is 28, and that for $\log \cos \alpha$ is 8, the given logarithmic tangent exceeding 9.79354 by 7. Hence $28 : 7 = 8 : x$; $\therefore\ x = \frac{8}{28} \times 7 = 2 =$ correction to 9.92905, giving $\log \cos \alpha = 9.92903$.

In the margin are tables to facilitate the process. In the column headed $\frac{8}{28}$, the numerator is the tabular difference for the column headed *L. Cos*, and the denominator that for the logarithmic tangents. The numbers in the marginal column are the values of Δ, — the excess of $\log \tan \alpha$ over the next smaller logarithmic tangent, found in the tables, — such that $\frac{8}{28}\Delta$ shall be 0.5, 1.5, 2.5, etc.; and the numbers on the left are the corrections x to be applied to the numbers in the column headed *L. Cos*. Thus, if Δ is between 1.8 and 5.2, x is between 0.5 and 1.5, and is called 1. Note that 1 is opposite the *space* between 1.8 and 5.2. In the example above, the excess Δ is 7, which lies between 5.2 and 8.8; hence x is 2, the number on the left opposite the space between 5.2 and 8.8.

For example, if we have given the logarithms of the sides of a right-angled triangle, $\log a = 2.98227$ and $\log b = 2.90255$, to find the hypotenuse, we use the formulas

$$\tan \alpha = \frac{a}{b}, \text{ and } c = \frac{a}{\sin \alpha} = \frac{b}{\cos \alpha}.$$

$$\begin{aligned} \log a &= 2.98227 \ (1) \\ \therefore \log \sin \alpha &= 9.88571 \ (4) \\ \log b &= 2.90255 \ (2) \\ \therefore \log \tan \alpha &= 0.07972 \ (3) \\ \therefore \log c &= 3.09656 \ (5) \end{aligned}$$

The value of $\log \tan \alpha$ being found in the column marked *L. Tan* at the bottom, the right column will contain the logarithmic sine of the corresponding angle. Also, the correction to 9.88563 is $20 \times \frac{10}{26}$, which we find to be 8 from the table headed $\frac{10}{26}$.

TABLE III.

43. This table (pages 72–94) contains the "natural" or actual numerical values of the trigonometric sines, cosines, tangents, and cotangents of angles from 0° to 90°, for each minute, while Table II. contains the logarithms of these numbers.

NOTE.—The secant is the reciprocal of the cosine, and the cosecant of the sine.

The arrangement and method of using the table are the same as those for Table II., except that the tabular differences are not printed. For the sake of clearness the first figures of the functions are given only opposite each fifth minute, and also when they change. Arts. 26, 27, 29, 30, 31, 32, and 34* will explain the table if the word "logarithmic" be changed to "natural," and "*L. Sin*," etc., to "*N. Sin*," etc.

$\sin 20° 10' = 0.34475$; $\cos 20° 10' = 0.93869$;
$\tan 20° 10' = 0.36727$; $\cot 20° 10' = 2.7228$.
$\sin 68° 10' = 0.92827$; $\cos 68° 10' = 0.37191$;
$\tan 68° 10' = 2.4960$; $\cot 68° 10' = 0.4006\bar{5}$.

In finding $\sin 68° 24'.4$ we see that the required sine lies between 0.92978 and 0.92988, the tabular difference being 10; the correction for $0'.4$ is $10 \times .4 = 4$; hence $\sin 68° 24'.4 = 0.92978 + 4$ units in the fifth place $= 0.92982$.

In finding $\cot 68° 20'.4$ we see that the required cotangent lies between 0.39727 and 0.39694, the tabular difference being 33; the correction for $0'.4$ is $33 \times .4 = 13.2$; hence $\cot 68° 20'.4 = 0.39727 - 13$ units in the fifth place $= 0.39714$.

NOTE.—The correction is added for the sine and tangent because they increase as the angle increases, and subtracted for the cosine and cotangent since they decrease as the angle increases.

In finding the angle whose tangent is 0.39870, the required angle will lie between 21° 44′ and 21° 45′, the tabular difference being $39896 - 39862 = 34$, while the given tangent exceeds that of 21° 44′ by $39870 - 39862 = 8$. Hence $8 \div 34$ or $0'.2+$ is the correction to be added to 21° 44′, giving 21° 44′.2 for the angle required.

In finding the angle whose cosine is 0.36850, the required angle will lie between 68° 22′ and 68° 23′, the tabular difference being $36867 - 36839 = 28$, while the given cosine is less than cos 68° 22′ by $36867 - 36850$ or 17. Hence $17 \div 28$ or $0'.6+$ is the correction to be added to 68° 22′, giving 68° 22′.6 for the angle required.

* The examples in these articles apply only to Table II.

$a_{\overline{n}|} = \frac{1-v^n}{i}$ $\quad a_{\overline{n}|} = v^n s_{\overline{n}|}$

$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}$ or $s_{\overline{n}|} = (1+i)^n a_{\overline{n}|}$

TABLE IV.

44. *Circular arcs with radius unity.* (Page 95.) — To find the length of the arc of 61° 42′ 35″.2 in a circle whose radius is 200 feet, we find that in the circle whose radius is unity,

Arc of 61°	=	1.06465 08
Arc of 42′	=	0.01221 73
Arc of 35″	=	0.00016 97
Arc of 0″.2	=	0.00000 10

∴ Arc of 61° 42′ 35″.2 = 1.07703 88* feet in the circle whose radius is 1 foot, and if the radius is 200 feet the length of the arc will be 1.07703 88 × 200.

To find the angle at the center of a circle with radius 3, the length of its intercepted arc being 13.39410 00: the length of its intercepted arc in the circle whose radius is unity will be $\frac{1}{3}$ × 13.39410 00 = 4.46470 00.

	4.46470 00	
Next less =	3.14159 27	corresponding to 180°.
Difference =	1.32310 73	
Next less =	1.30899 69	corresponding to 75°.
	.01411 04	
	.01396 26	corresponding to 48′.
	.00014 78	
	.00014 54	corresponding to 30″.
	.00000 24	corresponding to 0″.5.

∴ 255° 48′ 30″.5.

* Owing to the incompleteness of the table the last figure will probably be erroneous.

$a^{(p)}_{\overline{n}|} = \frac{1 - \frac{1}{(1+J/p)^{pn}}}{J(p)} = \frac{1 - \text{colog}(1+J/p)^{pn}}{Jp}$

$a_{\overline{n}|} = \frac{1-(1+i)^{-n}}{p\left[(1+i)^{1/p}-1\right]}$

TABLE V.

45. *Conversion of common logarithms into Naperian, and vice versâ* (page 96). — We have

$$\log_{10} N = M \log_e N, \text{ and } \log_e N = \frac{1}{M}\log_{10} N.$$

Table V. contains the multiples of M and $\frac{1}{M}$ by numbers from 1 to 100.

Find the common logarithm of 2, its Naperian logarithm being 0.69314 718056.

$M \times .69$	$=$	0.29966 31925
$M \times .0031$	$=$	.00134 63128 94
$M \times .000047$	$=$	.00002 04118 41
$M \times .00000018$	$=$	.00000 00781 73
$M \times .0000000005$	$=$	.00000 00002 17
$M \times .00000000006$	$=$	.00000 00000 26
$\therefore \log_{10} 2$	$=$	0.30102 99956 51

(True value = 0.30102 99957)

Find the Naperian logarithm of 0.2, its common logarithm being 9.30102 99957 − 10. Hence the true logarithm is

$$\log_{10} 0.2 = -1 + .30102\ 99957 = -0.69897\ 00043.$$

$\frac{1}{M} \times .69$	$=$	1.58878 37142
$\frac{1}{M} \times .0089$	$=$	.02049 30073 28
$\frac{1}{M} \times .000070$	$=$	.00016 11809 57
$\frac{1}{M} \times .0000000043$	$=$	.00000 00099 01
$\therefore \log_e 0.2$	$=$	− 1.60943 79123 86

(True value = − 1.60943 79124)

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$\cos 120 = \cos 90 + 30 =$

$\frac{a}{b} \cdot \frac{c}{a}$ $\quad c/b$

use c/a