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University of California - Berkeley

From the Papers of

Prof. Edmund Pinney

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## THE ELEMENTS

## OF

## PLANE AND SPHERICAL

## TRIGONOMETRY

DESIGNED FOR
THE USE OF STUDENTS

Int the anibrrsity.

## By JOHN HIND, M.A.

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## PREFACE.

The Work now laid before the Public consists of two Treatises, the former on Plane and the latter on Spherical Trigonometry, and in the execution of them both the Author has adopted that Arrangement which appeared to him the most natural, and at the same time the most elementary. The whole of the former Treatise with the exception of the last chapter, has been made to depend solely upon the Propositions usually read in Geometry, and the first Principles of Algebra. For an account of the particular Articles which may be found in the work the reader is referred to the Table of Contents, but the following brief outline will put him in possession of the general plan upon which the performance has been conducted.

The first Chapter contains the Definitions of the subject and the Terms made use of in it, accompanied by various Observations and Deductions of great importance to the complete understanding of the subsequent parts of the work. With respect to the explanation and elucidation of the geometrical Lines which form the leading feature of this part of Mathe-
matical Science, it may be observed that a Notation has been adopted, in which small Figures are suffixed to, or placed under, the Letters employed: this does away with the necessity of introducing a greater number of different letters, and has also the advantage of establishing and pointing out a Connection between the geometrical Lines and their algebraical Affections, which cannot fail greatly to assist the progress of the student; and it is of course always to be understood, when neither of the above-mentioned objects is to be attained, that the suffixes may be altogether suppressed.

The second Chapter comprises what is generally called the Arithmetic of Sines, and commences with a geometrical Demonstration of a Proposition which forms the basis of the whole Doctrine: from this proposition and the definitions of the preceding Chapter, all the other parts of it are either directly or indirectly derived, various Examples of great utility, or at least remarkable either for the frequency of their occurrence or for the singularity of their results, being occasionally introduced.

The third Chapter is a short treatise on the Construction and Verification of sets of Tables adapted to practical purposes, and commonly known by the name of the Trigonometrical Canon. It is this part of the subject which is most laborious, and renders it
available in the concerns of life. In this Chapter some Approximations to the numerical value of the Circumference, or to the Rectification, of the Circle, have been made.

The fourth Chapter contains the Application of Trigonometry to the determination of the Relations between the different Parts and Properties of Triangles and other rectilinear Figures, and it will be seen that a variety of Problems has been expeditiously solved by it, in which the operations of common Geometry would have been long and tedious. The Properties of regular Polygons have been here introduced, but for the subject of Polygonometry in general, the reader is referred to 'Polygonometrie, ou de la mesure des figures rectilignes par Simon Lhuilier,' or to a masterly extract from it, contained in the third volume of Dr. Hutton's Course.

The fifth Chapter exhibits the Solutions of all cases of triangles that usually occur, points out briefly the methods to be preferred under different circumstances, and concludes with several examples of their application in the Mensuration of Heights, Distances, \&c. This Chapter with the assistance of the tables of whose formation a short account has been given in the third, constitutes the general practical use of Trigonometry.

The sixth Chapter presents the subject in a more general and analytical point of view, and treats of what
was termed by Vieta and the Mathematicians of the old School, Angular Sections: this is, in fact, a generalization of the Arithmetic of Sines, and in a work not designed for the purpose of affording elementary Information, might have rendered unnecessary many of the propositions demonstrated in the second Chapter of the present.

The last Chapter on Plane Trigonometry is made up of such Propositions as could not without a violation of method be disposed of in any of the preceding divisions of the work.

The Treatise on Spherical Trigonometry is also comprised in seven Chapters, and in a great degree a similar plan of arrangement has been adhered to. Of course this Treatise has been made to depend almost entirely upon the preceding one, and its division into Chapters seemed so obvious that it is unnecessary here to attempt to assign any reasons for it, the only novelty in addition to the substance of the generality of works on the same subject being a Chapter on Polyhedrons. The reader will readily learn what he may expect to find in it by casting his eye over the Table of Contents.

Throughout the whole of both Treatises it has been the Author's object to present to his reader every proposition proved in a plain concise form ; and with the view of forwarding the purposes of Academical

Instruction, for which the work is principally intended, the leading propositions are stated in Italics, though it may be observed that the Corollaries and Deductions sometimes involve results of no less importance than the articles which have been so distinguished.

A collection of Theorems and Problems comnected with the substance of each Treatise has been annexed in two Appendices at the end of the work, and they have been partially allotted to the respective Chapters in order to direct the student in some degree to the knowledge necessary to enable him to attempt their solution.

In the Table of Contents asterisks have been prefixed to such articles as may be reserved for the student's perusal after he has made a partial progress in some of the other subjects of his Academical Education.

Cambridge,
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The ELEMENTS of ALGEBRA.

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## PLANE TRIGONOMETRY.

## CHAP. I.

## DEFINITIONS AND INTRODUCTORY OBSERVATIONS.

## Article I. Definition I.

Plane Trigonometry in its original acceptation is that part of Mathematical Science which treats of the mensuration of the sides and angles of plane rectilinear triangles; but it is here used in a more comprehensive sense, and includes the general doctrine of plane rectilinear angles, as well as their relations to one another, and to the straight lines by which they are formed, or with which they are in any manner connected.
2. In a circle of given radius, the arcs may be considered as the measures of the angles which they subtend at the centre.

For, let $C$ be the centre of the circle, of which $P Q, P^{\prime} Q^{\prime}$, are any two arcs subtending the angles $P C Q, P^{\prime} C Q^{\prime}$ respectively: draw the diameters $A C D, B C E$ at right angles to each other, which divide the circumference into four equal parts called Quadrants: then (Euclid, 6. 33.) we have

$\operatorname{arc} P Q: \operatorname{arc} P^{\prime} Q^{\prime}::$ angle $P C Q:$ angle $P^{\prime} C Q^{\prime}$;
that is, the arcs are proportional to the angles which they subtend, and may therefore be taken as the measures of them to the given radius $C A$.
3. If the radii of circles be supposed to be of different magnitudes, the angles at their centres wiill be directly proportional to the arcs which subtend them, and inversely proportional to the radii; and every angle may be measured by the fraction, ( $\left.\frac{\operatorname{arc}}{\text { radius }}\right)$.

For, let $P Q, P^{\prime} Q^{\prime}$ be two arcs of different circles subtending at the common centre $C$, the angles $P C Q, P^{\prime} C Q^{\prime}$ respectively:

draw the diameters $A C D, B C E$ at right angles to each other : then
$\operatorname{arc} P Q: \operatorname{arc} A B::$ angle $P C Q:$ angle $A C B$,
or angle $P C Q=\frac{P Q}{A B}$ angle $A C B$; again,
$\operatorname{arc} P^{\prime} Q^{\prime}: \operatorname{arc} A^{\prime} B^{\prime}::$ angle $P^{\prime} C Q^{\prime}:$ angle $A^{\prime} C B^{\prime}$,
or angle $P^{\prime} C Q^{\prime}=\frac{P^{\prime} Q^{\prime}}{A^{\prime} B^{\prime}}$ angle $A^{\prime} C B^{\prime}$ :
$\therefore$ angle $P C Q$ : angle $P^{\prime} C Q^{\prime}::$

$$
\frac{P Q}{A B} \text { angle } A C B: \frac{P^{\prime} Q^{\prime}}{A^{\prime} B^{\prime}} \text { angle } A^{\prime} C B^{\prime} ;
$$

but the angle $A C B$ is the same as the angle $A^{\prime} C \cdot B^{\prime}$, and the quadrants of circles are proportional to their radii, therefore we have

$$
\text { angle } P C Q: \text { angle } P^{\prime} C Q^{\prime}:: \frac{P Q}{A C}: \frac{P^{\prime} Q^{\prime}}{\Lambda^{\prime} C}
$$

that is, the angles are as the arcs directly and the radii inversely : and the fractions, $\frac{P Q}{A C}, \frac{P^{\prime} Q^{\prime}}{A^{\prime} C}$ may therefore be taken as the measures of the angles $P C Q$ and $P^{\prime} C Q^{\prime}$ respectively.
4. In Article 2. we have seen that an angle may be measured by the corresponding arc of a circle, whose radius is given; and in Article 3. that an angle may be measured generally by the fraction, ( $\frac{\operatorname{arc}}{\text { radius }}$ ) ; now it is manifest that these measures will not be upon the same scale, unless the given radius in Art. 2. be supposed to be 1 : hence, therefore, adopting this hypothesis, we conclude generally that any angle may be measured by the corresponding arc of the circle whose radius is 1 , or by $\left(\frac{1}{r}\right)^{\text {th }}$ part of the corresponding circular arc whose radius is $r$; or generally by the fraction, $\left(\frac{\text { arc }}{\text { radius }}\right)$ : that is, angle $=\frac{\operatorname{arc}}{\text { radius }}$.
5. Cor.1. Hence, therefore, if a represent the length of the arc which measures a given angle to the radius 1 , and $\alpha^{\prime}$ to the radius $r$, we shall have $a=\left(\frac{a^{\prime}}{r}\right)$, or $a^{\prime}=r \alpha$.
6. Cor. . If the angles at the centres of different circles be of the same given magnitude, the arcs by which they are measured will be proportional to the radii; and, if the arcs be of the same given magnitude, the angles will be proportional to the reciprocals of the radii.

Ex. 1. If the arc whose length is $a$ measure a given angle denoted by $A$ to the radius 1 , then will an arc whose length is $r a$ measure the same angle when the radius is supposed to be $r$.

Ex. 2. If the arc $a$ of given length measure an angle $A$ to the radius 1 , then will an arc of the same length be the measure of an angle expressed by $\left(\frac{A}{r}\right)$ when the radius used is $r$.
7. Def. 2. If the radius of the circle be supposed $=1$, the circumference (hereafter proved $=6.28318, \& c$.) is represented by $2 \pi$, and is supposed to be divided into 360 equal parts called degrees: each degree is again supposed to be divided into 60 equal parts called minutes : each minute into 60 equal parts called seconds, and so on. These are expressed by the characters ${ }^{0}, \prime^{\prime}, \prime \prime \prime \prime$, \&c. placed above the line to the right of the numbers: thus $45^{\circ} 35^{\prime} \simeq 5^{\prime \prime} 14^{\prime \prime \prime}$ represent 45 degrees, 35 minutes, 25 seconds, 14 thirds.

In the following pages we shall always suppose the radius to be 1 , unless the contrary be expressed.
8. Cor.1. It appears from the last definition that one right angle will be measured by $\frac{\pi}{2}$ or $90^{\circ}$;

$$
\begin{aligned}
& \text { Two right angles by } \pi \text { or } 180^{\circ} \text {; } \\
& \text { Three........... by } \frac{3 \pi}{2} \text { or } 270^{\circ} \text {; } \\
& \text { Four........... by } 2 \pi \text { or } 360^{\circ} \text {. }
\end{aligned}
$$

9. Cor. 2. If we suppose the circumference of the circle to be taken a second, third, \&c. time, we may in the same manner represent the sum of any number of right angles whatever : thus,

Five right angles will be measured by $\frac{5 \pi}{2}$ or $450^{\circ}$;

Six right angles will be measured by $3 \pi$ or $540^{\circ}$;
Seven..................................... $\frac{7 \pi}{2}$ or $630^{\circ}$;
\&c................................................... \&c.;
$n \ldots \ldots . . . . . . . . . . . . . . . . . . . . . . . . \frac{n \pi}{2}$ or $n .90^{\circ}$.
10. Most modern mathematicians, with the exception of the English, divide the circumference of the circle into 400 equal parts, which they call degrees; each degree into 100 equal parts, which they call minutes, and so on.

In this division of the circle
One right angle will be measured by $\frac{\pi}{2}$ or $100^{\circ}$;

11. Cor. Hence, we may easily investigate rules for reducing degrees, \&c. in either of these scales into degrees, \&c. in the other.

For, let $N=$ the number of degrees, \&c. in the English scale, then since 360 English degrees $=400$ foreign,
we shall have 1 English degree $=\frac{400}{360}$ foreign

$$
=\frac{10}{9} \text { foreign ; }
$$

$\therefore N$ English degrees $=\frac{10 N}{9}$ foreign

$$
=N+\frac{N}{9} \text { foreign, }
$$

which gives the first Rule:

To the number of English degrees, \&c. add one-ninth part, and the sum will be the number of foreign degrees, \&c.

Ex. 1. Represent $31^{\circ} 45^{\prime} 57^{\prime \prime}$ English in the foreign scale.

Here $\quad N=31^{0} 45^{\prime} 57^{\prime \prime}=31^{\circ} .765833 \ldots$

$$
\therefore \frac{N}{9}=\ldots \ldots \ldots=3^{0} .529537 \ldots
$$

$\therefore N+\frac{N}{9}$, or the number of foreign degrees, \&ic.

- $\quad=35^{\circ} \cdot 29537 \ldots=35^{\circ} 29^{\prime} 53^{\prime \prime} \& c$.

Ex. 2. Hence also, $1^{0}$ English $=1^{0} 11^{\prime} 11^{\prime \prime}$ \&c. foreign;
$1^{\prime}$ English $=1^{\prime} 85^{\prime} 18^{\prime \prime}$ \&c. foreign ;
$1^{\prime \prime}$ English $=3^{\prime \prime} 08^{\prime \prime \prime} 64^{\prime \prime \prime \prime}$ \&c. foreign.
Again, let $n=$ the number of degrees, \&c. in the foreign scale, then

$$
1 \text { foreign degree }=\frac{9}{10} \text { English }
$$

$\therefore n$ foreign degrees $=\frac{9 n}{10}$ English $=n-\frac{n}{10}$ English,
which gives the second Rule :
From the number of foreign degrees, \&c. subtract onetenth part, and the remainder will be the number of English degrees, \&c.

Ex. 1. Express $25^{\circ} 44^{\prime} 89^{\prime \prime}$ foreign, in the English scale.

$$
\text { In this case } n=25^{\circ} .4489 ;
$$

$$
\therefore \frac{n}{10}=2^{0} .54489
$$

$\therefore n-\frac{n}{10}$, or the number of English degrees, \&c.

$$
=22^{0} \cdot 90401=22^{\circ} 54^{\prime} 14^{\prime \prime} \& \mathrm{c}
$$

Ex. 2. Similarly we find that

$$
\begin{aligned}
& 1^{0} \text { foreign }=0^{0} 54^{\prime} \text { English; } \\
& 1^{\prime} \text { foreign }=0^{\prime} 32^{\prime \prime} .4 \text { English; } \\
& 1^{\prime \prime} \text { foreign }=0^{\prime \prime} .394 \text { English. }
\end{aligned}
$$

We may here observe that in both cases minutes, seconds, $\& c$. must be expressed as decimals of a degree, and that it is usual in practice to express all inferior demominations as decimals of seconds.
12. Def. 3. If $A$ represent any angle or arc, then $90^{\circ}$ $-A$, or $\left(\frac{\pi}{2}-A\right)$ is called its Complement.

Ex. 1. The complement of $34^{\circ} 15^{\prime}=90^{\circ}-34^{\circ} 15^{\prime}=55^{\circ} 45^{\prime}$ : and the complement of $23^{\circ} 27^{\prime} 53^{\prime \prime} .67=90^{\circ}-23^{\circ} 27^{\prime} 53^{\prime \prime} .67$ $=66^{0} 32^{\prime} 6^{\prime \prime} .33$.

Ex. 2. The complement of

$$
\left(\frac{\pi}{4} \pm A\right)=\frac{\pi}{2}-\left(\frac{\pi}{4} \pm A\right)=\left(\frac{\pi}{4} \mp A\right)
$$

Ex. 3. The complement of

$$
\left(\frac{\pi}{2} \pm A\right)=\frac{\pi}{2}-\left(\frac{\pi}{2} \pm A\right)=\mp A
$$

Ex. 4. If the angles of a plane rectilinear triangle be $A$, $B, C$, whereof $C$ is a right angle, then since $A+B+C=\pi$, we have $A+B=\pi-C=\pi-\frac{\pi}{2}=\frac{\pi}{2}$, and $\therefore A=\frac{\pi}{2}-B$,
$B=\frac{\pi}{2}-A$, or each of the acute angles of a right-angled triangle is the complement of the other.
13. Def. 4. If $A$ be any angle or arc, $180^{\circ}-A$, or $\pi-A$ is called its Supplement.

Ex. 1. The supplement of $44^{0} \quad 16^{\prime}=180^{\circ}-44^{0} \quad 16^{\prime}$ $=135^{\circ} 44^{\prime}$, and the supplement of $173^{\circ} 3^{\prime} 13^{\prime \prime} .81=180^{\circ}-$ $173^{0} 3^{\prime} 13^{\prime \prime} .81=6^{0} 56^{\prime} 46^{\prime \prime} .19$.

Ex. 2. The supplement of

$$
\left(\frac{\pi}{2} \pm A\right)=\pi-\left(\frac{\pi}{2} \pm A\right)=\frac{\pi}{2} \mp A
$$

Ex. 3. The supplement of

$$
(\pi \pm A)=\pi-(\pi \pm A)=\mp A
$$

Ex. 4. If $A, B, C$ be the angles of a triangle, and, therefore $A+B+C=\pi$, we shall have $A=\pi-(B+C)$, $B=\pi-(A+C)$, and $C=\pi-(A+B):$ that is, each of the angles of a triangle is the supplement of the sum of the two others.
14. If in the last two articles, the foreign scale be used, the complement of $A$ will $\mathrm{be}=100^{\circ}-A$, and the supplement of $A=200^{\circ}-A$.
15. Since the magnitudes of angles or of the arcs by which they are subtended, cannot without inconvenience be determined by actual measurement, and since all measures found by means of instruments are subject to error, both on account of the unavoidable defects in their construction and the necessary inaccuracy of their application, angular magnitudes as well as the relations of angles to one another, are more easily, and on that account more generally, found by means of certain straight lines which are supposed to belong to all arcs and angles, and are termed trigonometrical functions of the same:
these are the Sine, Co-sine, Versed sine, Chord, Tangent, Cotangent, Secant, and Co-secant : to which are sometimes added, though but little used, the Co-versed sine, Su-versed sine, Cochord, and Su-chord. These lines are properly called the Natural Sine, Natural Co-sine, Sic. when the arc is supposed to be the measure of the angle, or the radius is supposed to be 1 .
16. If positive quantities be represented by lines measured in any direction from a given point, it is easily shewn that negative quantities will be represented by lines taken in the opposite direction; and it therefore follows conversely, that if lines drawn in any direction be considered positive, those lines which are drawn in the opposite direction must be considered negative. Again if lines drawn from 'the centre of a circle through any point in its circumference be called positive, those lines which are drawn from any point in the circumference, through the centre, must be supposed negative. Also, if a point be taken in the circumference of a circle, and through it a diameter be drawn, the positive arcs being measured in one direction from this point, the negative arcs will be measured in the other.

The algebraical signs of lines are also sometimes determined from the principle, that every quantity which admits of different magnitudes has its sign changed.in passing through zero or infinity.

In the following pages, arcs of the circle which we begin to measure upwards from the extremity of a diameter are termed positive, and those which commence from the same point downwards, negative. Also, all lines, in whatever manner drawn, are termed positive for any arc not greater than a quadrant.
17. Def. 5. The sine (sin) of an arc is the straight line drawn from the end of the arc, perpendicular to the diameter passing through the beginning of it. Thus,


> The sine of $A P_{1}$ is $N_{1} P_{1}$, which is positive; $A P_{2}$ is $N_{2} P_{2}, \ldots \ldots .$. positive;
> $A P_{3}$ is $N_{3} P_{3}, \ldots \ldots \ldots$ negative;
> $A P_{4}$ is $N_{4} P_{4}, \ldots \ldots .$. negative.

It appears therefore, that the sine is positive in the first and second quadrants, and negative in the third and fourth.

Hence it is also manifest, that if the arc $A P_{1}$ be called $A$, $N_{1} P_{1}$ is also the sine of the arcs denoted by $2 \pi+A, 4 \pi+A$, $6 \pi+A$, \&c. $(2 n \pi+A)$.
18. Ex. ${ }^{*}$ From the definition, we have $\sin 0=0$;

$$
\begin{aligned}
& \sin 90^{\circ}=\sin \left(\frac{\pi}{2}\right)=C B=1 \\
& \sin 180^{\circ}=\sin \quad(\pi)=0 \\
& \sin 270^{\circ}=\sin \left(\frac{3 \pi}{2}\right)=C E=-1 \\
& \sin 360^{\circ}=\sin (2 \pi)=0
\end{aligned}
$$

Hence, in the first quadrant, the magnitude of the sine lies between 0 and 1 ; in the second, between 1 and 0 ; in the third, between 0 and -1 ; and in the fourth, between -1 and 0 .
19. Cor. 1. $\operatorname{Sin}(-A)=-\operatorname{Sin} A$.

For, take $A P_{1}=A, A P_{4}=-A$; then it is manifest, that $P_{4} N_{4}=P_{1} N_{1}$, or $\sin (-A)=-\sin A$, by (16), that is, the
algebraical sign of the sine of an arc changes with that of the arc itself.

By means of this corollary, we have $\sin A=-\sin (-A)$ $=-\sin (2 \pi-A)=-\sin (4 \pi-A)=\& \mathrm{c} .=-\sin (2 n \pi-A)$.

Also, from this corollary and (17) we have

$$
\sin (2 n \pi+A)=-\sin (2 n \pi-A)
$$

20. Cor. 2. $\operatorname{Sin}(\pi-A)=\operatorname{Sin} A$.

For, let $A P_{1}=A, A P_{2}=(\pi-A)$; therefore $D C P_{2}=\pi$ $-A C P_{2}=\pi-(\pi-A)=A=A C P_{1}$, and $\therefore P_{2} N_{2}=P_{1} N_{1}$, or $\sin (\pi-A)=\sin A$; in other words, the sine of an angle or arc is equal to the sine of its supplement.

From this corollary we have immediately, $\sin A=\sin (\pi-A)$ $=\sin (3 \pi-A)=\& \mathrm{c} .=\sin \{(2 n-1) \pi-A\}$, and therefore from (17) we conclude that

$$
\sin (2 n \pi+A)=\sin \{(2 n-1) \pi-A\} .
$$

21. Def. 6. The cosine (cos) of an arc is the sine of its complement, and is therefore equal to that part of the diameter which is intercepted between the centre and the sine. Thus,


The co-sine of $A P_{1}$ is $P_{1} M_{1}=C N_{1}$, which is positive; $A P_{2}$ is $P_{2} M_{2}=C N_{2}, \ldots \ldots .$. negative; $A P_{3}$ is $P_{3} M_{3}=C N_{3}, \ldots \ldots .$. negative; $A P_{4}$ is $P_{4} M_{4}=C N_{4}, \ldots \ldots \ldots$. positive.

Hence, the cosine is positive in the first and fourth quadrants, and negative in the second and third.

Here, as before, $C N_{1}=\cos A=\cos (2 \pi+A)=\cos (4 \pi+A)$ $=\& c .=\cos (2 n \pi+A)$.
22. Ex. We have, therefore, $\cos 0=C A=1$;

$$
\begin{aligned}
& \cos 90^{\circ}=\cos \left(\frac{\pi}{2}\right)=0 \\
& \cos 180^{\circ}=\cos (\pi)=C D=-1 \\
& \cos 270^{\circ}=\cos \left(\frac{3 \pi}{2}\right)=0 \\
& \cos 360^{\circ}=\cos (2 \pi)=C A=1
\end{aligned}
$$

By these Examples it is seen, that in the first quadrant the magnitude of the cosine lies between 1 and 0 ; in the second, between 0 and -1 ; in the third, between -1 and 0 ; and in the fourth, between 0 and 1 .
23. $\operatorname{Cor}$. 1. $\operatorname{Cos}(-\mathrm{A})=\operatorname{Cos} \mathrm{A}$.

For, take $A P_{1}=A, A P_{4}=-A$; then $\cos (-A)=C N_{4}$ $=C N_{1}=\cos A$; that is, whether an arc be considered positive or negative, the algebraical sign of the cosine is the same.

From this and (21) it follows, that $\cos A=\cos (-A)=$ $\cos (2 \pi \pm A)=\cos (4 \pi \pm A)=\& c .=\cos (2 n \pi \pm A)$.
24. $\operatorname{Cor}$ 2. $\operatorname{Cos}(\pi-\mathrm{A})=-\operatorname{Cos} \mathrm{A}$.

For, let $A P_{1}=A, A P_{2}=(\pi-A)$; therefore, as before, we shall have $C N_{2}=C N_{1}$, or $\cos (\pi-A)=-\cos A$, by (16), that is, the cosine of any arc is equal to the cosine of its supplement with a different algebraical sign.

Hence also, $\cos A=-\cos (\pi-A)=-\cos (3 \pi-A)=$
\&c. $=-\cos \{(2 n-1) \pi-A\}$; and by the last corollary, $\cos (2 n \pi \pm A)=-\cos \{(2 n-1) \pi-A\}$.
25. Cor. 3. From the right-angled triangle $C N_{1} P_{1}$, we have (Euclid, I. 47.)

$$
\begin{gathered}
P_{1} N_{1}^{2}+C N_{1}^{2}=C P_{1}^{2}, \text { or } \sin ^{2} A+\cos ^{2} A=1 ; \\
\therefore \text { also } \sin ^{2} A=1-\cos ^{2} A=(1+\cos A)(1-\cos A) ; \\
\text { and } \cos ^{2} A=1-\sin ^{2} A=(1+\sin A)(1-\sin A)
\end{gathered}
$$

Ex. If $A=45^{\circ}$, we shall have

$$
\begin{gathered}
1=\sin ^{2} 45^{\circ}+\cos ^{2} 45^{\circ}=\sin ^{2} 45^{\circ}+\sin ^{2} 45^{\circ} \\
=2 \sin ^{2} 45^{\circ}=2 \cos ^{2} 45^{\circ}, \text { by }(21), \\
\text { and } \therefore \sin 45^{\circ}=\frac{1}{\sqrt{2}}=\cos 45^{\circ} .
\end{gathered}
$$

26. Def. 7. The versed sine (vers) of an arc is the part of the diameter, intercepted between the beginning of the arc and the sine. It is sometimes called the Sagitta. Thus,

The versed sine of $A P_{1}$ is $A N_{1}$, which is positive; $A P_{2}$ is $A N_{2}, \ldots \ldots \ldots .$. positive ;
$A P_{3}$ is $A N_{3}, \ldots \ldots \ldots$. positive ;
$A P_{4}$ is $A N_{4}, \ldots \ldots \ldots .$. positive.
Hence, the versed sine is positive in every quadrant.
Also, it is clear that vers $A=\operatorname{vers}(2 \pi+A)=\operatorname{vers}(4 \pi+A)$ $=\& \mathrm{c} .=\operatorname{vers}(2 n \pi+A)$.
27. Ex. It follows therefore, that vers $0=0$;

$$
\begin{aligned}
& \text { vers } 90^{\circ}=\operatorname{vers} \quad\left(\frac{\pi}{2}\right)=A C=1 ; \\
& \text { vers } 180^{\circ}=\operatorname{vers} \quad(\pi)=A D=2 ; \\
& \text { vers } 270^{\circ}=\operatorname{vers}\left(\frac{3 \pi}{2}\right)=A C=1 ; \\
& \text { vers } 360^{\circ}=\operatorname{vers}(Q \pi)=0 .
\end{aligned}
$$

From these Examples it appears that in the first quadrant, the versed sine lies between 0 and 1 ; in the second, between 1 and 2 ; in the third, between 2 and 1 ; and in the fourth, between 1 and 0 .
28. Cor. 1. Vers $(-\mathrm{A})=$ Vers $A$.

For, let $A P_{1}=A, A P_{4}=-A$; then is $A N_{4}=A N_{1}$, or $\operatorname{vers}(-A)=\operatorname{vers} A$.

From this we have vers $A=\operatorname{vers}(-A)=\operatorname{vers}(2 \pi-A)$ $=\operatorname{vers}(4 \pi-A)=\& \mathrm{c} .=\operatorname{vers}(2 n \pi-A):$ and therefore also, vers $(2 n \pi+A)=\operatorname{vers}(2 n \pi-A)$.
29. Cor. 2. Vers $(\pi-A)=2-$ Vers $A$.

For, let $A P_{1}=A, A P_{2}=(\pi-A)$; then vers $(\pi-A)$ $=A N_{2}=D N_{1}=A D-A N_{1}=2-\operatorname{vers} A$.

This is called the Su-versed sine of $A$, because it is the versed sine of its supplement.

Hence also, vers $A=2-\operatorname{vers}(\pi-A)=2-\operatorname{vers}(3 \pi-A)$ $=\& \mathrm{c} .=2-\mathrm{vers}\{(2 n-1) \pi-A\}$.
30. Cor. 3. Since $A N_{1}=A C-C N_{1}$, we have vers $A=1-\cos A$, and $\cos A=1-$ vers $A$. Also, vers $(\pi-A)$ $=2-(1-\cos A)=1+\cos A$.
31. Cor.4. The versed sine of $\left(\frac{\pi}{2}-A\right)$ is $B M_{1}$, which $=B C-C M_{1}=B C-N_{1} P_{1}=1-\sin A$, and is called the Co-versed sine of $A$, since it is the versed sine of its complement.

Ex. Hence vers $45^{\circ}=1-\cos 45^{\circ}=1-\frac{1}{\sqrt{2}}$;

$$
\begin{aligned}
& \text { su-vers } 45^{\circ}=1+\cos 45^{\circ}=1+\frac{1}{\sqrt{2}} ; \\
& \text { and co-vers } 45^{\circ}=1-\sin 45^{\circ}=1-\frac{1}{\sqrt{2}} .
\end{aligned}
$$

32. Def. 8. The Chord (chd) of an arc is the straight line which joins the beginning and end of the arc. Thus,


The chord of $A P_{1}$ is $A P_{1}$, which is positive;
............... $A P_{2}$ is $A P_{2}, \ldots \ldots .$. positive ;
............... $A P_{3}$ is $A P_{3}, \ldots \ldots .$. positive;
................ $A P_{4}$ is $A P_{4}, \ldots \ldots \ldots$ positive.
Hence, the chord is positive in every quadrant.
And we likewise observe, that chd $A=\operatorname{chd}(2 \pi+A)=\operatorname{chd}$ $(4 \pi+A)=\& c .=\operatorname{chd}(2 n \pi+A)$.
33. Ex. From the last Article we have chd $0=0$;

$$
\begin{aligned}
& \text { chd } 90^{\circ}=\operatorname{chd} \quad\binom{\pi}{2}=A B=\sqrt{A C^{2}+B C^{2}}=\sqrt{2} ; \\
& \text { chd } 180^{\circ}=\operatorname{chd} \quad(\pi)=A D=2 ; \\
& \text { chd } 270^{\circ}=\text { chd }\left(\frac{3 \pi}{2}\right)=A E=\sqrt{A C^{2}+E C^{2}}=\sqrt{2} ; \\
& \text { chd } 360^{\circ}=\operatorname{chd}(2 \pi)=0 .
\end{aligned}
$$

Hence, then, in the first quadrant the chord lies between 0 and $\sqrt{2}$; in the second, between $\sqrt{2}$ and 2 ; in the third, between 2 and $\sqrt{2}$; and in the fourth, between $\sqrt{2}$ and 0 .
34. Cor. 1. $\quad C h d(-\mathrm{A})=\operatorname{Chd} \mathrm{A}$.

For, let $A P_{1}=A, A P_{4}=-A$; then it is manifest, that $A P_{4}=A P_{1}$, or $\operatorname{chd}(-\Lambda)=\operatorname{chd} A$.

Therefore we have chd $A=\operatorname{chd}(-A)=\operatorname{chd}(2 \pi-A)$ $=\operatorname{chd}(4 \pi-A)=\& c .=\operatorname{chd}(2 n \pi-A) ;$ and by (32) chd $(2 n \pi+A)=\operatorname{chd}(2 n \pi-A)$.
35. Cor. 2. $\quad \operatorname{Chd}(\pi-\mathrm{A})=\sqrt{4-\operatorname{Chd}^{2} \mathrm{~A}}$.

For, let $A P_{1}=A, A P_{2}=(\pi-A) ; \therefore$ the straight line $A P_{2}=D P_{1}=\sqrt{A D^{2}-A P_{1}{ }^{2}}$, or chd $(\pi-A)=\sqrt{4-\text { chd }^{2} A}$. This is called the Su-chord of $A$.
36. Cor. S. From the right-angled triangles, $A N_{1} P_{1}$, $A P_{1} D$, we have $A P_{1}{ }^{2}=A N_{1}^{2}+N_{1} P_{1}{ }^{2}$, or chd ${ }^{2} A=\operatorname{vers}^{2} A$ $+\sin ^{2} A$, and $\therefore$ chd $A=\sqrt{\text { vers }^{2} A+\sin ^{2} A}$.

Also, $A P_{1}{ }^{2}=A D . A N_{1}$, or chd ${ }^{2} A=2$ vers $A=2-2 \cos A$, and $\operatorname{chd} A=\sqrt{2-2 \cos A}$; and therefore $\operatorname{ch}(\pi-A)$ $=\sqrt{2+2 \cos A}$, from (35).
37. Cor. 4. Chd $\left(\frac{\pi}{2}-A\right)=B P_{1}=\sqrt{B M_{1}^{2}+M_{1} P_{1}{ }^{2}}$

$$
=\sqrt{(1-\sin A)^{8}+\cos ^{2} A}=\sqrt{2-2 \sin A}, \text { by }(25) .
$$

This is called the Co-chord of $A$.
Ex. If the arc $A P$ be taken equal to $60^{\circ}$, and $C P, A P, P N$ and $P M$ be drawn, it is manifest that the triangle $A C P$ is equilateral, and that $C A$ is bisected by $P N$ : hence it follows that

chd $60^{\circ}=A P=A C=$ radius $=1$;
vers $60^{\circ}=A N=\frac{T}{2} A C=\frac{\mathrm{r}}{2}$ radius $=\frac{\mathrm{r}}{2}$.

Again, we have
$\cos 60^{\circ}=C N=\frac{1}{2} A C=\frac{1}{2}$ radius $=\frac{1}{2}=\sin 30^{\circ}$;
$\sin 60^{\circ}=P N=\sqrt{C P^{2}-C N^{2}}=\sqrt{1-\frac{1}{4}}=\frac{\sqrt{ } 3}{2}=\cos 30^{\circ}$.
Hence also,
su-chd $60^{\circ}=\sqrt{4-\operatorname{chd}^{2} 60^{\circ}}=\sqrt{4-1}=\sqrt{3}=\operatorname{chd} 120^{\circ}$; co-chd $60^{\circ}=\sqrt{2-2 \sin 60^{\circ}}=\sqrt{2-\sqrt{3}}=\operatorname{chd} 30^{\circ}$.
38. Def. 9. The tangent (tan) of an are is the straight line touching the arc at the beginning, and terminated by the radius through the end of it, produced. Thus,


The tangent of $A P_{1}$ is $A T_{1}$, which is positive; $A P_{2}$ is $A T_{2}, \ldots . . .$. negative;
$\Lambda P_{3}$ is $\Lambda T_{3}, \ldots \ldots \ldots$ positive;
$A P_{4}$ is $A T_{4}^{\prime}, \ldots \ldots .$. negative.
We observe, therefore, that the tangent is positive in the first and third quadrants, and negative in the second and fourth.

Hence likewise, $\tan A=\tan (2 \pi+A)=\tan (4 \pi+A)=\mathbb{E} c$. $=\tan (2 n \pi+A)$.
39. Ex. From this definition, we have $\tan 0=0$;

$$
\tan 90^{\circ}=\tan \left(\frac{\pi}{q}\right)=\infty
$$

$$
\begin{aligned}
& \tan 180^{\circ}=\tan (\pi)=0 ; \\
& \tan 270^{\circ}=\tan \left(\frac{3 \pi}{2}\right)=-\infty ; \\
& \tan 360^{\circ}=\tan (2 \pi)=0 .
\end{aligned}
$$

These two articles prove that in the first quadrant, the magnitude of the tangent lies between 0 and $\infty$; in the second, between $-\infty$ and 0 ; in the third, between 0 and $\infty$; and in the fourth, between $-\infty$ and 0 .
40. Cor. 1. Tan $(-\mathrm{A})=-\operatorname{Tan} \mathrm{A}$.

For, take $A P_{1}=A, A P_{4}=-A$; then it is manifest that $A T_{4}=A T_{1}$, or $\tan (-A)=-\tan A$, by (16); that is, the algebraical sign of the tangent of an arc changes with that of the arc itself.

Hence therefore, $\tan A=-\tan (-A)=-\tan (2 \pi-A)$ $=-\tan (4 \pi-A)=\& c .=-\tan (2 n \pi-A)$.

And, from (38) we have likewise $\tan (2 n \pi+A)=-$ $\tan (2 n \pi-A)$.
41. Cor.2. Tan $(\pi-\mathrm{A})=-\operatorname{Tan} \mathrm{A}$.

For, take $A P_{1}=A, A P_{2}=\pi-A$; therefore $\angle A C T_{2}$ $=\angle A C T_{1}$, and $A T_{2}=A T_{1}$, that is, $\tan (\pi-A)=-\tan A$, by (16); or the tangent of an arc is equal to the tangent of its supplement with a different algebraical sign.

From this we have also, $\tan A=-\tan (\pi-A)=-$ $\tan (3 \pi-A)=-\tan (5 \pi-A)=\& c .=-\tan \{(2 n-1) \pi-A\} ;$ and also from (38), $\tan (2 n \pi+A)=-\tan \{(2 n-1) \pi-A\}$.
42. Cor. 3. By the similar triangles $C N_{1} P_{1}, C A T_{1}$, we have

$$
\begin{array}{r}
C N_{1}: N_{1} P_{1}:: C A: A T_{1}, \\
\text { or } \cos A: \sin A: 1: \tan A, \\
\text { and therefore } \tan A=\frac{\sin A}{\cos A} .
\end{array}
$$

Ex. From this corollary and the preceding pages, it follows that

$$
\begin{aligned}
& \tan 30^{\circ}=\frac{\sin 30^{\circ}}{\cos 30^{\circ}}=\frac{1}{\sqrt{3}} \\
& \tan 45^{\circ}=\frac{\sin 45^{\circ}}{\cos 45^{\circ}}=1 \\
& \tan 60^{\circ}=\frac{\sin 60^{\circ}}{\cos 60^{\circ}}=\sqrt{3} .
\end{aligned}
$$

43. Def. 10. The co-tangent (cot) of an arc is the tangent of its complement, and is therefore the straight line touching the circle at the end of the first quadrant, and terminated by the radius through the end of the arc, produced. Thus,


The co-tangent of $A P_{1}$ is $B t_{1}$, which is positive; $A P_{2}$ is $B t_{2}, \ldots \ldots .$. negative;
$\ldots \ldots \ldots . . . . . . . . \begin{aligned} & \\ & \end{aligned} P_{3}$ is $B t_{3}, \ldots \ldots .$. positive;
.................... $A P_{4}$ is $B t_{4}, \ldots . . .$. negative.
The co-tangent is therefore positive in the first and third quadrants, and negative in the second and fourth.

For the same reason as before, we shall have $\cot A=$ $\cot (2 \pi+A)=\cot (4 \pi+A)=\& c .=\cot (2 n \pi+A)$.
44. Ex. Hence therefore, $\cot 0=\infty$;

$$
\cot 90^{\circ}=\cot \left(\frac{\pi}{2}\right)=0 ;
$$

$$
\begin{aligned}
& \cot 180^{\circ}=\cot (\pi)=-\infty ; \\
& \cot 270^{\circ}=\cot \left(\frac{3 \pi}{2}\right)=0 ; \\
& \cot 360^{\circ}=\cot (2 \pi)=\infty
\end{aligned}
$$

From these two articles, it appears that the co-tangent in the first quadrant is between $\infty$ and 0 ; in the second, between 0 and $-\infty$; in the third, between $\infty$ and 0 ; and in the fourth, between 0 and $-\infty$.
45. Con. 1. $\operatorname{Cot}(-\mathrm{A})=-\operatorname{Cot} \mathrm{A}$.

For, let $A P_{1}=A, A P_{4}=-A$; then, it is evident that $B t_{4}=B t_{1}$, and therefore $\cot (-A)=-\cot A$, by $(16) ;$ or the algebraical sign of the co-tangent of an arc changes with that of the are itself.

Therefore also, $\cot A=-\cot (-A)=-\cot (2 \pi-A)$ $=-\cot (4 \pi-A)=\& i c .=-\cot (2 n \pi-A) ;$ and thence by (43) we have $\cot (2 n \pi+A)=-\cot (2 n \pi-A)$.
46. $\operatorname{Cor} .9 . \operatorname{Cot}(\pi-A)=-\operatorname{Cot} A$.

For, let $A P_{1}=A, A P_{2}=\pi-A$; then, it is manifest that $B t_{2}=B t_{1}$, or $\cot (\pi-A)=-\cot A$, by (16); that is, the co-tangent of an arc is of the same magnitude as the co-tangent of its supplement, but with a different algebraical sign.

From this corollary, we have likewise $\cot A=-\cot (\pi-A)$ $=-\cot (3 \pi-A)=\& c .=-\cot \{(2 n-1) \pi-A\} ;$ and therefore also by (43), $\cot (2 n \pi+A)=-\cot \{(2 n-1) \pi-A\}$.
47. Cor. 3. By the similar triangles, $C M_{1} P_{1}, C B t_{1}$, we have

$$
\begin{aligned}
C M_{1} & : M_{1} P_{1}:: C B: B t_{1}, \\
\text { or } \sin A & : \cos A:: 1: \cot A, \\
\text { and } \therefore \cot A & =\frac{\cos A}{\sin A}=\frac{1}{\tan A}, \text { by }(4 Q) .
\end{aligned}
$$

Ex. It follows from what has been already proved, that

$$
\begin{aligned}
& \cot 30^{\circ}=\frac{\cos 30^{\circ}}{\sin 30^{\circ}}=\frac{1}{\tan 30^{\circ}}=\sqrt{3} \\
& \cot 45^{\circ}=\frac{\cos 45^{\circ}}{\sin 45^{\circ}}=\frac{1}{\tan 45^{\circ}}=1 ; \\
& \cot 60^{\circ}=\frac{\cos 60^{\circ}}{\sin 60^{\circ}}=\frac{1}{\tan 60^{\circ}}=\frac{1}{\sqrt{3}} .
\end{aligned}
$$

48. Def. 11. The secant (sec) of an arc is the straight line drawn from the centre through the end of the arc, and terminated by the tangent. Thus,


The secant of $A P_{1}$ is $C T_{1}$, which is positive; $A P_{2}$ is $C T_{2}^{\prime}, \ldots \ldots .$. negative;
$A P_{3}$ is $C T_{3}, \ldots \ldots \ldots$ negative;
$A P_{4}$ is $C T_{4}, \ldots \ldots \ldots$ positive.
Therefore the secaut is positive in the first, and fourth quadrants, and negative in the second and third.

Hence also as before, $\sec A=\sec (2 \pi+A)=\sec (4 \pi+A)$ $=\& \mathrm{c} .=\sec (2 n \pi+A)$.
49. Ex. We shall therefore have $\sec 0=1$;

$$
\begin{aligned}
& \sec 90^{\circ}=\sec \binom{\pi}{2}=\infty \\
& \sec 180^{\circ}=\sec \quad(\pi)=-1 \\
& \sec 270^{\circ}=\sec \left(\frac{3 \pi}{2}\right)=-\infty \\
& \sec 360^{\circ}=\sec \quad(2 \pi)=1
\end{aligned}
$$

We conclude then, that the magnitude of the secant in the first quadrant lies between 1 and $\infty$; in the second, between $-\infty$ and -1 ; in the third, between -1 and $-\infty$; and in the fourth, between $\infty$ and 1 .
50. Cor. 1. $\operatorname{Sec}(-A)=\operatorname{Sec} A$.

For, let $A P_{1}=A, A P_{4}=-A$; then $C T_{4}=C T_{1}$, or $\sec (-A)=\sec A$; that is, the magnitude and algebraical sign of the secant is the same whether the arc be positive or negative.

Hence also, sec $A=\sec (-A)=\sec (2 \pi-A)=\sec$ $(4 \pi-A)=\& c .=\sec (2 n \pi-A) ;$ and therefore by (48) we have $\sec (2 n \pi+A)=\sec (2 n \pi-A)$.
51. Cor.2. $\operatorname{Sec}(\pi-A)=-\operatorname{Sec} A$.

For, let $A P_{1}=A, A P_{2}=\pi-A$; then, $C T_{2}=C T_{1}$, and $\therefore \sec (\pi-A)=-\sec A$, by (16); or the secants of an arc and of its supplement are of the same magnitude, but have different algebraical signs.

And as before, $\sec A=-\sec (\pi-A)=-\sec (3 \pi-A)$ $=\& \mathrm{c} .=-\sec \{(2 n-1) \pi-A\}:$ also by (48), $\sec (2 n \pi+A)$ $=-\sec \{(2 n-1) \pi-A\}$.
52. Cor. 3. From the similar triangles $C N_{1} P_{1}, C A T_{1}$, we get

$$
\begin{aligned}
C T_{1} & : C A
\end{aligned}: C P_{1}: C N_{1},
$$

$$
\text { and therefore } \sec A=\frac{1}{\cos A} .
$$

Also, from the triangle $C A T_{1}$, we have $C T_{1}{ }^{2}=C A^{2}+A T_{1}{ }^{2}$, that is, $\sec ^{2} A=1+\tan ^{2} A$, and $\therefore \tan ^{2} A=\sec ^{2} A-1$.

Ex. From either of these formulæ, we shall have

$$
\sec 30^{\circ}=\frac{2}{\sqrt{3}}, \sec 45^{\circ}=\sqrt{2}, \text { and } \sec 60^{\circ}=2
$$

53. Def. 12. The co-secant (cosec) of an arc is the secant of its complement, and is therefore the straight line drawn from the centre through the end of the arc, and terminated by the co-tangent. Thus,


The co-secant of $A P_{1}$ is $C t_{1}$, which is positive;
$A P_{2}$ is $C t_{2}, \ldots \ldots .$. positive;
................... $A P_{3}$ is $C t_{3}, \ldots . . . .$. negative;
.................. $A P_{4}$ is $C t_{4}, \ldots \ldots .$. negative.
Hence, the co-secant is positive in the first and second quadrants, and negative in the third and fourth.

Also, $\operatorname{cosec} A=\operatorname{cosec}(2 \pi+A)=\operatorname{cosec}(4 \pi+A)=\& c . \ldots$ $=\operatorname{cosec}(2 n \pi+A)$.
54. Ex. This definition gives $\operatorname{cosec} 0=\infty$;

$$
\operatorname{cosec} 90^{\circ}=\operatorname{cosec}\left(\frac{\pi}{g}\right)=1
$$

$$
\begin{aligned}
& \operatorname{cosec} 180^{\circ}=\operatorname{cosec} \quad(\pi)=\infty \\
& \operatorname{cosec} 270^{\circ}=\operatorname{cosec}\left(\frac{3 \pi}{2}\right)=-1 ; \\
& \operatorname{cosec} 360^{\circ}=\operatorname{cosec} \quad(2 \pi)=\infty .
\end{aligned}
$$

In the first quadrant therefore, the co-secant is between $\infty$ and 1 ; in the second, between 1 and $\infty$; in the third, between $-\infty$ and -1 ; and in the fourth, between -1 and $-\infty$.
55. Cor.1. $\operatorname{Cosec}(-\mathrm{A})=-\operatorname{Cosec} \mathrm{A}$.

For, let $A P_{1}=A, A P_{4}=-A$; then $C t_{4}=C t_{1}$, or cosec $(-A)=-\operatorname{cosec} A$, by (16); that is, the algebraical sign of the cosecant changes with that of the arc.

Hence also, $\operatorname{cosec} A=-\operatorname{cosec}(-A)=-\operatorname{cosec}(2 \pi-A)$ $=-\operatorname{cosec}(4 \pi-\Lambda)=\& c .=-\operatorname{cosec}(2 n \pi-A)$.

And therefore by (53), cosec $(2 n \pi+\Lambda)=-\operatorname{cosec}$ $(2 n \pi-A)$.
56. Cor. 2. $\operatorname{Cosec}(\pi-A)=\operatorname{Cosec} A$.

For, let $A P_{1}=A, A P_{2}=\pi-A$; then $C t_{2}=C t_{1}$, as is evident; that is, $\operatorname{cosec}(\pi-A)=\operatorname{cosec} A$, by (16); or the cosecant of an arc is equal to that of its supplement.

So also, we have $\operatorname{cosec} A=\operatorname{cosec}(\pi-A)=\operatorname{cosec}(3 \pi-A)$ $=\& c .=\operatorname{cosec}\{(2 n-1) \pi-\Lambda\}$; and therefore by (53), cosec $(2 n \pi+A)=\operatorname{cosec}\{(2 n-1) \pi-A\}$.
57. Cor.3. From the triangle $C B t_{1}, C t_{1}{ }^{2}=C B^{2}+B t_{1}{ }^{2}$, that is, $\operatorname{cosec}^{2} A=1+\cot ^{2} A$, and $\therefore \cot ^{2} A=\operatorname{cosec}^{2} A-1$.

Also, $\operatorname{cosec}^{2} A=1+\cot ^{2} A$

$$
=1+\frac{\cos ^{2} A}{\sin ^{2} A}=\frac{\sin ^{2} A+\cos ^{2} A}{\sin ^{2} A}=\frac{1}{\sin ^{2} \Lambda} \text {, by (25), }
$$

and therefore $\operatorname{cosec} \Lambda=\frac{1}{\sin \Lambda}$.

Ex. These formulæ, with what has gone before, give $\operatorname{cosec} 30^{\circ}=2, \operatorname{cosec} 45^{\circ}=\sqrt{2}$, and $\operatorname{cosec} 60^{\circ}=\frac{2}{\sqrt{9}}$.
58. In the preceding articles, we have determined the algebraical signs of the tangent, co-tangent, secant, and cosecant, from an examination of the lines which represent them in the figures, according to the principles assumed in (16); but it may be observed that they are all easily deducible from those of the sine and co-sine previonsly found.

Thus, $\tan A P_{1}=\frac{\sin A P_{1}}{\cos A P_{1}}$ must be positive, $\operatorname{since} \sin A P_{1}$ and $\cos A P_{1}$ are both positive :
and, $\cot A P_{2}=\frac{\cos A P_{2}}{\sin A P_{2}}$ must be negative, since $\cos A P_{2}$ is negative, and $\sin A P_{2}$ positive :
also, sec $A P_{3}=\frac{1}{\cos A P_{3}}$ must be negative, since $\cos A P_{3}$ is negative :
and, $\operatorname{cosec} A P_{4}=\frac{1}{\sin A P_{4}}$ must be negative, since $\sin A P_{4}$ is negative.

The same method may be used to determine the magnitudes of the same functions. Thus,

$$
\begin{aligned}
& \tan 0=\frac{\sin 0}{\cos 0}=\frac{0}{1}=0 \\
& \tan \left(\frac{\pi}{2}\right)=\frac{\sin \left(\frac{\pi}{2}\right)}{\cos \left(\frac{\pi}{2}\right)}=\frac{1}{0}=\infty ; \\
& \sec 0=\frac{1}{\cos 0}=\frac{1}{1}=1 ;
\end{aligned}
$$

$$
\sec \left(\frac{\pi}{2}\right)=\frac{1}{\cos \left(\frac{\pi}{2}\right)}=\frac{1}{0}=\infty ;
$$

and so of the rest, as already found.
59. To transform trigonometrical formula constructed to the radius 1 , into others which shall be adapted to any radius r .


Let $C A=1, C A^{\prime}=r$, and let $A C P=A^{\prime} C P^{\prime}$ be any proposed angle, which is represented by $A$ : then, to the radius I, we have

$$
\begin{aligned}
& P N=\sin \quad A \\
& C N=\cos \quad A \\
& A N=\operatorname{vers} \quad A, \\
& A P=\operatorname{chd} \quad A \\
& A T=\tan \quad A \\
& B t=\cot \quad A \\
& C T=\sec \quad A \\
& C t=\operatorname{cosec} A
\end{aligned}
$$

and to the radius $r$, we have

$$
\begin{aligned}
& P^{\prime} N^{\prime}=\sin A, \\
& C N^{\prime}=\cos A, \\
& A^{\prime} N^{\prime}=\operatorname{vers} A, \\
& A^{\prime} P^{\prime}=\operatorname{chd} A, \\
& A^{\prime} T^{\prime}=\tan A, \\
& B^{\prime} \iota^{\prime}=\cot A, \\
& C T^{\prime}=\sec A, \\
& C t^{\prime}=\operatorname{cosec} A
\end{aligned}
$$

hence, denoting these lines on the latter scale by accents placed over them, we shall have (Eucl. 6.4.)
$\sin A: \sin ^{\prime} A:: P N: P^{\prime} N^{\prime}:: C P: C P^{\prime}:: 1: r ;$
$\therefore \sin A=\frac{1}{r} \sin ^{\prime} A$, and $\sin ^{\prime} A=r \sin A:$
$\operatorname{Cos} A: \cos ^{\prime} A:: C N: C N^{\prime}:: C P: C P^{\prime}:: 1: r ;$
$\therefore \cos A=\frac{1}{r} \cos ^{\prime} A$, and $\cos ^{\prime} A=r \cos A:$
Vers $A: v^{\prime} A$ :: $A N: A^{\prime} N^{\prime}:: C P: C P^{\prime}:: 1: r$;
$\therefore$ vers $A=\frac{1}{r}$ vers' $^{\prime} A$, and vers' $A=r$ vers $A$ :
Chd $A: \operatorname{chd}^{\prime} A:: A P: A^{\prime} P^{\prime}:: C P: C P^{\prime}:: 1: r$;
$\therefore$ chd $A=\frac{1}{r} \operatorname{chd}^{\prime} A$, and chd $A=r$ chd $A:$
$\operatorname{Tan} A: \tan ^{\prime} A:: A^{\prime}: A^{\prime} T^{\prime}:: C A: C A^{\prime}:: 1: r ;$
$\therefore \tan A=\frac{1}{r} \tan ^{\prime} A$, and $\tan ^{\prime} A=r \tan A:$
$\operatorname{Cot} A: \cot ^{\prime} A:: B t: B^{\prime} t^{\prime}:: C B: C B^{\prime}:: 1: r ;$
$\therefore \cot A=\frac{1}{r} \cot ^{\prime} A$, and $\cot ^{\prime} A=r \cot A:$
Sec $A: \sec ^{\prime} A:: C T: C T^{\prime}:: C A: C A^{\prime}:: 1: r$;
$\therefore \sec A=\frac{1}{r} \sec ^{\prime} A$, and $\sec ^{\prime} A=r \sec A$ :
$\operatorname{Cosec} A: \operatorname{cosec}^{\prime} A:: C t: C t^{\prime}:: C B: C B^{\prime}:: 1: r ;$
$\therefore \operatorname{cosec} A=\frac{1}{r} \operatorname{cosec}^{\prime} A$, and $\operatorname{cosec}^{\prime} A=r \operatorname{cosec} A$.

Hence, therefore, if we wish to make use of the radius $r$ instead of the radius 1, we have only to substitute in any proposed
formula which are true on the supposition of the radius being 1 , the quantities $\frac{\sin A}{r}, \frac{\cos A}{r}, \& c$. in the places of $\sin A, \cos A$, \&c. respectively, and the results will be adapted to the radius $r$.

Ex. 1. We have seen in (25) that $\sin ^{2} A+\cos ^{2} A=1$, to the radius 1;
$\therefore\left(\frac{\sin A}{r}\right)^{2}+\left(\frac{\cos A}{r}\right)^{2}=1$, to the radius $r$, or $\sin ^{2} A+\cos ^{2} A=r^{2}$, when the radius is $r$.

Ex. 2. By article (42), tan $A=\frac{\sin A}{\cos A}$, to the radius 1 ;

$$
\begin{aligned}
\therefore\left(\frac{\tan A}{r}\right) & =\frac{\left(\frac{\sin A}{r}\right)}{\left(\frac{\cos A}{r}\right)}, \text { to the radius } r, \\
\text { or } \tan A & =r\left(\frac{\sin A}{\cos A}\right), \text { if the radius be } r .
\end{aligned}
$$

Ex. 3. If the formula, $\cos m A=a \cos ^{m} A+b \cos ^{m-1} A$ $+c \cos ^{m-2} A+\& c$. were true on the supposition of the radius being 1 , then according to the article, we have
$\left(\frac{\cos m A}{r}\right)=a\left(\frac{\cos A}{r}\right)^{m}+b\left(\frac{\cos A}{r}\right)^{m-1}+c\left(\frac{\cos A}{r}\right)^{m-2}+\& c$.
or $r^{m-1} \cos m A=a \cos ^{m} A+r b \cos ^{m-1} A+r^{2} c \cos ^{m-2} A+\& c$. which would be true if the radius were represented by $r$.
60. Cor. 1. From the last example, which involves a general expression, may be deduced the following Rule:

Render all the terms of any formula homogeneous, by multiplying each by such a power of $r$ as shall make its dimensions equal to the highest involved in it, and the result will be adapted to the radius $r$.

Ex. To the radius 1, we have seen in (52) that $\sec ^{2} A$ $=1+\tan ^{2} A$; therefore to the radius $r$, we shall have $\sec ^{2} A$ $=r^{2}+\tan ^{2} A$, by making each of the terms a quantity of two dimensions.
61. Cor. 2. By similar substitutions, formulæ deduced on the supposition of the radius being $R$, may be transformed into others which shall be true when the radius $r$ is made use of.

For, let $\sin A$ represent the sine of $A$ to the radius $R$, $\sin ^{\prime} A . \ldots \ldots . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .$.

$$
\text { then, } \sin A: \sin ^{\prime} A:: R: r \text {, as before, }
$$

$$
\text { and } \therefore \sin A=\frac{R}{r} \sin ^{\prime} A ;
$$

similarly, $\cos A=\frac{R}{r} \cos ^{\prime} A, \&<c .=\& c$.
hence, for $\sin A, \cos A, \& c$. we have only to substitute

$$
\frac{R}{r} \sin A, \frac{R}{r} \cos A, \text { \&c. respectively, }
$$

and the results will be adapted to the radius $r$.
Ex. 1. The formula $\tan A=Q \frac{\sin A}{\cos A}$, is true for the radius 2 , by (59); therefore if we wish to use the radius 3 , we shall have

$$
\frac{2}{3} \tan A=2 \frac{\frac{2}{3} \sin A}{\frac{2}{3} \cos A}, \text { and } \therefore \tan A=3 \frac{\sin A}{\cos A} .
$$

Ex.2. $\operatorname{Sec}^{2} A=16+\tan ^{2} A$, is true by (60) for the radius 4 ; therefore if the radius be 10 , we shall have

$$
\left(\frac{4}{10}\right)^{2} \sec ^{2} A=16+\left(\frac{4}{10}\right)^{2} \tan ^{2} A
$$

and $\therefore \sec ^{2} A=100+\tan ^{2} A$, to the radius 10 ; and so on.
The same methods are applicable to transform any other similar and similarly situated lines, from one radius to another.
62. By means of the relations between the trigonometrical functions of an arc established in this chapter, we are enabled to prove divers theorems, and to solve a variety of problems.

Ex. 1. It is required to prove that

$$
\sin A=\sqrt{2 \text { vers } A-\text { vers }^{2} A}, \text { to the radius } 1 .
$$

By (25) we have $\sin ^{2} A=1-\cos ^{2} A=1-(1-\text { vers } A)^{2}$ from (30), $=1-1+2$ vers $A-\operatorname{vers}^{2} A=2$ vers $A-$ vers $^{2} A$;
$\therefore \sin A=\sqrt{2 \text { vers } A-\text { vers }^{2} A}$, to the radius 1 .
Also, to the radius $r$, by (59) we shall manifestly have

$$
\begin{aligned}
& \frac{\sin A}{r}=\sqrt{2\left(\frac{\text { vers } A}{r}\right)-\left(\frac{\text { vers } A}{r}\right)^{2}}, \\
& \text { and } \therefore \sin A=\sqrt{2 r \text { vers } A-\text { vers }^{2} A} .
\end{aligned}
$$

Ex. 2. To prove that $\frac{\tan A+\tan B}{\cot A+\cot B}=\tan A \tan B$, to the radius 1 .

From (42) we have
$\tan A+\tan B=\frac{\sin A}{\cos A}+\frac{\sin B}{\cos B}=\frac{\sin A \cos B+\cos A \sin B}{\cos A \cos B}:$
also, from (47) we get
$\cot A+\cot B=\frac{\cos A}{\sin A}+\frac{\cos B}{\sin B}=\frac{\cos A \sin B+\sin A \cos B}{\sin A \sin B}:$
and the numerators of these fractions being the same, it follows that

$$
\frac{\tan A+\tan B}{\cot A+\cot B}=\frac{\sin A \sin B}{\cos A \cos B}=\left(\frac{\sin A}{\cos A}\right)\left(\frac{\sin B}{\cos B}\right)
$$

$=\tan A \tan B$, by (42), to the radius 1 .

Likewise $\frac{\tan A+\tan B}{\cot A+\cot B}=\frac{\tan A \tan B}{r^{2}}$ to the radius $r$, as easily appears by means of the rule laid down in (60).

Ex. 3. Given $m \sin A=u \cos A$, to find the values of $\sin A$ and $\cos A$.

Here, $m^{2} \sin ^{2} A=u^{2} \cos ^{2} A=n^{2}\left(1-\sin ^{2} A\right)$, by (25), $=n^{2}-n^{2} \sin ^{2} A$;
$\therefore\left(m^{2}+n^{2}\right) \sin ^{2} A=n^{2}$, whence $\sin A= \pm \frac{n}{\sqrt{m^{2}+n^{2}}}:$
also $\cos ^{2} A=1-\sin ^{2} A=1-\frac{n^{2}}{m^{2}+n^{2}}=\frac{m^{2}}{m^{2}+n^{2}}$,
and $\therefore \cos A= \pm \frac{m}{\sqrt{m^{2}+n^{2}}}$, to the radius 1 .
If the radius $r$ be used, we have by what is proved in (59),

$$
\frac{\sin A}{r}= \pm \frac{n}{\sqrt{m^{2}+n^{2}}}, \text { and } \frac{\cos A}{r}= \pm \frac{m}{\sqrt{m^{2}+n^{2}}}
$$

whence $\sin A= \pm \frac{n r}{\sqrt{m^{2}+n^{2}}}$, and $\cos A= \pm \frac{m r}{\sqrt{m^{2}+n^{2}}}$.
Ex. 4. Given $\sin A=m \sin B$, and $\tan A=n \tan B$, to find the values of $\sin A$ and $\sin B$.

Since $\tan A=n \tan B$, we have

$$
\begin{aligned}
& \frac{\sin A}{\cos A}=n \frac{\sin B}{\cos B}, \text { by } \\
\therefore & \frac{\sin A}{\sin B}=n \frac{\cos A}{\cos B}, \text { that is, } m=n \frac{\cos A}{\cos B} \\
& \frac{m^{2}}{n^{2}}=\frac{\cos ^{2} A}{\cos ^{2} B}=\frac{1-\sin ^{2} A}{1-\sin ^{2} B}=\frac{1-m^{2} \sin ^{2} B}{1-\sin ^{2} B}
\end{aligned}
$$

whence $m^{2}-m^{2} \sin ^{2} \boldsymbol{B}=n^{2}-m^{2} n^{2} \sin ^{2} \boldsymbol{B}$, and

$$
m^{2}\left(1-n^{2}\right) \sin ^{2} \boldsymbol{B}=m^{2}-n^{2}, \text { or } \sin ^{2} \boldsymbol{B}=\frac{m^{2}-n^{2}}{m^{2}\left(1-n^{2}\right)},
$$

$$
\text { and } \therefore \sin B= \pm \frac{1}{m} \sqrt{\frac{m^{2}-n^{2}}{1-n^{2}}}
$$

wherefore $\sin A=n \sin B= \pm \sqrt{\frac{m_{b}^{2}-n^{2}}{1-n^{2}}}$, in both of which the radius is 1 .

Adapting these values to the radius $r$, since $m$ and $n$ are merely numerical magnitudes and therefore not considered of any dimensions, we shall have by (60),

$$
\sin A= \pm r \sqrt{\frac{m^{2}-n^{2}}{1-n^{2}}}, \text { and } \sin B= \pm \frac{r}{m} \sqrt{\frac{m^{2}-n^{2}}{1-n^{2}}}
$$

From these two equations all the other trigonometrical functions of $A$ and $B$ are easily deduced.

## CHAP. II.

On the relations between the Trigonometrical Functions of arcs or angles, and those of their sums and differences, and also of some of their multiples, sub-multiples and powers.
63. To express the sines and cosines of the sum and difference of two arcs, in terms of the sines and cosines of the arcs themselves.

Let the arcs $A P, P Q$, be the measures of any two angles $A C P, P C Q$ denoted by $A$ and $B$ to the radius $C A$ : draw $Q M$ perpendicular to $C P$, and $Q R, M S, P N$ perpendicular to $C A$. draw also $M T$ parallel to $C A$. Then,

$P N=\sin A C P, C N=\cos A C P ;$
$Q M=\sin P C Q, C M=\cos P C Q$;
$Q R=\sin A C Q=\sin (A C P \pm P C Q) ;$
$C R=\cos A C Q=\cos (A C P \pm P C Q):$
Now $Q R=R T \pm Q T=M S \pm Q T$, the upper and lower sigus being used for the first and second figures respectively ;
but by similar triangles,

$$
\begin{gathered}
M S: C M:: P N: C P \\
\text { and } Q T: Q M: C N: C P \\
\text { from which } M S=C M \frac{P N}{C P}, \text { and } Q T=Q M \frac{C N}{C P} \\
\therefore \text { we have } Q R=C M \frac{P N}{C P} \pm Q M \frac{C N}{C P} \\
=\frac{1}{C P}(P N \cdot C M \pm C N \cdot Q M)=P N \cdot C M \pm C N \cdot Q M,
\end{gathered}
$$

if the radius be supposed to be 1 ; that is,

$$
\sin (A \pm B)=\sin A \cos B \pm \cos A \sin B \ldots \ldots(\alpha)
$$

Again, $C R=C S \mp R S=C S \mp M T$;
but by similar triangles,

$$
\begin{array}{r}
C S: C M:: C N: C P \\
\text { and } M T: Q M: P N: C P
\end{array}
$$

whence $C S=C M \frac{C N}{C P}$, and $M T=Q M \frac{P N}{C P}$;
$\therefore$ we have $C R=C M \frac{C N}{C P} \mp Q M \frac{P N}{C P}$

$$
=\frac{1}{C P}(C N \cdot C M \mp P N \cdot Q M)=C N . C M \mp P N \cdot Q M
$$

if $C P=1$, as before; that is,

$$
\cos (A \pm B)=\cos A \cos B \mp \sin A \sin B
$$

Ex. 1. Let $B=\frac{\pi}{2}$, or $A+B=\frac{\pi}{a}+A$ :
then, $\sin \left(\frac{\pi}{2}+A\right)=\sin \frac{\pi}{2} \cos A+\cos \frac{\pi}{2} \sin A$

$$
=\cos A, \text { by }(18) \text { and (22): }
$$

$$
\text { and } \begin{aligned}
\cos \left(\frac{\pi}{2}+A\right) & =\cos \frac{\pi}{2} \cos A-\sin \frac{\pi}{2} \sin A \\
& =-\sin A, \text { by (18) and (22). }
\end{aligned}
$$

$$
\begin{aligned}
& \text { Ex. 2. Let } B=\pi \text {, or } A+B=\pi+A \text { : } \\
& \text { therefore } \sin (\pi+A)=\sin \pi \cos A+\cos \pi \sin A \\
& =-\sin A, \text { by (18) and (22) : } \\
& \text { and } \cos (\pi+A)=\cos \pi \cos A-\sin \pi \sin A \\
& =-\cos A, \text { by (18) and (22). }
\end{aligned}
$$

64. Cor. 1. The construction and investigation above given hold good whatever be the magnitudes of the angles $A C P, P C Q$, due regard being had to the algebraical signs of the trigonometrical lines as determined in the preceding clapter : and any three of the functions just mentioned may with great facility be deduced from the remaining one. Thus,

$$
\begin{aligned}
& \sin (A-B)=\sin \{\pi-(A-B)\}, \text { by }(20), \\
& =\sin \{(\pi-A)+B\} \\
& =\sin (\pi-A) \cos B+\cos (\pi-A) \sin B, \text { by ( } \alpha), \\
& =\sin A \cos B-\cos A \sin B, \text { from (20) and (24): }
\end{aligned}
$$

again,

$$
\begin{aligned}
& \cos (A+B)=\sin \left\{\frac{\pi}{2}-(A+B)\right\}, \text { by }(21) \\
& =\sin \left\{\left(\frac{\pi}{2}+A\right)+B\right\}, \text { by }(22), \\
& =\sin \left(\frac{\pi}{2}+A\right) \cos B+\cos \left(\frac{\pi}{2}+A\right) \sin B, \text { by }(a), \\
& =\cos A \cos B-\sin A \sin B, \text { from }(63):
\end{aligned}
$$

$$
\text { and, } \begin{aligned}
& \cos (A-B)=\sin \left\{\frac{\pi}{2}-(A-B)\right\}, \text { by }(21) \\
&=\sin \left\{\left(\frac{\pi}{2}-A\right)+B\right\} \\
&=\sin \left(\frac{\pi}{2}-A\right) \cos B+\cos \left(\frac{\pi}{2}-A\right) \sin B, \text { by }(\alpha) \\
&=\cos A \cos B+\sin A \sin B, \text { from }(21)
\end{aligned}
$$

The same values of $\cos (A \pm B)$ are also easily deducible from the equation $\cos (A \pm B)=\sqrt{1-\sin ^{2}(A \pm B)}$.
65. Cor. 2. If the radius $C P$ be supposed $=r$, we shall have

$$
\begin{aligned}
\sin (A \pm B) & =\frac{1}{r}(\sin A \cos B \pm \cos A \sin B) \\
\text { and } \cos (A \pm B) & =\frac{1}{r}(\cos A \cos B \mp \sin A \sin B),
\end{aligned}
$$

which are the same as would have been derived from those just found by means of the rule laid down in article (60).
66. Cor. 3. Let $S$ and $s$ be the sines of any two arcs, $C$ and $c$ their cosines; then by (63) we have,

The arc whose sine is $S \pm$ the arc whose sine is $s=$ the arc whose sine is $(S c \pm s C)$ : also, the arc whose cosine is $C \pm$ the arc whose cosine is $c=$ the arc whose cosine is ( $C c \mp S s$ ).

These are usually written abbreviatedly as follows:

$$
\begin{aligned}
& \operatorname{Sin}^{-1} S \pm \sin ^{-1} s=\sin ^{-1}(S c \pm s C) \\
& \operatorname{Cos}^{-1} C \pm \cos ^{-1} c=\cos ^{-1}(C c \mp S s)
\end{aligned}
$$

Ex. $\sin ^{-1}\left(\frac{3}{5}\right)+\sin ^{-1}\left(\frac{4}{5}\right)=\sin ^{-1}\left\{\frac{3}{5} \frac{3}{5}+\frac{4}{5} \frac{4}{5}\right\}$

$$
\begin{aligned}
& =\sin ^{-1}\left\{\frac{9}{25}+\frac{16}{25}\right\}=\sin ^{-1}\{1\} \\
& =\left(\frac{\pi}{2}\right), \text { as appears from (18). }
\end{aligned}
$$

67. From (63), we obtain by addition and subtraction the following formulæ:

$$
\begin{aligned}
& \sin (A+B)+\sin (A-B)=2 \sin A \cos B \\
& \sin (A+B)-\sin (A-B)=2 \cos A \sin B \\
& \cos (A-B)+\cos (A+B)=2 \cos A \cos B \\
& \cos (A-B)-\cos (A+B)=2 \sin A \sin B
\end{aligned}
$$

These expressions furnish the following useful equations:

1. The sum of the sines of any two arcs is equal to twice the rectangle of the sine of their semi-sum, and the cosine of their semi-difference.
2. The difference of the sines of any two arcs is equal to twice the rectangle of the cosine of their semi-sum, and the sine of their semi-difference.
3. The sum of the cosines of any two arcs is equal to twice the rectangle of the cosine of their semi-sum, and the cosine of their semi-difference.
4. The difference of the cosines of any two arcs is equal to twice the rectangle of the sine of their semi-sum, and the sine of their semi-difference.
5. From the same article we obtain by multiplication,

$$
\sin (A+B) \sin (A-B)
$$

$=(\sin A \cos B+\cos A \sin B)(\sin A \cos B-\cos A \cdot \sin B)$
$=\sin ^{2} A \cos ^{2} B-\cos ^{2} A \sin ^{2} B$
$=\sin ^{2} A\left(1-\sin ^{2} B\right)-\left(1-\sin ^{2} A\right) \sin ^{2} B$
$=\sin ^{2} A-\sin ^{2} A \sin ^{2} B-\sin ^{2} B+\sin ^{2} A \sin ^{2} B$
$=\sin ^{2} A-\sin ^{2} B=(\sin A+\sin B)(\sin A-\sin B):$
or

$$
=\cos ^{2} B-\cos ^{2} A=(\cos B+\cos A)(\cos B-\cos A)
$$

## Similarly,

$$
\begin{aligned}
& \cos (A+B) \cos (A-B) \\
= & (\cos A \cos B-\sin A \sin B)(\cos A \cos B+\sin A \sin B) \\
= & \cos ^{2} A \cos ^{2} B-\sin ^{2} A \sin ^{2} B \\
= & \cos ^{2} A\left(1-\sin ^{2} B\right)-\left(1-\cos ^{2} A\right) \sin ^{2} B \\
= & \cos ^{2} A-\cos ^{2} A \sin ^{2} B-\sin ^{2} B+\cos ^{2} A \sin ^{2} B \\
= & \cos ^{2} A-\sin ^{2} B=(\cos A+\sin B)(\cos A-\sin B) \\
& \text { or } \\
= & \cos ^{2} B-\sin ^{2} A=(\cos B+\sin A)(\cos B-\sin A) .
\end{aligned}
$$

69. By division, we have from the same article,

$$
\begin{aligned}
\frac{\sin (A+B)}{\sin (A-B)} & =\frac{\sin A \cos B+\cos A \sin B}{\sin A \cos B-\cos A \sin B} \\
& =\frac{\left(\frac{\sin A}{\cos A}\right)+\left(\frac{\sin B}{\cos B}\right)}{\left(\frac{\sin A}{\cos A}\right)-\left(\frac{\sin B}{\cos B}\right)}
\end{aligned}
$$

(by dividing both numerator and denominator by $\cos A \cos B$ )

$$
=\frac{\tan A+\tan B}{\tan A-\tan B}, \text { from (42). }
$$

In like manner,

$$
\begin{aligned}
\frac{\cos (A+B)}{\cos (A-B)} & =\frac{\cos A \cos B-\sin A \sin B}{\cos A \cos B+\sin A \sin B} \\
& =\frac{1-\tan A \tan B}{1+\tan A \tan B}, \text { by proceeding as }
\end{aligned}
$$

above.

The former of these expressions furnishes the following useful proportion:

The sine of the sum of two arcs : the sine of their difference :: the sum of their tangents : the difference of their tangents.
70. To express the sines and cosines of twoे arcs in terms of the sines and cosines of their semi-sum and semidifference.

$$
A=\left(\frac{A+B}{2}\right)+\left(\frac{A-B}{2}\right), \text { and } B=\left(\frac{\Lambda+B}{2}\right)-\left(\frac{A-B}{2}\right)
$$

therefore we have by means of (63),

$$
\begin{gathered}
\sin A=\sin \left\{\left(\frac{A+B}{2}\right)+\left(\frac{A-B}{2}\right)\right\} \\
=\sin \left(\frac{A+B}{2}\right) \cos \left(\frac{A-B}{2}\right)+\cos \left(\frac{A+B}{2}\right) \sin \left(\frac{A-B}{2}\right) ; \\
\text { and } \sin B=\sin \left\{\left(\frac{A+B}{2}\right)-\left(\frac{A-B}{2}\right)\right\} \\
=\sin \left(\frac{A+B}{2}\right) \cos \left(\frac{A-B}{2}\right)-\cos \left(\frac{A+B}{2}\right) \sin \left(\frac{A-B}{2}\right) .
\end{gathered}
$$

Similarly, $\cos A=\cos \left\{\left(\frac{A+B}{2}\right)+\left(\frac{A-B}{2}\right)\right\}$

$$
=\cos \left(\frac{A+B}{2}\right) \cos \left(\frac{A-B}{2}\right)-\sin \left(\frac{A+B}{2}\right) \sin \left(\frac{A-B}{2!}\right)
$$

and $\cos B=\cos \left\{\left(\frac{A+B}{2}\right)-\left(\frac{A-B}{2}\right)\right\}$
$=\cos \left(\frac{A+B}{2}\right) \cos \left(\frac{A-B}{2}\right)+\sin \left(\frac{A+B}{2}\right) \sin \left(\frac{A-B}{2}\right)$.
71. This article, by addition and subtraction, gives

$$
\begin{aligned}
& \sin A+\sin B=2 \sin \left(\frac{A+B}{2}\right) \cos \left(\frac{A-B}{2}\right) \\
& \sin A-\sin B=2 \cos \left(\frac{A+B}{2}\right) \sin \left(\frac{A-B}{2}\right) \\
& \cos B+\cos A=2 \cos \left(\frac{A+B}{2}\right) \cos \left(\frac{A-B}{2}\right) \\
& \cos B-\cos A=2 \sin \left(\frac{A+B}{2}\right) \sin \left(\frac{A-B}{2}\right)
\end{aligned}
$$

in which expressions are comprised the Rules laid down in (67).
72. From the same article, we obtain by division,

$$
\begin{aligned}
\frac{\sin A+\sin B}{\sin A-\sin B} & =\frac{2 \sin \left(\frac{A+B}{2}\right) \cos \left(\frac{A-B}{2}\right)}{2 \cos \left(\frac{A+B}{2}\right) \sin \left(\frac{A-B}{2}\right)} \\
& =\frac{\sin \left(\frac{A+B}{2}\right) \cos \left(\frac{A-B}{2}\right)}{\cos \left(\frac{A+B}{2}\right) \sin \left(\frac{A-B}{2}\right)} \\
& =\frac{\tan \left(\frac{A+B}{2}\right)}{\tan \left(\frac{A-B}{2}\right)}, \text { by }(42) \text { and }(47) ;
\end{aligned}
$$

$$
\text { and } \begin{aligned}
\frac{\cos B+\cos A}{\cos B-\cos A} & =\frac{2 \cos \left(\frac{A+B}{2}\right) \cos \left(\frac{A-B}{2}\right)}{2 \sin \left(\frac{A+B}{2}\right) \sin \left(\frac{A-B}{2}\right)} \\
& =\frac{\cos \left(\frac{A+B}{2}\right) \cos \left(\frac{A-B}{Q}\right)}{\sin \left(\frac{A+B}{2}\right) \sin \left(\frac{A-B}{2}\right)} \\
& =\cot \left(\frac{A+B}{2}\right) \cot \left(\frac{A-B}{2}\right), \text { by (47). }
\end{aligned}
$$

From the former of these formulx, we have the following useful Proportion.

The sum of the sines of any two arcs: the difference of the sines :: the tangent of their semi-sum : the tangent of their semi-difference.

By a similar process it is easily proved that

$$
\frac{\sin A+\sin B}{\cos A+\cos B}=\tan \left(\frac{A+B}{2}\right):
$$

and if $B=0$, we shall find by reduction that

$$
\tan \frac{A}{Q}=\sqrt{\frac{1-\cos A}{1+\cos A}}, \text { and } \therefore \cos A=\frac{1-\tan ^{2} \frac{A}{Q}}{1+\tan ^{2} \frac{A}{Q}}
$$

73. To express the sine and cosine of $(11+1) \mathrm{A}$, in terms of the sines and cosines of $\mathrm{nA},(\mathrm{n}-1) \mathrm{A}$, and A .

Here, attending to the formula of (63) we have

$$
\begin{aligned}
\sin (n+1) A & =\sin (n A+A) \\
& =\sin n A \cos A+\cos n A \sin A \\
\text { and } \sin (n-1) A & =\sin (n A-A) \\
& =\sin n A \cos A-\cos n A \sin A
\end{aligned}
$$

hence by addition, we have

$$
\begin{aligned}
& \sin (n+1) A+\sin (n-1) A=2 \sin n A \cos A, \\
& \text { and } \therefore \sin (n+1) A=2 \sin n A \cos A-\sin (n-1) A . \\
& \text { Again, } \cos (n+1) A=\cos (n A+A) \\
&=\cos n A \cos A-\sin n A \sin A, \\
& \text { and } \cos (n-1) A=\cos (n A-A) \\
&=\cos n A \cos A+\sin n A \sin A ;
\end{aligned}
$$

whence as before, we get

$$
\begin{gathered}
\cos (n+1) A+\cos (n-1) A=2 \cos n A \cos A \\
\text { and } \therefore \cos (n+1) A=2 \cos n A \cos A-\cos (n-1) A .
\end{gathered}
$$

74. Ex. Let $n$ be taken equal to $1,2,3,4$, \&c. successively, and we shall have

$$
\begin{aligned}
& \sin 2 A=2 \sin A \cos A ; \\
& \sin 3 A=2 \sin 2 A \cos A-\sin A ; \\
& \sin 4 A=2 \sin 3 A \cos A-\sin 2 A ; \\
& \& c . . . .=\& c . . . . . . . . . . . . . . . . . . . . \\
& \cos 2 A=2 \cos A \cos A-1 ; \\
& \cos 3 A=2 \cos 2 A \cos A-\cos A ; \\
& \cos 4 A=2 \cos 3 A \cos A-\cos 2 A ; \\
& \text { \&c..... }=\text { \&c.......................... }
\end{aligned}
$$

75. To express the sine and cosine of twice an arc in terms of the sine and cosine of the arc itself:

Here, by means of (63), we have

$$
\begin{aligned}
\sin 2 A & =\sin (A+A)=\sin A \cos A+\cos A \sin A \\
& =2 \sin A \cos A
\end{aligned}
$$

$$
\text { or }=2 \sin A \sqrt{1-\sin ^{2} A} ; \text { or }=2 \cos A \sqrt{1-\cos ^{2} A} .
$$

$$
\text { Also, } \begin{aligned}
\cos 2 A & =\cos (A+A)=\cos A \cos A-\sin A \sin A \\
& =\cos ^{2} A-\sin ^{2} A ; \\
\text { or } & =2 \cos ^{2} A-1 ; \text { or }=1-2 \sin ^{2} A .
\end{aligned}
$$

76. Cor. Putting $A$ and $\frac{A}{2}$ in the places of $2 A$ and $A$ respectively, we have from the last article, .

$$
\sin A=2 \sin \frac{A}{2} \cos \frac{A}{2}
$$

and $\cos A=\cos ^{2} \frac{A}{2}-\sin ^{2} \frac{A}{2}=2 \cos ^{2} \frac{A}{2}-1=1-2 \sin ^{2} \frac{A}{2} ;$
also, from the latter of these we obtain

$$
\begin{gathered}
2 \sin ^{2} \frac{A}{2}=1-\cos A=\operatorname{vers} A=\frac{1}{2} \operatorname{chd}^{2} A, \text { by }(36), \\
\text { and } \therefore 2 \sin \frac{A}{2}=\operatorname{chd} A, \text { or } \sin \frac{A}{2}=\frac{1}{2} \operatorname{chd} A
\end{gathered}
$$

that is, the chord of an arc is equal to twice the sine of its half, or the sine of an arc is equal to half the chord or its double.
77. By some writers, the properties just mentioned are made the basis of many of the fundamental propositions of Trigonometry, and they may be proved merely by a comparison of the respective lines in the following figure: thus,


Let $A P=A, A Q=Q P=\frac{A}{2}$; then it appears from the
definitions laid down in the preceding chapter, that $A P=$ chd $A$, and $Q M=\sin \frac{A}{\mathcal{Q}}:$ it is also manifest that $A L=L P$, or $A P$ $=Q A L$, and $A L=Q M$ : whence it follows that $A P=2 Q M$, or $\operatorname{chd} A=\Omega \sin \frac{A}{\Omega}$, and $\therefore \sin \frac{A}{\varrho}=\frac{1}{2}$ chd $A$.
78. To express the sine and cosine of half an arc in terms of the sine of the arc itself.

$$
\begin{aligned}
& \text { Since } \cos ^{2} \frac{A}{2}+\sin ^{2} \frac{A}{2}=1, \text { by }(25), \\
& \text { and } 2 \sin \frac{A}{2} \cos \frac{A}{9}=\sin A, \text { by }(76)
\end{aligned}
$$

we have by addition,

$$
\cos ^{2} \frac{A}{2}+2 \sin \frac{A}{2} \cos \frac{A}{2}+\sin ^{2} \frac{A}{2}=1+\sin A
$$

and by subtraction,

$$
\cos ^{2} \frac{A}{2}-2 \sin \frac{A}{2} \cos \frac{A}{2}+\sin ^{2} \frac{A}{2}=1-\sin A
$$

whence, by extracting the square roots of both sides of these equations, we obtain
$\cos \frac{A}{2}+\sin \frac{A}{2}=\sqrt{1+\sin A}, \cos \frac{A}{2}-\sin \frac{A}{2}= \pm \sqrt{1-\sin A} ;$ in the latter of which the positive or negative sign must be used according as $\cos \frac{A}{\Omega}$ is greater or less than $\sin \frac{A}{2}$;
$\therefore$ by addition and subtraction and division by 2 ,

$$
\cos \frac{A}{2}=\frac{1}{9}\{\sqrt{(1+\sin A)} \pm \sqrt{(1-\sin A)}\}
$$

$$
\text { and } \sin \frac{A}{2}=\frac{1}{2}\{\sqrt{(1+\sin A)} \mp \sqrt{(1-\sin A)}\}
$$

These two expressions are frequently used to examine the accuracy of results deduced by other means, and on that account are termed Formule of Verification.

Ex. Let $A=30^{\circ}$, then $\frac{A}{2}=15^{\circ}$, and $\sin A=\frac{1}{2}$, from (37), wherefore $\sin 15^{\circ}=\frac{1}{\Omega}\left\{\sqrt{1+\sin 30^{\circ}}-\sqrt{1-\sin 30^{\circ}}\right\}$

$$
=\frac{1}{2}\left\{\sqrt{1+\frac{1}{2}}-\sqrt{1-\frac{1}{2}}\right\}=\frac{\sqrt{3}-1}{2 \sqrt{2}}:
$$

and $\cos 15^{\circ}=\frac{1}{2}\left\{\sqrt{1+\sin 30^{\circ}}+\sqrt{1-\sin 30^{\circ}}\right\}$

$$
=\frac{1}{2}\left\{\sqrt{1+\frac{1}{2}}+\sqrt{1-\frac{1}{2}}\right\}=\frac{\sqrt{3}+1}{2 \sqrt{2}}
$$

79. To express the sine and cosine of half an arc in terms of the cosine of the arc itself.

From (76), it appears that

$$
\begin{aligned}
\cos ^{2} \frac{A}{2}-\sin ^{2} \frac{A}{2} & =\cos A \\
\text { also, } \cos ^{2} \frac{A}{2}+\sin ^{2} \frac{A}{Q} & =1, \text { by }(25) \\
\therefore \text { by addition, } 2 \cos ^{2} \frac{A}{2} & =1+\cos A
\end{aligned}
$$

and by subtraction, $Q \sin ^{2} \frac{A}{2}=1-\cos A$;
whence, dividing each of these equations by $\Omega$, and extracting. the square roots, we have

$$
\begin{aligned}
\sin \frac{A}{2} & = \pm \sqrt{\left(\frac{1-\cos A}{2}\right)} \\
\text { and } \cos \frac{A}{2} & = \pm \sqrt{\left(\frac{1+\cos A}{2}\right)}
\end{aligned}
$$

Ex. Let $A=45^{\circ}$, then $\frac{A}{2}=22^{\circ} 30^{\prime}$, and $\cos A=\frac{1}{\sqrt{2}}$ from (25), whence we have
$\sin 22^{\circ} 30^{\prime}=\sqrt{\frac{1-\cos 45^{\circ}}{2}}=\sqrt{\frac{1}{2}-\frac{1}{2 \sqrt{ } 2}}=\sqrt{\frac{\sqrt{2-1}}{2 \sqrt{ } 2}}:$
$\cos 22^{\circ} 30^{\prime}=\sqrt{\frac{1+\cos 45^{\circ}}{2}}=\sqrt{\frac{1}{2}+\frac{1}{2 \sqrt{2}}}=\sqrt{\frac{\sqrt{2}+1}{2 \sqrt{2}}}$.
80. From articles (63) and (75) may easily be deduced what is called Delambre's formula.

For, $\sin \left(A+1^{\circ}\right)=\sin A \cos 1^{\circ}+\cos A \sin 1^{0}$, and $\sin \left(A-1^{0}\right)=\sin A \cos 1^{0}-\cos A \sin 1^{0}$;
whence by addition,

$$
\begin{aligned}
& \sin \left(A+1^{0}\right)+\sin \left(A-1^{0}\right)=2 \sin A \cos 1^{0}, \\
& \text { and } \therefore \sin \left(A+1^{\circ}\right)=2 \sin A \cos 1^{0}-\sin \left(A-1^{\circ}\right) \\
& \\
& =2 \sin A\left(1-2 \sin ^{2} 30^{\prime}\right)-\sin \left(A-1^{\circ}\right) \\
& \\
& =2 \sin A-\sin \left(A-1^{0}\right)-4 \sin A \sin ^{2} 30^{\prime} \\
& =\sin A+\left\{\sin A-\sin \left(A-1^{0}\right)\right\}-4 \sin A \sin ^{2} 30^{\prime} .
\end{aligned}
$$

81. We have seen in the last article but one, that

$$
\begin{aligned}
\sin \frac{A}{\mathscr{Q}} & =\sqrt{\frac{1}{\underset{\sim}{1}-\frac{1}{2} \cos A}} \\
& =\sqrt{\frac{1}{2}-\frac{1}{2} \sqrt{1-\sin ^{2} A}}
\end{aligned}
$$

and from these are readily derived the following equations:

$$
\begin{aligned}
\sin \frac{A}{2^{2}} & =\sqrt{\frac{1}{2}-\frac{1}{2} \cos \frac{A}{2}} \\
& =\sqrt{\frac{1}{2}-\frac{1}{2} \sqrt{1-\sin ^{2} \frac{A}{2}}}
\end{aligned}
$$

$$
\begin{aligned}
\sin \frac{A}{2^{3}} & =\sqrt{\frac{1}{2}-\frac{1}{2} \cos \frac{A}{2^{2}}} \\
& =\sqrt{\frac{1}{2}-\frac{1}{2} \sqrt{1-\sin ^{2} \frac{A}{2^{2}}}}
\end{aligned}
$$

$\& c . . . . . . . .=\& c . . . . . . . . . . . . . . . . .$.

$$
\begin{aligned}
\sin \frac{A}{2^{n}} & =\sqrt{\frac{1}{2}-\frac{1}{2} \cos \frac{A}{2^{n-1}}} \\
& =\sqrt{\frac{1}{2}-\frac{1}{2} \sqrt{1-\sin ^{2} \frac{A}{2^{n-1}}}} .
\end{aligned}
$$

Also, from the same article it appears that

$$
\begin{aligned}
& \cos \frac{A}{2}=\sqrt{\frac{1}{2}+\frac{1}{2} \cos A} \\
& \quad=\sqrt{\frac{1}{2}+\frac{1}{2} \sqrt{1-\sin ^{2} A}}
\end{aligned}
$$

whence we similarly obtain

$$
\begin{aligned}
\cos \frac{A}{Q^{2}} & =\sqrt{\frac{1}{2}+\frac{1}{2} \cos \frac{A}{2}} \\
& =\sqrt{\frac{1}{2}+\frac{1}{2} \sqrt{1-\sin ^{2} \frac{A}{2}}} \\
\cos \frac{A}{2^{3}} & =\sqrt{\frac{1}{2}+\frac{1}{2} \cos \frac{A}{Q^{2}}} \\
& =\sqrt{\frac{1}{2}+\frac{1}{2} \sqrt{1-\sin ^{2} \frac{A}{Q^{2}}}}
\end{aligned}
$$

$$
\$ c \ldots \ldots \ldots . .=\& c \ldots
$$

$$
\begin{aligned}
\cos \frac{A}{2^{n}} & =\sqrt{\frac{1}{2}+\frac{1}{2} \cos \frac{A}{2^{n-1}}} \\
& =\sqrt{\frac{1}{2}+\frac{1}{2} \sqrt{1-\sin ^{2} \frac{A}{2^{n-1}}}}
\end{aligned}
$$

82. From the last article, by substitution we shall have

$$
\sin \frac{A}{2^{2}}=\sqrt{\frac{1}{2}-\frac{1}{2} \sqrt{\frac{1}{2}+\frac{1}{2} \cos A}}
$$

$$
\sin \frac{A}{2^{3}}=\sqrt{\frac{1}{2}-\frac{1}{2} \sqrt{\frac{1}{2}+\frac{1}{2} \sqrt{\frac{1}{2}+\frac{1}{2} \cos A}}}
$$

$\& c . . . . . .=\$ c$

$$
\sin \frac{A}{2^{\prime \prime}}=
$$


the radical sign being repeated $n$ times.
In the same manner, we get

$$
\begin{aligned}
& \cos \frac{A}{Q^{2}}=\sqrt{\frac{1}{2}+\frac{1}{2} \sqrt{\frac{1}{2}+\frac{1}{2} \cos A}} \\
& \cos \frac{A}{Q^{3}}=\sqrt{\frac{1}{2}+\frac{1}{2} \sqrt{\frac{1}{2}+\frac{1}{2} \sqrt{\frac{1}{2}}+\frac{1}{2} \cos A}}
\end{aligned}
$$

\&c........ = \&c.

$$
\cos \frac{A}{2^{n}}=
$$


where the radical sign is repeated $n$ times.

Ex. Let $A=\left(\frac{\pi}{2}\right)$; and $\therefore \cos A=0$, and we have
$\sin \left\{\frac{1}{2^{n}} \frac{\pi}{2}\right\}=\sqrt{\frac{1}{2}-\frac{1}{2} \sqrt{\frac{1}{2}+\frac{1}{2} \sqrt{8 c \ldots \ldots \sqrt{\frac{1}{2}}}} ;}$
$\cos \left\{\frac{1}{2^{n}} \frac{\pi}{2}\right\}=\sqrt{\frac{1}{2}+\frac{1}{2} \sqrt{\frac{1}{2}+\frac{1}{2} \sqrt{\& c \ldots \ldots \sqrt{\frac{1}{2}}}}}$
the radical sign in each case occurring $n$ times.
83. To express the sine and cosine of thrice an arc in terms of the sine and cosine of the arc itself.

Here we have from (63),

$$
\sin 3 A=\sin (2 A+A)=\sin 2 A \cos A+\cos 2 A \sin A
$$

and from the same article,

$$
\begin{aligned}
\cos 3 A & =\cos (2 A+A)=\cos 2 A \cos A-\sin 2 A \sin A \\
& =\left(2 \cos ^{2} A-1\right) \cos A-2 \sin ^{2} A \cos A, \text { by }(75) \\
& =2 \cos ^{3} A-\cos A-2 \cos A+2 \cos ^{3} A \\
& =4 \cos ^{3} A-3 \cos A .
\end{aligned}
$$

84. From the two formulæ just proved, we have immediately $\sin ^{5} A=\frac{1}{4}(3 \sin A-\sin 3 A)$, and $\cos ^{3} A=\frac{1}{4}(3 \cos A+\cos 3 A)$.
85. By substituting in the formulæ just investigated, $A$ and $\frac{A}{3}$ in the places of $3 A$ and $A$ respectively, we get $\sin A=3 \sin \frac{A}{3}-4 \sin ^{3} \frac{A}{3}$, and $\cos A=4 \cos ^{3} \frac{A}{3}-3 \cos \frac{A}{3} ;$ and thence the equations,

$$
4 \sin ^{3}\left(\frac{A}{3}\right)-3 \sin \left(\frac{A}{3}\right)+\sin \Lambda=0
$$

$$
\text { and } 4 \cos ^{3}\left(\frac{A}{3}\right)-3 \cos \left(\frac{A}{3}\right)-\cos A=0
$$

by the solution of which, $\sin \left(\frac{A}{3}\right)$ and $\cos \left(\frac{A}{3}\right)$ may be found in terms of $\sin A$ and $\cos A$ respectively.

Ex. 1. Let $A=180^{\circ}$; then $\sin A=0$, and $\frac{A}{3}=60^{\circ}$;

$$
\begin{aligned}
& \text { therefore, } 4 \sin ^{3} 60^{\circ}-3 \sin 60^{\circ}=0 \\
& \text { and } \sin 60^{\circ}=\frac{\sqrt{ } 3}{2}, \text { and } \therefore \cos 60^{\circ}=\frac{1}{2}
\end{aligned}
$$

Ex. 2. Suppose $A=90^{\circ}$; then $\cos A=0$, and $\frac{A}{3}=30^{\circ}$;

$$
\begin{gathered}
\text { hence, } 4 \cos ^{3} 30^{\circ}-3 \cos 30^{\circ}=0 \\
\text { and } \cos 30^{\circ}=\frac{V 3}{2}, \text { and } \therefore \sin 30^{\circ}=\frac{1}{2}
\end{gathered}
$$

86. From the last examples, combined with some of the preceding articles, we are enabled to find the sines and cosines of $15^{0}, 75^{0}, 105^{\circ}, 165^{\circ}, \& c$.

For, $\sin 15^{\circ}=\sin \left(45^{\circ}-30^{\circ}\right)=\sin 45^{\circ} \cos 30^{\circ}-\cos 45^{\circ} \sin 30^{\circ}$

$$
=\frac{1}{\sqrt{2}} \frac{\sqrt{ } 3}{2}-\frac{1}{\sqrt{2}} \frac{1}{2}=\frac{1}{2 \sqrt{ } 2}(\sqrt{ } 3-1)=\cos 75^{\circ}
$$

similarly, $\cos 15^{\circ}=\frac{1}{2 \sqrt{2}}(\sqrt{3}+1)=\sin 75^{\circ}$.
Again, $\sin 105^{\circ}=\sin \left(60^{\circ}+45^{\circ}\right)=\sin 60^{\circ} \cos 45^{\circ}+\cos 60^{\circ} \sin 45^{\circ}$

$$
=\frac{\sqrt{ } 3}{2} \frac{1}{\sqrt{ } 2}+\frac{1}{2} \frac{1}{\sqrt{ } 2}=\frac{1}{2 \sqrt{ } 2}(\sqrt{ } 3+1)=-\cos 165^{\circ}
$$

similarly, $\cos 105^{\circ}=-\frac{1}{2 \sqrt{2}}(\sqrt{ } 3-1)=-\sin 165^{\circ}$.
87. Articles (75) and (83) afford us the means of determining the sines and cosines of $18^{\circ}$ and $72^{\circ}$.

$$
\text { For, } \begin{aligned}
\text { since } \sin 2 A & =2 \sin A \cos A, \\
\text { and } \cos 3 A & =4 \cos ^{3} A-3 \cos A:
\end{aligned}
$$

if $A=18^{\circ}$, we have $2 A=36^{\circ}$, and $3 A=54^{\circ}$;
also, $\sin 2 A=\sin 36^{\circ}=\cos \left(90^{\circ}-36^{\circ}\right)=\cos 54^{\circ}=\cos 3 A$ :
hence $2 \sin 18^{\circ} \cos 18^{\circ}=4 \cos ^{3} 18^{\circ}-3 \cos 18^{\circ}$, and $2 \sin 18^{\circ}=4 \cos ^{2} 18^{\circ}-3=1-4 \sin ^{2} 18^{\circ}$;
$\therefore 4 \sin ^{2} 18^{\circ}+2 \sin 18^{\circ}=1$, which gives

$$
\sin 18^{\circ}=\frac{V 5-1}{4}=\cos 72^{\circ}
$$

and $\cos 18^{\circ}=\sqrt{1-\sin ^{2} 18^{\circ}}=\frac{\sqrt{10+2 \sqrt{5}}}{4}=\sin 72^{\circ}$.
88. Hence the sines and cosines of $36^{\circ}$ and $54^{\circ}$ are easily found.

For, $\sin 36^{\circ}=2 \sin 18^{\circ} \cos 18^{\circ}$

$$
\begin{gathered}
=2\left(\frac{\sqrt{5}-1}{4}\right)
\end{gathered} \begin{gathered}
\frac{\sqrt{10+2 \sqrt{5}}}{4}=\frac{(\sqrt{ } 5-1) \sqrt{10+2 \sqrt{5}}}{8} \\
=\frac{\sqrt{10-2 \sqrt{5}}}{4}=\cos 54^{\circ}:
\end{gathered}
$$

and $\cos 36^{\circ}=2 \cos ^{2} 18^{\circ}-1$

$$
=\frac{2(10+2 \sqrt{ } 5)}{16}-1=\frac{\sqrt{ } 5+1}{4}=\sin 54^{\circ} .
$$

89. From the last two articles is derived Euler's Formula of Verification, which is

$$
\begin{aligned}
\sin A & =\sin \left(36^{\circ}+A\right)+\sin \left(79^{\circ}-\Lambda\right) \\
& -\sin \left(36^{\circ}-A\right)-\sin \left(72^{\circ}+\Lambda\right):
\end{aligned}
$$

For, $\sin \left(36^{\circ}+A\right)-\sin \left(36^{\circ}-A\right)=2 \cos 36^{\circ} \sin A$, and $\sin \left(72^{\circ}+A\right)-\sin \left(72^{\circ}-A\right)=2 \cos 72^{\circ} \sin A$ :
therefore by subtraction, we have

$$
\begin{aligned}
\sin \left(36^{\circ}+A\right)+ & \sin \left(72^{\circ}-A\right)-\sin \left(36^{\circ}-A\right)-\sin \left(72^{\circ}+A\right) \\
& =2 \sin A\left(\cos 36^{\circ}-\cos 72^{\circ}\right) \\
= & 2 \sin A\left\{\frac{\sqrt{ } 5+1}{4}-\frac{\sqrt{ } 5-1}{4}\right\}=\sin A
\end{aligned}
$$

90. Legendre's Formula of Verification is, in fact, the same as Euler's, though different in form, that is,

$$
\begin{aligned}
\cos A & =\sin \left(54^{\circ}+A\right)+\sin \left(54^{\circ}-A\right) \\
& -\sin \left(18^{\circ}+A\right)-\sin \left(18^{\circ}-A\right) .
\end{aligned}
$$

Here, $\sin \left(54^{\circ}+A\right)+\sin \left(54^{\circ}-A\right)=2 \sin 54^{\circ} \cos A$, and $\sin \left(18^{\circ}+A\right)+\sin \left(18^{\circ}-A\right)=2 \sin 18^{\circ} \cos A ;$
therefore by subtraction, as before, we have

$$
\begin{gathered}
\sin \left(54^{\circ}+A\right)+\sin \left(54^{\circ}-A\right)-\sin \left(18^{\circ}+A\right)-\sin \left(18^{\circ}-A\right) \\
=2 \cos A\left\{\sin 54^{\circ}-\sin 18^{\circ}\right\} \\
=2 \cos A\left\{\frac{\sqrt{ } 5+1}{4}-\frac{\sqrt{ } 5-1}{4}\right\}=\cos A
\end{gathered}
$$

91. By means of (78) the sine and cosine of $9^{\circ}$ and $81^{\circ}$ are easily obtained from the sine of $18^{\circ}$; from the sines and cosines of $9^{0}$ and $15^{\circ}$, the sines and cosines of $6^{\circ}$ and $24^{\circ}$, and therefore of $84^{\circ}$ and $66^{\circ}$ are very readily deduced; from the sines and cosines of $6^{\circ}$ and $84^{\circ}$ may be found those of $78^{\circ}$ and $12^{\circ}$; and so on.
92. By a process in every respect similar to that used in article (83), we readily obtain the following results:

$$
\begin{aligned}
& \sin 4 A=\left(4 \sin A-8 \sin ^{3} A\right) \cos A ; \\
& \sin 5 A=5 \sin A-20 \sin ^{3} A+16 \sin ^{5} A \text {; } \\
& \sin 6 A=\left(0 \sin A-32 \sin ^{3} A+32 \sin ^{5} A\right) \cos A \text {; } \\
& \text { \&c..... } \pm \& c .
\end{aligned}
$$

$$
\begin{aligned}
& \cos 4 A=8 \cos ^{4} A-8 \cos ^{2} A+1 ; \\
& \cos 5 A=16 \cos ^{5} A-20 \cos ^{3} A+5 \cos A ; \\
& \cos 6 A=32 \cos ^{6} A-48 \cos ^{4} A+18 \cos ^{2} A-1 \text {; } \\
& \text { \&c..... }=\text { \& } c \text {. }
\end{aligned}
$$

93. To express the versed sines of the sum and difference of two arcs in terms of the versed sines of the arcs themselves.

From (30) we have

$$
\text { vers } \begin{aligned}
(A \pm B) & =1-\cos (A \pm B) \\
& =1-\cos A \cos B \pm \sin A \sin B
\end{aligned}
$$

which, by (30) and (62),

$$
\left.\begin{array}{l}
\quad=1-(1-\operatorname{vers} A)(1-\text { vers } B) \\
\pm \sqrt{\left(2 \text { vers } A-\text { vers }^{2} A\right)\left(2 \text { vers } B-\text { vers }^{2} B\right)} \\
\quad=\text { vers } A+\text { vers } B-\text { vers } A \text { vers } B
\end{array}\right] \begin{aligned}
& \left(2 \text { vers } A-\text { vers }^{2} A\right)\left(2 \text { vers } B-\text { vers }^{2} B\right)
\end{aligned}
$$

94. From the expressions just proved we have immediately,

$$
\begin{aligned}
& \operatorname{vers}(A+B)+\operatorname{vers}(A-B) \\
& =2 \text { vers } A+2 \text { vers } B-2 \text { vers } A \text { vers } B ; \\
& \text { or }=2(1-\cos A \cos B) ; \\
& \text { also, vers }(A+B)-\operatorname{vers}(A-B) \\
& =2 \sqrt{\left(2 \text { vers } A-\text { vers }^{2} A\right)\left(2 \text { vers } B-\text { vers }^{2} B\right)} ; \\
& \text { or }=2 \sin A \sin B ; \\
& \text { and vers }(A+B) \text { vers }(A-B)=(\operatorname{vers} A-\operatorname{vers} B)^{2} .
\end{aligned}
$$

95. The value of vers $(A \pm B)$ may however be exhibited in a much neater form than that in which it is given in the last article but one. Thus,

$$
\begin{aligned}
& \text { vers }(A \pm B)=\text { vers } A+\text { vers } B-\text { vers } A \text { vers } B \\
& \pm \sqrt{\left(2 \text { vers } A-\text { vers }^{2} A\right)\left(2 \text { vers } B-\text { vers }^{9} B\right)}
\end{aligned}
$$

$$
\begin{aligned}
= & \frac{1}{2}\{\operatorname{vers} A(2-\operatorname{vers} B)+\text { vers } B(2-\text { vers } A) \\
& \left. \pm 2 \sqrt{\left(2 \text { vers } A-\text { vers }^{2} A\right)\left(2 \text { vers } B-\operatorname{vers}^{2} B\right)}\right\} \\
= & \frac{1}{2}\{\text { vers } A \text { vers }(\pi-B)+\text { vers } B \text { vers }(\pi-A) \\
& \pm 2 \sqrt{\text { vers } A \text { vers }(\pi-A) \text { vers } B \text { vers }(\pi-B)}\} \\
= & \frac{1}{2}\{\sqrt{\text { vers } A \text { vers }(\pi-B)} \pm \sqrt{\text { vers } B \text { vers }(\pi-A)}\}^{2} .
\end{aligned}
$$

96. Cor. 1. In (93), suppose $B=A$, and we have immediately

$$
\text { vers } \begin{aligned}
2 A & =2 \text { vers } A-\text { vers }^{2} A+2 \text { vers } A-\text { vers }^{2} A \\
& =4 \text { vers } A-2 \text { vers }^{2} A=2 \text { vers } A(\mathcal{2}-\operatorname{vers} A) \\
& =2 \text { vers } A \text { vers }(\pi-A), \text { by }(29) .
\end{aligned}
$$

97. Cor. 2. In (95) for $B$ put $2 A$, and we have vers $3 A$
$=\frac{1}{2}\{\sqrt{\text { vers } A \text { vers }(\pi-Q A)}+\sqrt{\text { vers } 2 A \text { vers }(\pi-A)}\}^{2}$
$=\frac{1}{2}\{\sqrt{\text { vers } A(2-\operatorname{vers} 2 A)}+\sqrt{\text { vers } 2 A(2-\text { vers } A)}\}^{2}$
$=\frac{1}{2}\left\{\sqrt{2 \operatorname{vers} A(1-\operatorname{vers} A)^{2}}+\sqrt{2 \text { vers } A(2-\operatorname{vers} A)^{2}}\right\}^{2}$
$=\frac{1}{2}\{(1-\operatorname{vers} A) \sqrt{2 \text { vers } A}+(2-\operatorname{vers} A) \sqrt{2 \text { vers } A}\}^{2}$
$=\frac{1}{2}\{(3-2 \text { vers } A) \sqrt{2 \text { vers } A}\}^{2}=$ vers $A(3-2 \text { vers } A)^{2}$.
A similar process may be used to express the versed sines of $4 A, 5 A, \& c$. in terms of vers $A$.
98. Since vers $2 A=4$ vers $A-2$ vers $^{2} A$, by substituting $A$ and $\frac{A}{2}$ in the places of $2 A$ and $A$, we shall have

$$
\begin{gathered}
\text { vers } A=4 \text { vers } \frac{A}{2}-2 \operatorname{vers}^{2} \frac{A}{2}, \\
\text { and } \therefore \operatorname{vers}^{2} \frac{A}{2}-2 \text { vers } \frac{A}{2}=-\frac{\text { vers } A}{2},
\end{gathered}
$$

$$
\begin{aligned}
& \text { whence vers } \frac{A}{2}-2 \text { vers } \frac{A}{2}+1=\frac{2-\text { vers } A}{2} \\
& \text { and } \therefore \text { vers } \frac{A}{2}=1 \pm \sqrt{\left(\frac{2-\operatorname{vers} A}{2}\right)}
\end{aligned}
$$

Ex. 1. Let $A=90^{\circ} ; \therefore$ vers $A=1$, by (27), and $\frac{A}{2}=45^{\circ}$, wheuce vers $45^{\circ}=1-\sqrt{\frac{1}{2}}=1-\frac{1}{\sqrt{2}}=\frac{1}{\sqrt{2}}(\sqrt{ } 2-1)$.

Ex. 2. Let $A=60^{\circ} ; \therefore$ vers $A=\frac{1}{2}$, by (37), and $\frac{A}{2}=30^{\circ}$,
$\therefore$ vers $30^{\circ}=1-\sqrt{\frac{3}{4}}=1-\frac{\sqrt{ } 3}{2}=\frac{1}{2}\{2-\sqrt{ } 3\}$.
Similarly the versed sines of $\frac{A}{3}, \frac{A}{4}$, \&c. may be found in terms of vers $A$, by the solutions of a cubic, biquadratic, \&c. equation respectively.
99. By means of the last article we have the following equations:

$$
\begin{aligned}
& \operatorname{vers} \frac{A}{Q}=1-\sqrt{1-\frac{1}{2} \operatorname{vers} A} \\
& \operatorname{vers} \frac{A}{Q^{2}}=1-\sqrt{1-\frac{1}{2} \operatorname{vers} \frac{A}{2}} \\
& \operatorname{vers} \frac{A}{Q^{3}}=1-\sqrt{1-\frac{1}{Q} \operatorname{vers} \frac{A}{Q^{2}}} \\
& \& c \ldots . .
\end{aligned}
$$

and hence

$$
\text { vers } \frac{A}{2^{2}}=1-\sqrt{1-\frac{1}{\underset{\sim}{2}} \operatorname{vers} \frac{A}{Q}}
$$

$$
\begin{aligned}
& =1-\sqrt{\frac{1}{2}+\frac{1}{2} \sqrt{1-\frac{1}{2} \operatorname{vers} A}} ; \\
& \operatorname{vers} \frac{A}{2^{3}}=1-\sqrt{1-\frac{1}{2} \operatorname{vers} \frac{A}{2^{2}}} \\
& =1-\sqrt{\frac{1}{2}+\frac{1}{2} \sqrt{\frac{1}{2}+\frac{1}{2} \sqrt{1-\frac{1}{2} \text { vers } A}}} ; \\
& \text { \&c..... }=\& c . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . \\
& \text { and generally, vers } \frac{A}{2^{n}}=1-\sqrt{1-\frac{1}{2} \text { vers } \frac{A}{2^{n-1}}}
\end{aligned}
$$

$$
=1-\sqrt{\frac{1}{2}+\frac{1}{2} \sqrt{\frac{1}{2}+\frac{1}{2} \sqrt{\frac{1}{2}+\& c \ldots \sqrt{1-\frac{1}{2}} \operatorname{vers} A}}}
$$

where the radical sign is repeated $n$ times.
100. By substituting in the expressions for vers $(A \pm B$ found in (93) or (95), the quantities $\left(\frac{A+B}{2}\right)$ and $\left(\frac{A-B}{2}\right)$ in the places of $A$ and $B$ respectively, the values of the versed sines of $A$ and $B$ will be exhibited in terms of the versed sines of $\left(\frac{A+B}{2}\right)$ and $\left(\frac{A-B}{2}\right)$.
101. To express the chords of the sum and difference of two arcs in terms of the chords of the arcs themselves.

From (36) we have

$$
\begin{aligned}
\operatorname{chd}^{2}(A \pm B) & =2 \text { vers }(A \pm B)=2\{1-\cos (A \pm B)\} \\
& =2\{1-\cos A \cos B \pm \sin A \sin B\}:
\end{aligned}
$$

now, from the same article it is easily proved that

$$
\cos A=\frac{2-\operatorname{chd}^{2} A}{2}, \text { and } \sin A=\frac{\sqrt{4 \operatorname{chd}^{2} A-\operatorname{chd}^{4} A}}{2} ;
$$

$$
\cos B=\frac{2-\operatorname{chd}^{2} B}{2}, \text { and } \sin B=\frac{\sqrt{4 \operatorname{chd}^{2} B-\operatorname{chd}^{4} B}}{2} ;
$$

therefore $\operatorname{chd}^{2}(A \pm B)$

$$
\begin{aligned}
= & 2\left\{1-\frac{\left(2-\operatorname{chd}^{2} A\right)\left(Q-\operatorname{chd}^{2} B\right)}{4}\right. \\
& \left. \pm \frac{\sqrt{\left(4 \operatorname{chd}^{2} A-\operatorname{chd}^{4} A\right)\left(4 \operatorname{chd}^{2} B-\operatorname{chd}^{4} B\right)}}{4}\right\} \\
= & \frac{1}{2}\left\{4-4+2 \operatorname{chd}^{2} A+2 \operatorname{chd}^{2} B-\operatorname{chd}^{2} A \operatorname{chd}^{2} B\right. \\
& \left. \pm \sqrt{\left(4 \operatorname{chd}^{2} A-\operatorname{chd}^{4} A\right)\left(4 \operatorname{chd}^{2} B-\operatorname{chd}^{4} B\right)}\right\} \\
= & \frac{1}{2}\left\{2 \operatorname{chd}^{2} A+2 \operatorname{chd}^{2} B-\operatorname{chd}^{2} A \operatorname{chd}^{2} B\right. \\
& \left. \pm \sqrt{\left(4 \operatorname{chd}^{2} A-\operatorname{chd}^{4} A\right)\left(4 \operatorname{chd}^{2} B-\operatorname{chd}^{4} B\right)}\right\}
\end{aligned}
$$

and chd $(A \pm B)$

$$
\begin{aligned}
= & \frac{1}{\sqrt{ }}\left\{2 \operatorname{chd}^{2} A+2 \operatorname{chd}^{2} B-\operatorname{chd}^{2} A \operatorname{chd}^{2} B\right. \\
& \left. \pm \sqrt{\left(4 \operatorname{chd}^{2} A-\operatorname{chd}^{4} A\right)\left(4 \operatorname{chd}^{2} B-\operatorname{chd}^{2} B\right)}\right\}^{\frac{1}{2}}
\end{aligned}
$$

102. From the result above found, we shall have

$$
\operatorname{chd}^{2}(A+B)+\operatorname{chd}^{2}(A-B)
$$

$$
=2 \operatorname{chd}^{2} A+2 \operatorname{chd}^{2} B-\operatorname{chd}^{2} A \operatorname{chd}^{\circ} B
$$

$$
\text { or }=4 \text { vers } A+4 \text { vers } B-2 \text { vers } A 2 \text { vers } B
$$

$$
\text { or }=4-4 \cos A+4-4 \cos B
$$

$$
-4(1-\cos A-\cos B+\cos A \cos B)
$$

$$
=4-4 \cos A \cos B=4(1-\cos A \cos B)
$$

$$
\text { and } \operatorname{chd}^{2}(A+B)-\operatorname{chd}^{2}(A-B)
$$

$$
=\sqrt{\left(4 \operatorname{chd}^{2} A-\operatorname{chd}^{4} A\right)\left(4 \operatorname{chd}^{2} B-\operatorname{chd}^{4} B\right)}
$$

$$
\text { or }=\operatorname{chd} A \text { chd } B \sqrt{\left(4-\operatorname{chd}^{2} A\right)\left(4-\operatorname{chd}^{2} B\right)} ;
$$

$$
\text { or }=4 \sin A \sin B
$$

103. A more convenient expression for the chord of $(A \pm B)$ may easily be deduced from that found above.

$$
\begin{aligned}
& \text { Thus, } \operatorname{chd}^{2}(A \pm B) \\
& =\frac{1}{4}\left\{\operatorname{chd}^{2} A\left(4-\operatorname{chd}^{2} B\right)+\operatorname{chd}^{2} B\left(4-\operatorname{chd}^{2} A\right)\right. \\
& \left.\quad \pm \sqrt{\operatorname{chd}^{2} A\left(4-\operatorname{chd}^{2} A\right) \operatorname{chd}^{2} B\left(4-\operatorname{chd}^{2} B\right)}\right\} \\
& =\frac{1}{4}\left\{\operatorname{chd}^{2} A \operatorname{chd}^{2}(\pi-B)+\operatorname{chd}^{2} B \operatorname{chd}^{2}(\pi-A)\right. \\
& \left.\quad \pm 2 \sqrt{\operatorname{chd}^{2} A \operatorname{chd}^{2}(\pi-A) \operatorname{chd}^{2} B \operatorname{chd}^{2}(\pi-B)}\right\} \\
& =\frac{1}{4}\{\operatorname{chd} A \operatorname{chd}(\pi-B) \pm \operatorname{chd} B \operatorname{chd}(\pi-A)\}
\end{aligned}
$$

and $\therefore$ chd $(A \pm B)$

$$
=\frac{1}{2}\{\operatorname{chd} A \text { chd }(\pi-B) \pm \operatorname{chd} B \text { chd }(\pi-A)\}
$$

104. Hence, by the common operations of arithmetic, we have immediately,
chd $(A+B)+\operatorname{chd}(A-B)=\operatorname{chd} A$ chd $(\pi-B) ;$
$\operatorname{chd}(A+B)-\operatorname{chd}(A-B)=\operatorname{chd} B \operatorname{chd}(\pi-A) ;$
and chd $(A+B)$ chd $(A-B)$
$=\frac{1}{4}\left\{\operatorname{chd}^{2} A \operatorname{chd}^{2}(\pi-B)-\operatorname{chd}^{2} B \operatorname{chd}^{2}(\pi-A)\right\} ;$
or $=\frac{1}{4}\left\{\operatorname{chd}^{2} A\left(4-\operatorname{chd}^{2} B\right)-\operatorname{chd}^{2} B\left(4-\operatorname{chd}^{2} A\right)\right\}$
$=\operatorname{chd}^{2} A-\operatorname{chd}^{2} B=(\operatorname{chd} A+\operatorname{chd} B)(\operatorname{chd} A-\operatorname{chd} B)$.
105. In (101) if we suppose $B=A$, we shall obtain chd $2 A=\sqrt{4 \operatorname{chd}^{2} A-\operatorname{chd}^{4} A}=\operatorname{chd} A \sqrt{4-\operatorname{chd}^{2} A}$;

$$
\text { or }=\operatorname{chd} A \operatorname{chd}(\pi-A)
$$

Hence also, by the solution of a quadratic equation, $\operatorname{chd} \frac{A}{Q}=\sqrt{2-\sqrt{4-\operatorname{chd}^{2} A}}$ or $=\sqrt{Q-\operatorname{chd}(\pi-A)}$.

Ex. Let $A=90^{\circ}$; then we have chad $(\pi-A)=$ hd $\frac{\pi}{2}$ $=\sqrt{2}$, by (33); and therefore chad $45^{\circ}=\sqrt{2-\sqrt{ } 2}$.

By making $B$ successively equal to $2 A, 3 A$, \&c. the chords of $3 A, 4 A$, \&c. will be found in terms of chi $A$, and by the solutions of a cubic, biquadratic, \&ic. equation, chi $\frac{A}{3}$, chi $\frac{A}{4}$, Sc. may be expressed in terms of the same line.
106. As in the last article, we obtain the following results:
and therefore by substitution, we have

$$
\begin{aligned}
& \text { hd } \frac{A}{Q^{2}}=\sqrt{2-\sqrt{2+\operatorname{chd}(\pi-A)}} \\
& \text { chad } \frac{A}{Q^{5}}=\sqrt{2-\sqrt{2+\sqrt{2+\operatorname{chd}(\pi-A)}}} ;
\end{aligned}
$$

$$
\& c \ldots \ldots=\& c .
$$

$$
\begin{aligned}
& \text { che } \frac{A}{Q^{2}}=\sqrt{2-\sqrt{4-\operatorname{chd}^{2} \frac{\Lambda}{2}}} \\
& =\sqrt{2-\operatorname{chd}\left(\pi-\frac{A}{Q}\right)} ; \\
& \text { che } \frac{A}{2^{5}}=\sqrt{2-\sqrt{4-\operatorname{chd}^{2} \frac{A}{2^{5}}}} \\
& =\sqrt{2-\operatorname{chd}\left(\pi-\frac{A}{2^{2}}\right)} ; \\
& \& c . \ldots \ldots=\& c . . \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\
& \text { che } \begin{aligned}
\frac{A}{2^{n}} & =\sqrt{2-\sqrt{4-\operatorname{chd}^{2} \frac{A}{2^{n-1}}}} \\
& =\sqrt{2-\operatorname{chd}\left(\pi-\frac{A}{2^{n-1}}\right)} ;
\end{aligned}
\end{aligned}
$$

$$
\operatorname{chd} \frac{A}{2^{n}}=\sqrt{2-\sqrt{a+\sqrt{Q+\& c \ldots \cdot \sqrt{2+\operatorname{chd}(\pi-A})}}}
$$

in which the radical sign occurs $n$ times.
Ex. Let $A=\pi$; therefore $\operatorname{chd}(\pi-A)=\operatorname{chd} 0=0$, and

$$
\operatorname{chd} \frac{\pi}{2^{n}}=\sqrt{2-\sqrt{2+\sqrt{2+8 c \ldots \sqrt{2}}}}
$$

107. By pursuing the method pointed out in (100), the chords of $A$ and $B$ may be expressed in terms of the chords of $\left(\frac{A+B}{2}\right)$ and $\left(\frac{A-B}{2}\right)$; but as the results possess no elegance, and are at the same time, of little use, the operations in this case as well as those in the Article just alluded to, are omitted.
108. To express the tangents and co-tangents of the sum and difference of two arcs in terms of the tangents and cotangents of the arcs themselies.

From (4Q) we have

$$
\begin{aligned}
& \tan (A \pm B)=\frac{\sin (A \pm B)}{\cos (A \pm B)}=\frac{\sin A \cos B \pm \cos A \sin B}{\cos A \cos B \mp \sin A \sin B} \\
&=\frac{\frac{\sin A}{\cos A} \pm \frac{\sin B}{\cos B}}{1 \mp \frac{\sin A}{\cos A} \frac{\sin B}{\cos B}}, \text { as in }(69),=\frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}
\end{aligned}
$$

Again, from (47) we get
$\cot (A \pm B)=\frac{\cos (A \pm B)}{\sin (A \pm B)}=\frac{\cos A \cos B \mp \sin A \sin B}{\sin A \cos B \pm \cos A \sin B}$
$=\frac{\frac{\cos A}{\sin A} \frac{\cos B}{\sin B} \mp_{1}}{\frac{\cos B}{\sin B} \pm \frac{\cos A}{\sin A}}$, by a similar process, $=\frac{\cot A \cot B \mp_{1}}{\cot B \pm \cot A}$
109. Either of the expressions given in the last article, might have been deduced from the other.

Thus, $\cot (A \pm B)=\frac{1}{\tan (A \pm B)}$, by (47),

$$
\begin{aligned}
& =\frac{1}{\left(\frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}\right)}=\frac{1 \mp \tan A \tan B}{\tan A \pm \tan B} \\
& =\frac{1 \mp \frac{1}{\cot A} \frac{1}{\cot B}}{\frac{1}{\cot A} \pm \frac{1}{\cot B}}=\frac{\cot A \cot B \mp 1}{\cot B \pm \cot A} .
\end{aligned}
$$

Ex. 1. $\operatorname{Tan}\left(45^{\circ} \pm A\right)=\frac{\tan 45^{\circ} \pm \tan A}{1 \mp \tan 45^{\circ} \tan A}$

$$
=\frac{1 \pm \tan A}{1 \bar{\mp} \tan A}, \text { as appears from (4Q). }
$$

Ex. 2. Tan $\left(90^{\circ} \pm A\right)=\frac{\tan 90^{\circ} \pm \tan A}{1 \mp \tan 90^{\circ} \tan A}$ $=\frac{\infty \pm \tan A}{1 \mp \tan A}=\frac{1 \pm \frac{\tan A}{\infty}}{\frac{1}{\infty} \mp \tan A}=\mp \frac{1}{\tan A}$
$=\mp \cot A$, as is manifest from (39) and (47).
110. Cor. Let $T$ and $t$ be the tangents of any two arcs, $T^{\prime}$ and $t^{\prime}$ their co-tangents; then, using the kind of notation adopted in (66), we shall have

$$
\begin{aligned}
& \tan ^{-1} T \pm \tan ^{-1} t=\tan ^{-1}\left(\frac{T \pm t}{1 \mp T t}\right) \\
& \cot ^{-1} T^{\prime \prime} \pm \cot ^{-1} t^{\prime}=\cot ^{-1}\left(\frac{T^{\prime} t^{\prime} \mp 1}{t^{\prime} \pm T^{\prime}}\right)
\end{aligned}
$$

Ex. 1. $\operatorname{Tan}^{-1} \frac{1}{2}+\tan ^{-1} \frac{1}{3}=\tan ^{-1}\left\{\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2} \frac{1}{3}}\right\}$
$=\tan ^{-1}(1)=45^{\circ}$, from (42).
$=\cot ^{-1}(-1)=-45^{\circ}$ or $=135^{\circ}$, from (47) and (46).

$$
\begin{aligned}
& \text { Ex. 3. } \operatorname{Tan}^{-1} \frac{1}{3}+\tan ^{-1} \frac{1}{5}+\tan ^{-1} \frac{1}{7}+\tan ^{-1} \frac{1}{8} \\
& =\left\{\tan ^{-1} \frac{1}{3}+\tan ^{-1} \frac{1}{5}\right\}+\left\{\tan ^{-1} \frac{1}{7}+\tan ^{-1} \frac{1}{8}\right\} \\
& =\tan ^{-1} \frac{4}{7}+\tan ^{-1} \frac{3}{11}=\tan ^{-1}(1)=45^{\circ} .
\end{aligned}
$$

111. To express the tangents and co-tangents of two arcs in terms of the tangents and co-tangents of their semi-sum and semi-difference.

$$
\begin{aligned}
\text { Since } A & =\left(\frac{A+B}{2}\right)+\left(\frac{A-B}{2}\right) \\
\text { and } B & =\left(\frac{A+B}{2}\right)-\left(\frac{A-B}{2}\right)
\end{aligned}
$$

from article (108) we have

$$
\tan A=\frac{\tan \left(\frac{A+B}{2}\right)+\tan \left(\frac{A-B}{2}\right)}{1-\tan \left(\frac{A+B}{2}\right) \tan \left(\frac{A+B}{2}\right)}
$$

$$
\begin{aligned}
& \tan B=\frac{\tan \left(\frac{A+B}{2}\right)-\tan \left(\frac{A-B}{2}\right)}{1+\tan \left(\frac{A+B}{2}\right) \tan \left(\frac{A+B}{2}\right)} \\
& \cot A=\frac{\cot \left(\frac{A+B}{2}\right) \cot \left(\frac{A-B}{2}\right)-1}{\cot \left(\frac{A-B}{2}\right)+\cot \left(\frac{A+B}{2}\right)} \\
& \cot B=\frac{\cot \left(\frac{A+B}{2}\right) \cot \left(\frac{A-B}{2}\right)+1}{\cot \left(\frac{A-B}{2}\right)-\cot \left(\frac{A+B}{2}\right)}
\end{aligned}
$$

112. To express the tangent and co-tangent of twice an arc in terms of the tangent and co-tangent of the arc itself.

Here, $\tan \mathcal{Q} A=\tan (A+A)=\frac{\tan A+\tan A}{1-\tan A \tan A}$, from (108),

$$
\begin{aligned}
& =\frac{2 \tan A}{1-\tan ^{2} A}=\frac{2}{\frac{1}{\tan A}-\tan A} \\
& =\frac{2 \cot A}{\cot ^{2} A-1}=\frac{2}{\cot A-\tan A} .
\end{aligned}
$$

Also, $\cot 2 A=\cot (A+A)=\frac{\cot A \cot A-1}{\cot A+\cot A}$, from (108),

$$
\begin{aligned}
& =\frac{\cot ^{2} A-1}{\mathcal{Q} \cot A}=\frac{\cot A}{\mathcal{Q}}-\frac{1}{\mathcal{Q} \cot A} \\
& =\frac{1-\tan ^{2} A}{2 \tan A}=\frac{\cot A-\tan A}{Q}
\end{aligned}
$$

113. To express the tangent and co-tangent of half an arc in terms of the tangent and co-tangent of the arc itself.

By substituting in the expressions found in the last article, $A$ and $\frac{A}{2}$ in the places of $2 A$ and $A$ respectively, we shall have

$$
\tan A=\frac{2 \tan \frac{A}{2}}{1-\tan ^{2} \frac{A}{2}}, \text { and } \cot A=\frac{\cot ^{2} \frac{A}{2}-1}{2 \cot \frac{A}{2}}:
$$

the former of which gives

$$
\begin{gathered}
\tan ^{2} \frac{A}{2}+\frac{2}{\tan A} \tan \frac{A}{2}=1 \\
\text { and } \therefore \tan \frac{A}{2}=\frac{-1 \pm \sqrt{1+\tan ^{2} A}}{\tan A} \\
\text { or }=\frac{-1 \pm \sec A}{\tan A} ; \text { or }= \pm \operatorname{cosec} A-\cot A ;
\end{gathered}
$$

and from the latter we get, $\cot ^{2} \frac{A}{2}-2 \cot \Lambda \cot \frac{A}{2}=1$,

$$
\begin{aligned}
& \text { whence, } \cot \frac{\Lambda}{2}=\cot A \pm \sqrt{1+\cot ^{2} A} ; \\
& \text { or }=\cot A \pm \operatorname{cosec} A .
\end{aligned}
$$

Ex. 1: Let $A=90^{\circ} ; \therefore \tan A=\infty$, and $\cot A=0 ;$ and from the equations above given we get

$$
\tan 45^{\circ}=1=\cot 45^{\circ}
$$

Ex. 2. Let $A=45^{\circ} ; \therefore \tan A=1=\cot A$;
whence, $\tan 22^{\circ} 30^{\prime}=\frac{-1+\sqrt{1+\tan ^{2} 45^{\circ}}}{\tan 45^{\circ}}=\sqrt{ } 2-1=\cot 67^{\circ} 30^{\prime}$; and $\cot 22^{\circ} 30^{\prime}=\cot 45^{\circ}+\sqrt{1+\cot ^{2} 45^{\circ}}=\sqrt{ } 2+1=\tan 67^{\circ} 30^{\prime}$.
114. To express the tangent and co-tangent of thrice an arc in terms of the tangent and colangent of the arc itself.

As before we have,

$$
\begin{gathered}
\tan 3 A=\tan (2 A+A) \\
=\frac{\tan 2 A+\tan A}{1-\tan 2 A \tan A}, \text { by (108), } \\
=\frac{\left(\frac{2 \tan A}{1-\tan ^{2} A}\right)+\tan A}{1-\left(\frac{2 \tan A}{1-\tan ^{2} A}\right) \tan A}, \text { by }(112),=\frac{3 \tan A-\tan ^{3} A}{1-3 \tan ^{2} A} \\
=\frac{\tan ^{3} A-3 \tan A}{3 \tan ^{2} A-1}=\frac{3 \cot ^{2} A-1}{\cot ^{3} A-3 \cot A}=\frac{1-3 \cot ^{2} A}{3 \cot A-\cot ^{3} A} ;
\end{gathered}
$$

$$
\text { also, } \cot 3 A=\cot (2 A+A)
$$

$$
=\frac{\cot 2 A \cot A-1}{\cot 2 A+\cot A}, \text { by (108), }
$$

$$
=\frac{\left(\frac{\cot ^{2} A-1}{2 \cot A}\right) \cot A-1}{\left(\frac{\cot ^{2} A-1}{2 \cot A}\right)+\cot A}, \text { by }(112),=\frac{\cot ^{3} A-3 \cot A}{3 \cot ^{2} A-1}
$$

$$
=\frac{3 \cot A-\cot ^{3} A}{1-3 \cot ^{2} A}=\frac{1-3 \tan ^{2} A}{3 \tan A-\tan ^{3} A}=\frac{3 \tan ^{2} A-1}{\tan ^{3} A-3 \tan A} .
$$

115. By substitutions similar to those used in some of the preceding articles, we readily obtain,

$$
\tan A=\frac{3 \tan \frac{A}{3}-\tan ^{3} \frac{A}{3}}{1-3 \tan ^{2} \frac{A}{3}}, \text { and } \cot A=\frac{3 \cot \frac{A}{3}-\cot ^{3} \frac{A}{3}}{1-3 \cot ^{2} \frac{A}{3}}:
$$

and thence the equations,

$$
\tan ^{3} \frac{A}{3}-3 \tan A \tan ^{2} \frac{A}{3}-3 \tan \frac{A}{3}+\tan A=0
$$

and $\cot ^{3} \frac{A}{3}-3 \cot A \cot ^{2} \frac{A}{3}-3 \cot \frac{A}{3}+\cot A=0 ;$
by the solution of which, $\tan \frac{A}{3}$ and $\cot \frac{A}{3}$ will be expressed in terms of $\tan A$ and $\cot A$ respectively.

Ex. 1. If $A=90^{\circ}$, we have $\tan A=\infty$, and $\frac{A}{3}=30^{\circ}$; therefore, $1-3 \tan ^{2} 30^{\circ}=0$, and $\tan 30^{\circ}=\frac{1}{\sqrt{3}}=\cot 60^{\circ}$.

Ex. 2. Let $A=180^{\circ}$; then $\tan A=0$, and $\frac{A}{3}=60^{\circ}$; therefore, $\tan ^{3} 60^{\circ}-3 \tan 60^{\circ}=0$, and $\tan 60^{\circ}=\sqrt{ } 3=\cot 30^{\circ}$.
116. The method used in the last article to determine the tangent and cotangent of $3 A$, may be applied to express the tangents and cotangents of $4 A, 5 A, 6 A, \& c$. in terms of $\tan A$ and $\cot A$; and there will result

$$
\tan 4 A=\frac{4 \tan A-4 \tan ^{3} A}{1-6 \tan ^{2} A+\tan ^{4} A}
$$

$$
\tan 5 A=\frac{5 \tan A-10 \tan ^{3} A+\tan ^{5} A}{1-10 \tan ^{2} A+5 \tan ^{4} A} ;
$$

$$
\tan 6 A=\frac{6 \tan A-20 \tan ^{3} A+6 \tan ^{5} A}{1-15 \tan ^{2} A+15 \tan ^{4} A-\tan ^{6} A} ;
$$

$$
\& c . \ldots . .=\& c .
$$

$$
\begin{aligned}
& \cot 4 A=\frac{\cot ^{4} A-6 \cot ^{2} A+1}{4 \cot ^{3} A-4 \cot A} ; \\
& \cot 5 A=\frac{\cot ^{5} A-10 \cot ^{3} A+5 \cot A}{5 \cot ^{4} A-10 \cot ^{2} A+1} ; \\
& \cot 6 A=\frac{\cot ^{6} A-15 \cot ^{4} A+15 \cot ^{2} A-1}{6 \cot ^{5} A-20 \cot ^{3} A+6 \cot A} ; \\
& \& c . \ldots .=\& c .
\end{aligned}
$$

117. To express the secants and cosecants of the sum and difference of two arcs in terms of the secants and cosecants of the arcs themselves.

From (52) we have

$$
\begin{gathered}
\sec (A \pm B)=\frac{1}{\cos (A \pm B)}=\frac{1}{\cos A \cos B \mp \sin A \sin B} \\
=\frac{\frac{1}{\cos A} \cdot \frac{1}{\cos B}}{\frac{\cos A}{\cos A}} \frac{\cos B}{\cos B} \mp \frac{\sin A}{\cos A} \frac{\sin B}{\cos B}
\end{gathered} \text {, as in (69), }=\frac{\sec A \sec B}{1 \mp \tan A \tan B}
$$

also, from (57)
$\operatorname{cosec}(A \pm B)=\frac{1}{\sin (A \pm B)}=\frac{1}{\sin A \cos B \pm \cos A \sin B}$
$=\frac{\frac{1}{\sin A} \frac{1}{\sin B}}{\frac{\sin A}{\sin A} \frac{\cos B}{\sin B} \pm \frac{\cos A}{\sin A} \frac{\sin B}{\sin B}}$, similarly,$=\frac{\operatorname{cosec} A \operatorname{cosec} B}{\cot B \pm \cot A}$

$$
=\frac{\operatorname{cosec} A \operatorname{cosec} B}{\sqrt{\operatorname{cosec}^{2} B-1} \pm \sqrt{\operatorname{cosec}^{2} A-1}}, \text { by }(57)
$$

118. The functions in the last article may be expressed in terms somewhat different.

Thus,

$$
\begin{aligned}
\sec (A \pm B) & =\frac{1}{\cos (A \pm B)}=\frac{1}{\cos A \cos B \mp \sin A \sin B} \\
& =\frac{1}{\frac{1}{\sec A} \frac{1}{\sec B} \mp \frac{1}{\operatorname{cosec} A} \frac{1}{\operatorname{cosec} B}} \\
& =\frac{\sec A \sec B \operatorname{cosec} A \operatorname{cosec} B}{\operatorname{cosec} A \operatorname{cosec} B \mp \sec A \sec B}
\end{aligned}
$$

and

$$
\begin{aligned}
\operatorname{cosec}(A \pm B) & =\frac{1}{\sin (A \pm B)}=\frac{1}{\sin A \cos B \pm \cos A \sin B} \\
& =\frac{1}{\frac{1}{\operatorname{cosec} A} \frac{1}{\sec B} \pm \frac{1}{\sec A} \frac{1}{\operatorname{cosec} B}} \\
& =\frac{\operatorname{cosec} A \operatorname{cosec} B \sec A \sec B}{\operatorname{cosec} B \sec A \pm \operatorname{cosec} A \sec B}
\end{aligned}
$$

119. By means of the substitutions used in articles (72) and (107), the secants and cosecants of two arcs are expressed in terms of the secants and cosecants of their semi-sum and semi-difference.
120. T'o express the secant and cosecant of twice an arc in terms of the secant and cosecant of the arc itself.

Here, from (117), we have
$\sec 2 A=\sec (A+A)=\frac{\sec A \sec A}{1-\sec ^{2} A+1}=\frac{\sec ^{2} A}{2-\sec ^{2} A}$;
and

$$
\begin{aligned}
\operatorname{cosec} 2 A & =\operatorname{cosec}(A+A)=\frac{\operatorname{cosec} A \operatorname{cosec} A}{2 \sqrt{\operatorname{cosec}^{2} A-1}} \\
& =\frac{\operatorname{cosec}^{2} A}{2 \sqrt{\operatorname{cosec}^{2} A-1}} .
\end{aligned}
$$

Or thus, by (118), we get

$$
\sec 2 A=\frac{\sec A \sec A \operatorname{cosec} A \operatorname{cosec} A}{\operatorname{cosec} A \operatorname{cosec} A-\sec A \sec A}
$$

and

$$
\begin{aligned}
\operatorname{cosec} 2 A & =\frac{\operatorname{cosec} A \operatorname{cosec} A \sec A \sec A}{\operatorname{cosec} A \sec A+\operatorname{cosec} A \sec A} \\
& =\frac{\operatorname{cosec}^{2} A \sec ^{2} A}{2 \operatorname{cosec} A \sec A}=\frac{1}{2} \operatorname{cosec} A \sec A .
\end{aligned}
$$

121. To express the secant and cosecant of half an arc in terms of the secant and cosecant of the arc itself:

By the requisite substitutions in the last article we obtain

$$
\sec \frac{A}{2}= \pm \sqrt{\frac{2 \sec A}{1+\sec A}}
$$

and

$$
\operatorname{cosec} \frac{A}{2}= \pm \sqrt{2 \operatorname{cosec}^{2} A \pm 2 \operatorname{cosec} A \sqrt{\operatorname{cosec}^{2} A-1}}
$$

Ex. Let $A=90^{\circ}$; therefore since sec $A=\infty, \operatorname{cosec} A=1$, and $\frac{A}{2}=45^{\circ}$, we shall have

$$
\sec 45^{\circ}=V 2=\operatorname{cosec} 45^{\circ}
$$

122. To express the secant and cosecant of thrice an arc in terms of the secant and cosecant of the arc itself.

From (117) we have

$$
\begin{aligned}
\sec 3 A & =\sec (2 A+A) \\
& =\frac{\sec 2 A \sec A}{1-\sqrt{\left(\sec ^{2} 2 A-1\right)\left(\sec ^{2} A-1\right)}} \\
& =\frac{\sec ^{3} A}{4-3 \sec ^{2} A}, \text { by substitution and reduction } ;
\end{aligned}
$$

also,

$$
\begin{aligned}
\operatorname{cosec} 3 A & =\operatorname{cosec}(2 A+A) \\
& =\frac{\operatorname{cosec} 2 A \operatorname{cosec} A}{\sqrt{\operatorname{cosec}^{2} A-1}+\sqrt{\operatorname{cosec}^{2} Q A-1}} \\
& =\frac{\operatorname{cosec}^{5} A}{3 \operatorname{cosec}^{2} A-4}, \text { by the same process. }
\end{aligned}
$$

123. Let $A$ and $\frac{A}{3}$ be put for $3 A$ and $A$ respectively in the last article, and we get

$$
\begin{aligned}
\sec A & =\frac{\sec ^{3} \frac{A}{3}}{4-3 \sec ^{2} \frac{A}{3}} \\
\text { and } \operatorname{cosec} A & =\frac{\operatorname{cosec}^{3} \frac{A}{3}}{3 \operatorname{cosec}^{2} \frac{A}{3}-4}
\end{aligned}
$$

which give the following equations,

$$
\sec ^{3} \frac{A}{3}+3 \sec A \sec ^{2} \frac{A}{3}-4 \sec A=0
$$

and $\operatorname{cosec}^{3} \frac{A}{3}-3 \operatorname{cosec} \Lambda \operatorname{cosec}^{2} \frac{A}{3}+4 \operatorname{cosec} A=0 ;$
by means of which the secant and cosecant of $\frac{A}{3}$ are expressed in terms of the secant and cosecant of $A$.

Ex. 1. Let $A=90^{\circ}$; then $\sec A=\infty, \operatorname{cosec} A=1$, and $\frac{A}{3}=30^{\circ}$;

$$
\begin{aligned}
& \text { therefore } 3 \sec ^{2} 30^{\circ}-4=0 \\
& \text { and } \sec 30=\frac{2}{\sqrt{3}}=\operatorname{cosec} 60^{\circ}
\end{aligned}
$$

Ex. 2. Let $A=180^{\circ}$; therefore $\sec A=-1, \operatorname{cosec}$ $A=\infty$, and $\frac{A}{3}=60$;

$$
\text { hence } \sec ^{3} 60^{\circ}-3 \sec ^{2} 60^{\circ}+4=0
$$

from which we obtain $\sec 60^{\circ}=2=\operatorname{cosec} 30^{\circ}$.
It may be observed that the formulæ in these two articles might have been deduced from those in (83) and (84), by means of (52) and (57); and by continuing the process we should in a similar manner obtain the values of

$$
\sec 4 A, \operatorname{cosec} 4 A, \& c \cdot \sec \frac{A}{4}, \operatorname{cosec} \frac{A}{4}, \& c
$$

124. To express the sine and cosine of the sum of three arcs in terms of the sines and cosines of the arcs themselves.

By considering the sum of two of the proposed arcs as one, we have

$$
\begin{aligned}
\sin & (A+B+C)=\sin \{(A+B)+C\} \\
= & \sin (A+B) \cos C+\cos (A+B) \sin C \\
= & (\sin A \cos B+\cos A \sin B) \cos C \\
& +(\cos A \cos B-\sin A \sin B) \sin C, \text { by }(63), \\
= & \sin A \cos B \cos C+\sin B \cos A \cos C \\
& +\sin C \cos A \cos B-\sin A \sin B \sin C
\end{aligned}
$$

> and $\cos (A+B+C)=\cos \{(A+B)+C\}$ $=\cos (A+B) \cos C-\sin (A+B) \sin C$ $=(\cos A \cos B-\sin A \sin B) \cos C$ $\quad-(\sin A \cos B+\cos A \sin B) \sin C$, by $(63)$, $=$ $=\cos A \cos B \cos C-\cos A \sin B \sin C$ $\quad-\cos B \sin A \sin C-\cos C \sin A \sin B$

Ex. 1. If we have $(A+B+C)=2 n \frac{\pi}{2}$, or $n \pi$, then will $\sin A \sin B \sin C=\sin A \cos B \cos C$ $+\sin B \cos A \cos C+\sin C \cos A \cos B$.

Ex. 2. If $A+B+C=(2 n-1) \frac{\pi}{2}$, we shall have $\cos A \cos B \cos C=\cos A \sin B \sin C$ $+\cos B \sin A \sin C+\cos C \sin A \sin B$.
125. Cor. Let $A=B=C$, then the formulæ in the last article become

$$
\sin 3 A=3 \sin A \cos ^{2} A-\sin ^{3} A=3 \sin A-4 \sin ^{3} A,
$$

and $\cos 3 A=\cos ^{5} A-3 \cos A \sin ^{2} A=4 \cos ^{5} A-3 \cos A$; which have been already proved in (83).
126. By a process similar to that used in (124), we may prove that
$\sin (A+B) \sin (B+C)=\sin A \sin C+\sin B \sin (A+B+C)$.
For,

$$
\sin (A+B+C)=\sin A \cos (B+C)+\cos A \sin (B+C)
$$

$=\sin A \cos B \cos C-\sin A \sin B \sin C+\cos A \sin (B+C) ;$

$$
\begin{aligned}
& \quad \therefore \sin B \sin (A+B+C)+\sin A \sin C \\
& =\sin A \cos B \cos C \sin B+\sin A \sin C\left(1-\sin ^{2} B\right) \\
& \quad+\cos A \sin B \sin (B+C) \\
& =\sin A \cos B(\sin B \cos C+\cos B \sin C) \\
& \quad \quad+\cos A \sin B \sin (B+C) \\
& =\sin A \cos B \sin (B+C)+\cos A \sin B \sin (B+C) \\
& =(\sin A \cos B+\cos A \sin B) \sin (B+C) \\
& =\sin (A+B) \sin (B+C) . \\
& \text { Similarly, } \sin (A-B) \sin (C-B)=\sin A \sin C \\
& \quad \quad-\sin B \sin (A-B+C) .
\end{aligned}
$$

Ex. 1. Let $A+B+C=\pi$; then $\sin (A+B+C)=0$, and it follows that

$$
\sin (A+B) \sin (B+C)=\sin A \sin C
$$

Ex. 2. Let $A-B+C=0$; then $\sin (A-B+C)=0$, and we have

$$
\sin (A-B) \sin (C-B)=\sin A \sin C
$$

127. From (93) and (101) the versed sine and chord of $(A+B+C)$ are obtained after the same manner.
128. To express the tangent and cotangent of the sum of three arcs in terms of the tangents and cotangents of the arcs themselves.

Proceeding as in (124) we have

$$
\begin{aligned}
& \tan (A+B+C)=\tan \{(A+B)+C\} \\
= & \frac{\tan (A+B)+\tan C}{1-\tan (A+B) \tan C}, \text { by (108), } \\
= & \frac{\left(\frac{\tan A+\tan B}{1-\tan A \tan B}\right)+\tan C}{1-\left(\frac{\tan A+\tan B}{1-\tan A \tan B}\right) \tan C}, \text { by (108), }
\end{aligned}
$$

$$
\begin{aligned}
& =\frac{\tan A+\tan B+\tan C-\tan A \tan B \tan C}{1-(\tan A \tan B+\tan A \tan C+\tan B \tan C)} \\
& \text { and } \cot (A+B+C)=\cot \{(A+B)+C)\} \\
& =\frac{\cot (A+B) \cot C-1}{\cot (A+B)+\cot C}, \text { by }(108) \\
& =\frac{\left(\frac{\cot A \cot B-1}{\cot A+\cot B}\right) \cot C-1}{\left(\frac{\cot A \cot B-1}{\cot A+\cot B}\right)+\cot C}, \text { by (108), } \\
& =\frac{\cot A \cot B \cot C-(\cot A+\cot B+\cot C)}{\cot A \cot B+\cot A \cot C+\cot B \cot C-1}
\end{aligned}
$$

Ex. 1. Let $(A+B+C)=(2 n-1) \frac{\pi}{2}$; therefore $\tan (A+B+C)=\infty$, and $\cot (A+B+C)=0$; and hence $\tan A \tan B+\tan A \tan C+\tan B \tan C=1 ;$
also, $\cot A \cot B \cot C=\cot A+\cot B+\cot C$.
Ex.2. Let $(A+B+C)=2 n \frac{\pi}{2}=n \pi$; then $\tan (A+B+C)=0$, and $\cot (A+B+C)=\infty ;$
therefore $\tan A+\tan B+\tan C=\tan A \tan B \tan C$; and $\cot A \cot B+\cot A \cot C+\cot B \cot C=1$.
129. Cor. In the last article, suppose $A=B=C$; then $\tan 3 A=\frac{3 \tan A-\tan ^{3} A}{1-3 \tan ^{2} A}$, and $\cot 3 A=\frac{3 \cot A-\cot ^{5} A}{1-3 \cot ^{2} A} ;$ which are the formulæ proved in (114).
130. The same method leads to expressions for the secant and cosecant of $(A+B+C)$ in terms of the secants and cosecants of $A, B$ and $C$.
131. By separating the arcs into two parts as has been done in some of the preceding articles, we are enabled to determine the sine, cosine, \&c. of the sums of 4,5 , \&c. $n$ arcs: but as the methods are so very simple, notwithstanding the prolixity of some of the results, we shall not pursue the subject further in this place except to notice a curious property of the tangent and cotangent of the sum of any number of arcs, which shall be the subject of most of the remainiug articles of this chapter.
132. If $S_{1}$ denote the sum of the tangents of $n$ arcs, $A, B, C, D$, \&c. $K, L ; S_{2}$ the sum of their products taken two and two together; $S_{3}$ the sum of their products taken three and three together; and so on : then will

$$
\begin{aligned}
& \tan (A+B+C+\& c \cdot+K+L)=\frac{S_{1}-S_{3}+S_{5}-\& c}{1-S_{2}+S_{4}-\& c} \\
& \text { For, by }(108), \tan (A+B)=\frac{\tan A+\tan B}{1-\tan A \tan B}=\frac{S_{1}}{1-S_{2}}
\end{aligned}
$$

again, by (128), $\tan (A+B+C)$

$$
=\frac{\tan A+\tan B+\tan C-\tan A \tan B \tan C}{1-(\tan A \tan B+\tan A \tan C+\tan B \tan C)}=\frac{S_{1}-S_{3}}{1-S_{2}}
$$

and so on: and generally, if

$$
\tan (A+B+C+\& \mathrm{c} .+K)=\frac{S_{1}-S_{3}+S_{5}-\& \mathrm{cc}}{1-S_{2}+S_{4}-\& \mathrm{c} .}
$$

we shall have, from (108),

$$
\begin{aligned}
& \tan (A+B+C+\& c .+K+L) \\
&= \frac{\tan (A+B+C+\& c .+K)+\tan L}{1-\tan (A+B+C+\& c .+K) \tan L} \\
&= \frac{\left(\frac{S_{1}-S_{3}+S_{5}-\& c .}{1-S_{2}+S_{4}-\& c}\right)+\tan L}{1-\left(\frac{S_{1}-S_{3}+S_{5}-\& c .}{1-S_{2}+S_{4}-\delta c .}\right) \tan L}
\end{aligned}
$$

$$
\begin{aligned}
& =\frac{\left(S_{1}-S_{3}+S_{5}-\& c .\right)+\left(1-S_{2}+S_{4}-\& c .\right) \tan L}{\left(1-S_{2}+S_{4}-\& c .\right)-\left(S_{1}-S_{3}+S_{5}-\& c .\right) \tan L} \\
& =\frac{\left(S_{1}+\tan L\right)-\left(S_{3}+S_{2} \tan L\right)+\left(S_{5}+S_{4} \tan L\right)-\& c .}{1-\left(S_{2}+S_{1} \tan L\right)+\left(S_{4}+S_{3} \tan L\right)-\& c .}
\end{aligned}
$$

which expression is manifestly formed after the same law as the preceding one:

Therefore, if the form be true for the tangent of the sum of $n-1$ arcs, it will also be true for the tangent of the sum of $n$ arcs. Now it has been shewn that the law obtains for the tangents of the sums of two and three arcs: hence it obtains also for the tangents of the sums of $4,5, \& c$. arcs; that is, generally for the tangent of the sum of $n$ arcs.

Ex. 1. If $(A+B+C+\& c .+K+L)=2 n \frac{\pi}{2}$, or $n \pi$, we have

$$
0=\frac{S_{1}-S_{3}+S_{5}-\& c}{1-S_{2}+S_{4}-\& c}
$$

and therefore $S_{1}+S_{5}+\& c \ldots=S_{3}+S_{7}+\& c$.
Ex. 2. If $(A+B+C+\& \mathrm{c} .+K+L)=(2 n-1) \frac{\pi}{2}$, we have

$$
\infty=\frac{S_{1}-S_{3}+S_{5}-\& c}{1-S_{2}+S_{4}-\& c}
$$

and thence $1+S_{4}+8 c .=S_{2}+S_{6}+\& c$.
133. Cor. Supposing $A=B=C=\& c$. to $n$ terms, we shall manifestly have

$$
\frac{n \tan A-n\left(\frac{n-1}{2}\right)\left(\frac{n-2}{3}\right) \tan ^{3} A+\& \mathrm{c}}{1-n\left(\frac{n-1}{2}\right) \tan ^{2} A+\& \mathrm{c} .}
$$

134. In the last article but one, let $A=\frac{\pi}{2}-A^{\prime}$,

$$
\begin{gathered}
B=\frac{\pi}{2}-B^{\prime}, C=\frac{\pi}{2}-C^{\prime}, \& \mathrm{c} .=\& \mathrm{c} ., \text { then will } \\
A+B+C+\& \mathrm{c} . \text { to } n \text { terms } \\
=\frac{n \pi}{2}-\left\{A^{\prime}+B^{\prime}+C^{\prime}+\& \mathrm{c} . \text { to } n \text { terms }\right\}
\end{gathered}
$$

and $\tan A=\cot A^{\prime}, \tan B=\cot B^{\prime}, \tan C=\cot C^{\prime}, \& c .=\& c$. also, on this hypothesis, we have
$S_{1}=$ the sum of the cotangents of $A^{\prime}, B^{\prime}, C^{\prime}, \& c$.
$S_{2}=$ the sum of their products taken two and two together; $\& c .=\& c$.

Hence if $n$ be even,

$$
\begin{gathered}
\cot \left(A^{\prime}+B^{\prime}+C^{\prime}+\& c .\right)=\cot \left\{\frac{n \pi}{2}-(A+B+C+\& \mathrm{c} .)\right\} \\
=-\frac{1}{\tan (A+B+C+\& c .)}=-\frac{1-S_{2}+S_{4}-\& c .}{S_{1}-S_{3}+S_{5}-\& c}
\end{gathered}
$$

and if $u$ be odd,

$$
\begin{gathered}
\cot \left(A^{\prime}+B^{\prime}+C^{\prime}+\& c .\right)=\cot \left\{\frac{n \pi}{2}-(A+B+C+\& c .)\right\} \\
=\tan (A+B+C+\& c .)=\frac{S_{1}-S_{3}+S_{5}-\& c}{1-S_{2}+S_{4}-\& c}
\end{gathered}
$$

135. Cor. If $A^{\prime}=B^{\prime}=C^{\prime}=\& c$. to $n$ terms, we shall have, when $n$ is even,

$$
\cot n A^{\prime}=-\frac{1-n\left(\frac{n-1}{2}\right) \cot ^{2} A^{\prime}+\& c .}{n \cot A^{\prime}-n\left(\frac{n-1}{2}\right)\left(\frac{n-2}{3}\right) \cot ^{3} A^{\prime}+\& c}
$$

and when $n$ is odd,

$$
\cot n A^{\prime}=\frac{n \cot A^{\prime}-n\left(\frac{n-1}{2}\right)\left(\frac{n-2}{3}\right) \cot ^{3} A^{\prime}+\& c .}{1-n\left(\frac{n-1}{2}\right) \cot ^{2} A^{\prime}+\& c .}
$$

136. In the various articles of this chapter, the trigonometrical functions of $(A+B)$ and $(A-B)$ have each been deduced by a separate process; but this is unnecessary, for in fact the corresponding functions of both are contained in the same expressions.

Thus, if we put $-B$ in the place of $B$, and $-\sin B,-\tan B$, $-\cot B,-\operatorname{cosec} B$ in the places of $\sin B, \tan B, \cot B$, and $\operatorname{cosec} B$ respectively, the rest remaining unchanged agreeably to what has been proved in Chap. 1; any trigonometrical function of either $(A+B)$ or $(A-B)$ will be changed into the corresponding one of the other. Thus,

$$
\text { since } \sin (A+B)=\sin A \cos B+\cos A \sin B
$$

by changing $B$ into $-B$, and $\sin B$ into $-\sin B$, we have

$$
\sin (A-B)=\sin A \cos B-\cos A \sin B:
$$

again, because $\tan (A-B)=\frac{\tan A-\tan B}{1+\tan A \tan B} ;$
$\therefore$ by putting $-B$ for $B$, and $-\tan B$ for $\tan B$, we get

$$
\tan (A+B)=\frac{\tan A+\tan B}{1-\tan A \tan B} ; \text { and so on. }
$$

## CHAP. III.

On the computation of the sines, cosines, \&c. of one, two, three, \&c. minutes, and succeeding arcs, and on the construction of the Trigonometrical Canon. On the uses of Formula of Verification. On the Logarithmic sines, cosines, \&c. of arcs. On the ratio of the circumference of a circle to its diameter, $\& c$.
137. To express the sine and cosine of one minute in terms of the radius 1 .

In the last chapter at (81), it has been proved that

$$
\begin{aligned}
\sin \frac{A}{2} & =\sqrt{\frac{1}{2}-\frac{1}{2} \sqrt{1-\sin ^{2} A}}, \\
\text { and } \sin \frac{A}{2^{n}} & =\sqrt{\frac{1}{2}-\frac{1}{2} \sqrt{1-\sin ^{2} \frac{A}{Q^{n-1}}}}:
\end{aligned}
$$

if $\therefore A=30^{\circ}$, we have $\sin A=\frac{1}{\Omega}$, from (37), and thence

$$
\begin{aligned}
& \sin 15^{\circ}=\frac{1}{2} \sqrt{2-\sqrt{ } 3}=.2588190 \mathrm{Kc} . \\
& \sin 7^{\circ} 30^{\prime}=\frac{1}{2} \sqrt{\frac{4-\sqrt{2}-\sqrt{6}}{2}}=.1305262 \& c . \\
& \& c . . .=\& c \ldots \ldots . \ldots \ldots \ldots . . .
\end{aligned}
$$

and it is manifest that by this process we shall obtain successively the sines of $3^{0} 45^{\prime}, 1^{0} 52^{\prime} 30^{\prime \prime}$, \&c.

Now, at the end of the tenth division from $30^{\circ}$, the arc becomes $1^{\prime} 45^{\prime \prime} 28^{\prime \prime \prime} 7^{\prime \prime \prime \prime} 30^{\prime \prime \prime \prime \prime}$ and its sine .0005113269 \&c.; also at the end of the eleventh, the arc becomes $52^{\prime \prime} 44^{\prime \prime \prime} 3^{\prime \prime \prime \prime} 45^{\prime \prime \prime \prime \prime}$ and its sine .0002556634 \&c.: from which it appears, that wheu the operation above-mentioned has been repeated so many times, the sine of the arc is halved at the same time that the arc itself is bisected; that is, the sines become then proportional to the arcs: hence

$$
\begin{array}{r}
\sin 52^{\prime \prime} 44^{\prime \prime \prime} 3^{\prime \prime \prime \prime} 45^{\prime \prime \prime \prime \prime}: \sin 1^{\prime}:: 52^{\prime \prime} 4^{\prime \prime \prime} 3^{\prime \prime \prime \prime} 45^{\prime \prime \prime \prime \prime}: 1^{\prime} \\
:: .0002556634 \& c .: .0002908882 \& c .
\end{array}
$$

and therefore $\sin 1^{\prime}=.0002908882 \& c$.

$$
\text { also, } \cos 1^{\prime}=\sqrt{1-\sin ^{2} 1^{\prime}}=.999999957 \& c
$$

138. Cor. From what has been proved in the preceding article, it is clear that the sine of any number $u$ of seconds may be obtained simply by a proportion. Thus,

$$
\sin n^{\prime \prime}: \sin 1^{\prime}:: n^{\prime \prime}: 1^{\prime}:: n: 60,
$$

and therefore $\sin n^{\prime \prime}=\frac{n}{60} \sin 1^{\prime}$; and the cosine may be determined by means of the equation, $\cos A=\sqrt{1-\sin ^{2} A}$, as before.
139. To express the sine and cosine of $2,3,4,5, \& c$. minutes in terms of the radius 1 .

From (73), we have the equation

$$
\sin (n+1) A=2 \cos A \sin n A-\sin (n-1) A
$$

and if in this we suppose $A$ to be $1^{\prime}$, and $n$ to be taken equal to the numbers $1,2,3,4 \& c$. successively, we get

$$
\begin{aligned}
\sin 2^{\prime} & =2 \cos 1^{\prime} \sin 1^{\prime} \\
& =.0005817764 \& c .=\cos 89^{\circ} 58^{\prime}
\end{aligned}
$$

$$
\begin{aligned}
& \sin 3^{\prime}=2 \cos 1^{\prime} \sin 2^{\prime}-\sin 1^{\prime} \\
& =.0008726645 \& \mathrm{c} .=\cos 89^{\circ} 57^{\prime} \text {; } \\
& \sin 4^{\prime}=2 \cos 1^{\prime} \sin 3^{\prime}-\sin 2^{\prime} \\
& =.0011635526 \& \mathrm{c} .=\cos 89^{\circ} 56^{\prime} \text {; } \\
& \sin 5^{\prime}=2 \cos 1^{\prime} \sin 4^{\prime}-\sin 3^{\prime} \\
& =.0014544406 \& \mathrm{c} .=\cos 89^{\circ} 55^{\prime} ; \\
& \text { \&c.... }=\text { \&c.................. }=\text { \&c........... }
\end{aligned}
$$

Again, from the other formula proved in the same article,

$$
\cos (n+1) A=2 \cos A \cos n A-\cos (n-1) A
$$

we shall have by the same substitutions,

$$
\begin{aligned}
\cos 2^{\prime} & =2 \cos 1^{\prime} \cos 1^{\prime}-1 \\
& =.9999998308 \& c .=\sin 89^{\circ} 58^{\prime} ; \\
\cos 3^{\prime} & =2 \cos 1^{\prime} \cos 2^{\prime}-\cos 1^{\prime} \\
& =.9999996192 \& c .=\sin 89^{\circ} 57^{\prime} ; \\
\cos 4^{\prime} & =2 \cos 1^{\prime} \cos 3^{\prime}-\cos 2^{\prime} \\
& =.9999993231 \& c .=\sin 89^{\circ} 56^{\prime} ; \\
\cos 5^{\prime} & =2 \cos 1^{\prime} \cos 4^{\prime}-\cos 3^{\prime} \\
& =.9999989493 \& c .=\sin 89^{\circ} 55^{\prime} ; \\
\& c \ldots . & =\& c \ldots \ldots \ldots \ldots \ldots=\$ c \ldots \ldots \ldots \ldots
\end{aligned}
$$

We may observe from the latter of these sets of equations, that when an arc becomes very nearly equal to $90^{\circ}$ and $0^{\circ}$, the changes which the sine and cosine respectively undergo, are of no value as far as five or six places of decimals.
140. The process adopted in the last article being confinued would enable us to determine the sines and cosines of all
arcs whatsoever; but by reason of the long and tedious numerical operations that are required, expedients of various kinds have been had recourse to, to facilitate the computation : thus, by means of the formula,
$\sin (A+B)=2 \sin A \cos B-\sin (A-B)$

$$
\begin{aligned}
& =2 \sin A\left(1-2 \sin ^{2} \frac{B}{2}\right)-\sin (A-B) \\
& =2 \sin A-\sin (A-B)-4 \sin A \sin ^{2} \frac{B}{2}
\end{aligned}
$$

if we suppose $B=1^{\prime}$, and $A$ to take the values $1^{\prime}, 2^{\prime}, 3^{\prime}, \& c$. in succession, we shall have,

$$
\begin{aligned}
\sin 2^{\prime} & =2 \sin 1^{\prime}-\sin 0^{\prime}-4 \sin 1^{\prime} \sin ^{2} 30^{\prime \prime} \\
\sin 3^{\prime} & =2 \sin 2^{\prime}-\sin 1^{\prime}-4 \sin 2^{\prime} \sin ^{2} 30^{\prime \prime} \\
\sin 4^{\prime} & =2 \sin 3^{\prime}-\sin 9^{\prime}-4 \sin 3^{\prime} \sin ^{2} 30^{\prime \prime} \\
\& c & =\& \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots
\end{aligned}
$$

Again, if $B=1^{\prime}$, and $A$ assume the values $1^{0}, 1^{0} 1^{\prime}, 1^{0} Q^{\prime}$, \&c. we get

$$
\begin{aligned}
\sin 1^{0} 1^{\prime} & =2 \sin 1^{0}-\sin 59^{\prime}-4 \sin 1^{0} \sin ^{2} 30^{\prime \prime} \\
\sin 1^{0} 2^{\prime} & =2 \sin 1^{0} 1^{\prime}-\sin 1^{0}-4 \sin 1^{0} 1^{\prime} \sin ^{2} 30^{\prime \prime} \\
\sin 1^{0} 3^{\prime} & =2 \sin 1^{0} 2^{\prime}-\sin 1^{0} 1^{\prime}-4 \sin 1^{0} 2^{\prime} \sin ^{2} 30^{\prime \prime} \\
\& c & =\& \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots
\end{aligned}
$$

and so on ;
and these operations are somewhat less laborious than those which would be necessary by the former method.
141. The formulæ,
$\sin (A+B) \sin (A-B)=(\sin A+\sin B)(\sin A-\sin B)$, and
$\cos (A+B) \cos (A-B)=(\cos A+\sin B)(\cos A-\sin B)_{,}$
proved in (68), will also enable us to deduce the sines and cosines of arcs from the sines and cosines of others previously determined, and may likewise be the means of verifying the results found by the preceding methods: thus by making $B=1^{0}$, and $A=2^{0}, 3^{0}$, \&c. in order, we obtain

$$
\begin{aligned}
& \sin 3^{0}=\frac{\left(\sin 2^{0}+\sin 1^{0}\right)\left(\sin 2^{0}-\sin 1^{0}\right)}{\sin 1^{0}} ; \\
& \sin 4^{\circ}=\frac{\left(\sin 3^{\circ}+\sin 1^{\circ}\right)\left(\sin 3^{\circ}-\sin 1^{\circ}\right)}{\sin 2^{\circ}} ; \\
& \text { \&c. }=\text { \&c...................................... } \\
& \cos 3^{0}=\frac{\left(\cos 2^{0}+\sin 1^{0}\right)\left(\cos 2^{0}-\sin 1^{0}\right)}{\cos 1^{0}} ; \\
& \cos 4^{0}=\frac{\left(\cos 3^{0}+\sin 1^{0}\right)\left(\cos 3^{\circ}-\sin 1^{0}\right)}{\cos 2^{0}} ; \\
& \text { \&c. }=\& c . . . . . . . . . . . . . . . . . . . . . . . .
\end{aligned}
$$

and the values this found may be checked by assigning different values to $A$ and $B$, so that their sum may still remain the same.

By one or other of these methods we may proceed to determine the values of the sines and cosines of all arcs as far as $30^{\circ}$, after which the tediousness of the numerical operations may in a great degree be avoided, by means of certain formule which have already been investigated in the second chapter.
142. To express the sines and cosines of arcs greater than $30^{\circ}$ and less than $45^{\circ}$ in terms of the radius 1.

It has been shewn in (67), that

$$
\sin (A+B)+\sin (A-B)=2 \sin A \cos B
$$

therefore $\sin (A+B)=2 \sin A \cos B-\sin (A-B)$,
and if $A$ be made equal to $30^{\circ}$, and $B$ be assumed equal to $1^{\prime}$, $2^{\prime}, 3^{\prime}$, \&c...successively, we get, since $\sin 30^{\circ}=\frac{1}{2}$, from (37),

$$
\begin{aligned}
\sin 30^{\circ} 1^{\prime} & =\cos 1^{\prime}-\sin 29^{\circ} 59^{\prime} ; \\
\sin 30^{\circ} 2^{\prime} & =\cos 2^{\prime}-\sin 29^{\circ} 58^{\prime} ; \\
\sin 30^{\circ} 3^{\prime} & =\cos 3^{\prime}-\sin 29^{\circ} 57^{\prime} ; \\
\& c \ldots \ldots & =\& c \ldots \ldots \ldots \ldots \ldots
\end{aligned}
$$

and thus the sines of all arcs as far as $45^{\circ}$ may be derived from the sines and cosines of those previously found :
again, since

$$
\cos (A-B)-\cos (A+B)=2 \sin A \sin B, \text { by }(67)
$$

we have

$$
\cos (A+B)=\cos (A-B)-2 \sin A \sin B:
$$

and if $A$ be supposed $=30^{\circ}$ as above, and $B$ equal to $1^{\prime}, 2^{\prime}, 3^{\prime}$, \&c. in succession,

$$
\begin{aligned}
\cos 30^{\circ} 1^{\prime} & =\cos 29^{\circ} 59^{\prime}-\sin 1^{\prime} \\
\cos 30^{\circ} 2^{\prime} & =\cos 29^{\circ} 58^{\prime}-\sin 2^{\prime} \\
\cos 30^{\circ} 3^{\prime} & =\cos 29^{\circ} 57^{\prime}-\sin 3^{\prime} \\
\& c \ldots \ldots & =\& c \ldots \ldots \ldots \ldots \ldots
\end{aligned}
$$

and hence the cosines of all arcs up to $45^{\circ}$ may be determined by means of the sines and cosines of those which are less than $30^{\circ}$.
143. To express the sines and cosines of arcs greater than $45^{\circ}$ and less than $90^{\circ}$ in terms of the radius 1.

Since by (12) and (21), $\sin \left(45^{\circ}+A\right)=\cos \left(45^{\circ}-A\right)$, we have

$$
\begin{aligned}
\sin 45^{\circ} 1^{\prime} & =\cos 44^{0} 59^{\prime} \\
\sin 45^{\circ} 2^{\prime} & =\cos 44^{\circ} 58^{\prime} \\
\sin 45^{\circ} 3^{\prime} & =\cos 44^{\circ} 57^{\prime} \\
\{c \ldots \ldots & =\& c \ldots \ldots \ldots
\end{aligned}
$$

and since $\cos \left(45^{\circ}+A\right)=\sin \left(45^{\circ}-A\right)$, by the same articles, we get

$$
\begin{aligned}
\cos 45^{\circ} 1^{\prime} & =\sin 44^{\circ} 59^{\prime} ; \\
\cos 45^{\circ} 2^{\prime} & =\sin 44^{\circ} 58^{\prime} ; \\
\cos 45^{\circ} 3^{\prime} & =\sin 44^{\circ} 57^{\prime} ; \\
\& c \ldots . & =\operatorname{sc} \ldots \ldots \ldots
\end{aligned}
$$

and thus the sines and cosines of all arcs as far as $90^{\circ}$ may be found.

From this it is manifest that if the sines and cosines of all arcs up to $45^{\circ}$ were formed into a table, such a table would serve for the sines and cosines of all arcs as far as $90^{\circ}$.
144. T'o express the sines and cosines of arcs greater than $90^{\circ}$ in terms of the radius 1.

From (63), we have

$$
\begin{aligned}
& \sin \left(90^{\circ}+A\right)=\sin 90^{\circ} \cos A+\cos 90^{\circ} \sin A=\cos A \\
& \cos \left(90^{\circ}+A\right)=\cos 90^{\circ} \cos A-\sin 90^{\circ} \sin A=-\sin A
\end{aligned}
$$

again,

$$
\begin{aligned}
& \sin \left(180^{\circ}+A\right)=\sin 180^{\circ} \cos A+\cos 180^{\circ} \sin A=-\sin A ; \\
& \cos \left(180^{\circ}+A\right)=\cos 180^{\circ} \cos A-\sin 180^{\circ} \sin A=-\cos A:
\end{aligned}
$$

and,

$$
\begin{aligned}
& \sin \left(270^{\circ}+A\right)=\sin 270^{\circ} \cos A+\cos 270^{\circ} \sin A=-\cos A \\
& \cos \left(270^{\circ}+A\right)=\cos 270^{\circ} \cos A-\sin 270^{\circ} \sin A=\sin A \\
& \& c \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots
\end{aligned}
$$

therefore the values of the sines and cosines of all arcs greater than $90^{\circ}$, will be the same as the sines and cosines of corresponding arcs less than $90^{\circ}$ : and if a table be formed to contain the sines and cosines of all arcs less than a quadrant, such table will contain the sines and cosines of all arcs greater than a
quadrant, proper regard being paid to the algebraical signs of the quantities according to the principles laid down in (16) and applied in the subsequent articles of the first chapter.
145. The sines and cosines of all arcs being determined by the methods just explained, the tangents, cotangents, secants, and cosecants are immediately deduced from the following equations :

$$
\begin{gathered}
\tan A=\frac{\sin A}{\cos A}, \cot A=\frac{\cos A}{\sin A}, \sec A=\frac{1}{\cos A} \\
\text { and } \operatorname{cosec} A=\frac{1}{\sin A}
\end{gathered}
$$

and the versed sines and chords, if necessary, from the equations, vers $A=1-\cos A$, and $\operatorname{chd} A=\sqrt{2-2 \cos A}$, or $=2 \sin \frac{A}{2}$.
146. The tangents of arcs greater than $45^{\circ}$ may however be easily found from the tangents of those that are less, by simple addition only.

$$
\begin{aligned}
& \text { For, since } \tan A-\cot A=\frac{\sin A}{\cos A}-\frac{\cos A}{\sin A} \\
& =\frac{\sin ^{2} A-\cos ^{2} A}{\sin A \cos A}=-2 \frac{\cos 2 A}{\sin 2 A}=-2 \cot 2 A:
\end{aligned}
$$

$$
\text { if we suppose, } A=45^{\circ}+B \text {, and } \therefore \Omega A=90^{\circ}+2 B \text {, }
$$

we shall have
$\tan \left(45^{\circ}+B\right)-\tan \left(45^{\circ}-B\right)=-2 \cot \left(90^{\circ}+2 B\right)=2 \tan 2 B ;$

$$
\therefore \tan \left(45^{\circ}+B\right)=2 \tan 2 B+\tan \left(45^{\circ}-B\right):
$$

hence, assuming $B$ to be equal to $1^{0}, 2^{0}, 3^{0}$, \&c. successively, we have

$$
\begin{aligned}
& \tan 46^{\circ}=2 \tan 2^{\circ}+\tan 44^{\circ} ; \\
& \tan 47^{\circ}=2 \tan 4^{\circ}+\tan 43^{\circ} ;
\end{aligned}
$$

$$
\begin{aligned}
& \tan 48^{\circ}=2 \tan 6^{\circ}+\tan 42^{\circ} \\
& \& \operatorname{con}=\& \mathrm{c} \ldots \ldots \ldots \ldots \ldots
\end{aligned}
$$

These expressions may also be used to try the correctness of the values of the tangents deduced by the other method.
147. The sines, cosines, \&c. of all arcs being thus calculated and tabulated, form what is called the Trigonometrical Canon; and it is easily seen that the sole difficulty in constructing such tables arises from the application of the fundamental rules of arithmetic to numbers consisting of many places of figures; and some of the expedients generally resorted to, to remove this difficulty, have already been explained. As a check upon such computations, Formulce of Verification have been introduced, which involving the dependance of the trigonometrical functions of arcs upon one another, may be applied to ascertain the correctness of a numerical calculation from the known accuracy of one or more others.

Formulæ of Verification might be multiplied indefinitely, but the most useful and those most generally used, have been proved in (78), (89), and (90), and their utility will be manifest from the two following articles.
148. In article (78) it has been proved that

$$
\begin{aligned}
\sin \frac{A}{2} & =\frac{1}{2}\{\sqrt{(1+\sin A)} \mp \sqrt{(1-\sin A)}\}, \\
\text { and } \cos \frac{A}{2} & =\frac{1}{2}\{\sqrt{(1+\sin A)} \pm \sqrt{(1-\sin A)}\}:
\end{aligned}
$$

now if we assign any value as $25^{\circ}$ to $A$, we shall have

$$
\sin 12^{\circ} 30^{\prime}=\frac{1}{2}\left\{\sqrt{\left(1+\sin 25^{\circ}\right)}-\sqrt{\left(1-\sin 25^{\circ}\right)}\right\}
$$

and $\cos 12^{\circ} 30^{\prime}=\frac{1}{\underset{\sim}{\alpha}}\left\{\sqrt{\left(1+\sin 25^{\circ}\right)}+\sqrt{\left(1-\sin 95^{\circ}\right)}\right\} ;$
hence if the results of these equations be the same as the sine and cosine of $12^{0} 30^{\prime}$ calculated by the method before given, we may conclude with a considerable degree of certainty that all the operations concerned are correct.

The formulæ just mentioned might manifestly have been likewise employed to deduce the sine and cosine of $\frac{A}{2}$ immediately from the sine of $A$.
149. In Euler's formula proved in (89) we have seen that $\sin A=\sin \left(36^{\circ}+\boldsymbol{A}\right)+\sin \left(72^{\circ}-A\right)-\sin \left(36^{\circ}-A\right)-\sin \left(72^{\circ}+A\right):$

> and if $A$ be taken equal to $5^{\circ}$, we shall have $\sin 5^{\circ}=\sin 41^{\circ}+\sin 67^{\circ}-\sin 31^{\circ}-\sin 77^{\circ}$ :
therefore if the values of the sines of these arcs already computed satisfy this equation, they may each be reasonably presumed to be correct, and the contrary.

Again, in Legendre's formula,
$\cos A=\sin \left(54^{\circ}+A\right)+\sin \left(54^{\circ}-A\right)-\sin \left(18^{\circ}+A\right)-\sin \left(18^{\circ}-A\right)$,
which is proved in (90), if we suppose $A=7^{0}$, we get

$$
\cos 7^{\circ}=\sin 61^{\circ}+\sin 47^{\circ}-\sin 25^{\circ}-\sin 11^{\circ}
$$

from which the same inferences may be drawn as before.
Similarly, of the sines and cosines of other arcs.
150. In the Trigonometrical Canon, constructed and verified by these methods, the radius has been supposed to be equal to 1 ; but as the logarithms of quantities afford great facilities in the multiplication, division, involution and evolution of large numbers, it is desirable that the logarithms of the sines, cosines, \&c. of arcs should also be tabulated, many of which from their nature would to this radius be negative. On this account the Tabular Radius has been assumed equal to ten
thousand millions, and consequently each of the sines, cosines, \&c. thus computed, must be increased in the same proportion, and their logarithms will then become positive quantities. Thus,
since to the radius 1 , we have $\sin 1^{\prime}=.0002908882$ \& c. $\therefore$ to the radius $10^{10}$, we shall have $\sin 1^{\prime}=2908882$. \&c.

$$
\text { and hence, } \begin{aligned}
\log \sin 1^{\prime} & =\log 2908882 . \text { \&c. } \\
& =6.4637961 \text { \&c.and so on: }
\end{aligned}
$$

and a table constructed on this principle, is called a table of logarithmic sines, cosines, \&c. by the use of which most of the practical applications of trigonometry are greatly facilitated and generally performed.
151. If the logarithmic sines and cosines of all arcs be found as in the last article, the logarithmic tangents, cotangents, secants and cosecants, as also the versed sines and chords, may be deduced from them by the operations of addition and subtraction only. Thus,
$\log \tan A=\log \left(r \frac{\sin A}{\cos A}\right)=\log r+\log \sin A-\log \cos A$

$$
=10+\log \sin A-\log \cos A
$$

$\log \cot A=\log \left(r \frac{\cos A}{\sin A}\right)=\log r+\log \cos A-\log \sin A$

$$
=10+\log \cos A-\log \sin A
$$

$\log \sec A=\log \left(\frac{r^{2}}{\cos A}\right)=2 \log r-\log \cos A$

$$
=20-\log \cos A
$$

$\log \operatorname{cosec} A=\log \left(\frac{r^{2}}{\sin A}\right)=2 \log r-\log \sin A$

$$
=20-\log \sin A
$$

$\log$ vers $A=\log \left(\frac{2 \sin ^{2} \frac{A}{2}}{r}\right)=\log \alpha+2 \log \sin \frac{A}{\alpha}-\log r$

$$
=\log 2+2 \log \sin \frac{A}{2}-10
$$

and $\log \operatorname{ch} \mathrm{d} A=\log \left(2 \sin \frac{A}{2}\right)=\log 2+\log \sin \frac{A}{2}$.
152. To find the mumerical ratio of the circumference of a circle to its radius and diameter.

In (197) where the radius is supposed to be 1 , the sine of $1^{\prime}$ has been shewn to be .0002908882 \&c. and ithas been proved also that the sines of arcs so small as $1^{\prime}$ are very nearly equal to the arcs themselves: hence, since the number of minutes in the whole circumference is $360 \times 60=6 \times 60 \times 60$, we shall have

$$
\begin{aligned}
\text { the whole circumference } & =.0002908882 \text { \&c. } \times 6 \times 60 \times 60 \\
& =.0017453292 \text { \&c. } \times 60 \times 60 \\
& =.1047197520 \text { \&c. } \times 60 \\
& =6.28318512 \& c .
\end{aligned}
$$

which was assumed in (7) to be represented by $2 \pi$; therefore the circumference of a circle : the radius

$$
:: 6.28318512 \text { \&c. : } 1
$$

and the circumference of a circle: the diameter

$$
: 6.28318512 \text { \&c. : } 2 \text { :: 3. } 14159256 \text { \&c. : } 1 .
$$

153. By the method of converging fractions, approximations to the ratio just found are $3: 1 ; 22: 7 ; 393: 106$, \&c. which are alternately less and greater than, but more and more nearly equal to, the true ratio, and may be adopted in most cases of practice without sensible error.
154. To find the magnitude of the angle which is subtended by an arc of the circle equal to the radius.

Since $6.28318512 \& c$. or the whole circumference subtends four right angles, or is equivalent to $360^{\circ}$ on the same scale on which 1 represents the radius, we shall have

$$
360^{\circ}: \text { the required } \angle:: 6.283185128 \mathrm{c} .: 1
$$

and therefore the required angle will be

$$
=\frac{360^{\circ}}{6.28318512 \& c .}=57^{\circ} .2957795 \& \mathrm{cc}=57^{\circ} 17^{\prime} 44^{\prime \prime} 48^{\prime \prime \prime} \& c .
$$

155. To express the length of the arc which measures a given angle, in terms of the radius.

Since the arc subtending an angle of $57^{\circ} 17^{\prime} 44^{\prime \prime} 48^{\prime \prime \prime}$ \&c. is in every circle equal to the radius, because the arcs are proportional to the radii, when the angles which they subtend at the centres are equal; if $A^{0}$ be the magnitude of any angle, and $a$ the arc subtending it, we have

$$
r: \alpha:: 57^{0} .2957795 \& c .: A^{0}
$$

and $\therefore$ the arc $a$ expressed in terms of the radius $r$

$$
=r \frac{A^{0}}{57^{0} .2957795 \& c}
$$

Ex. Let $A^{0}$ be taken equal to $1^{0},\left(\frac{1}{60}\right)^{0},\left(\frac{1}{60^{2}}\right)^{0}$, successively, then we have

$$
\begin{aligned}
\text { the length of one degree } & =r\left(\frac{1^{0}}{57^{0} .2957795 \mathrm{\& c} .}\right) \\
& =r(.017453292 \& c .)
\end{aligned}
$$

the length of one minute $=r(.0002908882$ \&ic. $)$, the length of one second $=r(.00000484813 \& c$.$) .$
156. Cor. If $57^{\circ} .2957795$ \&c. be represented by $r^{0}$, we shall have $a=r \frac{A^{0}}{r^{0}}$, and thence $\frac{a}{r}=\frac{A^{0}}{r^{0}}$, which agrees with what is assumed in (4): also if $r=1$, then will $a=\frac{A^{0}}{r^{0}}$.

## CHAP. IV.

On the relations between the sides, angles, areas, circumscribed and inscribed circles, $\& c$. of plane triangles. On the relations between the sides, angles, diagonals, areas and circumscribed circles, \&c. of certain quadrilaterals. On the perimeters, areas, \&c. of regular polygons. On the periphery, area, \&c. of a circle.
157. THE sides of a plane triangle are proportional to the sines of the angles which they respectively subtend.

Let $A B C$ be a plane triangle, of which the angles

are $A, B$ and $C$; with centres $A, B$, and radius 1 , describe circular arcs cutting $C A$ and $C B$, or these lines produced in the points $\alpha$ and $\beta$; draw $\alpha \mu, \beta \nu$ and $C D$ perpendicular to $A B$ (produced if necessary); then by similar triangles

$$
\begin{aligned}
A C: C D:: A \alpha & : a \mu:: \quad 1 \quad: \sin A, \text { by }(17) \\
\text { and } C D & : B C:: \beta \nu: B \beta:: \sin B: 1, \text { by }(17) \text { or }(20)
\end{aligned}
$$

$\therefore$ by compounding these proportions, we have

$$
A C: B C:: \sin B: \sin A
$$

$$
\begin{array}{r}
\text { similarly, } A C: A B:: \sin B: \sin C \\
\text { and } A B: B C:: \sin C: \sin A
\end{array}
$$

and therefore generally

$$
B C: A C: A B:: \sin A: \sin B: \sin C
$$

158. This fundamental property of plane triangles may likewise be proved as follows:


Suppose a circle to be described about the triangle $A B C$, and let its centre be $O$, and its radius equal to $R$; then it is manifest that the sides of the triangle are the chords of the arcs they respectively cut off, to the radius $R$; join $A O, B O, C O$, and by (59) we have

$$
\begin{aligned}
\frac{C B}{R} & =\operatorname{chd} B O C=2 \sin \left(\frac{B O C}{2}\right)=2 \sin A ; \text { by }(76), \\
\frac{C A}{R} & =\operatorname{chd} A O C=2 \sin \left(\frac{A O C}{2}\right)=2 \sin B ; \\
\text { and } \frac{A B}{R} & =\operatorname{chd} A O B=2 \sin \left(\frac{A O B}{2}\right)=2 \sin C
\end{aligned}
$$

whence $\frac{C B}{R}: \frac{C A}{R}: \frac{A B}{R}:: 2 \sin A: 2 \sin B: 2 \sin C$ :
that is, $C B: C A: A B:: \sin A: \sin B: \sin C$.

If the sides which subtend the angles $A, B, C$ be called $a, b, c$ respectively, we have

$$
\begin{gathered}
a: b: c:: \sin A: \sin B: \sin C \\
\text { or } \frac{a}{b}=\frac{\sin A}{\sin B}, \frac{a}{c}=\frac{\sin A}{\sin C}, \frac{b}{c}=\frac{\sin B}{\sin C}
\end{gathered}
$$

159. Cor.1. By either of the last two articles, we have

$$
a: b \quad:: \quad \sin A \quad: \sin B
$$

$\therefore a+b: a-b:: \sin A+\sin B: \sin A-\sin B$

$$
:: \tan \left(\frac{A+B}{2}\right): \tan \left(\frac{A-B}{2}\right), \text { by }(72)
$$

similarly

$$
\begin{aligned}
& a+c: a-c:: \tan \left(\frac{A+C}{2}\right): \tan \left(\frac{A-C}{2}\right) ; \\
& \text { and } b+c: b-c:: \tan \left(\frac{B+C}{2}\right): \tan \left(\frac{B-C}{2}\right) .
\end{aligned}
$$

Hence, in a plane triangle, the sum of any two sides : the difference :: the tangent of the semi-sum of their opposite angles : the tangent of the semi-difference.
160. Cor. 2. Let $C E$ drawn to bisect the angle $A C B$

meet the base $A B$ in $E$, then by (157) we have

$$
\begin{aligned}
& A E: A C:: \sin A C E: \sin A E C \\
&:: \sin B C E: \sin B E C: B E: B C ; \\
& \therefore A E: B E:: A C \quad: B C:
\end{aligned}
$$

that is, the segments of the base have the same ratio which the other sides of the triangle have to one another.

Also, if $A E=a^{\prime}, B E=b^{\prime}$, we have $a^{\prime}: b^{\prime}:: b: a$,
whence we find $a^{\prime}=\frac{b c}{a+b}$, and $b^{\prime}=\frac{a c}{a+b}$;

$$
\begin{aligned}
& \text { and } \therefore a^{\prime} b^{\prime}=a b\left(\frac{c}{a+b}\right)^{2}: \\
& \text { or } a^{\prime} b^{\prime}: a b:: c^{2}:(a+b)^{2} .
\end{aligned}
$$

A similar process may be used if the exterior angle be bisected, and it will appear that

$$
a^{\prime} b^{\prime}: a b:: c^{2}:(a \sim b)^{2}
$$

161. Cor. 3. If $C F$ be supposed to bisect the side $A B$, we have from (157),


$$
\sin A C F: \sin C A F:: A F: F C
$$

$$
:: B F: F C:: \sin B C F: \sin C B F
$$

$\therefore \sin A C F: \sin B C F:: \sin C A F: \sin C B F$ :
or the sines of the segments of the vertical angle are proportional to the sines of the corresponding angles at the base.

Also, if $A^{\prime}$ and $B^{\prime}$ represent the segments of the angle $C$, we have

$$
\begin{aligned}
& \sin A^{\prime}: \sin B^{\prime}:: \sin A: \sin B \\
& \text { and } \therefore \sin A^{\prime}+\sin B^{\prime}: \sin A^{\prime}-\sin B^{\prime}:: \\
& \sin A+\sin B: \sin A-\sin B
\end{aligned}
$$

or $\tan \left(\frac{A^{\prime}+B^{\prime}}{2}\right): \tan \left(\frac{A^{\prime}-B^{\prime}}{2}\right):: \tan \left(\frac{A+B}{2}\right): \tan \left(\frac{A-B}{2}\right) ;$
$\therefore \tan \left(\frac{A^{\prime}-B^{\prime}}{2}\right)=\frac{\tan \left(\frac{A-B}{2}\right)}{\tan \left(\frac{A+B}{2}\right)} \tan \left(\frac{A^{\prime}+B^{\prime}}{2}\right)$

$$
=\frac{\tan \left(\frac{A-B}{2}\right)}{\tan \left(\frac{A+B}{2}\right)} \tan \frac{C}{2}, \text { or }=\tan \left(\frac{A-B}{2}\right) \tan ^{\circ} \frac{C}{2}
$$

from which, and the equation $A^{\prime}+B^{\prime}=C$, the values of $A^{\prime}$ and $B^{\prime}$ become known.
162. To find the relations between the sides and angles of right-angled triangles.

Let $A C B$ be a triangle having its sides represented by $a, b, c$, as before, and the angle at $C$ a right angle, then


$$
\begin{aligned}
B C: A B & :: \sin B A C: \sin A C B \\
& :: \sin A: \sin \frac{\pi}{2} \\
& :: \sin A: 1, \text { by }(18)
\end{aligned}
$$

whence $B C=A B \sin A=A B \sin \left(\frac{\pi}{2}-B\right)=A B \cos B ;$
similarly, $A C=A B \sin B=A B \sin \left(\frac{\pi}{2}-A\right)=A B \cos A ;$

$$
\begin{aligned}
\text { again, } B C: A C & :: \sin B A C: \sin A B C \\
& :: \sin A: \cos A \\
& :: \tan A: 1, \text { by }(42) \\
\therefore B C=A C \tan A & =A C \tan \left(\frac{\pi}{2}-B\right)=A C \cot B ;
\end{aligned}
$$

similarly, $A C=B C \tan B=B C \tan \left(\frac{\pi}{2}-A\right)=B C \cot A:$ whence we have

$$
\begin{aligned}
A B=\frac{B C}{\sin A} & =\frac{B C}{\cos B}=B C \operatorname{cosec} A=B C \sec B \\
\text { also, } \sin A & =\frac{B C}{A B}=\cos B, \text { and } \tan A=\frac{B C}{A C}=\cot B
\end{aligned}
$$

163. To find the relations between the sides and angles of oblique-angled triangles.

Let $A C B$ be an oblique-angled triangle, draw $C D$ perpendicular to $A B$, and let the sides subtending the angles $A, B, C$ be called $a, b, c$ respectively:

then, $c=A D+B D$
$=A C \cos A+B C \cos B$, by the last article, $=b \cos A+a \cos B ;$
similarly, $b=a \cos C+c \cos A$;
and $a=b \cos C+c \cos B$ :
and from these equations any one of the quantities involved may be found in terms of the rest.
164. Cor. The last article combined with the property proved in (157), is sometimes applied to express the sine of the sum of two angles in terms of the sines and cosines of the angles themselves. Thus,

$$
\text { since } c=b \cos A+a \cos B \text {, we have } \frac{c}{a}=\frac{b}{a} \cos A+\cos B
$$

but $\frac{c}{a}=\frac{\sin C}{\sin A}=\frac{\sin (\pi-C)}{\sin A}=\frac{\sin (A+B)}{\sin A}$, and $\frac{b}{a}=\frac{\sin B}{\sin A}$;
$\therefore$ we shall have $\frac{\sin (A+B)}{\sin A}=\frac{\sin B}{\sin A} \cos A+\cos B$,
and thence, $\sin (A+B)=\sin B \cos A+\sin A \cos B$

$$
=\sin A \cos B+\cos A \sin B
$$

as has been already proved in (63).
Since $A+B$ is less than $\pi$, the proof just given may at first sight seem partial; but by means of the relations established in the first chapter, it is easily extended to the sine of the sum of any two arcs whatever.
165. To express the cosines of the angles of a plane triangle in terms of the sides.

If $A, B, C$ be the angles of any plane triangle, $a, b, c$ the corresponding sides which subtend them, we have seen that

$$
\begin{aligned}
& a=b \cos C+c \cos B \\
& b=a \cos C+c \cos A \\
& c=a \cos B+b \cos A
\end{aligned}
$$

and multiplying both sides of these equations by $a, b, c$ respectively, we obtain

$$
\begin{aligned}
& a^{2}=a b \cos C+a c \cos B \\
& b^{2}=a b \cos C+b c \cos A \\
& c^{2}=a c \cos B+b c \cos A
\end{aligned}
$$

therefore, by addition,

$$
a^{2}+b^{2}+c^{2}=2 a b \cos C+2 a c \cos B+2 b c \cos A:
$$

from this equation, subtract successively $2 a^{2}, 2 b^{2}, \varrho c^{2}$ and their equals, and we have

$$
\begin{aligned}
& b^{2}+c^{2}-a^{2}=2 b c \cos A ; \\
& a^{2}+c^{2}-b^{2}=2 a c \cos B ; \\
& a^{2}+b^{2}-c^{2}=2 a b \cos C:
\end{aligned}
$$

from which equations immediately result

$$
\begin{aligned}
& \cos A=\frac{b^{2}+c^{2}-a^{2}}{2 b c} ; \\
& \cos B=\frac{a^{2}+c^{2}-b^{2}}{2 a c} ; \\
& \cos C=\frac{a^{2}+b^{2}-c^{2}}{2 a b}
\end{aligned}
$$

166. The values above found are frequently deduced by means of the twelfth or thirteenth Propositions of the Second Book of Euclid's Elements.

$$
\begin{gathered}
\text { For, } B C^{2}=A C^{2}+A B^{2} \mp 2 A B \cdot A D, \\
\text { but } A D=A C \cos A, \text { by }(162), \\
\text { or }=A C \cos (\pi-A) \text { by }(162),=-A C \cos A, \text { by }(24) ; \\
\therefore B C^{2}=A C^{2}+A B^{2}-2 A B \cdot A C \cos A, \\
\text { or } a^{2}=b^{2}+c^{2}-2 b c \cos A, \\
\text { and } \therefore \cos A=\frac{b^{2}+c^{2}-a^{2}}{2 b c}, \text { as before. }
\end{gathered}
$$

It may here be observed, that the Propositions of Euclid above referred to, are in reality proved in the last article.

For, since $a^{2}=b^{2}+c^{2}-\varrho b c \cos A$, we have

$$
\begin{aligned}
B C^{2} & =A C^{2}+A B^{2}-2 A B \cdot A C \cos A \\
& =A C^{2}+A B^{2} \mp 2 A B \cdot A D,
\end{aligned}
$$

as appears from (162).
Ex. 1. Let $a=b$, or the triangle be isosceles: then

$$
\cos A=\frac{c^{2}}{2 a c}=\frac{c}{2 a}=\frac{c}{2 b}=\cos B ;
$$

$$
\text { also, } \cos C=\frac{2 a^{2}-c^{2}}{2 a^{2}}=1-\frac{c^{2}}{2 a^{8}}, \text { and vers } C=\frac{c^{2}}{2 a^{2}} .
$$

Ex. 2. Let $a=b=c$, or the triangle be equilateral: then $\cos A=\frac{1}{2}=\cos 60^{\circ}=\cos B=\cos C$.
167. Cor. 1. Since $\cos C=\frac{a^{2}+b^{2}-c^{2}}{2 a b}$, we have $c^{2}=a^{2}-2 a b \cos C+b^{2}$, and thence $c=\sqrt{a^{2}-2 a b \cos C+b^{2}}$, which is the value of one side expressed in terms of the two others and their included angle.

Ex. 1. Let $C=90^{\circ}$, then $\cos C=0$, and $\therefore c^{2}=a^{2}+b^{2}$, which is the 47 th Proposition of the first book of Euclid's Elements established by the Principles of Trigonometry.

Ex. 2. If $C=60^{\circ}$, we have $\cos C=\frac{1}{2}$, from (37),

$$
\text { and } \therefore c^{2}=a^{2}-a b+b^{2}=\frac{a^{3}+b^{5}}{a+b}
$$

Ex. 3. If $C=120^{\circ}$, we have $\cos C=-\frac{1}{2}$, by (24),

$$
\text { and } \therefore c^{2}=a^{2}+a b+b^{2}=\frac{a^{2}-b^{3}}{a-b} .
$$

168. Cor. 2. From (165), we have immediately

$$
\begin{gathered}
2 b c \cos A=b^{2}+c^{2}-a^{2} \\
\text { or } a^{2}-b^{2}=c^{2}-2 b c \cos A=2 c\left(\frac{c}{2}-b \cos A\right):
\end{gathered}
$$

that is, the difference of the squares of the sides is equal to twice the rectangle contained by the base, and the distance of its middle point from the perpendicular.

$$
\begin{aligned}
& \text { Again, }(a+b)(a-b)=c(c-2 b \cos A) \\
& \quad \text { or } c: a+b:: a-b: c-2 b \cos A:
\end{aligned}
$$

that is, the base : the sum of the sides :: the difference of the sides : the difference or sum of the segments of the base made by a perpendicular let fall upon it from the opposite angle, according as it falls within or without the triangle.
169. Cor. 3. From the preceding articles it is seen, that

$$
\begin{aligned}
c^{2} & =a^{2}+b^{2}-2 a b \cos C \\
& =a^{2}-a b \cos C+b^{2}-a b \cos C \\
& =a(a-b \cos C)+b(b-a \cos C):
\end{aligned}
$$

suppose now $A F$ and $B G$ to be drawn from the angles $A$ and $B$ respectively perpendicular to the subtending sides;

$$
\begin{aligned}
& \text { then } a-b \cos C=B C-C F=B F \\
& \text { and } b-a \cos C=A C-C G=A G
\end{aligned}
$$

hence replacing $a, b, c$ by $B C, A C$ and $A B$ respectively, we have

$$
A B^{2}=B C . B F+A C . A G:
$$

or the square described upon any side of a triangle is equal to the sum of the rectangles contained by the two others and their segments respectively cut off by perpendiculars let fall upon them (produced if necessary) from the opposite angles.
170. Cor. 4. If the angle $C$ be bisected by the straight line $C E$ meeting the opposite side in $E$, the value of this line may be found; for by (166) we have


$$
\begin{aligned}
C E^{2} & =A C^{2}+A E^{2}-2 A C \cdot A E \cos A \\
& =b^{2}+\frac{b^{2} c^{2}}{(a+b)^{2}}-\frac{2 b^{2} c}{a+b} \cos A, \text { by }(160), \\
& =b^{2}+\frac{b^{2} c^{2}}{(a+b)^{2}}-\frac{b}{a+b}\left(b^{2}+c^{2}-a^{2}\right) \\
& =b^{2}+\frac{b^{2} c^{2}}{(a+b)^{2}}+\frac{\left(a^{2}-b^{2}\right) b}{a+b}-\frac{b c^{2}}{a+b}
\end{aligned}
$$

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$$
\begin{aligned}
& =a b+\frac{b^{2} c^{2}}{(a+b)^{2}}-\frac{b c^{2}}{a+b} \\
& =a b-\frac{a b c^{2}}{(a+b)^{2}}
\end{aligned}
$$

and hence $a b=\frac{a b c^{2}}{(a+b)^{2}}+C E^{2}=\left(\frac{a c}{a+b}\right)\left(\frac{b c}{a+b}\right)+C E^{2}$ :

$$
\text { or } A C \cdot B C=A E . E B+C E^{2}:
$$

that is, if any angle of a triangle be bisected by a straight line which cuts the opposite side, the rectangle of the two other sides is equal to the rectangle of the segments of the divided side together with the square of the dividing line.
171. Cor. 5. Supposing $C F$ to bisect the side $A B$ in $F$, we shall have by (166),

$C F^{2}=C A^{2}+\Lambda F^{2}-2 A C \cdot A F \cos A=b^{2}+\left(\frac{c}{2}\right)^{2}-b c \cos A$
$=b^{2}+\frac{c^{2}}{4}-\frac{b^{2}+c^{2}-a^{2}}{2}=\frac{a^{2}+b^{2}}{2}-\frac{c^{2}}{4} ;$
therefore $a^{2}+b^{2}=2\left(\frac{c}{2}\right)^{2}+2 C F^{2} ;$
which shews that the sum of the squares of any two sides of a triangle is equal to twice the square of half the other side, and twice the square of the straight line which is drawn from the opposite angle to bisect it.
172. Cor.6. If we suppose $C F=h$, and the corresponding lines drawn from the angles $A$ and $B$ to bisect the opposite sides, equal to $k$ and $l$ respectively, we shall have from the last article,

$$
\begin{aligned}
& a^{2}+b^{2}=2\left(\frac{c}{2}\right)^{2}+2 h^{2} \\
& a^{2}+c^{2}=2\left(\frac{b}{2}\right)^{2}+2 l^{2} \\
& b^{2}+c^{2}=2\left(\frac{a}{2}\right)^{2}+2 k^{2}
\end{aligned}
$$

and therefore by addition,

$$
\begin{aligned}
2\left(a^{2}+b^{2}+c^{2}\right) & =2\left\{\left(\frac{a}{2}\right)^{2}+\left(\frac{b}{2}\right)^{2}+\left(\frac{c}{2}\right)^{2}+h^{2}+k^{2}+l^{2}\right\} \\
& =2\left\{\frac{a^{2}+b^{2}+c^{2}}{4}+h^{2}+k^{2}+l^{2}\right\},
\end{aligned}
$$

and thence

$$
3\left(a^{2}+b^{2}+c^{2}\right)=4\left(h^{2}+k^{2}+l^{2}\right):
$$

that is, three times the sum of the squares of the sides of a plane triangle is equal to four times the sum of the squares of the lines drawn from the angles to bisect the opposite sides.
173. To express the sines of the angles of a plane triangle in terms of the sides.

From (25), we have $\sin ^{2} A=1-\cos ^{2} A$

$$
\begin{aligned}
& =1-\left(\frac{b^{2}+c^{2}-a^{2}}{2 b c}\right)^{2}=\frac{(2 b c)^{2}-\left(b^{2}+c^{2}-a^{2}\right)^{2}}{4 b^{2} c^{2}} \\
& =\frac{\left(2 b c+b^{2}+c^{2}-a^{2}\right)\left(2 b c-b^{2}-c^{2}+a^{2}\right)}{4 b^{2} c^{2}} \\
& =\frac{\left\{(b+c)^{2}-a^{2}\right\}\left\{a^{2}-(b-c)^{2}\right\}}{4 b^{2} c^{2}}
\end{aligned}
$$

$$
=\frac{(a+b+c)(b+c-a)(a+c-b)(a+b-c)}{4 b^{2} c^{2}}
$$

and therefore
$\sin A=\frac{1}{2 b c} \sqrt{(a+b+c)(b+c-a)(a+c-b)(a+b-c)} ;$
similarly,
$\sin B=\frac{1}{2 a c} \sqrt{(a+b+c)(b+c-a)(a+c-b)(a+b-c)} ;$ and
$\sin C=\frac{1}{2 a b} \sqrt{(a+b+c)(b+c-a)(a+c-b)(a+b-c)}$.
assume now, $2 S=a+b+c=$ the sum of the sides;

$$
\text { then } \begin{aligned}
2(S-a) & =b+c-a \\
2(S-b) & =a+c-b \\
2(S-c) & =a+b-c
\end{aligned}
$$

whence by substitution we obtain

$$
\begin{aligned}
& \sin A=\frac{2}{b c} \sqrt{S(S-a)(S-b)(S-c)} \\
& \sin B=\frac{2}{a c} \sqrt{S(S-a)(S-b)(S-c)} \\
& \sin C=\frac{2}{a b} \sqrt{S(S-a)(S-b)(S-c)}
\end{aligned}
$$

Ex. 1. Let $a=b$, or the triangle be isosceles; then

$$
\begin{aligned}
\sin A & =\frac{2}{a c} \sqrt{S(S-a)(S-a)(S-c)} \\
& =\frac{2(S-a)}{a c} \sqrt{S(S-c)} \\
& =\frac{1}{a} \sqrt{\left(a+\frac{c}{2}\right)\left(a-\frac{c}{2}\right)}
\end{aligned}
$$

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$$
=\frac{1}{2 a} \sqrt{4 a^{2}-c^{2}}=\frac{1}{2 b} \sqrt{4 b^{2}-c^{2}}=\sin B:
$$

and $\sin C=\frac{2(S-a)}{a^{2}}$

$$
\sqrt{S(S-c)}=\frac{c}{a^{2}} \sqrt{\left(a+\frac{c}{2}\right)\left(a-\frac{c}{2}\right)}
$$

$$
=\frac{c}{2 a^{2}} \sqrt{4 a^{2}-c^{2}}
$$

Ex. 2. Let $a=b=c$, or the triangle be equilateral, then

$$
\begin{aligned}
\sin A & =\frac{2}{a^{2}} \sqrt{S(S-a)^{5}}=\frac{2}{a^{2}} \sqrt{\left(\frac{3 a}{2}\right)\left(\frac{a}{2}\right)^{3}} \\
& =\frac{\sqrt{ } 3}{2}=\sin 60^{\circ}=\sin B=\sin C
\end{aligned}
$$

Ex. S. If the sides of the triangle $a, b, c$ be respectively equal to $3,4,5$, we shall have

$$
\begin{gathered}
2 S=3+4+5=12, \text { and } S=6, \\
\therefore S-a=3, S-b=2, \text { and } S-c=1: \\
\text { and } \sin A=\frac{2}{20} \sqrt{6.3 \cdot 2 \cdot 1}=\frac{12}{20}=\frac{3}{5} ; \\
\sin B=\frac{2}{15} \sqrt{6.3 \cdot 2 \cdot 1}=\frac{19}{15}=\frac{4}{5} ; \\
\sin C=\frac{2}{12} \sqrt{6.3 \cdot 2 \cdot 1}=\frac{12}{12}=1=\sin \frac{\pi}{2}:
\end{gathered}
$$

hence $C=\frac{\pi}{\boldsymbol{a}}$, or the triangle is right-angled at $C$ :
174. Cor. From the last article we have

$$
\begin{gathered}
\sqrt{S(S-a)(S-b)(S-c)}=\frac{a b}{2} \sin C, \text { which, if } C=\frac{\pi}{9}, \\
\text { gives } 2 \sqrt{S(S-a)(S-b)(S-c)}=a b \\
0
\end{gathered}
$$

and by substituting for $a, b, c$, the quantities $n^{2}-1,2 n$ and $n^{2}+1$ respectively, it will be found that this equation is verified; and therefore the sides of any rational right-angled triangle may be represented by these quantities, $n$ being assumed at pleasure equal to any quantity greater than unity.
175. To express the sine, co-sine, tangent, \&c. of half an angle of a triangle in terms of the sides.

From (79) we have

$$
\begin{aligned}
& 2 \sin ^{2} \frac{A}{2}=1-\cos A=1-\frac{b^{2}+c^{2}-a^{2}}{2 b c}=\frac{2 b c-b^{2}-c^{2}+a^{2}}{2 b c} \\
&= \frac{a^{2}-(b-c)^{2}}{2 b c}=\frac{(a+b-c)(a+c-b)}{2 b c}=\frac{2(S-b) 2(S-c)}{2 b c} ; \\
& \therefore \sin \frac{A}{2}=\sqrt{\frac{(S-b)(S-c)}{b c}}=\frac{1}{\operatorname{cosec} \frac{A}{2}} ;
\end{aligned}
$$

similarly,

$$
\begin{aligned}
& \sin \frac{B}{2}=\sqrt{\frac{(S-a)(S-c)}{a c}}=\frac{1}{\operatorname{cosec} \frac{B}{2}} \\
& \text { and } \sin \frac{C}{2}=\sqrt{\frac{(S-a)(S-b)}{a b}}=\frac{1}{\operatorname{cosec} \frac{C}{2}}
\end{aligned}
$$

Again, from the same article, we have

$$
\begin{aligned}
& 2 \cos ^{2} \frac{A}{2}=1+\cos A=1+\frac{b^{2}+c^{2}-a^{2}}{2 b c}=\frac{2 b c+b^{2}+c^{2}-a^{2}}{2 b c} \\
& =\frac{(b+c)^{2}-a^{2}}{2 b c}=\frac{(a+b+c)(b+c-a)}{2 b c}=\frac{2 S 2(S-a)}{2 b c} ;
\end{aligned}
$$

$$
\therefore \cos \frac{A}{2}=\sqrt{\frac{S(S-a)}{b c}}=\frac{1}{\sec \frac{1}{2}} ;
$$

$$
\text { similarly, } \cos \frac{B}{2}=\sqrt{\frac{S(S-b)}{a c}}=\frac{1}{\sec \frac{B}{2}}
$$

$$
\text { and } \cos \frac{C}{2}=\sqrt{\frac{S(S-c)}{a b}}=\frac{1}{\sec \frac{C}{2}}
$$

Hence $\therefore$ we shall have by (42)

$$
\begin{array}{r}
\tan \frac{A}{2}=\sqrt{\frac{(S-b)(S-c)}{S(S-a)}}=\frac{1}{\cot \frac{A}{2}}, \\
\tan \frac{B}{2}=\sqrt{\frac{(S-a)(S-c)}{S(S-b)}}=\frac{1}{\cot \frac{B}{2}} ; \\
\text { and } \tan \frac{C}{2}=\sqrt{\frac{(S-a)(S-b)}{S(S-c)}}=\frac{1}{\cot \frac{C}{2}} .
\end{array}
$$

176. Cor.1. If the augle $C$ be a right angle, we shall have

$$
\sin \frac{C}{2}=\sin 45^{\circ}=\frac{1}{\sqrt{ } 2}=\cos 45^{\circ}=\cos \frac{C}{2} ;
$$

$$
\text { hence } \frac{1}{\sqrt{ } a}=\sqrt{\frac{(S-a)(S-b)}{a b}}=\sqrt{\frac{S(S-c)}{a b}} ;
$$

and therefore

$$
\begin{gathered}
a b=2(S-a)(S-b)=2 S(S-c) \\
\quad \text { and }(S-a)(S-b)=S(S-c)
\end{gathered}
$$

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177. Cor. 2. From (175) we may easily deduce what is proved in (173).
For, $\sin A=\Omega \sin \frac{A}{2} \cos \frac{A}{2}$, by (76),

$$
\begin{aligned}
& =o \sqrt{\frac{(S-b)(S-c)}{b c} \frac{S(S-a)}{b c}} \\
& =\frac{2}{b c} \sqrt{S(S-a)(S-b)(S-c)},
\end{aligned}
$$

as before: similarly of the others.
178. To express the area of a plane triangle in terms of the sides.


The area of the triangle $A B C=\frac{1}{2} A B \cdot C D$

$$
\begin{aligned}
& =\frac{1}{2} A B \cdot A C \sin A, \text { by }(162), \\
& =\frac{b c}{2} \frac{2}{b c} \sqrt{S(S-a)(S-b)(S-c)}, \text { by }(173), \\
& =\sqrt{S(S-a)(S-b)(S-c)} .
\end{aligned}
$$

179. The area above found might easily have been determined without assuming the expression for the sine of an angle of the triangle. Thus,

$$
\begin{aligned}
& \text { since } a^{2}=b^{2}+c^{2}-2 c A D, \text { by }(166), \therefore A D=\frac{b^{2}+c^{2}-a^{2}}{2 c} \\
& \therefore C D^{2}=A C^{i}-A D^{2}=b^{2}-\left(\frac{b^{2}+c^{2}-a^{2}}{2 c}\right)^{2}
\end{aligned}
$$

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$$
\begin{gathered}
=\frac{4 b^{2} c^{2}-\left(b^{2}+c^{2}-a^{2}\right)^{2}}{4 c^{2}}=\frac{\left\{(b+c)^{2}-a^{2}\right\}\left\{a^{2}-(b-c)^{2}\right\}}{4 c^{2}} \\
=\frac{(a+b+c)(b+c-a)(a+c-b)(a+b-c)}{4 c^{2}} \\
=\frac{2 S 2(S-a) 2(S-b) 2(S-c)}{4 c^{2}}:
\end{gathered}
$$

hence the area $=\frac{A B \cdot C D}{2}=\sqrt{S(S-a)(S-b)(S-c)}$.
From either of these articles, we obtain the following Itule :

From the semi-sum of the sides, subtract each side separately; multiply the semi-sum and the three remainders together, and the square root of the product will be the area.

Ex. 1. Let $a=b$, then the area of an isosceles triangle whose base is $c=(S-a) \sqrt{S(S-c)}$

$$
\begin{gathered}
=\frac{c}{2} \sqrt{\left(a+\frac{c}{2}\right)\left(a-\frac{c}{2}\right)} \\
=\frac{c}{4} \sqrt{(2 a+c)(2 a-c)}=\frac{c}{4} \sqrt{4 a^{2}-c^{2}} .
\end{gathered}
$$

Ex. Q. If $a=b=c$, the area of an equilateral triangle
whose side is $a=S^{\frac{1}{2}}(S-a)^{\frac{3}{2}}=\left(\frac{3 a}{2}\right)^{\frac{1}{2}}\left(\frac{a}{2}\right)^{\frac{3}{2}}=\frac{a^{2} \sqrt{ } 3}{4}$.
Ex. 3. If $a, b, c$ be equal to 18,24 , and 30 respectively, we shall have

$$
\begin{gathered}
S=\frac{1}{2}(18+24+30)=\frac{1}{2}(72)=36 ; \\
\therefore S-a=36-18=18, \\
S-b=36-24=12, \\
S-c=36-30=6
\end{gathered}
$$

$$
\text { and } \begin{aligned}
\therefore \text { the area } & =\sqrt{36.18 \cdot 12.6}=\sqrt{36.36 .36} \\
& =6 \cdot 6 \cdot 6=216 .
\end{aligned}
$$

180. Cor. 1. Hence the perpendicular drawn from any angle to the opposite side is easily expressed in terms of the sides of the triangle : for

$$
\begin{aligned}
C D^{2} & =\frac{2 S 2(S-a) 2(S-b) 2(S-c)}{4 c^{2}} \\
& =\frac{4 S(S-a)(S-b)(S-c)}{c^{2}} ; \\
\text { and } \therefore C D & =\frac{2 \sqrt{S(S-a)(S-b)(S-c)}}{c}
\end{aligned}
$$

181. Cor. 2. The area of the triangle may very easily be expressed in different terms.

Thus, by (178) the area $=\frac{1}{2} b c \sin A$;

$$
\begin{aligned}
\text { or } & =\sqrt{\frac{(S-b)(S-c)}{b c} \sqrt{S(S-a) b c}} \\
& =\sin \frac{A}{2} \sqrt{S(S-a) b c} ; \\
\text { or } & =\sqrt{\frac{S(S-a)}{b c} \sqrt{(S-b)(S-c) b c}} \\
& =\cos \frac{A}{2} \sqrt{(S-b)(S-c) b c} ; \\
\text { or } & =\sqrt{\frac{(S-b)(S-c)}{S(S-a)}} \sqrt{S^{2}(S-a)^{2}} \\
& =\tan \frac{A}{2} S(S-a) ; \& c .
\end{aligned}
$$

182. Cor. 3. If the triangle be right-angled at $C$, we shall have

$$
\tan \frac{C}{2} \text { or } 1=\sqrt{\frac{(S-a)(S-b)}{S(S-c)}}
$$

and $\therefore$ the area $=S(S-c)$, or $=(S-a)(S-b):$
that is, the area of a right-angled triangle is equal to the rectangle contained by the semi-perimeter and its excess above the hypothenuse; or to the rectangle contained by the excesses of the semi-perimeter above each of the sides containing the right angle.
183. Cor. 4. From the values of the area above determined, it may be demonstrated that the areas of similar triangles are in the duplicate ratio of their homologous sides.

Let $A, B, C ; a, b, c$, and $A^{\prime}, B^{\prime}, C^{\prime} ; a^{\prime}, b^{\prime}, c^{\prime}$ be the corresponding angles and sides of two similar triangles;

$$
\text { then if } 2 S=a+b+c \text {, and } 2 S^{\prime}=a^{\prime}+b^{\prime}+c^{\prime}
$$

we have $\sin \frac{A}{2}=\sqrt{\frac{(S-b)(S-c)}{b c}}=\sin \frac{A^{\prime}}{2}=\sqrt{\frac{\left(S^{\prime}-b^{\prime}\right)\left(S^{\prime}-c^{\prime}\right)}{b^{\prime} c^{\prime}}} ;$
and $\cos \frac{A}{Q}=\sqrt{\frac{S\left(S^{\prime}-a\right)}{b c}}=\cos \frac{A^{\prime}}{2}=\sqrt{\frac{S^{\prime}\left(S^{\prime}-a^{\prime}\right)}{b^{\prime} c^{\prime}}} ;$
$\therefore$ area of the triangle $A B C$ : area of the triangle $A^{\prime} B^{\prime} C^{\prime}$

$$
\begin{aligned}
& \left.:: \sqrt{S(S-a)(S-b)(S-c)}: \sqrt{S^{\prime}\left(S^{\prime}-a^{\prime}\right)\left(S^{\prime}-b^{\prime}\right)\left(S^{\prime}-c^{\prime}\right.}\right) \\
& :: b c: b^{\prime} c^{\prime}:: c \frac{c \sin B}{\sin C}: c^{\prime} \frac{c^{\prime} \sin B^{\prime}}{\sin C^{\prime}}, \text { by }(157),:: c^{2}: c^{\prime 2} .
\end{aligned}
$$

184. Cor. 5. Since $A B C: A^{\prime} B^{\prime} C^{\prime}:: b c: l^{\prime} c^{\prime}$, if we suppose $A B C=A^{\prime} B^{\prime} C^{\prime}$, we shall have $b c=b^{\prime} c^{\prime}$,

$$
\text { and } \therefore b: b^{\prime}:: c^{\prime}: c:
$$

or, if the areas of two triangles which have one angle of the one
equal to one angle of the other, be equal, the sides about the equal angles are reciprocally proportional : and conversely.
185. To express the radius of the circle inscribed in a plane triangle in terms of the sides.

Let $A B C$ be the triangle, its angles and corresponding opposite sides being denoted by $A, B, C ; a, b, c$ as before:

bisect the angles $\boldsymbol{A}$ and $\boldsymbol{B}$ by the straight lines $\boldsymbol{A} 0, \boldsymbol{B}_{0}$ meeting in $o$, draw $o a$, ob, oc perpendicular to $B C, A C$ and $A B$ respectively; then $o$ is the centre, and $o a=o b=o c$ the radius of the inscribed circle; let this be called $r$ :
now by (157), we have

$$
\begin{gathered}
\frac{A c}{o c}=\frac{\sin A \circ c}{\sin o A c}=\frac{\cos o A c}{\sin o A c}=\frac{1}{\tan o A c}=\frac{1}{\tan \frac{A}{2}}, \\
\therefore A c=\frac{o c}{\tan \frac{A}{2}}=\frac{r}{\tan \frac{A}{2}} ; \operatorname{similarly}, B c=\frac{o c}{\tan \frac{B}{2}}=\frac{r}{\tan \frac{B}{2}} ; \\
=r\left\{\sqrt{\left.\frac{S(S-a)}{(S-b)(S-c)}+\sqrt{\frac{S(S-b)}{(S-a)(S-c)}}\right\}, \text { by }(175),}\right. \\
\text { and } \therefore c=A c+B c=r\left\{\frac{1}{\tan \frac{A}{Q}}+\frac{1}{\tan \frac{B}{2}}\right\} \\
\quad=r\left\{\frac{S(S-a)+S(S-b)}{\sqrt{S(S-a)(S-b)(S-c)}}\right\}
\end{gathered}
$$

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$$
\begin{aligned}
& =r S\left\{\frac{2 S-(a+b)}{\sqrt{S(S-a)(S-b)(S-c)}}\right\} \\
& =\frac{r c S}{\sqrt{S(S-a)(S-b)(S-c)}} ;
\end{aligned}
$$

whence we obtain

$$
\begin{aligned}
r & =\frac{\sqrt{S(S-a)(S-b)(S-c)}}{S} \\
& =\sqrt{\frac{(S-a)(S-b)(S-c)}{S}}
\end{aligned}
$$

Ex. 1. Let $a=b$, or the triangle be isosceles, then

$$
r=(S-a) \sqrt{\frac{S-c}{S}}=\frac{c}{2} \sqrt{\frac{2 a-c}{2 a+c}} .
$$

Ex. 2. If $a=b=c$, or the triangle be equilateral, we shall have

$$
r=\sqrt{\frac{(S-a)^{3}}{S}}=\sqrt{\frac{a^{2}}{12}}=\frac{a}{2 \sqrt{3}} .
$$

186. Cor. 1. Since $r S=\sqrt{S(S-a)(S-b)(S-c)}$, we have

$$
r\left(\frac{a+b+c}{2}\right)=\text { the area of the triangle. }
$$

This is also manifest from the consideration that the triangle $A B C=$ the sum of the triangles $A o B, A o C, B o C$,

$$
=\frac{A B . o c}{2}+\frac{A C . o b}{2}+\frac{B C . o a}{2}=r\left(\frac{a+b+c}{2}\right) ;
$$

and from this property the value of $r$ is very easily obtained; thus

$$
r=\frac{2 \text { area }}{a+b+c}=\frac{\sqrt{S(S-a)(S-b)(S-c)}}{S}, \text { as before. }
$$

187. Cor. 2. We may hence find the segments of the sides of the triangle made by the points of contact with the inscribed circle.

From (185) and (175) we have

$$
\frac{A c}{B c}=\frac{\tan \frac{B}{2}}{\tan \frac{A}{2}}=\frac{S-a}{S-b}, \therefore \frac{A c}{A B}=\frac{S-a}{2 S-(a+b)}=\frac{S-a}{c}
$$

wherefore $A c=S-a=\frac{1}{2}(b+c-a)$ :
In the same manner $B c=S-b=\frac{1}{2}(a+c-b)$ : and similarly of the rest.

Hence also, $A c . B c=(S-a)(S-b) ;$ and $\frac{A c}{B c}=\frac{S-a}{S-b} ;$ and so of the rest.
188. To express the radius of the circle circumscribed about a plane triangle in terms of the sides.

Let $A B C$ be the triangle, the angles and sides being $A, B, C ; a, b, c$ as before : bisect the sides $A C$ and $B C$

in the points $b$ and $a$; draw $b o$, $a o$ at right angles to $A C$ and $B C$ respectively, meeting in $o$; then is $o$ the centre, and

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$A o=B o=C o$ the radius of the circumscribed circle : call this $R$. Then

$$
\begin{aligned}
\frac{A o}{A b}=\frac{1}{\sin A o b} & =\frac{1}{\sin \frac{A o c}{2}}=\frac{1}{\sin B},(\text { Eucl. 3. 20.) }
\end{aligned} \quad \begin{aligned}
\therefore R=\frac{A b}{\sin B} & =\frac{b}{2} \frac{1}{\frac{2}{a c} \sqrt{S(S-a)(S-b)(S-c)}} \\
& =\frac{a b c}{4 \sqrt{S(S-a)(S-b)(S-c)}}
\end{aligned}
$$

Ex. 1. Let $a=b$, then in an isosceles triangle we have

$$
\begin{gathered}
R=\frac{a^{2} c}{4(S-a) \sqrt{S(S-c)}}=\frac{a^{2}}{2 \sqrt{\left(a+\frac{c}{2}\right)\left(a-\frac{c}{2}\right)}} \\
=\frac{a^{2}}{\sqrt{4 a^{2}-c^{2}}} .
\end{gathered}
$$

Ex. 2. If $a=b=c$, we shall have for an equilateral triangle,

$$
R=\frac{a^{3}}{4 \sqrt{S(S-a)^{3}}}=\frac{a^{3}}{\sqrt{3 a^{4}}}=\frac{a}{\sqrt{3}} .
$$

189. Cor. 1. By means of the last article, we have

$$
\begin{aligned}
2 R & =\frac{a b c}{2 \sqrt{S(S-a)(S-b)(S-c)}} \\
& =\frac{a \dot{b}}{\frac{2}{c} \sqrt{S(S-a)(S-b)(S-c)}} \\
& =\frac{a b}{C D}, \text { as appears from }(180)
\end{aligned}
$$

$$
\text { and } \therefore 2 R . C D=a b=A C \cdot B C:
$$

or the rectangle contained by any two sides of a plane triangle is equal to the rectangle contained by the diameter of the circumscribed circle, and the perpendicular let fall upon the remaining side from its opposite angle.
190. Cor.2. The property just mentioned which may be proved by means of similar triangles, is frequently made use of to determine the radius of the circumscribed circle.

$$
\text { For, since } C D=\frac{a b}{2 R}, \text { therefore } \frac{A B \cdot C D}{2}=\frac{a b c}{4 R}
$$

$$
\text { but } \begin{aligned}
\frac{A B \cdot C D}{2} & =\text { the area of the triangle } \\
& =\sqrt{S(S-a)(S-b)(S-c)}
\end{aligned}
$$

$$
\begin{aligned}
\text { hence } \begin{aligned}
\frac{a b c}{4 R} & =\sqrt{S(S-a)(S-b)(S-c)} \\
\text { and } R & =\frac{a b c}{4 \sqrt{S(S-a)(S-b)(S-c)}}
\end{aligned},=\text {, }
\end{aligned}
$$

as before proved.
191. Cor. 3. The segments of the angles $A, B, C$ made by the radii of the circumscribed circle may easily be found.

$$
\text { For, } \angle o A b=\frac{\pi}{2}-A o b=\frac{\pi}{2}-B=\angle o C b ;
$$

similarly, $\angle o A c=\frac{\pi}{2}-A o c=\frac{\pi}{2}-C=\angle o B c$,

$$
\text { and } \angle o B a=\frac{\pi}{2}-B o a=\frac{\pi}{2}-A=\angle o C a \text {. }
$$

192. To express the cosines of the angles of a quadrilateral in terms of the sides, two opposite angles being supplemental to each other.

Let the angles of the quadrilateral be denoted by the letters at its angular points, $A, B, C, D$ : also join $A D$, and suppose
$A B=a, B C=b, C D=c$, and $D A=d$ : draw the diagonals

$A C, B D$ and let $B D=\alpha, A C=\beta$ : then by (165) we have 2 ad $\cos A=a^{2}+d^{2}-\alpha^{2}$, from the triangle $A B D ;$
also,
$2 b c \cos C=b^{2}+c^{2}-a^{2}$, from the triangle $B C D$
now $\cos C=\cos (\pi-A)=-\cos A$;
therefore $-2 b c \cos A=b^{2}+c^{2}-a^{2}$ :
hence, by the elimination of $a$, we obtain

$$
\begin{gathered}
2(a d+b c) \cos A=a^{2}+d^{2}-b^{2}-c^{2} \\
\text { and } \cos A=\frac{a^{2}+d^{2}-b^{2}-c^{2}}{2(a d+b c)}=-\cos C:
\end{gathered}
$$

similarly, $\cos B=\frac{a^{2}+b^{2}-c^{2}-d^{2}}{2(a b+c d)}=-\cos D$.
193. To express the sines of the angles in terms of the sides of the quadrilateral.

As in (179) we have $\sin ^{2} A$

$$
\begin{aligned}
& =1-\cos ^{2} A=1-\left(\frac{a^{2}+d^{2}-b^{2}-c^{2}}{2(a d+b c)}\right)^{2} \\
& =\frac{1}{4(a d+b c)^{2}}\left\{4(a d+b c)^{2}-\left(a^{2}+d^{2}-b^{2}-c^{2}\right)^{2}\right\} \\
& =\frac{1}{4(a d+b c)^{2}}
\end{aligned}
$$

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$$
\begin{gathered}
\left\{2(a d+b c)+a^{2}+d^{2}-b^{2}-c^{2}\right\}\left\{2(a d+b c)-a^{2}-d^{9}+b^{2}+c^{2}\right\} \\
=\frac{1}{4(a d+b c)^{2}}\left\{(a+d)^{2}-(b-c)^{2}\right\}\left\{(b+c)^{2}-(a-d)^{2}\right\} \\
=\frac{1}{4(a d+b c)^{2}} \\
\{(a+b+d-c)(a+c+d-b)(a+b+c-d)(b+c+d-a)\} \\
\text { now let } a+b+c+d=2 S ; \\
\therefore b+c+d-a=2(S-a), \\
a+c+d-b=2(S-b), \\
a+b+d-c=2(S-c), \\
a+b+c-d=2(S-d) ;
\end{gathered}
$$

$$
\therefore \sin ^{2} A=\frac{1}{4(a d+b c)^{2}}\{2(S-a) 2(S-b) 2(S-c) 2(S-d)\}
$$

$$
=\frac{4}{(a d+b c)^{2}}\{(S-a)(S-b)(S-c)(S-d)\}
$$

$$
\text { and } \sin A=\frac{2}{a d+b c} \sqrt{(S-a)(S-b)(S-c)(S-d)}:
$$

and similarly of the rest.
194. By a process very similar we may express the sine, cosine, tangent, \&c. of half an angle of the quadrilateral in terms of the sides.

$$
\begin{aligned}
& \text { Thus, } 2 \sin ^{2} \frac{A}{2}=1-\cos A=1-\frac{a^{2}+d^{2}-b^{2}-c^{2}}{2(a d+b c)} \\
& =\frac{2 a d+2 b c-a^{2}-d^{2}+b^{2}+c^{2}}{2(a d+b c)}=\frac{(b+c)^{2}-(a-d)^{2}}{2(a d+b c)} \\
& =\frac{(a+b+c-d)(b+c+d-a)}{2(a d+b c)}=\frac{2(S-a) 2(S-d)}{2(a d+b c)} ; \\
& \text { and } \therefore \sin \frac{A}{2}=\sqrt{\frac{(S-a)(S-d)}{a d+b c}}=\frac{1}{\operatorname{cosec} \frac{A}{2}}
\end{aligned}
$$

similarly, $\cos \frac{A}{2}=\sqrt{\frac{(S-b)(S-c)}{a d+b c}}=\frac{1}{\sec \frac{A}{2}} ;$

$$
\text { and } \therefore \tan \frac{A}{2}=\sqrt{\frac{(S-a)(S-d)}{(S-b)(S-c)}}=\frac{1}{\cot \frac{A}{2}}
$$

similarly of the others.
195. On the same supposition to express the diagonals of the quadrilateral in terms of the sides.

The construction and notation remaining the same as in (192), we have

$$
\begin{gathered}
\cos A=\frac{a^{2}+d^{2}-a^{2}}{2 a d}, \\
\text { and } \cos C=-\cos A=\frac{b^{2}+c^{2}-a^{2}}{2 b c} ; \\
\text { therefore } \frac{a^{2}+d^{2}-a^{2}}{2 a d}=-\frac{b^{2}+c^{2}-a^{2}}{2 b c} ; \\
\text { hence } a^{2}(a d+b c)=\left(a^{2}+d^{2}\right) b c+\left(b^{2}+c^{2}\right) a d, \\
\text { and } \alpha=\sqrt{\frac{\left(a^{2}+d^{2}\right) b c+\left(b^{2}+c^{2}\right) a d}{a d+b c}} \\
=\sqrt{\frac{a^{2} b c+d^{2} b c+b^{2} a d+c^{2} a d}{a d+b c}}=\sqrt{\frac{(a c+b d)(a b+c d}{a d+b c}}
\end{gathered}
$$

similarly, since $\cos B=-\cos D$, we shall have

$$
\beta=\sqrt{\frac{\left(a^{2}+b^{2}\right) c d+\left(c^{2}+d^{2}\right) a b}{a b+c d}}
$$

$=\sqrt{\frac{a^{2} c d+b^{2} c d+c^{2} a b+d^{2} a b}{a b+c d}}=\sqrt{\frac{(a c+b d)(a d+b c)}{a b+c d}}$.
196. Cor. 1. From the last article, we have immediately

$$
a \beta=a c+b d, \text { or } A C \cdot B D=A B \cdot C D+B C \cdot A D:
$$

that is, the rectangle of the diagonals is equal to the sum of the rectangles of the opposite sides.

$$
\text { Also, } \frac{\alpha}{\beta}=\frac{a b+c d}{a d+b c} \text {, or } \frac{A C}{B D}=\frac{A B \cdot B C+C D \cdot D A}{A B \cdot A D+B C \cdot C D} \text { : }
$$

that is, the diagonals are to each other as the sums of the rectangles of the conterminous sides respectively meeting their extremities.
197. Cor. 2. The former property deduced in the last article, which may be proved geometrically, is sometimes made use of to express the sine of the sum of two arcs in terms of the sines and cosines of the arcs themselves.

For, if $A$ and $B$ be the proposed arcs, take $A P=2 A, P Q$ $=2 B$; draw the diameter $P R$ and join $A P, A Q, A R, P Q$, QR:

then $A P=\operatorname{chd} 2 A=2 \sin A, P Q=\operatorname{chd} 2 B=2 \sin B$,

$$
\begin{aligned}
& A R=\operatorname{chd}(\pi-2 A)=2 \sin \left(\frac{\pi}{2}-A\right)=2 \cos A \\
& Q R=\operatorname{chd}(\pi-2 B)=2 \sin \left(\frac{\pi}{2}-B\right)=2 \cos B
\end{aligned}
$$

$$
\text { and } A Q=\operatorname{chd}(2 A+2 B)=2 \sin (A+B):
$$

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now $P R . A Q=A P . Q R+A R . P Q$, by the last article;
$\therefore 4 \sin (A+B)=2 \sin A 2 \cos B+2 \cos A 2 \sin B$,
or $\sin (A+B)=\sin A \cos B+\cos A \sin B$, as before.
198. On the same hypothesis, to express the area of the quadrilateral in terms of the sides.

The area of $A B C D=$ the area of the triangle $A B C+$ the area of the triangle $A C D$

$$
\begin{aligned}
& =\frac{a b}{2} \sin B+\frac{c d}{2} \sin D, \text { by }(178), \\
& =\frac{a b}{2} \sin B+\frac{c d}{2} \sin (\pi-B)=\frac{a b+c d}{2} \sin B \\
& =\frac{a b+c d}{2} \frac{2}{a b+c d} \sqrt{(S-a)(S-b)(S-c)(S-d)} \\
& =\sqrt{(S-a)(S-b)(S-c)(S-d)} .
\end{aligned}
$$

199. Cor. From the last article, it appears that

$$
\Delta A B C=\frac{a b}{a b+c d} \sqrt{(S-a)(S-b)(S-c)(S-d)},
$$

and

$$
\Delta A C D=\frac{c d}{a b+c d} \sqrt{(S-a)(S-b)(S-c)(S-d)} .
$$

Also, if $\phi$ denote the angle in which the diagonals intersect each other, and $A G, C H$ be drawn perpendicular to the diagonal $B D$, we manifestly have
the area of $A B C D=\triangle A B D+\triangle C B D=\frac{B D \cdot A G}{2}+\frac{B D \cdot C H}{2}$

$$
\begin{aligned}
=\frac{B D}{2}\{A G+C H\} & =\frac{B D}{2}\{A E \sin \phi+C E \sin \phi\} \\
& =\frac{B D \cdot A C}{2} \sin \phi
\end{aligned}
$$

whence

$$
\sin \phi=\frac{2 A B C D}{A C \cdot B D}=\frac{2 \sqrt{(S-a)(S-b)(S-c)(S-d)}}{a c+b d} .
$$

200. A circle may be circumscribed about the above-mentioned quadrilateral, and its radius may be determined.

Let $R$ be the radius of the circle circumscribed about the triangle $A B D, R^{\prime}$ that of the circle circumscribed about the triangle $B C D$ : then by (188) we have

$$
R=\frac{a a d}{4 \Delta A B D}=\frac{a a d}{4 \frac{a d}{2} \sin A}=\frac{a}{2 \sin A}
$$

similarly,

$$
R^{\prime}=\frac{a b c}{4 \Delta B C D}=\frac{a b c}{4 \frac{b c}{2} \sin C}=\frac{a}{2 \sin C}
$$

now $\sin C=\sin (\pi-A)=\sin A$, and therefore $R^{\prime}=R$ :
or the circle which can be circumscribed about the triangle $A B D$, will also be circumscribed about the triangle $A C D$, and therefore about the quadrilateral $A B C D$.

$$
\begin{gathered}
\text { Also, } R=\frac{a a d}{4 \Delta A B D} \\
=\frac{a(a d+b c)}{4 \sqrt{(S-a)(S-b)(S-c)(S-d)}}, \text { by (198), } \\
=\frac{1}{4} \sqrt{\frac{(a b+c d)(a c+b d)(a d+b c)}{(S-a)(S-b)(S-c)(S-d)}}, \text { from (195). }
\end{gathered}
$$

By making any one of the sides of the quadrilateral equal to nothing, all the formulæ just proved will manifestly be true of a triangle.
201. To express the area of a regular polygon in terms of the side.

Let $A B, B C$ be two adjacent sides of the polygon each $=a$ : bisect the angles at $A$ and $B$, by the straight lines $A O$ and $B O$ meeting in $O$; then if $n$ be the number of sides of the polygon, and therefore the number of angles, we have (Euc. 1.32) the sum of the angles

$\therefore$ each of the angles $=\left(\frac{n-2}{n}\right) \pi$ :
Draw $O a$ perpendicular to the side $A B$, then $A a=B a$, and the area of the polygon

$$
\begin{gathered}
=n \triangle A O B=n \frac{A B \cdot O a}{2}=n \frac{a}{2} \frac{a}{2} \tan O A a=\frac{n a^{2}}{4} \tan \left(\frac{n-2}{n}\right) \frac{\pi}{2} \\
=\frac{n a^{2}}{4} \tan \left(\frac{\pi}{2}-\frac{\pi}{n}\right)=\frac{n a^{2}}{4} \cot \frac{\pi}{n}=\frac{n a^{2}}{4 \tan \frac{\pi}{n}} .
\end{gathered}
$$

Ex. Let $n$ be taken equal to 3, 4, 5, \&c. successively, and we shall have

$$
\text { area of a triangle }=\frac{3 a^{2}}{4} \tan \frac{1}{3} \frac{\pi}{2}=\frac{3 a^{2}}{4} \tan 30^{\circ}=\frac{a^{2} \sqrt{ } s}{4}:
$$

$$
\text { area of a square }=\frac{4 a^{2}}{4} \tan \frac{1}{2} \frac{\pi}{2}=a^{2} \tan 45^{\circ}=a^{2}:
$$

area of a pentagon $=\frac{5 a^{2}}{4} \tan \frac{3}{5} \frac{\pi}{2}=\frac{5 a^{2}}{4} \tan 54^{\circ}$ : and so on.
202. I'o express the radius of the circle inscribed in a regular polygon, in terms of the side.

Let $A B$ and $B C$ as before be two adjacent sides of the polygon, $n$ the number of sides: bisect the angles at $A$ and $B$ by the straight lines $A O$ and $B O$ meeting in $O$; draw $O a, O b$ perpendicular to $A B$ and $B C$ respectively, and therefore bisecting them ; then will $O$ be the centre and $O a=O b$ the radius of the inscribed circle; let this $=r$ :

now

$$
\begin{aligned}
\frac{O a}{B a} & =\frac{\sin O B a}{\sin a O B}=\frac{\sin O B a}{\cos O B a}=\tan O B a \\
& =\tan \left(\frac{\pi}{2}-\frac{\pi}{n}\right)=\cot \frac{\pi}{n} ; \\
\therefore r & =\frac{a}{2} \cot \frac{\pi}{n}=\frac{a}{2 \tan \frac{\pi}{n}} .
\end{aligned}
$$

203. Cor. Hence from article (201), we have
the area $=\frac{n a^{2}}{4 \tan \frac{\pi}{n}}=\frac{n a}{2} \frac{a}{\rho \tan \frac{\pi}{n}}=\frac{n a r}{2}=\frac{r}{2}$ perimeter.
204. To express the radius of the circle circumscribed about a regular polygon, in terms of the sides.

Bisect the angles $A$ and $B$ by the straight lines $A O$ and $B O$ meeting in $O$; then it is manifest from the fourth book of

Euclid's Elements, that $O$ is the centre of the circumscribed circle, and $O A=O B$ the radius, which call $R$; then


$$
\begin{gathered}
\frac{O A}{A a}=\frac{\sin A a O}{\sin A O a}=\frac{1}{\sin \left(\frac{\pi}{2}-O A a\right)}=\frac{1}{\cos O A a} \\
=\frac{1}{\cos \left(\frac{n-2}{n}\right) \frac{\pi}{2}}=\frac{1}{\cos \left(\frac{\pi}{2}-\frac{\pi}{n}\right)}=\frac{1}{\sin \frac{\pi}{n}} \\
\therefore R=\frac{a}{2 \sin \frac{\pi}{n}}
\end{gathered}
$$

205. Cor. 1. Hence also the area of the polygon

$$
\begin{aligned}
=\frac{n a^{2}}{4 \tan \frac{\pi}{n}} & =\frac{n a \cos \frac{\pi}{n}}{2} \frac{a}{2 \sin \frac{\pi}{n}}=\frac{n a R \cos \frac{\pi}{n}}{2} \\
& =\frac{R \cos \frac{\pi}{n}}{2} \text { perimeter. }
\end{aligned}
$$

206. Cor. 2. From articles (202) and (204) we have directly

$$
\frac{R}{r}=\frac{a}{2 \sin \frac{\pi}{n}} \div \frac{a}{\Omega \tan \frac{\pi}{n}}=\frac{\tan \frac{\pi}{n}}{\sin \frac{\pi}{2}}=\frac{1}{\cos \frac{\pi}{n}}
$$

Ex. Let $n$ be taken successively equal to $3,4,5, \& c$., then

$$
\begin{aligned}
\text { in a triangle } \frac{R}{r} & =\frac{1}{\cos 60^{\circ}}=2: \\
\text { in a square } \frac{R}{r} & =\frac{1}{\cos 45^{\circ}}=\sqrt{2}: \\
\text { in a pentagon } \frac{R}{r} & =\frac{1}{\cos 36^{\circ}}=\sqrt{5}-1:
\end{aligned}
$$

and so on.
207. To express the perimeter and area of a regular polygon inscribed in a circle, in terms of the radius.

Let $r$ be the radius of the circle, $n$ the number of sides of the polygon; then the angle at the centre of the circle subtended by each side $A B$ is $\frac{2 \pi}{n}$;

now by article (59), we have

$$
\begin{gathered}
\frac{A B}{r}=\operatorname{chd} A O B=\operatorname{chd} \frac{2 \pi}{n}=2 \sin \frac{\pi}{n}, \text { by }(76) \\
\therefore A B=2 r \sin \frac{\pi}{n}
\end{gathered}
$$

and the perimeter of the polygon $=n A B=2 n r \sin \frac{\pi}{n}$;
Again,
$\triangle A O B=\frac{A O \cdot B O}{2} \sin A O B$, by $(178),=\frac{r^{2}}{2} \sin \frac{2 \pi}{n} ;$
and the area of the polygon $=n \Delta A O B=\frac{n r^{2}}{2} \sin \frac{2 \pi}{n}$

$$
=\frac{r}{2} \cos \frac{\pi}{n} 2 n r \sin \frac{\pi}{n}=\frac{r}{2} \cos \frac{\pi}{n} \text { perimeter. }
$$

208. To express the perimeter and area of a regular. polygon circumscribed about a circle, in terms of the radius.

Let $r$ be the radius, $n$ the number of sides of the polygon, then the angle at the centre subtended by each side $A B$ is $\frac{2 \pi}{n}$; draw $O a$ perpendicular to $A B$ which is therefore bisected in $a$ :

then, $A B=2 A a=2 O a \tan A O a=2 r \tan \frac{\pi}{n}$;
therefore the perimeter $=2 n r \tan \frac{\pi}{n}$.
Also, $\triangle A O B=\frac{A B \cdot O a}{2}=A a \cdot O a=r^{2} \tan \frac{\pi}{n}$;
$\therefore$ the area of the pologon $=u r^{2} \tan \frac{\pi}{n}$

$$
=\frac{r}{2} 2 n r \tan \frac{\pi}{n}=\frac{r}{2} \text { perimeter. }
$$

209. Cor. From the last two articles, if $P, P^{\prime}$ and $A_{\text {, }}$ $A^{\prime}$ represent respectively the perimeters and areas of the inscribed and circumscribed regular polygons of the same number $n$ of sides, we perceive that

$$
\frac{P}{P^{\prime}}=\frac{\sin \frac{\pi}{n}}{\tan \frac{\pi}{n}}=\cos \frac{\pi}{n} ; \text { and } \frac{A}{A^{\prime}}=\frac{\sin \frac{2 \pi}{n}}{2 \tan \frac{\pi}{n}}=\cos ^{\circ} \frac{\pi}{n}
$$

210. To express the periphery of a circle in terms of the radius.

Let $p$ and $p^{\prime}$ represent the perimeters of two regular polygons of $n$ sides, the former inscribed in, the latter circumscribed about, a circle whose radius is 1 ;

$$
\begin{aligned}
& \text { then } p=2 n \sin \frac{\pi}{n}, \text { by (207), } \\
& \text { and } p^{\prime}=2 n \tan \frac{\pi}{n} \text {, by (208); } \\
& \text { therefore } \frac{p}{p^{\prime}}=\frac{\sin \frac{\pi}{n}}{\tan \frac{\pi}{n}}=\cos \frac{\pi}{n}:
\end{aligned}
$$

and if we suppose the value of $n$ to be increased indefinitely, the value of $\frac{\pi}{16}$ will be indefinitely diminished,

$$
\text { and } \therefore \cos \frac{\pi}{n}=1 \text {, or } p=p^{\prime} \text { : }
$$

now the periphery of the circle evidently lies between $p$ and $p^{\prime}$, and therefore in this case is equal to either of them; hence on this supposition $\frac{1}{n}$ th part of the perimeter of the polygon is equal to $\frac{1}{n}$ th part of the periphery of the circle;

$$
\text { that is, } 2 \sin \frac{\pi}{n}=\frac{2 \pi}{n}=2 \tan \frac{\pi}{n} \text {, or } \sin \frac{\pi}{n}=\frac{\pi}{n}=\tan \frac{\pi}{n} \text {; }
$$

therefore the perimeter of a polygon described about the circle whose radius is $r$

$$
=2 n r \tan \frac{\pi}{n},
$$

and the circumference of the circle $=2 n r \tan \frac{\pi}{n}$ when $n$ is infinite, $=2 n r \frac{\pi}{n}=2 \pi r$.
211. Cor. 1. Let $a$ be an arc of a circle whose radius is $r, A^{0}$ the angle subtended by it at the centre; then by (Eucl. 6.33) we have

$$
2 \pi r: a:: 2 \pi: A^{0},
$$

and thence $A^{0}=\frac{2 \pi \alpha}{2 \pi r}=\frac{\alpha}{r}$ :
or an angle is equal to the corresponding arc divided by the radius.
212. Cor. 2. From what has been proved in (210), if $r$ and $r^{\prime}$ be the radii of two circles, $D$ and $D^{\prime}$ their diameters, $C$ and $C^{\prime}$ their circumferences, it appears that

$$
\begin{gathered}
C=2 \pi r, \text { and } C^{\prime}=2 \pi r^{\prime} ; \\
\text { and } \therefore \frac{C}{C^{\prime}}=\frac{2 \pi r}{2 \pi r^{\prime}}=\frac{r}{r^{\prime}}=\frac{2 r}{2 r^{\prime}}=\frac{D}{D^{\prime}} \text { : }
\end{gathered}
$$

that is, the circumferences of circles are proportional to their radii or diameters.

The properties proved in this and the preceding article were assumed in articles (S) and (4); but it may be observed that no conclusion was drawn from them, upon which any of the propositions on the trigonometrical functions of arcs or angles in any way depénd.
213. Cor. 3. From the demonstration of (210) it appears that if a circular arc be continually diminished, it approaches continually to a ratio of equality with its sine or tangent: also, since chd $A=2 \sin \frac{A}{2}$ by (76) $=2 \frac{A}{2}$ (if $A$ be indefinitely diminished) $=A$, we conclude generally that the sine,
chord, and tangent of a circular arc are all ultimately equal to one another, and to the arc itself.
214. To express the area of a circle in terms of the radius.

If $A$ and $A^{\prime}$ denote the areas of two regular polygons of the same number $n$ of sides, described, in and about the circle whose radius is $r$, we have seen by (207) and (208) that

$$
\begin{gathered}
A=\frac{n r^{2}}{2} \sin \frac{2 \pi}{n}, \text { and } A^{\prime}=n r^{2} \tan \frac{\pi}{n} \\
\text { hence } \frac{A}{A^{\prime}}=\frac{1}{2} \frac{\sin \frac{2 \pi}{n}}{\tan \frac{\pi}{n}} \\
=\cos ^{2} \frac{\pi}{n}=1, \text { if } n \text { be indefinitely increased }
\end{gathered}
$$

$\therefore A=A^{\prime}$; and on this supposition the area of the circle is equal to either of them, that is,
the area of the circle $=\frac{n r^{2}}{2} \sin \frac{2 \pi}{n}$ when $n$ is infinite,

$$
=\frac{n r^{2}}{2} \frac{2 \pi}{n} \text { by }(210)=\pi r^{2}:
$$

the area also $=\frac{r}{2} 2 \pi r=\frac{r}{2}$ the circumference, from (210).
If the radius $=1$, we have the area $=\pi$ : that is, $\pi$ which represents the semi-circumference of a circle whose radius is 1 , will also represent the area.
215. Cor. 1. By means of the last article, the area of a circular sector is easily found.

For, let $A^{0}$ be the angle of the sector, $a$ the arc; then (Eucl.6.33) the area of the sector: the area of the quadrant

$$
:: \alpha: \frac{2 \pi r}{4}:: \frac{\alpha}{r}: \frac{\pi}{2}:: A^{0}: \frac{\pi}{2}, \text { from (211); }
$$

$\therefore$ the area of the sector $=\frac{A}{\frac{\pi}{2}}$ area of the quadrant $=\frac{2 A}{\pi} \frac{\pi r^{2}}{4}$ from $(\underline{Q} \mid 4)=\frac{A r^{2}}{2}=\frac{r}{2} A r=\frac{r}{\mathcal{Q}}$ the $\operatorname{arc}$, from (Q11).
216. Cor. 2. Hence it is easily shewn that the areas of circles are proportional to the squares of their radii, diameters, or circumferences.

For, let $r, r^{\prime}$ be the radii of two circles,
$D, D^{\prime}$ the diameters, $C, C^{\prime}$ the circumferences; then by (214) we have $A=\pi r^{2}$ and $A^{\prime}=\pi r^{\prime 2}$;
and $\therefore \frac{A}{A^{\prime}}=\frac{\pi r^{2}}{\pi r^{\prime 2}}=\frac{r^{2}}{r^{\prime 2}}=\frac{4 r^{2}}{4 r^{\prime 2}}=\frac{D^{2}}{D^{\prime 2}}=\frac{C^{2}}{C^{\prime 2}}$, by (212).

## CHAP. V.

On the Solution of Triangles and the Application of Trigonometry to the Mensuration of Heights, Distances, \&c.
217. IN every triangle there are six parts, the three sides and the three angles; and if $a, b, c$ represent the former, and $A, B, C$ the angles subtended by them respectively, it has been proved in (163) that their mutual dependance upon one another is expressed by the equations

$$
\begin{aligned}
& a=b \cos C+c \cos B \\
& b=a \cos C+c \cos A \\
& c=a \cos B+b \cos A:
\end{aligned}
$$

now, since $u$ independent equations are in general necessary and sufficient for the determination of $n$ unknown quantities, it is manifest that if three of the above-mentioned quantities be given, the other three may generally be found:

On further examination however, it will appear that when the three parts given are the angles, the magnitudes of the sides will be indeterminate, though their ratios to one another may be found; for, in addition to the dependance expressed in the equations just mentioned, the sides and angles are further connected by the equations proved in (158), namely

$$
\frac{a}{b}=\frac{\sin A}{\sin B}, \frac{a}{c}=\frac{\sin A}{\sin C}, \text { and } \frac{b}{c}=\frac{\sin B}{\sin C}
$$

hence, by division and substitution we have immediately

$$
\frac{a}{b}=\cos C+\frac{c}{b} \cos B=\cos C+\frac{\sin C}{\sin B} \cos B
$$

$$
\begin{aligned}
& \frac{b}{c}=\cos A+\frac{a}{c} \cos C=\cos A+\frac{\sin A}{\sin C} \cos C \\
& \frac{c}{a}=\cos B+\frac{b}{a} \cos A=\cos B+\frac{\sin B}{\sin A} \cos A:
\end{aligned}
$$

therefore if $A, B, C$ be given, and the latter sides of either of these two sets of equations be called $m, n, p$, respectively, we have

$$
\frac{a}{b}=m, \frac{b}{c}=n \text { and } \frac{c}{a}=p
$$

from which it is evident that the magnitudes of $a, b, c$ cannot be determined, though their ratios to each other are found.

From what has been said, it follows that in every triangle, if any three parts not all angles be given, the remaining parts can be found; and the reason of the exception above stated is still further apparent from the circumstance that the lengths of the sides of triangles may be increased or diminished, while the magnitudes of the angles remain the same.
218. If one of the angles of the triangle be equal to $90^{\circ}$, or the triangle be right-angled, it follows that this angle may in all cases be considered as one of the parts which are given, and therefore that only two other parts will be necessary and sufficient for the determination of all the rest; the same exception being made, and for the same reason as in obliqueangled triangles.
219. From the considerations of the last two articles, it is manifest that the solutions of all right-angled triangles are comprised in those of the two following cases:
I. When one side and one angle are given :
II. When two sides are given :
and the solutions of all oblique-angled triangles in the following four :
I. When one side and two angles are given:
II. When two sides and the angle opposite one of them, are given:
III. When two sides and the angle included by them, are given:
IV. When the three sides are given :
and the investigations of the solutions of these cases in order, will be contained in the following articles.

## Solution of right-angled triangles.

Case I, in which one side and one angle are given.
220. Let $a, b, c$ be the sides of the triangle, $A, B, C$ the angles subtended by them respectively, $C$ being the right angle ; then since $A+B=\frac{\pi}{2}$, (Euc. 1. 32), if one of these angles be known, the other is likewise found:

$$
\begin{aligned}
& \text { also, } \begin{aligned}
& \frac{a}{b}=\frac{\sin A}{\sin B}=\frac{\sin A}{\cos A}=\tan A=\cot B \\
& \frac{a}{c}=\frac{\sin A}{\sin C}=\frac{\sin A}{\sin \frac{\pi}{2}}=\sin A=\cos B \\
& \text { and } \frac{b}{c}=\frac{\sin B}{\sin C}=\frac{\sin B}{\sin \frac{\pi}{2}}=\sin B=\cos A
\end{aligned},
\end{aligned}
$$

and from these equations, if $A$ or $B$ and any one of the quantities $a, b, c$, be given, all the rest may be determined.
221. Ex. 1. Given the side $a$ and the opposite angle $A$, to find the rest.

$$
\text { Here } B=\frac{\pi}{2}-A, \text { and is therefore found : }
$$

also,

$$
\begin{aligned}
& \frac{a}{b}=\frac{\sin A}{\sin B}=\frac{\sin A}{\cos A}=\tan A, \text { or } b=\frac{a}{\tan A}=a \cot A \\
& \text { and } \frac{c}{a}=\frac{\sin C}{\sin A}=\frac{1}{\sin A}, \text { or } c=\frac{a}{\sin A}=a \operatorname{cosec} A
\end{aligned}
$$

hence $b$ and $c$ are also found.
These values of $b$ and $c$ being adapted to the radius $r$ by means of ( 60 ), become

$$
b=\frac{r a}{\tan A}, c=\frac{r a}{\sin A}
$$

and taking the logarithms of both sides of each, we have

$$
\begin{aligned}
& \log b=\log r+\log a-\log \tan A ; \\
& \log c=\log r+\log a-\log \sin A ;
\end{aligned}
$$

from which, by means of logarithmic tables, the logarithms of $b$ and $c$, and therefore $b$ and $c$ themselves, may be found.
222. In the expressions for the logarithms of $b$ and $c$ just found, the radius $r$ has been introduced, because the natural ${ }^{-}$ sines, cosines, \&c. being all calculated to the radius 1 , the logarithms of many of them would of course be negative or decimals; and to avoid this, the radius used in the tables of logarithms as has been observed in (150) is generally supposed to be ten thousand millions, and consequently its logarithm to be 10 .

Hence, therefore the equations above given become

$$
\begin{aligned}
& \log b=10+\log a-\log \tan A \\
& \log c=10+\log a-\log \sin A
\end{aligned}
$$

To illustrate what has just been said, let us suppose

$$
\begin{aligned}
a & =4, \text { and } A=53^{\circ} 7^{\prime} 54^{\prime \prime} ; \\
\therefore B & =90^{\circ}-53^{\circ} 7^{\prime} 54^{\prime \prime}=36^{\circ} 52^{\prime} 6^{\prime \prime}:
\end{aligned}
$$

$$
\text { also } \begin{aligned}
\log b & =10+\log 4-\log \tan 53^{0} 7^{\prime} 54^{\prime \prime} \\
& =10+0.60206-10.12494 \\
& =0.47712 \\
& =\log 3
\end{aligned}
$$

therefore $b=3$ :

$$
\text { and } \begin{aligned}
\log c & =10+\log 4-\log \sin 53^{\circ} 7^{\prime} 54^{\prime \prime} \\
& =10+0.60206-9.90309 \\
& =0.69897 \\
& =\log 5
\end{aligned}
$$

therefore $c=5$.
223. The quantities, $10-\log \sin A, 10-\log \tan A, \& c$. are called the Arithmetic Complements of $\log \sin A, \log \tan A$, \&c. and it is manifest that if we denote these complements by $\operatorname{colog} \sin A, \operatorname{colog} \tan A, \& c$. we shall from (222) have

$$
\begin{aligned}
& \log b=\log a+\operatorname{colog} \tan A \\
& \log c=\log a+\operatorname{colog} \sin A:
\end{aligned}
$$

and it may here be further observed that to obtain the arithmetic complement of a logarithm, it is necessary merely to subtract the first digit to the right-hand from 10 , and all the rest from 9 in succession.
224. Ex.2. Given the side $c$ and the adjacent angle $A$, to find the rest.

Here, we have $B=90^{\circ}-A$, which is therefore known :

$$
\begin{aligned}
& \text { and } \frac{c}{a}=\frac{\sin C}{\sin A}=\frac{1}{\sin A}, \text { or } a=c \sin A ; \\
& \text { also } \frac{c}{b}=\frac{\sin C}{\sin B}=\frac{1}{\sin B}=\frac{1}{\cos A}, \text { or } b=c \cos A ;
\end{aligned}
$$

whence $a$ and $b$ are known.

Adapting the expressions for $a$ and $b$ to the radius $r$, and taking the logarithms of both sides as before, we have

$$
a=c \frac{\sin A}{r}, b=c \frac{\cos A}{r} ;
$$

$\therefore \log a=\log c+\log \sin \Lambda-\log r=\log c+\log \sin \Lambda-10$

$$
=\log c-(10-\log \sin A)=\log c-\operatorname{colog} \sin A ;
$$

$$
\log b=\log c+\log \cos A-\log r=\log c+\log \cos A-10
$$

$$
=\log c-(10-\log \cos A)=\log c-\operatorname{colog} \cos A:
$$

and from these equations, the logarithms of $a$ and $b$, and thence $a$ and $b$ themselves, are found.

Case II, in which two sides are given.
225. Using the same notation as before, we have (Euc. I. 47.), $c^{2}=a^{2}+b^{2}$, whence if any two of the quantities $a, b, c$ be given, the remaining one is found:

$$
\text { also, } \begin{aligned}
\frac{a}{b} & =\frac{\sin A}{\sin B}=\frac{\sin A}{\cos A}=\tan A=\cot B ; \\
\frac{a}{c} & =\frac{\sin A}{\sin C}=\frac{\sin A}{\sin \frac{\pi}{2}}=\sin A=\cos B ; \\
\frac{b}{c} & =\frac{\sin B}{\sin C}=\frac{\sin B}{\sin \frac{\pi}{2}}=\sin B=\cos A ;
\end{aligned}
$$

from which, if any two of the quantities $a, b, c$ be given, the angles of the triangle will be found.
226. Ex. 1. Given the sides $a$ and $b$, to find the rest.

$$
\begin{gathered}
\text { Here, } c=\sqrt{a^{2}+b^{2}} \text { is found } \\
\text { also, } \frac{a}{b}=\frac{\sin A}{\sin B}=\tan A=\frac{1}{\tan B},
\end{gathered}
$$

$$
\therefore \tan A=\frac{a}{b}, \text { and } \tan B=\frac{b}{a} ;
$$

and hence $A$ and $B$ are found.
Adapting the expressions above deduced to the radius $r$, and taking the logarithms of both sides of the equations, we shall have

$$
\begin{aligned}
& \qquad \tan A=r \frac{a}{b}, \tan B=r \frac{b}{a}, \\
& \quad \log \tan A=\log r+\log a-\log b \\
& =(10-\log b)+\log a=\operatorname{colog} b+\log a ; \\
& \text { and } \log \tan B=\log r+\log b-\log a \\
& =(10-\log a)+\log b=\operatorname{colog} a+\log b ;
\end{aligned}
$$

and thus, by means of the tables, the logarithms of $\tan A$ and $\tan B$, and therefore $A$ and $B$ themselves, are found.
227. If the values of $a$ and $b$ be expressed in numbers, we have only to add together their squares, and by extracting the square root, to obtain the value of $c$; but if these quantities involve trigonometrical functions of angles, the value of $c$ may be adapted to logarithmic computation by the following process :

$$
\text { since } c=\sqrt{a^{2}+b^{2}}=a \sqrt{1+\frac{b^{2}}{a^{2}}}
$$

assume a subsidiary angle $\theta$ such that $\tan \theta=\frac{b}{a}$;

$$
\text { therefore } c=a \sqrt{1+\tan ^{2} \theta}=a \sec \theta ;
$$

and to the radius $r$ we have $\tan \theta=\frac{r b}{a} ; c=\frac{a \sec \theta}{r}$;
whence $\log \tan \theta=\log r+\log b-\log a=10+\log b-\log a$, from which $\theta$ is found :
also $\log c=\log a+\log \sec \theta-\log r=\log a+\log \sec \theta-10$, from which the value of $c$ is obtained;
or if the logarithmic secants be not found in the tables,

$$
\begin{aligned}
\log c & =\log a+20-\log \cos \theta-10, \text { by }(151) \\
& =10+\log a-\log \cos \theta
\end{aligned}
$$

from which the value of $c$ is readily determined.
228. Ex. 2. Given the sides $a$ and $c$, to find the rest.

In this we have, $b=\sqrt{c^{2}-a^{2}}$, which is found; and by (158), $\frac{a}{c}=\frac{\sin A}{\sin C}=\sin A=\cos B ;$
whence $A$ and $B$ are determined.

As in (226), we have to the radius $r, \sin A=r \stackrel{a}{c}=\cos B$, $\therefore \log \sin A=\log r+\log a-\log c=10+\log a-\log c=\log \cos B$, from which, by the tables, $A$ and $B$ are found:
also, as in (227), since $b=\sqrt{c^{2}-a^{2}}=c \sqrt{1-\frac{a^{2}}{c^{2}}}$;
assume $\cos \theta=\frac{a}{c}$, then $b=c \sqrt{1-\cos ^{2} \theta}=c \sin \theta$;
and to the radius $r$ we have $\cos \theta=\frac{r a}{c}$, and $b=\frac{c \sin \theta}{r}$ :
hence $\log \cos \theta=\log r+\log a-\log c=10+\log a-\log c$, which gives the value of $\theta$;
and $\log b=\log c+\log \sin \theta-\log r=\log c+\log \sin \theta-10$, from which $b$ is determined.

## Solution of oblique-angled Triangles.

Case I, in which one side and two angles are given.
229. In the triangle, let $a, b, c ; A, B, C$ be the sides and opposite angles respectively; then since $A+B+C=\pi$, (Euc. 1 . 32), if any two of the angles $A, B, C$ be given, the remaining one is found :

$$
\text { also, } \frac{a}{b}=\frac{\sin A}{\sin B}, \frac{a}{c}=\frac{\sin A}{\sin C}, \frac{b}{c}=\frac{\sin B}{\sin C}
$$

from which equations it is manifest that if any one of the quantities $a, b, c$, and any two of the quantities $A, B, C$ be given, the rest may be found.
230. Ex. Given the side $a$ and the angles $A, B$, to find the rest.

Since $A+B+C=\pi$, we have $C=\pi-(A+B)$, which is found;

$$
\begin{aligned}
& \text { also, } \frac{a}{b}=\frac{\sin A}{\sin B}, \therefore b=a \frac{\sin B}{\sin A} \\
& \text { and } \frac{a}{c}=\frac{\sin A}{\sin C}=\frac{\sin A}{\sin (A+B)}, \therefore c=a \frac{\sin (A+B)}{\sin A}
\end{aligned}
$$

and thence $b$ and $c$ are also determined.
The valnes of $b$ and $c$ just found being already adapted to any radius, we have immediately,

$$
\begin{aligned}
& \log b=\log a+\log \sin B-\log \sin A \\
& \log c=\log a+\log \sin (A+B)-\log \sin A
\end{aligned}
$$

therefore, by means of the tables, the logarithms of $b$ and $c$, and thence $b$ and $c$ themselves are obtained.

Case II, in which two sides and the angle opposite one of them, are given.
231. Retaining the notation of (229), we have

$$
\frac{a}{b}=\frac{\sin A}{\sin B}, \frac{a}{c}=\frac{\sin A}{\sin C}, \frac{b}{c}=\frac{\sin B}{\sin C}
$$

and from these equations, if two of the quantities $a, b, c$ and $a$ corresponding one of the quantities $A, B, C$ be given, all the rest may be found.
232. Ex. Given the sides $a, b$ and the angle $A$, to find the rest.

$$
\text { Here, } \frac{a}{b}=\frac{\sin A}{\sin B}, \therefore \sin B=\frac{b}{a} \sin A
$$

from which $B$ is found, and hence $C=\pi-(A+B)$ is determined:

$$
\text { also, } \frac{a}{c}=\frac{\sin A}{\sin C}, \quad \therefore c=a \frac{\sin C}{\sin A},
$$

which is therefore found.
The side $c$ as here found involves the value of $C$ which is not one of the quantities given, though it has been determined in the previous part of the solution. The same side may however be found in terms of $a, b$ and $A$ only:

$$
\text { for, } \cos A=\frac{b^{2}+c^{2}-a^{2}}{2 b c}, \text { by (165), }
$$

$\therefore 2 b c \cos A=b^{2}+c^{2}-a^{2}$, and $c^{2}-2 b c \cos A=a^{2}-b^{2}$,
from which $c=b \cos A \pm \sqrt{a^{2}-b^{2} \sin ^{2} A}$.
The equations $\sin B=\frac{b}{a} \sin A$, and $c=a \frac{\sin C}{\sin A}$,
are already adapted to any 1 adius;

$$
\begin{aligned}
\therefore \log \sin B & =\log b-\log a+\log \sin A ; \\
\quad \text { and } \log c & =\log a+\log \sin C-\log \sin A,
\end{aligned}
$$

which, by the tables, give the values of $B$ and $c$.
The equation $c=b \cos A \pm \sqrt{a^{2}-b^{2} \sin ^{2} A}$, is not much used, owing to the difficulty of adapting it to logarithnic computation, and is indeed rendered almost unnecessary by the facility with which the angles $B$ and $C$ are determined.
233. Cor. In the last example we have seen that $\sin B=\frac{b}{a}$ $\sin A ;$ and because the sine of an angle and the sine of its supplement (20) are equal, we are left in doubt whether the angle $B$ should be acute or obtuse. If however the side $b$ adjacent to the given angle $A$ be less than the side $a$ which is opposite to it, it follows (Euc. 1. 18) that the angle $B$ is less than the angle $A$, and therefore the ambiguity is in this case removed. But if $b$ be greater than $a$, the case remains ambiguous, as is also easily shewn by geometrical construction.


For, at the point $A$ in the indefinite straight line $A D$ make the angle $C A D$ equal to the given angle $A$; take $A C$ equal to the side $b$, and with centre $C$ and radius equal to the side $a$, describe a circular arc, which, since $a$ is less than $b$, will, cut $A D$ in two points $B, B$ on the same side of $A$ : therefore each of the triangles $A B C$ possesses the same data, and consequently each of the required parts admits of two different values.

The same construction shews that if $a$ be greater than $b$, there is no ambiguity, the intersections $B, B$ being then on opposite sides of $A$.

Case III, in which two sides and the angle included by them, are given.
234. The same notation remaining, we have from (159)

$$
\frac{a+b}{a-b}=\frac{\tan \left(\frac{A+B}{2}\right)}{\tan \left(\frac{A-B}{2}\right)}=\frac{\cot \frac{C}{2}}{\tan \left(\frac{A-B}{Q}\right)}
$$

$$
\begin{aligned}
& \frac{a+c}{a-c}=\frac{\tan \left(\frac{A+C}{2}\right)}{\tan \left(\frac{A-C}{2}\right)}=\frac{\cot \frac{B}{2}}{\tan \left(\frac{A-C}{2}\right)} \\
& \frac{b+c}{b-c}=\frac{\tan \left(\frac{B+C}{2}\right)}{\tan \left(\frac{B-C}{2}\right)}=\frac{\cot \frac{A}{2}}{\tan \left(\frac{B-C}{2}\right)}
\end{aligned}
$$

from which, if any one of the angles $A, B, C$ and the two sides containing it be given, the difference of the two remaining angles is found: also the sum of the same angles being the supplement of the given one is known, whence the angles themselves are found: and the remaining side is then determined from (157).
235. Ex. Given the sides $a, b$ and the included angle $C$, to find the rest.

Here we have $\frac{a+b}{a-b}=\frac{\cot \frac{C}{2}}{\tan \left(\frac{A-B}{2}\right)}$,
$\therefore \tan \left(\frac{A-B}{2}\right)=\left(\frac{a-b}{a+b}\right) \cot \frac{C}{2}=\tan \frac{D}{2}$, suppose,
from which $\left(\frac{A-B}{2}\right)$ is found $=\frac{D}{2}$; then we have

$$
\frac{A+B}{2}=\frac{\pi}{2}-\frac{C}{2}, \text { and } \frac{A-B}{2}=\frac{D}{2} ;
$$

whence, by addition and subtraction, are obtained

$$
A=\frac{\pi}{2}-\left(\frac{C-D}{2}\right), \text { and } B=\frac{\pi}{2}-\left(\frac{C+D}{2}\right)
$$

which therefore both become known :

$$
\begin{gathered}
\text { also, } \frac{c}{a}=\frac{\sin C}{\sin A}=\frac{\sin C}{\sin \left(\frac{\pi}{2}-\frac{C-D}{2}\right)}=\frac{\sin C}{\cos \left(\frac{C-D}{2}\right)}, \\
\therefore c=\frac{a \sin C}{\cos \left(\frac{C-D}{2}\right)}, \text { which becomes known. }
\end{gathered}
$$

The value of $c$ may also be found directly, without previously determining the angles $A$ and $B$, for in (167) it has been proved that

$$
c=\sqrt{a^{2}+b^{2}-c} \cos C
$$

The equation, $\tan \left(\frac{A-B}{2}\right)=\left(\frac{a-b}{a+b}\right) \cot \frac{C}{2}$,
is already adapted to any radius, and if $a$ and $b$ be numerical magnitudes, it is also adapted to logarithmic computation; thus, $\log \tan \left(\frac{A-B}{2}\right)=\log (a-b)-\log (a+b)+\log \cot \frac{C}{2}$, from which by the tables, $\left(\frac{A-B}{2}\right)$ becomes known.

If however $a$ and $b$ be not numbers, but involve trigonometrical functions of angles, assume $\frac{a}{b}=\tan \theta$,

$$
\therefore \tan \left(\frac{A-B}{2}\right)=\left(\frac{a-b}{a+b}\right) \cot \frac{C}{2}=\left\{\frac{\frac{a}{b}-1}{\frac{a}{b}+1}\right\} \cot \frac{C}{2}
$$

$=\left(\frac{\tan \theta-1}{\tan \theta+1}\right) \cot \frac{C}{2}=\tan \left(\theta-45^{\circ}\right) \cot \frac{C}{2}$ to the radius 1 ;
$\therefore$ to the radius $r$ we have $\frac{a}{b}=\frac{\tan \theta}{r}$,
and $\tan \left(\frac{A-B}{2}\right)=\frac{\tan \left(\theta-45^{\circ}\right)}{r} \cot \frac{C}{2}$ :

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hence $\log \tan \theta=\log r+\log a-\log b=10+\log a-\log b$, from which $\theta$ is found;
and $\log \tan \left(\frac{\Lambda-B}{2}\right)=\log \tan \left(\theta-45^{\circ}\right)+\log \cot \frac{C}{2}-\log r$

$$
=\log \tan \left(\theta-45^{\circ}\right)+\log \cot \frac{C}{2}-10
$$

from which $\left(\frac{A-B}{2}\right)$ is determined :

$$
\text { again, } \log c=\log a+\log \sin C-\log \cos \left(\frac{C-D}{2}\right)
$$

from which, by the tables, $c$ becomes known.
The other expression for $c$ is also easily adapted to logarithmic computation, for

$$
\begin{aligned}
& \left.c=\sqrt{a^{2}+b^{2}-2 a b \cos C}=\sqrt{(a-b)^{2}+2 a b(1-\cos C}\right) \\
& =\sqrt{(a-b)^{2}+4 a b \sin ^{2} \frac{C}{2}}=(a-b) \sqrt{1+\frac{4 a b}{(a-b)^{2}} \sin ^{2} \frac{C}{2}}:
\end{aligned}
$$

let $\therefore$ the subsidiary angle $\theta$ be such that $\tan ^{2} \theta=\frac{4 a b}{(a-b)^{2}} \sin ^{2} \frac{C}{2}$,

$$
\text { whence } c=(a-b) \sqrt{1+\tan ^{2} \theta}=(a-b) \sec \theta ;
$$

and to the radius $r$, we have

$$
\tan ^{2} \theta=\frac{4 a b}{(a-b)^{2}} \sin ^{2} \frac{C}{2}, \text { and } c=(a-b) \frac{\sec \theta}{r}
$$

therefore taking the logarithms of both sides, we get

$$
\log \tan \theta=\frac{1}{2}\left\{\log 4 a b+2 \log \sin \frac{C}{2}-2 \log (a-b)\right\}
$$

which determines the value of $\theta$;

$$
\text { and } \begin{aligned}
\log c & =\log (a-b)+\log \sec \theta-\log r \\
& =\log (a-b)+\log \sec \theta-10, \\
& =\log (a-b)+10-\log \cos \theta,
\end{aligned}
$$

from which $c$ is found.
236. In practice, when two sides $A C, A B$ of a triangle and the included angle $A$ are given, a perpendicular $C D$ is generally let fall upon one of the given sides $A B$ from the opposite angle $C$; the triangle is thus divided into two right-

angled triangles $A C D, B C D$, of which the sides and angles are easily found by the methods already laid down; and hence the remaining side and angles of the proposed triangle may be determined.

Case IV, in which the three sides are given.
237. The notation of the preceding articles remaining, we have seen by (173), that if $2 S=a+b+c$,

$$
\begin{aligned}
& \sin A=\frac{2}{b c} \sqrt{S(S-a)(S-b)(S-c)} \\
& \sin B=\frac{2}{a c} \sqrt{S(S-a)(S-b)(S-c)} \\
& \sin C=\frac{2}{a b} \sqrt{S(S-a)(S-b)(S-c)}:
\end{aligned}
$$

also, in (175), it has been shewn that

$$
\begin{aligned}
& \sin \frac{A}{2}=\sqrt{\frac{(S-b)(S-c)}{b c}} \\
& \sin \frac{B}{2}=\sqrt{\frac{(S-a)(S-c)}{a c}} ; \\
& \sin \frac{C}{2}=\sqrt{\frac{(S-a)(S-b)}{a b}}
\end{aligned}
$$

$$
\begin{aligned}
\text { and } \cos \frac{A}{2} & =\sqrt{\frac{S(S-a)}{b c}} ; \\
\cos \frac{B}{2} & =\sqrt{\frac{S(S-b)}{a c} ;} \\
\cos \frac{C}{2} & =\sqrt{\frac{S(S-c)}{a b}:} \\
\text { and } \therefore \tan \frac{A}{2} & =\sqrt{\frac{(S-b)(S-c)}{S(S-a)}} ; \\
\tan \frac{B}{2} & =\sqrt{\frac{(S-a)(S-c)}{S(S-b)}} ; \\
\tan \frac{C}{2} & =\sqrt{\frac{(S-a)(S-b)}{S(S-c)}}:
\end{aligned}
$$

and from any of these sets of equations the values of the angles $A, B, C$ may be determined.
238. Ex. Given the sides $a, b, c$, to find the angle $A$.

From the first set of equations given in the last article, we have

$$
\sin A=\frac{2}{b c} \sqrt{S(S-a)(S-b)(S-c)},
$$

which being adapted to the radius $r$ by means of (59) becomes

$$
\sin A=\frac{2 r}{b c} \sqrt{S(S-a)(S-b)(S-c)},
$$

and in logarithms gives

$$
\begin{gathered}
\log \sin A=10+\log 2-\log b-\log c \\
+\frac{1}{2}\{\log S+\log (S-a)+\log (S-b)+\log (S-c)\}
\end{gathered}
$$

from which the value of $A$ is found.
This solution might at first sight seem to be all that is necessary, and sufficient for the determination of an angle
in all cases; but upon examination of the tables of logarithmic sines, and from (139), it appears that when an angle becomes nearly equal to $90^{\circ}$, its logarithmic sine does not differ by any significant figure from those of several other angles nearly equal to it, though this does not happen in other cases. If therefore, from the relations of the sides of the triangle, we perceive that no one of its angles is nearly equal to $90^{\circ}$, this method of solution will not be liable to objection, owing to any imperfection in the construction of the trigonometrical and logarithmic tables, and may therefore be applied without apprehension of great inaccuracy in the result.
239. The second set of equations mentioned in (237), gives

$$
\sin \frac{A}{2}=\sqrt{\frac{(S-b)(S-c)}{b c}}, \text { to the radius } 1 \text {, }
$$

and $\therefore \sin \frac{A}{2}=r \sqrt{\frac{(S-b)(S-c)}{b c}}$, to the radius $r$;
$\therefore \log \sin \frac{A}{2}=10+\frac{1}{2}\{\log (S-b)+\log (S-c)-\log b-\log c\}$, from which $\frac{A}{Q}$, and therefore $A$, may be found.

To determine in what cases it may be expedient to make use of this solution.

Since $\sin a^{\prime}-\sin a=2 \cos \left(\frac{a^{\prime}+a}{2}\right) \sin \left(\frac{a^{\prime}-a}{2}\right)$, by (67),

$$
\text { we have } \frac{\sin a^{\prime}-\sin a}{a^{\prime}-a}=\cos \left(\frac{a^{\prime}+a}{2}\right) \frac{\sin \left(\frac{a^{\prime}-a}{2}\right)}{\left(\frac{a^{\prime}-a}{2}\right)}
$$

which, if $a^{\prime}$ and $a$ be very nearly equal to one another, becomes

$$
\frac{\sin a^{\prime}-\sin a}{a^{\prime}-a}=\cos a, \text { nearly, as appears from (213): }
$$

that is, corresponding to a given change in the angle, the change of the sine will be nearly proportional to the cosine, or the greater as the angle is the less; and hence it follows that this method of solution is to be preferred when the angle required is acute.

From this it is also manifest why the changes in the sines of angles nearly equal to $90^{\circ}$ are very small, the cosines being then nearly equal to 0 ; and also, that this second method of solution may be employed when the angle of the triangle which is required is nearly equal to a right augle.
240. From the third set of equations enumerated in (237), we have

$$
\begin{aligned}
\cos \frac{A}{2} & =\sqrt{\frac{S(S-a)}{b c}}, \text { to the radius } 1, \\
\text { and } \therefore \cos \frac{A}{2} & =r \sqrt{\frac{S(S-a)}{b c}}, \text { to the radius } r:
\end{aligned}
$$

hence $\log \cos \frac{A}{2}=10+\frac{1}{2}\{\log S+\log (S-a)-\log b-\log c\}$, from which $\frac{A}{2}$, and therefore $A$, becomes known.

To determine the eligibility of this solution in any case. Since $\cos a^{\prime} \sim \cos a=2 \sin \left(\frac{a^{\prime}+a}{2}\right) \sin \left(\frac{a^{\prime}-a}{2}\right)$, by (67),

$$
\text { we have } \frac{\cos a^{\prime} \sim \cos a}{a^{\prime}-a}=\sin \left(\frac{a^{\prime}+a}{2}\right) \frac{\sin \left(\frac{a^{\prime}-a}{2}\right)}{\left(\frac{a^{\prime}-a}{2}\right)}
$$

$$
\text { which becomes } \frac{\cos a^{\prime} \sim \cos a}{a^{\prime}-a}=\sin a \text {, nearly, }
$$

when $a$ and $a^{\prime}$ are nearly equal to one another: hence the change in the cosine corresponding to a given change in the angle varies nearly as the sime of the angle, and will therefore be the
greatest when that angle is nearly equal to $90^{\circ}$. This solution will therefore be best adapted to those cases in which the angle considerably exceeds a right angle, and consequently when its half differs not greatly from a right angle.

Hence also, this solution and the next preceding one will respectively have the advantage, according as the angle sought is obtuse or acute.
241. The fourth set of equations given in (237), has

$$
\begin{gathered}
\tan \frac{A}{2}=\sqrt{\frac{(S-b)(S-c)}{S(S-a)}} \text {, to the radius } 1, \\
\text { and } \therefore \tan \frac{A}{2}=r \sqrt{\frac{(S-b)(S-c)}{S(S-a)}}, \text { to the radius } r ; \\
\therefore \log \tan \frac{A}{2} \\
=10+\frac{1}{2}\{\log (S-b)+\log (S-c)-\log S-\log (S-a)\},
\end{gathered}
$$

and hence $\frac{A}{2}$ and $A$ are found.
As in the two preceding methods of solution,

$$
\begin{gathered}
\text { since } \tan a^{\prime}-\tan a=\frac{\sin a^{\prime}}{\cos a^{\prime}}-\frac{\sin a}{\cos a} \\
=\frac{\sin a^{\prime} \cos a-\cos a^{\prime} \sin a}{\cos a^{\prime} \cos \alpha}=\frac{\sin \left(a^{\prime}-a\right)}{\cos a^{\prime} \cos \alpha}
\end{gathered}
$$

we have

$$
\frac{\tan a^{\prime}-\tan \alpha}{a^{\prime}-a}=\frac{1}{\cos \alpha^{\prime} \cos \alpha} \frac{\sin \left(a^{\prime}-\alpha\right)}{\left(\alpha^{\prime}-\alpha\right)}
$$

$$
=\frac{1}{\cos ^{2} \alpha}=\sec ^{2} \alpha \text {, nealy, if } \alpha^{t} \text { be very nearly equal to } \alpha:
$$

hence, if the change in the angle be given, the change in its tangent will be proportional to the square of its secant, and therefore very great when that angle is nearly equal to $90^{\circ}$. This method of solution therefore, owing to the want of accuracy in the proportional parts, will be least eligible when the angle required considerably exceeds a right angle, but in other cases may generally be used with advantage.
242. This case, like the preceding, is in practice generally solved by drawing a perpendicular from one of the angles $C$ upon the side which subtends it, and thus dividing the triangle into two others $A C D, B C D$ having each a right angle at $D$ :

then, since by (168), the base $A B$ : the sum of the sides $(A C+B C)::$ the difference of the sides $(A C-B C):$ the difference or sum of the segments of the base $(A D \mp D B)$ made by a perpendicular let fall upon it from the opposite angle, according as it falls within or without the triangle: therefore the difference or sum of the segments of the base may be found, and their sum or difference $A B$ being given, the segments $A D$ and $D B$ become known, and consequently two parts in each of the right-angled triangles are determined, from which all the rest are immediately derived.

## Mensuration of Heights, Distances, \&c.

243. The solutions of triangles given in this chapter, will in all cases enable us to determine the relations between their different parts, and if the number of quantities which are given be sufficient, to find the rest ; and the mensuration of heights, distances, \&c. is merely the application of these solutions to particular instances, together with the use of certain instruments, by which the lengths of lines and the measures of angles are ascer-
tained. Gunter's chain of 4 poles or 22 yards, or common tape of 50 or 100 feet is used to measure distances; a graduated quadrant furnished with a plumb-line to measure angles of elevation or depression; a theodolite, to measure horizontal angles, and a sextant or quadrant, such as are oblique. This application of trigonometry involves no principles but what have been already explained, and no general rules can be laid down, except that such lines must be measured and such angles observed, as may most easily and conveniently lead to the determination of those which are required. The following examples will be sufficient to make this part of the subject understood.
244. Ex. 1. To find the height of an accessible object standing upon a horizontal plane.

Let $A B$ be the object standing upon the horizontal plane $B D$; measure off $B C=a$ feet, and at $C$ let the angle $A C B$ be observed : then

$$
\frac{A B}{B C}=\frac{\sin C}{\sin A}=\frac{\sin C}{\cos C}=\tan C
$$

$\therefore A B=B C \tan C=a \tan C$, which is the height of the object:

$$
\text { and } \frac{A C}{B C}=\frac{\sin B}{\sin A}=\frac{1}{\cos C}=\sec C \text {, }
$$

$\therefore A C=B C \sec C=a \sec C$, which is the distance of its summit from the place of observation.

If the observed angle were $C^{\prime}$ instead of $C$, and $h, h^{\prime}$ the heights deduced from these two angles, we have

$$
\begin{gathered}
h^{\prime}=a \tan C^{\prime}, \text { and } h=a \tan C \\
\therefore h^{\prime}-h=a\left(\tan C^{\prime}-\tan C\right)=a\left\{\frac{\sin C^{\prime}}{\cos C^{\prime}}-\frac{\sin C}{\cos C}\right\} \\
=a\left\{\frac{\sin C^{\prime} \cos C-\cos C^{\prime} \sin C}{\cos C^{\prime} \cos C}\right\}=a \frac{\sin \left(C^{\prime}-C\right)}{\cos C^{\prime} \cos C},
\end{gathered}
$$

and $\therefore \frac{h^{\prime}-h}{C^{\prime}-C}=\frac{a}{\cos C^{\prime} \cos C} \frac{\sin \left(C^{\prime}-C\right)}{C^{\prime}-C}=\frac{a}{\cos ^{2} C}$, by (187),
(if $C^{\prime}$ be very nearly equal to $C$ )

$$
=\frac{h}{\tan C \cos ^{2} C}=\frac{h}{\sin C \cos C}=\frac{2 h}{\sin 2 C}
$$

hence, if a small error of given magnitude be made in the observation of the angle, the error in the computed height will be inversely as the sine of double that angle, and therefore the least when that angle is $45^{\circ}$; which consequently points out the place in which it is desirable that the observation should be made.
245. Ex. 2. I'o find the height of an inaccessible object above a horizontal plane.

Let the point $A$ denote the place of the object : draw $A B$ perpendicular to the horizontal line $B D$, and in this line take two positions $C, D$ distant $a$ feet from each other, at which observe the angles $A C B, A D B$ equal to $C$ and $D$ respectively; then $\angle D A C=\angle B C A-\angle B D A=C-D$ :

$$
\begin{aligned}
\text { and } \frac{A C}{C D} & =\frac{\sin A D C}{\sin C A D}=\frac{\sin D}{\sin (C-D)} \\
\therefore A C & =\frac{C D \sin D}{\sin (C-D)}=\frac{a \sin D}{\sin (C-D)} \\
\text { also, } \frac{A B}{A C} & =\frac{\sin A C B}{\sin A B C}=\sin C \\
\therefore A B & =A C \sin C=\frac{a \sin C \sin D}{\sin (C-D)}
\end{aligned}
$$

which is the height required :
Likewise

$$
A C=\frac{a \sin D}{\sin (C-D)}, \text { and } A D=\frac{a \sin C}{\sin (C-D)},
$$

which are the distances of the object from the places of observation.

This example determines the distances of an inaccessible object from two stations in the same plane with it.
246. Ex. 3. To find the height of an accessible object standing upon an inclined plane.

Let $A B$ be the object upon the inclined plane, in which take any two positions $C, D$ in a line with it: suppose $B C=a$, $C D=b$, and let the angles $A C B, A D B$ be observed, and called $C$ and $D$ : then
$\frac{A C}{C D}=\frac{\sin D}{\sin (C-D)}$, and $\therefore A C=\frac{C D \sin D}{\sin (C-D)}=\frac{b \sin D}{\sin (C-D)}$ :
also, in the triangle $A C B$, we have by (234),

$$
\begin{gathered}
\frac{A C+C B}{A C-C B}=\frac{\cot \frac{C}{2}}{\tan \left(\frac{B-A}{2}\right)} ; \\
\therefore \tan \left(\frac{B-A}{2}\right)=\frac{A C-C B}{A C+C B} \cot \frac{C}{2} \\
=\frac{b \sin D-a \sin (C-D)}{b \sin D+a \sin (C-D)} \cot \frac{C}{2}=\tan \frac{E}{2}, \text { suppose, } \\
\therefore \frac{B-A}{2}=\frac{E}{2}, \text { also } \frac{B+A}{2}=\frac{\pi}{2}-\frac{C}{2}, \\
\text { whence } B=\frac{\pi}{2}-\left(\frac{C-E}{2}\right) \text { and } A=\frac{\pi}{2}-\left(\frac{C+E}{2}\right): \\
\therefore \frac{A B}{A C}=\frac{\sin C}{\sin B}=\frac{\sin C}{\cos \left(\frac{C-E}{2}\right)}
\end{gathered}
$$

$$
\text { and } A B=A C \frac{\sin C}{\cos \left(\frac{C-E}{2}\right)}=\frac{b \sin C \sin D}{\sin (C-D) \cos \left(\frac{C-E}{2}\right)}
$$

which is the height required.
247. Ex. 4. To find the height of an object standing upon a hill contiguous to a horizontal plane.

Let $A B$ be the object, $C, D$ two stations on the plane in a line with it ; produce $A B$ to meet the plane in $E$; at $C$ observe the angles $A C E, B C E$, and let them be $C$ and $C^{\prime}$ respectively; suppose $C D=a$, and let the angles $A D E, B D E$ be called $D$ and $D^{\prime}$ : then
$\frac{A C}{C D}=\frac{\sin D}{\sin (C-D)}$, and $\therefore A C=\frac{C D \sin D}{\sin (C-D)}=\frac{a \sin D}{\sin (C-D)} ;$
also, $\frac{B C}{C D}=\frac{\sin D^{\prime}}{\sin \left(C^{\prime}-D^{\prime}\right)}$, and $\therefore B C=\frac{C D \sin D^{\prime}}{\sin \left(C^{\prime}-D^{\prime}\right)}=\frac{a \sin D^{\prime}}{\sin \left(C^{\prime}-D^{\prime}\right)}$;
hence in the triangle $A C B$ we have found the two sides $A C$, $B C$ and the included angle $A C B=C-C^{\prime}$, from which $A B$ the height of the object may be determined as in the last example.

Precisely in the same manner the distance between two objects at $A$ and $B$ which are inaccessible to the observer at $C$ and $D$, and to each other, may be ascertained.
248. Ex. 5. From the top of an eminence of given height the angles of depression of two objects on the horizon in the same plane with it are observed, to find the distance between them.

Let $A B$ be the given eminence and $=a, C, D$ the objects in the horizon: draw $A E$ parallel to the horizon and
let the angles of depression $E A C, E A D$ of the objects be $C$ and $D$;

$$
\begin{aligned}
& \text { then } \begin{aligned}
\frac{A C}{A B} & =\frac{\sin A B C}{\sin A C D}=\frac{1}{\sin E A C}=\frac{1}{\sin C} \\
\therefore A C & =\frac{A B}{\sin C}=\frac{a}{\sin C} \\
\text { and } \frac{C D}{C A} & =\frac{\sin C A D}{\sin A D C}=\frac{\sin (C-D)}{\sin D} \\
\therefore C D & =\frac{a \sin (C-D)}{\sin C \sin D}
\end{aligned}
\end{aligned}
$$

which is the distance between the objects.
Hence also the distances of the objects from the point of observation are $\frac{a}{\sin C}$ and $\frac{a}{\sin D}$.
249. Ex. 6. To determine the height of an object standing upon a horizontal plane, by means of observations made at the top of another object of given height on the same plane.

Let the height of the given object $A B$ be $a$; observe the angles of depression or elevation of the bottom and top of the other $C D$, and let these be $\beta$, a respectively; then $A E$ being drawn horizontal to meet $C D$ or $C D$ produced in $E$, we have

$$
\begin{gathered}
\angle A D B=D A E=\beta \\
\therefore \frac{A D}{A B}=\frac{1}{\sin A D B}=\frac{1}{\sin \beta}, \text { whence } A D=\frac{a}{\sin \beta}: \\
\text { and } \frac{C D}{A D}=\frac{\sin C A D}{\sin A C D}=\frac{\sin (\beta \pm \alpha)}{\sin \left(\frac{\pi}{2} \mp a\right)}
\end{gathered}
$$

$$
\therefore C D=\frac{A D \sin (\beta \pm a)}{\sin \left(\frac{\pi}{2} \mp a\right)}=\frac{a \sin (\beta \pm \alpha)}{\cos \alpha \sin \beta},
$$

which is the height required.
Hence also the distances of the bottom and top of the object $C D$ from the place of observation $A$ may be found:

$$
\begin{gathered}
\text { for } A D=\frac{a}{\sin \beta}, \text { as above, } \\
\text { and } \frac{A C}{A D}=\frac{\cos \beta}{\cos \alpha}, \therefore A C=\frac{A D \cos \beta}{\cos \alpha}=\frac{a \cos \beta}{\cos \alpha \sin \beta}
\end{gathered}
$$

250. Ex. 7. To determine the height and distance of an object standing on the horizon, from two observations of its altitude, one made on the horizon, and the other at a given height above it.

Let $A$ and $B$ be the two points of observation in the same vertical line, at which the angles of elevation are $\alpha$, $\beta$ respectively; $A B=a, C D$ the object whose height and distance are required; draw $A E$ parallel to the horizon, meeting it in $\boldsymbol{E}$, then

$$
\begin{gathered}
C D=B D \tan \beta, \text { and } C E=A E \tan \alpha \\
\therefore A B=E D=C D-C E=B D(\tan \beta-\tan \alpha), \\
\text { and } B D=\frac{B A}{\tan \beta-\tan \alpha}=\frac{a \cos \alpha \cos \beta}{\sin (\beta-a)},
\end{gathered}
$$

which is the required distance :
also, $C D=B D \tan \beta=\frac{a \cos \alpha \cos \beta}{\sin (\beta-\alpha)} \tan \beta=\frac{a \cos \alpha \sin \beta}{\sin (\beta-\alpha)}$, which is the height required.
251. Ex. 8. Given the distances between three stations in a straight line with an object standing upon a horizontal plane,
and the angles of its elevation at these points $\theta, \frac{\pi}{2}-\theta$, and $2 \theta$ in order, $\theta$ being unknown, to find its height.

Let $A B$ be the object, $C, D, E$ the stations in a straight line with it, $C D=a, D E=b$ : then

$$
\angle A E B=\theta, \angle A D B=\frac{\pi}{2}-\theta, \text { and } \angle A C B=2 \theta
$$

$\therefore \angle E A C=\angle A C B-\angle A E B=2 \theta-\theta=\theta=\angle A E C$,
and $\therefore A C=C E ;$ also, $\angle E A D=\angle A D B-\angle A E B$

$$
=\frac{\pi}{2}-\theta-\theta=\frac{\pi}{2}-2 \theta
$$

hence, in the triangle $A D E$, we have

$$
\begin{aligned}
\frac{D A}{D E} & =\frac{\sin A E D}{\sin D A E}=\frac{\sin \theta}{\sin \left(\frac{\pi}{2}-2 \theta\right)} \\
& =\frac{\sin \theta}{\cos 2 \theta}, \therefore D A=\frac{b \sin \theta}{\cos 2 \theta}
\end{aligned}
$$

again, in the triangle $A C D$, we have

$$
\frac{A C}{A D}=\frac{\sin A D C}{\sin A C D}=\frac{\sin \left(\frac{\pi}{2}-\theta\right)}{\sin 2 \theta}=\frac{\cos \theta}{\sin 2 \theta}
$$

$\therefore A C=A D \frac{\cos \theta}{\sin 2 \theta}=\frac{b \sin \theta \cos \theta}{\sin 2 \theta \cos 2 \theta}=\frac{b}{2 \cos 2 \theta}=E C=a+b$,

$$
\text { whence } \cos 2 \theta=\frac{b}{2(a+b)} \text {, and is } \therefore \text { found }
$$

and $A B=\Lambda C \sin 2 \theta=(a+b) \sqrt{1-\frac{b^{2}}{4(a+b)^{2}}}$

$$
=\frac{\sqrt{4(a+b)^{2}-b^{2}}}{2} \text {, the required height. }
$$

252. Ex. 9. Given the distances between three objects, and the angles contained by the lines drawn from each of them to a certain station in the same plane with them, to find its distance from each.

Let $A, B, C$ be the objects, $D$ the station, at which let the angular distances of $A$ and $C, C$ and $B$ be $\alpha, \beta$ respectively; then calling the sides of the triangle $a, b, c$ as before, if the angles $D A C, D B C$ be $\theta$ and $\theta^{\prime}$,

$$
\begin{gathered}
\text { we shall have } \frac{\sin \alpha}{\sin \theta}=\frac{b}{D C}, \text { and } \frac{\sin \theta^{\prime}}{\sin \beta}=\frac{D C}{a}, \\
\therefore \\
\frac{\sin \alpha \sin \theta^{\prime}}{\sin \theta \sin \beta}=\frac{b}{a}, \text { or } a \sin \alpha \sin \theta^{\prime}=b \sin \beta \sin \theta:
\end{gathered}
$$

but the angles of the quadrilateral $A D B C$ being together equal to four right-angles, we have

$$
\begin{aligned}
& \theta^{\prime}=2 \pi-(\alpha+\beta+C)-\theta=\phi-\theta, \text { suppose, } \\
& \therefore b \sin \beta \sin \theta=a \sin \alpha \sin (\phi-\theta) \\
& \quad=a \sin \alpha(\sin \phi \cos \theta-\cos \phi \sin \theta)
\end{aligned}
$$

and $b \sin \beta \tan \theta=a \sin \alpha(\sin \phi-\cos \phi \tan \theta)$,

$$
\text { whence } \tan \theta=\frac{a \sin a \sin \phi}{a \sin a \cos \phi+b \sin \beta}
$$

and therefore the angles $C A D, C B D$ become known, and consequently the distances $A D, B D$ and $C D$.

This problem may be solved geometrically, by describing upon $A C$ and $B C$ two segments of circles containing angles equal to $\alpha, \beta$ respectively, and intersecting each other in the point $D$.
253. Ex. 10. To determine the height and distance of an object from its observed equal elevations at two points whose distance from each other is given, and its elevation at the middle point between them.

Let $A B$ be the object, $C, D$ the two points whose distance from each other is $a$, and $E$ the middle point between them :

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$\alpha$ the elevations at $C$ and $D$, and $\beta$ the elevation at $E:$ join $B C, B D, B E, A C, A D, A E$; then

$$
\frac{\tan \alpha}{\tan \beta}=\frac{B E}{B C}=\sin B C E
$$

since $B C E$ is manifestly right angled at $E$, because $C B D$ is isosceles;
$\therefore B E=C E \tan B C E=\frac{a}{2} \frac{\sin B C E}{\cos B C E}=\frac{a}{2} \frac{\tan \alpha}{\sqrt{\tan ^{2} \beta-\tan ^{2} \alpha}} ;$ whence $A B=B E \tan \beta$

$$
=\frac{a}{2} \frac{\tan \alpha \tan \beta}{\sqrt{\tan ^{2} \beta-\tan ^{2} \alpha}}=\frac{a}{2} \frac{\sin \alpha \sin \beta}{\sqrt{\sin (\beta-\alpha) \sin (\beta+\alpha)}},
$$

which is the height of the object:
also, $B C=A B \cot \alpha=\frac{a}{2} \frac{\cos \alpha \sin \beta}{\sqrt{\sin (\beta-\alpha) \sin (\beta+\alpha)}}=B D$;
and $B E=A B \cot \beta=\frac{a}{2} \frac{\sin \alpha \cos \beta}{\sqrt{\sin (\beta-\alpha) \sin (\beta+\alpha)}}$,
which are the horizontal distances of the object from the places of observation.
254. Ex. 11. Given the elevations of an object above a horizontal plane at three points at given distances from one another in the same straight line, to find its height.

Let $A B$ be the object standing at the point $B$ on the horizontal plane; and at the three stations $C, D, E$ in a straight line, let the observed elevations be $\alpha, \beta, \gamma$, and suppose the height $A B$ to be represented by $h$ : then

$$
B C=h \cot \alpha, B D=h \cot \beta \text { and } B E=h \cot \gamma:
$$

draw $B F$ perpendicular to $C E$, then from (166), if $C D$ and $D E$ be called $a$ and $b$, we shall have

$$
\begin{aligned}
h^{2} \cot ^{2} \alpha & =a^{2}+h^{2} \cot ^{2} \beta+2 a D F \\
\text { and } h^{2} \cot ^{2} \gamma & =b^{2}+h^{2} \cot ^{2} \beta-2 b D F
\end{aligned}
$$

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$$
\text { hence } b h^{2} \cot ^{2} \alpha=a^{2} b+b h^{2} \cot ^{2} \beta+2 a b D F, ~ \begin{aligned}
& \text { and } a h^{2} \cot ^{2} \gamma
\end{aligned}=a b^{2}+a h^{2} \cot ^{2} \beta-2 a b D F, ~ \$
$$

$\therefore$ by addition, we get

$$
\left(a \cot ^{2} \gamma+b \cot ^{2} a\right) h^{2}=a^{8} b+a b^{2}+(a+b) \cot ^{2} \beta h^{2},
$$

and consequently,

$$
\begin{aligned}
& \left\{a \cot ^{2} \gamma-(a+b) \cot ^{2} \beta+b \cot ^{2} a\right\} h^{2}=a b(a+b) \\
& \text { whence } h=\sqrt{\frac{a b(a+b)}{a \cot ^{2} \gamma-(a+b) \cot ^{2} \beta+b \cot ^{2} \alpha}}
\end{aligned}
$$

the required height :
also, $B C=h \cot \alpha=\cot \alpha \sqrt{\frac{a b(a+b)}{a \cot ^{2} \gamma-(a+b) \cot ^{2} \beta+b \cot ^{2} \alpha}}$, which is the horizontal distance of the object from one of the stations.

If we suppose $b=a$, and $\gamma=a$, the perpendicular height and horizontal distance of the object will be the same as determined in the last example.
255. Ex. 12. Four objects situated at unequal but given distances in the same straight line, appear to a spectator in the same plane with them to be at equal distances from each other, it is required to determine his position.

Let $A, B, C, D$ be the objects, $E$ the place of the eye; draw $E F$ perpendicular to $D A$ produced if necessary, and suppose $A B=a, B C=b, C D=c$;
also let $\angle A E B=\angle B E C=\angle C E D=\phi$, and $\angle E A F=\theta$ :

$$
\begin{aligned}
\text { then } \frac{\sin 3 \phi}{\sin E A D}=\frac{\sin A E D}{\sin E A D}=\frac{A D}{E D}, \\
\text { and } \frac{\sin E A D}{\sin \phi}=\frac{\sin E A D}{\sin A E B}=\frac{E B}{A B},
\end{aligned}
$$

$$
\begin{gathered}
\text { whence } \frac{\sin 3 \phi}{\sin \phi} \text { or } 3-4 \sin ^{2} \phi=\frac{A D}{A B} \frac{E B}{E D} \\
=\frac{A D}{A B} \frac{B C}{C D} \text { by }(160),=\frac{(a+b+c) b}{a c}
\end{gathered}
$$

$$
\text { and } \therefore \sin \phi=\sqrt{\frac{3 a c-(a+b+c) b}{4 a c}}
$$

consequently $\phi$ is determined :
again,

$$
\begin{gathered}
\sin \theta=\sin E A F=\frac{E F}{A E}, \text { and } \sin (\theta-2 \phi)=\sin E C F=\frac{E F}{E C} \\
\therefore \frac{\sin \theta}{\sin (\theta-2 \phi)}=\frac{C E}{A E}=\frac{B C}{A B} \text { by }(160),=\frac{b}{a}
\end{gathered}
$$

and $a \sin \theta=b \sin (\theta-2 \phi)=b(\sin \theta \cos 2 \phi-\cos \theta \sin 2 \phi)$, whence $\tan \theta=\frac{b \sin 2 \phi}{b \cos 2 \phi-a}$, and $\therefore \theta$ is found: $\therefore \frac{A E}{A B}=\frac{\sin E B A}{\sin A E B}=\frac{\sin (\theta-\phi)}{\sin \phi}$, and $A E=\frac{a \sin (\theta-\phi)}{\sin \phi}$;
whence also $A F=A E \cos \theta=\frac{a \sin (\theta-\phi) \cos \theta}{\sin \phi}$,

$$
\text { and } E F=A E \sin \theta=\frac{a \sin (\theta-\phi) \sin \theta}{\sin \phi}
$$

and thus the position of $E$ is complétely determined.
It is obvious that if $(a+b+c) b$ be greater than $3 a c$, the problem is impossible: also, if we suppose $a=b=c$, we shall have $\sin \phi=0$, and $A F$ and $E F$ indefinitely great; that is, equidistant objects in the same straight line appear to be so to a spectator indefinitely distant.
256. It would be no difficult matter to extend the number of examples on this subject, but from what has been already
done, the method of proceeding in other cases cannot but be manifest, though few or no general rules have been given.

The Trigonometrical Survey of a Country or large tract of land is conducted in a similar manner, by selecting conspicuous places which may be seen from one another as stations at which angles are observed, for instance, the mutual bearings and directions of such objects as it is intended to include; and the distances between two or more of these stations being found by actual measurement, each of the other parts of the triangles employed may be determined: this may also be verified by taking the measure of a different line and proceeding with it as with the first, so that if the whole be correct, the conclusions will necessarily be the same by each process.

In Navigation, the computations used in what is called Plane Sailing are nothing more than the solutions of rightangled triangles, the hypothenuse being termed the Distance, the other two sides the Difference of Latitude and Departure, and one of the acute angles the Course: and if any two of these quantities be given, the remaining two may be found by the methods already explained.

## CHAP. VI.

On Algebraical Expressions for the sines, cosines, \&cc. of arcs, and their sums, differences, multiples, \&c. On the general relations between the sine, cosine, \&c. of an arc, and those of its multiples and submultiples. On the general relations between the powers of the sine, cosine, \&c. of an arc, and the sines, cosines, \&c. of its multiples. On general expressions for the sine, cosine, \&c. of an arc in terms of the arc, and impossible exponential quantities.
257. T'o express the sine and cosine of an arc by means of algebraical binomials.

Since, by article (25),
$1=\cos ^{2} A+\sin ^{2} A=(\cos A+\sqrt{-1} \sin A)(\cos A-\sqrt{-1} \sin A)$,
we have $\cos A-\sqrt{-1} \sin A=\frac{1}{\cos A+\sqrt{-1} \sin A}$ :
let then $\cos A+\sqrt{-1} \sin A=x, \therefore \cos A-\sqrt{-1} \sin A=\frac{1}{x}$, and by addition and subtraction we get

$$
\begin{gathered}
2 \cos A=x+\frac{1}{x}, \text { or } \cos A=\frac{1}{2}\left(x+\frac{1}{x}\right) ; \text { and } \\
2 \sqrt{-1} \sin A=x-\frac{1}{x}, \text { or } \sin A=\frac{1}{2 \sqrt{-1}}\left(x-\frac{1}{x}\right) .
\end{gathered}
$$

By means of the relations established in the first chapter, the versed sime, chord, \&c. of $A$ might be expressed in terms of $x$, if it were necessary.
258. To express the sines and cosines of the sum and difference of two arcs by means of algebraical binomials.

Let $\cos A+\sqrt{-1} \sin A=x, \therefore \cos A-\sqrt{-1} \sin A=\frac{1}{x} ;$ and $\cos B+\sqrt{-1} \sin B=y, \therefore \cos B-\sqrt{-1} \sin B=\frac{1}{y}$ :

$$
\begin{gathered}
\text { hence } \cos A=\frac{1}{2}\left(x+\frac{1}{x}\right), \sin A=\frac{1}{2 \sqrt{-1}}\left(x-\frac{1}{x}\right) ; \\
\text { and } \cos B=\frac{1}{2}\left(y+\frac{1}{y}\right), \sin B=\frac{1}{2 \sqrt{-1}}\left(y-\frac{1}{y}\right): \\
\therefore \cos (A+B)=\cos A \cos B-\sin A \sin B \\
=\frac{1}{2}\left(x+\frac{1}{x}\right) \frac{1}{2}\left(y+\frac{1}{y}\right)-\frac{1}{2 \sqrt{-1}}\left(x-\frac{1}{x}\right) \frac{1}{2 \sqrt{-1}}\left(y-\frac{1}{y}\right) \\
=\frac{1}{4}\left\{2 x y+\frac{2}{x y}\right\}=\frac{1}{2}\left(x y+\frac{1}{x y}\right) ; \\
\cos (A-B)=\cos A \cos B+\sin A \sin B \\
=\frac{1}{2}\left(x+\frac{1}{x}\right) \frac{1}{2}\left(y+\frac{1}{y}\right)+\frac{1}{2 \sqrt{-1}}\left(x-\frac{1}{x}\right) \frac{1}{2 \sqrt{-1}}\left(y-\frac{1}{y}\right) \\
=\frac{1}{4}\left\{\frac{2 x}{y}+\frac{2 y}{x}\right\}=\frac{1}{2}\left\{\frac{x}{y}+\frac{y}{x}\right\} ;
\end{gathered}
$$

$\sin (A+B)=\sin A \cos B+\cos A \sin B$
$=\frac{1}{2 \sqrt{-1}}\left(x-\frac{1}{x}\right) \frac{1}{2}\left(y+\frac{1}{y}\right)+\frac{1}{2}\left(x+\frac{1}{x}\right) \frac{1}{2 \sqrt{-1}}\left(y-\frac{1}{y}\right)$
$=\frac{1}{4 \sqrt{-1}}\left\{2 x y-\frac{2}{x y}\right\}=\frac{1}{2 \sqrt{-1}}\left\{x y-\frac{1}{x y}\right\} ;$
$\sin (A-B)=\sin A \cos B-\cos A \sin B$

$$
\begin{gathered}
=\frac{1}{2 \sqrt{-1}}\left(x-\frac{1}{x}\right) \frac{1}{2}\left(y+\frac{1}{y}\right)-\frac{1}{2}\left(x+\frac{1}{x}\right) \frac{1}{2 \sqrt{-1}}\left(y-\frac{1}{y}\right) \\
=\frac{1}{4 \sqrt{-1}}\left\{\frac{2 x}{y}-\frac{2 y}{x}\right\}=\frac{1}{2 \sqrt{-1}}\left\{\frac{x}{y}-\frac{y}{x}\right\} .
\end{gathered}
$$

259. Cor. 1. If $\mathbb{C}$ be another arc, and $\cos C+\sqrt{-1} \sin C$ be assumed equal to $z$, we shall, by a process exactly similar, have

$$
\begin{aligned}
\cos (A+B+C) & =\frac{1}{2}\left\{x y z+\frac{1}{x y z}\right\} \\
\text { and } \sin (A+B+C) & =\frac{1}{2 \sqrt{-1}}\left\{x y z-\frac{1}{x y z}\right\} ;
\end{aligned}
$$

and generally, if there be any number of arcs whatever,

$$
\begin{aligned}
& \cos \left(A+B+C+\& c_{0}\right)=\frac{1}{2}\left\{x y z \& c_{0}+\frac{1}{x y z \& c_{0}}\right\} \\
& \sin \left(A+B+C+\& c_{0}\right)=\frac{1}{2 \sqrt{-1}}\left\{x y z \& c_{0}-\frac{1}{x y z \& c_{0}}\right\}
\end{aligned}
$$

260. Cor.2. By multiplication and addition, we obtain from the expressions just deduced,

$$
\begin{gathered}
\cos (A+B+C+8 c .)+\sqrt{-1} \sin (A+B+C+\& c .)=x y z \& c \\
=(\cos A+\sqrt{-1} \sin A)(\cos B+\sqrt{-1} \sin B) \\
(\cos C+\sqrt{-1} \sin C) \& c
\end{gathered}
$$

and by multiplication and subtraction,

$$
\begin{gathered}
\cos \left(A+B+C+8 c_{0}\right)-\sqrt{-1} \sin \left(A+B+C+\& c_{.}\right)=\frac{1}{x y z \& c} \\
=\frac{1}{(\cos A+\sqrt{-1} \sin A)} \frac{1}{(\cos B+\sqrt{-1} \sin B)} \\
\frac{1}{(\cos C+\sqrt{-1} \sin C)}, \& c
\end{gathered}
$$

$$
\begin{gathered}
=(\cos A-\sqrt{-1} \sin A)(\cos B-\sqrt{-1} \sin B) \\
\\
(\cos C-\sqrt{-1} \sin C) \& c .
\end{gathered}
$$

261. The properties proved in the last article are frequently deduced by a direct process; thus, by actual multiplication we have

$$
\begin{gathered}
(\cos A \pm \sqrt{-1} \sin A)(\cos B \pm \sqrt{-1} \sin B) \\
=(\cos A \cos B-\sin A \sin B) \pm \sqrt{-1}(\sin A \cos B+\cos A \sin B) \\
=\cos (A+B) \pm \sqrt{-1} \sin (A+B):
\end{gathered}
$$

again,

$$
\begin{aligned}
& (\cos A \pm \sqrt{-1} \sin A)(\cos B \pm \sqrt{-1} \sin B)(\cos C \pm \sqrt{-1} \sin C) \\
& =\{\cos (A+B) \pm \sqrt{-1} \sin (A+B)\}\{\cos C \pm \sqrt{-1} \sin C\} \\
& =\cos (A+B+C) \pm \sqrt{-1} \sin (A+B+C), \text { as before } ;
\end{aligned}
$$

and by the principle that if the formula be true for the sum of $n-1$ arcs, it will also be true for the sum of $n$ arcs, we have generally

$$
\begin{aligned}
& (\cos A \pm \sqrt{-1} \sin A)(\cos B \pm \sqrt{-1} \sin B) \& c \\
& =\cos (A+B+\& c .) \pm \sqrt{-1} \sin (A+B+\& c .)
\end{aligned}
$$

262. Cor. From the last article is easily deduced what was proved in (132).

For, $\cos \{A+B+C+\& \mathrm{c}\}+.\sqrt{-1} \sin \{A+B+C+\& c$.

$$
\begin{gathered}
=(\cos A+\sqrt{-1} \sin A)(\cos B+\sqrt{-1} \sin B) \\
(\cos C+\sqrt{-1} \sin C) \& c .
\end{gathered}
$$

$=\cos A \cos B \cos C \& c .(1+\sqrt{-1} \tan A)(1+\sqrt{-1} \tan B)$

$$
(1+\sqrt{-1} \tan C) \& c .=\cos A \cos B \cos C \& c
$$

$$
\left\{1+\sqrt{-1} S_{1}-S_{2}-\sqrt{-1} S_{3}+S_{4}+\sqrt{-1} S_{5}-\& c .\right\}
$$

the notation used in article (132) being retained:
hence, equating the possible and impossible parts respectively on both sides of this equation, we get
$\cos \{A+B+C+\& c\}=.\cos A \cos B \cos C \& c .\left\{1-S_{2}+S_{4}-\& c.\right\} ;$
and
$\sin \{A+B+C+\& c\}=.\cos A \cos B \cos C \& c .\left\{S_{1}-S_{3}+S_{5}-\& c.\right\}$

$$
\begin{gathered}
\therefore \tan \left(A+B+C+\& c_{.}\right)=\frac{\sin (A+B+C+\& c .)}{\cos (A+B+C+\& c .)} \\
=\frac{S_{1}-S_{3}+S_{5}-\& c .}{1-S_{2}+S_{4}-\& c .}
\end{gathered}
$$

as before.
263. To express the sine and cosine of the multiple of an arc by means of algebraical binomials involving the same quantities as the sine and cosine of the arc itself.

Let $\cos A+\sqrt{-1} \sin A=x, \therefore \cos A-\sqrt{-1} \sin A=\frac{1}{x}$,

$$
\text { and } 2 \cos A=x+\frac{1}{x}, \varrho \sqrt{-1} \sin A=x-\frac{1}{x}
$$

hence, $\cos 2 A=2 \cos ^{2} A-1=2 \frac{1}{4}\left(x+\frac{1}{x}\right)^{2}-1$

$$
=\frac{1}{2}\left(x^{2}+\frac{1}{x^{2}}\right), \text { or } 2 \cos 2 A=x^{2}+\frac{1}{x^{2}}
$$

and
$\sin 2 A=2 \sin A \cos A=2 \frac{1}{2 \sqrt{-1}}\left(x-\frac{1}{x}\right) \frac{1}{2}\left(x+\frac{1}{x}\right)$
$=\frac{1}{2 \sqrt{-1}}\left(x^{2}-\frac{1}{x^{2}}\right)$, or $2 \sqrt{-1} \sin 2 \Lambda=x^{2}-\frac{1}{x^{2}}:$ and so on :
if then,

$$
\begin{gathered}
2 \cos (n-1) A=x^{n-1}+\frac{1}{x^{n-1}} \\
\text { and } \therefore 2 \sqrt{-1} \sin (n-1) A=x^{n-1}-\frac{1}{x^{n-1}},
\end{gathered}
$$

we shall have
$2 \cos n A=\frac{1}{2}\{2 \cos (n-1) A 2 \cos A-2 \sin (n-1) A 2 \sin A\}$

$$
=\frac{1}{2}\left\{\left(x^{n-1}+\frac{1}{x^{n-1}}\right)\left(x+\frac{1}{x}\right)-\frac{1}{\sqrt{-1}}\left(x^{n-1}-\frac{1}{x^{n-1}}\right) \frac{1}{\sqrt{-1}}\left(x-\frac{1}{x}\right)\right\}
$$

$=\frac{1}{2}\left(2 x^{n}+\frac{2}{x^{n}}\right)=x^{n}+\frac{1}{x^{n}}$, and $\therefore 2 \sqrt{-1} \sin n A=x^{n}-\frac{1}{x^{n}}$;
hence therefore it appears that if these formulæ be true for any one value of $n$, they are necessarily true for the next superior value: also, it has been just shewn that they are true when $n=2$, therefore they will likewise be true when $n$ is equal to $3,4,5, \& c$. ; that is, they will be generally true.

It may here be remarked, that these two formule are immediately derivable from (259) by making $A=B=C=\$$ c. and therefore $x=y=z=\& c$.
264. If $n$ be an odd number, $n+1$ will be an even one, and therefore, as we have seen by the last article, we shall have

$$
\begin{aligned}
& 2 \cos \left(\frac{n+1}{2}\right) A=x^{\frac{n+1}{2}}+\frac{1}{x^{\frac{n+1}{2}}} \\
& \text { and } 2 \sqrt{-1} \sin \left(\frac{n+1}{2}\right) A=x^{\frac{n+1}{2}}-\frac{1}{\frac{n+1}{2}}
\end{aligned}
$$

but the formulæ are equally true when $n$ is even, and therefore $\frac{n+1}{\Omega}$ an improper fraction.

Since $2 \cos ^{2} \frac{A}{2}=1+\cos A=1+\frac{1}{2}\left(x+\frac{1}{x}\right)=\frac{1}{2}\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^{2}$,

$$
\therefore 2 \cos \frac{A}{\varrho}=\sqrt{ } x+\frac{1}{\sqrt{ } x}, \text { and } \varrho \sqrt{-1} \sin \frac{A}{\varrho}=\sqrt{x}-\frac{1}{\sqrt{x}}
$$

$$
\text { hence, if } 2 \cos \left(\frac{n-1}{2}\right) A=x^{\frac{n-1}{2}}+\frac{1}{x^{\frac{n-1}{2}}}
$$

$$
\text { and } \therefore 2 \sqrt{-1} \sin \left(\frac{n-1}{2}\right) A=x^{\frac{n-1}{2}}-\frac{1}{x^{\frac{n-1}{2}}}
$$

we shall, as in the last article, have

$$
2 \cos \left(\frac{n+1}{2}\right) A=x^{\frac{n+1}{2}}+\frac{1}{x^{\frac{n+1}{2}}}
$$

and $2 \sqrt{-1} \sin \left(\frac{n+1}{2}\right) A=x^{\frac{n+1}{2}}-\frac{1}{x^{\frac{n+1}{2}}}$ :
and therefore, as before, the formula will be generally true.
265. Cor. From the preceding articles of this chapter, it may easily be proved, on the suppositions there made, that we shall have
$\mathfrak{y} \cos (n A+m \boldsymbol{B})=x^{n} y^{m}+\frac{-1}{x^{n} y^{m}} ; \bumpeq \cos (n \boldsymbol{A}-m \boldsymbol{B})=\frac{x^{n}}{y^{n}}+\frac{y^{m}}{x^{n}} ;$

$$
2 \sqrt{-1} \sin (n A+m B)=x^{n} y^{m}-\frac{1}{x^{n} y^{m}}
$$

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$$
\text { and } 2 \sqrt{-1} \sin (n A-m B)=\frac{x^{n}}{y^{m}}-\frac{y^{n \dot{c}}}{x^{n}}:
$$

these formulæ are however more curious than useful.
266. The following singular property of the chords of a circle, the discovery of which has been attributed to Vieta, Waring, \&c. may be immediately deduced from (263).

Let $P A$ be the diameter of a circle, and let there be taken any number of contiguous equal arcs $A B, B C, C D$, \&c. then, if the chords $P B, P C, P D, \& c$. be drawn, and $P B$ be assumed equal to $x+\frac{1}{x}$; we shall have

$$
P C=x^{2}+\frac{1}{x^{2}}, P D=x^{3}+\frac{1}{x^{3}}, \& c .=\& c .
$$



For, $P B=\operatorname{chd}(\pi-A B)=2 \sin \left(\frac{\pi}{2}-\frac{A B}{2}\right)=2 \cos \frac{A B}{2}=x+\frac{1}{x} ;$

$$
\begin{aligned}
P C & =\operatorname{chd}(\pi-A C)=2 \sin \left(\frac{\pi}{2}-\frac{A C}{2}\right)=2 \cos \frac{A C}{2} \\
& =2 \cos 2\left(\frac{A B}{2}\right)=x^{2}+\frac{1}{x^{2}}, \text { by }(263)
\end{aligned}
$$

$$
P D=\operatorname{chd}(\pi-A D)=2 \sin \left(\frac{\pi}{2}-\frac{A D}{2}\right)=2 \cos \frac{A D}{2}
$$

$$
=2 \cos 3\left(\frac{A B}{2}\right)=x^{3}+\frac{1}{x^{3}} ; \text { and so on. }
$$

267. Demoiore's formulx, which are

$$
(\cos A \pm \sqrt{-1} \sin A)^{n}=\cos n A \pm \sqrt{-1} \sin n A
$$

and

$$
(\cos A \pm \sqrt{-1} \sin A)^{\frac{n}{m}}=\cos \frac{n}{m} A \pm \sqrt{-1} \sin \frac{n}{m} A
$$

may also be proved by means of the expressions investigated in (263).

$$
\text { Let } x=\cos A+\sqrt{-1} \sin A, \therefore \frac{1}{x}=\cos A-\sqrt{-1} \sin A \text {; }
$$

then we shall have
$x^{n}=(\cos A+\sqrt{-1} \sin A)^{n}$, and $\frac{1}{x^{n}}=(\cos A-\sqrt{-1} \sin A)^{n}:$
but on the assumption above made we have shewn in (263) that

$$
x^{n}+\frac{1}{x^{n}}=2 \cos n A \text {, and } x^{n}-\frac{1}{x^{n}}=2 \sqrt{-1} \sin n A \text {; }
$$

therefore, by addition and subtraction, we get
$x^{n}=\cos n A+\sqrt{-1} \sin n A$, and $\frac{1}{x^{n}}=\cos n A-\sqrt{-1} \sin n A ;$
whence

$$
(\cos A+\sqrt{-1} \sin A)^{n}=\cos n A+\sqrt{-1} \sin n A
$$

and

$$
(\cos A-\sqrt{-1} \sin A)^{n}=\cos n A-\sqrt{-1} \sin n A .
$$

Again, let $n A=m B$, then $(\cos A \pm \sqrt{-1} \sin A)^{n}$ $=\cos n A \pm \sqrt{-1} \sin n A=\cos m B \pm \sqrt{-1} \sin m B$
$=(\cos B \pm \sqrt{-1} \sin B)^{m}=\left(\cos \frac{n}{m} A \pm \sqrt{-1} \sin \frac{n}{m} A\right)^{m}$, and $\therefore(\cos A \pm \sqrt{-1} \sin A)^{\frac{n}{m}}=\cos \frac{n}{m} A \pm \sqrt{-1} \sin \frac{n}{m} A$.
268. Cor. 1. If the indices $n$ and $\frac{n}{m}$ be negative, the formulæ will still hold good, by changing the algebraical sign of the arc in the latter sides of the equations.

$$
\begin{aligned}
& \text { For, }(\cos A \pm \sqrt{-1} \sin A)^{-n}=\frac{1}{(\cos A \pm \sqrt{-1} \sin A)^{n}} \\
& =\frac{1}{\left(\frac{1}{\cos A \mp \sqrt{-1} \sin A}\right)^{n}}, \text { by }(257) \text {, } \\
& =(\cos A \mp \sqrt{-1} \sin A)^{n}=\cos n A \mp \sqrt{-1} \sin n A \\
& =\cos (-n A) \pm \sqrt{-1} \sin (-n A) ; \\
& \text { and similarly of the other case. }
\end{aligned}
$$

269. Cor. 2. We may here observe that the formulæ of Demoivre contained in the last two articles are in reality only particular cases of those which were demonstrated in (260) and (261), and may be immediately derived from them by supposing $A=B=C=\& \mathrm{c}$.

Also, from the truth of the formulæ $(\cos A \pm \sqrt{-1} \sin A)^{\frac{3_{3}}{m_{6}}}$

$$
=\cos \frac{n}{m} A \pm \sqrt{-1} \sin \frac{n}{m} A, \text { if we put } 2 \cos A=x+\frac{1}{x}
$$

we may prove conversely, that $2 \cos \frac{n}{m} A=x^{\frac{n}{m}}+\frac{1}{x^{\frac{n}{m}}}$,
and $2 \sqrt{-1} \sin \frac{n}{m} A=x^{\frac{n}{m}}-\frac{1}{x^{\frac{n}{m}}}$ : and thence conclude that these formulæ are general.
270. To express the sine and cosine of the multiple of an arc in terms of the powers of the sine and cosine of the arc itself.

By Denoiure's formulæ proved in (267), we have
$\cos n A+\sqrt{-1} \sin n A=(\cos A+\sqrt{-1} \sin A)^{n}$
$=\cos ^{n} A+n \cos ^{n-1} A \sqrt{-1} \sin A-n\left(\frac{n-1}{2}\right) \cos ^{n-2} A \sin ^{2} A$
$-n\left(\frac{n-1}{2}\right)\left(\frac{n-2}{3}\right) \cos ^{n-3} A \sqrt{-1} \sin ^{5} A+8 \mathrm{c} .+(-1)^{\frac{n}{2}} \sin ^{n} A$,
by the binomial theorem :
hence, equating the impossible and possible parts of these expressions respectively, we obtain the following equations;
$=n \cos ^{n-1} A \sqrt{-1} \sin A-n\left(\frac{n-1}{2}\right)\left(\frac{n-2}{3}\right) \cos ^{n-3} A \sqrt{-1} \sin ^{3} A+\& c$.
$\therefore$ dividing both sides by $\sqrt{-1}$, we get
$\sin n A=n \cos ^{n-1} A \sin A-n\left(\frac{n-1}{2}\right)\left(\frac{n-2}{3}\right) \cos ^{n-5} A \sin ^{3} A$
$+\& c$. which will be continued to $\frac{n+1}{2}$ terms if $n$ be odd and to $\frac{n}{2}$ terms if $n$ be even:
and $\cos n A=\cos ^{n} A-n\left(\frac{n-1}{2}\right) \cos ^{n-2} A \sin ^{2} A+8 \mathrm{c}$.
which will be continued to $\frac{n+1}{2}$ terms if $n$ be odd, and to $\frac{n}{2}+1$ terms if $n$ be even.

Ex. Making $n$ equal to the numbers $2,3,4$, \&c. successively, we shall have

$$
\begin{aligned}
\sin 2 A & =2 \cos A \sin A \\
\cos 2 A & =\cos ^{2} A-\sin ^{2} A \\
\sin 3 A & =3 \cos ^{2} A \sin A-\sin ^{3} A \\
\cos 3 A & =\cos ^{3} A-3 \cos A \sin ^{2} A \\
\sin 4 A & =4 \cos ^{3} A \sin A-4 \cos A \sin ^{3} A \\
\cos 4 A & =\cos ^{4} A-6 \cos ^{2} A \sin ^{2} A+\sin ^{4} A: \\
\& c . & =\& \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots
\end{aligned}
$$

The versed sines and chords of the same multiples may be expressed in terms of the sines and cosines, and therefore in terms of the versed sines and chords of the same arcs, by means of the observations made in Chap. i.
271. Cor. From the last article, the tangent of the multiple of an arc may be expressed in terms of the powers of the tangent of the arc itself.

For, $\tan n A=\frac{\sin n A}{\cos n A}$
$=\frac{n \cos ^{n-1} A \sin A-n\left(\frac{n-1}{2}\right)\left(\frac{n-9}{3}\right) \cos ^{n-3} A \sin ^{3} A+\mathcal{\&} c .}{\cos ^{n} A-n\left(\frac{n-1}{2}\right) \cos ^{n-2} A \sin ^{2} A+\mathbb{S} c .}$

$$
=\frac{n \frac{\sin A}{\cos A}-n\left(\frac{n-1}{2}\right)\left(\frac{n-9}{3}\right) \frac{\sin ^{3} A}{\cos ^{3} A}+\delta c .}{1-n\left(\frac{n-1}{2}\right) \frac{\sin ^{2} A}{\cos ^{2} A}+\& c .}
$$

(by dividing both numerator and denominator by $\cos ^{n} A$ )

$$
=\frac{n \tan A-n\left(\frac{n-1}{2}\right)\left(\frac{n-2}{3}\right) \tan ^{3} A+8 \mathrm{c} .}{1-n\left(\frac{n-1}{2}\right) \tan ^{2} A+8 c .} \text {, as has }
$$

been proved in (133), and in which if $n$ be odd, the numerator and denominator will each be continued to $\frac{n+1}{2}$ terms, and if $n$ be even, to $\frac{n}{2}$ and $\frac{n}{2}+1$ terms respectively.

This expression for tan $n A$ might however have been found without taking for granted those for $\sin n A$ and $\cos n A$.

Thus, $\tan n A=\frac{\sin n A}{\cos n A}$
$=\frac{1}{\sqrt{-1}}\left\{\frac{(\cos n A+\sqrt{-1} \sin n A)-(\cos n A-\sqrt{-1} \sin n A)}{(\cos n A+\sqrt{-1} \sin n A)+(\cos n A-\sqrt{-1} \sin n A)}\right\}$
$\left.=\frac{1}{\sqrt{-1}}\left\{\frac{(\cos A+\sqrt{-1} \sin A)^{n}-(\cos A-\sqrt{-1} \sin A)^{n}}{(\cos A+\sqrt{-1} \sin A)^{n}-(\cos A-\sqrt{-1}} \sin A\right)^{n}\right\}$
$=\frac{1}{\sqrt{-1}}\left\{\frac{(1+\sqrt{-1} \tan A)^{n}-(1-\sqrt{-1} \tan A)^{n}}{(1+\sqrt{-1} \tan A)^{n}+(1-\sqrt{-1} \tan A)^{n}}\right\}$
$=\frac{n \tan A-n\left(\frac{n-1}{2}\right)\left(\frac{n-2}{3}\right) \tan ^{3} A+\& \mathrm{c} .}{1-n\left(\frac{n-1}{2}\right) \tan ^{2} A+\mathbb{N} .}$, as before.

The expressions just found for the sine, cosine and tangent of $n A$, have been deduced on the supposition that $n$ is a whole number; they are however true whether $n$ be whole or fractional, but in the latter case the number of terms will be indefinitely great, and consequently they are approximations to the true value only when the series converge; and by substitutions similar to those made above, we can find expressions for the co-tangent, secant, and co-secant of $n A$, whether $n$ be whole or fractional.
272. To express the powers of the cosine of an arc in terms of the cosines of its multiples.

Assume $\cos A+\sqrt{-1} \sin A=x$, then $2 \cos A=x+\frac{1}{x}$ by (257):
$\therefore$ by the binomial theorem we have $2^{n} \cos ^{n} A=\left(x+\frac{1}{x}\right)^{n}$
$=x^{n}+n x^{n-2}+n\left(\frac{n-1}{2}\right) x^{n-4}+\& c .+n\left(\frac{n-1}{2}\right) \frac{1}{x^{n-4}}+n \frac{1}{x^{n-2}}+\frac{1}{x^{n}}$
$=\left(x^{n}+\frac{1}{x^{n}}\right)+n\left(x^{n-2}+\frac{1}{x^{n-2}}\right)+n\left(\frac{n-1}{2}\right)\left(x^{n-4}+\frac{1}{x^{n-4}}\right)+8 \mathrm{cc}$.

First let $u$ be odd, therefore the number of terms in the expanded binomial will be $n+1$ which is even, and there will be an exact number $\frac{n+1}{2}$ of pairs of terms; hence by (263) we shall have

$$
\begin{gathered}
2^{n} \cos ^{n} A=2 \cos n A+2 n \cos (n-2) A+2 n\left(\frac{n-1}{2}\right) \cos (n-4) A \\
+\& c \cdot \text { to } \frac{n+1}{2} \text { terms; }
\end{gathered}
$$

and $\therefore \cos ^{n} A$
$=\frac{1}{2^{n-1}}\left\{\cos n A+n \cos (n-2) A+n\left(\frac{n-1}{2}\right) \cos (n-4) A+\& \mathrm{c}\right.$. to $\frac{n+1}{2}$ terms $\}:$

Next let $u$ be even, then the number of terms of the expanded binomial being $n+1$ will be odd, so that in addition to the pairs of terms above, there will be an additional one which is the middle or $\left(\frac{n}{2}+1\right)$ th term of the expansion, and is equal to

$$
\frac{n(n-1)(n-2) \& c \cdot\left(\frac{n}{2}+1\right)}{1 \cdot 2.3 \& c \cdot \frac{n}{2}}=2^{\frac{n}{2}} \frac{1.3 \cdot 5 \& c \cdot(n-1)}{1.2 \cdot 3 \& c \cdot \frac{n}{2}} ;
$$

hence $2^{n} \cos ^{n} A=\left(x^{n}+\frac{1}{x^{n}}\right)+n\left(x^{n-2}+\frac{1}{x^{n-2}}\right)+\& \cos$ to $\frac{n}{2}$ terms

$$
+2^{\frac{n}{2}} \frac{1 \cdot 3 \cdot 5 \& \mathrm{cc} \cdot(n-1)}{1.2 .3 \& \mathrm{sc} \cdot \frac{n}{2}}
$$

$=2 \cos n A+2 n \cos (n-2) A+\& c$. to $\frac{n}{2}$ terms

$$
+2^{\frac{n}{2}} \frac{1.3 \cdot 5 \text { \&c. }(n-1)}{1.2 .3 \& c \cdot \frac{n}{2}}
$$

and $\therefore \cos ^{n} A=\frac{1}{2^{n-1}}\left\{\cos n A+n \cos (n-2) A+\& c\right.$. to $\frac{n}{2}$ terms

$$
\left.+2^{\frac{n}{2}-1} \frac{1.3 .5 \& c \cdot(n-1)}{1.2 .3 \& c \cdot \frac{n}{2}}\right\}
$$

273. Ex. Let $n$ be taken equal to the numbers 2, 3, 4,5 , \&c. successively, and the formulæ just demonstrated alternately give

$$
\begin{aligned}
\cos ^{2} A & =\frac{1}{2}\{\cos 2 A+1\} \\
\cos ^{3} A & =\frac{1}{4}\{\cos 3 A+3 \cos A\} \\
\cos ^{4} A & =\frac{1}{8}\{\cos 4 A+4 \cos 2 A+3\} \\
\cos ^{5} A & =\frac{1}{16}\{\cos 5 A+5 \cos 3 A+10 \cos A\} \\
\& c . & \& \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots
\end{aligned}
$$

274. To express the powers of the sine of an arc in terms of the sines or cosines of its multiples.

If $\cos A+\sqrt{-1} \sin A=x$,
we have $2 \sqrt{-1} \sin A=x-\frac{1}{x}$, by (257);
therefore $2^{n}(\sqrt{-1})^{n} \sin ^{n} A=\left(x-\frac{1}{x}\right)^{n}$
$=x^{n}-n x^{n-2}+n\left(\frac{n-1}{2}\right) x^{n-4}-\& \mathrm{c} . \pm n\left(\frac{n-1}{2}\right) \frac{1}{x^{n-4}} \mp n \frac{1}{x^{n-2}} \pm \frac{1}{x^{n}}$
$=\left(x^{n} \pm \frac{1}{x^{n}}\right)-n\left(x^{n-2} \pm \frac{1}{x^{n-2}}\right)+n\left(\frac{n-1}{2}\right)\left(x^{n-4} \pm \frac{1}{x^{n-4}}\right)-\& c . ;$
now $n$ must necessarily be of one or other of the forms $4 m$, $4 m+1,4 m+2,4 m+3$, and therefore $(\sqrt{-1})^{n}$ must admit of four different values :

In the first place, let $n$ be equal to $4 m$, and therefore $(\sqrt{-1})^{n}=(\sqrt{-1})^{4 m}=\left((\sqrt{-1})^{4}\right)^{m}=1^{m}=1 ;$ then $2^{n} \sin ^{n} A$

$$
\begin{gathered}
=\left(x^{n}+\frac{1}{x^{n}}\right)-n\left(x^{n-2}+\frac{1}{x^{n-2}}\right)+n\left(\frac{n-1}{2}\right)\left(x^{n-4}+\frac{1}{x^{n-4}}\right) \\
-\& \text { c. to } \frac{n}{2} \text { terms }+2^{\frac{n}{2}} \frac{1.3 .5 \& \mathrm{c} \cdot(n-1)}{1.2 .3 \& \mathrm{c} \cdot \frac{n}{2}} \\
=2 \cos n A-2 n \cos (n-2) A+2 n\left(\frac{n-1}{2}\right) \cos (n-4) A \\
\\
-\& \mathrm{cc} . \text { to } \frac{n}{2} \text { terms }+2^{\frac{n}{2}} \frac{1.3 .5 \& \mathrm{c} \cdot(n-1)}{1.2 .3 \& \mathrm{c} \cdot \frac{1}{2}} ;
\end{gathered}
$$

$\therefore \sin ^{n} A$

$$
\begin{gathered}
=\frac{1}{2^{n-1}}\left\{\cos n A-n \cos (n-2) A+n\left(\frac{n-1}{2}\right) \cos (n-4) A\right. \\
\left.-\& c \cdot \text { to } \frac{n}{2} \text { terms }+2^{\frac{n}{2-1}} \frac{1.3 .5 \& \mathrm{c} \cdot(n-1)}{1.2 \cdot 3 \& \mathrm{c} \cdot \frac{n}{2}}\right\}
\end{gathered}
$$

Secondly, let $n=4 m+1$,

$$
\therefore(\sqrt{-1})^{n}=(\sqrt{-1})^{4 m+1}=(\sqrt{-1})^{4 m} \sqrt{-1}=\sqrt{-1} ;
$$

$$
\text { hence } \mathscr{2}^{n} \sqrt{-1} \sin ^{n} A=\left(x^{n}-\frac{1}{x^{n}}\right)-n\left(x^{n-2}-\frac{1}{x^{n-2}}\right)
$$

$$
+n\left(\frac{n-1}{2}\right)\left(x^{n-4}-\frac{1}{x^{n-4}}\right)-\& \mathrm{c} . \text { to } \frac{n+1}{2} \text { terms }
$$

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$$
\begin{aligned}
& \quad=2 \sqrt{-1} \sin n A-2 n \sqrt{-1} \sin (n-2) A \\
& +2 n\left(\frac{n-1}{2}\right) \sqrt{-1} \sin (n-4) A-\& c \cdot \text { to } \frac{n+1}{2} \text { terms } \\
& \therefore \sin ^{n} A \\
& =\frac{1}{2^{n-1}}\left\{\sin n A-n \sin (n-2) A+n\left(\frac{n-1}{2}\right) \sin (n-4) A\right. \\
& \left.\quad \quad-\& \text { c. to } \frac{n+1}{2} \text { terms }\right\}:
\end{aligned}
$$

Thirdly, if $n=4 m+2$, we have

$$
\begin{aligned}
& (\sqrt{-1})^{n}=(\sqrt{-1})^{4 m+2}=(\sqrt{-1})^{4 n}(\sqrt{-1})^{2}=-1 ; \\
& \therefore-2^{n} \sin ^{n} A
\end{aligned}
$$

$$
\begin{gathered}
=\left(x^{n}+\frac{1}{x^{n}}\right)-n\left(x^{n-2}+\frac{1}{x^{n-2}}\right)+n\left(\frac{n-1}{2}\right)\left(x^{n-4}+\frac{1}{x^{n-4}}\right) \\
-\& \mathrm{c} . \text { to } \frac{n}{2} \text { terms }-2^{\frac{n}{2}} \frac{1.3 .5 \& \mathrm{c} \cdot(n-1)}{1.2 .3 \& \mathrm{c} \cdot \frac{n}{2}}
\end{gathered}
$$

$$
=2 \cos n A-2 n \cos (n-2) A+2 n\left(\frac{n-1}{2}\right) \cos (n-4) A
$$

$$
- \text { \&ic. to } \frac{n}{2} \text { terms }-2^{\frac{n}{2}} \frac{1.3 .5 \text { \&c. }(n-1)}{1.2 .3 \text { \&c. } \frac{n}{2}}
$$

$\therefore \sin ^{n} A$

$$
\begin{aligned}
=- & \frac{1}{2^{n-1}}\left\{\cos n A-n \cos (n-2) A+n\left(\frac{n-1}{2}\right) \cos (n-4) A\right. \\
& \left.-\& c \cdot \text { to } \frac{n}{2} \text { terms }-2^{\frac{n}{2}-1} \frac{1 \cdot 3.5 \& c \cdot(n-1)}{1.2 .3 \& c \cdot \frac{n}{2}}\right\}:
\end{aligned}
$$

Lastly, if $n=4 m+9$, we have

$$
\begin{aligned}
& (\sqrt{-1})^{n}=(\sqrt{-1})^{4 n+3}=(\sqrt{-1})^{4 n}(\sqrt{-1})^{3}=-\sqrt{-1} \\
& \quad \therefore-2^{n} \sqrt{-1} \sin ^{n} A
\end{aligned}
$$

$$
=\left(x^{n}-\frac{1}{x^{n}}\right)-n\left(x^{n-2}-\frac{1}{x^{n-2}}\right)+n\left(\frac{n-1}{2}\right)\left(x^{n-4}-\frac{1}{x^{n-4}}\right)
$$

- \&c. to $\frac{n+1}{2}$ terms

$$
=2 \sqrt{-1} \sin n A-2 n \sqrt{-1} \sin (n-2) A
$$

$$
+2 n \frac{n-1}{2} \sqrt{-1} \sin (n-4) A-\& c . \text { to } \frac{n+1}{2} \text { terms }
$$

$\therefore \sin ^{n} A$

$$
\begin{gathered}
=-\frac{1}{2^{n-1}}\left\{\sin n A-n \sin (n-2) A+n\left(\frac{n-1}{2}\right) \sin (n-4) A\right. \\
\left.-\& c . \text { to } \frac{n+1}{2} \text { terms }\right\} .
\end{gathered}
$$

275. Cor. The first and third cases of the proposition proved in the last article are both comprehended in the formula,

$$
\begin{aligned}
& \sin ^{n} A \\
= & \pm \frac{1}{2^{n-1}}\left\{\cos n A-n \cos (n-2) A+n\left(\frac{n-1}{2}\right) \cos (n-4) A-\& c .\right. \\
& \text { to } \left.\frac{n}{2} \text { terms } \pm 2^{\frac{n}{2}-1} \frac{1.3 .5 \& c \cdot(n-1)}{1.2 .3 \& c \cdot \frac{n}{2}}\right\}
\end{aligned}
$$

and the second and fourth in the formula,

$$
\begin{gathered}
\sin ^{n} A \\
= \pm \frac{1}{2^{n-1}}\left\{\sin n A-u \sin (n-2) A+n\left(\frac{n-1}{2}\right) \sin (n-4) A-\& \mathrm{c}\right. \\
\text { to } \left.\frac{n+1}{2} \text { terms }\right\}
\end{gathered}
$$

and attention to the algebraical signs might have made it sufficient to divide the proposition into the two cases only, in which $n$ is even and odd.

Ex. If we suppose $n$ to assume the particular values $2,3,4,5$, \&c. successively, we shall, by attending to the different forms, immediately obtain from these formulæ,

$$
\begin{aligned}
& \sin ^{2} A=-\frac{1}{2}(\cos 2 A-1)=\frac{1}{2}(1-\cos 2 A) ; \\
& \sin ^{3} A=-\frac{1}{4}(\sin 3 A-3 \sin A)=\frac{1}{4}(3 \sin A-\sin 3 A) ; \\
& \sin ^{4} A=\frac{1}{8}(\cos 4 A-4 \cos 2 A+3)=\frac{1}{8}(3-4 \cos 2 A+\cos 4 A) ;
\end{aligned}
$$

$$
\sin ^{5} A=\frac{1}{16}(\sin 5 A-5 \sin 3 A+10 \sin A)=\frac{1}{16}(10 \sin A-5 \sin 3 A+\sin 5 A)
$$

$$
\& \mathrm{c} .=\& \mathrm{c}
$$

276. To express the sine of an arc in terms of the arc itself.

In article (270) we have proved that $\sin n A=n \cos ^{n-1} A \sin A-n\left(\frac{n-1}{2}\right)\left(\frac{n-2}{3}\right) \cos ^{n-3} A \sin ^{3} A+\& c$.

$$
\begin{aligned}
& =\cos ^{n} A\left\{n \frac{\sin A}{\cos A}-n\left(\frac{n-1}{2}\right)\left(\frac{n-2}{3}\right) \frac{\sin ^{3} A}{\cos ^{3} A}+\mathbb{S c} .\right\} \\
& =\cos ^{n} A\left\{n \tan A-n\left(\frac{n-1}{2}\right)\left(\frac{n-2}{3}\right) \tan ^{3} A+\mathbb{S} .\right\}
\end{aligned}
$$

$$
\text { assume now } n A=\theta, \text { or } A=\frac{\theta}{n} \text {, }
$$

$\therefore \sin \theta=\cos ^{n} \frac{\theta}{n}\left\{n \tan \frac{\theta}{n}-n\left(\frac{n-1}{2}\right)\left(\frac{n-2}{3}\right) \tan ^{3} \frac{\theta}{n}+\& c.\right\} ;$
then if $n$ become indefinitely great, $\frac{\theta}{n}$ will be indefinitely small,

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$$
\therefore \text { we have } \cos ^{n} \frac{\theta}{n}=1, \tan \frac{\theta}{n}=\frac{\theta}{n} \text {, }
$$

$$
\text { and } n(n-1)(n-2)=n^{3}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)=n^{5}, \& \mathrm{c} .
$$

hence $\sin \theta=n \frac{\theta}{n}-\frac{n^{3}}{1.2 .3} \frac{\theta^{3}}{n^{5}}+\frac{n^{5}}{1.2 .3 .4 .5} \frac{\theta^{5}}{n^{5}}-\& \mathbf{c . i n}$ infinitum

$$
=\theta-\frac{\theta^{3}}{1.2 .3}+\frac{\theta^{5}}{1.2 .3 .4 .5}-\& c . \text { in infinitum } .
$$

277. Since in the expression for the sine of $n A$, the quantities involved are trigonometrical functions of the arc and therefore expressed in terms of the radius 1, it follows that before the expression just investigated can be applied the value of $\theta$ must be found in terms of the same radius: hence therefore if $r^{0}=57^{0} .2957795 \mathrm{\& c}$. the number of degrees contained in an arc equal to the radius, and $\theta^{0}$ be the number in any proposed arc, we shall have
$r^{0}: \theta^{0}:: 1: \frac{\theta^{0}}{r^{0}}=$ the value of $\theta^{0}$ in terms of the radius,
and $\therefore \sin \theta^{0}=\left(\frac{\theta^{0}}{r^{0}}\right)-\frac{1}{1.2 .3}\left(\frac{\theta^{0}}{r^{0}}\right)^{3}+\frac{1}{1.2 .3 .4 .5}\left(\frac{\theta^{0}}{r^{0}}\right)^{5}-8 \mathrm{c}$.
Ex.1. Suppose $\theta^{0}=1^{0}$, then, in (155) we have seen that

$$
\frac{\theta^{0}}{r^{0}}=.0174532 \& c .,
$$

$\therefore \sin 1^{0}=.0174532 \& \mathrm{c} .-\frac{1}{1.2 .3}(.0174532 \& \mathrm{c} .)^{3}+\& \mathrm{c}$. $=.0174524 \& c$.
Ex. 2. Let $\theta^{0}=\left(\frac{1}{60}\right)^{0}=1^{\prime}$; then $\frac{\theta^{0}}{r^{0}}=.00029088 \& c$.
from (155),

$$
\text { and } \begin{aligned}
\sin 1^{\prime} & =.00029088 \& c .-\frac{1}{1.2 \cdot 3}(.00029088 \& \cdot .)^{3}+\& c . \\
& =.0002909 \& c .
\end{aligned}
$$

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Ex. 3. If $\theta^{0}=\left(\frac{1}{60^{2}}\right)^{0}=1^{\prime \prime}$, we have $\frac{\theta^{0}}{r^{0}}=.000004848 \& c$.

$$
\text { and } \begin{aligned}
\therefore \sin 1^{\prime \prime} & =.000004848 \& c .-\frac{1}{1.2 .3}(.000004848)^{3}+\& c . \\
& =.000004848 \& c .
\end{aligned}
$$

and so on.
278 . Cor. From the last of the examples above given, it appears that the arc of one second and its sine do not differ by any significant figure, and therefore in all practical applications of trigonometry, we may without sensible error assume $\sin 1^{\prime \prime}=1^{\prime \prime}:$ similarly, $\sin 9^{\prime \prime}=2 \sin 1^{\prime \prime}=2^{\prime \prime}, \sin 3^{\prime \prime}=3 \sin 1^{\prime \prime}=3^{\prime \prime}$ \&c. very nearly, by (74).

Also, if $\phi(A)$ denote any trigonometrical function of $A$ expressed in terms of the radius, we shall have

$$
\sin 1^{\prime \prime}: \phi(A):: 1^{\prime \prime}:
$$

the value of the same function expressed in seconds, which

$$
\therefore=\frac{1^{\prime \prime} \phi(A)}{\sin 1^{\prime \prime}}=\frac{\phi(A)}{\sin 1^{\prime \prime}} \text { seconds. }
$$

279. To express the cosine of an arc in terms of the arc itself.

It has been proved in (270) that

$$
\cos n A=\cos ^{n} A+n\left(\frac{n-1}{2}\right) \cos ^{n-2} A \sin ^{2} A+\& c
$$

$$
=\cos ^{n} A\left\{1-n\left(\frac{n-1}{2}\right) \tan ^{2} A+\delta \mathrm{c} \cdot\right\}
$$

$\therefore$ as before $\cos \theta=\cos ^{n} \frac{\theta}{n}\left\{1-n\left(\frac{n-1}{2}\right) \tan ^{2} \frac{\theta}{n}+8 c.\right\}$

$$
=1-\frac{n^{2}}{1.2} \frac{\theta^{2}}{n^{2}}+\frac{n^{4}}{1.2 .3 .4} \frac{\theta^{4}}{n^{4}}-\text { \&c. in infinitum, }
$$

by reasoning as in (276),

$$
=1-\frac{\theta^{2}}{1.2}+\frac{\theta^{4}}{1.2 \cdot 3.4}-\text { \&c. in infinitum, }
$$

and pursuing the steps taken in article (277), we get

$$
\cos \theta^{0}=1-\frac{1}{1 \cdot 2}\left(\frac{\theta^{0}}{r^{0}}\right)^{2}+\frac{1}{1 \cdot 2 \cdot 3 \cdot 4}\left(\frac{\theta^{0}}{r^{0}}\right)^{4}-8 c .
$$

Ex. If the same values of $\theta^{0}$ be assumed as in the last examples, we shall find

$$
\begin{aligned}
& \cos 1^{0}=.9998477 \& c . \\
& \cos 1^{\prime}=.9999998 \& c . \\
& \cos 1^{\prime \prime}=.9999999 \& c .
\end{aligned}
$$

280. By means of the expressions for $\sin \theta$ and $\cos \theta$, found in articles (276) and (279), formulæ for the other trigonometrical functions are easily obtained. Thus,

$$
\text { vers } \theta=1-\cos \theta=1-\left(1-\frac{\theta^{2}}{1.2}+\frac{\theta^{1}}{1.2 .3 .4}-\& c .\right)
$$

$=\frac{\theta^{2}}{1.2}-\frac{\theta^{4}}{1.2 .3 .4}+\& c$. in infinitum;
$\operatorname{chd} \theta=2 \sin \frac{\theta}{2}=2\left\{\frac{\theta}{2}-\frac{1}{1.2 .3} \frac{\theta^{5}}{2^{5}}+\frac{1}{1.2 .3 .4 .5} \frac{\theta^{5}}{2^{5}}-8 c.\right\}$
$=\theta-\frac{1}{1.3} \frac{\theta^{5}}{2^{3}}+\frac{1}{1.3 .4 .5} \frac{\theta^{5}}{2^{5}}-\& \mathrm{c}$. in infinitum;
$\tan \theta=\frac{\sin \theta}{\cos \theta}=\frac{\theta-\frac{\theta^{3}}{1.2 .3}+\frac{\theta^{5}}{1 \cdot 2.3 .4 \cdot 5}-\& c .}{1-\frac{\theta^{2}}{1.2}+\frac{\theta^{4}}{1.2 .3 .4}-\& c .}$
$=\theta+\frac{\theta^{3}}{1.3}+\frac{2 \theta^{5}}{1.3 .5}+$ \&c. in infinitum;

$$
\begin{aligned}
& \cot \theta=\frac{\cos \theta}{\sin \theta}=\frac{1-\frac{\theta^{2}}{1.2}+\frac{\theta^{4}}{1.2 .3 .4}-\& c .}{\theta-\frac{\theta^{3}}{1.2 .3}+\frac{\theta^{5}}{1.2 .3 .4 .5}-\& c .} \\
&= \frac{1}{\theta}-\frac{\theta}{1.3}-\frac{\theta^{3}}{1.5 .9}-\text { \&c. in infinitum; } \\
& \sec \theta=\frac{1}{\cos \theta}=\frac{1}{1-\frac{\theta^{2}}{1.2}+\frac{\theta^{4}}{1.2 .3 .4}-8 c .} \\
&= 1+\frac{\theta^{2}}{1.2}+\frac{5 \theta^{4}}{1.2 .3 .4}+8 c . \text { in infinitum; } \\
& \operatorname{cosec} \theta=\frac{1}{\sin \theta}=\frac{1}{\theta-\frac{\theta^{3}}{1.2 .3}+\frac{\theta^{5}}{1.2 .3 .4 .5}-\& c .} \\
&= \frac{1}{\theta}+\frac{\theta}{1.2 .3}+\frac{14 \theta^{5}}{1.2 .3 .4 .5 .6}+\text { \&c. in infinituin. }
\end{aligned}
$$

281. Cor. If the arc be small, approximate values of some of the preceding functions are readily obtained. Thus, if $\theta$ be very small, we shall have

$$
\sin \theta=\theta-\frac{\theta^{3}}{1.2 \cdot 3}=\theta\left(1-\frac{\theta^{2}}{1.2 .3}\right)=\theta\left(1-\frac{\theta^{2}}{1.2}\right)^{\frac{1}{3}}=\theta(\cos \theta)^{\frac{1}{t}}
$$

$$
\text { hence, } \begin{aligned}
\log \sin \theta & =\log \theta+\frac{1}{3} \log \cos \theta \\
\text { and } \log \theta & =\log \sin \theta-\frac{1}{3} \log \cos \theta
\end{aligned}
$$

$$
\text { again, } \tan \theta=\theta+\frac{\theta^{3}}{1.3}=\theta\left(1+\frac{\theta^{2}}{1.3}\right)=\frac{\theta}{\left(1-\frac{\theta^{2}}{2}\right)^{\frac{2}{3}}}=\frac{\theta}{(\cos \theta)^{\frac{2}{3}}}
$$

$\therefore \log \tan \theta=\log \theta-\frac{2}{3} \log \cos \theta$,
and $\log \theta=\log \tan \theta+\frac{2}{3} \log \cos \theta$ :
also, $2 \sin \theta+\tan \theta=2 \theta-\frac{2 \theta^{3}}{1.2 .3}+\theta+\frac{\theta^{3}}{1.3}=3 \theta$ :

$$
\begin{gathered}
\text { and } 8 \operatorname{chd} \frac{\theta}{2}-\operatorname{chd} \theta=8\left(\frac{\theta}{2}-\frac{1}{1.3} \frac{\theta^{3}}{4^{3}}\right)-\left(\theta-\frac{1}{1.3} \frac{\theta^{5}}{2^{3}}\right) \\
\quad=4 \theta-\frac{1}{1.3} \frac{\theta^{3}}{2^{3}}-\theta+\frac{1}{1.3} \frac{\theta^{3}}{2^{5}}=3 \theta
\end{gathered}
$$

$$
\text { whence } \theta=\frac{1}{3}\left(8 \operatorname{chd} \frac{\theta}{2}-\operatorname{chd} \theta\right)
$$

which is an useful approximate formula for practice; and in all these instances, the arc is of course supposed to be expressed in terms of the radius.
282. I'o express the sine and cosine of an arc in terms of impossible exponential functions of the arc itself.

In the expression $e^{x}=1+\frac{x}{1}+\frac{x^{2}}{1.2}+\frac{x^{3}}{1.2 .3}+\& c$.
where $e=2.71828$ \&cc. the base of hyperbolic logarithms, let $\theta \sqrt{-1}$ and $-\theta \sqrt{-1}$ be successively substituted in the place of $x$, and we shall have

$$
\begin{aligned}
& e^{\theta \sqrt{-1}}=1+\frac{\theta \sqrt{-1}}{1}-\frac{\theta^{2}}{1.2}-\frac{\theta^{3} \sqrt{-1}}{1.2 .3}+\frac{\theta^{4}}{1.2 .3 .4}+\& c . \\
& e^{-\theta \sqrt{-1}}=1-\frac{\theta \sqrt{-1}}{1}-\frac{\theta^{2}}{1.2}+\frac{\theta^{3} \sqrt{-1}}{1.2 .3}+\frac{\theta^{4}}{1.2 .3 .4}-\& c .
\end{aligned}
$$

hence by addition and from (279), we get

$$
\begin{gathered}
e^{\theta \sqrt{-1}}+e^{-\theta \sqrt{-1}}=2\left\{1-\frac{\theta^{2}}{1 \cdot 2}+\frac{\theta^{4}}{1 \cdot 2 \cdot 3 \cdot 4}-\& c \cdot\right\}=2 \cos \theta \\
\text { and } \therefore \cos \theta=\frac{e^{\theta \sqrt{-1}}+e^{-\theta \sqrt{-1}}}{2}
\end{gathered}
$$

and by subtraction and from (276), we obtain

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$$
\begin{aligned}
e^{\theta \sqrt{-1}}-e^{-\theta \sqrt{-1}} & =2 \sqrt{-1}\left\{\theta-\frac{\theta^{3}}{1.2 .3}+\frac{\theta^{5}}{1.2 \cdot 3 \cdot 4.5}-\& c .\right\} \\
& =2 \sqrt{-1} \sin \theta \\
\text { hence } & \therefore \sin \theta=\frac{e^{\theta \sqrt{-1}}-e^{-\theta \sqrt{-1}}}{2 \sqrt{-1}} .
\end{aligned}
$$

283. These two formulæ, as before, enable us to find expressions of the same kind for all the other trigonometrical functions. Thus,
$\operatorname{vers} \theta=1-\cos \theta=1-\frac{e^{\theta \sqrt{-1}}+e^{-\theta \sqrt{-1}}}{2}=\frac{2-e^{\theta \sqrt{-1}}-e^{-\theta \sqrt{-1}}}{2}$

$$
=-\frac{1}{2}\left(e^{\frac{\theta}{2} \sqrt{-1}}-e^{-\frac{\theta}{2} \sqrt{-1}}\right)^{2}
$$

$\operatorname{chd} \theta=2 \sin \frac{\theta}{2}=2 \frac{e^{\frac{\theta}{2} \sqrt{-1}}-e^{-\frac{\theta}{2} \sqrt{-1}}}{2 \sqrt{-1}}=\frac{e^{\frac{\theta}{2} \sqrt{-1}}-e^{-\frac{\theta}{2} \sqrt{-1}}}{\sqrt{-1}} ;$
$\tan \theta=\frac{\sin \theta}{\cos \theta}=\frac{1}{\sqrt{-1}} \frac{e^{\theta \sqrt{-1}}-e^{-\theta \sqrt{-1}}}{e^{\theta \sqrt{-1}}+e^{-\theta \sqrt{-1}}}=\frac{1}{\sqrt{-1}} \frac{e^{2 \theta \sqrt{-1}}-1}{e^{2 \theta \sqrt{-1}+1}} ;$
$\cot \theta=\frac{\cos \theta}{\sin \theta}=\sqrt{-1} \frac{e^{\theta \sqrt{-1}}+e^{-\theta \sqrt{-1}}}{e^{\theta \sqrt{-1}}-e^{-\theta \sqrt{-1}}}=\sqrt{-1} \frac{e^{2 \theta \sqrt{-1}}+1}{e^{2 \theta \sqrt{-1}}-1} ;$
$\sec \theta=\frac{1}{\cos \theta}=\frac{2}{e^{\theta \sqrt{-1}}+e^{-\theta \sqrt{-1}}}=2 \frac{e^{\theta \sqrt{-1}}-e^{-\theta \sqrt{-1}}}{e^{2 \theta \sqrt{-1}}-e^{-2 \theta \sqrt{-1}}} ;$
$\operatorname{cosec} \theta=\frac{1}{\sin \theta}=\frac{2 \sqrt{-1}}{e^{\theta \sqrt{-1}}-e^{-\theta \sqrt{-1}}}=2 \sqrt{-1} e^{\theta \sqrt{-1}}+e^{-\theta \sqrt{-1}}$.
284. If the equations investigated in the last two articles were established by any independent method, they might be used to determine the relations between other different trigonometrical expressions. Thus,

$$
2 \sin \theta \cos \theta=2\left(\frac{e^{\hat{\sqrt{-1}}}-e^{-\theta \sqrt{-1}}}{2 \sqrt{-1}}\right)\left(\frac{e^{\theta \sqrt{-1}}+e^{-\theta \sqrt{-1}}}{2}\right)
$$

$$
\begin{aligned}
& =\frac{1}{2 \sqrt{-1}}\left(e^{\theta \sqrt{-1}}-e^{-\theta \sqrt{-1}}\right)\left(e^{\theta \sqrt{-1}}+e^{-\theta \sqrt{-1}}\right) \\
& =\frac{1}{2 \sqrt{-1}}\left(e^{2 \theta \sqrt{-1}}-e^{-2 \theta \sqrt{-1}}\right)=\sin 2 \theta, \text { as in }(76) ;
\end{aligned}
$$

$$
\text { also, } \cos ^{3} \theta=\left(\frac{e^{\theta \sqrt{-1}}+e^{-\theta \sqrt{-1}}}{2}\right)^{3}
$$

$$
=\frac{1}{8}\left\{e^{3 \theta \sqrt{-1}}+3 e^{\theta \sqrt{-1}}+3 e^{-\theta \sqrt{-1}}+e^{-3 \theta \sqrt{-1}}\right\}
$$

$$
=\frac{1}{4}\left\{\frac{e^{3 \theta \sqrt{-1}}+e^{-5^{\theta} \sqrt{-1}}}{2}+3 \frac{e^{\theta \sqrt{-1}}+e^{-\theta \sqrt{-1}}}{2}\right\}
$$

$$
=\frac{1}{4}\{\cos 3 \theta+3 \cos \theta\}
$$

as has been before proved in (273):

$$
\begin{gathered}
\text { again, }(\cos \theta \pm \sqrt{-1} \sin \theta)^{m} \\
=\left(\frac{e^{\theta \sqrt{-1}}+e^{-\theta \sqrt{-1}}}{2} \pm \frac{e^{\theta \sqrt{-1}}-e^{-\theta \sqrt{-1}}}{2}\right)^{m}=e^{ \pm m \theta \sqrt{-1}} \\
=\frac{e^{m \theta \sqrt{-1}}+e^{-m^{\theta \sqrt{-1}}}}{2} \pm \frac{e^{m \theta \sqrt{-1}}-e^{-m \theta \sqrt{-1}}}{2} \\
=\cos m \theta \pm \sqrt{-1} \sin m \theta,
\end{gathered}
$$

which is Demoivre's formula already established by a different process.

## CHAP. VII.

On the application of Trigonometrical Formule to the solution of quadratic and cubic equations. On Theorems for the decomposition of certain Functions into their simple and quadratic Factors. On Expressions for the sine and cosine of an arc by means of continued products, and their use. On Expressions for an arc in terms of its tangent, sines of its multiples, \&c. On the Solution of certain cases of triangles by means of series, and without the aid of Tables, \&c. On Expressions for the cosine and sine of the multiple of an arc in terms of the cosine and sine of the arc itself. On the Summation of the T'rigonometrical Functions of certain series of arcs, \&c.
285. To find the roots of a quadratic equation by means of a table of sines, cosines, \&c.

First, let the proposed equation be $x^{2} \pm p x+q=0$,

$$
\text { then } \begin{aligned}
x=\frac{\mp p \pm \sqrt{p^{2}-4 q}}{2} & =\mp \frac{p}{2}\left\{1 \mp \sqrt{1-\frac{4 q}{p^{2}}}\right\} \\
\text { let } \sin ^{2} \theta & =\frac{4 q}{p^{2}}
\end{aligned}
$$

$$
\text { and } \therefore x=\mp \frac{p}{2}\left\{1 \mp \sqrt{1-\sin ^{2} \theta}\right\}=\mp \frac{p}{2}\{1 \mp \cos \theta\}:
$$

$$
\text { now, } 1-\cos \theta=2 \sin ^{2} \frac{\theta}{2} \text {, and } 1+\cos \theta=2 \cos ^{2} \frac{\theta}{\mathscr{g}} ;
$$

$\therefore$ the values of $x$ will be $\mp p \sin ^{2} \frac{\theta}{\Omega}$, and $\mp p \cos ^{2} \frac{\theta}{\Omega}$.

These values may be exhibited in a different form: for since $p=\frac{2 \sqrt{ } q}{\sin \theta}$, we have

$$
\begin{aligned}
& \mp p \sin ^{2} \frac{\theta}{2}=\mp \frac{2 \sqrt{ } q}{\sin \theta} \sin ^{2} \frac{\theta}{2}=\mp \sqrt{ } q \tan \frac{\theta}{2} \text {, } \\
& \text { and } \mp p \cos ^{2} \frac{\theta}{2}=\mp \frac{2 \sqrt{ } q}{\sin \theta} \cos ^{2} \frac{\theta}{2}=\mp \vee q \cot \frac{\theta}{2} \text {. }
\end{aligned}
$$

If these solutions be applied to practice, the formulæ must be adapted to the radius $r$ by ( 60 ), and then we get

$$
\begin{aligned}
& \log \sin \theta=\frac{1}{2}\{20+2 \log 2+\log q-2 \log p\}, \\
& \log x=\mp\left\{\log p+2 \log \sin \frac{\theta}{2}-20\right\}, \\
& \text { and } \log x=\mp\left\{\log p+2 \log \cos \frac{\theta}{2}-20\right\}:
\end{aligned}
$$

similarly of the other solutions.
Next, let the equation proposed be $x^{2} \pm p x-q=0$, from which we have $x=\mp \frac{p}{2}\left\{1 \mp \sqrt{1+\frac{4 q}{p^{2}}}\right\}$;

$$
\text { let } \tan ^{2} \theta=\frac{4 q}{p^{2}}
$$

and $\therefore x=\mp \frac{p}{2}\left\{1 \mp \sqrt{1+\tan ^{2} \theta}\right\}=\mp \frac{p}{2}\{1 \mp \sec \theta\}$ :
now, $1-\sec \theta=-\tan \theta \tan \frac{\theta}{2}$, and $1+\sec \theta=\tan \theta \cot \frac{\theta}{2}$;
$\therefore$ the values of $x$ ane $\pm \frac{p}{2} \tan \theta \tan \frac{\theta}{2}$, and $\mp \frac{p}{2} \tan \theta \cot \frac{\theta}{2}$ :
and these, by substituting for $p$ its value $\frac{2 \sqrt{ } q}{\tan \theta}$, become

$$
\pm \sqrt{ } q \tan \frac{\theta}{2}, \text { and } \mp \sqrt{ } q \cot \frac{\theta}{2}, \text { respectively. }
$$

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Each of these sets of formule must be adapted to practice as before; and it may be observed that all these solutions can be advantageously employed in those cases only, in which $p$ and $q$ are very large or very complicated quantities.
286. To find the roots of a cubic equation by means of a table of sines, cosines, \& $c$.

Let the equation be reduced to $x^{3}-q x \pm r=0$, by taking away the second term, and assume $\sin \theta=x$,
$\therefore \sin 3 \theta=3 \sin \theta-4 \sin ^{5} \theta$, to the radius 1 :
hence $\sin ^{3} \theta-\frac{3 R^{2}}{4} \sin \theta+\frac{R^{2}}{4} \sin 3 \theta=0$, to the radius $R$,

$$
\text { that is, } x^{3}-\frac{3 R^{2}}{4} x+\frac{R^{2}}{4} \sin 3 \theta=0
$$

therefore equating the coefficients of the corresponding terms of this and the proposed equation, we have

$$
\frac{3 R^{2}}{4}=q, \text { and } \frac{R^{2}}{4} \sin 3 \theta= \pm r
$$

whence $R=2 \sqrt{\frac{q}{3}}$, and $\sin 3 \theta= \pm \frac{3 r}{q}$ :
$\therefore$ to the radius $2 \sqrt{\frac{q}{3}}$ find an angle $3 \theta$ whose sine is $\pm \frac{3 r}{q}$, and thus $\sin \theta$ or $x$ will be determined:

$$
\begin{aligned}
& \text { also, since } \pm \frac{3 r}{q}=\sin 3 \theta=\sin (2 \pi+3 \theta)=-\sin (2 \pi-3 \theta) \\
& =\sin (4 \pi+3 \theta)=-\sin (4 \pi-3 \theta)=\& c . \text { by (17) and }(20) \text {, the } \\
& \text { values of } x \text { will be }
\end{aligned}
$$

$$
\begin{aligned}
\sin \theta, \sin \left(\frac{2 \pi}{3}+\right. & \theta),-\sin \left(\frac{2 \pi}{3}-\theta\right), \sin \left(\frac{4 \pi}{3}+\theta\right) \\
& -\sin \left(\frac{4 \pi}{3}-\theta\right), \text { \&c. }
\end{aligned}
$$

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$$
\begin{aligned}
\text { but } \sin \left(\frac{4 \pi}{3}+\theta\right) & =-\sin \left(2 \pi-\frac{4 \pi}{3}-\theta\right)=-\sin \left(\frac{2 \pi}{3}-\theta\right) \\
-\sin \left(\frac{4 \pi}{3}-\theta\right) & =\sin \left(2 \pi-\frac{4 \pi}{3}+\theta\right)=\sin \left(\frac{2 \pi}{3}+\theta\right) \\
\& c \ldots \ldots & =\text { sc.............................................................. }
\end{aligned}
$$

therefore all the different values of $x$ are $\sin \theta, \sin \left(\frac{2 \pi}{3}+\theta\right)$, and $-\sin \left(\frac{2 \pi}{3}-\theta\right)$ to the radius $2 \sqrt{\frac{9}{3}}$, since after these three, the same values continually recur.

To the radius 1 , the values of $x$ will manifestly be
$2 \sqrt{\frac{q}{3}} \sin \theta, 2 \sqrt{\frac{q}{3}} \sin \left(\frac{2 \pi}{3}+\theta\right)$, and $-2 \sqrt{\frac{q}{3}} \sin \left(\frac{2 \pi}{3}-\theta\right)$,
because $\sin \theta$ to the radius $R=R \sin \theta$ to the radius 1 .
If we had assumed $\cos \theta=x$, the roots might have been obtained in a similar manner.

Ex. Let it be required to determine the roots of the equation $x^{5}-3 x-1=0$.

In this case $q=3, r=1$,
$\therefore R=2 \sqrt[3]{\frac{q}{3}}=2$, and $\sin 3 \theta=-\frac{3 r}{q}=-1$, to the radius 2 :
hence to the radius $1, \sin 3 \theta=-\frac{1}{2}=\sin 210^{\circ}$, and $\theta=70^{\circ}$;
$\therefore$ the values of $x$ are $2 \sin 70^{\circ}, 2 \sin 190^{\circ}$, and $-2 \sin 50^{\circ}$.
287. In the solution above given, we have assumed $R=2 \sqrt{\frac{q}{3}}$, and therefore if $q$ be negative, $R$ will be impossible : again, if $q$ be positive and $\pm \frac{3 r}{q}$ be greater than
$2 \sqrt{\frac{q}{3}}$, we shall have $\sin 3 \theta$ greater than $R$, which is also impossible: hence therefore in both these cases this solution by the trisection of an arc fails, but Cardan's solution does not, two roots being then impossible, and it is observable that both solutions succeed when $\pm \frac{3 r}{q}=2 \sqrt{\frac{q}{3}}$, or two roots are equal.
288. Though the solution just given fails in the instances above enumerated, trigonometrical formulæ may still be applied in finding the only root which is possible: thus, if the proposed cubic be $x^{3}+q x+r=0$, we have by Cardan's rule,

$$
\begin{aligned}
& x=\sqrt[3]{-\frac{r}{2}+\sqrt{\frac{r^{2}}{4}+\frac{q^{3}}{27}}}+\sqrt{-\frac{r}{2}-\sqrt{\frac{r^{2}}{4}+\frac{q^{3}}{27}}} \\
& =\sqrt[3]{-\frac{r}{2}}\left\{\sqrt[3]{1-\sqrt{1+\frac{4 q^{3}}{27 r^{2}}}}+\sqrt[3]{1+\sqrt{1+\frac{4 q^{3}}{27 r^{2}}}}\right\} ; \\
& \text { assume } \tan ^{2} \theta=\frac{4 q^{3}}{27 r^{2}} \text {, or } \frac{r^{2}}{4}=\frac{q^{3}}{27 \tan ^{2} \theta}, \\
& \therefore x=\sqrt[5]{-\frac{r}{2}}\{\sqrt[3]{1-\sec \theta}+\sqrt[3]{1+\sec \theta}\} \\
& =\sqrt[3]{-\frac{r}{2}}\{\sqrt[5]{\sec \theta+1}-\sqrt[3]{\sec \theta-1}\} \\
& =\sqrt{\frac{q}{3}}\left\{\sqrt[3]{\frac{\sec \theta+1}{\tan \theta}}-\sqrt[3]{\frac{\sec \theta-1}{\tan \theta}}\right\} \\
& =\sqrt{\frac{q}{3}}\left\{\sqrt[3]{\cot \frac{\theta}{q}}-\sqrt[3]{\tan \frac{\theta}{2}}\right\}: \\
& \text { let } \sqrt[3]{\cot \frac{\theta}{\underline{Q}}}=\cot \phi, \therefore \sqrt[3]{\tan \frac{\theta}{2}}=\tan \phi ; \\
& \text { and } x=\sqrt{\frac{q}{3}}\{\cot \phi-\tan \phi\}=2 \sqrt{\frac{q}{9}} \cot \varrho \phi,
\end{aligned}
$$

which is the possible root.

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If the equation proposed be $x^{3}-q x+r=0$, and $\frac{3 r}{q}$ be greater than $2 \sqrt{\frac{q}{3}}$, we have as before, $x=\sqrt[3]{-\frac{r}{2}}\left\{\sqrt[5]{1-\sqrt{1-\frac{4 q^{3}}{27 r^{2}}}}+\sqrt[3]{1+\sqrt{1-\frac{4 q^{3}}{27 r^{2}}}}\right\} ;$

$$
\begin{aligned}
& \text { assume } \sin ^{2} \theta=\frac{4 q^{5}}{27 r^{2}}, \text { or } \frac{r^{2}}{4}=\frac{q^{3}}{27 \sin ^{2} \theta} \\
& \therefore x=\sqrt[3]{-\frac{r}{2}}\{\sqrt[3]{1-\cos \theta}+\sqrt[3]{1+\cos \theta}\} \\
&=\sqrt{\frac{q}{3}}\left\{\sqrt{\frac{1-\cos \theta}{\sin \theta}}+\sqrt[3]{\frac{1+\cos \theta}{\sin \theta}}\right\} \\
&=\sqrt{\frac{q}{3}}\left\{\sqrt[3]{\tan \frac{\theta}{2}}+\sqrt[3]{\cot \frac{\theta}{2}}\right\} \\
&=\sqrt{\frac{q}{3}}\{\tan \phi+\cot \phi\}=2 \sqrt{\frac{q}{3}} \operatorname{cosec} 2 \phi
\end{aligned}
$$

the possible root.
289. The latter solution alluded to in article (286) might without much difficulty have been deduced from that of Cardan, but the process will be less simple than the one which would result immediately from the assumption there pointed out.

$$
\begin{gathered}
\text { For, since } x=\sqrt[3]{-\frac{r}{2}+\sqrt{\frac{r^{2}}{4}-\frac{q^{3}}{27}}+\sqrt[3]{-\frac{r}{2}-\sqrt{\frac{r^{2}}{4}-\frac{q^{3}}{27}}},} \begin{array}{c}
\text { let }-\frac{r}{2}=\alpha, \text { and } \sqrt{\frac{r^{2}}{4}-\frac{q^{3}}{Q 7}}=\beta \sqrt{-1} \text {, and } \therefore a^{2}+\beta^{2}=\frac{q^{5}}{27} \\
\text { hence } x=\sqrt[3]{a+\beta \sqrt{-1}}+\sqrt[3]{a-\beta \sqrt{-1}} \\
\text { now } a \pm \beta \sqrt{-1}=\sqrt{a^{2}+\beta^{2}}\left\{\frac{a}{\sqrt{\alpha^{2}+\beta^{2}}} \pm \frac{\beta}{\sqrt{a^{2}+\beta^{2}}} \sqrt{-1}\right\}
\end{array},
\end{gathered}
$$

and if we assume $\cos 3 \theta=\frac{a}{\sqrt{a^{2}+\beta^{2}}}$, and $\therefore \sin 3 \theta=\frac{\beta}{\sqrt{\theta^{2}+\beta^{2}}}$, we shall have $a \pm \beta \sqrt{-1}=\sqrt{\frac{q^{3}}{27}}\{\cos 3 \theta \pm \sqrt{-1} \sin 3 \theta\}$;

$$
\begin{aligned}
\therefore \sqrt[5]{a+\beta \sqrt{-1}} & =\sqrt{\frac{q}{3}}\{\cos \theta+\sqrt{-1} \sin \theta\} \\
\text { and } \sqrt[3]{a-\beta \sqrt{-1}} & =\sqrt{\frac{q}{3}}\{\cos \theta-\sqrt{-1} \sin \theta\} \\
\text { whence } x & =2 \sqrt{\frac{q}{3}} \cos \theta
\end{aligned}
$$

$$
\text { also since } \cos 3 \theta=\cos (2 \pi+3 \theta)=\cos (2 \pi-3 \theta)
$$

the two remaining values of $x$ will be

$$
2 \sqrt{\frac{q}{3}} \cos \left(\begin{array}{c}
2 \pi \\
3
\end{array}+\theta\right), \text { and } 2 \sqrt{\frac{q}{3}} \cos \left(\frac{2 \pi}{3}-\theta\right),
$$

as would have been found from (286).
This is the solution of what is called the Irreducible Case of Cardan's Rule.

By an assumption similar to the one just made, that is, if $\tan \theta=\frac{\beta}{\alpha}$, it is easily proved that

$$
\sqrt[n]{a \pm \beta \sqrt{-1}}=\sqrt[2 n]{a^{2}+\beta^{2}}\left\{\cos \frac{\theta}{n} \pm \sqrt{-1} \sin \frac{\theta}{n}\right\}
$$

290. To decompose $\mathrm{x}^{\mathrm{s}}-2 \cos \theta x^{\mathrm{n}}+1$ into its simple and quadratic factors.

$$
\begin{aligned}
& \text { Assuming } x^{2 n}-2 \cos \theta \quad x^{n}+1=0, \text { we obtain } \\
& \begin{aligned}
x^{n} & =\cos \theta \pm \sqrt{-1} \sin \theta, \text { and also by (17) and (21), } \\
& =\cos (2 \pi+\theta) \pm \sqrt{-1} \sin (2 \pi+\theta) \\
& =\cos (4 \pi+\theta) \pm \sqrt{-1} \sin (4 \pi+\theta) \\
& =\text { sc............................ }
\end{aligned}
\end{aligned}
$$

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$$
=\cos (2(n-1) \pi+\theta) \pm \sqrt{-1} \sin (2(n-1) \pi+\theta):
$$

but, by (267), we shall have from these equations,

$$
\begin{aligned}
& x=\cos \frac{\theta}{n} \pm \sqrt{-1} \sin \frac{\theta}{n} \\
& x=\cos \frac{2 \pi+\theta}{n} \pm \sqrt{-1} \sin \frac{2 \pi+\theta}{n} \\
& x=\cos \frac{4 \pi+\theta}{n} \pm \sqrt{-1} \sin \frac{4 \pi+\theta}{n}
\end{aligned}
$$

$$
\& c,=\& c . . \ldots \ldots \ldots \ldots \ldots . .
$$

$$
x=\cos \frac{2(n-1) \pi+\theta}{n} \pm \sqrt{-1} \sin \frac{2(n-1) \pi+\theta}{n}
$$

hence, by the nature of equations, we have

$$
\begin{aligned}
= & \left\{x-2 \cos \theta x^{n}+1\right. \\
& \left.\left\{x-\cos \frac{\theta}{n}+\sqrt{-1} \sin \frac{\theta}{n}\right)\right\}\left\{x-\left(\cos \frac{\theta}{n}-\sqrt{-1} \sin \frac{\theta}{n}\right)\right\} \\
& \left\{x-\left(\cos \frac{2 \pi+\theta}{n}+\sqrt{-1} \sin \frac{2 \pi+\theta}{n}\right)\right\} \\
& \left\{x-\left(\cos \frac{2 \pi+\theta}{n}-\sqrt{-1} \sin \frac{2 \pi+\theta}{n}\right)\right\} \& c \\
& \left\{x-\left(\cos \frac{2(n-1) \pi+\theta}{n}+\sqrt{-1} \sin \frac{2(n-1) \pi+\theta}{n}\right)\right\} \\
& \left\{x-\left(\cos \frac{2(n-1) \pi+\theta}{n}-\sqrt{-1} \sin \frac{2(n-1) \pi+\theta}{n}\right)\right\}
\end{aligned}
$$

and combining each of these pairs of simple factors, we get

$$
\begin{gathered}
x^{2 n}-2 \cos \theta x^{n}+1 \\
=\left(x^{2}-2 \cos \frac{\theta}{n} x+1\right)\left(x^{2}-2 \cos \frac{2 \pi+\theta}{n} x+1\right)
\end{gathered}
$$

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$\left(x^{2}-2 \cos \frac{4 \pi+\theta}{n} x+1\right) \& c \cdot\left(x^{2}-2 \cos \frac{2(n-1) \pi+\theta}{n} x+1\right)$,
the number of factors being $n$.
This theorem of Demoire contains the solution of the equation $x^{2 n}-2 \cos \theta x^{n}+1=0$, all the quadratic factors of which appear to be possible, and all the roots impossible, unless some extreme value be assigned to $\cos \theta$.
291. Cor. 1. In the formula just investigated suppose $\theta=0$, or $\cos \theta=1$; then we have

$$
\begin{gathered}
x^{2 n}-2 x^{n}+1=\left(x^{2}-2 x+1\right)\left(x^{2}-2 \cos \frac{2 \pi}{n} x+1\right) \\
\left(x^{2}-2 \cos \frac{4 \pi}{n} x+1\right) \& \mathrm{c} \cdot\left(x^{2}-2 \cos \frac{2(n-1) \pi}{n} x+1\right),
\end{gathered}
$$

to $n$ factors:

$$
\begin{gathered}
\text { now } \cos \frac{2(n-1) \pi}{n}=\cos \left(2 \pi-\frac{2 \pi}{n}\right)=\cos \frac{2 \pi}{n} \\
\cos \frac{2(n-2) \pi}{n}=\cos \left(2 \pi-\frac{4 \pi}{n}\right)=\cos \frac{4 \pi}{n} \\
8 c \ldots \ldots \ldots \ldots=\& \operatorname{con} \ldots \ldots \ldots=\& c \ldots \\
\therefore x^{2 n}-2 x^{n}+1=\left(x^{2}-2 x+1\right)\left(x^{2}-2 \cos \frac{2 \pi}{n} x+1\right)^{2} \\
\left(x^{2}-2 \cos \frac{4 \pi}{n} x+1\right)^{2} \& c .
\end{gathered}
$$

First let $n$ be odd, then

$$
x^{2 n}-2 x^{n}+1=\left(x^{2}-2 x+1\right)\left(x^{2}-2 \cos \frac{2 \pi}{n} x+1\right)^{2}
$$

$$
\left(x^{2}-2 \cos \frac{4 \pi}{n} x+1\right)^{2} \& c \cdot\left(x^{2}-2 \cos \frac{(n-1) \pi}{n} x+1\right)^{2}
$$

the number of factors being manifestly $\frac{n+1}{\Omega}$; and extracting the square roots of both sides, we have

$$
\begin{gathered}
x^{n}-1=(x-1)\left(x^{2}-2 \cos \frac{2 \pi}{n} x+1\right) \\
\left(x^{2}-2 \cos \frac{4 \pi}{n} x+1\right) \& c \cdot\left(x^{2}-2 \cos \frac{(n-1) \pi}{n} x+1\right):
\end{gathered}
$$

Next let $n$ be even,

$$
\therefore x^{2 n}-2 x^{n}+1=\left(x^{2}-2 x+1\right)
$$

$$
\begin{aligned}
& \left(x^{2}-2 \cos \frac{2 \pi}{n} x+1\right)^{2}\left(x^{2}-2 \cos \frac{4 \pi}{n} x+1\right)^{2} \& c \\
& \left(x^{2}-2 \cos \frac{(n-2) \pi}{n} x+1\right)^{2}\left(x^{2}-2 \cos \pi x+1\right) \\
& =\left(x^{2}-2 x+1\right)\left(x^{2}-2 \cos \frac{2 \pi}{n} x+1\right)^{2} \& c \cdot\left(x^{2}+2 x+1\right)
\end{aligned}
$$

the number of factors being $\frac{n}{2}$; and extracting the square roots as before, we get

$$
\begin{gathered}
x^{n}-1=(x-1)\left(x^{2}-2 \cos \frac{2 \pi}{n} x+1\right) \& \mathrm{c} \cdot(x+1) \\
=\left(x^{2}-1\right)\left(x^{2}-2 \cos \frac{2 \pi}{n} x+1\right) \& \mathrm{c} \cdot\left(x^{2}-2 \cos \frac{(n-2) \pi}{n} x+1\right) .
\end{gathered}
$$

In the same manner, if we make $\theta=\pi, x^{n}+1$ may be decomposed into its simple and quadratic factors, and the roots of the equation $x^{n}+1=0$, will be determined.

These formulæ which are from their inventor called Cotes's Theorems, include the solution of the equations $x^{n} \mp 1=0$, the roots of which are the $n$ roots of $\pm 1$, and it is evident that only one of them is possible when $n$ is odd, and two or none when $n$ is even.
292. Cor.2. By means of the formula of (290), we are enabled to prove also Cotes's Properties of the Circle.


For, since $x^{2 n}-2 \cos \theta x^{n}+1$
$=\left(x^{2}-2 \cos \frac{\theta}{n} x+1\right)\left(x^{2}-2 \cos \frac{2 \pi+\theta}{n} x+1\right)$ \& c. to $n$ factors,
if we suppose $\theta=2 \pi$, we shall have
$\cos \theta=1, \frac{\theta}{n}=\frac{2 \pi}{n}, \frac{2 \pi+\theta}{n}=\frac{4 \pi}{n}, \frac{4 \pi+\theta}{n}=\frac{6 \pi}{n}, \& \mathrm{c} .=\mathbb{s c}$.
and $\therefore x^{2 n}-2 x^{n}+1$
$=\left(x^{2}-2 \cos \frac{2 \pi}{n} x+1\right)\left(x^{2}-2 \cos \frac{4 \pi}{n} x+1\right) \& c$. to $n$ factors:
now if $P$ be any point in the diameter (produced if necessary) of a circle whose radius is 1 , and the whole circumference be divided into $n$ equal parts, $A B, B C, C D, \& c$. we have, if $O P=x$,

$$
x^{2 n}-2 x^{n}+1=O P^{2} n-2 O P^{n}+1=\left(O P^{n} \sim O A^{n}\right)^{2}
$$

C c

$$
\begin{gathered}
x^{2}-2 \cos \frac{2 \pi}{n} x+1=O P^{2}-2 O P \cos A O B+1=P B^{2} ; \\
x^{2}-2 \cos \frac{4 \pi}{n} x+1=O P^{2}-2 O P \cos A O C+1=P C^{2} ; \\
x^{2}-2 \cos \frac{6 \pi}{n} x+1=O P^{2}-2 O P \cos A O D+1=P D^{2} ; \\
\& c \ldots \ldots \ldots \ldots \ldots \ldots=8 c \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\
\therefore\left(O P^{n} \sim O A^{n}\right)^{2}=P B^{2} . P C^{2} . P D^{2} . \& c . \\
\text { and } O P^{n} \sim O A^{n}=P B . P C . P D . \& c .
\end{gathered}
$$

Again, if the arcs $A B, B C, C D, \& c$. be bisected in the points $a, b, c, \& c$. and $P a, P b, P c, \& c \cdot ; O a, O b, O c, \& c$. be joined, we shall have as before

$$
O P^{2 n} \sim O A^{2 n}=P a . P B . P b \cdot P C . \& c .
$$

$=P a \cdot P b . \& c \cdot P B \cdot P C . \& c=P a \cdot P b \cdot \& c \cdot\left(O P^{n} \sim O A^{n}\right)$,
and $\therefore P a . P b . P c . \& c .=\frac{O P^{2 n} \sim O A^{2 n}}{O P^{n} \sim O A^{n}}=O P^{n}+O A^{n}$.
293. Cor. If the point $P$ be supposed to coincide with $A, P a, P b, \& c$. will become the chords of $\frac{\pi}{n}, \frac{3 \pi}{n}, \& c$.

$$
\text { and } \therefore \text { chd } \frac{\pi}{n} \text { chd } \frac{3 \pi}{n} \text { chd } \frac{5 \pi}{n} \& \text { c. to } n \text { factors }=2 \text {, }
$$

$$
\text { or } 2 \sin \frac{\pi}{2 n} 2 \sin \frac{3 \pi}{2 n} 2 \sin \frac{5 \pi}{2 n} \& c .=2,
$$

$$
\text { and } \therefore \sin \frac{\pi}{2 n} \sin \frac{3 \pi}{2 n} \sin \frac{5 \pi}{2 n} \& c .=\frac{1}{Q^{n-1}} 1 .
$$

294. To express the sine and cosine of an arc by means of continued products.

Since, $\sin \theta=\theta\left\{1-\frac{\theta^{2}}{1.2 .3}+\frac{\theta^{4}}{1.2 .3 .4 .5}-\& c.\right\}$, if we assume $\sin \theta=0$, the corresponding values of $\theta$ will be

$$
0, \pm \pi, \pm 2 \pi, \pm 3 \pi, \& c .
$$

$\therefore$ by the nature of equations, we have

$$
\begin{aligned}
& \sin \theta=C \theta(\pi-\theta)(\pi+\theta)(2 \pi-\theta)(2 \pi+\theta) \& c . \\
& =C \pi^{2} 2^{2} \pi^{2} s^{2} \pi^{2} \& c . \theta\left(1-\frac{\theta^{2}}{\pi^{2}}\right)\left(1-\frac{\theta^{2}}{2^{2} \pi^{2}}\right) \& c .
\end{aligned}
$$

but when $\theta$ is indefinitely small, we have seen in (213) that

$$
\frac{\sin \theta}{\theta}=1, \text { and } \therefore \text { we have } C \pi^{2} 2^{2} \pi^{2} 3^{2} \pi^{2} \& c .=1 ;
$$

$$
\text { hence, } \sin \theta=\theta\left(1-\frac{\theta^{2}}{\pi^{2}}\right)\left(1-\frac{\theta^{2}}{2^{2} \pi^{2}}\right) \& c
$$

Again, since $\cos \theta=1-\frac{\theta^{2}}{1.2}+\frac{\theta^{4}}{1.2 .3 .4}-\& \mathrm{c}$. if we suppose $\cos \theta=0$, the values of $\theta$ will be

$$
\begin{gathered}
\pm \frac{\pi}{2}, \pm \frac{3 \pi}{2}, \pm \frac{5 \pi}{2}, \text { \&c. } \\
\therefore \cos \theta=C^{\prime}\left(\frac{\pi}{2}-\theta\right)\left(\frac{\pi}{2}+\theta\right)\left(\frac{3 \pi}{2}-\theta\right)\left(\frac{3 \pi}{2}+\theta\right) \& \mathrm{sc} . \\
=C^{\prime} \frac{\pi^{2}}{2^{2}} \frac{3^{2} \pi^{2}}{2^{2}} \text { \&c. }\left(1-\frac{2^{2} \theta^{2}}{\pi^{2}}\right)\left(1-\frac{2^{2} \theta^{2}}{3^{2} \pi^{2}}\right) \& s c
\end{gathered}
$$

and if $\theta$ be indefinitely small, we shall have

$$
\cos \theta=1, \text { or } C^{\prime} \frac{\pi^{2}}{2^{2}} \frac{3^{2} \pi^{2}}{2^{2}} \& c .=1
$$

$$
\text { hence, } \cos \theta=\left(1-\frac{Q^{2} \theta^{2}}{\pi^{2}}\right)\left(1-\frac{\frac{Q}{}^{2} \theta^{2}}{3^{2} \pi^{2}}\right) \text { \&c. }
$$

295. Cor.1. It is manifest that if we suppose $C=C^{\prime}$, there will be the same number of factors in the expressions for both $\sin \theta$ and $\cos \theta$ : omitting therefore one of the first of the equal factors in the latter to make the number correspond with that of the former, we have

$$
\frac{\pi}{2}\left\{\frac{3^{2} \pi^{2}}{2^{2}} \frac{5^{2} \pi^{2}}{2^{2}} \frac{7^{2} \pi^{2}}{2^{2}} \& c .\right\}=\pi^{2} 2^{2} \pi^{2} 3^{2} \pi^{2} \& c
$$

and $\therefore \frac{\pi}{2}=\frac{2^{2} 4^{2} 6^{2} \text { \&c. in inf. }}{3^{2} 5^{2} 7^{2} \text { \&c. in inf. }}$, which is Wallis's expression for the circumference of a circle whose diameter is 1 .

Hence also from (214) it appears that the area of a circle : the square of its diameter

$$
:: \frac{2^{2} 4^{2} 6^{2} \& c . \text { in inf. }}{3^{2} 5^{2} 7^{2} \& c . \operatorname{in~inf.}}: 1:: \frac{8}{9} \times \frac{24}{25} \times \frac{48}{49} \times \& c .: 1
$$

296. Cor. 2. From the two theorems above proved, the logarithmic sines and cosines of arcs are easily derived without previously computing their natural sines and cosines.

For, let $\theta=\frac{m}{n} \frac{\pi}{Q}$; therefore we have

$$
\begin{aligned}
& \sin \frac{m}{n} \frac{\pi}{2}=\frac{m}{n} \frac{\pi}{2}\left(1-\frac{m^{2}}{2^{2} n^{2}}\right)\left(1-\frac{m^{2}}{4^{2} n^{2}}\right) \& c \\
& \cos \frac{m}{n} \frac{\pi}{2}=\left(1-\frac{n^{2}}{n^{2}}\right)\left(1-\frac{m^{2}}{3^{2} n^{2}}\right) \& c
\end{aligned}
$$

and thence

$$
\begin{aligned}
& \log \sin \frac{m}{n} \frac{\pi}{2}=\log \pi+\log \left(\frac{m}{2 n}\right)+\log \left(1-\frac{m^{2}}{2^{2} n^{2}}\right)+\& c . \\
& \log \cos \frac{m}{n} \frac{\pi}{2}=\log \left(1-\frac{m^{2}}{n^{2}}\right)+\log \left(1-\frac{m^{2}}{3^{2} n^{2}}\right)+\& c .
\end{aligned}
$$

297. To express the length of an arc in terms of its tangent.

From (282) we have
$\cos \theta+\sqrt{-1} \sin \theta=e^{\theta \sqrt{-1}}$, and $\cos \theta-\sqrt{-1} \sin \theta=e^{-\theta \sqrt{-1}} ;$
$\therefore \frac{e^{\theta \sqrt{-1}}}{e^{-\theta \sqrt{-1}}}$, or $e^{8 \theta \sqrt{-1}}=\frac{\cos \theta+\sqrt{-1} \sin \theta}{\cos \theta-\sqrt{-1} \sin \theta}=\frac{1+\sqrt{-1} \tan \theta}{1-\sqrt{-1} \tan \theta}$;
hence, taking the logarithms of both sides, we get

$$
\begin{aligned}
& 2 \theta \sqrt{-1}=\log (1+\sqrt{-1} \tan \theta)-\log (1-\sqrt{-1} \tan \theta) \\
& =\sqrt{-1} \tan \theta+\frac{1}{2} \tan ^{2} \theta-\frac{1}{3} \sqrt{-1} \tan ^{3} \theta-\frac{1}{4} \tan ^{4} \theta+\& c . \\
& +\sqrt{-1} \tan \theta-\frac{1}{2} \tan ^{2} \theta-\frac{1}{3} \sqrt{-1} \tan ^{3} \theta+\frac{1}{4} \tan ^{4} \theta-\& c . \\
& \quad \text { and } \therefore \theta=\tan \theta-\frac{1}{3} \tan ^{3} \theta+\frac{1}{5} \tan ^{5} \theta-\& c .
\end{aligned}
$$

Ex. 1. Let $\theta=45^{\circ}$, or $\tan \theta=1$,
$\therefore$ the arc of $45^{\circ}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\& c$. in inf.

$$
=\left(1-\frac{1}{3}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\left(\frac{1}{9}-\frac{1}{11}\right)+\& c . \text { in infinitum }
$$

$$
=\frac{2}{1.3}+\frac{2}{5.7}+\frac{2}{9.11}+\& c . \text { in infinitum: }
$$

aud the whole circumference of the circle whose radius is $r$

$$
=16 r\left\{\frac{1}{1.3}+\frac{1}{5.7}+\frac{1}{9.11}+\& c . \text { in infinitum }\right\} .
$$

Ex. 2. Let $\tan \theta$ and $\tan \theta^{\prime}$ be taken respectively equal to $\frac{1}{2}$ and $\frac{1}{3}$, then

$$
\theta=\frac{1}{2}-\frac{1}{3} \frac{1}{2^{5}}+\frac{1}{5} \frac{1}{9^{5}}-\& c ., \theta^{\prime}=\frac{1}{3}-\frac{1}{3} \frac{1}{3^{3}}+\frac{1}{5} \frac{1}{3^{5}}-\& c .
$$

$\therefore \theta+\theta^{\prime}=\left(\frac{1}{2}+\frac{1}{3}\right)-\frac{1}{3}\left(\frac{1}{2^{3}}+\frac{1}{3^{3}}\right)+\frac{1}{5}\left(\frac{1}{2^{5}}+\frac{1}{3^{5}}\right)-\& c$.
but by (110), $\theta+\theta^{\prime}=\tan ^{-1} \frac{1}{2}+\tan ^{-1} \frac{1}{3}=45^{\circ}$;
$\therefore$ the $\operatorname{arcof} 45^{0}=\left(\frac{1}{2}+\frac{1}{3}\right)-\frac{1}{3}\left(\frac{1}{2^{3}}+\frac{1}{3^{5}}\right)+\frac{1}{5}\left(\frac{1}{2^{5}}+\frac{1}{3^{5}}\right)-$ \& c.
298. Cor. 1. Since $\log u=\left(u-u^{-1}\right)-\frac{1}{2}\left(u^{2}-u^{-2}\right)$
$+\frac{1}{3}\left(u^{3}-u^{-3}\right)-\& c$. , if we suppose $u=\sqrt{-1}$, we shall have $\log \sqrt{-1}$
$=\sqrt{-1}-\frac{1}{\sqrt{-1}}-\frac{1}{2}\{-1+1\}+\frac{1}{3}\left\{-\sqrt{-1}+\frac{1}{\sqrt{-1}}\right\}-\& c$.
$=2 \sqrt{-1}\left\{1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+8 \mathrm{cc}.\right\}=2 \sqrt{-1} \frac{\pi}{4}$, by Ex. 1 ;
$\therefore 2 \pi$ or the circumference of the circle whose radius is 1

$$
=\frac{4 \log \sqrt{-1}}{\sqrt{-1}}
$$

299. Cor. 2. Since $u^{u}=e^{u \log u}$, if $u=\sqrt{-1}$, we have

$$
\begin{gathered}
(\sqrt{-1})^{\sqrt{-1}}=e^{\sqrt{-1} \log \sqrt{-1}}=e^{-\frac{\pi}{2}} \\
=1-\frac{\pi}{2}+\frac{1}{1 \cdot 2} \frac{\pi^{2}}{4}-\frac{1}{1.2 \cdot 3} \frac{\pi^{3}}{8}+\& \mathrm{c} .=.207879 \& \mathrm{c}
\end{gathered}
$$

The discovery of these singular formula is due to the celebrated John Bernoulli.

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300. To express the length of an arc in terms of its sine, and the sines of its successive multiples.

Since $\log x=\left(x-x^{-1}\right)-\frac{1}{2}\left(x^{2}-x^{-2}\right)+\frac{1}{3}\left(x^{3}-x^{-3}\right)-\& c$.
let $x=e^{\theta \sqrt{-1}}, \therefore \theta \sqrt{-1}=\left(e^{\theta \sqrt{-1}}-e^{-\theta \sqrt{-1}}\right)-\frac{1}{\Omega}\left(e^{2 \theta \sqrt{-1}}-e^{-2^{\theta \sqrt{-1}}}\right)$

$$
\text { and } \begin{aligned}
& \frac{\theta}{2}= \frac{+\frac{1}{3}\left(e^{3 \theta \sqrt{-1}}-e^{-3^{\theta \sqrt{ } /-1}}\right)-\& \mathbf{c} .}{2 \sqrt{-1}}-e^{-\theta \sqrt{-1}} \\
& 2 \sqrt{2}\left(\frac{1}{2 \sqrt{-1}}\right) \\
&+\frac{1}{3}\left(\frac{e^{3 \theta \sqrt{-1}}-e^{-2 \theta \sqrt{-1}}}{2 \sqrt{-1}}\right)-\& c . \\
&= \sin \theta-\frac{1}{2} \sin 2 \theta+\frac{1}{3} \sin 3 \theta-\& \overline{-1} .
\end{aligned}
$$

301. To express the length of an arc in terms of its sine, and the secants of successive submultiples.

By article (76) we have immediately,

$$
\begin{aligned}
\sin \theta & =2 \sin \frac{\theta}{2} \cos \frac{\theta}{Q^{2}} \\
\sin \frac{\theta}{2} & =2 \sin \frac{\theta}{2^{2}} \cos \frac{\theta}{2^{2}} \\
\sin \frac{\theta}{2^{2}} & =2 \sin \frac{\theta}{2^{5}} \cos \frac{\theta}{2^{5}} \\
\& c . & =\& c \ldots \ldots \ldots \ldots \ldots \\
\sin \frac{\theta}{2^{n-1}} & =2 \sin \frac{\theta}{2^{n}} \cos \frac{\theta}{2^{n}}
\end{aligned}
$$

therefore by multiplication, we get

$$
\sin \theta=2^{n} \sin \frac{\theta}{2^{n}} \cos \frac{\theta}{2} \cos \frac{\theta}{2^{2}} \cos \frac{\theta}{2^{3}} \& c \cdot \cos \frac{\theta}{2^{n}},
$$

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and $2^{n} \sin \frac{\theta}{2^{n}}=\sin \theta \sec \frac{\theta}{2} \sec \frac{\theta}{2^{2}} \sec \frac{\theta}{2^{3}}$ \&c. $\sec \frac{\theta}{2^{n}}$ :
suppose now $n$ to be infinite, in which case $\sin \frac{\theta}{2^{n}}=\frac{\theta}{2^{n}}$ by (213),

$$
\therefore \theta=\sin \theta \sec \frac{\theta}{2} \sec \frac{\theta}{2^{2}} \& c . \text { in infinitum } ;
$$

which is generally known by the name of Euler's formula.
302. Given two sides a, b of a triangle, and the included angle $\mathbf{C}$, to find the remaining side and angles by means of infinite series.

By (167) we have

$$
c^{2}=a^{2}+b^{2}-2 a b \cos C
$$

$=a^{2}+b^{2}-a b\left(e^{C^{\sqrt{-1}}}+e^{-C^{\sqrt{-1}}}\right)=\left(a-b e^{C^{\sqrt{-1}}}\right)\left(a-b e^{-C^{\sqrt{-1}}}\right)$,
$\therefore$ as before, $2 \log c=\log \left(a-b e^{\left.c^{\sqrt{-1}}\right)}+\log \left(a-b e^{-C^{\sqrt{-1}}}\right)\right.$

$$
=2 \log a+\log \left(1-\frac{b}{a} e^{C \sqrt{-1}}\right)+\log \left(1-\frac{b}{a} e^{-C^{\sqrt{-1}}}\right)
$$

$=2 \log a-\frac{b}{a}\left(e^{C^{\sqrt{-1}}}+e^{-C^{\sqrt{-1}}}\right)-\frac{b^{2}}{2 a^{2}}\left(e^{2 \boldsymbol{C}^{\sqrt{-1}}}+e^{-2 C^{\sqrt{-1}}}\right)-\& \mathbf{c}$.
and $\log c=\log a-\frac{b}{a} \cos C-\frac{b^{2}}{2 a^{2}} \cos 2 C-\frac{b^{3}}{3 a^{3}} \cos 3 C-\& c$.
the logarithms being taken in the system whose base is $e$ or 2.71828 \&c.

$$
\begin{gathered}
\text { also, since } \frac{a}{b}=\frac{\sin A}{\sin B}=\frac{\sin (B+C)}{\sin B}=\cos C+\frac{\sin C}{\tan B}, \\
\text { we have tan } B=\frac{b \sin C}{a-b \cos C}, \\
\text { and } \therefore \frac{e^{B \sqrt{-1}}-e^{-B \sqrt{-1}}}{e^{B \sqrt{-1}}+e^{-B \sqrt{-1}}}=\frac{b\left(e^{C \sqrt{-1}}-e^{-C \sqrt{-1}}\right)}{2 a-b\left(e^{C \sqrt{-1}}+e^{-C \sqrt{-1}}\right)}
\end{gathered}
$$

$$
\text { or } e^{2 B^{\sqrt{-1}}}=\frac{a-b e^{-C^{\sqrt{-1}}}}{a-b e^{C^{\sqrt{-1}}}}
$$

and taking the logarithms of both sides, we get

$$
\begin{gathered}
2 B \sqrt{-1}=\log \left(a-b e^{-C \sqrt{-1}}\right)-\log \left(a-b e^{C^{\sqrt{-1}}}\right) \\
=-\frac{b}{a} e^{-C \sqrt{-1}}-\frac{b^{2}}{2 a^{2}} e^{-2 C \sqrt{-1}}-\frac{b^{3}}{3 a^{3}} e^{-3 C \sqrt{-1}}-\& c . \\
+\frac{b}{a} e^{C \sqrt{-1}}+\frac{b^{2}}{2 a^{2}} e^{2 C \sqrt{-1}}+\frac{b^{3}}{3 a^{3}} e^{3 C \sqrt{-1}}+\& \mathrm{c} . \\
=\frac{b}{a}\left(e^{C \sqrt{-1}}-e^{-C \sqrt{-1}}\right)+\frac{b^{2}}{2 a^{2}}\left(e^{2 C \sqrt{-1}}-e^{-2 C \sqrt{-1}}\right) \\
+\frac{b^{3}}{3 a^{3}}\left(e^{3 C \sqrt{-1}}-e^{-3 C \sqrt{-1}}\right)+\& c . \\
\text { and } \therefore B=\frac{b}{a} \sin C+\frac{b^{2}}{2 a^{2}} \sin 2 C+\frac{b^{3}}{3 a^{3}} \sin 3 C+\& c .
\end{gathered}
$$

which is expressed in terms of the radius 1: and by (278)

$$
\begin{aligned}
B & =\frac{b}{a} \frac{\sin C}{\sin 1^{\prime \prime}}+\frac{b^{2}}{2 a^{2}} \frac{\sin 2 C}{\sin 1^{\prime \prime}}+\frac{b^{3}}{3 a^{3}} \frac{\sin 3 C}{\sin 1^{\prime \prime}}+\& \mathrm{c} \\
& =\frac{b}{a} \frac{\sin C}{\sin 1^{\prime \prime}}+\frac{b^{2}}{a^{2}} \frac{\sin 2 C}{\sin 2^{\prime \prime}}+\frac{b^{3}}{a^{3}} \frac{\sin 3 C}{\sin 3^{\prime \prime}}+\& \mathrm{c}
\end{aligned}
$$

which is the value of $B$ in seconds; also, if $b$ be much less than $a$, a few terms of these series will be sufficient.
303. By processes similar to those pursued in the last frticle, we shall find that the equation

$$
\begin{gathered}
\tan \theta=\frac{n \sin \phi}{1+n \cos \phi}, \text { gives } \\
\theta=n \sin \phi-\frac{n^{2}}{\varrho} \sin 2 \phi+\frac{n^{3}}{3} \sin 3 \phi-\mathbb{N} c \\
D_{n}
\end{gathered}
$$

$\tan \theta=1$ tan $\phi$ gives

$$
\begin{aligned}
& \theta=\phi+\left(\frac{n-1}{n+1}\right) \sin 8 \phi+\frac{1}{2}\left(\frac{n-1}{n+1}\right)^{2} \sin 4 \phi+\& c \\
& \phi=\theta-\left(\frac{n-1}{n+1}\right) \sin 2 \theta+\frac{1}{2}\left(\frac{n-1}{n+1}\right)^{2} \sin 4 \theta-\& c_{0} \\
& \quad \tan \theta=\cos \alpha \tan \phi \text { gives } \\
& \theta=\phi-\tan ^{2} \frac{a}{2} \sin 2 \phi+\frac{1}{2} \tan ^{4} \frac{a}{2} \sin 4 \phi-\& c \\
& \quad \text { and } \sin \theta=\sin \alpha \sin (\beta+\theta) \text { gives } \\
& \theta=\sin a \sin \beta+\frac{1}{2} \sin ^{2} \alpha \sin 2 \beta+\frac{1}{3} \sin ^{3} \alpha \sin 3 \beta+\& c .
\end{aligned}
$$

304. T'o solve triangles without the aid of tables.

In (224) we have seen that $a=c \sin A, \therefore$ by (277) we have

$$
a=c\left\{\left(\frac{A^{0}}{r^{0}}\right)-\frac{1}{1.2 .3}\left(\frac{A^{0}}{r^{0}}\right)^{3}+\frac{1}{1.2 .9 \cdot 4.5}\left(\frac{A^{0}}{r^{0}}\right)^{5}-\& c_{0}\right\}:
$$

also taw $A=\frac{a}{b}$, and $\therefore$ by (297) we obtain

$$
A^{0}=r^{0}\left\{\left(\frac{a}{b}\right)-\frac{1}{3}\left(\frac{a}{b}\right)^{3}+\frac{1}{5}\left(\frac{a}{b}\right)^{5}-\& \mathrm{c} .\right\}:
$$

again, since $\cos C=\frac{a^{2}+b^{2}-c^{2}}{2 a b}$, we shall have from (279)
$C^{0}=r^{0}\left\{1-\frac{1}{2}\left(\frac{a^{2}+b^{2}-c^{2}}{2 a b}\right)^{2}+\frac{1}{1.2 .3 .4}\left(\frac{a^{2}+b^{2}-c^{2}}{2 a b}\right)^{4}-\& c.\right\}$, and so on for all other cases.
305. By a similar process if $r$ be the radius of the circle circumscribed about a triangle whose sides are $2 a, 2 b, 2 c$, it will readily appear that

$$
\left(\frac{a+b+c}{r}\right)+\frac{1}{2.3}\left(\frac{a^{8}+b^{3}+c^{3}}{r^{3}}\right)+\frac{1.3}{2.4 .5}\left(\frac{a^{5}+b^{5}+c^{5}}{r^{5}}\right)
$$

$+\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7}\left(\frac{a^{7}+b^{7}+c^{7}}{r^{7}}\right)+\& \mathbf{c} .=$ the semi-circumference of a circle whose radius is 1 .
306. To solve triangles, two of whose angles are very acute.

First, let the two very acute angles $A, B$ and the side $c$ be given, then will the remaining angle $C$ be found to be very obtuse: also by (276) we have

$$
\begin{gathered}
\qquad \sin A=A-\frac{A^{3}}{1.2 \cdot 3}, \sin B=B-\frac{B^{3}}{1.2 \cdot 3} \\
\text { and } \sin C=\sin (A+B)=(A+B)-\frac{(A+B)^{3}}{1.2 .3} \text { nearly } \\
\text { hence } a=\frac{c \sin A}{\sin (A+B)}=\frac{c A}{A+B}\left\{1+\frac{2 A B+B^{2}}{1.2 .3}\right\} \\
\text { and } b=\frac{c \sin B}{\sin (A+B)}=\frac{c B}{A+B}\left\{1+\frac{A^{2}+2 A B}{1.2 .3}\right\}
\end{gathered}
$$

swhich are the true values after neglecting such terms as contain four or more dimensions of $A$ and $B$.

Hence the excess of the sum of the two other sides above that which subtends the greatest angle $=a+b-c=\frac{c A B}{2}$, and it may be observed that $A$ and $B$ are both understood to be expressed in terms of the radius.

Next, let the two sides $a, b$ and the included angle $C$ which is very obtuse be given : assume $C=\pi-\alpha$, then by (167) we have

$$
\begin{aligned}
& c^{2}=a^{2}+b^{2}-2 a b \cos C=a^{2}+b^{2}+2 a b \cos a \\
& =a^{2}+b^{2}+2 a b\left(1-\frac{a^{2}}{1.2}\right)=(a+b)^{2}-a b a^{2}
\end{aligned}
$$

$$
\begin{aligned}
& \therefore c=\sqrt{(a+b)^{2}-a b a^{2}}=a+b-\frac{1}{2} \frac{a b a^{2}}{a+b}: \\
& \text { again, } \sin A=\frac{a}{c} \sin C=\frac{a}{c} \sin a=\frac{a}{c}\left\{a-\frac{a^{3}}{1.2 \cdot 3}\right\} \\
& \quad=\frac{a}{a+b}\left\{\alpha+\frac{1}{2} \frac{a b a^{3}}{(a+b)^{2}}-\frac{a^{3}}{1.2 .3}\right\} \\
& \quad=\frac{a a}{a+b}\left\{1-\frac{\left(a^{2}+b^{2}\right)-a b}{(a+b)^{2}} \frac{a^{2}}{1.2 \cdot 3}\right\}
\end{aligned}
$$

whence $A=\sin A+\frac{\sin ^{3} A}{1.2 .3}=\frac{a a}{a+b}+\frac{a b(a-b)}{(a+b)^{2}} \frac{a^{3}}{1.2 .3}$

$$
=\frac{a a}{a+b}\left\{1+\frac{b(a-b)}{(a+b)^{2}} \frac{a^{2}}{1.2 .3}\right\} ;
$$

$$
\text { similarly, } B=\frac{b a}{a+b}\left\{1-\frac{a(a-b)}{(a+b)^{2}} \frac{a^{2}}{1 \cdot 2 \cdot 3}\right\}
$$

which would have been found to be of the same value from the equation $B=a-A$.
307. To express the cosine of the mulliple of an arc in terms of descending pozers of the cosine of the arc itself.

Since
$(1-a x)\left(1-\frac{x}{a}\right)=1-\left(a+\frac{1}{a}\right) x+x^{2}=1-x\left(a+\frac{1}{a}-x\right)$,
if we assume $a+\frac{1}{a}=p$, and take the logarithms of both sides of this equation, we have

$$
\begin{gathered}
\log (1-a x)+\log \left(1-\frac{x}{a}\right)=\log \{1-x(p-x)\}, \text { and } \\
a x+\frac{a^{2}}{2} x^{2}+\frac{a^{3}}{3} x^{3}+\& c+\frac{a^{\prime \prime}}{n} x^{n}+\alpha c
\end{gathered}
$$

$$
\begin{gathered}
\quad+\frac{1}{a} x+\frac{1}{2 a^{2}} x^{2}+\frac{1}{3 a^{3}} x^{3}+\& \mathrm{c} .+\frac{1}{n a^{n}} x^{n}+\& \mathrm{c} \\
=(p-x) x+\frac{(p-x)^{2}}{2} x^{3}+\frac{(p-x)^{3}}{3} x^{3}+\& \mathrm{c} .+\frac{(p-x)^{n}}{n} x^{n}+\& \mathrm{c} .
\end{gathered}
$$

therefore equating the coefficients of $x^{n}$ in both sides, we get

$$
\begin{aligned}
a^{n}+\frac{1}{a^{n}} & =p^{n}-n p^{n-2}+\frac{n(n-3)}{1.2} p^{n-4}-\frac{n(n-4)(n-5)}{1.2 \cdot 3} p^{n-6}+\& \mathrm{c} . \\
& \pm \frac{n(n-m-1) \& \mathrm{c} \cdot(n-2 m+1)}{1.2 .3 \& \mathrm{c} . m} p^{n-2 n} \mp \& \mathrm{c} .
\end{aligned}
$$

in which the last term will be 2 or $n p$ according as $n$ is even or odd :

$$
\text { now if } p=a+\frac{1}{a}=2 \cos A, \text { we have } 2 \cos n A=a^{n}+\frac{1}{a^{n}}
$$

and if $n$ be even, we shall have

## $2 \cos n A$

$=(2 \cos A)^{n}-n(2 \cos A)^{n-2}+\frac{n(n-3)}{1.2}(2 \cos A)^{n-4}-8 \mathrm{c} . \pm 2 ;$ also if $n$ be odd, we shall find
$2 \cos n A$
$=(2 \cos A)^{n}-n(2 \cos A)^{n-2}+\frac{n(n-3)}{1.2}(2 \cos A)^{n-4}-\& \mathrm{c}$. $\pm n(2 \cos A)$.

If we differentiate both sides of the equations just investigated, we shall obtain similar expressions for $\sin n A$.
308. To express the cosine of the multiple of an arc in terms of ascending powers of the cosine of the arc itself.

Reversing the order of the terms of the expressions found in the last article, we get immediately, if $n$ be even,

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$$
\cos n A= \pm\left\{1-\frac{n^{2}}{1 \cdot 2} \cos ^{2} A+\frac{n^{2}\left(n^{2}-4\right)}{1 \cdot 2 \cdot 3 \cdot 4} \cos ^{4} A-8 c .\right\}
$$

and if $n$ be odd,
$\cos n A= \pm n \cos A\left\{1-\frac{\left(n^{2}-1\right)}{1.2 \cdot 3} \cos ^{2} A+\frac{\left(n^{2}-1\right)\left(n^{2}-9\right)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \cos ^{4} A-\& \mathrm{c}.\right\}$,
in which the upper signs must be used when $n$ is of the forms $4 m$ and $4 m+1$, and the lower when of the forms $4 m+2$ and $4 m+3$.

Differentiating both sides of these equations, we shall immediately obtain expressions of the same kind for $\sin n A$.
309. To find the sum of the sines of a series of arcs in arithmetical progression.

Let $\sin \theta+\sin (\theta+\delta)+\sin (\theta+2 \delta)+8 c .+\sin (\theta+(n-1) \delta)$ be the proposed series, then by (67), we have

$$
\begin{aligned}
2 \sin \frac{\delta}{2} \sin \theta & =\cos \left(\theta-\frac{\delta}{2}\right)-\cos \left(\theta+\frac{\delta}{2}\right) \\
2 \sin \frac{\delta}{2} \sin (\theta+\delta) & =\cos \left(\theta+\frac{\delta}{2}\right)-\cos \left(\theta+\frac{3 \delta}{2}\right) \\
\delta c \ldots \ldots \ldots \ldots & =\& c \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots
\end{aligned}
$$

$2 \sin \frac{\delta}{2} \sin (\theta+(n-1) \delta)=\cos \left(\theta+\frac{2 n-3}{2} \delta\right)-\cos \left(\theta+\frac{2 n-1}{.2} \delta\right) ;$
and denoting the sum of the proposed series by $S$,
we shall have by addition,

$$
\begin{aligned}
2 \sin \frac{\delta}{2} S & =\cos \left(\theta-\frac{\delta}{2}\right)-\cos \left(\theta+\frac{2 n-1}{2} \delta\right) \\
& =2 \sin \left(\theta+\frac{n-1}{2} \delta\right) \sin \frac{n \delta}{2}, \text { by }(67),
\end{aligned}
$$

$$
\text { and } \therefore S=\frac{\sin \left(\theta+\frac{n-1}{2} \delta\right) \sin \frac{n \delta}{2}}{\sin \frac{\delta}{2}}
$$

310. Ex. If $\delta$ be taken equal to $\theta, 2 \theta, 3 \theta, \& c$. successively, we shall have

$$
\sin \theta+\sin 2 \theta+\sin 3 \theta+\mathcal{K c}+\sin n \theta
$$

$$
=\frac{\sin \left(\frac{n+1}{2}\right) \theta \sin \frac{n \theta}{2}}{\sin \frac{\theta}{2}}
$$

$\sin \theta+\sin 3 \theta+\sin 5 \theta+\mathcal{s c} .+\sin (2 n-1) \theta=\frac{\sin ^{2} n \theta}{\sin \theta}:$
$\sin \theta+\sin 4 \theta+\sin 7 \theta+\mathcal{k} c .+\sin (3 n-2) \theta$

$$
=\frac{\sin \left(\frac{3 n-1}{2}\right) \theta \sin \frac{3 n \theta}{2}}{\sin \frac{3 \theta}{2}}
$$

$\& c \ldots \ldots=\& c$.
311. Cor. Hence also,

$$
\sin \theta-\sin (\theta+\delta)+\sin (\theta+2 \delta)-\varepsilon c
$$

$=\{\sin \theta+\sin (\theta+2 \delta)+\delta c\}-.\{\sin (\theta+\delta)+\sin (\theta+3 \delta)+\delta c$. may be found; and if $-\delta$ be substituted in the place of $\delta$, the sum of the series $\sin \theta \pm \sin (\theta-\delta)+\sin (\theta-2 \delta) \pm \delta c$. will be obtained.

312 By proper substitutions in the formula above deduced, the summation of the sines of any series of arcs in arithmetical progression may be effected : also, if
$\theta+\frac{2 n-1}{2} \delta=(2 m-1) \frac{\pi}{2}$, the sum will be

$$
\frac{\cos \left(\theta-\frac{\delta}{2}\right)}{2 \sin \frac{\delta}{2}}
$$

and this has been erroneously called the sum of the series continued in infinitum, but which in fact cannot be determined.
313. To find the sum of the cosines of a series of arcs in arithmetical progression.

Let $\cos \theta+\cos (\theta+\delta)+\cos (\theta+2 \delta)+\& c .+\cos (\theta+(n-1) \delta)$ be the proposed series, then as before,

$$
\begin{aligned}
& 2 \sin \frac{\delta}{2} \cos \theta=\sin \left(\theta+\frac{\delta}{2}\right)-\sin \left(\theta-\frac{\delta}{2}\right), \\
& 2 \sin \frac{\delta}{2} \cos (\theta+\delta)=\sin \left(\theta+\frac{3 \delta}{2}\right)-\sin \left(\theta+\frac{\delta}{2}\right), \\
& \delta c \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots
\end{aligned}
$$

$2 \sin \frac{\delta}{2} \cos (\theta+(n-1) \delta)=\sin \left(\theta+\frac{2 n-1}{2} \delta\right)-\sin \left(\theta+\frac{2 n-3}{2} \delta\right) ;$
whence by addition, we have

$$
\begin{aligned}
2 \sin \frac{\delta}{2} S & =\sin \left(\theta+\frac{2 n-1}{2} \delta\right)-\sin \left(\theta-\frac{\delta}{2}\right) \\
& =2 \cos \left(\theta+\frac{n-1}{2} \delta\right) \sin \frac{n \delta}{2}, \text { by }(67), \\
\text { and } \therefore S & =\frac{\cos \left(\theta+\frac{n-1}{2} \delta\right) \sin \frac{n \delta}{2}}{\sin \frac{\delta}{2}}
\end{aligned}
$$

314. Ex. Let $\delta$ be taken equal to $\theta, 2 \theta, 3 \theta, \& c$. in succession, and

$$
\begin{gathered}
\cos \theta+\cos 2 \theta+\cos 3 \theta+\& c+\cos n \theta \\
=\frac{\cos \left(\frac{n+1}{2}\right) \theta \sin \frac{n \theta}{2}}{\sin \frac{\theta}{2}}
\end{gathered}
$$

$$
\begin{gathered}
\cos \theta+\cos 3 \theta+\cos 5 \theta+\& c+\cos (2 n-1) \theta \\
=\frac{\cos n \theta \sin n \theta}{\sin \theta}
\end{gathered}
$$

$$
\cos \theta+\cos 4 \theta+\cos 7 \theta+8 c \cdot+\cos (3 n-2) \theta
$$

$$
=\frac{\cos \left(\frac{3 n-1}{2}\right) \theta \sin \frac{3 n \theta}{2}}{\sin \frac{3 \theta}{2}}
$$

$$
\& c \ldots \ldots=\& c
$$

315. Cor. The sums of the cosines of the series mentioned in article (311) may be found by the formula above deduced, as indeed may the sums of the cosines of any series of arcs whatsoever in arithmetical progression.
316. As in the preceding articles, the sums of the squares, cubes, \&c. of the sines and cosines of the same arcs may be obtained by means of (272) and (274). Thus

$$
\sin ^{2} \theta+\sin ^{2}(\theta+\delta)+8 c \cdot+\sin ^{2}(\theta+(n-1) \delta)=
$$

$$
\frac{n}{2}-\frac{1}{2}\{\cos 2 \theta+\cos 2(\theta+\delta)+\varepsilon c \cdot+\cos 2(\theta+(n-1) \delta)\}
$$

$$
=\frac{n}{2}-\frac{1}{2}\left\{\frac{\cos (2 \theta+(n-1) \delta) \sin n \delta}{\sin \delta}\right\},
$$

by (313), and so on.
Also, by successive differentiations of the examples given in (310) and (314), the sums of such series as

$$
\sin \theta+2^{m} \sin 2 \theta+3^{m} \sin 3 \theta+\& c
$$

and $\cos \theta+2^{m} \cos 2 \theta+3^{m} \cos 3 \theta+\& c$.
will be obtained.
Similarly, by multiplying by $d \theta$, and integrating successively, the sums of such series as

$$
\sin \theta+\frac{1}{2^{n n}} \sin 2 \theta+\frac{1}{3^{n n}} \sin 3 \theta+8 c
$$

EE

$$
\text { and } \cos \theta+\frac{1}{2^{m}} \cos 2 \theta+\frac{1}{3^{m}} \cos 3 \theta+\& c
$$

may be determined.
317. The series mentioned in the preceding articles might have been summed by means of the expressions for the sine and cosine of an arc investigated in (263) and (282). Thus, if we suppose $\sin \theta=\frac{1}{2 \sqrt{-1}}\left(x-\frac{1}{x}\right)$, we shall have

$$
\begin{aligned}
& \quad \sin \theta+\sin 2 \theta+\sin 3 \theta+\& c . \text { to } n \text { terms } \\
& =\frac{1}{2 \sqrt{-1}}\left\{x+x^{2}+x^{3}+\& c . \text { to } n \text { terms }\right\} \\
& -\frac{1}{2 \sqrt{-1}}\left\{\frac{1}{x}+\frac{1}{x^{2}}+\frac{1}{x^{3}}+8 \mathrm{c} . \text { to } n \text { terms }\right\} \\
& =\frac{1}{2 \sqrt{-1}}\left\{\frac{x\left(x^{n}-1\right)}{x-1}+\frac{1-x^{n}}{x^{n}(x-1)}\right\} \\
& =\frac{1}{2 \sqrt{-1}}\left\{\frac{\left(x^{n}-1\right)\left(x^{n+1}-1\right)}{x^{n}(x-1)}\right\} \\
& =\frac{1}{2 \sqrt{-1}}\left\{\frac{\left(x^{\frac{n}{2}}-\frac{1}{x^{\frac{n}{2}}}\right)\left(x^{\frac{n+1}{2}}-\frac{1}{x^{\frac{n+1}{2}}}\right)}{\left.x^{\frac{2}{2}}-\frac{1}{x^{\frac{1}{2}}}\right\}}\right. \\
& \left.=\frac{\sin \frac{n \theta}{2} \sin }{\sin \frac{\theta}{2}+1} \frac{n}{2}\right) \theta
\end{aligned}
$$

Again, since $\cos \theta=\frac{e^{\theta \sqrt{-1}}+e^{-\theta \sqrt{-1}}}{2}$, we have

$$
\cos \theta+\cos 2 \theta+\cos 3 \theta+8 c . \text { to } n \text { terms }
$$

$$
\begin{aligned}
& =\frac{1}{2}\left\{e^{\theta \sqrt{-1}}+e^{2 \theta \sqrt{-1}}+e^{3 \theta \sqrt{-1}}+\& c . \text { to } n \text { terms }\right\} \\
& +\frac{1}{2}\left\{e^{-\theta \sqrt{-1}}+e^{-2 \theta \sqrt{-1}}+e^{-5 \theta \sqrt{-1}}+\& \mathbf{c} . \text { to } n \text { terms }\right\} \\
& =\frac{1}{2}\left\{\frac{\left(e^{\frac{n \theta}{2} \sqrt{-1}}-e^{-\frac{n \theta}{2} \sqrt{-1}}\right)\left(e^{\frac{(n+1) \theta}{2} \sqrt{-1}}+e^{-\frac{(n+1) \theta}{2} \sqrt{-1}}\right)}{e^{\frac{\theta}{2}}-e^{-\frac{\theta}{2}}}\right\} \\
& =\frac{\sin \frac{n \theta}{2} \cos \left(\frac{n+1}{2}\right) \theta}{\sin \frac{\theta}{2}}, \text { as before. }
\end{aligned}
$$

It moreover appears from (73) that these are recurring series, and may therefore be summed by the rules laid down for that purpose; but there still remains to be explained another method not inferior to any that have yet been given.

Let $S=\sin \theta+\sin 3 \theta+\sin 5 \theta+\& c .+\sin (2 n-1) \theta$,
$\therefore S \sin 2 \theta=\sin \theta \sin 2 \theta+\sin 2 \theta \sin 3 \theta+\sin 2 \theta \sin 5 \theta+\& c$. $+\sin 2 \theta \sin (2 n-1) \theta$, and as appears from $(67)=$

$$
\begin{aligned}
& \frac{1}{2}\{\cos \theta-\cos 3 \theta+\cos \theta-\cos 5 \theta+\cos 3 \theta-\cos 7 \theta+8 c \\
& +\cos (2 n-5) \theta-\cos (2 n-1) \theta+\cos (2 n-3) \theta-\cos (2 n+1) \theta\}
\end{aligned}
$$

$$
=\frac{1}{2}\{2 \cos \theta-\cos (2 n-1) \theta-\cos (2 n+1) \theta\}
$$

$$
=\frac{1}{2}\{2 \cos \theta-2 \cos 2 n \theta \cos \theta\}, \text { by }(67)
$$

$$
\therefore S=\frac{\cos \theta-\cos 2 n \theta \cos \theta}{2 \sin \theta \cos \theta}=\frac{1-\cos 2 n \theta}{2 \sin \theta}=\frac{\sin ^{2} n \theta}{\sin \theta} .
$$

318. To find the sum of the series, $\operatorname{cosec} \theta+\operatorname{cosec} 2 \theta$ $+\operatorname{cosec} 2^{2} \theta+8 c$. to n terms.

$$
\text { Here, } \begin{aligned}
\operatorname{cosec} \theta & =\cot \frac{\theta}{2}-\cot \theta \\
\operatorname{cosec} 2 \theta & =\cot \theta-\cot 2 \theta \\
\& c & =\ldots \& \operatorname{con} \\
\operatorname{cosec} 2^{n-1} \theta & =\cot 2^{n-2} \theta-\cot 2^{n-1} \theta
\end{aligned}
$$

$\therefore$ by addition, we shall have

$$
\begin{aligned}
\operatorname{cosec} \theta & +\operatorname{cosec} 2 \theta+8 \mathrm{c} .+\operatorname{cosec} 2^{n-1} \theta \\
& =\cot \frac{\theta}{2}-\cot 2^{n-1} \theta
\end{aligned}
$$

319. To find the sum of the series, $\tan \theta+2 \tan 2 \theta+$ $2^{2} \tan 2^{2} \theta+8$ c. to $n$ terms.

It is easily proved that

$$
\begin{aligned}
\tan \theta & =\cot \theta-2 \cot 2 \theta \\
2 \tan 2 \theta & =2 \cot 2 \theta-2^{2} \cot 2^{2} \theta \\
2^{2} \tan 2^{2} \theta & =2^{2} \cot 2^{2} \theta-2^{3} \cot 2^{3} \theta \\
\& c . \ldots & =\& c \ldots \ldots \ldots \ldots \ldots \\
2^{n-2} \tan 2^{n-1} \theta & =2^{n-1} \cot 2^{n-1} \theta-2^{n} \cot 2^{n} \theta
\end{aligned}
$$

$\therefore$ by addition, we get
$\tan \theta+2 \tan 2 \theta+8 c .+2^{n-1} \tan 2^{n-1} \theta=\cot \theta-2^{n} \cot 2^{n} \theta$.
320. To find the sum of the series

$$
\frac{1}{2} \tan \frac{\theta}{2}+\frac{1}{2^{2}} \tan \frac{1}{2^{2}}+\frac{\theta}{2^{3}} \tan \frac{\theta}{2^{3}} \text { \&c. to } \mathrm{n} \text { terms. }
$$

We have seen in (112) that $\tan \frac{\theta}{2}=\cot \frac{\theta}{Q}-2 \cot \theta$;

$$
\begin{aligned}
& \therefore \frac{1}{2} \tan \frac{\theta}{2}=\frac{1}{2} \cot \frac{\theta}{2}-\cot \theta, \\
& \frac{1}{2^{2}} \tan \frac{\theta}{2^{2}}=\frac{1}{2^{2}} \cot \frac{\theta}{2^{2}}-\frac{1}{2} \cot \frac{\theta}{2}, \\
& \{c \ldots \ldots . .=\& c \ldots \ldots .
\end{aligned}
$$

$$
\frac{1}{2^{n}} \tan \frac{\theta}{2^{n}}=\frac{1}{2^{n}} \cot \frac{\theta}{2^{n}}-\frac{1}{2^{n-1}} \cot \frac{\theta}{2^{n-1}}
$$

hence by addition, the required $\operatorname{sum}=\frac{1}{2^{n}} \cot \frac{\theta}{2^{n}}-\cot \theta$; and if $n$ be infinite, this becomes $\frac{1}{\theta}-\cot \theta$.
321. To find the sum of the series, $(\tan \theta+\cot \theta)$ $+(\tan 2 \theta+\cot 2 \theta)+\left(\tan 2^{2} \theta+\cot 2^{2} \theta\right)+\& c$. to $n$ terms.

In this case, we have

$$
\begin{aligned}
& \tan \theta+\cot \theta=2 \cot \theta-2 \cot 2 \theta \\
& \tan 2 \theta+\cot 2 \theta=2 \cot 2 \theta-2 \cot 2^{2} \theta \\
& \& c \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots
\end{aligned}
$$

$\tan 2^{n-1} \theta+\cot 2^{n-1} \theta=2 \cot 2^{n-1} \theta-2 \cot 2^{n} \theta ;$
$\therefore$ the required sum $=2 \cot \theta-2 \cot 2^{n} \theta$.
322. To find the sum of the series $x \sin \theta+x^{2} \sin 2 \theta$ $+\mathrm{x}^{3} \sin 3 \theta+\delta c$. to n terms.

By means of the formulæ investigated in (282), if $S$ be the required sum, we shall have

$$
\begin{aligned}
S & =x\left(\frac{e^{\theta \sqrt{-1}}-e^{-\theta_{\sqrt{ }}^{-1}}}{2 \sqrt{-1}}\right)+x^{2}\left(\frac{e^{2 \theta^{-1}}-e^{-2 \theta \sqrt{-1}}}{2 \sqrt{-1}}\right) \\
& +x^{3}\left(\frac{e^{36 \sqrt{-1}}-e^{-36 \sqrt{-1}}}{2 \sqrt{-1}}\right)+\text { \&c. to } n \text { terms }
\end{aligned}
$$

$$
\begin{aligned}
& \left.=\frac{1}{2 \sqrt{-1}}\left\{x e^{\theta \sqrt{-1}}+x^{2} e^{2 \theta \sqrt{-1}}+x^{3} e^{3 \theta \sqrt{-1}}+\& \mathbf{c} \cdot+x^{n} e^{n \theta \sqrt{-1}}\right)\right\} \\
& -\frac{1}{2 \sqrt{-1}}\left\{x e^{-\theta \sqrt{-1}}+x^{2} e^{-2_{\theta} \sqrt{-1}}+x^{3} e^{-3 \theta^{2} \sqrt{-1}}+\& \mathbf{c} \cdot+x^{n} e^{-n \theta \sqrt{-1}}\right\} \\
& =\frac{x e^{\theta \sqrt{-1}}}{2 \sqrt{-1}}\left\{\frac{\left(x e^{\theta \sqrt{-1}}\right)^{n}-1}{x e^{\theta \sqrt{-1}}-1}\right\}-\frac{x e^{-\theta \sqrt{-1}}}{2 \sqrt{-1}}\left\{\frac{\left(x e^{-\theta \sqrt{-1}}\right)^{n}-1}{x e^{-\theta \sqrt{-1}}-1}\right\} \\
& \quad=\frac{1}{2 \sqrt{-1}}\left\{\frac{x^{n+2}\left(e^{n \theta \sqrt{-1}}-e^{-n \theta \sqrt{-1}}\right)}{x^{2}-\left(e^{\theta \sqrt{-1}}+e^{-\theta \sqrt{-1}}\right) x+1}\right\} \\
& \quad-\frac{1}{2 \sqrt{-1}\left\{\frac{x^{n+1}\left(e^{(n+1) \theta \sqrt{-1}}-e^{-(n+1) \theta \sqrt{-1}}\right)}{x^{2}-\left(e^{\theta \sqrt{-1}}+e^{-\theta^{\sqrt{-1}}}\right) x+1}\right\}} \\
& \quad+\frac{1}{2 \sqrt{-1}\left\{\frac{x\left(e^{\theta \sqrt{-1}}-e^{-\theta \sqrt{-1}}\right)}{x^{2}-\left(e^{\theta \sqrt{-1}}+e^{-\theta \sqrt{-1}}\right) x+1}\right\}} \\
& \quad=\frac{x^{n+2} \sin n \theta-x^{n+1} \sin (n+1) \theta+x \sin \theta}{x^{2}-2 \cos \theta x+1}
\end{aligned}
$$

If $x$ be a proper fraction and $n$ iudefinitely great, we shall have the sum of the series continued in infinitum

$$
=\frac{x \sin \theta}{x^{2}-2 \cos \theta x+1} .
$$

323. To find the sum of the series $\mathrm{x} \cos \theta+\mathrm{x}^{2} \cos 2 \theta$ $+\mathrm{x}^{3} \cos 3 \theta+\mathcal{E} c$ to n terms.

As before,

$$
\begin{aligned}
S & =x\left(\frac{e^{\theta \sqrt{-1}}+e^{-\theta \sqrt{-1}}}{2}\right)+x^{2}\left(\frac{e^{2 \theta \sqrt{-1}}+e^{-26 \sqrt{-1}}}{2}\right) \\
& +x^{3}\left(\frac{e^{9 \theta \sqrt{-1}}+e^{-5 \theta \sqrt{-1}}}{2}\right)+\& c . \text { to } n \text { terms } \\
= & \frac{1}{2}\left\{x e^{\theta \sqrt{-1}}+x^{2} e^{2 \theta \sqrt{-1}}+x^{3} e^{3^{\theta \sqrt{-1}}}+\& c .+x^{n} e^{n \theta \sqrt{-1}}\right\}
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{1}{2}\left\{x e^{-\theta \sqrt{-1}}+x^{2} e^{-2 \theta \sqrt{-1}}+x^{3} e^{-3 \theta \sqrt{-1}}+\& c_{0}+x^{n} e^{-n \theta \sqrt{-1}}\right\} \\
& \begin{aligned}
&= \frac{x e^{\theta \sqrt{-1}}}{2}\left\{\frac{\left(x e^{\theta \sqrt{-1}}\right)^{n}-1}{x e^{\theta \sqrt{-1}}-1}\right\}+\frac{x e^{-\theta \sqrt{-1}}}{2}\left\{\frac{\left(x e^{-\theta \sqrt{-1}}\right)^{n}-1}{x e^{-\theta \sqrt{-1}}-1}\right\} \\
&=\frac{1}{2}\left\{\frac{x^{n+2}\left(e^{n \theta \sqrt{-1}}+e^{-n \theta \sqrt{-1}}\right)}{x^{2}-\left(e^{\theta \sqrt{-1}}+e^{-\theta \sqrt{-1}}\right) x+1}\right\} \\
&-\frac{1}{2}\left\{\frac{x^{n+1}\left(e^{(n+1) \theta \sqrt{-1}}+e^{-(n+1) \theta \sqrt{-1}}\right)}{x^{2}-\left(e^{\theta \sqrt{-1}}+e^{-\theta \sqrt{-1}}\right) x+1}\right\} \\
&+\frac{1}{2}\left\{\frac{x\left(e^{\theta \sqrt{-1}}+e^{-\theta \sqrt{-1}}\right)-2 x^{2}}{x^{2}-\left(e^{\theta \sqrt{-1}}+e^{-\theta \sqrt{-1}}\right) x+1}\right\} \\
&=\frac{x^{n+\varepsilon} \cos n \theta-x^{n+1} \cos (n+1) \theta+x \cos \theta-x^{2}}{x^{2}-2 \cos \theta x+1}
\end{aligned} .
\end{aligned}
$$

and as in the last article, the sum of the series indefinitely continued

$$
=\frac{x \cos \theta-x^{2}}{x^{2}-2 \cos \theta x+1}
$$

324. By means of the operations of differentiation and integration as pointed out in article (316), the sums of various other trigonometrical series may easily be determined; but the almost entire absence of utility renders it unnecessary now to devote more time to the subject. We shall however conclude this Chapter with two or three instances in which some of the preceding series are made available to the solution of more important Problems.
325. From (322) it appears that the sum of the series $\sin \theta+x \sin 2 \theta+x^{2} \sin 3 \theta+\& c$ c continued in infinitum is

$$
\frac{\sin \theta}{x^{2}-2 \cos \theta x+1}=\frac{\sin \theta}{1-x(2 \cos \theta-x)}
$$

and therefore by actual division we shall have

$$
\begin{aligned}
& \sin \theta+x \sin 2 \theta+x^{2} \sin 3 \theta+\& \mathrm{c} .+x^{n-1} \sin n \theta+\& \mathrm{c} \\
& =\sin \theta\left\{1+x(2 \cos \theta-x)+x^{2}(2 \cos \theta-x)^{2}+\& \mathrm{c}\right. \\
& \left.+x^{n-2}(2 \cos \theta-x)^{n-2}+x^{n-1}(2 \cos \theta-x)^{n-1}+\& \mathrm{c} \cdot\right\}:
\end{aligned}
$$

whence by expanding the binomials and equating the coefficients of $x^{n-1}$ on both sides, we shall obtain

$$
\begin{aligned}
& \sin n \theta=\sin \theta\left\{(2 \cos \theta)^{n-1}-\frac{(n-2)}{1}(2 \cos \theta)^{n-5}\right. \\
& \left.+\frac{(n-3)(n-4)}{1}(2 \cos \theta)^{n-5}-\& c .\right\} .
\end{aligned}
$$

Similarly by means of the series summed in (323), it may be shewn that
$\cos n \theta=\frac{1}{2}\left\{(2 \cos \theta)^{n}-u(2 \cos \theta)^{n-2}+\frac{n(n-3)}{1.2}(2 \cos \theta)^{n-4}-\& c \cdot\right\}$,
which might have also been readily derived from the preceding by the operation of differentiation.
326. If we suppose $\theta=\frac{2 \pi}{5}$ or $72^{\circ}$, the second example of (314) gives
$\cos \theta+\cos 3 \theta=\frac{\cos 2 \theta \sin 2 \theta}{\sin \theta}=\frac{1}{2} \frac{\sin 4 \theta}{\sin \theta}=\frac{1}{2} \frac{\sin (2 \pi-\theta)}{\sin \theta}=-\frac{1}{2} ;$
also from (67) we have $\sin \theta \cos 3 \theta=\frac{1}{2}\{\cos 4 \theta+\cos 2 \theta\}$
$=\frac{1}{2}\left\{\cos (2 \pi-\theta)+\cos (2 \pi-3 \theta\}=\frac{1}{2}\{\cos \theta+\cos 3 \theta\}=-\frac{1}{4}:\right.$
and from these two equations are immediately deduced
$\cos \theta=\cos 72^{\circ}=\frac{\sqrt{5}-1}{4}$, and $\cos 3 \theta=\cos 216^{\circ}=\frac{-\sqrt{5}-1}{4}$,
which are the same as would have been found by the methods pointed out in the second chapter.
327. Let $\theta=\frac{\pi}{17}$ or $17 \theta=\pi$, then from the example referred to in the last article we have

$$
\begin{aligned}
& \cos \theta+\cos 3 \theta+\cos 5 \theta+\& c \cdot+\cos 15 \theta=\frac{\cos 8 \theta \sin 8 \theta}{\sin \theta} \\
&=\frac{1}{2} \frac{\sin 16 \theta}{\sin \theta}=\frac{1}{2} \frac{\sin (\pi-\theta)}{\sin \theta}=\frac{1}{2}
\end{aligned}
$$

assume now $x=\cos \theta+\cos 9 \theta+\cos 13 \theta+\cos 15 \theta$,

$$
\text { and } y=\cos 3 \theta+\cos 5 \theta+\cos 7 \theta+\cos 11 \theta
$$

then if these two quantities be multiplied together, and their product be reduced by (67), we shall obtain

$$
\begin{aligned}
x y & =2\{\cos 2 \theta+\cos 4 \theta+\cos 6 \theta+\& c .+\cos 16 \theta\} \\
& =-2\{\cos 15 \theta+\cos 13 \theta+\cos 11 \theta+\& c .+\cos \theta\}=-1,
\end{aligned}
$$

by what has just been proved :
whence the equations $x+y=\frac{1}{2}$, and $x y=-1$, give

$$
x=\frac{1+\sqrt{17}}{4} \text { and } y=\frac{1-\sqrt{17}}{4}
$$

Again, let $s=\cos \theta+\cos 13 \theta$, and $t=\cos 9 \theta+\cos 15 \theta$, also $u=\cos 3 \theta+\cos 5 \theta$, and $v=\cos 7 \theta+\cos 11 \theta$,

$$
\text { so that } s+t=\frac{1+\sqrt{17}}{4}, \text { and } u+v=\frac{1-\sqrt{17}}{4} \text {; }
$$

whence proceeding as before we shall obtain

$$
s t=-\frac{1}{4}, \text { and } u v=-\frac{1}{4} ;
$$

and thus the four quantities $s, t, u, v$, may be determined: hence since $\cos \theta+\cos 13 \theta=s$, and by (67), $\cos \theta \cos 13 \theta$

$$
=\frac{1}{2}\{\cos 12 \theta+\cos 14 \theta\}=-\frac{\mathrm{I}}{2}\{\cos 3 \theta+\cos 5 \theta\}=-\frac{u}{\mathrm{Q}},
$$

the values of $\theta$ and $13 \theta$ are readily obtained.

$$
\mathrm{F}_{\mathrm{F}}
$$

This article enables us to determine the side of a regular polygon of 17 sides inscribed in a circle whose radius is 1 , which is manifestly $=\operatorname{chd} 2 \theta=2 \sin \theta=2 \sqrt{1-\cos ^{2} \theta}$.

In what we have just been doing, no reason has been assigned for the assumptions there made : and in fact no reasons can be given without entering upon a theory much too difficult for a place in an elementary Treatise like the present. The invention of such a theory is due to M. Guuss, Professor of Mathematics at Strasburgh, and it may be seen fully developed in his work entitled Disquisitiones Arithmetica, which has been translated into French by M. Poullet-Delisle under the title of Recherches Arithmetiques. On this subject the reader is referred also to the last chapter of Barlow's Elementary Investigation of the Theory of Numbers.

## SPHERICAL TRIGONOMETRY.

СНАР. I.<br>DEFINITIONS AND PRELIMINARY PROPOSITIONS.

## Article I. Definition I.

Spherical Trigonometry treats of the relations between the sides and angles, \&c. of figures formed by the intersections of three or more planes with the surface of a sphere.
2. Every section of the surface of a sphere made by a plane cutting $i t$, is the arc of a circle.


Let $O$ be the centre of the sphere, $A B C$ the section made by a plane passing through it ; draw $O D$ perpendicular to this plane and produce it hoth ways to meet the surface in $E$ and $F$, join $A D, B D, C D$, and draw the radii of the sphere $O A$, $O B, O C$ : then by Euclid xı. Def. 3, ODA, ODB, ODC are right angles;

$$
\therefore O A^{2}-O D^{2}=O B^{2}-O D^{2}=O C^{2}-O D^{2}=\& \mathrm{c} .
$$

that is $D A^{2}=D B^{2}=D C^{2}=\& c$. or $D A=D B=D C=\& c$. and therefore the section $A B C$ is a circle whose centre is $D$, and radius $=D A=D B=D C=\& c$.
3. Cor. If the distance of the cutting plane from the centre of the sphere be called $d$, and the radius of the sphere $r$, we shall have the radius $D A$ of the section $=\sqrt{O A^{2}-O D^{2}}$ $=\sqrt{r^{2}-d^{3}}$ : and if $d=0$, or the cutting plane pass through the centre of the sphere, the radius of the section is equal to the radius of the sphere, and its centre coincides with the centre of the sphere.
4. Def. 2. The pole of a circle of the sphere is a point in the surface of the sphere from which all straight lines drawn to the circumference of the circle are equal.
5. Cor. Hence if the line $O D$ be produced both ways to meet the surface of the sphere in $E$ and $F$, these points a e called the poles of the circle $A B C$, the former the near, the latter the remote pole.
6. Def.3. When the cutting plane passes through the centre of the sphere, the radius of the section being equal to the radius of the sphere is the greatest possible, and the circle is called a Great Circle of the sphere: in all other cases the section is termed a Small Circle.
7. Cor.1. If the section pass through the centre of the sphere, the points $O$ and $D$ coincide, and the poles of a great circle are the points of intersection with the surface of the sphere made by a perpendicular to the circle passing through its centre ; and it is manifest that the arc of the sphere intercepted between the circumference of a great circle and either of its poles is a quadrant.
8. Cor. 2. Hence two great circles of the sphere bisect one another, because they have a common centre, and their
common section being a diameter of each therefore bisects them.
9. Def.4. The arcs on the surface of a sphere are always understood to be portions of great circles unless the contrary be expressed: a figure formed by three such arcs is a spherical triangle: by four a spherical quadrilateral, \&c : and by $n$ such arcs a spherical polygon of $u$ sides.
10. Def. 5. The angles of spherical triangles, \&c. are those on the surface of the sphere contained by the arcs of the great circles which form the sides, and are the same with the inclinations of the planes of those great circles to one another.
11. Any two sides of a spherical triangle are together greater than the third, and the three sides are together less than the circumference of a great circle.

Let $A B C$ be a spherical triangle on the surface of a sphere whose centre is $O$; draw the radii of the sphere $O A, O B$,

$O C$ to the angular points: then since the solid angle at $O$ is contained by the three plane angles $A O B, A O C, B O C$, any two of which are by Euclid xı. 20. together greater than the third, it follows that any two of the arcs which measure these angles are together greater than the third : that is, $A B+A C$ is greater than $B C, A B+B C$ greater than $A C$, and $A C+B C$ greater than $A B$.

Also, since the solid angle at $O$ is contained by the three plane angles $A O B, A O C, B O C$, which by Euclid xı. 21 . are together
less than four right angles, it is manifest that the three arcs $A B$, $A C$ and $B C$ are together less than the circumference of a great circle.

Hence if the sides be denoted by $a, b, c$, and the radius of the sphere be 1 , then $a+b>c, a+c>b, b+c>a$, and $a+b+c<2 \pi$.
12. Cor. For the same reason, since (Euclid xı. 21.) every solid angle is contained by plane angles which are together less than four right angles, it follows that all the sides of a spherical polygon are together less than the circumference of a great circle.
13. Def.6. If with the angular points of a spherical triangle as poles, great circles of the sphere be described, the figure formed by the intersections of these circles is called the Polar Triangle, in contradistinction to which, the proposed one is styled the Primitive Triangle.
14. I'he angular points of the polar triangle are the poles of the sides of the primitive triangle.

Let $\triangle B C$ be the primitive triangle, $D F E$ the polar triangle

described according to the definition, and let the great circles be produced as in the figure: then
since $A$ is the pole of $D E, A D$ is a quadrant, and since $B$ is the pole of $D F, B D$ is a quadrant:
$\therefore$ the distances of the points $A$ and $B$ from $D$ being quadrants are equal to one another, and consequently $D$ is the pole of $A B$ : for the same reason the angular points $E$ and $F$ of the polar triangle are the poles of the sides $A C$ and $B C$ of the primitive triangle.
15. The sides and angles of the polar triangle are the supplements of the angles and sides respectively of the primitive triangle.

The same construction remaining, and the radius of the sphere being supposed $=1$, so that an angle may be equal to the arc which measures it, we have

$$
\begin{gathered}
\angle A=H M=D H-D M=D H+M E-D E=\pi-D E: \\
\text { similarly } \angle B=\pi-D F \text {, and } \angle C=\pi-E F \text { : } \\
\text { again } \angle D=G H=G B+B H=G B+A H-A B=\pi-A B: \\
\text { similarly } \angle E=\pi-A C \text {, and } \angle F=\pi-B C .
\end{gathered}
$$

Hence if $a, b, c, A, B, C$ be the sides and angles respectively of the primitive triangle, and $a^{\prime}, b^{\prime}, c^{\prime}, A^{\prime}, B^{\prime}, C^{\prime}$ those of the polar triangle, we shall have

$$
\begin{aligned}
\quad a^{\prime}=\pi-A, \quad b^{\prime}=\pi-B, \quad c^{\prime}=\pi-C \\
\text { and } \quad A^{\prime}=\pi-a, \quad B^{\prime}=\pi-b, \quad C^{\prime}=\pi-c ;
\end{aligned}
$$

and from these properties the polar triangle is frequently styled the supplemental triangle.
16. Cor. 1. If one or more of the sides or angles of the primitive triangle be quadrants or right angles, the corresponding angles or sides of the polar triangle will be right angles or quadrants.
17. Cor.2. Hence the sum of the three angles of a spherical triangle lies between two and six right angles.

For, since by (11) $a^{\prime}+b^{\prime}+c^{\prime}$ is less than $2 \pi$, it follows that $A+B+C=3 \pi-\left(a^{\prime}+b^{\prime}+c^{\prime}\right)$ is greater than $\pi$ or two
right angles: and we manifestly have $A+B+C=3 \pi-\left(a^{\prime}+b^{\prime}+c^{\prime}\right)$ less than $3 \pi$ or six right angles.

A spherical triangle may therefore have two or three right angles, or two or three obtuse angles.
18. Cor.3. Hence also the sum of any two angles of a spherical triangle exceeds the third by less than two right angles.

For, since by (11) $a^{\prime}+b^{\prime}$ is greater than $c^{\prime}$, we have $\pi-A+\pi-B$ greater than $\pi-C$, or $\pi$ greater than $A+B-C$, $\therefore A+B-C$ is less than $\pi$ : similarly $A+C-B$, and $B+C-A$ are each less than $\pi$.
19. Cor. 4. In the same manner if the sides of a spherical polygon be each less than a semicircle, and with its angular points as poles great circles be described, another spherical polygon will be formed which will be supplemental to the former.
20. Def. 7. If one of the angles of a spherical triangle be a right angle, it is called a right-angled triangle; if one of the sides be a quadrant, it is called a quadrantal triangle, and all others are called oblique-angled triangles.

## СНАР. II.

On the relations between the sides and angles, \&c. of spherical triangles.
21. To express the cosines of the angles of a spherical triangle in terms of the sides.

Let $A B C$ be a triangle on the surface of a sphere whose centre is $O$ and radius $=1$, the angles being $A, B, C$ and the

corresponding opposite sides $a, b, c$ : let $A D, A E$ touching the $\operatorname{arcs} A B, A C$ at the point $A$, meet the radii $O B, O C$ produced in $D, E$, and join $D E$ : then we have

$$
\begin{aligned}
& A E=\tan A C=\tan b, A D=\tan A B=\tan c \\
& O E=\sec A C=\sec b, O D=\sec A B=\sec c:
\end{aligned}
$$

now in the triangle $D O E$, we have from (165) Pl. Trig.

$$
\begin{aligned}
D E^{2} & =O E^{2}+O D^{2}-2 O E . O D \cos D O E \\
& =\sec ^{2} b+\sec ^{2} c-2 \sec b \sec c \cos a \\
& =2+\tan ^{2} b+\tan ^{2} c-2 \sec b \sec c \cos a
\end{aligned}
$$

G $\mathbf{G}$
also, in the triangle $E A D$, we have by the same article

$$
\begin{aligned}
D E^{2} & =A E^{2}+A D^{2}-2 A E . A D \cos D A E \\
& =\tan ^{2} b+\tan ^{2} c-2 \tan b \tan c \cos A:
\end{aligned}
$$

whence equating and transposing we get
$2 \tan b \tan c \cos A=2 \sec b \sec c \cos a-2$,
and $\therefore \cos A=\frac{2 \sec b \sec c \cos a-2}{2 \tan b \tan c}=\frac{\cos a-\cos b \cos c}{\sin b \sin c}$ :
similarly $\cos B=\frac{\cos b-\cos a \cos c}{\sin a \sin c}$, and $\cos C=\frac{\cos c-\cos a \cos b}{\sin a \sin b}$.
Ex. 1. If $a=b$, we have

$$
\begin{gathered}
\cos A=\frac{\cos a-\cos a \cos c}{\sin a \sin c}=\frac{\cos a(1-\cos c)}{\sin a \sin c} \\
=\frac{\cos a}{\sin a} \frac{2 \sin ^{2} \frac{c}{2}}{\sin c}=\cot a \tan \frac{c}{2}=\cot b \tan \frac{c}{2}=\cos B \\
\text { and } \cos C=\frac{\cos c-\cos ^{2} a}{\sin ^{2} a}=\cos c \operatorname{cosec}^{2} a-\cot ^{2} a
\end{gathered}
$$

Hence we have $A=B$, or the angles at the base of an isosceles spherical triangle are equal to one another.

Ex. 2. Let $a=b=c$, then we shall have

$$
\cos A=\frac{\cos a-\cos ^{2} a}{\sin ^{2} a}=\frac{\cos a(1-\cos a)}{\sin a \sqrt{1-\cos ^{2} a}}
$$

$=\cot a \sqrt{\frac{1-\cos a}{1+\cos a}}=\cot a \tan \frac{a}{2}=\frac{\tan \frac{a}{2}}{\tan a}=\cos B=\cos C$.
Hence every equilateral spherical triangle is also equiangular.
22. Cor. If the angle at $C$ be a right angle, we have $0=\cos c-\cos a \cos b$, and therefore according as $\cos a$ and $\cos b$ have the same or different signs, $\cos c$ will be positive or negative; that is, according as the sides are of the same or different affections, the hypothenuse is less or greater than a quadrant.
23. To express the sines of the angles of a spherical triangle in terms of the sides.

$$
\begin{aligned}
& \text { Since } \sin A=\sqrt{1-\cos ^{2} A}=\sqrt{(1-\cos A)(1+\cos A)}, \\
& \text { and } \cos A=\frac{\cos a-\cos b \cos c}{\sin b \sin c}, \\
& \text { we have } 1-\cos A=1-\frac{\cos a-\cos b \cos c}{\sin b \sin c} \\
& =\frac{\sin b \sin c-\cos a+\cos b \cos c}{\sin b \sin c}=\frac{\cos (b-c)-\cos a}{\sin b \sin c} \\
& =\frac{2 \sin \left(\frac{a+b-c}{2}\right) \sin \left(\frac{a+c-b}{2}\right)}{\sin b \sin c}, \text { by }(67), P l . \text { Trig. } \\
& =\frac{\sin b \sin c+\cos a-\cos b \cos c}{\sin b \sin c}=\frac{\cos a-\cos (b+c)}{\sin b \sin c} \\
& =\frac{2 \sin \left(\frac{a+b+c}{2}\right) \sin \left(\frac{b+c-a}{2}\right)}{\sin b \sin c}, \mathrm{by}(67) ; \\
& =\frac{\cos A=1+\frac{\cos a-\cos b \cos c}{\sin b \sin c}}{\operatorname{let} a+b+c=2 S, b+c-a=2(S-a)} \\
& a+c-b=2(S-b), \text { and } a+b-c=2(S-c) ; \\
& \therefore 1-\cos A=\frac{2 \sin (S-b) \sin (S-c)}{\sin b \sin c}, \\
& \operatorname{sind} 1+\cos A=\frac{2 \sin S \sin (S-a)}{\sin b \sin c} ;
\end{aligned}
$$

whence $\sin A=\frac{2}{\sin b \sin c} \sqrt{\sin S \sin (S-a) \sin (S-b) \sin (S-c)}$ : similarly $\sin B=\frac{2}{\sin a \sin c} \sqrt{\sin S \sin (S-a) \sin (S-b) \sin (S-c)}$,

$$
\text { and } \sin C=\frac{2}{\sin a \sin b} \sqrt{\sin S \sin (S-a) \sin (S-b) \sin (S-c)}
$$

Ex. 1. Let $a=b$, or the triangle be isosceles, then

$$
\begin{aligned}
& \sin A=\frac{2 \sin (S-a)}{\sin a \sin c} \sqrt{\sin S \sin (S-c)} \\
& =\frac{2 \sin \frac{c}{2}}{\sin a \sin c} \sqrt{\sin \left(a+\frac{c}{2}\right) \sin \left(a-\frac{c}{2}\right)}
\end{aligned}
$$

$$
=\frac{\sqrt{\sin \left(a+\frac{c}{2}\right) \sin \left(a-\frac{c}{2}\right)}}{\sin a \cos \frac{c}{2}}
$$

$$
=\frac{\sqrt{\sin \left(b+\frac{c}{2}\right) \sin \left(b-\frac{c}{2}\right)}}{\sin b \cos \frac{c}{2}}=\sin B:
$$

and $\sin C=\frac{2 \sin (S-a)}{\sin ^{2} a} \sqrt{\sin S \sin (S-c)}$

$$
=\frac{2 \sin \frac{c}{2}}{\sin ^{2} a} \sqrt{\sin \left(a+\frac{c}{2}\right) \sin \left(a-\frac{c}{2}\right)} .
$$

Ex. 2. Let $a=b=c$, then in an equilateral triangle we have

$$
\begin{aligned}
& \sin A= \\
&=\frac{2 \sqrt{\sin S \sin ^{3}(S-a)}}{\sin ^{2} a}=\frac{2 \sqrt{\sin \frac{3 a}{2} \sin ^{3} \frac{a}{2}}}{4 \sin ^{2} \frac{a}{2} \cos ^{2} \frac{a}{2}} \\
& \frac{1}{2 \cos ^{2} \frac{a}{2}} \sqrt{\frac{\sin \frac{3 a}{2}}{\sin \frac{a}{2}}}=\frac{1}{2 \cos ^{2} \frac{a}{2}} \sqrt{3-4 \sin ^{2} \frac{a}{2}}=\sin B=\sin C .
\end{aligned}
$$

24. Cor. 1. Hence rejecting the common factors, we have $\sin A: \sin B: \sin C=\sin a: \sin b: \sin c ;$
or the sines of the sides of a spherical triangle are to one another as the sines of the opposite angles.
25. Cor. Q. Since $1-\cos A=2 \sin ^{2} \frac{A}{2}$, and $1+\cos A$ $=2 \cos ^{2} \frac{A}{2}$, we have from (23) by reduction,

$$
\begin{aligned}
\sin \frac{A}{2} & =\sqrt{\frac{\sin (S-b) \sin (S-c)}{\sin b \sin c}} \\
\text { and } \cos \frac{A}{2} & =\sqrt{\frac{\sin S \sin (S-a)}{\sin b \sin c}}
\end{aligned}
$$

and thence $\tan \frac{A}{2}=\sqrt{\frac{\sin (S-b) \sin (S-c)}{\sin S \sin (S-a)}}$.
26. Cor. 3. Hence also $\frac{\sin \frac{A}{2}}{\sin \frac{B}{2}}=\sqrt{\frac{\sin a \sin (S-b)}{\sin b \sin (S-a)}}$, and therefore according as $a$ is greater or less than $b, \sin \frac{A}{2}$
will be greater or less than $\sin \frac{B}{2}$, and thence $A$ greater or less than $B$ : that is, the greater side of every spherical triangle is opposite the greater angle, and the contrary.
27. To express the cosines of the sides of a spherical triangle in terms of the angles.

Let $a, b, c, A, B, C$ be the sides and angles of the proposed triangle, $a^{\prime}, b^{\prime}, c^{\prime}, A^{\prime}, B,{ }^{\prime} C^{\prime}$ those of the polar triangle, then by (21), we have

$$
\cos A^{\prime}=\frac{\cos a^{\prime}-\cos b^{\prime} \cos c^{\prime}}{\sin b^{\prime} \sin c^{\prime}}
$$

but $\cos A^{\prime}=\cos (\pi-a)=-\cos a, \cos a^{\prime}=\cos (\pi-A)=-\cos A$,

$$
\cos b^{\prime}=\cos (\pi-B)=-\cos B, \cos c^{\prime}=\cos (\pi-C)=-\cos C
$$

$$
\sin b^{\prime}=\sin (\pi-B)=\sin B, \sin c^{\prime}=\sin (\pi-C)=\sin C ;
$$

$\therefore$ the formula just given becomes $\cos a=\frac{\cos A+\cos B \cos C}{\sin B \sin C}$ :

$$
\begin{aligned}
\text { similarly, } \cos b & =\frac{\cos B+\cos A \cos C}{\sin A \sin C} \\
\text { and } \cos c & =\frac{\cos C+\cos A \cos B}{\sin A \sin B}
\end{aligned}
$$

$$
\begin{aligned}
& \text { Ex. 1. Let } A=B, \therefore \cos a=\frac{\cos A}{\sin A} \frac{1+\cos C}{\sin C} \\
& =\cot A \cot \frac{C}{2}=\cot B \cot \frac{C}{2}=\cos b \\
& \text { and } \cos c=
\end{aligned}
$$

Hence if two angles of a spherical triangle be equal to one another, the sides which subtend them are also equal.

Ex. 2. Let $A=B=C, \therefore \cos a=\cot A \cot \frac{A}{2}=\cos b=\cos c$ : wherefore equiangular spherical triangles are also equilateral.
28. Cor. If $C=90^{\circ}$, we have $\cos a=\frac{\cos A}{\sin B} ; \therefore \cos a$ will be positive or negative according as $\cos A$ is positive or negative: that is, $a$ will be less or greater than a quadrant according as $A$ is less or greater than a right angle: or the sides of right-angled triangles are of the same affections as their opposite angles.
29. To express the sines of the sides of a spherical triangle in terms of the angles.

From the last article but one we have

$$
\begin{gathered}
1-\cos a=1-\frac{\cos A+\cos B \cos C}{\sin B \sin C} \\
=\frac{\sin B \sin C-\cos A-\cos B \cos C}{\sin B \sin C}=-\frac{\cos A+\cos (B+C)}{\sin B \sin C} \\
=-\frac{2 \cos \left(\frac{A+B+C}{2}\right) \cos \left(\frac{B+C-A}{2}\right)}{\sin B \sin C} ; \\
=\frac{\sin B \sin C+\cos A+\cos B \cos C}{\sin B \sin C}=\frac{\cos A+\cos (B-C)}{\sin B \sin C} \\
\text { and } 1+\cos a=1+\frac{\cos A+\cos B \cos C}{\sin B \sin C} \\
=\frac{2 \cos \left(\frac{A+B-C}{2}\right) \cos \left(\frac{A+C-B}{2}\right)}{\sin B \sin C}:
\end{gathered}
$$

$\therefore$ assume $2 S^{\prime}=A+B+C$, whence we shall have

$$
1-\cos a=-\frac{2 \cos S^{\prime} \cos \left(S^{\prime}-A\right)}{\sin B \sin C}
$$

$$
\text { and } 1+\cos a=\frac{2 \cos \left(S^{\prime}-B\right) \cos \left(S^{\prime}-C\right)}{\sin B \sin C}
$$

whence $\sin a$
$=\frac{2}{\sin B \sin C} \sqrt{-\cos S^{\prime} \cos \left(S^{\prime}-A\right) \cos \left(S^{\prime}-B\right) \cos \left(S^{\prime}-C\right)} ;$
similarly $\sin b$

$$
=\frac{2}{\sin A \sin C} \sqrt{-\cos S^{\prime} \cos \left(S^{\prime}-A\right) \cos \left(S^{\prime}-B\right) \cos \left(S^{\prime}-C\right)}
$$

and $\sin c$

$$
=\frac{2}{\sin A \sin B} \sqrt{-\cos S^{\prime} \cos \left(S^{\prime}-A\right) \cos \left(S^{\prime}-B\right) \cos \left(S^{\prime}-C\right)}
$$

It may here be remarked that since by (17) the sum of the three angles of a spherical triangle is greater than two right angles and less than six, $S^{\prime}$ is manifestly greater than one right angle and less than three, and consequently $\cos S^{\prime}$ is a negative quantity; also since by (18) the excess of the sum of any two angles of a spherical triangle above the remaining one is less than two right angles, it follows that $S^{\prime}-A, S^{\prime}-B$ and $S^{\prime}-C$ are all less than one right angle, and therefore that $\cos \left(S^{\prime}-A\right)$, $\cos \left(S^{\prime}-B\right)$ and $\cos \left(S^{\prime}-C\right)$ are all positive, from which it results that all the expressions just investigated are possible, though they appear in an imaginary form.
30. Cor. 1. From the demonstration of the last article we have
$2 \sin ^{2} \frac{a}{2}=1-\cos a=-\frac{2 \cos \left(\frac{A+B+C}{2}\right) \cos \left(\frac{B+C-A}{2}\right)}{\sin B \sin C}$,
and $2 \cos ^{2} \frac{a}{2}=1+\cos a=\frac{2 \cos \left(\frac{A+B-C}{2}\right) \cos \left(\frac{A+C-B}{2}\right)}{\sin B \sin C}$;
whence $\sin \frac{a}{2}=\sqrt{\frac{-\cos \left(\frac{A+B+C}{2}\right) \cos \left(\frac{B+C-A}{2}\right)}{\sin B \sin C}}$

$$
\begin{gathered}
=\sqrt{\frac{-\cos S^{\prime}\left(\cos S^{\prime}-A\right)}{\sin B \sin C},} \\
\text { and } \cos \frac{a}{2}=\sqrt{\frac{\cos \left(\frac{A+B-C}{2}\right) \cos \left(\frac{A+C-B}{2}\right)}{\sin B \sin C}} \\
=\sqrt{\frac{\cos \left(S^{\prime}-B\right) \cos \left(S^{\prime}-C\right)}{\sin B \sin C}}, \\
\text { and } \therefore \tan \frac{a}{2}=\sqrt{\frac{-\cos S^{\prime} \cos \left(S^{\prime}-A\right)}{\cos \left(S^{\prime}-B\right) \cos \left(S^{\prime}-C\right)}} .
\end{gathered}
$$

31. To express the tangents of the semi-sum and semidifference of tzoo angles of a spherical triangle in terms of their opposite sides and the remaining angle.
Since $\cos A=\frac{\cos a-\cos b \cos c}{\sin b \sin c}$, and $\cos C=\frac{\cos c-\cos a \cos b}{\sin a \sin b}$,

$$
\begin{aligned}
& \therefore \cos A \sin c=\frac{\cos a-\cos b \cos c}{\sin b} \\
& =\frac{\cos a}{\sin b}-\frac{\cos b}{\sin b}(\sin a \sin b \cos C+\cos a \cos b) \\
& =\frac{\cos a}{\sin b}--\sin a \cos b \cos C-\cos a \frac{\cos ^{2} b}{\sin b} \\
& =\frac{\cos a}{\sin b}-\sin a \cos b \cos C-\frac{\cos a}{\sin b}+\cos a \sin b \\
& =\cos a \sin b-\sin a \cos b \cos C
\end{aligned}
$$

similarly $\cos B \sin c=\cos b \sin a-\sin b \cos a \cos C$;

$$
\therefore(\cos A+\cos B) \sin c=\sin (a+b)(1-\cos C):
$$

but since from (24) $\frac{\sin A}{\sin a}=\frac{\sin B}{\sin b}=\frac{\sin C}{\sin c}$, we shall have

$$
(\sin A \pm \sin B) \sin c=(\sin a \pm \sin b) \sin C:
$$

hence using the upper sign we obtain

$$
\begin{gathered}
\frac{\sin A+\sin B}{\cos A+\cos B}=\frac{\sin a+\sin b}{\sin (a+b)} \frac{\sin C}{1-\cos C} \\
\text { or } \tan \frac{A+B}{2}=\frac{\cos \left(\frac{a-b}{2}\right)}{\cos \left(\frac{a+b}{2}\right)} \cot \frac{C}{2}
\end{gathered}
$$

and by means of the lower we get

$$
\begin{aligned}
& \frac{\sin A-\sin B}{\cos A+\cos B}=\frac{\sin a-\sin b}{\sin (a+b)} \frac{\sin C}{1-\cos C} \\
& \text { or } \tan \left(\frac{A-B}{2}\right)=\frac{\sin \left(\frac{a-b}{2}\right)}{\sin \left(\frac{a+b}{2}\right)} \cot \frac{C}{2}
\end{aligned}
$$

These equations converted into proportions constitute what are from their inventor called Napier's first and second Analogies.

Ex. If $C=90^{\circ}$, we shall have for a right-angled triangle, $\tan \left(\frac{A+B}{2}\right)=\frac{\cos \left(\frac{a-b}{2}\right)}{\cos \left(\frac{a+b}{2}\right)}$, and $\tan \left(\frac{A-B}{2}\right)=\frac{\sin \left(\frac{a-b}{2}\right)}{\sin \left(\frac{a+b}{2}\right)}$.
32. Cor. From the two equations just investigated we have

$$
\cot \frac{C}{2}=\frac{\cos \left(\frac{a+b}{2}\right)}{\cos \left(\frac{a-b}{2}\right)} \tan \left(\frac{A+B}{2}\right)=\frac{\sin \left(\frac{a+b}{2}\right)}{\sin \left(\frac{a-b}{2}\right)} \tan \left(\frac{A-B}{2}\right) .
$$

33. To express the tangents of the semi-sum and semidifference of two sides of a spherical triangle in terms of their opposite angles and the remaining side.

Retaining the notation before used, we shall by the last article but one have in the supplemental triangle

$$
\begin{aligned}
& \tan \left(\frac{A^{\prime}+B^{\prime}}{2}\right)=\frac{\cos \left(\frac{a^{\prime}-b^{\prime}}{2}\right)}{\cos \left(\frac{a^{\prime}+b^{\prime}}{2}\right)} \cot \frac{C^{\prime}}{2} \\
& \text { and } \tan \left(\frac{A^{\prime}-B^{\prime}}{2}\right)=\frac{\sin \left(\frac{a^{\prime}-b^{\prime}}{2}\right)}{\sin \left(\frac{a^{\prime}+b^{\prime}}{2}\right)} \cot \frac{C^{\prime}}{2}
\end{aligned}
$$

and by effecting the proper substitutions as in (27) we shall obtain

$$
\begin{aligned}
\tan \left(\frac{a+b}{2}\right) & =\frac{\cos \left(\frac{A-B}{2}\right)}{\cos \left(\frac{A+B}{2}\right)} \tan \frac{c}{2} \\
\text { and } \tan \left(\frac{a-b}{2}\right) & =\frac{\sin \left(\frac{A-B}{2}\right)}{\sin \left(\frac{A+B}{2}\right)} \tan \frac{c}{2}
\end{aligned}
$$

which converted into proportions as before are Napier's third and fourth Analogies.
34. Cor. Hence in the same manner as in (32), we have

$$
\tan \frac{c}{2}=\frac{\cos \left(\frac{A+B}{2}\right)}{\cos \left(\frac{A-B}{2}\right)} \tan \left(\frac{a+b}{2}\right)=\frac{\sin \left(\frac{A+B}{2}\right)}{\sin \left(\frac{A-B}{2}\right)} \tan \left(\frac{a-b}{2}\right) .
$$

Ex. Let $c=\frac{\pi}{2}$, then in a quadrantal triangle we have $\tan \left(\frac{a+b}{2}\right)=\frac{\cos \left(\frac{A-B}{2}\right)}{\cos \left(\frac{A+B}{2}\right)}$, and $\tan \left(\frac{a-b}{2}\right)=\frac{\sin \left(\frac{A-B}{2}\right)}{\sin \left(\frac{A+B}{2}\right)}$.
35. To express the co-tangent of an angle of a spherical triangle in terms of another angle, and the sides which include it. Since $\cos A=\frac{\cos a-\cos b \cos c}{\sin b \sin c}$, and $\cos C=\frac{\cos c-\cos a \cos b}{\sin a \sin b}$;

$$
\therefore \sin b \sin c \cos A=\cos a-\cos b \cos c
$$

but $\sin c=\frac{\sin C}{\sin A} \sin a$, and $\cos c=\sin a \sin b \cos C+\cos a \cos b$;
$\therefore \cot A \sin a \sin b \sin C=\cos a-\sin a \sin b \cos b \cos C$
$-\cos a \cos ^{2} b=\cos a \sin ^{2} b-\sin a \sin b \cos b \cos C$,

$$
\begin{aligned}
\therefore \cot A & =\frac{\cos a}{\sin a} \frac{\sin b}{\sin C}-\cos b \frac{\cos C}{\sin C} \\
& =\cot a \sin b \operatorname{cosec} C-\cos b \cot C
\end{aligned}
$$

36. To express the co-tangent of the side of a spherical triangle in terms of another side, and the angles which are adjacent to it.
Since $\cos a=\frac{\cos A+\cos B \cos C}{\sin B \sin C}$, and $\cos c=\frac{\cos C+\cos A \cos B}{\sin A \sin B}$;
we have $\sin B \sin C \cos a=\cos A+\cos B \cos C$;
but $\sin C=\frac{\sin c}{\sin a} \sin A$, and $\cos C=\sin A \sin B \cos c-\cos A \cos B$;
$\therefore \cot a \sin A \sin B \sin c=\cos A+\sin A \sin B \cos B \cos c$
$-\cos A \cos ^{2} B=\cos A \sin ^{2} B+\sin A \sin B \cos B \cos c$, whence $\cot a=\frac{\cos A}{\sin A} \frac{\sin B}{\sin c}+\cos B \frac{\cos c}{\sin c}$
$=\cot A \sin B \operatorname{cosec} c+\cos B \cot c$.

## CHAP. III.

## On the Solution of Spherical Triangles.

37. From the preceding chapter it appears that the following relations between the sides and angles of spherical triangles have been established; namely, from (21)

$$
\begin{gathered}
\cos A=\frac{\cos a-\cos b \cos c}{\sin b \sin c}, \cos B=\frac{\cos b-\cos a \cos c}{\sin a \sin c}, \\
\cos C=\frac{\cos c-\cos a \cos b}{\sin a \sin b} ; \text { and from (27), } \\
\cos a=\frac{\cos A+\cos B \cos C}{\sin B \sin C}, \cos b=\frac{\cos B+\cos A \cos C}{\sin A \sin C}, \\
\cos c=\frac{\cos C+\cos A \cos B}{\sin A \sin B}:
\end{gathered}
$$

and since each of these sets contains three independent equations, of the six quantities involved in them any three being given, the remaining three may be found; but if one of the parts of the triangle be a right angle or a quadraital arc, it is manifest that only two other parts will be necessary for the discovery of all the rest.

On this account therefore the solutions of spherical triangles are distributed under the three following heads:
I. Solution of right-angled triangles.
II. Solution of quadrantal triangles.
III. Solution of oblique-angled triangles.
and a proper application of the propositions contained in the last chapter, will enable us to effect the solutions of all their particular cases.

## I. Solution of Right-Angled Triangles.

38. From what has been said, it appears that all the cases of right-angled spherical triangles may be solved by means of either of the sets of formulæ above given; but it may also be observed that the substitutions and eliminations necessary to effect these solutions would in many cases be too tedious for practice, and their results burdensome to the memory. To remedy this inconvenience, Baron Napier the celebrated inventor of logarithms devised two rules easy to be remembered, which are sufficient for the solutions of all cases of right-angled spherical triangles, and of which the following explanation may be given.

If $C$ be supposed to be the right angle, there remain five other parts belonging to every triangle, namely, the two sides or legs $a, b$, the hypothenuse $c$, and the two angles $A, B$ : now the two legs $a, b$, the complement of the hypothenuse $\frac{\pi}{2}-c$, and the complements of the two angles $\frac{\pi}{2}-A, \frac{\pi}{2}-B$ are by Napier termed Circular Parts, the right angle being left entirely out of the consideration, and any one of these parts may be assumed to be what he calls a Middle Part: then the two parts which lie close on each side of it are called Adjacent Extremes, and the two remaining parts which are farthest off from it and separated from it by an adjacent part are termed Opposite Extremes. This being premised, the two following equations are found universally to obtain, and are called Napier's Rules:
(1) Radius $\times$ the sine of the middle part $=$ the rectangle of the tangents of the adjacent extremes.
(2) Radius $\times$ the sine of the middle part $=$ the rectangle of the cosines of the opposite extremes:
and from these equations if any two of the quantities involved be given, the remaining parts may be immediately derived.
39. To prove Napier's Rules.

First, let one of the legs $a$ be the middle part, then $\frac{\pi}{2}-B$ and $b$ are the adjacent, and $\frac{\pi}{2}-A$ and $\frac{\pi}{2}-c$ the opposite extremes :
now by (27) $\cos b=\frac{\cos B+\cos A \cos C}{\sin A \sin C}=\frac{\cos B}{\sin A}$, since $C=90^{\circ}$,
$\therefore \cos B=\sin A \cos b=\frac{\sin B \sin a}{\sin b} \cos b$, by (24);

$$
\begin{align*}
& \text { whence } \sin a=\frac{\sin b}{\cos b} \frac{\cos B}{\sin B}=\tan b \cot B, \\
& \text { or } r \sin a=\tan b \tan \left(\frac{\pi}{2}-B\right) \ldots \ldots \ldots \ldots \tag{1}
\end{align*}
$$

similarly $r \sin b=\tan a \cot A=\tan a \tan \left(\frac{\pi}{2}-A\right)$
again by (24) $\frac{\sin a}{\sin c}=\frac{\sin A}{\sin C}=\sin A, \therefore \sin a=\sin c \sin A$,

$$
\begin{equation*}
\text { or } r \sin a=\cos \left(\frac{\pi}{2}-c\right) \cos \left(\frac{\pi}{2}-A\right) . \tag{3}
\end{equation*}
$$

similarly $r \sin b=\sin c \sin B=\cos \left(\frac{\pi}{2}-c\right) \cos \left(\frac{\pi}{2}-B\right)$
Secondly, let the complement of the hypothenuse $\frac{\pi}{2}-c$ be the middle part, then the adjacent extremes are $\frac{\pi}{2}-A$ and $\frac{\pi}{2}-B$, and the opposite extremes $a$ and $b$ :
now by (27) $\cos c=\frac{\cos C+\cos A \cos B}{\sin A \sin B}$

$$
=\frac{\cos A \cos B}{\sin A \sin B}=\cot A \cot B
$$

$$
\therefore r \sin \left(\frac{\pi}{2}-c\right)=\tan \left(\frac{\pi}{2}-A\right) \tan \left(\frac{\pi}{2}-B\right) \ldots \ldots(5)
$$

and by (21) $\cos C=\frac{\cos c-\cos a \cos b}{\sin b \sin c}=0, \therefore \cos c=\cos a \cos b$,

$$
\begin{equation*}
\text { or } r \sin \left(\frac{\pi}{2}-c\right)=\cos a \cos b \tag{6}
\end{equation*}
$$

Lastly, let the complement of one of the angles $\frac{\pi}{2}-A$ be the middle part, then the adjacent extremes are $b$ and $\frac{\pi}{2}-c$, and the opposite extremes $a$ and $\frac{\pi}{2}-B$ :
now by (21) $\cos A=\frac{\cos a-\cos b \cos c}{\sin b \sin c}=\frac{\frac{\cos c}{\cos b}-\cos b \cos c}{\sin b \sin c}$ $=\frac{\cos c\left(1-\cos ^{2} b\right)}{\sin b \cos b \sin c}=\frac{\cos c \sin ^{2} b}{\sin b \cos b \sin c}=\frac{\sin b}{\cos b} \frac{\cos c}{\sin c}=\tan b \cot c ;$

$$
\begin{equation*}
\text { or } r \sin \left(\frac{\pi}{2}-A\right)=\tan b \tan \left(\frac{\pi}{2}-c\right) \tag{7}
\end{equation*}
$$

similarly $r \cos B=\tan a \cot c$,

$$
\begin{equation*}
\text { or } r \sin \left(\frac{\pi}{2}-B\right)=\tan a \tan \left(\frac{\pi}{2}-c\right) \text {. } \tag{8}
\end{equation*}
$$

again by (27) $\cos a=\frac{\cos A+\cos B \cos C}{\sin B \sin C}=\frac{\cos A}{\sin B}$, since $C=90^{\circ}$,

$$
\therefore \cos A=\cos a \sin B
$$

$$
\begin{equation*}
\text { or } r \sin \left(\frac{\pi}{2}-A\right)=\cos a \cos \left(\frac{\pi}{2}-B\right) \text {. } \tag{9}
\end{equation*}
$$

$$
\begin{gathered}
\text { similarly } \cos B=\cos b \sin A \\
\text { or } r \sin \left(\frac{\pi}{2}-B\right)=\cos b \cos \left(\frac{\pi}{2}-A\right) \ldots \ldots \ldots(10)
\end{gathered}
$$

and since out of five things taken two and two together, there can be formed $\frac{5.4}{1.2}$ or 10 combinations, it follows that the ten equations above deduced include all the cases that can possibly occur: moreover they are all adapted to logarithmic computation.

We will now illustrate the use of these rules by the following examples in which the radius is supposed to be 1 .

Ex. 1. Given $a$ and $b$, to find the rest.
From (1), $\sin a=\tan b \cot B, \therefore \cot B=\frac{\sin a}{\tan b}=\sin a \cot b$ :
(2), $\sin b=\tan a \cot A, \therefore \cot A=\frac{\sin b}{\tan a}=\sin b \cot a$ :
(6), $\cos c=\cos a \cos b ;$
whence $B, A$ and $c$ may be found, and it is manifest that there is no ambiguity.

Ex. 2. Given $a$ and $c$, to find the rest.
From (3), $\sin a=\sin c \sin A, \therefore \sin A=\frac{\sin a}{\sin c}$ :
(6), $\cos c=\cos a \cos b, \therefore \cos b=\frac{\cos c}{\cos a}$ :
(8), $\cos B=\tan a \cot c$ :
whence $A, c, B$ may be determined, and there can be no ambiguity except in the value of $A$, and this is removed by means of the circumstance stated in (28).

Ex. 3. Given $a$ and $A$, to find the rest.
From (3), $\sin a=\sin c \sin A, \therefore \sin c=\frac{\sin a}{\sin A}$ :
(2), $\sin b=\tan a \cot A$ :
(9), $\cos A=\cos a \sin B, \therefore \sin B=\frac{\cos A}{\cos a}$ :
hence the sines of $c, b$, and $B$, and therefore the parts themselves may be found: but it may be observed that there is nothing to decide whether $c, b$, and $B$ should be greater or less than $\frac{\pi}{2}$ or $90^{\circ}$, and therefore the solution is ambiguous; and as in Plane Trigonometry (233), it is readily shewn that there may be two right-angled spherical triangles, which possess the proposed data, and in which the required parts are supplemental to each other.

Ex. 4. Given $a$ and $B$, to find the rest.
From (1), $\sin a=\tan b \cot B, \therefore \tan b=\frac{\sin a}{\cot B}=\sin a \tan B$ :
(8), $\cos B=\tan a \cot c, \therefore \cot c=\frac{\cos B}{\tan a}=\cos B \cot a$ :
(9), $\cos A=\cos a \sin B ;$
therefore $b, c$ and $A$ may be determined, and there is no ambiguity in the solution.

Ex.5. Given $c$ and $A$, to find the rest.
From (3), $\sin a=\sin c \sin A$ :
(5), $\cos c=\cot A \cot B, \therefore \cot B=\frac{\cos c}{\cot A}=\cos c \tan A:$
(7), $\cos A=\tan b \cot c, \therefore \tan b=\frac{\cos A}{\cot c}=\cos A \tan c$;
whence $a, B$ and $b$ may be found, and there can be no ambiguity except in the first, which may be removed by means of the considerations noticed in (28).

Ex.6. Given $A$ and $B$, to find the rest.
From (5), $\cos c=\cot A \cot B:$

$$
\begin{aligned}
& (9), \cos A=\cos a \sin B, \therefore \cos a=\frac{\cos A}{\sin B} \\
& (10), \cos B=\cos b \sin A, \therefore \cos b=\frac{\cos B}{\sin A}
\end{aligned}
$$

therefore $c, a$, and $b$ may be found without ambiguity.

## II. Solution of Quadrantal Triangles.

40. Let $A B C$ be a spherical triangle whose sides and angles are denoted by $a, b, c, A, B, C$ respectively, whereof $c$ is a quadrantal arc: construct the polar triangle, and let its sides and angles be expressed by $a^{\prime}, b^{\prime}, c^{\prime}, A^{\prime}, B^{\prime}, C^{\prime \prime}$ respectively as in (27), then it is manifest that $C^{\prime}$ will be a right angle. Now by the last article we have

$$
r \sin a^{\prime}=\tan b^{\prime} \cot B^{\prime}, \text { and } r \sin a^{\prime}=\sin c^{\prime} \sin A^{\prime}:
$$

whence by substitution we get $r \sin (\pi-A)=\tan (\pi-B) \cot (\pi-b)$, or $r \sin A=\tan B \cot b ;$ and
$r \sin (\pi-A)=\sin (\pi-C) \sin (\pi-a)$, or $r \sin A=\sin C \sin a:$
In the same manner all the ten cases of the polar triangle as enumerated in the last article being resolved, those of the primitive triangle will be immediately deduced from them, and it will readily be observed that the two Rules of Napier above explained will be applicable to the solution of all the cases of quadrantal triangles, if the two angles adjacent to the quadrantal side, the complements of the two other sides, and the complement of the hypothenusal augle, or angle subtended by the quadrant be considered as the circular parts; and to all ambiguity of solution whether real or apparent, the remarks
made in the different examples at the end of the preceding article may be applied.

These solutions like the preceding are already adapted to logarithmic computation.

## III. Solution of Oblique-Angled Triangles.

41. Since every oblique-angled triangle has six distinct parts, the three sides and the three angles, it follows that the number of solutions in which from three parts given a fourth may be found, will be equal to the number of combinations that can be formed out of six quantities taken four at a time, that is $=15$; but from a little consideration it will appear that all the solutions essentially different will be comprised in the six following cases :
I. When two sides and the angle opposite one of them are given.
II. When two angles and the sides subtending one of them are given.
III. When two sides and the included angle are given.
IV. When two angles and the adjacent side are given.
V. When the three sides are given.
VI. When the three angles are given;
and the resolution of oblique-angled spherical triangles will be complete, if we can show that the expressions already investigated can be applied to effect the solution of each particular case.
42. Case I , in which two sides a , b , and the angle A opposite one of them are given, to find the rest.

$$
\text { Since } \frac{\sin B}{\sin A}=\frac{\sin b}{\sin a} \text { by (24), we have } \sin B=\frac{\sin b}{\sin a} \sin A
$$ which is therefore found:

$$
\text { again from (32), } \cot \frac{C}{2}=\frac{\cos \left(\frac{a+b}{2}\right)}{\cos \left(\frac{a-b}{2}\right)} \tan \left(\frac{A+B}{2}\right)
$$

which is also determined;
and from (34), $\tan \frac{c}{2}=\frac{\cos \left(\frac{A+B}{2}\right)}{\cos \left(\frac{A-B}{2}\right)} \tan \left(\frac{a+b}{2}\right)$ is obtained,
or $\sin c=\frac{\sin C}{\sin A} \sin a=\frac{\sin C}{\sin B} \sin b$ becomes known.
All these formulæ are adapted to logarithmic computation; but it must be observed that $C$ and $c$ are here expressed in terms involving $B$ which was not originally given, but has been determined in the previous part of the solution. This is however by no means necessary, for in (35) we have seen that $\cot A \sin C=\cot a \sin b-\cos b \cos C$, and to adapt it to logarithms, assume the subsidiary angle $\theta$ such that

$$
\begin{aligned}
& \tan \theta=\cos b \tan A ; \text { then we shall have } \\
& \frac{\cos b \sin C}{\tan \theta}=\cot a \sin b-\cos b \cos C,
\end{aligned}
$$

$\therefore \cos b \sin C \cos \theta=\cot a \sin b \sin \theta-\cos b \cos C \sin \theta$,
whence
$\cot a \sin b \sin \theta=\cos b(\sin C \cos \theta+\cos C \sin \theta)=\cos b \sin (C+\theta)$, and $\therefore \sin (C+\theta)=\cot a \tan b \sin \theta$, from which $C+\theta$, and therefore $C$ may be determined.

Again, from (21), $\sin b \sin c \cos A=\cos a-\cos b \cos c ;$
$\therefore$ if we assume $\tan \theta=\cos A \tan b$, we shall have $\cos b \sin c \tan \theta=\cos a-\cos b \cos c$,
and $\cos b \sin c \sin \theta=\cos a \cos \theta-\cos b \cos c \cos \theta$,
whence
$\cos b \cos (c-\theta)=\cos a \cos \theta$, or $\cos (c-\theta)=\frac{\cos a}{\cos b} \cos \theta$, which is adapted to logarithms, and gives the value of $c-\theta$ and therefore of $c$.
43. Case II, in which two angles $\mathrm{A}, \mathrm{B}$ and the side a subtending one of them are given, to find the rest.

Since $\frac{\sin b}{\sin a}=\frac{\sin B}{\sin A}$ by (24), we get $\sin \bar{b}=\frac{\sin B}{\sin A} \sin a$, which is found:
and from (34) $\tan \frac{c}{2}=\frac{\cos \left(\frac{A+B}{2}\right)}{\cos \left(\frac{A-B}{2}\right)} \tan \left(\frac{a+b}{2}\right)$ becomes known: also from (32) $\cot \frac{C}{2}=\frac{\cos \left(\frac{a+b}{2}\right)}{\cos \left(\frac{a-b}{2}\right)} \tan \left(\frac{A+B}{2}\right)$ is determined,

$$
\text { or } \sin C=\frac{\sin c}{\sin a} \sin A=\frac{\sin c}{\sin b} \sin B \text { is found. }
$$

These solutions are all adapted to logarithmic computation, and to these methods of finding $c$ and $C$ the same observations may be applied as in the last case ; but articles (36) and (27) by the introduction of subsidiary angles may as above be the means of expressing in logarithmic forms the values of these two parts without the previous determination of the side $b$.
44. Case III, in which two sides $\mathrm{a}, \mathrm{b}$, and the included angle C are given, to find the rest.

In (31) we have seen that

$$
\tan \left(\frac{A+B}{2}\right)=\frac{\cos \left(\frac{a-b}{2}\right)}{\cos \left(\frac{a+b}{2}\right)} \cot \frac{C}{2}
$$

$$
\text { and } \tan \left(\frac{A-B}{2}\right)=\frac{\sin \left(\frac{a-b}{2}\right)}{\sin \left(\frac{a+b}{2}\right)} \cot \frac{C}{2}
$$

whence $\frac{A+B}{2}$ and $\frac{A-B}{2}$ become known and therefore $A$ and $B$ :
also $\sin c=\frac{\sin C}{\sin A} \sin a=\frac{\sin C}{\sin B} \sin b$ is thence found.
These forms are all adapted to logarithms, but the side $c$ may likewise be expressed in terms of $a, b, C$ immediately and in a form fitted for practice.

For, since $\cos C=\frac{\cos c-\cos a \cos b}{\sin a \sin b}$, we have

$$
\begin{aligned}
\cos c & =\cos a \cos b+\sin a \sin b \cos C \\
& =\cos a \cos b+\sin a \sin b-\sin a \sin b \text { vers } C \\
& =\cos (a-b)-\sin a \sin b \text { vers } C
\end{aligned}
$$

$\therefore$ vers $c=$ vers $(a-b)+\sin a \sin b$ vers $C$

$$
=\operatorname{vers}(a-b)\left\{1+\frac{\sin a \sin b \text { vers } C}{\operatorname{vers}(a-b)}\right\},
$$

which, by assuming $\frac{\sin a \sin b \text { vers } C}{\text { vers }(a-b)}=\tan ^{2} \theta$, becomes vers $c=\frac{\text { vers }(a-b) \sec ^{2} \theta}{r^{2}}$ to the radius $r$, and is adapted as before.
45. Case IV, in which two angles $\mathrm{A}, \mathrm{B}$ and the adjacent side c are given, to find the rest.

In (33) it has been proved that

$$
\tan \left(\frac{a+b}{2}\right)=\frac{\cos \left(\frac{A-B}{2}\right)}{\cos \left(\frac{A+B}{2}\right)} \tan \frac{c}{2}
$$

$$
\text { and } \tan \left(\frac{a-b}{2}\right)=\frac{\sin \left(\frac{A-B}{2}\right)}{\sin \left(\frac{A+B}{2}\right)} \tan \frac{c}{2}
$$

from which $\frac{a+b}{2}$ and $\frac{a-b}{2}$, and therefore $a$ and $b$ are known:
and $\sin C=\frac{\sin c}{\sin a} \sin A=\frac{\sin c}{\sin b} \sin B$ is thus determined.
These formulee are all ready for logarithmic computation, but as in the last case, the value of $C$ may be expressed in terms of the given parts $A, B$ and $c$ alone, and adapted to practice.

For, since by (27) $\cos c=\frac{\cos C+\cos A \cos B}{\sin A \sin B}$, we have $\cos C=\sin A \sin B \cos c-\cos A \cos B ;$
$\therefore$ vers $C=1-\sin \Lambda \sin B(1-\operatorname{vers} c)+\cos A \cos B$

$$
\begin{aligned}
& =1-\sin A \sin B+\sin A \sin B \text { vers } c+\cos A \cos B \\
& =1+\cos (A+B)+\sin A \sin B \text { vers } c,
\end{aligned}
$$

$$
\therefore 2 \sin ^{2} \frac{C}{2}=2 \cos ^{2}\left(\frac{A+B}{2}\right)+\sin A \sin B \text { vers } c
$$

$$
\begin{aligned}
& =2 \cos ^{2}\left(\frac{A+B}{2}\right)\left\{1+\frac{\sin A \sin B \text { vers } c}{2 \cos ^{2}\left(\frac{A+B}{2}\right)}\right\} \\
& =2 \cos ^{2}\left(\frac{A+B}{2}\right) \sec ^{2} \theta, \text { if } \frac{\sin A \sin B \text { vers } c}{2 \cos ^{2}\left(\frac{A+B}{2}\right)} \text { be }
\end{aligned}
$$

assumed $=\tan ^{2} \theta$, and thus the value of $C$ may be found from the tables.
46. Case V , in which the three sides $\mathrm{a}, \mathrm{b}, \mathrm{c}$ are given, to find the rest.

In (23) and (25) it has been demonstrated that
$\sin A=\frac{2}{\sin b \sin c} \sqrt{\sin S \sin (S-a) \sin (S-b) \sin (S-c)}$,

$$
\begin{gathered}
\sin \frac{A}{2}=\sqrt{\frac{\sin (S-b) \sin (S-c)}{\sin b \sin c}, \cos \frac{A}{2}=\sqrt{\frac{\sin S \sin (S-a)}{\sin b \sin c}}}, \\
\text { and } \tan \frac{A}{2}=\sqrt{\frac{\sin (S-b) \sin (S-c)}{\sin S \sin (S-a)}}
\end{gathered}
$$

all of which when adapted to the radius $r$ will be prepared for logarithmic computation : and to which of these the preference ought to be given above the rest, must be decided by means of the remarks made in (238), (239), (240) and (241), of the Plane Trigonometry.
47. Case VI, in which the three angles $A, B, C$ are given, to find the rest.

In (29) and (30), we have seen that

$$
\begin{gathered}
\sin a=\frac{2}{\sin B \sin C} \sqrt{-\cos S^{\prime} \cos \left(S^{\prime}-A\right) \cos \left(S^{\prime}-B\right) \cos \left(S^{\prime}-C\right)}, \\
\sin \frac{a}{2}=\sqrt{\frac{-\cos S^{\prime} \cos \left(S^{\prime}-A\right)}{\sin B \sin C}}, \cos \frac{a}{2}=\sqrt{\frac{\cos \left(S^{\prime}-B\right) \cos \left(S^{\prime}-C\right)}{\sin B \sin C}}, \\
\quad \text { and } \tan \frac{a}{2}=\sqrt{\frac{-\cos S^{\prime} \cos \left(S^{\prime}-A\right)}{\cos \left(S^{\prime}-B\right) \cos \left(S^{\prime}-C\right)}}
\end{gathered}
$$

from any of which when adapted to the radius $r$, the value of $a$ or $\frac{a}{2}$ may be logarithmically determined, the preference as to method being given according to the remarks referred to in the last article.
48. Though there has just been given a solution of every one of the six cases above enumerated, and which, it was observed, are all that are essentially different from each other, it may still be added that by supposing an arc of a great circle to be drawn from one of the angles perpendicular to the opposite side, Napier's Rules for the solution of right-angled triangles are sufficient for the solution of all spherical triangles whatsoever. For brevity's sake, we will exemplify their application in the last two cases only.

First, let the three sides $a, b, c$ be given, to find the three angles, and suppose the arc $C D$ of a great circle to be drawn from the angle $C$ perpendicular to the side $A B$; then by Napier's Rules we have

$r \cos a=\cos B D \cos C D$, and $r \cos b=\cos A D \cos C D$,
whence $\frac{\cos a}{\cos b}=\frac{\cos B D}{\cos A D}$, or the cosines of the segments of the base are proportional to the cosines of the adjacent sides;

$$
\begin{aligned}
\therefore \frac{\cos a-\cos b}{\cos a+\cos b} & =\frac{\cos B D-\cos A D}{\cos B D+\cos A D} \\
& =\frac{\sin \frac{A D+B D}{2} \sin \frac{A D-B D}{2}}{\cos \frac{A D+B D}{2} \cos \frac{A D-B D}{2}} \\
& =\tan \frac{c}{2} \tan \left(\frac{A D-B D}{2}\right)
\end{aligned}
$$

whence $\frac{A D-B D}{2}$ is found, and $\frac{A D+B D}{2}$ being given, two parts in each of the right-angled triangles $A C D, B C D$ become known, and consequently all the angles of the triangle may be determined.

Hence also the perpendicular $C D$ may be found from either of the equations, $\cos C D=\frac{r \cos a}{\cos B D}$, or $\cos C D=\frac{r \cos b}{\cos A D}$.

Next, let the three angles $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ be given, to find the rest; then the same construction remaining, we have by Napier's Rules,
$r \cos A=\cos C D \sin A C D$, and $r \cos B=\cos C D \sin B C D ;$ and therefore $\frac{\cos A}{\cos B}=\frac{\sin A C D}{\sin B C D}$, or the sines of the segments of the vertical angle are proportional to the sines of the corresponding angles at the base;
hence if these segments be called $\alpha, \beta$ respectively, we have

$$
\begin{aligned}
\frac{\cos A-\cos B}{\cos A+\cos B} & =\frac{\sin \alpha-\sin \beta}{\sin \alpha+\sin \beta} \\
& =\frac{\tan \left(\frac{a-\beta}{2}\right)}{\tan \left(\frac{a+\beta}{2}\right)}=\frac{\tan \left(\frac{a-\beta}{2}\right)}{\tan \frac{C}{2}}
\end{aligned}
$$

from which $\frac{\alpha-\beta}{2}$ may be found, and $\frac{\alpha+\beta}{2}$ or $\frac{C}{2}$ being given, the values of the segments $\alpha, \beta$ may easily be determined; and thus in each of the right-angled triangles $A C D, B C D$ two angles being known, the sides of the proposed triangle may be found.

As before, the magnitude of $C D$ may be found from either of the equations

$$
\cos C D=\frac{r \cos A}{\sin A C D}, \text { or } \cos C D=\frac{r \cos B}{\sin B C D} .
$$

## CHAP. IV.

On the Areas of Spherical Triangles, \&c. and the Spherical Excess. On the Measures of solid Angles, \&c.
49. THE surface of a spherical Lane is proportional to the angle contuined between the planes of the two semi-circles by which it is formed.

Let $A P N Q$ be the lune formed by the two great semi-circles of the sphere $P A Q, P N Q$ : then it is obvious that if the arc

$A N$ which measures the angle $A P N$ be doubled, or increased in any other ratio, the surface $A P N Q$ will be doubled, or increased in the same ratio, because equal portions of surface will manifestly correspond to equal parts of the arc: that is, the surface $A P N Q$ is proportional to the arc $A N$, or to the angle $A P N$.
50. Cor. 1. Hence, if $S$ represent the whole surface of the sphere, the surface $A P B Q=\frac{S}{2}$, and we shall have
the area of the lune $A P N Q: \frac{S}{2}:: \angle A P N: 180^{\circ}$;
$\therefore$ the area of the lune $A P N Q=\frac{\angle A P N}{180^{\circ}} \frac{S}{2}$ :
and if the radius $=1$, we have seen in (214) Pl. Trig. that the area of the circle $=\pi$, and $\therefore \frac{S}{2}=2 \pi=180^{\circ}$, whence the area of the lune $A P N Q=\frac{\angle A P N}{180^{\circ}} 2 \pi=\angle A P N$.
51. Cor. 2. The area of the lune may be expressed in other terms.

For since the spherical angle $A P N=$ the plane angle $A O N=\angle a O n=\frac{a n}{a o}=\frac{a n}{\sin P a}$, we have the area of the lune $A P N Q=\frac{a n}{\sin P a}$.
52. To express the area of a spherical triangle in terms of its angles.

Let $A B C$ be a spherical triangle on the surface of a sphere whose radius is 1 , and produce the sides $A C, B C$ till they meet

in $c$ on the opposite hemisphere; then it is manifest that the arcs $C a c, C b c$ are semi-circles; whence, since $A C a, B C b$ are also semi-circles, it follows that $A C=a c$ and $B C=b c$ : therefore the angles at $C$ and $c$ which measure the inclination of the same planes being also equal, we have the triangles $A C B$, $a c b$ in every respect equal to one another:

Now if the area of the triangle $A C B$ be called $x$, and $B C a$, $A C b, C B a$ be assumed equal to $a, \beta, \gamma$ respectively, we shall have

$$
x+\alpha=\frac{A}{180^{\circ}} \frac{S}{2}, x+\beta=\frac{B}{180^{\circ}} \frac{S}{2}, x+\gamma=\frac{C}{180^{\circ}} \frac{S}{2} ;
$$

$\therefore$ by addition, observing that $x+\alpha+\beta+\gamma=\frac{S}{2}$, we get

$$
2 x+\frac{S}{2}=\frac{A+B+C}{180^{\circ}} \frac{S}{2} ;
$$

whence $x$, or the area of the triangle $=\frac{A+B+C-180^{\circ}}{180^{\circ}} \frac{S}{4}$ $=A+B+C-180^{\circ}$, as appears from (214).
53. Cor. 1. Hence the area of the triangle is equal to the excess of the sum of its three angles above the two right angles, which is called the Spherical Excess.
54. Cor.2. It follows therefore by (17), that the area of a spherical triangle may be represented by any number of degrees between 0 and 360 , and also that if two of the angles be right angles, the area varies as the third.
55. Cor.3. If the radius of the sphere be supposed $=r$, we shall have $S=4 \pi r^{2}$, Diff. Cal., and therefore the area of the triangle $=\frac{A+B+C-180^{\circ}}{180^{\circ}} \pi r^{2}$ : and to the radius $1, \pi$ represents $180^{\circ}$ expressed in terms of that radius, therefore the area of the triangle expressed in seconds

$$
=\left(A+B+C-180^{\circ}\right) r^{2} \sin 1^{\prime \prime} .
$$

56. Cor. 4. By means of this article the area of a spherical polygon may likewise be expressed in terms of its angles.

For let $A B C D \& c$. be a polygon of $n$ sides whose angles are $A, B, C, D \& c \cdot$ : take any point $F$ in its surface, and from
it to the angular points draw arcs of great circles of the sphere: then the area of the polygon $A B C D \& c$. = the sum of the areas

of the triangles $A F B, B F C, C F D, D F E, \& c .=$ the sum of the angles of the polygou, together with the angles at $F-n 180^{\circ}$

$$
=A+B+C+D+8 \mathrm{c} \cdot+2.180^{\circ}-n 180^{\circ}
$$

$=A+B+C+D+8 c .-(n-2) 180^{\circ}$, the radius of the sphere being supposed $=1$.
57. To express the area of a spherical triangle in terms of two sides and the included angle.

Let $a, b$ be the proposed sides and $C$ their included angle, and suppose the area of the triangle or the spherical excess $A+B+C-180^{\circ}$ to be represented by $E$ : then in (31) we have seen that

$$
\begin{gathered}
\tan \left(\frac{A+B}{2}\right)=\frac{\cos \left(\frac{a-b}{2}\right)}{\cos \left(\frac{a+b}{2}\right)} \cot \frac{C}{2} \\
\text { whence } \cot \frac{E}{2}=-\tan \left(\frac{A+B+C}{2}\right)=-\frac{\tan \left(\frac{A+B}{2}\right)+\tan \frac{C}{2}}{1-\tan \left(\frac{A+B}{2}\right) \tan \frac{C}{2}} \\
= \\
=\frac{\cos \left(\frac{a-b}{2}\right) \cot \frac{C}{2}+\cos \left(\frac{a+b}{2}\right) \tan \frac{C}{2}}{\cos \left(\frac{a-b}{2}\right)-\cos \left(\frac{a+b}{2}\right)}
\end{gathered}
$$

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$$
\begin{gathered}
=\frac{\cos \left(\frac{a-b}{2}\right) \cos ^{2} \frac{C}{2}+\cos \left(\frac{a+b}{2}\right) \sin ^{2} \frac{C}{2}}{2 \sin \frac{a}{2} \sin \frac{b}{2} \sin \frac{C}{2} \cos \frac{C}{2}} \\
=\frac{\cos \frac{a}{2} \cos \frac{b}{2}+\sin ^{\frac{a}{2}} \sin \frac{b}{2}\left(\cos ^{2} \frac{C}{2}-\sin ^{2} \frac{C}{2}\right)}{\sin \frac{a}{2} \sin \frac{b}{2} \sin C} \\
=\frac{\cot \frac{a}{2} \cot \frac{b}{2}+\cos C}{\sin C} .
\end{gathered}
$$

58. To express the area of a spherical triangle in terms of the sides.

It has been shewn in the last article that

$$
\cot \frac{E}{2}=\frac{\cot \frac{a}{2} \cot \frac{b}{2}+\cos C}{\sin C}
$$

$$
\begin{gathered}
\text { now } \cot \frac{a}{2} \cot \frac{b}{2}=\frac{(1+\cos a)(1+\cos b)}{\sin a \sin b} \\
=\frac{1+\cos a+\cos b+\cos a \cos b}{\sin a \sin b}, \cos C=\frac{\cos c-\cos a \cos b}{\sin a \sin b}
\end{gathered}
$$

and $\sin C=\frac{2}{\sin a \sin b} \sqrt{\sin S \sin (S-a) \sin (S-b) \sin (S-c)}$, according to the notation adopted in (23);
$\therefore \cot \frac{E}{2}=\frac{1+\cos a+\cos b+\cos c}{2 \sqrt{\sin S \sin (S-a) \sin (S-b) \sin (S-c)}}$.
This formula for the spherical excess was discovered by De Gua, but it is not adapted to logarithmic computation.
59. Cor. 1. Since $\cot \frac{E}{2}=\frac{\cot \frac{a}{2} \cot \frac{b}{2}+\cos C}{\sin C}$, we shall have
$\operatorname{cosec}^{2} \frac{E}{2}=1+\cot ^{2} \frac{E}{2}=\frac{1+\cot ^{2} \frac{a}{2} \cot ^{2} \frac{b}{2}+2 \cot \frac{a}{2} \cot \frac{b}{2} \cos C}{\sin ^{2} C}:$

$$
\begin{aligned}
& \text { but } \cot ^{2} \frac{a}{2} \cot ^{2} \frac{b}{2}=\frac{\left(1-\sin ^{2} \frac{a}{2}\right)\left(1-\sin ^{2} \frac{b}{2}\right)}{\sin ^{2} \frac{a}{2} \sin ^{2} \frac{b}{2}} \\
&= \frac{1-\sin ^{2} \frac{a}{2}-\sin ^{2} \frac{b}{2}+\sin ^{2} \frac{a}{2} \sin ^{2} \frac{b}{2}}{\sin ^{2} \frac{a}{2} \sin ^{2} \frac{b}{2}}
\end{aligned}
$$

$$
\text { also } \cos C=\frac{\cos c-\cos a \cos b}{\sin a \sin b}
$$

$$
=\frac{1-2 \sin ^{2} \frac{c}{2}-\left(1-2 \sin ^{2} \frac{a}{2}\right)\left(1-2 \sin ^{2} \frac{b}{2}\right)}{4 \sin \frac{a}{2} \sin \frac{b}{2} \cos \frac{a}{2} \cos \frac{b}{2}}
$$

$\therefore 2 \cot \frac{a}{2} \cot \frac{b}{2} \cos C=\frac{1-2 \sin ^{2} \frac{c}{2}-\left(1-2 \sin ^{2} \frac{a}{2}\right)\left(1-2 \sin ^{2} \frac{b}{2}\right)}{2 \sin ^{2} \frac{a}{2} \sin ^{2} \frac{b}{2}}$

$$
=\frac{\sin ^{2} \frac{a}{2}+\sin ^{2} \frac{b}{2}-\sin ^{2} \frac{c}{2}-2 \sin ^{2} \frac{a}{2} \sin ^{2} \frac{b}{2}}{\sin ^{2} \frac{a}{2} \sin ^{2} \frac{b}{2}}:
$$

$\therefore$ by substitution we get $\operatorname{cosec}^{2} \frac{E}{\Omega}$

$$
=\frac{\sin ^{2} \frac{a}{2} \sin ^{2} \frac{b}{2}+1-\sin ^{2} \frac{a}{2}-\sin ^{2} \frac{b}{2}+\sin ^{2} \frac{a}{2} \sin ^{2} \frac{b}{2}+\sin ^{2} \frac{a}{2}+\sin ^{2} \frac{b}{2}-\sin ^{2} \frac{c}{2}-2 \sin ^{2} \frac{a}{2} \sin ^{2} \frac{b}{2}}{\sin ^{2} \frac{a}{2} \sin ^{2} \frac{b}{2} \sin ^{2} C}
$$

L L
$=\frac{1-\sin ^{2} \frac{c}{2}}{\sin ^{2} \frac{a}{2} \sin ^{2} \frac{b}{2} \sin ^{2} C}=\frac{4 \cos ^{2} \frac{a}{2} \cos ^{2} \frac{b}{2} \cos ^{2} \frac{c}{2}}{\sin S \sin (S-a) \sin (S-b) \sin (S-c)} ;$
whence $\sin \frac{E}{2}=\frac{\sqrt{\sin S \sin (S-a) \sin (S-b) \sin (S-c)}}{2 \cos \frac{a}{2} \cos \frac{b}{2} \cos \frac{c}{2}}$.
The discovery of this formula is due to Cagnoli, and it has the advantage of being easily adapted to logarithms.
60. Cor. . . Since $\cos \frac{E}{\Omega}=\cot \frac{E}{Q} \sin \frac{E}{\Omega}$, we have from the last two articles
$\cos \frac{E}{2}=\frac{1+\cos a+\cos b+\cos c}{4 \cos \frac{a}{2} \cos \frac{b}{2} \cos \frac{c}{2}}=\frac{\cos ^{2} \frac{a}{2}+\cos ^{2} \frac{b}{2}+\cos ^{2} \frac{c}{2}-1}{2 \cos \frac{a}{2} \cos \frac{b}{2} \cos \frac{c}{2}}$,
which is rational, but not adapted to logarithmic computation.
61. Cor. 3. Because

$$
\begin{aligned}
& \frac{1-\cos \frac{E}{2}}{\sin \frac{E}{2}}=\frac{1-\cos \frac{E}{2}}{\sqrt{\left(1+\cos \frac{E}{2}\right)\left(1-\cos \frac{E}{2}\right)}}=\sqrt{\frac{1-\cos \frac{E}{2}}{1+\cos \frac{E}{2}}} \\
& =\tan \frac{E}{4}, \text { we shall have by substitution, } \\
& \tan \frac{E}{4}=\frac{1-\cos ^{2} \frac{a}{2}-\cos ^{2} \frac{b}{2}-\cos ^{2} \frac{c}{2}+2 \cos \frac{a}{2} \cos \frac{b}{2} \cos \frac{c}{2}}{\sqrt{\sin S \sin (S-a) \sin (S-b) \sin (S-c)}}: \\
& \quad \text { but } 1-\cos ^{2} \frac{a}{2}-\cos ^{2} \frac{b}{2}-\cos ^{2} \frac{c}{2}+2 \cos \frac{a}{2} \cos \frac{b}{2} \cos \frac{c}{2} \\
& \quad=\left(1-\cos ^{2} \frac{a}{2}\right)\left(1-\cos ^{2} \frac{b}{2}\right)-\left(\cos \frac{a}{2} \cos \frac{b}{2}-\cos \frac{c}{2}\right)^{2}
\end{aligned}
$$

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$$
\begin{aligned}
& =\left(\sin \frac{a}{2} \sin \frac{b}{2}\right)^{2}-\left(\cos \frac{a}{2} \cos \frac{b}{2}-\cos \frac{c}{2}\right)^{2} \\
& =\left\{\sin \frac{a}{2} \sin \frac{b}{2}+\cos \frac{a}{2} \cos \frac{b}{2}-\cos \frac{c}{2}\right\} \times \\
& \quad\left\{\sin \frac{a}{2} \sin \frac{b}{2}-\cos \frac{a}{2} \cos \frac{b}{2}+\cos \frac{c}{2}\right\} \\
& =\left\{\cos \left(\frac{a-b}{2}\right)-\cos \frac{c}{2}\right\}\left\{\cos \frac{c}{2}-\cos \left(\frac{a+b}{2}\right)\right\} \\
& =4 \sin \frac{S}{2} \sin \left(\frac{S-a}{2}\right) \sin \left(\frac{S-b}{2}\right) \sin \left(\frac{S-c}{2}\right),
\end{aligned}
$$

if we adopt the notation of (23);

$$
\begin{aligned}
\therefore \tan \frac{E}{4} & =\frac{4 \sin \frac{S}{2} \sin \left(\frac{S-a}{2}\right) \sin \left(\frac{S-b}{2}\right) \sin \left(\frac{S-c}{2}\right)}{\sqrt{\sin S \sin (S-a) \sin (S-b) \sin (S-c)}} \\
& =\sqrt{\tan \frac{S}{2} \tan \left(\frac{S-a}{2}\right) \tan \left(\frac{S-b}{2}\right) \tan \left(\frac{S-c}{2}\right)}
\end{aligned}
$$

which being transformed to the radius $r$ will be adapted to logarithmic computation.

This singular formula for the spherical excess was discovered by Simon Lhuillier of Geneva.
62. A solid angle being the angular space included between the several planes by which it is formed, will manifestly have the same relation to the corresponding spherical surface whose centre is the angular point, as plane angles have to their corresponding circular arcs, and therefore the magnitudes of solid angles may be compared by determining the ratios between the spherical surfaces by which they are, as it were, respectively subtended. Now we have seen (52) that the area of a spherical triangle is measured by the excess of the sum of its angles above $180^{\circ}$, and (56) the area of a spherical polygon of $n$ sides by the excess of the sum of all its angles above ( $n-2$ ) $180^{\circ}$; hence it follows that these quantities may be assumed as
the measures of the solid angles formed by the planes whose inclinations to one another are the same as the angles of the triangle or polygon. The maximum limit of solid angular space will manifestly be a hemisphere and its measure the surface of the hemisphere in the same manner as the maximum limit of plane angular space is a semi-circle and its measure the arc of the semi-circle. Representing therefore the maximum solid angular space by the content of the hemisphere whose radius is 1 , or by $\frac{2 \pi}{3}$, (Diff. Cal.) we shall have its measure equal to the corresponding hemispherical surface $2 \pi$ or $360^{\circ}$.

Ex. 1. In a cube each of the solid angles is formed by three plane right angles, and thence we have
$\frac{\text { the solid angle of a cube }}{\text { the maximum solid angle }}=\frac{90^{\circ}+90^{\circ}+90^{\circ}-180^{\circ}}{360^{\circ}}=\frac{90^{\circ}}{360^{\circ}}=\frac{1}{4}$,
which we also know to be true from the circumstance that if four cubes be placed together upon a plane, they exactly fill up the angular space about a common point.

Ex. 2. In a regular right prism with a triangular base two of the plane angles which form each solid angle are manifestly $90^{\circ}$, and the remaining one $60^{\circ}$ : therefore
$\frac{\text { the solid angle of this prism }}{\text { the maximum solid angle }}=\frac{90^{\circ}+90^{\circ}+60^{\circ}-180^{\circ}}{360^{\circ}}=\frac{60^{\circ}}{360^{\circ}}=\frac{1}{6}$.
Hence by means of the last Example, we have

$$
\frac{\text { the solid angle of a cube }}{\text { the solid angle of this prism }}=\frac{1}{4} \div \frac{1}{6}=\frac{6}{4}=\frac{3}{2} \text {. }
$$

Ex. 3. Let there be two regular right prisms of $m$ and $n$ sides respectively, then each of the angles of their bases will be

$$
\left(\frac{m-2}{m}\right) 180^{\circ} \text { and }\left(\frac{n-2}{n}\right) 180^{\circ} \text { respectively: hence }
$$

$\frac{\text { the solid angle of the first prism }}{\text { the maximum solid angle }}=\frac{\left(\frac{m-2}{m}\right) 180^{\circ}}{360^{\circ}}=\frac{m-2}{2 m}:$
$\frac{\text { the solid angle of the second prism }}{\text { the maximum solid angle }}=\frac{\left(\frac{n-2}{n}\right) 180^{\circ}}{360^{\circ}}=\frac{n-2}{2 n}$;

$$
\text { and } \therefore \frac{\text { the solid angle of the first prism }}{\text { the solid angle of the second prism }}=\frac{(m-2) n}{(n-2) m} \text {; }
$$

and by means of this example, the solid angles of all regular right prisms whatsoever may be compared.

Ex. 4. If there be any two prisms whatever, whose numbers of sides are $m$ and $n$, we shall manifestly have
$\frac{\text { the sum of all the solid angles of the first prism }}{\text { the maximum solid angle }}=\frac{(m-2) 180^{\circ}}{360^{\circ}}=\frac{m-2}{2}$;
$\frac{\text { the sum of all the solid angles of the second prism }}{\text { the maximum solid angle }}=\frac{(n-2) 180^{\circ}}{360^{\circ}}=\frac{n-2}{2}$ :
$\therefore \frac{\text { the sum of all the solid angles of the first prism }}{\text { the sum of all the solid angles of the second prism }}=\frac{m-2}{n-2}$.
63. The vertical angles of pyramids whether regular or irregular may be ascertained and compared by the same methods: thus if there be two regular pyramids of $m$ and $n$ sides having the inclinations of two contiguous sides to each other represented by $\alpha$ and $\beta$ respectively, then according to the principles above explained, we shall have
$\frac{\text { the vertical angle of the first pyramid }}{\text { the vertical angle of the second pyramid }}=\frac{m \alpha-(m-2) 180^{\circ}}{n \beta-(n-2) 180^{\circ}}$.
64. The same principles enable us to compare the vertical angles of cones, by comparing the areas of corresponding spherical segments of spheres of equal radii, whose centres are the angular points.

Hence since (Diff. Cal.) the surface of a spherical segment varies as its altitude, if $H$ and $h$ be the altitudes of two cones corresponding to $L$ the common length of their sides, we shall have
$\frac{\text { the vertical angle of the first cone }}{\text { the vertical angle of the second cone }}=\frac{\text { the height of the first segment }}{\text { the height of the second segment }}$

$$
=\frac{L-H}{L-h}=\frac{1-\cos \theta}{1-\cos \phi}=\frac{\operatorname{vers} \theta}{\operatorname{vers} \phi}
$$

if $\theta$ and $\phi$ be the vertical angles of their generating triangles.
Ex. For the equilateral and right-angled cones we have

$$
\theta=30^{\circ}, \text { and } \phi=45^{\circ}, \text { respectively }
$$

$\therefore \frac{\text { the vertical angle of the equilateral cone }}{\text { the vertical angle of the right-angled cone }}=\frac{\text { vers } 30^{\circ}}{\text { vers } 45^{\circ}}=\frac{1-\frac{\sqrt{3}}{2}}{1-\frac{1}{\sqrt{2}}}=\frac{2-\sqrt{ } 3}{2-\sqrt{2}}$.
Whenever the spherical surface by which any solid angle is measured can be divided into $n$ parts either equal to one another, or having any assigned ratios, the solid angle itself can be divided into $n$ parts having to each other the same ratios.

This method of measuring and comparing solid angles by means of the positions, and not the magnitudes, of its plane angles, was first suggested by Albert Girard in his Invention nouvelle en Algèbre published about the year 1629, and has been extended and exemplified by several modern writers.

## CHAP. V.

On the regular Polyhedrons, and on the Parallelopiped and triangular Pyramid.
65. IN every Polyhedron, the numbers of Solid Angles and Plane Faces together exceed the number of Edges by 2.

Let $x$ be the number of solid angles, $y$ the number of plane faces and $z$ the number of edges; then since every edge is common to two plane faces, $2 z$ will be the number of sides of all the faces :

Within the polyhedron suppose a point to be assumed from which to all the angular points straight lines may be drawn, and with this point as a centre let a spherical surface be described cutting these lines, and let the points of intersection be joined by arcs of great circles of the sphere so as to form as many spherical polygons as the solid has faces: let $A B C D$ \&c. be one of such polygons, the number of whose sides is $n$;

then by (56), its area $=A+B+C+D+\& c .-(n-2) 180^{\circ}$ : and the same being found for all the polygons, we shall have the whole surface of the sphere or $720^{\circ}$ equal to the sum of all the angles of all the polygons $-(2 z-2 y) 180^{\circ}$, or the sum of all
the angles of all the polygons $=720^{\circ}+(z-y) 360^{\circ}$ : but the sum of all the angles about any point as $A$ being $=360^{\circ}$, we have the sum of all the angles of all the polygons $=x 360^{\circ}$ :

$$
\begin{gathered}
\text { whence } x 360^{\circ}=720^{\circ}+(z-y) 360^{\circ} \text {, or } x=2+z-y, \\
\text { and } \therefore x+y=z+2 .
\end{gathered}
$$

66. Cor. 1. Hence the sum of all the plane angles forming all the solid angles of any polyhedron $=(x-2) 360^{\circ}$.

For if any face have $n$ sides, the sum of all its angles $=(2 n-4) 90^{\circ}=(n-2) 180^{\circ}$; hence the sum of all the angles of all the faces $=(2 z-2 y) 180^{\circ}=(z-y) 360^{\circ}=(x-2) 360^{\circ}$.
67. Cor. 2. In a regular polyhedron, if $n$ be the number of sides of each plane face, $m$ the number of plane angles constituting each solid angle, then will $\left(\frac{n-2}{n}\right) 180^{\circ}$ be the magnitude of each plane angle, and $\frac{m(n-Q)}{n} 180^{\circ}$ the sum of all the plane angles forming each solid angle: hence retaining the notation above used, we shall have the sum of all the plane angles forming all the solid angles $=\frac{m(n-2) x}{n} 180^{\circ}$ :
therefore from the last article, we get

$$
\begin{aligned}
& (x-2) 360^{\circ}=\frac{m(n-2) x}{n} 180^{\circ}, \\
& \text { and } \therefore 2 n x-4 n=m n x-2 m x,
\end{aligned}
$$

whence the number of solid angles $x=\frac{4 n}{2(m+n)-m n}$ :

$$
\text { also since } y=2+z-x=2+\frac{n y}{2}-\frac{4 n}{2(m+n)-m n},
$$

we have the number of plane faces $y=\frac{4 m}{2(m+n)-m n}$.
68. Cor.3. From what has just been proved, we immediately obtain the number of edges $z=\frac{n y}{2}=\frac{2 m n}{2(m+n)-m n}$ : and therefore the number of solid angles, the number of plane faces, and the number of edges of any regular polyhedron, are to one another respectively as $4 n, 4 m$ and $2 m n$, or as $2 n$, $2 m$ and $m n$.
69. There can be only five regular polyhedrons.

For, the notation above adopted being retained, if $x, y, z$ be finite positive quantities, the denominator $2(m+n)-m n$ must be positive, and $\therefore 2 m+2 n$ must be greater than $m n$, from which we have $\frac{m}{m-2}$ greater than $\frac{n}{2}$.

First, let $n=3$, or each solid angle be formed by three plane angles, and $n=3$, or the faces be triangular, then $\frac{3}{3-2}$ being greater than $\frac{3}{2}$, a solid will be formed having $\frac{12}{2.6-3.3}$ or 4 solid angles; $\frac{12}{2.6-3.3}$ or 4 plane triangular faces; and $\frac{18}{2.6-3.3}$ or 6 edges, and is therefore called a Tetrahedron.

Again, let $m=3$ and $n=4$, and we shall have $\frac{3}{3-2}$ greater than $\frac{4}{2}$, and thus a solid will be formed having eight solid angles, six plane square faces, and twelve edges, and is therefore called a Hexuhedron, which is the same as a cube.

Next, let $m=3$ and $n=5$, which gives $\frac{3}{2-3}$ greater than $\frac{5}{2}$, and therefore a solid will be formed having twenty solid angles, twelve plane pentagonal faces, and thirty edges, and on this account is termed a Dodecahedron.

If $n=3$, and $n$ be any number greater than 5 , the condition that $\frac{m}{m-2}$ must be greater than $\frac{n}{2}$ will not be satisfied; and thence it appears that there can be constructed only three regular polyhedrons in which each solid angle is formed by three plane angles.

Secondly, let $m=4$ and $n=3$, and we have $\frac{4}{4-2}$ greater than $\frac{3}{2}$; wherefore there will be formed a solid having six solid angles, eight plane triangular faces, and twelve edges, which is therefore called an Octahedron.

Again, let $m=4$ and $n=4$, or any larger number, and the condition will no longer be fulfilled; and consequently there can be constructed only one regular polyhedron in which each of the solid angles is formed by four plane angles.

Thirdly, let $m=5$ and $n=4$, then $\frac{5}{5-2}$ being greater than $\frac{3}{2}$, there will be formed a solid having twelve solid angles, twenty plane triangular faces, and thirty edges: this solid is therefore called an Icosahedron.

If $m=5$, or any larger number, and $n$ be greater than 3 , the specified condition cannot be fulfilled, and thence it appears that there can exist five, and only five regular polyhedrons.

By supposing $2(m+n)-m n=0$, we find the values of $x, y, z$ indefinitely great: and a sphere may be considered as a regular polyhedron of an infinite number of solid angles, \&c.
70. To find the inclination of two contiguous faces of a regular polyhedron to one another.

Let $A B$ be the side common to the two contiguous faces, $C$ and $E$ being their centres, from which let $C D$ and $E D$ be drawn perpendicular to it, then will the angle contained between $C D$ and $E D$ be the inclination of these two faces to each other: in the plane in which $C D$ and $E D$ lie, let
$C O$ and $E O$ be drawn perpendicular to them respectively and meeting in $O$, join $O A, O B, O D$, and from $O$ as a centre

suppose a spherical surface to be described cutting the lines $O A, O C, O D$ in the points $p, q, r$, and let these points be joined by arcs of great circles: then it is manifest that the angle $p r q$ is a right angle: hence retaining the notation before used, we shall have

$$
\angle q p r=\frac{\pi}{m}, \text { and } \angle p q r=\frac{\pi}{n}:
$$

now by Napier's Rules, the spherical triangle $p q r$ gives

$$
\begin{gathered}
\cos q p r=\sin p q r \cos q r, \text { and } \therefore \cos q r=\frac{\cos \frac{\pi}{m}}{\sin \frac{\pi}{n}} ; \\
\text { but } \cos q r=\cos C O D=\sin C D O=\sin \frac{C D E}{2} ; \\
\therefore \sin \frac{C D E}{2}=\frac{\cos \frac{\pi}{m}}{\sin \frac{\pi}{n}},
\end{gathered}
$$

from which the required angle $C D E$ may be found.
(1) In the Tetrahedron $m=3$ and $n=3$,

$$
\therefore \sin \frac{C D E}{2}=\frac{\cos 60^{\circ}}{\sin 60^{\circ}}=\frac{1}{\sqrt{3}}, \text { and } \cos C D E=\frac{1}{3} \text {. }
$$

(2) In the Hexahedron $m=3$ and $n=4$,
$\therefore \sin \frac{C D E}{2}=\frac{\cos 60^{\circ}}{\sin 45^{\circ}}=\frac{1}{\sqrt{2}}$, and $\cos C D E=0$, or $C D E=90^{\circ}$.
(3) In the Octahedron $n=4$ and $n=3$,
$\therefore \sin \frac{C D E}{2}=\frac{\cos 45^{\circ}}{\sin 60^{\circ}}=\sqrt{\frac{2}{3}}$, and $\cos C D E=-\frac{1}{3}$.
(4) In the Dodecahedron $m=3$ and $n=5$,
$\therefore \sin \frac{C D E}{2}=\frac{\cos 60^{\circ}}{\sin 36^{\circ}}=\frac{2}{\sqrt{10-2 \sqrt{5}}}$, and $\cos C D E=\frac{1-\sqrt{ } 5}{5-\sqrt{5}}$.
(5) In the Icosahedron $m=5$ and $n=3$,
$\therefore \sin \underset{2}{C D E}=\frac{\cos 36^{\circ}}{\sin 60^{\circ}}=\frac{1+\sqrt{ } 5}{2 \sqrt{3}}$, and $\cos C D E=-\frac{\sqrt{5}}{3}$.
71. To find the radii of the spheres inscribed in and circumscribed about a regular polyhedron.

Retaining the construction and notation of the last article, it is manifest that $C O$ and $E O$ are respectively perpendicular to the planes $A B C, A B E$ and equal to each other; and the same being true of any other two contiguous faces, it follows that $C$ is the centre of the inscribed and circumscribed spheres whose radii are $O C$ and $O A$ respectively :
now $\frac{O C}{O A}=\cos p q=\cot q p r \cot p q r=\cot \frac{\pi}{m} \cot \frac{\pi}{n}$, by Napier's Rules: but if a side of one of the faces $=a$, we have from (204)

$$
C A=\frac{a}{2 \sin \frac{\pi}{n}} \text {, and } \therefore O A^{2}=O C^{2}+\frac{a^{2}}{4 \sin ^{2} \frac{\pi}{n}} ;
$$

wherefore $r$ and $R$ representing these radii, we have the two following equations:

$$
\frac{r}{R}=\cot \frac{\pi}{m} \cot \frac{\pi}{n}, \text { and } R^{2}-r^{2}=\frac{a^{2}}{4 \sin ^{2} \frac{\pi}{n}}
$$

to determine their values.
72. Cor. If the angle $C D E$ be found as in the last article but one, we shall manifestly have

$$
r=\frac{a}{2} \cot \frac{\pi}{n} \tan \frac{C D E}{2} \text { and } R=\frac{a}{2} \tan \frac{\pi}{m} \tan \frac{C D E}{2}
$$

Ex. 1. In the Tetrahedron $m=3$ and $n=3$,

$$
\therefore \frac{r}{R}=\cot 60^{\circ} \cot 60^{\circ}=\frac{1}{3}, \text { whence } R=3 r:
$$

also $R^{2}-r^{2}=\frac{a^{2}}{4 \sin ^{2} 60^{\circ}}=\frac{a^{2}}{3}$, from which are obtained

$$
r=\frac{a}{2 \sqrt{6}} \quad \text { and } R=\frac{3 a}{2 \sqrt{6}} .
$$

Ex.2. In the Hexahedron and Octahedron, since in the former $m=3$ and $n=4$, and in the latter $m=4$ and $n=3$, we shall have $\frac{r}{R}=\cot 60^{\circ} \cot 45^{\circ}=\frac{1}{\sqrt{3}} ;$ which shews that if these two solids were inscribed in one sphere, they might both be circumscribed about another sphere, and the contrary: also, in the former it is easily proved that $r=\frac{a}{2}, R=\frac{a}{2} \sqrt{3}$, in the latter $r=\frac{a}{\sqrt{6}}$ and $R=\frac{a}{\sqrt{Q}}$.

Ex. 3. In the Dodecahedron and Icosahedron $m=3, n=5$, and $m=5, n=3$ respectively,

$$
\therefore \frac{r}{R}=\cot 60^{\circ} \cot 36^{\circ}=\sqrt{\frac{5+2 \sqrt{5}}{15}}
$$

and the same remark may be made as in the last example :

$$
\text { hence also in the former } r=\frac{a}{20} \sqrt{250+110 \sqrt{5}}
$$

$$
\begin{gathered}
R=\frac{a}{4}(\sqrt{ } 15+\sqrt{ } 3) ; \text { in the latter } \\
r=\frac{a}{12} \sqrt{42+18 \sqrt{ } 5}, \quad R=\frac{a}{4} \sqrt{10+2 \sqrt{ } 5} .
\end{gathered}
$$

73. To find the content of a regular polyhedron.

From the centre $O$ of the inscribed sphere, let straight lines $O A, O B, O C, \& c$. be drawn to all the angular points, then will the polyhedron be divided into as many pyramids with equilateral bases as there are plane faces, and whose altitudes are each equal to the radius of the sphere $r$ : now if we retain the notation which we have before adopted, we shall by (201) have the area of each face $=\frac{n a^{2}}{4 \tan \frac{\pi}{n}}$;

$$
\text { and therefore the whole surface }=\frac{n a^{2} y}{4 \tan \frac{\pi}{n}} \text {, }
$$

whence the content $=\frac{1}{3}$ whole surface $\times r=\frac{n a^{2} r y}{12 \tan \frac{\pi}{n}}$.
Ex. In the Tetrahedron $n=3, y=4, r=\frac{a}{2 \sqrt{6}}$, and $\tan \frac{\pi}{n}=\tan 60^{\circ}=\sqrt{ } 3 ;$
$\therefore$ (1) the content of the Tetrahedron $=\frac{a^{3}}{2 \sqrt{18}}=\frac{a^{3}}{12} \sqrt{ } 2$. Similarly,
(2) the content of the Hexahedron $=a^{3}$ :
(3) $\ldots \ldots \ldots \ldots \ldots$ Octahedron $=\frac{a^{3}}{3} \sqrt{ } 2$ :
(4) $\ldots \ldots \ldots \ldots \ldots$ Dodecahedron $=\frac{a^{3}}{4} \sqrt{470+210 \sqrt{5}}$ :
(5) Icosahedron $=\frac{5 a^{3}}{12} \sqrt{14+6 \sqrt{5}}$.

By means of the last proposition there will be no difficulty in expressing the content in terms of the radii of the inscribed or circumscribed spheres.
74. Given the three edges of a parallelopiped which meet, and the angles included between them, to find its perpendicular altitude.

Let $O A, O B, O C$, the three edges of a parallelopiped meeting at the same angle be represented by $a, b, c$; draw $C D$ perpendicular to the plane $A O B$, join $O D$, and let the angles $A O B, A O C, B O C$ be called $a, \beta, \gamma$ respectively, then

$C D=O C \sin C O D$; now to find the sine of the angle $C O D$, with centre $O$ and radius $=1$, suppose the surface of a sphere to be described cutting the edges $O A$ and $O C$ in $p$ and $q$, and $O D$ in $r$; then if $2 S=\alpha+\beta+\gamma$, we shall manifestly have $\sin C O D=\sin q r=\sin p q \sin q p r$, by Napier's Rules,

$$
\begin{aligned}
& =\sin \beta \frac{2}{\sin \alpha \sin \beta} \sqrt{\sin S \sin (S-\alpha) \sin (S-\beta) \sin (S-\gamma)} \\
& =\frac{2}{\sin \alpha} \sqrt{\sin S \sin (S-\alpha) \sin (S-\beta) \sin (S-\gamma)},
\end{aligned}
$$

whence $C D=\frac{\alpha c}{\sin \alpha} \sqrt{\sin S \sin (S-\alpha) \sin (S-\beta) \sin (S-\gamma)}$.
75. Cor. The whole surface of the parallelopiped will manifestly $=2\{a b \sin a+a c \sin \beta+b c \sin \gamma\}$.
76. On the same hypothesis, to find the content of the parallelopiped.

The content $=$ area of the base $\times$ the perpendicular altitude

$$
\begin{aligned}
& =O A . O B \sin A O B \cdot C D \\
& =a b \sin a \frac{2 c}{\sin \alpha} \sqrt{\sin S \sin (S-\alpha) \sin (S-\beta) \sin (S-\gamma)} \\
& =2 a b c \sqrt{\sin S \sin (S-\alpha) \sin (S-\beta) \sin (S-\gamma)}
\end{aligned}
$$

77. On the same supposition, to find the diagonal of the parallelopiped.

Let $2 O D$ (which, if the figure were completed, would be the diagonal of one of the faces) $=d$, then

$$
d^{2}=a^{2}+b^{2}+2 a b \cos a:
$$

also if $D$ be the diagonal of the parallelopiped, we shall easily perceive that
$D^{2}=d^{2}+c^{2}+2 c d \cos C O D=a^{2}+b^{2}+c^{2}+2 a b \cos \alpha+2 c d \cos C O D:$ now the angle $C O D$ is manifestly measured by the arc $q r$, and $\cos q r=\cos p q \cos p r+\sin p q \sin p r \cos q p r$

$$
\begin{aligned}
& =\cos \beta \cos p r+\sin \beta \sin p r\left\{\frac{\cos \gamma-\cos \alpha \cos \beta}{\sin \alpha \sin \beta}\right\} \\
& =\cos \beta \cos p r+\sin p r\left\{\frac{\cos \gamma-\cos \alpha \cos \beta}{\sin \alpha}\right\} \\
& =\frac{\cos \gamma \sin p r}{\sin \alpha}+\frac{\cos \beta(\sin \alpha \cos p r-\cos \alpha \sin p r)}{\sin a} \\
& =\frac{\cos \gamma \sin p r}{\sin \alpha}+\frac{\cos \beta \sin (\alpha-p r)}{\sin \alpha} \\
& =\cos \gamma \frac{\sin A O D}{\sin \alpha}+\cos \beta \frac{\sin B O D}{\sin \alpha}:
\end{aligned}
$$

but in the parallelogram whose diagonal is $2 O D$, it is evident that

$$
\frac{\sin A O D}{\sin a}=\frac{O B}{2 O D}=\frac{b}{d}, \text { and } \frac{\sin B O D}{\sin a}=\frac{O A}{2 O D}=\frac{a}{d}
$$

$$
\therefore \cos q r \text { or } \cos C O D=\frac{a}{d} \cos \beta+\frac{b}{d} \cos \gamma ;
$$

whence by substitution we get

$$
D=\sqrt{a^{2}+b^{2}+c^{2}+2 a b \cos a+2 a c \cos \beta+2 b c \cos \gamma} .
$$

78. Cor. By proper substitutions in the last expression, the other diagonals are determined : and thence it may easily be shewn that the sum of the squares of the four diagonals of a parallelopiped is equal to the sum of the squares of the twelve edges.
79. Given three edges of a triangular pyramid which meet, and the angles which they make with each other, to find its content.

Since by Euclid xi1. 7, every prism having a triangular base may be divided into three pyramids that have triangular bases and are equal to one another, it follows that the content of the triangular pyramid will $=\frac{1}{6}$ of the corresponding parallelopiped: that is, retaining the notation of the last article, we shall have the content of the triangular pyramid

$$
=\frac{a b c}{3} \sqrt{\sin S \sin (S-\alpha) \sin (S-\beta) \sin (S-\gamma)}
$$

80. Cor. The sum of the areas of the three sides of the pyramid manifestly $=\frac{1}{2}\{a b \sin \alpha+a c \sin \beta+b c \sin \gamma\}$.
81. Given the six edges of a triangular pyramid, to find its content.

Retaining the notation of the preceding articles, and in addition thereto representing the sides $A B, A C, B C$ of the base by $h, k$ and $l$ respectively, we have

$$
\cos \alpha=\frac{a^{2}+b^{2}-h^{2}}{2 a b}, \cos \beta=\frac{a^{2}+c^{2}-k^{2}}{2 a c}, \cos \gamma=\frac{b^{2}+c^{2}-l^{2}}{2 b c} ;
$$

bence the content $=\frac{a b c}{3} \sqrt{\sin S \sin (S-\alpha) \sin (S-\beta) \sin (S-\gamma)}$

$$
\begin{aligned}
& =\frac{a b c}{6} \sqrt{1-\cos ^{2} a-\cos ^{2} \beta-\cos ^{2} \gamma+2 \cos \alpha \cos \beta \cos \gamma,} \operatorname{asin}(61) \\
& =\frac{a b c}{6} \sqrt{1-\frac{H^{2}}{4 a^{2} b^{2}}-\frac{K^{2}}{4 a^{2} c^{2}}-\frac{L^{2}}{4 b^{2} c^{2}}+\frac{H K L}{4 a^{2} b^{2} c^{2}}} \\
& =\frac{1}{12} \sqrt{4 a^{2} b^{2} c^{2}-c^{2} H^{2}-b^{2} K^{2}-a^{2} L^{2}+H K L}
\end{aligned}
$$

the quantities $H, K, L$ representing the numerators of the values of $\cos \alpha, \cos \beta$ and $\cos \gamma$ respectively.
82. We shall conclude this chapter with a few remarks upon certain consequences resulting from Euler's Theorem demonstrated in (65), extracted and slightly altered from Note 8, of the Elements de Geometrie of $M$. Legendre, to which work the student is further referred for the geometrical construction of the regular polyhedrons treated of in the preceding part of the chapter.
(1) Let $y_{3}$ be the number of triangles, $y_{4}$ the number of quadrilaterals, $y_{5}$ the number of pentagons, \&c. which form the surface of a polyhedron: then the whole number of faces is

$$
y_{3}+y_{4}+y_{5}+y_{6}+\& c
$$

and the whole number of their sides is

$$
3 y_{3}+4 y_{4}+5 y_{5}+6 y_{6}+8 c
$$

which is also twice the number of edges: hence as in (65) if $x$ be the number of solid angles, $y$ the number of plane faces, and $z$ the number of edges, we shall have
$y=y_{3}+y_{4}+y_{5}+y_{6}+\& c .$, and $2 z=3 y_{3}+4 y_{4}+5 y_{5}+6 y_{6}+\& c$.
but since by (65), $x+y=z+2$, we get

$$
2 x=4+2 z-2 y=4+y_{3}+2 y_{4}+3 y_{5}+4 y_{6}+\& c
$$

from which we conclude that the number of faces of a polyhedron having odd numbers of sides is always even.
(2) Since $z=\frac{3 y_{3}}{2}+2 y_{4}+\frac{5 y_{5}}{2}+3 y_{6}+\& c$.

$$
\begin{aligned}
& =\frac{3}{2}\left\{y_{3}+\frac{4 y_{4}}{3}+\frac{5 y_{5}}{3}+2 y_{6}+\& \mathrm{c} .\right\} \\
\text { and } x & =2+\frac{y_{3}}{2}+y_{4}+\frac{3 y_{5}}{2}+2 y_{6}+8 \mathrm{c} \\
& =2+\frac{\mathrm{I}}{2}\left\{y_{3}+2 y_{4}+3 y_{5}+4 y_{6}+\& \mathrm{c} .\right\}
\end{aligned}
$$

it follows that $z$ cannot be less than $\frac{3}{2} y$, nor $x$ less than $2+\frac{1}{2} y$.
Also the whole number of plane angles being $2 z$, and the number of solid angles $x$, it is manifest that the mean number of plane angles constituting each solid angle $=\frac{2 z}{x}=\frac{4 z}{2 x}$

$$
=\frac{2\left(3 y_{3}+4 y_{4}+5 y_{5}+6 y_{6}+8 \mathrm{sc} .\right)}{4+y_{3}+2 y_{4}+3 y_{5}+4 y_{6}+8 c .}
$$

also since no solid angle can be formed by fewer than three plane angles, $\frac{2 z}{x}$ cannot be less than 3 , or $2 z$ cannot be less than $3 x$ : hence
$3 y_{3}+4 y_{4}+5 y_{5}+6 y_{6}+\& c$ cannot be less than

$$
6+\frac{3 y_{3}}{2}+3 y_{4}+\frac{9 y_{5}}{2}+6 y_{6}+\& c
$$

$$
\text { nor } 6 y_{3}+8 y_{4}+10 y_{5}+12 y_{6}+\& c
$$

less than $12+3 y_{3}+6 y_{4}+9 y_{5}+12 y_{6}+\& c .$, and therefore $3 y_{3}+2 y_{4}+y_{5}$ cannot be less than

$$
12+y_{7}+2 y_{8}+3 y_{9}+\& c
$$

from which we learn that $y_{3}, y_{4}, y_{5}$ cannot all be zero at the same time; or in other words, that there cannot exist a polyhedron all of whose faces have more than five sides.

From what has been proved, it appears that $y$ cannot be less than

$$
4+\frac{y_{4}+2 y_{5}+3 y_{6}+8 c}{3}
$$

and therefore $x$ not less than

$$
4+\frac{2 y_{4}+4 y_{5}+6 y_{6}+\& c}{3}
$$

and consequently $z$ not less than

$$
6+y_{4}+9_{5}+3 y_{6}+\& c
$$

also since $3 y$ is not less than $12+y_{4}+2 y_{5}+3 y_{6}+8 c$. we shall evidently have $2 y$ not less than $x+4$, and $3 y$ not less than $z+6$, for all polyhedrons whatever.
(3) If we suppose $2 z$ greater than $4 x$, so that all the solid angles shall be formed by four or more plane angles, we shall manifestly have

$$
y_{3}+y_{4}+y_{5}+y_{6}+\& c . \text { not less than } 8+y_{4}+2 y_{5}+3 y_{6}+\& c
$$

or $y_{3}$ not less than $8+y_{5}+2 y_{6}+3 y_{7}+\& c$. and therefore we infer that such a polyhedron must have at least eight triangular faces.

This likewise gives $x$ not less than

$$
6+y_{4}+2 y_{5}+3 y_{6}+8 c
$$

and $z$ not less than

$$
12+2 y_{4}+4 y_{5}+8 y_{6}+\& c
$$

from which it follows that $x$ is not greater than $y-2$, and $z$ not greater than $2 y-4$.
(4) Let $2 z$ be greater than $5 x$, so that all the solid angles shall be formed by at least 5 plane angles, then we have $y_{3}+y_{4}+y_{5}+y_{6}+\& c$. not less than $20+3 y_{4}+6 y_{5}+9 y_{6}+\& c$. and $\therefore y_{3}$ not less than $20+2 y_{4}+5 y_{5}+8 y_{6}+\& c$.
and thence it appears that the solid must have at least twenty triangular faces.

Also on the same hypothesis $x$ cannot be less than

$$
12+2 y_{4}+4 y_{5}+6 y_{6}+\& c .
$$

nor $z$ less than

$$
30+5 y_{4}+10 y_{5}+15 y_{6}+\& c
$$

from whence we conclude that $x$ cannot be greater than $\frac{2}{3}(y-2)$, nor $z$ greater than $\frac{5}{3}(y-2)$.
(5) Since $2 z+2 y_{4}+4 y_{5}+6 y_{6}+8 \mathrm{c} .+12$

$$
\begin{aligned}
& =3 y_{3}+4 y_{4}+5 y_{5}+6 y_{6}+\& c .+2 y_{4}+4 y_{5}+6 y_{6}+\& \mathrm{c} .+12 \\
& =12+3 y_{3}+6 y_{4}+9 y_{5}+12 y_{6}+\& c . \\
& =6\left\{2+\frac{y_{3}}{2}+y_{4}+\frac{3}{2} y_{5}+2 y_{6}+\& c .\right\}=6 x
\end{aligned}
$$

therefore it is evident that $2 z$ must always be less than $6 x$; or in other words, that there cannot exist a polyhedron, all of whose solid angles are constituted by six or more plane angles: and in fact six angles of equilateral triangles $=360^{\circ}$ which exceeds the sum of the plane angles forming any solid angle whatever.
(6) If all the faces of a polyhedron be triangular, we have $y_{4}+2 y_{5}+3 y_{6}+\& c .=0$, and thence we find that $z=\frac{3}{2} y$, and $x=2+\frac{1}{2} y$.

If all the solid angles of a polyhedron be formed by five and six plane angles, the number of the former being $x_{5}$, and of the latter $x_{6}$, then $x=x_{5}+x_{6}$, and $2 z=5 x_{5}+6 x_{6}$, whence $6 x-2 z=x_{5}$ : also $z=\frac{3}{2} y$, and $x=2+\frac{1}{2} y$, from which it follows that $x_{5}=6 x-2 z=12$, and this shews us that the number of solid angles formed by five plane angles will always be twelve, and $x_{6}$ being indeterminate proves that the number formed by six may be any whatever.

## CHAP. VI.

## On the Variations of the Sides and Angles of Spherical Triangles.

83. In the practical applications of Spherical Trigonometry to philosophical subjects, wherein certain parts of spherical triangles are determined from instrumental observation, and the remaining parts deduced from them by arithmetical or logarithmic calculation, it is manifest that the effect of any error however small in the observed part or parts, will be entailed upon the results as determined by computation. If then we suppose the instrumental or original error to be of given magnitude, we may by means of the Calculus of Finite Differences or by Taylor's Theorem be enabled to determine what relation the resulting error bears to it, when these errors are of considerable magnitude, and the operation of Differentiation will be sufficient for the same purpose when they are very small. Thus, since in every spherical triangle there are six distinct parts, any three of which would be sufficient for the determination of all the rest, we may suppose two of them to remain constant, and then find the ratios of the simultaneous changes of all the rest. We will illustrate these principles by an example of each method, and then proceed to the consideration of such particular cases as are most commonly met with in practice.
84. Assume the equation $\cos A=\frac{\cos a-\cos b \cos c}{\sin b \sin c}$, and suppose that in consequence of $a$ becoming $a+\Delta a, A$ becomes $A+\Delta A$, whilst $b, c$ remain constant, then we have

$$
\cos (A+\Delta A)=\frac{\cos (a+\Delta a)-\cos b \cos c}{\sin b \sin c}
$$

$\therefore$ by subtraction, $\cos (A+\Delta A)-\cos A=\frac{\cos (a+\Delta a)-\cos a}{\sin b \sin c}$,

$$
\text { or } \sin \left(A+\frac{\Delta A}{2}\right) \sin \frac{\Delta A}{2}=\frac{\sin \left(a+\frac{\Delta a}{2}\right) \sin \frac{\Delta a}{2}}{\sin b \sin c} ;
$$

and if one of the quantities $\Delta a, \Delta A$ were given, the other might be determined by the solution of this equation ; but it may be observed that it would be no easy matter to disentangle either $\Delta A$ or $\Delta a$ from the other quantities with which they are combined.

Again, since Taylor's Theorem gives
$\cos (A+\Delta A)-\cos A=-\sin A \Delta A-\cos A \frac{(\Delta A)^{2}}{1.2}+\& c$.
and $\cos (a+\Delta a)-\cos a=-\sin a \Delta a-\cos a \frac{(\Delta a)^{2}}{1.2}+\& \mathrm{c} .$,
$\sin A \Delta A+\cos A \frac{(\Delta A)^{2}}{1.2}-\& c .=\frac{\sin a \Delta a+\cos a \frac{(\Delta a)^{2}}{1.2}-\& c .}{\sin b \sin c}$,
in each side of which the number of terms is indefinite, and consequently in this form the difficulty of determining the ratio $\frac{\Delta A}{\Delta a}$ is in no degree diminished, and recourse must finally be had to some such method as approximation.

If however any dependance can be placed upon the accuracy of instrumental observations, it will follow that the errors above alluded to are very small quantities, so small indeed, that they may be neglected when connected with finite quantities by the operations of addition or subtraction, or at most, that one or two terms of such expressions as those above given will ensure a sufficient degree of correctness.

Thus, on the first hypothesis, we obtain
$\sin A \sin \frac{\Delta A}{2}=\frac{\sin a \sin \frac{\Delta a}{2}}{\sin b \sin c}$, and $\therefore \frac{\sin \frac{\Delta A}{2}}{\sin \frac{\Delta a}{2}}=\frac{\sin a}{\sin b \sin c \sin A}$,

or $\frac{\Delta A}{\Delta a}=\frac{\sin a}{\sin b \sin c \sin A}$,
since the arc and sine are ultimately equal: and this is the same result as would be obtained by retaining on each side only the first terms of the expansious given by T'aylor's Theorem.

A greater degree of accuracy will however be ensured on the second hypothesis by retaining two terms of the expansions, and rejecting all the powers of the increments above the second, so that

$$
\sin A \Delta A+\cos A \frac{(\Delta A)^{2}}{1.2}=\frac{\sin a \Delta a+\cos a \frac{(\Delta a)^{2}}{1.2}}{\sin b \sin c}
$$

from which the value of $\frac{\Delta A}{\Delta a}$ may manifestly be obtained by the solution of a quadratic; and if three or more terms were retained on each side, the ratio $\frac{\Delta A}{\Delta a}$ might be still more accurately found by the solution of an equation of three or more dimensions.

Reverting to the equation $\frac{\Delta A}{\Delta a}=\frac{\sin a}{\sin b \sin c \sin A}$, we observe that it is immediately derived from the proposed one by the operation of differentiation; that is, replacing $\Delta$ by $d$ we have

$$
\frac{d A}{d a}=\frac{\sin a}{\sin b \sin c \sin A}
$$

and for the reasons above assigned, the ratio of the differentials may in all practical cases be substituted for the ratio of the errors introduced.
85. Let $A B C$ be a right-angled triangle whose sides and angles are $a, b, c, A, B, C$ respectively; then if any one of
these quantities remain constant, we may find the ratios of the small contemporaneous increments of the rest.

We shall divide this into the three following cases:
(1) When $a$ is constant:
(2) When $c$ is constant:
(3) When $\boldsymbol{A}$ is constant.
86. Let one of the legs a remain constant.

Since $\cos A=\cos a \sin B$, we get
$-\sin A d A=\cos a \cos B d B ;$
$\therefore \frac{d A}{d B}=-\frac{\cos a \cos B}{\sin A}=-\frac{\cos A \cos B}{\sin B \sin A}=-\frac{\cot B}{\tan A} \ldots$ (1).
Since $\sin b=\tan a \cot A$, we shall have

$$
\begin{gather*}
\cos b d b=-\tan a \operatorname{cosec}^{2} A d A \\
\therefore \frac{d A}{d b}=-\frac{\cos b}{\tan a \operatorname{cosec}^{2} A} \\
=-\frac{\cos b}{\sin b \tan A \operatorname{cosec}^{2} A}=-\frac{\sin 2 A}{2 \tan b} . \tag{2}
\end{gather*}
$$

Since $\sin a=\sin c \sin A, \therefore \sin c=\sin a \operatorname{cosec} A$,
whence $\cos c d c=-\sin a \operatorname{cosec} A \cot A d A ;$
$\therefore \frac{d A}{d c}=-\frac{\cos c}{\sin a \operatorname{cosec} A \cot A}=-\frac{\cos c}{\sin c \cot A}=-\frac{\tan A}{\tan c} \ldots$ (3)
Since $\sin a=\tan b \cot B, \therefore \tan b=\sin a \tan B$, whence $\sec ^{2} b d b=\sin a \sec ^{2} B d B ;$
$\therefore \frac{d B}{d b}=\frac{\sec ^{2} b}{\sin a \sec ^{2} B}=\frac{\sec ^{2} b}{\tan b \cot B \sec ^{2} B}=\frac{\sin 2 B}{\sin 2 b}$.
Since $\cos c=\cos a \cos b, \therefore$ we have

$$
\begin{gathered}
\sin c d c=\cos a \sin b d b \\
\therefore \frac{d c}{d b}=\frac{\cos a \sin b}{\sin c}=\frac{\cos c \sin b}{\cos b \sin c}=\frac{\tan b}{\tan c} \ldots \ldots \ldots \text { (5). } \\
\text { Oo }
\end{gathered}
$$

87. Let the hypothenuse c be considered constant.

Since $\cos c=\cot A \cot B, \therefore \cot A=\cos c \tan B$, whence $-\operatorname{cosec}^{2} A d A=\cos c \sec ^{2} B d B ;$
$\therefore \frac{d A}{d B}=-\frac{\cos c \sec ^{2} B}{\operatorname{cosec}^{2} A}=-\frac{\cot A \cot B \sec ^{2} B}{\operatorname{cosec}^{2} A}=-\frac{\sin 2 A}{\sin 2 B} \ldots$ (1),
Since $\sin a=\sin c \sin A$, we obtain

$$
\cos a d a=\sin c \cos A d A
$$

$$
\therefore \frac{d A}{d a}=\frac{\cos a}{\sin c \cos A}=\frac{\cos a \sin A}{\sin a \cos A}=\frac{\tan A}{\tan a} \ldots \ldots \ldots(2) .
$$

Since $\cos A=\tan b \cot c$, we have

$$
-\sin \Lambda d A=\cot c \sec ^{2} b d b
$$

$$
\therefore \frac{d A}{d b}=-\frac{\sec ^{2} b \cot c}{\sin A}=-\frac{\sec ^{2} b \cos A}{\sin A \tan b}=-\frac{2 \cot A}{\sin 2 b} \ldots(3)
$$

Since $\cos c=\cos a \cos b$, we have $\cos a=\cos c \sec b$, whence $-\sin a d a=\cos c \sec b \tan b d b$,
$\therefore \frac{d a}{d b}=-\frac{\cos c \sec b \tan b}{\sin a}=-\frac{\cos a \sin b}{\sin a \cos b}=-\frac{\tan b}{\tan a} \ldots \ldots$ (4).
88. Let one of the angles A remain constant.

Since $\cos A=\cos a \sin B$, we have $\sin B=\cos A \sec a$, whence $\cos B d B=\cos A \sec a \tan a d a ;$
$\therefore \frac{d B}{d a}=\frac{\cos A \sec a \tan a}{\cos B}=\frac{\cos a \sin B \sec a \tan a}{\cos B}=\frac{\tan B}{\cot a}$. (1).
Since $\cos B=\cos b \sin A$, we get $\sin B d B=\sin A \sin b d b$;

$$
\therefore \frac{d B}{d b}=\frac{\sin b \sin A}{\sin B}=\frac{\sin b \cos B}{\sin B \cos b}=\frac{\tan b}{\tan B} \ldots \ldots \text { (2). }
$$

Since $\cos c=\cot A \cot B$, we have $\sin c d c=\cot A \operatorname{cosec}^{2} B d B$;

$$
\begin{equation*}
\therefore \frac{d B}{d c}=\frac{\sin c}{\cot A \operatorname{cosec}^{2} B}=\frac{\sin c \cot B}{\cos c \operatorname{cosec}^{2} B}=\frac{\sin 2 B}{2 \cot c} . \tag{3}
\end{equation*}
$$

Since $\sin b=\tan a \cot A$, we have

$$
\begin{gathered}
\cos b d b=\cot A \sec ^{2} a d a ; \\
\therefore \frac{d a}{d b}=\frac{\cos b}{\cot A \sec ^{2} a}=\frac{\cos b \tan a}{\sin b \sec ^{2} a}=\frac{\sin 2 a}{2 \tan b} \ldots \ldots \text { (4). }
\end{gathered}
$$

Since $\sin a=\sin c \sin A$, we have $\cos a d a=\sin A \cos c d c$;

$$
\therefore \frac{d a}{d c}=\frac{\sin A \cos c}{\cos a}=\frac{\sin a \cos c}{\cos a \sin c}=\frac{\tan a}{\tan c} \ldots \ldots(5) .
$$

89. Thus the ratio of the evanescent increments has been determined on each of the suppositions above made, and the last three articles include all the different cases that can occur in right-angled spherical triangles, the right angle not being supposed to undergo any change; and it may be observed that the two parts for which this ratio has been found increase or decrease at the same time when the differential coefficient is positive, and the contrary when it is negative.

Moreover, if the small change in any one of the parts be considered given and constant, the greatest or least values of the contemporaneous small changes in any of the rest may be determined by putting the corresponding differential coefficient equal to zero, according to the principles of Maxima and Minima.
90. Let $A B C$ be any spherical triangle whatever whose sides and angles are $a, b, c, A, B, C$ respectively: then the consideration of the corresponding small changes in the parts may be comprehended in the four following cases:
(1) When $A$ and $a$ are constant:
(2) When $A$ and $b$ are constant:
(3) When $a$ and $b$ are constant:
(4) When $A$ and $B$ are constaut.
91. Let the angle A and its opposite side a remain constant. Since $\cos a=\frac{\cos A+\cos B \cos C}{\sin B \sin C}$, we have $\cos a \sin B \sin C=\cos A+\cos B \cos C ;$

$$
\begin{aligned}
& \therefore \frac{d B}{d C}=-\frac{\sin B \cos C \cos a+\cos B \sin C}{\cos B \sin C \cos a+\sin B \cos C} \\
& \text { but } \sin B \cos C \cos a=\frac{\cos A+\cos B \cos C}{\tan C} \\
& \text { and } \cos B \sin C \cos a=\frac{\cos A+\cos B \cos C}{\tan B}
\end{aligned}
$$

therefore by substitution we obtain

$$
\begin{gathered}
\frac{d B}{d C}=-\frac{\sin B}{\sin C}\left\{\frac{\cos B+\cos A \cos C}{\cos C+\cos A \cos B}\right\}=-\frac{\cos b}{\cos c} \ldots(1) . \\
\text { Since } \frac{\sin B}{\sin b}=\frac{\sin A}{\sin a}, \text { we have } \sin B=\frac{\sin A}{\sin a} \sin b ; \\
\therefore \cos B d B=\frac{\sin A}{\sin a} \cos b d b, \text { whence } \\
\frac{d B}{d b}=\frac{\sin A}{\sin a} \frac{\cos b}{\cos B}=\frac{\sin B \cos b}{\cos B \sin b}=\frac{\tan B}{\tan b} \ldots \ldots(2) . \\
\text { Since } \cos A=\frac{\cos a-\cos b \cos c}{\sin b \sin c},
\end{gathered}
$$

we have $\sin b \sin c \cos A=\cos a-\cos b \cos c$; whence

$$
\frac{d b}{d c}=-\frac{\sin b \cos c \cos A-\cos b \sin c}{\cos b \sin c \cos A-\sin b \cos c},
$$

from which by substituting for $\cos A$, its value, and reducing, we get

$$
\begin{equation*}
\frac{d b}{d c}=-\frac{\cos B}{\cos C} . \tag{3}
\end{equation*}
$$

Since $\frac{d B}{d b}=\frac{\tan B}{\tan b}$, and $\frac{d b}{d c}=-\frac{\cos B}{\cos C}$, we have

$$
\frac{d B}{d c}=-\frac{\tan B}{\tan b} \frac{\cos B}{\cos C}=-\frac{\sin B}{\tan b \cos C} \ldots \ldots(4)
$$

92. Let the angle A and its adjacent side b remain constant.

$$
\text { Since } \cos b=\frac{\cos B+\cos A \cos C}{\sin A \sin C}
$$

$$
\begin{aligned}
& \text { we have } \sin A \cos b \sin C=\cos B+\cos A \cos C \\
& \begin{aligned}
\therefore \frac{d B}{d C} & =-\frac{\sin A \cos b \cos C+\cos A \sin C}{\sin B} \\
& =-\frac{(\cos B+\cos A \cos C) \cos C+\cos A \sin ^{2} C}{\sin B \sin C}
\end{aligned}
\end{aligned}
$$

by substitution, whence

$$
\begin{equation*}
\frac{d B}{d C}=-\frac{\cos A+\cos B \cos C}{\sin B \sin C}=-\cos a . \tag{1}
\end{equation*}
$$

Since $\frac{\sin B}{\sin b}=\frac{\sin A}{\sin a}$, we have $\sin a \sin B=\sin A \sin b$;

$$
\text { whence } \frac{d B}{d a}=-\frac{\sin B \cos a}{\cos B \sin a}=-\frac{\tan B}{\tan a} \ldots \ldots \ldots \text { (2). }
$$

$$
\text { Since } \cos A=\frac{\cos a-\cos b \cos c}{\sin b \sin c}
$$

$$
\therefore \cos A \sin b \sin c=\cos a-\cos b \cos c
$$

whence $\frac{d a}{d c}=-\frac{\cos A \sin b \cos c-\cos b \sin c}{\sin a}$

$$
=-\frac{(\cos a-\cos b \cos c) \cos c-\cos b \sin ^{2} c}{\sin a \sin c}
$$

by substitution,

$$
\therefore \frac{d a}{d c}=\frac{\cos b-\cos a \cos c}{\sin a \sin c}=\cos B \ldots \ldots \ldots \text { (3). }
$$

Since $\frac{d C}{d B}=-\frac{1}{\cos a}, \frac{d B}{d a}=-\frac{\tan B}{\tan a}$, and $\frac{d a}{d c}=\cos B$, we have

$$
\begin{equation*}
\frac{d C}{d c}=\frac{\tan B}{\tan a} \frac{\cos B}{\cos a}=\frac{\sin B}{\sin a} \tag{4}
\end{equation*}
$$

93. Let the two sides a and b remain constant.

Since $\frac{\sin A}{\sin B}=\frac{\sin a}{\sin b}$, we have $\sin A=\frac{\sin a}{\sin b} \sin B$;

$$
\therefore \frac{d A}{d B}=\frac{\sin a \cos B}{\sin b \cos A}=\frac{\sin A \cos B}{\sin B \cos A}=\frac{\tan A}{\tan B} \ldots(1)
$$

Since $\cos C=\frac{\cos c-\cos a \cos b}{\sin a \sin b}$, we have $\sin C d C=\frac{\sin c d c}{\sin a \sin b}$;

$$
\text { whence } \frac{d C}{d c}=\frac{\sin c}{\sin a \sin b \sin C}=\frac{1}{\sin b \sin A}=\frac{1}{\sin a \sin B} . \text { (2). }
$$

$$
\text { Since } \cos A=\frac{\cos a-\cos b \cos c}{\sin b \sin c}
$$

$$
\therefore \cos A \sin b \sin c=\cos a-\cos b \cos c
$$

$$
\text { whence } \frac{d A}{d c}=\frac{\cos A \sin b \cos c-\cos b \sin c}{\sin b \sin c \sin A}
$$

$$
=\frac{\cos a \cos c-\left(\cos b \cos ^{2} c+\cos b \sin ^{2} c\right)}{\sin b \sin ^{2} c \sin A}, \text { or }
$$

$$
\frac{d A}{d c}=-\frac{\cos B \sin a}{\sin b \sin c \sin A}=-\frac{\cos B}{\sin B \sin c}=-\frac{\cot B}{\sin c} \ldots \ldots \text { (3). }
$$

$$
\text { Since } \frac{d A}{d c}=-\frac{\cot B}{\sin c}, \text { and } \frac{d c}{d C}=\sin a \sin B
$$

$$
\text { we have } \frac{d A}{d C}=-\frac{\cot B \sin a \sin B}{\sin c}=-\frac{\cos B \sin a}{\sin c} \ldots \text { (4). }
$$

94. Let the two angles A and B remain constant.

$$
\text { Since } \cos a=\frac{\cos A+\cos B \cos C}{\sin B \sin C}
$$

we shall have as before

$$
\frac{d C}{d a}=\frac{\sin a \sin B \sin C}{\cos a \sin B \cos C+\cos B \sin C}=\frac{\sin a \sin B \sin ^{2} C}{\cos B+\cos A \cos C}
$$

$$
\text { or } \frac{d C}{d a}=\frac{\sin a \sin B \sin C}{\sin A \cos b}=\frac{\sin b \sin C}{\cos b}=\frac{\sin C}{\cot b} \ldots \ldots \ldots \text { (1). }
$$

$$
\text { Since } \cos c=\frac{\cos C+\cos A \cos B}{\sin \Lambda \sin B}, \therefore \sin c d c=\frac{\sin C d C}{\sin A \sin B}
$$

$$
\text { whence } \frac{d C}{d c}=\frac{\sin A \sin B \sin c}{\sin C}=\sin a \sin B=\sin b \sin A \ldots \text { (2). }
$$

$$
\text { Since } \frac{\sin a}{\sin b}=\frac{\sin A}{\sin B} \text {, we have } \sin a=\frac{\sin A}{\sin B} \sin b \text {; }
$$

$$
\text { whence } \frac{d a}{d b}=\frac{\sin A \cos b}{\sin B \cos a}=\frac{\sin a \cos b}{\sin b \cos a}=\frac{\tan a}{\tan b} \ldots \text { (3). }
$$

Since $\frac{d a}{d C}=\frac{\cot b}{\sin C}$, and $\frac{d C}{d c}=\sin b \sin A$, we shall have

$$
\frac{d a}{d c}=\frac{\cot b \sin A \sin b}{\sin C}=\frac{\cos b \sin A}{\sin C} \ldots \ldots(4)
$$

95. Cor. The ratios which have been deduced in the same manner as the last in the preceding article, might like the rest have been found by an independent process, and all the ratios determined in the last three articles may be expressed in different terms according to the nature of the case in which they are employed: thus in (93), we have seen that

$$
\begin{gathered}
\frac{d A}{d c}=-\frac{\cot B}{\sin c} \\
\text { but by (35) } \cot B=\frac{\cot b \sin c}{\sin A}-\frac{\cos c}{\tan A}, \\
\therefore \frac{d A}{d c}=\frac{\cot c}{\tan A}-\frac{\cot b}{\sin A} .
\end{gathered}
$$

This instance has been selected because it includes the solution of an important astronomical problem, but it is clear that in every one of the other cases similar substitutions might have been made.

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The principles explained in this chapter may with great facility be applied to compare the corresponding small changes in the parts of plane triangles; and indeed the observations made towards the ends of articles (239), (240), (241) and (244) in the Plane Trigonometry, are merely examples of the same principles without introducing the notation of the Differential Calculus.

This subject was first treated of by Roger Cotes, in his tract entitled Estimatio Errorum in mixtâ Mathesi, \&c. which is the first of his Opera Miscellanea, and may be found at the end of the Harmonia Mensurarum.

## CHAP. VII.

Containing some miscellaneous Propositions.
96. To express the sum of the angles of a spherical triangle in terms of the sides.

Since by (58), $\cot \frac{E}{2}=\frac{1+\cos a+\cos b+\cos c}{2 \sqrt{\sin S \sin (S-a) \sin (S-b) \sin (S-c)}}$,
we have $\tan \left(\frac{A+B+C}{2}\right)=\tan \left(\frac{E}{2}+90^{\circ}\right)=-\cot \frac{E}{2}$

$$
=-\frac{1+\cos a+\cos b+\cos c}{2 \sqrt{\sin S \sin (S-a) \sin (S-b) \sin (S-c)}}
$$

97. To express the perimeter of a spherical triangle in terms of the angles.

Resuming the notation of (15) we shall have

$$
\begin{aligned}
& \sin \left(\frac{a+b+c}{Q}\right)=\sin \left\{\frac{\pi-A^{\prime}+\pi-B^{\prime}+\pi-C^{\prime}}{Q}\right\} \\
= & \sin \left\{\frac{3 \pi}{Q}-\frac{A^{\prime}+B^{\prime}+C^{\prime}}{Q}\right\}=-\cos \left(\frac{A^{\prime}+B^{\prime}+C^{\prime}}{Q}\right) \\
= & \frac{\sqrt{-\cos S^{\prime} \cos \left(S^{\prime}-A\right) \cos \left(S^{\prime}-B\right) \cos \left(S^{\prime}-C\right)}}{2 \sin \frac{A}{Q} \sin \frac{B}{Q} \sin \frac{C}{Q}}
\end{aligned}
$$

by substitution and reduction, as in (59).

$$
P_{r}
$$

98. T'o express the excess of the sum of two angles of a spherical triangle above the third in terms of the sides.

Retaining the notation hitherto used, we have

$$
\begin{gathered}
\tan \left(\frac{A+B-C}{2}\right)=\frac{\tan \left(\frac{A+B}{2}\right)-\tan \frac{C}{2}}{1+\tan \left(\frac{A+B}{Q}\right) \tan \frac{\tan \frac{a}{2} \tan \frac{b}{2}+\cos C}{\sin C}}=\frac{1+\cos c-\cos a-\cos b}{\sqrt{\sin S \sin (S-a) \sin (S-b) \sin (S-c)}}
\end{gathered}
$$

by substitution and reduction as in (58).
Similarly, $\tan \left(\frac{a+b-c}{2}\right)$ may be expressed in terms of the angles.
99. To express the perpendicular from an angle of a spherical triangle upon its opposite side, in terms of the sides and angles respectively.

By Napier's Rules, $\sin C D=\sin B C \sin B$

$$
\begin{gathered}
=\sin a \frac{2}{\sin a \sin c} \sqrt{\sin S \sin (S-a) \sin (S-b) \sin (S-c)} \\
\quad=\frac{2 \sqrt{\sin S \sin (S-a) \sin (S-b) \sin (S-c)}}{\sin c}
\end{gathered}
$$

Also, $\sin C D=\sin B \sin B C$
$=\sin B \frac{2}{\sin B \sin C} \sqrt{-\cos S^{\prime} \cos \left(S^{\prime}-A\right) \cos \left(S^{\prime}-B\right) \cos \left(S^{\prime}-C\right)}$

$$
=\frac{2 \sqrt{-\cos S^{\prime} \cos \left(S^{\prime}-A\right) \cos \left(S^{\prime}-B\right) \cos \left(S^{\prime}-C\right)}}{\sin C}
$$

100. To find the position of the pole of the small circle of the sphere, which may be inscribed in a given spherical triangle.

Referring to the figure of (185) Pl. Trig. and supposing all the lines employed to be arcs of great circles instead of straight lines, we have by means of the same construction,

$$
A b=A c, \quad B a=B c, \text { and } C a=C b,
$$

whence $A b+B a+C a=A c+B c+C b$, or $A b+a=c+C b$, $\therefore 2 A b+a=c+A b+C b=b+c$, and $A b=\frac{b+c-a}{2}=S-a$ :
similarly, $B c=S-b$, and $C a=S-c$.
Now by Napier's Rules, we have $\sin A b=\tan o b \cot o A b$, from which if $r$ be the circular radius required, we get

$$
\begin{aligned}
& \tan r=\sin A b \tan o A b=\sin (S-a) \tan \frac{A}{2} \\
& =\sin (S-a) \sqrt{\frac{\sin (S-b) \sin (S-c)}{\sin S \sin (S-a)}} \\
& =\sqrt{\frac{\sin (S-a) \sin (S-b) \sin (S-c)}{\sin S}} \\
& =\frac{\sqrt{\sin S \sin (S-a) \sin (S-b) \sin (S-c)}}{\sin S}
\end{aligned}
$$

Hence also the segments of the sides are found, and thus the position of the pole is determined.
101. To find the position of the pole of the small circle of the sphere, which may be circumscribed about a given spherical triangle.

Referring back to (188), Pl. Trig., and making the same supposition as in the last article, we have
$\angle B A o=\angle A B o, \angle B C o=\angle C B O, \angle A C o=\angle C A o$,
$\therefore$ by addition, $\angle B A o+C=B+\angle C A o$, whence $2 \angle B A o+C=B+\angle C A o+\angle B A o=A+B$,

$$
\text { and } \therefore \angle B A o=\frac{A+B-C}{2}=S^{\prime}-C \text { : }
$$

similarly, $\angle C B o=S^{\prime}-A$, and $\angle A C o=S^{\prime}-B$.
Now by Napier's Rules, we obtain

$$
\begin{gathered}
\cos B A o=\cot A \circ \tan \frac{c}{2}, \text { whence if } A o=R, \\
\text { we have tan } R=\sec \left(S^{\prime}-C\right) \tan \frac{c}{2}: \\
=1+\left\{\frac{1+\cos c-\cos a-\cos b}{\sqrt{1-\cos ^{2} a-\cos ^{2} b-\cos ^{2} c+2 \cos a \cos b \cos c}}\right\}^{2}, \text { by }(98), \\
=\frac{(1+\cos c)(1-\cos a)(1-\cos b)}{2 \sin S \sin (S-a) \sin (S-b) \sin (S-c)}, \text { by reduction, } \\
\quad=\frac{4 \cos ^{2} \frac{c}{2} \sin ^{2} \frac{a}{2} \sin ^{2} \frac{b}{2}}{\sin S \sin (S-a) \sin (S-b) \sin (S-c)}, \\
\therefore \sec \left(S^{\prime}-C\right)=\frac{2 \cos \frac{c}{2} \sin \frac{a}{2} \sin \frac{b}{2}}{\sqrt{\sin S \sin (S-a) \sin (S-b) \sin (S-c)}}, \\
\text { and } \tan R=\frac{2 \sin \frac{a}{2} \sin \frac{b}{2} \sin \frac{c}{2}}{\sqrt{\sin S \sin (S-a) \sin (S-b) \sin (S-c)}}
\end{gathered}
$$

The segments of the angles have been found above, and thus the pole is determined.

The articles of the Plane 'Trigonometry just alluded to, readily show how great is the similarity of the formulæ found in this and the preceding articles to those investigated there: and it is manifest that the methods here pursued would lead immediately to the resulis before obtained.
102. Giien two sides and the included angle of a spherical triangle, to find the angle coittained between the chords of these sides.

Let the two given sides and included angle be $a, b, C$, and let $\alpha, \beta, \gamma$ be the chords of the sides $a, b, c$ respectively, then from (21) we have

$$
\begin{gathered}
\cos c=\sin a \sin b \cos C+\cos a \cos b, \text { that is, } \\
1-2 \sin ^{2} \frac{c}{2}=4 \sin \frac{a}{2} \cos \frac{a}{2} \sin \frac{b}{2} \cos \frac{b}{2} \cos C \\
\quad+\left(1-2 \sin ^{2} \frac{a}{2}\right)\left(1-2 \sin ^{2} \frac{b}{2}\right), \\
\text { or } 1-\frac{\gamma^{2}}{2}=a \beta \cos \frac{a}{2} \cos \frac{b}{2} \cos C+1-\frac{\alpha^{2}}{2}-\frac{\beta^{2}}{2}+\frac{a^{2} \beta^{2}}{4}, \\
\therefore \frac{a^{2}+\beta^{2}-\gamma^{2}}{2}=a \beta \cos \frac{a}{2} \cos \frac{b}{2} \cos C+\frac{a^{2} \beta^{2}}{4},
\end{gathered}
$$

but if $C^{\prime}$ be the angle contained between the chords $a, \beta$, we have

$$
\begin{aligned}
\cos C^{\prime} & =\frac{a^{2}+\beta^{2}-\gamma^{2}}{2 a \beta}=\cos \frac{a}{2} \cos \frac{b}{2} \cos C+\frac{a \beta}{4} \\
& =\cos \frac{a}{2} \cos \frac{b}{2} \cos C+\sin \frac{a}{2} \sin \frac{b}{2} .
\end{aligned}
$$

From this it is not difficult to shew that $C^{\prime}$ is greater than $C$ when it is an acute angle, but less when it is either an obtuse or a right angle.
103. Cor. If $D$ be the pole of the circle circumscribed about the triangle $A B C$, then the angle $A D B$ will be measured

by the are of the circle included between $A$ and $B$ : also the
angle between the chords of $A C, B C$ stands upon the same circumference, and therefore by Euclidini. 20, $C^{\prime}=\frac{1}{2} \angle A D B$ : and hence it also follows that the angle $A D B$ at the centre is greater or less than twice the angle at the circumference according as that angle is acute or obtuse.
104. Given the chords of two sides of a spherical triangle and the included angle, to find the angle contained between the sides.

$$
\begin{aligned}
& \text { Since } \cos C^{\prime}=\cos \frac{a}{2} \cos \frac{b}{2} \cos C+\sin \frac{a}{2} \sin \frac{b}{2} \text {, we have } \\
& \cos C=\frac{\cos C^{\prime}-\sin \frac{a}{2} \sin \frac{b}{2}}{\cos \frac{a}{2} \cos \frac{b}{2}} \\
& =\frac{\cos C^{\prime}-\frac{a \beta}{4}}{\sqrt{\left(1-\frac{a^{2}}{4}\right)\left(1-\frac{\beta^{2}}{4}\right)}}=\frac{4 \cos C^{\prime}-\alpha \beta}{\sqrt{\left(4-a^{2}\right)\left(4-\beta^{2}\right)}}
\end{aligned}
$$

in which the radius of the sphere is supposed to be 1 .
Ex. If $a=b$, and $\therefore a=\beta$, we shall have

$$
\cos C^{\prime}=\cos ^{2} \frac{a}{2} \cos C+\sin ^{2} \frac{a}{2}, \text { and } \cos C=\frac{4 \cos C^{\prime}-a^{2}}{4-a^{2}}
$$

105. Given the oblique angle contained between two given objects above the horizon, to find the horizontal angle.

Let $a$ and $n$ be the objects whose angular distance $a n$ is observed from the point $O$ in the horizon: then if straight lines were drawn from $a$ and $n$ to $O$, the angle $a O n$ would be the oblique angle, and $A O N$ is the corresponding horizontal
angle, which is the same as the spherical angle $A P N$ : let $a n=c, A a=H, N n=h$, then by (25) we have


$$
\sin ^{2} \frac{A P N}{2}=\frac{\sin \left(\frac{c+H-h}{2}\right) \sin \left(\frac{c+h-H}{2}\right)}{\cos H \cos h}
$$

from which the horizontal angle may be found.
106. Given two sides of a spherical triangle very nearly equal to quadrants, to find the difference between the remaining side and the measure of the included angle.

Retaining the notation of the last article,

$$
\text { let } c+\theta=A N=\angle A O N \text { to the radius } 1 \text {, }
$$

$$
\text { then } a=\frac{\pi}{2}-H, \text { and } b=\frac{\pi}{2}-h ;
$$

whence, $H$ and $h$ being very small, we have

$$
\begin{aligned}
& \cos (c+\theta)=\frac{\cos c-\sin H \sin h}{\cos H \cos h}=\frac{\cos c-H h}{\left(1-\frac{H^{2}}{2}\right)\left(1-\frac{h^{2}}{2}\right)} \\
& =\frac{\cos c-H h}{1-\frac{1}{2}\left(H^{2}+h^{2}\right)}=(\cos c-H h)\left\{1+\frac{1}{2}\left(H^{2}+h^{2}\right)\right\} \text { very nearly : } \\
& \text { that is, } \cos c-\theta \sin c=\cos c-H h+\frac{1}{2}\left(H^{2}+h^{2}\right) \cos c \\
& \text { since } \theta \text { is very small, }
\end{aligned}
$$

$$
\therefore \theta=\frac{H h-\frac{1}{2}\left(H^{2}+h^{2}\right) \cos c}{\sin c}
$$

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$$
=\frac{2 H h-\left(H^{2}+h^{2}\right) \cos c}{2 \sin c}
$$

$=\frac{2 H h\left(\cos ^{2} \frac{c}{2}+\sin ^{2} \frac{c}{2}\right)-\left(H^{2}+h^{2}\right)\left(\cos ^{2} \frac{c}{2}-\sin ^{2} \frac{c}{2}\right)}{4 \sin \frac{c}{2} \cos \frac{c}{2}}$

$$
\begin{gathered}
\frac{\left(H^{2}+2 H h+h^{2}\right) \sin ^{2} \frac{c}{2}-\left(H^{2}-2 H h+h^{2}\right) \cos ^{2} \frac{c}{2}}{4 \sin \frac{c}{2} \cos \frac{c}{2}} \\
=\left(\frac{H+h}{2}\right)^{2} \tan \frac{c}{2}-\left(\frac{H-h}{2}\right)^{2} \cot \frac{c}{2}
\end{gathered}
$$

This expression is called the approximate reduction to the horizon, and was first given by $M$. Legendre in 1787.
107. The sides of a spherical triangle being small with respect to the radius of the sphere, it is required to find the angles of a plane triangle whose sides are of the same magnitudes.

If $a, b, c$ be the sides of the proposed triangle to the radius $r$, the sides of a similar triangle to the radius 1 , will manifestly be

$$
\begin{gathered}
\frac{a}{r}, \frac{b}{r}, \frac{c}{r}: \\
\text { now } \cos \Lambda=\frac{\cos \frac{a}{r}-\cos \frac{b}{r} \cos \frac{c}{r}}{\sin \frac{b}{r} \sin \frac{c}{r}}: \\
\text { but } \cos \frac{a}{r}=1-\frac{a^{2}}{1.2 r^{2}}+\frac{a^{4}}{1.2 .3 .4 r^{4}} \text { nearly, } \\
\text { and similarly of } \cos \frac{b}{r} \text { and } \cos \frac{c}{r} ;
\end{gathered}
$$

also $\sin \frac{b}{r}=\frac{b}{r}-\frac{l^{3}}{1.2 \cdot 3 r^{5}}$ nearly, and similarly of $\sin \frac{c}{r}$;
therefore by substitution and reduction we get

$$
\begin{gathered}
\cos A=\frac{b^{2}+c^{2}-a^{2}}{2 b c}+\frac{a^{4}+b^{4}+c^{4}-2\left(a^{2} b^{2}+a^{2} c^{2}+b^{2} c^{2}\right)}{24 b c r^{2}} \\
=\cos A^{\prime}-\frac{b c}{6 r^{2}} \sin ^{2} A^{\prime}
\end{gathered}
$$

if $A^{\prime}$ be the corresponding angle of the plane triangle whose sides are $a, b, c$ :
but if $A=A^{\prime}+\theta$, we have $\cos A=\cos A^{\prime}-\theta \sin A^{\prime}$, nearly;

$$
\begin{gathered}
\therefore \theta=\frac{b c}{6 r^{2}} \sin A^{\prime}=\frac{b c \sin A^{\prime}}{2.3 r^{2}}=\frac{1}{3 r^{2}} \text { area, } \\
\text { whence } A^{\prime}=A-\theta=A-\frac{1}{3 r^{2}} \text { area }
\end{gathered}
$$

similarly $B^{\prime}=B-\theta=B-\frac{1}{3 r^{2}}$ area, and $C^{\prime}=C-\theta=C-\frac{1}{3 r^{2}}$ area :
also since $\pi=A^{\prime}+B^{\prime}+C^{\prime}=A+B+C-\frac{1}{r^{2}}$ area, we have
$\frac{1}{r^{2}}$ area $=A+B+C-\pi=E$, the spherical excess,
$\therefore A^{\prime}=A-\frac{E}{3}, \quad B^{\prime}=B-\frac{E}{3}$, and $C^{\prime}=C-\frac{E}{3}:$
that is, a spherical triangle under the above-mentioned circumstances may be treated as a plane triangle having the same sides, and each of its angles less by one-third of the spherical excess, than the corresponding angle of the proposed triangle.

The discovery of this beautiful Theorem is also due to M. Legendre, and is alike remarkable for the simplicity and conciseness of its application, and the accuracy of its results.
108. Cor. If we suppose the radius $r$ of the sphere to be indefinitely increased so that any finite portion of its surface may be considered as a plane, we shall have for a plane triangle

$$
Q_{Q}
$$

$$
\cos A=\frac{b^{2}+c^{2}-a^{2}}{2 b c}, \text { as in (165), Pl. Trig. }
$$

Also since the arc, sine and tangent are all ultimately equal, it may be observed generally that all formulæ for spherical triangles involving the sines and tangents of the sides, will be true for plane triangles, when for the sines and tangents are substituted the sides themselves.

Since the sides and angles of spherical triangles are measured by arcs of great circles on the surface of the sphere, if the radius be indefinitely increased, the measures of both sides and angles will become right lines, whereas we have always supposed angular magnitude to be measured by a circular arc. On this account when we suppose the radius of the sphere infinite, the angles must still be measured by arcs of a circle of finite radius which may remain the same whatever be the magnitude of the sphere on which the sides are described. This circumstance will therefore render the magnitudes of the sides indeterminate when only the angles are given : and, in fact, whenever the magnitude of the side of a spherical triangle has been expressed in terms of its angles, the radius of the sphere has been supposed finite and determinate. This additional element constitutes the whole difference between plane and spherical triangles; for if the radius of the inscribed or circumscribed circle of a plane triangle, or any other line given in species, be known, the sides of a plane triangle may be determined by means of the angles.

## APPENDIX I.

CONTAINING MISCELLANEOUS THEOREMS AND PROBLEMS IN PLANE TRIGONOMETRY.

## On CHAP. I.

## I. Theorems.

1. IF ${ }^{*}$ the radii of two circles be $R$ and $r$, and the arcs $A$ and $a$, the corresponding angles will be as $A r: a R$.
2. The sum of the angles of any polygon of $n$ sides $=(n-2) 180^{\circ}$ English, or $(n-2) 200^{\circ}$ foreign.
3. Each of the angles of a regular polygon of $2 n$ sides $=\left(\frac{n-1}{n}\right) 180^{\circ}$ English, or $\left(\frac{n-1}{n}\right) 200^{\circ}$ foreign.
4. The greater the number of sides of a regular polygon, the greater is the magnitude of each of its angles.
5. The ratios of the lengths of a foreign and English degree, minute and second are expressed by the fractions $\frac{3.3}{2.5}, \frac{3.3^{2}}{2.5^{2}}$ and $\frac{3.3^{3}}{2.5^{3}}$ respectively.
6. The difference of an arc and its complement is equal to the complement of twice the arc.
7. The complement of $\frac{m-n}{m+n} 90^{\circ}$ English $=\frac{n}{m+n} 180^{\circ}$.
8. The supplement of $\frac{p-q}{p+q} 200^{\circ}$ foreign $=\frac{q}{p+q} 400^{\circ}$.
9. $\operatorname{Sin}\left(\frac{\pi}{2}+A\right)=\cos A$, and $\cos \left(\frac{\pi}{2}+A\right)=-\sin A$.
10. $\operatorname{Sin}(\pi+A)=-\sin A$, and $\cos (\pi+A)=-\cos A$.
11. $\operatorname{Sin}\left(\frac{3 \pi}{2} \pm A\right)=-\cos A$, and $\cos \left(\frac{3 \pi}{2} \pm A\right)= \pm \sin A$.
12. $\operatorname{Sin}\left((4 n+1) \frac{\pi}{2} \pm A\right)=\cos A$,

$$
\text { and } \cos \left((4 n+1) \frac{\pi}{2} \pm A\right)=\mp \sin A
$$

13. $\operatorname{Tan}\left((4 n+3) \frac{\pi}{2} \pm A\right)=\mp \cot A$,

$$
\text { and } \cot \left((4 n+3) \frac{\pi}{2} \pm A\right)=\mp \tan A
$$

14. $\operatorname{Sin}\left((3 n+1) \frac{\pi}{2} \pm A\right)= \pm \cos A$,

$$
\text { and } \cos \left((3 n+1) \frac{\pi}{2} \pm A\right)=\mp \sin A
$$

15. $\operatorname{Tan}\left((3 n-1) \frac{\pi}{2} \pm A\right)=\mp \cot A$,
and $\cot \left((3 n-1) \frac{\pi}{2} \pm A\right)=\mp \tan A$.
16. $\operatorname{Sin} A=\frac{\cos A}{\cot A}=\frac{\tan A}{\sec A}=\frac{\cos A \sec A}{\operatorname{cosec} A}=\frac{\tan A \cot A}{\operatorname{cosec} A}$.
17. $\operatorname{Cos} A=\frac{\sin A}{\tan A}=\frac{\cot A}{\operatorname{cosec} A}=\frac{\sin A \operatorname{cosec} A}{\sec A}=\frac{\tan A \cot A}{\sec A}$.
18. Vers $A=\frac{\tan A-\sin A}{\tan A}=\frac{\operatorname{cosec} A-\cot A}{\operatorname{cosec} A}=\frac{\sec A-1}{\sec A}$.
19. $\operatorname{Tan} A=\frac{\sec A}{\operatorname{cosec} A}=\frac{\sin A \operatorname{cosec} A}{\cot A}=\frac{\cos A \sec A}{\cot A}$

$$
=\frac{\cos A}{\sin A \cot ^{2} A}
$$

20. $\operatorname{Cot} A=\frac{\operatorname{cosec} A}{\sec A}=\frac{\sin A \operatorname{cosec} A}{\tan \Lambda}=\frac{\cos A \sec A}{\tan A}$

$$
=\frac{\sin A}{\cos A \tan ^{2} A} .
$$

21. $\operatorname{Sec} A=\frac{\tan A}{\sin A}=\frac{\operatorname{cosec} A}{\cot A}=\frac{\sin A \operatorname{cosec} A}{\cos A}$

$$
=\frac{\tan A \cot A}{\cos A}
$$

22. $\operatorname{Cosec} A=\frac{\cot A}{\cos A}=\frac{\sec A}{\tan A}=\frac{\cos A \sec A}{\sin A}=\frac{\tan A \cot A}{\sin A}$.
23. Vers $\left\{\left(\frac{m}{m+n}\right) \pi-A\right\}+\operatorname{vers}\left\{\left(\frac{n}{m+n}\right) \pi+A\right\}=\operatorname{vers} \pi$.
24. Chd $A$ chd $(\pi-A)=$
$\left\{\operatorname{chd} \frac{\pi}{2}+\operatorname{chd}\left(\frac{\pi}{2}-A\right)\right\}\left\{\operatorname{chd} \frac{\pi}{2}-\operatorname{chd}\left(\frac{\pi}{2}-A\right)\right\}$.
25. $\operatorname{Chd}^{2} A=1+\sin ^{2} A \cot ^{2} A-2 \sin A \cot A+\cos ^{2} A \tan ^{2} A$.
26. $1-2 \sin ^{2} 30^{\circ}=2 \cos ^{2} 30^{\circ}-1=\sin 60^{\circ}$, and $\left(\sin 30^{\circ}+\cos 30^{\circ}\right)\left(\sin 60^{\circ}-\cos 60^{\circ}\right)=\sin 30^{\circ}$.
27. $2 \sin 30^{\circ} \cos 30^{\circ}=\sin 60^{\circ}, \tan 60^{\circ}=2 \sin 60^{\circ}=3 \tan 30^{\circ}$ $=\operatorname{chd} 120^{\circ}$, and $\operatorname{cosec} 60^{\circ}=2 \tan 30^{\circ}$.
28. $\operatorname{Sin} 45^{\circ}+\cos 45^{\circ}=\operatorname{chd} 90^{\circ}$, and $2 \sin 45^{\circ} \cos 45^{\circ}=\sin 90^{\circ}$.
29. $\frac{\sin 45^{\circ}-\sin 30^{\circ}}{\sin 45^{\circ}+\sin 30^{\circ}}=\frac{\sec 45^{\circ}-\tan 45^{\circ}}{\sec 45^{\circ}+\tan 45^{\circ}}$,

$$
\text { and } \frac{\sin 60^{\circ}-\sin 30^{\circ}}{\sin 60^{\circ}+\sin 30^{\circ}}=\frac{\tan 60^{\circ}-\tan 45^{\circ}}{\tan 60^{\circ}+\tan 45^{\circ}}
$$

30. If $S$ and $s$ be the sines of two arcs, $C$ and $c$ their cosines: then

$$
\left(S^{2}-S^{2} s^{2}\right)\left(C^{2}-C^{2} c^{2}\right)=\left(s^{2}-S^{2} s^{2}\right)\left(c^{2}-C^{2} c^{2}\right)
$$

31. If $S$ and $s$ be the secants of two arcs, $T$ and $t$ their kangents, then

$$
\frac{S^{2}\left(s^{2}-1\right)}{s^{2}\left(S^{2}-1\right)}=\frac{t^{2}\left(T^{2}+1\right)}{T^{2}\left(t^{2}+1\right)} .
$$

32. If $S$ and $s$ be the sines or secants of two arcs, $T$ and $t$ their tangents, then is $\frac{T}{t}$ greater or less than $\frac{S}{s}$, according as $T$ is greater or less than $t$.
33. If $C$ and $c$ be the cosines or cosecants of two arcs, $T$ and $t$ their cotangents, then $\frac{T}{t}$ will be less or greater than $\frac{C}{c}$, according as $T$ is greater or less than $t$.

## II. Problems.

1. Compare the magnitudes of two angles, when the arcs which subtend them are inversely as the radii.
2. If to the radius $r$ an angle be measured by an arc whose length is $\alpha$, required the length of the arc which will measure an angle $m$ times as great to the radius $\frac{r}{n}$.
3. One regular figure has twice as many sides as another, and each of its angles greater than each of the angles of the other in the ratio of $4: 3$ : find the number of sides of each.
4. The number of sides of one regular polygon exceeds the number of those of another by 1 , and an angle of one exceeds an angle of the other by $4^{0}$ : find the number of sides of each.
5. The interior angles of a rectilinear figure are in arithmetical progression, the least angle being $120^{\circ}$, and the common difference $5^{\circ}$ : find the number of sides.
6. Represent in the foreign scale, the English arcs: $11^{\circ} 15^{\prime}, 22^{\circ} 50^{\prime}, 30^{\circ}, 45^{\circ}, 60^{\circ}, 75^{\circ}$, and $78^{\circ} 45^{\prime}$.
7. Express in the English scale, the foreign arcs: $10^{\circ}$, $20^{0}, 33 \frac{1}{3}^{0}, 40^{\circ}, 50^{\circ}, 66 \frac{20}{3}, 70^{\circ}$ and $95^{\circ}$.
8. Find the complements of the following arcs in the English and foreign scales : $15^{\circ}, 18^{0}, 22^{0} 30^{\prime}, 36^{\circ}, 54^{\circ}, 72^{\circ}, 75^{\circ}$, $95^{\circ}$, and $120^{\circ}$.
9. What are the supplements of the following arcs in the English and foreign divisions of the circle: $33^{\circ} 45^{\prime}, 78^{\circ} 30^{\prime}$, $150^{\circ}, 180^{\circ}, 210^{\circ}$, and $270^{\circ}$ ?
10. Point out where the sine increases or decreases, and shew that it changes its algebraical sign only when it passes through 0 .
11. Trace the increase or decrease of the tangent in each of the four quadrants, and prove that this line changes its sign either by passing through 0 or $\infty$.
12. Trace the changes of algebraical sign in the secant of an arc, and find whether $\sec A$ and $\sec (\pi+A)$ have the same or different signs.
13. Given the algebraical signs of the sine and cosine of $A$ in each of the four quadrants, to determine the sign of the tangent, co-tangent, secant, and co-secant.
14. Transform from the radius 1 to the radius $r$, the formulæ,

$$
\tan A=\frac{1}{\cot A}, \sec A=\frac{1}{\cos A}, \operatorname{cosec} A=\frac{1}{\sin A},
$$

chd $A=\sqrt{2-2 \cos A}, \sin ^{2} A=$ vers $A$ vers $(\pi-A)$,
chd ${ }^{2}\left(\frac{\pi}{2}+A\right)$ chd $^{2}\left(\frac{\pi}{2}-A\right)=4$ vers $\left(\frac{\pi}{2}+A\right) \operatorname{vers}\left(\frac{\pi}{2}-A\right)$.
15. Deduce the sine, cosine, $\& \mathrm{cc}$. of $150^{\circ}, 225^{\circ}, 270^{\circ}$ and $315^{\circ}$.
16. Given any one of the trigonometrical lines defined in this chapter, to deduce all the rest, and adapt them to the radius $r$.
17. Given $2 \sin A=\tan A$, to find the sine, cosine, $\& c$. of $A$.
18. Given vers $\left(\frac{\pi}{2}-A\right)=v$, to find all the rest, and adapt the results to the radius $r$.
19. Given chd $(\pi-A)=k$, to find all the rest in forms adapted to the radius $r$.
20. If $\sin A \cos A=m$, find the values of $\sin A$ and $\cos A$.
21. Given $m \sin A=n \cos ^{2} A$, to find $\tan A$ and $\operatorname{cosec} A$.
22. Given $m \sin A \pm n^{\prime} \cos A=p$, to find $\tan A$ and $\sec A$.
23. If $m \operatorname{chd} A \pm n \cos A=p$, what is the value of chd $A$ ?
24. If $\sin A(\sin A-\cos A)=m$, find the value of $\sin A$.
25. If $m$ vers $\Lambda \pm n$ vers $(\pi-A)=p$, what is the value of vers $A$ ?
26. Given $\tan A+\cot A=4$, to find the value of $\tan A$.
27. Given $m \tan A \pm n \cot A=p \sec A$, to find $\operatorname{cosec} A$.
28. If $\sin A+\sin B=m$, and $\sin A \sin B=n$, required the values of $\sin A$ and $\sin B$.
29. Given $\sin A \sec B=m$, and $\cos A \operatorname{cosec} B=n$, to find the value of $\sec A$ and $\sec B$.
30. Given $\sin A+\cos B=m$ and $\sin B+\cos \Lambda=n$, to find $\sin A$ and $\cos B$.
31. Given $\tan A+\tan B=m$, and $\sec \Lambda-\sec B=n$, to find $\tan A$ and $\sec B$.
32. If in a right-angled triangle the sine of one of the acute angles be given, it is required to find the versed sire and chord of the other.
33. Give two general formula, one including all the arcs whose sines are positive, and the other all the arcs whose cosines are negative.
34. Express the radius of a circle in which the length of $45^{\circ}=L$, in terms of the radius of a circle in which the length of $60^{\circ}=l$.
35. Given chd $A=m$ and vers $A=n$, find vers $B$ when chd $B=p$, the radius being unknown.

## On CHAP. II. and III.

## I. Theorems

Intolving the Trigonometrical Functions of one Arc and of some of its Multiples and Submultiples.

1. $\operatorname{Sin} A=\frac{2 \sin \frac{A}{Q}}{\sec \frac{A}{Q}}=\frac{2 \cos \frac{A}{Q}}{\operatorname{cosec} \frac{A}{2}}=\frac{2 \sin ^{2} \frac{A}{Q}}{\tan \frac{A}{Q}}=\frac{2 \cos ^{2} \frac{A}{Q}}{\cot \frac{A}{Q}}$
$=\frac{2 \tan \frac{A}{2}}{\sec ^{2} \frac{A}{2}}=\frac{2 \tan \frac{A}{2}}{1+\tan ^{2} \frac{A}{2}}=\frac{2 \cot \frac{A}{2}}{\operatorname{cosec}^{2} \frac{A}{2}}=\frac{2 \cot \frac{A}{2}}{1+\cot ^{2} \frac{A}{2}}$
$=\frac{\mathrm{Q}}{\tan \frac{A}{Q}+\cot \frac{A}{Q}}=\frac{1}{\tan \frac{A}{Q}+\cot A}=\frac{1}{\cot \frac{A}{Q}-\cot A}=\frac{1}{\operatorname{cosec} A}$.
2. $\operatorname{Cos} A=\cos ^{1} \frac{A}{2}-\sin ^{4} \frac{A}{2}=\frac{\sin 2 A}{2 \sin A}=\frac{\operatorname{cosec} A}{2 \operatorname{cosec} 2 A}$

$$
\begin{aligned}
& =\frac{2 \cos \frac{A}{2}-\sec \frac{A}{2}}{\sec \frac{A}{2}}=\frac{\operatorname{cosec} \frac{A}{2}-2 \sin \frac{A}{2}}{\operatorname{cosec} \frac{A}{2}}=\frac{\cot \frac{A}{2}-\tan \frac{A}{2}}{\cot \frac{A}{2}+\tan \frac{A}{2}} \\
& =\frac{1-\tan ^{2} \frac{A}{2}}{1+\tan ^{2} \frac{A}{2}}=\frac{\cot ^{2} \frac{A}{2}-1}{\cot ^{2} \frac{A}{2}+1}=\frac{\sin \frac{A}{\tan \frac{A}{2}}-1}{1+\frac{1}{1+\tan A \tan \frac{A}{2}}=\frac{1}{\sec A}} . \\
& 3 . \quad \text { Tan } A=\frac{2 \sin \frac{A}{2} \cos \frac{A}{2}}{1-2 \sin ^{2} \frac{A}{2}}=\frac{2 \sin \frac{A}{2}}{\sec \frac{A}{2}-2 \sin \frac{A}{2} \tan \frac{A}{2}} \\
& =\frac{2 \tan \frac{A}{2}}{2-\sec \frac{A}{2}}=\frac{2 \cot \frac{A}{2}}{\operatorname{cosec}^{2} \frac{A}{2}-Q}=\frac{2}{\cot \frac{A}{2}-\tan \frac{A}{2}} \\
& =\frac{2+2 \tan A \tan \frac{A}{2}}{\tan \frac{A}{2}+\cot \frac{A}{2}}=(1+\sec A) \tan \frac{A}{2}=\frac{1}{\cot A} .
\end{aligned}
$$

4. $2 \tan A=(\sin A+\tan A) \sec ^{2} \frac{A}{2}=\sec ^{2} A \sin 2 A$.
5. Tan $2 A-\tan A=\frac{\sin A}{2 \cos ^{5} A-\cos A}=\frac{2 \sin A}{\cos A+\cos 3 A}$.
6. Tan $A=\cot A-2 \cot 2 A=\frac{\sin 2 A}{1+\cos 2 A}=\frac{1-\cos 2 A}{\sin 2 A}$

$$
=\sqrt{\frac{1-\cos Q A}{1+\cos Q A}} .
$$

7. $\operatorname{Tan} \frac{A}{2}=\frac{\sin 2 A}{1+\cos 2 A} \frac{\cos A}{1+\cos \Lambda}$.
8. $2 \sin ^{2} \frac{A}{2}$ suvers $A=\sin ^{2} A$.
9. $\quad(2 \sin A+\sin 2 A) \tan ^{2} \frac{A}{2}=2 \sin A-\sin 2 A$.
10. $\left(1-\tan \frac{A}{2}\right) \tan A=1-\sec A+\tan \frac{A}{\Omega}(1+\sec A)$.
11. $\operatorname{Sin} 3 A \sin A=\sin ^{2} 2 A-\sin ^{2} A$.
12. $\operatorname{Sin} 5 A \sin A=\sin ^{2} 3 A-\sin ^{2} 2 A$.
13. $\operatorname{Chd} \frac{3 A}{2} \operatorname{chd} \frac{A}{2}=\operatorname{chd}^{2} A-\operatorname{chd}^{2} \frac{A}{2}$.
14. $\operatorname{Tan} \frac{A}{2} \tan \frac{3 A}{2}=\frac{\cos A-\cos 2 A}{\cos A+\cos 2 A}$.
15. Tan $3 A \tan A=\frac{\tan ^{2} 2 A-\tan ^{2} A}{1-\tan ^{2} A \tan ^{2} 2 A}$.
16. Tan $2 A+\sec 2 A=\frac{\cos A+\sin A}{\cos A-\sin A}$.
17. $\frac{1-2 \sin ^{2} A}{1+\sin 2 A}=\frac{1-\tan A}{1+\tan A}=\frac{1}{\tan 2 A+\sec 2 A}$.
18. $\operatorname{Cos}^{2} 2 A-\sin ^{2} A=\cos A \cos 3 A$.
19. $\operatorname{Tan}^{2} 2 A-\tan ^{2} A=\frac{\sin 3 A \sin A}{\cos ^{2} 2 A \cos ^{2} A}$.
20. $\operatorname{Tan}^{4} A=\frac{\sin ^{2} 2 A-4 \sin ^{2} A}{\sin ^{2} 2 A+4 \sin ^{2} A-4}=\frac{\cos ^{2} 2 A-4 \cos ^{2} A+3}{\cos ^{2} 2 A+4 \cos ^{2} A-1}$.
21. $\operatorname{Cos} A(1-\tan 2 \Lambda \tan A)=\cos 3 A(1+\tan 2 A \tan A)$.
22. $\operatorname{Sin} A=\cos ^{2} \frac{A}{2}\left(1+\tan \frac{A}{2}+\sec \frac{A}{2}\right)$

$$
\left(1+\tan \frac{A}{2}-\sec \frac{A}{2}\right)
$$

23. $2 \sin A=\sqrt[3]{-\sin 9 A+\sqrt{\sin ^{2} 3 A-1}}$

$$
+\sqrt[3]{-\sin 3 A-\sqrt{\sin ^{2} 3 A-1}}
$$

24. $2 \cos A=\sqrt{\cos 3 A+\sqrt{\cos ^{2} 3 A-1}}$

$$
+\sqrt[3]{\cos 3 A-\sqrt{\cos ^{2} 3 A-1}}
$$

## II. Theorems

Involving the Trigonometrical Functions of two Arcs and of some of their Multiples and Submultiples.

1. $\sin A=\cos \left(30^{\circ}-A\right)-\cos \left(30^{\circ}+A\right)$

$$
=\frac{1}{\sqrt{3}}\left\{\sin \left(30^{\circ}+A\right)-\sin \left(30^{\circ}-A\right)\right\}
$$

$$
=\frac{1}{\sqrt{2}}\left\{\sin \left(45^{\circ}+A\right)-\sin \left(45^{\circ}-A\right)\right\}=2 \sin ^{2}\left(45^{\circ}+\frac{A}{2}\right)-1
$$

$$
=\frac{1-\tan ^{2}\left(45^{\circ}-\frac{A}{2}\right)}{1+\tan ^{2}\left(45^{\circ}-\frac{A}{2}\right)}=\frac{1-\cot ^{2}\left(45^{\circ}+\frac{A}{2}\right)}{1+\cot ^{2}\left(45^{\circ}+\frac{A}{2}\right)}
$$

$$
=\frac{\tan \left(45^{\circ}+\frac{A}{2}\right)-\tan \left(45^{\circ}-\frac{A}{2}\right)}{\tan \left(45^{\circ}+\frac{A}{2}\right)+\tan \left(45^{\circ}-\frac{A}{2}\right)}
$$

$$
=\sin \left(60^{\circ}+A\right)-\sin \left(60^{\circ}-A\right)=\frac{1}{\operatorname{cosec} \Lambda}
$$

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2. $\operatorname{Cos} A=\sin \left(30^{\circ}+A\right)+\sin \left(30^{\circ}-A\right)$

$$
\begin{gathered}
=\frac{1}{\sqrt{2}}\left\{\cos \left(45^{\circ}+A\right)+\cos \left(45^{\circ}-A\right)\right\} \\
=\varrho \sin \left(45^{\circ}+\frac{A}{2}\right) \sin \left(45^{\circ}-\frac{A}{2}\right)=\frac{2}{\tan \left(45^{\circ}+\frac{A}{2}\right)+\cot \left(45^{\circ}+\frac{A}{2}\right)} \\
=\frac{1}{\sqrt{3}}\left\{\sin \left(60^{\circ}+A\right)+\sin \left(60^{\circ}-A\right)\right\}=\frac{1}{\sec A} . \\
\text { 3. Tan } A=\frac{1}{\sqrt{3}} \frac{\sin \left(30^{\circ}+A\right)-\sin \left(30^{\circ}-A\right)}{\sin \left(30^{\circ}+A\right)+\sin \left(30^{\circ}-\Lambda\right)}
\end{gathered}
$$

$$
=\frac{\sin \left(45^{\circ}+A\right)-\cos \left(45^{\circ}+A\right)}{\sin \left(45^{\circ}+A\right)+\cos \left(45^{\circ}+\Lambda\right)}=\frac{\tan \left(45^{\circ}+\frac{A}{2}\right)-\tan \left(45^{\circ}-\frac{A}{2}\right)}{2}
$$

$$
\begin{aligned}
& =\frac{\tan \left(45^{\circ}+\frac{A}{2}\right)+\cot \left(45^{\circ}-\frac{A}{2}\right)}{\tan \frac{A}{2}+\cot \frac{A}{2}} \\
& =\frac{\sin \left(60^{\circ}+A\right)-\sin \left(60^{\circ}-A\right)}{\cos \left(60^{\circ}+A\right)+\cos \left(60^{\circ}-\Lambda\right)}=\frac{1}{\cot A} .
\end{aligned}
$$

4. $\operatorname{Sin}\left(30^{\circ} \pm A\right)=\cos \left(60^{\circ} \mp A\right)=\frac{\cos A \pm \sqrt{ } 3 \sin A}{2}$.
5. $\operatorname{Sin}\left(45^{\circ} \pm A\right)=\cos \left(45^{\circ} \mp A\right)=\sqrt{\frac{1 \pm \sin 2 A}{2}}$.
6. $\operatorname{Tan}\left(30^{\circ} \pm A\right)=\frac{1}{2}\left\{\cot \left(30^{\circ} \mp \frac{A}{Q}\right)-\tan \left(30^{\circ} \mp \frac{A}{Q}\right)\right\}$.
7. $\operatorname{Tan}\left(30^{\circ}+\frac{A}{2}\right) \tan \left(30^{\circ}-\frac{A}{2}\right)=\frac{2 \cos A-1}{2 \cos A+1}$.
8. $\operatorname{Tan}\left(45^{\circ} \pm A\right)=\frac{\cos A}{1 \mp \sin A}=\sqrt{\frac{1 \pm \sin 2 A}{1 \mp \sin 2 A}}$.
9. $\operatorname{Tan}\left(45^{\circ} \pm \frac{A}{2}\right)=\frac{1 \pm \sin A}{\cos A}=\sec A \pm \tan A$.
10. $\sin A=4 \sin \frac{A}{3} \sin \left(60^{\circ}-\frac{A}{3}\right) \sin \left(60^{\circ}+\frac{A}{3}\right)$.
11. $\operatorname{Cos} A=4 \cos \frac{A}{3} \sin \left(30^{\circ}-\frac{A}{3}\right) \sin \left(30^{\circ}+\frac{A}{3}\right)$.
12. $\operatorname{Tan} \Lambda=\tan \frac{A}{3} \tan \left(60^{\circ}-\frac{A}{3}\right) \tan \left(60^{\circ}+\frac{A}{3}\right)$.
13. $3 \cot A=\cot \frac{A}{3}-\cot \left(60^{\circ}-\frac{A}{3}\right)+\cot \left(60^{\circ}+\frac{A}{3}\right)$.
14. $\operatorname{Tan}\left(45^{\circ}+\frac{A}{2}\right)-\tan \left(45^{\circ}-\frac{A}{2}\right)=2 \tan A$.
15. $\operatorname{Sec}\left(45^{\circ}+\frac{A}{2}\right) \sec \left(45^{\circ}-\frac{A}{2}\right)=2 \sec A$.
16. $\operatorname{Sec} A=\tan \left(45^{\circ}+\frac{A}{2}\right)-\tan A=\cot \left(45^{\circ}-\frac{A}{2}\right)-\tan A$.
17. $2 \operatorname{vers}\left(\frac{\pi+A}{2}\right) \operatorname{vers}\left(\frac{\pi-A}{2}\right)=\operatorname{vers}(\pi-A)$.
18. $\sin B=\sin (A+B) \cos A-\cos (A+B) \sin A$.
19. $\operatorname{Cos} A+\cos (A+2 B)=\left(2 \cos \frac{B}{2}-1\right) \cos (A+B)$.
20. $\operatorname{Sin}(A+B) \sin 3(A-B)=\sin ^{2}(2 A-B)-\sin ^{2}(2 B-A)$.
21. $\operatorname{Sin}(A \pm B)=\cos A \cos B(\tan A \pm \tan B)$.
22. $\operatorname{Cos}(A \pm B)=\cos A \cos B(1 \mp \tan A \tan B)$.
23. Vers $(A-B)$ vers $\{\pi-(A+B)\}=(\sin A-\sin B)^{2}$.
24. $\sin ^{2}(A+B)-\sin ^{2}(A-B)=\sin 2 A \sin 2 B$.
25. $\operatorname{Cos}^{2}(A+B)-\sin ^{2} A=\cos B \cos (2 \Lambda+B)$.
26. $\operatorname{Sin}^{2}(A+B)=\sin ^{2} A+\sin ^{2} B+2 \sin A \sin B \cos (A+B)$.
27. $\operatorname{Cot} A \pm \tan B=\frac{\cos (A \mp B)}{\sin A \cos B}$.
28. $\frac{\operatorname{Tan} A \pm \tan B}{\operatorname{Cot} A \mp \tan B}=\tan A \tan (A \pm B)$.
29. $\frac{\sin 2(A+B)}{\sin 2 A+\sin 2 B}=\frac{\cos (A+B)}{\cos (A-B)}=\frac{\cot B-\tan A}{\cot B+\tan A}$.
30. $\operatorname{Tan}^{2} A-\tan ^{2} B=\frac{\sin (A+B) \sin (A-B)}{\cos ^{2} A \cos ^{2} B}$.
31. $\frac{\operatorname{Tan}(A+B)+\tan (A-B)}{\operatorname{Tan}(A+B)-\tan (A-B)}=\frac{\tan A\left(1+\tan ^{2} B\right)}{\tan B\left(1+\tan ^{2} A\right)}$.
32. $1+\cos 2 A \cos 2 B=2\left(\sin ^{2} A \cos ^{2} B+\cos ^{2} A \sin ^{2} B\right)$.
33. If $\frac{m \tan (A-B)}{\cos ^{2} B}=\frac{n \tan B}{\cos ^{2}(A-B)}$, then will

$$
\tan (A-\Omega B)=\frac{n-m}{n+m} \tan A
$$

34. If $\cos B(m+\cos A)=1+m \cos A$, then will

$$
\tan \frac{A}{2}+\tan \frac{B}{2}=m\left(\tan \frac{A}{2}-\tan \frac{B}{2}\right)
$$

35. If $\tan \frac{B}{2}=\frac{1-\tan ^{3} A}{1+\tan ^{3} A}$, then will $2 \cot Q A=\sqrt[5]{\tan B+\sec B}+\sqrt[3]{\tan B-\sec B}$.

## III. Theorems

Envolving the Trigonometrical Functions of three or more Arcs and of some of their Multiples and Submultiples.

1. Chd $n A \operatorname{chd}(\pi-A)=\operatorname{chd}(n+1) A+\operatorname{chd}(n-1) A$.
2. $\sin A+\cos B=2 \sin \left(45^{\circ}+\frac{A-B}{2}\right) \cos \left(45^{\circ}-\frac{A+B}{2}\right)$.
3. $\operatorname{Cos} A+\sin B=2 \cos \left(45^{\circ}+\frac{A-B}{2}\right) \cos \left(45^{\circ}-\frac{A+B}{2}\right)$.
4. $\frac{\operatorname{Cos} B-\sin A}{\operatorname{Cos} B+\sin A}=\tan \left(45^{\circ}-\frac{A+B}{2}\right) \tan \left(45^{\circ}-\frac{A-B}{2}\right)$.
5. If $A+B+C=\pi$, then $\sin A+\sin B+\sin C=4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$.
6. On the same supposition, $\sin 2 A+\sin 2 B+\sin 2 C=4 \sin A \sin B \sin C$.
7. The same hypothesis remaining,

$$
\sin ^{2} \frac{A}{2}+\sin ^{2} \frac{B}{2}+\sin ^{2} \frac{C}{2}+2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{Q}=1
$$

8. If $A+B+C=45^{\circ}$, then

$$
\begin{aligned}
& \tan A+\tan B+\tan C-\tan A \tan B \tan C \\
= & 1-\tan A \tan B-\tan A \tan C-\tan B \tan C
\end{aligned}
$$

9. If $A+B+C=(2 n+1) \frac{\pi}{2}$, then $\cos 2 A+\cos 2 B+\cos 2 C=4 \cos A \cos B \cos C$.
10. If $A+B+C=\frac{\pi}{9}$, then
$\tan A+\tan B+\tan C=\tan A \tan B \tan C+\sec A \sec B \sec C$.
11. If the arcs $A, B, C$ be in arithmetical progression, then

$$
\sin A-\sin C=2 \sin (A-B) \cos B
$$

12. Again, $\tan (A-B)=\frac{\cos A-\cos C}{\sin A+\sin C}=\frac{\sin A-\sin C}{\cos A+\cos C}$.
13. And $\sin A \sin C$

$$
=\{\sin B+\sin (A-B)\}\{\sin B-\sin (A-B)\}
$$

15. If $A+B+C=\frac{\pi}{4}$, and $A, B, C$ be in arithmetical progression, then

$$
\sqrt{3}-\tan A=(1+\sqrt{ } 3 \tan A) \tan C
$$

16. If $A+B+C=\pi$, and the simes of $A, B, C$ be in arithmetical progression, then

$$
\sin \frac{A}{2} \sin \left(\frac{B-C}{2}\right)=\sin \frac{C}{2} \sin \left(\frac{A-B}{9}\right)
$$

17. On the same supposition, if the cosines of $A, B, C$ be in arithmetical progression, then

$$
2 \cos \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2}=\cos ^{2} \frac{B}{2} .
$$

18. $\operatorname{Cos}^{-1}\left(\sqrt{\frac{2}{3}}\right)-\cos ^{-1}\left(\frac{1}{\sqrt{2}}+\frac{1}{2 \sqrt{3}}\right)=30^{\circ}$.
19. $2 \tan ^{-1}\left(\frac{1}{3}\right)+\tan ^{-1}\left(\frac{1}{7}\right)=4.5^{\circ}$.

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20. $\operatorname{Tan}^{-1}\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)-\tan ^{-1}\left(\frac{1}{\sqrt{2}}\right)=\frac{\pi}{4}$.
21. If $\tan A=\frac{1}{\sqrt{3}}$, and $\tan B=\frac{1}{\sqrt{15}}$, then

$$
\sin (A+B)=\sin 60^{\circ} \cos 36^{\circ}
$$

22. $\operatorname{Sin}(A-B) \sin C-\sin (A-C) \sin B+\sin (B-C) \sin A=0$.
23. If $\cos A=\cos B \cos C$, then

$$
\tan \left(\frac{A+B}{2}\right) \tan \left(\frac{A-B}{2}\right)=\tan ^{\mathrm{e}} \frac{C}{2}
$$

24. If $\sin (A+B) \cos C=2 \cos (B-C) \sin A$, then

$$
\cot A-\cot B=2 \tan C
$$

25. $\quad \sin (A+B) \sin (B+C+D)=\sin A \sin (C+D)$

$$
+\sin B \sin (A+B+C+D)
$$

26. $\sin \mathcal{Q}(A-C)+\sin 2(B-C)-\sin 2(A-B)$

$$
=4 \cos (A-C) \sin (B-C) \cos (A-B)
$$

27. $\operatorname{Cos}(A-B-C)+\cos (A-B+C)+\cos (A+B-C)$ $+\cos (A+B+C)=4 \cos A \cos B \cos C$.
28. If $A, B, C, D, \& c . L$, be any arcs, then $\sin (A+B) \sin (A-B)+\sin (B+C) \sin (B-C)+\& c$.
$+\sin (L+A) \sin (L-A)=0$, and $\cos (A+B) \sin (A-B)$ $+\cos (B+C) \sin (B-C)+\& \mathrm{c} .+\cos (L+A) \sin (L-A)=0$.
29. If $\frac{\cos A-\sin B \sin C}{\cos B \cos C}=\frac{\cos D-\sin E \sin F}{\cos E \cos F}$,

$$
\text { then } \frac{\cos B \cos C}{\cos E \cos F}=\frac{\text { vers } A-\operatorname{vers}(B-C)}{\text { vers } D-\operatorname{vers}(E-F)} .
$$

## IV. Theorems

Involving the Numerical Values of the Trigonometrical Functions of certain given Arcs to the Radius 1.

1. $\operatorname{Sin} 7^{0} 30^{\prime}=\frac{\sqrt{4-\sqrt{6-\sqrt{2}}}}{2 \sqrt{2}}$.
2. $\operatorname{Sin} 9^{0}=\frac{\sqrt{3+\sqrt{5}}-\sqrt{5-\sqrt{2}}}{4}$.
3. $\operatorname{Cos} 11^{\circ} 10^{\prime}=\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{Q}$.
4. $\operatorname{Cos} 33^{\circ} 45^{\prime}=\frac{\sqrt{2+\sqrt{2-\sqrt{2}}}}{2}$.
5. $V \operatorname{ers} 15^{\circ}=\frac{2 \sqrt{2}-\sqrt{ } 3-1}{2 \sqrt{2}}$.
6. Vers $78^{\circ} 45^{\prime}=\frac{2-\sqrt{2-\sqrt{2+\sqrt{2}}}}{2}$.
7. Chd $30^{\circ}=\frac{\sqrt{3}-1}{\sqrt{2}}=\frac{\sqrt{ }( }{1+\sqrt{3}}$.
8. Chd $67^{\circ} 30^{\prime}=\sqrt{2-\sqrt{2-\sqrt{2}}}$.
9. $\operatorname{Tan} 18^{\circ}=\sqrt{\frac{3-\sqrt{5}}{5+\sqrt{5}}}$.
10. $\operatorname{Tan} 37^{\circ} 30^{\prime}=\frac{2 \sqrt{2+\sqrt{3}}}{4+\sqrt{6-\sqrt{2}}}$.
11. $\operatorname{Cot} 78^{\circ} 45^{\prime}=\sqrt{\frac{2-\sqrt{2+\sqrt{2}}}{2+\sqrt{2+\sqrt{2}}}}$.

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12. $\operatorname{Cot} 81^{\circ}=\frac{\sqrt{3+\sqrt{5}}-\sqrt{5-\sqrt{5}}}{\sqrt{3+\sqrt{5}}+\sqrt{5-\sqrt{5}}}$.
13. $\operatorname{Sec} 27^{\circ}=\frac{2 \sqrt{5+\sqrt{5}}-2 \sqrt{3-\sqrt{5}}}{1+\sqrt{5}}$.
14. $\operatorname{Sec} 56^{\circ} 15^{\prime}=\frac{2 \sqrt{2+\sqrt{2-\sqrt{2}}}}{\sqrt{2+\sqrt{2}}}$.
15. $\operatorname{Cosec} 22^{\circ} 30^{\prime}=\frac{2}{\sqrt{2-\sqrt{2}}}$.
16. $\quad \operatorname{Cosec} 52^{\circ} 30^{\prime}=\frac{2 \sqrt{ } 2}{\sqrt{4-\sqrt{2}+\sqrt{6}}}$.
17. $\operatorname{Sin} 6^{\circ}=-\frac{1}{8}(\sqrt{5}+1)+\frac{\sqrt{3}}{4 \sqrt{2}} \sqrt{5-\sqrt{5}}$.
18. $\operatorname{Sin} 12^{0}=-\frac{\sqrt{ } 3}{8}(\sqrt{5}-1)+\frac{1}{4 \sqrt{2}} \sqrt{5+\sqrt{5}}$.
19. $\operatorname{Sin} 21^{\circ}=-\frac{\sqrt{ } 3-1}{8 \sqrt{2}}(\sqrt{ } 5+1)+\frac{\sqrt{ } 3+1}{8} \sqrt{5-\sqrt{5}}$.
20. $\quad \operatorname{Sin} 39^{\circ}=-\frac{\sqrt{ } 3+1}{8 \sqrt{2}}(\sqrt{ } 5+1)-\frac{\sqrt{ } 3-1}{8} \sqrt{5-\sqrt{5} 5}$.
21. $\quad \operatorname{Sin} 42^{\circ}=-\frac{1}{8}(\sqrt{5}-1)+\frac{\sqrt{ } 3}{4 \sqrt{2}} \sqrt{5+\sqrt{5}}$.
22. $\operatorname{Sin} 57^{\circ}=-\frac{\sqrt{ } 3-1}{8 \sqrt{2}}(\sqrt{5}-1)+\frac{\sqrt{ } 3+1}{8} \sqrt{5+\sqrt{5}}$.
23. $\operatorname{Sin} 66^{0}=\frac{1}{9}(\sqrt{ } 5+1)+\frac{\sqrt{ } 3}{4 \sqrt{ } 9} \sqrt{5-\sqrt{5} 5}$.
24. $\operatorname{Sin} 87^{\circ}=\frac{\sqrt{ } 3-1}{8 \sqrt{2}}(\sqrt{5}-1)+\frac{\sqrt{ } 3+1}{8} \sqrt{5+\sqrt{ } 5}$.

## V. Problems

Involving the Trigonometrical Functions of one or more Arcs.

1. Express each of the Trigonometrical Functions of $A$ in terms of $\sin 2 A$ and $\sin \frac{A}{2}$, and adapt the results to radius $r$.
2. In terms of $\cos 2 A$ and $\cos \frac{A}{2}$.
3. In terms of vers $2 A$ and vers $\frac{A}{2}$.
4. In terms of chd $2 A$ and chd $\frac{A}{2}$.
5. In terms of $\tan 2 A$ and $\tan \frac{A}{2}$.
6. In terms of $\cot 2 A$ and $\cot \frac{A}{2}$.
7. In terms of $\sec 2 A$ and sec $\frac{A}{2}$.
8. In terms of $\operatorname{cosec} 2 \Lambda$ and $\operatorname{cosec} \frac{\Lambda}{2}$.
9. Given $\tan Q A=3 \tan A$, to find $A$.
10. Given $\frac{1+\tan A}{1-\tan A}=\frac{1}{2} \sec Q A$, to find $A$.
11. From the equation, $\tan \frac{A}{2}=\operatorname{cosec} A-\sin A$, find the value of $A$.
12. Find the value of $A$ which satisfies the equation,

$$
\tan A+2 \cot 2 A=\sin A\left(1+\tan A \tan \frac{A}{2}\right)
$$

13. If $\sin 3 A-2 \sin 2 A+\sin ^{2} A+4 \sin ^{3} A=0$, find the value of $\sin A$.
14. Given $\sin 4 A+4 \sin 9 A \cos A=0$, to find $\tan A$.
15. Given $\sin 2 A \cos 2 A+3 \sin A \cos 3 A=0$, to find $\cos A$.
16. Find general formula including all the values of $\Lambda$ which fulfil the conditions of the equation,

$$
2 \sin ^{2} 9 A+\sin ^{2} 6 A=2
$$

17. If $\sin (A-B)=\cos (A+B)=\frac{1}{2}$, find the values of $A$ and $B$.
18. Given $\sin (A+B)+\sin (A-B)=\cos (A+B)$ $+\cos (A-B)$, to find the value of $A$.
19. Given $\sin (A+B)-\sin (A-B)=\tan 60^{\circ} \sin B$, to find the value of $A$.
20. If $\sin (2 A+B)-\sin (2 A-B)=\sin (A+B)$ $-\sin (A-B)-\sin B$, find the value of $\sin \Lambda$.
21. Given $\tan A+\tan B=\sec A$, find the relation between the values of $A$ and $B$.
22. Given $\tan C+2 \tan (A-C)=\tan (A+B-C)$, to find $\tan C$.
23. Given $2 \tan C+\tan (A-C)=\tan (C-B)$, to find $\tan C$.
24. Divide a given angle into two parts $A$ and $B$ so that

$$
\frac{\sin A}{\sin B}=\frac{\operatorname{cosec} B}{\operatorname{cosec} A}=\frac{m}{n} .
$$

25. So that $\frac{\cos A}{\cos B}=\frac{\sec B}{\sec A}=\frac{m}{n}$.
26. So that $\frac{\text { vers } A}{\text { vers } B}=\frac{m}{u}$.
27. So that $\frac{\operatorname{chd} A}{\operatorname{chd} B}=\frac{m}{n}$.
28. So that $\frac{\tan A}{\tan B}=\frac{\cot B}{\cot A}=\frac{m}{n}$.
29. So that $\frac{\sec A}{\tan B}=\frac{m}{n}$.
30. If $\tan A+\tan B+\tan C=\tan A \tan B \tan C$, it is required to find the general value of $(A+B+C)$.
31. If $\sin A-\mathcal{q} \sin B+\sin C=2 \sin B$ vers $(A-B)$, find the relation between $A, B$ and $C$.
32. Given $\cos A+\cos B=m$, and $\cos 5 A+\cos 5 B=n$, to find $A$ and $B$.
33. If $A+B+C=180^{\circ}$, and $\tan A \tan B=m, \tan A$ $\tan C=n$, find the values of $\tan A, \tan B$ and $\tan C$.

## On CHAP. IV.

## I. Theorems.

1. In a right-angled triangle, if $A, B, C, a, b, c$ be the angles and sides, and $C$ the right angle, then

$$
\sin \frac{A}{2}=\sqrt{\frac{c-b}{Q c}}, \cos \frac{A}{Q}=\sqrt{\frac{c+b}{2 c}}, \tan \frac{A}{Q}=\sqrt{\frac{c-b}{c+b}} .
$$

2. $\sin \varrho A=\frac{\varrho a b}{a^{2}+b^{2}}, \quad \cos \varrho A=\frac{b^{2}-a^{2}}{b^{2}+a^{2}}, \tan 2 A=\frac{9 a b}{b^{2}-a^{2}}$.
3. $\quad \operatorname{Sin}\left(45^{\circ} \pm A\right) \neq \frac{1}{\sqrt{2}}\left(\frac{b \pm a}{c}\right), \cos \left(45^{\circ} \pm A\right)$

$$
=\frac{1}{\sqrt{ } g}\left(\frac{b \mp a}{c}\right), \quad \tan \left(45^{\circ} \pm A\right)=\frac{b \pm a}{b \mp a}
$$

4. $\operatorname{Sin}(A-B)=\frac{a^{2}-b^{2}}{c^{2}}, \cos (A-B)=\frac{2 a b}{c^{2}}$,

$$
\tan (A-B)=\frac{a^{2}-b^{2}}{2 a b}
$$

5. The area $=\frac{a b}{2}=\frac{a}{2} \sqrt{c^{2}-a^{2}}=\frac{b}{2} \sqrt{c^{2}-b^{2}}=\frac{c^{2}}{4} \sin 2 A$

$$
=\frac{c^{2}}{4} \sin 2 B=\frac{a^{2}}{2} \tan B=\frac{\bar{b}^{2}}{2} \tan A .
$$

6. If $a^{\prime}, b^{\prime}$ be the segments of the hypothenuse made by a line bisecting the right angle, then

$$
\frac{a^{\prime} b^{\prime}}{a b}=\frac{a^{2}+b^{2}}{(a+b)^{2}}
$$

7. If the lines drawn from the acute angles to bisect the opposite sides be $\alpha, \beta$, the tangents of these angles are

$$
\sqrt{\frac{4 a^{2}-\beta^{2}}{4 \beta^{2}-a^{2}}}, \text { and } \sqrt{\frac{4 \beta^{2}-a^{2}}{4 \alpha^{2}-\beta^{2}}}
$$

8. The radius of the inscribed circle is

$$
\frac{1}{2}(a+b-c)=\frac{1}{2} \frac{(a+c-b)(b+c-a)}{a+b+c}=\frac{a b}{a+b+c} .
$$

9. The radius of the cirumscribed circle is

$$
\frac{a b c}{(a+b+c)(a+b-c)}=\frac{a b c}{(b+c-a)(a+c-b)}=\frac{c}{\Omega} .
$$

10. The sum of the diameters of the inscribed and circumscribed circles is $a \nmid b$.
11. If $A, B, C$ be the angles of a triangle, and $2 \cos B$ $=\frac{\sin A}{\sin C}$, the triangle is isosceles.
12. If $\tan A=\frac{\sin ^{2} A}{\sin ^{2} B}$, then will the triangle be either isosceles, or right-angled at $C$.
13. In any oblique-angled triangle, if $A, B, C, a, b, c$ be the angles and sides respectively, then
$\frac{a+b}{a-b}=\frac{\cot \frac{C}{2}}{\tan \left(\frac{A-B}{2}\right)}$, and $\frac{c}{a-b}=\frac{\tan \frac{A}{Q}+\tan \frac{B}{Q}}{\tan \frac{A}{2}-\tan \frac{B}{2}}$.
14. $\frac{\text { Vers } A}{\operatorname{Vers} B}=\frac{a(S-b)}{b(S-a)}$, and $\frac{\operatorname{vers}(A+B)}{\operatorname{vers} C}=\frac{(S-a)(S-b)}{S(S-c)}$.
15. $\operatorname{Tan} \frac{A}{2} \tan \frac{B}{2}=\frac{S-c}{S}$, and $\frac{\tan \frac{A}{2}}{\tan \frac{B}{Q}}=\frac{S-b}{S-a}$.
16. $\operatorname{Sin}\left(\frac{A-B}{2}\right)=\left(\frac{a-b}{c}\right) \cos \frac{C}{2}, \cos \left(\frac{A-B}{2}\right)=\left(\frac{a+b}{c}\right) \sin \frac{C}{2}$,

$$
\text { and } \sin (A-B)=\left(\frac{a^{2}-b^{2}}{c^{2}}\right) \sin (A+B)
$$

17. $\operatorname{Tan}\left(\frac{A+B-C}{2}\right)=\frac{a^{2}+b^{2}-c^{2}}{4 \sqrt{S(S-a)(S-b)(S-c)}}$,
and $\frac{\tan \left(\frac{A+B-C}{2}\right)}{\tan \left(\frac{A+C-B}{Q}\right)}=\frac{a^{2}+b^{2}-c^{2}}{a^{2}+c^{2}-b^{2}}=\frac{\tan B}{\tan C}$.
T
18. The perpendicular from the angle $\mathbb{C}$ upon the opposite side

$$
=\frac{a+b+c}{\cot \frac{A}{2}+\cot \frac{B}{2}}=\frac{c}{2} \frac{\cos (A-B)-\cos (A+B)}{\sin C} .
$$

19. The distance of the perpendicular from the middle of the base

$$
=\frac{c}{2} \frac{\sin (A-B)}{\sin C}=\frac{c}{2} \frac{\tan A-\tan B}{\tan A+\tan B}
$$

20. The area of the triangle $=\frac{c^{2}}{2} \frac{\sin A \sin B}{\sin C}$
$=\frac{c^{2}}{4} \frac{\cos (A-B)-\cos (A+B)}{\sin (A+B)}=\frac{Q a b c}{a+b+c} \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$.
21. Four times the area of the triangle

$$
\begin{gathered}
=\frac{(a+b+c)^{2}\{\cos A+\cos B+\cos C-1\}}{\sin A+\sin B+\sin C} \\
=\left(a^{2}+b^{2}-c^{2}\right) \cot \left(\frac{A+B-C}{2}\right)
\end{gathered}
$$

$=\sqrt{\frac{4\left(a^{4}+b^{4}+c^{4}\right)-\left(a^{2}+b^{2}+c^{2}\right)^{2}}{\cot ^{2} A+\cot ^{2} B+\cot ^{2} C}}=\frac{a^{2}+b^{2}+c^{2}}{\cot A+\cot B+\cot C}$.
22. The sum of the perpendiculars from the angles upon the opposite sides $=2$ area $\left(\frac{a b+a c+b c}{a b c}\right)$.
23. If $R$ and $r$ be the radii of the circumscribed and inscribed circles, then

$$
2 R r=\frac{a b c}{a+b+c}, \text { or } \frac{1}{2 R r}=\frac{1}{a b}+\frac{1}{a c}+\frac{1}{b c} .
$$

24. The diameter of the inscribed circle

$$
=(a+b+c) \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{a}
$$

$=\frac{2 \sqrt[3]{a^{2} b^{2} c^{2} \sin A \sin B \sin C}}{a+b+c}=\frac{a b c(\sin A+\sin B+\sin C)}{2(a+b+c)^{2}}$.
25. The diameter of the circumscribed circle

$$
\begin{gathered}
=\frac{a+b+c}{\sin A+\sin B+\sin C}=\sqrt[5]{\frac{a b c}{\sin A \sin B \sin C}} \\
=\frac{a+b+c}{4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}} .
\end{gathered}
$$

26. The sum of the squares of the distances of the centre of the inscribed circle from the angular points

$$
=a b+a c+b c-\frac{6 a b c}{a+b+c}
$$

27. If $d$ be the line drawn to bisect the angle $C$, and meeting the opposite side in $E$, then

$$
\begin{gathered}
\tan A E C=\frac{a+b}{a-b} \tan \frac{C}{2} \\
\cos A C E=\frac{(a+b) d}{2 a b}, \text { and } d^{2}=\frac{a b}{(a+b)^{2}}\left\{(a+b)^{2}-c^{2}\right\}
\end{gathered}
$$

28. If $P$ and $p$ be the perpendiculars from the extremities of the base of a triangle upon the line bisecting the vertical angle at distances $D$ and $d$ from it, then

$$
\begin{gathered}
4 P p=(a-b+c)(b+c-a), \text { and } 4 D d=(a+b+c)(a+b-c), \\
\text { and the area }=P d=p D .
\end{gathered}
$$

29. The perpendiculars drawn from the angles of a triangle upon the opposite sides meet in one point, and the rectangles of the segments of the perpendiculars are equal in each.
30. If three straight lines be drawn from the angles of a triangle to bisect the opposite sides, they meet in the same point : and the sum of the squares of the sides of the triangle is equal to three times the sum of the squares of the distances of the point of intersection from the angles.
31. If lines be drawn from the angles of a triangle to any point, the products of the sines of the angles thus formed taken alternately are equal, as are also the products of the alternate segments of the sides.
32. If two angles of a triangle be bisected by lines meetiug in a point, the remaining angle will be bisected by the line joining it with this point.
33. The perpendiculars to the three sides of a triangle at their middle points, meet in one point.
34. If $a, b, c$ be the sides of a triangle, and $a^{\prime}, b^{\prime}, c^{\prime}$ the perpendiculars drawn from a point within the triangle, to bisect the sides, then

$$
4\left\{\frac{a}{a^{\prime}}+\frac{b}{b^{\prime}}+\frac{c}{c^{\prime}}\right\}=\frac{a b c}{a^{\prime} b^{\prime} c^{\prime}} .
$$

35. The side of an equilateral triangle inscribed in a circle : the side of a square inscribed in the same circle $:: \sqrt{3}: \sqrt{2}$, and the area of the triangle : the area of the square $:: 3 \sqrt{3}$ : $2 \sqrt{2}$.
36. The square of the side of a pentagon inscribed in a circle is equal to the sum of the squares of the sides of a regular hexagon and decagon inscribed in the same circle.
37. If a point be assumed in a regular polygon of $n$ sides, from which perpendiculars are drawn to each of the sides or sides produced; the sum of these perpendiculars : the radius of the inscribed circle :: $n: 1$.
38. If the external angles of a quadrilateral figure be denoted by $a, \beta, \gamma, \delta$, and the sides by $a, b, c, d$, then

$$
\begin{aligned}
& a \sin \alpha+b \sin (a+\beta)+c \sin (a+\beta+\gamma)+d \sin (a+\beta+\gamma+\delta)=0 \\
& a \cos a+b \cos (a+\beta)+c \cos (a+\beta+\gamma)+d \cos (a+\beta+\gamma+\delta)=0 .
\end{aligned}
$$

39. The area of a regular polygon inscribed in a circle is a mean proportional between the areas of an inscribed, and of a circumscribed regular polygon of half the number of sides.
40. The area of a regular polygon circumscribed about a circle is an harmonical mean between the areas of an inscribed regular polygon of the same number of sides, and of a circumscribed regular polygon of half that number.
41. If an equilateral polygon of $2^{n}$ sides be inscribed in a circle whose radius is 1 , the side $=\sqrt{2-\sqrt{2+\sqrt{2+\& c}}}$. the radical sign being repeated $n$ times.
42. If the diagonals of a quadrilateral whose opposite angles are supplemental to each other, intersect at right angles, their segments are proportional to the rectangles of the sides which are terminated at their extremities.
43. In every polygon, any one side is equal to the sum of the products of each of the other sides and the cosine of the angle made by it with the aforesaid side.
44. In every polygon, the perpendicular upon a side from any of the angular points is equal to the sum of the products of the sides comprised between that point and side, and the sines of their respective inclinations to that side.
45. The square of a side of any polygon is equal to the sum of the squares of all the other sides, diminished by twice the sum of the products of all those sides, taken two and two together, and the cosines of the included angles.
46. Twice the area of any polygon is equal to the sum of the products of its sides except one taken two and two together, and the sines of the sums of the exterior angles contained by those sides produced.
47. A circle is inscribed in an equilateral triangle, an equilateral triangle in the circle, a circle in the last triangle, and so on, in infinitum: then the radius of any one of these circles is equal to the sum of the radii of all those within it.
48. If lines be drawn from all the angles of a polygon to any point, the products of the sines of the angles so formed taken alternately are equal.
49. In a right-angled triangle, a perpendicular is drawn from the right angle to the opposite side: then the areas of the circles inscribed in the triangles made by it, are proportional to the corresponding segments of the side.
50. In a plane triangle, the differences of the segments of a side made by a perpendicular from the opposite angle, by the contact of the inscribed circle, and by the line bisecting the opposite angle, are in geometrical progression.
51. If through any point $O$ within a triangle, straight lines be drawn from the angles $A, B, C$ to meet the opposite sides in $a, b, c$ respectively, then

$$
\frac{O a}{A a}+\frac{O b}{B b}+\frac{O c}{C c}=1 .
$$

52. In any triangle, the rectangle contained by the excess of the semi-perimeter above each of the sides including any angle, is equal to the rectangle of the radius of the inscribed circle and the radius of the circle which touches the base and the two sides produced.
53. If $r$ be the radius of a circle inscribed in a triangle, $r_{1}, r_{2}, r_{3}$ the radii of three other circles touching the sides and sides produced, of the same triangle, then $\frac{1}{r}=\frac{1}{r_{1}}+\frac{1}{r_{2}}+\frac{1}{r_{3}}$, and the area of the triangle $=\sqrt{r r_{1} r_{2} r_{3}}$.
54. If $R$ be the radius of the circle circumscribed about a triangle, $r$ the radius of the circle inscribed in it, the distance between the centres of these circles is $\sqrt{R^{2}-2 R r}$.
55. If $r_{1}, r_{\Omega}, r_{3}$ be the radii of the circles touching one side of a triangle and the two others produced, the distances of their centres from that of the circumscribed circle whose radius is $R$ are

$$
\sqrt{R^{2}+2 R r_{1}}, \sqrt{R^{2}+2 R r_{2}}, \text { and } \sqrt{R^{2}+2 R r_{3}}
$$

## II. Problems.

1. Given the three sides of a triangle, to find the perpendicular upon one side from the opposite angle, and the segments into which that side is divided.
2. Given the perimeter and area of a right-angled triangle, to find the sides.
3. Given the perimeter of a triangle, to find the sides, when a perpendicular from one of the angles to the opposite side divides that side in a given ratio.
4. Given one angle of a triangie, and the straight lines drawn from each of the other angles to bisect the opposite sides, to find the sides of the triangle.
5. Express the perimeter of a triangle in terms of two of the angles, and the perpendicular from the remaining angle upon the opposite side.
6. Given the lines bisecting the acute angles of a rightangled triangle and terminated by the opposite sides, to find the area of the triangle.
7. Given the perimeter of a triangle and the ratios of its angles, to find the sides.
8. In a right-angled triangle, given one of the sides containing the right angle and the radius of the inscribed circle, to find the sides.
9. Given the hypothenuse of a right-angled triangle and the radius of the inscribed circle, to find the sides.
10. Given the perimeter of a right-angled triangle and the radius of the inscribed circle, to find the sides.
11. Given the area of a right-angled triangle and the radius of the circle inscribed in it, to find the sides.
12. Given the three angles of a triangle and the radius of the inscribed circle, to find the sides.
13. Express the area of a triangle in terms of the radius of the inscribed circle and the three angles.
14. Given the three angles of a triangle and the radius of the circumscribed circle, to find the sides.
15. Express the area of a triangle as a function of the radius of the circumscribed circle and the three angles.
16. Investigate an expression for the area of a triangle involving all the sides, and the tangents of all the semi-angles.
17. In a right-angled triangle, given the radii of the inscribed and circumscribed circles, to find the sides and area.
18. In any triangle, given the vertical angle, the radius of the inscribed circle, and the sum of the lines drawn from its centre to the angles at the base, to find the sides.
19. Given the perimeter, the area and one angle of a triangle, to find the side opposite to it.
20. Given the area, the vertical angle and the sum of the including sides, to find the sides of the triangle.
21. Given the radius of the circumscribed circle, the vertical angle and the ratio of the sides containing it, to find the sides of the triangle.
22. If a circle be described about a triangle, find the distances of the bisections of the sides from the circumference.
23. Given the three straight lines drawn from the angles of a triangle to bisect the opposite sides, to find the sides of the triangle.
24. Express the area of a triangle in terms of two of its sides and an angle opposite one of them: also, in terms of two of its angles and a side opposite one of them.
25. Determine the triangle whose sides are three consecutive natural numbers, and whose greatest angle is double of the least.
26. Find the angles of a triangle, when the base, the sides, and the perpendicular are in continued geometrical progression.
$2 \%$. Given three straight lines, to find the radius of the circle so that they shall be the chords of three contiguous arcs which together make a semi-circle.
27. In a given scalene triangle, it is required to draw from one side to another produced, a straight line which shall be bisected by the third side.
28. Given the perpendiculars from the angles upon the opposite sides of a triangle, to find the angles and sides.
29. Given the area, the base and the sum of the angles at the base of a triangle, to find the angles.
30. Given the angles of a triangle and the perpendiculars upon the sides from a given point within it, to find the sides.
31. Given the augles of a triangle and the perpendiculars upon the sides from a given point without it, to find the sides.
32. Compare the sides and areas of the squares and regular octagons described in, and about, the same circle.

$$
\mathrm{U}_{\mathrm{u}}
$$

34. Given the ratio of the side of a regular polygon inscribed in a circle to the radius, to find the number of sides and the magnitude of each angle.
35. The alternate angles of a regular pentagon being joined, it is required to compare the sum of the isosceles triangles so formed with the pentagon.
36. Find the side and area of a regular decagon inscribed in a given circle.
37. Determine the sides of a regular hexagon and dodecagon inscribed in the same circle, and compare their perimeters and areas.
38. The area of a regular polygon of $n$ sides in a circle : the area of another regular polygon of $3 n$ sides in the same circle $:: p: q$ : find the values of the angles subtended by a side of each at the centre.
39. The area of a regular polygon inscribed in a circle : the area of a similar figure circumscribed about it :: $3: 4$; find the number of sides.
40. Find the side of a regular quindecagon inscribed in a circle of given radius.
41. The area of a regular polygon inscribed in a circle being given, and the area of one circumscribed with the same number of sides, it is required to find the areas of the inscribed and circumscribed polygons of half the number of sides.
42. Find the area included between two regular polygons of the same number of sides, one being inscribed in, and the other circumscribed about, a circle of given radius; and determine the number of sides when this area has a given ratio to either.
43. Given two sides and the included angle of a quadrilateral, to find the sides and diagonals, when two opposite angles are right augles.
44. Express the area of any quadrilateral in terms of all the sides, and two of the opposite angles.
45. Given one side of a polygon and the angles made by it with the lines drawn from its extremities to all the other angles, to find the area of the polygon.
46. A circle has an equilateral triangle inscribed in it; a circle is iuscribed in the triangle which also has an equilateral triangle inscribed in it, and so on: find the sums of the perimeters and the areas of all the circles and triangles.

> ON CHAP. V. VI. VII.

Theorems and Problems.

1. $c=\sqrt{(a-b)^{2}+4 a b \sin ^{2} \frac{C}{2}}=\sqrt{(a+b)^{2}-4 a b \cos ^{2} \frac{C}{2}}$

$$
\begin{aligned}
& =\sqrt{(a+b)^{2} \sin ^{2} \frac{C}{2}+(a-b)^{2} \cos ^{2} \frac{C}{2}} \\
& =\frac{a}{\cos B+\sin B \cot C}=b(\cos A+\sin A \cot B) .
\end{aligned}
$$

2. $\operatorname{Sin} C=\frac{c \sin \Lambda}{\sqrt{b^{2}+c^{2}-2 b c \cos \Lambda}}$
$=\frac{\sin B\left(a \cos B \pm \sqrt{b^{2}-a^{2} \sin ^{2} B}\right.}{b}$
$=\frac{a}{\varrho b} \sin 2 B \pm \sin B \sqrt{1-\left(\frac{a}{b}\right)^{2} \sin ^{2} B}$.

$$
\begin{aligned}
& \text { 3. } \operatorname{Cos} C=\frac{a-c \cos B}{\sqrt{a^{2}+c^{2}-2 a c \cos B}} \\
& \quad=\frac{a}{b} \sin ^{2} B \mp \cos B \sqrt{1-\left(\frac{a}{b}\right)^{2} \sin ^{2} B}
\end{aligned}
$$

$$
\text { 4. } \operatorname{Tan} C=\frac{c \sin A}{\sqrt{a^{2}-c^{2} \sin ^{2} A}}=\frac{c \sin A}{b-c \cos A}
$$

$$
=\frac{b \cos A+\sqrt{a^{2}-b^{2} \sin ^{2} A}}{b \cos A-\cot A \sqrt{a^{2}-b^{2} \sin ^{2} A}}
$$

$$
=\frac{1+\sec A \sqrt{\left(\frac{a}{b}\right)^{2}-\sin ^{2} A}}{1-\operatorname{cosec} A \sqrt{\left(\frac{a}{b}\right)^{2}-\sin ^{2} A}}
$$

5. If $p$ be the perpendicular upon the side $c$ from the opposite angle $C$, then are the other two sides respectively equal to

$$
\begin{aligned}
& \sqrt{c^{2}+p c \cot C}+\sqrt{c^{2}+p c \tan C}, \\
& \text { and } \sqrt{c^{2}+p c \cot C}-\sqrt{c^{2}+p c \tan C}
\end{aligned}
$$

6. Given the angle $B$, the side $c$ and the sum of the remaining sides, to solve the triangle
7. Given the angle $B$, the side $c$ and the ratio of the remaining sides, to solve the triangle.
8. Given the angle $B$, the side $a$ and the area, to solve the triangle.
9. Given the area, the base and the sum of the angles at the base, to solve the triangle.
10. Given the logathms of the three sides of a plane friangle, to determine the logarithms of the segments of one of them made by a line bisecting the opposite angle.
11. If the base of an isosceles triangle be $c$ and the perpendicular from one of the equal angles upon the opposite side $p$, then

$$
\log \text { area }=\log p+2 \log c-Q \log Q-\frac{1}{2} \log \{(c+p)(c-p)\} .
$$

12. If $\tan A \tan B=3$, and $\frac{\sin A}{\sin B}=\frac{a}{b}$, and we assune

$$
\tan 2 \phi=\frac{2 a b}{3\left(a^{2}-b^{2}\right)} \text {, then will }
$$

$$
\tan A=\sqrt{\frac{3 a \cot \phi}{b}} \text {, and } \tan B=\sqrt{\frac{3 b \tan \phi}{a}} \text {. }
$$

13. In finding the sine of half an are, shew that when $\theta$ is small, a large error may be expected in applying the formula

$$
\sin \frac{\theta}{2}=\sqrt{\frac{1-\cos \theta}{2}},
$$

and a small one in using the formula

$$
\sin \frac{\theta}{2}=\frac{1}{2} \sqrt{1+\sin \theta}-\frac{1}{2} \sqrt{1-\sin \theta} .
$$

14. The Sun's altitude being $30^{\circ}$, find the position of a stick of given length that the shadow may be the longest possible, and determine the shadow's length.
15. The aspect of a wall is due south and the Sun is in the south east at an altitude of $30^{\circ}$ : find the breadth of the wall's shadow.
16. A person attempts to swim directly across a stream of given breadth, where will he reach the opposite side, if he swim $n$ times as far as he would have done, had there been no current, and what angle does his course make with it?
17. Three objects $A, B, C$ form an isosceles triangle whose vertex is $B$ and whose angles are as numbers $4,1,1$ : a person walking from $A$ towards $C$ meastres a base $\Lambda D=a$ feet and observes the angle $B D C$ : he then advances to $E, b$ feet
farther, and finds the angle $B E C$ the supplement of $B D C$ : find the sides of the triangle.
18. Coasting along shore observed two headlands, the first bore N. N. W., and the second N. E. by E.: then steering 12 miles E. N. E., the first bore N. W. and the second N. E.: shew how the distance and bearing of the two headlands from each other may be found.
19. A person on a tower can see the top of a pillar of known altitude from which he wishes to know his distance and the height of the tower: he can see also an object on the horizontal plane from which he has formerly observed the angular distance of the tops of the tower and pillar: shew how he may find the required distances.
20. For determining the distance between two inaccessible objects $A$ and $B$, two positions $C$ and $D$ are taken such that the triangles $A C D, B C D$ are not in the same plane: state the requisite observations for determining their distance, and the bearing and elevation of one as seen from the other, and give the solutions of the triangles in logarithms.
21. A person wishing to ascertain the horizontal distance of two inaccessible objects from each other, can find no point from which they are visible together: he finds however two stations the distance between which he can determine, from which the objects may be separately seen: explain what observations and measurements it will be necessary for him to make, and how they must be applied to effect his purpose.
22. The top of a tower is visible from three stations $A, B, C$ in the same horizontal plane: at each of the stations the angular distance of the top of the tower from each of the other two stations is observed: given the distance between $A$ and $B$, and the height of the tower, to find the distance of $C$ from each of the other stations and from the tower.
23. A hill rises due north at an angle of $45^{\circ}$, and a shaft was discovered in it making an angle of $60^{\circ}$ with the horizon, and extending 100 feet in a north cast direction which led into a cavern stretching horizontally to the north cast. At the
foot of the hill, 300 feet in a south west direction from the mouth of the shaft, another opening was found extending horizontally 120 feet due north: find the length and direction of the least shaft that can be cut from the extremity of this opening to reach the line of the cavern.
24. Prove that $\frac{60^{\circ}}{12}=\left(2^{2}-\frac{1}{2^{2}}\right)^{3}$ seconds $=52^{\prime \prime} 44^{\prime \prime \prime} 3^{1 \mathrm{v}} 45^{\mathrm{v}}$.
25. If the sides of a triangle be $a, b, c$,

$$
\text { and } x+\frac{1}{x}=2 \cos A, y+\frac{1}{y}=2 \cos B \text {, then } \frac{a}{y}+b x=c .
$$

26. If $c_{0}=\cos A \cos B \cos C \& c ., c_{n}=$ the sum of the products of all the cosines but $n$, multiplied by the sines of those $n$, then

$$
\begin{aligned}
\cos (A+B+C+\& c .) & =c_{0}-c_{2}+c_{4}-\& c \\
\sin (A+B+C+\& c .) & =c_{1}-c_{3}+c_{5}-\& c
\end{aligned}
$$

27. Prove that
$2 \sqrt{-1} \sin n A=(\cos A+\sqrt{-1} \sin A)^{n}-(\cos A-\sqrt{-1} \sin A)^{n}$, and $2 \cos n A=(\cos A+\sqrt{-1} \sin A)^{n}+(\cos A-\sqrt{-1} \sin A)^{n} ;$ and adapt them to the radius $r$.
28. Shew that
$2 \sin A=\sqrt[n]{\sqrt{\sin ^{2} n A-1}+\sin n A}-\frac{1}{\sqrt[n]{\sqrt{\sin ^{2} n A-1}+\sin n A}}$, and
$2 \cos A=\sqrt[n]{\sqrt{\cos ^{2} n A-1}+\cos n A}+\frac{1}{\sqrt[n]{\sqrt{\cos ^{2} n A-1}+\cos n A}} ;$ and adapt them to the radius $r$.
29. Find the sum of the $n^{\text {th }}$ powers of the tangent and cotangent of an arc.
30. Solve the equation $x^{5}-6 x-4=0$, by means of a table of natural sines and cosines.
31. Solve $x^{3}-49 x-120=0$, by means of trigonometrical formula.
32. Solve the equation $x^{3}-\frac{1}{2} x^{2}-\frac{1}{2} x+\frac{1}{4}=0$, by the trisection of an arc.
33. Solve $x^{3}-3 a x^{2}-3 x+a=0$, by means of an arc whose tangent is $a$.
34. Solve $x^{4}+4 x^{3}-6 x^{2}-4 x+1=0$, by means of an arc of $45^{\circ}$.
35. Determine the roots of $x^{4}+2 x^{3}-x^{2}-2 x+1=0$, by Trigonometry.
36. If $\cos \theta=\frac{a}{\sqrt{a^{2}+b^{2}}}$, then will
$(a+b \sqrt{-1})^{\frac{m}{n}}+(a-b \sqrt{-1})^{\frac{m}{n}}=2\left(a^{2}+b^{2}\right)^{\frac{m}{2 n}} \cos \frac{m}{n} \theta$.
37. If the quadrant of a circle be divided into an odd number of equal parts so that $\theta=\frac{\pi}{2(2 n-1)}$, then to the radius $r$,
$\sin \theta \sin 3 \theta \sin 5 \theta$ \&c. $\sin (2 n-3) \theta=\left(\frac{r}{2}\right)^{n-1}$,
and $\cos \theta \cos 3 \theta \cos 5 \theta$ \&c. $\cos (2 n-3) \theta=\sqrt{2 n-1}\left(\frac{r}{2}\right)^{n-1}$.
38. If the semi-circumference of a circle be divided into an odd number of equal parts so that $\theta=\frac{\pi}{2 n-1}$, then
$\sin \theta \sin 2 \theta \sin 3 \theta$ \&c. $\sin (n-1) \theta=\sqrt{2 n-1}\left(\frac{r}{2}\right)^{n-1}$,
and $\cos \theta \cos 2 \theta \cos 3 \theta \& c .(n-1) \theta=\binom{r}{2}^{n-1}$.
39. If the semi-circumference of a circle be divided into an even number of equal parts so that $\theta=\frac{\pi}{2 n}$, then $\operatorname{chd} \theta \operatorname{chd} 3 \theta \operatorname{chd} 5 \theta \& c$. chd $(2 n-1) \theta=\sqrt{2 r^{n}}$.
40. If the circumference of a circle be divided into any number of equal parts as $n$ of which $\theta$ is one, then $\operatorname{chd} \theta$ chd $2 \theta$ chd $3 \theta$ \& c. chd $(n-1) \theta=n r^{n-1}$.
41. Prove that $\sin \theta+\sin (\theta+\delta)+\sin (\theta+2 \delta)+\& c$. is a recurring series, find the scale of relation, and by means of it the sum of $n$ terms.
42. Prove that

$$
\begin{aligned}
& \quad \frac{\sin \theta+\sin 2 \theta+\sin 3 \theta+8 c . \text { to } n \text { terms }}{\cos \theta+\cos 2 \theta+\cos 3 \theta+\& c . \text { to } n \text { terms }}=\tan (n+1) \frac{\theta}{2}, \\
& \text { and } \frac{\sin \theta+\sin 3 \theta+\sin 5 \theta+\& c . \text { to } n \text { terms }}{\cos \theta+\cos 3 \theta+\cos 5 \theta+\& c . \text { to } n \text { terms }}=\tan n \theta .
\end{aligned}
$$

43. Find the sum of vers $\theta+\operatorname{vers} 2 \theta+\operatorname{vers} 3 \theta+\& c$. to $n$ terms.
44. If the circumnference of a circle be divided into any odd number of equal parts so that $\theta=\frac{2 \pi}{(2 n-1)}$, then to the radius $r$, $\operatorname{chd}^{2} \theta+\operatorname{chd}^{2} 2 \theta+\operatorname{chd}^{2} 3 \theta+\& c .+\operatorname{chd}^{2}(2 n-2) \theta=(4 n-2) r^{2}$.
45. If the circumference of a circle be divided into an even number of equal parts as $2 n$, then the sums of the squares of the alternate chords are equal to each other and to $2 n r^{2}$.
46. On the same supposition, the product of the squares of the odd chords together with the product of the squares of the even chords $=4 r^{2 n}$.
47. If the circumference of a circle whose radius is $r$, be divided into $2 n$ equal parts and from one of the points straight lines be drawn to all the rest, the sum of all these lines

$$
=2 r \cot \frac{45^{\circ}}{u}
$$

48. If $A, B, C, D, \& c$. be the angular points of an equitateral polygon of $m$ sides inscribed in a circle whose radius is $r$, and $P$ be any point in the circumference, then $P A^{2 n}+P B^{2 n}+P C^{2 n}+\mathcal{E} \mathbf{c}=m$ times the middle term of $\left(1+r^{2}\right)^{2 n}$, if $n$ be less than $m$.
49. Find the sum of all the natural sines to every minate in the quadrant.
50. Sum the series
$\cos \theta+\frac{1}{2} \cos 2 \theta+\frac{1}{3} \cos 3 \theta+\& c$. to $n$ terms.
51. Sum the series
$\cos ^{2} \theta+2 \cos ^{2} 2 \theta+3 \cos ^{2} 3 \theta+8 c$. to $n$ terms.
52. Sum the series
$\sin \theta \cos \phi+\sin 2 \theta \cos 3 \phi+\sin 3 \theta \cos 5 \phi+\& c$. to $n$ terms.
53. Sum the series
$\tan \theta \sec ^{2} \theta+\frac{1}{2} \tan \frac{\theta}{2}\left(\frac{1}{2} \sec \frac{\theta}{2}\right)^{2}+\frac{1}{4} \tan \frac{\theta}{4}\left(\frac{1}{4} \sec \frac{\theta}{4}\right)^{2}+s c$ to $n$ terms.
54. Sum the series

$$
\sin \theta\left(\sin \frac{\theta}{2}\right)^{2}+2 \sin \frac{\theta}{2}\left(\sin \frac{\theta}{4}\right)^{2}+4 \sin \frac{\theta}{4}\left(\sin \frac{\theta}{8}\right)^{2}+\varepsilon c .
$$

to $n$ terms and to infinity.
55. Sum the series

$$
\tan \theta\left(\tan \frac{\theta}{2}\right)^{2}+2 \tan \frac{\theta}{2}\left(\tan \frac{\theta}{4}\right)^{2}+4 \tan \frac{\theta}{4}\left(\tan \frac{\theta}{8}\right)^{2}+\& c .
$$

to $n$ terms and to infinity.
56. Sum $\sec ^{2} \theta+4 \sec ^{2} 2 \theta+16 \sec ^{2} 4 \theta+64 \sec ^{2} 8 \theta+\& c$. to $n$ terms.
57. Prove that $x \sin \theta+\frac{x^{2}}{2} \sin 2 \theta+\frac{x^{3}}{3} \sin 3 \theta+\& c$. in inf.

$$
=\tan ^{-1}\left\{\frac{x \sin \theta}{1-x \cos \theta}\right\} .
$$

58. Sum the series

$$
e^{\theta} \sin \theta-\frac{e^{2 \theta}}{2} \sin 2 \theta+\frac{e^{3 \theta}}{3} \sin 3 \theta-\& c . \text { to infinity. }
$$

59. Resolve $\left(a^{2}-2 a b \cos \theta+b^{2}\right)^{-2 m}$ into a series of cosines of $\theta$ and its multiples, by means of the equation

$$
2 \cos n \theta=x^{n}+\frac{1}{x^{n}}
$$

and the binomial theorem.

## APPENDIX II.

CONTAINING MISCELLANEOUS THEOREMS AND PROBLEMS IN SPHERICAL TRIGONOMETRY.

## Theorems and Problems.

1. In a right-angled spherical triangle wherein $C$ is the right angle,

$$
\begin{aligned}
& \frac{\sin (a-b)}{\sin (a+b)}=\tan \frac{A+B}{2} \tan \frac{A-B}{2}, \text { and } \frac{\sin (c-b)}{\sin (c+b)}=\tan ^{2} \frac{A}{2} \\
& \text { 2. } \operatorname{Tan}^{2} \frac{a}{2}=\frac{\tan \left(\frac{B+A}{2}-45^{\circ}\right)}{\tan \left(\frac{B-A}{2}+45^{\circ}\right)}=\tan \left(\frac{c+b}{2}\right) \tan \left(\frac{c-b}{2}\right) .
\end{aligned}
$$

3. $\operatorname{Tan}^{2} \frac{c}{2}=-\frac{\cos (A+B)}{\cos (A-B)}=\frac{\tan \Lambda \tan B-1}{\tan A \tan B+1}$, and $2 \cos c=\cos (a+b)+\cos (a-b)$.
4. $\operatorname{Tan}\left(45^{\circ}+\frac{A}{2}\right)=\frac{\tan \left(\frac{c+a}{2}\right)}{\tan \frac{b}{2}}$,

$$
\begin{aligned}
\cot \left(45^{\circ}+\frac{A}{2}\right) & =\frac{\tan \left(\frac{c-a}{2}\right)}{\tan \frac{b}{2}} \\
\text { and } \cot ^{2}\left(45^{\circ}+\frac{A}{2}\right) & =\frac{\tan \left(\frac{c-a}{2}\right)}{\tan \left(\frac{c+a}{2}\right)}
\end{aligned}
$$

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5. $\operatorname{Tan}^{2}\left(45^{\circ}+\frac{a}{2}\right)=\frac{1+\sin c \sin A}{1-\sin c \sin A}=\frac{\sin (B+b)}{\sin (B-b)}$,
$\tan ^{2}\left(45^{\circ}+\frac{c}{2}\right)=\frac{\tan \left(\frac{B+b}{2}\right)}{\tan \left(\frac{B-b}{2}\right)}, \tan ^{2}\left(45^{\circ}+\frac{A}{2}\right)=\frac{\cot \left(\frac{B+b}{2}\right)}{\tan \left(\frac{B-b}{2}\right)}$.
6. In any spherical triangle whose angles and sides are $A, B, C, a, b, c$,
$\sin \left(\frac{A+B}{2}\right)=\frac{\cos \left(\frac{a-h}{2}\right)}{\cos \frac{c}{2}} \cos \frac{C}{2}, \cos \left(\frac{A+B}{2}\right)=\frac{\cos \left(\frac{a+b}{2}\right)}{\cos \frac{c}{2}} \sin \frac{C}{2}:$
$\sin \left(\frac{A-B}{2}\right)=\frac{\sin \left(\frac{a-\dot{b}}{2}\right)}{\sin \frac{c}{2}} \cos \frac{C}{2}, \cos \left(\frac{A-B}{2}\right)=\frac{\sin \left(\frac{a+b}{2}\right)}{\sin \frac{c}{2}} \sin \frac{C}{2}$.
7. On the same hypothesis,

$$
\begin{aligned}
& \sin \left(\frac{a+b}{2}\right)=\frac{\cos \left(\frac{A-B}{2}\right)}{\sin \frac{C}{2}} \sin \frac{c}{Q}, \cos \left(\frac{a+b}{2}\right)=\frac{\cos \left(\frac{A+B}{2}\right)}{\sin \frac{C}{2}} \cos \frac{c}{2}: \\
& \sin \left(\frac{a-b}{2}\right)=\frac{\sin \left(\frac{A-B}{2}\right)}{\cos \frac{C}{2}} \sin \frac{c}{2}, \cos \left(\frac{a-b}{2}\right)=\frac{\sin \left(\frac{A+B}{2}\right)}{\cos \frac{C}{2}} \cos \frac{c}{2}
\end{aligned}
$$

8. $\operatorname{Sin}(A+B)=\frac{\cos \left(\frac{a+b}{2}\right) \cos \left(\frac{a-b}{2}\right)}{\cos ^{2} \frac{c}{2}} \sin C$,

$$
\sin (A-B)=\frac{\sin \left(\frac{a+b}{2}\right) \sin \left(\frac{a-b}{2}\right)}{\sin ^{2} \frac{c}{2}} \sin C
$$

9. $\operatorname{Sin}(a+b)=\frac{\cos \left(\frac{A+B}{2}\right) \cos \left(\frac{A-B}{2}\right)}{\sin ^{2} \frac{C}{2}} \sin c$,

$$
\sin (a-b)=\frac{\sin \left(\frac{A+B}{2}\right) \sin \left(\frac{A-B}{2}\right)}{\cos ^{2} \frac{C}{2}} \sin c
$$

10. $\operatorname{Sin}^{9} C \sin (a+b) \sin (a-b)=\sin ^{\circ} c \sin (A+B) \sin (A-B)$.
11. $\operatorname{Sin} a \sin c+\cos a \cos c \cos B=\sin A \sin C-\cos A$ $\cos C \cos b$.
12. $\operatorname{Sin} A=\frac{\cos C \sqrt{\sin ^{2} B-\sin ^{2} C \sin ^{2} b+\sin C \cos B \cos b}}{1-\sin ^{2} b \sin ^{2} C}$.
13. $\operatorname{Cos} C=-\cos (A-B) \sin ^{2} \frac{c}{2}-\cos (A+B) \cos ^{2} \frac{c}{2}$.
14. $\operatorname{Sin}^{2} \frac{C}{2}=\cos ^{2}\left(\frac{A-B}{2}\right) \sin ^{2} \frac{c}{2}+\cos ^{2}\left(\frac{A+B}{2}\right) \cos ^{2} \frac{c}{2}$.
15. $\operatorname{Cos}^{2} \frac{C}{2}=\sin ^{2}\left(\frac{A-B}{2}\right) \sin ^{2} \frac{c}{2}+\sin ^{2}\left(\frac{A+B}{2}\right) \cos ^{2} \frac{c}{2}$.
16. $\operatorname{Cos} c=\cos (a-b) \cos ^{2} \frac{C}{2}+\cos (a+b) \sin ^{2} \frac{C}{2}$.
17. $\operatorname{Sin}^{2} \frac{c}{2}=\sin ^{2}\left(\frac{a-b}{2}\right) \cos ^{8} \frac{C}{2}+\sin ^{2}\left(\frac{a+b}{2}\right) \sin ^{2} \frac{C}{2}$.
18. $\operatorname{Cos}^{2} \frac{c}{2}=\cos ^{2}\left(\frac{a-b}{2}\right) \cos ^{2} \frac{C}{2}+\cos ^{2}\left(\frac{a+b}{2}\right) \sin ^{2} \frac{C}{2}$.
19. $\sin ^{2} S=\frac{\left(\sin ^{2} a \sin ^{2} b \sin ^{2} c \sin A \sin B \sin C\right)^{\frac{1}{2}}}{2 \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2}}$.
20. $\operatorname{Cos}^{2} S^{\prime}=\frac{\left(\sin a \sin b \sin c \sin ^{2} A \sin ^{2} B \sin ^{2} C\right)^{\frac{1}{3}}}{2 \cot \frac{a}{2} \cot \frac{b}{2} \cot \frac{c}{2}}$.
21. If $a^{\prime}, b^{\prime}$ be the segments of the base contiguous to the angles $A$ and $B$ respectively, made by a perpendicular arc from the angle $C$, then

$$
\tan a^{\prime}=\frac{\cos a-\cos b \cos c}{\cos b \sin c}, \tan b^{\prime}=\frac{\cos b-\cos a \cos c}{\cos a \sin c},
$$

$\tan \left(\frac{a^{\prime}-b^{\prime}}{2}\right)=\frac{\tan \left(\frac{a+b}{2}\right) \tan \left(\frac{a-b}{2}\right)}{\tan \frac{c}{2}}=\frac{\sin (A-B)}{\sin (A+B)} \tan \frac{c}{2}$.
22. If $A^{\prime}, B^{\prime}$ be the corresponding segments of the vertical angle, then
$\cot A^{\prime}=\frac{\cos B+\cos A \cos C}{\cos A \sin C}, \cot B^{\prime}=\frac{\cos A+\cos B \cos C}{\cos B \sin C}$,
$\tan \left(\frac{A^{\prime}-B^{\prime}}{2}\right)=\frac{\tan \left(\frac{A+B}{2}\right) \tan \left(\frac{A-B}{2}\right)}{\cot \frac{C}{2}}=\frac{\sin (a-b)}{\sin (a+b)} \cot \frac{C}{2}$.
23. In an isosceles triangle, wherein $b=c$, prove that

$$
\sin b=\frac{\sin \frac{a}{2}}{\sin \frac{A}{2}}, \text { and } \sin B=\frac{\cos \frac{A}{2}}{\cos \frac{a}{2}}
$$

24. If the two sides $a, b$ of a spherical triangle be supplemental to each other, then $\sin 2 A+\sin 2 B=0$.
25. In a right-angled spherical triangle whose right angle is $C$,

$$
\tan \frac{A}{2}=\frac{\sin (c-b)}{\sin a \cos b}=\frac{\sin (c-b)}{\tan a \cos c} .
$$

26. On the same hypothesis

$$
\sin ^{2} \frac{c}{2}=\sin ^{2} \frac{a}{2} \cos ^{2} \frac{b}{2}+\cos ^{2} \frac{a}{2} \sin ^{2} \frac{b}{2}
$$

27. The sines of the arcs drawn from the angles of a spherical triangle perpendicular to the opposite sides, are to each other inversely as the sines of the sides upon which they fall, or of the angles from which they are drawn.
28. If $d$ be the length of an arc bisecting the angle $C$ and terminated by the opposite side, then

$$
\tan d=\frac{2 \sin a \sin b}{\sin (a+b)} \cos \frac{C}{2} .
$$

29. If $D$ be the length of the arc drawn from the angle $C$ to bisect the opposite side, then

$$
\cos D=\frac{\sin (A+B)}{\sin C} \cos \frac{c}{2}
$$

30. Draw through a given point in the side of a spherical triangle, an arc of a great circle which shall cut off a given portion of it.
31. Find the whole number of equal and regular figures which may be described upon the surface of a sphere so as exactly to cover it.
32. If the sides of a spherical triangle $A B, A C$ be produced to $b, c$ so that $B b, C c$ shall be the semi-supplements of $A B$ and $A C$ respectively: prove that the arc $b c$ subtends an angle at the centre of the sphere equal to the angle between the chords of $A B$ and $A C$.
33. If each of the sides of a spherical triangle be produced till they meet, three triangles will be formed; and if $r_{1}, r_{2}, r_{3}$ be the circular radii of their inscribed circles, then

$$
\begin{gathered}
\tan r \tan r_{1} \tan r_{2} \tan r_{3} \\
=\sin S \sin (S-a) \sin (S-b) \sin (S-c)
\end{gathered}
$$

34. On the same hypothesis, if $R_{1}, R_{2}, R_{3}$ be the circular radii of their circumscribed circles, then

$$
\begin{gathered}
\cot R \cot R_{1} \cot R_{2} \cot R_{3} \\
=-\cos S^{\prime} \cos \left(S^{\prime}-A\right) \cos \left(S^{\prime}-B\right) \cos \left(S^{\prime}-C\right)
\end{gathered}
$$

35. If a spherical triangle be inscribed in a circle whose pole is in its base, the angle at the vertex of the triangle will be equal to the sum of the angles at the base.
36. If two arcs of great circles terminated by a circle on the surface of the sphere cut one another, the rectangle of the tangents of the semi-segments of one of them is equal to the rectangle of the tangents of the semi-segments of the other.
37. The sums of the opposite angles of a spherical quadrilateral inscribed in a circle are equal to one another.
38. If a spherical quadrilateral be inscribed in a circle, the rectangles of the sines of the semi-diagonals is equal to the sum of the rectangles of the sines of half the opposite sides.
39. In a spherical quadrilateral inscribed in a circle, whose sides are $a, b, c, d$, if $D$ be the diagonal joining $A$ and $C$,

$$
\sin ^{2} \frac{D}{2}=\frac{\left(\sin \frac{a}{2} \sin \frac{d}{2}+\sin \frac{b}{2} \sin \frac{c}{2}\right)\left(\sin \frac{a}{2} \sin \frac{c}{2}+\sin \frac{b}{2} \sin \frac{d}{2}\right)}{\sin \frac{a}{2} \sin \frac{b}{2}+\sin \frac{c}{2} \sin \frac{d}{2}}
$$


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 cas) cruccentices cresermerta mor.

 Cu.

