THE ELEMENTS

OF

PLANE TRIGONOMETRY

BY

WILLIAM P. DURFEE

PROFESSOR OF MATHEMATICS IN HOBART COLLEGE

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PREFACE

In preparing this book the author had two ends in view: First, to give the student an elementary knowledge of the science of Trigonometry, together with an introduction to the theory of functions as illustrated by the trigonometric ratios.

Second, to give him practice in the art of computation. Especial stress has been laid on this side of the subject for the reason that many students have no other experience with the calculation of approximate numbers. The value of preliminary estimates of results and the necessity of frequent *checking* are constantly insisted on. This feature is believed to be novel.

The chapter on the Right Triangle is an informal introduction to the subject. The use of natural functions is advised here that the student may become familiar with them. In the remainder of the book logarithms are used in all computations. In order to meet what seems to be the demand at the present time, the author has worked the illustrative problems with five-place logarithms. Personally he prefers four-place, as they are sufficiently accurate for most practical purposes, and their use permits the student to solve more problems in the limited time at his disposal. By dropping the fractional part of the minute and the fifth figure of a number, where they occur in the data, four-place tables may be used without trouble.

The author is under great obligation to Professor E. W. Davis, of the University of Nebraska, for his criticism and suggestion. Much of the chapter on Computation is due to him.

W. P. D.

GENEVA, August, 1900.

CONTENTS

CHAPTER I — COMPUTATION

CHAPTER II — THE RIGHT TRIANGLE

CHAPTER III — UNLIMITED ANGLES

CHAPTER IV — REDUCTION FORMULAS

CHAPTER V — THE ADDITION THEOREM

CHAPTER VI — THE TRIANGLE

CHAPTER VII — SOLUTION OF THE TRIANGLE

APPENDIX

PLANE TRIGONOMETRY

CHAPTER I

ON COMPUTATION

1. Suppose I measure this table and find it 3 ft. 6 in. long. Is it exactly that? not a shade more nor less? Obviously no one can be certain of this. We find the table to be 3 ft. 6 in. long within some limit of accuracy beyond which we do not care or are not able to go.

Should we, by the refined methods known to science, attempt to get the length to within, say, a millionth of an inch, we should probably find that two different measurements would give discordant results, while neither result would agree with our first rude measurement of 3 ft. 6 in.

This inaccuracy holds of all numbers got by measurement; that is, with the great bulk of numbers with which we have to deal in practical computation.

Nor can we altogether avoid this approximation when the numbers are ideal. For example, the square root of 2 is 1.4 to the nearest tenth, 1.41 to the nearest hundredth, 1.4142 to the nearest ten thousandth; that is to say, the square of each of these numbers is nearer 2 than the square of any other with the same number of places. Similar remarks apply to all surds, to nearly all logarithms, to π, and to the various trigonometric ratios.

It is plain that an error of a foot in a mile is of far less relative importance than an error of an inch in a yard. It is indeed a wonderful triumph of measurement and calculation to have determined the sun's distance within some hundred thousand miles, the whole distance being 92,900,000. This being the accuracy of the sun's distance, and, furthermore, the earth's orbit being only approximately circular, it would be impossible to determine the length of the orbit with an uncertainty less than some hundred thousand miles, even though we knew the value of π to a million places.

2. The sum of a number of approximate numbers cannot be accurate beyond the place where accuracy ceases in any one of them.

Suppose, for example, two men had measured parts of the same line, one finding his end 307.492 ft. long and the other his end 602.43 ft. The length of the whole line is

$$\begin{array}{r} 307.492 \\ 602.43 \\ \hline 909.92 \end{array}$$

Since the second man did not measure to thousandths, the result cannot be accurate beyond hundredths.

A product cannot have a greater degree of accuracy than that of its least accurate factor. Suppose we wish the product of the approximate numbers 23.57 and 612.3. The approximate number 23.57 may have any value from 23.565 to 23.575, while 612.3 lies between 612.25 and 612.35. This product may be anything

from $\quad 23.565 \times 612.25 = 14427.67125$

to $\quad 23.575 \times 612.35 = 14436.15125.$

The mean of these results is 14432, but we cannot be sure of the last figure. We do feel sure, however, that the fourth figure is nearer 3 than any other digit. The product to four figures is 14430.

The labor of obtaining useless figures can be avoided by simply not getting those partial products that are not to be retained in the final result.

A convenient arrangement is as below, keeping all decimal points in line and multiplying from left to right. The place of the right-hand figure of the product of the multiplicand by any multiplier digit is as many places to the left or right of the right-hand figure of the multiplicand as the multiplier digit is to the left or right of unit's place. In the example given above, 2, the first figure of the partial product arising when the multiplicand is multiplied by 6, is put two places to the left of 7, since 6 is two places to the left of unit's place. The stars indicate places of figures not obtained and would usually be omitted. We *carry* to the first figure retained as we would carry were the work done in full; moreover, if the first figure omitted is 5 or more than 5, we carry 1 to the first figure retained. This is illustrated in the second and the fourth partial products found above.

```
   23.57
  612.3
 ------
14142.
  236. *
   47. **
    7. ***
 ---------
14432.
```

The student can test the result by interchanging multiplier and multiplicand. If the multiplier or the multiplicand has only three-figure accuracy, it will readily be seen that the product can have only three-figure accuracy.

In division also the accuracy of the quotient cannot exceed the least accurate of the numbers which are dividend and divisor.

The arrangement for division is given below.

To find the quotient of 14432, divided by 612.3.

Move the decimal point in both dividend and divisor as many places to the right as are necessary to make the

```
         23.57
6123.|144320.
      12246
       2186
       1837
        349
        306
         43
         43
```

divisor an integer. The left-hand figure of the quotient is placed over the last figure of the first product, and the decimal point of the quotient is in line with the decimal point of the dividend. After the last significant figure of the dividend has been used, the partial products are formed by omitting one, then two, then three final figures of the divisor, remembering to carry to the first figure retained, as in multiplication.

3. So much for the ordinary arithmetical processes. The general principle underlying it all, that accuracy of results is limited by accuracy of data, continues applicable when tables or other labor-saving devices are used. With four-place tables seven-place accuracy is not to be looked for, and when our data are only four-place it is a foolish waste of time and increases the liability to error to use seven-place tables.

LOGARITHMS

The labor of computation is very greatly abridged by the use of logarithms. The principal facts concerning the practical use of logarithms are recapitulated below:

The *logarithm* of a number is the power to which 10 must be raised to produce the number. Since

$$10^0 = 1, \quad 10^1 = 10, \quad 10^2 = 100, \quad 10^3 = 1000, \text{ etc.},$$

we have by definition, logarithm 1, written

$$\log 1 = 0, \ \log 10 = 1, \ \log 100 = 2, \ \log 1000 = 3, \text{ etc.}$$

If m and n are any two numbers, we have by definition,

$$m = 10^x, \quad \text{or} \quad \log m = x,$$
$$n = 10^y, \quad \text{or} \quad \log n = y.$$

Multiplying these two equations,

$$mn = 10^{x+y}, \quad \text{or} \quad \log mn = x + y.$$

$$\therefore \log mn = \log m + \log n.$$

Similarly,

$$\log mnpq \cdots = \log m + \log n + \log p + \log q + \cdots.$$

I. *The logarithm of the product of several numbers is equal to the sum of the logarithms of the factors.*

Dividing the first of the two equations above by the second,

$$\frac{m}{n} = \frac{10^x}{10^y} = 10^{x-y}, \quad \text{or} \quad \log \frac{m}{n} = x - y = \log m - \log n.$$

II. *The logarithm of the quotient of two numbers is equal to the logarithm of the dividend minus the logarithm of the divisor.*

Raising $m = 10^x$ to the kth power,

$$m^k = 10^{kx}, \quad \text{or} \quad \log m^k = kx = k \log m.$$

III. *The logarithm of any power of a number is equal to the logarithm of the number multiplied by the exponent of the power.*

Since a root is a fractional power, the logarithm of the kth root of a number is $\frac{1}{k}$ times the logarithm of the number.

The following series of equations illustrates these principles:

$$y = \sqrt[3]{\frac{ab^2}{c^4}}$$

$$\log y = \log \sqrt[3]{\frac{ab^2}{c^4}} = \log \left(\frac{ab^2}{c^4}\right)^{\frac{1}{3}} = \frac{1}{3} \log \frac{ab^2}{c^4} \quad \text{by III}$$

$$= \tfrac{1}{3}[\log ab^2 - \log c^4] \quad \text{by II}$$

$$= \tfrac{1}{3}[\log a + \log b^2 - \log c^4] \quad \text{by I}$$

$$= \tfrac{1}{3}[\log a + 2 \log b - 4 \log c] \quad \text{by III}$$

4. What is the logarithm of 22738? Since the number lies between 10000 and 100000, its logarithm lies between 4 and 5. By calculation it has been found to be 4.35675. The integral part, 4, is its *characteristic;* the decimal part, .35675, is the *mantissa.*

$$\begin{aligned}
\log 22738 &= 4.35675, \text{ or } 22738 = 10^{4.35675}.\\
\log 2273.8 &= 3.35675, \text{ or } 2273.8 = 10^{3.35675}.\\
\log 227.38 &= 2.35675, \text{ or } 227.38 = 10^{2.35675}.\\
\log 22.738 &= 1.35675, \text{ or } 22.738 = 10^{1.35675}.\\
\log 2.2738 &= 0.35675, \text{ or } 2.2738 = 10^{0.35675}.\\
\log .22738 &= \bar{1}.35675, \text{ or } .22738 = 10^{\bar{1}.35675}.\\
\log .022738 &= \bar{2}.35675, \text{ or } .022738 = 10^{\bar{2}.35675}.\\
\log .002738 &= \bar{3}.35675, \text{ or } .002738 = 10^{\bar{3}.35675}.
\end{aligned}$$

Inspection of this algorithm shows us, 1st, that the characteristic depends solely on the position of the decimal point and can always be determined by inspection; 2d, that the mantissa is independent of the decimal point and depends on the sequence of digits constituting the number.

The student can easily make for himself a set of rules for determining the characteristic. In case of a decimal, say .000473, he can determine the characteristic of 473, and then move the point six places to the left; by so doing the characteristic is diminished by six and is $2 - 6 = -4$, written $\bar{4}$. The minus sign is written *above* the characteristic because it is the characteristic alone that is negative, the mantissa being always positive.

The mantissas are found from a table. Mantissas are all *approximate* numbers, and tables are published giving mantissas to four, five, six, and seven places. The kind of table to use depends entirely on the accuracy of the numbers which constitute our data. Four-place tables are accurate enough for ordinary data obtained by the use of field instruments, five-place tables for all data except such as are obtained by the use of the most delicate instruments. We shall use the latter.

USE OF THE TABLE

To find the mantissa of 2273 follow down the left-hand column of your table of the logarithms of numbers, passing from page to page until you reach 227; run your eye across the page on this line to the column headed 3; the number so reached, 35660, is the mantissa sought. In some tables the first two figures, 35, are printed only in the column headed 0.

Verify:

$\log 3748 = 3.57380.$	$\log 165 = 2.21748.$
$\log 9741 = 3.98860.$	$\log 17 = 1.23045.$
$\log 112.1 = 2.04961.$	$\log 1624 = 3.21059.$
$\log 32.40 = 1.51055.$	$\log .0034 = \bar{3}.53148.$

5. The mantissas of five-figure numbers may be obtained from the table by *interpolation*. The mantissa of 37423 lies between the mantissas of 37420 and 37430. We assume that it lies $\frac{3}{10}$ of the way from the first mantissa to the second; *i.e.*, $\frac{3}{10}$ of the way from 57310 to 57322. The difference of these numbers is 12 and $\frac{3}{10}$ of $12 = 3.6 = 4$. The mantissa of 37423 is $57310 + 4 = 57314$.

We may formulate the process of finding the mantissa of a five-figure number thus: Enter the table with the first four figures; subtract the corresponding mantissa from the next larger mantissa to find the *tabular difference;* multiply the tabular difference by the fifth figure, considered as a decimal, to find the *correction;* add the correction to the mantissa first found; the result is the mantissa of the five-figure number.

In most tables the multiplication spoken of above is performed in the tables of *proportional parts* printed on the margin of the page.

Verify the following logarithms:

$\log 127.34 = 2.10497.$	$\log 8964.3 = 3.95252.$
$\log 34.876 = 1.54253.$	$\log 90002 = 4.95425.$
$\log .42748 = \bar{1}.63092.$	$\log (.42748)^6 = \bar{3}.78552.$

The last problem will present no difficulty if we remember that the mantissa is always positive. Division of the logarithm when the characteristic is negative requires care. Suppose we wish to divide $\bar{3}.78552$ by 6. We write it $\bar{6} + 3.78552$. The division is now a simple matter. If we wish to divide by 5, we write it $\bar{5} + 2.78552$. We make the negative characteristic a multiple of the divisor.

When the logarithm is given and we wish to find the number, the process is the inverse of the one just considered. We may formulate it thus: Find in the table the mantissa equal to or next less than the given mantissa. The corresponding number will be the first four figures of the number sought. Subtract this mantissa from the given mantissa to find the *correction.* Divide the correction by the tabular difference, obtaining a one-figure quotient. Annex this figure to the four already found. The result is the five-figure number corresponding to the given mantissa. Place the decimal point at the place indicated by the characteristic. The result is the number corresponding to the given logarithm.

Verify the following:

$3.14216 = \log 1386.3.$	$\bar{2}.37489 = \log .023708.$
$2.15362 = \log 142.44.$	$.96756 = \log 9.2803.$
$1.87460 = \log 74.92.$	

The logarithm of $\frac{1}{m}$ is called the cologarithm of m, written either colog m or col m. In computing, it often saves labor to add the cologarithm instead of subtracting the logarithm.

If $$\log m = 3.27463$$
$$\text{colog}\, m = \log 1 - \log m = 0 - 3.27463$$
$$= 6.72537 - 10.$$

The subtraction is readily performed from left to right by taking each digit, except the last, from 9. The -10 is used to avoid negative characteristics. Some computers increase all negative characteristics by 10 and take account of these 10's in the final result.

6. Accuracy in computing can be attained only by practice and by constant care. When the computer has made his interpolation, he should glance back at the table and see that his result lies between the proper tabular numbers and nearest the right one. This takes but an instant and corrects many errors. The importance of carefully planning a computation before entering upon it can hardly be overestimated. The plan should be written out. The computer is then free to devote his whole attention to the mechanical details of the work. Paper ruled in squares conduces to accuracy. If the computation be confined to one column, it can be repeated or a similar one inserted in a parallel column without repeating the plan. If any given number occurs repeatedly in a computation, it may be written down once for all on a separate piece of paper and held over any number with which it is to be combined.

The computer will avoid many errors if he accustoms himself to making rough estimates of results. When the nature of the subject permits, these estimates may be obtained by *graphic* methods.

To insure accuracy the computer must continually *check* his work. Every operation, every step in every operation must be tested before going on. If two numbers are added, subtract one of them from the sum. If two numbers are subtracted, add the difference to the smaller. If two numbers are multiplied, interchange multiplier and multiplicand and compare products. Test every step, and when the computation is finished check the final result if the nature of the problem furnishes a test; if not, work the problem by a second method and compare results.

The computer who aims at rapidity should train himself to do all he safely can mentally. He should early acquire the habit of remembering a number of six or seven figures long enough to transcribe it. He should perform his interpolations mentally. He should add and subtract two numbers from left to right. Other devices will come to him with practice.

The most important habit to be acquired is that of being constantly on the watch for errors and of constantly checking results. The computer who makes no mistakes can hardly be said to exist. Such a one would be a marvel. The ordinary man who forms the habit of not letting a mistake go uncorrected is more trustworthy than the marvel who does not verify his work.

CHAPTER II

THE RIGHT TRIANGLE

7. The three sides x, y, r of the right triangle ABC furnish six ratios:

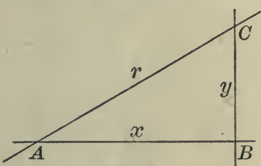


FIG. 1.

$$\frac{y}{r},\ \frac{x}{r},\ \frac{y}{x},\ \frac{x}{y},\ \frac{r}{x},\ \frac{r}{y}.$$

If a second right triangle $A'B'C'$ be constructed with angle A' equal to angle A, it will be similar to the first and we have:

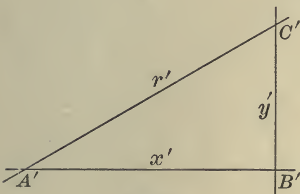


FIG. 2.

$$x' : y' : r' = x : y : r.$$

Therefore,

$$\frac{y'}{r'} = \frac{y}{r},\quad \frac{x'}{r'} = \frac{x}{r},\quad \frac{y'}{x'} = \frac{y}{x},$$

$$\frac{x'}{y'} = \frac{x}{y},\quad \frac{r'}{x'} = \frac{r}{x},\quad \frac{r'}{y'} = \frac{r}{y}.$$

Each ratio of one triangle is equal to the corresponding ratio in the other.

Let us construct a third right triangle $A''B''C''$, making angle $A'' > A$ and side $A''C'' = AC$.

From the construction

$$r'' = r,\quad x'' < x,\quad y'' > y,$$

and

$$\frac{y''}{r''} > \frac{y}{r},\quad \frac{x''}{r''} < \frac{x}{r},\quad \frac{y''}{x''} > \frac{y}{x},$$

$$\frac{x''}{y''} < \frac{x}{y},\quad \frac{r''}{x''} > \frac{r}{x},\quad \frac{r''}{y''} < \frac{r}{y}.$$

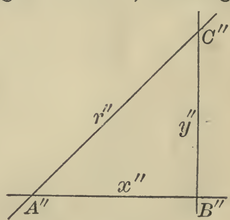


FIG. 3.

The ratios in this triangle are not equal to the corresponding ratios in the first triangle.

The foregoing considerations lead to the conclusion that these ratios depend for their values solely on the angle A; *i.e.*, they change when A changes, they are constant when A is constant. This dependence is expressed mathematically by saying that the ratios are *functions* of the angle A. To distinguish them from other functions they are called *Trigonometric Functions.*

The six trigonometric functions of A are named as follows:

$\frac{y}{r}$ = sine of A, written $\sin A$.

$\frac{x}{r}$ = cosine of A, " $\cos A$.

$\frac{y}{x}$ = tangent of A, " $\tan A$.

$\frac{x}{y}$ = cotangent of A, " $\cot A$.

$\frac{r}{x}$ = secant of A, " $\sec A$.

$\frac{r}{y}$ = cosecant of A, " $\csc A$.

8. The foregoing equations *define* the trigonometric functions. They are fundamental and should be carefully memorized. These definitions may be put into words:

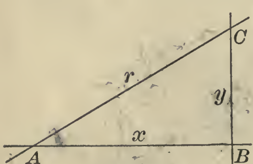


Fig. 4.

$$\sin A = \frac{y}{r} = \frac{\text{side opposite}}{\text{hypotenuse}}.$$

$$\cos A = \frac{x}{r} = \frac{\text{side adjacent}}{\text{hypotenuse}}.$$

$$\tan A = \frac{y}{x} = \frac{\text{side opposite}}{\text{side adjacent}}.$$

$$\cot A = \frac{x}{y} = \frac{\text{side adjacent}}{\text{side opposite}}.$$

$$\sec A = \frac{r}{x} = \frac{\text{hypotenuse}}{\text{side adjacent}}.$$

$$\csc A = \frac{r}{y} = \frac{\text{hypotenuse}}{\text{side opposite}}.$$

EXERCISES

Find the six functions of each of the acute angles in the right triangle whose sides are:

1. 5, 12, 13.
2. 3, 4, 5.
3. 8, 15, 17.
4. 9, 12, 15.
5. 5, 8, $\sqrt{89}$.
6. 2, 3, $\sqrt{13}$.
7. a, $\sqrt{1 - a^2}$, 1.
8. a, b, $\sqrt{a^2 + b^2}$.
9. 5, 5, $5\sqrt{2}$.
10. a, $\sqrt{2ax + x^2}$, $a + x$.
11. $a + b$, $a - b$, $\sqrt{2(a^2 + b^2)}$.
12. $m^2 - n^2$, $2mn$, $m^2 + n^2$.

9. Inverse Functions. Suppose we have the expression $\sin A = \frac{4}{5}$; how may we describe A? A is the angle whose sine is $\frac{4}{5}$. It is customary to express this by writing

$$A = \sin^{-1} \tfrac{4}{5}.$$

This is read "A is the angle whose sine is $\frac{4}{5}$." So, too, the expressions $\cos^{-1} \frac{2}{3}$, $\tan^{-1} \frac{3}{2}$, $\sec^{-1} \frac{5}{3}$ are read the angle whose cosine is $\frac{2}{3}$, the angle whose tangent is $\frac{3}{2}$, the angle whose secant is $\frac{5}{3}$. These functions are called *inverse* functions. They are distinguished from the trigonometric functions by the exponent -1.

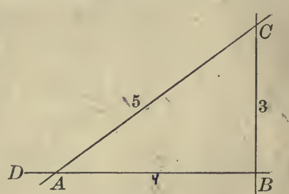


FIG. 5.

Let it be required to construct $\sin^{-1}\frac{3}{5}$.

Construction. At B erect BC perpendicular to DB and equal to 3. With C as a center, and with a radius equal to 5, describe an arc cutting BD at A. Draw CA; then is A the required angle. For by definition

$$\sin A = \tfrac{3}{5}, \quad \text{or} \quad A = \sin^{-1}\tfrac{3}{5}.$$

EXERCISES

Construct the following angles:

1. $\sin^{-1}\frac{1}{2}$, $\sin^{-1}\frac{3}{4}$, $\sin^{-1}\frac{2}{5}$.
2. $\cos^{-1}\frac{1}{2}$, $\cos^{-1}\frac{2}{3}$, $\cos^{-1}\frac{2}{9}$.
3. $\tan^{-1}\frac{1}{2}$, $\tan^{-1}\frac{3}{2}$, $\tan^{-1}1$.
4. $\cot^{-1}\frac{2}{3}$, $\cot^{-1}2$, $\cot^{-1}5$.
5. $\sec^{-1}\frac{5}{4}$, $\sec^{-1}2$, $\sec^{-1}5$.
6. $\csc^{-1}\frac{5}{2}$, $\csc^{-1}\frac{4}{3}$, $\csc^{-1}3$.

Show by constructing a figure that:

7. $\sin^{-1}\frac{3}{5} = \cos^{-1}\frac{4}{5} = \tan^{-1}\frac{3}{4}$.

Show by construction that the following angles are impossible.

8. $\sin^{-1}\frac{3}{2}$, $\cos^{-1}2$, $\sec^{-1}\frac{2}{3}$, $\csc^{-1}\frac{1}{2}$.
9. $\sin^{-1}\dfrac{a}{b}$, $\cos^{-1}\dfrac{a}{b}$, $a > b$.
10. $\sec^{-1}\dfrac{b}{a}$, $\csc^{-1}\dfrac{b}{a}$, $b < a$.

10. Functions of Complementary Angles. The angles A and C are complementary. By definition

$$\sin A = \frac{y}{r} = \cos C. \qquad \cot A = \frac{x}{y} = \tan C.$$

$$\cos A = \frac{x}{r} = \sin C. \qquad \sec A = \frac{r}{x} = \csc C.$$

$$\tan A = \frac{y}{x} = \cot C. \qquad \csc A = \frac{r}{y} = \sec C.$$

We may summarize these relations by saying that any function of an angle is equal to the co-function of the complementary angle. In this statement we assume that the co-function of the cosine is the sine, etc. The cosine, cotangent, cosecant are contractions for *complement's* sine, *complement's* tangent, *complement's* secant.

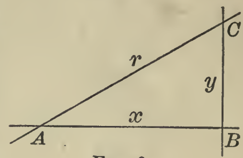


FIG. 6.

EXERCISES

1. The functions of 30° are:

$$\sin 30^\circ = \tfrac{1}{2}, \qquad \cos 30^\circ = \tfrac{1}{2}\sqrt{3}, \qquad \tan 30^\circ = \tfrac{1}{3}\sqrt{3}.$$
$$\cot 30^\circ = \sqrt{3}, \qquad \sec 30^\circ = \tfrac{2}{3}\sqrt{3}, \qquad \csc 30^\circ = 2;$$

write the functions of 60°.

2. $\sin 40^\circ = \cos 50^\circ$; express the relations between the other functions of these angles.

3. The angles $45^\circ + A$ and $45^\circ - A$ are complementary; express the functions of $45^\circ + A$ in terms of the functions of $45^\circ - A$.

4. A and $90^\circ - A$ are complementary; express the functions of $90^\circ - A$ in functions of A.

5. 45° is its own complement; show that $\sin 45^\circ = \cos 45^\circ$, $\tan 45^\circ = \cot 45^\circ$, $\sec 45^\circ = \csc 45^\circ$.

11. Fundamental Relations of the Trigonometric Functions. The six functions sine, cosine, etc., are connected by a number of equations. The more important of these are derived below. The first five depend immediately on the definitions, the other three on a well-known property of the right triangle. These last involve the squares of the functions. By

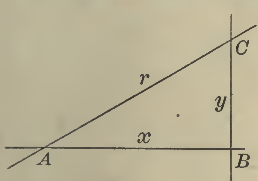


FIG. 7.

universal usage powers are indicated by affixing the exponent to the functional symbol. *E.g.*, $(\sin A)^2$ is written $\sin^2 A$, $(\cos A)^3$ is written $\cos^3 A$.

Since $\frac{y}{r} \times \frac{r}{y} = 1$ we have $\sin A \csc A = 1$ [1]

$\frac{x}{r} \times \frac{r}{x} = 1$ " $\cos A \sec A = 1$ [2]

$\frac{y}{x} \times \frac{x}{y} = 1$ " $\tan A \cot A = 1$ [3]

$\frac{y}{r} \div \frac{x}{r} = \frac{y}{x}$ " $\frac{\sin A}{\cos A} = \tan A$ [4]

$\frac{x}{r} \div \frac{y}{r} = \frac{x}{y}$ " $\frac{\cos A}{\sin A} = \cot A.$ [5]

From the figure $y^2 + x^2 = r^2.$

Dividing this equation by r^2, by x^2, and by y^2,

$\frac{y^2}{r^2} + \frac{x^2}{r^2} = 1$; $\left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2 = 1.$ $\therefore \sin^2 A + \cos^2 A = 1.$ [6]

$\frac{y^2}{x^2} + 1 = \frac{r^2}{x^2}$; $\left(\frac{y}{x}\right)^2 + 1 = \left(\frac{r}{x}\right)^2.$ $\therefore 1 + \tan^2 A = \sec^2 A.$ [7]

$1 + \frac{x^2}{y^2} = \frac{r^2}{y^2}$; $1 + \left(\frac{x}{y}\right)^2 = \left(\frac{r}{y}\right)^2.$ $\therefore 1 + \cot^2 A = \csc^2 A.$ [8]

These eight identities constitute the fundamental relations of the trigonometric functions. They are very important and should be committed to memory.

By means of these relations, when we know one function of an angle, we can find all the others.

Suppose, for example, $\sin A = \frac{1}{2}$.

[6] $\cos A = \sqrt{1 - \sin^2 A} = \sqrt{1 - \frac{1}{4}} = \frac{1}{2}\sqrt{3}.$

[4] $\tan A = \frac{\sin A}{\cos A} = \frac{\frac{1}{2}}{\frac{1}{2}\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3}.$

[3] $$\cot A = \frac{1}{\tan A} = \sqrt{3}.$$

[2] $$\sec A = \frac{1}{\cos A} = \frac{2}{\sqrt{3}} = \tfrac{2}{3}\sqrt{3}.$$

[1] $$\csc A = \frac{1}{\sin A} = 2.$$

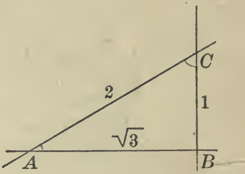


FIG. 8.

The values of these functions may also be found by constructing $\sin^{-1}\frac{1}{2}$ and finding the third side of the triangle geometrically. This side is $\sqrt{3}$. The functions can now be written from the definitions (§ 8).

EXERCISES

Find all the functions of the following angles, using each of the methods illustrated above:

1. $\cot^{-1} 2$.
2. $\tan^{-1} 3$.
3. $\cot^{-1} \frac{1}{2}$.
4. $\sec^{-1} \frac{3}{2}$.
5. $\cos^{-1} \frac{2}{3}$.
6. $\csc^{-1} \frac{5}{3}$.
7. $\cos^{-1} 2$.
8. $\sin^{-1} \frac{1}{3}$.

12. We can express any function of A in terms of any other function of A by making use of formulas [1] to [8]. As an illustration let us express each of the functions in terms of the tangent.

$$\tan A = \tan A.$$

[3] $$\cot A = \frac{1}{\tan A}.$$

[7] $$\sec A = \sqrt{1 + \tan^2 A}.$$

[8] $$\csc A = \sqrt{1 + \cot^2 A} = \sqrt{1 + \frac{1}{\tan^2 A}} = \frac{\sqrt{1 + \tan^2 A}}{\tan A}.$$

[1] $$\sin A = \frac{1}{\csc A} = \frac{\tan A}{\sqrt{1 + \tan^2 A}}.$$

[2] $$\cos A = \frac{1}{\sec A} = \frac{1}{\sqrt{1 + \tan^2 A}}.$$

EXERCISES

Express each of the functions of A in terms of

1. $\sin A$. 2. $\cos A$. 3. $\cot A$. 4. $\sec A$. 5. $\csc A$.

6. Tabulate the results.

Prove the following identities by means of formulas [1] to [8].

7. $\sin A \sec A = \tan A$.

8. $(\sin A + \cos A)^2 = 1 + 2 \sin A \cos A$.

9. $(\sec A + \tan A)(\sec A - \tan A) = 1$.

10. $\dfrac{1 - \sin A}{\cos A} = \dfrac{\cos A}{1 + \sin A}$.

11. $(1 + \tan A)^2 + (1 - \tan A)^2 = 2 \sec^2 A$.

12. $(\sin A + \cos A)^2 + (\sin A - \cos A)^2 = 2$.

13. Functions of 45°, 30°, and 60°.

Construct angle $A = 45°$, lay off $AB = 1$, and complete the right triangle.

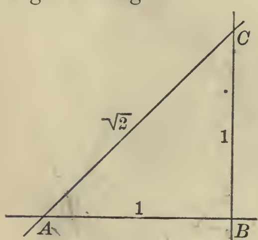


FIG. 9.

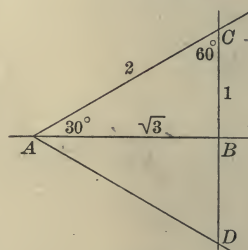


FIG. 10.

Angle $C = 45°$. $\therefore BC = 1$ and $AC = \sqrt{2}$.

From the definitions

$$\sin 45° = \cos 45° = \frac{1}{\sqrt{2}} = \tfrac{1}{2}\sqrt{2}.$$

$$\tan 45° = \cot 45° = 1.$$

$$\sec 45° = \csc 45° = \frac{\sqrt{2}}{1} = \sqrt{2}.$$

Construct the equilateral triangl $.4462 = 26.$
Bisect angle A by AB. The triangle A $= 18.2 = 18.$
with angle $BAC = 30°$, angle $C = 60°$, $= .4470.$
side $AB = \sqrt{3}$.

By definition

$$\sin 30° = \tfrac{1}{2}. \qquad \sin 60° = \frac{\sqrt{3}}{2} = \tfrac{1}{2}\sqrt{3}.$$

$$\cos 30° = \frac{\sqrt{3}}{2} = \tfrac{1}{2}\sqrt{3}. \qquad \cos 60° = \tfrac{1}{2}.$$

$$\tan 30° = \frac{1}{\sqrt{3}} = \tfrac{1}{3}\sqrt{3}. \qquad \tan 60° = \frac{\sqrt{3}}{1} = \sqrt{3}.$$

$$\cot 30° = \frac{\sqrt{3}}{1} = \sqrt{3}. \qquad \cot 60° = \frac{1}{\sqrt{3}} = \tfrac{1}{3}\sqrt{3}.$$

$$\sec 30° = \frac{2}{\sqrt{3}} = \tfrac{2}{3}\sqrt{3}. \qquad \sec 60° = \tfrac{2}{1} = 2.$$

$$\csc 30° = \tfrac{2}{1} = 2. \qquad \csc 60° = \frac{2}{\sqrt{3}} = \tfrac{2}{3}\sqrt{3}.$$

14. Trigonometric Tables. In the preceding paragraph we have found the functions of 30°, 45°, and 60° by simple geometrical expedients. The functions of other angles are not found so easily. For purposes of computation, tables of trigonometric functions are used. Such tables give the values of the sine, cosine, tangent, and cotangent of all angles from 0° to 90° at intervals of 10′. Examine such a table. You will find in the left-hand column the angles; their sines, cosines, tangents, and cotangents are opposite in appropriately headed columns. The column at the extreme right also contains angles. Inspection will show you that these angles are the complements of the corresponding angles at the left. We learned in § 10 that the function of any angle was the co-function of its complementary angle. The sine of an angle at the right is the cosine of the

at the left. You will find this indi-
Express each of ty the word *cosine* at the bottom of the
1. sin A. 2. ne. Similarly for the other functions. If
6. Tabulate your table carefully, you will find that the
Prov run *down* the left side of the page till 45° is reached; they then run *up* the right side of the page till 90° is reached. If the angle is less than 45°, you look for it at the left; if more than 45°, at the right. If the angle is at the *left*, the name of the required function is at the *top* of the page; while if the angle is at the *right* the name of the function is at the bottom of the page.

The tables do not contain secants and cosecants. These functions are reciprocals of cosine and sine, and can readily be found by taking advantage of this fact. You will find by experience that it is never necessary to use them in computation.

Take your table and run down the column of sines. They increase with the angle. So do the tangents. Examine the cosines and cotangents. They *decrease* as the angle *increases.*

15. The table gives the functions of angles which are multiples of 10′. To find the functions of other angles we *interpolate,* as explained in § 5. Care must be taken to *add* the correction in finding sines and tangents, to *subtract* it in finding cosines and cotangents.

Find the sine of 27° 34′.

$$\sin 27° \, 30' = .4617.$$

The tabular difference $= .4643 - .4617 = 26.$

The correction $= .4 \text{ of } 26 = 10.4 = 10.$

$$\sin 27° \, 34' = .4617 + 10 = .4627.$$

Find the cosine of 63° 27′.

$$\cos 63° \, 20' = .4488.$$

The tabular difference $= .4488 - .4462 = 26.$

The correction $= .7 \text{ of } 26 = 18.2 = 18.$

$$\cos 63° 27' = .4488 - 18 = .4470.$$

Find the tangent of 84° 28′.

$$\tan 84° 20' = 10.078.$$

The tabular difference $= 10.385 - 10.078 = 307.$

The correction $= .8 \text{ of } 307 = 245.6 = 246.$

$$\tan 84° 28' = 10.078 + 246 = 10.324.$$

The work which is here done out in full should be performed mentally as far as possible. In case your table has a column of differences, the operation of finding the tabular difference is unnecessary; if your table is provided with a table of *proportional parts*, the operation of finding the *correction* is much simplified.

EXERCISES

Verify the following:

$\sin 0° 42' = .0122.$	$\sin 58° 38' = .8539.$
$\cos 0° 42' = .9999.$	$\cos 58° 38' = .5220.$
$\tan 0° 42' = .0122.$	$\tan 58° 38' = 1.6405.$
$\cot 9° 42' = 5.8505.$	$\cot 58° 38' = .6096.$
$\sin 43° 01' = .6822.$	$\cos 28° 13' = .8812.$
$\tan 38° 29' = .7949.$	$\cot 81° 31' = .1492.$

To find $\sin^{-1} .4327$.

The next smaller sine is .4305, the sine of 25° 30′.

The difference $= .4327 - .4305 = 22.$

The tabular difference $= .4331 - .4305 = 26.$

Correction $= \frac{22}{26} = 8.$

$\therefore \sin^{-1} .4327 = 25° 38'.$

Find $\cos^{-1} .8826$.

The next larger cosine is .8829, the cosine of 28° 00′.

The difference $= .8829 - .8826 = 3.$

The tabular difference $= .8829 - .8816 = 13.$

The correction $= \frac{3}{13} = 2.$

$\therefore \cos^{-1} .8826 = 28° 02'.$

Here the next larger cosine is taken because the cosine is a *decreasing* function.

EXERCISES

Verify the following:

$\tan^{-1} .4329 = 23° 24'.$ $\quad$ $\tan^{-1} 3.4268 \doteq 73° 44'.$

$\cot^{-1} .3721 = 69° 35'.$ $\quad$ $\cos^{-1} .4268 = 64° 44'.$

$\sin^{-1} .8523 = 58° 28'.$ $\quad$ $\cot^{-1} 1.4823 = 34° 00'.$

16. The Solution of the Right Triangle. To *solve* a right triangle is to find numerical values for the unknown parts. This is possible when two parts, one of which is a side, are known. Two methods of solution are open to us, — the *Graphic* and the *Trigonometric*.

Graphic Solution. It is desirable to solve all problems by this method before proceeding to the more accurate trigonometric method. It gives rough approximations and enables the student to detect his grosser mistakes in the application of the trigonometric formulas. The solution consists in accurately constructing the figure from the data given. The required or unknown parts may now be carefully measured by scale and protractor. The use of paper ruled in squares facilitates this work. The only instruments needed are a scale, a protractor, a straight-edge, and a pair of dividers. Two-figure accuracy is all that should be aimed at in this method of solution.

17. Trigonometric Solution. In using this method we *compute* the values of the unknown parts. The first four definitions (p. 12) furnish formulas sufficient for this purpose.

These formulas are:

(a) $\sin A = \frac{y}{r}.$ (c) $\cos A = \frac{x}{r}.$

(b) $\tan A = \frac{y}{x}.$ (d) $\cot A = \frac{x}{y}.$

No matter what two parts are given, one of these four formulas includes them both. This statement assumes that when A is known B is known, since the two angles are complementary; and it further assumes that when A is known any of its functions are known, and *vice versa*, since the tables enable us to find the one from the other.

The student should satisfy himself of the truth of this statement by selecting all possible combinations of two parts as known parts. To effect the solution we proceed as follows:

Select a formula containing the two known parts, and substitute in it the values of these parts; the resulting equation will give a third part. Of the three parts now known, one is an angle and two are sides. To find the remaining side, select a formula containing it. Where possible, a formula should be selected which does not contain the computed part. Experience will show that this is possible when one of the given parts is an angle.

Checks. These computations, like all others, should be checked. A convenient formula for this purpose is

$$r^2 = x^2 + y^2,$$

or

$$y^2 = r^2 - x^2 = (r + x)(r - x),$$

or

$$x^2 = r^2 - y^2 = (r + y)(r - y).$$

The problems that follow illustrate the process of solution.

1°. The hypotenuse of a right triangle is 36, and one of its angles is 32° 14′; find the other parts.

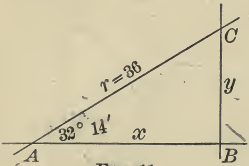


FIG. 11.

Graphic Solution. Construct the angle $A = 32° \, 14'$, or as nearly so as your protractor admits; lay off $AC = 36$; drop $CB \perp$ to AB. The triangle ABC is the required triangle. Measure AB and BC.

Trigonometric Solution. $C = 90° - 32° \, 14' = 57° \, 46'$.

Formula (b) $\cos A = \frac{x}{r}$. $\therefore x = r \cos A$.

[table] $x = 36\,(.8459) = 30.45$.

Formula (a) $\sin A = \frac{y}{r}$. $\therefore y = r \sin A$.

[table] $y = 36\,(.5334) = 19.20$.

Check. $x^2 = (r + y)(r - y)$.

$$(30.45)^2 = (55.20)\,(16.80).$$

$$927.2 = 927.4.$$

This shows that our work is fairly accurate.

2°. One leg of a right triangle is 27, and the adjacent angle is 67° 23′; find the other parts.

Graphic Solution. Construct angle $A = 67° \, 23'$, lay off $AB = 27$, erect the $\perp$ BC. ABC is the required triangle. Measure AC and BC.

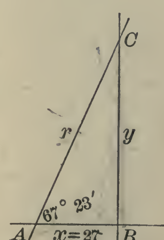


FIG. 12.

Trigonometric Solution.

$$C = 90° - A = 22° \, 37'.$$

Formula (b) $\cos A = \frac{x}{r}$. $\therefore r = \frac{x}{\cos A}$.

$$r = \frac{27}{.3846} = 70.21.$$

Formula (c) $\tan A = \frac{y}{x}$. $\therefore y = x \tan A$.

$$y = 27\,(2.4004) = 64.81.$$

Check. $y^2 = (r + x)(r - x)$.

$$(64.81)^2 = (97.21)(43.21).$$

$$4200.2 = 4200.4.$$

3°. The hypotenuse of a right triangle is 48, and one leg is 37; find the other parts.

Graphic Solution. Construct the right angle ABC, lay off $BA = 37$, from A as center, with radius 48, draw an arc, cutting BC in C; draw AC. ABC is the required triangle. Measure BC and the angle BAC.

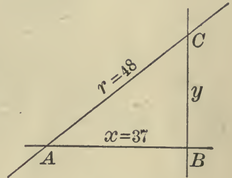


FIG. 13.

Trigonometric Solution.

Formula (b) $\cos A = \frac{x}{r} = \frac{37}{48} = .7708.$

$$A = 39°\ 34'.$$

$$C = 90° - A = 50°\ 26'.$$

Formula (a) $\sin A = \frac{y}{r}$. $\therefore y = 48 \sin 39°\ 34'$.

$$y = 48\,(.6370) = 30.58.$$

Check. $x^2 = (r + y)(r - y)$.

$$(37)^2 = (78.58)(17.42).$$

$$1369 = 1368.9.$$

4°. The two legs of a right triangle are 487 and 756; find the other parts.

Graphic Solution. In the right angle ABC lay off $BA = 487$ and $BC = 756$; draw AC. ABC is the required triangle. Measure AC and the angle A.

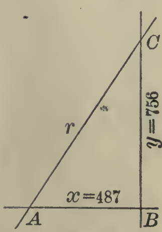


FIG. 14.

Trigonometric Solution.

Formula (d) $\cot A = \frac{487}{756} = .6442.$

$$A = 57° \, 13'.$$

$$C = 90° - A = 32° \, 47'.$$

Formula (a) $\sin C = \frac{487}{r}. \quad \therefore r = \frac{487}{\sin C} = \frac{487}{.5415}.$

$$r = 899.4.$$

Check. $y^2 = (r + x)(r - x).$

$$(756)^2 = (1386.4)(412.4).$$

$$571536 = 571740.$$

Four-figure accuracy is all that we expect, and this we probably have in r but not in $(r + x)(r - x)$.

EXERCISES

Exercises 1–6 refer to Fig. 15; 7–16 refer to Fig. 16.

1. $x = 20, \; r = 30.$
2. $y = 17, \; r = 60.$
3. $x = 34, \; y = 45.$
4. $A = 30° \, 24', \; r = 207.$
5. $C = 38° \, 47', \; r = 103.4$
6. $A = 64° \, 23', \; x = 20.32.$

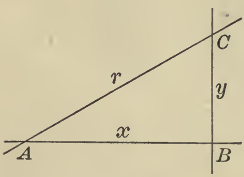


FIG. 15.

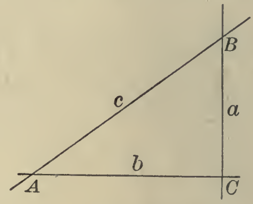


FIG. 16.

7. $a = 20, \quad A = 30°.$
8. $b = 16, \quad A = 45°.$
9. $c = 75, \quad B = 60°.$
10. $a = 12, \quad b = 15.$
11. $a = 407, \quad c = 609.$
12. $b = 1.306, \quad c = 2.501.$
13. $A = 15° \, 17', \; c = 163.$
14. $A = 81° \, 17', \; b = .0143.$
15. $a = 137.4, \quad b = 101.2.$
16. $B = 65° \, 8', \quad c = 3.145.$

18. The Solution of Problems. The problems that complete this chapter can all be solved by right triangles. While different problems demand different methods of solution, the following general method of procedure will be found very useful:

1°. Carefully construct a diagram to some convenient scale and find the graphic solution by proper measurements.

2°. Examine the diagram for a right triangle with two parts given; if this triangle contains the required part, solve it; if not, consider all the parts of this triangle as known, and find another right triangle with two parts known; if this second triangle contains the required part, the method of solution is obvious; if it does not contain the required part, repeat the process until a triangle is found that does contain it. It may be necessary to draw auxiliary lines. When you have found the several steps that lead to the solution, review the work to make sure that all of them are necessary.

3°. Proceed to the computation, being careful to *check* each step. No computation should be made until the whole process of solution is determined upon and written out.

Definitions. If O denote an observer, P an object above the horizon, and POH' a vertical plane intersecting the horizon in HH', the angle POH' is called the *Elevation* (or Altitude) of P. If the object be below the horizontal plane, as at Q, the angle QOH' is called the *depression* of Q.

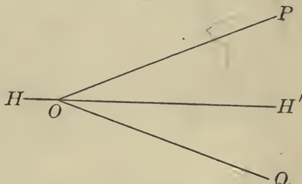


FIG. 17.

The *bearing* of an object is its direction from the observer. The use of the word is obvious from the following illustrations. If O be the observer, the bearing of P is $E\ 20°\ N$, of Q is $N\ 25°\ E$, of R is $W\ 30°\ N$, of T is $S\ 25°\ W$.

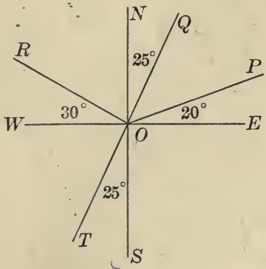


FIG. 18.

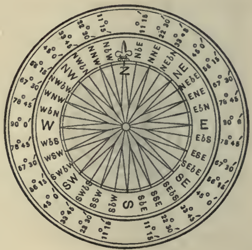


FIG. 19.

Bearing is also often given in terms of the divisions of a mariner's compass. The circle is divided into 32 equal parts, the points of division being named as indicated in the figure.

EXERCISES

1. The center pole of a tent is 20 ft. high, and its top is stayed by ropes 40 ft. long; what is the inclination of the ropes to the ground?

2. A man standing 140 ft. from the foot of a tower finds that the elevation of its top is 28° 25′; what is the height of the tower?

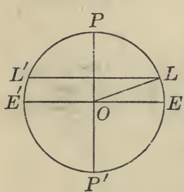


FIG. 20.

3. At what latitude is the circumference of a parallel of latitude equal to two-thirds of the circumference of the equator?

Suggestion. Let $PEP'E'$ be a section of the earth through its axis, PP' its axis, EE' the equator, LL' the circle of latitude. Then will EOL be the latitude.

4. The length of a degree of longitude at the equator is 69.16 mi.; find a formula for the length of a degree of longitude at latitude λ.

5. A ladder 40 ft. long reaches a window 33 ft. from the ground. Being turned on its foot to the opposite side of the street, it reaches a window 21 ft. from the ground; how wide is the street?

6. From a window the top of a house on the opposite side of a street 30 ft. wide has an elevation of 60°, while the bottom of the house has a depression of 30°; what is the height of the house?

7. A pole stands on top of a knoll. From a point at a distance of 200 ft. from the foot of the knoll, the elevations of the top and the bottom of the pole are 60° and 30°, respectively; prove that the pole is twice as high as the knoll.

8. A regular hexagon is circumscribed about a circle whose radius is 20 ft.; find the length of the side of this hexagon.

9. The radius of a circle is 1. Find the side, the perimeter, the apothem, and the area of a regular inscribed polygon of 5 sides, of 8 sides, of 9 sides, of 12 sides, of n sides.

10. A person at the top of a tower 100 ft. high observes two objects on a straight road running by its foot. The depression of the nearer is 45° 36′, of the more remote is 30° 24′; what is their distance apart?

11. If the edge of a regular tetrahedron is 10 ft., find the length of a face altitude; the length of the altitude, and the angle between two faces.

12. The roof rises from the adjacent sides of a square house at an angle of 30°; find the angle which the corner of the roof makes with the horizon.

13. At a certain port the seacoast runs N. N. E., and a vessel 10 mi. out is making 12 mi. per hour S. S. W. At 2.30 P.M. she is due east; what is her bearing at 2 P.M.? at 3 P.M.? At 2 P.M. a vessel sailing 10 mi. per hour is dispatched to intercept her; what course must the latter vessel take?

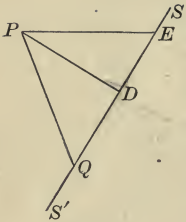


FIG. 21.

Suggestions. Let P be the port and SS' the course of the vessel, E its position at 2.30 P.M. Let PD be perpendicular to SS'; find when she will be at D. The bearings at 2 P.M. and at 3 P.M. will be easily found. To find course of second vessel let Q be point of meeting and t the time after 2 P.M., when they meet. PQ and DQ can now be calculated in terms of t, which can be easily found.

14. A smokestack is secured by wires running from points 35 ft. from its base to within 3 ft. of its top. These wires are inclined at an angle of 40° to the ground. What is the height of the smokestack? the length of the wires? What is the least number of wires necessary to secure the stack? If they are symmetrically placed, how far apart are their ground ends? How far are the lines joining their ground ends from the foot of the stack? from the top of the stack? What angle do the wires make with these lines? with each other? What angle does the plane of two wires make with the ground? What angle does the perpendicular from the foot of the stack on this plane make with the ground? what is its length?

15. On the U. S. Coast Survey an observation platform 50 ft. high was built. The platform was 8 ft. square. The four legs spread to the corners of a 12-foot square at the base. They were braced together by three sets of cross-pieces, as represented in the illustration. If the cross-pieces are equidistant and the lowest is 3 ft. from the ground, find the length of each piece required for the construction.

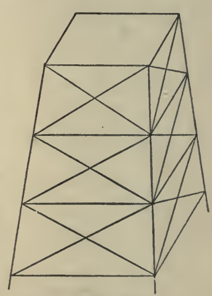


FIG. 22.

16. A man wishing to know the width of a river selects a point, A, one bank, directly opposite a tree, TR, on the other bank. He finds its elevation to be 10° 30′; going back 150 ft. to B, he finds its elevation to be 9°. What is the width of the river? (Find AC, perpendicular to BR; then AR and AT.)

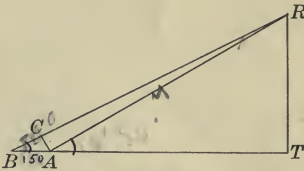


FIG. 23.

17. Upon the top of a shaft 125 ft. high stands a statue which subtends an angle of 3° at a point 200 ft. from the shaft; how tall is the statue?

18. A wheel 1 ft. in diameter is driven by a belt from a wheel 4 ft. in diameter. If the shafts bearing these wheels are parallel and 10 ft. apart, how long will the belt be (α) if crossed? (β) if not crossed?

19. In a circle whose radius is 15 ft., what angle will a chord of 20 ft. subtend at the center? a chord of 25 ft.? of 10 ft.? In a circle of radius r, what angle will a chord, a, subtend?

20. In a circle of 15 ft. radius, find the area of the segment cut off by a chord of 18 ft.; the area of the segment included between this chord and a chord of 25 ft.

21. The base of a quadrilateral is 60 ft., the adjacent sides are 30 ft. and 40 ft., the corresponding adjacent angles are 110° and 130°, respectively; find the fourth side and the other two angles.

22. The elevation of a balloon due north from A is 60°; from B, 1 mi. west of A, its elevation is 45°; what is the height of the balloon?

23. One of the equal sides of an isosceles triangle is 47 ft., and one of the equal angles is 38° 24′; what is the base of the triangle?

CHAPTER III

THE TRIGONOMETRIC FUNCTIONS OF UNLIMITED ANGLES

19. A *directed* line is a straight line generated by a point moving in a given direction. It possesses two qualities — *length* and *direction.*

The lines AB and DC are of equal length but of opposite direction or sign. We indicate the direction of a line by the order of naming its extremities. For example:

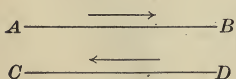


FIG. 24.

$$AB = -BA,$$

or

$$AB + BA = 0.$$

Parallel directed lines may be added by placing the initial point of the second on the terminal point of the first. Their sum is the line defined by the initial point of the first and the terminal point of the second. They may be subtracted by placing their terminal points together. The remainder is the line defined by the initial points of the first and second.

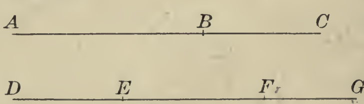


FIG. 25.

$$AB + BC = AC. \qquad DE + EF + FG = DG.$$

$$AC + CB = AB. \qquad DG - FG + FE = DE.$$

$$AC - BC = AB. \qquad GF + FE = GE.$$

$$AB - CB = AC. \qquad GE - FE = GF.$$

The difference may also be obtained by putting the initial points together. The remainder is defined by the terminal points of the second and the first.

$$AC - AB = BC.$$

$$AB - AC = CB.$$

By general agreement horizontal lines are positive when they make to the right, negative when they make to the left. Vertical lines are positive when they make upwards, negative when they make downwards.

Measurement. A directed line possesses two qualities — length and direction. Measurement takes account of both. Its length is the number of times it contains the unit line, and its direction is indicated by its sign.

20. Angles. We conceive the angle LVM to be generated by the revolution of LV about V till it comes into coincidence with VM. This revolution may be performed in two ways: 1st, as indicated by the arrow marked α; 2d, as indicated by the arrow marked β. In α the motion is *counter-clockwise* (opposite to the motion of a clock hand), in β the motion is clockwise. Mathematicians have agreed to call the former motion positive. The angle denoted by α is positive, β is negative.

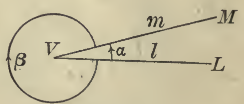


FIG. 26.

Nomenclature. Capital letters denote points; small letters, lines; Greek letters, angles. The angle above may be named, indifferently, LVM, lm, α.

The angle ml is the angle described when m turns in a positive direction to coincidence with l. The angle described by m when it turns negatively into coincidence with l is $-lm = -\alpha$.

The turning line is the *initial* line of the angle; the other bounding line is the *terminal* line. In naming an angle, the initial line is always put first.

The angle lm is generated by the revolution of l, in a positive direction, until it comes into coincidence with m. If l continues to revolve, it will again come into coincidence with m. The angle it has described is still called lm. It differs from the former lm by a whole revolution, 360°. The two angles are *congruent*. If l continues to revolve, it will pass m repeatedly. The angles described when it passes m are all denoted by lm. They differ from each other by some multiple of 360°. They are congruent angles. While lm denotes any one of these congruent angles, the smallest is always understood.

The student may get a clearer conception of what an angle is by considering the motion of the minute hand of a clock. In one hour this hand describes an angle of 360°; in an hour and a half, an angle of 540°; in a half day, an angle of 4320°, and so on. At 12.15, 1.15, 2.15, 3.15, etc., this hand is in the same position. Counting from 12 o'clock, it has described angles of 90°, 450°, 810°, and 1170°. These angles are congruent. It is to be remembered that in the case under consideration all the angles are negative.

21. Addition and Subtraction of Angles. Angles are added and subtracted in the same way that lines are.

The sum of two angles is found by placing their vertices together and bringing the initial line of the second into coincidence with the terminal line of the first, preserving the direction of both. The angle determined by the initial line of the first angle and the terminal line of the second angle is their *sum*.

Two angles are subtracted by bringing together their terminal lines. The angle determined by the initial lines of the first and second angles is their difference. The same result may be obtained by placing their initial lines together. Their difference is defined by the terminals of the second and the first. It is to be noted that the difference defined above is the difference obtained by subtracting the second angle from the first.

$$LVM + MVN = LVN.$$
$$LVN + NVM = LVM.$$
$$LVN - MVN = LVM.$$
$$LVN - LVM = MVN.$$
$$LVM - LVN = NVM.$$

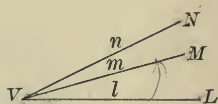


FIG. 27.

22. Measurement of Angles. The measure of an angle is the number of times it contains the unit angle, and this measure will be positive or negative, according as the angle is positive or negative.

Definition. A *Perigon* is the angle generated by a single, complete revolution of a line about a point in a plane. Two unit angles are in common use:

The *degree*, which is $\frac{1}{360}$ of a perigon. This unit is too familiar to require further comment.

The *radian*, which is the angle whose arc is equal to the radius.

The *radian* is a definite angle. For the circumference of any circle is $2\pi (\pi = 3.1416)$ times its radius. The angle whose arc is equal in length to the radius is therefore the angle whose intercepted arc is $\frac{1}{2\pi}$th of the circumference. Since angles are proportional to their arcs the radian is $\frac{1}{2\pi}$th of the perigon.

Radians are denoted by the letter r, *e.g.*, 1^r, 2^r, 6^r, π^r. Generally, however, this symbol is omitted unless such omission gives rise to ambiguity.

Formulas for changing from degree measure to radian measure, and *vice versa*, are readily obtained.

$$2\,\pi^r = 360°.$$

$$1^r = \frac{360°}{2\,\pi} = \frac{180°}{\pi} = 57°.2958.$$

To reduce degrees to radians, divide by $\dfrac{180°}{\pi}$.

To reduce radians to degrees, multiply by $\dfrac{180°}{\pi}$.

When the angle bears a simple ratio to the perigon, its radian measure is expressed as a multiple of π.

E.g., $180° = \pi, \quad 90° = \dfrac{\pi}{2}, \quad 270° = \dfrac{3}{2}\pi, \quad 45° = \dfrac{\pi}{4}.$

EXERCISES

1. Express the following angles in terms of π radians: 30°, 45°, 60°, 75°, 90°, 105°, 120°, 135°, 150°, 165°, 210°, 225°, 240°, 300°, 330°, 450°, 600°.

2. Express the following angles in degrees: $\frac{1}{6}\pi$, $\frac{2}{3}\pi$, $\frac{8}{9}\pi$, $\frac{1}{5}\pi$, $\frac{7}{12}\pi$, $\frac{8}{15}\pi$, $\frac{9}{10}\pi$, $\frac{7}{8}\pi$.

3. Express the following angles in radians: 130°, 36° 4′, 147° 21′, 200°, 340° 36′, 38° 35′.

4. Express the following angles in degrees: 1^r, 2^r, 5^r, $1^r.4$, $3^r.6$, $5^r.47$, $8^r.1$, 10^r, $1^r.1$.

Degree measure is used in all practical applications of trigonometry, while radian measure is used in analytical work. In this book both systems are used indiscriminately.

23. Quadrants. It is customary to place the angle in such a position that the initial line is horizontal and the vertex of the angle toward the left.

The initial line and a line through the vertex perpendicular to this divide the perigon into four equal parts called *quadrants.* These quadrants are numbered I, II, III, IV, as in the accompanying figure. An angle is said to *belong to*, or to be *of*, the quadrant in which its terminal line lies. Thus,

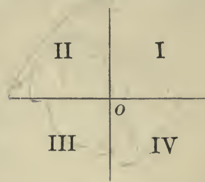


FIG. 28.

angles	$> 0°$	and	$< 90°$	are of the	Ist	quadrant.
"	$> 90°$	"	$< 180°$	"	IId	"
"	$> 180°$	"	$< 270°$	"	IIId	"
"	$> 270°$	"	$< 360°$	"	IVth	"

Angles greater than 360° may be said to belong either to the quadrant of their smallest congruent angle, or to the quadrant determined by counting the number of quadrants passed over in the generation of the angle.

E.g., the angle 800° is congruent to 80°, since $800° = 2 \cdot \times 360° + 80°$.

This angle belongs to either the 1st quadrant or to the 9th since $800° = 8 \times 90° + 80°$.

EXERCISES

To what quadrant do each of the following angles belong: 50°, 150°, 200°, 300°, 400°, 500°, 600°, 700°, 1000°, 2000°, 10000°, 100000°, − 40°, − 100°, − 200°, − 300°, − 600°, $\frac{\pi}{3}$, $\frac{2\pi}{3}$, $3\frac{1}{4}\pi$, $7\frac{2}{3}\pi$, $\frac{6}{5}\pi$, $\frac{5}{6}\pi$, $\frac{25}{6}\pi$?

24. Ordinate and Abscissa. The position of any point in the plane is uniquely determined as soon as we know its distance and direction from each of the two perpendicular axes XX' and YY'. The distance from XX' (SP in the figure) is called the *ordinate* of the point P. Its distance from YY' (OS in the figure) is the *abscissa* of P. Together they are the *coördinates* of P. The abscissa is usually denoted by x, and the ordinate by y. We write $P \equiv (x, y)$, where the abscissa is always put first. These coördinates are directed lines, and according to the convention mentioned in § 19 (p. 33), abscissas which make to the right are positive, to the left, negative; ordinates which make upwards are positive, downwards, negative. The initial extremity of the abscissa is on the y-axis, of the ordinate on the x-axis. The signs of the coördinates in the several quadrants are therefore:

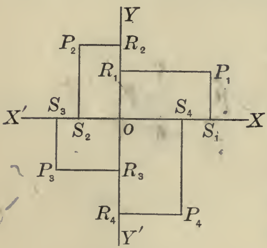


FIG. 29.

	I	II	III	IV
x	+	−	−	+
y	+	+	−	−

EXERCISES

Draw a pair of axes (preferably on coördinate or cross-section paper) and fix the following points:

3, 5; − 3, 7; 3, − 7; − 6, 10; 4, − 8; − 3, − 5; − 1, 2; − 2, − 6; 3, 0; − 2, 0; 0, 4; 0, − 5; 0, 0.

NOTE. Since $RP = OS$ and $OR = SP$, the coördinates of P may be taken as RP and OR instead of OS and SP.

25. The Trigonometric Functions of any Angle. We are now in a position to define the trigonometric functions of any angle. These definitions are more general than those given in § 7 and include them.

Let lm (or XOP) be any angle. Through O, its vertex, draw YY' perpendicular to OX. Take XX' and YY' as axes of coördinates. Let P be any point on m, and x, y its coördinates. Let $OP = r$, and let us agree that r shall be positive when it lies on m and negative when it lies on the backward extension of m. Denote the angle lm or XOP by ϕ.

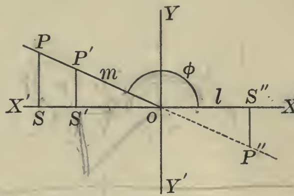


FIG. 30.

The functions are defined as follows:

$$\text{sine of } \phi \equiv \sin\phi \equiv y/r.$$

$$\text{cosine of } \phi \equiv \cos\phi \equiv x/r.$$

$$\text{tangent of } \phi \equiv \tan\phi \equiv y/x.$$

$$\text{cotangent of } \phi \equiv \cot\phi \equiv x/y.$$

$$\text{secant of } \phi \equiv \sec\phi \equiv r/x.$$

$$\text{cosecant of } \phi \equiv \csc\phi \equiv r/y.$$

NOTE. These definitions do not differ from those in § 7 except in generality.

These are all the possible ratios of the three lines x, y, and r.

These ratios are independent of the position of P on m. For if P be taken at any other point, as P', the signs of x, y, and r are unchanged, while the ratios of the lengths are the same in both cases, since the triangles OSP and $OS'P'$ are similar. If the point be taken at P'' on the backward extension of m, the signs of x, y, and r are all changed. The triangles OSP and $OS''P''$ are similar. The ratios of x, y, and r are therefore the same as before in both magnitude and sign.

26. These ratios, being independent of the position of P on m, are functions of the angle ϕ. Their algebraic signs depend upon the quadrant to which ϕ belongs.

Draw an angle of each quadrant and verify the following table, taking r positive.

	x	y	sin	cos	tan	cot	sec	csc
I	+	+	+	+	+	+	+	+
II	−	+	+	−	−	−	−	+
III	−	−	−	−	+	+	−	−
IV	+	−	−	+	−	−	+	−

In quadrant I all functions are positive.
" II " negative except sin, csc.
" III " " tan, cot.
" IV " " cos, sec.

EXERCISES

1. Write down the signs of the several functions of the following angles:

40°, 100°, 160°, 200°, 250°, 300°, 340°, − 40°, − 80°, − 130°, − 190°, − 240°, − 300°.

2. In Fig. 31 the lines CC' and BB' are drawn equally inclined to XX', forming the angles ϕ_1, ϕ_2, ϕ_3, ϕ_4. If we take

$$OP_1 = OP_2 = OP_3 = OP_4,$$

the coördinates of the points P_1, P_2, P_3, and P_4 will be equal in magnitude but not in sign. We shall have

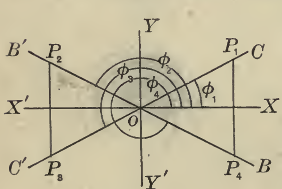


FIG. 31.

$$\sin \phi_1 = \sin \phi_2 = - \sin \phi_3 = - \sin \phi_4.$$

Find the corresponding relations between the other functions.

3. Show by Fig. 31 that there are always two angles less than a perigon which have the same sine: 1st, when the sine is positive; 2d, when it is negative. Show the same thing for each of the other functions.

4. Construct the following angles: (See § 9.)

$\sin^{-1}\frac{2}{3}$;	$\sec^{-1}3$;	$\cos^{-1}-\frac{1}{3}$;	$\tan^{-1}-2$;
$\cot^{-1}-1$;	$\sec^{-1}-2$;	$\sin^{-1}-\frac{1}{2}$;	$\csc^{-1}\frac{1}{2}$;
$\tan^{-1}2$;	$\cos^{-1}\frac{2}{3}$;	$\sin^{-1}\frac{3}{2}$;	$\csc^{-1}3$.

Remember that in each case there are two solutions.

5. Prove the following equations by means of a diagram:

$\sin 60° = \sin 120°$;	$\tan 225° = \tan 45°$;
$\cos 30° = -\cos 150°$;	$\cos 45° = \sin 135°$;
$\cos 120° = -\sin 30°$;	$\tan 150° = -\tan 30°$;
$\sec 40° = -\sec 140°$;	$\cot 130° = -\cot 50°$;
$\sin 210° = -\sin 30°$;	$\tan 135° = -\tan 45°$.

6. What angle has the same sine as 35°, 130°, 190°, 350°, 47°, − 40°, − 140°, − 230°, − 340°, ϕ? What angle has the same cosine as each of the preceding angles? the same tangent? the same cotangent? the same secant? the same cosecant?

7. Draw a diagram and find the functions of 120°. (See § 13.)

8. Find the functions of each of the following angles:

135°, 150°, 210°, 225°, 240°, 300°, 315°, 330°.

27. Fundamental Relation of the Trigonometric Functions. The relations [1] to [8], which were proved in § 11 for acute angles, can be readily shown to hold for all angles. The proof is left for the student. For convenience of reference they are repeated here.

$$\sin\phi \cdot \csc\phi = 1. \qquad [1]$$

$$\cos\phi \cdot \sec\phi = 1. \qquad [2]$$

$$\tan\phi \cdot \cot\phi = 1. \qquad [3]$$

$$\tan\phi = \frac{\sin\phi}{\cos\phi}. \qquad [4]$$

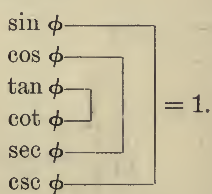
$$\cot\phi = \frac{\cos\phi}{\sin\phi}. \qquad [5]$$

$$\sin^2\phi + \cos^2\phi = 1. \qquad [6]$$

$$1 + \tan^2\phi = \sec^2\phi. \qquad [7]$$

$$1 + \cot^2\phi = \csc^2\phi. \qquad [8]$$

The following scheme may assist in remembering the first three of these formulas:



EXERCISES

By means of the relations [1] to [8] verify the following equations:

1. $\sin\phi = \tan\phi \cos\phi$.
2. $\sin\phi = \dfrac{\tan\phi}{\sec\phi}$.
3. $\sin\phi = \sqrt{1 - \cos^2\phi}$.
4. $\cos\phi = \sqrt{1 - \sin^2\phi}$.
5. $\tan\phi = \dfrac{\sin\phi}{\sqrt{1 - \sin^2\phi}}$.
6. $\tan\phi = \sqrt{\sec^2\phi - 1}$.
7. $\cot\phi = \dfrac{\sqrt{1 - \sin^2\phi}}{\sin\phi} = \dfrac{\cos\phi}{\sqrt{1 - \cos^2\phi}} = \dfrac{1}{\tan\phi}$

$$= \frac{1}{\sqrt{\sec^2\phi - 1}} = \sqrt{\csc^2\phi - 1}.$$

8. Express each of the functions in terms of the sine.

9. $\cos^2\phi - \sin^2\phi = 1 - 2\sin^2\phi = 2\cos^2\phi - 1.$

10. $\sec^2\phi + \csc^2\phi = \sec^2\phi\csc^2\phi.$

11. $\dfrac{\sin\phi}{1 \pm \cos\phi} = \dfrac{1 \mp \cos\phi}{\sin\phi}.$

12. $\dfrac{\cos\phi}{1 \mp \sin\phi} = \dfrac{1 \pm \sin\phi}{\cos\phi}.$

13. $\dfrac{\sec\phi \pm 1}{\tan\phi} = \dfrac{\tan\phi}{\sec\phi \mp 1}.$

14. $\tan\phi + \cot\phi = \sec\phi\csc\phi.$

15. $\sin\phi = \frac{1}{2}$; find all the other functions analytically.

16. $\cos\phi = -\frac{3}{5}$; " " " "

17. $\tan\phi = \frac{2}{3}$; " " " "

18. $\sec\phi = \frac{4}{3}$; " " " "

19. $\sin\phi + \cos\phi = 1.2$; find $\sin\phi$.

20. $\tan^2\phi - \sin^2\phi = \tan^2\phi\sin^2\phi.$

28. The functions of 0°, 90°, 180°, 270°, 360°.

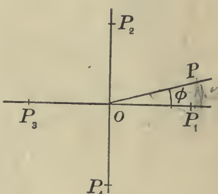


FIG. 32.

Let P be a point on the terminal line of ϕ at a distance r from the origin.

When $\phi = 0$, P coincides with the point P_1, and its coördinates are $x = r$, $y = 0$.

$$\sin 0° = \frac{0}{r} = 0, \qquad \cos 0° = \frac{r}{r} = 1,$$

$$\tan 0° = \frac{0}{r} = 0, \qquad \cot 0° = \frac{r}{0} = \infty,$$

$$\sec 0° = \frac{r}{r} = 1, \qquad \csc 0° = \frac{r}{0} = \infty.$$

When $\phi = 90°$, P coincides with P_2, and its coördinates are $x = 0$, $y = r$.

$$\sin 90° = \frac{r}{r} = 1, \quad \cos 90° = \frac{0}{r} = 0, \quad \tan 90° = \frac{r}{0} = \infty,$$

$$\cot 90° = \frac{0}{r} = 0, \quad \sec 90° = \frac{r}{0} = \infty, \quad \csc 90° = \frac{r}{r} = 1.$$

When $\phi = 180°$, P coincides with P_3, and its coördinates are $x = -r$, $y = 0$.

$$\sin 180° = \frac{0}{r} = 0, \qquad \cos 180° = \frac{-r}{r} = -1,$$

$$\tan 180° = \frac{0}{-r} = 0, \qquad \cot 180° = \frac{-r}{0} = \infty,$$

$$\sec 180° = \frac{r}{-r} = -1, \qquad \csc 180° = \frac{r}{0} = \infty.$$

When $\phi = 270°$, P coincides with P_4, and its coordinates are $x = 0$, $y = -r$.

$$\sin 270° = \frac{-r}{r} = -1, \qquad \cos 270° = \frac{0}{r} = 0,$$

$$\tan 270° = \frac{-r}{0} = \infty, \qquad \cot 270° = \frac{0}{-r} = 0,$$

$$\sec 270° = \frac{r}{0} = \infty, \qquad \csc 270° = \frac{r}{-r} = -1.$$

When $\phi = 360°$, P coincides with P_1, and the functions of 360° are identical with those of 0°.

It is customary to prefix a double sign to the zero and infinity values of the functions, the upper sign being that of the function in the preceding quadrant, the lower that of the function in the following quadrant.

The results obtained are tabulated in the first table on the opposite page.

$\cos LOP = \frac{OS}{OP} =$ length of OS, $\because OP =$ unit of length.

$\tan LOP = \frac{SP}{OS} = \frac{LT}{OL} =$ " LT, $\because OL =$ "

$\cot LOP = \frac{OS}{SP} = \frac{NH}{ON} =$ " NH, $\because ON =$ "

$\sec LOP = \frac{OP}{OS} = \frac{OT}{OL} =$ " OT, $\because OL =$ "

$\csc LOP = \frac{OP}{SP} = \frac{OH}{ON} =$ " OH, $\because ON =$ "

If we agree that secants and cosecants shall be positive when measured on the terminal line and negative when measured on the backward extension of this line, it will be found on examination that these lines represent the functions in *sign* as well as in *magnitude*. For example, LT, the tangent, is positive in quadrants I and III, negative in quadrants II and IV.

30. The March of the Functions. We will now study the variation or march of each of the several functions as the angle increases from 0° to 360°. As the angle increases the point P travels in the positive direction around the circumference of the circle. As P passes through the 1st, 2d, 3d, and 4th quadrants:

The sine, SP, increases from 0 to 1, decreases to 0, decreases to -1, increases to 0.

The cosine, OS, decreases from 1 to 0, decreases from 0 to -1, increases to 0, increases to 1.

The tangent, LT, increases from 0 to ∞, changes sign and increases from $-\infty$ to 0, increases to ∞, changes sign and increases from $-\infty$ to 0.

The cotangent, NH, decreases from ∞ to 0, decreases to $-\infty$, changes sign and decreases from ∞ to 0, decreases to $-\infty$ and changes sign.

The secant, OT, increases from 1 to ∞, changes sign and increases from $-\infty$ to -1, decreases from -1 to $-\infty$, changes sign and decreases from ∞ to 1.

The cosecant, OH, decreases from ∞ to 1, increases from 1 to ∞, changes sign and increases from $-\infty$ to -1, increases from -1 to $-\infty$ and changes sign.

These results are tabulated below:

	0°	1st Quad.	90°	2d Quad.	180°	3d Quad.	270°	4th Quad.	0°
sin	∓ 0	inc.	1	dec.	± 0	dec.	-1	inc.	∓ 0
cos	1	dec.	± 0	dec.	-1	inc.	∓ 0	inc.	1
tan	∓ 0	inc.	$\pm\infty$	inc.	∓ 0	inc.	$\pm\infty$	inc.	∓ 0
cot	$\mp\infty$	dec.	± 0	dec.	$\mp\infty$	dec.	± 0	dec.	$\mp\infty$
sec	1	inc.	$\pm\infty$	inc.	-1	dec.	$\mp\infty$	dec.	1
csc	$\mp\infty$	dec.	1	inc.	$\pm\infty$	inc.	-1	dec.	$\mp\infty$

31. Graphic Representation of the Functions. The nature of the variations which we have just been studying may be exhibited by the following constructions.

Divide the circumference of the unit circle into any number of equal parts. In the figure the points of division are marked 0, 1, 2, 3 ... 12. Lay off the same number of equal parts on a horizontal line, and number the points of division in the same way. Make the divisions of the line approximately equal to the divisions of the circumference.

At the points 0, 1, 2, 3 on the line erect perpendiculars equal in sign and length to the sine (SP) of the corresponding point on the circle. Join the ends of these perpendiculars by a continuous line. The resulting curve is the *curve* of *sines*. As P moves along the circle, SP changes continuously, *i.e.*, it changes from one value to another by passing through all intermediate values. If now we conceive S' as

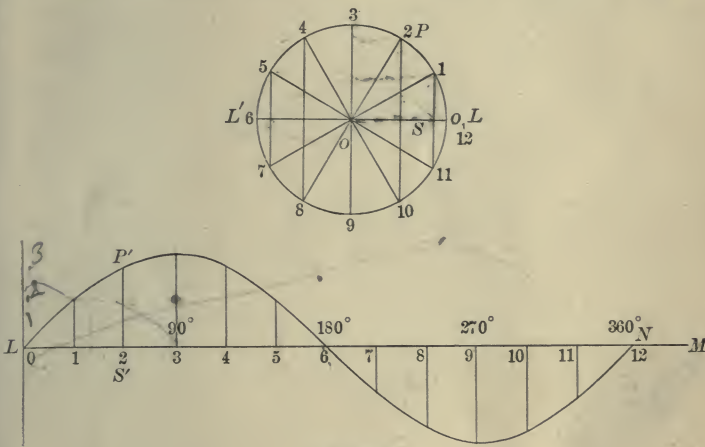


FIG. 34.

moving along LM, keeping pace with P, while $S'P'$ is equal to $S'P$, the point P' will trace the curve of sines. Our construction is an attempt to realize this conception.

32. If the angle increases beyond 360°, *i.e.*, if P makes a second revolution, the values of SP would repeat themselves in the same order. If we plot these values, we shall have the curve between L and N repeated beyond N, and this curve will be repeated as many times as P makes revolutions. The sine curve will take this form. (Fig. 35.)

The student should construct the curve of cosines, the curve of tangents, and the curve of secants in a similar manner. To find the tangents and secants, the construction of the preceding section should be used. The sum or the difference of two functions may be plotted. To plot

$\sin\phi + \cos\phi$, erect at 0, 1, 2, 3, etc., on MN (Fig. 33), perpendiculars equal to $SP + OS$ at the several points 0, 1, 2, 3 on the fundamental circle in that figure. The resulting curve will represent $\sin\phi + \cos\phi$.

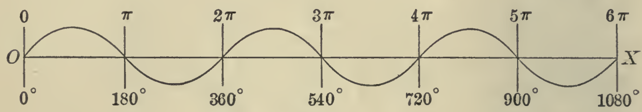


FIG. 35.

Periodic Functions. Functions which repeat themselves as the variable or argument increases are called *periodic functions.* The *period* is the amount of change in the variable which produces the repetition in the values of the function. The sine, as is evident from Fig. 35, is a periodic function with a period of 360°, or 2π. The tangent has π for its period.

EXERCISES

Plot the following functions and determine their periods:

1. $\sin\phi - \cos\phi$.
2. $\tan\phi - \sin\phi$.
3. $\sec\phi - \tan\phi$.
4. $\sin(90° + \phi)$.
5. $\sin(-\phi)$.
6. $\cos(-\phi)$.
7. $\cos\phi$ and $\sec\phi$ on the same axes.

CHAPTER IV

REDUCTION FORMULAS

33. Negative Angles. The object of this chapter is to obtain a set of formulas which will enable us to express any function of an angle greater than 90° as a function of an angle less than 90°.

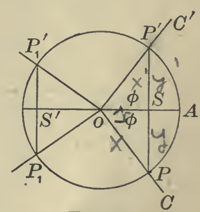


FIG. 36.

Let *AOC* be a negative angle and *AOC'* an equal positive angle. Lay off $OP = OP'$ and draw PP'. $SP' = -SP$. Let the coördinates of P be x, y, of $P' = x'$, y'. Now $x = x'$, $y = -y'$.

$$\sin(-\phi) = \frac{y}{r} = -\frac{y'}{r} = -\sin\phi.$$

$$\cos(-\phi) = \frac{x}{r} = \frac{x'}{r} = \cos\phi.$$

$$\tan(-\phi) = \frac{y}{x} = -\frac{y'}{x'} = -\tan\phi. \qquad [9]$$

$$\cot(-\phi) = \frac{x}{y} = \frac{x'}{-y'} = -\cot\phi.$$

$$\sec(-\phi) = \frac{r}{x} = \frac{r}{x'} = \sec\phi.$$

$$\csc(-\phi) = \frac{r}{y} = \frac{r}{-y'} = -\csc\phi.$$

A little reflection will show that this proof is independent of the magnitude of ϕ and is therefore *general.* Its results may be summed up by saying that in passing from $-\phi$ to ϕ the functions do not change name but do change sign except the cosine and secant.

34. Let $AOB = \phi$ and $AOC = 90° + \phi$. Lay off $OP' = OP$. The two triangles OPS and $OP'S'$ are congruent (equal). Let coördinates of P be x, y; of P', $x'y'$. *Neglecting algebraic signs*, we have

$$x = y', \qquad x' = y.$$

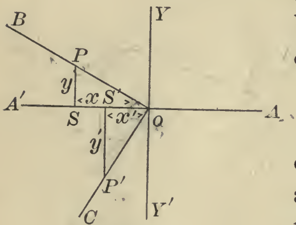


FIG. 37.

No matter what the magnitude of ϕ, it is obvious that we shall always have this relation between the coördinates of P and P'.

By definition we have, neglecting algebraic signs,

$$\sin(90° + \phi) = \frac{y'}{r} = \frac{x}{r} = \cos\phi.$$

$$\cos(90° + \phi) = \frac{x'}{r} = \frac{y}{r} = \sin\phi.$$

$$\tan(90° + \phi) = \frac{y'}{x'} = \frac{x}{y} = \cot\phi.$$

$$\cot(90° + \phi) = \frac{x'}{y'} = \frac{y}{x} = \tan\phi.$$

$$\sec(90° + \phi) = \frac{r}{x'} = \frac{r}{y} = \csc\phi.$$

$$\csc(90° + \phi) = \frac{r}{y'} = \frac{r}{x} = \sec\phi.$$

By studying this table (p. 53) of the signs of the functions in the several quadrants, it appears that $\sin(90° + \phi)$ and $\cos\phi$ always have the same algebraic sign. For if $(90° + \phi)$

falls in quadrant IV, ϕ falls in quadrant III, and $\sin(90° + \phi)$ and $\cos\phi$ are both negative. If $(90° + \phi)$ falls in III, ϕ falls in II, and both are negative, etc. We conclude: The sine of an angle in any quadrant and the cosine of an angle in the preceding quadrant have the same algebraic sign.

II	I
sin +	sin +
cos −	cos +
tan −	tan +
cot −	cot +
sec −	sec +
csc +	csc +
sin −	sin −
cos −	cos +
tan +	tan −
cot +	cot −
sec −	sec +
csc −	csc −
III	IV

Cos $(90° + \phi)$ and $\sin\phi$ have different signs. For the sign of the cosine in any quadrant in the table is different from the sign of the sine in the preceding quadrant.

By employing the same method of reasoning we can show that

$\tan(90° + \phi)$ and $\cot\phi$ have different signs
$\cot(90° + \phi)$ " $\tan\phi$ " " "
$\sec(90° + \phi)$ " $\csc\phi$ " " "
$\csc(90° + \phi)$ " $\sec\phi$ " the same "

The preceding formulas (p. 52), written with the proper signs, are:

$$
\begin{aligned}
\sin(90° + \phi) &= \cos\phi.\\
\cos(90° + \phi) &= -\sin\phi.\\
\tan(90° + \phi) &= -\cot\phi.\\
\cot(90° + \phi) &= -\tan\phi.\\
\sec(90° + \phi) &= -\csc\phi.\\
\csc(90° + \phi) &= \sec\phi.
\end{aligned}
\qquad [10]
$$

Since $180° + \phi = 90° + \overline{90° + \phi}$,

$$
\begin{aligned}
\sin(180° + \phi) &= \cos(90° + \phi) = -\sin\phi.\\
\cos(180° + \phi) &= -\sin(90° + \phi) = -\cos\phi.\\
\tan(180° + \phi) &= -\cot(90° + \phi) = \tan\phi.\\
\cot(180° + \phi) &= -\tan(90° + \phi) = \cot\phi.\\
\sec(180° + \phi) &= -\csc(90° + \phi) = -\sec\phi.\\
\csc(180° + \phi) &= \sec(90° + \phi) = -\csc\phi.
\end{aligned}
\qquad [11]
$$

Functions of $270° + \phi$ are found by putting $270° + \phi = 180° + \overline{90° + \phi}$. The results are tabulated below. The student is advised to verify these results by drawing diagrams.

	$90° - \phi$	$90° + \phi$	$180° - \phi$	$180° + \phi$	$270° - \phi$	$270° + \phi$	$360° - \phi$
sin	$\cos \phi$	$\cos \phi$	$\sin \phi$	$-\sin \phi$	$-\cos \phi$	$-\cos \phi$	$-\sin \phi$
cos	$\sin \phi$	$-\sin \phi$	$-\cos \phi$	$-\cos \phi$	$-\sin \phi$	$\sin \phi$	$\cos \phi$
tan	$\cot \phi$	$-\cot \phi$	$-\tan \phi$	$\tan \phi$	$\cot \phi$	$-\cot \phi$	$-\tan \phi$
cot	$\tan \phi$	$-\tan \phi$	$-\cot \phi$	$\cot \phi$	$\tan \phi$	$-\tan \phi$	$-\cot \phi$
sec	$\csc \phi$	$-\csc \phi$	$-\sec \phi$	$-\sec \phi$	$-\csc \phi$	$\csc \phi$	$\sec \phi$
csc	$\sec \phi$	$\sec \phi$	$\csc \phi$	$-\csc \phi$	$-\sec \phi$	$-\sec \phi$	$-\csc \phi$

This table includes, beside the cases we have already discussed, the functions of $90° - \phi$, $180° - \phi$, $270° - \phi$, and $360° - \phi$. These are reduced as follows:

$\sin(90° - \phi) = \sin[90° + (-\phi)] = \cos(-\phi) = \cos \phi$, by [9]
$\sin(180° - \phi) = \sin[180° + (-\phi)] = -\sin(-\phi) = \sin \phi$, "
$\sin(270° - \phi) = \sin[270° + (-\phi)] = -\cos(-\phi) = -\cos \phi$, "
$\sin(360° - \phi) = \sin(-\phi) = -\sin \phi$.

The other functions of these angles are derived in a similar manner.

35. If we inspect the table carefully, we find that it can be summed up in the two rules that follow:

1°. If 90° or 270° is involved, the function changes name (from sine to cosine, from tangent to cotangent, from secant to cosecant, and *vice versa*), while if 180° or 360° is involved the function does not change name.

The second rule has to do with the algebraic sign.

When we write

$$\cos(90° + \phi) = -\sin \phi,$$
$$\tan(180° - \phi) = -\tan \phi,$$

both terms must have the same sign. If ϕ is less than 90°, $\sin\phi$ is positive and $\cos(90° + \phi)$ is negative. The equality is secured by putting the minus sign before $\sin\phi$. Since these formulas are general, the signs are the same, no matter what the value of ϕ. Our rule is then:

2°. Assume that ϕ is less than 90° and make the signs of both terms alike.

36. Applications. Any angle greater than 90° can be expressed in two of the following forms:

$90° + \phi$, $180° - \phi$, $180° + \phi$, $270° - \phi$, $270° + \phi$, $360° - \phi$,

where ϕ is less than 90°.

E.g.,

$$200° = 180° + 20°, \text{ or } 270° - 70°.$$
$$300° = 270° + 30°, \text{ or } 360° - 60°.$$
$$135° = 90° + 45°, \text{ or } 180° - 45°.$$

The functions of 200° are:

$$\sin 200° = \sin(180° + 20°) = -\sin 20°.$$
$$\cos 200° = \cos(180° + 20°) = -\cos 20°.$$
$$\tan 200° = \tan(180° + 20°) = \tan 20°.$$
$$\cot 200° = \cot(180° + 20°) = \cot 20°.$$
$$\sec 200° = \sec(180° + 20°) = -\sec 20°.$$
$$\csc 200° = \csc(180° + 20°) = -\csc 20°.$$

They may also be written:

$$\sin 200° = \sin(270° - 70°) = -\cos 70°.$$
$$\cos 200° = \cos(270° - 70°) = -\sin 70°.$$
$$\tan 200° = \tan(270° - 70°) = \cot 70°.$$
$$\cot 200° = \cot(270° - 70°) = \tan 70°.$$
$$\sec 200° = \sec(270° - 70°) = -\csc 70°.$$
$$\csc 200° = \csc(270° - 70°) = -\sec 70°.$$

EXERCISES

1. Express the following functions as functions of angles less than 90°: tan 130°, sin 160°, cos 100°, cot 215°, sec 260°, csc 280°, sin 310°, cos 310°.

2. Express each of the preceding functions as the function of an angle less than 45°.

3. Express each of the following functions as the function of an angle less than 45° $\left[=\frac{\pi}{4}\right]$.

$$\sin\frac{2\pi}{3},\ \tan\frac{3\pi}{4},\ \cos\frac{5\pi}{6},\ \sec\frac{4\pi}{3},\ \cot\frac{5\pi}{3},\ \csc\frac{11\pi}{6}.$$

4. By using formulas [9] express the following functions as functions of positive angles less than 90°:

$$\sin(-160°),\quad \cos(-30°),\quad \tan(-300°),\quad \sec(-140°),$$
$$\cot(-240°),\quad \csc(-100°),\quad \sin(-300°).$$

5. The angle $-\phi$ is obviously congruent to $360° - \phi$, and their functions are identical. Reduce the functions in problem 4 by making use of this identity.

6. Express the following functions as functions of angles less than 45°:

$$\cos 117°\,17',\quad \sin 143°\,21'\,16'',\quad \tan 317°\,29'\,31'',$$
$$\cot 90°\,46'\,12'',\quad \sec(-135°\,14'\,11''),\quad \cos(-71°\,23').$$

CHAPTER V

THE ADDITION FORMULA

37. Projection. The projection of a *point* on a *line* is the foot of the perpendicular from the point to the line.

The projection of a line-segment on a given line in the same plane is the portion of the second line bounded by the projections of the ends of the first line.

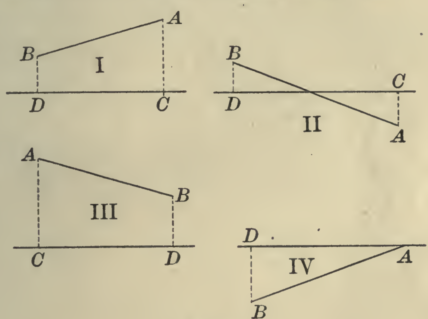


FIG. 38.

The projection of AB is CD in I, II, and III, and AD in IV.

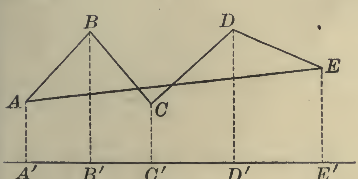


FIG. 39.

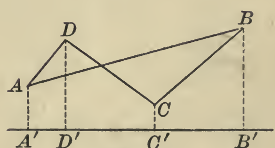


FIG. 40.

The projection of a broken line is the sum of the projections of its parts.

The projection of $ABCDE$ (Fig. 39) is $A'B' + B'C' + C'D' + D'E' = A'E'$; and the projection of $ABCD$ (Fig. 40) is $A'B' + B'C' + C'D' = A'D'$. (See § 19.)

It is obvious that the projection of a broken line is equal to the projection of the straight line connecting the ends of the broken line. It is to be noted that here we take the *direction* of the lines into account. The projection of $ABCD$ (Fig. 40) is equal to the projection of AD, and is the negative of the projection of DA.

38. The projection of a line-segment on any line in its plane is equal, in length and direction, to the length of the segment multiplied by the cosine of the angle which the segment makes with the line. In the figure the line-segment is AC, the line of projection is LM, and the angle, measured according to § 20, is ϕ.

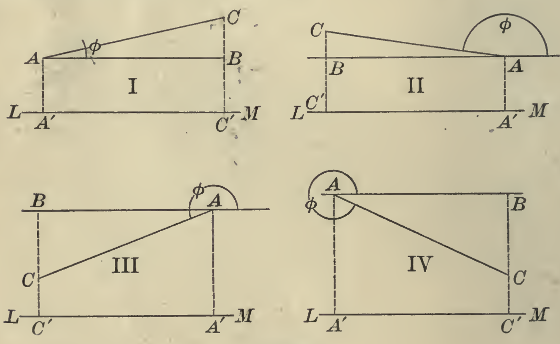


FIG. 41.

The projection of AC is $A'C' = AB$.

Now $\cos\phi = \dfrac{AB}{AC}$. $\therefore AB = A'C' = AC\cos\phi$.

The projection is positive when ϕ is an angle of the 1st or 4th quadrant; it is negative when ϕ is an angle of the 2d or 3d quadrant.

39. Projection on Coördinate Axes. The projections of AC on XX' and YY' may be called the x-projection of AC and the y-projection of AC. Let ϕ be the angle which AC makes with XX'. Draw OD parallel to AC.

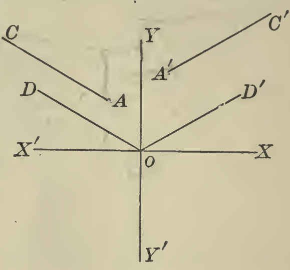


FIG. 42.

$$\phi = XOD, \quad \therefore YOD = \phi - 90°.$$

If ϕ is less than 90°, as in the case of $A'C'$, the angle YOD' is negative and equal to $90° - \phi$.

But $-(90° - \phi) = \phi - 90°$.

In any case the angle which AC makes with YY' is 90° less than the angle it makes with XX'.

$$x\text{-projection of } AC = AC \cos \phi.$$

$$\begin{aligned} y\text{-projection of } AC &= AC \cos(\phi - 90°) \\ &= AC \cos(90° - \phi) \qquad \text{by [9]} \\ &= AC \sin \phi. \end{aligned}$$

40. The Addition Formulas. These formulas enable us to express the functions of the sum or the difference of two angles in terms of the functions of the constituent angles. Without examining the matter, the student might make the mistake of writing:

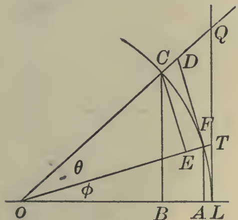


FIG. 43.

$$\sin(\phi + \theta) = \sin \phi + \sin \theta.$$

$$\tan(\phi + \theta) = \tan \phi + \tan \theta, \text{ etc.}$$

In the accompanying figure the points F and C, on the terminal lines of ϕ and θ, respectively, are taken on the circumference of the unit circle. We have, therefore, in line representatives (see § 29):

$$\sin\phi = AF,\ \sin\theta = EC,\ \sin(\phi+\theta) = BC.$$
$$\tan\phi = LT,\ \tan\theta = FD,\ \tan(\phi+\theta) = LQ.$$

It is evident that $AF + EC > BC$,

or $$\sin\phi + \sin\theta > \sin(\phi+\theta),$$

and $$LT + FD < LQ,$$

or $$\tan\phi + \tan\theta < \tan(\phi+\theta).$$

Since the formulas fail in this particular case, they are obviously untrue.

41. The Sine and Cosine of $\phi + \theta$.

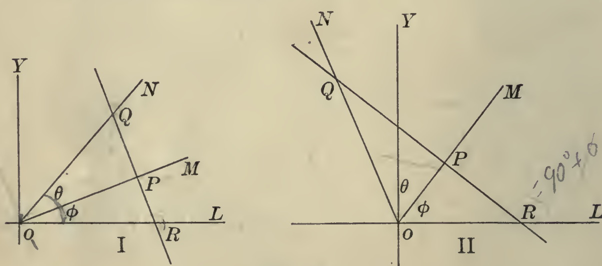


FIG. 44.

Let $LOM = \phi$ and $MON = \theta$, then $LON = \phi + \theta$.

In I, $\phi + \theta < 90°$; in II, $\phi + \theta > 90°$; in both, $\phi < 90°$, $\theta < 90°$.

Through P, any point in OM, draw RPQ perpendicular to OM.

Angle $LRQ = 90 + \phi$, being the exterior angle of the triangle OPR.

Since OQ is a line connecting the extremities of the broken line OPQ, we have, by § 37:

y-projection of OQ
$$= y\text{-projection of } OP + y\text{-projection of } PQ, \quad [12]$$
x-projection of OQ
$$= x\text{-projection of } OP + x\text{-projection of } PQ. \quad [13]$$

(x + y) = sin x cos y + cos x sin y.
(x + y) = cos x cos y − sin x sin y.

Applying § 39, these equations become:

$$OQ \sin(\phi + \theta) = OP \sin\phi + PQ \sin(90^\circ + \phi), \quad [14]$$

$$OQ \cos(\phi + \theta) = OP \cos\phi + PQ \cos(90^\circ + \phi). \quad [15]$$

But $OP = OQ \cos\theta$, $PQ = OQ \sin\theta$.

$$\sin(90^\circ + \phi) = \cos\phi, \ \cos(90^\circ + \phi) = -\sin\phi.$$

Substituting these values in [14] and [15], we have

$$OQ \sin(\phi + \theta) = OQ \sin\phi \cos\theta + OQ \cos\phi \sin\theta, \quad [16]$$

$$OQ \cos(\phi + \theta) = OQ \cos\phi \cos\theta - OQ \sin\phi \sin\theta. \quad [17]$$

$$\therefore \sin(\phi + \theta) = \sin\phi \cos\theta + \cos\phi \sin\theta, \quad [18]$$

$$\cos(\phi + \theta) = \cos\phi \cos\theta - \sin\phi \sin\theta, \quad [19]$$

where ϕ and θ are both less than $90^\circ \left(\frac{\pi}{2}\right)$.

42. To establish the truth of these formulas where ϕ and θ are unlimited we proceed as follows:

Let $\phi = 90^\circ + \beta$, where $\beta < 90^\circ$.

$$\sin(\phi + \theta) = \sin(90^\circ + \beta + \theta) = \cos(\beta + \theta), \quad [20]$$

$$\cos(\phi + \theta) = \cos(90^\circ + \beta + \theta) = -\sin(\beta + \theta). \quad [21]$$

Since β and θ are each less than 90°,

$$\sin(\phi + \theta) = \cos(\beta + \theta) = \cos\beta \cos\theta - \sin\beta \sin\theta, \quad [22]$$

$$\cos(\phi + \theta) = -\sin(\beta + \theta) = -\sin\beta \cos\theta - \cos\beta \sin\theta. \quad [23]$$

But $\sin\beta = \sin(\phi - 90^\circ) = -\sin(90^\circ - \phi) = -\cos\phi$,

$\cos\beta = \cos(\phi - 90^\circ) = \cos(90^\circ - \phi) = \sin\phi$.

Substituting these values in [22] and [23],

$$\sin(\phi + \theta) = \sin\phi \cos\theta + \cos\phi \sin\theta, \quad [18]$$

$$\cos(\phi + \theta) = \cos\phi \cos\theta - \sin\phi \sin\theta. \quad [19]$$

Here ϕ is an angle of the 2d quadrant, θ an angle of the 1st quadrant. By a repetition of this process we can show that [18] and [19] hold when ϕ is of the 3d quadrant, etc. Treating θ similarly, we prove these formulas true for all positive values of ϕ and θ.

43. They also hold when one or both the angles are negative.

Let $\phi = \beta - 90°$ where $\beta < 90°$.

$$\sin(\phi + \theta) = \sin(\beta - 90° + \theta)$$
$$= -\sin(90° - \beta - \theta) = -\cos(\beta + \theta), \quad [24]$$
$$\cos(\phi + \theta) = \cos(\beta - 90° + \theta)$$
$$= \cos(90° - \beta - \theta) = \sin(\beta + \theta). \quad [25]$$

Since β and θ are positive,

$$\sin(\phi + \theta) = -\cos(\beta + \theta) = -\cos\beta\cos\theta + \sin\beta\sin\theta, \quad [26]$$
$$\cos(\phi + \theta) = \sin(\beta + \theta) = \sin\beta\cos\theta + \cos\beta\sin\theta. \quad [27]$$

But $\sin\beta = \sin(\phi + 90°) = +\cos\phi,$

$\cos\beta = \cos(\phi + 90°) = -\sin\phi.$

Substituting these values in [26] and [27],

$$\sin(\phi + \theta) = \sin\phi\cos\theta + \cos\phi\sin\theta, \quad [18]$$
$$\cos(\phi + \theta) = \cos\phi\cos\theta - \sin\phi\sin\theta. \quad [19]$$

A similar process of reasoning would show that these formulas remain unchanged when both ϕ and θ are negative. They are true for all positive and negative values of ϕ and θ.

These formulas are so important that they should be carefully memorized. They may be translated into words as follows:

I. The sine of the sum of two angles is equal to the sine of the first into the cosine of the second, plus the cosine of the first into the sine of the second.

II. The cosine of the sum of two angles is equal to the product of their cosines minus the product of their sines.

44. The sine and the cosine of $\phi - \theta$.

Putting $-\theta$ for θ in [18] and [19], we have

$$\sin(\phi - \theta) = \sin\phi\cos(-\theta) + \cos\phi\sin(-\theta), \quad [28]$$

$$\cos(\phi - \theta) = \cos\phi\cos(-\theta) - \sin\phi\sin(-\theta). \quad [29]$$

But $\sin(-\phi) = -\sin\phi$, $\cos(-\phi) = \cos\phi$.

Substituting these values in [28] and [29], we have

$$\sin(\phi - \theta) = \sin\phi\cos\theta - \cos\phi\sin\theta, \quad [30]$$

$$\cos(\phi - \theta) = \cos\phi\cos\theta + \sin\phi\sin\theta. \quad [31]$$

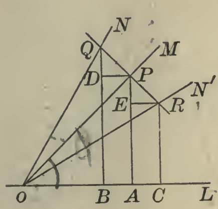
Formulas [18] and [30], and [19] and [31], may be combined as follows:

$$\sin(\phi \pm \theta) = \sin\phi\cos\theta \pm \cos\phi\sin\theta, \quad [32]$$

$$\cos(\phi \pm \theta) = \cos\phi\cos\theta \mp \sin\phi\sin\theta. \quad [33]$$

It should be noted that in [32] the double sign in the second member is like the double sign in the first member, while in [33] it is unlike.

45. Formulas [18] and [19] are so important that other geometrical proofs are added.



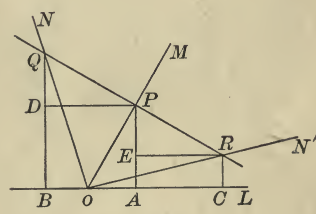


FIG. 45.

Let $LOM = \phi$ and $MON = MON' = \theta$,

then $LON = \phi + \theta$ and $LON' = \phi - \theta$.

Through P, any point in OM, draw QPR perpendicular to OM. Draw PA, QB, and RC perpendicular to OL. Draw PD and RE parallel to OL.

$\angle DQP = \angle EPR = \phi$, since their sides are perpendicular to LO and OM.

$$\sin\phi = \frac{AP}{OP} = \frac{DP}{QP} = \frac{ER}{PR}, \quad \cos\phi = \frac{OA}{OP} = \frac{QD}{QP} = \frac{EP}{PR}.$$

$$\sin\ \theta = \frac{PQ}{OQ} = \frac{PR}{OR}, \quad \cos\ \theta = \frac{OP}{OQ} = \frac{OP}{OR}.$$

$$\begin{aligned}\sin(\phi+\theta) &= \frac{BQ}{OQ} = \frac{AP+QD}{OQ}\\ &= \frac{AP}{OQ} + \frac{QD}{OQ} = \frac{AP}{OP}\cdot\frac{OP}{OQ} + \frac{QD}{QP}\cdot\frac{QP}{OQ}\\ &= \sin\phi\cos\theta + \cos\phi\sin\theta.\end{aligned} \qquad [18]$$

$$\begin{aligned}\cos(\phi+\theta) &= \frac{OB}{OQ} = \frac{OA-DP}{OQ}\\ &= \frac{OA}{OQ} - \frac{DP}{OQ} = \frac{OA}{OP}\cdot\frac{OP}{OQ} - \frac{DP}{QP}\cdot\frac{QP}{OQ}\\ &= \cos\phi\cos\theta - \sin\phi\sin\theta.\end{aligned} \qquad [19]$$

$$\begin{aligned}\sin(\phi-\theta) &= \frac{RC}{OR} = \frac{AP-EP}{OR}\\ &= \frac{AP}{OR} - \frac{EP}{OR} = \frac{AP}{OP}\cdot\frac{OP}{OR} - \frac{EP}{PR}\cdot\frac{PR}{OR}\\ &= \sin\phi\cos\theta - \cos\phi\sin\theta.\end{aligned} \qquad [30]$$

$$\begin{aligned}\cos(\phi-\theta) &= \frac{OC}{OR} = \frac{OA+ER}{OR}\\ &= \frac{OA}{OR} + \frac{ER}{OR} = \frac{OA}{OP}\cdot\frac{OP}{OR} + \frac{ER}{PR}\cdot\frac{PR}{OR}\\ &= \cos\phi\cos\theta + \sin\phi\sin\theta.\end{aligned} \qquad [31]$$

46. Still another proof of [18] and [19] is given below.

Construction. Lay off OA = unity. Draw AB and AQ perpendicular to OM and ON, respectively. Draw BC perpendicular to ON. Draw AD perpendicular to BC.

$\angle ABD = \theta$; their sides are perpendicular.

Let $\sin\phi, \cos\phi = s_1, c_1,$

$\sin\theta, \cos\theta = s_2, c_2.$

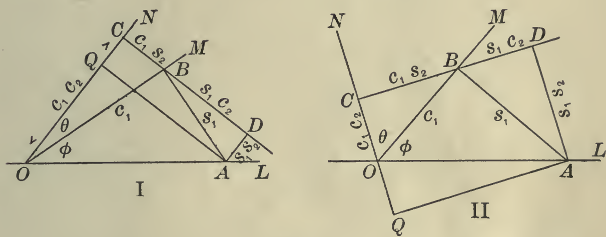


FIG. 46.

The lines in the figure evidently have the lengths indicated. For example, $BC = c_1 s_2$, etc.

$$\sin(\phi+\theta) = \frac{AQ}{OA} = AQ = CD = BD + CB = s_1c_2 + c_1s_2$$

$$= \sin\phi\cos\theta + \cos\phi\sin\theta. \qquad [18]$$

$$\cos(\phi+\theta) = \frac{OQ}{OA} = OQ = OC - AD = c_1c_2 - s_1s_2$$

$$= \cos\phi\cos\theta - \sin\phi\sin\theta.$$

47. Tangent and Cotangent of $\phi + \theta$ and $\phi - \theta$.

$$\tan(\phi+\theta) = \frac{\sin(\phi+\theta)}{\cos(\phi+\theta)} = \frac{\sin\phi\cos\theta + \cos\phi\sin\theta}{\cos\phi\cos\theta - \sin\phi\sin\theta}.$$

Cf. [4], [18], [19]

Dividing both numerator and denominator by $\cos\phi\cos\theta$, we have

$$\tan(\phi+\theta)=\frac{\tan\phi+\tan\theta}{1-\tan\phi\tan\theta}. \qquad [34]$$

Other forms for $\tan(\phi+\theta)$ may be found by dividing by $\sin\phi\sin\phi$, $\sin\phi\cos\theta$, $\cos\phi\sin\theta$, instead of $\cos\phi\cos\theta$. Find them. Why is [34] preferred? In like manner we find

$$\tan(\phi-\theta)=\frac{\sin(\phi-\theta)}{\cos(\phi-\theta)}=\frac{\tan\phi-\tan\theta}{1+\tan\phi\tan\theta}. \qquad [35]$$

NOTE. [35] might be obtained from [34] by putting $-\phi$ for ϕ. Verify this statement.

Similarly

$$\cot(\phi+\theta)=\frac{\cos(\phi+\theta)}{\sin(\phi+\theta)}=\frac{\cos\phi\cos\theta-\sin\phi\sin\theta}{\sin\phi\cos\theta+\cos\phi\sin\theta}.$$

Dividing numerator and denominator by $\sin\phi\sin\theta$, we have

$$\cot(\phi+\theta)=\frac{\cot\phi\cot\theta-1}{\cot\theta+\cot\phi}. \qquad [36]$$

In like manner

$$\cot(\phi-\theta)=\frac{\cot\phi\cot\theta+1}{\cot\theta-\cot\phi}. \qquad [37]$$

Find other forms for [36] and [37] by dividing by $\cos\phi\cos\theta$, by $\cos\phi\sin\theta$, by $\sin\phi\cos\theta$, instead of by $\sin\phi\sin\theta$.

EXERCISES

1. Deduce [36] and [37] from [34] and [35] by using [3].

2. Deduce [36] and [37] from [34] and [35] by substituting $(90+\phi)$ for ϕ in the latter.

3. Prove

$$\sin(\phi+\theta)\sin(\phi-\theta)=\sin^2\phi-\sin^2\theta=\cos^2\theta-\cos^2\phi.$$

4. Prove

$$\cos(\phi+\theta)\cos(\phi-\theta)=\cos^2\phi-\sin^2\theta=\cos^2\theta-\sin^2\phi.$$

5. Find formulas for $\sec(\phi+\theta)$, $\sec(\phi-\theta)$, $\csc(\phi+\theta)$, $\csc(\phi-\theta)$ in terms of the secants and cosecants of ϕ and θ.

6. Find the sine of 75°.

$$\begin{aligned}\sin 75^\circ = \sin(45^\circ+30^\circ) &= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ\\ &= \tfrac{1}{2}\sqrt{2}\cdot\tfrac{1}{2}\sqrt{3}+\tfrac{1}{2}\sqrt{2}\cdot\tfrac{1}{2} \quad \text{(p. 18)}\\ &= \tfrac{1}{4}(\sqrt{6}+\sqrt{2}).\end{aligned}$$

7. Find the other functions of 75°.

8. Find all the functions of 15°. $(15^\circ = 45^\circ - 30^\circ.)$

9. Find all the functions of 180°. $(180^\circ = 90^\circ + 90^\circ.)$

10. Find all the functions of 135°. $(135^\circ = 90^\circ + 45^\circ.)$

11. $\sin\phi=\frac{1}{2}$, $\sin\theta=\frac{1}{3}$; find the functions of $\phi+\theta$ and $\phi-\theta$.

12. Prove $\sin^{-1}x+\sin^{-1}y=\sin^{-1}(x\sqrt{1-y^2}+y\sqrt{1-x^2})$.

13. Prove

$$\cos^{-1}x+\cos^{-1}y=\cos^{-1}(xy-\sqrt{(1-x^2)(1-y^2)}).$$

14. Prove $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\dfrac{x+y}{1-xy}$.

48. Functions of Double Angles. If we put $\theta=\phi$ in formulas [24], [25], [34], and [36], we shall have

$$\sin(\phi+\phi)=\sin\phi\cos\phi+\cos\phi\sin\phi.$$

$$\therefore \sin 2\phi = 2\sin\phi\cos\phi. \qquad [38]$$

$$\cos(\phi+\phi)=\cos\phi\cos\phi-\sin\phi\sin\phi.$$

$$\left.\begin{aligned}\cos 2\phi &= \cos^2\phi-\sin^2\phi & \text{I}\\ &= 2\cos^2\phi-1, \quad \text{since } \sin^2\phi=1-\cos^2\phi & \text{II}\\ &= 1-2\sin^2\phi, \quad \text{"} \quad \cos^2\phi=1-\sin^2\phi. & \text{III}\end{aligned}\right\}[39]$$

$$\tan(\phi + \phi) = \frac{\tan\phi + \tan\phi}{1 - \tan\phi\tan\phi}.$$

$$\tan 2\phi = \frac{2\tan\phi}{1 - \tan^2\phi}. \qquad [40]$$

$$\cot 2\phi = \frac{\cot^2\phi - 1}{2\cot\phi}. \qquad [41]$$

EXERCISES

1. Given the functions of 30°, find those of 60°, of 120°, of 240°.

2. Given the functions of 45°, find those of 90°, of 180°, of 360°.

Prove the following:

3. $\dfrac{2 - \sec^2 x}{\sec^2 x} = \cos 2x.$ 4. $\tan x + \cot x = 2\csc 2x.$

5. $(\sin x \pm \cos x)^2 = 1 \pm \sin 2x.$

6. $\dfrac{2\tan x}{1 + \tan^2 x} = \sin 2x.$ *Cf.* [40]. 7. $\dfrac{1 + \tan^2 x}{1 - \tan^2 x} = \sec 2x.$

8. Find formulas for $\sec 2\theta$ and $\csc 2\theta$.

9. $2\sin(45° + \phi)\sin(45° - \phi) =$ ~~$\cos\phi$~~ $\cos 2\phi$

49. Functions of Half-angles. If in III and II of [39] we substitute $\frac{1}{2}\phi$ for ϕ,

$$\cos\phi = 1 - 2\sin^2\tfrac{1}{2}\phi.$$

$$\cos\phi = 2\cos^2\tfrac{1}{2}\phi - 1.$$

$$\therefore \sin\tfrac{1}{2}\phi = \pm\sqrt{\frac{1 - \cos\phi}{2}}. \qquad [42]$$

$$\cos\tfrac{1}{2}\phi = \pm\sqrt{\frac{1 + \cos\phi}{2}}. \qquad [43]$$

By formula [4]

$$\left.\begin{aligned}\tan \tfrac{1}{2}\phi &= \sqrt{\frac{1-\cos\phi}{1+\cos\phi}} && \text{I}\\ &= \frac{1-\cos\phi}{\sin\phi} && \text{II}\\ &= \frac{\sin\phi}{1+\cos\phi}. && \text{III}\end{aligned}\right\} [44]$$

II is derived from I by multiplying numerator and denominator by $1-\cos\phi$; while

III is derived from I by using $1+\cos\phi$ as multiplier.

By formula [3]

$$\left.\begin{aligned}\cot \tfrac{1}{2}\phi &= \sqrt{\frac{1+\cos\phi}{1-\cos\phi}} && \text{I}\\ &= \frac{\sin\phi}{1-\cos\phi} && \text{II}\\ &= \frac{1+\cos\phi}{\sin\phi}. && \text{III}\end{aligned}\right\} [45]$$

EXERCISES

1. Given the functions of 60°, find those of 30°, of 15°.
2. Given the functions of 45°, find those of 22° 30′.
3. Given $\sin\phi = \frac{1}{2}$, find the functions of $\frac{\phi}{2}$.
4. $\cos\phi = x$; find the functions of $\frac{\phi}{2}$.

Verify the following:

5. $\dfrac{1+\sec\phi}{\sec\phi} = 2\cos^2\dfrac{\phi}{2}$.
6. $\cos^2\dfrac{\phi}{2}\left(1+\tan\dfrac{\phi}{2}\right)^2 = 1+\sin\phi$.
7. $\csc x - \cot x = \tan\dfrac{x}{2}$.
8. $\sin^2\dfrac{x}{2}\left(\cot\dfrac{x}{2}-1\right)^2 = 1-\sin x$.

50. Functions of Three Angles.

$$\begin{aligned}\sin(\alpha+\beta+\gamma) &= \sin(\overline{\alpha+\beta}+\gamma)\\ &= \sin(\alpha+\beta)\cos\gamma+\cos(\alpha+\beta)\sin\gamma\\ &= (\sin\alpha\cos\beta+\cos\alpha\sin\beta)\cos\gamma\\ &\quad +(\cos\alpha\cos\beta-\sin\alpha\sin\beta)\sin\gamma\\ &= \sin\alpha\cos\beta\cos\gamma+\cos\alpha\sin\beta\cos\gamma\\ &\quad +\cos\alpha\cos\beta\sin\gamma-\sin\alpha\sin\beta\sin\gamma. \quad [46]\end{aligned}$$

Similarly

$$\begin{aligned}\cos(\alpha+\beta+\gamma) &= \cos\alpha\cos\beta\cos\gamma-\sin\alpha\sin\beta\cos\gamma\\ &\quad -\sin\alpha\cos\beta\sin\gamma-\cos\alpha\sin\beta\sin\gamma. \quad [47]\end{aligned}$$

$$\tan(\alpha+\beta+\gamma)=\frac{\tan\alpha+\tan\beta+\tan\gamma-\tan\alpha\tan\beta\tan\gamma}{1-\tan\beta\tan\gamma-\tan\gamma\tan\alpha-\tan\alpha\tan\beta}. \quad [48]$$

Formula [48] may also be obtained by dividing [46] by [47], and then dividing numerator and denominator by $\cos\alpha \cos\beta \cos\gamma$.

If now in [46], [47], and [48] we put $\beta = \gamma = \alpha$, we shall have

$$\begin{aligned}\sin 3\alpha &= 3\cos^2\alpha\sin\alpha-\sin^3\alpha\\ &= 3\sin\alpha-4\sin^3\alpha. \quad [49]\end{aligned}$$

$$\begin{aligned}\cos 3\alpha &= \cos^3\alpha-3\sin^2\alpha\cos\alpha\\ &= 4\cos^3\alpha-3\cos\alpha. \quad [50]\end{aligned}$$

$$\tan 3\alpha=\frac{3\tan\alpha-\tan^3\alpha}{1-3\tan^2\alpha}. \quad [51]$$

EXERCISES

1. Put the last six formulas into words.

2. Find the sine, the cosine, and the tangent of $\alpha+\beta-\gamma$, of $\alpha-\beta-\gamma$.

3. Find the cotangent of $\alpha+\beta+\gamma$ in terms of the cotangents of the constituent angles.

4. Find $\sin 3\phi$ by developing $\sin(2\phi + \phi)$ by formula [18] and simplifying the result.

5. Find $\sin 4\phi$ by putting 2ϕ for ϕ in [38].

6. Deduce formula [47] from [46] by putting $90 + a$ for a.

7. Find $\cos 4\phi$.

8. Find $\sin 5\phi$ and $\cos 5\phi$.

9. Given the functions of 30°, find those of 90°.

51. Conversion Formulas. By adding and subtracting [18] and [30], and [19] and [31], we have

$$\sin(\phi + \theta) + \sin(\phi - \theta) = 2 \sin\phi \cos\theta.$$
$$\sin(\phi + \theta) - \sin(\phi - \theta) = 2 \cos\phi \sin\theta.$$
$$\cos(\phi + \theta) + \cos(\phi - \theta) = 2 \cos\phi \cos\theta.$$
$$\cos(\phi + \theta) - \cos(\phi - \theta) = -2 \sin\phi \sin\theta.$$

Putting $(\phi + \theta) = a, (\phi - \theta) = \beta$,

whence $\phi = \frac{1}{2}(a + \beta), \theta = \frac{1}{2}(a - \beta)$, we have

$$\sin a + \sin\beta = 2 \sin\tfrac{1}{2}(a + \beta) \cos\tfrac{1}{2}(a - \beta), \quad [52]$$
$$\sin a - \sin\beta = 2 \cos\tfrac{1}{2}(a + \beta) \sin\tfrac{1}{2}(a - \beta), \quad [53]$$
$$\cos a + \cos\beta = 2 \cos\tfrac{1}{2}(a + \beta) \cos\tfrac{1}{2}(a - \beta), \quad [54]$$
$$\cos a - \cos\beta = -2 \sin\tfrac{1}{2}(a + \beta) \sin\tfrac{1}{2}(a - \beta). \quad [55]$$

These formulas enable us to express the sum or the difference of two sines or two cosines as a product.

EXERCISES

1. Express the last four formulas in words.

Verify these formulas when

2. $a = 60°, \beta = 30°$.
3. $a = 90°, \beta = 60°$.
4. $a = 180°, \beta = 90°$.
5. $a = 270°, \beta = 180°$.

Verify the following identities:

6. $\dfrac{\sin \alpha + \sin \beta}{\sin \alpha - \sin \beta} = \dfrac{\tan \frac{1}{2}(\alpha + \beta)}{\tan \frac{1}{2}(\alpha - \beta)}$.

7. $\dfrac{\sin \alpha + \sin \beta}{\cos \alpha + \cos \beta} = \tan \frac{1}{2}(\alpha + \beta)$.

8. $\dfrac{\cos \alpha + \cos \beta}{\cos \alpha - \cos \beta} = - \cot \frac{1}{2}(\alpha + \beta) \cot \frac{1}{2}(\alpha - \beta)$.

9. $\sin 60° + \sin 30° = 2 \sin 45° \cos 15°$.

10. $\sin 40° - \sin 10° = 2 \cos 25° \sin 15°$.

11. $\cos 75° + \cos 15° = 2 \cos 45° \cos 30°$.

12. $\sin 5x + \sin 3x = 2 \sin 4x \cos x$.

13. $\dfrac{\sin 3x + \sin 2x}{\cos 2x - \cos 3x} = \cot \dfrac{x}{2}$.

14. $\cos(60° + x) + \cos(60° - x) = \cos x$.

15. $\tan 50° + \cot 50° = 2 \sec 10°$.

16. $\sin 2 \cos^{-1} x = 2x\sqrt{1 - x^2}$.

17. $\cos 2 \sin^{-1} x = 1 - 2x^2$.

18. $\cos 2 \cos^{-1} x = 2x^2 - 1$.

19. $\tan 2 \tan^{-1} x = \dfrac{2x}{1 - x^2}$.

20. $\tan^{-1} x + \tan^{-1} y = \tan^{-1} \dfrac{x + y}{1 - xy}$. (Take the tangent of both members.)

21. $\sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2}$.

22. $\sin^{-1} x + \cos^{-1} y = \sin^{-1}(xy + \sqrt{(1 - x^2)(1 - y^2)})$.

23. $\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = \dfrac{\pi}{4}$, or $\dfrac{5\pi}{4}$. *Cf.* example 20.

24. $\tan^{-1} \frac{1}{3} = \tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{8}$.

$$\therefore \frac{\pi}{4} = \tan^{-1} \tfrac{1}{2} + \tan^{-1} \tfrac{1}{5} + \tan^{-1} \tfrac{1}{8}.$$

52. Trigonometric Equations. Trigonometric equations are generally best solved by expressing all the functions involved in terms of some one function and solving the resulting equation.

Illustrations. 1. $\sin \phi = \tan \phi$.

$$\sin \phi = \frac{\sin \phi}{\cos \phi}. \quad \therefore \sin \phi \left(1 - \frac{1}{\cos \phi}\right) = 0.$$

$$\therefore \sin \phi = 0, \quad \text{and} \quad \cos \phi = 1.$$

$$\phi = 0 \text{ and } \pi, \qquad \phi = 0.$$

The solutions are therefore $\phi = 0, \pi$.

2. $\tan \phi = \csc \phi$.

$$\frac{\sin \phi}{\cos \phi} = \frac{1}{\sin \phi}, \quad \sin^2 \phi = \cos \phi, \quad 1 - \cos^2 \phi = \cos \phi,$$

$$\cos^2 \phi + \cos \phi = 1,$$

$$\cos \phi = \tfrac{1}{2}(-1 \pm \sqrt{5}),$$

$$\phi = \cos^{-1} \tfrac{1}{2}(-1 \pm \sqrt{5});$$

but since $\frac{1}{2}(-1 - \sqrt{5})$ is numerically greater than unity, this solution is impossible; and

$$\phi = \cos^{-1} \tfrac{1}{2}(-1 + \sqrt{5}).$$

3. $\sin \theta + \cos \theta = 1$.

$$\sin \theta + \sqrt{1 - \sin^2 \theta} = 1.$$

$$1 - \sin^2 \theta = (1 - \sin \theta)^2.$$

$$(1 - \sin \theta)^2 - (1 - \sin^2 \theta) = 0.$$

$$(1 - \sin \theta)(1 - \sin \theta - 1 - \sin \theta) = 0.$$

$$(1 - \sin \theta)(-2 \sin \theta) = 0.$$

$$\therefore \sin \theta = 1 \text{ and } 0.$$

$$\theta = \frac{\pi}{2}, \ 0, \ \pi.$$

The solution $\theta = \pi$ does not satisfy the original equation.

EXERCISES

Find the values of ϕ that satisfy each of the following equations:

1. $\cos 2\phi + \cos\phi = 0.$
2. $\tan\phi = n\cot\phi.$
3. $\sec\phi - \tan\phi = \cos\phi.$
4. $\sin\phi + \cos\phi = \tan\phi.$
5. $3\sin\theta + 4\cos\theta = 5.$
6. $\tan\phi + \cot\phi = 2\frac{1}{2}.$
7. $\cot\theta = 2\cos\theta.$
8. $\tan\phi + \cot\phi = m.$
9. $\tan\phi + \sec\phi = a.$
10. If $\sin\theta + \cos\theta = a$, then $\sin 2\theta = a^2 - 1.$
11. $l\cos\theta + m\sin\theta = 0$, find $\tan\frac{\theta}{2}$.

MISCELLANEOUS EXERCISES

1. From $\sin 30° = \cos 60° = \frac{1}{2}$, $\cos 30° = \sin 60° = \frac{1}{2}\sqrt{3}$, $\sin 45° = \cos 45° = \frac{1}{2}\sqrt{2}$, find all the functions of 15°, of 75°, of 105°.

2. From $\sin\alpha = \frac{4}{5}$, $\sin\beta = \frac{3}{5}$, find all the functions of $\alpha + \beta$ and $\alpha - \beta$.

Prove the following:

3. $$\frac{\sin(a+b)+\sin(a-b)}{\cos(a+b)+\cos(a-b)} = \tan a.$$

4. $$\frac{\sin(a \pm b)}{\cos a\cos b} = \tan a \pm \tan b.$$

5. $$\frac{\cos(a \mp b)}{\sin a\cos b} = \cot a \pm \tan b.$$

6. $$\frac{\tan x + \tan y}{\tan x - \tan y} = \frac{\sin(x+y)}{\sin(x-y)}.$$

7. $$\frac{1 - \tan x\tan y}{1 + \tan x\tan y} = \frac{\cos(x+y)}{\cos(x-y)}.$$

8. $$\frac{\cot y + \cot x}{\cot y - \cot x} = \csc(x-y)\sin(x+y).$$

9. $\dfrac{\tan x \cot y + 1}{\tan x \cot y - 1} = \dfrac{\sin(x+y)}{\sin(x-y)}.$

10. $\dfrac{\sin(x+y)\sin(x-y)}{\cos^2 x \cos^2 y} = \tan^2 x - \tan^2 y.$

11. $\dfrac{\tan^2 x - \tan^2 y}{1 - \tan^2 x \tan^2 y} = \tan(x+y)\tan(x-y).$

12. $\sqrt{2}\sin(a \pm 45°) = \sin a \pm \cos a.$

13. $\sin(x+y)\cos x - \cos(x+y)\sin x = \sin y.$

14. $\sin(x-y)\cos y + \cos(x-y)\sin y = \sin x.$

15. $\cos(x+y)\cos x + \sin(x+y)\sin x = \cos y.$

16. $\dfrac{\tan(x-y) + \tan y}{1 - \tan(x-y)\tan y} = \dfrac{\tan(x+y) - \tan y}{1 + \tan(x+y)\tan y} = \tan x.$

17. $2\sin(45° + a)\cos(45° - b) = \cos(a-b) + \sin(a+b).$
Cf. exercise 12.

18. $2\sin(45° - a)\cos(45° + b) = \cos(a-b) - \sin(a+b).$

19. $2\sin(45° + a)\cos(45° + b) = \cos(a+b) + \sin(a-b).$

20. $2\sin(45° - a)\cos(45° - b) = \cos(a+b) - \sin(a-b).$

21. $\tan x = \frac{1}{2}$, $\tan y = \frac{1}{4}$; find $\tan(x+y)$ and $\tan(x-y)$.

22. $\tan x = 3$, $\tan y = \frac{1}{3}$; find $\tan(x+y)$ and $\tan(x-y)$.

23. $\tan x = k$, $\tan y = \dfrac{1}{k}$; find $\cot(x+y)$ and $\cot(x-y)$.

24. $\cot(x + 45°) = \dfrac{\cot x - 1}{\cot x + 1} = \sqrt{\dfrac{1 - \sin 2x}{1 + \sin 2x}} = \dfrac{1 - \sin 2x}{\cos 2x}.$

25. $\cot(x - 45°) = \dfrac{\cot x + 1}{1 - \cot x} = \dfrac{\tan x + 1}{\tan x - 1}.$

26. $\tan(x \mp 45°) + \cot(x \pm 45°) = 0.$

27. $\tan x = \dfrac{m}{m+1}$, $\cot y = 2m + 1$;
find $\tan(x+y)$ and $\cot(x-y)$.

28. If $x + y + z = 90°$, then $\tan z = \dfrac{1 - \tan x \tan y}{\tan x + \tan y}$.

29. $\sin 7x - \sin 5x = 2 \sin x \cos 6x$.

30. $\cos 5x + \cos 9x = 2 \cos 7x \cos 2x$.

31. $\cos x - \cos 2x = 2 \sin \frac{3}{2} x \sin \frac{1}{2} x$.

32. $\dfrac{\sin 2x - \sin x}{\cos x - \cos 2x} = \cot \frac{3}{2} x$.

33. $\dfrac{\sin 3x - \sin 2x}{\cos 2x - \cos 3x} = \cot \dfrac{5x}{2}$.

34. $\dfrac{\sin x + \sin y}{\cos x - \cos y} = \dfrac{\cos x + \cos y}{\sin y - \sin x}$.

35. $\cos\left(\dfrac{\pi}{6} - x\right) - \cos\left(\dfrac{\pi}{6} + x\right) = \sin x$.

36. $\cos\left(\dfrac{\pi}{4} + x\right) + \cos\left(\dfrac{\pi}{4} - x\right) = \cos x \cdot \sqrt{2}$.

Express each of the following products as the sum or difference of two trigonometric functions:

37. $2 \sin x \cos y$.

38. $2 \cos x \cos y$.

39. $2 \sin 2x \cos 3y$.

40. $2 \sin 3x \cos 5x$.

41. $2 \cos (x + y) \cos (x - y)$.

42. $2 \cos \frac{3}{2} x \cos \frac{1}{2} x$.

43. $2 \sin 50° \cos 10°$.

44. $2 \cos \dfrac{\pi}{4} \sin \dfrac{\pi}{12}$.

Simplify:

45. $2 \cos 3x \cos x - 2 \sin 4x \sin 2x$.

46. $\dfrac{\cos x - \cos 5x}{\sin x + \sin 5x}$.

47. $\dfrac{\sin 3x - \sin x}{\cos 3x + \cos x} - \dfrac{\sin 3x - \sin x}{\cos 3x - \cos x}$.

48. $\dfrac{(\sin 4x - \sin 2x)(\cos x - \cos 3x)}{(\cos 4x + \cos 2x)(\sin x + \sin 3x)}$.

Verify:

49. $\dfrac{\cos x + \cos 3x}{\cos 3x + \cos 5x} = \dfrac{\cos 2x}{\cos 4x}.$

50. $\tan \dfrac{x+y}{2} - \tan \dfrac{x-y}{2} = \dfrac{2 \sin y}{\cos x + \cos y}.$

51. $2 \sin 2x \cos x + 2 \cos 4x \sin x = \sin 5x + \sin x.$

52. $\dfrac{\csc^2 x}{\csc^2 x - 2} = \sec 2x.$

53. $\cos^2 x\,(1 - \tan^2 x) = \cos 2x.$

54. $\cot x - \tan x = 2 \cot 2x.$

55. $\dfrac{\cos 2x}{1 + \sin 2x} = \dfrac{1 - \tan x}{1 + \tan x}.$

56. $\cos^2 x + \cos^2\left(\dfrac{\pi}{2} + x\right) + \cos^2(\pi + x) + \cos^2\left(\dfrac{3\pi}{2} + x\right) = 2.$

57. $\sin x + \sin 3x + \sin 5x + \sin 7x = 4 \sin 4x \cos 2x \cos x.$

58. $\cos x + \cos(120 + x) + \cos(120^\circ - x) = 0.$

59. $\dfrac{\sin 3x}{\sin x} - \dfrac{\cos 3x}{\cos x} = 2.$

60. $\dfrac{\cos 3x}{\sin x} + \dfrac{\sin 3x}{\cos x} = 2 \cot 2x.$

61. $\dfrac{\sin 5x}{\sin x} - \dfrac{\cos 5x}{\cos x} = 4 \cos 2x.$

62. $\tan x = \frac{1}{7}$, $\tan y = \frac{2}{11}$; find $\tan(2x + y)$.

63. $\sin(y + z - x) + \sin(z + x - y) + \sin(x + y - z) - \sin(x + y + z) = 4 \sin x \sin y \sin z.$

64. $\cos(y + z - x) + \cos(z + x - y) + \cos(x + y - z) + \cos(x + y + z) = 4 \cos x \cos y \cos z.$

65. $\sin x \sin(y - z) + \sin y \sin(z - x) + \sin z \sin(x - y) = 0.$

66. $\cos x \sin(y - z) + \cos y \sin(z - x) + \cos z \sin(x - y) = 0.$

67. $\cos x \cos(y - z) - \sin y \sin(z - x) - \cos z \cos(x - y) = 0.$

68. $\sin x \cos(y - z) + \cos y \sin(z - x) - \sin z \cos(x - y) = 0.$

69. $\cot^{-1}(x - y) - \cot^{-1}(x + y) = \cot^{-1}\left(\frac{x^2 - y^2 + 1}{2y}\right).$

70. $\tan^{-1}\frac{a}{a-1} - \tan^{-1}\frac{a+1}{a} = \tan^{-1}\frac{1}{2a^2}.$

71. $2 \sin^{-1} a = \tan^{-1}\frac{2a\sqrt{1-a^2}}{1-2a^2}.$

72. $\tan^{-1} a + 2\tan^{-1} b = \tan^{-1}\frac{a(1-b^2)+2b}{1-b^2-2ab}.$

73. $\tan^{-1} a + \cos^{-1}\frac{1}{a} = \sin^{-1}\frac{a+\sqrt{a^2-1}}{a\sqrt{a^2+1}}.$

74. $\cos^4 x - \sin^4 x = \cos 2x.$

75. $\sin^2(x + y) - \sin^2(x - y) = \sin 2x \sin 2y.$

76. $(\sin x - \sin y)^2 + (\cos x - \cos y)^2 = 4 \sin^2\frac{x-y}{2}.$

77. $\frac{1 + \sin x - \cos x}{1 + \sin x + \cos x} = \tan \frac{1}{2} x.$

78. $\frac{\tan x \pm \tan y}{\cot x \pm \cot y} = \pm \tan x \tan y.$

79. $(\sin\phi + \sin\theta)(\sin\phi - \sin\theta) = \sin(\phi + \theta)\sin(\phi - \theta).$

80. $(\sqrt{1 + \sin \alpha} - \sqrt{1 - \sin \alpha})^2 = 4 \sin^2 \frac{1}{2}\alpha.$

81. $(\sqrt{1 + \sin \alpha} + \sqrt{1 - \sin \alpha})^2 = 4 \cos^2 \frac{1}{2}\alpha.$

82. $\sec 2a + \tan 2a + 1 = \frac{2}{1 - \tan a}.$

83. $\sin A = \frac{\sin(30° + A) - \sin(30° - A)}{\sqrt{3}}.$

84. $\cot(x + y) = \frac{1}{\tan x + \tan y} - \frac{1}{\cot x + \cot y}.$

CHAPTER VI

THE TRIANGLE

53. The object of this chapter is to study the relations between the sides of a triangle and the trigonometric functions of its angles. Other properties of the triangle are also considered.

NOTATION

$A, B, C \equiv$ the vertices of the triangle.

$a, b, c \equiv$ the sides opposite A, B, C, respectively.

$\alpha, \beta, \gamma \equiv$ the interior angles at A, B, C, respectively.

$s \equiv \frac{1}{2}(a + b + c)$, the semi-perimeter.

$R, r \equiv$ the radii of the circumscribed and inscribed circles.

$r_a, r_b, r_c \equiv$ the radii of the escribed circles opposite A, B, C, respectively.

$p_a, p_b, p_c \equiv$ the altitudes from A, B, C to a, b, c.

$K \equiv$ area of the triangle.

MEMORANDA

$$\alpha + \beta + \gamma = 180° = \pi.$$

$$\alpha, \beta, \gamma = \pi - (\beta + \gamma),\ \pi - (\gamma + \alpha),\ \pi - (\alpha + \beta).$$

$$\tfrac{1}{2}\alpha, \tfrac{1}{2}\beta, \tfrac{1}{2}\gamma = \frac{\pi}{2} - \frac{1}{2}(\beta + \gamma),\ \frac{\pi}{2} - \tfrac{1}{2}(\gamma + \alpha),\ \frac{\pi}{2} - \frac{1}{2}(\alpha + \beta).$$

$$\sin\alpha, \sin\beta, \sin\gamma = \sin(\beta + \gamma),\ \sin(\gamma + \alpha),\ \sin(\alpha + \beta).$$

$$\cos\alpha, \cos\beta, \cos\gamma = -\cos(\beta + \gamma),\ -\cos(\gamma + \alpha),\ -\cos(\alpha + \beta).$$

$$\tan \alpha = -\tan(\beta + \gamma), \text{ etc.}$$
$$\cot \alpha = -\cot(\beta + \gamma), \text{ etc.}$$
$$\sin \tfrac{1}{2} \alpha = \cos \tfrac{1}{2}(\beta + \gamma), \text{ etc.}$$
$$\cos \tfrac{1}{2} \alpha = +\sin \tfrac{1}{2}(\beta + \gamma), \text{ etc.}$$
$$\tan \tfrac{1}{2} \alpha = +\cot \tfrac{1}{2}(\beta + \gamma), \text{ etc.}$$
$$\cot \tfrac{1}{2} \alpha = +\tan \tfrac{1}{2}(\beta + \gamma), \text{ etc.}$$
$$2K = ap_a = bp_b = cp_c = 2rs.$$
$$b + c - a = 2(s - a).$$
$$c + a - b = 2(s - b).$$
$$a + b - c = 2(s - c).$$

54. The Law of Sines.

In either figure, let $AE = q$, then, § 19, $EB = AB - AE$.

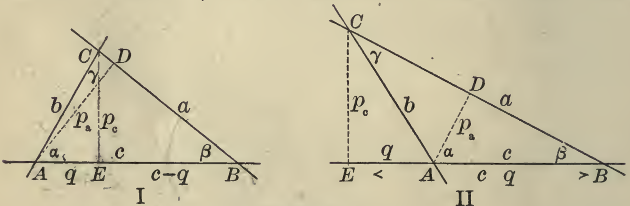


FIG. 47.

$$\sin \alpha = \frac{p_c}{b}, \qquad \sin \beta = \frac{p_c}{a}.$$

$$\therefore \frac{\sin \alpha}{\sin \beta} = \frac{p_c}{b} \div \frac{p_c}{a} = \frac{a}{b}.$$

This may be written

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta}.$$

Similarly $$\frac{b}{\sin \beta} = \frac{c}{\sin \gamma}.$$

$$\therefore \frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}. \qquad [56]$$

This is the *law of sines.* It may be stated in words as follows: The ratio of the side of any triangle to the sine of its opposite angle is constant.

Let us denote this constant by M. The formula becomes

$$\frac{a}{\sin\alpha}=\frac{b}{\sin\beta}=\frac{c}{\sin\gamma}=M. \qquad [57]$$

55. The Law of Tangents. From [57],

$$\frac{a}{\sin\alpha}=\frac{b}{\sin\beta}, \text{ or } \frac{a}{b}=\frac{\sin\alpha}{\sin\beta};$$

by composition and division

$$\frac{a+b}{a-b}=\frac{\sin\alpha+\sin\beta}{\sin\alpha-\sin\beta}$$

$$=\frac{2\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)}{2\cos\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta)} \quad \text{by [52] and [53]}$$

$$=\frac{\tan\frac{1}{2}(\alpha+\beta)}{\tan\frac{1}{2}(\alpha-\beta)}.$$

$$\therefore \frac{a+b}{a-b}=\frac{\tan\frac{1}{2}(\alpha+\beta)}{\tan\frac{1}{2}(\alpha-\beta)}=\frac{\cot\frac{1}{2}\gamma}{\tan\frac{1}{2}(\alpha-\beta)}. \qquad [58]$$

If b is greater than a we can avoid negative signs by writing $b-a$ and $\beta-\alpha$ instead of $a-b$ and $\alpha-\beta$.

Similar formulas may be derived involving b and c, and c and a.

This is the *law of tangents.* In words it is: The ratio of the sum of any two sides of a triangle to their difference is equal to the ratio of the tangent of one-half the sum of the opposite angles to the tangent of one-half their difference.

56. The Law of Cosines. From Fig. 47,

$$a^2=(c-q)^2+p_c^{\,2},\; p_c^{\,2}=b^2-q^2.$$

$$\therefore\; a^2=(c-q)^2+b^2-q^2=b^2+c^2-2\,cq.$$

But q is the projection of b and, therefore,

$$q = b \cos \alpha.$$

Substituting in the preceding equation,

$$\left.\begin{aligned} a^2 &= b^2 + c^2 - 2\,bc\,\cos \alpha. \\ b^2 &= c^2 + a^2 - 2\,ca\cos \beta. \\ c^2 &= a^2 + b^2 - 2\,ab\cos \gamma. \end{aligned}\right\} \quad [59]$$

Similarly

Solving for cos α, etc.,

$$\cos \alpha = \frac{b^2 + c^2 - a^2}{2\,bc}. \quad [60]$$

This is the *law of cosines.*

57. Functions of the Half-angles in Terms of the Sides.

Substituting $\frac{1}{2}\alpha$ for ϕ in [39] III,

$$\cos \alpha = 1 - 2 \sin^2 \tfrac{1}{2}\alpha.$$

$$\begin{aligned} 2\sin^2 \tfrac{1}{2}\alpha &= 1 - \cos \alpha \\ &= 1 - \frac{b^2 + c^2 - a^2}{2\,bc} \qquad \text{by [60]} \\ &= \frac{a^2 - (b^2 + c^2 - 2\,bc)}{2\,bc} \\ &= \frac{a^2 - (b - c)^2}{2\,bc} \\ &= \frac{(a + b - c)(a - b + c)}{2\,bc} \\ &= \frac{2(s-c)2(s-b)}{2\,bc}. \quad \text{(See Memoranda.)} \end{aligned}$$

$$\therefore \sin \tfrac{1}{2}\alpha = \sqrt{\frac{(s-b)(s-c)}{bc}}. \quad [61]$$

Similarly

$$\sin \tfrac{1}{2}\beta = \sqrt{\frac{(s-c)(s-a)}{ca}}. \quad [61]$$

$$\sin \tfrac{1}{2}\gamma = \sqrt{\frac{(s-a)(s-b)}{ab}}. \quad [61]$$

Substituting $\frac{1}{2}\alpha$ for ϕ in [39] II,

$$\cos\alpha = 2\cos^2\tfrac{1}{2}\alpha - 1.$$

$$\begin{aligned} 2\cos^2\tfrac{1}{2}\alpha &= 1 + \cos\alpha \\ &= 1 + \frac{b^2 + c^2 - a^2}{2bc} \\ &= \frac{2bc + b^2 + c^2 - a^2}{2bc} \\ &= \frac{(b+c)^2 - a^2}{2bc} \\ &= \frac{(b+c+a)(b+c-a)}{2bc} \\ &= \frac{2s\,2(s-a)}{2bc}. \quad \text{(See Memoranda.)} \end{aligned}$$

$$\left.\begin{aligned} \therefore \cos\tfrac{1}{2}\alpha &= \sqrt{\frac{s(s-a)}{bc}}. \\ \cos\tfrac{1}{2}\beta &= \sqrt{\frac{s(s-b)}{ca}}. \\ \cos\tfrac{1}{2}\gamma &= \sqrt{\frac{s(s-c)}{ab}}. \end{aligned}\right\} \quad [62]$$

Dividing [61] by [62],

$$\left.\begin{aligned} \frac{\sin\frac{1}{2}\alpha}{\cos\frac{1}{2}\alpha} = \tan\tfrac{1}{2}\alpha &= \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}. \\ \cot\tfrac{1}{2}\alpha &= \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}. \end{aligned}\right\} \quad [63]$$

Since $\frac{1}{2}\alpha$, $\frac{1}{2}\beta$, $\frac{1}{2}\gamma < 90°$, the functions of these angles are positive and the radicals in [61], [62], [63] are also positive.

EXERCISES

Verify the following relations :

1. $\dfrac{s}{a}=\dfrac{\cos\frac{1}{2}\beta\cos\frac{1}{2}\gamma}{\sin\frac{1}{2}\alpha}.$ 2. $\dfrac{s-a}{b}=\dfrac{\cos\frac{1}{2}\alpha\sin\frac{1}{2}\gamma}{\cos\frac{1}{2}\beta}.$

3. $\dfrac{s-a}{a}=\dfrac{\sin\frac{1}{2}\beta\sin\frac{1}{2}\gamma}{\sin\frac{1}{2}\alpha}.$

4. $\cos\alpha+\cos\beta\cos\gamma=\sin\beta\sin\gamma.$

5. $a\cos\beta+b\cos\alpha=c.$

   $c\cos\alpha+a\cos\gamma=b.$

   $c\cos\beta+b\cos\gamma=a.$

6. $a\cos\beta-b\cos\alpha=\dfrac{a^2-b^2}{c}.$

7. $a\cos\beta\cos\gamma+b\cos\gamma\cos\alpha+c\cos\alpha\cos\beta$

   $=\frac{1}{2}[a\cos\alpha+b\cos\beta+c\cos\gamma]$

   $=a\sin\beta\sin\gamma=b\sin\gamma\sin\alpha=c\sin\alpha\sin\beta.$

8. $a\sin(\beta-\gamma)+b\sin(\gamma-\alpha)+c\sin(\alpha-\beta)=0.$

58. Circumscribed and Inscribed Circles.

Circumscribe the circle O about the triangle ABC. Draw CD, a diameter. Angle α = angle D. (Fig. 48.)

$$\sin\alpha=\sin D=\frac{a}{CD}=\frac{a}{2R}.$$

$$\therefore 2R=\frac{a}{\sin\alpha}=\frac{b}{\sin\beta}=\frac{c}{\sin\gamma}=M. \qquad [64]$$

Inscribe the circle O in the triangle ABC. By geometry

$$a_1=c_2,\ b_1=a_2,\ c_1=b_2. \quad \text{(Fig. 49.)}$$

$$\therefore s=\tfrac{1}{2}(a+b+c)=a_1+b_1+c_1=a_1+a_2+c_1=a+c_1.$$

$$\therefore AF=c_1=s-a.$$

Now $\angle FAO = \frac{1}{2}\alpha.$

$$\tan\tfrac{1}{2}\alpha = \frac{OF}{AF} = \frac{r}{s-a}. \qquad [65]$$

Similarly $$\tan\tfrac{1}{2}\beta,\ \tan\tfrac{1}{2}\gamma = \frac{r}{s-b},\ \frac{r}{s-c}.$$

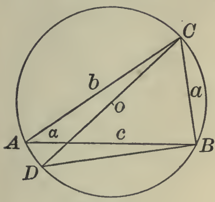


FIG. 48.

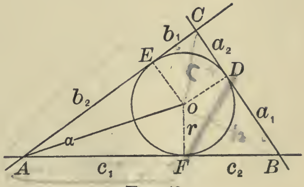


FIG. 49.

Combining [63] and [65],

$$\frac{r}{s-a} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

$$\therefore r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}. \qquad [66]$$

From [65] we have

$$r = (s-a)\tan\tfrac{1}{2}\alpha = (s-b)\tan\tfrac{1}{2}\beta = (s-c)\tan\tfrac{1}{2}\gamma. \qquad [67]$$

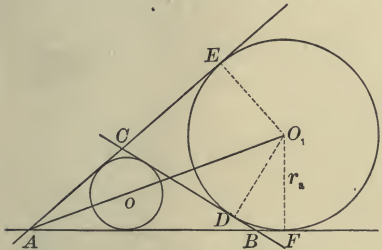


FIG. 50.

Let O be escribed to the triangle ABC opposite A. We have by geometry

$$BD = BF,\ CD = CE,$$

$$CB = BF + CE.$$

$$\therefore 2s = AE + AF,$$

$$s = AE.$$

Now $$\tan\tfrac{1}{2}\alpha = \frac{O_1E}{AE} = \frac{r_a}{s}. \qquad [68]$$

Similarly $$\tan\tfrac{1}{2}\beta,\ \tan\tfrac{1}{2}\gamma = \frac{r_b}{s},\ \frac{r_c}{s}.$$

Combining [63] and [68],

$$\frac{r_a}{s} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

$$\therefore r_a,\ r_b,\ r_c = \sqrt{\frac{s(s-b)(s-c)}{s-a}}, \text{ etc.} \qquad [69]$$

Comparing [65] and [68],

$$rs = r_a(s-a) = r_b(s-b) = r_c(s-c). \qquad [70]$$

EXERCISES

Verify the following identities:

1. $r_a + r_b + r_c - 3r = \dfrac{ar_a + br_b + cr_c}{s}.$

2. $\dfrac{1}{r} = \dfrac{1}{r_a} + \dfrac{1}{r_b} + \dfrac{1}{r_c}.$

3. $\tan \frac{1}{2}\alpha = \dfrac{r_a - r}{a}.$

4. $OO_1 = (r_a - r)\csc \frac{1}{2}\alpha.$

5. $OO_1 = a \sec \frac{1}{2}\alpha = b \sec \frac{1}{2}\beta = c \sec \frac{1}{2}\gamma.$

6. $\tan \frac{1}{2}\alpha \tan \frac{1}{2}\beta \tan \frac{1}{2}\gamma = \dfrac{r}{s}.$

7. $\sin \alpha + \sin \beta + \sin \gamma = \dfrac{s}{R}.$

58. Area of the Triangle.

The area of a triangle may be expressed in different ways, depending upon the parts known. We have from geometry (Fig. 51)

$$2K = ap_a = bp_b = cp_c, \qquad [71]$$

$$p_a = c \sin \beta = b \sin \gamma.$$

$$\therefore 2K = ac \sin \beta = ab \sin \gamma = bc \sin \alpha. \qquad [72]$$

From [57],
$$c = \frac{a \sin \gamma}{\sin \alpha}.$$

Substituting this value in [72],

$$2K = \frac{a^2 \sin \beta \sin \gamma}{\sin \alpha} = \frac{b^2 \sin \gamma \sin \alpha}{\sin \beta}$$
$$= \frac{c^2 \sin \alpha \sin \beta}{\sin \gamma}. \qquad [73]$$

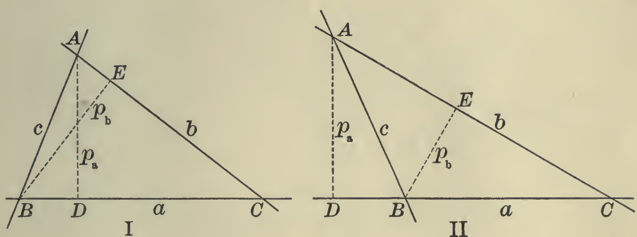


FIG. 51.

We have from geometry

$$K = rs. \qquad [74]$$

Combining [74] and [66],

$$K = \sqrt{s(s-a)(s-b)(s-c)}. \qquad [75]$$

EXERCISES

Find the areas of the following triangles:

1. $a = 13$, $b = 10$, $c = 17$.
2. $a = 143$, $b = 100$, $\gamma = 74° 16'$.
3. $b = 200$, $\alpha = 47° 24'$, $\gamma = 63° 25'$.
4. The sides of a triangle are 175, 120, 215; find its area and the radii of its inscribed and escribed circles.
5. Prove $K = \frac{abc}{4R}$.

$\frac{b^2 \text{ Sin A Sin C}}{2 \text{ Sin b}}$ when 2 angles & one si[de]

Verify the following identities:

6. $\cos \frac{1}{2} \alpha \cos \frac{1}{2} \beta \cos \frac{1}{2} \gamma = \frac{Ks}{abc}$. (Use [62].)

7. $\cot \frac{1}{2} \alpha \cot \frac{1}{2} \beta \cot \frac{1}{2} \gamma = \frac{s^2}{K}$. (Use [63].)

8. $\cot \frac{1}{2} \alpha + \cot \frac{1}{2} \beta + \cot \frac{1}{2} \gamma = \frac{s^2}{K}$.

CHAPTER VII

THE SOLUTION OF THE TRIANGLE

60. We have learned in geometry that a triangle can be constructed when we are given three of its parts, of which one, at least, is a side. The formulas of the preceding chapter enable us to compute the values of the unknown parts when we know the measures of the given parts.

The three given parts may be:

I. One side and two angles.

II. Two sides and the included angle.

III. Two sides and the angle opposite one of them.

IV. Three sides.

Formulas [57], [58], and [60] are sufficient to solve all four cases. In the computations in Chapter II we used natural functions; here we propose to use logarithms, and formula [60] is not adapted to logarithmic calculation. In its place we shall use formulas [65] and [66], which are derived from it.

The necessary formulas are:

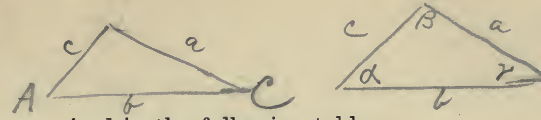
$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma} = M, \qquad [57]$$

$$\tan \tfrac{1}{2}(\alpha - \beta) = \frac{a-b}{a+b} \tan \tfrac{1}{2}(\alpha + \beta), \qquad [58]$$

$$\tan \tfrac{1}{2} \alpha = \frac{r}{s-a}, \qquad [65]$$

where

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}. \qquad [66]$$



The solutions of the four cases are summarized in the following table:

Case	I	II	III	IV
Data	a, β, γ	a, b, γ	a, b, α	a, b, c
Solution	$\alpha = 180° - (\beta + \gamma)$	$\alpha + \beta = 180° - \gamma$	$M = \dfrac{a}{\sin\alpha}$	$r = \sqrt{\dfrac{(s-a)(s-b)(s-c)}{s}}$
	$M = \dfrac{a}{\sin\alpha}$	$\tan\frac{1}{2}(\alpha - \beta) = \dfrac{a-b}{a+b}\tan\frac{1}{2}(\alpha+\beta)$	$\sin\beta = \dfrac{b}{M}$	$\tan\frac{1}{2}\alpha = \dfrac{r}{s-a}$
	$b = M\sin\beta$	$\alpha, \beta = \frac{1}{2}(\alpha+\beta) \pm \frac{1}{2}(\alpha-\beta)$	β may have two values, say β_1, β_2	$\tan\frac{1}{2}\beta = \dfrac{r}{s-b}$
	$c = M\sin\gamma$	$c = a\dfrac{\sin\gamma}{\sin\alpha}$	$\gamma_1 = 180° - (\alpha + \beta_1)$	$\tan\frac{1}{2}\gamma = \dfrac{r}{s-c}$
			$\gamma_2 = 180 - (\alpha + \beta_2)$ $c_1 = M\sin\gamma_1$ $c_2 = M\sin\gamma_2$	
Check	$\dfrac{b+c}{b-c} = \dfrac{\tan\frac{1}{2}(\beta+\gamma)}{\tan\frac{1}{2}(\beta-\gamma)}$	$\dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta} = \dfrac{c}{\sin\gamma}$	$\dfrac{b+c}{b-c} = \dfrac{\tan\frac{1}{2}(\beta+\gamma)}{\tan\frac{1}{2}(\beta-\gamma)}$	$\alpha + \beta + \gamma = 180°$
Area	$\frac{1}{2}a^2\dfrac{\sin\beta\sin\gamma}{\sin\alpha}$	$\frac{1}{2}ab\sin\gamma$	$\frac{1}{2}ab\sin\gamma$	$\frac{1}{2}rs$

LOGARITHMIC SOLUTIONS

Case	I	II
Data	a, β, γ	a, b, γ
Solution	$\alpha = 180° - (\beta + \gamma)$ $\log M = \log a - \log \sin \alpha$ $\log b = \log M + \log \sin \beta$ $\log c = \log M + \log \sin \gamma$	$\alpha + \beta = 180° - \gamma$ $\log \tan \frac{1}{2}(\alpha - \beta) = \log(a - b) + \log \tan \frac{1}{2}(\alpha + \beta)$ $- \log(a + b)$ $\alpha, \beta = \frac{1}{2}(\alpha + \beta) \pm \frac{1}{2}(\alpha - \beta)$ $\log c = \log a + \log \sin \gamma - \log \sin \alpha$
Check	$\log(b + c) - \log(b - c) = \log \tan \frac{1}{2}(\beta + \gamma)$ $- \log \tan \frac{1}{2}(\beta - \gamma)$	$\log a - \log \sin \alpha = \log b - \log \sin \beta = \log c$ $- \log \sin \gamma$, or check under I

Case	III	IV
Data	a, b, α	a, b, c
Solution	$\log M = \log a - \log \sin \alpha$ $\log \sin \beta = \log b - \log M$ β may have two values, β_1, β_2 $\gamma_1 = 180° - (\alpha + \beta_1)$; $\gamma_2 = 180° - (\alpha + \beta_2)$ $\log c_1 = \log M + \log \sin \gamma_1$ $\log c_2 = \log M + \log \sin \gamma_2$	$\log r = \frac{1}{2}[\log(s - a) + \log(s - b)$ $+ \log(s - c) - \log s]$ $\log \tan \frac{1}{2}\alpha = \log r - \log(s - a)$ $\log \tan \frac{1}{2}\beta = \log r - \log(s - b)$ $\log \tan \frac{1}{2}\gamma = \log r - \log(s - c)$
Check	$\log(b + c) - \log(b - c) = \log \tan \frac{1}{2}(\beta + \gamma)$ $- \log \tan \frac{1}{2}(\beta - \gamma)$	$\alpha + \beta + \gamma = 180°$

61. Logarithmic Functions. Tables of logarithmic functions are arranged like tables of natural functions. They consist of the logarithms of the natural functions. When, however, the characteristic is negative, 10 is added. For this reason the characteristics of all sines and cosines, of tangents of angles less than 45°, and of cotangents of angles greater than 45°, are 10 too large. This fact must be kept in mind when computing. A little experience will correct any liability to error from this source. Sines and tangents of very small angles, cosines and cotangents of angles near 90°, cannot be accurately obtained by *interpolation.* Supplementary tables are generally furnished for this purpose.

62. The actual work of computation in each case will now be illustrated by the solution of specific problems. The first step in the solution of every problem is the careful construction of the figure and the *graphic* solution by measurement. The results so obtained serve as a rough estimate of what is to be more accurately determined by computation.

In the following illustrative problems the work is arranged in convenient form, and this form should be followed by the student.

CASE I. **Two Angles and a Side.**

Given $a = 571$, $\alpha = 57°\ 21'.3$, $\beta = 43°\ 16'.8$, find the other parts.

$$\text{Data}\begin{cases} a = 571. \\ \alpha = 57°\ 21'.3. \\ \beta = 43°\ 16'.8. \end{cases}$$

$\gamma = 79°\ 21'.9.$

$\log a = 2.75664.$

$\log \sin \alpha = 9.92532 - 10.$

Check

$c + b = 1131.41.$

$c - b = 201.59.$

$\frac{1}{2}(\gamma + \beta) = 61°\ 19'.35.$

$\log M = 2.83132.$	$\frac{1}{2}(\gamma - \beta) = 18° \ 2'.55.$
$\log \sin \beta = 9.83605 - 10.$	$\log (c + b) = 3.05362.$
$\log \sin \gamma = 9.99248 - 10.$	$\log (c - b) = 2.30447.$
$\log b = 2.66737.$	$\log \text{quotient} = .74915.$
$\log c = 2.82380.$	$\log \tan \frac{1}{2}(\gamma + \beta) = .26204.$
$b = 464.91.$	$\log \tan \frac{1}{2}(\gamma - \beta) = 9.51288 - 10.$
$c = 666.5.$	$\log \text{quotient} = .74916.$

EXERCISES

1.	$a = 137.43,$	$\alpha = 43° \ 21'.3,$	$\beta = 65° \ 23'.5.$
2.	$a = 437.18,$	$\beta = 83° \ 25'.7,$	$\gamma = 73° \ 32'.8.$
3.	$b = 943.49,$	$\alpha = 12° \ 17'.6,$	$\gamma = 121° \ 07'.2.$
4.	$c = 349.44,$	$\beta = 102° \ 35'.3,$	$\gamma = 80° \ 12'.1.$
5.	$c = 637.23,$	$\alpha = 46° \ 46',$	$\beta = 56° \ 56'.$
6.	$a = 63.72,$	$\alpha = 1° \ 20',$	$\beta = 75° \ 40'.$
7.	$b = 6.372,$	$\alpha = 88° \ 14'.5,$	$\gamma = 88° \ 14'.2.$
8.	$b = .0641,$	$\alpha = 36° \ 17'.1,$	$\gamma = 53° \ 43'.6.$
9.	$c = .0037,$	$\beta = 36° \ 17',$	$\gamma = 72° \ 34'.$
10.	$a = 4.003,$	$\alpha = 36° \ 17',$	$\beta = 108° \ 51'.$

63. CASE II. **Two Sides and the Included Angle.**

Given $a = 1371$, $b = 1746$, $\gamma = 46° \ 30'$, find the other parts.

$$\text{Data} \begin{cases} a = 1371. \\ b = 1746. \\ \gamma = 46° \ 30'. \end{cases}$$

$b + a = 3117.$

$b - a = 375.$

$\frac{1}{2}(\beta + \alpha) = 66° \ 45'.$

$\log (b - a) = 2.57403.$

Check

$\log a = 3.13704$

$\log \sin \alpha = 9.89116 - 10$

3.24588

$$\log \tan \tfrac{1}{2}(\beta + \alpha) = .36690.$$
$$\text{colog}\,(b + a) = 6.50626.$$
$$\log \tan \tfrac{1}{2}(\beta - \alpha) = 9.44719 - 10.$$
$$\tfrac{1}{2}(\beta - \alpha) = 15^\circ\ 38'.6.$$
$$\alpha = 51^\circ\ 6'.4.$$
$$\beta = 82^\circ\ 23'.6.$$
$$\log a = 3.13704.$$
$$\text{colog}\,\sin \alpha = 0.10884.$$
$$\log \sin \gamma = 9.86056.$$
$$\log c = 3.10644.$$
$$c = 1276.7.$$

$$\log b = 3.24204$$
$$\log \sin \beta = \underline{9.99616 - 10}$$
$$3.24588$$

$$\log c = 3.10644$$
$$\log \sin \gamma = \underline{9.86056 - 10}$$
$$3.24588$$

EXERCISES

1. $a = 127,\quad b = 145,\quad \gamma = 24^\circ\ 37'.2.$
2. $a = 127,\quad b = 145,\quad \gamma = 84^\circ\ 13'.6.$
3. $a = 127,\quad b = 145,\quad \gamma = 173^\circ\ 28'.5.$
4. $b = 231,\quad c = 31,\quad \alpha = 74^\circ\ 15'.2.$
5. $a = 231,\quad b = 221,\quad \gamma = 100^\circ\ 14'.5.$
6. $c = 347,\quad a = 34,\quad \beta = 10^\circ\ 46'.3.$
7. $b = 12.473,\quad c = 34.257,\quad \alpha = 146^\circ\ 24'.1.$
8. $a = 100,\quad b = 200,\quad \gamma = 100^\circ.$
9. $a = 100,\quad b = 200,\quad \gamma = 10^\circ.$
10. The line AB is divided at D into two segments, $AD = 200$, $DB = 100$; from C each of these segments subtends an angle of 35°. Find the angles CAB and CBA.

64. Case III. **Two Sides and an Angle Opposite One of Them.** This case sometimes admits of two solutions. Let the given parts be a, b, α. Construct the angle α. On one side lay off $AC = b$; from C as center with radius a, describe an arc, cutting the other side AM at B_1 and B_2.

The triangles AB_1C and AB_2C both satisfy the conditions, and both are therefore solutions. Study of the diagram will show that we shall have two solutions when, and only when,

$$\alpha < 90°, \; b > a > p.$$

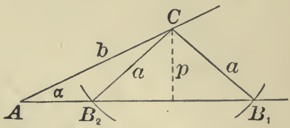


FIG. 52.

In any particular case the graphic solution will determine whether there is one or two solutions.

The angles at B_1 and B_2 are obviously supplementary. In the computation we find sin β. Now we learned in § 26 that there were two angles less than 180° with the same sine, the one the supplement of the other; when we find β from sin β, we must therefore take not only the value given in the table but also the supplement of this value. If there is but one solution, later steps in the computation will compel the rejection of the second of these values.

Given, 1. $a = 44.243$, $b = 30.347$, $\alpha = 34° \, 23'.2$.

2. $a = 44.243$, $b = 60.347$, $\alpha = 34° \, 23'.2$.

	1.	2.	2′.
Data a	44.243,	44.243.	
Data b	30.347,	60.347.	
Data α	34° 23′.2,	34° 23′.2.	
log a	1.64585,	1.64585.	
log sin α	9.75188 − 10,	9.75188 − 10.	
log M	1.89397,	1.89397.	
log b	1.48212,	1.78066.	
log sin β	9.58815 − 10,	9.88669 − 10.	
β	22° 47′.5,	50° 23′.1,	129° 36′.9.

γ	122° 49′.3,	95° 13′.7,	15°59′.9.
log sin γ	9.92447 − 10,	9.99819 − 10,	9.44030 − 10.
log c	1.81844,	1.89216,	1.33427.
c	65.833,	78.012,	21.591.

Check

$c + b$	96.180,	138.36,	81.938.
$c - b$	35.486,	17.665,	38.756.
$\frac{1}{2}(\gamma + \beta)$	72° 48′.4,	72° 48′.4,	72° 48′.4.
$\frac{1}{2}(\gamma - \beta)$	50° 00′.9,	22° 25′.3,	56° 48′.5.
$\log(c + b)$	1.98308,	2.14101,	1.91349.
$\log(c - b)$	1.55006,	1.24712,	1.58833.
log quotient	**.43302**,	**.89389**,	**.32516**.
$\log\tan\frac{\gamma+\beta}{2}$	.50945,	.50945,	.50945.
$\log\tan\frac{\gamma-\beta}{2}$	.07641,	9.61555 − 10,	.18431.
log quotient	**.43304**,	**.89390**,	**.32514**.

EXERCISES

1. $a = 145$, $b = 160$, $\alpha = 47° 38'$.
2. $a = 2.37$, $c = 3.14$, $\gamma = 65° 23'$.
3. $b = 147.3$, $a = 124.2$, $\beta = 142° 17'$.
4. $a = 32.14$, $b = 270$, $\beta = 75° 48'.3$.
5. $b = 13.47$, $c = 18.75$, $\beta = 110° 43'$.
6. $b = .149$, $c = .137$, $\gamma = 38° 47'$.
7. $a = 1.243$; $b = 2.345$, $\alpha = 10° 57'.5$.
8. $a = 432.1$, $b = 321.4$, $\beta = 28° 47'$.
9. $c = .0027$, $a = .0031$, $\alpha = 84° 21'.6$.
10. $a = 124$, $b = 83$, $\beta = 68° 43'$.

11.	$l = 241$,	$m = 214$,	$\mu = 43° 27'$.
12.	$p = 13.17$,	$q = 17.13$,	$Q = 71° 31'$.
13.	$a = 187.5$,	$b = 201.1$,	$\alpha = 67° 47'.4$.
14.	$a = 5872$,	$b = 7857$,	$\beta = 78° 5'$.
15.	$a = 1$,	$b = 2$,	$\alpha = 23° 32'$.
16.	$a = .0003$,	$b = .0004$,	$\alpha = 50° 5'$.
17.	$a = 3000$,	$b = 4000$,	$\alpha = 5° 50'$.
18.	$a = 1241$,	$b = 2114$,	$\alpha = 63° 36'$.
19.	$a = 1899$,	$b = 2004$,	$\alpha = 73° 1'$.

20. $b = 173$, $\alpha = 74° 12'$; find the limits of a for two solutions.

21. $a = 127$, $b = 143$; find the limits of α for two solutions.

65. Case IV. **Three Sides.** Given $a = 1573$, $b = 2044$, $c = 2736$.

Data $\begin{cases} a = 1573, \\ b = 2044, \\ c = 2736, \end{cases}$	$\text{colog } s = 6.49805.$ $\log (s - a) = 3.20507.$ $\log (s - b) = 3.05404.$
$2s = 6353$,	$\log (s - c) = 2.64395.$
$s = 3176.5$,	$\log r^2 = 5.40111.$
$s - a = 1603.5$,	$\log r = 2.70056.$
$s - b = 1132.5$,	$\log \tan \frac{1}{2} \alpha = 9.49549 - 10.$
$s - c = 440.5$,	$\log \tan \frac{1}{2} \beta = 9.64652 - 10.$
$\frac{1}{2} \alpha = 17° 22'.7$,	$\log \tan \frac{1}{2} \gamma = .05661.$
$\frac{1}{2} \beta = 23° 53'.9$,	
$\frac{1}{2} \gamma = 48° 43'.4$,	*Check*
$\alpha = 34° 45'.4$,	$\alpha + \beta + \gamma = 180° 00'.0.$
$\beta = 47° 47'.8.$	
$\gamma = 97° 26'.8.$	

EXERCISES

1.	$a = 51$,	$b = 65$,	$c = 60$.
2.	$a = 51$,	$b = 65$,	$c = 20$.
3.	$a = 431$,	$b = 440$,	$c = 25$.
4.	$a = 78.43$,	$b = 101.67$,	$c = 29.82$.
5.	$a = 111.1$,	$b = 120$,	$c = 130$.
6.	$a = .003$,	$b = .007$,	$c = .011$.
7.	$a = .431$,	$b = .34$,	$c = .7$.
8.	$a = 6$,	$b = 6$,	$c = 2$.
9.	$a = 6$,	$b = 6$,	$c = 11$.
10.	$a = 12$,	$b = 14$,	$c = 16$.
11.	$a = 4$,	$b = 6$,	$c = 9$.
12.	$a = 4$,	$b = 6$,	$c = 8$.
13.	$a = 4$,	$b = 6$,	$c = 11$.

EXERCISES

1. One side of a triangular lot is 1427 ft.; the adjacent angles are 48° 15′ and 75° 35′; find the perimeter and the area.

2. Prove that the area of a quadrilateral is one-half the product of its diagonals into the sine of the angle between them.

3. The diagonals of a parallelogram are 17 ft. and 30 ft., and the angle between them is 64° 27′; find the sides of the parallelogram and its area.

4. A balloon is directly over a straight road. From two points 3 mi. apart and on opposite sides its elevation was found to be 30° 28′ and 47° 22′; what was its height? If the two points of observation had been on the same side of the balloon, what would its height have been?

5. What is the angle between two faces of a regular tetrahedron? of a regular octahedron?

6. Three circles whose radii are 12, 17, and 19 are tangent, two and two externally; find the area of the surface enclosed by them.

7. From two successive mile posts on a straight and level road the elevation of the top of a hill in line with them is 8° and 10°; find the distance and height of the hill.

8. The longer sides of a parallelogram are 18, the shorter sides 10, and one diagonal is 12; find the other diagonal and the angles.

9. The parallel sides of a trapezoid are 34 and 50, the non-parallel sides 20 and 25; find the angles and the diagonals.

10. Two sides of a triangle are 20 and 30, and the median from their intersection is 16; find the base and the angles of the triangle.

11. A field is 500 ft. square; a post stands 350 ft. from one corner and 400 ft. from an adjacent corner; what are its distances from the other two corners, 1°, when it is within the field; 2°, when it is outside? If the second corner were opposite the first instead of adjacent to it, what would the distances be?

12. Wishing to find the height of a mountain, I measure a line of 600 yds. in the same vertical plane with the top of the mountain. The upper end of this line is 40 ft. higher than the lower end, and the elevation of the mountain top at the former is 6° 23′, at the latter 3° 23′; what is the height of the mountain above the lower end of the base line? If the lower end of the base line were next to the mountain, what would its height be?

13. A straight and level road runs along a seacoast. From two points on this road, 2 mi. apart, the top of a lofty mountain is visible; what measurements must I make to find its height without leaving the road?

14. The parallel sides of a trapezoid are 42 and 32, one oblique side is 20, and it makes an angle of 65° with the longer parallel side; find the other side, the diagonals, and the angles; find the same parts if the oblique side makes an angle of 65° with the shorter parallel side.

15. A tower 50 ft. high has a mark 20 ft. from the ground. At what distance from its foot do the two parts of the tower subtend equal angles? at what distance does the lower part subtend twice the angle that the upper does?

16. The altitude of a certain rock is observed to be 47°, and after walking 1000 ft. towards it, up a slope of 22°, the observer finds its altitude to be 77°; find the height of the rock above the first point of observation.

17. From two points A and B, 5000 ft. apart, two inaccessible points P and Q are visible. I find the angles

$$PAB = 107° 37',$$
$$PBA = 34° 23',$$
$$QAB = 43° 46',$$
$$QBA = 81° 11';$$

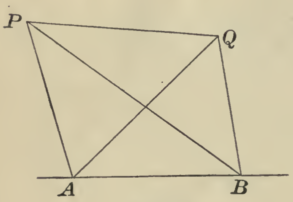


Fig. 53.

what is the distance from P to Q, 1°, when both are on the same side of AB; 2°, when they are on opposite sides?

18. Two flag-poles are 203 ft. apart. From the middle point of the line joining them the elevation of the taller is double that of the shorter; but on going $43\frac{1}{2}$ ft. nearer the shorter, their elevations are equal. What is the height of each?

19. From the top of a hill the depressions of the top and bottom of a flagstaff 25 ft. high, standing at the foot of the hill, are 45° 13′ and 47° 12′, respectively. What is the height of the hill above the foot of the flagstaff?

20. A column on a pedestal 20 ft. high subtends an angle of 30°; on approaching 20 ft. nearer, it again subtends an angle of 30°. What is the height of the column?

21. From the middle point of the longest side of the triangle, whose sides are 10, 14, 17, a circle is described with radius 12; where will it cut the other sides?

22. Two towers stand near each other in a plane. Their altitudes, each measured from the base of the other, are 46° 6′ and 33° 45′, respectively, and the distance between their summits is 87 ft. What is the height of each, and what is their distance apart?

23. Three circles with radii 16, 7, 5 touch each other externally; what is the area of the curvilinear triangle so formed? If the two smaller circles are within the larger, what is the area of the curvilinear triangle?

24. The sides of a triangle are 20, 30, 40; find the lengths of, 1°, the three altitudes; 2°, the three medians; 3°, the bisectors of the three interior angles; 4°, the bisectors of the three exterior angles; 5°, the radii of the circumscribed circle, the inscribed circle, the escribed circles.

25. Near the foot of a flagstaff, 150 ft. high, are two posts, A 70 ft. north, B 100 ft. east. What is the shortest distance from T, the top of the staff, to the line AB? what angle does this line make with the ground?

26. Three sides of a convex quadrilateral inscribed in a circle 30 ft. in diameter are $l = 14$ ft., $m = 18$ ft., $n = 12$ ft.; find the fourth side and the angles when, 1°, l is the middle one of the three given sides; 2°, when m is the middle one; 3°, when n is the middle one.

27. Standing on a headland 250 ft. high, I observe a ship. At first it bears N.N.W., and its angle of depression is 16° 8′, ten minutes later it bears E. by S. and its depression is 32° 18′; find what direction the ship is sailing, its speed, and how near its course lies to the headland.

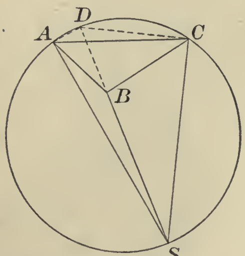


FIG. 54.

28. A, B, and C are three buoys; $AB = 320$ yds., $BC = 435$ yds., $CA = 600$ yds. A ship S finds that AB subtends an angle of 8° and BC an angle of 26°. How far is the ship from each of the buoys?

Suggestion. Draw a circle through A, C, and S, cutting SB produced in D. Draw AD and CD.