ELEMENTS OF PLANE AND SPHERICAL TRIGONOMETRY

DANIEL A. MURRAY

ELEMENTS

OF

PLANE TRIGONOMETRY

ELEMENTS

OF

PLANE TRIGONOMETRY

BY

DANIEL A. MURRAY, Ph.D.
Professor of Applied Mathematics in McGill University

LONGMANS, GREEN, AND CO.
FOURTH AVENUE & 30TH STREET, NEW YORK
LONDON, BOMBAY, CALCUTTA, AND MADRAS
1919

First Edition, July, 1911
Reprinted, August, 1912
October, 1912; September, 1913
July, 1916; September, 1919

PRESS OF
BRAUNWORTH & CO.
PRINTERS AND BOOKBINDERS
BROOKLYN, N. Y.

PREFACE

THIS text-book is shorter than my former text-book, entitled "Plane Trigonometry," by the omission of many of the notes and several of the topics in that book and by the more condensed treatment of other topics. There is also a marked difference in arrangement. Thus, radian measure, the periodicity of the trigonometric functions, their general values, and their graphs, and the inverse trigonometric functions, which are discussed in the later chapters of the "Plane Trigonometry," are treated in the earlier chapters of the "Elements of Plane Trigonometry." The line definitions of the functions are explained more fully, and the unit circle is used to a greater extent, in this book than in the former one.

D. A. MURRAY.

May 1, 1911.

CONTENTS

CHAPTER I

Trigonometric Functions of Acute Angles

CHAPTER II

Solution of Right-angled Triangles

CHAPTER III

Angles in General and Their Trigonometric Functions

CHAPTER IV

General Values. Inverse Trigonometric Functions. Trigonometric Equations

CHAPTER V

Trigonometric Functions of the Sum and Difference of Two Angles

GENERAL FORMULAS

CHAPTER VI

Relations between the Sides and Angles of a Triangle

CHAPTER VII

Solution of Oblique Triangles

CHAPTER VIII

Miscellaneous Theorems

ELEMENTS

OF

PLANE TRIGONOMETRY

CHAPTER I

TRIGONOMETRIC FUNCTIONS OF ACUTE ANGLES

1. Angle defined. *An angle XOP is the amount of turning which a line makes when it revolves about O from*

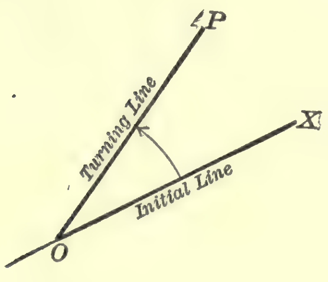


FIG. 1.

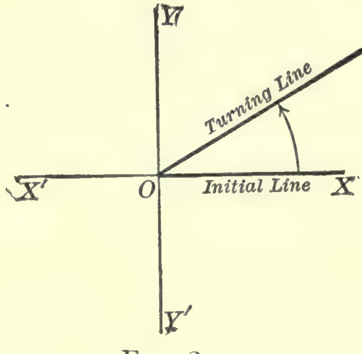


FIG. 2.

the position OX into the position OP (Figs. 1, 2). Accordingly (Fig. 2),

one-fourth of a revolution = a right angle;
a complete revolution = four right angles.

When the amount of turning is less than one-fourth a revolution (i.e., less than right angle XOY), the angle is called *an acute angle.*

2. Degree (or sexagesimal) measure of angles. Since all right angles are equal, a right angle may be chosen as the unit of measurement. A right angle, however, is inconveniently large as a unit. Accordingly *a ninetieth part of a right angle,* called *a degree,* is taken for unit. Degrees are divided into minutes, and minutes into seconds, according to the following table of angular measure:

60 seconds = 1 minute,
60 minutes = 1 degree,
90 degrees = 1 right angle.

Degrees, minutes, and seconds are denoted by symbols: thus, 23 degrees 17 minutes 20 seconds is written $23° \, 17' \, 20''$.

3. Trigonometric functions (defined for acute angles). From any point P in one of the lines bounding an angle A (Figs. 3, 4, 5) draw a perpendicular PM to the other bounding line. (The angles A in Figs. 3, 4, 5 are equal.)

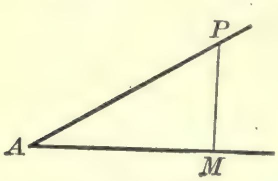


FIG. 3.

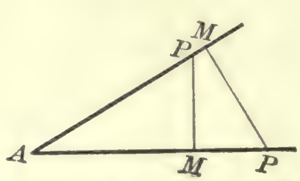


FIG. 4.

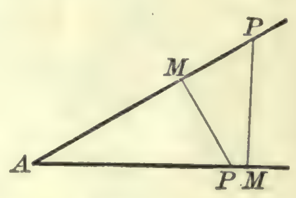


FIG. 5.

In any of these triangles AMP there can be formed six ratios with the lines AM, MP, AP, viz.:

$$\frac{MP}{AP}, \quad \frac{AM}{AP}, \quad \frac{MP}{AM}, \quad \frac{AM}{MP}, \quad \frac{AP}{AM}, \quad \frac{AP}{MP}.$$

Since all the triangles AMP above are similar, each of these ratios has the same value whatever be the position of P on a bounding line of the angle. *These six ratios are called* **trigonometric functions** *of the angle A, and are given names as follows:*

$$\left.\begin{array}{lll} \dfrac{MP}{AP} \text{ is called the} & \textbf{sine} & \text{of the angle } A; \\[2ex] \dfrac{AM}{AP} \text{ is called the} & \textbf{cosine} & \text{of the angle } A; \\[2ex] \dfrac{MP}{AM} \text{ is called the} & \textbf{tangent} & \text{of the angle } A; \\[2ex] \dfrac{AM}{MP} \text{ is called the} & \textbf{cotangent} & \text{of the angle } A; \\[2ex] \dfrac{AP}{AM} \text{ is called the} & \textbf{secant} & \text{of the angle } A; \\[2ex] \dfrac{AP}{MP} \text{ is called the} & \textbf{cosecant} & \text{of the angle } A. \end{array}\right\} . \qquad (1)$$

Short symbols for these functions and definitions applicable for any right-angled triangle are given in (2):

$$\left.\begin{array}{l} \sin A \left(=\dfrac{MP}{AP}\right)=\dfrac{\text{opposite side}}{\text{hypotenuse}}, \\[2ex] \cos A \left(=\dfrac{AM}{AP}\right)=\dfrac{\text{adjacent side}}{\text{hypotenuse}}, \\[2ex] \tan A \left(=\dfrac{MP}{AM}\right)=\dfrac{\text{opposite side}}{\text{adjacent side}}, \\[2ex] \cot A \left(=\dfrac{AM}{MP}\right)=\dfrac{\text{adjacent side}}{\text{opposite side}}, \\[2ex] \sec A \left(=\dfrac{AP}{AM}\right)=\dfrac{\text{hypotenuse}}{\text{adjacent side}}, \\[2ex] \operatorname{cosec} A \left(=\dfrac{AP}{MP}\right)=\dfrac{\text{hypotenuse}}{\text{opposite side}}. \end{array}\right\} . \qquad (2)$$

The symbol *csc* A is also used for cosecant A.

From the definitions of these functions and the properties of similar triangles the following properties are easily deduced:

(1) *To each value of an angle there corresponds but one value of each trigonometric function.*

(2) *To each value of a trigonometric function there corresponds but one value of an acute angle.*

(3) *Two unequal acute angles have different values for each trigonometric function.*

The values of the trigonometric functions for angles from 0° to 90° are arranged in tables. These values, which may be given to four, five, six or seven places of decimals, are called the *Natural sines, Natural cosines,* etc. The logarithms of these values of sines and cosines (with 10 added) are called *Logarithmic sines, Logarithmic cosines, etc.*

In addition to functions (1), (2), the following are occasionally used:

$$\textbf{versed sine of } A = \text{vers } A = 1 - \cos A;$$
$$\textbf{coversed sine of } A = \text{covers } A = 1 - \sin A.$$

EXAMPLES

1. Suppose that the line OP (Fig. 2) revolves about O in a *counter-clockwise direction,* starting from the position OX; show that, as the angle XOP increases, its sine, tangent, and secant increase, and its cosine, cotangent, and cosecant decrease. *Test this conclusion by an inspection of a table of Natural functions.*

2. Find by tables, sin 17° 40′, sin 76° 43′, cos 18° 10′, cos 61° 37′, tan 79° 37′ 30″, cot 72° 25′ 30″. Log sin 37° 20′, log cos 71° 25′, log tan 79° 30′ 20″.

3. Find the angles corresponding to the following Natural and Logarithmic functions:

$$\text{sine}=.15327, \qquad \text{sine}=.62175,$$
$$\text{cosine}=.85970, \qquad \text{cosine}=.61497,$$
$$\text{tangent}=.42482, \qquad \text{tangent}=.60980,$$

$$\text{Log sine}=9.79230,$$
$$\text{Log cosine}=9.96611,$$
$$\text{Log tangent}=9.82120.$$

4. Problems. The student is recommended **to try to** solve Exs. 1, 2, 3, 4 without help from the book.

EXAMPLES

1. Construct the acute angle whose cosine is $\frac{2}{3}$. What are its other trigonometric functions? Find the number of degrees in the angle.

The required angle is equal to an angle in a right-angled triangle, in which "the side adjacent to the angle is to the hypotenuse in the ratio 2:3." Construct a right-angled triangle AST which has side $AS=2$, and hypotenuse $AT=3$. The angle A is the angle required, for $\cos A=\frac{2}{3}$.

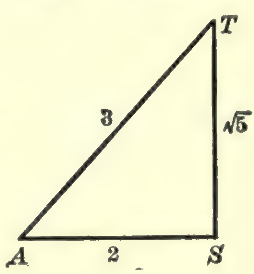


FIG. 6.

Now $\qquad ST=\sqrt{3^2-2^2}=\sqrt{5}=2.2361.$

Hence, *the other functions* are

$$\sin A=\frac{\sqrt{5}}{3}=.7454, \quad \tan A=\frac{\sqrt{5}}{2}=1.1180, \quad \cot A=\frac{2}{\sqrt{5}}=.8944,$$

$$\sec A=\frac{3}{2}=1.5000, \quad \operatorname{cosec} A=\frac{3}{\sqrt{5}}=1.3416.$$

The *measure of the angle* can be found in either one of two ways, viz.: (*a*) by measuring the angle with the protractor; (*b*) by finding in the table the angle whose cosine is $\frac{2}{3}$ or .6667. The latter method shows that $A=48°\ 11.4'$. [Compare the result obtained by method (*a*) with the value given by method (*b*).]

2. A right-angled triangle has an angle whose cosine is $\frac{2}{3}$, and the length of the hypotenuse is 50 ft. Find the angles and the lengths of the two sides.

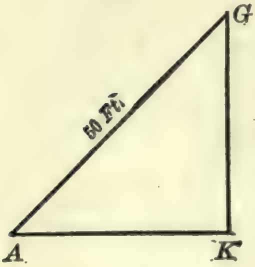


FIG. 7.

By method shown in Ex. 1, construct an angle A whose cosine is $\frac{2}{3}$. On one boundary line of the angle take a length AG to represent 50 ft. Draw GK perpendicular to the other boundary line.

$$\cos A = \tfrac{2}{3} = .6666\ldots,$$

$$\therefore\quad A = 48^\circ\ 11.4',$$

$$\therefore\quad B = 90 - A = 41^\circ\ 48.6'.$$

$$\cos A = \frac{AK}{AG} = .6666\ldots, \qquad \sin A = \frac{\sqrt{5}}{3}, \qquad \text{(Ex. 1)}$$

$$\therefore\quad AK = 50 \times .6666\ldots, \qquad \therefore\quad \frac{KG}{AG} = \frac{\sqrt{5}}{3},$$

$$= 33.333\ldots, \qquad \therefore\quad KG = \frac{\sqrt{5}}{3} \times 50 = 37.27\ldots$$

The problem may also be solved graphically as follows: Measure angles A, G with the protractor. Measure AK, KG directly in the figure.

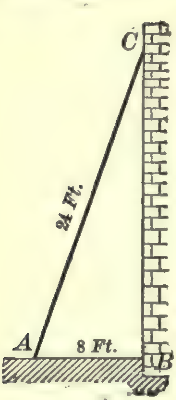


FIG. 8.

3. A ladder 24 ft. long is leaning against the side of a building, and the foot of the ladder is distant 8 ft. from the building in a horizontal direction. What angle does the ladder make with the wall? How far is the end of the ladder from the ground?

Graphical method. Let AC represent the ladder, and BC the wall. Draw AC, AB, to scale, to represent 24 ft. and 8 ft. respectively. Measure angle ACB with the protractor. Measure BC directly in the figure.

Method of computation.

$$BC = \sqrt{\overline{AC}^2 - \overline{AB}^2} = \sqrt{576 - 64} = \sqrt{512} = 22.63 \text{ ft.}$$

$$\sin ACB = \frac{AB}{AC} = \frac{8}{24} = .33333,$$

$$\therefore \quad ACB = 19° \, 28.2'.$$

4. Find tan 40° by construction and measurement. With the protractor lay off an angle SAT equal to 40°. From any point P in AT draw PR perpendicularly to AS. Then measure AR, RP, and substitute the values in the ratio, $\tan 40° = \frac{RP}{AR}$. Compare the result thus obtained with the value given for tan 40° in the tables.

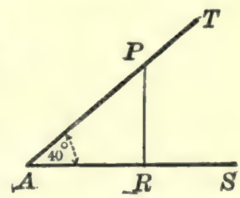


FIG. 9.

5. Construct the angle whose tangent is $\frac{5}{8}$. Find its other functions. Measure the angle approximately, and compare the result with that given in the tables. Draw a number of right-angled, obtuse-angled, and acute-angled triangles, each of which has an angle equal to this angle.

6. Similarly for the angle whose sine is $\frac{2}{5}$; and for the angle whose cotangent is 3.

7. Similarly for the angle whose secant is $2\frac{1}{3}$; and for the angle whose cosecant is $3\frac{1}{8}$.

8. Find by measurement of lines the approximate values of the trigonometric functions of 30°, 40°, 45°, 50°, 55°, 60°, 70°; compare the results with the values given in the tables.

If any of the following constructions asked for is impossible, explain why it is so.

9. Construct the acute angles in the following cases: (*a*) when the sines are $\frac{1}{2}$, 2, $\frac{2}{7}$; (*b*) when the cosines are $\frac{1}{2}$, $\frac{4}{5}$, 3; (*c*) when the tangents are 3, 4, $\frac{2}{3}$; (*d*) when the cotangents are 4, $\frac{2}{5}$; (*e*) when the secants are 2, 3, $\frac{1}{2}$; (*f*) when the cosecants are 3, 4.

10. Find the other trigonometric functions of the angles in Ex. 9. Find the measures of these angles, (*a*) with the protractor, (*b*) by means of the tables.

11. What are the other trigonometric functions of the angles: (1) whose sine is $\frac{a}{b}$; (2) whose cosine is $\frac{a}{b}$; (3) whose tangent is $\frac{a}{b}$; (4) whose cotangent is $\frac{a}{b}$; (5) whose secant is $\frac{a}{b}$; (6) whose cosecant is $\frac{a}{b}$?

12. A ladder 32 ft. long is leaning against a house, and reaches to a point 24 ft. from the ground. Find the angle between the ladder and the wall.

13. A man whose eye is 5 ft. 8 in. from the ground is on a level with, and 120 ft. distant from the foot of a flag pole 45 ft. 8 in. high. What angle does the direction of his gaze, when he is looking at the top of the pole, make with a horizontal line from his eye to the pole?

14. Find the functions of 45°, 60°, 30°, 0°, 90°, *before* reading the next article.

5. Trigonometric functions of 45°, 60°, 30°, 0°, 90°. Variation of functions.

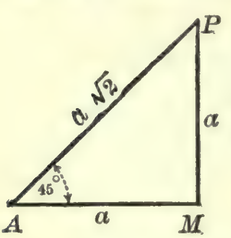


FIG. 10.

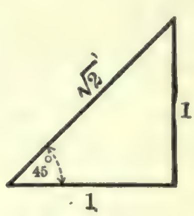


FIG. 11.

A. **Functions of 45°.** Let *AMP* be an isosceles right-angled triangle, and let each of the sides about the right angle be equal to *a*. Then angle $A=45°$, and $AP=a\sqrt{2}$.

$$\therefore \quad \sin 45° = \sin A = \frac{MP}{AP} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

Thus, by definitions Art. 3 and Fig. 10,

$$\sin 45° = \frac{1}{\sqrt{2}}, \qquad \tan 45° = 1, \qquad \sec 45° = \sqrt{2},$$

$$\cos 45° = \frac{1}{\sqrt{2}}, \qquad \cot 45° = 1, \qquad \operatorname{cosec} 45° = \sqrt{2}.$$

The sides of triangle AMP are proportional to 1, 1, $\sqrt{2}$. Hence, in order to produce the ratios of 45° quickly, it is merely necessary to draw Fig. 11; from this figure the ratios of 45° can be read off at once.

B. **Functions of 30° and 60°.** Let ABC be an equilateral triangle. From any vertex B draw a perpendicular BD to the opposite side AC. Then angle $DAB = 60°$, angle $ABD = 30°$.

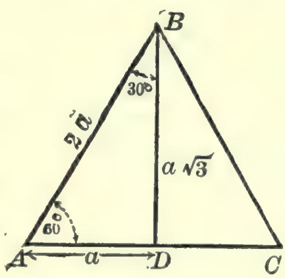


FIG. 12.

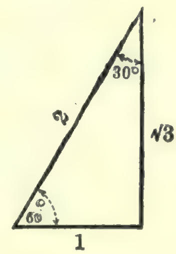


FIG. 13.

If $AB = 2a$, then $AD = a$, and $DB = \sqrt{4a^2 - a^2} = a\sqrt{3}$.

$$\therefore \quad \sin 60° = \sin DAB = \frac{DB}{AB} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2}.$$

Thus, from Fig. 12,

$$\sin 60° = \frac{\sqrt{3}}{2}, \qquad \tan 60° = \sqrt{3}, \qquad \sec 60° = 2,$$

$$\cos 60° = \frac{1}{2}, \qquad \cot 60° = \frac{1}{\sqrt{3}}, \qquad \operatorname{cosec} 60° = \frac{2}{\sqrt{3}}.$$

Also, $\qquad \sin 30° = \sin ABD = \dfrac{AD}{AB} = \dfrac{a}{2a} = \dfrac{1}{2}.$

Thus,

$$\sin 30° = \frac{1}{2}, \qquad \tan 30° = \frac{1}{\sqrt{3}}, \qquad \sec 30° = \frac{2}{\sqrt{3}},$$

$$\cos 30° = \frac{\sqrt{3}}{2}, \qquad \cot 30° = \sqrt{3}, \qquad \operatorname{cosec} 30° = 2.$$

In ADB the sides opposite to the angles 30°, 60°, 90°, are respectively proportional to 1, $\sqrt{3}$, 2. Hence, in order to produce the functions of 30°, 60°, at a moment's notice, it is merely necessary to draw Fig. 13, from which these functions can be immediately read off.

C. **Functions of 0° and 90°.** Let the hypotenuse in each of the right-angled triangles in Fig. 14 be equal to a.

$$\sin MAP = \frac{MP}{AP},$$

$$\cos MAP = \frac{AM}{AP}.$$

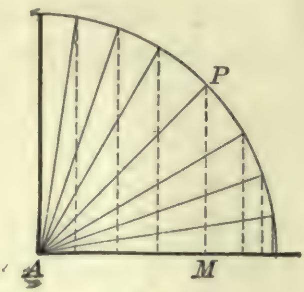


FIG. 14.

It is apparent from this figure that if the angle MAP approaches zero, then the perpendicular MP approaches zero, and the hypotenuse AP approaches to an equality with AM; so that, finally, if $MAP=0$, then $MP=0$, and $AP=AM$. Therefore, when $MAP=0$, it follows that

$$\sin 0° = \frac{0}{a} = 0, \qquad \tan 0° = \frac{0}{a} = 0,$$

$$\cos 0° = \frac{a}{a} = 1, \qquad \sec 0° = \frac{a}{a} = 1.$$

$$\text{Also } \cot 0° = \left(\frac{AM \text{ approaching } a}{MP \text{ approaching } 0}\right) = \text{unlimited number} = \infty.$$

$$\operatorname{cosec} 0° = \left(\frac{AP \text{ equal to } a}{MP \text{ approaching } 0}\right) = \text{unlimited number} = \infty.$$

As MAP approaches 90°, AM approaches zero, and MP

approaches to an equality with AP. Therefore, when $MAP = 90°$, it follows that

$$\sin 90° = \frac{a}{a} = 1, \quad \tan 90° = \infty, \qquad \sec 90° = \infty,$$

$$\cos 90° = \frac{0}{a} = 0, \quad \cot 90° = \frac{0}{a} = 0, \quad \operatorname{cosec} 90° = \frac{a}{a} = 1.$$

Thus as the angle increases from 0° to 90°, its sine increases from 0 to 1; its cosine decreases from 1 to 0; its tangent increases from 0 to ∞; its cotangent decreases from ∞ to 0; its secant increases from 1 to ∞; its cosecant decreases from ∞ to 1.

EXAMPLES

N.B. *Exponents.* In trigonometry $(\sin x)^n$ is usually written $\sin^n x$, and similarly for the other functions. This is not done when $n = -1$. The reason for this exception is given in Art. 29. Find the numerical value of

1. $\sin 60° + 2\cos 45°$.

2. $\sec^2 30° + \tan^3 45°$.

3. $\sin^3 60° + \cot^3 30°$.

4. $\cos 0° \sin 45° + \sin 90° \sec^2 30°$.

5. $4\cos^2 30° \sin^2 60° \cos^2 0°$.

6. $3\tan^3 30° \sec^3 60° \sin^2 90° \tan^2 45°$.

7. $10\cos^4 45° \sec^6 30°$.

8. $2\sin^5 30° \tan^3 60° \cos^3 0°$.

9. $x \cot^3 45° \sec^2 60° = 11 \sin^2 90°$; find x.

10. $x(\cos 30° + 2\sin 90° + 3\cos 45° - \sin^2 60°) = 2\sec 0° - 5\sin 90°$; find x.

6. Relations between the trigonometric functions of an acute angle and those of its complement. When two angles added together make a right angle, the two angles are said to be *complementary*, and each angle is called *the complement* of the other.

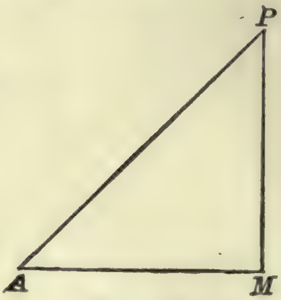


FIG. 15.

Thus, in Fig. 15, $P = 90° - A$ and P is the complement of A. Now

$$\sin A = \frac{MP}{AP} = \cos P = \cos(90° - A);$$

$$\cos A = \frac{AM}{MP} = \sin P = \sin(90° - A);$$

$$\tan A = \frac{MP}{AM} = \cot P = \cot(90° - A);$$

$$\cot A = \frac{AM}{MP} = \tan P = \tan(90° - A);$$

$$\sec A = \frac{AP}{AM} = \operatorname{cosec} P = \operatorname{cosec}(90° - A);$$

$$\operatorname{cosec} A = \frac{AP}{MP} = \sec P = \sec(90° - A)$$

These six relations can be expressed briefly:

Each trigonometric function of an angle is equal to the corresponding co-function of its complement.

EXAMPLES

1. Compare the functions of 30° and 60°; of 0° and 90°.
2. Express the following as functions of angles less than 45°: sin 78° 20′, cos 80° 30′, tan 50°, cot 65°, sec 71°, cosec 80°.

3. If $\sin x = \cos (2x + 40°)$, find a value of x.

4. If $\cot 2x = \tan (x - 30°)$, find a value of x.

5. Show that in a triangle ABC, $\sin \frac{1}{2}B = \cos \frac{1}{2}(A + C)$.

7. Relations between the trigonometric functions of an acute angle.

A. Reciprocal relations between the functions.

Inspection of the definitions (2), Art. 3, shows that:

$$\left.\begin{array}{llll} (a)\ \sin A = \dfrac{1}{\operatorname{cosec} A}, & \operatorname{cosec} A = \dfrac{1}{\sin A}, & \text{or,} & \mathbf{\sin}\ \boldsymbol{A}\ \mathbf{cosec}\ \boldsymbol{A} = \mathbf{1}; \\ (b)\ \cos A = \dfrac{1}{\sec A}, & \sec A = \dfrac{1}{\cos A}, & \text{or,} & \mathbf{\cos}\ \boldsymbol{A}\ \mathbf{sec}\ \boldsymbol{A} = \mathbf{1}; \\ (c)\ \tan A = \dfrac{1}{\cot A}, & \cot A = \dfrac{1}{\tan A}, & \text{or,} & \mathbf{\tan}\ \boldsymbol{A}\ \mathbf{cot}\ \boldsymbol{A} = \mathbf{1}. \end{array}\right\} \quad (1)$$

B. The tangent and cotangent in terms of the sine and cosine.

In the triangle AMP (Fig. 3),

$$\mathbf{\tan}\ \boldsymbol{A} = \frac{MP}{AM} = \frac{\dfrac{MP}{AP}}{\dfrac{AM}{AP}} = \frac{\mathbf{\sin}\ \boldsymbol{A}}{\mathbf{\cos}\ \boldsymbol{A}}; \quad (2)$$

$$\mathbf{\cot}\ \boldsymbol{A} = \frac{AM}{MP} = \frac{\dfrac{AM}{AP}}{\dfrac{MP}{AP}} = \frac{\mathbf{\cos}\ \boldsymbol{A}}{\mathbf{\sin}\ \boldsymbol{A}}. \quad (3)$$

C. Relations between the squares of certain functions.

In the triangle AMP (Fig. 3), indicating by $\overline{MP}^2$ the square of the length of MP,

$$\overline{MP}^2 + \overline{AM}^2 = \overline{AP}^2.$$

On dividing each member of this equation by $\overline{AP}^2$, $\overline{AM}^2$, $\overline{MP}^2$, in turn, there is obtained

$$\left(\frac{MP}{AP}\right)^2+\left(\frac{AM}{AP}\right)^2=\left(\frac{AP}{AP}\right)^2,$$

$$\left(\frac{MP}{AM}\right)^2+\left(\frac{AM}{AM}\right)^2=\left(\frac{AP}{AM}\right)^2,$$

$$\left(\frac{MP}{MP}\right)^2+\left(\frac{AM}{MP}\right)^2=\left(\frac{AP}{MP}\right)^2.$$

In reference to the angle A, these equations can be written:

$$\left.\begin{aligned}\sin^2 A+\cos^2 A&=1,\\ \tan^2 A+1&=\sec^2 A,\\ 1+\cot^2 A&=\operatorname{cosec}^2 A.\end{aligned}\right\} \qquad (4)$$

NOTE. An equation involving trigonometric functions is a *trigonometric equation.* Thus, for example, $\tan A=1$. *One* angle which satisfies this equation is the *acute* angle $A=45°$. Other solutions can be found after Art. 28 has been taken up.

EXAMPLES

In the following examples, the positive values of the radicals are to be taken. The meaning of the negative values is shown in Art. 20.

1. Given that $\sin A=\frac{1}{2}$, find the other trigonometric ratios of A by means of the relations shown in this article.

$$\operatorname{cosec} A=\frac{1}{\sin A}=2;\quad \cos A=\sqrt{1-\sin^2 A}=\frac{\sqrt{3}}{2};$$

$$\sec A=\frac{1}{\cos A}=\frac{2}{\sqrt{3}};\quad \tan A=\frac{\sin A}{\cos A}=\frac{1}{\sqrt{3}};$$

$$\cot A=\frac{1}{\tan A}=\sqrt{3}.$$

These results may be verified by the method used in solving Exs. 1, 5–7, Art. 4.

2. Express all the ratios of angle A in terms of $\sin A$.

$$\sin A = \sin A; \qquad \cos A = \sqrt{1-\sin^2 A};$$

$$\tan A = \frac{\sin A}{\cos A} = \frac{\sin A}{\sqrt{1-\sin^2 A}}; \qquad \cot A = \frac{1}{\tan A} = \frac{\sqrt{1-\sin^2 A}}{\sin A};$$

$$\sec A = \frac{1}{\cos A} = \frac{1}{\sqrt{1-\sin^2 A}}; \qquad \operatorname{cosec} A = \frac{1}{\sin A}.$$

3. Prove that $\dfrac{1}{1-\sin A} + \dfrac{1}{1+\sin A} = 2\sec^2 A.$

$$\frac{1}{1-\sin A} + \frac{1}{1+\sin A} = \frac{2}{1-\sin^2 A} = \frac{2}{\cos^2 A} = 2\sec^2 A.$$

4. Prove that $\sec^4 A - 1 = 2\tan^2 A + \tan^4 A.$

$$\sec^4 A - 1 = (\sec^2 A)^2 - 1 = (1+\tan^2 A)^2 - 1 = 2\tan^2 A + \tan^4 A.$$

5. Solve the equation $4\sin\theta - 3\operatorname{cosec}\theta = 0.$

$$4\sin\theta - \frac{3}{\sin\theta} = 0.$$

$$\therefore\quad 4\sin^2\theta - 3 = 0.$$

$$\therefore\quad \sin^2\theta = \frac{3}{4}, \quad \therefore\quad \sin\theta = +\frac{\sqrt{3}}{2}, \quad \text{and} \quad \sin\theta = -\frac{\sqrt{3}}{2}.$$

On taking the *plus* sign, *one* solution is the acute angle $\theta = 60°$; other solutions will be found later. For the *minus* sign there is also a set of solutions; these will be found later.

6. Solve $2\sin^2\theta \operatorname{cosec}\theta - 5 + 2\operatorname{cosec}\theta = 0,$

$$\frac{2\sin^2\theta}{\sin\theta} - 5 + \frac{2}{\sin\theta} = 0,$$

$$2\sin^2\theta - 5\sin\theta + 2 = 0,$$

$$(2\sin\theta - 1)(\sin\theta - 2) = 0.$$

$$\therefore \quad \sin\theta = \tfrac{1}{2}, \quad \text{and} \quad \sin\theta = 2.$$

The acute angle whose sine is $\frac{1}{2}$ is 30°; hence $\theta = 30°$ is *one* solution. The sine cannot exceed unity; hence $\sin\theta = 2$ does not afford any solution.

7. Given $\cos A = \frac{3}{4}$, $\sin B = \frac{2}{3}$, $\tan C = 2$, $\cot D = \frac{4}{3}$, $\sec E = 3$, $\operatorname{cosec} F = 2.5$; find the other trigonometric ratios of A, B, C, D, E, F, by the algebraic method. Verify the results by the method used in Art. 4.

8. Find by the algebraic method the ratios required in Exs. 1, 5–7, 10, 11, Art. 4.

9. Express all the trigonometric ratios of an angle A in terms of: (a) $\cos A$; (b) $\tan A$; (c) $\cot A$; (d) $\sec A$; (e) $\operatorname{cosec} A$. Arrange the results and those of Ex. 2 neatly in tabular form.

Prove the following identities:

10. $(\sec^2 A - 1)\cot^2 A = 1$; $\cos A \tan A = \sin A$; $(1 - \sin^2 A)\sec^2 A = 1$.

11. $\sin^2\theta \sec^2\theta = \sec^2\theta - 1$; $\tan^2\theta - \cot^2\theta = \sec^2\theta - \operatorname{cosec}^2\theta$.

12. $$\frac{1}{\sec^2 A} + \frac{1}{\operatorname{cosec}^2 A} = 1; \quad \frac{\sin A}{\operatorname{cosec} A} + \frac{\cos A}{\sec A} = 1;$$

$$(\tan\theta + \sec\theta)^2 = \frac{1 + \sin\theta}{1 - \sin\theta}.$$

13. $\sec^2 A+\operatorname{cosec}^2 A=\tan^2 A+\cot^2 A+2;$

$$\frac{1+\tan^2 A}{1+\cot^2 A}=\frac{\sin^2 A}{\cos^2 A}; \quad \frac{\operatorname{cosec} A}{\cot A+\tan A}=\cos A.$$

14. $$\frac{\cos A}{1-\tan A}+\frac{\sin A}{1-\cot A}=\sin A+\cos A;$$

$$\frac{1}{\sec A-\tan A}=\sec A+\tan A; \quad \sec^4 A-\sec^2 A=\tan^4 A+\tan^2 A.$$

Solve the following equations:

15. $2\sin\theta=2-\cos\theta.$

16. $\tan\theta+\cot\theta=2.$

17. $\tan\theta+3\cot\theta=4.$

18. $6\sec^2\theta-13\sec\theta+5=0.$

19. $8\sin^2\theta-10\sin\theta+3=0.$

20. $\sin\theta+2\cos\theta=2.2.$

CHAPTER II

SOLUTION OF RIGHT-ANGLED TRIANGLES

Applications

8. Solution of a triangle. There are two methods for finding the unknown parts of triangle (sides and angles) when a sufficient number of parts are given, viz.:

(*a*) **The graphical method;**

(*b*) **The method of computation.**

The graphical method consists in drawing a triangle which has angles equal to the given angles, and sides proportional to the given sides, and then measuring the remaining parts directly from the drawing.

The method of computation consists in writing the formulas by which the unknown parts can be found and doing the arithmetical work.

The latter method is the more exact. The graphical method affords a rough check on the results obtained by computation and leads to the detection of large errors.

General suggestions for solving problems.

(1) Make an off-hand estimate as to what the magnitude required may be, and write this estimate down;

(2) Solve the problem by the graphical method;

(3) Solve the problem by the method of computation;

(4) Check the accuracy of the results arithmetically, using formulas not used in process (3).

9. Cases in the solution of right-angled triangles. All the possible sets of two elements that can be made from the three sides and the two acute angles of a right-angled triangle are the following:

(1) The two sides about the right angle.

(2) The hypotenuse and one of the sides about the right angle.

(3) The hypotenuse and an acute angle.

(4) One of the sides about the right angle, and an acute angle.

(5) The two acute angles. (An unlimited number of triangles can have two given acute angles.)

Relations (2), Art. 3, for the triangle AMP (Fig. 3) show that if any two sides of a right-angled triangle be given, or any side and an acute angle be given, the remaining parts of the triangle can be found.

In solving a triangle, the *general method of procedure*, after making an off-hand estimate and finding an approximate solution by the graphical method, is as follows:

First: *Write all the relations (or formulas) which are to be used in solving the problem.*

Second: *Write the check formulas.*

Third: *In making the computations arrange the work as neatly as possible.*

This last is important, because, by attention to this rule, the work is presented clearly, and mistakes are less likely to occur. The computations may be made either with or without the help of logarithms. The calculations can generally be made more easily and quickly by using logarithms.

EXAMPLES

1. In the triangle ABC, right-angled at C, $a=42$ ft., $b=56$ ft. Find the hypotenuse and the acute angles.

I. *Computation without logarithms.* [Four-place tables.]

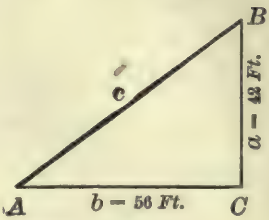


FIG. 16.

$$\tan A=\frac{a}{b}=\frac{42}{56}=.7500. \quad \therefore\ A=36^\circ\ 52'.2.$$

$$B=90-A. \qquad \therefore\ B=53^\circ\ 7'.8.$$

$$c=\sqrt{a^2+b^2}=\sqrt{1764+3136}. \qquad \therefore\ c=70 \text{ ft.}$$

Check: $a=c\cos B=70\times\cos 53^\circ\ 7'.8=70\times .6000=42$ ft.

II. *Computation with logarithms.*

Given: $a=42$ ft.
$b=56$ ft.

To find:* $A=$
$B=$
$c=$

Formulas: $\tan A=\dfrac{a}{b}$. (1)

$B=90^\circ-A.$ (2)

$c=\dfrac{a}{\sin A}$. (3)

Checks: $\tan B=\dfrac{b}{a}$.

$$a^2=c^2-b^2 = (c+b)(c-b).$$

Logarithmic formulas: $\log\tan A=\log a-\log b.$
$\log c=\log a-\log\sin A.$

$$\begin{aligned} \log a&=1.62325\\ \log b&=1.74819\\ \hline \therefore\ \log\tan A&=9.87506-10\\ \therefore\ A&=36^\circ\ 52'\ 12''\\ \therefore\ B&=53^\circ\ 7'\ 48'' \end{aligned}$$

$$\begin{aligned} \log a&=1.62325\\ \log\sin A&=9.77815-10\\ \hline \therefore\ \log c&=1.84510\\ \therefore\ c&=70 \end{aligned}$$

---

* This is to be filled after the values of the unknown quantities have been found. It is advisable to indicate the given parts and the unknown parts clearly.

The work can be more compactly arranged, as follows:

	Checks:
$\log a = 1.62325$	$\log \tan B = 10.12494 - 10$
$\log b = 1.74819$	$\therefore\ B = 53°\ 7'\ 48''$
$\therefore\ \log \tan A = 9.87506 - 10$	$c + b = 126$
$\therefore\ A = 36°\ 52'\ 12''$	$c - b = 14$
$\therefore\ B = 53°\ 7'\ 48''$	$\log (c+b) = 2.10037$
$\log \sin A = 9.77815 - 10$	$\log (c-b) = 1.14613$
$\therefore\ \log c = 1.84510$	$\therefore\ \log a^2 = 3.24650$
$\therefore\ c = 70$	$\therefore\ \log a = 1.62325$

NOTE. *In every example it is advisable to make a complete* **skeleton scheme** *of the solution, before using the tables and proceeding with the actual computation.* In this exercise, for instance, such a skeleton scheme can be seen on erasing all the numerical quantities in the equations that follow the logarithmic formulas.

2. In a triangle ABC right angled at C, $c = 60$ ft., $b = 50$ ft.; find side a and the acute angles.

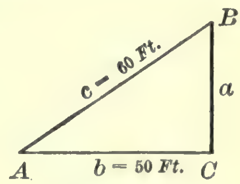


FIG. 17.

I. *Computation without logarithms.*

$$\cos A = \frac{b}{c} = \frac{50}{60} = .8333. \quad \therefore\ A = 33°\ 33'.75.$$

$$B = 90° - A. \quad \therefore\ B = 56°\ 26'.25.$$

$$a = c \sin A = 60 \times .5528 = 33.17 \text{ ft.}$$

Check: $a = b \tan A = 50 \times .6635 = 33.17.$

II. *Computation with logarithms.*

Given: $c = 60$ ft. To find: $A =$

$b = 50$ ft. $B =$

$a =$

Formulas: $\cos A = \dfrac{b}{c}$. *Checks:* $a^2 = c^2 - b^2 = (c+b)(c-b)$.

$B = 90° - A$. $a = b \tan A$.

$a = c \sin A$.

Logarithmic formulas: $\log \cos A = \log b - \log c$.

(If necessary.) $\log a = \log c + \log \sin A$.

$\log b = 1.69897$ (1)

$\log c = 1.77815$ (2)

$\therefore \log \cos A = 9.92082 - 10$ (3)

$= (1) - (2)$

$\therefore A = 33° 33' 27''$

$\therefore B = 56° 26' 33''$

$\log \sin A = 9.74255 - 10$ (4)

$\therefore \log a = 1.52070$ (5)

$= (2) + (4)$

$\therefore a = 33.16$

$\log \tan A = 9.82173 - 10$ (6)

$\therefore \log a = 1.52070$ (7)

$= (1) + (6)$

$c + b = 110$

$c - b = 10$

$\log (c+b) = 2.04139$

$\log (c-b) = 1$

$\therefore \log a^2 = 3.04139$

$\therefore \log a = 1.52070$

NOTE. There is a slight difference between the results obtained by the two methods. This is due to the fact that the calculations have been made with a four-place table in one case, and with a five-place table in the other. A four-place table will give an angle correctly to within *one minute;* a five-place table will give it correctly to within *six seconds,* and sometimes to within a second.

3. In a triangle right angled at C, the hypotenuse is 250 ft., and angle A is $67° 30'$. Solve the triangle.

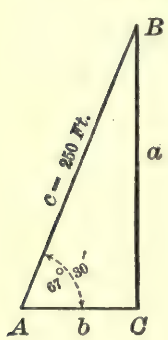


FIG. 18.

I. *Computation without logarithms.*

$B = 90° - A = 90° - 67° 30' = 22° 30'$.

$a = c \sin A = 250 \times \sin 67° 30' = 250 \times .9239 = 230.98$.

$b = c \cos A = 250 \times \cos 67° 30' = 250 \times .3827 = 95.68$.

Checks: $a^2 = c^2 - b^2$, or $a = b \tan A$.

II. *Computation with logarithms.*

Given $c = 250$ ft. To find: $B =$

$A = 67° 30'$. $a =$

$b =$

Formulas: $B = 90° - A$.

$a = c \sin A$.

$b = c \cos A$.

Checks: $a^2 = c^2 - b^2$

$= (c+b)(c-b)$.

Logarithmic formulas: $\log a = \log c + \log \sin A.$
$\log b = \log c + \log \cos A.$

$\therefore\ B = 22^\circ\ 30'$
$\log c = 2.39794$
$\log \sin A = 9.96562 - 10$
$\log \cos A = 9.58284 - 10$
$\therefore\ \log a = 2.36356$
$\therefore\ \log b = 1.98078$
$\therefore\ a = 230.97$
$\therefore\ b = 95.67$

$c + b = 345.67$
$c - b = 154.33$
$\log (c+b) = 2.53866$
$\log (c-b) = 2.18845$
$\therefore\ \log a^2 = 4.72711$
$\therefore\ \log a = 2.36356$

4. In a triangle ABC right angled at C, $b = 300$ ft. and $A = 37^\circ\ 20'$. Solve the triangle.

I. *Computation without logarithme.*

$B = 90^\circ - A = 90^\circ - 37^\circ\ 30' = 52^\circ\ 40'.$

$$c = \frac{b}{\cos A} = \frac{300}{.7951} = 377.3.$$

$a = b \tan A = 300 \times .7627 = 228.8.$

Checks: $a^2 = c^2 - b^2$, $a = c \sin A$.

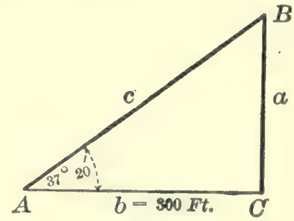


Fig. 19.

II. *Computation with logarithms.*

Given $A = 37^\circ\ 20'.$ To find: $B =$
$b = 300$ ft. $c =$
Formulas: $B = 90^\circ - A.$ $a =$

$$c = \frac{b}{\cos A}.$$

$a = b \tan A.$

Checks: $a^2 = c^2 - b^2$
$= (c+b)(c-b)$

$\therefore\ B = 52^\circ\ 40'$
$\log b = 2.47712$
$\log \cos A = 9.90043 - 10$
$\log \tan A = 9.88236 - 10$
$\therefore\ \log c = 2.57669$
$\therefore\ \log a = 2.35948$
$\therefore\ c = 377.3$
$\therefore\ a = 228.8$

$c + b = 677.3$
$c - b = 77.3$
$\log (c+b) = 2.83078$
$\log (c-b) = 1.88818$
$\therefore\ \log a^2 = 4.71896$
$\therefore\ \log a = 2.35948$

N.B. *Check all results in the following examples.* The given elements belong to a triangle ABC which is right angled at C.

From the given elements solve the following triangles:

5. $c=18.7, \quad a=16.98.$
6. $a=194.5, \quad b=233.5.$
7. $c=2934, \quad A=31° 14' 12''.$
8. $a=36.5, \quad B=68° 52'.$
9. $a=58.5, \quad b=100.5.$
10. $c=45.96, \quad a=1.095.$
11. $c=324, \quad A=48° 17'.$
12. $b=250, \quad A=51° 19'.$
13. $c=1716, \quad A=37° 20' 30''.$
14. $a=2314, \quad b=1768.$
15. $b=3741, \quad A=27° 45' 20''.$
16. $c=50.13, \quad a=24.62.$

Solve Exs. 17–24 by two methods, viz.: (1) with logarithms; (2) without logarithms.

17. $a=40, \quad B=62° 40'.$
18. $c=9, \quad a=5.$
19. $a=4.5, \quad b=7.5.$
20. $c=15, \quad A=39° 40'.$
21. $c=12, \quad B=71° 20'.$
22. $c=12, \quad a=8.$
23. $b=15, \quad B=42° 30'.$
24. $a=8, \quad b=12.$

10. Projection of a straight line upon another straight line. Let AB, of length l, be inclined at an angle α to LR. If perpendiculars AM, BN, are drawn to LR, MN is called *the* (orthogonal) *projection* of AB on LR. Through A draw AD parallel to LR. Then

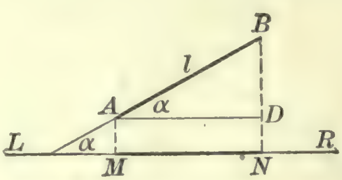


FIG. 20.

$$\textbf{Projection} = MN = AD = AB \cos DAB = l \cos \alpha.$$

That is, *the projection of one straight line upon another straight line is equal to the product of the length of the first line and the cosine of the angle of inclination of the two lines.*

EXAMPLES

In working these examples use logarithms or not, as appears most convenient. Check the results.

1. A ladder 28 ft. long is leaning against the side of a house, and makes an angle 27° with the wall. Find its projections upon the wall and upon the ground.

2. What is the projection of a line 87 in. long upon a line inclined to it at an angle 47° 30′?

3. What are the projections: (*a*) of a line 10 in. long upon a line inclined 22° 30′ to it? (*b*) of a line 27 ft. 6 in. long upon a line inclined 37° to it? (*c*) of a line 43 ft. 7 in. long upon a line inclined 67° 20′ to it? (*d*) of a line 34 ft. 4 in. long upon a line inclined 55° 47′ to it?

11. Measurement of heights and distances. When an object is above the observer's eye, the angle between the line from the eye to the object, and the horizontal line through

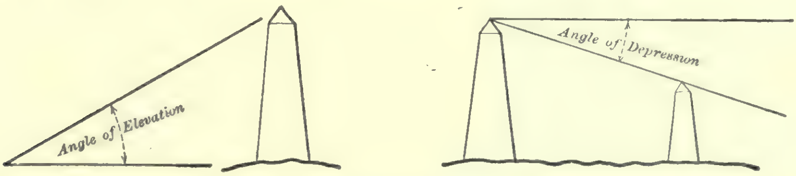


FIG. 21.

the eye and in the same vertical plane as the first line, is called **the angle of elevation** of the object, or simply *the elevation* of the object. When the object is below the observer's eye, this angle is called **the angle of depression** of the object, or simply *the depression* of the object.

EXAMPLES

1. At a point 150 ft. from, and on a level with, the base of a tower, the angle of elevation of the top of the tower is observed to be 60°. Find the height of the tower.

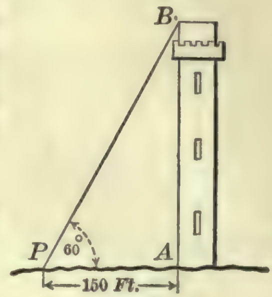


FIG. 22.

Let AB be the tower, and P the point of observation.

By the observations,

$$AP = 150 \text{ ft.}, \qquad APB = 60°.$$

$$AB = AP \tan 60° = 150 \times \sqrt{3} = 150 \times 1.7321 = 259.82 \text{ ft.}$$

2. In order to find the height of a hill, a line was measured equal to 100 ft., in the same level with the base of the hill, and in the same vertical plane with its top. At the ends of this line the angles of elevation of the top of the hill were 30° and 45°. Find the height of the hill.

Let P be the top of the hill, and AB the base line. The vertical line through P will meet AB produced in C. $AB = 100$ ft. $CAP = 30°$, $CBP = 45°$; the height CP is required. Let $BC = x$, and $CP = y$.

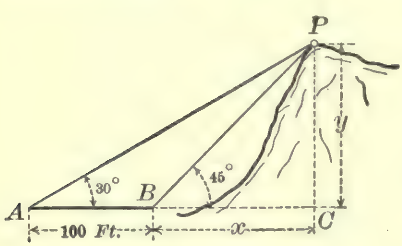


FIG. 23.

In triangle CAP,

$$\frac{CP}{AC} = \tan 30°;$$

in CBP,
$$\frac{CP}{BC} = \tan 45°.$$

Hence,
$$\frac{y}{x+100} = \tan 30° = .57735, \tag{1}$$

and
$$\frac{y}{x} = \tan 45° = 1. \tag{2}$$

From (2), $x = y$. Substitution in (1) gives

$$y = (y + 100) \times .57735.$$

$$\therefore \quad y(1-.57735)=57.735.$$

$$\therefore \quad CP=y=\frac{57.735}{.42265}=136.6 \text{ ft.}$$

3. A flagstaff 30 ft. high stands on the top of a cliff, and from a point on a level with the base of the cliff the angles of elevation of the top and bottom of the flagstaff are observed to be 40° 20′ and 38° 20′, respectively. Find the height of the cliff.

Let BP be the flagstaff on the top of the cliff BL, and let C be the place of observation. $BP=30$ ft., $LCB=38° 20'$, $LCP=40° 20'$. Let $CL=x$, $LB=y$.

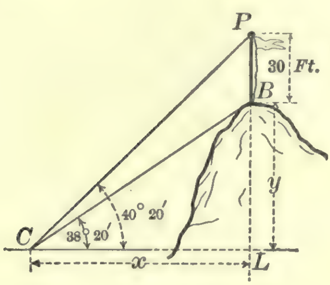


FIG. 24.

In LCB, $\frac{LB}{LC}=\tan 38° 20';$

i.e., $\frac{y}{x}=.7907.$

In LCP, $\frac{LP}{LC}=\tan 40° 20';$

i.e., $\frac{y+30}{x}=.8491.$

Hence, on division,

$$\frac{y}{y+30}=\frac{7907}{8491}.$$

On solving for y, $LB=y=406.18$ ft.

4. At a point 180 ft. from a tower, and on a level with its base, the elevation of the top of the tower is found to be 65° 40.′5. What is the height of the tower?

5. From the top of a tower 120 ft. high the angle of depression of an object on a level with the base of the tower is 27° 43′. What is the distance of the object from the top and bottom of the tower?

6. From the foot of a post the elevation of the top of a column is 45°, and from the top of the post, which is 27 ft. high, the elevation is 30°. Find the height and distance of the column.

7. From the top of a cliff 120 ft. high the angles of depression of two boats, which are due south of the observer, are 20° 20′ and 68° 40′. Find the distance between the boats.

8. From the top of a hill 450 ft. high, the angle of depression of the top of a tower, which is known to be 200 ft. high, is 63° 20′. What is the distance from the foot of the tower to the top of the hill?

9. From the top of a hill the angles of depression of two consecutive mile-stones, which are in a direction due east, are 21° 30′ and 47° 40′. How high is the hill?

10. For an observer standing on the bank of a river, the angular elevation of the top of a tree on the opposite bank is 60°; when he retires 100 ft. from the edge of the river the angle of elevation is 30°. Find the height of the tree and the breadth of the river.

11. Find the distance in space travelled in an hour, in consequence of the earth's rotation, by an object in latitude 44° 20′. [Take earth's diameter equal to 8000 mi.]

12. At a point straight in front of one corner of a house, its height subtends an angle 34° 45′, and its length subtends an angle 72° 30′; the height of the house is 48 ft. Find its length.

12. Solution of isosceles triangles. An isosceles triangle can often be solved on dividing it into two equal right-angled triangles.

EXAMPLES

1. The base of an isosceles triangle is 24 in. long, and the vertical angle is 48°; find the other angles and sides, the perpendicular from the vertex and the area. Only the steps in the solution will be indicated. Let ABC be an isosceles triangle having base $AB=24$ in., angle $C=48°$. Draw CD at right angles to base; then CD bisects the angle ACB and base AB. Hence, in the right-angled triangle ADC, $AD=\frac{1}{2}AB=12$, $ACD=\frac{1}{2}ACB=24°$. Hence, angle A, sides AC, DC, and the area, can be found.

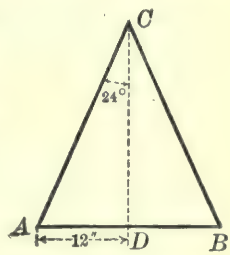


FIG. 25.

2. In an isosceles triangle each of the equal sides is 363 ft., and each of the equal angles is 75°. Find the base, perpendicular on base, and the area.

3. In an isosceles triangle each of the equal sides is 241 ft., and their included angle is 96°. Find the base, angles at the base, height, and area.

4. In an isosceles triangle the base is 65 ft., and each of the other sides is 90 ft. Find the angles, height, and area.

5. In an isosceles triangle the base is 40 ft., height is 30 ft. Find sides, angles, area.

6. In an isosceles triangle the height is 60 ft., one of equal sides is 80 ft. Find base, angles, area.

7. In an isosceles triangle the height is 40 ft., each of equal angles is 63°. Find sides and area.

8. In an isosceles triangle the height is 63 ft., vertical angle is 75°. Find sides and area.

13. Related regular polygons and circles. Among the problems on regular polygons which can be solved with the help of right-angled triangles are the following:

(*a*) Given the length of the side of a regular polygon of a given number of sides, to find its area; also, to find the radii of the inscribed and circumscribing circles of the polygon;

(*b*) To find the length of the sides of regular polygons of a given number of sides which are inscribed in, and circumscribed about, a circle of given radius.

For example, in Fig. 26, let AB be a side, equal to $2a$, of a regular polygon of n sides, and let C be the centre of the inscribed circle. Draw CA, CB, and draw CD at right angles to AB. Then D is the middle point of AB.

By geometry, angle $ACD = \frac{1}{2} ACB = \frac{1}{2} \cdot \frac{360°}{n} = \frac{180°}{n}$.

Also, angle $DAC = 90° - ACD$.

Thus, in the triangle ADC, the side AD and the angles are known; therefore CD, the radius of the circle inscribed in the polygon, can be found. On making similar constructions, the solution of the other problems referred to above will be

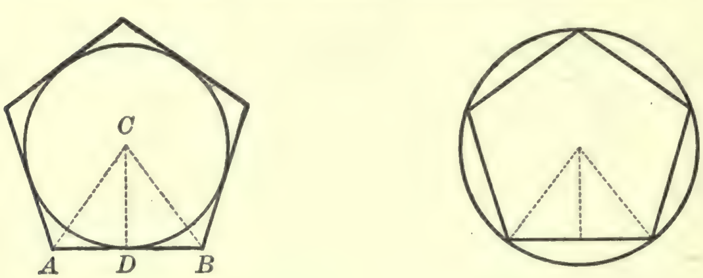


Fig. 26.

apparent. The perpendicular from the centre of the circle to a side of the inscribed polygon is called the apothem of the polygon.

EXAMPLES

1. The side of a regular heptagon is 14 ft.: find the radii of the inscribed and circumscribing circles; also, find the difference between the areas of the heptagon and the inscribed circle, and the difference between the area of the heptagon and the area of the circumscribing circle.

2. The radius of a circle is 24 ft. Find the lengths of the sides and apothems of the inscribed regular triangle, quadrilateral, pentagon, hexagon, heptagon, and octagon. Compare the area of the circle and the areas of these regular polygons; also compare the perimeters of the polygons and the circumference of the circle.

3. For the same circle as in Ex. 2, find the lengths of the sides of the circumscribing regular figures named in Ex. 2. Compare their areas and perimeters with the area and circumference of the circle.

4. If a be the side of a regular polygon of n sides, show that R, the radius of the circumscribing circle, is equal to $\frac{1}{2}a \operatorname{cosec} \frac{180°}{n}$; and that r, the radius of the circle inscribed, is equal to $\frac{1}{2}a \cot \frac{180°}{n}$.

5. If r be the radius of a circle, show that the side of the regular inscribed polygon of n sides is $2r \sin \frac{180°}{n}$; and that the side of the regular circumscribing polygon is $2r \tan \frac{180°}{n}$.

6. If a be the side of a regular polygon of n sides. R the radius of the circumscribing circle, and r the radius of the circle inscribed, show that area of polygon $= \frac{1}{4}na^2 \cot \frac{180°}{n}$

$$= \frac{1}{2}nR^2 \sin \frac{360°}{n} = nr^2 \tan \frac{180°}{n}.$$

14. Problems requiring a knowledge of the points of the Mariner's Compass. The circle in the Mariner's Compass is divided into 32 equal parts, each part being thus equal to $360° \div 32$, i.e., $11\frac{1}{4}°$. The points of division are named as indicated on the figure.

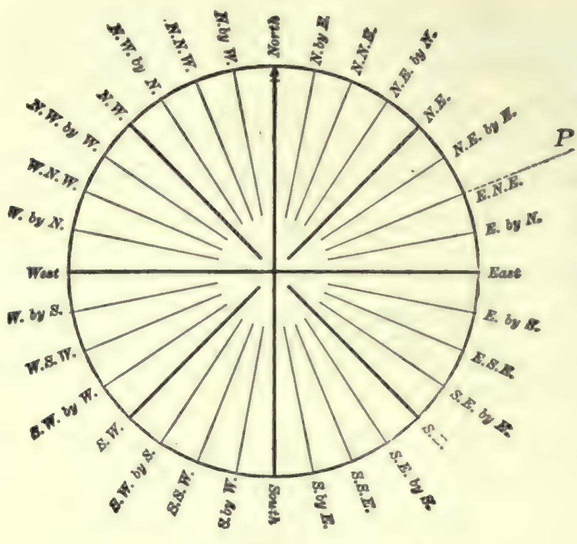


FIG. 27.

It will be observed that the points are named with reference to the points North, South, East, and West, which are called the *cardinal points*. Direction is indicated in a variety of ways. For instance, suppose C were the centre of the circle; then the point P in the figure is said to *bear* E.N.E. from C, or, from C the *bearing* of P is E.N.E. Similarly, C bears W.S.W. from P, or, the bearing of C from P is W.S.W. The point E.N.E. is 2 points North of East, and 6 points East of North. Accordingly, the phrases E. $22\frac{1}{2}°$ N., N. $67\frac{1}{2}°$ E., are sometimes used instead of E.N.E.

EXAMPLES

1. Two ships leave the same dock at 8 A.M. in directions S.W. by S., and S.E. by E. at rates of 9 and $9\frac{1}{2}$ mi. an hour respectively. Find their distance apart, and the bearing of one from the other at 10 A.M. and at noon.

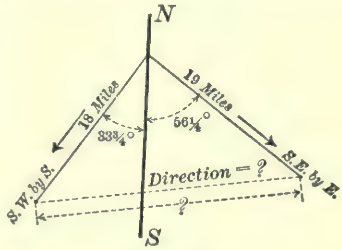


FIG. 28.

2. From a lighthouse L two ships A and B are observed in a direction N.E. and N. 20° W. respectively. At the same time A bears S.E. from B. If LA is 6 mi., what is LB?

15. Examples in the measurement of land. In order to find the area of a piece of ground, a surveyor measures distances and angles sufficient to provide *data* for the computation. An account of his method of doing this belongs to works on surveying. This article merely gives some examples which can be solved without any knowledge of professional details. In solving these problems, the student should make the plotting or mapping an important feature of his work.

The Gunter's chain is generally used in measuring land. It is 4 rods or 66 feet in length, and is divided into 100 links.

An acre = 10 square chains = 4 roods = 160 square rods or poles.

EXAMPLES

1. A surveyor starting from a point A runs S. 70° E. 20 chains, thence N. 10° W. 20 chains, thence N. 70° W. 10 chains, thence S. 20° W. 17.32 chains to the place of beginning. What is the area of the field which he has gone around?

Make a plot or map of the field, namely, $ABCD$. Here AB represents 20 chains, and the bearing of B from A is S. 70° E. BC represents 20 chains, and the bearing C from B is N. 10° W., and so on. Through the most westerly point of the field draw a north-and-south line. This line is called *the meridian*. In the case of each line measured, find the distance that one end of the line is east or west from the other end. This easting or westing is called *the departure of the line.* Also find the distance that one end of the line is north or south of the other end. This northing or southing is called *the latitude of the line.* For example, in Fig. 29, the departures of AB, BC, CD, DA, are B_1B, BL, CH, DD_1, respectively; the latitudes of the boundary lines are AB_1, B_1C_1, C_1D_1, D_1A, respectively. The following formulas are easily deduced:

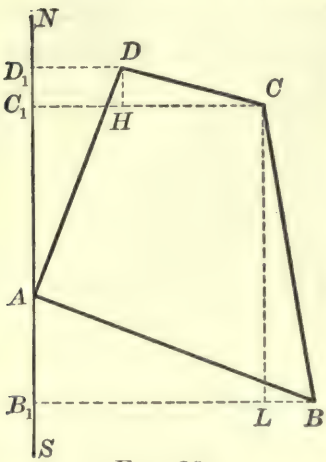


FIG. 29.

Departure of a line = length of line × sine of the bearing;
Latitude of a line = length of line × cosine of the bearing.

By means of the departures, *the meridian distance of a point* (i.e., its distance from the north-and-south line) can be found. Thus the meridian distance of C is C_1C, and $C_1C = D_1D + HC$. Hence in Fig. 2.9, AB_1, B_1B, B_1C_1, C_1C, C_1D_1, D_1D can be computed. Now

$$\text{area } ABCD = \text{trapezoid } D_1DCC_1 + \text{trapezoid } C_1CBB_1 - \text{triangle } ADD_1 - \text{triangle } ABB_1.$$

The areas in the second member can be computed; it will be found that area $ABCD = 26$ acres.

NOTE. Sometimes the bearing and length of one of the lines enclosing the area is also required. These can be computed by means of the latitudes and departures of the given lines.

2. In Ex. 1, deduce the length and bearing of *DA* from the lengths and bearings of *AB*, *BC*, *CD*.

3. A surveyor starts from *A* and runs 4 chains S. 45° E. to *B*, thence 5 chains E. to *C*, thence 6 chains N. 40° E. to *D*. Find the distance and bearing of *A* from *D*; also, the area of the field *ABCD*. Verify the results by going around the field in the reverse direction, and calculating the length and bearing of *BA* from the lengths and directions of *AD*, *DC*, *CB*.

4. A surveyor starts from one corner of a pentagonal field, and runs N. 25° E. 433 ft., thence N. 76° 55′ E. 191 ft., thence S. 6° 41′ W. 539 ft., thence S. 25° W. 40 ft., thence N. 65° W. 320 ft. Find the area of the field. Deduce the length and direction of one of the sides from the lengths and directions of the other four.

5. From a station within a hexagonal field the distances of each of its corners were measured, and also their bearings; required its plan and area, the distances in chains and the bearings of the corners being as follows: 7.08 N.E., 9.57 N. ½ E., 7.83 N.W. by W., 8.25 S.W. by S., 4.06 S.S.E. 7° E., 5.89 E. by S. 3½° E.

CHAPTER III

ANGLES IN GENERAL AND THEIR TRIGONOMETRIC FUNCTIONS

16. General definition of an angle. The **amount of rotation** that a line OP makes in turning about a point O from an initial position OX until it comes to rest in a terminal position OQ is said to be *the* **angle** *described* (or generated) by the line OP.

Angles unlimited in magnitude. Since the revolving line may make any number of complete revolutions before coming to rest, angles can be of any magnitude and can be unlimited in magnitude.

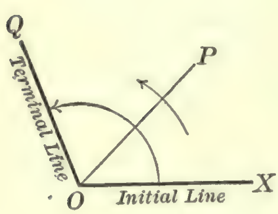


FIG. 30.

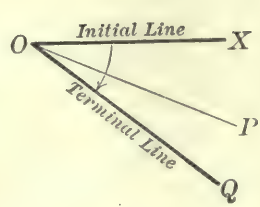


FIG. 31.

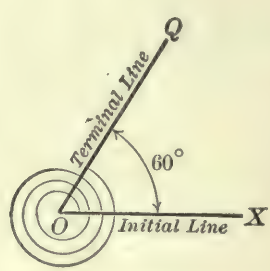


FIG. 32.

When necessary the number of revolutions can be indicated as in Fig. 32, which represents the angle 60° and the angle $3 \cdot 360° + 60°$. The lines in this figure may represent any angle $n \cdot 360° + 60°$, n denoting any whole number.

Ex. Taking the initial line in the same position as OX in Fig. 30 draw the terminal lines of the angles 40°, 160°, 220°, 325°, 437°, 860°, 1020°, 180°, 360°, 720°.

Positive and negative angles. When the turning line revolves in the *counterclockwise* direction (as in Fig. 30), the angle described is called *positive* and is given the plus sign; when the turning line revolves in the *clockwise* direction (as in Fig. 31), the angle described is called *negative* and is given the minus sign.

Ex. Taking the initial line in the same position as OX in Fig. 30 draw the terminal lines of the angles 140°, $-200°$, $-430°$, 335°, $-850°$, 820°.

Coterminal angles. Angles having the same initial and terminal lines respectively are called *coterminal* angles.

Ex. Show that, if the same initial line be taken, the angles 40°, $-320°$, 400°, 760°, $-1040°$ are coterminal.

Congruent angles. Angles differing by multiples of 360° are called *congruent* angles.

Ex. Show that the angles in the preceding exercise are congruent.

Complementary angles. Supplementary angles. Angles whose sum is 90° are called complementary angles; angles whose sum is 180° are called supplementary. In other words,

$$\text{the complement of angle } A = 90° - A;$$

$$\text{the supplement of angle } A = 180° - A.$$

Angles in the various quadrants. In Fig. 33, XX' and

YY' are at right angles and OX is the initial line. In this figure XOY, YOX', $X'OY'$, $Y'OX$, are called *the first, second, third, and fourth quadrants*, respectively. When the turning line ceases its revolution at some position between OX and OY, the angle described is said to be *an angle in the first guadrant;* when the final position of the turning line is between OY and OX' the angle described is said to be *in the second quadrant*; and so on for the third and fourth quadrants.

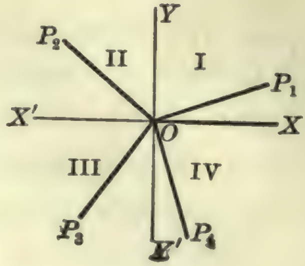


Fig. 33.

EXAMPLES

1. Lay off the following angles: In the case of each angle name the least positive angle that has the same terminal line. Name the quadrants in which the angles are situated. In the case of each angle name the four smallest positive angles that have the same terminal line.

(*a*) 137°, 785°, 321°, 930°, 840°, 1060°, 1720°, 543°, 3657°.

(*b*) −240°, −337°, −967°, −830°, −750°, −1050°, −7283°.

2. What are the complements and supplements of 40°, 227°, −40°?

$$\text{complement of } 40° = 90° - 40° = 50°;$$

$$\text{supplement of } 40° = 180° - 40° = 140°.$$

$$\text{complement of } 227° = 90° - 227° = -137°;$$

$$\text{supplement of } 227° = 180° - 227° = -47°.$$

$$\text{complement of } -40° = 90° - (-40°) = 130°;$$

$$\text{supplement of } -40° = 180° - (-40°) = 220°.$$

3. What are the complements of −230°, 150°, −40°, 340°, 75°, 83°, 12°, −295°, −324°, 200°, 240°, −110°, −167°?

4. What are the supplements of the angles in Ex. 3?

17. Measurement of angles. There are three systems of angular measure, *the sexagesimal, the centesimal,* and *the circular.*

In the **sexagesimal** system the unit angle is one-ninetieth of a right angle, called *a degree,* and the subdivisions are minutes and seconds, as described in Art. 2. This measure is used in the solution of triangles and in elementary mathematics generally.

In the **centesimal** system the unit angle is one-hundredth of a right angle, called *a grade.* The table of centesimal measure is

1 right angle = 100 grades.

1 grade = 100 minutes.

1 minute = 100 seconds.

This system, which is used in France, has not come into general use.

In the **circular** system the unit angle is the angle which is subtended at the centre of a circle by an arc equal in length to the radius. This angle is called **a radian.** *The radian*

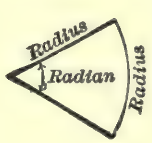


FIG. 34.

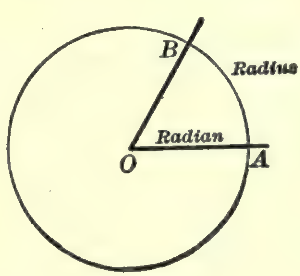


FIG. 35.

measure (often called *the circular measure*) *of an angle* is the number of radians it contains. Radian measure is the measure used in higher mathematics. It is also used in various problems and discussions in elementary mathematics.

18. Value of a radian. Relation between radian measure and degree measure. In Fig. 35 arc AB = radius, and thus angle AOB = a radian. By geometry,

$$\frac{\text{angle } AOB}{\text{complete angle about } O} = \frac{\text{arc } AB}{\text{length of circle}},$$

i.e.,

$$\frac{\text{a radian}}{360°} = \frac{\text{radius}}{2\pi \times \text{radius}} = \frac{1}{2\pi}.$$

$$\therefore \quad \textbf{a radian} = \frac{\mathbf{180°}}{\boldsymbol{\pi}} = 57°\ 17'\ 44''.8. \tag{1}$$

Thus a radian has a constant value, and accordingly can be used as a unit. From (1)

$$\boldsymbol{\pi}\ \textbf{radians} = \mathbf{180°}. \tag{2}$$

Also, from (1) or (2),

$$1° = \frac{\pi}{180} \text{ radians.}$$

By means of relation (2) an angle expressed in the one measure can be expressed in the other.

Notation. An angle 2 radians is expressed 2^r or 2^c (r from radian, c from circular). The symbol r, or c, is often omitted. For instance, angle π denotes the angle π radians i.e., 180°; angle $\frac{\pi}{2}$ denotes the angle $\frac{\pi}{2}$ radians, i.e., 90°; angle θ denotes an angle containing θ radians.

Radian measure as ratio of subtended arc to radius. Length of arc. Since the arc of a circle equal to the radius subtends at the centre an angle equal to a radian, *the number of times an arc contains the radius is the radian measure of its subtended angle*; *i.e.*,

$$\textbf{Number of radians in angle} = \frac{\textbf{length of subtended arc}}{\textbf{length of radius}}. \tag{3}$$

If a = length of arc, r = length of radius, and θ = radian measure of the angle, (3) may be written

$$\boldsymbol{\theta} = \frac{a}{r}. \tag{4}$$

From this $$a = r\boldsymbol{\theta}. \tag{5}$$

In words, **arc = radius × radian measure of angle.**

Formula (4) *derived otherwise.* In Fig. 36 AOB is a radian. Let r = length of radius, a = length of arc AP, θ = radian measure of angle AOP.

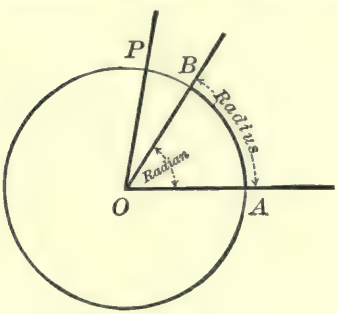


Fig. 36.

By geometry,

$$\frac{\text{angle } AOP}{\text{angle } AOB} = \frac{\text{arc } AP}{\text{arc } AB},$$

i.e., $$\frac{\text{angle } AOP}{1 \text{ radian}} = \frac{a}{r}.$$

$\therefore$ number of radians in AOP, $\boldsymbol{\theta} = \frac{a}{r}.$

EXAMPLES

1. How many degrees are there in 2.5 radians?

2. How many radians are there in 231°?

3. Express $\frac{1}{2}^r$, 4^r, $\frac{1}{3}^r$ in degrees.

4. Express the angles $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}, \frac{2}{3}\pi, 3\pi, -\frac{4}{3}\pi, -10\pi$, in degrees.

5. Express in radians 83° 20′, 142° 30′.

6. Express in radians (in terms of π) 45°, 210°, 300°, 120°, 225°.

7. What is the radian measure of the angle which at the centre of a circle of radius $1\frac{1}{2}$ yds. subtends an arc of 8 in.? Also express the angle in degrees.

8. Find the radius of the circle in which an arc 12 in. subtends an angle 2 radians at the centre.

9. The radius of a circle is 10 in. How long is the arc subtended by an angle of 4 radians at the centre?

10. How long does it take the minute hand of a clock to turn through $-1\frac{2}{3}$ radians?

18a. Motion of a particle in a circle.

Let r ft.* = radius of the circle on which the particle is moving;
s ft. = length of arc traversed; and
θ radians = angle swept over by the radius joining the moving point to the centre of the circle.

Then, by Art. 18, Eq. (5),

$$s = r\theta.$$

It should be noted that this equation is not true unless the radius and the arc are measured in the same unit of length and the angle is measured in radians.

Again, suppose the particle is moving at a uniform rate on a circle of radius r ft.*

Let v ft. per sec. = the linear velocity of the particle in its path,
and ω radians per sec. = its angular velocity.†

Then $$v = r\omega;$$

and thus $$\omega = \frac{v}{r}.$$

---

* Or whatever may be the unit of length.

† The *angular velocity* of the particle is the number of radians swept over in a unit of time—in this instance a second—by the radius joining the moving point to the centre of the circle.

Suppose the angular velocity $= N$ revolutions per minute.

Since $\qquad 1 \text{ revolution} = 2\pi \text{ radians},$

$$N \text{ revolutions per minute} = 2\pi N \text{ radians per min.}$$

$$= \frac{\pi N}{30} \text{ radians per sec.}$$

EXAMPLES

1. A wheel of a carriage which is travelling at the rate of 8 miles per hour is 3 ft. in diameter. Find the angular velocity of any point of the wheel about the axle, in radians per second.

2. Compare the angular velocities of the hour, minute, and second hands of a watch.

3. Express in terms of radians per minute an angular velocity of 25° per second.

4. A wheel makes 300 revolutions per hour. Express its angular velocity (*a*) in radians per sec.; (*b*) in degrees per min.

5. An automobile having a 28-in. wheel travels at a speed of 15 miles an hour. What is the angular speed of the wheel about the axle (*a*) in revolutions per minute? (*b*) in radians per second?

6. A belt travels over a 30-in. pulley which is carried on a shaft making 300 revolutions per minute. Find the linear speed of the belt in feet per second.

19. Convention for signs of lines in a plane. In Fig. 37 $X'OX$ is a horizontal line to which YOY' is drawn at right angles. Lines measured horizontally *towards the right* are taken as *positive*, and accordingly lines measured horizontally *towards the left* are *negative*. Thus OM_1, OM_4, M_2O are positive, OM_2, OM_3, M_1O are negative. Lines measured vertically *upward*, as M_1P_1, M_2P_2, are taken as *positive*, and lines measured vertically *downward*, as M_3P_3, M_4P_4, are *negative*.

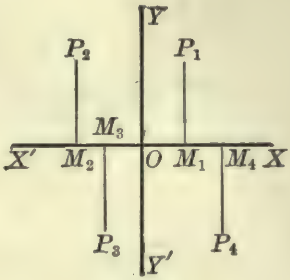


FIG. 37.

20. Trigonometric functions defined for angles in general. In Figs. 38–41 the horizontal line OX is taken as the initial line, and angles in the first, second, third, and fourth quad-

FIG. 38.

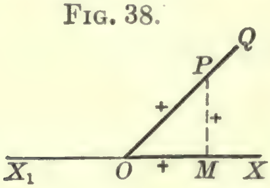
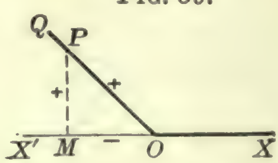


FIG. 39.



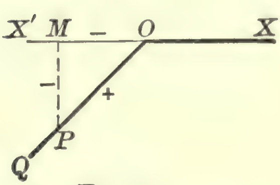


FIG. 40.

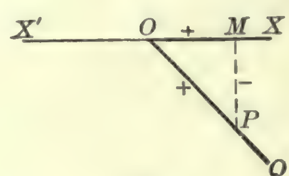


FIG. 41.

rants respectively are described. In each angle any point P in the terminal line OQ is taken, and MP is drawn at right angles to OX. Thus (Art. 10) in each figure

OM is *the horizontal projection* of ***OP***,

MP is *the vertical projection* of ***OP***.

The lines OM, MP receive signs in accordance with the convention in Art. 19 and OP is taken as positive. For each figure, the angle XOQ being denoted as A, the *six ratios* formed by means of the lines OM, MP, OP are called *trigonometric functions* of A, as follows:

the *sine* of angle $A = \sin A = \frac{MP}{OP}$,

the *cosine* of angle $A = \cos A = \frac{OM}{OP}$,

the *tangent* of angle $A = \tan A = \frac{MP}{OM}$,

the *cotangent* of angle $A = \cot A = \frac{OM}{MP}$,

the *secant* of angle $A = \sec A = \frac{OP}{OM}$,

the *cosecant* of angle $A = \csc A = \frac{OP}{MP}$.

Two other trigonometric functions are occasionally used, viz.:

the *versed sine* of angle $A =$ *vers* $A = 1 - \cos A$,

the *coversed sine* of angle $A =$ *covers* $A = 1 - \sin A$.

Inspection of Figs. 38–41, in connection with the definitions, shows that the trigonometric functions of angles in the various quadrants have the signs stated in the following table:

Quadrant.	sin A	cos A	tan A	cot A	sec A	cosec A
I.	+	+	+	+	+	+
II.	+	−	−	−	−	+
III.	−	−	+	+	−	−
IV.	−	+	−	−	+	−

EXAMPLES

1. Tell the signs of the following functions:

(*a*) sin 100°; (*b*) cos 220°; (*c*) tan 230°;

(*d*) sec 340°; (*e*) cot (−130°); (*f*) csc 560°.

2. Describe the angle 480°, and find the values of its sine, cosine, and tangent.

3. Describe the angle 945°, and find the values of its sine, cosine, and tangent.

4. In which quadrants may an angle lie, if:

(*a*) its sine is positive; (*b*) its cosine is negative;
(*c*) its tangnet is positive; (*d*) its tangent is negative.

5. In what quadrant must an angle lie, if:

(*a*) its sine and cosine are negative;
(*b*) its sine is positive and tangent negative;
(*c*) its sine is negative and tangent positive;
(*d*) its cosine is positive and tangent negative.

6. Construct angle A, when $\sin A = \frac{3}{4}$. Find the remaining functions of A. The definition of the sine, Art. 20, shows that for this angle

$$MP = +3, \qquad OP = 4.$$

The construction then is as indicated in Fig. 42. There are thus *two sets of angles* whose sines are $\frac{3}{4}$, viz., the angles whose terminal line is OP, and the angles whose terminal line is OP_1.

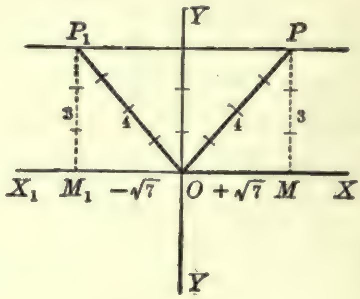


FIG. 42.

Each set of angles is unlimited in number, and the angles in it differ from one another by multiples of 360°.

For an angle having OP for terminal line,

$$\cos A=\frac{\sqrt{7}}{4},\quad \tan A=\frac{3}{\sqrt{7}},\quad \cot A=\frac{\sqrt{7}}{3},$$

$$\sec A=\frac{4}{\sqrt{7}},\quad \csc A=\frac{4}{3}.$$

For an angle having OP_1 for terminal line,

$$\cos A=-\frac{\sqrt{7}}{4},\quad \tan A=-\frac{3}{\sqrt{7}},\quad \cot A=-\frac{\sqrt{7}}{3},$$

$$\sec A=-\frac{4}{\sqrt{7}},\quad \csc A=\frac{4}{3}.$$

7. Construct angle A, given $\tan A=-\frac{3}{4}$. $\left[\text{Note } -\frac{3}{4}=\frac{-3}{4}=\frac{3}{-4}.\right]$ The definition of the tangent, Art. 20, shows that for this angle,

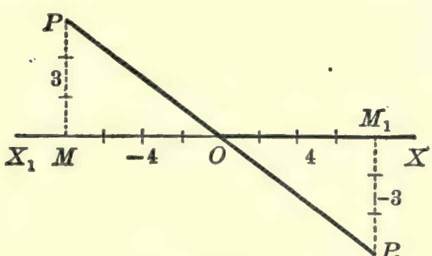


FIG. 43.

$$MP=-3 \text{ and } OM=4,$$

or

$$MP=3 \text{ and } OM=-4.$$

The construction then is as indicated in Fig. 43. There are thus *two sets of angles* whose tangents are $-\frac{3}{4}$, viz., the angles whose terminal line is OP and the angles whose terminal line is OP_1. The angles in each set are congruent.

8. Calculate the remaining functions of the angles in Ex. 7.

9. Construct the angles which have the following functions, and calculate the remaining functions:

(*a*) $\cos x=-\frac{2}{3}$; (*b*) $\sin x=-\frac{3}{5}$; (*c*) $\cos x=\frac{2}{5}$;

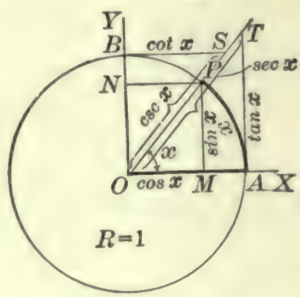
(*d*) $\sec x = \frac{5}{3}$; (*e*) $\tan x = \frac{4}{3}$. $\left[\text{Note. } \frac{4}{3} = \frac{+4}{+3} = \frac{-4}{-3}\cdot\right]$

10. Construct the angles and find their remaining functions in the following cases:

(*a*) $\sin x = \frac{3}{7}$ and $\cos x$ negative; (*b*) $\cos x = \frac{2}{5}$ and $\sin x$ negative;

(*c*) $\tan x = \frac{3}{2}$ and $\sin x$ negative; (*d*) $\sec x = 2$ and $\tan x$ negative.

21. Line definitions of the trigonometric functions. Geometrical representation of the functions. In Fig. 44 XOP is an angle of x radians. A circle, whose radius is a unit in length, is described about O. This circle is called a **unit circle.** Angle $XOY = 90°$. From A and B tangents AT, BS are drawn to meet the terminal line OP. From P, the intersection of the terminal line with the circle, PM is drawn at right angles to OX. Then



Angle in First Quadrant.

FIG. 44.

$$\sin x = \frac{MP}{OP(=1)} = \boldsymbol{MP},$$

$$\cos x = \frac{OM}{OP(=1)} = \boldsymbol{OM},$$

$$\tan x = \frac{AT}{OA(=1)} = \boldsymbol{AT},$$

$$\cot x = \frac{OM}{MP} = \frac{BS\,*}{OB(=1)} = \boldsymbol{BS},$$

---

* Since the triangles OMP, OBS, are similar.

† Thus also, $\operatorname{vers} x = 1 - \cos x = OA - OM = MA$;
$\operatorname{covers} x = 1 - \sin x = OB - MP = NB$.

$$\sec\ x = \frac{OT}{OA(=1)} = \boldsymbol{OT},$$

$$\csc\ x = \frac{OP}{MP} = \frac{OS*}{OB(=1)} = \boldsymbol{OS}†.$$

Also
$$x = \frac{\text{arc } AP}{OA(=1)} = \textbf{arc } \boldsymbol{AP}.$$

Since the radius is of unit length the lengths of these lines are expressed in *the same numbers* as the ratios which are the trigonometric functions. Accordingly **the lines can represent the functions.**

Also, on the same scale, *the arc represents the radian measure of the angle.* (Compare definition (3), Art. 18.)

The *line definitions* of the trigonometric functions of an angle may be thus expressed, the angle being at the centre of a unit circle:

The **sine** *is the length of the perpendicular drawn from the extremity of the terminal radius to the initial radius.*

The **cosine** *is the length of the line from the centre of the circle to the foot of this perpendicular.*

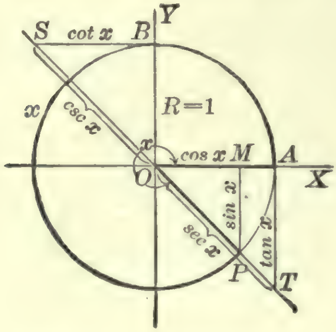
The **tangent** *is the length of the tangent drawn at the extremity of the initial radius from this extremity to where the tangent meets the terminal radius produced.*

The **secant** *is the length from the centre along the terminal line to where it meets the tangent drawn at the extremity of the initial radius.*

The **cotangent** *is the length of the tangent drawn at the extremity of the radius which makes an angle* $+90°$ *with the initial radius, from this extremity to where this tangent meets the terminal radius produced.*

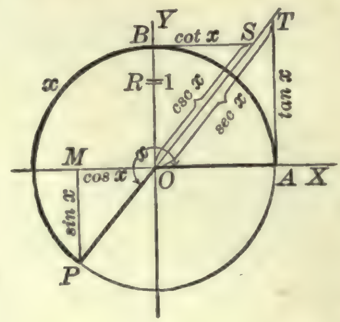
The **cosecant** *is the length from the centre along the terminal line to where it meets the tangent drawn at the extremity of the radius which makes an angle* $+90°$ *with the initial radius.*

Figs. 45, 46, 47 show the lines which represent the trigonometric functions of angles in the second, third, and fourth quadrants respectively. The lines are named as in Fig. 44. The line representing the tangent is always drawn from A,



Angle in Second Quadrant.

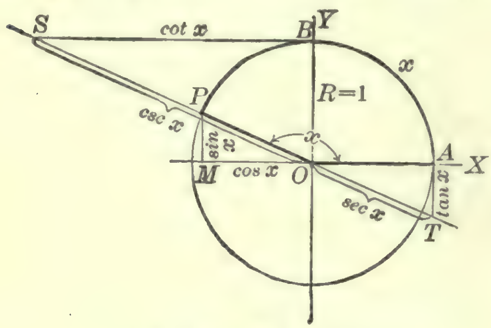
FIG. 45.



Angle in Third Quadrant.

FIG. 46.

and the line representing the cotangent is always drawn from B, to the terminal line of the angle. The signs of OM, MP, AT, BS follow the convention in Art. 19, and the



Angle in Fourth Quadrant.

FIG. 47.

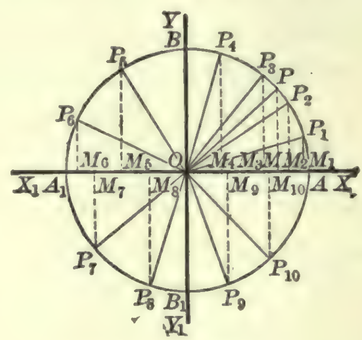


FIG. 48.

direction from O towards P is taken as positive. Inspection of Figs. 44–47 shows that the functions of angles in the various quadrants have the signs set down in the table in Art. 20.*

* The line definitions of the trigonometric functions were employed before the ratio definitions were suggested.

22. Changes in the values of the functions when the angle varies. Limiting values of the functions. In Fig. 48 AOP is a variable angle x, and a unit circle is described about O.

The sine. Now $$\sin x = \frac{MP}{OP(=1)} = MP.$$

When angle x approaches 0° as a limit, MP passes through the values M_2P_2, M_1P_1 to zero as a limit. Thus

$$\sin 0° = 0.$$

When OP revolves positively from OA toward OB as a limit, x increases from 0° to 90° as a limit, and MP increases from zero through M_1P_1, M_2P_2, M_4P_4 to $OB(=1)$ as a limit. Thus

$$\sin 90° = 1.$$

When OP revolves from OB to OA_1, x increases from 90° to 180°, and MP decreases from $OB(=1)$ through M_5P_5, M_6P_6 to zero. Thus

$$\sin 180° = 0.$$

When OP revolves from OA_1 to OB_1, x increases from 180° to 270° and MP changes from zero through M_7P_7, M_8P_8 to $OB_1(=-1)$, and thus

$$\sin 270° = -1.$$

When OP revolves from OB_1 to OA, x varies from 270° to 360°, and MP changes from $OB_1(=-1)$ through M_9P_9, $M_{10}P_{10}$ to zero. Thus

$$\sin 360° = 0.$$

The cosine. Now

$$\cos x = \frac{OM}{OP(=1)} = OM.$$

When angle x approaches $0°$ as a limit, OM passes through the values OM_2, OM_1 to $OA(=1)$ as a limit. Thus

$$\cos 0° = 1.$$

When OP revolves positively from OA to OB, x increases from $0°$ to $90°$ and OM decreases from $OA(=1)$ through OM_1, OM_2, OM_3, OM_4, to zero. Thus

$$\cos 90° = 0.$$

When OP revolves from OB to OA_1, x increases from $90°$ to $180°$ and OM changes from zero through OM_5, OM_6 to $OA_1(=-1)$. Thus

$$\cos 180° = -1.$$

When OP revolves from OA_1 to OB_1, x increases from $180°$ to $270°$ and OM changes from $OA_1(=-1)$ through OM_7, OM_8 to zero. Thus

$$\cos 270° = 0.$$

When OP revolves from OB_1 to OA, x increases from $270°$ to $360°$ and OM changes from zero through OM_9, OM_{10}, to $OA = 1$. Thus

$$\cos 360° = 1.$$

The tangent. *First method, using the ratio definitions.* Now

$$\tan x = \frac{MP}{OM}.$$

When angle x approaches $0°$ as a limit, MP approaches zero and OM approaches $OA(=1)$. Thus

$$\tan 0° = 0.$$

When x increases from 0° to 90°, MP is approaching $OB = (1)$ and OM is approaching zero. Thus $\frac{MP}{OM}$, which is positive during this change, is constantly increasing and becomes unlimited in value. Thus

$$\tan 90° = \infty .$$

When x is increasing from 90° to 180°, MP is changing from $OB(=1)$ to zero and OM is changing from zero to $OA_1(=-1)$. Thus $\frac{MP}{OM}$, which is negative during this change, changes from $-\infty$ to 0. Thus

$$\tan 180° = 0.$$

N.B. When x is increasing from 0° to 180°, $\tan x = +\infty$ when x reaches 90°, and $\tan x = -\infty$ when x leaves 90°. Thus $\tan x$ changes sign when x is passing through the value 90°. If x is decreasing from 180° to 0°, $\tan x = -\infty$ when x reaches 90°, and $\tan x = +\infty$ when x leaves 90°. Thus the sign of $\tan x$ when $x = 90°$, depends on the way in which x has approached the value 90°.

When x is increasing from 180° to 270°, MP is changing from zero to OB_1 and OM is changing from OA_1 to zero. Thus $\frac{MP}{OM}$, which is positive during this change, changes from 0 to an unlimited value. So

$$\tan 270° = \infty .$$

When x is increasing from 270° to 360°, MP is changing from OB_1 to zero, and OM is changing from zero to OA.

Thus $\frac{MP}{OM}$, which is negative during this change, changes from $-\infty$ to 0. So

$$\tan 360° = 0.$$

Tan 270°, like tan 90°, has an unlimited value whose sign is positive or negative, according to the way in which the angle has approached the value 270°.

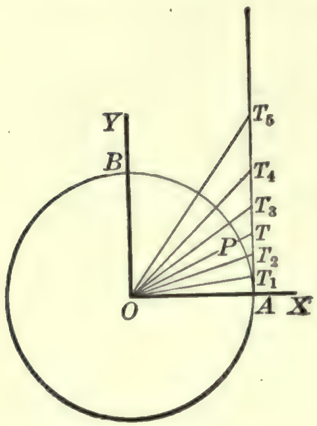


FIG. 49.

The tangent. *Second method, using the line definitions.* In Fig. 49, the variable angle $AOP = x$. A unit circle is drawn and the tangent drawn at A. Now

$$\tan x = \frac{AT}{OA(=1)} = AT.$$

When x approaches 0°, AT passes through AT_2, AT_1 to zero; and thus

$$\tan 0° = 0.$$

When x changes from 0° to 90°, AT passes through AT_1, AT_2, AT_3, AT_4, AT_5 to an unlimited value; and thus

$$\tan 90° = \infty.$$

The student can trace by means of Figs. 45, 46, 47 the further changes in tan x when x increases from 90° to 360°.

The tracing of the changes in *cot x, sec x, csc x,* as x changes from 0° to 360°, is left as an exercise to the student.

The changes for the above six functions when the angle is increasing from 0° *to* 360° *are indicated in the following table:*

Y

In the second quadrant the

sine	decreases from	1	to	0
cosine	decreases from	0	to	-1
tangent	increases from	$-\infty$	to	0
cotangent	decreases from	0	to	$-\infty$
secant	increases from	$-\infty$	to	-1
cosecant	increases from	1	to	∞

In the first quadrant the

sine	increases from	0	to	1
cosine	decreases from	1	to	0
tangent	increases from	0	to	∞
cotangent	decreases from	∞	to	0
secant	increases from	1	to	∞
cosecant	decreases from	∞	to	1

X_1 O X

In the third quadrant the

sine	decreases from	0	to	-1
cosine	increases from	-1	to	0
tangent	increases from	0	to	∞
cotangent	decreases from	∞	to	0
secant	decreases from	-1	to	$-\infty$
cosecant	increases from	$-\infty$	to	-1

In the fourth quadrant the

sine	increases from	-1	to	0
cosine	increases from	0	to	1
tangent	increases from	$-\infty$	to	0
cotangent	decreases from	0	to	$-\infty$
secant	decreases from	∞	to	1
cosecant	decreases from	-1	to	$-\infty$

Y_1

23. Periodicity of the trigonometric functions. When the angles 360° to 720° are described the sine goes through all its changes in the same order as when the angles 0° to 360° are described. Also the sine repeats the same value every time the angle changes by 360°, i.e., 2π. That is, n denoting a whole number

$$\sin(n\cdot 360^\circ + x) = \sin x,$$

i.e.,

$$\sin(2n\pi + x) = \sin x.$$

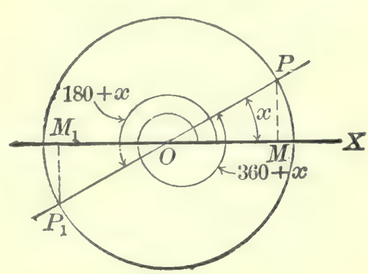


FIG. 50.

Accordingly *the sine* is said to be *a periodic function,* and *its period* is 2π.

Similarly, the *cosine, secant, cosecant have the period* 2π.

When the angles 180° to 360° are described the tangent goes through all its changes in the same order as when the angles 0° to 180° are described. Also the tangent repeats the same values every time the angle changes by 180°, i.e., π. That is, n denoting any whole number,

$$\tan(n \cdot 180° + x) = \tan x,$$

i.e.,

$$\tan(n\pi + x) = \tan x.$$

Accordingly *the tangent is a periodic function*, and *its period is* π. Similarly, *the cotangent has the period* π.

24. Graphs of the functions.

Graph of $\sin x$. From the equation,

$$y = \sin x$$

form corresponding pairs of x and y. Convenient values to take for x are 30°, 45°, 60°, 90°, 120°, ..., i.e., $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \ldots,$ Thus:

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7}{6}\pi$	$\frac{3}{2}\pi$	$\frac{7}{4}\pi$	2π	$\frac{9\pi}{4}$	$-\frac{\pi}{4}$
y	0	.5	.71	.87	1	.87	.71	.5	0	−.5	−1	−.71	0	.71	−.71

Choose any convenient length to represent the angle 2π (360°);* mark the points corresponding to angles $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \ldots$; at these points erect ordinates representing the

* E.g., one such length that may be taken is the length of a unit circle.

corresponding values of y; draw a smooth curve through the ends of these ordinates.

This curve, Fig. 51, is called "*The curve of sines*" or "*the sine curve.*"

A very convenient way to draw the ordinates corresponding to values of x is to take the sines from the unit circle (Art. 21), as indicated in Fig. 52.

Graph of cos x. From the equation

$$y = \cos x$$

form corresponding values of x and y, and proceed as in the case of the sine. The ordinates can also be obtained readily by using a unit circle. The cosine curve is in Fig. 53.

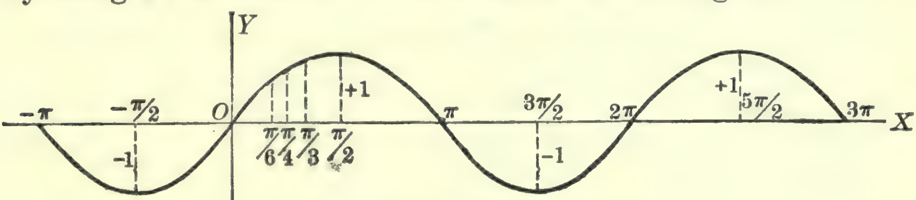


FIG. 51.—Graph of sin x.

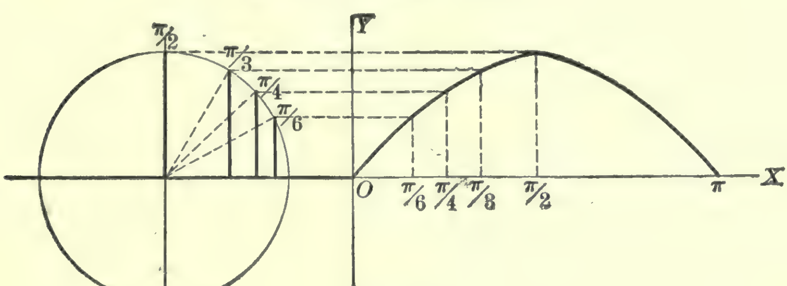


FIG. 52.

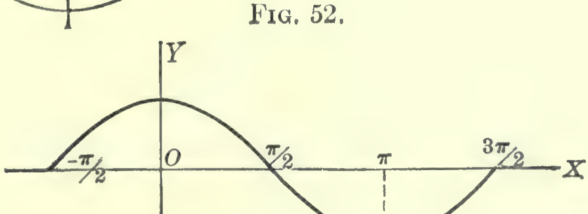


FIG. 53.—Graph of cos x.

Graph of tan x. From the equation

$$y = \tan x$$

form corresponding values of x and y; thus

x	0	30°	45°	60°	90°	120°	135°	180°	210°	270°	300°	....	360°
	·	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2}{3}\pi$	$\frac{3}{4}\pi$	π	$\frac{7}{6}\pi$	$\frac{3\pi}{2}$	$\frac{5}{3}\pi$	....	2π
y	0	.58	1	1.73	∞	−1.73	−1	0	.58	∞	−1.73	....	0

Then, on proceeding as in the case of the sine the graph in Fig. 54 is obtained.

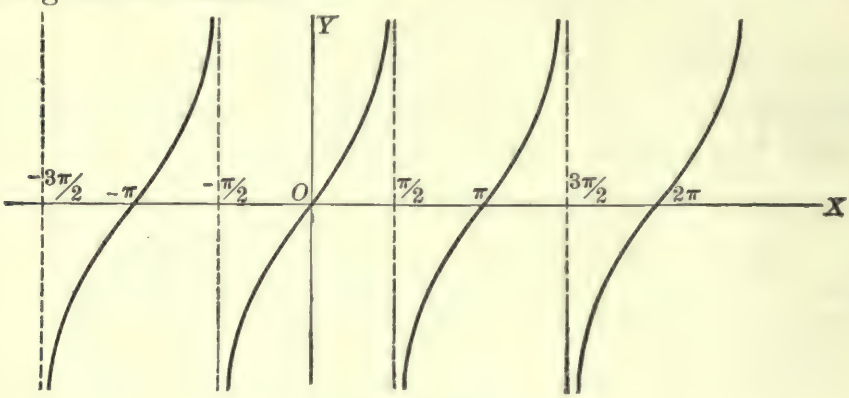


FIG. 54.—Graph of tan x.

By using a unit circle, see Fig. 49, the ordinates can be drawn quickly.

Graphs of cot x, sec x, cosec x. These can be obtained by proceeding as in the case of the preceding graphs. They are shown in Figs. 55, 56, 57.

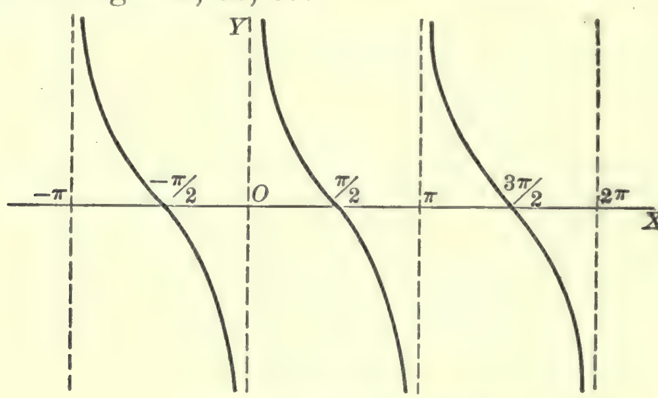


FIG. 55.—Graph of cot x.

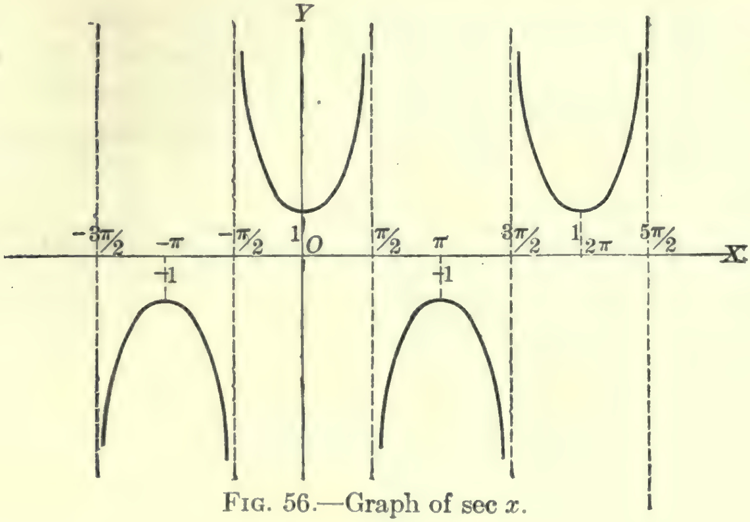


FIG. 56.—Graph of sec x.

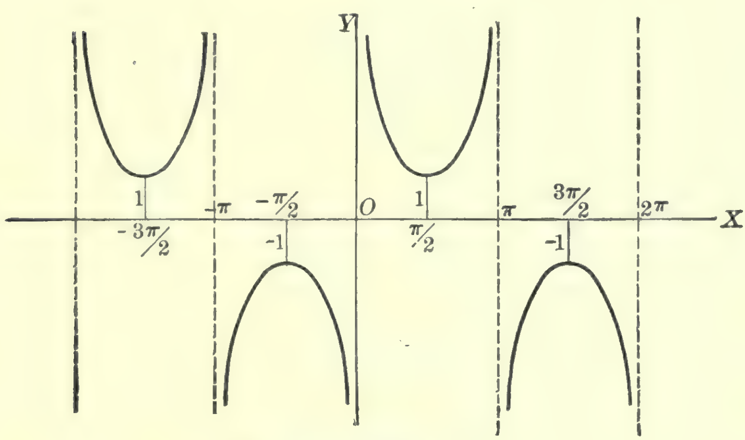


FIG. 57.—Graph of csc x.

EXAMPLES

1. Draw various graphs for $\sin\theta$, $\cos\theta$, $\tan\theta$, by varying the scales used in representing radians and trigonometric functions.

2. Draw graphs of the following:

(*a*) $\sin\theta+\cos\theta$; (*b*) $\sin\theta-\cos\theta$; (*c*) $\sin 2\theta$; (*d*) $\cos 3\theta$;

(*e*) $\sin\frac{\theta}{2}$; (*f*) $\tan\frac{\theta}{2}$.

3. Draw, with the same axes of reference, graphs of $\sin\theta$ and $\cos\theta$; and from your figure obtain values of θ between 0° and 360° for which (1) $\sin\theta=\cos\theta$; (2) $\sin\theta+\cos\theta=0$.

Also with the help of this figure draw the graph of $\sin\theta+\cos\theta$.

25. Relations between the trigonometric functions of an angle. The relations between the trigonometric functions of *an acute angle* were set forth in Art. 7. These relations hold for the ratios of *any angle*.

A. Inspection of the definitions, Art. 20, shows *the reciprocal relations*, namely:

$$\tan A\cot A=1;\quad \cos A\sec A=1;\quad \sin A\csc A=1, \tag{1}$$

i.e., $\cot A=\dfrac{1}{\tan A}$; $\sec A=\dfrac{1}{\cos A}$; $\csc A=\dfrac{1}{\sin A}$.

B. In each of the figures in Art. 20,

$$\tan A=\frac{MP}{OM}=\frac{\dfrac{MP}{OP}}{\dfrac{OM}{OP}}=\frac{\sin A}{\cos A};\quad \cot A=\frac{OM}{MP}=\frac{\dfrac{OM}{OP}}{\dfrac{MP}{OP}}=\frac{\cos A}{\sin A}. \tag{2}$$

C. In each of the figures in Art. 20,

$$\overline{MP}^2+\overline{OM}^2=\overline{OP}^2.$$

On dividing both members of this equation by, $\overline{OP}^2$, $\overline{OM}^2$, $\overline{MP}^2$, in turn, and following the same process as that adopted in Art. 7, it results that

$$\left.\begin{aligned}\sin^2 A+\cos^2 A&=1;\\ \sec^2 A&=1+\tan^2 A;\\ \operatorname{cosec}^2 A&=1+\cot^2 A.\end{aligned}\right\} \tag{3}$$

Relations C also follow directly from the line definitions, Art. 21, as shown in Fig. 44 or Figs. 45–47.

EXAMPLES

1. Given that $\sin A = \frac{3}{4}$; find the other functions of A by means of the relations shown in this article.

[In Ex. 6, Art. 20, this problem is solved *geometrically;* here it is solved *algebraically.*]

$$\cos A = \pm\sqrt{1-\sin^2 A} = \frac{\pm\sqrt{7}}{4}; \quad \sec A = \frac{1}{\cos A} = \frac{4}{\pm\sqrt{7}};$$

$$\operatorname{cosec} A = \frac{1}{\sin A} = \frac{4}{3}; \quad \tan A = \frac{\sin A}{\cos A} = \frac{3}{\pm\sqrt{7}};$$

$$\cot A = \frac{1}{\tan A} = \frac{\pm\sqrt{7}}{3}.$$

Since the given sine is positive, the corresponding angles are in the first and second quadrants. Hence the double values of the calculated ratios may be paired as follows:

$\sin A$	$\cos A$	$\tan A$	$\cot A$	$\sec A$	$\operatorname{cosec} A$
$\frac{3}{4}$	$\frac{+\sqrt{7}}{4}$	$\frac{3}{\sqrt{7}}$	$\frac{\sqrt{7}}{3}$	$\frac{4}{\sqrt{7}}$	$\frac{4}{3}$
$\frac{3}{4}$	$\frac{-\sqrt{7}}{4}$	$-\frac{3}{\sqrt{7}}$	$\frac{-\sqrt{7}}{3}$	$\frac{4}{\sqrt{7}}$	$\frac{4}{3}$

Find the other ratios *algebraically,* and verify the results *geometrically,* when:

2. $\cos A = -\frac{2}{3}$. **3.** $\tan A = \frac{5}{7}$.

4. $\operatorname{cosec} A = -5$. **5.** $\cot A = -3$.

Find the other ratios algebraically, and verify the results geometrically, when angle A satisfies the following *pairs of conditions*:

6. $\sin A = \frac{1}{2}$ and $\tan A$ negative.

7. $\tan A = \sqrt{3}$ and $\sec A$ negative.

8. $\cos A = -\frac{2}{3}$ and $\sin A$ positive.

9. $\sin A = -\frac{3}{5}$ and $\tan A$ positive.

10. If $\sin\theta=\dfrac{m^2+2mn}{m^2+2mn+2n^2}$, prove that

$$\tan\theta=\pm\frac{m^2+2mn}{2mn+2n^2}.$$

11. If $\cos\theta=\dfrac{a^2-b^2}{a^2+b^2}$, find $\sin\theta$ and $\tan\theta$.

Verify each of the following relations by reducing the first member to the second:

12. $\cos x\tan x=\sin x$.

13. $\sec x-\tan x\cdot\sin x=\cos x$.

14. $\dfrac{\cot^2 A}{1+\cot^2 A}=\cos^2 A$.

15. $(1+\tan^2 z)\cos^2 z=1$.

16. $\cos^4\theta-\sin^4\theta+1=2\cos^2\theta$.

17. $\dfrac{1}{\tan^2 B+1}+\dfrac{1}{\cot^2 B+1}=1$.

18. $\sec^4\theta-\tan^4\theta=2\sec^2\theta-1$.

26. Functions of $-A$, $90°\mp A$, $180°\mp A$ in terms of functions of A, A being any angle.

Functions of $-A$. Describe the angles A, $-A$, OP being

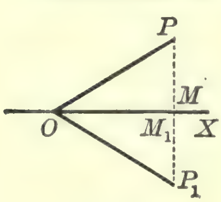


Fig. 58.

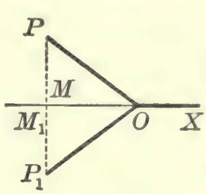


Fig. 59.

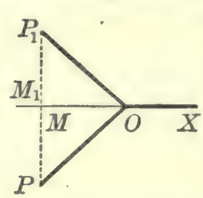


Fig. 60.

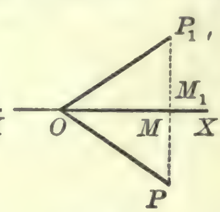


Fig. 61.

the terminal line of A, and OP_1 the terminal line of $-A$. In Figs. 58, 59, 60, 61, A is in the first, second, third, fourth quadrants respectively.

Take $OP_1 = OP$, and draw PM, P_1M_1 at right angles to OX. Then in each figure

$$OM_1 = OM, \qquad M_1P_1 = -MP.$$

Thus, for an angle A in any quadrant,

$$\sin(-A) = \frac{M_1P_1}{OP_1} = -\frac{MP}{OP} = -\sin A;$$

$$\cos(-A) = \frac{OM_1}{OP_1} = \frac{OM}{OP} = \cos A.$$

So also $\tan(-A) = -\tan A$; $\cot(-A) = -\cot A$;

$\sec(-A) = \sec A$; $\csc(-A) = -\csc A$.

Functions of $90° - A$. Describe the angles A, $90° - A$, OP being the terminal line of A, and OP_1 the terminal line of $90° - A$. In Figs. 62, 63, 64, 65, A is in the first, second, third, fourth quadrants respectively.

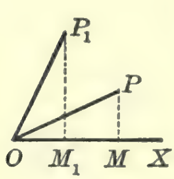


FIG. 62.

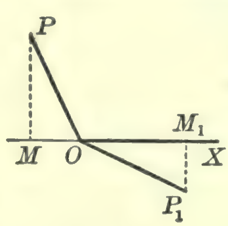


FIG. 63.

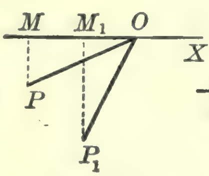


FIG. 64.

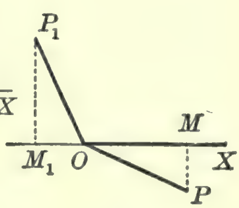


FIG. 65.

Take $OP_1 = OP$, and draw PM, P_1M_1 at right angles to OX. Then in each figure

$$OM_1 = MP, \qquad M_1P_1 = OM.$$

Thus, for A in any quadrant, and so for all angles A,

$$\sin(90° - A) = \frac{M_1P_1}{OP_1} = \frac{OM}{OP} = \cos A;$$

$$\cos(90° - A) = \frac{OM_1}{OP_1} = \frac{MP}{OP} = \sin A;$$

$$\tan (90^\circ - A) = \frac{M_1P_1}{OM_1} = \frac{OM}{MP} = \cot A;$$

$$\cot (90^\circ - A) = \frac{OM_1}{M_1P_1} = \frac{MP}{OM} = \tan A;$$

$$\sec (90^\circ - A) = \frac{OP_1}{OM_1} = \frac{OP}{MP} = \csc A;$$

$$\csc (90^\circ - A) = \frac{O_1P_1}{M_1P_1} = \frac{OP}{OM} = \sec A.$$

Hence, the function of any angle is the same as the co-function of its complement.

Functions of $90^\circ + A$. Describe the angles A, $90^\circ + A$, OP being the terminal line of A and OP_1 the terminal line of $90^\circ + A$. In Figs. 66, 67, 68, 69, A is in the first, second, third, fourth quadrants respectively.

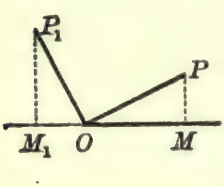


FIG. 66.

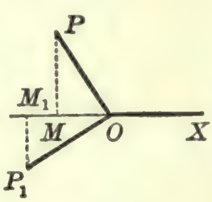


FIG. 67.

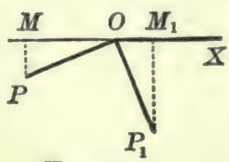


FIG. 68.

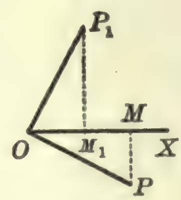


FIG. 69.

Take $OP_1 = OP$, and draw PM, P_1M_1 at right angles to OX. Then in each figure

$$OM_1 = -MP, \qquad M_1P_1 = OM.$$

Hence, for any angle A,

$$\sin (90^\circ + A) = \frac{M_1P_1}{OP} = \frac{OM}{OP} = \cos A;$$

$$\cos (90^\circ + A) = \frac{OM_1}{OP_1} = -\frac{MP}{OP} = -\sin A.$$

So also, $\tan (90^\circ + A) = -\cot A$; $\cot (90^\circ + A) = -\tan A$;

$\sec (90^\circ + A) = -\csc A$; $\csc (90^\circ + A) = \sec A$.

Functions of $180° - A$. Describe the angles A, $180° - A$, OP being the terminal line of A, and OP_1 the terminal line of $180° - A$. In Figs. 70, 71, 72, 73, A is an angle in the first, second, third, fourth quadrants respectively.

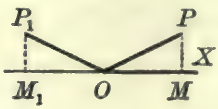


FIG. 70.

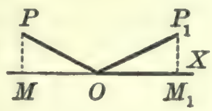


FIG. 71.

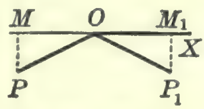


FIG. 72.

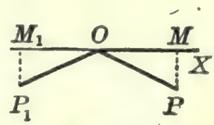


FIG. 73.

Take $OP_1 = OP$, and draw PM, P_1M_1 at right angles to OX. Then in each figure

$$OM_1 = -OM, \qquad M_1P_1 = MP.$$

Hence, for any angle A,

$$\sin(180° - A) = \frac{M_1P_1}{OP_1} = \frac{MP}{OP} = \sin A;$$

$$\cos(180° - A) = \frac{OM_1}{OP_1} = \frac{-OM}{OP} = -\cos A.$$

So also,

$$\tan(180° - A) = -\tan A; \quad \cot(180° - A) = -\cot A;$$

$$\sec(180° - A) = -\sec A; \quad \csc(180° - A) = \csc A.$$

Thus, *a function of the supplement of an angle and the same function of the angle itself are numerically equal; the sines and cosecants respectively of supplementary angles have the same sign, but the cosines and the remaining functions respectively have opposite signs.*

Functions of $180° + A$. On proceeding as in the preceding cases it will be found that

$$\sin(180° + A) = -\sin A, \quad \cos(180° + A) = -\cos A,$$

$$\tan(180° + A) = \tan A, \quad \cot(180° + A) = \cot A,$$

$$\sec(180° + A) = -\sec A, \quad \csc(180° + A) = -\csc A.$$

Since, OX being the initial line, all angles having the same terminal line have the same value for a function, it follows that

any function of $360°+A$, and of $n \cdot 360°+A$ (n=any whole number) is equal to the same function of A.

Also,

any function of $360°-A$, and of $n \cdot 360°-A$ (n=any whole number) is equal to the same function of $-A$.

EXAMPLES

1. Express the functions of $270°-A$ in terms of functions of A.

2. Express the functions of $270°+A$ in terms of functions of A.

27. Reduction of trigonometric functions of any angle to functions of acute angles. By means of the relations in Art. 26 the functions of any angle can be expressed in terms of the functions of an angle between 0° and 90°, and in terms of the functions of an angle between 0° and 45°.

EXAMPLES

1. $\sin 700° = \sin(360°+340°) = \sin 340° = \sin(-20°)$
$= -\sin 20° = -.3420.$

2. $\tan 975° = \tan(2 \cdot 360°+255°) = \tan 255° = \tan(180°+75°)$
$= \tan 75° = \cot 15° = 3.7321.$

3. $\csc(-1160°) = -\csc(1160°) = -\csc(3 \cdot 360°+80°)$
$= -\csc 80° = -\sec 10°.$

Hence

$$\csc(-1160°) = -\sec 10° = -\frac{1}{\cos 10°} = \frac{-1}{.9848} = -1.015.$$

4. Express the following as functions of acute angles:

(*a*) $\sin 287°$.	(*d*) $\sec 925° 10'$.	(*g*) $\tan (-1055°)$.
(*b*) $\cos 332°$.	(*e*) $\sin 2150°$.	(*h*) $\cos (-2055°)$.
(*c*) $\tan 218° 30'$.	(*f*) $\cot (-487°)$.	(*i*) $\csc 310° 30'$.

5. Find the values of the following:

(*a*) $\sin 346° 10'$.	(*d*) $\sec (-310°)$	(*g*) $\csc 876°$.
(*b*) $\cos 231° 30'$.	(*e*) $\cot 950°$.	(*h*) $\cos (-1131°)$.
(*c*) $\tan 174° 15'$.	(*f*) $\sin (-2830°)$	(*i*) $\tan (-1487°)$.

6. Prove the following:

(*a*) $\cos 240° \cos 120° - \sin 120° \cos 150° = 1$.

(*b*) $\tan 675° \sec 540° + \cot 495° \csc 450° = 0$.

CHAPTER IV

GENERAL VALUES. INVERSE TRIGONOMETRIC FUNCTIONS. TRIGONOMETRIC EQUATIONS

28. General values. All angles having the same initial and terminal lines have the same values for each trigonometric function. *The general value* of an angle having a given trigonometric function is an expression, or formula, which includes *all* the angles that have that function.

General expression for all angles having the same sine and cosecant. Let α be an acute angle having a given sine, a, say. Let its terminal line be OP. All angles having OP

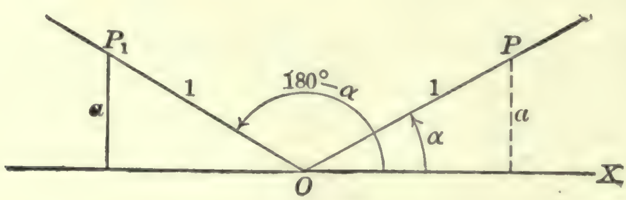


FIG. 74.

for terminal line have the same sine a. These angles are the angles in the expression,

$$m\cdot 360°+\alpha, \text{ i.e., } 2m\cdot 180°+\alpha, \quad (m=\text{any integer}). \qquad (i)$$

Also $$\sin(180°-\alpha)=\sin\alpha=a.$$

In Fig. 74, OP_1 is the terminal line of $180°-\alpha$. All angles having OP_1 for terminal line have the same sine a. They are the angles n the expression,

$$m\cdot 360°+(180°-\alpha), \text{ i.e., } (2m+1)180°-\alpha, \quad (m \text{ an integer}). \qquad (ii)$$

Expressions (*i*) and (*ii*) are both included in the expression,

$$\boldsymbol{n \cdot 180° + (-1)^n \alpha}, \qquad \text{i.e.,} \quad \boldsymbol{n\pi + (-1)^n \alpha}, \qquad n \text{ an integer.}$$

For, if n is even, α has the sign $+$, as in (*i*); if n is odd, α has the sign $-$, as in (*ii*).

Since $\csc \alpha = \dfrac{1}{\sin \alpha}$, this expression includes all angles that have the same cosecant as α.

Otherwise expressed:

$$\sin \alpha = \sin [n \cdot 180° + (-1)^n \alpha] = \sin [n\pi + (-1)^n \alpha];$$

$$\csc \alpha = \csc [n \cdot 180° + (-1)^n \alpha] = \csc [n\pi + (-1)^n \alpha].$$

The proof is similar for α in any quadrant.

Ex. 1. Give the general expression for α if $\sin \alpha = \dfrac{\sqrt{3}}{2}$.

The least positive value of α is 60°.

$\therefore$ The general value of α is $n \cdot 180° + (-1)^n\, 60°$, i.e., $n\pi + (-1)\dfrac{\pi}{3}$.

On giving n the values 0, 1, 2, 3, . . . , particular values of α are obtained, viz., 60°, 120°, 420°, 480°, . . .

General expression for all angles having the same cosine and secant. Let α be an acute angle having a given cosine, a, say. Let its terminal line be OP. All angles having OP for terminal line have the same cosine a. These angles are the angles in the expression,

$$n \cdot 360° + \alpha, \qquad (n = \text{an integer.}) \qquad (iii)$$

Also $\qquad \cos(-\alpha) = \cos \alpha = a.$

In Fig. 75, OP_1 is the terminal line of $-\alpha$. All angles having OP_1 for terminal line have the same cosine a. They are the angles in the expression,

$$n \cdot 360° - \alpha, \qquad (n = \text{an integer.}) \qquad (iv)$$

Expressions (*iii*) and (*iv*) are both included in the expression,

$$\boldsymbol{n \cdot 360° \pm \alpha}, \qquad \text{i.e., } \boldsymbol{2n\pi \pm \alpha}, \qquad (n \text{ an integer.})$$

Since $\sec \alpha = \dfrac{1}{\cos \alpha}$, this expression includes all angles that have the same secant as α.

Otherwise expressed:

$$\cos \alpha = \cos (n \cdot 360° \pm \alpha) = \cos (2n\pi \pm \alpha);$$

$$\sec \alpha = \sec (n \cdot 360° \pm \alpha) = \sec (2n\pi \pm \alpha).$$

The proof is similar for α in any quadrant.

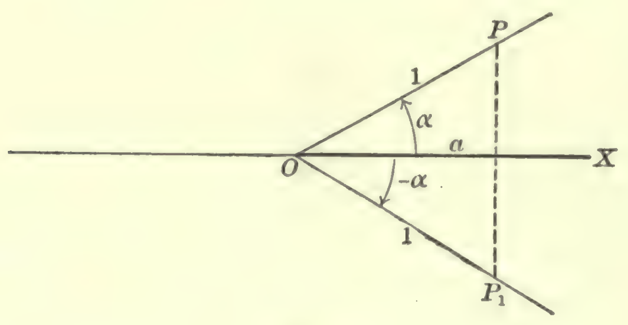


FIG. 75.

Ex. 2. Give the general expression for α when $\cos \alpha = \frac{1}{2}$. The least positive value of α is 60°.

$\therefore$ The general value of α is $n \cdot 360° \pm 60°$, i.e., $2n\pi \pm \frac{\pi}{3}$.

On giving n the values 0, 1, 2, 3, . . . , particular values of α are obtained, viz., $\pm 60°$, 420°, 300°, 780°, 660°, . . .

General expression for all angles having the same tangent and cotangent. Let α (Fig. 76) be an acute angle having a given tangent, a, say. Let its terminal line be OP. All angles having OP for terminal line have the same tangent a. These angles are the angles in the expression,

$$m \cdot 360° + \alpha, \quad \text{i.e.,} \quad 2m \cdot 180° + \alpha, \quad (m \text{ an integer.}) \qquad (v)$$

Also $$\tan(180° + \alpha) = \tan \alpha.$$

In Fig. 76, OP_1 is the terminal line of $180° + \alpha$. All angles having OP_1 for terminal line have the same tangent a. They are the angles in the expression

$$m \cdot 360° + (180° + \alpha) \quad \text{i.e.,} \quad (2m+1)180° + \alpha, \quad (m \text{ an integer}) \quad (vi)$$

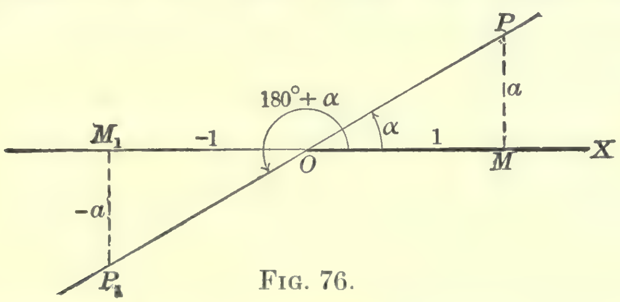


FIG. 76.

Expressions (v) and (vi) are both included in the expression,

$$\boldsymbol{n \cdot 180° + \alpha}, \qquad \text{i.e.,} \quad \boldsymbol{n\pi + \alpha}, \qquad n \text{ an integer.}$$

Since $\cot \alpha = \dfrac{1}{\tan \alpha}$, this expression includes all angles that have the same cotangent as α.

Otherwise expressed:

$$\tan \alpha = \tan(n \cdot 180° + \alpha) = \tan(n\pi + \alpha);$$

$$\cot \alpha = \cot(n \cdot 180° + \alpha) = \cot(n\pi + \alpha).$$

The proof is similar for α in any quadrant.

Ex. 3. Give the general expression for α when $\tan\alpha=\sqrt{3}$. The least positive value of α is 60°.

$\therefore$ The general value of α is $n\cdot 180°+60°$, i.e., $n\pi+\frac{\pi}{3}$.

On giving n the values 0, 1, 2, 3, . . . , particular values of α are obtained, viz., 60°, 240°, 420°, 600°, . . .

EXAMPLES

4. Given $\sin A=\frac{1}{2}$, find the general value of A. Also find the four least positive values of A.

5. Given $\cos A=-\frac{1}{2}$, find the general value of A and the three least positive values of A.

6. Find the general value of θ which satisfies each of the following equations:

(*a*) $\sin\theta=0$.

(*b*) $\cos\theta=0$.

(*c*) $\tan\theta=0$.

(*d*) $\cot\theta=\frac{1}{\sqrt{3}}$.

(*e*) $\sec\theta=1$.

(*f*) $\csc\theta=\sqrt{2}$.

(*g*) $\sin\theta=-\frac{\sqrt{3}}{2}$.

(*h*) $\cos\theta=-\frac{1}{\sqrt{2}}$.

(*i*) $\tan\theta=1$.

(*j*) $\cot\theta=-1$.

(*k*) $\sec\theta=\frac{2}{\sqrt{3}}$.

(*l*) $\tan\theta=-\frac{1}{\sqrt{3}}$.

29. Inverse trigonometric functions. It has been seen that *the sine is a function of the angle.* On the other hand, the value of the angle depends on the value of the sine, and thus *the angle is a function of the sine.* This function of the sine is called *the inverse sine* or *the anti-sine*. Similarly, the angle is a function of each of the other trigonometric functions. Thus there arise other *inverse trigonometric functions* or *anti-trigonometric functions*, as they are sometimes called, viz., *the inverse cosine* or *the anti-cosine*, *the inverse tangent* or *the anti-tangent*, etc.

E.g., since $\sin 30° = \frac{1}{2}$,

30° is an angle whose sine is $\frac{1}{2}$.

This line is also expressed thus:

30° is an inverse sine of $\frac{1}{2}$ (or an anti-sine of $\frac{1}{2}$),

and more briefly, on using a symbol invented for the inverse sine,

$$30° = \sin^{-1} \tfrac{1}{2}.$$

In general: the two statements,

"the sine of an angle θ" is m,

θ is "an angle whose sine is m,"

are briefly expressed,

$$\sin \theta = m,$$

$$\theta = \sin^{-1} m = \text{arc} \sin m.*$$

The expressions,

$$\sin^{-1} m, \ \cos^{-1} m, \ \tan^{-1} m, \ \text{etc.}, \ldots,$$

are the symbols for the inverse trigonometric functions.

The expression $\tan^{-1} 2$, for instance, is read "the inverse

* This notation is the one used on the Continent.

tangent of 2," "the anti-tangent of 2," and may also be read "an angle (or the set of angles) whose tangent is 2."

N.B. *The trigonometric functions* are *ratios,* and thus are *pure numbers; the inverse trigonometric functions* are *angles.*

For instance (see Art. 28, Exs. 1, 2, 3),

$$\sin^{-1}\frac{\sqrt{3}}{2}=n\cdot 180^\circ+(-1)^n\, 60^\circ,\ n \text{ an integer};$$

$$\cos^{-1}\frac{1}{2}=n\cdot 360^\circ \pm 60^\circ,\ n \text{ an integer};$$

$$\tan^{-1}\sqrt{3}=n\cdot 180^\circ+60^\circ,\ n \text{ an integer}.$$

Thus, while each of the trigonometric functions has a *single definite value,* each inverse trigonometric function has *an infinite number of values.* The angle having the smallest *numerical value* in an inverse function is called **the principal value** of the inverse function.

E.g. The principal value of $\tan^{-1}\sqrt{3}$ is 60°;
The principal value of $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ is -45°.

When the smallest numerical value has both positive and negative signs the positive sign is taken; thus
The principal value of $\cos^{-1}\frac{1}{2}$ is 60°.

N.B. It should be noted that the "-1" in the symbol for an inverse trigonometric function *is not an algebraic exponent.* See Art. 5, N.B. in Examples.

Thus: $\sin^{-1} m$ does not denote $(\sin m)^{-1}$, i.e., $\frac{1}{\sin m}$;

$(\tan x)^{-1}$, i.e., $\frac{1}{\tan x}$, should not be written $\tan^{-1} x$.

EXAMPLES

1. Prove that $\sin^{-1} x = \cos^{-1}\sqrt{1-x^2}$.

Let $\theta = \sin^{-1} x$.

Then $\sin\theta = x$.

Hence $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2}$.

Thus $\theta = \cos^{-1}\sqrt{1-x^2}$; i.e., $\sin^{-1} x = \cos^{-1}\sqrt{1-x^2}$.

2. Construct the following:

(*a*) $\sin^{-1}\frac{2}{3}$. (*c*) $\tan^{-1}(-2)$. (*e*) $\sec^{-1} 2$.

(*b*) $\cos^{-1}\left(-\frac{3}{5}\right)$. (*d*) $\cot^{-1}\left(\frac{1}{4}\right)$. (*f*) $\csc^{-1}\left(-\frac{7}{3}\right)$.

Read the following identities, and prove them:

3. $\sin\left(\tan^{-1}\frac{5}{12}\right) = \pm\frac{5}{13}$.

4. $\tan\left(\sin^{-1}\frac{12}{13}\right) = \pm\frac{12}{5}$.

5. $\cot^{-1} 1 = n\pi^* + \frac{\pi}{4}$.

6. $\cos^{-1}\left(-\frac{1}{2}\right) = 2n\pi^* \pm \frac{2\pi}{3}$.

7. $\cos^{-1}\frac{63}{65} = \csc^{-1}\left(\pm\frac{65}{16}\right)$.

8. $\sin^{-1}\frac{5}{13} = \tan^{-1}\left(\pm\frac{5}{12}\right)$.

9. $\sin\left(\cos^{-1}\frac{4}{5}\right) = \tan\left(\sin^{-1}\frac{3}{\sqrt{34}}\right)$.

10. $\tan^{-1} m = \cos^{-1}\frac{1}{\sqrt{1+m^2}}$.

30. Trigonometric equations. Trigonometric identities. *Trigonometric equations* are equations in which one or more trigonometric functions or inverse trigonometric functions are involved. These equations are true only under certain conditions, viz., for certain values of the angles. To solve these equations is to find these values.

---

* n a whole number.

E.g. The equation

$$\tan x = 1.$$

Here $x = \tan^{-1} 1 = 45,^\circ\ 225,^\circ$ etc.

The general, or complete, solution of this equation is

$$x = n \cdot 180^\circ + 45^\circ, \quad \text{i.e.,} \quad n\pi + \frac{\pi}{4}.$$

Trigonometric identities are in the form of equations and are unconditionally true, i.e., are true for all values of the angles involved.

E.g. $\sin^2 x + \cos^2 x = 1,$

and the other relations in Art. 25, (1), (2), (3), are identities.

EXAMPLES

1. Solve the equation

$$\sin^2 x - 2 \cos x + \tfrac{1}{4} = 0.$$

Here $1 - \cos^2 x - 2 \cos x + \tfrac{1}{4} = 0.$

$$\therefore\ 4 \cos^2 x + 8 \cos x - 5 = 0;$$

i.e., $(2 \cos x + 5)\ (2 \cos x - 1) = 0.$

$$\therefore\ 2 \cos x + 5 = 0, \quad \text{or} \quad 2 \cos x - 1 = 0.$$

$$\therefore\ \cos x = -\tfrac{5}{2}, \quad \text{or} \quad \cos x = \tfrac{1}{2}.$$

There is no solution for $\cos x = -\frac{5}{2}$, since the cosine of an angle lies between -1 and $+1$.* From $\cos x = \frac{1}{2}$ comes the solution,

$$x = n \cdot 360^\circ \pm 60^\circ.$$

In solving an equation containing several functions the general method is to reduce the equation to a form in which only one function appears.

* In other words: The solution is impossible.

2. Prove the identity

$$\cos^4 A - \sin^4 A = 1 - 2\sin^2 A.$$

$$\cos^4 A - \sin^4 A = (\cos^2 A + \sin^2 A)(\cos^2 A - \sin^2 A)$$
$$= 1 \cdot (1 - \sin^2 A - \sin^2 A) = 1 - 2\sin^2 A.$$

Solve each of the following equations:

3. $2\cos^2 x + 5\sin x - 4 = 0.$

4. $\sin x + \csc x = 2.$

5. $\sin y + \cos y = \sqrt{2}.$

6. $\sin^2 x = 1.$

7. $2\cos A + \sec A = 3.$

8. $\sin x = \tan^2 x.$

9. $2\sin^2 x + \sqrt{3}\cos x + 1 = 0.$

10. $4\sec^2 B - 7\tan^2 B = 3.$

11. $4\tan x - \cot x = 3.$

12. $\sec^2 y - 5\tan y + 5 = 0.$

13. Given $4\sin^2\theta = 3$, find the values of θ which are between 0° and 500°.

14. Given $\sin x + \cos x \cot x = 2$, find the values of x which are between 0° and 360°.

Prove the following identities:

15. $\sin^3\theta + \cos^3\theta = (\sin\theta + \cos\theta)(1 - \sin\theta\cos\theta).$

16. $\sin x(\cot x + 2)(2\cot x + 1) = 2\operatorname{cosec} x + 5\cos x.$

17. $\cos^6 A + \sin^6 A = 1 - 3\cos^2 A\sin^2 A.$

18. $\cos^6 x + 2\cos^4 x\sin^2 x + \cos^2 x\sin^4 x + \sin^2 x = 1.$

19. $\dfrac{\sec\theta + \csc\theta}{\sec\theta - \csc\theta} = \dfrac{1 + \cot\theta}{1 - \cot\theta} = \dfrac{\tan\theta + 1}{\tan\theta - 1}.$

20. $\tan\theta + \cot\theta = \dfrac{\sec^2\theta + \csc^2\theta}{\sec\theta\csc\theta}.$

CHAPTER V

TRIGONOMETRIC FUNCTIONS OF THE SUM AND DIFFERENCE OF TWO ANGLES

General Formulas

31. To deduce sin $(A+B)$, cos $(A+B)$. N.B. In Arts. 31, 32 the conventions (Arts. 16, 19, 20) regarding the signs of angles and lines are followed. The formulas can be derived, however, for Figs. 77–79 by using the definitions in Art. 3.

Case I. A and B both acute.

In Figs. 77, 78,

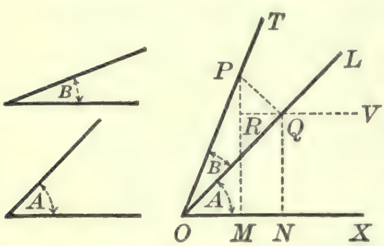


Fig. 77.

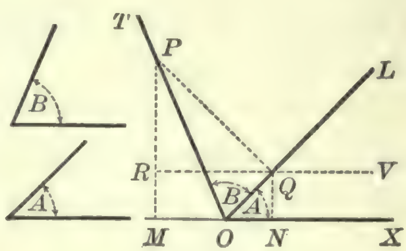


Fig. 78.

$$XOL=A, \quad LOT=B, \quad XOT=A+B.$$

(In Fig. 77, $A+B$ is acute; in Fig. 78, $A+B$ is obtuse.)

From any point P in OT, PM, PQ are drawn at right angles to OX, OL respectively; QN is drawn at right angles to OX; VQR is drawn parallel to OX. Now

$$VQL=XOQ=A.$$

$$\therefore \; VQP=VQL+LQP=90°+A.$$

$$\sin(A+B)=\sin XOP=\frac{MP}{OP}=\frac{NQ+RP}{OP}=\frac{NQ}{OP}+\frac{RP}{OP}.$$

Now, $\dfrac{NQ}{OQ}=\sin A;\qquad \therefore\ NQ=OQ\sin A.$

Also $\dfrac{RP}{QP}=\sin VQP=\sin(90°+A)=\cos A.$

$$\therefore\ RP=QP\cos A.$$

$$\therefore\ \sin(A+B)=\frac{OQ\sin A}{OP}+\frac{QP\cos A}{OP}.$$

But $\dfrac{OQ}{OP}=\cos QOP=\cos B;\quad \dfrac{QP}{OP}=\sin QOP=\sin B.$

$$\therefore\ \mathbf{\sin(A+B)=\sin A\cos B+\cos A\sin B.} \qquad (1)$$

$$\cos(A+B)=\cos XOP=\frac{OM}{OP}=\frac{ON+QR^{*}}{OP}=\frac{ON}{OP}+\frac{QR}{OP}.$$

Now $\dfrac{ON}{OQ}=\cos A;\qquad \therefore\ ON=OQ\cos A$

Also, $\dfrac{QR}{QP}=\cos VQP=\cos(90°+A)=-\sin A.$

$$\therefore\ QR=-QP\sin A.$$

$$\therefore\ \cos(A+B)=\frac{OQ\cos A}{OP}-\frac{QP}{OP}\sin A;$$

$$\therefore\ \mathbf{\cos(A+B)=\cos A\cos B-\sin A\sin B.} \qquad (2)$$

Case II. A and B any angles. First suppose B acute and A obtuse, viz., $90°+A'$, A' thus being acute. Since

$$A=90°+A',$$

$$A'=-(90°-A).$$

---

* Note that ON, QR, are in opposite directions.

$$\therefore \sin A' = -\sin(90° - A)^* = -\cos A;$$
$$\cos A' = \cos(90° - A)^* = \sin A.$$

Then $\sin(A+B) = \sin(90° + A' + B) = \cos(A' + B)^*$
$$= \cos A' \cos B - \sin A' \sin B$$
$$= \sin A \cos B + \cos A \sin B.$$

Also, $\cos(A+B) = \cos(90° + A' + B) = -\sin(A' + B)^*$
$$= -\sin A' \cos B - \cos A' \sin B.$$
$$= \cos A \cos B - \sin A \sin B.$$

The above procedure shows that formulas (1), (2) still hold true when one of the angles A, B, in Case II, is increased by 90°, and that these formulas will continue to hold true as the angles continue to be increased by 90°. *Hence these formulas are true for angles in any quadrants, and thus for all angles.*

Note. Formulas (1), (2) can also be derived from a figure, as in Case I, for A and B in any quadrants.

EXAMPLES

1. Derive $\sin 75° = \sin(30° + 45°) = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$.

2. Derive $\cos 75° = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$.

3. Given $\sin x = \frac{1}{4}$, $\sin y = \frac{2}{5}$, and x and y both acute, find $\sin(x+y)$, $\cos(x+y)$.

4. If $\tan x = \frac{3}{4}$ and $\tan y = \frac{7}{24}$, find $\sin(x+y)$ and $\cos(x+y)$ when x and y are acute angles.

5. Prove $\sin(60° + x) - \cos(30° + x) = \sin x$.

---

* See Art. 26.

6. Prove $\cos(60°+B)+\sin(30°+B)=\cos B$.

7. Prove $\cos(x+y)\cos x+\sin(x+y)\sin x=\cos y$.

8. Find $\sin(\sin^{-1}\frac{1}{2}+\sin^{-1}\frac{1}{3})$ when the angles are between 0° and 90°.

9. Prove $\sin(\sin^{-1} m+\sin^{-1} n)=m\sqrt{1-n^2}\pm n\sqrt{1-m^2}$.

10. Prove $\cos(\sin^{-1} m+\sin^{-1} n)=\sqrt{1-m^2}\sqrt{1-n^2}\pm mn$.

32. To deduce $\sin(A-B)$, $\cos(A-B)$. First method. Formulas (1), (2), Art. 31, are true for all angles. On taking $-B$ instead of B, there results:

$$\sin(A-B)=\sin A\cos(-B)+\cos A\sin(-B);$$

$$\therefore \sin(A-B)=\sin A\cos B-\cos A\sin B. \qquad (1)$$

Also, $\cos(A-B)=\cos A\cos(-B)-\sin A\sin(-B)$.

$$\therefore \cos(A-B)=\cos A\cos B+\sin A\sin B. \qquad (2)$$

Second method. *Case I.* A and B, both acute, $A>B$. In Fig. 79,

$$XOL=A, \quad LOT=-B,$$
$$XOT=A-B.$$

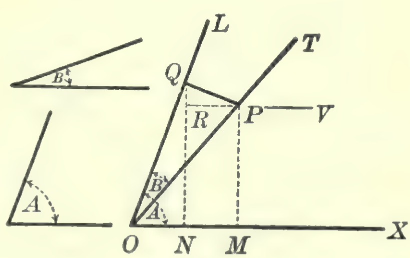


FIG. 79.

From any point P in OT, PM, PQ are drawn at right angles to OX, OL, respectively; QN is at right angles to OX; VPR is drawn parallel to OX. Now

$$RQP=\ 90°-OQN=A;$$

$$\therefore QPR=\ 90°-RQP=90°-A;$$

$$\therefore\ VPQ = 180° - QPR = 90° + A.$$

$$\sin(A-B) = \sin XOP = \frac{MP}{OP} = \frac{NQ-RQ}{OP} = \frac{NQ}{OP} - \frac{RQ}{OP}.$$

But $\dfrac{NQ}{OQ} = \sin A;$ $\qquad \therefore\ NQ = OQ \sin A.$

Also $\dfrac{RQ}{PQ} = \sin VPQ = \sin(90° + A) = \cos A.$

$$\therefore\ RQ = PQ \cos A.$$

$$\therefore\ \sin(A-B) = \frac{OQ \sin A}{OP} - \frac{PQ \cos A}{OP}.$$

But $\dfrac{OQ}{OP} = \cos QOP = \cos(-B) = \cos B;$

$$-\frac{PQ}{OP} = \frac{QP}{OP} = \sin QOP = \sin(-B) = -\sin B;$$

$$\therefore\ \sin(A-B) = \sin A \cos B - \cos A \sin B.$$

$$\cos(A-B) = \cos XOP = \frac{OM}{OP} = \frac{ON+RP}{OP} = \frac{ON}{OP} - \frac{PR^*}{OP}.$$

Now $\dfrac{ON}{OQ} = \cos A;$ $\qquad \therefore\ ON = OQ \cos A.$

Also $\dfrac{PR}{PQ} = \cos VPQ = \cos(90° + A) = -\sin A.$

$$\therefore\ PR = -PQ \sin A.$$

$$\therefore\ \cos(A-B) = \frac{OQ \cos A}{OP} + \frac{PQ \sin A}{OP}.$$

$$\therefore\ \cos(A-B) = \cos A \cos B + \sin A \sin B.$$

---

* Since $RP = -PR$.

Case II. A and B any angles. The methods of proof are similar to those shown in Art. 31, Case II, and Note.

EXAMPLES

1. Derive $\sin 15° = \sin(45° - 30°) = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$.

2. Derive $\cos 15° = \cos(60° - 45°) = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$.

3. If $\cos A = \frac{40}{41}$ and $\cos B = \frac{4}{5}$, find $\sin(A-B)$ and $\cos(A-B)$ when A and B are acute angles.

4. If $\tan x = \frac{3}{4}$ and $\tan y = \frac{7}{24}$, find $\sin(x-y)$ and $\cos(x-y)$ when x and y are acute angles.

5. Find $\sin(\sin^{-1}\frac{1}{2} - \sin^{-1}\frac{1}{3})$ when the angles are acute.

Verify the following identities:

6. $\cos(A+45°) + \sin(A-45°) = 0.$

7. $\cos(30° + x) - \cos(30° - x) = -\sin x.$

8. $\cos(x+y)\cos(x-y) = \cos^2 x - \sin^2 y.$

9. $\sin(x+y)\sin(x-y) = \cos^2 y - \cos^2 x.$

10. $\dfrac{\sin(A \pm B)}{\cos A \cos B} = \tan A \pm \tan B.$

33. Fundamental formulas. Formulas (1), (2), Art. 31, are called *the addition formulas* or *theorems* in trigonometry; formulas (1), (2), Art. 32, are called *the subtraction formulas* or *theorems.* These four formulas are also called *the fundamental formulas of trigonometry.* For convenience they are brought together:

$$\sin(A+B)=\sin A\cos B+\cos A\sin B. \tag{1}$$

$$\sin(A-B)=\sin A\cos B-\cos A\sin B. \tag{2}$$

$$\cos(A+B)=\cos A\cos B-\sin A\sin B. \tag{3}$$

$$\cos(A-B)=\cos A\cos B+\sin A\sin B. \tag{4}$$

In words: Of any two angles,

sine *sum* = sine *first* · cosine *second* + cosine *first* · sine *second*

sine *difference* = sine *first* · cosine *second* − cosine *first* · sine *second*

cosine *sum* = cosine *first* · cosine *second* − sine *first* · sine *second*

cosine *difference* = cosine *first* · cosine *second* + sine *first* · sine *second*

34. To deduce $\tan(A+B)$, $\tan(A-B)$, $\cot(A+B)$, $\cot(A-B)$.

$$\tan(A+B)=\frac{\sin(A+B)}{\cos(A+B)}=\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}.$$

On dividing each term of the numerator and the denominator of the second member by $\cos A\cos B$ there is obtained

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}. \tag{1}$$

In a similar way it can be shown that

$$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}. \tag{2}$$

$$\cot(A+B)=\frac{\cos(A+B)}{\sin(A+B)}=\frac{\cos A\cos B-\sin A\sin B}{\sin A\cos B+\cos A\sin B}.$$

On dividing each term of the numerator and the denominator of the second member by $\sin A\sin B$ there is obtained

$$\cot(A+B)=\frac{\cot A\cot B-1}{\cot B+\cot A}. \tag{3}$$

In a similar way it can be shown that

$$\cot(A-B)=\frac{\cot A\cot B+1}{\cot B-\cot A}. \tag{4}$$

EXAMPLES

1. If $\tan P=2$, $\tan Q=\frac{1}{3}$, show that $\tan(P+Q)=7$, $\tan(P-Q)=1$.

2. Derive $\tan 75°$ from $\tan 45°$ and $\tan 30°$.

3. Derive $\tan 15°$ from $\tan 60°$ and $\tan 45°$.

4. Prove: $\tan(45°+x)=\dfrac{1+\tan x}{1-\tan x}$; $\tan(45°-x)=\dfrac{1-\tan x}{1+\tan x}$.

5. Prove: $\cot(45°+x)=\dfrac{\cot x-1}{\cot x+1}$; $\cot(45°-x)=\dfrac{\cot x+1}{\cot x-1}$.

6. If $\tan A=\frac{1}{2}$, $\tan B=\frac{1}{3}$, find $\tan(A+B)$ and $\tan(A-B)$.

7. **Show that $\tan^{-1} m+\tan^{-1} n=\tan^{-1}\dfrac{m+n}{1-mn}$.**

Let $\qquad x=\tan^{-1} m,\quad$ and $\quad y=\tan^{-1} n.$

Then $\qquad \tan x=m,\qquad \tan y=n.$

Now $\qquad \tan(x+y)=\dfrac{\tan x+\tan y}{1-\tan x\tan y}=\dfrac{m+n}{1-mn}.$

$\therefore\ x+y=\tan^{-1}\dfrac{m+n}{1-mn}$; i.e., $\tan^{-1} m+\tan^{-1} n=\tan^{-1}\dfrac{m+n}{1-mn}$.

8. **Show that $\tan^{-1} m-\tan^{-1} n=\tan^{-1}\dfrac{m-n}{1+mn}$.**

9. Find $\tan^{-1} 7+\tan^{-1} 3$, and $\tan^{-1} 7-\tan^{-1} 3$.

10. Find $\tan^{-1} 2+\tan^{-1} \cdot 5$, and $\cot^{-1} 2+\cot^{-1} \cdot 5$.

35. To deduce $\sin 2A$, $\cos 2A$, $\tan 2A$. On putting $B=A$ in formulas (1), (3), Art. 33, there results:

$$\sin 2A = 2 \sin A \cos A; \tag{1}$$

$$\cos 2A = \cos^2 A - \sin^2 A; \tag{2}$$

i.e.,
$$= 1 - 2 \sin^2 A^*; \tag{3}$$

i.e.,
$$= 2 \cos^2 A - 1.^* \tag{4}$$

On putting $B = A$ in formula (1), Art. 34:

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}. \tag{5}$$

On putting $2A = x$, then $A = \frac{1}{2}x$; and these formulas are expressed:

$$\sin x = 2 \sin \tfrac{1}{2}x \cdot \cos \tfrac{1}{2}x,$$

$$\cos x = \cos^2 \tfrac{1}{2}x - \sin^2 \tfrac{1}{2}x,$$

etc.

In words:

sine any *angle* = 2 sine *half-angle* · cosine *half-angle*,

cosine any *angle* = (cosine *half-angle*)^2 − (sine *half-angle*)^2,

= 1 − 2(sine *half-angle*)^2,

= 2(cosine *half-angle*)^2 − 1.

$$\text{tangent any } angle = \frac{2 \text{ tangent } half\text{-}angle}{1 - (\text{tangent } half\text{-}angle)^2}.$$

EXAMPLES

1. Find $\cos 22\frac{1}{2}°$ from $\cos 45°$.

$$2 \cos^2 22\tfrac{1}{2}° = 1 + \cos 45°.$$

$$\therefore \cos^2 22\tfrac{1}{2}° = \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right) = \frac{1+\sqrt{2}}{2\sqrt{2}} = \frac{1+1.4142}{2 \times 1.4142} = .8536;$$

$$\therefore \cos 22\tfrac{1}{2}° = .9239.$$

* Since $\cos^2 A + \sin^2 A = 1$.

2. (*a*) Deduce $\sin 22\frac{1}{2}°$ from $\cos 45°$.

(*b*) Deduce $\tan 22\frac{1}{2}°$ from $\tan 45°$.

(*c*) From functions of 180° derive $\sin 90°$, $\cos 90°$, $\tan 90°$.

3. Derive $\cot 2A = \dfrac{\cot^2 A - 1}{2 \cot A}$.

4. Express $\cos 3x$ and $\sin 3x$ in terms of functions of $\dfrac{3}{2}x$.

5. Express $\cos 3x$ and $\sin 3x$ in terms of functions of $6x$.

6. Express $\cos 6x$ and $\sin 6x$ in terms of functions of $3x$.

7. Derive the following:

$$\sin \tfrac{1}{2}x = \sqrt{\frac{1 - \cos x}{2}}; \qquad \cos \tfrac{1}{2}x = \sqrt{\frac{1 + \cos x}{2}};$$

$$\tan \tfrac{1}{2}x = \sqrt{\frac{1 - \cos x}{1 + \cos x}}.$$

8. Derive the following:

$$\sin 3A = 3 \sin A - 4 \sin^3 A;$$

$$\cos 3A = 4 \cos^3 A - 3 \cos A;$$

$$\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}.$$

[Suggestion: $\sin 3A = \sin (2A + A)$

$= \sin 2A \cos A + \cos 2A \sin A =$ etc.]

9. Verify the following identities:

(1) $\cot A - \cot 2A = \operatorname{cosec} 2A$. (2) $1 + \tan 2A \tan A = \sec 2A$.

(3) $\left(\sin \dfrac{A}{2} \pm \cos \dfrac{A}{2}\right)^2 = 1 \pm \sin A$. (4) $\dfrac{\sin 2A}{1 + \cos 2A} = \tan A$.

(5) $\dfrac{1-\cos 2A}{\sin 2A}=\tan A.$

(6) $\dfrac{1+\cos A}{\sin A}=\cot \dfrac{A}{2}.$

(7) $\dfrac{2 \tan A}{1+\tan^2 A}=\sin 2A.$

(8) $\dfrac{2-\sec^2 A}{\sec^2 A}=\cos 2A.$

(9) $\cos^4 \theta-\sin 4\theta=\cos 2\theta.$

(10) $\cot \theta-\tan \theta =2 \cot 2\theta.$

(11) $\dfrac{\sin 2\theta}{\sin \theta}-\dfrac{\cos 2\theta}{\cos \theta}=\sec \theta.$

(12) $\dfrac{\sin 3x}{\sin x}-\dfrac{\cos 3x}{\cos x}=2.$

(13) $\cos A=\dfrac{1-\tan^2 \dfrac{A}{2}}{1+\tan^2 \dfrac{A}{2}}.$

(14) $\tan \dfrac{x}{2}=\dfrac{\sin x}{1+\cos x}=\dfrac{1-\cos x}{\sin x}.$

(15) $\tan 4x=\dfrac{4 \tan x-4 \tan^3 x}{1-6 \tan^2 x+\tan^4 x}.$

(16) $\sin 4x=4 \sin x \cos x-8 \sin^3 x \cos x$
$=8 \cos^3 x \sin x-4 \cos x \sin x.$

(17) $\cos 4x=1-8 \sin^2 x+8 \sin^4 x=1-8 \cos^2 x+8 \cos^4 x.$

10. (*a*) If $\sin A=\dfrac{4}{5}$, calculate $\cos A$, $\sin 2A$, $\cos 2A$, $\tan 2A$.

(*b*) If $\cos 2A=\dfrac{2}{3}$, prove $\tan A=\dfrac{1}{5}\sqrt{5}$.

(*c*) If $\tan A=\dfrac{1}{3}$, prove $\cos 2A= +\dfrac{4}{5}$.

36. Transformation formulas. From formulas (1)–(4), Art. 33, there follow, on making the add tions and subtractions indicated:

$$\sin (A+B)+\sin (A-B)= 2 \sin A \cos B. \qquad (1)$$

$$\sin (A+B)-\sin (A-B)= 2 \cos A \sin B. \qquad (2)$$

$$\cos (A+B)+\cos (A-B)= 2 \cos A \cos B. \qquad (3)$$

$$\cos (A+B)-\cos (A-B)= -2 \sin A \sin B. \qquad (4)$$

By this set of formulas *products* of sines and cosines can be transformed into *sums* and *differences*. Thus:

In words (reading the second members first): Of any two angles,

2 *sin* one · *cos* the other = *sin* sum + *sin* difference,* (1′)

2 *cos* one · *sin* the other = *sin* sum − *sin* difference, (2′)

2 *cos* one · *cos* the other = *cos* difference + *cos* sum, (3′)

2 *sin* one · *sin* the other = *cos* differenee − *cos* sum. (4′)

On putting

$$A+B=P,$$

$$A-B=Q,$$

and solving for A and B,

$$A=\tfrac{1}{2}(P+Q), \qquad B=\tfrac{1}{2}(P-Q).$$

Formulas (1)–(4) then take the forms:

$$\sin P+\sin Q= \ \ 2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2}. \tag{5}$$

$$\sin P-\sin Q= \ \ 2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}. \tag{6}$$

$$\cos P+\cos Q= \ \ 2\cos\frac{P+Q}{2}\cos\frac{P-Q}{2}. \tag{7}$$

$$\cos P-\cos Q= -2\sin\frac{P+Q}{2}\sin\frac{P-Q}{2}. \tag{8}$$

By this set of formulas *sums* and *differences* of sines and cosines can be transformed into products.

In words: Of any two angles,

the *sum of the sines* = 2 *sin* half sum · *cos* half difference,* (5′)

the *difference of the sines* = 2 *cos* half sum · *sin* half difference, (6′)

* The difference is taken: first angle − the second.

the *sum of the cosines* $=2$ *cos* half sum $\cdot$ *cos* half difference, (7′)

the *difference of the cosines* $=-2$ *sin* half sum $\cdot$ *sin* half difference. (8′)

EXAMPLES

1. Show that $\dfrac{\cos x-\cos y}{\cos x+\cos y}=-\tan\frac{1}{2}(x+y)\tan\frac{1}{2}(x-y)$.

$$\frac{\cos x-\cos y}{\cos x+\cos y}=\frac{-2\sin\frac{1}{2}(x+y)\sin\frac{1}{2}(x-y)}{2\cos\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y)}$$
$$=-\tan\tfrac{1}{2}(x+y)\tan\tfrac{1}{2}(x-y).$$

2. Show that $2\sin(A+45^\circ)\sin(A-45^\circ)=\sin^2 A-\cos^2 A$.

$2\sin(A+45^\circ)\sin(A-45^\circ)$

$$=\cos(\overline{A+45^\circ}-\overline{A-45^\circ})-\cos(\overline{A+45^\circ}+\overline{A-45^\circ}),$$
$$=\cos 90^\circ-\cos 2A=\sin^2 A-\cos^2 A.$$

3. Show that $\dfrac{\sin A+\sin 3A}{\cos A+\cos 3A}=\tan 2A$.

$$\frac{\sin A+\sin 3A}{\cos A+\cos 3A}=\frac{2\sin\frac{1}{2}(3A+A)\cos\frac{1}{2}(3A-A)}{2\cos\frac{1}{2}(3A+A)\cos\frac{1}{2}(3A-A)},$$
$$=\frac{\sin 2A}{\cos 2A}=\tan 2A.$$

4. Solve the equation $\sin 5\theta+\sin\theta=\sin 3\theta$.

$\therefore\ 2\sin 3\theta\cos 2\theta=\sin 3\theta.\qquad \therefore\ \sin 3\theta(2\cos 2\theta-1)=0.$

$$\therefore\ (a)\ \sin 3\theta=0;\qquad (b)\ 2\cos 2\theta-1=0.$$

From (a), $3\theta=0^\circ$, 180°, etc.; the general value of 3θ is $n\pi$ (n being any integer).

$\therefore\ \theta=0^\circ$, 60°, etc.; the general value of θ is $\dfrac{n\pi}{3}$.

From (b), $\cos 2\theta=\frac{1}{2}$.

$$\therefore\ 2\theta=\pm 60^\circ,\text{ etc.; its general value is } 2n\pi\pm\frac{\pi}{3}.$$

$$\therefore\ \theta=\pm 30^\circ,\text{ etc.; its general value is } n\pi\pm\frac{\pi}{6}.$$

5. Transform each of the following sums and differences into a product:

(1) $\sin 3x+\sin 5x$.
(2) $\sin 7A-\sin 5A$.
(3) $\cos 2x-\cos 6x$.
(4) $\cos 5x+\cos 9x$.
(5) $\sin mA+\sin nB$.
(6) $\cos mx-\cos ny$.
(7) $\sin 3x+\cos 5x$.
(8) $\cos 4x-\sin 2x$.

6. Transform each of the following products into a sum or a difference:

(1) $\sin 5x \cos 3x$.
(2) $\cos 7x \sin 5x$.
(3) $\sin 2x \sin 6x$.
(4) $\cos 5x \cos 9x$.
(5) $\sin mA \sin nB$.
(6) $\cos nx \cos my$.
(7) $\sin 4x \sin 2x$.
(8) $\cos 7x \cos 3x$.

7. Show that the value of

$$\sin (n+1)B \sin (n-1)B+\cos (n+1)B \cos (n-1)B$$

is independent of n.

Verify the following identities:

8. (*a*) $\sin (n+1)A+\sin (n-1)A=2 \sin nA \cos A$.

(*b*) $\cos (n+1)A+\cos (n-1)A=2 \cos nA \cos A$.

9. $\cos (A+B) \cos A+\sin (A+B) \sin A=\cos B$.

10. $\csc 2A+\cot 2A=\cot A$.

11. $\sin 5A \sin A=\sin^2 3A-\sin^2 2A$.

12. $\dfrac{\sin 3x-\sin x}{\cos 3x+\cos x}=\tan x$.

13. $\dfrac{\sin A+\sin B}{\sin A-\sin B}=\dfrac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}$.

14. $\dfrac{\sin (x+y)}{\sin (x-y)}=\dfrac{\tan x+\tan y}{\tan x-\tan y}$.

15. $\dfrac{\cos (x+y)}{\cos (x-y)}=\dfrac{1-\tan x \tan y}{1+\tan x \tan y}$.

16. $\sin 3x+\sin 5x=8 \sin x \cos^2 x \cos 2x$.

17. $\cos 20^\circ \cos 40^\circ \cos 80^\circ=.125$ (without use of tables).

18. Given $\tan A = \frac{1}{2}$, $\tan B = \frac{1}{5}$, $\tan C = \frac{1}{8}$, find $\tan (A+B+C)$.

Prove the following:

19. $\sin\left(2\tan^{-1}\frac{9}{40}\right) = \frac{720}{1681}$.

20. $\sin\left[\frac{1}{2}\sin^{-1}\left(-\frac{24}{25}\right)\right] = \pm\frac{3}{5}$ and $\pm\frac{4}{5}$.

21. $3\sin^{-1} x = \sin^{-1}(3x - 4x^3)$.

22. $3\cos^{-1} x = \cos^{-1}(4x^3 - 3x)$.

23. $\sec^{-1} 3 = 2\cot^{-1}\sqrt{2}$.

24. $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\frac{x+y+z-xyz}{1-xy-yz-zx}$.

25. $\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{16}{65} = \frac{\pi}{2}$.

26. $\tan^{-1}\frac{a-b}{1+ab} + \tan^{-1}\frac{b-c}{1+bc} + \tan^{-1}\frac{c-a}{1+ca} = 0$.

27. If $\tan\frac{A}{2} = \frac{a}{b}$, show that $\sin A = \frac{2ab}{a^2+b^2}$, $\sin 2A = \frac{\pm 4ab(a^2-b^2)}{(a^2+b^2)^2}$.

Solve the following equations:

28. $\cos\theta - \cos 7\theta = \sin 4\theta$.

29. $\sin 2\theta + \sin 4\theta = \sqrt{2}\cos\theta$.

30. $\sin 2\alpha + \cos 2\alpha = 1$.

31. $\sin 2\alpha + 2\cos 2\alpha = 1$.

32. $\sin 2\theta + 2\sin 4\theta + \sin 6\theta = 0$.

33. $4\sin\theta\cos 2\theta = 1$.

34. $\tan^{-1} 2x + \tan^{-1} 3x = \frac{3\pi}{4}$.

35. $\cos^{-1} x - \sin^{-1} x = \cos^{-1} x\sqrt{3}$.

36. $\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}\frac{8}{31}$.

37. Find $\tan(A+B+C)$ in terms of $\tan A$, $\tan B$, $\tan C$.

Thence show:

(*a*) If $A+B+C = 90°$,

$$\tan A \tan B + \tan B \tan C + \tan C \tan A = 1;$$

(b) If $A+B+C=180°$,

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

38. Prove the following, given that $A+B+C=180°$:

(a) $\cos A + \cos B + \cos C = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$;

(b) $\sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$.

CHAPTER VI

RELATIONS BETWEEN THE SIDES AND ANGLES OF A TRIANGLE

37. Notation. Simple geometrical relations.

Notation:

In stating and deriving the relations in Arts. 37–41 the triangle is denoted as ABC, and the sides opposite the angles A, B, C, as a, b, c, rsepectively.

Simple geometrical relations:

(*a*) $A+B+C=180°$.

(*b*) The greater side is opposite the greater angle, and conversely.

38. The law of sines. From C in the triangle ABC draw CD at right angles to opposite side AB, and meeting AB or AB produced in D. (In Fig. 80 B is acute, in Fig. 81

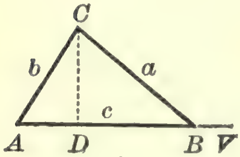


Fig. 80.

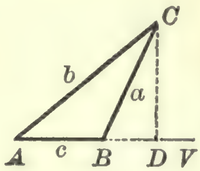


Fig. 81.

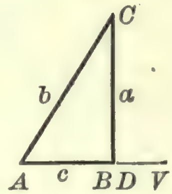


Fig. 82.

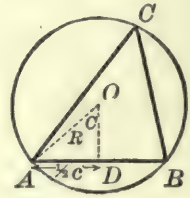


Fig. 83.

B is obtuse, and in Fig. 82 B is a right angle.) Produce AB to V. In what follows, AB is taken as the positive direction.

In CDA, $$DC = b \sin A.$$

In CDB (Figs. 80, 81),

$$DC = a \sin VBC$$
$$= a \sin B.$$

In Fig. 82, $$DC = BC = a = a \sin B.*$$

Therefore, in all three triangles,

$$a \sin B = b \sin A.$$

Hence, $$\frac{a}{\sin A} = \frac{b}{\sin B}, \quad \text{and} \quad \frac{a}{b} = \frac{\sin A}{\sin B}. \tag{1}$$

Similarly, on drawing a line from B at right angles to AC, it can be shown that

$$\frac{a}{\sin A} = \frac{c}{\sin C}, \quad \text{and} \quad \frac{a}{c} = \frac{\sin A}{\sin C}. \tag{2}$$

Hence, in any triangle ABC,

$$\boldsymbol{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}}. \tag{3}$$

In words: *The sides of any triangle are proportional to the sines of the opposite angles.*

The circle described about $\boldsymbol{ABC}$. *Each fraction in* (3) *gives the diameter of this circle.* Let O (Fig. 83) be its centre and R its radius. Draw OD at right angles to any side, say AB. Then

$$AD = \tfrac{1}{2}c, \qquad AOD = C,$$

and $$AD = AO \sin AOD; \quad \text{i.e.,} \quad \tfrac{1}{2}c = R \sin C.$$

Hence $$\boldsymbol{2R = \frac{c}{\sin C}}. \tag{4}$$

---

* $\because$ $B = 90°$, and $\sin 90° = 1$.

39. The law of cosines. The angle A is acute in Fig. 84, obtuse in Fig. 85, right in Fig. 86. From C draw CD at right angles to AB. The direction AB is taken as positive.

In Figs. 84, 85,

$$\overline{BC}^2 = \overline{DC}^2 + \overline{DB}^2.$$

In Fig. 84, $\quad DB = AB - AD;$

in Fig. 85, $\quad DB = DA + AB = -AD + AB.$

Hence, in both figures,

$$\begin{aligned}\overline{BC}^2 &= \overline{DC}^2 + (AB - AD)^2 \\ &= \overline{DC}^2 + \overline{AD}^2 + \overline{AB}^2 - 2AB \cdot AD.\end{aligned}$$

In Fig. 84, $\quad AD = AC \cos BAC;$

in Fig. 85, $\quad AD = AC \cos BAC$ (Art. 20).

Also,

$$\overline{DC}^2 + \overline{AD}^2 = \overline{AC}^2.$$

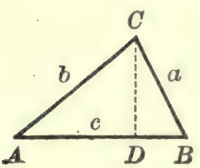


FIG. 84.

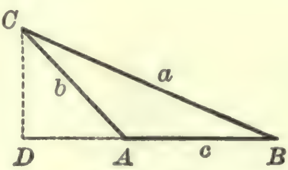


FIG. 85.

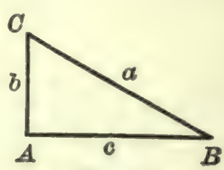


FIG. 86.

Hence, in both figures,

$$\overline{BC}^2 = \overline{AC}^2 + \overline{AB}^2 - 2AC \cdot AB \cos A;$$

that is,
$$\boldsymbol{a^2 = b^2 + c^2 - 2bc \cos A.} \tag{1}$$

This formula also holds for Fig. 86; for there,

$$\cos A = \cos 90° = 0.$$

Similar formulas for b, c, can be derived in like manner, or can be obtained from (1) by symmetry, viz.:

$$b^2 = c^2 + a^2 - 2ca \cos B, \quad c^2 = a^2 + b^2 - 2ab \cos C.$$

In words: *In any triangle, the square of any side is equal to the sum of the squares of the other two sides minus twice the product of these two sides multiplied by the cosine of their included angle.*

Relation (1) may be expressed as follows:

$$\boldsymbol{\cos A = \frac{b^2 + c^2 - a^2}{2bc}}. \tag{2}$$

Similarly:

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca}, \qquad \cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

40. The law of tangents. In any triangle ABC, for any two sides, say a, b,

$$\frac{a}{b} = \frac{\sin A}{\sin B}.$$

From this, on composition and division,

$$\frac{a-b}{a+b} = \frac{\sin A - \sin B}{\sin A + \sin B}$$

$$= \frac{2 \cos \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B)}{2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)}. \qquad \text{[Art. 36, (5), (6).]}$$

$$\therefore \boldsymbol{\frac{a-b}{a+b} = \frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)}}. \qquad \text{[Art. 25, A, B.]} \tag{1}$$

In words: *The difference of two sides of a triangle is to their sum as the tangent of half the difference of the opposite angles is to the tangent of half their sum.*

Now $A + B = 180° - C$. $\qquad \therefore \frac{1}{2}(A+B) = 90° - \frac{C}{2}$.

$$\therefore \tan \frac{1}{2}(A+B) = \tan \left(90° - \frac{C}{2}\right) = \cot \frac{C}{2}.$$

Hence, relation (1) may be expressed:

$$\tan \tfrac{1}{2}(A-B)=\frac{a-b}{a+b}\cot \tfrac{1}{2}C.$$

41. Functions of the half-angles of a triangle in terms of its sides. Let s denote half the sum of the sides of the triangle. Then

$$2s=a+b+c,$$

and $$2s-2a=2(s-a)=-a+b+c.$$

Similarly, $$2(s-b)=a-b+c,$$

$$2(s-c)=a+b-c.$$

By Art. 39 (2), $$\cos A=\frac{b^2+c^2-a^2}{2bc}.$$

By Art. 35, $$2\sin^2 \tfrac{1}{2}A=1-\cos A,$$

$$2\cos^2 \tfrac{1}{2}A=1+\cos A.$$

$$\therefore\ 2\sin^2 \tfrac{1}{2}A=1-\frac{b^2+c^2-a^2}{2bc}=\frac{2bc-b^2-c^2+a^2}{2bc}=\frac{a^2-(b-c)^2}{2bc}=\frac{(a-b+c)(a+b-c)}{2bc}=\frac{2(s-b)\cdot 2(s-c)}{2bc}.$$

$$\therefore\ 2\cos^2 \tfrac{1}{2}A=1+\frac{b^2+c^2-a^2}{2bc}=\frac{2bc+b^2+c^2-a^2}{2bc}=\frac{(b+c)^2-a^2}{2bc}=\frac{(b+c+a)(b+c-a)}{2bc}=\frac{2s\cdot 2(s-a)}{2bc}.$$

$$\therefore\ \sin^2 \tfrac{1}{2}A=\frac{(s-b)(s-c)}{bc}; \tag{1}$$

$$\cos^2 \tfrac{1}{2}A=\frac{s(s-a)}{bc}. \tag{2}$$

Since $\tan^2 \frac{1}{2}A=\sin^2 \frac{1}{2}A\div\cos^2 \frac{1}{2}A$, it follows that

$$\tan^2 \tfrac{1}{2}A = \frac{(s-b)(s-c)}{s(s-a)}. \tag{3}$$

$$\therefore \quad \sin \tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{bc}}; \quad \cos \tfrac{1}{2}A = \sqrt{\frac{s(s-a)}{bc}}; \tag{4}$$

$$\tan \tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}. \tag{5}$$

Similar formulas hold for $\frac{1}{2}B$, $\frac{1}{2}C$, viz.:

$$\sin^2 \tfrac{1}{2}B = \frac{(s-a)(s-c)}{ac}; \qquad \cos^2 \tfrac{1}{2}B = \frac{s(s-b)}{ac};$$

$$\sin^2 \tfrac{1}{2}C = \frac{(s-a)(s-b)}{ab}. \qquad \cos^2 \tfrac{1}{2}C = \frac{s(s-c)}{ab}.$$

$$\tan^2 \tfrac{1}{2}B = \frac{(s-a)(s-c)}{s(s-b)};$$

$$\tan^2 \tfrac{1}{2}C = \frac{(s-a)(s-b)}{s(s-c)}.$$

Formula (5) can be given a more symmetrical form. For, on multiplying the numerator and denominator in the second member of (3) by $(s-a)$,

$$\tan^2 \tfrac{1}{2}A = \frac{(s-a)(s-b)(s-c)}{s(s-a)^2};$$

whence
$$\tan \tfrac{1}{2}A = \frac{1}{s-a}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}. \tag{6}$$

If
$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}, \tag{7}$$

then
$$\tan \tfrac{1}{2}A = \frac{r}{s-a}. \tag{8}$$

Similarly, $\tan \frac{1}{2}B = \frac{r}{s-b}$, $\tan \frac{1}{2}C = \frac{}{s-c}$.

The circle inscribed in *ABC*. *The r in the formulas above is equal to the radius of this circle.* In Fig. 87 O is the centre of the circle, and L, M, N, its points of contact with the sides. Let r denote the radius.

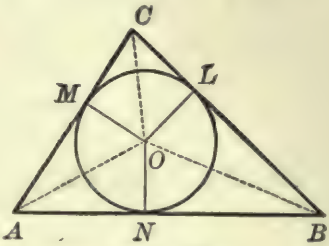


FIG. 87.

By geometry,

$$AN = MA, \quad BL = NB, \quad CM = LC.$$

$$\therefore\ AN + BL + LC = s, \text{ i.e., } AN + a = s.$$

$$\therefore\ AN = s - a.$$

Now $$\tan \tfrac{1}{2}A = \tan NAO = \frac{NO}{AN} = \frac{r}{s-a}. \tag{9}$$

Comparison of (8) and (9) shows that the r in (8) must have the same value as the r in (9). Accordingly,

$$\textbf{radius } r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

EXAMPLES

1. Prove: (*a*) with a figure, (*b*) without a figure, that in any triangle ABC,

$$a = b \cos C + c \cos B;$$

and write corresponding formulas for b and c.

[Suggestion: Draw the perpendicular to BC or BC produced.]

2. If the sines of the angles of a triangle are in the ratios of 13:14:15, prove that the cosines are in the ratios 39:33:25.

3. In ABC, if $a:b:c = 8:7:5$, find the angles.

4. The sides of a triangle are proportional to the numbers 4, 5, 6; find the least angle.

5. Prove that $a \cos B - b \cos A = \dfrac{a^2 - b^2}{c}$.

CHAPTER VII

SOLUTION OF OBLIQUE TRIANGLES

42. Cases for solution. General remarks on methods of solution. In order that a triangle may be constructed, three elements, one of which must be a side, are required. Hence, there are **four cases** for construction and solution, namely, when the given parts are as follows:

I. **One side** and **two angles.**

II. **Two sides** and **the angle opposite to one of them.**

III. **Two sides** and **their included angle.**

IV. **Three sides.**

Careful attention should now be paid to the remarks in Art. 8 on the methods of solution of triangles and to the general suggestions made there for solving problems and checking results. These remarks and suggestions apply also to the problems in this chapter.

Oblique triangles can be solved (by computation) in the following ways:

(*a*) By dividing them conveniently into right-angled triangles, solving these triangles, and combining the results.

This method is not discussed here, but is left as an exercise for the student. Full details are in Murray, *Plane Trigonometry*, Art. 34.

(*b*) By means of certain relations in Chap. VI, logarithms not being used.

(*c*) By means of certain relations in Chap. VI, logarithms being used.

Cases I–IV are solved in manner (*b*) in Arts. 43–46; these cases are solved in manner (*c*) in Arts. 48–50.

43. Case I. Given one side and two angles. In triangle *ABC*, suppose that A, B, a are known; it is required to find C, b, c. In this case (see Fig. 80, Art. 38),

$$C=180^\circ-(A+B);$$

$$\frac{b}{\sin B}=\frac{a}{\sin A}, \quad \text{whence} \quad b=\frac{a}{\sin A}\cdot\sin B;$$

$$\frac{c}{\sin C}=\frac{a}{\sin A}, \quad \text{whence} \quad c=\frac{a}{\sin A}\cdot\sin C.$$

Checks: $a^2=b^2+c^2-2\,bc\cos A$; $\dfrac{b}{\sin B}=\dfrac{c}{\sin A}$; or other results in Chap. VI which have not been used in the solution.

EXAMPLES

1. Solve the triangle PQR, given:

$PQ=12$ in.,	Solution:* $R=$
$Q=40^\circ$,	$PR=$
$P=75^\circ$.	$RQ=$

$$R=180^\circ-(P+Q)=180^\circ-(40^\circ+75^\circ)=65^\circ.$$

$$\frac{PR}{\sin Q}=\frac{PQ}{\sin R}. \qquad \frac{RQ}{\sin P}=\frac{PQ}{\sin R}.$$

$$\therefore\ PR=\frac{PQ}{\sin R}\cdot\sin Q, \qquad RQ=\frac{PQ}{\sin R}\cdot\sin P,$$

$$=\frac{12}{\sin 65^\circ}\cdot\sin 40^\circ, \qquad =\frac{12}{\sin 65^\circ}\cdot\sin 75^\circ,$$

---

* Results to be written here.

$$=\frac{12}{.9063}\times .6428, \qquad\qquad =\frac{12}{.9063}\times .9659,$$

$$=13.24\times .6428, \qquad\qquad =13.24\times .9659,$$

$$=8.51 \text{ in.} \qquad\qquad =12.8 \text{ in.}$$

Check: Take some relation of Chap. VI not involving PQ, P, Q, as above; e.g.:

$$\frac{PR}{RQ}=\frac{\sin Q}{\sin P}.$$

According to this,

$$\frac{8.51}{12.8}=\frac{\sin 40^\circ}{\sin 75^\circ}=\frac{.6428}{.9659};$$

which gives, on multiplying up,

$$8.2198\ldots=8.2281.$$

This shows that the results are very nearly accurate. They are as accurate as can be obtained with four-place tables.

Another check: Ex. Use relation (1), Art. 40, as a check.

2. In ABC, $A=50^\circ$, $B=75^\circ$, $c=60$ in. Solve the triangle.

3. In ABC, $A=131^\circ\ 35'$, $B=30^\circ$, $b=5\frac{1}{3}$ ft. Find a.

4. In ABC, $B=70^\circ\ 30'$, $C=78^\circ\ 10'$, $a=102$. Solve the triangle.

5. In ABC, $B=98^\circ\ 22'$, $C=41^\circ\ 1'$, $a=5.42$. Solve the triangle.

44. Case II. Given two sides and an angle opposite to one of them. In the triangle ABC let a, b, A be known, and C, B, c be required. The triangle will first be constructed with the given elements. At any point A of a straight line LM, unlimited in length, make angle MAC equal to angle A, and cut off AC equal to b. About C as a centre, and with a radius equal to a, describe a circle. This circle will either:

(1) Not reach to LM, as in Fig. 88.

(2) Just reach to LM, thus having LM for a tangent, as in Fig. 89.

(3) Intersect LM in two points, as in Figs. 90, 91.

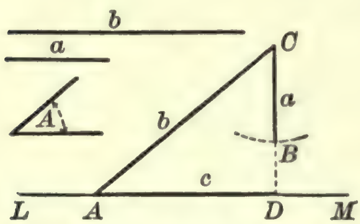


FIG. 88.

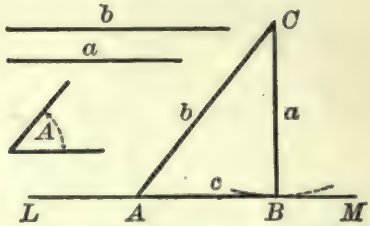


FIG. 89.

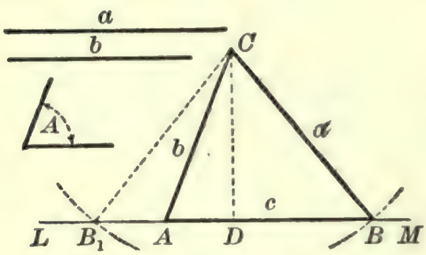


FIG. 90.

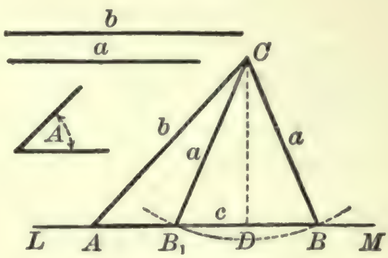


FIG. 91.

Each of these possible cases must be considered. In each figure, from C draw CD at right angles to AM; then $CD = b \sin A$.

In case (1), Fig. 88, $CB < CD$, and there is no triangle which can have the given elements. Hence *the triangle is impossible when* $a < b \sin A$.

In case (2), Fig. 89, $CB = CD$. Hence, *the triangle which has elements equal to the given elements is right-angled when* $a = b \sin A$.

In case (3), Figs. 90, 91, $CB > CD$; that is, $a > b \sin A$. If $a > b$, then the points B, B_1, in which the circle intersects LM, are on opposite sides of A, as in Fig. 90, and *there is one triangle which has three elements equal to the given elements,* namely, ABC. If $a < b$, then the points of intersection B, B_1, are on the same side of A, as in Fig. 91, and *there are two triangles which have elements equal to the given elements,* namely, ABC, AB_1C. For, in ABC, angle $BAC = A$, $AC = b$, $BC = a$; in AB_1C, angle $B_1AC = A$, $AC = b$, $B_1C = a$. *Both triangles must be solved.* In this case, Fig. 91, the given angle is opposite to the smaller of the two given sides. Hence, *there may be two solutions* **when the given angle is opposite to the smaller of the two given sides.*** Accordingly case II is sometimes called **the ambiguous case** in the solution of triangles.

Checks: As in Case I.

EXAMPLES

1. Solve the triangle STV, given: $ST = 15$, $VT = 12$, $S = 52°$.

$$\frac{\sin V}{ST} = \frac{\sin S}{VT};$$

$$\frac{\sin V}{15} = \frac{\sin 52°}{12} = \frac{.7880}{12}.$$

$$\therefore \sin V = .9850.$$

$$\therefore V = 80° 4', \quad \text{or} \quad 180° - 80° 4', \quad \text{i.e.,} \quad 99° 56'.$$

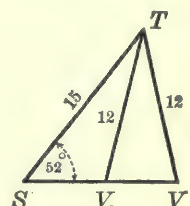


FIG. 92.

Both values of V must be taken, since the given angle is opposite to the smaller of the given sides. The two triangles

* *Summary.* When the given angle is opposite to the smaller of the two given sides there may be no solution as in Fig. 88, or one solution as in Fig. 89, or two solutions, as in Fig. 91.

When the given angle is opposite to the greater of the two given sides there is one solution, as in Fig. 90.

corresponding to the two values of V are STV, STV_1, Fig. 92, in which

$$SVT = 80° \, 4'; \quad SV_1T = 99° \, 56'.$$

In STV_1:

angle $STV_1 = 180° - (S + SV_1T)$
$= 28° \, 4'$.

$$\frac{SV_1}{\sin STV_1} = \frac{V_1T}{\sin S};$$

$$\frac{SV_1}{.4705} = \frac{12}{.7880};$$

$$\therefore \; SV_1 = 7.165.$$

In STV:

angle $STV = 180° - (S + SVT)$
$= 47° \, 56'$.

$$\frac{SV}{\sin STV} = \frac{VT}{\sin S};$$

$$\frac{SV}{.7423} = \frac{12}{.7880};$$

$$\therefore \; SV = 11.3.$$

The solutions are:

$$\left.\begin{aligned} V_1 &= 99° \, 56' \\ STV_1 &= 28° \;\; 4' \\ SV_1 &= 7.165 \end{aligned}\right\}; \qquad \left.\begin{aligned} V &= 80° \;\; 4' \\ STV &= 47° \, 56' \\ SV &= 11.3 \end{aligned}\right\}.$$

Check: Of several possible checks use relations (1), Art. 40.

In STV_1:

$$\frac{V_1T - SV_1}{V_1T + SV_1} = \frac{\tan \frac{1}{2}(S - STV_1)}{\tan \frac{1}{2}(S + STV_1)}.$$

In STV:

$$\frac{VT - SV}{VT + SV} = \frac{\tan \frac{1}{2}(S - STV)}{\tan \frac{1}{2}(S + STV)}.$$

On substituting the values above there comes,

$$\frac{4.835}{19.165} = \frac{\tan 11° \, 58'}{\tan 40° \;\; 2'}. \qquad \frac{.7}{23.3} = \frac{\tan \;\; 2° \;\; 2'}{\tan 49° \, 58'}.$$

$$\frac{4.835}{19.165} = \frac{.2100}{.8401}; \qquad \frac{.7}{23.3} = \frac{.0355}{1.1904}.$$

From these, on clearing fractions,

$$4.0619 = 4.0246, \qquad .83328 = .82715.$$

N.B. Had five-place or six-place tables been used, the calculated values would have approximated still more closely to absolute correctness.

This shows that the values found very nearly satisfy the check, and accordingly closely approximate to correctness.

2. Solve ABC, given: $a=29$ ft., $b=34$ ft., $A=30° 20'$.

3. Solve ABC when $a=30$ ft., $b=24$ ft., $B=65°$.

4. Solve ABC when $a=30$ in., $b=24$ in., $A=65°$.

5. Solve ABC when $a=15$ ft., $b=8$ ft., $B=23° 25'$.

45. Case III. Given two sides and their included angle. In the triangle ABC, a, b, C, say, are known, and it is required to find A, B, c. In this case, c can be determined from the relation

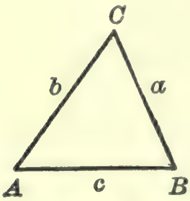


FIG. 93.

$$c^2=a^2+b^2-2ab\cos C;$$

angle A can be determined from the relation

$$\frac{\sin A}{a}=\frac{\sin C}{c};$$

angle B can be determined from the relation

$$A+B+C=180°, \text{ or from } \frac{\sin B}{b}=\frac{\sin C}{c}.$$

Checks: Any relations in Chap. VI which have not been used in the above solution.

EXAMPLES

1. In triangle PQR, $p=8$ ft., $r=10$ ft., $Q=47°$. Find q, P, R.

$$q^2=p^2+r^2-2pr\cos Q$$

$$=64+100-2\times 8\times 10\times .6820=54.88.$$

$\therefore\ q=7.408.$

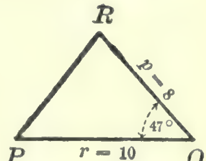


FIG. 94.

$$\sin P = \frac{p \sin Q}{q} = \frac{8 \times .7314}{7.408} = .7898. \qquad \therefore\ P = 52^\circ\ 10'.$$

$$\sin R = \frac{r \sin Q}{q} = \frac{10 \times .7314}{7.408} = .9873. \qquad \therefore\ R = 80^\circ\ 50'.$$

Check: Of the available checks use $P+Q+R=180^\circ$.

Here $52^\circ\ 10' + 47^\circ + 80^\circ\ 50' = 180^\circ;$

and the test is thus satisfied.

2. Solve ABC, given: $a=34$ ft., $b=24$ ft., $C=59^\circ\ 17'$.

3. Solve ABC, given: $a=33$ ft., $c=30$ ft., $B=35^\circ\ 25'$.

4. Solve RST, given: $r=30$ ft., $s=54$ ft., $T=46^\circ$.

5. Solve PQR, given: $p=10$ in., $q=16$ in., $R=97^\circ\ 54'$.

46. Case IV. Three sides given. If the sides a, b, c are known in the triangle ABC, then the angles A, B, C can be found by means of the relations (2), Art. 39, or the relations in Art. 41.

Checks: Any relations in Chap. VI which have not been used in the solution.

EXAMPLES

1. In ABC, $a=4$, $b=7$, $c=10$; find A, B, C.

$$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{49+100-16}{2 \times 7 \times 10} = \frac{133}{140} = .9500. \quad \therefore\ A = 18^\circ\ 12'.$$

$$\cos B = \frac{c^2+a^2-b^2}{2ca} = \frac{100+16-49}{2 \times 10 \times 4} = \frac{67}{80} = .8375.$$

$$\therefore\ B = 33^\circ\ 7'.5$$

$$\cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{16+49-100}{2 \times 4 \times 7} = \frac{-35}{56} = -.6250.$$

$$\therefore\ C = 128^\circ\ 40'.8$$

Angle C is in the *second* quadrant since its cosine is negative.

Check: $18° \ 12' + 33° \ 7'.5 + 128° \ 40'.8 = 180° \ 0'.3$. The discrepancy is due to the fact that four-place tables were used in the computation. Had five-place tables been used, the discrepancy would have been less.

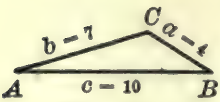


FIG. 95.

2. In PQR, $p = 9$, $q = 24$, $r = 27$. Find P, Q, R.

3. In RST, $r = 21$, $s = 24$, $t = 27$. Find R, S. T.

4. In ABC, $a = 12$, $b = 20$, $c = 28$. Find A, B, C.

5. In ABC, $a = 80$, $b = 26$, $c = 74$. Find, A, B, C.

6. Solve Ex. 1, using five-place tables.

7. Solve several of Exs. 1–5, using relations in Art. 41.

47. Use of logarithms in the solution of triangles. If logarithms are not employed all the relations in Arts. 38–41 are available for the solution of triangles. If logarithms are employed, only those relations which are adapted to logarithmic computation can be used, viz., the relations in Arts. 38, 40, 41. The relations in Art. 39 are not adapted for logarithmic computation.

The examples worked in Arts. 48–50 will give sufficient explanation of how logarithms are used in the solution of triangles.

48. Cases, I, II, logarithms used.

EXAMPLES

1. In ABC, given: $a = 447$, To find: $B =$ (Write the results

$b = 576$, $C =$ here.)

$A = 47° \ 35'$. $c =$

Since $a<b$, there may be two solutions. Construction shows there are two solutions.

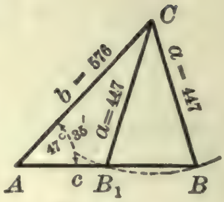


FIG. 96.

Formulas: $\sin ABC = \frac{b}{a} \sin A = \sin AB_1C$.

$$ACB = 180° - (A + ABC).$$

$$ACB_1 = 180° - (A + AB_1C).$$

$$AB = \frac{a}{\sin A} \sin ACB. \qquad AB_1 = \frac{a}{\sin A} \sin ACB_1.$$

$$\therefore \log \sin ABC = \log b + \log \sin A - \log a = \log \sin AB_1C;$$

$$\log AB = \log a + \log \sin ACB - \log \sin A;$$

$$\log AB_1 = \log a + \log \sin ACB_1 - \log \sin A.$$

$$\log a = 2.65031$$

$$\log b = 2.76042$$

$$\log \sin A = 9.86821 - 10$$

$$\therefore \log \sin B = 9.97832 - 10$$

$\therefore ABC = 72° \; 2' \; 45''$ and $AB_1C = 107° \; 57' \; 15''$

$\therefore ACB = 60° \; 22' \; 15''$ $\quad$ $\therefore ACB_1 = 24° \; 27' \; 45''$

$\log \sin ACB = 9.93914 - 10$ $\quad$ $\log \sin ACB_1 = 9.61710 - 10$

$\therefore \log AB = 2.72124$ $\quad$ $\therefore \log AB_1 = 2.39920$

$\therefore AB = 526.3$ $\quad$ $\therefore AB_1 = 250.7$

In obtaining $\log AB$, for instance, $\log \sin ACB$ may be written on the margin of a slip of paper, placed under $\log a$, the addition made, $\log \sin A$ placed beneath, and the subtraction made.

Solve the triangle ABC when the following elements are given, and check the results:

2. $A = 63° \; 48'$, $B = 49° \; 25'$, $a = 825$ ft.

3. $B=128°\ 3'\ 49''$, $C=33°\ 34'\ 47''$, $a=240$ ft.

4. $A=78°\ 30'$, $b=137$ ft., $a=65$ ft.

5. $a=275.48$, $b=350.55$, $B=60°\ 0'\ 32''$.

6. $c=690$, $a=464$, $A=37°\ 20'$.

7. $a=690$, $b=1390$, $A=21°\ 14'\ 25''$.

49. Case III, logarithms used.

EXAMPLES

1. In triangle ABC, given: $b=472$, $c=324$, $A=78°\ 40'$. Find: $B=$, $C=$, $a=$

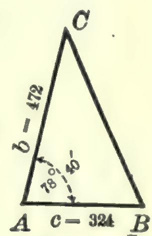


Fig. 97.

Formulas: $\tan\frac{1}{2}(B-C)=\dfrac{b-c}{b+c}\cot\frac{1}{2}A.$

$$\tfrac{1}{2}(B+C)=90°-\tfrac{1}{2}A.$$

$$B=\tfrac{1}{2}(B+C)+\tfrac{1}{2}(B-C).$$

$$C=\tfrac{1}{2}(B+C)-\tfrac{1}{2}(B-C).$$

$$a=\frac{b\sin A}{\sin B};\quad \text{or}\quad =\frac{c\sin A}{\sin C}.$$

$$\log\tan\tfrac{1}{2}(B-C)=\log(b-c)+\log\cot\tfrac{1}{2}A-\log(b+c),$$

$$\log a=\log b+\log\sin A-\log\sin B;\quad \text{or}$$

$$=\log c+\log\sin A-\log\sin C.$$

$b=472$	$\log(b-c)=2.17026$	$\log b=2.67394$
$c=324$	$\log(b+c)=2.90091$	$\log\sin A=9.99145-10$
$A=78°\ 40'$	$\log\cot\frac{1}{2}A=10.08647-10$	$\log\sin B=9.95160-10$
$b-c=148$	$\therefore \log\tan\frac{1}{2}(B-C)=9.35582-10$	$\therefore \log a=2.71379$
$b+c=796$	$\therefore \frac{1}{2}(B-C)=12°\ 47'\ 1''$	$\therefore a=517.36$
$\frac{1}{2}A=39°\ 20'$	$\frac{1}{2}(B+C)=50°\ 40'$	

$$\therefore B=63°\ 27'\ 1''.$$

$$\therefore C=37°\ 52'\ 59''.$$

Check: $a = c \sin A \div \sin C$.

Solve the following triangles and check the results:

2. ABC, given $b = 352$, $a = 266$, $C = 73°$.

3. PQR, given $p = 91.7$, $q = 31.2$, $R = 33°\ 7'\ 9''$.

4. ABC, given $a = 960$, $b = 720$, $C = 25°\ 40'$.

5. ABC, given $b = 9.081$, $c = 3.6545$, $A = 68°\ 14'\ 24''$.

50. Case IV, logarithms used.

EXAMPLES

1. In triangle ABC, $a = 25.17$, $b = 34.06$, $c = 22.17$. Find A, B, C.

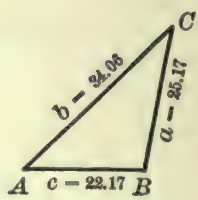


FIG. 98.

Formulas: $$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

$$\tan \tfrac{1}{2}A = \frac{r}{s-a}; \quad \tan \tfrac{1}{2}B = \frac{r}{s-b}; \quad \tan \tfrac{1}{2}C = \frac{r}{s-c}.$$

$$\therefore \log r = \tfrac{1}{2}[\log (s-a) + \log (s-b) + \log (s-c) - \log s].$$

$$\log \tan \tfrac{1}{2}A = \log r - \log (s-a); \quad \log \tan \tfrac{1}{2}B = \log r - \log (s-b);$$

$$\log \tan \tfrac{1}{2}C = \log r - \log (s-c).$$

Check: $A + B + C = 180°$.

$a = 25.17$	$\log s = 1.60959$	$\log \tan \frac{1}{2}A = 9.64465 - 10$
$b = 34.06$	$\log (s-a) = 1.19117$	$\frac{1}{2}A = 23°\ 48'\ 28''$
$c = 22.17$	$\log (s-b) = 0.82217$	$\log \tan \frac{1}{2}B = 10.01365 - 10$
$2s = 81.40$	$\log (s-c) = 1.26788$	$\frac{1}{2}B = 45°\ 54'$
$s = 40.70$		$\log \tan \frac{1}{2}C = 9.56794 - 10$

$s-a=15.53 \qquad \therefore \log r^2=1.67163 \qquad \frac{1}{2}C=20°\ 17'\ 35''$

$s-b=\ 6.64 \qquad \therefore \log r=0.83582$

$s-c=18.53 \qquad \therefore A=47°\ 36'\ 56'',\ B=91°\ 48',\ C=40°\ 35'\ 10''$

Check: $\qquad A+B+C=180°\ 0'\ 6''.$

Solve the following triangles and check the results:

2. ABC, given $a=260$, $b=280$, $c=300$.

3. ABC, when $a=26.19$, $b=28.31$, $c=46.92$.

4. PQR, given $p=650$, $q=736$, $r=914$.

5. RST, given $r=1152$, $s=2016$, $t=2592$.

51. Problems in heights and distances. Some problems in heights and distances have been solved in Art. 11 by the aid of right-angled triangles. Additional problems of the same kind will now be given, in the solution of which oblique-angled triangles may be used. It is advisable to draw the figures neatly and accurately. The graphical method should also be employed.

EXAMPLES

1. Another solution of Ex. 2, Art. 11.

In the triangle ABP (Fig. 23), $AB=100$ ft., $BAP=30°$, $PBA=180°-45°=135°$. Hence the triangle can be solved, and BP can be found. When BP shall have been found, then in the triangle CBP, BP is known and $CBP=45°$; hence CP can be found. The computation is left to the student.

2. Another solution of Ex. 3, Art. 11. In the triangle CBP (Fig. 24), $BP=30$ ft., $BCP=40°\ 20'-38°\ 20'=2°$, $PBC=90°+LCB=128°\ 20'$. Hence CBP can be solved and the length of CB can be found. When CB shall have been found, then, in the triangle LCB, angle $C=38°\ 20'$, CB is known, and hence LB can be found. The computation is left to the student.

3. Find the distance between two objects that are invisible from each other on account of a wood, their distances from a station at which they are visible being 441 and 504 yd., and the angle at the station subtended by the distance of the objects being $55° 40'$.

4. The distance of a station from two objects situated at opposite sides of a hill is 1128 and 936 yd., and the angle subtended at the station by their distance, is $64° 28'$. What is their distance?

5. Find the distance between a tree and a house on opposite sides of a river, a base of 330 yd. being measured from the tree to another station, and the angles at the tree and the station formed by the base line and lines in the direction of the house being $73° 15'$ and $68° 2'$, respectively. Also find the distance between the station and the house.

6. Find the height of a tower on the opposite side of a river, when a horizontal line in the same level with the base and in the same vertical plane with the top is measured and found to be 170 ft., and the angles of elevation of the top of the tower at the extremities of the line are $32°$ and $58°$, the height of the observer's eye being 5 ft.

7. Find the height of a tower on top of a hill, when a horizontal base line on a level with the foot of the hill and in the same vertical plane with the top of the tower is measured and found to be 460 ft.; and at the end of the line nearer the hill the angles of elevation of the top and foot of the tower are $36° 24'$, $24° 36'$, and at the other end the angle of elevation of the top of the tower is $16° 40'$.

8. A church is at the top of a straight street having an inclination of $14° 10'$ to the horizon; a straight line 100 ft. in length is measured along the street in the direction of the church; at the extremities of this line the angles of elevation of the top of the steeple are $40° 30'$, $58° 20'$. Find the height of the steeple.

9. The distance between the houses C, D, on the right bank of a river and invisible from each other, is required. A straight line AB, 300 yd. long, is measured on the left bank of the river, and angular measurements are taken as follows: $ABC = 53° 30'$, $CBD = 45° 15'$, $CAD = 37°$, $DAB = 58° 20'$. What is the length CD?

10. A tower CD, C being the base, stands in a horizontal plane; a horizontal line AB on the same level with the base is measured and found to be 468 ft.; the horizontal angles BAC, ABC, are equal to $125° 40'$, $12° 35'$, respectively, and the vertical angles CAD, CBD, are equal to $38° 20'$, $11° 50'$, respectively. Find the height of the tower and its distances from A and B.

11. A base line AB 850 ft. long is measured along the straight bank of a river; C is an object on the opposite bank; the angles BAC, ABC, are observed to be $63° 40'$, $37° 15'$, respectively. Find the breadth of the river.

12. A tower subtends an angle α at a point on the same level as the foot of the tower and, at a second point, h feet above the first, the depression of the foot of the tower is β. Show that the height of the tower is $h \tan \alpha \cot \beta$.

13. The elevation of a steeple at a place due south of it is 45°; at another place due west of the steeple the elevation is 15°. If the distance between the two places be a, prove that the height of the steeple is

$$a(\sqrt{3}-1) \div 2\sqrt{2}.$$

14. The elevation of a steeple at a place due south of it is 45°; at another place due west of the first place the elevation is 15°. If the distance between the two places be a, prove that the height of the steeple is

$$a(\sqrt{3}-1) \div 2\sqrt[4]{3}.$$

15. The elevation of the summit of a hill from a station A is α; after walking c feet toward the summit up a slope inclined at an angle β to the horizon the elevation is γ. Show that the height of the hill above A is $c \sin \alpha \sin (\gamma-\beta) \operatorname{cosec} (\gamma-\alpha)$ ft.

CHAPTER VIII

MISCELLANEOUS THEOREMS

52. Area of a triangle. The area of ABC is required. Let the length of the perpendicular DC from C to AB, or

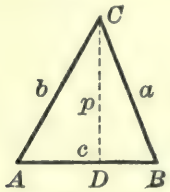


FIG. 99.

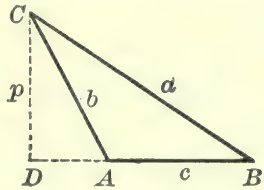


FIG. 100.

AB produced, be denoted by p, the semi-perimeter by s, and the area by S. The following cases may occur:

I. *One side and the perpendicular on it from the opposite angle known,* say (c, p).

$$\boldsymbol{S = \tfrac{1}{2}cp.} \tag{1}$$

II. *Two sides and their included angle known,* say, b, c, A.

$$S = \tfrac{1}{2}cp = \tfrac{1}{2}c \cdot AC \sin BAC.$$

$$\therefore \quad \boldsymbol{S = \tfrac{1}{2}bc \sin A.} \tag{2}$$

Similarly, $\qquad S = \tfrac{1}{2}ca \sin B = \tfrac{1}{2}ab \sin C.$

Problems which do not fall under Cases I or II or III directly may be solved on finding a perpendicular or a side or an angle.

Eg. Let a, A, B, C, be known.

Now $S=\frac{1}{2}ab\sin C$. But $b=\dfrac{a\sin B}{\sin A}$.

$$\therefore\ S=\tfrac{1}{2}a^2\cdot\frac{\sin C\sin B}{\sin A}.$$

III. *Three sides known.*

$$S=\tfrac{1}{2}bc\sin A=\tfrac{1}{2}bc\cdot 2\sin\tfrac{1}{2}A\cos\tfrac{1}{2}A$$

$$=bc\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{s(s-a)}{bc}}.$$

$$\therefore\quad \boldsymbol{S=\sqrt{s(s-a)(s-b)(s-c)}}. \qquad (3)$$

EXAMPLES

1. Find the area of the following triangles:

(*a*) ABC in which $a=30$ ft., $b=36$ ft., $c=44$ ft.
(*b*) PQR in which $p=22$ ft., $q=31$ ft., $r=43$ ft.

2. Find the area of the following triangles:

(*a*) ABC in which $a=37$ ft., $b=53$ ft., $C=43°$.
(*b*) PQR in which $q=23$ ft., $r=48$ ft., $P=65°$.

3. An isosceles triangle whose vertical angle is 78° contains 400 sq.yd.; find the lengths of the sides.

4. Find two triangles each of which has sides 63 and 55 ft. long, and an area of 874 sq.ft.

5. Two roads form an angle of 27° 10′. At what distance from their intersection must a fence at right angles to one of them be placed so as to inclose an acre of land?

6. The angles at the base of a triangle are 22° 30′ and 112° 30′, respectively: show that the area of the triangle is equal to the square of half the base.

53. Area of a circular sector. Let r be the radius of the circle and θ the number of radians in the angle of the sector AOB. Now

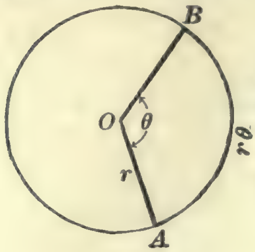


FIG. 101.

$$\frac{\text{area of sector}}{\text{area of circle}}=\frac{\text{angle } AOB}{4 \text{ right angles}}=\frac{\theta \text{ radians}}{2\pi \text{ radians}},$$

i.e.,
$$\frac{\text{area of sector}}{\pi r^2}=\frac{\theta}{2\pi}.$$

$$\therefore \textbf{ area of sector}=\tfrac{1}{2}\boldsymbol{r^2\theta}.$$

This formula is not true unless the angle is measured in radians.

Otherwise: Area sector $=\frac{1}{2}r\cdot$arc AB.

Now arc $AB=r\theta$. [Art. 18 (5).]

$\therefore$ area sector $=\frac{1}{2}r^2\theta$.

EXAMPLES

1. Draw the following sectors and calculate their arcs and areas:

(*a*) Radius $=10$ in., angle $=\frac{2}{3}$ radian.
(*b*) Radius $=24$ in., angle $=1\frac{1}{4}$ radians.
(*c*) Radius $=18$ in., angle $=2$ radians.
(*d*) Radius $=20$ in., angle $=4\frac{1}{2}$ radians.

2. The arc of a sector $=24$ in. and its angle $=\frac{7}{8}$ radian: find the radius and the area of the sector.

3. The area of a sector is 124 sq.in. and its angle is 2 radians: find the lengths of its radius and arc.

4. The area of a sector is 236 sq.in. and its arc $=32$ in.: find the radius and the angle of the sector in radians and in degrees.

54. Circles connected with a triangle. A. The circumscribing circle. Let S denote the area of the triangle ABC

and R the radius of its circumscribing circle. Then by Art. 38 (3), (4),

$$R = \frac{a}{2 \sin A} = \frac{b}{2 \sin B} = \frac{c}{2 \sin C}. \tag{1}$$

From (2), Art. 52, $\sin A = \frac{2S}{bc}$. Substitution of this in the first of equations (1), gives

$$R = \frac{abc}{4S}. \tag{2}$$

B. The inscribed circle. Let the radius of the circle inscribed in a triangle ABC be denoted by r. Join the centre O and the points of contact L, M, N. By geometry, the angles at L, M, N are right angles. Draw OA, OB, OC.

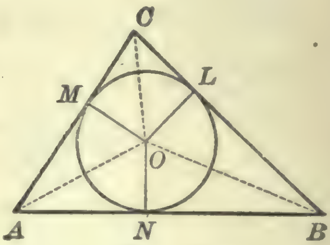


FIG. 102.

Area BOC + area COA

$$+ \text{area } AOB = \text{area } ABC.$$

$$\therefore \ \tfrac{1}{2}ar + \tfrac{1}{2}br + \tfrac{1}{2}cr = \sqrt{s(s-a)(s-b)(s-c)}, \text{ or } S.$$

$$\therefore \ \tfrac{1}{2}(a+b+c)r = S,$$

i.e.,

$$sr = S.$$

$$\therefore \ r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} = \frac{S}{s}. \tag{3}$$

This was shown in another way in Art. 41.

C. The escribed circle. An *escribed circle* of a triangle is a circle that touches one of the sides of the triangle and the other two sides produced.

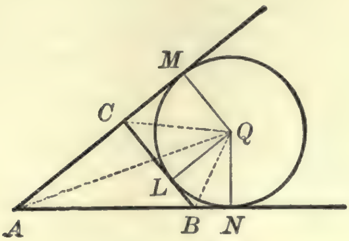


FIG. 103.

Let r_a denote the radius of the escribed circle touching the side BC opposite to the angle A. Join the centre Q and the points of contact L, M, N. By geometry, the angles at L, M, N are right angles. Draw QA, QB, QC.

$$\text{Area } ABQ + \text{area } CAQ - \text{area } BCQ = \text{area } ABC.$$

$$\therefore\ \tfrac{1}{2}r_a c + \tfrac{1}{2}r_a b - \tfrac{1}{2}r_a a = S,$$

$$\therefore\ \tfrac{1}{2}(c+b-a)r_a = S;$$

$$(s-a)r_a = S.$$

$$\therefore\ r_a = \frac{S}{s-a}.$$

Similarly, $$r_b = \frac{S}{s-b}; \qquad r_c = \frac{S}{s-c}.$$

EXAMPLES

1. Find the radii of the inscribed, circumscribed, and escribed circles of the following triangles:

(a) ABC in which $a=22$ in., $b=35$ in., $c=43$ in.

(b) PQR in which $p=10$ in., $q=13$ in., $r=8$ in.

(c) RST in which $r=32$ in., $s=40$ in., $t=50$ in.

2. Prove that in an equilateral triangle the radii of the inscribed, circumscribed, and escribed circles are as 1:2:3.

3. The sides of a triangle are 17 in., 25 in., 36 in.; show that the radii of the escribed circles are as 21:33:154.

4. Prove: (*a*) $r_a+r_b+r_c-r=4R$; (*b*) $\sqrt{r\cdot r_a\cdot r_b\cdot r_c}=S$.

5. Prove: (*a*) $\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}$; (*b*) $Rr=\frac{abc}{4(a+b+c)}$;

(*c*) $r=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$.

6. Prove $r_a\cot\frac{A}{2}=r_b\cot\frac{B}{2}=r_c\cot\frac{C}{2}=r\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

$$=4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

7. Prove $R=\frac{a}{2\sin A}$, $r=(s-a)\tan\frac{A}{2}$, $r_a=s\tan\frac{A}{2}$. Write two other similar formulas for R and r. Write similar formulas for r_b and r_c.

8. Prove $r=\dfrac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}$. Write two similar formulas involving b, c.

9. (*a*) Show that the area of a regular polygon inscribed in a circle is a mean proportional between the areas of an inscribed and circumscribing polygon of half the number of sides. (*b*) The sides of a triangle are as $2:3:4$; show that the radii of the escribed circles are as $\frac{1}{5}:\frac{1}{3}:1$.

10. If the altitude of an isosceles triangle is equal to its base, the radius of the circumscribing circle is $\frac{5}{8}$ of the base.

11. An equilateral triangle and a regular hexagon have the same perimeter. Show that the areas of their inscribed circles are as $4:9$.

12. If the sides of a triangle are 51, 68, and 85 ft., show that the shortest side is divided by the point of contact of the inscribed circle into two segments, one of which is double the other.

55. Relations between the radian measure, the sine, and the tangent of certain angles.

A. If θ be the radian measure of an acute angle,

$$\sin \theta < \theta < \tan \theta.$$

B. When an angle θ radians approaches the limit zero,

$$\frac{\sin \theta}{\theta} \quad \text{and} \quad \frac{\tan \theta}{\theta}$$

each approaches unity as a limit.

Proof of A. Let Angle $AOP = \theta$ radians.

Make angle $AOR = AOP$. With radius r describe the arc QBR about O as centre. Draw the chord QR intersecting OA at M. Draw the tangents QT, RT, intersecting OA at T. Draw the chord QB.* Then

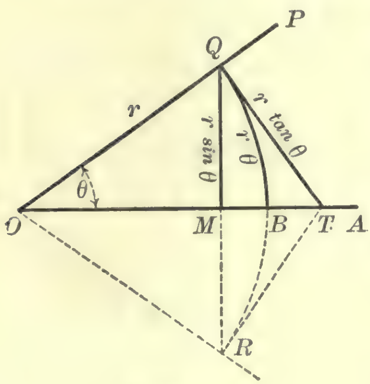


FIG. 10?.

area triangle $OQB <$ area sector OQB
$< $ area triangle OQT.

That is,

$$\tfrac{1}{2}OB \cdot MQ < \tfrac{1}{2}OQ \cdot \text{arc } BQ < \tfrac{1}{2}OQ \cdot QT;$$

or,
$$\tfrac{1}{2}r \cdot r \sin \theta < \tfrac{1}{2}r \cdot r\theta < \tfrac{1}{2}r \cdot r \tan \theta.$$

Hence,
$$\sin \theta < \theta < \tan \theta. \qquad (1)$$

Otherwise: It can be proved that

$$QMR < \text{arc } QBR < QTR.$$

From this, on dividing by 2,

---

* Or suppose it is drawn.

$$MQ < \text{arc } QB < QT.$$

That is, $$r \sin \theta < \quad r\theta \quad < r \tan \theta.$$

$$\therefore \sin \theta < \quad \theta \quad < \tan \theta.$$

Proof of B. Division of each member of (1) by $\sin \theta$ gives

$$1 < \frac{\theta}{\sin \theta} < \frac{1}{\cos \theta}.$$

Thus, for θ between $0°$ and $90°$, $\frac{\theta}{\sin \theta}$ lies between 1 and $\frac{1}{\cos \theta}$. Now when θ approaches zero, $\cos \theta$ approaches 1 as a limit and thus $\frac{1}{\cos \theta}$ approaches 1 as a limit. Accordingly the limit of $\frac{\theta}{\sin \theta}$, which must lie between 1 and the limit of $\frac{1}{\cos \theta}$, viz., 1, is itself 1. Hence the limit of the reciprocal $\frac{\sin \theta}{\theta}$ is 1.

Division of each member of (1) by $\tan \theta$ gives

$$\cos \theta < \frac{\theta}{\tan \theta} < 1.$$

Thus, for θ between $0°$ and $90°$, $\frac{\theta}{\tan \theta}$ lies between $\cos \theta$ and 1. Now when θ approaches zero, $\cos \theta$ approaches 1 as a limit. It follows, on reasoning as in the preceding case, that $\frac{\theta}{\tan \theta}$, and consequently $\frac{\tan \theta}{\theta}$, approaches 1 as a limit.

These results may be briefly expressed:

$$\underset{\theta=0}{\text{Limit}}\left(\frac{\sin \theta}{\theta}\right) = 1; \quad \underset{\theta=0}{\text{Limit}}\left(\frac{\tan \theta}{\theta}\right) = 1. \tag{2}$$

These are two of the most important theorems in elementary trigonometry; they are frequently employed both in practical work and in pure mathematics.

A very important corollary to (2) is the following:

If θ be the radian measure of a very small angle, then θ can be used for $\sin\theta$ and $\tan\theta$ in calculations.

For instance, sin 10″ to 12 places of decimals is .000048481368. This is also the radian measure of 10″ to 12 places of decimals. The radian measures, sines, and tangents, of angles from 0° to 6°, agree in the first three places of decimals. For

$$\text{radian measure } 6° = (.10472) = .105; \quad \sin 6° = (.10453) = .105;$$

$$\tan 6° = (.10510) = .105.$$

EXAMPLES

1. Find the angle subtended by a man 6 ft. high at a distance of half a mile.

FIG. 105.

Here, $\theta = \tan\theta = \dfrac{6}{2640} = \dfrac{1}{440}$.

Now

$$\frac{1^r}{440} = \frac{1}{440} \times \frac{180°}{\pi} = \frac{7 \times 108°}{22 \times 440} = 7'\ 48''.6.$$

2. What must be the height of a tower, in order that it subtend an angle 1° at a distance of 4000 ft.?

$$\frac{x}{4000} = \tan 1° = \text{radian measure } 1° = \frac{\pi}{180}$$

$$= \frac{22}{7 \times 180}.$$

FIG. 106.

$$\therefore\ x = \frac{22 \times 4000}{7 \times 180} = 69.84 \text{ ft.}$$

3. Verify the following statements:

An angle 1° is subtended by 1 in. at a distance 4 ft. 9.3 in., and by 1 ft. at a distance 57.3 ft. An angle 1′ is subtended by 1 in. at a distance 286.5 ft., and by 1 ft. at a distance 3437.6 ft., about two-thirds of a mile. An angle 1″ is subtended by 1 in. at a distance of nearly $3\frac{1}{3}$ mi., by 1 ft. at a distance a little greater than 39 mi., by a horizontal line 200 ft. long on the other side of the world, nearly 8000 mi. away.

4. The moon's mean angular diameter as observed at the earth is 31′ 5″, and its actual diameter is about 2160 miles. Find the mean distance of the moon. How many full moons would make a chaplet across the sky?

5. Taking the earth's equatorial radius as 3963 mi., find the angular semi-diameter of the earth as it would appear if observed from the moon. Compare the relative apparent sizes of the moon as seen from the earth, and the earth as seen from the moon.

6. The semi-diameter of the earth as seen from the sun is very nearly 8″.8. What is the sun's distance from the earth, the radius of the earth being assumed as 4000 miles?

7. At least how many times farther away than the sun is the nearest fixed star α Centauri, at which the mean distance between the earth and the sun (about 92,897,000 miles) subtends an angle something less than 1″? How long, at least, will it take light to come from this star to the earth?

8. Find approximately the distance at which a coin an inch in diameter must be placed so as just to hide the moon, the latter's angular diameter being taken 31′ 5″.

9. The inclination of a railway to a horizontal plane is 50′. Find how many feet it rises in a mile.

10. Find the angle subtended by a circular target 4 ft. in diameter at a distance of 1000 yd.

11. Find the height of an object whose angle of elevation at a distance of 900 yd. is 1°.

12. Find the angle subtended by a pole 20 ft. high at a distance of a mile.

ANSWERS

Art. 3, Pages 4, 5

2. .3025, .9732, .9502, .4754, 5.4623, .3167, 9.78280, 9.50336, 10.73227.

3. $8° 48'.8$, $38° 26'.7$,
$30° 43'$, $52° 3'.2$,
$23° 0'.9$, $31° 22'.5$,
$28° 18' 22''.5$,
$22° 20' 30''$,
$33° 31' 31''.1$.

Art. 4, Pages 7, 8

In the following examples the functions are given in the order on page 3.

5. .8575, .5145, 1.6667, .6, 1.9437, 1.1662, $59° 2'.2$.

6. .4, .9165, .4364, 2.2913, 1.0911, 2.5, $23° 35'$; .3162, .9487, .3333, 3, 1.0541, 3.1623, $18° 25'.9$.

7. 9035, .4286, 2.1082, .4743, 2.3333, 1.1068, $64° 37'.3$.

11. (1) $a:b$, $\sqrt{b^2-a^2}:b$, $a:\sqrt{b^2-a^2}$, $\sqrt{b^2-a^2}:a$, $b:\sqrt{b^2-a^2}$, $b:a$;
(2) $\sqrt{b^2-a^2}:b$, $a:b$, $\sqrt{b^2-a^2}:a$, $a:\sqrt{b^2-a^2}$, $b:a$, $b:\sqrt{b^2-a^2}$;
(3) $a:\sqrt{a^2+b^2}$, $b:\sqrt{a^2+b^2}$, $a:b$, $b:a$, $\sqrt{a^2+b^2}:b$, $\sqrt{a^2+b^2}:a$;
(4) $b:\sqrt{a^2+b^2}$, $a:\sqrt{a^2+b^2}$, $b:a$, $a:b$, $\sqrt{a^2+b^2}:a$, $\sqrt{a^2+b^2}:b$;
(5) $\sqrt{a^2-b^2}:a$, $b:a$, $\sqrt{a^2-b^2}:b$, $b:\sqrt{a^2-b^2}$, $a:b$, $a:\sqrt{a^2-b^2}$;
(6) $b:a$, $\sqrt{a^2-b^2}:a$, $b:\sqrt{a^2-b^2}$, $\sqrt{a^2-b^2}:b$, $a:\sqrt{a^2-b^2}$, $a:b$.

12. $41° 24' 35''$.

13. $19° 28' 16''$.

Art. 5, Page 11

1. 2.28025.
2. 2.3333.
3. 5.846.
4. 2.0404.
5. 2.25.
6. 4.619.
7. 5.9259.
8. .3248.
9. 2.75.
10. $-.708$.

Art. 6, Pages 12, 13

2. cos 11° 40′, sin 9° 30′, cot 40°, tan 25°, cosec 19°, sec 10°.
3. 16° 40′.
4. 40°.

Art. 7, Pages 16, 17

9.

	sin A	cos A	tan A	cot A	sec A	cosec A
sin A	sin A	$\sqrt{1-\cos^2 A}$	$\frac{\tan A}{\sqrt{1+\tan^2 A}}$	$\frac{1}{\sqrt{1+\cot^2 A}}$	$\frac{\sqrt{\sec^2 A-1}}{\sec A}$	$\frac{1}{\operatorname{cosec} A}$
cos A	$\sqrt{1-\sin^2 A}$	cos A	$\frac{1}{\sqrt{1+\tan^2 A}}$	$\frac{\cot A}{\sqrt{1+\cot^2 A}}$	$\frac{1}{\sec A}$	$\frac{\sqrt{\operatorname{cosec}^2 A-1}}{\operatorname{cosec} A}$
tan A	$\frac{\sin A}{\sqrt{1-\sin^2 A}}$	$\frac{\sqrt{1-\cos^2 A}}{\cos A}$	tan A	$\frac{1}{\cot A}$	$\sqrt{\sec^2 A-1}$	$\frac{1}{\sqrt{\operatorname{cosec}^2 A-1}}$
cot A	$\frac{\sqrt{1-\sin^2 A}}{\sin A}$	$\frac{\cos A}{\sqrt{1-\cos^2 A}}$	$\frac{1}{\tan A}$	cot A	$\frac{1}{\sqrt{\sec^2 A-1}}$	$\sqrt{\operatorname{cosec}^2 A-1}$
sec A	$\frac{1}{\sqrt{1-\sin^2 A}}$	$\frac{1}{\cos A}$	$\sqrt{1+\tan^2 A}$	$\frac{\sqrt{1+\cot^2 A}}{\cot A}$	sec A	$\frac{\operatorname{cosec} A}{\sqrt{\operatorname{cosec}^2 A-1}}$
cosec A	$\frac{1}{\sin A}$	$\frac{1}{\sqrt{1-\cos^2 A}}$	$\frac{\sqrt{1+\tan^2 A}}{\tan A}$	$\sqrt{1+\cot^2 A}$	$\frac{\sec A}{\sqrt{\sec^2 A-1}}$	cosec A

15. 90°, 36° 52′ 12″.
16. 45°.
17. 45°, 71° 34′.
18. 53° 7′ 48″.
19. 30°, 48° 35′ 25″.
20. 36° 52′ 12″, 16° 15′ 36″.

Art. 9, Page 24

5. $A=65°\ 14'$, $B=24°\ 46'$, $b=7.834$.
6. 303.9, 39° 47′.6, 50° 12′.4.
7. 58° 45′ 48″, $a=1521.5$, $b=2508.6$.
8. $A=21°\ 8'$, $b=94.43$, $c=101.24$.
9. $A=30°\ 12'.2$, $B=59°\ 47'.8$, $c=116.25$.
10. $A=1°\ 21'.9$, $B=88°\ 38'.1$, $b=45.95$.
11. $B=41°\ 43'$, $a=241.85$, $b=215.6$.
12. $B=38°\ 41'$, $a=312.2$, $c=400$.
13. $B=52°\ 39'\ 30''$, $a=1040.9$, $b=1364.3$.
14. $A=52°\ 37'.1$, $B=37°\ 22'.9$, $c=2912.1$.
15. $B=62°\ 14'\ 40''$, $a=1968.7$, $c=4227.4$.
16. $A=29°\ 24'.9$, $B=60°\ 35'.1$, $b=43.67$.
17. $A=27°\ 20'$, $b=77.4$, $c=87.1$.
18. $b=7.4833$, $A=33°\ 44'.6$.
19. $c=8.75$, $A=30°\ 57'.8$.
20. $a=9.57$, $b=11.54$.
21. $a=3.84$, $b=11.37$.
22. $b=8.9433$, $A=41°\ 48'.6$.
23. $a=16.4$, $c=22.2$.
24. $c=14.42$, $A=33°\ 41'.4$.

Art. 10, Page 25

1. 24. 95 ft., 12.71 ft. **2.** 58.78 in.

3. (*a*) 9.239 in. (*b*) 21.96 ft. (*c*) 16.8 ft. (*d*) 19.3 ft.

Art. 11, Pages 27, 28

4. 398.2 ft. **7.** 276.95 ft. **10.** 86.6 ft., 50 ft.

5. 228 4 ft., 258 ft. **8.** 463.7 ft. **11.** 749 mi.

6. 63.9 ft., 63.9 ft. **9.** 3243.8 ft. **12.** 219.45 ft.

Art. 12, Page 29

2. Base = 187.9 ft.; height = 350.63 ft.; area = 32,943 sq.ft.

3. Base = 358.21 ft.; height = 161.26 ft.; area = 28,881 sq.ft.

4. 68° 50′ 5″.4, 68° 50′ 5″.4, 42° 19′ 49″.2; height = 83.93 ft.; area = 2727.7 sq.ft.

5. 56° 18′.6, 56° 18′.6, 67° 22′.8; 36.06 ft.; 600 sq.ft.

6. 105.83 ft.; 48° 35′, 48° 35′, 82° 50′, 3175 sq.ft.

7. 40.76 ft., 44.9 ft., 44.9 ft.; area = 815.2 sq.ft.

8. 96.68 ft., 79.4 ft., 79.4 ft.; area = 3045.4 sq.ft.

Art. 13, Page 31

1. 14.54 ft., 16.13 ft., 48.45 sq.ft., 105.2 sq.ft.

Art. 14, Page 33

1. 26.172, 52.345 miles; second ship bears E. 19° 42′.1 N. from first.

2. $LB = 14.197$ miles.

Art. 15, Page 35

4. 2.852 acres. **5.** 12 acres, 3 roods, 6.45 poles.

Art. 16, Page 38

1. *a.* For 137°: second; 137°, 497°, 857°, 1217°.
For 785°: first; 65°, 425°, 785°, 1145°.
For 3657°: first; 57°, 417°, 777°, 1137°.
b. For −240°; second; 120°, 480°, 840°, 1200°.
For −337°: first; 23°, 383°, 743°, 1103°.
For −7283°; fourth; 277°, 637°, 997°, 1357°.

3. 320°, −60°, 130°, −250°, 15°, 7°, 78°, 385°, 414°, −110°, −150°, 200°, 257°.

4. 410°, 30°, 220°, −160°, 105°, 97°, 168°, 475°, 504°, −20°, −60° 290°, 347°.

Art. 18, Pages 41, 42

1. $143° 14' 22''$. 2. 4.03.

3. $28° 38' 52''.4$, $229° 10' 59''.2$, $19° 5' 54''.9$.

4. $90°, 60°, 45°, 30°, 120°, 540°, -240°, -1800°$.

5. 1.454, 2.487. 6. $\frac{\pi}{4}, \frac{7\pi}{6}, \frac{5\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{4}$.

7. $\frac{4r}{27}$; $8° 29'$. 8. 6 in.

9. 40 in. 10. $\frac{50}{\pi}$ mins.

Art. 18a, Page 43

1. $7\frac{37}{45}$ radians per sec. 2. 1:12:720.

3. $\frac{25\pi}{3}$ radians per sec.

4. (*a*) $\frac{\pi}{6}$ radians per sec.; (*b*) 1800° per minute.

5. (*a*) $\frac{3960}{7\pi}$ revolutions per min.; (*b*) $\frac{132}{7}$ radians per sec.

6. 39.27 ft. per sec.

Art. 20, Pages 46, 47, 48

1. (*a*) +; (*b*) −; (*c*) +; (*d*) +; (*e*) +; (*f*) −.

2. $\frac{\sqrt{3}}{2}, -\frac{1}{2}, -\sqrt{3}$. 3. $-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 1$.

4. (*a*) First and second; (*b*) second and third;
(*c*) first and third; (*d*) second and fourth.

5. (*a*) Third; (*b*) second; (*c*) third; (*d*) fourth.

In the following examples the functions are given in the order on page 40.

8. $\frac{3}{5}, -\frac{4}{5}, -\frac{3}{4}, -\frac{4}{3}, -\frac{5}{4}, \frac{5}{3}; -\frac{3}{5}, \frac{4}{5}, -\frac{3}{4}, -\frac{4}{3}, \frac{5}{4}, -\frac{5}{3}$.

9. (*a*) $+\frac{\sqrt{5}}{3}, -\frac{2}{3}, -\frac{\sqrt{5}}{2}, -\frac{2}{\sqrt{5}}, -\frac{3}{2}, +\frac{3}{\sqrt{5}}$;

$-\frac{\sqrt{5}}{3}, -\frac{2}{3}, +\frac{\sqrt{5}}{2}, +\frac{2}{\sqrt{5}}, -\frac{3}{2}, -\frac{3}{\sqrt{5}}$.

(*b*) $-\frac{3}{5}, -\frac{4}{5}, \frac{3}{4}, \frac{4}{3}, -\frac{5}{4}, -\frac{5}{3}; -\frac{3}{5}, \frac{4}{5}, -\frac{3}{4}, -\frac{4}{3}, \frac{5}{4}, -\frac{5}{3}$.

(c) $+\frac{\sqrt{21}}{5}, \frac{2}{5}, +\frac{\sqrt{21}}{2}, +\frac{2}{\sqrt{21}}, \frac{5}{2}, +\frac{5}{\sqrt{21}};$

$-\frac{\sqrt{21}}{5}, \frac{2}{5}, -\frac{\sqrt{21}}{2}, -\frac{2}{\sqrt{21}}, \frac{5}{2}, -\frac{5}{\sqrt{21}}.$

(d) $\frac{4}{5}, \frac{3}{5}, \frac{4}{3}, \frac{3}{4}, \frac{5}{3}, \frac{5}{4}; \quad -\frac{4}{5}, \frac{3}{5}, -\frac{4}{3}, -\frac{3}{4}, \frac{5}{3}, -\frac{5}{4}.$

(e) $\frac{4}{5}, \frac{3}{5}, \frac{4}{3}, \frac{3}{4}, \frac{5}{3}, \frac{5}{4}; \quad -\frac{4}{5}, -\frac{3}{5}, \frac{4}{3}, \frac{3}{4}, -\frac{5}{3}, -\frac{5}{4}.$

10. (a) $\frac{3}{7}, -\frac{2\sqrt{10}}{7}, -\frac{3}{2\sqrt{10}}, -\frac{2\sqrt{10}}{3}, -\frac{7}{2\sqrt{10}}, \frac{7}{3}.$

(b) $-\frac{\sqrt{21}}{5}, \frac{2}{5}, -\frac{\sqrt{21}}{2}, -\frac{2}{\sqrt{21}}, \frac{5}{2}, -\frac{5}{\sqrt{21}}.$

(c) $-\frac{3}{\sqrt{13}}, -\frac{2}{\sqrt{13}}, \frac{3}{2}, \frac{2}{3}, -\frac{\sqrt{13}}{2}, -\frac{\sqrt{13}}{3}.$

(d) $-\frac{\sqrt{3}}{2}, \frac{1}{2}, -\sqrt{3}, -\frac{1}{\sqrt{3}}, 2, -\frac{2}{\sqrt{3}}.$

Art. 25, Page 62

11. $\sin\theta = \pm\frac{2ab}{a^2+b^2}$, $\tan = \theta \pm \frac{2ab}{a^2-b^2}$.

Art. 26, Page 66

1. $\sin(270° - A) = -\cos A$, $\cos(270° - A) = -\sin A$,
$\tan(270° - A) = \cot A$, $\cot(270° - A) = \tan A$,
$\sec(270° - A) = -\csc A$, $\csc(270° - A) = -\sec A$.

2. $\sin(270° + A) = -\cos A$, $\cos(270° + A) = \sin A$,
$\tan(270° + A) = -\cot A$, $\cot(270° + A) = -\tan A$,
$\sec(270° + A) = \csc A$, $\csc(270° + A) = -\sec A$.

Art. 27, Page 67

4. (a) $-\sin 73°$, $-\cos 17°$. (e) $-\sin 10°$, $-\cos 80°$.
(b) $\cos 28°$, $\sin 62°$. (f) $\cot 53°$, $\tan 37°$.
(c) $\tan 38° 30'$, $\cot 51° 30'$. (g) $\tan 25°$, $\cot 65°$.
(d) $-\sec 25° 10'$, $-\csc 64° 50'$. (h) $-\sin 15°$, $-\cos 75°$.
(i) $-\csc 49° 30'$, $-\sec 40° 30'$.

5. (a) $-.2391$. (d) 1.555. (g) 2.458.
(b) $-.6225$. (e) .8391. (h) .6293.
(c) $-.1007$. (f) .7660. (i) -1.0724

Art. 28, Page 72

4. $n \cdot 180° + (-1)^n\, 30°$, $n\pi + (-1)^n \frac{\pi}{6}$; $30°, 150°, 390°, 510°$.

5. $n \cdot 360° \pm 120°$, $2n\pi \pm \frac{2}{3}\pi$; $120°, 240°, 480°$.

6. (*a*) $n \cdot 180°$, $n\pi$.

*(*b*) $n \cdot 360° \pm 90°$, $2n\pi \pm \frac{\pi}{2}$.

(*c*) $n \cdot 180°$, $n\pi$.

(*d*) $n \cdot 180° + 60$, $n\pi + \frac{\pi}{3}$.

(*e*) $n \cdot 180° \pm 90°$, $n\pi \pm \frac{\pi}{2}$.

(*f*) $n \cdot 180° + (-1)^n\, 45°$, $n\pi + (-1)^n \frac{\pi}{4}$.

(*g*) $n \cdot 180° - (-1)^n\, 60°$, $n\pi - (-1)^n \frac{\pi}{3}$.

(*h*) $n \cdot 360° \pm 135°$, $2n\pi \pm \frac{3\pi}{4}$.

(*i*) $n \cdot 180° + 45°$; $n\pi + \frac{\pi}{4}$.

(*j*) $n \cdot 180° + 135°$, $n\pi + \frac{3\pi}{4}$.

(*k*) $n \cdot 360° \pm 30°$, $2n\pi \pm \frac{\pi}{6}$.

(*l*) $n \cdot 180° + 150°$, $n\pi + \frac{5\pi}{6}$.

Art. 30, Page 77

3. $x = n\pi + (-1)^n \frac{\pi}{6}$; $x = \sin^{-1} 2$, impossible.

4. $x = 2n\pi + \frac{\pi}{2}$.

5. $y = 2n\pi + \frac{\pi}{4}$.

6. $x = n\pi \pm \frac{\pi}{2}$.

7. $A = 2n\pi \pm \frac{\pi}{3}$, $A = 2n\pi$.

8. $x = n\pi$, $x = n \cdot 360° + (-1)^n\, 38°\ 10'.3$; $x = \sin^{-1}(-1.618)$, impossible.

9. $x = 2n\pi \pm \frac{5\pi}{6}$; $x = \cos^{-1}\sqrt{3}$, impossible.

10. $B = n\pi \pm \frac{\pi}{6}$.

11. $x = n\pi + \frac{\pi}{4}$, $x = n \cdot 180° + 165°\ 47'.7$.

12. $y = n \cdot 180° + 71°\ 33'.8$, $y = n \cdot 180° + 63°\ 26'$.

13. $60°, 120°, 240°, 300°, 420°, 480°$.

14. $x = 30°$, $x = 150°$.

Art. 31, Pages 80, 81

3. $\frac{+\sqrt{21} + 2\sqrt{15}}{20}$; $\frac{+\sqrt{315} - \sqrt{2}}{20}$.

4. $\frac{4}{5}$; $\frac{3}{5}$.

8. $\frac{2\sqrt{2} + \sqrt{3}}{6}$.

Art. 32, Page 83

3. $-\frac{84}{205}$; $\frac{187}{205}$.

4. $\frac{44}{125}$; $\frac{117}{125}$.

5. $\frac{2\sqrt{2} - \sqrt{3}}{6}$.

---

* In this instance the general value may be expressed; $m \cdot 180° \pm 90°$, $m\pi \pm \frac{\pi}{2}$.

Art. 34, Page 85

6. $1;\ \frac{1}{7}$.

9. $\tan^{-1}\left(-\frac{1}{2}\right)$; $\tan^{-1}\frac{2}{11}$.

10. $\tan^{-1}\infty$, i.e., $(2n+1)\frac{\pi}{2}$; $\cot^{-1}0$, i.e., $(2n+1)\frac{\pi}{2}$.

Art. 35, Pages 87, 88

4. $\cos^2\frac{3}{2}x-\sin^2\frac{3}{2}x$, $2\cos^2\frac{3}{2}x-1$, $1-2\sin^2\frac{3}{2}x$; $2\sin\frac{3}{2}x\cos\frac{3}{2}x$

5. $\sqrt{\frac{1+\cos 6x}{2}}$; $\sqrt{\frac{1-\cos 6x}{2}}$.

6. $\cos^2 3x-\sin^2 3x$, $2\cos^2 3x-1$, $1-2\sin^2 3x$; $2\sin 3x\cos 3x$.

10. (a) $\pm\frac{3}{5}$; $\pm\frac{24}{25}$; $-\frac{7}{25}$; $\mp\frac{24}{7}$.

Art. 36, Pages 91, 92, 93

5. (1) $2\sin 4x\cos x$. (2) $2\cos 6A\sin A$.
(3) $2\sin 4x\sin 2x$. (4) $2\cos 7x\cos 2x$.
(5) $2\sin\frac{mA+nB}{2}\cos\frac{mA-nB}{2}$. (6) $-2\sin\frac{mx+ny}{2}\sin\frac{mx-ny}{2}$.
(7) $2\sin(45°-x)\cos(45°-4x)$. (8) $2\cos(45°-x)\sin(45°-3x)$.

6. (1) $\frac{1}{2}(\sin 8x+\sin 2x)$. (2) $\frac{1}{2}(\sin 12x-\sin 2x)$.
(3) $\frac{1}{2}(\cos 4x-\cos 8x)$. (4) $\frac{1}{2}(\cos 14x+\cos 4x)$.
(5) $\frac{1}{2}\{\cos(mA-nB)-\cos(mA+nB)\}$.
(6) $\frac{1}{2}\{\cos(nx+my)+\cos(nx-my)\}$.
(7) $\frac{1}{2}(\cos 2x-\cos 6x)$. (8) $\frac{1}{2}(\cos 10x+\cos 4x)$.

18. 1.

28. $\theta=\frac{n\pi}{4}$, $\theta=\frac{n\pi}{3}+(-1)^n\frac{\pi}{18}$.

29. $\theta=(2n+1)\,90°$, $\theta=\{4n+(-1)^n\}15°$.

30. $\alpha=\frac{n\pi}{2}$, $\alpha=n\cdot 90°+(-1)^n\,45°$.

31. $\alpha=n\cdot 90°+(-1)^n\,45°$, $\alpha=\frac{1}{2}\sin^{-1}(-\frac{3}{5})=n\cdot 90°+(-1)^n(-18°\ 26'\cdot 1)$

32. $\theta=n\frac{\pi}{4}$, $\theta=n\cdot 180°\pm 90°$.

33. $\theta=\{6n+(-1)^n\}30°$, $\theta=\{10n+(-1)^n\}18°$, $\theta=\{10n-3(-1)^n\}18°$.

34. $x=-\frac{1}{6}$, $x=1$. 35. $x=0$, $x=\pm\frac{1}{2}$. 36. $x=\frac{1}{4}$, $x=-8$.

37. $\tan(A+B+C)=\frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}$.

Art. 41, Page 100

3. 38° 12′.8, 60°, 81° 47′.4. 4. 41° 24′.7.

Art. 43, Page 103

2. $b = 70.8$, $a = 56.1$
3. 7.98 ft
4. $b = 185$, $c = 192$.
5. $b = 8.237$, $c = 5.464$.

Art. 44, Page 107

2. $B = 36° 18'.4$ or $143° 41'.6$, $c = 52.71$ or 5.98.
3. Triangle impossible.
4. $B = 46° 28'$, $C = 68° 32'$, $c = 30.8$ in.
5. $A = 48° 10'$, $C = 108° 25'$, $c = 19.1$ ft.; or
   $A = 131° 50'$, $C = 24° 45'$, $c = 8.4$ ft.

Art. 45, Page 108

2. $A = 77° 12'.9$, $B = 43° 30'.1$, $c = 29.97$.
3. $A = 80° 46'.4$, $C = 63° 48'.6$, $b = 19.4$.
4. $R = 33° 3'.3$, $S = 100° 56'.7$, $t = 39.6$ ft.
5. $P = 29° 41'.2$, $Q = 52° 24'.4$, $r = 20$ in.

Art. 46, Page 109

2. $P = 19° 12'$, $Q = 61° 13'$, $R = 99° 35'$.
3. $48° 11'.4$, $58° 24'.7$, $73° 23'.9$.
4. $28° 22'$, $49° 43'$, $101° 56'$, nearly.
5. $93° 41'$, $67° 23'$, $18° 56'$.

Art. 48, Pages 110, 111

2. $C = 66° 47'$, $b = 698.3$, $c = 845$.
3. 600, 421.5.
4. Triangle impossible.
5. $A = 42° 53' 34''$, $C = 77° 5' 54''$, $c = 394.53$.
6. $C = 64° 24'$, $B = 78° 16'$, $b = 749.1$.
7. $B = 46° 52' 10''$, $C = 111° 53' 25''$, $c = 1767.3$; or
   $B = 133° 7' 50''$, $C = 25° 37' 45''$, $c = 823.8$.

Art. 49, Page 112

2. $B = 64° 9' 3''$, $A = 42° 50' 57''$, $c = 374$.
3. $P = 132° 18' 27''$, $Q = 14° 34' 24''$, $r = 67.75$.
4. $A = 109° 15'.5$, $B = 45° 4'.5$, $c = 440.5$.
5. $B = 88° 2'.6$, $C = 23° 43'$, $a = 8.439$.

Art. 50, Page 113

2. $53° 7'.8$, $59° 29'.4$, $67° 22'.8$.
3. $A = 29° 17' 16''$, $B = 31° 55' 31''$.
4. $44° 48' 15''$, $52° 55' 56''$, $82° 15' 49''$.
5. $R = 25° 12' 32''$, $S = 48° 11' 22.''$

Art. 51, Pages 114, 115

3. 444.72 yd.
4. 1112.8 yd.
5. 489.29 yd.; 505.3 yd.
6. 179.28 ft.
7. 87.88 ft.
8. 104.08 ft.
9. 479.3 yd.
10. Height $=119.6$ ft., $AC=153.1$ ft.; $BC=571$ ft.
11. 469.6 ft.

Art. 52, Page 118

1. (*a*) 536.06 sq.ft.
   (*b*) 325.7 sq.ft.
2. (*a*) 668.7 sq.ft.
   (*b*) 500.3 sq.ft.
3. Each of the equal sides $=28.6$ yd.
4. The triangles in which the angles between the given sides are respectively $30° 17'.8$ and $149° 42'.2$.
5. 154.37 yd. on the road opposite the right angle.

Art. 53, Page 119

1. (*a*) $6\frac{2}{3}$ in., $33\frac{1}{3}$ sq.in.
   (*b*) 30 in.; 360 sq.in.
   (*c*) 36 in.; 324 sq.in.
   (*d*) 90 in.; 900 sq.in.
2. $27\frac{3}{7}$ in.; $329\frac{1}{7}$ sq.in.
3. 11.1 in.; 22.2 in.
4. Radius $=14\frac{3}{4}$ in.: angle $=2\frac{10}{59}$ radians $=124°.3$.

Art. 54, Page 121

1. (*a*) 7.67; 21.6; 13.7, 25.5, 54.8 in. respectively.
   (*b*) 2.6; 6.5, 7.3, 16, 5.3 in. respectively.
   (*c*) 10.48; 25.03; 22.04, 30.44, 58.11 in. respectively.

Art. 55, Pages 126, 127

4. 238,890 mi. (approx)., 347.5.
5. About $57' 2''$; about 1:13.5.
6. About 93,757,000 mi.
7. 206,265 times the distance of the earth from the sun; 3.26 yr.
8. 9 ft. 2.6 in.
9. 76 ft. 9.5 in.
10. $4' 35''$.
11. 15.708 yd.
12. $13' 1''.3$.

SPHERICAL TRIGONOMETRY

SPHERICAL TRIGONOMETRY

FOR

COLLEGES AND SECONDARY SCHOOLS

BY

DANIEL A. MURRAY, PH.D.

PROFESSOR OF MATHEMATICS IN MCGILL UNIVERSITY

LONGMANS, GREEN, AND CO.

FOURTH AVENUE & 30TH STREET, NEW YORK

LONDON, BOMBAY, CALCUTTA, AND MADRAS

1919

First Edition, June, 1900.
Reprinted, September, 1900; July, 1902; August, 1903;
April, 1905; February, 1906; August, 1907; March, 1908;
June, 1909; June, 1912; September, 1913; July, 1916;
September, 1919.

PREFACE.

This book contains little more than what is required for the solution of spherical triangles and related simple practical problems. The articles on spherical geometry are necessary for those who have not already studied that subject; for others, they provide a useful review. More than usual attention has been given to the measurement of solid angles. The explanations in connection with the astronomical problems are somewhat fuller than is customary in elementary text-books on spherical trigonometry.

I am indebted to Mr. W. B. Fite, Ph.B., Fellow in Mathematics at Cornell University, for his kind assistance in reading the proof-sheets; and to Mr. A. T. Bruegel, M.M.E., of the Pratt Institute, Brooklyn, N.Y., for the pleasing character of the diagrams.

D. A. MURRAY.

Cornell University,
May, 1900.

CONTENTS.

CHAPTER I.

Review of Solid and Spherical Geometry.

CHAPTER II.

Right-angled Spherical Triangles.

CHAPTER III.

Relations between the Sides and Angles of Spherical Triangles.

CHAPTER IV.

Solution of Triangles.

CHAPTER V.

Circles connected with Spherical Triangles.

CHAPTER VI.

Areas and Volumes connected with Spheres.

CHAPTER VII.

Practical Applications.

APPLICATIONS TO ASTRONOMY.

APPENDIX.

SPHERICAL TRIGONOMETRY.

CHAPTER I.

REVIEW OF SOLID AND SPHERICAL GEOMETRY.

On beginning the study of spherical trigonometry it is advisable to recall to mind or learn some of the definitions and propositions of solid geometry. A clear and vivid conception of the principal properties of the sphere is especially necessary. The definitions and theorems which will be used frequently in the following pages, are quoted in this chapter.*

Planes and Lines in Space. Diedral Angles. Solid Angles.

1. ***a.*** Two planes which are not parallel intersect in a straight line. (Euc. XI. 3.)

b. The angle which one of two planes makes with the other is called a diedral angle. Thus, in Fig. 1, the two planes *BD* and

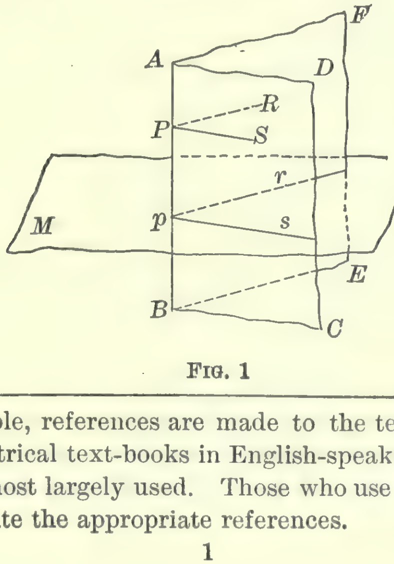


Fig. 1

* As far as possible, references are made to the text of Euclid; since, of the numerous geometrical text-books in English-speaking countries, his work is the one which is most largely used. Those who use a text-book other than Euclid's can substitute the appropriate references.

AE intersect in the straight line *AB*, and form the diedral angle *FABC*.

c. The planes *AE* and *AC* are called the *faces*, and the line *AB* is called the *edge*, of the diedral angle. The faces are unlimited in extent. The magnitude of the diedral angle depends, not upon the extent of its faces, but only upon their relative position. (Just as the magnitude of a plane angle depends, not upon the lengths of its boundary lines, but upon their relative position.)

d. If *PR* be drawn perpendicular to *AB* in the plane *AE*, and *PS* be drawn perpendicular to *AB* in the plane *AC*, the angle *RPS* is called *the plane angle of the diedral angle.*

e. If a plane is drawn perpendicular to the edge of a diedral angle, the intersections of this plane with the faces of the diedral angle form the plane angle of the diedral angle. (See Euc. XI. 4.) Thus, if the plane *M* be passed through *p* perpendicular to *AB*, the intersections, *pr*, *ps*, of the plane *M* and the planes *AE*, *AC*, form the angle *rps* which is the plane angle of *FABC*.

f. All plane angles of the same diedral angle are equal. (See Euc. XI. 10.) Hence, *the plane angle can be taken as the measure of the diedral angle.*

2. ***a.*** If a straight line be at right angles to a plane, every plane which passes through the line is at right angles to that plane. (Euc. XI. 18.)

b. If two planes which cut one another be each of them perpendicular to a third plane, their common section is perpendicular to the same plane. (Euc. XI. 19.)

3. ***a.*** When three or more planes meet in a common point, they are said to form a *solid angle*, or a *polyedral angle*, at that point.

The point in which the planes meet is called the *vertex* of the solid angle; the intersections of the planes are called its *edges;* the portions of the planes between the edges are called its *faces;* the plane angles formed by the edges are called its *face angles;* and the diedral angles formed at the edges by the planes are called the *diedral angles* (or the *edge angles*) of the solid angle.

Thus, in Fig. 2, for the solid angle formed at S: the vertex is S; SB, SC, SD, SE, are the edges; BSE, ESD, etc., are the faces; the face angles are the angles BSE, ESD, DSC, CSB; the diedral (or edge) angles are $BESD$, $EDSC$, etc.

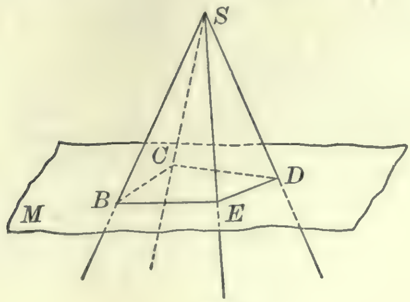


FIG. 2

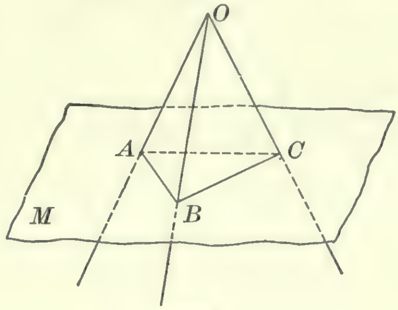


FIG. 3

b. A solid angle with three faces is called a *triedral* angle. Thus, the solid angle at O (Fig. 3) is a triedral angle.

(The measurement of solid angles is discussed in Art. 61. The magnitude of the solid angle in nowise depends upon the lengths of its edges.)

4. ***a.*** The sum of any two face angles of a triedral angle is greater than the third. (See Euc. XI. 20.)

b. The sum of the face angles of any solid angle is less than four right angles (Euc. XI. 21). (This is true, in general, only when the polygon, say $BEDC$ (Fig. 2), formed by the intersections of the faces with a cutting plane M, does not have a re-entrant angle; in other words, when the polygon $BEDC$ is convex.

Geometry of the Sphere.

For the benefit of those who have not studied the geometry of the sphere, proofs of a few of its propositions are either outlined, or given in detail. Some propositions can be proved very easily; hence, only their enunciations are given. Other properties of the sphere will be proved when they are required. (See Arts. 53, 54, 57, 62, 65.) The use of a globe on which figures can be drawn, will be of great assistance to the student. If such a globe is not at hand, a terrestrial or celestial globe can afford some service.

5. The sphere and its plane sections.

***a.* Definitions.** A *spherical surface* is a surface all points of which are equidistant from a point called the *centre.* A *sphere* is a solid bounded by a spherical surface. The surface of a sphere can be generated by the revolution of a semicircle about its diameter. A *radius* of a sphere is a straight line joining the centre to any point on the surface. According to the definition of a sphere, all the radii of a sphere are equal. A *diameter* of a sphere is a straight line passing through the centre and terminated at both ends by the surface. A *plane section of a sphere* is a figure whose boundary is the intersection of a plane and the surface of the sphere.

***b.* Proposition.** *The boundary of every plane section of a sphere is a circle.*

Let the sphere whose centre is at O be cut by a plane in the section ABD; then ABD is a circle. Through O draw OC perpendicular to the plane ABD. Let A and B be any two points in the boundary of the section ABD. Draw OA, OB, CA, and CB. In the two triangles OCA and OCB, the angles at C are equal (both being right angles), the side OC is common, and the side OA is equal to the side OB, since both are radii of the sphere. Hence the triangles are equal in every respect, and CA is equal to CB. But A and B are any two points on the boundary of the section; hence all points on the boundary are equidistant from C. Therefore ABD is a circle whose centre is at C, the foot of the perpendicular let fall from the centre O to the cutting plane ABD.

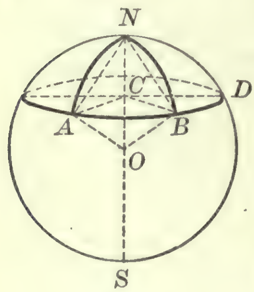


FIG. 4

6. Great and small circles on a sphere.

***a.* Definitions.** The section in which a sphere is cut by a plane is called a *Great Circle* when the plane passes through the centre of the sphere; the section is called a *Small Circle* when the cutting plane does not pass through the centre of the sphere. Thus, on a terrestrial globe the meridians and equator are great circles; the parallels of latitude are small circles. The *Axis* of a circle of a sphere is the diameter of the sphere perpendicular to the plane of the circle; the extremities of the axis are called the *Poles* of the circle and any of its arcs. Thus, in Fig. 4, Art. 5, N and S are the poles of the circle ABD and of the arcs AB and BD. It is obvious that all circles made by the intersections of parallel planes with a sphere have the same axis and poles. For instance, all parallels of latitude have the same axis and poles, namely, the polar axis of the earth and the North and South Poles.

***b.* Propositions relating to great circles.**

Every great circle bisects the surface of the sphere; *e.g.* the equator bisects the surface of a terrestrial globe.

Any two great circles bisect each other; *e.g.* the meridians bisect one another at the poles. All great circles of a sphere are equal; since their radii are radii of the sphere.

A great circle can be passed through any two points on a sphere; since a plane can be made to pass through these two points and the centre of the sphere, and this plane intersects the surface of a sphere in a great circle. In general, only one great circle can be drawn through two points on a sphere, since these points and the centre determine a plane; but, when the two given points are at the ends of a diameter an infinite number of great circles can be drawn through them; *e.g.* the meridians passing through the North and South Poles.

***c.* Definitions.** By **distance between two points on a sphere** is meant the shorter arc of the great-circle passing through them. It is shown in Art. 20 that this arc is the *shortest line* that can be drawn on the surface of the sphere from the one point to the other. For example, the arc NA in Fig. 4 measures the distance between the points N and A. [Ex. Distance between N and S?]

NOTE. The theorem in Art. 20 can be shown *mechanically* by taking two points on a parallel of latitude on a globe and letting a string be stretched taut from one point to the other. The string will not lie on the parallel, but will evidently be in a plane which passes through the centre of the sphere. If the two points be on a meridian, the stretched string will lie on the meridian.

By **angular distance** between two points on a sphere is meant the angle subtended at the centre of the sphere by the arc joining the given points. Thus in Fig. 4 the angle NOA is the angular distance of A from N.

d. Propositions and definitions relating to small and great circles. In Fig. 4 all the arcs of great circles, as NA, NB, ND, drawn from points on the circle ABD to the pole N, are equal. Thus the arcs of meridians on a terrestrial globe drawn from a parallel of latitude to the North Pole are equal. The chords NA, NB, ND, are all equal; the angles AON, BON, DON, are likewise equal. It thus appears that all points in the circumference of a circle on a sphere are equally distant from a pole of the circle, whether the distance be measured by the arc of a great circle joining one of the points and the pole, or by the straight line joining the point and the pole, or by the angle which such an arc or chord subtends at the centre of the sphere.

Definitions. The last mentioned angle is called the *angular radius* of the circle. The angular radius of a great circle is evidently a right angle. The *polar distance* of a circle on a sphere is its distance from its pole, the distance being measured along an arc of a great circle passing through the pole. Thus the north polar distance of a parallel of latitude is its distance from the North Pole measured along a meridian. The term *quadrant*, when used in connection with a sphere, usually means an arc equal in length to one-fourth of a great circle. The polar distance of each point on a great circle is evidently a quadrant; *e.g.* a point on the equator is at a quadrant's distance from the North or South Pole. Points on a great circle are equidistant from both its poles. The polar distance of a circle may be called the *radius* of the circle.

7. To draw circles upon the surface of a sphere about a given point as pole.

(*a*) *With a pair of compasses.* Open the compasses until the distance between the points of the compasses is equal to the chord of the polar distance (or, what is the same thing, the chord subtended by the angular radius) of the required circle. Then, one point being placed and kept fixed at the pole, the other can describe the circle.

(*b*) *With a string.* Take a string equal in length to the polar distance of the required circle. If the string be kept stretched taut, and one end be fixed at the pole while the other end moves on the sphere, the required circle will be described.

In order to describe a great circle the polar distance must be taken equal to a quadrant of the sphere.

8. Proposition. *If a point on the surface of a sphere lies at a quadrant's distance from each of two points, it is the pole of the great circle passing through these points.*

If the point P be at a quadrant's distance from each of the points A and B, then P is the pole of the great circle passing through A and B. Let O be the centre of the sphere, and draw OA, OB, OP. Since PA and PB are quadrants, the angles POA and POB are right angles. Hence PO is perpendicular to the plane AOB (Euc. XI. 4); therefore P is the pole of the great circle ABL.

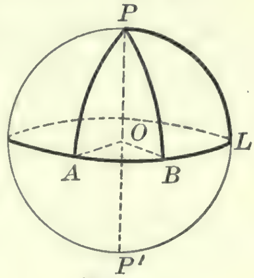


FIG. 5

9. Problem. *Through two given points to draw an arc of a great circle.* About each point as a pole draw a great circle (Art. 7). The two points of intersection of the great circles thus drawn are each at a quadrant's distance from the two given points; and hence, by Art. 8, are the poles of the great circle through the two given points. Accordingly, the required arc will be obtained by describing a great circle about either of these poles.

NOTE. If the two given points are diametrically opposite, an infinite number of great circles can be drawn through them. (Art. 6. *b*.)

10. Lines and planes which are tangent to a sphere.

***a*. Definitions.** A straight line or a plane is said to be *tangent* to a sphere when it has but one point in common with the surface of the sphere. The common point is called the *point of contact* or *point of tangency*.

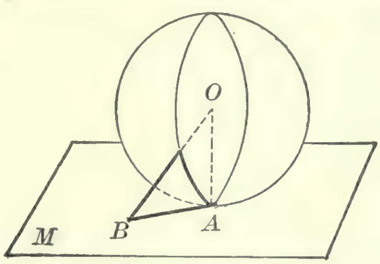


FIG. 6

b. Propositions. (See Fig. 6.)

A plane or a line perpendicular to a radius at its extremity is tangent to the sphere. [Suggestion for proof: The perpendicular is the shortest line that can be drawn from a point to a plane.]

A tangent to an arc of a great circle at any point of the arc is perpendicular to the radius (of the sphere) drawn to the point.

11. On spherical angles.

a. Definitions. *The angle made by any two curves* meeting in a common point is the angle formed by the two tangents to the curves at that point. Thus in Fig. 7, the angle made by the curves C_1 and C_2 at the point P, is the angle T_1PT_2 between the tangents to C_1 and C_2 at P. (This definition applies to all curves, whether they are in the same plane or not.)

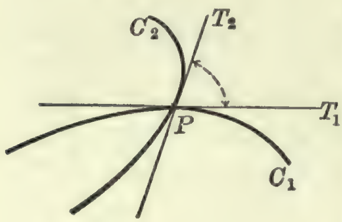


Fig. 7

A spherical angle is the angle formed by two intersecting arcs of great circles on the surface of a sphere. Thus the angle formed by the arcs CA and CB (Fig. 8) is a spherical angle. This angle is the angle ECD between the tangents CE and CD. But ECD is the plane angle of the diedral angle between the planes COA and COB which are the planes of the arcs CA and CB. Thus *the spherical angle is equal to the diedral angle of the planes of the arcs forming the angle.*

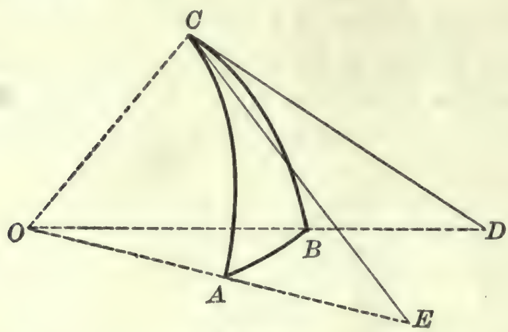


Fig. 8

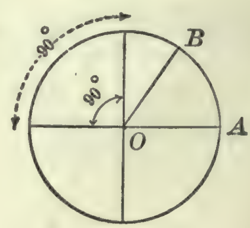


Fig. 9

b. Propositions. (1) If two arcs of great circles intersect, the opposite vertical angles thus formed are equal. Thus in Fig. 49, Art. 57, the angles BAC and $B'AC'$ are equal.

(2) If one arc of a great circle meets another arc of a great circle, the sum of the adjacent spherical angles is equal to two right angles. Thus in Fig. 49, $CAB + CAB' = 2$ right angles.

NOTE. It is shown in plane geometry that angles at the centre of a circle are proportional to their intercepted arcs; hence, the angles can be measured by the arcs. Accordingly, if each right angle at the centre of a circle (Fig. 9) be divided into 90 equal parts called degrees, and the circle be divided into 360 equal parts, also called degrees, then the number of degrees (of angle) in any angle AOB is equal to the number of degrees (of arc) in AB, the arc subtended by AOB. [When it is necessary to distinguish between *degrees of angle* and *degrees of arc*, the former may be called *angular degrees;* and the latter *arcual degrees.*]

***c.* Proposition.** *A spherical angle is measured by the arc of a great circle described with its vertex as a pole and included between its boundary arcs, produced if necessary:*

Let ABC and $AB'C$ be two intersecting arcs of great circles on the sphere S whose centre is at O. Pass the plane BOB' through O perpendicular to AC, and let this plane intersect the planes ABC and $AB'C$ in the radii OB and OB', and intersect the sphere in the great circle $B'BL$. From the construction, A is the pole of the great circle $B'BL$. By Art. 1. *e.* BOB' is the plane angle of the diedral angle $BACB'$, and, accordingly (Art. 11. *a*), is equal to the spherical angle BAB'. Now, by the preceding note, the number of degrees in the arc BB' is equal to the number of degrees in the angle BOB'. Hence, the number of degrees in the arc BB' is equal to the number of degrees in the angle BAB'. In other words, the spherical angle BAB' is measured by the arc BB' of which A is the pole.

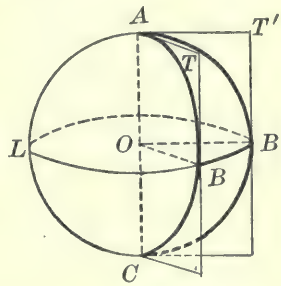


FIG. 10

This can be illustrated on a terrestrial globe. For instance, the angle at the North Pole between the meridians of Paris and New York is $76° \, 2' \, 25.5''$; and this is the number of degrees of arc intercepted by these meridians on the equator.

d. The great circles drawn through any point on a sphere are perpendicular to the great circle of which the point is the pole.

For instance, the meridians of longitude cross the equator at right angles.

e. The distance of any point on the surface of a sphere, from a circle traced thereon, is measured by the shorter arc of a great circle passing through the point and perpendicular to the given circle; that is, by the shorter arc of the great circle passing through the given point and the pole of the given circle. For example, on a globe the latitude of any place (*i.e.* its distance in degrees from the equator) is measured by the arc of the meridian intercepted between the place and the equator.

N.B. When an *arc on a sphere* is referred to, an arc of a great circle is meant, unless expressly stated otherwise.

ON SPHERICAL TRIANGLES.

12. Definitions. A *spherical polygon* is a portion of the surface of a sphere bounded by three or more arcs of great circles. The bounding arcs are the *sides* of the polygon; the points of intersection of the sides are the *vertices* of the polygon, and the angles which the sides make with one another are the *angles* of the polygon. A *diagonal* of a spherical polygon is an arc of a great circle joining any two vertices which are not consecutive.

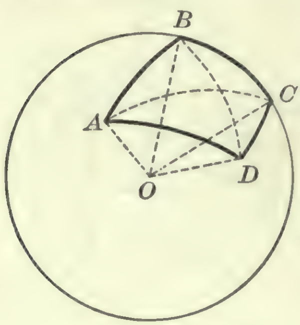


Fig. 11

A *spherical triangle* is a spherical polygon of three sides.

Thus, in Fig. 11, $ABCD$ is a spherical polygon; its sides are AB, BC, CD, DA; its angles are ABC, BCD, CDA, DAB; its diagonals are BD and AC; ADC and ABC are spherical triangles. Since the sides of a spherical polygon are arcs of great circles, their magnitudes are expressed in degrees.* The lengths of the sides can be calculated in terms of *linear* units when the radius of the sphere is known.

A spherical triangle is *right-angled, oblique, scalene, isosceles,* or *equilateral,* in the same cases as a plane triangle. The notation

* The reason for expressing the sides of spherical polygons in degrees is considered more fully in Art. 14.

adopted in discussing the plane triangle will be used for the spherical triangle; namely, the triangle will be denoted by ABC, and the sides opposite the angles A, B, C, will be denoted by a, b, c, respectively.

Two spherical polygons are **equal** if they can be applied one to the other so as to coincide. They are said to be **symmetrical** when the sides and angles of the one are respectively equal to the sides and angles of the other, but arranged in *the reverse order.*

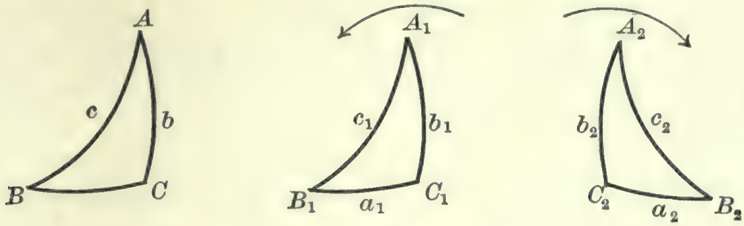


FIG. 12

Thus, the spherical triangles ABC and $A_1B_1C_1$ (Fig. 12) are *equal* if they can be brought into coincidence, say, by sliding one of them, as ABC, over the surface of the sphere until it exactly covers the surface $A_1B_1C_1$. Accordingly, it is evident that if these triangles are equal, the angles A, B, C, are respectively equal to the angles A_1, B_1, C_1, and the sides a, b, c, are respectively equal to the sides a_1, b_1, c_1.* On the other hand, the triangles ABC and $A_2B_2C_2$ are *symmetrical* if the angles A, B, C, are respectively equal to the angles A_2, B_2, C_2, and the sides a, b, c, to the sides a_2, b_2, c_2. In this case, the triangle ABC cannot be brought into coincidence with $A_2B_2C_2$ by a sliding motion over the surface of the sphere.

NOTE 1. Two symmetrical spherical triangles can be brought into coincidence if the surface be covered very thinly with some flexible material. For then ABC can be lifted up, turned over, and the surface bent (or made to 'spring back') in the opposite direction; after this treatment, ABC can be made to coincide with $A_2B_2C_2$.

NOTE 2. The meaning of the phrase *reverse order* can be seen clearly on considering the triangles $A_1B_1C_1$ and $A_2B_2C_2$ above. In $A_1B_1C_1$, on

* Some of the sets of *minimum* conditions necessary for equality of spherical triangles are stated in Art. 13.

going from A_1 to B_1, thence to C_1, and thence to A_1, one goes around any point within the triangle in a *counter-clockwise* direction. In $A_2B_2C_2$, on the other hand, on taking the respective equal angles in the same order as before, that is, on going from A_2 to B_2, thence to C_2, and thence to A_2, one goes round any point within the triangle $A_2B_2C_2$ in a *clockwise* direction. The directions are indicated by the arrows.

13. Propositions. (1) *Two spherical triangles which are on the same sphere, or on equal spheres, and whose parts are in the same order* (as ABC and $A_1B_1C_1$, Fig. 12) *are* **equal** *under the same conditions as plane triangles*, viz.:

(*a*) When two sides and their included angle in the one triangle are respectively equal to two sides and their included angle in the other;

(*b*) When a side and its two adjacent angles in the one triangle are respectively equal to a side and its two adjacent angles in the other;

(*c*) When the three sides of the one triangle are respectively equal to the three sides of the other.

[SUGGESTION FOR PROOFS. Equality can be shown by the same methods as in plane geometry.]

(2) *Two spherical triangles which are on the same sphere, or on equal spheres, and whose parts are in the reverse order* (as ABC and $A_2B_2C_2$, Fig. 12), are **symmetrical** under the conditions (*a*), (*b*), (*c*), above.

[SUGGESTIONS FOR PROOF. Construct * a triangle $A_1B_1C_1$ which is symmetrical to $A_2B_2C_2$. Under the given conditions, according to the preceding proposition, ABC and $A_1B_1C_1$ have all their parts respectively equal, and hence ABC and $A_2B_2C_2$ have all their parts respectively equal, and are accordingly symmetrical.]

On a plane two triangles may have three angles of the one respectively equal to three angles of the other and yet not be equal. On the other hand, as will be made apparent in Arts. 16, 24:

(3) *On the same sphere, or on equal spheres, two triangles which have three angles of the one respectively equal to three angles of the other, are either equal or symmetrical.*

---

* For the construction of spherical triangles under various conditions, see Art. 24.

14. Correspondence between the face angles and the diedral angles of a triedral angle on the one hand, and the sides and angles of a spherical triangle on the other.

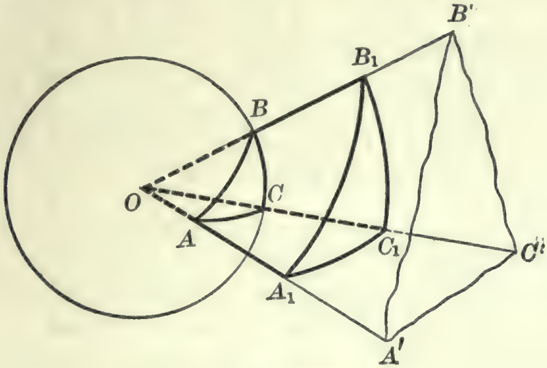


FIG. 13

Take any triedral angle $O\text{–}A'B'C'$; let a sphere of any radius, OA say, be described about O as centre; and let the intersections of this sphere with the faces $OA'B'$, $OB'C'$, and $OC'A'$, be the arcs AB, BC, and CA respectively. The sides of the spherical triangle ABC, namely, AB, BC, CA, measure the face angles, AOB, BOC, COA, of the solid angle $O\text{–}A'B'C'$ (Art. 11. *b*, Note). By Art. 11 the angles CAB, ABC, BCA, of the spherical triangle ABC are the diedral angles between the planes of the sides, that is, the diedral angles of the solid angle $O\text{–}A'B'C'$.

Hence, to find the relations existing between the face angles and the edge angles of a triedral angle, is the same thing as to find the relations between the sides and angles of the spherical triangle, intercepted by the faces, upon the surface of any sphere whose centre is at the vertex of the triedral angle.

NOTE 1. The number of degrees in the intercepted arcs does not depend upon the radius of the sphere. Thus, in Fig. 13, if a sphere is described with a radius OA_1, about O as a centre, the number of degrees in the intercepted arc A_1B_1 is the same as the number of degrees in the intercepted arc AB, for each number is the same as the number of degrees in the angle $A'OB'$.

Since the face angles and diedral angles of a triedral angle are not altered by varying the radius of the sphere, the relations between the sides and angles of the corresponding spherical triangle are independent of the length of the radius.

Note 2. Since the side of a spherical triangle measures the angle subtended by it at the centre, the side is measured in degrees or radians. (See Art. 12.) By "sin AB," for example, is meant the sine of the angle AOB, subtended by AB at the centre O.

Note 3. A three-sided spherical figure, one or more of whose sides is not an arc of a great circle, is not regarded as a spherical triangle. For example, the figure bounded by an arc of a parallel of latitude and the arcs of two meridians does not correspond to a triedral angle at the centre of the sphere, and is not a spherical triangle as defined in Art. 12.

Note 4. A triedral angle, and its corresponding spherical triangle, can be easily constructed. From stiff cardboard cut out a circular sector having any arc between 0° and 360°. On this sector draw any two radii, taking care, however, that no one of the three sectors thus formed shall be greater than the sum of the other two. Along these radii cut the cardboard partly through. Bend the two outer sectors over until their edges meet; a figure like O–ABC (Fig. 13) will be obtained. (Find what happens if the above precaution in drawing the radii is not taken.)

This perfect correspondence between the sides and angles of a spherical triangle on the one hand, and the face angles and diedral angles of the solid angle subtended at the centre of the sphere by the triangle on the other hand, **is very important,** both for the deduction of the relations between these sides and angles and for the solution of practical problems. This correspondence holds in the case of any spherical polygon and the solid angle subtended by it at the centre of the sphere. (The student may inspect Fig. 11.) Hence, *from any property of polyedral angles an analogous property of spherical polygons can be inferred, and vice versa.*

15. Propositions. (1) *Any side of a spherical triangle is less than the sum of the other two sides.* This follows from Arts. 14 and 4. *a*.

Cor. Any side of a spherical polygon is less than the sum of the remaining sides.

(2) *The sum of the sides of a spherical polygon (not re-entrant) is less than 360°.* In other words: *The perimeter of any (non-re-entrant) spherical polygon is less than the length of a great circle.* This important proposition follows from Arts. 14 and 4. *b*.

(3) In an isosceles spherical triangle the angles opposite the equal sides are equal.

(4) The arc of a great circle drawn from the vertex of an isosceles spherical triangle to the middle of the base is perpendicular to the base, and bisects the vertical angle.

(5) If two angles of a spherical triangle are unequal, the opposite sides are unequal, and the greater side is opposite the greater angle.

Cor. If two edge angles of a triedral angle are unequal, the opposite face angles are unequal, and the greater face angle is opposite the greater diedral angle.

(6) If two sides of a spherical triangle are unequal, the opposite angles are unequal, and the greater angle is opposite the greater side.

Ex. Give the corresponding proposition for a triedral angle.

Propositions (3)–(6) can be proved in the same way as the corresponding propositions in plane geometry.

ON POLAR TRIANGLES.

16. ***a.*** Note. Three straight lines on a plane, no two of which are parallel, intersect in three points, and form one triangle. Three great circles on a sphere have six points of intersection, and form eight spherical triangles. Thus, on a globe, the equator and any two great circles through the poles have as intersections the two poles and the four points where the two great circles cross the equator; and there are eight triangles formed, namely, four in the northern hemisphere and four in the southern.

b. **Definitions.** If great circles be described with the vertices of a spherical triangle, say ABC (Fig. 14), as poles; and if there be taken that intersection of the circles described with B and C as poles which lies on the same side of BC as does A, namely A_1; and similarly for the other intersections; then a spherical triangle is formed, which is called **the polar triangle** of the first triangle ABC.

Two spherical polygons are *mutually equilateral* when the sides of the one are respectively equal to the sides of the other, whether taken in the same or in the reverse order; the polygons are *mutually equiangular* when the angles of the one are respectively equal to the angles of the other, whether taken in the same or in the reverse order.

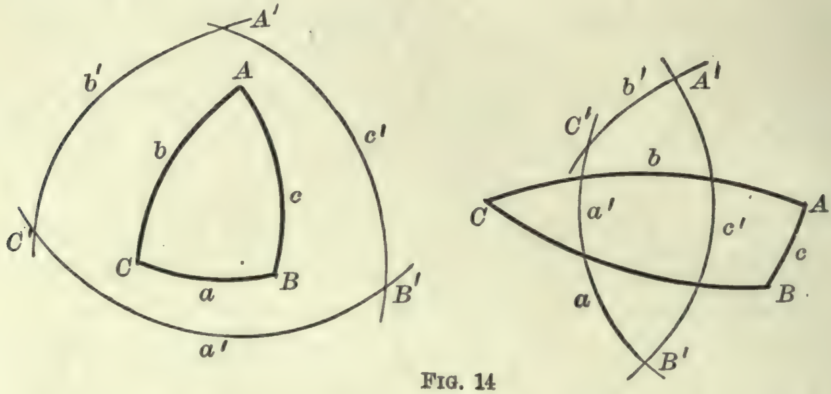


FIG. 14

***c.* Proposition.** *If the first of two spherical triangles is the polar triangle of the second, then the second is the polar triangle of the first.*

If $A'B'C'$ (Fig. 14) is the polar triangle of ABC, then ABC is the polar triangle of $A'B'C'$. Since A is the pole of the arc $B'C'$, the point A is a quadrant's distance from B'. Also, since C is the pole of $B'A'$, the point C is a quadrant's distance from B'. Since B' is thus a quadrant's distance from both A and C, it is the pole of the arc AC (Art. 8). Similarly it can be shown that A' is the pole of the arc BC, and that C' is the pole of the arc AB. Hence ABC is the polar triangle of $A'B'C'$.

***d.* Proposition.** *In two polar triangles, each angle of the one is the supplement of the side opposite to it in the other.*

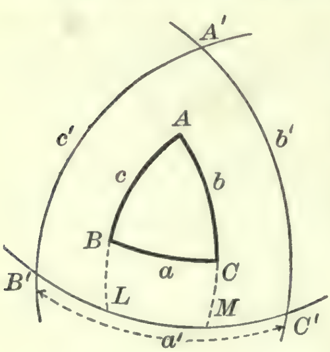


FIG. 15

Let ABC and $A'B'C'$ (Fig. 15) be a pair of polar triangles, in which A, B, C, A', B', C', are the angles, and a, b, c, a', b', c', are the sides. Then

$$A = 180 - a', \qquad A' = 180 - a,$$
$$B = 180 - b', \qquad B' = 180 - b,$$
$$C = 180 - c', \qquad C' = 180 - c$$

Produce the arcs AB and AC to meet $B'C'$ in L and M respectively.

Since B' is the pole of ACM, $\quad B'M = 90°$; and

since C' is the pole of ABL, $\quad LC' = 90°$.

Hence $$B'M + LC' = 180°;$$

that is, $$B'M + MC' + LM = 180°,$$

or $$B'C' + LM = 180°. \qquad (1)$$

Since A is the pole of the arc $B'C'$, the arc LM measures the angle A (Art. 11. c).

Hence, (1) becomes $A + a' = 180°$, or $A = 180° - a'$.

The other relations can be proved in a similar manner.

Cor. If two spherical triangles are mutually equiangular, their polar triangles are mutually equilateral. If two spherical triangles are mutually equilateral, their polar triangles are mutually equiangular.

Note. On account of the properties in (d), a triangle and its polar are sometimes called *supplemental triangles.*

***e.* The use of the polar triangle.** Because of the fact that the sides and angles of a triangle are respectively supplementary to the angles and sides of its polar triangle, many relations can be easily derived by reference to the polar triangle. For, if a relation is true for spherical triangles in general, then it is true for the polar of any triangle. Let the relation be stated for the polar triangle; in this statement express the values of the sides and angles of the polar triangle in terms of the angles and sides of the original triangle; the statement thus derived expresses a new relation between the parts of the original triangle. This will be exemplified in later articles.

17. Proposition. *The sum of the angles of a spherical triangle is greater than two, and less than six, right angles.*

Let ABC be any spherical triangle; it is required to show that

$$180° < A + B + C < 540°.$$

Construct the polar triangle $A'B'C'$. Then, by Art. 16. d,

$$A + a' = 180°, \; B + b' = 180°, \; C + c' = 180°.$$

Hence, on adding, $A + B + C + a' + b' + c' = 540°$,

or, $$A + B + C = 540° - (a' + b' + c').$$

Now [Art. 15 (2)] $a' + b' + c'$ is less than 360°, and greater than 0°.

$\therefore (A + B + C) = 540° -$ (something less than 360° and greater than 0°).

$$\therefore A + B + C > 540° - 360°, \text{ i.e. } A + B + C > 180°;$$

and $$A + B + C < 540° - 0°, \text{ i.e. } A + B + C < 540°.$$

18. Definitions. ***a.*** The amount by which the sum of the three angles of a spherical triangle is greater than 180° is called its **spherical excess.** It is shown in Art. 57 that the area of a triangle depends upon its spherical excess.

b. A spherical triangle may have two right angles, three right angles, two obtuse angles, or three obtuse angles. For example, on a globe the spherical triangle bounded by any arc (not 90°) on the equator and the arcs of the meridians joining the extremities of the former arc to the North Pole, has two right angles; if the arc on the equator is a quadrant, the triangle has three right angles. The polar of the triangle whose sides are 35°, 25°, 15°, has three obtuse angles. A spherical triangle having two right angles is called a *bi-rectangular triangle,* and a spherical triangle having three right angles is called a *tri-rectangular* triangle. A triangle having one side equal to a quadrant is called a *quadrantal* triangle; one having two sides each a quadrant is said to be *bi-quadrantal,* and one having each of its three sides equal to a quadrant is said to be *tri-quadrantal.*

c. **A lune** is a spherical surface bounded by the halves of two great circles. The *angle of the lune* is the angle made by the two great circles. For instance, on a globe the surface between the meridians 10° W. and 40° W. is a lune; the angle of this lune is equal to 30°. On the same circle or on equal circles lunes having equal angles are equal. (For they can evidently be made to coincide.)

19. On the convention that each side of a spherical triangle be less than 180°. In spherical geometry and trigonometry it is found convenient to restrict attention to triangles the sides of which are each less than a semicircle or 180°. (This convention can be set aside when it is necessary to consider what is called the *general spherical triangle*, in which an element may have any value from 0° to 360°.) A triangle such as $ADBC$ (Fig. 16) which has a side ADB greater than 180°, need not be considered; for its parts can be immediately deduced from the parts of ACB, each of whose sides is less than 180°. It is easily proved that if an angle of a spherical triangle is greater than 180°, the opposite side is also greater than 180°, and *vicê versâ*. Thus, in the triangle $ADBC$, if the angle ACB is greater than 180°, so is the side ADB; and if ADB is greater than 180°, so is the opposite angle. [*Suggestion for proof:* Produce the arc AC to meet the arc ADB.]

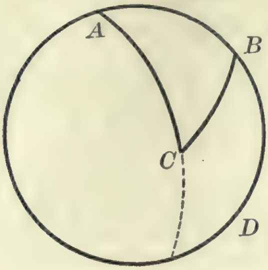


FIG. 16

20. Proposition. The shortest line *that can be drawn on the surface of a sphere between two given points is the arc of a great circle, not greater than a semicircle, which joins the points.*

Let A and B be any two points on a sphere, and let ACB be a great-circle arc not greater than a semicircle; then ACB is the shortest line that can be drawn from A to B on the sphere.

About A as a pole describe a circle DCE with radius AC, and about B as a pole describe a circle FCG with radius BC. It will be shown (1) that C is the only point which is common to both these circles; (2) that the shortest line that can be drawn from A to B on the surface must pass through C.

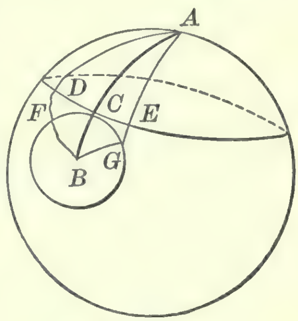


FIG. 17

(1) Take any point G, other than C, on the circle FCG. Draw the great-circle arcs AEG and BG. By Art. 15 (1),

$$AG + GB > AB; \text{ i.e. } AG + GB > AC + CB.$$

Now $\qquad AE = AC, \text{ and } GB = CB.$

Hence $\qquad AE + GB = AC + CB;$

and, accordingly, $\qquad AG > AE.$

Therefore G is outside of the circle DCE. But G is any point (other than C) on the circle FCG. Hence C is the only point common to the circles DCE and FCG.

(2) Let $ADFB$ be any line drawn on the surface from A to B, but not passing through C. Whatever the character of the line AD may be, a line exactly like it can be drawn from A to C; and a line like BF can be drawn from B to C.

[This can be seen by regarding A–DCE as a cap fitting closely to the sphere, and supposing that this cap revolves about A until D is at C. Then a line exactly like AD is drawn from A to C.]

These lines being drawn, there will be a line from A to B which is less than $ADFB$ by the part DF. It has thus been proved that a line can be drawn from A to B through C which is shorter than any other line from A to B which does not pass through C. But C is *any* point on the great-circle arc from A to B. Hence the shortest line from A to B must pass through *every* point in ACB, and, accordingly, must be the arc ACB itself.

Note. This proposition can also be proved by the *method of limits*. It is shown that the length of any arc on a sphere is equal to the limit of the sum of the lengths of an infinite number of infinitesimal great-circle arcs inscribed in the given arc. (See Rouché et De Comberousse, *Traité de Géométrie*.) See Art. 6. *c*.

PROBLEMS OF CONSTRUCTION.

21. The actual making of the following constructions will add much to the clearness and vividness of the notions of most students about the surface of a sphere. An easy familiarity with the problems of Arts. 23, 24, which discuss the construction of triangles, will place the student in an advantageous position with respect to spherical trigonometry. This position is similar to that occupied by him, through his knowledge of the construction of plane triangles, when he entered upon the study of plane trigonometry. (See *Plane Trigonometry*, p. 20, Note, Art. 21, Art. 34 (to Case I.), Art. 53.)

N.B. The student should try to make these constructions for himself, and should fall back upon the book only as a last resort.

22. Problems on great circles.

(1) *To find the poles of a given great circle.* About any two points of the given circle as poles, describe great circles; their intersections will be the poles required (Art. 8).

(2) *To draw a great circle through two given points.* About the two given points as poles, describe great circles; about either of the intersections of these circles as a pole, describe a great circle; this will be the circle required. (See Arts. 8, 9.)

(3) *To cut from a great circle an arc $n°$ long.* Separate the points of the compasses by a distance equal to a chord which subtends a central angle of $n°$ in a circle whose radius is equal to the radius of the sphere; place the points of the compass on the great circle; the intercepted arc will be the one required.

(4) *To draw a great circle through a given point perpendicular to a given great circle.* Find a pole of the given circle by (1); draw a great circle through this pole and the given point by (2); this circle will be the one required (Art. 11. *d*).

(5) *To construct a great circle making a given angle with a given great circle, the point of intersection being given.* About the given point of intersection as pole, describe a great circle; on this circle lay off an arc, measured from the given circle, having as many (arcual) degrees as there are (angular) degrees in the given angle; draw a great circle through the extremity of this arc and the given point of intersection; this will be the circle required (Art. 11. *c*).

(6) *To construct a great circle passing through a given point and making a given angle with a given great circle.* [When the given point is on the given circle this problem reduces to problem (5).] It is easily shown that the angle between two great circles is equal to the angular distance (Art. 6. *c*) between their poles. Hence, find a pole of the given circle by (1); about this point as pole describe a second circle whose angular radius (Art. 6. *d*) is equal to the given angle; the pole of the required circle must be on this second circle. About the given point as pole describe a great circle; if the required problem is possible, this circle will either touch or intersect the second circle. The points of contact or intersection are the poles of two great circles, each of which will satisfy the given conditions.

Ex. Discuss the case in which the given point is the pole of the given circle

23. Construction of triangles. The three sides and the three angles of a spherical triangle constitute its *six parts* or *elements*. *If any three of these six parts be known, the triangle can be constructed.* The *construction* belongs to geometry; the *computation* of the three remaining parts, when three parts are given, is an important part of spherical trigonometry. The sets of three parts that can be taken from the six parts of a spherical triangle are as follows:

I. **Three sides.**
II. **Three angles.**
III. **Two sides** and **their included angle.**
IV. **One side** and **the two adjacent angles.**
V. **Two sides** and **the angle opposite one of them.**
VI. **Two angles** and **the side opposite one of them.**

Note. There are *four* construction problems in the case of plane triangles (Plane Trig., Art. 53). When three angles of a spherical triangle are given, there is only one spherical triangle (with the triangle symmetrical to it), as will presently appear, which satisfies the given conditions. When three angles of a plane triangle are given, there is an infinite number of triangles, of the same shape, but of different magnitudes, which have angles equal to the three given angles. Cases IV. and VI. above, in which two angles are given, reduce to a single case in plane trigonometry, namely, the case in which one side and two angles are given; since the sum of the three angles of any plane triangle is 180°.

24. To construct a spherical triangle.

I. **Given the three sides.** On any great circle lay off an arc equal to one of the given sides [Art. 22 (3)]. About one extremity of this arc as pole, describe a circle with a radius (arcual) equal to the second of the given sides; about the other extremity of the arc as pole, describe a circle with a radius equal to the third of the given sides. By arcs of great circles join either of the points of intersection of the last two circles to the extremities of the arc first laid off; the triangle thus formed satisfies the given conditions.

Ex. 1. Compare the construction in the corresponding case in plane triangles.

Ex. 2. How many triangles are possible when the first arc is laid off? Are these triangles equal or symmetrical?

Ex. 3. Construct ABC: (*a*) Given $a = 70°$, $b = 65°$, $c = 40°$; (*b*) Given $a = 120°$, $b = 115°$, $c = 80°$.

Ex. 4. Determine approximately the angles of these triangles. (See Arts. 11. *c*, 34.)

II. **Given the three angles.** Calculate the sides of the polar triangle (Art. 16. *d*); construct it by I. above; construct *its* polar (Art. 16. *b*); the latter triangle is the one required.

Ex. 1. How many triangles can be drawn when one side of the polar triangle is fixed? Are these triangles equal or symmetrical?

Ex. 2. Discuss the corresponding case in plane triangles.

Ex. 3. Construct ABC: (*a*) Given $A = 85°$, $B = 75°$, $C = 55°$; (*b*) Given $A = 75°$, $B = 105°$, $C = 100°$.

Ex. 4. Determine approximately the sides of these triangles.

III. **Given two sides and their included angle.** Take any point on any great circle; through this point draw a circle making with the first circle an angle equal to the given included angle [Art. 22 (5)]; from the chosen point and on the first circle bounding this angle, lay off an arc equal to one of the given sides; from the same point and on the second circle bounding the angle, lay off an arc equal to the other given side. Join the extremities of these arcs by the arc of a great circle; the triangle thus formed is the one required.

Ex. 1. How many triangles can be made when the first circle and the point are chosen? Are these possible triangles equal or symmetrical?

Ex. 2. Discuss the corresponding case in plane triangles.

Ex. 3. Construct ABC: (*a*) Given $a = 75°$, $b = 120°$, $C = 65°$; (*b*) Given $b = 35°$, $c = 70°$, $A = 110°$.

Ex. 4. Determine approximately the remaining parts of these triangles.

IV. **Given a side and its two adjacent angles.**

Either: ***a.*** On any arc of a great circle lay off an arc equal to the given side; its extremities will be taken as two vertices of the required triangle. Through one extremity of this arc draw a great circle making with the arc an angle equal to one of the given angles; through the other extremity of the arc draw a great circle making with the arc (and on the same side as the angle first constructed) an angle equal to the other of the given angles. The point of intersection of these two circles which is on the same side of the arc as the two angles, is the third vertex of the required triangle.

Or: ***b.*** Calculate the corresponding parts of the polar triangle; construct it by III.; construct *its* polar; this will be the triangle required.

Ex. **1.** How many triangles are possible when the first arc is chosen? Are these triangles equal or symmetrical?

Ex. **2.** Discuss the corresponding case in plane triangles.

Ex. **3.** Solve Problem III. by means of IV. a, and the polar triangle.

Ex. **4.** Construct ABC: (a) Given $a = 75°$, $B = 65°$, $C = 110°$; (b) Given $b = 110°$, $A = 40°$, $C = 63°$.

Ex. **5.** Determine approximately the remaining parts of these triangles.

V. **Given two sides and the angle opposite to one of them.** [First, review the corresponding case in plane geometry.]

To construct a triangle ABC when a, b, A, are known: Through any point A of a great circle $ALA'A$ draw the semicircle, ACA', making an angle $A'LAC$ equal to the given angle A. From this semicircle cut off an arc AC equal to b. About C as a pole, with an arc equal (in degrees) to the side a, describe a circle. The intersection B of this circle with ALA' will be the third vertex of the required triangle, A and C being the other two vertices.

Four cases arise, viz.: —

(1) When the circle described about C fails to intersect ALA';

(2) When it just reaches to ALA';

(3) When it intersects the semicircle ALA' in but *one* point;

(4) When it intersects the semicircle ALA' in *two* points.

Case (1) is represented in Figs. 18, 22; case (2), in Figs. 19, 23; case (3), in Figs. 20, 24; and case (4), in Figs. 21, 25. In Figs. 18 and 22 the angle A is respectively acute and obtuse; and similarly for each of the other pairs of figures.

NOTE. In Figs. 18–25 AKA' is a great circle in the plane of the paper, and $ALA'A$ is a great circle at right angles to that plane, ALA' being above the surface of the paper, and the dotted AA' being below. In Figs. 18–21,

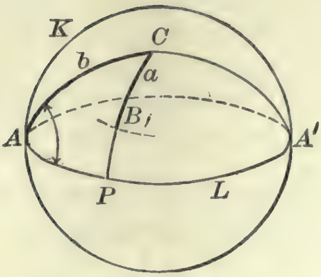


FIG. 18

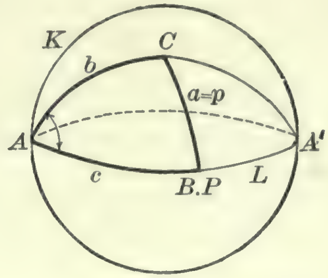


FIG. 19

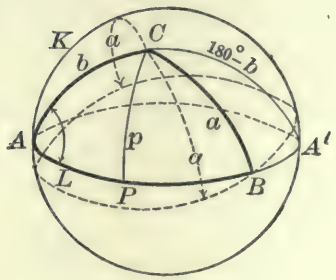


FIG. 20

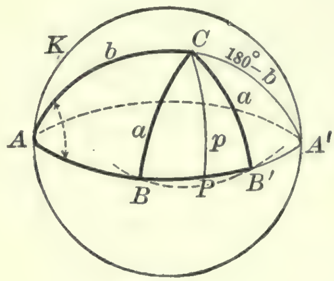


FIG. 21

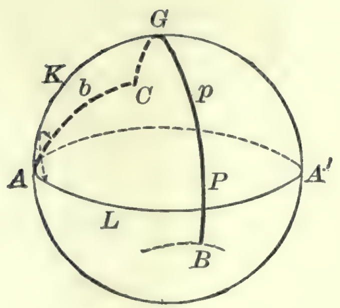


FIG. 22

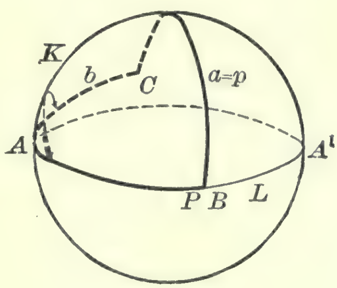


FIG. 23

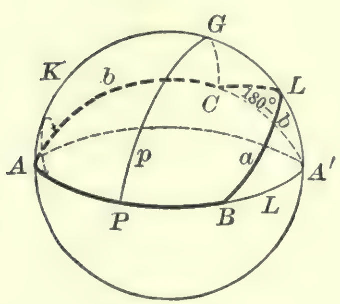


FIG. 24

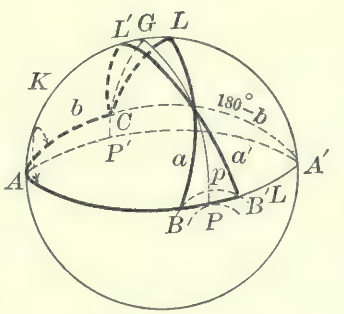


FIG. 25

angle A is acute [equal to $PAK(90°) - KAC$], and the arc AC is in front of the paper. In Figs. 22–25, angle A is obtuse [equal to $PAK(90°) + KAC$), and the arc AC is behind the paper.

In Fig. 21 there are two triangles (not equal or symmetrical) that satisfy the given conditions; and likewise in Fig. 25. Hence V. is **an ambiguous case** in spherical geometry.

In each figure let the perpendicular arc CP be drawn from C to the semicircle ALA', and let its length be called p. [See Ex. 1, p. 101.]

A. ***When angle A is acute:***

Fig. 18 shows that the triangle required is impossible, if $CB < CP$, *i.e.* if $a < p$.

Fig. 19 shows that the triangle required is right angled if $CB = CP$, *i.e.* if $a = p$.

Fig. 20 shows that there is but one triangle which satisfies the given conditions, if

$$CB > CP,\ CB > CA, \text{ and } CB < CA';$$

$$\textit{i.e.} \text{ if } a > p,\ a > b, \text{ and } a < 180° - b.$$

Similarly, there is only one triangle if $a > p$, $a < b$, and $a > 180° - b$.

Fig. 21 shows that there are two triangles which satisfy the given conditions, if

$$CB > CP,\ CB < CA, \text{ and } CB < CA';$$

$$\textit{i.e.} \text{ if } a > p,\ a < b, \text{ and } a < 180° - b.$$

B. ***When angle A is obtuse:***

Fig. 22 shows that the triangle required is impossible, if $CGB > CGP$, *i.e.* if $a > p$.

Fig. 23 shows that the triangle required is right angled, if $CGB = CGP$, *i.e.* if $a = p$.

Fig. 24 shows that there is but one triangle which satisfies the given conditions, if

$$CLB < CGP,\ CLB < CA \text{ and } CLB > CA';$$

$$\textit{i.e.} \text{ if } a < p,\ a < b, \text{ and } a > 180° - b.$$

Similarly, there is only one triangle if $a < p$, $a > b$, and $a < 180° - b$.

Fig. 25 shows that there are two triangles which satisfy the given conditions, if

$$CLB < CGP,\ CLB > CA, \text{ and } CLB > CA';$$

$$\textit{i.e.} \text{ if } a < p,\ a > b, \text{ and } a > 180° - b.$$

In Fig. 25 produce PGC to meet the great circle $ALA'A$ in P'. Then $CP' = 180° - p$. Since AC and CA' are each greater than CP', it follows that $a > 180° - p$.

It is also apparent from Figs. 20 and 21 that the triangle is impossible,

if A is acute, $a > b$, and $a > 180° - b$;

and it is apparent from Figs. 24 and 25 that the triangle is impossible,

if A is obtuse, $a < b$, and $a < 180° - b$.

Some special cases which may be investigated by the student, are indicated in the exercises on this chapter at page 101.

VI. **Given two angles and the side opposite one of them.** Calculate the corresponding parts of the polar triangle; construct it by V.; construct *its* polar; this is the required triangle. There may be two solutions, since the triangle first constructed may have two solutions.

Ex. 1. Construct ABC: (a) Given $a = 52°$, $b = 71°$, $A = 46°$; (b) Given $a = 99°$, $b = 64°$, $A = 95°$.

Ex. 2. Construct ABC: (a) Given $A = 46°$, $B = 36°$, $a = 42°$; (b) Given $A = 36°$, $B = 46°$, $a = 42°$.

Ex. 3. Determine approximately the remaining parts of these triangles.

N.B. *Questions and exercises on Chapter I. will be found at pages* 101–102.

CHAPTER II.

RIGHT-ANGLED SPHERICAL TRIANGLES.

25. Spherical trigonometry. Spherical trigonometry treats of the relations between the six parts of a triedral angle, or, what is the same thing (Art. 14), the relations between the six parts of the corresponding spherical triangle intercepted on the surface of the sphere. In Art. 24 it has been seen how a triangle can be *constructed* when any three parts are given; Chapters II. and III. are concerned with showing how the remaining parts can be *computed* when any three parts are known. In this chapter the relations between the sides and angles of a right-angled spherical triangle are deduced.* Throughout the book the relations between trigonometric ratios, discussed in plane trigonometry, will be employed.

26. Relations between the sides and angles of a right-angled spherical triangle.

Case I. *The sides about the right angle both less than* 90°.

Let ABC be a spherical triangle which is right angled at C and on a sphere whose centre is at O. First suppose that a and b are

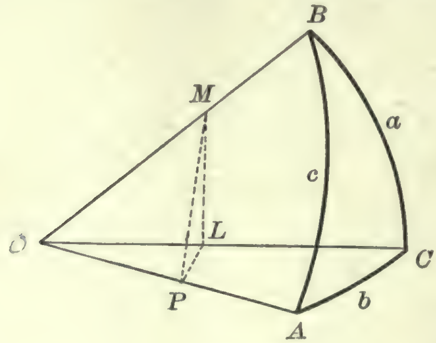


FIG. 26

* These relations can also be obtained from the relations, derived in Chapter III., between the parts of any spherical triangle or triedral angle.

each less than 90°. (It is easily shown, geometrically, that c is then less than 90°.) Draw OA, OB, OC. Take any point P on OA; in the plane OAC draw PL at right angles to OA, and let it meet OC in L; in the plane OAB draw PM at right angles to OA, and let it meet OB in M; and draw ML. Since PL and PM are perpendicular to the line OA, the line OA is perpendicular to the plane LPM (Euc. XI. 4); and, therefore, the plane LPM is perpendicular to the plane OAC (Euc. XI. 18). Also, the plane OCB is perpendicular to the plane OAC, since C is a right angle. Hence, LM, the intersection of the planes LPM and OCB, is perpendicular to the plane OAC (Euc. XI. 19); and hence, MLP and MLO are right angles.

In the triangle OPM, the angle OPM being right,

$$\cos POM = \frac{OP}{OM} = \frac{OP}{OL} \cdot \frac{OL}{OM}.$$

Now, $$\text{angle } POM = c, \quad \frac{OP}{OL} = \cos POL = \cos b,$$

and $$\frac{OL}{OM} = \cos LOM = \cos a.$$

$$\therefore \cos c = \cos a \cos b. \qquad (1)$$

In the triangle LPM, angle $PLM = 90°$, and angle $LPM = A$;

$$\sin LPM = \frac{LM}{PM} = \frac{\dfrac{LM}{OM}}{\dfrac{PM}{OM}} = \frac{\sin LOM}{\sin POM}.$$

$$\therefore \sin A = \frac{\sin a}{\sin c}. \qquad (2)$$

Also, $$\cos LPM = \frac{PL}{PM} = \frac{\dfrac{PL}{OP}}{\dfrac{PM}{OP}} = \frac{\tan POL}{\tan POM};$$

whence, $$\cos A = \frac{\tan b}{\tan c}. \qquad (3)$$

Also, $$\tan LPM = \frac{LM}{PL} = \frac{\dfrac{LM}{OL}}{\dfrac{PL}{OL}} = \frac{\tan LOM}{\sin POL};$$

whence, $$\tan A = \frac{\tan a}{\sin b}. \tag{4}$$

On remarking that A, a, denote an angle and its opposite side, and that b denotes the other side, the relations for angle B corresponding to (2), (3), (4), can be written immediately; viz.:

$$\sin B = \frac{\sin b}{\sin c} \quad (2'); \quad \cos B = \frac{\tan a}{\tan c} \quad (3'); \quad \tan B = \frac{\tan b}{\sin a} \quad (4').$$

These relations for B can also be deduced directly, by taking any point on OB and making a construction similar to that shown in Fig. 26.

Moreover,

$$\sin A = \tan A \cos A = \frac{\tan a}{\sin b} \cdot \frac{\tan b}{\tan c} = \frac{\tan a}{\tan c} \cdot \frac{1}{\cos b}. \quad \text{[By (3), (4).]}$$

$$\therefore \sin A = \frac{\cos B}{\cos b}. \quad \text{[By (3').]} \tag{5}$$

Similarly, $$\sin B = \frac{\cos A}{\cos a}. \tag{5'}$$

Also, $$\cos c = \cos a \cos b = \frac{\cos A}{\sin B} \cdot \frac{\cos B}{\sin A}. \quad \text{[By (1), (5), (5').]}$$

$$\therefore \cos c = \cot A \cot B. \tag{6}$$

For convenience of reference, relations (1)–(6) may be grouped together:

$$\mathbf{\cos c = \cos a \cos b.} \quad (1)$$

$$\mathbf{\sin A = \frac{\sin a}{\sin c}.} \qquad \mathbf{\sin B = \frac{\sin b}{\sin c}.} \qquad (2), (2')$$

$$\mathbf{\cos A = \frac{\tan b}{\tan c}.} \qquad \mathbf{\cos B = \frac{\tan a}{\tan c}.} \qquad (3), (3')$$

$$\mathbf{\tan A = \frac{\tan a}{\sin b}.} \qquad \mathbf{\tan B = \frac{\tan b}{\sin a}.} \qquad (4), (4')$$

$$\mathbf{\sin A = \frac{\cos B}{\cos b}.} \qquad \mathbf{\sin B = \frac{\cos A}{\cos a}.} \qquad (5), (5')$$

$$\mathbf{\cos c = \cot A \cot B.} \quad (6)$$

Case II. *The sides about the right angle both greater than* 90°.

In Fig. 27, C denotes the right angle, and the sides a, b, are each greater than a quadrant.

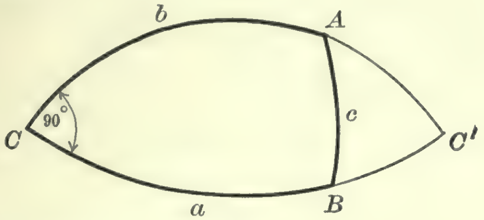


FIG. 27

Form the lune CC' by producing the sides a and b to meet in C'. Then ABC' is a right triangle in which the sides about the right angle are each less than 90°.

$$\therefore \ \cos c = \cos BC' \cos AC' = \cos(180 - a)\cos(180 - b).$$

Hence $$\cos c = \cos a \cos b.$$

Also,

$$\cos BAC' = \frac{\tan AC'}{\tan AB}; \ \textit{i.e.} \ \cos(180° - BAC) = \frac{\tan(180° - AC)}{\tan AB};$$

whence, $$\cos A = \frac{\tan b}{\tan c}.$$

In a similar manner the other relations in (1)–(6) can be shown to be true for ABC (Fig. 27).

NOTE. ABC' is said to be *co-lunar* with ABC. Every triangle has *three* co-lunar triangles, one corresponding to each angle.

Case III. *One of the sides about the right angle less than* 90°, *and the other side greater than* 90°.

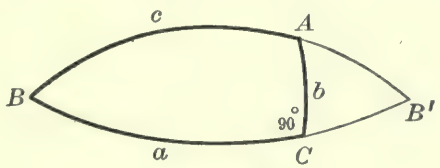


FIG. 28

In ACB let $C = 90°$, $a > 90°$, and $b < 90°$. Complete the lune BB'. Then ACB' is a right-angled triangle in which the sides about the right angle are each less than 90°.

In ACB', $$\cos AB' = \cos AC \cos CB';$$

$$i.e.\ \cos(180^\circ - c) = \cos b \cos(180^\circ - a);$$

whence $$\cos c = \cos a \cos b.$$

Again,

$$\cos CAB' = \frac{\tan AC}{\tan AB'};\ i.e.\ \cos(180^\circ - BAC) = \frac{\tan AC}{\tan(180^\circ - BA)};$$

whence, $$\cos A = \frac{\tan b}{\tan c}.$$

In a similar way the other relations in (1)–(6) can be shown to be true for ABC (Fig. 28).

27. On species. Two parts of a spherical triangle are said to be *of the same species* (*or of the same affection*) when both are less than 90°, both greater than 90°, or both equal to 90°. Formula (1), Art. 26, shows that the *hypotenuse* of a right-angled spherical triangle is *less than* 90° when the *sides* about the right angle are *both greater or both less than* 90°; and it shows that the *hypotenuse is greater than* 90° when the *sides are of different species.* Formulas (4) and (4′) show that in a right-angled spherical triangle (since the sines of the sides are positive) *an angle and its opposite side are of the same species.* These important properties can also be deduced geometrically.

EXAMPLES.

N.B. It is advisable to remember the result of **Ex. 1.**

1. State relations (1)–(6), Art. 26, in words.

(1). *cos* hyp. = product of *cosines* of sides.

(6). *cos* hyp. = product of *cotangents* of angles.

(2), (2′). *sin* angle = *sin* opposite side ÷ *sin* hyp.

(3), (3′). *cos* angle = *tan* adjacent side ÷ *tan* hyp.

(4), (4′). *tan* angle = *tan* opposite side ÷ *sin* adjacent side.

(5), (5′). *cos* angle = *cos* opposite side × *sin* other angle.

[Compare (2), (3), (4), with the corresponding formulas in plane triangles.]

2. Deduce formulas (1)–(4) by means of a figure in which P is anywhere on OB (see Fig. 26).

3. Deduce formulas (1)–(4) by means of a figure in which P (see Fig. 26) is: (*a*) in OA produced; (*b*) in OB produced; (*c*) at the point A; (*d*) at the point B.

4. The two sides about the right angle of a spherical triangle are 60° and 75°; find the hypotenuse and the other angles by means of relations (1), (4), (4′), Art. 26. Check (or test) the result by means of other formulas.

5. In ABC, given $A = 47° 30'$, $B = 53° 40'$, $C = 90°$; find the remaining parts, and check the results.

6. Solve some of the examples in Art. 31, and check the results.

28. Solution of a right-angled triangle.

N.B. The student is advised to investigate this subject independently; and, *before* reading this article, to *put in writing* in an orderly manner his ideas about the solution of right triangles. These ideas will thus be made clearer in his mind, and his subsequent reading will be easier.

In a right triangle there are five elements beside the right angle. These five elements can be taken in groups of three in ten different ways. Each of these ten groups is involved in the ten relations derived in Art. 26; the three elements of each group are, accordingly, connected by one relation.

Ex. (*a*) Write all the groups of three that can be formed from a, b, c, A, B, such as a, b, c; a, b, A; etc.

(*b*) Write the relation connecting the elements of each group.

It follows that if any two elements of a right-angled spherical triangle besides the right angle be given, then the remaining three elements can be determined. The method of finding a third element is as follows:

Write the relation involving the two given elements and the required element; solve this equation for the required element.

Check (or *test*). When the required elements are obtained, the results can be checked by examining whether they satisfy relations which have not been employed in the solution, and, preferably, the relation involving the newly found elements.

E.g., suppose that A, b, are known, C being 90°; then a, c, B, are required. Side a can be found by (4); side c, by (3); and angle B, by (5). The values found for a, c, B, can be checked by (3′).

NOTE 1. The cosine, tangent, and co-tangent of sides and angles greater than 90°, are *negative. Careful attention must be paid to the algebraic signs of the trigonometric functions appearing in the work.*

NOTE 2. The properties stated in Art. 27 are very useful.

NOTE 3. Determine each element from the given elements alone. If an element is found erroneously and then used in finding a second element, the second element will also be wrong.

The ten possible groups of three elements correspond to the following *six* cases for solution, in which the given elements are respectively:

(1) Two sides.

(2) Hypotenuse and a side.

(3) Hypotenuse and an angle.

(4) Two angles.

(5) Side and opposite angle.

(6) Side and adjacent angle.

Before proceeding to the solution of numerical examples, it is necessary to refer more particularly to one of these cases; and also to call attention to the fact that the ten formulas for right triangles (Art. 26) may be grouped in two very simple and convenient rules.

29. The ambiguous case. When the given parts are a side and its opposite angle, there are two triangles which satisfy the given conditions. For, in ABC (Fig. 29), let $C = 90°$, and let A and CB (equal to a) be the given parts. Then, on completing the lune AA', it is evident that the triangle $A'BC$ also satisfies the given conditions, since $BCA' = 90°$, $A' = A$, and $CB = a$. The remaining parts in $A'BC$ are respectively supplementary to the remaining parts in ABC; thus $A'B = 180° - c$, $A'C = 180° - b$, $A'BC = 180° - ABC$.

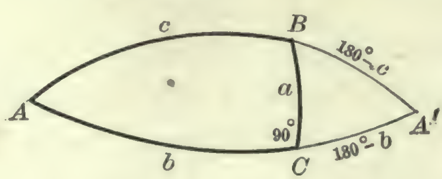


FIG. 29

This ambiguity is also apparent from the relations (1)–(6), Art. 26. For, if a, A, are given, then the remaining parts, namely,

c, b, B, are all determined from their sines [see (2), (4), (5'),]; and, accordingly, c, b, B, may each be less or greater than 90°. On the other hand, if, for instance, a and c are given, then b is determined from its cosine by (1); and there is no ambiguity, because b is less or greater than 90° according as $\cos b$ is respectively positive or negative.

N.B. Both solutions should be given in the ambiguous case, unless some information is given which serves to indicate the particular solution that is suitable.

30. Napier's rules of circular parts. The ten relations derived in Art. 26 are all included in two statements, which are called *Napier's rules of circular parts*, after the man who first announced them, Napier, the inventor of logarithms.

Let ABC be a triangle right-angled at C. Either draw a right-angled triangle, and mark the sides and angles as in Fig. 31, or draw a circle and mark successive arcs as in Fig. 32, in which b, a, *Co-B*, *Co-c*, *Co-A*, are

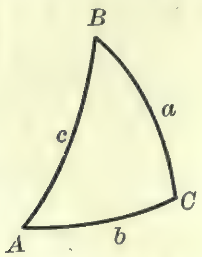


FIG. 30

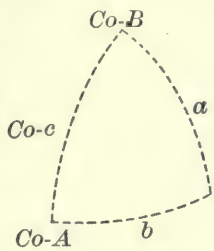


FIG. 31

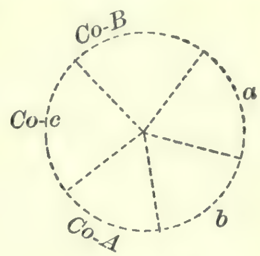


FIG. 32

arranged in the same order as b, a, B, c, A, in Fig. 30. (Here *Co-B*, *Co-c*, *Co-A*, denote the complements of B, c, and A, respectively.) The five quantities, a, b, *Co-B*, *Co-c*, *Co-A*, are known as *circular parts.* That is, the right angle being omitted, the two sides and the complements of the hypotenuse and the other angles are called the circular parts of the triangle.

In Figs. 31 and 32 each part has two circular parts *adjacent* to it, and two circular parts *opposite* to it. Thus, on taking a, for instance, the adjacent parts are b, *Co-B*, and the opposite parts are *Co-c*, *Co-A*. If any three parts be taken, one of them is midway between the other two, and the latter are either its two adjacent parts or its two opposite parts. Thus, if a, b, *Co-A*, be taken, then b is the middle part, and a, *Co-A*, are the adjacent parts; if a, b, *Co-c*, be taken, then *Co-c* is the middle part, and a, b, are opposite parts.

Ex. Take each of the circular parts in turn, write its opposite parts and adjacent parts, thus getting *ten* sets in all.

Napier's rules of circular parts are as follows:

I. *The sine of the middle part is equal to the product of the tangents of the adjacent parts.*

II. *The sine of the middle part is equal to the product of the cosines of the opposite parts.*

(The *i*'s, *a*'s, and *o*'s are lettered thus, in order to aid the memory.)

These rules are easily *verified.* For example, on taking a for the middle part,

$$\sin a = \tan b \tan (90^\circ - B) = \tan b \cot B. \qquad \text{[See Art. 26 (4').]}$$

$$\sin a = \cos (90^\circ - A) \cos (90^\circ - c) = \sin A \sin c. \qquad \text{[See Art. 26 (2).]}$$

Again, on taking *Co-A* for the middle part,

$$\sin (90^\circ - A) = \tan b \tan (90^\circ - c), \text{ i.e. } \cos A = \tan b \cot c. \qquad \text{[See Art. 26 (3).]}$$

$$\sin (90^\circ - A) = \cos a \cos (90^\circ - B), \text{ i.e. } \cos A = \cos a \sin B. \qquad \text{[See Art. 26 (5').]}$$

In a similar way each of the remaining three parts can be taken in turn for the middle part, and the remaining six relations of Art. 26 shown to agree with Napier's rules.*

NOTE. One may either memorize the relations in Art. 26 (or Ex. 1, Art. 27), or use Napier's rules. Opinions differ as to which is the better thing to do.

Ex. **1.** Verify Napier's rules by showing that they include the 10 relations in Art. 26.

Ex. **2.** Prove Napier's rules.

31. Numerical problems. In solving right triangles the procedure is as follows:

(1) Indicate the two given parts and the three required parts.

---

* This is only a *verification* of Napier's rules. One *proof* of the rules would consist of the derivation of the relations in Art. 26 *plus* this verification. These rules were first published by Napier in his work *Mirifici Logarithmorum Canonis Descriptio* in 1614. Napier indicated a geometrical method of proof, and deduced the rules as special applications of a more general proposition. They are something more than mere technical aids to the memory. For an explanation of this and of their wider geometrical interpretation, see Charles Hutton, *Course in Mathematics* (edited by T. S. Davies, London, 1843), Vol. II. pp. 24–26; Todhunter, *Spherical Trigonometry*, Art. 68; E. O. Lovett, *Note on Napier's Rules of Circular Parts* (Bulletin Amer. Math. Soc., 2d Series, Vol. IV. No. 10, July, 1898).

(2) Write the relations that will be employed in the solution, and note carefully the algebraic signs of the functions involved The noting of these signs will serve to show (unless a part is determined from its sine) whether a required part is less or greater than 90°.

(3) For use as a *check,* write the relation involving the three required parts.

(4) Arrange the work as neatly and clearly as possible.

N.B. Attention may be directed to the notes in Art. 28. Also see *Plane Trigonometry*, Art. 27 (particularly p. 45, notes 2, 4–6), and Art. 59, p. 107.

NOTE. The trigonometric function of any angle can be expressed in terms of some trigonometric function of an angle less than 90°. See *Plane Trigonometry*, Art. 45.

EXAMPLES.

1. Solve the triangle ABC, given:

$$C = 90°, \quad a = 44° 30', \quad b = 71° 40'$$

Solution*: $c =$, $A =$, $B =$

Formulas:

$$\cos c = \cos a \cos b,$$
$$\tan A = \tan a \div \sin b,$$
$$\tan B = \tan b \div \sin a.$$

Check: $\cos c = \cot A \cot B$.

Logarithmic formulas: [If necessary; see *Pl. Trig.*, Art. 27, Note 6.]

$$\log \cos c = \log \cos a + \log \cos b,$$
$$\log \tan A = \log \tan a - \log \sin b,$$
$$\log \tan B = \log \tan b - \log \sin a,$$
$$\log \cos c = \log \cot A + \log \cot B \ (check).$$

$$\begin{aligned}
\log \sin a &= 9.84566 - 10 & \therefore \log \cos c &= 9.35092 - 10\\
\log \cos a &= 9.85324 - 10 & \log \tan A &= 0.01504\\
\log \tan a &= 9.99242 - 10 & \log \tan B &= 0.63403\\
\log \sin b &= 9.97738 - 10 & \therefore c &= 77° \ 2'.1.\\
\log \cos b &= 9.49768 - 10 & A &= 45° 59'.5.\\
\log \tan b &= 0.47969 & B &= 76° 55'.5.
\end{aligned}$$

Check:
$$\begin{aligned}
\therefore \log \cot A &= 9.98497 - 10\\
\log \cot B &= 9.36595 - 10\\
\therefore \log \cos c &= 9.35092 - 10
\end{aligned}$$

* To be filled in.

Note. Spherical triangles, like plane triangles, can also be solved without the use of logarithms. (See *Plane Trigonometry*, examples in Arts. 27, 55–62.)

2. Solve the triangle ABC, given:

$$C = 90^\circ, \quad A = 57^\circ 40', \quad a = 48^\circ 30'.$$

Solution: $c =$
$b =$
$B =$

Formulas:

$$\sin c = \frac{\sin a}{\sin A}, \qquad \sin b = \frac{\tan a}{\tan A}, \qquad \sin B = \frac{\cos A}{\cos a}.$$

Check: $\sin B = \dfrac{\sin b}{\sin c}.$

$\log \sin a = 9.87446 - 10$ $\therefore \log \sin c = 9.94763 - 10$

$\log \cos a = 9.82126 - 10$ $\log \sin b = 9.85459 - 10$

$\log \tan a = 0.05319$ $\log \sin B = 9.90697 - 10$

$\log \sin A = 9.92683 - 10$ $\therefore c = 62^\circ 25'.4$, or $117^\circ 34'.6$.

$\log \cos A = 9.72823 - 10$ $b = 45^\circ 40'.9$, or $134^\circ 19'.1$.

$\log \tan A = 0.19860$ $B = 53^\circ 49'.3$, or $126^\circ 10'.7$.

The check gives $\log \sin B = 9.90696$.

On combining the results according to the principles of Art. 27, the solutions are:

(1) $c = 62^\circ 25'.4,\ b = 45^\circ 40'.9,\ B = 53^\circ 49'.3$;

(2) $c = 117^\circ 34'.6,\ b = 134^\circ 19'.1,\ B = 126^\circ 10'.7$.

3. Solve Ex. 1 without using logarithms.

4. Given $c = 90^\circ$, $A = 57^\circ 40'$, $a = 108^\circ 30'$. Show both by geometry and trigonometry why the solution is impossible.

5. Solve the triangle ABC, in which $C = 90^\circ$, and *check the results*, given:

(1) $a = 36^\circ 25' 30''$, $b = 85^\circ 40'$; (2) $c = 120^\circ 20' 30''$, $a = 47^\circ 30' 40''$;

(3) $c = 78^\circ 25'$, $A = 36^\circ 42' 30''$; (4) $A = 63^\circ 18'$, $B = 37^\circ 47'$;

(5) $a = 76^\circ 48'$, $A = 82^\circ 36'$; (6) $b = 39^\circ 50' 20''$, $A = 47^\circ 50'$;

(7) $a = 47^\circ 40'$, $A = 30^\circ 43'$; (8) $b = 70^\circ 30' 30''$, $B = 80^\circ 40' 20''$;

(9) $a = 108^\circ 45'$, $B = 37^\circ 42'$; (10) $c = 78^\circ 20'$, $B = 47^\circ 50'$;

(11) $A = 110^\circ 27'$, $B = 74^\circ 36'$; (12) $a = 108^\circ 42'$, $b = 63^\circ 26'$;

(13) $A = 124^\circ 30'$, $b = 25^\circ 40'$; (14) $c = 84^\circ 47'$, $b = 39^\circ 43'$.

32. Solution of isosceles triangles and quadrantal triangles.

Isosceles Triangles. Plane isosceles triangles can be solved by means of right triangles, as shown in *Plane Trigonometry,* Art. 32. A spherical isosceles triangle can be solved in a similar way, namely, by dividing it into two right triangles by an arc drawn from the vertex at right angles to the base. This arc bisects the base and the vertical angle.

Quadrantal Triangles. The polar triangle of a quadrantal triangle (Art. 18) is right-angled (Art. 16. d). Hence, a quadrantal triangle may be solved by solving its polar triangle by Arts. 28, 31, and then computing the required parts of the quadrantal triangle by Art. 16. d.

EXAMPLES.

1. Solve the triangle ABC, in which A and B are equal, and *check the results*, given:

(1) $a = 54° 20'$, $c = 72° 54'$; (2) $a = 66° 29'$, $A = 50° 17'$;

(3) $a = 54° 30'$, $C = 71°$; (4) $c = 156° 40'$, $C = 187° 46'$.

2. Solve the triangle ABC, given:

(1) $c = 90°$, $C = 67° 12'$, $a = 123° 48' 4''$;

(2) $c = 90°$, $A = 136° 40'$, $B = 105° 47'$.

33.* Solution of oblique spherical triangles. It has been seen (*Plane Trigonometry,* Art. 34) that oblique plane triangles can be solved by means of right triangles. Oblique spherical triangles can also be solved by means of right spherical triangles. Relating to spherical triangles there are *six problems of computation;* these correspond to the six problems of construction discussed in Arts. 23, 24. If any three parts of a triangle are given, the triangle can be constructed and the remaining parts can be computed. The several cases for computation will now be solved with the help of right-angled triangles.†

(In the figures in this article the given parts are indicated by crosses.)

N.B. The student is advised to try to solve Cases II.–VI. *before* reading the text.

---

* When time is limited this article may be omitted, or merely glanced over.

† Other methods of solving triangles are shown in Chap. IV.

Case I. Given the three sides. In ABC (Figs. 33, 34) let a, b, c, be given, and A, B, C, be required. From C draw CD at right angles to AB, or AB produced. Let the segments AD and DB be denoted by m and n, respectively. If the direction from A to B is taken as the positive direction along the arc AB, then m is positive in Fig. 33 and negative in Fig. 34, while n is positive in both figures.

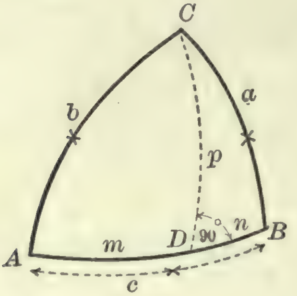


FIG. 33

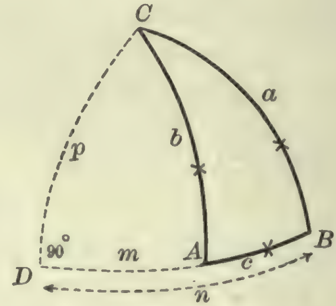


FIG. 34

Special formula. In each figure,

$$\cos a = \cos n \cos p, \text{ and } \cos b = \cos m \cos p.$$

$$\therefore \cos p = \frac{\cos a}{\cos n} = \frac{\cos b}{\cos m}.$$

$$\therefore \frac{\cos n}{\cos m} = \frac{\cos a}{\cos b}.$$

$$\therefore \frac{\cos n - \cos m}{\cos n + \cos m} = \frac{\cos a - \cos b}{\cos a + \cos b}. \quad \text{[Composition and division.]}$$

From this, on applying *Plane Trigonometry*, Art. 52, formulas (7), (8),

$$\tan \tfrac{1}{2}(n + m) \tan \tfrac{1}{2}(n - m) = \tan \tfrac{1}{2}(a + b) \tan \tfrac{1}{2}(a - b). \qquad (1)$$

Now $n + m = c$; hence, $n - m$ can be found by (1). Then the segments m and n can each be determined. The triangles ADC and BDC can then be solved by Arts. 28, 31; and the solution of ABC can be obtained therefrom.

Ex. 1. Solve Exs. 1, 2, Art. 42, by the method outlined above.

Ex. 2. Show how to solve this case when the perpendicular is drawn from A to BC.

Case II. Given the three angles. Solve the polar triangle by the method used in Case I.; and therefrom (Art. 16. *d*) compute the parts of the original triangle.

Ex. Solve Exs. 1, 2, Art. 43, by this method.

Case III. Given two sides and their included angle. In *ABC* (Fig. 35) let *a, c, B,* be given. Draw *AD* at right angles to *BC,* or *BC* produced.

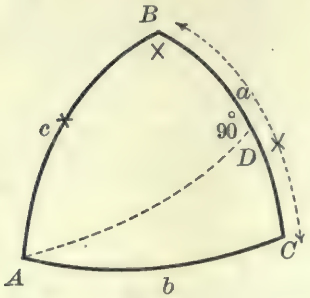


Fig. 35

In *ABD, c* and *B* are known; hence, *BAD, AD,* and *BD* can be found. In *ADC, AD* and *DC* (equal to $a - BD$) are now known; hence *DAC, ACD,* and *AC* can be found. Also, $CAB = CAD + DAB$. The student can examine the case in which *AD* falls outside *ABC.*

Ex. 1. Show how to solve the triangle by drawing a perpendicular arc from *C* to *AB.*

Ex. 2. Solve Exs. 1, 2, Art. 44, by means of right triangles.

Case IV. Given a side and the two adjacent angles. Two methods of solution may be employed.

Either: (1) Solve the polar triangle by the method used in Case III.; and therefrom compute the parts of the original triangle.

Or: (2) In *ABC* (Fig. 36) let *A, B, c,* be given. Draw the arc *BD* at right angles to *AC.* In *ADB, AD, DB,* and *ABD* can be found, since *A* and *AB* are known. Now $DBC = ABC - ABD$. In *DBC, DB* and *DBC* are now known; hence *BC, CD,* and *C* can be found. Then $AC = AD + DC$.

The student can examine the case in which *BD* falls outside *ABC.*

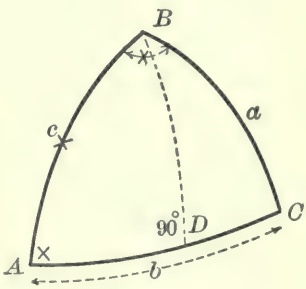


Fig. 36

Ex. 1. Solve the triangle by drawing a different perpendicular.

Ex. 2. How may solution (2) aid in the solution of Case III.?

Ex. 3. Solve Exs. 1, 2, Art. 45, by means of right triangles.

Case V. Given two sides and the angle opposite one of them. (*This may be an ambiguous case;* see Art. 24, V.)

In ABC (Fig. 37) let a, c, A, be given. From B draw the arc BD at right angles to AC to meet AC or AC produced. In ABD, c and A are known; hence AD, DB, and ABD can be found. In DBC, DB and a are now known; hence DBC, C, and DC can be found. Then $AC = AD + DC$, and $ABC = ABD + DBC$.

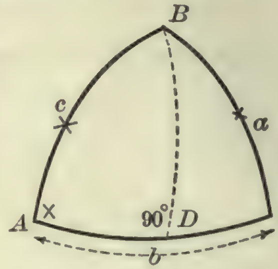


FIG. 37

Ex. 1. Examine the cases in which BD falls outside ABC.

Ex. 2. Examine the case in which *two* triangles satisfy the given conditions.

Ex. 3. Solve Exs. 1, 2, Art. 46, by means of right triangles.

Case VI. Given two angles and the side opposite one of them. Like Case V. this may be ambiguous; see Art. 24, VI. Two methods of solution may be employed.

Either: (1) Solve the polar triangle by the method used in Case V.; and therefrom compute the parts of the original triangle.

Or: (2) In ABC (Fig. 38) let A, C, c, be known. From B draw BD at right angles to AC to meet AC, or AC produced. Solve the triangle ABD; then solve the triangle DBC. The parts of ABC can be computed from these solutions.

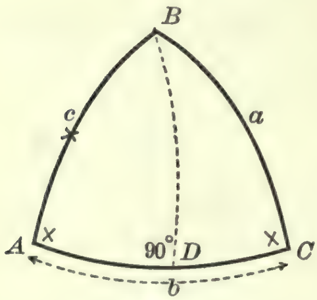


FIG. 38

Ex. 1. How may (2) aid in the solution of Case V.?

Ex. 2. Solve Exs. 1, 2, Art. 47, by means of right triangles.

Ex. 3. Solve the numerical examples in Art. 24.

34. Graphical solution of (oblique and right) spherical triangles. A plane triangle can be solved graphically by drawing to scale a triangle that satisfies the given conditions, and then measuring the required parts directly from the figure (*Plane Trigonometry,* Arts. 10, 21). A spherical triangle can be solved graphically by drawing (Art. 24) upon any sphere a triangle that satisfies the given conditions, and then measuring the required parts of the triangle. The sides and angles (see Art. 11. *c*) can be measured with a thin, flexible, brass ruler, on which a length equal to a quadrant or a semicircle of the sphere is graduated from 0° to 90° or 180° respectively.

Small slated globes can be obtained fitting into hemispherical cups, whose rims are graduated from 0° to 180° in both directions. With such a globe, cup, and a pair of compasses, the constructions discussed in Art. 24 and the measurements referred to in this article are easily made.

If the student has the means at hand, it is advisable for him to solve some of the numerical problems graphically.

N.B. *Questions and exercises on Chapter II. will be found at p.* 102.

CHAPTER III.

RELATIONS BETWEEN THE SIDES AND ANGLES OF SPHERICAL TRIANGLES.

35. In this chapter some relations between the sides and angles of any spherical triangle (whether right-angled or oblique) will be derived. In the next chapter these relations will be used in the solution of practical numerical problems. The first two general relations (namely, the *Law of Sines* and the *Law of Cosines*), which are by far the most important, can be derived in various ways. In a short course it may be best to deduce these laws by means of the properties of right-angled triangles as set forth in Art. 26; and, accordingly, this method is adopted here. These laws are also derived directly from geometry in Note A at the end of the book. It may be stated here that the geometrical derivation will strengthen the student's understanding of the subject, and will show more clearly the correspondence (Art. 14) between the parts of a spherical triangle and the parts of a triedral angle.

36. The *Law of Sines* and the *Law of Cosines* deduced by means of the relations of right-angled triangles.

***A*. Derivation of the Law of Sines.**

Let ABC (Figs. 39, 40) be any spherical triangle. From B draw the arc BD at right angles to AC to meet AC, or AC produced, in D.

In ABD, $\sin BD = \sin c \sin A$;

in CBD, $\sin BD = \sin a \sin C$ (Fig. 39)

$= \sin a \sin BCD$ (Fig. 40) $= \sin a \sin BCA$.

Hence, in both figures, $\sin a \sin C = \sin c \sin A$.

$$\therefore \frac{\sin a}{\sin A} = \frac{\sin c}{\sin C}.$$

Similarly, by drawing an arc from C at right angles to AB, it can be shown that $\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B}$.

$$\therefore \frac{\sin \boldsymbol{a}}{\sin \boldsymbol{A}} = \frac{\sin \boldsymbol{b}}{\sin \boldsymbol{B}} = \frac{\sin \boldsymbol{c}}{\sin \boldsymbol{C}}. \qquad (1)$$

In words: *In a spherical triangle the sines of the sides are proportional to the sines of the opposite angles.* (Compare *Plane Trigonometry*, Art. 54, I.)

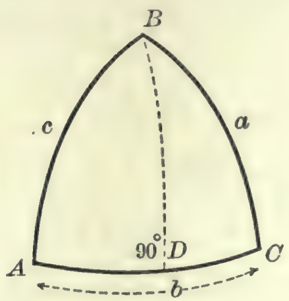


FIG. 39

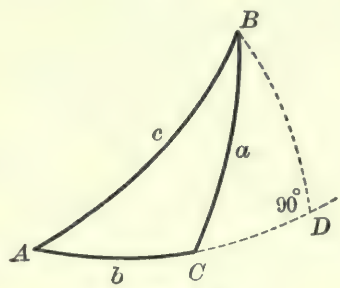


FIG. 40

***B.* Derivation of the Law of Cosines.**

$$\begin{aligned}\cos BC &= \cos CD \cos DB \\ &= \cos (b - AD) \cos DB, \text{ or } \cos (AD - b) \cos DB \\ &= \cos b \cos AD \cos DB + \sin b \sin AD \cos DB. \qquad (a)\end{aligned}$$

But $$\cos AD \cos DB = \cos c;$$

and $$\sin AD \cos DB = \frac{\cos c \sin AD}{\cos AD} = \cos c \tan AD$$

$$= \cos c \tan c \cos A = \sin c \cos A.$$

Hence, on substituting in (a),

$$\cos \boldsymbol{a} = \cos \boldsymbol{b} \cos \boldsymbol{c} + \sin \boldsymbol{b} \sin \boldsymbol{c} \cos \boldsymbol{A}. \qquad (2)$$

Similarly, or by taking the sides in turn,

$$\cos b = \cos c \cos a + \sin c \sin a \cos B,$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$

In words: *In a spherical triangle the cosine of any side is equal to the product of the cosines of the other two sides plus the product of the sines of these two sides and the cosine of their included angle.* (Compare *Plane Trigonometry*, Art. 54, II.)

NOTE 1. **The law of cosines, (2), is the fundamental and the most important relation in spherical trigonometry.** For, as shown in Note A, it can be deduced directly; the law of sines, (1), can be deduced from it; all other relations follow from these; and the relations for right triangles, Art. 26, can be deduced from the relations for triangles in general, on letting C be a right angle. The formulas for $\cos a$, $\cos b$, $\cos c$, were known to the Arabian astronomer Al Battani in the ninth century. (See *Plane Trigonometry*, p. 166.)

***C.* The Law of Cosines for the angles.** Relation (2) holds for all triangles, and, accordingly, for $A'B'C'$, the polar triangle of ABC. (See Fig. 14, Art. 16.) That is,

$$\cos a' = \cos b' \cos c' + \sin b' \sin c' \cos A'.$$

$$\therefore \cos(180° - A) = \cos(180° - B)\cos(180° - C)$$
$$+ \sin(180° - B)\sin(180° - C)\cos(180° - a). \quad \text{[Art. 16. } d.]$$

$$\therefore -\cos A = (-\cos B)(-\cos C) + \sin B \sin C(-\cos a).$$

$$\therefore \mathbf{\cos A = -\cos B \cos C + \sin B \sin C \cos a.} \quad (3)$$

Similarly, $\cos B = -\cos C \cos A + \sin C \sin A \cos b,$

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c.$$

Relation (3) can also be derived by means of right-angled triangles.

NOTE 2. From (2), $\quad \cos A = \dfrac{\cos a - \cos b \cos c}{\sin b \sin c}.$

The denominator in the second member is always positive. *If a differs more from 90° than does b*, then $\cos a$ is numerically greater than $\cos b$, and, accordingly, greater than $\cos b \cos c$; hence $\cos A$ and $\cos a$ have the same sign, and thus, *a and A are in the same quadrant.*

Similarly, a and A are in the same quadrant when a differs more from 90° than does c.

From (3), in which $\quad \cos a = \dfrac{\cos A + \cos B \cos C}{\sin B \sin C},$

it can be shown in a similar way that *if A differs more from 90° than does B or C, then a and A are in the same quadrant.*

Ex. **1.** Derive $\cos b$ and $\cos c$ by means of right triangles.

Ex. **2.** Derive $\cos A$ and $\cos B$ by means of right triangles.

Ex. **3.** Derive $\cos C$ from $\cos c$ by means of the polar triangle.

37. Formulas for the half-angles and the half-sides.

[Compare the method and results of this article with those of Art. 62, *Plane Trigonometry*].

I. The half-angles.

From Art. 36, (2), $\cos A = \dfrac{\cos a - \cos b \cos c}{\sin b \sin c}$. (1)

$$\therefore\ 1 - \cos A = 1 - \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

$$= \frac{\cos b \cos c + \sin b \sin c - \cos a}{\sin b \sin c}$$

$$= \frac{\cos (b - c) - \cos a}{\sin b \sin c}.$$

$$\therefore\ 2 \sin^2 \tfrac{1}{2} A = \frac{2 \sin \frac{1}{2} (a - b + c) \sin \frac{1}{2} (a + b - c)}{\sin b \sin c}.$$

[*Plane Trigonometry*, Art. 52, (8).]

On putting $a + b + c = 2s$, then $-a + b + c = 2(s - a)$,

$a - b + c = 2(s - b)$, and $a + b - c = 2(s - c)$.

$$\therefore\ \sin^2 \tfrac{1}{2} A = \frac{\sin (s - b) \sin (s - c)}{\sin b \sin c}. \qquad (2)$$

Similarly, from (1),

$$1 + \cos A = 1 + \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

$$= \frac{\cos a + \sin b \sin c - \cos b \cos c}{\sin b \sin c}$$

$$= \frac{\cos a - \cos (b + c)}{\sin b \sin c}.$$

$$\therefore\ 2 \cos^2 \tfrac{1}{2} A = \frac{2 \sin \frac{1}{2} (a + b + c) \sin \frac{1}{2} (b + c - a)}{\sin b \sin c}.$$

$$\therefore\ \cos^2 \tfrac{1}{2} A = \frac{\sin s \sin (s - a)}{\sin b \sin c}. \qquad (3)$$

$$\left.\begin{aligned} \sin\tfrac{1}{2}A &= \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b\sin c}}; \\ \cos\tfrac{1}{2}A &= \sqrt{\frac{\sin s\sin(s-a)}{\sin b\sin c}}; \\ \text{and hence,}\qquad \tan\tfrac{1}{2}A &= \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}}. \end{aligned}\right\}\tag{4}$$

Therefore, $\tan\frac{1}{2}A = \dfrac{1}{\sin(s-a)}\sqrt{\dfrac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}};$

$$\left.\begin{aligned} \text{hence, if}\qquad \tan r &= \sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}, \\ \text{then}\qquad \tan\tfrac{1}{2}A &= \frac{\tan r}{\sin(s-a)}. \end{aligned}\right\}\tag{5}$$

Like formulas can be similarly derived for $\frac{1}{2}B$ and $\frac{1}{2}C$; or they may be written immediately on observing the symmetry in formulas (4) and (5); namely,

$$\left.\begin{aligned} \sin\tfrac{1}{2}B &= \sqrt{\frac{\sin(s-a)\sin(s-c)}{\sin a\sin c}}, & \sin\tfrac{1}{2}C &= \sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a\sin b}}, \\ \cos\tfrac{1}{2}B &= \sqrt{\frac{\sin s\sin(s-b)}{\sin a\sin c}}, & \cos\tfrac{1}{2}C &= \sqrt{\frac{\sin s\sin(s-c)}{\sin a\sin b}}, \\ \tan\tfrac{1}{2}B &= \sqrt{\frac{\sin(s-a)\sin(s-c)}{\sin s\sin(s-b)}}, & \tan\tfrac{1}{2}C &= \sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin s\sin(s-c)}}. \end{aligned}\right\}\tag{6}$$

$$\tan\tfrac{1}{2}B = \frac{\tan r}{\sin(s-b)}, \qquad \tan\tfrac{1}{2}C = \frac{\tan r}{\sin(s-c)}. \tag{7}$$

It is shown in Art. 50 that r is the radius of the circle inscribed in the triangle ABC. Article 50 may be read at this time.

NOTE. By geometry, $2s < 360°$ and $b + c > a$. Hence, $s - a$ is positive and less than 180°. Similarly, $s - b$, $s - c$, are positive. Therefore, the quantities under the radical signs are positive. The positive signs must be given to each radical, for A, B, C, are each less than 180°, and, consequently, $\frac{1}{2}A$, $\frac{1}{2}B$, $\frac{1}{2}C$, are each between 0° and 90°.

EXAMPLES.

1. Derive each of the above formulas.

2. Given $a = 58°$, $b = 80°$, $c = 96°$. Find A, B, C.

3. Given $a = 46° 30'$, $b = 62° 40'$, $c = 83° 20'$. Find A, B, C.

The results in Exs. 2, 3, may be *checked* by Art. 36, (1).

II. The half-sides. From Art. 36, (3),

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C}.$$

On finding $1 - \cos a$ and $1 + \cos a$, combining and simplifying in the manner followed for the half-angles, and putting

$$A + B + C = 2S,$$

the following formulas are obtained:

$$\left.\begin{aligned}
\sin \tfrac{1}{2} a &= \sqrt{\frac{-\cos S \cos (S - A)}{\sin B \sin C}};\\
\cos \tfrac{1}{2} a &= \sqrt{\frac{\cos (S - B) \cos (S - C)}{\sin B \sin C}};\\
\tan \tfrac{1}{2} a &= \sqrt{\frac{-\cos S \cos (S - A)}{\cos (S - B) \cos (S - C)}}.
\end{aligned}\right\} \quad (8)$$

$$\left.\begin{aligned}
\text{Let}\qquad \tan R &= \sqrt{\frac{-\cos S}{\cos (S - A) \cos (S - B) \cos (S - C)}},\\
\text{then}\qquad \tan \tfrac{1}{2} a &= \tan R \cos (S - A).
\end{aligned}\right\} \quad (9)$$

Similarly, or from (8) and (9) by symmetry,

$$\left.\begin{aligned}
\sin \tfrac{1}{2} b &= \sqrt{\frac{-\cos S \cos (S - B)}{\sin A \sin C}}, & \sin \tfrac{1}{2} c &= \sqrt{\frac{-\cos S \cos (S - C)}{\sin A \sin B}},\\
\cos \tfrac{1}{2} b &= \sqrt{\frac{\cos (S - A) \cos (S - C)}{\sin A \sin C}}, & \cos \tfrac{1}{2} c &= \sqrt{\frac{\cos (S - A) \cos (S - B)}{\sin A \sin B}},\\
\tan \tfrac{1}{2} b &= \sqrt{\frac{-\cos S \cos (S - B)}{\cos (S - A) \cos (S - C)}}, & \tan \tfrac{1}{2} c &= \sqrt{\frac{-\cos S \cos (S - C)}{\cos (S - A) \cos (S - B)}}.
\end{aligned}\right\} \quad (10)$$

$$\tan \tfrac{1}{2} b = \tan R \cos (S - B), \qquad \tan \tfrac{1}{2} c = \tan R \cos (S - C). \qquad (11)$$

It is shown in Art. 49 that R is the radius of the circumscribing circle of the triangle ABC. Article 49 may be read at this time.

NOTE 1. Formulas (8)–(11) can also be derived from formulas (4)–(7) by making use of the polar triangle, as done in Art. 36. C.

NOTE 2. Since $A + B + C$ lies between 180° and 540° (Art. 17), S lies between 90° and 270°; hence, $\cos S$ is negative, and, accordingly, $-\cos S$ is positive. Since all the other functions under the radical signs are positive, the whole expression under each radical sign is positive.

NOTE 3. The positive value of the radical is taken, since each side (Art. 19) is less than 180°

EXAMPLES.

1. Derive formulas (10) from the values of $\cos b$ and $\cos c$.

2. Derive formulas (10) from formulas (6) by means of the polar triangle.

3. In ABC, given $A = 78° 40'$, $B = 63° 50'$, $C = 46° 20'$. Find a, b, c.

[SUGGESTION. Either use formulas (8)-(10); or, solve the polar triangle, and thence obtain the parts of the original triangle. [*The results may be checked by using both these methods*, or by Art. 36, (1).]

4. In ABC, given $A = 121°$, $B = 102°$, $C = 68°$. Find a, b, c.

5. Show that $\cos(S - A)$ is positive.

38. Napier's Analogies. On *dividing* $\tan \frac{1}{2} A$ by $\tan \frac{1}{2} B$ (Art. 37), there is obtained,

$$\frac{\tan \frac{1}{2} A}{\tan \frac{1}{2} B} = \frac{\sin (s - b)}{\sin (s - a)}.$$

From this, by composition and division,

$$\frac{\tan \frac{1}{2} A + \tan \frac{1}{2} B}{\tan \frac{1}{2} A - \tan \frac{1}{2} B} = \frac{\sin (s - b) + \sin (s - a)}{\sin (s - b) - \sin (s - a)}.$$

This, by *Plane Trigonometry*, Arts. 44. *B*, 52 (also, see Art. 61), reduces to

$$\frac{\sin \frac{1}{2} A \cos \frac{1}{2} B + \cos \frac{1}{2} A \sin \frac{1}{2} B}{\sin \frac{1}{2} A \cos \frac{1}{2} B - \cos \frac{1}{2} A \sin \frac{1}{2} B} = \frac{2 \sin \frac{1}{2} (2s - a - b) \cos \frac{1}{2} (a - b)}{2 \cos \frac{1}{2} (2s - a - b) \sin \frac{1}{2} (a - b)};$$

and thence, to
$$\frac{\sin \frac{1}{2} (A + B)}{\sin \frac{1}{2} (A - B)} = \frac{\tan \frac{1}{2} c}{\tan \frac{1}{2} (a - b)}. \tag{1}$$

On *multiplying* $\tan \frac{1}{2} A$ by $\tan \frac{1}{2} B$, there is obtained

$$\tan \frac{1}{2} A \tan \frac{1}{2} B = \frac{\sin (s - c)}{\sin s},$$

i.e.
$$\frac{\sin \frac{1}{2} A \sin \frac{1}{2} B}{\cos \frac{1}{2} A \cos \frac{1}{2} B} = \frac{\sin (s - c)}{\sin s}.$$

From this, by composition and division,

$$\frac{\cos \frac{1}{2} A \cos \frac{1}{2} B - \sin \frac{1}{2} A \sin \frac{1}{2} B}{\cos \frac{1}{2} A \cos \frac{1}{2} B + \sin \frac{1}{2} A \sin \frac{1}{2} B} = \frac{\sin s - \sin (s - c)}{\sin s + \sin (s - c)}$$
$$= \frac{2 \cos \frac{1}{2} (2s - c) \sin \frac{1}{2} c}{2 \sin \frac{1}{2} (2s - c) \cos \frac{1}{2} c}.$$

Whence,
$$\frac{\cos \frac{1}{2} (A + B)}{\cos \frac{1}{2} (A - B)} = \frac{\tan \frac{1}{2} c}{\tan \frac{1}{2} (a + b)}. \tag{2}$$

Either, on proceeding in a similar way with $\tan\frac{1}{2}a$ and $\tan\frac{1}{2}b$ [Art. 37, (8), (10)], *or*, on applying (1) and (2) to the polar triangle, there is obtained,

$$\frac{\sin\frac{1}{2}(a+b)}{\sin\frac{1}{2}(a-b)} = \frac{\cot\frac{1}{2}C}{\tan\frac{1}{2}(A-B)} \tag{3}$$

and

$$\frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}(a-b)} = \frac{\cot\frac{1}{2}C}{\tan\frac{1}{2}(A+B)}. \tag{4}$$

Relations (1)–(4) are known as *Napier's Analogies.**

NOTE 1. Compare (3) with formula (2) Art. 61, *Plane Trigonometry.*

NOTE 2. The numerators in (3) are always positive, since $a+b+c<360°$ and $C<180°$. It follows, accordingly, that *$a-b$ and $A-B$ must have the same sign.* This also follows from the geometrical fact [Art. 15, (5)] that the greater angle is opposite the greater side.

NOTE 3. In relation (4), $\cot\frac{1}{2}C$ and $\cos\frac{1}{2}(a-b)$ are positive quantities; hence $\cos\frac{1}{2}(a+b)$ and $\tan\frac{1}{2}(A+B)$ have the same sign; and, accordingly, *$\frac{1}{2}(a+b)$ and $\frac{1}{2}(A+B)$ are of the same species* (Art. 27).

NOTE 4. *Derivation of* (3) *by applying* (1) *to the polar triangle.* On applying (1) to the polar triangle $A'B'C'$ (Fig. 14, Art. 16),

$$\frac{\sin\frac{1}{2}(A'+B')}{\sin\frac{1}{2}(A'-B')} = \frac{\tan\frac{1}{2}c'}{\tan\frac{1}{2}(a'-b')}.$$

$$\therefore \frac{\sin\frac{1}{2}(\overline{180°-a}+\overline{180°-b})}{\sin\frac{1}{2}(\overline{180°-a}-\overline{180°-b})} = \frac{\tan\frac{1}{2}(180°-C)}{\tan\frac{1}{2}(\overline{180°-A}-\overline{180°-B})}; \text{ [Art. 16. } d.]$$

i.e.
$$\frac{\sin(180°-\frac{1}{2}\overline{a+b})}{\sin\frac{1}{2}(b-a)} = \frac{\tan(90°-\frac{1}{2}C)}{\tan\frac{1}{2}(B-A)}.$$

Whence,
$$\frac{\sin\frac{1}{2}(a+b)}{\sin\frac{1}{2}(a-b)} = \frac{\cot\frac{1}{2}C}{\tan\frac{1}{2}(A-B)}.$$

NOTE 5. For a *geometrical deduction* of Napier's Analogies and the formulas in Art. 39, see M'Clelland and Preston, *Treatise on Spherical Trigonometry*, Part I., Art. 56, and the article *Trigonometry* in the Encyclopedia Britannica (9th edition).

* That is, Napier's *proportions.* For a long time the word *analogy* was used in English in one of its original Greek meanings, namely, a proportion (*i.e.* an equality of ratios). This use of the word is now obsolete, and is only retained in a few phrases such as the above. Napier (see Art. 30, and *Plane Trigonometry*, Art. 1) discovered these proportions and gave them in his work, *Mirifici logarithmorum canonis descriptio*, in 1614

EXAMPLES.

1. Express Napier's Analogies in *words*.
2. *Write* the analogies involving B and C, A and C, b and c, a and c.
3. *Derive* some of the analogies in Ex. 2.

39. Delambre's Analogies or Gauss's Formulas.

By *Plane Trigonometry*, Art. 46, (1),

$$\sin \tfrac{1}{2}(A+B) = \sin \tfrac{1}{2} A \cos \tfrac{1}{2} B + \cos \tfrac{1}{2} A \sin \tfrac{1}{2} B.$$

By Art. 37, (4), (6),

$$\sin \tfrac{1}{2} A \cos \tfrac{1}{2} B = \frac{\sin(s-b)}{\sin c}\sqrt{\frac{\sin s \sin(s-c)}{\sin a \sin b}} = \frac{\sin(s-b)}{\sin c} \cos \tfrac{1}{2} C,$$

and $$\cos \tfrac{1}{2} A \sin \tfrac{1}{2} B = \frac{\sin(s-a)}{\sin c}\sqrt{\frac{\sin s \sin(s-c)}{\sin a \sin b}} = \frac{\sin(s-a)}{\sin c} \cos \tfrac{1}{2} C.$$

$$\therefore \sin \tfrac{1}{2}(A+B) = \frac{\sin(s-a)+\sin(s-b)}{\sin c} \cos \tfrac{1}{2} C$$

$$= \frac{2 \sin \tfrac{1}{2}(2s-a-b) \cos \tfrac{1}{2}(a-b)}{2 \sin \tfrac{1}{2} c \cos \tfrac{1}{2} c} \cos \tfrac{1}{2} C.$$

$$\therefore \sin \tfrac{1}{2}(A+B) = \frac{\cos \tfrac{1}{2}(a-b)}{\cos \tfrac{1}{2} c} \cos \tfrac{1}{2} C. \qquad (1)$$

In a similar way it may be shown that

$$\sin \tfrac{1}{2}(A-B) = \frac{\sin \tfrac{1}{2}(a-b)}{\sin \tfrac{1}{2} c} \cos \tfrac{1}{2} C, \qquad (2)$$

$$\cos \tfrac{1}{2}(A+B) = \frac{\cos \tfrac{1}{2}(a+b)}{\cos \tfrac{1}{2} c} \sin \tfrac{1}{2} C, \qquad (3)$$

$$\cos \tfrac{1}{2}(A-B) = \frac{\sin \tfrac{1}{2}(a+b)}{\sin \tfrac{1}{2} c} \sin \tfrac{1}{2} C. \qquad (4)$$

Formulas (1)–(4) are known as Delambre's Analogies, and also as Gauss's Formulas or Equations.*

---

* These formulas were discovered by Karl Friedrich Gauss (1777–1855), one of the greatest of German mathematicians and astronomers, and published without proof in his *Theoria Motus Corporum Cœlestium* in 1809; thus they bear his name. They were, however, published earlier by Karl Brandon Mollweide of Leipzig (1774–1825) in Zach's *Monatliche Correspondenz* for November, 1808. They were earliest discovered by Jean Baptiste Joseph Delambre (1749–1822), a great French astronomer, in 1807, and published in the *Connaissance des Temps* in 1808. The geometrical proof (see Note 5, Art. 38) was the one originally given by Delambre. This proof was rediscovered and announced by M. W. Crofton in 1869, and published in the *Proceedings of the London Math. Soc.*, Vol. III. (1869–1871), p. 13.

NOTE 1. Equations (3) and (4) can also be derived by applying (1) and (2) to the polar triangle.

NOTE 2. Delambre's Analogies can also be deduced by help of Napier's Analogies. (See Todhunter, *Spherical Trigonometry*, Art. 54; *Nature*, Vol. XL. (1889, Oct. 31), p. 644.)

NOTE 3. On the other hand, Napier's Analogies can be easily derived from Delambre's Analogies; namely, on dividing corresponding members, one by the other, in (1) and (3), (2) and (4), (4) and (3), (2) and (1).

EXAMPLES.

1. Write Delambre's Analogies involving B and C, and C and A.
2. Derive (3) and (4) from (1) and (2), using the polar triangle.
3. Derive Delambre's Analogies from Napier's Analogies.
4. Derive some of the analogies in Ex. 1 directly.

40. Other relations between the parts of a spherical triangle. The preceding articles of this chapter present few more relations than are required for the solution of spherical triangles. Between the parts of a spherical triangle there are many other relations which are interesting and useful for many purposes, and which either set forth, or lead to the discovery of, important geometrical properties* of spherical triangles.

For example, if in equation (2) Art. 36, the value of $\cos c$ in the second equation that follows, be substituted, then

$$\cos a = \cos a \cos^2 b + \sin a \sin b \cos b \cos C + \sin b \sin c \cos A;$$

whence, on putting for $\cos^2 b$ its value $1 - \sin^2 b$, dividing by $\sin b$, and transposing, it follows that

$$\cos a \sin b - \sin a \cos b \cos C = \sin c \cos A.$$

Five similar relations can be derived by permuting the letters; and on applying these six relations to the polar triangle, six others can be derived.

To pursue this topic further is beyond the scope of this book, which aims to give little more than the simplest elements of spherical trigonometry and what is absolutely required for the solution of spherical triangles. Those who are interested can refer to the works on spherical trigonometry by M'Clelland and Preston (Macmillan & Co.), Casey (Longmans, Green, & Co.), Bowser (D. C. Heath & Co.), and others.

N.B. *Questions and exercises on Chapter III. will be found on page* 104.

---

* Instances in which geometrical properties are deduced by means of trigonometry, are given in Art. 27, Art. 36, (Note 2), Art. 38, (Notes 2, 3).

CHAPTER IV.

SOLUTION OF TRIANGLES.

N.B. The student is recommended to work one or two examples in each set in this chapter *before* reading any of the text.

41. Cases for solution. This chapter is concerned with the numerical solution of spherical triangles. In all there are six cases for solution; these correspond respectively to the six cases for construction which were discussed in Arts. 23, 24. In these cases the given parts are as follows:

I. **Three sides.**

II. **Three angles.**

III. **Two sides** and **their included angle.**

IV. **One side** and **its two adjacent angles.**

V. **Two sides** and **the angle opposite one of them.**

VI. **Two angles** and **the side opposite one of them.**

With slight changes the procedure described in Art. 31 is recommended. Figures may be helpful. Of formulas adapted for logarithmic computation, the necessary ones are (1) Art. 36, (4)–(11) Art. 37, and (1)–(4) Art. 38. If the polar triangle is used in finding the solution, then I. and II. constitute one case, likewise III. and IV., and likewise V. and VI.; and the necessary formulas are (1) Art. 36 (4)–(7) *or* (8)–(11) Art. 37, and (1), (2), *or* (3), (4) Art. 38. Cases V. and VI. must be examined as to ambiguity; and accordingly, they give more trouble than the others. *Unless the triangle satisfies the conditions specified in Arts* 15, 17, 24 V., *its solution is impossible.*

Checks. **The results obtained should always be checked.** Delambre's Analogies and formulas which have not been used in the course of the solution, may be used as check formulas.

N.B. Before doing any of the numerical work the student should try to get a clear idea of the figure of the triangle upon a sphere. This geometrical conception will enable him to make a reasonable estimate of what the results will be; this estimate will help him to detect wild results that may be obtained by making numerical errors. For example, in ABC let $a = 110°$, $b = 114°$, $C = 10°$; and suppose that the result $c = 76°$ presents itself. A person who has drawn a figure of the triangle on a sphere, or one who has geometrical imagination sufficient to give him an idea of the look of the given triangle, will at once see that the result, $c = 76°$, must be wrong. In working spherical triangles it is much better not to proceed blindly.

42. Case I. Given the three sides.

EXAMPLES.

1. In ABC, given $a = 47° 30'$, $b = 55° 40'$, $c = 60° 10'$. Find A, B, C.

Formulas:

$$\tan r = \sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}},$$

$$\tan \tfrac{1}{2} A = \frac{\tan r}{\sin(s-a)}, \quad \tan \tfrac{1}{2} B = \frac{\tan r}{\sin(s-b)}, \quad \tan \tfrac{1}{2} C = \frac{\tan r}{\sin(s-c)}.$$

Check: Law of Sines, or Napier's Analogies, or Delambre's Analogies.

Logarithmic formulas:

$$\log \tan^2 r = \log \sin(s-a) + \log \sin(s-b) + \log \sin(s-c) - \log \sin s, \text{ etc.}$$

Check: $\log \sin a - \log \sin A = \log \sin b - \log \sin B = \log \sin c - \log \sin C.$

$a = 47° 30'$	$\log \sin s = 9.99539 - 10$	$\therefore \tfrac{1}{2} A = 28° 16' 2''$
$b = 55° 40'$	$\log \sin(s-a) = 9.74943 - 10$	$\tfrac{1}{2} B = 34° 33' 41.5''$
$c = 60° 10'$	$\log \sin(s-b) = 9.64184 - 10$	$\tfrac{1}{2} C = 39° 29' 12''$
$\therefore 2s = 163° 20'$	$\log \sin(s-c) = 9.56408 - 10$	$\therefore A = 56° 32' 4''$
$s = 81° 40'$	$\therefore \log \tan^2 r = 18.95996 - 20$	$B = 69° \; 7' 23''$
$s - a = 34° 10'$	$\therefore \log \tan r = 9.47998 - 10$	$C = 78° 58' 24''$
$s - b = 26°$	$\therefore \log \tan \tfrac{1}{2} A = 9.73055 - 10$	
$s - c = 21° 30'$	$\log \tan \tfrac{1}{2} B = 9.83814 - 10$	
	$\log \tan \tfrac{1}{2} C = 9.91590 - 10$	

Check:	$\log \sin a = 9.86763$	$\log \sin b = 9.91686$	$\log \sin c = 9.93826$
	$\log \sin A = 9.92128$	$\log \sin B = 9.97051$	$\log \sin C = 9.99191$
	9.94635	9.94635	9.94635

NOTE 1. *Directions for the numerical work:* Fill in the first column; turn up the first four logarithms in the second column (since these logarithms are required by the formulas); compute the last five logarithms in the second column according to the formulas (these computations may be made on another paper, if necessary); find the first three angles of the third column by the tables; thence compute A, B, C.

NOTE 2. If only one angle is required, say A, it can be found by one of formulas (4) Art. 37; preferably, the second. Angle A can also be found (without logarithms) by formula (1) Art. 37.

2. Solve ABC, given that $a = 43° 30'$, $b = 72° 24'$, $c = 87° 50'$.

3. Solve ABC, given that $a = 110° 40'$, $b = 45° 10'$, $c = 73° 30'$.

4. Solve ABC, given that $a = 120° 50'$, $b = 98° 40'$, $c = 74° 60'$.

5. Solve PQR, given that $p = 67° 40'$, $q = 47° 20'$, $r = 83° 50'$.

43. Case II. Given the three angles.

Either: Solve the polar triangle by the method used in Case I., and therefrom obtain the parts of the original triangle.

Or: Solve by means of formulas (8)–(11) Art. 37.

EXAMPLES.

Solve ABC, and check the results.

1. Given $A = 74° 40'$, $B = 67° 30'$, $C = 49° 50$.
2. Given $A = 112° 30'$, $B = 83° 40'$, $C = 70° 10'$.
3. Given $A = 130°$, $B = 98°$, $C = 64°$.
4. Given $P = 33° 40'$, $Q = 26° 10'$, $R = 20° 30'$. Find p, q, r.

NOTE. The results may also be checked by solving the examples by both the methods above.

44. Case III. Given two sides and their included angle.

EXAMPLES.

1. In ABC, $a = 64° 24'$, $b = 42° 30'$, $C = 58° 40'$; find A, B, c.

Formulas:

$$\tan \tfrac{1}{2}(A + B) = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)} \cot \tfrac{1}{2} C;$$

$$\tan \tfrac{1}{2}(A - B) = \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)} \cot \tfrac{1}{2} C;$$

$$\sin c = \frac{\sin a}{\sin A} \sin C.$$

Checks: Law of Sines, etc.

$C = 58° 40'$	$\log \cot \frac{1}{2} C = 0.25031$	$\therefore \log \tan \frac{1}{2}(A+B) = 0.46745$
$a = 64° 24'$	$\log \sin \frac{1}{2}(a+b) = 9.90490-10$	$\log \tan \frac{1}{2}(A-B) = 9.62405$
$b = 42° 30'$	$\log \cos \frac{1}{2}(a+b) = 9.77490-10$	$\therefore \frac{1}{2}(A+B) = 71° 10' 41''$
$a+b = 106° 54'$	$\log \sin \frac{1}{2}(a-b) = 9.27864-10$	$\frac{1}{2}(A-B) = 22° 49' 12''$
$a-b = 21° 54'$	$\log \cos \frac{1}{2}(a-b) = 9.99202-10$	$\therefore A = 93° 59' 53''$
$\frac{1}{2} C = 29° 20'$	$\log \sin a = 9.95513-10$	$B = 48° 21' 29''$
$\frac{1}{2}(a+b) = 53° 27'$	$\log \sin C = 9.93154-10$	$\therefore \log \sin A = 9.99894-10$
$\frac{1}{2}(a-b) = 10° 57'$		$\therefore \log \sin c = 9.88773-10$
		$\therefore c = 50° 33' 6''$

NOTE 1. Since $C<A$, then $c<a$; and hence, the acute value of c is taken.

NOTE 2. *Directions for the numerical work:* Fill in the first column; then turn up *all* the logarithms for the second column, these logarithms being required by the formulas; then compute the first two logarithms in the third column, according to the formulas; thence find the corresponding angles, and calculate A and B; turn up $\log \sin A$; compute $\log \sin c$ according to the formula; then find c in the Tables.

NOTE 3. In using formulas involving the difference of two sides or two angles, place the larger side or angle first.

2. Solve ABC, given $a = 93° 20'$, $b = 56° 30'$, $C = 74° 40'$.

3. Solve ABC, given $b = 76° 30'$, $c = 47° 20'$, $A = 92° 30'$.

4. Solve ABC, given $c = 40° 20'$, $a = 100° 30'$, $B = 46° 40'$.

5. Solve PQR, given $q = 76° 30'$, $r = 110° 20'$, $P = 46° 50'$.

45. Case IV. Given one side and its two adjacent angles.

Either: Solve the polar triangle by the method used in Case III.; and therefrom obtain the parts of the original triangle.

Or: Solve by using formulas (1), (2), Art. 38.

EXAMPLES.

1. Solve ABC, given $A = 67° 30'$, $B = 45° 50'$, $c = 74° 20'$.

2. Solve ABC, given $B = 98° 30'$, $C = 67° 20'$, $a = 60° 40'$.

3. Solve ABC, given $C = 110°$, $A = 94°$, $b = 44°$.

4. Solve PQR, given $R = 70° 20'$, $Q = 43° 50'$, $p = 50° 46'$.

46. Case V. Given two sides and the angle opposite one of them.

This is an **ambiguous case,*** since (Art. 24, V.) there may be two solutions. It may be well to examine this case (1) *geometrically*, that is, by an inspection of the figure; (2) *analytically*, that is, by an inspection of the formulas involved in its solution.

(1) *Geometrically.* In Art. 24, V. (Figs. 21, 25) it has been seen that, when two sides and an angle opposite one of them (say, a, b, A) of a triangle ABC are given, there are *two* triangles possible if either of the following sets of conditions holds, viz.:

$$A < 90°,\ a > p,\ a < b, \text{ and } a < 180° - b; \qquad (a)$$

$$A > 90°,\ a < p,\ a > b, \text{ and } a > 180° - b. \qquad (b)$$

* For a detailed discussion of the ambiguous case, see Todhunter, *Spherical Trigonometry*, pp. 53–58; M'Clelland and Preston, *Spherical Trigonometry*, pp. 137–143.

In order that the triangle be *possible*, it is apparent that: *either* $CB = CP$; or, in Fig. 21, $CB > CP$, *i.e.* $\sin CB > \sin CP$, *i.e.* $\sin a > \sin AC \sin CAP$,

$$\textit{i.e.}\ \sin a > \sin b \sin A;$$

and, in Fig. 25, $CLB < CP$, and $CLB > CP'$, *i.e.* $\sin a > CP'$, *i.e.* $\sin a > \sin AC \sin CAP'$,

$$\textit{i.e.}\ \sin a > \sin b \sin (180° - CAP),\ \textit{i.e.}\ \sin a > \sin b \sin A.$$

Art. 24 also shows that, when the triangle is possible, there is *one* solution if either of the following sets of conditions holds, viz.:

$$A < 90°,\ a > p,\ a \text{ between } b \text{ and } 180° - b; \tag{c}$$

$$A > 90°,\ a < p,\ a \text{ between } b \text{ and } 180° - b. \tag{d}$$

If $CB = CP$, *i.e.* if $a = p$, then there is *one* solution.

Art. 24 also shows that the triangle is impossible if either one of the following sets of conditions holds, viz.:

$$A < 90°,\ a \text{ greater than both } b \text{ and } 180° - b; \tag{e}$$

$$A > 90°,\ a \text{ less than both } b \text{ and } 180° - b.$$

Since the greater angle is opposite the greater side, B must be such that $A - B$ shall have the same sign as $a - b$.

(2) *Analytically.* The formulas used in solving this case are as follows:

$$\sin B = \frac{\sin b \sin A}{\sin a}, \tag{1}$$

$$\cot \tfrac{1}{2} C = \frac{\sin \frac{1}{2}(a + b)}{\sin \frac{1}{2}(a - b)} \tan \tfrac{1}{2}(A - B), \quad \text{[or, (4) Art. 38]} \tag{2}$$

$$\tan \tfrac{1}{2} c = \frac{\sin \frac{1}{2}(A + B)}{\sin \frac{1}{2}(A - B)} \tan \tfrac{1}{2}(a - b). \quad \text{[or, (2) Art. 38]} \tag{3}$$

Since B is determined from its sine, it may be in either the first or the second quadrant. If $\sin a = \sin b \sin A$, then $B = 90°$. If $\sin a < \sin b \sin A$, then $\sin B > 1$, and B has an impossible value, and, accordingly, the triangle is impossible. [Compare above.] Equation (2) shows that $A - B$ and $a - b$ have the same sign.

Hence, from the *analytical* inspection comes the following **rule:**

If $\sin a < \sin b \sin A$, *there is* **no** *solution; if* $\sin a = \sin b \sin A$, *there is* **one** *solution; if* $\sin a > \sin b \sin A$, *and if* **both** *values of* B *obtained from* (1) *be such that*

$$A - B \text{ and } a - b \text{ have } \textbf{like} \text{ signs,}$$

there are **two** *solutions; if only* **one** *of the values of* B *satisfies this condition, there is only* **one** *solution; if* **neither** *of the values of* B *satisfies this condition, the solution is* **impossible.**

From the *geometrical* inspection comes the following **rule:**

If $\sin a < \sin b \sin A$, *there is* **no** *solution; if* $\sin a = \sin b \sin A$, *there is* **one** *solution; if* $\sin a > \sin b \sin A$, *then:*

When A is less than 90°:

there are **two** *solutions if* a *is* **less** *than both* b *and* $180° - b$;
there is **one** *solution if* a *lies* **between** b *and* $180° - b$;
there is **no** *solution if* a *is* **greater** *than both* b *and* $180° - b$.

When A is greater than 90°:

there are **two** *solutions if* a *is* **greater** *than both* b *and* $180° - b$;
there is **one** *solution if* a *lies* **between** b *and* $180° - b$;
there is **no** *solution if* a *is* **less** *than both* b *and* $180° - b$.

NOTE 1. The second rule has one advantage over the first, in that it enables one to say, merely on calculating $\sin B$, but without finding B, whether the triangle is ambiguous or not.

NOTE 2. The property observed in Art. 36, Note 2, is frequently used in investigating the ambiguous case.

EXAMPLES.

1. In ABC, $a = 43° 20'$, $b = 48° 30'$, $A = 58° 40'$; find B, C, c.

Formulas:
$$\sin B = \frac{\sin b \sin A}{\sin a}.$$

$$\cot \tfrac{1}{2} C = \frac{\sin \frac{1}{2}(b + a)}{\sin \frac{1}{2}(b - a)} \tan \tfrac{1}{2}(B - A). \qquad \text{[Art. 38, (3)]}$$

$$\tan \tfrac{1}{2} c = \frac{\sin \frac{1}{2}(B + A)}{\sin \frac{1}{2}(B - A)} \tan \tfrac{1}{2}(b - a). \qquad \text{[Art. 38, (1)]}$$

Checks: Formulas (2), (4), Art. 38; or, formulas, Art. 37; or, Delambre's Analogies.

$A = 58° 40'$	$\log \sin A = 9.93154 - 10$	$\therefore B + A = 127° 27'$
$a = 43° 20'$	$\log \sin a = 9.83648 - 10$	$B - A = 10°\ 7'$
$b = 48° 30'$	$\log \sin b = 9.87446 - 10$	$\frac{1}{2}(B + A) = 63° 43' 30''$
$\therefore b + a = 91° 50'$	$\therefore \log \sin B = 9.96952 - 10$	$\frac{1}{2}(B - A) = 5°\ 3' 30''$
$b - a = 5° 10'$	$\therefore B = 68° 47'$	$\therefore B' + A = 169° 53'$
$\frac{1}{2}(b + a) = 45° 55'$	$B' = 111° 13'$	$B' - A = 52° 33'$
$\frac{1}{2}(b - a) = 2° 35'$	[According to the test for	$\frac{1}{2}(B' + A) = 84° 56' 30''$
	ambiguity.]	$\frac{1}{2}(B' - A) = 26° 16' 30''$

In ABC. (*See Fig.* 21, *Art.* 24.) *In AB′C.*

$\log \sin \frac{1}{2}(b + a) = 9.85632 - 10$	As in ABC.
$\log \sin \frac{1}{2}(b - a) = 8.65391 - 10$	
$\log \tan \frac{1}{2}(b - a) = 8.65435 - 10$	
$\log \sin \frac{1}{2}(B + A) = 9.95264 - 10$	$\log \sin \frac{1}{2}(B' + A) = 9.99830 - 10$
$\log \sin \frac{1}{2}(B - A) = 8.94532 - 10$	$\log \sin \frac{1}{2}(B' - A) = 9.64609 - 10$
$\log \tan \frac{1}{2}(B - A) = 8.94702 - 10$	$\log \tan \frac{1}{2}(B' - A) = 9.69345 - 10$
$\therefore \log \cot \frac{1}{2} C = 0.14943$	$\therefore \log \cot \frac{1}{2} C = 0.89586$
$\log \tan \frac{1}{2} c = 9.66167 - 10$	$\log \tan \frac{1}{2} c = 9.00656 - 10$
$\therefore \frac{1}{2} C = 35° 19' 55''.4,\ \frac{1}{2} c = 24° 38' 53''.$	$\therefore \frac{1}{2} C = 7° 14' 36'',\ \frac{1}{2} c = 5° 47' 49''.$
$\therefore C = 70° 39' 51'',\ c = 49° 17' 46''.$	$\therefore C = 14° 29' 12'',\ c = 11° 35' 38''.$

Hence, the solutions are :

$$ABC = 68° 47',\ ACB = 70° 39' 51'',\ AB = 49° 17' 46'';$$
$$AB'C = 111° 13',\ ACB' = 14° 29' 12'',\ AB' = 11° 35' 38''.$$

Note 3. *Directions for the numerical work:* Fill in the first of the three columns; turn up the first three logarithms in the second column, these being required by the first formula; compute log sin B according to the first formula; find B in the tables; *decide the question of ambiguity;* fill in the third column (only four lines when the triangle is not ambiguous). Turn up the first six logarithms in the first of the next two columns; compute the next two logarithms according to the formulas; find the corresponding values in the Tables; thence compute C and c. If the case is ambiguous, do the same work for the second triangle.

2. Solve ABC when $a = 56° 40'$, $b = 30° 50'$, $A = 103° 40'$.

3. Solve ABC when $a = 30° 20'$, $b = 46° 30'$, $A = 36° 40'$.

4. Solve ABC when $c = 74° 20'$, $a = 119° 40'$, $C = 88° 30'$.

5. Solve ABC when $b = 30° 10'$, $c = 44° 30'$, $B = 86° 50'$.

6. Solve PQR when $q = 42° 30'$, $r = 46° 50'$, $Q = 56° 30'$.

47. Case VI. Given two angles and the side opposite one of them. This is also an ambiguous case.

Either: Solve the polar triangle by the method used in Case V.; and therefrom obtain the parts of the original triangle.

Or: Solve by using formula (1) Art. 36, and Napier's Analogies.

The first rule (Art. 46) for determining ambiguity suits the case, if a, b, be substituted for A, B, therein. On making use of the polar triangle, it is found that the second rule can be adapted by substituting a, A, B, for A, a, b, respectively.

EXAMPLES.

1. Solve ABC when $A = 108° 40'$, $B = 134° 20'$, $a = 145° 36'$.
2. Solve ABC when $B = 36° 20'$, $C = 46° 30'$, $b = 42° 12'$.
3. Solve ABC when $C = 62° 10'$, $A = 23° 46'$, $c = 33° 50'$.
4. Solve STV when $T = 102° 50'$, $V = 81° 20'$, $t = 124° 30'$.

48. Subsidiary angles. Formulas can sometimes be adapted for logarithmic computation and the triangle solved, by the use of *subsidiary angles*. *For example*, in ABC let a, c, B be known, and b required. (See Fig. 35, Art. 33.)

$$\cos b = \cos a \cos c + \sin a \sin c \cos B \qquad \text{(Art. 36, } B)$$

$$= \cos c (\cos a + \sin a \tan c \cos B).$$

On putting $\tan c \cos B = \tan \phi$, this becomes

$$\cos b = \cos c (\cos a + \sin a \tan \phi)$$

$$= \frac{\cos c (\cos a \cos \phi + \sin a \sin \phi)}{\cos \phi}$$

$$= \frac{\cos c \cos (a - \phi)}{\cos \phi}.$$

On referring to Fig. 35 it is seen that $BD = \phi$, that $DC = a - \phi$, and $\cos AD = \dfrac{\cos c}{\cos \phi}$; so that solving as above is equivalent to solving the triangle by dividing it into right-angled triangles.

N.B. *Questions and exercises on Chapter IV. will be found on page* 105.

CHAPTER V.

CIRCLES CONNECTED WITH SPHERICAL TRIANGLES.

49. The circumscribing circle. The circle passing through the vertices of a spherical triangle is called the *circumscribing circle*, or *circum-circle*, of the triangle. This circle can be constructed in somewhat the same manner as the circumscribing circle of a plane triangle.

Let ABC (Fig. 41) be a spherical triangle, and let R denote the radius (*i.e.* the polar distance, Art. 6) of its circumscribing circle. Bisect the arcs BC, CA, in L, M, respectively; and at L, M, draw arcs at right angles to BC, CA, respectively. The point O, at which these arcs meet, is the pole of the circumscribing circle.

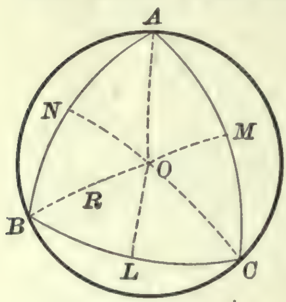


FIG. 41

For, draw OA, OB, OC, arcs of great circles. In the triangles OLB and OLC, $BL = LC$, LO is common, and the angles at L are right angles. Hence, $OB = OC$. In a similar way it can be shown that $OC = OA$. Hence O is the pole of the circumscribing circle.

Join O and N, the middle point of AB; then it is easily shown that ON is at right angles to AB.

In ABC, $\qquad A + B + C = 2S$.

Now (since $OA = OB = OC$),

$$OAB = OBA,\ OBC = OCB,\ OCA = OAC.$$

Hence, $OAB + OBC + OAC = S$.

$$\therefore\ OBC = S - (OAB + OAC) = S - A.$$

In the right-angled triangle OBL,

$$\tan OB = \frac{\tan BL}{\cos OBL}. \qquad \text{[Art. 26, Eq. (3)]}$$

$$i.e.\ \tan R = \frac{\tan \frac{1}{2} a}{\cos (S - A)}. \tag{1}$$

Similarly, $\tan R = \dfrac{\tan \frac{1}{2} b}{\cos (S - B)}$, $\tan R = \dfrac{\tan \frac{1}{2} c}{\cos (S - C)}$.

On substituting in (1) the value of $\tan \frac{1}{2} a$ in relation (8) Art. 37, equation (1) becomes

$$\tan R = \sqrt{\frac{-\cos S}{\cos (S - A) \cos (S - B) \cos (S - C)}}. \tag{2}$$

NOTE 1. Compare (1) with the corresponding case in plane triangles (*Plane Trig.*, Art. 68). (In plane triangles, $S = 90°$, and, hence, $\cos (S - A) = \sin A$.)

NOTE 2. On putting $N = \sqrt{-\cos S \cos(S - A) \cos(S - B) \cos(S - C)}$,

$$\tan R = -\frac{\cos S}{N}.$$

50. The inscribed circle. The circle which touches each of the sides of a spherical triangle is called the *inscribed circle*, or *in-circle*, of the triangle. This circle can be constructed in somewhat the same manner as the inscribed circle of a plane triangle.

Let ABC be a spherical triangle, and let r denote the radius (*i.e.* the polar distance) of its inscribed circle. Bisect angles A, B, by arcs of great circles, and let these arcs meet at O. Draw OL, OM, ON, at right angles to BC, CA, AB, respectively.

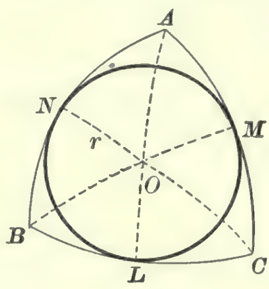


FIG. 42

In the triangles OAM and OAN, the angles at A are equal, the angles at N and M are right angles, and the side OA is common. Hence these triangles are symmetrical, and $OM = ON$. Similarly it can be shown that $ON = OL$. Hence O is the pole of the circle inscribed in ABC.

Since the triangles OAM and OAN are equal, $AM = AN$. Similarly, $BN = BL$, and $CL = CM$.

Now $\qquad AB + BC + CA = 2s$;

hence $\qquad AN + BL + CL = s$.

$$\therefore\ AN = s - (BL + LC) = s - a.$$

In the right-angled triangle AON,

$$\tan ON = \tan OAN \sin AN. \qquad \text{[Art. 26, (4)]}$$

$$\therefore \ \tan r = \tan \tfrac{1}{2} A \sin (s - a). \qquad (1)$$

Similarly, $\tan r = \tan \frac{1}{2} B \sin (s - b)$; $\tan r = \tan \frac{1}{2} C \sin (s - c)$.

On substituting in (1) the value of $\tan \frac{1}{2} A$ in (4) Art. 37, equation (1) becomes

$$\tan r = \sqrt{\frac{\sin (s-a) \sin (s-b) \sin (s-c)}{\sin s}}. \qquad (2)$$

On putting $\quad n = \sqrt{\sin s \sin (s-a) \sin (s-b) \sin (s-c)}, \qquad (3)$

$$\tan r = \frac{n}{\sin s}. \qquad (4)$$

NOTE 1. Compare (1) with *Plane Trigonometry*, Art. 69, Note; (2) with Art. 69, (3); n with S, Art. 66, (3); (4) with (3) Art. 69.

51. Escribed circles. A circle which touches a side of a spherical triangle, and the other two sides produced (that is, which is inscribed in a *co-lunar* triangle), is an *escribed circle,* or an *ex-circle,* of the triangle. There are three ex-circles, one corresponding to each side of the triangle.

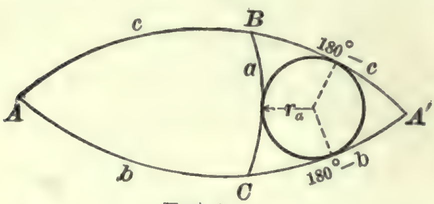


FIG. 43

Let ABC be a spherical triangle; and let the radii of the escribed circles, touching a, b, c, respectively, be denoted by r_a, r_b, r_c, respectively. Complete the lune whose angle is A. The escribed circle which touches a is the inscribed circle of the co-lunar triangle $A'BC$. Hence [Art. 50, (1)],

$$\tan r_a = \tan \tfrac{1}{2} A' \sin \tfrac{1}{2} [(a + \overline{180° - b} + \overline{180° - c}) - 2a];$$

i.e. $$\tan r_a = \tan \tfrac{1}{2} A \sin s. \qquad (1)$$

Similarly, $\tan r_b = \tan \frac{1}{2} B \sin s$; $\tan r_c = \tan \frac{1}{2} C \sin s$.

On substituting for $\tan \frac{1}{2} A$ its value in (4) Art. 37, equation (1) becomes

$$\tan r_a = \sqrt{\frac{\sin s \sin (s-b) \sin (s-c)}{\sin (s-a)}}, \tag{2}$$

i.e.

$$\tan r_a = \frac{n}{\sin (s-a)}. \qquad \text{[Art. 50, (3)]} \tag{3}$$

Similarly, $\tan r_b = \dfrac{n}{\sin (s-b)}$; $\tan r_c = \dfrac{n}{\sin (s-c)}$.

NOTE. Compare (3) with the corresponding result in *Plane Trigonometry*, Art. 70.

Some other relations between the sides and angles of a spherical triangle and the radii of the circles connected with it, are indicated in the exercises at the end of the book.

Ex. Find the radii of the circumscribing, inscribed, and escribed circles of some of the triangles in Chapters II., IV.

N.B. *For questions and exercises on Chapter V., see page* 107.

CHAPTER VI.

AREAS AND VOLUMES CONNECTED WITH SPHERES.

52. Preliminary propositions.

a. The lateral area of a frustum of a regular pyramid is equal to the product of the slant height of the frustum and half the sum of the perimeters of its bases.

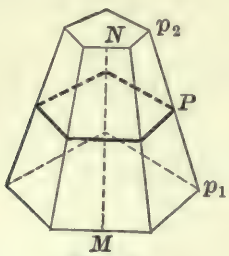


FIG. 44

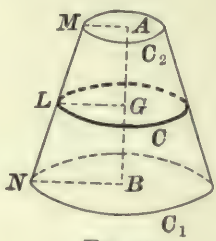


FIG. 45

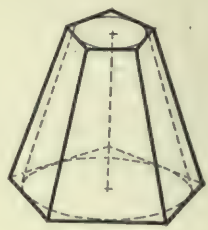


FIG. 46

The student can easily prove this (Fig. 44). It should be noted that the half sum of the perimeters of the bases of the frustum is equal to the perimeter of the section which is parallel to the bases and midway between them.

In symbols: If p_1, p_2, P, are the perimeters of the bases and the middle section of the frustum, and MN is its slant height, then

$$\text{lateral area of frustum} = \tfrac{1}{2}\, MN\,(p_1 + p_2) = MN \cdot P.$$

b. The lateral area of a frustum of a cone of revolution is equal to the product of the slant height of the frustum and half the sum of the circumferences of its bases.

[*Suggestion for proof:* If the number of the lateral faces of a frustum of a regular pyramid be indefinitely increased and each face be indefinitely decreased, then this frustum approaches the frustum of a cone of revolution as a limit (see Fig. 46). Accordingly, Proposition (*b*) follows at once from (*a*)]. It should be noted that half the sum of the circumferences of the bases of the frustum is equal to the circumference of the section which is parallel to the bases and midway between them.

In symbols: If C_1, C_2, C (Fig. 45) are the circumferences of the bases and the middle section of the frustum, and MN is its slant height, then lateral area of frustum

$$= \tfrac{1}{2}\, MN\, (C_1 + C_2) = MN \cdot C = 2\, \pi LG \cdot MN.$$

NOTE. The lateral surface of the frustum of the cone (Fig. 45) can be generated by the revolution of the line MN about the line AB which is in the same plane with MN.

53. To find the area of a sphere. The surface of a sphere can be generated by the revolution of a semicircle about its diameter. For example, the semicircle $ATKB$ of radius R on revolving about its diameter AB, will describe the surface of a sphere of radius OA.

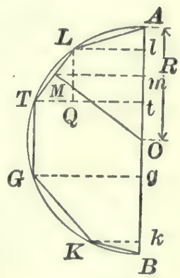


FIG. 47

Let a polygon $ALTGKB$ be inscribed in this semicircle. At M, the middle point of one of the chords LT, draw MO at right angles to LT. By geometry, MO will meet AB at O, the middle point of AB. Project LT on AB, the projection being lt; draw LQ at right angles to Tt.

By Art. 52. *b*, the area generated by LT in its revolution about AB

$$= 2\, \pi Mm \cdot LT. \tag{1}$$

Since the angles of the triangle LTQ are respectively equal to the angles of OMm, these triangles are similar; accordingly,

$$LT : LQ = OM : Mm.$$

$$\therefore\ Mm \cdot LT = LQ \cdot OM = lt \cdot OM.$$

Hence, from (1), area generated by $LT = 2\, \pi OM \cdot lt$. (2)

In words: When a chord of a semicircle revolves about the diameter, the area generated is equal to $2\, \pi$ times the product of the length of the perpendicular from the centre to the chord, and the projection of the chord upon the diameter.

∴ The area of the surface generated by the revolution of the polygon $ALTGKB$

$$
\begin{aligned}
= \quad & 2\pi \times (\text{perpendicular on } AL \text{ from } O) \times Al \\
& + 2\pi \times (\text{perpendicular on } LT \text{ from } O) \times lt \\
& + 2\pi \times (\text{perpendicular on } TG \text{ from } O) \times tg \\
& + 2\pi \times (\text{perpendicular on } GK \text{ from } O) \times gk \\
& + 2\pi \times (\text{perpendicular on } KB \text{ from } O) \times kB.
\end{aligned}
$$

If the number of sides in the polygon inscribed in the semicircle is indefinitely increased and each side is indefinitely decreased, then the broken line $ALTGKB$ approaches the semicircle as a limit, and each of the perpendiculars drawn from O to the middle points of the chords approaches R as a limit; while the sum of the projections of the chords remains equal to AB, the diameter of the circle. Hence, area of surface generated by revolution of semicircle $AGB = 2\pi \cdot R \cdot 2R$;

$$i.e. \text{ area of surface of sphere of radius } R = 4\pi R^2.$$

In words: The area of the surface of a sphere is four times the area of a great circle of the sphere.

Definition. A **zone** of a sphere is a portion of the surface included between two parallel planes, or, what comes to the same thing, is the portion of the surface included between two circles which have common poles; for example, the surface between the parallels of 30° N. latitude and 50° N. latitude.

The area of a zone. An infinite number of chords can be inscribed in the arc LT (Fig. 47). By reasoning similar to that employed above, it can be shown that

$$\text{area of surface generated by arc } LT = 2\pi R \cdot lt.$$

∴ *The area of a spherical zone is equal to the product of the length of a great circle of the sphere and the height of the zone.*

It follows that on a sphere or on equal spheres the areas of zones of equal heights are equal.

EXAMPLES.

1. Find the area of a sphere of radius 15 inches.

2. Find the surface of a spherical zone of height 2.5 inches on a sphere of diameter 50 inches.

3. Find the convex surface of a spherical segment of height 4.5 inches on a sphere of diameter 7 feet. [See definition, Art. 63.]

4. Suppose that the earth is a sphere whose radius is 3960 miles; find the area of the surface included between the North Pole and the parallel of 80° N. latitude; between the parallels of 49° N. and 50° N.; between 5° N. and 5° S.

54. Lunes. *Definition.* The spherical surface bounded by two halves of great-circles is called a *lune; e.g.* the surface between two meridians. The *angle of the lune* is the angle between the two semicircles; thus the angle of the lune between the meridians 70° W. and 80° W. is 10°.

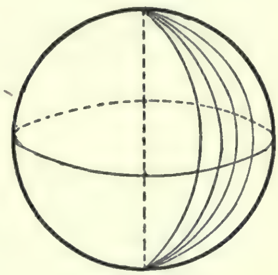


FIG. 48

Proposition. On the same circle or on equal circles the areas of lunes are proportional to their angles. This can be proved by a method similar to that which is used in proving that the angles at the centre of a circle are proportional to the arcs subtended by them.

55. A spherical degree defined. From the proposition in Art. 54 it follows that the area of a lune is to the area of the surface of the sphere as the angle of the lune is to four right angles. That is,

$$\text{area of lune of angle } A^\circ : \text{area of sphere} = A^\circ : 360^\circ.$$

Hence, $$\text{area of lune of angle } 1^\circ = \frac{\text{area of sphere}}{360}.$$

Let a great circle be drawn about one of the vertices of a lune of angle 1° as a pole. The lune is then divided into two equal bi-rectangular triangles; accordingly, each triangle contains $(\frac{1}{720})$th of the surface of the sphere, or $(\frac{1}{360})$th of the surface of the hemisphere. *The surface of each such triangle is called* **a spherical degree.**

For example, the part of the surface of a globe bounded by the meridians 43° W. and 63° W. longitude and the equator, contains 20 spherical degrees; the lune bounded by these meridians contains 40 spherical degrees.

A lune of angle $A°$ contains $2A$ spherical degrees.

The passage from spherical degrees of surface to the ordinary measure (of the area) of the surface is easily effected when the radius of the sphere is given.

A spherical degree $= (\frac{1}{720})$*th part of the surface of a sphere;*

hence, **on a sphere of radius r,**

a spherical degree contains $\dfrac{4\pi r^2}{720}$, *i.e.* $\dfrac{\pi r^2}{180}$ **square units of area.**

Thus,

area of a lune of angle 20° on a sphere of radius $r = \dfrac{40\pi r^2}{180} = \frac{2}{9}\pi r^2.$

EXAMPLES.

1. Find the area of a lune of angle 10° on a sphere of radius 2 feet.

2. Find the area of a lune of angle 37° 30′ on a sphere of radius 7 feet.

3. Find the area between the meridians 77° W. and 83° 20′ W.; and the area between the meridians 174° 20′ W. and 158° 35′ E. (Radius of earth = 3960 miles.) [Express areas in spherical degrees and in square miles.]

56. Spherical excess of a triangle. The sum of the angles of a plane triangle is always *equal to* 180°; the sum of the angles of a spherical triangle is always *greater than* 180° (Art. 17). The difference between the latter sum and 180° is called *the spherical excess* of the triangle. (This *excess* is due to the fact that the triangle is spherical and not plane; hence the excess is called *spherical.*) For example, in the triangle bounded by the meridians 47° W. and 48° W. longitude and the equator, the sum of the angles is 181°; and, accordingly, the spherical excess is 1°. In the triangle bounded by the meridians 43° W. and 63° W. and the equator the sum of the angles is 200°, and the spherical excess is 20°; in the spherical triangle having angles 50°, 65°, 125°, the spherical excess is $(50° + 65° + 125° - 180°)$, *i.e.* 60°.

If E denote the number of *degrees* in the spherical excess, and E_r denote the number of *radians* therein, then

in a triangle ABC, $\quad \boldsymbol{E^\circ = A^\circ + B^\circ + C^\circ - 180^\circ}$; $\qquad$ (1)

and [*Plane Trigonometry*, Art. 73, (7)],

$$\boldsymbol{E_r = \left(\frac{A + B + C - 180}{180}\right)\pi}. \qquad (2)$$

Ex. Find the spherical excess (in *degrees* and in *radians*) of the triangles described in Art. 42, Exs. 1, 2, 3; Art. 43, Exs. 1, 2; Art. 44, Exs. 1, 2, 3; Art. 45, Exs. 1, 2; Art. 46, Exs. 1, 2, 3; Art. 47, Exs. 1, 2.

57. The area of a spherical triangle.

Proposition: The number of spherical degrees (of surface) *in a spherical triangle is equal to the number of* (angular) *degrees in its spherical excess.**

Let ABC be a spherical triangle whose spherical excess is E°; then area ABC is equal to E spherical degrees. Complete the great circle $BCB'C'$, and produce the arcs BA, CA to meet this circle in B', C', respectively. Complete the great circles $BAB'B$ and $ACA'C'$. The triangle $AB'C'$ is equal to the triangle $A'BC$. For,

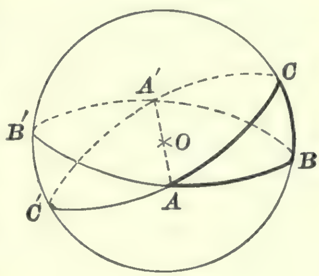


FIG. 49

$$B'A = 180^\circ - AB = BA',$$

$$C'A = 180^\circ - AC = CA',$$

$$C'B' = 180^\circ - B'C = CB.$$

Hence, *in area*, $ABC + AB'C' = \text{lune } ACA'BA$;

also $\qquad ABC + AB'C = \text{lune } BCB'AB$;

and $\qquad ABC + ABC' = \text{lune } CBC'AC$.

---

* This proposition is sometimes stated thus: *The area of a triangle is equal to its spherical excess;* but this enunciation is rather slipshod.

Hence, on addition,

$$2\,ABC + (ABC + AB'C' + AB'C + ABC')$$
$$= \text{lune } A + \text{lune } B + \text{lune } C;$$

$$\therefore \qquad 2\,ABC = \text{lune } A + \text{lune } B + \text{lune } C - \text{hemisphere.}$$

$$\therefore \text{ (by Art. 55) } 2\,ABC = (2\,A + 2\,B + 2\,C - 360) \text{ spherical degrees.}$$

$$\therefore \qquad ABC = (A + B + C - 180) \text{ spherical degrees}$$
$$= E \text{ spherical degrees.}$$

Since (Art. 55) a spherical degree on a sphere of radius r contains $\frac{1}{180}\pi r^2$ square units of area, then, on this sphere,

$$\textbf{area } ABC = \frac{A + B + C - 180}{180}\pi r^2 * = \frac{E}{180}\pi r^2, \qquad (1)$$
$$= E_r r^2, \qquad \text{[Art. 56 (2)]} \quad (2)$$

in which E denotes the number of degrees, and E_r denotes the number of radians in the spherical excess.

Hence, in order to find the area of a triangle, find the angles, calculate the spherical excess in degrees or radians, and use one of formulas (1), (2).

NOTE. It should be observed that [from Art. 14, Art. 56 (1), and the proposition above], *the number of spherical degrees* contained in the area subtended on a spherical surface by a solid angle at the centre of the sphere, *remains the same*, however the radius may vary. On the other hand, by (1) and (2), *the number of square units* in the subtended area *varies* as the square of the radius.

---

* This expression for the area of a spherical triangle was first given in 1629 by Albert Girard (1590–1634) (see *Plane Trigonometry*, pp. 22, 167); and it is often called *Girard's Theorem.* The method of proof used above was invented by John Wallis (1616–1703) professor of geometry at Oxford. (See Wallis, *Works*, Vol. II., p. 875.)

It follows from (1) that

$$\text{area } ABC : 2\,\pi R^2 = E^\circ : 360^\circ.$$

Hence, the above proposition may be expressed thus: *The area of a spherical triangle is to the surface of the hemisphere as the excess of its three angles above two right angles is to four right angles.*

EXAMPLES.

Find the areas of the following triangles (see examples, Art. 56):

1. Those described in Art. 42, Exs. 1, 2, 3, when on a sphere of radius 10 feet.

2. Those described in Art. 43, Exs. 1, 2, when on a sphere of radius 25 inches.

3. Those described in Art. 44, Exs. 1, 2, 3, when on a sphere of radius 30 yards.

4. Those described in Art. 45, Exs. 1, 2, when on a sphere of radius 4 feet.

5. Those described in Art. 46, Exs. 1, 2, 3, when on a sphere of radius 18 inches.

6. Those described in Art. 47, Exs. 1, 2, when on a sphere of radius 3960 miles.

58. Formulas for the spherical excess ($E°$) of a triangle. Since, in a spherical triangle ABC, $E° = A° + B° + C° - 180°$, and since there are many relations between the sides and angles of a triangle, it may be expected that there can be many formulas for the spherical excess; and, accordingly, for the area of a spherical triangle. [It will be remembered that there are several formulas for the area of a plane triangle (*Plane Trigonometry*, Art. 66).] Following are some of the most important of these (the deduction of some of them is given in Note B):

A. The spherical excess in terms of the three sides.

(*a*) *L'Huillier's* formula:

$$\tan \tfrac{1}{4} E° = \sqrt{\tan \tfrac{1}{2} s \tan \tfrac{1}{2}(s-a) \tan \tfrac{1}{2}(s-b) \tan \tfrac{1}{2}(s-c)}.$$

(*b*) *Cagnoli's* formula: $\sin \tfrac{1}{2} E° = \dfrac{n}{2 \cos \tfrac{1}{2} a \cos \tfrac{1}{2} b \cos \tfrac{1}{2} c}$,

in which $$n = \sqrt{\sin s \sin (s-a) \sin (s-b) \sin (s-c)}.$$

(*c*) *De Gua's* formula*: $\cot \tfrac{1}{2} E° = \dfrac{1 + \cos a + \cos b + \cos c}{2n}$.†

---

* *Simon L'Huillier* (1750–1810), a Swiss mathematician and philosopher; *Antoine Cagnoli* (1743–1816), an Italian astronomer; *L'abbé Jean Paul de Gua* (1712–1786), a French philosopher.

† For the deduction of this formula see Chauvenet, *Trigonometry*, p. 230, and Crawley, *Trigonometry*, p. 166.

B. The spherical excess in terms of two sides and their included angle.

$$(d)\ \tan \tfrac{1}{2} E^\circ = \frac{\tan \frac{1}{2} a \tan \frac{1}{2} b \sin C}{1 + \tan \frac{1}{2} a \tan \frac{1}{2} b \cos C};$$

$$(e)\ \cot \tfrac{1}{2} E^\circ = \frac{\cot \frac{1}{2} a \cot \frac{1}{2} b + \cos C}{\sin C}.$$

Ex. By these formulas find the spherical excess of some of the triangles referred to in Ex. 1, Art. 56.

59. ***a.*** **The number of spherical degrees in any figure on a sphere,** whatever may be its boundary, is the ratio of the area of the figure to the area of a spherical degree, that is, to $(\frac{1}{360})$th part of the area of the hemisphere (Art. 55). Thus, on a sphere of radius r, if A denotes the area of the figure, and E the number of spherical degrees therein, then, since area of a hemisphere $= 2\pi r^2$,

$$E = A : \frac{1}{360} \text{ of } 2\pi r^2 = \frac{180 A}{\pi r^2}. \tag{1}$$

[Compare Art. 57 (1), Art. 59 (2).]

The plane angle E° may be called the *spherical excess of the figure.* For example, the spherical excess of a lune of angle A° is $2A^\circ$.

b. **The spherical excess of a** (*non-re-entrant*) **spherical polygon.** On drawing diagonals from any vertex of a polygon of n sides to the other vertices, it will be seen that the polygon is divided into $n - 2$ triangles. The sum of the angles of all these triangles is the same as the sum of the angles of the polygon. Hence,

spherical excess (E°) *of polygon of n sides*

$$= \textit{sum of angles} - (n-2)\,180^\circ.$$

If the radius of the sphere is r, then (Art. 57)

$$\textit{area of the polygon} = \frac{E}{180}\pi r^2. \tag{2}$$

60. Given the area of a figure: to find its spherical excess. More fully: *To find the spherical excess of a figure on a sphere when the area of the figure is given in square units.*

Let r denote the radius of the sphere, A the area of the figure, E the number of degrees, n the number of seconds, and E_r the number of radians, in its spherical excess. Then, by (1) Art. 59,

$$E = \frac{180\,A}{\pi r^2}. \tag{1}$$

$$\therefore\ n = 3600\,E = 206265\frac{A}{r^2}. \tag{2}$$

Now $$1^\circ = \frac{\pi}{180} \text{ radians};$$

hence $$E^\circ = \frac{\pi}{180} E \text{ radians}$$

$$= \frac{A}{r^2} \text{ radians.} \qquad \text{[by (1)]}$$

$$\therefore\ E_r = \frac{A}{r^2}. \tag{3}$$

A particular application of (2) can be made to the following problem, viz.: *The area of a spherical triangle on the earth's surface being known, to derive a formula for computing the spherical excess.*

The length of a degree on the earth's surface is found to be 365155 feet. Accordingly,

$$R \text{ (the radius of the earth)} = \frac{365155 \times 180}{\pi} \text{ feet.} \tag{4}$$

From (2), $$\log n = \log A + \log 206265 - 2 \log R. \tag{5}$$

On expressing A in square feet, and substituting in (5) the value of R in (4), there is obtained,

$$\log n = \log A - 9.3267737. \tag{6}$$

Formula (6) is called *Roy's Rule*, as it was used by General William Roy (1726–1790) in the Trigonometrical Survey of the British Isles.* The area of the spherical triangle can be approximately determined to a sufficient degree of accuracy.

* The rule should probably be credited to Isaac Dalby (1744–1824), who was mathematical assistant to General Roy from 1787 to 1790, and later became professor of mathematics at the Royal Military College. [See *Phil. Trans.*, vol. 80 (1790).] This was the first practical application of Gerard's theorem (Art. 57).

61. The measure of a solid angle. A *plane angle* can be measured by *any circular arc which it subtends;* and the measure can be expressed in *radians* and in *degrees.* The *radian* (or circular) *measure of an angle* is the number of times any circular arc subtended by it contains the radius (*Plane Trig.*, Art. 73); and the *number of degrees in the angle* is equal to the number of degrees in the subtended circular arc. Thus, the radian measure of an angle of an equiangular triangle is $\frac{1}{3}\pi$, and its degree measure is 60.

A *solid angle* can be measured in a somewhat similar manner, namely, by means of *any spherical surface which it subtends.* What may be called the **spherical measure** *of a solid angle* is the number of times any spherical surface subtended by it contains an area equal to the square on the radius. For example, since the surface of a sphere is equal to $4\pi r^2$, the sum of all the solid angles about any point is 4π. The angle at the corner of a cube subtends one-eighth of the surface of the sphere; accordingly, its spherical measure is $\frac{4\pi r^2}{8} \div r^2$, *i.e.* $\frac{1}{2}\pi$.

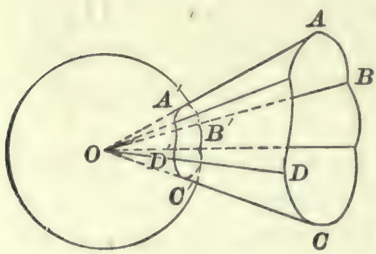


FIG. 50

A solid angle may also be measured in *spherical degrees*, a term that will be explained presently. What may be called the **spherical degree measure** *of a solid angle* (or, the number of spherical degrees in the angle) is a number equal to the number of spherical degrees of area in any spherical surface subtended by the angle. An angle that subtends a spherical degree of surface, contains what may be called *a solid spherical degree.* For example, the sum of all the solid angles about any point is 720 spherical degrees (of angle); the angle at the corner of a cube contains 90 spherical degrees (of angle). Thus the *spherical measure* of the angle at the corner of a cube is $\frac{1}{2}\pi$, and its *spherical degree measure* is 90. On comparing these definitions of solid angular measures with Art. 55 and equations (3) and (1) Art. 60, it is seen that these *measures of solid angles are equal to the measures, in radians and degrees respectively, of the spherical excess of the figures subtended on any sphere by the angle,* when the vertex of the angle is at the centre of the sphere.

NOTE 1. **The term degree.** In geometry and trigonometry the word *degree* is used in connection with *four* very different kinds of quantities; namely, *circular arcs*, *plane angles*, *spherical surfaces*, and *solid angles.*

A *degree of arc*, or *an arcual degree*, is $(\frac{1}{360})$th part of any circle;

A *degree of angle*, or *an angular degree*, is $(\frac{1}{360})$th part of four right angles;

A *degree of surface on a sphere*, or a *spherical degree of surface*, is $(\frac{1}{720})$th part of the surface of any sphere;

A *degree of solid angle*, or a *solid spherical degree*, is $(\frac{1}{720})$th part of the solid angles about any point.

NOTE 2. If two plane angles are equal, they can be superposed, the one on the other. On the other hand, just as two figures on a sphere may be equal in area and differ in every other respect, so two solid angles can be equal in measure and differ in every other respect.

NOTE 3. The following remarks relating to the measurement of solid angles are from Hutton's *Course in Mathematics*, Vol. II., p. 64:

"*Solid angles:* If about the angular point of a solid angle as centre, a sphere be described to radius unity, the portion of its surface intercepted between the planes which contain the solid angle is the *measure of the solid angle.* (This method of estimating the magnitude of solid angles appears to have been first given by Albert Girard in his *Invention Nouvelle en Algebre*, 1629; and it would very naturally suggest itself as one of the simplest applications of his theorem for the spherical excess.)" [Compare *Plane Trigonometry*, p. 126, Note 2.]

Ex. 1. The edge angles of a triedral angle are 74° 40′, 67° 30′, 49° 50′; calculate its spherical degree measure, and its spherical measure. (See Ex. 1, Art. 43.)

Ex. 2. The face angles of a triedral angle are 47° 30′, 55° 40′, 60° 10′; calculate its spherical degree measure, and its spherical measure. (See Ex. 1, Art. 42.)

Ex. 3. Two face angles of a triedral angle are 64° 24, 42° 30′, and the edge angle between their planes is 58° 40′; calculate its spherical degree measure, and its spherical measure. (See Ex. 1, Art. 44.)

Ex. 4. A face angle of a triedral angle is 74° 20′, and the two adjacent edge angles are 67° 30′ and 45° 50′; calculate its measure. (See Ex. 1, Art. 45.)

Ex. 5. Calculate the spherical degree measure, and the spherical measure, of the solid angles corresponding to the spherical triangles described in Art. 42, Exs. 2, 3; Art. 43, Ex. 2; Art. 44. Exs. 2, 3; Art. 45, Ex. 2; Art 46, Exs. 2, 3; Art. 47, Ex. 2. (See Ex., Art. 56.)

62. The volume of a sphere. In some works on solid geometry and in books on mensuration it is shown that the volume of a pyramid is equal to one third the product of its base and altitude. Now suppose that a polyedron (*i.e.* a solid bounded by plane faces) is circumscribed about a sphere, each of the faces of the polyedron, accordingly, touching the sphere. This polyedron may be regarded as made up of pyramids which have a common vertex (namely, the centre of the sphere), and a common altitude (namely, the radius of the sphere), and which have the faces of the polyedron as bases. Then, R being the radius of the sphere,

$$\text{Vol. of polyedron} = \tfrac{1}{3} R \times (\text{sum of faces of polyedron}). \quad (1)$$

If the number of faces of the polyedron be increased and the area of each face be decreased, then the sum of the faces becomes more nearly equal to the area of the surface of the sphere, and the volume of the polyedron becomes more nearly equal to the volume of the sphere. By increasing the number of faces and decreasing the area of each face, the difference between the sum of the faces of the polyedron and the area of the sphere can be made as small as one please; and, likewise, the difference between the volume of the polyedron and the volume of the sphere can be made as small as one please. In other words:

The area of the surface of the sphere is the limit of the area of the surface of the polyedron, and the volume of the sphere is the limit of the volume of the polyedron, when the faces of the latter are increased without limit, and each face is made to approach zero in area.

Hence, from (1),
$$\begin{aligned}\text{Vol. of sphere} &= \tfrac{1}{3} R \times \text{surface of sphere}\\ &= \tfrac{4}{3}\pi R^3.^* \qquad (2)\end{aligned}$$

63. Definitions. A **spherical pyramid** is a portion of a sphere bounded by a spherical polygon and the planes of the sides of the polygon. The polygon is called the *base* of the pyramid.

---

* For a note concerning the measurement of the circle and the sphere see *Plane Trigonometry*, Art. 72, and Note *C*, p. 171. For the proofs of Archimedes, see T. L. Heath, *The Works of Archimedes edited in modern notation, with introductory chapters* (Cambridge, University Press), pp. 39, 41, 93.

For example, in Fig. 11, Art. 12, *O–ABCD, O–ABC, O–ABD*, are spherical pyramids; their bases are *ABCD, ABC, ABD*.

A **spherical sector** is the portion of a sphere generated by the revolution of a sector of a circle about any diameter of the circle as axis. For example, in Fig. 47, Art. 53, when the semicircle *ATB* revolves about *AB*, each of the circular sectors *AOL, LOT, LOK*, etc., describes a spherical sector.

A **spherical segment** is the portion of a sphere bounded by two parallel planes and the zone intercepted between them. (One of the planes may be tangent to the sphere.)

64. Volume of a spherical pyramid; of a spherical sector. By reasoning analogous to that in Art. 62, it can be shown that, in a sphere of radius R,

$$\text{vol. of a spherical pyramid} = \tfrac{1}{3} R \times \text{area of its base};$$

$$\text{vol. of a spherical sector} = \tfrac{1}{3} R \times \text{area of its zone}.$$

Since the area of a zone of height $h = 2\pi Rh$ (Art. 53),

then $$\text{vol. of spherical sector} = \tfrac{2}{3}\pi R^2 h.$$

Thus in Fig. 11, Art. 12,

$$\text{vol. } O\text{–}ABCD = \tfrac{1}{3}\, OA \times \text{area } ABCD;$$

in Fig. 47, Art. 53,

vol. of sector described by $AOL = \frac{1}{3}\, OA \times$ area of zone described by arc $AL = \frac{2}{3}\pi R^2 \cdot Al$, and

vol. of sector described by $LOT = \frac{1}{3}\, OA \times$ area of zone described by arc $LT = \frac{2}{3}\pi R^2 \cdot lt$.

EXAMPLES.

1. Find the volumes of the spherical pyramids whose bases are the triangles described in Art. 57, Exs. 1–6.

2. Find the volumes of the following spherical sectors:

(*a*) The sector whose base is a zone of height 2 inches on a sphere of radius 18 inches.

(*b*) The sector whose base is a zone of height 3 feet on a sphere of radius 12 feet.

65. Volume of a spherical segment. Let AB be an arc of a semicircle of radius R having the diameter DD'. From A, B, draw Aa, Bb, at right angles to DD'. It is required to find the volume of the spherical segment generated by the revolution of $ABba$ about DD'.

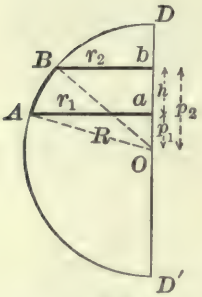


FIG. 51

Let h denote the height of the segment, and p_1, p_2, the lengths of the perpendiculars from the centre O to the parallel bases of the segment. On making the revolution of the semicircle DAD', it is seen that

segment generated by $ABba =$ cone generated by BOb + spherical sector generated by AOB − cone generated by AOa.

Now, $\qquad$ vol. cone generated by $BOb = \frac{1}{3}\pi r_2^2 p_2$;

vol. sector generated by $AOB = \frac{2}{3}\pi R^2 h$; $\qquad$ (Art. 64)

vol. cone generated by $AOa = \frac{1}{3}\pi r_1^2 p_1$.

$$\therefore \text{ vol segment} = \tfrac{1}{3}\pi(r_2^2 p_2 + 2R^2 h - r_1^2 p_1). \tag{1}$$

NOTE. The result (1) can be reduced to various forms. For example, since

$$p_1^2 = R^2 - r_1^2, \; p_2^2 = R^2 - r_2^2, \; p_2 - p_1 = h,$$

then $\quad$ vol. segment $= \frac{2}{3}\pi R^2(p_2 - p_1) + \frac{1}{3}\pi p_2(R^2 - p_2^2) - \frac{1}{3}\pi p_1(R^2 - p_1^2)$

$$= (p_2 - p_1)\pi R^2 - \tfrac{1}{3}\pi(p_2^3 - p_1^3) \tag{2}$$

$$= \frac{p_2 - p_1}{3}\pi[3R^2 - (p_2^2 + p_2 p_1 + p_1^2)]. \tag{3}$$

Since $\qquad h = p_2 - p_1$, then $h^2 = p_2^2 - 2p_2 p_1 + p_1^2$.

$$\therefore p_1 p_2 = \frac{p_2^2 + p_1^2 - h^2}{2}, \text{ and } p_2^2 + p_2 p_1 + p_1^2 = \frac{3}{2}(p_1^2 + p_2^2) - \frac{h^2}{2}.$$

On substituting the last result in (3), expressing p_1^2 and p_2^2 in terms of R, r_1, r_2, and reducing, the following formula is obtained, viz.:

$$\text{vol. segment} = \frac{\pi h}{2}\left(r_1^2 + r_2^2 + \frac{h^2}{3}\right). \tag{4}$$

EXAMPLES.

1. Show that if (in Fig. 51) angle $AOD = \alpha$, then the volume of the spherical sector generated by AOD is $\frac{2}{3}\pi R^3(1 - \cos\alpha)$.

2. Show that if angle $AOD = \alpha$, then the volume of the segment generated by the revolution of ADa is $\frac{4}{3}\pi R^3 \sin^4 \frac{1}{2}\alpha(1 + 2\cos^2\frac{1}{2}\alpha)$.

Suggestion. Segment generated by ADa = sector generated by AOD − cone generated by AOa.

3. Find the volume of a spherical segment, the diameters of its ends being 10 and 12 inches, and its height 2 inches.

4. The diameters of the ends of a spherical segment are 8 and 12 inches, and its height is 10 inches. Find its volume.

N.B. *For questions and exercises on Chapter VI., see page* 108.

CHAPTER VII.

PRACTICAL APPLICATIONS.

66. Geographical problem. *To find the distance between two places and the bearing* (*i.e.* the direction) *of each from the other, when their latitudes and longitudes are known.* An interesting application of spherical trigonometry can be made in solving this problem. In the following examples the earth is regarded as spherical, and its radius is taken to be 3960 miles.

EXAMPLES.

1. Find the shortest distance along the earth's surface between Baltimore (lat. 39° 17′ N., long. 76° 37′ W.) and Cape Town (lat. 33° 56′ S., long. 18° 26′ E.).

In Fig. 52 B and C represent Baltimore and Cape Town; EQ is the earth's equator; NGS, NBS, NCS are the meridians of Greenwich, Baltimore, and Cape Town respectively; BC is the great circle arc whose length is required.

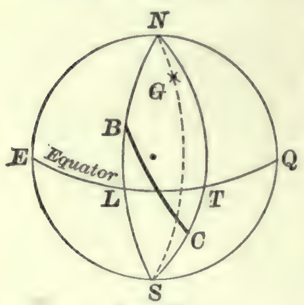


Fig. 52

In the spherical triangle BNC, NB, NC, and BNC are known. For

$$NB = 90° - BL = 90° - 39° \, 17' = 50° \, 43'$$

$$NC = 90° + TC = 90° + 33° \, 56' = 123° \, 56'$$

$$BNC = BNG + GNC = 76° \, 37' + 18° \, 26' = 95° \, 3'$$

Hence, BC can be determined in degrees by Art. 44; then, the radius of the sphere being given, BC can be determined in miles. The angles NBC, NCB, can also be found.

Answers: $BC = (65° \, 47' \, 48'') = 4685.8$ miles; $NBC = 115° \, 1' \, 35'$; $NCB = 57° \, 42' \, 23''$.

Note 1. The *bearing* of one place from a second place is the angle which the great circle arc joining the two places makes with the meridian of the second place. Thus, in Fig. 52 the bearing of Cape Town from Baltimore is the angle NBC, and the bearing of Baltimore from Cape Town is NCB.

Since $NBC = 115° \ 1' \ 35''$ the ship *sets out from Baltimore* on a course S. $64° \ 58' \ 25''$ E.; since $NCB = 57° \ 42' \ 23''$ the ship *approaches Cape Town* on a course S. $57° \ 42' \ 23''$ E.

Note 2. A ship that sails on a great circle (excepting the equator or a meridian) must be continually changing her course.

2. Find the latitude of the place where BC crosses the meridian 15° W.; also find the bearing of Cape Town from this place.

3. If a vessel sails from Baltimore and keeps constantly on the course (see Ex. 1) S. $64° \ 58' \ 25''$ E. (*i.e.* crosses every meridian at the angle $64° \ 58' \ 25''$), will she arrive at Cape Town? [*Answer.* No.]

4. What path will the vessel in Ex. 3 make on the sea? *Answer.* A path which is a spiral going round and round the earth and gradually approaching the south pole. This path is called the *loxodrome*, or *rhumb line*.

5. If a person leaves Boston, Mass. (lat. 42° 21′ N., long. 71° 4′ W.), starting due east, and keeps on a great circle: (*a*) Where will he be after he has passed over an arc of 90°, and in what direction will he be going? (*b*) Where will he be after he has passed over an arc of 180°, and in what direction will he be going? (*c*) Where will he be after he has passed over an arc of 270°, and in what direction will he be going? [Solve this example: (1) by spherical geometry; (2) by spherical trigonometry.]

6. What is the distance from New York (40° 43′ N., 74° 0′ W.) to Liverpool (53° 24′ N., 3° 4′ W.)? Find the bearing of each place from the other. In what latitude will a steamer sailing on a great circle from New York to Liverpool cross the meridian of 50° W.; and what will be her course at that point?

N.B. *Check the results in the following exercises:*

7. Find the distance and bearing of Liverpool from Montreal (45° 30′ N., 73° 33′ W.).

8. Find the distance and bearing of Liverpool from Halifax, N. S. (44° 40′ N., 63° 35′ W.).

9. Find the distance and bearing of Santiago de Cuba (20° N., 75° 50′ W.) from Rio de Janeiro (22° 54′ S., 43° 8′ W.).

10. Find the distance and bearing of San Francisco (37° 47′ 55″ N., 122° 24′ 32″ W.) from New York.

11. Find the distance of Victoria, B. C. (48° 25′ N., 123° 23′ W.) from Sydney, N. S. W. (33° 52′ S., 151° 13′ E.); and the bearing of each place from the other.

12. Find the distances between the following places: (*a*) San Francisco and Honolulu; (*b*) Cape Town and Cairo; (*c*) Honolulu and Manila; (*d*) Victoria, B. C., and Tokio.

13. Find the distances between other places, and their bearings from each other.

APPLICATIONS TO ASTRONOMY.

N.B. *In connection with his study of the following articles the student should consult some elementary text-book on astronomy. The numerical examples given here will supplement his outside reading on spherical astronomy.*

67. One of the most important applications of spherical trigonometry is to astronomy. Trigonometry was invented to aid astronomy, and for centuries was studied as an adjunct of the latter subject. (See *Plane Trigonometry*, pp. 165, 166.) A few of the simplest problems of spherical astronomy are introduced in Arts. 73, 74. In order to understand these problems a clear conception of a few astronomical terms and principles is necessary. These terms are explained in Arts. 68–72.

68. The celestial sphere. To a person on the surface of the earth, the sky above is like a great hemispherical bowl with himself at the centre. The stars seem to move from east to west across the spherical sky in parallel circles whose axis is the earth's polar axis prolonged. Each star makes a complete revolution about this axis in 23 hours 56 minutes ordinary clock time. The stars appear never to change their positions with reference to one another, being in this respect like places on the earth's surface.* Another way of describing the relations of the earth and the enveloping sky, is to say that the whole sky is turning, like an immense crystal sphere, about an axis which is the earth's polar axis prolonged, the motion being from east to west. The stars keep the same positions with respect to one another, and, accordingly, appear to be attached to the surface of the sphere. As the sphere turns, the stars fixed in it appear to trace parallel circles

* The positions of some of the stars suffer a very slight change which is perceptible in the course of centuries.

about the axis. The sphere turns completely in 23 hours 56 minutes ordinary clock time.* The *stars all seem to be at the same distance* from the observer because his eyes can judge their *directions only,* and not their *distances.*

The following considerations will show that it is natural enough for an observer on the earth to think that he is always *at the centre* of the sphere on which the stars appear to be. When a person changes his position, the direction of an object at which he is looking changes also, unless he moves directly towards or away from the object. For instance, from a certain point a tree may be in an easterly direction, and when the observer moves a little way the tree may be in a southeasterly direction. Moreover, the further away an object is, the less will be the change in its direction caused by any particular change in the observer's position. Thus, if a person is near a tree, a few steps on his part may change the direction of the tree from east to southeast, but if he is five miles from the tree, an equal number of steps taken by him will make very little difference in the direction of the tree. Now the earth's mean distance from the sun is about 93,000,000 miles. Hence, an observer who now looks at the stars from a certain position, in about six months from now will look at them from a point 186,000,000 miles distant from his present position.† Astronomers have succeeded in a few instances in determining the distances of the stars from the earth.‡ It has been found that the nearest star yet known, *Alpha Centauri*, is so far away that the *change in its direction* from the centre of the earth, due to the change of position of 186,000,000 miles on the part of the earth, *is less* than the change in the direction of an object $3\frac{1}{3}$ miles away when the observer moves his head a couple of inches at right angles to the line of sight. This being so in the case of the sun's *nearest* stellar neighbour, it is natural for an observer on the earth to think that he is always at the centre of the great sphere on which the stars appear to be; and it is perfectly proper

---

* The student probably knows that the apparent turning of the spherical sky from east to west about an axis which is the earth's polar axis prolonged, is really due to the rotation of the earth in an opposite direction. The observer is not conscious of any motion of the earth, and thinks that the sky with its bright points is revolving about the earth from east to west, while all the time the sky is motionless, and the earth is turning under it from west to east. Just as to a person in a swiftly moving train the objects outside seem to be rushing by him while the train appears to be at rest.

† This, moreover, does not take any account of the motion of the sun with his system through space.

‡ The first stellar distance determined was that of *61 Cygni* by Friedrich Wilhelm Bessel (1784–1846), one of the greatest of German astronomers, in 1838. Since then the distances of about 100 stars have been measured; about 50 of these distances are regarded as reliably determined.

for him to act in accordance with this notion when he makes astronomical observations and deductions.*

The sphere on which the stars appear to move in parallel circles, or, what comes to the same thing, the sphere which appears to have the stars attached to it and to revolve about the earth's polar axis prolonged, is called **the celestial sphere.**

69. Points and lines of reference on the celestial sphere. There will now be shown some methods for indicating the positions of the heavenly bodies on the celestial sphere — *their positions with respect to the observer* and *their positions with respect to one another.*

The positions of places on the terrestrial sphere are described by means of certain points and great circles on the sphere. There are various pairs of circles which are used for reference; for example, the equator (whose poles are the north and south poles of the earth) and the meridian passing through the Royal Observatory at Greenwich, the equator and the meridian passing through the observatory at Washington, etc. It will be observed that in each case *the reference circles are at right angles to each other, and, accordingly, each of them passes through the poles of the other.*

In an analogous manner the positions of bodies on the celestial sphere are described by means of, or by reference to, certain points and great circles on that sphere. There are *four* different systems of circles of reference. As in the case of the terrestrial sphere, each system consists of two circles, each of which passes through the pole of the other, and, accordingly, is at right angles to the other. Two of these systems are described in Arts. 70, 71, a third in Art. 76, and the fourth in Art. 77. A point which will be referred to in these systems is the **north celestial pole.** This is the point where the earth's axis, if prolonged, would pierce the celestial sphere. It is near the pole star, being about $1\frac{1}{4}°$ from it.

---

* ". . . imagine the entire solar system as represented by a tiny circle the size of the dot over this letter *i*." (Neptune the outermost planet known of the solar system is 2790 millions of miles from the sun; *i.e.* 30 times as far as the earth.) "Even the sun itself, on this exceedingly reduced scale, could not be detected with the most powerful microscope ever made. But on the same scale the vast circle centred at the sun and reaching to *Alpha Centauri* would be represented by the largest circle which could be drawn on the floor of a room 10 feet square." (Todd, *New Astronomy*, p. 438.)

70. The horizon system: Positions described by altitude and azimuth. For any place on the earth's surface, the point at which the plumb line extended upwards meets the celestial sphere is called **the zenith**; the diametrically opposite point is called **the nadir.** If a plane perpendicular to the plumb line be passed either immediately beneath the observer's feet, or through the centre of the earth, about 4000 miles below him, then the intersection of this plane with the celestial sphere is called **the horizon.** (Since the earth is so small and so far away from even the nearest star, two parallel planes 4000 miles apart and passing through the earth will appear, to a terrestrial observer, to intersect the celestial sphere in the same great circle.)

Great circles passing through the zenith are perpendicular to the horizon; they are called *vertical circles.* The *north point of the horizon* is the point which is directly north from the observer. It is where the vertical circle passing through the north pole intersects the horizon. *This circle which passes through the zenith and the pole is called* **the meridian** *of the observer. The horizon and the meridian* are the reference circles in the horizon system.

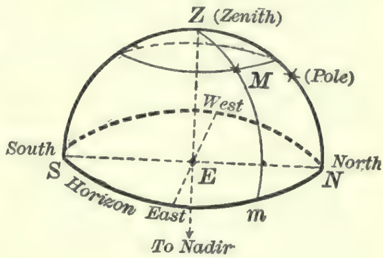


FIG. 53

The **altitude** (denoted by h) of a heavenly body is its angular distance above the horizon. Thus the altitude of M (Fig. 53, in which E is the earth and Z the zenith of the place of observation) is Mm. The altitude of the zenith is 90°. The distance of a star from the zenith is called its *zenith distance;* this is obviously the complement of the altitude.

The **azimuth** (denoted by A) of a heavenly body is the angle between its vertical circle and the meridian. This angle is measured usually along the horizon from the south point in the direction of the west point, to the foot of the star's vertical circle. Thus in Fig. 53 the azimuth of M is $180° + NZm$, which is measured by the arc $180° + Nm$ on the horizon.

NOTE. Any two points on the earth's surface have different zeniths. Hence, the above system of describing positions on the celestial sphere is peculiarly *local.* Moreover, a star rises in the eastern part of the horizon (altitude zero), mounts higher in the sky until it reaches the observer's meridian, then sinks towards, and sets in, the west; it is, accordingly, continually changing its altitude and azimuth.

71. The equator system: Positions described by declination and hour angle. The north celestial pole is the principal point of this system. **The celestial equator** is the great circle of which that point is the pole; it is evidently the projection of the earth's equator upon the celestial sphere. *The celestial equator and the meridian of the observer* are the reference circles in the system now being described. In Fig. 54, P is the north celestial pole, S the south celestial pole, EQ the celestial equator; also, HR is the horizon and Z the zenith for some particular place on the earth's surface. As said in Art. 68, the stars move in parallel circles whose axis is PS; these circles are, accordingly, parallel to the equator EQ. The angular distance of a star from the equator is called the **declination** (denoted by D or δ) of the star; *north* (or $+$) *declination* when the star is north of the equator, and *south* (or $-$) *declination* when the star is south. Thus the declination of S_3 is S_3s_3. The angular distance of a star from the north pole is called its *north polar distance;* this is evidently the complement of the star's declination.*

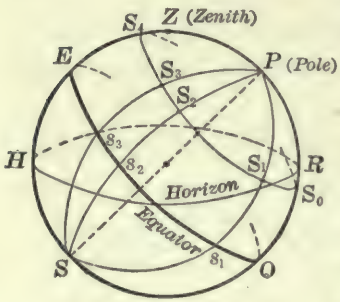


Fig. 54

In 24 (sidereal) hours a star appears to make a complete revolution (*i.e.* to pass over 360°) about the celestial polar axis; hence, *the star passes over 15° in 1 hour.*† The great circles passing through the poles are called *hour circles.* Thus PS_3S is the hour circle of S_3. **The hour angle** (denoted by *H. A.*) of a star is the angle between the meridian of the observer and the hour circle of the

* The declination of the stars change by an exceedingly small amount in the course of a year.

† The interval of time between two successive passages of the observer's meridian by the sun (*i.e.* from noon to noon) is about 4 minutes longer than the interval of time between two successive passages of the meridian by any particular star. (This difference is due to the yearly revolution of the earth about the sun. See text-books on astronomy.) The second interval is called a sidereal day; it is divided into 24 sidereal hours.

star. This angle is measured *towards the west.* Thus, suppose that a star is on the meridian at S_4; its hour angle is then zero. Twelve hours later the star will be at S_0, and will have an hour angle 180°. After a while it will be at S_1, just rising above the horizon, and its hour angle will be $180° + S_0PS_1$; later it will be at S_3, having the hour angle $180° + S_0PS_3$; later still it will be on the meridian at S_4, and its hour angle will be zero again. *The hour angle is usually reckoned in hours* from 1 to 24, 1 hour being equal to 15 degrees. Thus, when the star is at S_0 its hour angle is *12 h.* The hour angle of a star is partly *local;* for only places on the same meridian of longitude have the same celestial meridian. Moreover, the hour angle of a star is continually changing, and its magnitude depends upon the time of observation. In Arts. 76, 77, the positions of stars are described in terms which are independent of the time and place of observation.

In Arts. 73, 74, 75, the astronomical ideas so far obtained, are used in the solution of two simple problems.

72. The altitude of the pole is equal to the latitude of the place of observation. This theorem, which is necessary in Arts. 73, 74, is the fundamental and most important theorem of spherical astronomy.

In Fig. 55, C represents the centre of the earth, P its north pole, and EQ its equator; O is the place of observation, say some place in the northern hemisphere, Z is its zenith and HR its horizon; CPP_1 is the celestial polar axis, P_1 being the north celestial pole. Draw OP_2 parallel to CP_1, P_2 being on the celestial sphere. The angle ROP_2 is the altitude of the pole at O, since (see Arts. 68, 70) P_1 and P_2 are in the same direction from O.

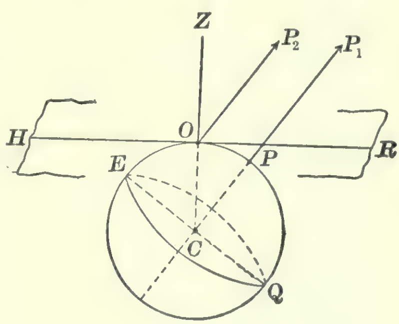


FIG. 55

The latitude of a place is equal to the angle between the plumb line and the plane of the equator. Thus, the latitude of O is equal to OCE. Since OR and OP_2 are respectively perpendicular to CZ and CE, the angle $ROP_2 = OCE$; that is, *the altitude of the pole as observed at O is equal to the latitude of O.*

73. The time of day *can be determined at any place whose latitude is known, if the declination and the altitude of the sun at that time and place are also known.*

NOTE 1. The sun, unlike the stars, changes in declination from $23\frac{1}{2}°$ south (about Dec. 22) to $23\frac{1}{2}°$ north (about June 21), and then returns south. Its declination is zero, that is, it is on the celestial equator, about March 20 and Sept. 22. This change in declination is due to the revolution of the earth about the sun, and to the fact that the plane of the earth's equator is inclined about $23\frac{1}{2}°$ to the plane of its orbit about the sun. The latter plane is called the plane of *the ecliptic.* The declination of the sun is given for each day of the year in the Nautical Almanac. The altitude of the sun can be observed with a sextant.

NOTE 2. The student should consult a text-book on astronomy for an account of the special precautions and corrections necessary in connection with this and similar astronomical problems.

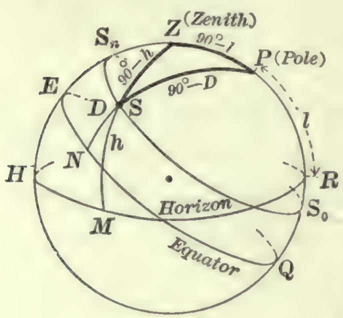


FIG. 56

In Fig. 56, P is the north celestial pole, EQ the celestial equator, S the sun, and S_0SS_n is the small circle on which the sun is moving at the given time; Z is the zenith, and HR the horizon, of the place of observation; ZSM is the sun's vertical circle, and PSN is its hour circle.

It is midnight when the sun is at S_0, and noon when the sun is at S_n. From noon to noon is 24 hours. Hence, *to find the time when the sun is at S, determine the angle ZPS in hours* ($15° = 1$ h.); *subtract* the number of hours from 12, if it is forenoon; and *add*, if it is afternoon.

Let l, h, D, respectively, denote the latitude of the place, and the altitude and declination of the sun.

Then $\qquad PR = l$ (Art. 72), $SM = h$, $SN = D$.

In ZPS, whose vertices are the *sun, zenith*, and *pole*,

$$ZP = 90° - l,\ ZS = 90° - h,\ SP = 90° - D.$$

Hence, the angle ZPS can be found.

EXAMPLES.

1. In New York (lat. 40° 43′ N.) the sun's altitude is observed to be 30° 40′. What is the time of day, given that the sun's declination is 10° N., and the observation is made in the forenoon?

2. In Montreal (lat. 45° 30′ N.) at an afternoon observation the sun's altitude is 26° 30′. Find the time of day, given that the sun's declination is 8° S.

3. In London (lat. 51° 30′ 48″ N.) at an afternoon observation the sun's altitude is 15° 40′. Find the time of day, given that the sun's declination is 12° S.

4. As in Ex. 2, given that the sun's declination is 18° N.

5. As in Ex. 3, given that the sun's declination is 22° N.

6. As in Ex. 1, given that the sun's declination is 10° S.

74. To find the time of sunrise *at any place whose latitude is known, when the sun's declination is also known.* This is a special case of the preceding problem; for at sunrise the sun is on the horizon and its altitude is zero. The problem can also be solved by means of the triangle RPS_1 (instead of ZPS_1, which is employed in Art 73). For, in RPS_1

$$S_1P = 90° - D, \; PR = l, \; PRS_1 = 90°.$$

$$\therefore \quad \cos RPS_1 = \frac{\tan PR}{\tan PS_1} = \frac{\tan l}{\cot D} = \tan l \tan D.$$

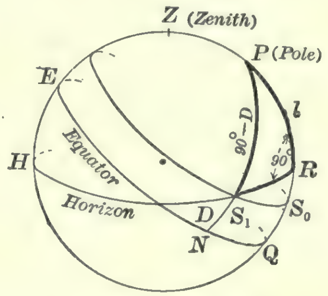


FIG. 57

The angle RPS_1 (*i.e.* S_0PS_1) reduced to hours, gives the time of sunrise (after midnight). If ZPS_1 is found, then ZPS_1 reduced to hours and subtracted from 12 (noon), gives the time of sunrise. The time of sunset is about as many hours *after* noon as the time of sunrise is before it.

In Fig. 57 the sun is north of the equator. When the sun is south of the equator, $PS_1 = 90° + D$, and $RPS_1 > 90°$ for places in the northern hemisphere. The student can make the figure and investigate this case, and also the case in which the place is in the southern hemisphere.

EXAMPLES.

Find the approximate time of sunrise at a place in latitude l, when the sun's declination is D, in the following cases:

1. $l = 40° 43'$ N. (latitude of New York), D equal to: (*a*) 4° 30′ N. (about April 1); (*b*) 15° 10′ N. (about May 1); (*c*) 23° N. (about June 10); (*d*) 5° N. (about Sept. 10); (*e*) 6° S. (about Oct. 8); (*f*) 15° S. (about Nov. 3); (*g*) 23° S.

2. $l = 51° 30' 48''$ N. (latitude of London), D as in Ex. 1.

3. $l = 60°$ N. (latitude of St. Petersburg), D as in Ex. 1.

4. $l = 70° 40' 7''$ N. (latitude of Hammerfest, Norway, D as in Ex. 1.

5. $l = 29° 58'$ N. (latitude of New Orleans), D as in Ex. 1.

6. $l = 33° 52'$ S. (latitude of Sydney, N. S. W.), D as in Ex. 1.

7. Find the approximate time of sunrise for other days and places.

75. Theorem. *If the latitude of the place of observation is known, then the declination and hour angle of a star can be determined from its altitude and azimuth, and vice versa.* For, in the triangle ZPS (Fig. 56), $ZP = 90° - l$, $SP = 90° - D$, $SZ = 90° - h$, $SPZ = 360° - H. A.$, $PZS = A - 180°$. Hence, if the latitude and any two of the four quantities, viz., altitude, azimuth, declination, hour angle, be known, then the remaining two can be found by solving the triangle SPZ.

76. The equator system: Positions described by declination and right ascension. In the system in Art. 71 the circles of reference were the equator and the meridian of the observer. In the system in this article the circles of reference are *the equator and the circle passing through the celestial poles and the vernal equinox.* **The vernal equinox** is one of the points where the ecliptic intersects the equator; namely, the point where the sun, in its (apparent) yearly path among the stars, crosses the equator in spring. (See text-book on astronomy.) This point may be called *the Greenwich of the celestial sphere.* (*The ecliptic* is the projection of the plane of the earth's orbit on the celestial sphere. The plane of the equator and the plane of the ecliptic are inclined to each other at an angle of $23\frac{1}{2}°$. See Art. 73, Note 1.)

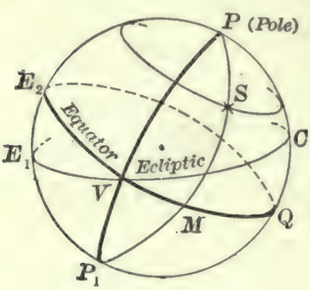


FIG. 58

The right ascension (denoted by R.A.) of a heavenly body is the angle at the north celestial pole between the hour circle of the body and the hour circle of the vernal equinox. This angle is measured from the latter circle *towards the east,* from 0° to 360° or 1 h. to 24 h.; it may be measured by the arc intercepted on the equator. Declination has been defined in Art. 71.

In Fig. 58, P is the north celestial pole, E_2Q the equator, E_1C the ecliptic, and V the vernal equinox. If S is any star, then for S

$$D = SM, \text{ and R.A.} = \text{angle } VPM = \text{arc } VM.$$

77. The ecliptic system: Positions described by latitude and longitude. In this system the point and circles of reference are *the pole of the ecliptic, the ecliptic,* and *the great circle passing through the pole of the ecliptic and the vernal equinox.* The **latitude** of a star is its angular (or arcual) distance from the ecliptic; its **longitude** is the angle at the pole of the ecliptic between the circle passing through this pole and the vernal equinox and the circle passing through this pole and the star. This angle may be measured by the arc intercepted on the ecliptic. It is always measured *towards the east* from the vernal equinox.

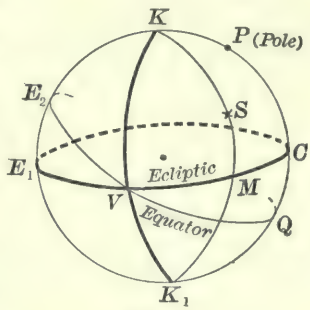


FIG. 59

In Fig. 59, K is the pole of the ecliptic, E_1C the ecliptic, P the pole of the equator, E_2Q the equator, and V the vernal equinox. If S is any star, then

$$\text{latitude of } S = SM, \text{ longitude of } S = VKM = VM.$$

When the latitude and longitude of a star are known, its declination and right ascension can be found, and vice versa. For, in the triangle KPS (the triangle whose vertices are the star and the poles of the equator and the ecliptic), $KP = 23\frac{1}{2}°$ (since $QVC = 23\frac{1}{2}°$), $KS = 90° - \text{lat.}$, $SKP = 90° - \text{long.}$, $SP = 90° - D$; $SPK = VPK - VPS = 90° - (360° - \text{R.A.})$, if S is west of VP; $SPK = 90° + \text{R.A.}$, if S is east of VP. If any two of these be known besides KP, the remaining two can be found by solving KPS.

N.B. *Questions and exercises on Chapter VII. will be found at page* 109.

APPENDIX.

NOTE A.

ON THE FUNDAMENTAL FORMULAS OF SPHERICAL TRIGONOMETRY.

1. The relations between the sides and angles of a right-angled spherical triangle were obtained in Art. 26. The law of sines and the law of cosines (Art. 36) for any spherical triangle have been derived by means of these relations. (See Note 1, Art. 36.) These two laws can also be derived directly by geometry; this is done in Arts. 2, 3, below. Moreover, the law of sines can be derived analytically from the law of cosines, as shown in Art. 4. In Art. 5 it is shown how the relations for right-angled triangles can be derived from these two laws. Other relations between the parts of a spherical triangle have been referred to in Art. 40; these relations can also be deduced by means of the law of cosines and the law of sines. *The law of cosines is, accordingly, the fundamental and most important formula in spherical trigonometry.*

2. Direct geometrical derivation of the law of cosines. Let O–ABC be a triedral angle, and ABC be the corresponding spherical triangle on a sphere of radius OA. It is required to find the cosine of the face angle COB, or, what is the same thing, the cosine of the side CB.

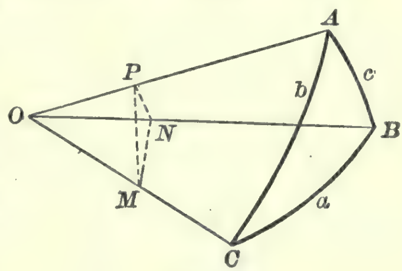


FIG. 60

In OA take any point P, and through P pass a plane MPN at right angles to the line OA. Then OPN and OPM are right angles, and angle MPN = angle A. Also, the measures (in degrees) of the sides AB, BC, CA, are the same as the measures of the face angles COB, BOA, AOC, respectively.

In MPN, $\quad \overline{MN}^2 = \overline{MP}^2 + \overline{PN}^2 - 2\,MP \cdot PN \cos MPN;$ (1)

In MON, $\quad \overline{MN}^2 = \overline{MO}^2 + \overline{ON}^2 - 2\,MO \cdot ON \cos MON.$ (2

Hence, on equating these values of $\overline{MN}^2$ and transposing,

$$2\,MO\cdot ON\cos MON = \overline{MO}^2 - \overline{MP}^2 + \overline{ON}^2 - \overline{PN}^2 + 2\,MP\cdot PN\cos MPN.$$

Now $\overline{OM}^2 - \overline{MP}^2 = \overline{OP}^2$, and $\overline{ON}^2 - \overline{PN}^2 = \overline{OP}^2$, since OPM and OPN are right angles.

$$\therefore\ 2\,MO\cdot ON\cos MON = 2\,OP^2 + 2\,MP\cdot PN\cos MPN.$$

$$\therefore\ \cos MON = \frac{OP}{MO}\,\frac{OP}{ON} + \frac{MP}{MO}\,\frac{PN}{ON}\cos MPN;$$

i.e.

$$\cos a = \cos b\cos c + \sin b\sin c\cos A. \tag{3}$$

Like formulas for $\cos b$, $\cos c$, can be derived in a similar manner; they can also be written immediately, on paying regard to the symmetry in (3). The formulas for $\cos A$, $\cos B$, and $\cos C$, can be derived by means of the polar triangle, as done in Art. 36, *C*.

EXERCISES.

1. Make the figure and derive the law of cosines: (*a*) when P is taken at A; (*b*) when P is taken in OA produced towards A.

2. Derive the formula for *cos b* geometrically. (Take any point in OB, and through this point pass a plane at right angles to OB.)

3. Derive the formula for *cos c* geometrically

3. Direct geometrical derivation of the law of sines. Let $O\text{-}ABC$ be a triedral angle, and ABC be the corresponding spherical triangle on a sphere of radius OA.

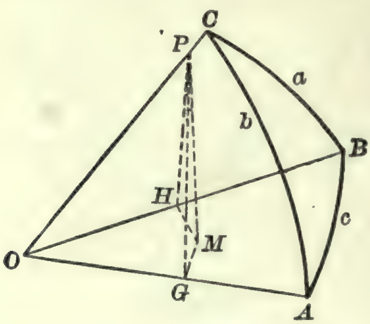


FIG. 61

In OC take any point P, and draw PM at right angles to the plane AOB, and intersecting this plane in M. Through M draw MG and MH, at right angles to OA and OB respectively. Pass a plane through the lines PM and MG.

Since PM is perpendicular to OAB, the plane PMG is perpendicular to OAB (Euc. XI. 18). Hence, since AGM is a right angle, AGP is also a right angle. Therefore angle PGM = angle A. Similarly it can be shown that angle PHM = angle B.

$$\therefore\ \sin A = \frac{PM}{PG} = \frac{PM}{OP\sin AOC} = \frac{PM}{OP\sin b}.\quad \therefore\ \sin A\sin b = \frac{PM}{OP}. \tag{1}$$

Also, $\sin B = \frac{PM}{PH} = \frac{PM}{OP \sin BOC} = \frac{PM}{OP \sin a}$. $\therefore \sin B \sin a = \frac{PM}{OP}$. (2)

$\therefore$ by (1), (2), $\sin A \sin b = \sin B \sin a$.

$$\therefore \qquad \frac{\sin A}{\sin a} = \frac{\sin B}{\sin b}.$$

In a similar way it can be shown that $\frac{\sin A}{\sin a} = \frac{\sin C}{\sin c}$. Hence

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

Ex. 1. Show geometrically:

(a) that $\frac{\sin A}{\sin a} = \frac{\sin C}{\sin c}$; (b) that $\frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$.

Ex. 2. Make the derivation when M is not in the sector AOB.

4. Analytical derivation of the law of sines from the law of cosines.

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}. \qquad \text{[From (3) Art. 2]}$$

$$\therefore 1 - \cos^2 A = 1 - \left(\frac{\cos a - \cos b \cos c}{\sin b \sin c}\right)^2$$

$$= \frac{\sin^2 b \sin^2 c - \cos^2 a - \cos^2 b \cos^2 c + 2 \cos a \cos b \cos c}{\sin^2 b \sin^2 c};$$

$$= \frac{(1 - \cos^2 b)(1 - \cos^2 c) - \cos^2 a - \cos^2 b \cos^2 c + 2 \cos a \cos b \cos c}{\sin^2 b \sin^2 c};$$

i.e. $$\sin^2 A = \frac{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}{\sin^2 b \sin^2 c}.$$

$$\therefore \frac{\sin^2 A}{\sin^2 a} = \frac{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}{\sin^2 a \sin^2 b \sin^2 c}. \qquad (1)$$

Similarly, $\frac{\sin^2 B}{\sin^2 b}$ and $\frac{\sin^2 C}{\sin^2 c}$ can each be shown to be equal to the second member of (1). Hence,

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} = \frac{2n}{\sin a \sin b \sin c}; \qquad (2)$$

in which $2n$ denotes the positive square root of the numerator of the second member of (1).

Ex. 1. Show the truth of the statement made above.

Ex. 2. Show that the numerator in the second member of (1) is equal to $4 \sin s \sin (s - a) \sin (s - b) \sin (s - c)$.

Suggestion. $\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2}$, and Art. 37, (4).

5. Formulas for right-angled triangles derived from the general formulas. In the triangle ABC let angle $C = 90°$. Then $\sin C = 1$, and relations (1), p. 45, become (2) and (2′), p. 30. Also, $\cos C = 0$, and the third formula in Art. 36, B becomes (1), p. 30. The three formulas in Art. 36, C reduce to (5), (5′), and (6), p. 30, respectively. Formulas (3), (3′), (4) and (4′), p. 30, can be derived from the others on that page. For

$$\cos A = \sin B \cos a \text{ [by (5')]} = \frac{\sin b}{\sin c} \cdot \frac{\cos c}{\cos b} \text{ [by (2'), (1)]} = \frac{\tan b}{\tan c};$$

similarly,
$$\cos B = \frac{\tan a}{\tan c}.$$

Also,
$$\tan A = \frac{\sin A}{\cos A} = \frac{\sin A}{\sin B \cos a} \text{ [by (5')]} = \frac{\sin a}{\sin b \cos a} \text{ [by (2), (2')]} = \frac{\tan a}{\sin b};$$

similarly,
$$\tan B = \frac{\tan b}{\sin a}.$$

Other relations in triangles (see Art. 40) can also be used in the derivation of the formulas for right-angled triangles.

EXERCISES.

1. Deduce the law of cosines: (1) directly, by geometry; (2) by means of the relations in a right-angled triangle.

2. Deduce the law of sines: (1) analytically, from the law of cosines (2) directly, by geometry; (3) by means of the relations in a right-angled triangle.

3. Deduce the ten relations between the sides and angles of a right-angled spherical triangle: (1) by means of the relations between the sides and angles of the general spherical triangle; (2) directly, by geometry.

NOTE B.

[Supplementary to Art. 58.]

DERIVATION OF FORMULAS FOR THE SPHERICAL EXCESS OF A TRIANGLE.

***I.* Cagnoli's Formula.** (*In terms of the sides.*)

$$\begin{aligned}
\sin \tfrac{1}{2} E &= \sin \tfrac{1}{2}(A + B + C - 180°) = -\cos \tfrac{1}{2}(A + B + C) \\
&= \sin \tfrac{1}{2}(A + B) \sin \tfrac{1}{2} C - \cos \tfrac{1}{2}(A + B) \cos \tfrac{1}{2} C \\
&= \frac{\sin \frac{1}{2} C \cos \frac{1}{2} C}{\cos \frac{1}{2} c}[\cos \tfrac{1}{2}(a - b) - \cos \tfrac{1}{2}(a + b)] \quad \text{[Art. 39, (1), (3)]} \\
&= \frac{\sin \frac{1}{2} a \sin \frac{1}{2} b \sin C}{\cos \frac{1}{2} c} \quad \text{[Arts. 50 (5), 52 (8), \textit{Plane Trig.}]} \quad (1) \\
&= \frac{\sin \frac{1}{2} a \sin \frac{1}{2} b}{\cos \frac{1}{2} c} \cdot \frac{2n}{\sin a \sin b} \quad \text{[Note A, Art. 4, Eq. (2)]} \\
&= \frac{n}{2 \cos \frac{1}{2} a \cos \frac{1}{2} b \cos \frac{1}{2} c}.
\end{aligned}$$

***II.* Lhuillier's Formula.** (*In terms of the sides.*)

$$\tan \tfrac{1}{4} E = \frac{\sin \frac{1}{4}(A + B + C - 180^\circ)}{\cos \frac{1}{4}(A + B + C - 180^\circ)}$$

$$= \frac{\sin \frac{1}{2}(A + B) - \sin \frac{1}{2}(180^\circ - C)}{\cos \frac{1}{2}(A + B) + \cos \frac{1}{2}(180^\circ - C)} \qquad [\textit{Plane Trig.}, \text{ p. } 94]$$

$$= \frac{\sin \frac{1}{2}(A + B) - \cos \frac{1}{2} C}{\cos \frac{1}{2}(A + B) + \sin \frac{1}{2} C}$$

$$= \frac{\cos \frac{1}{2}(a - b) - \cos \frac{1}{2} c}{\cos \frac{1}{2}(a + b) + \cos \frac{1}{2} c} \cdot \frac{\cos \frac{1}{2} C}{\sin \frac{1}{2} C} \qquad [\text{Art. 39, (1), (3)}]$$

$$= \frac{\sin \frac{1}{2}(s - b) \sin \frac{1}{2}(s - a)}{\cos \frac{1}{2} s \cos \frac{1}{2}(s - c)} \sqrt{\frac{\sin s \sin (s - c)}{\sin (s - a) \sin (s - b)}}$$

[Art. 37, (6); *Plane Trig.*, p. 94]

$$= \sqrt{\tan \tfrac{1}{2} s \tan \tfrac{1}{2}(s - a) \tan \tfrac{1}{2}(s - b) \tan \tfrac{1}{2}(s - c)}.$$

***III.* Formula in terms of two sides and their included angle.**

$$\cos \tfrac{1}{2} E = \cos \tfrac{1}{2}(A + B + C - 180^\circ) = \sin \tfrac{1}{2}(A + B + C)$$

$$= \cos \tfrac{1}{2}(A + B) \sin \tfrac{1}{2} C + \sin \tfrac{1}{2}(A + B) \cos \tfrac{1}{2} C$$

$$= [\cos \tfrac{1}{2}(a + b) \sin^2 \tfrac{1}{2} C + \cos \tfrac{1}{2}(a - b) \cos^2 \tfrac{1}{2} C] \sec \tfrac{1}{2} c$$

[Art. 39, (1), (3)]

$$= (\cos \tfrac{1}{2} a \cos \tfrac{1}{2} b + \sin \tfrac{1}{2} a \sin \tfrac{1}{2} b \cos C) \sec \tfrac{1}{2} c. \qquad (2)$$

Hence, from (1) and (2), on division and reduction,

$$\tan \tfrac{1}{2} E = \frac{\tan \frac{1}{2} a \tan \frac{1}{2} b \sin C}{1 + \tan \frac{1}{2} a \tan \frac{1}{2} b \cos C}.$$

On taking the reciprocals and reducing, this takes the form

$$\cot \tfrac{1}{2} E = \frac{\cot \frac{1}{2} a \cot \frac{1}{2} b + \cos C}{\sin C}.$$

QUESTIONS AND EXERCISES FOR PRACTICE AND REVIEW.

CHAPTER I.

1. On a sphere let N be the pole of a great circle ABC, and P be any point on the surface between N and ABC; also let $DPNG$ be a semicircle drawn through P at right angles to ABC, and let it intersect ABC in D and G: prove (*a*) that PD is the shortest great-circle arc that can be drawn from P to ABC; (*b*) that PNG is the longest great-circle arc that can be drawn from P to ABC.

2. Show that the greater the distance of the plane of a small circle from the centre of the sphere, the less is the circle.

3. The radius of a sphere is 10 inches, and the radius of a small circle upon it is 6 inches. Find: (*a*) the distance between the centre of the sphere and the centre of the small circle; (*b*) the angular radius of the small circle; (*c*) the polar distance (or arcual radius) of the small circle; (*d*) the distance on the sphere from the small circle to the great circle having the same axis.

4. Prove that if a spherical triangle has two right angles, the sides opposite them are quadrants, and the third angle has the same measure as its opposite side.

5. Prove that in any spherical right triangle an angle and its opposite side are always in the same quadrant.

6. Prove that any side of a spherical triangle is greater than the difference between the other two sides.

7. Prove that each angle of a spherical triangle is greater than the difference between 180° and the sum of the other two angles.

8. Show that the surface of a sphere is eight times the surface of a trirectangular triangle.

9. (*a*) Show that a trirectangular triangle is its own polar; (*b*) show that a triquadrantal triangle is its own polar.

10. Show that if two great circles are equally inclined to a third, their poles are equidistant from the pole of the third.

11. Show that the arc through the poles of two great circles cuts both circles at right angles.

12. A ship sails along the parallel of 45° N. a distance of 600 nautical miles. Find the difference of longitude that she has made.

13. Two places in latitude 60° N. are 150 statute miles apart. Find their difference of longitude. [Take the radius of the earth as 3960 miles.]

14. Compare the lengths of the parallels of 30° N., 45° N., and 60° N., with the length of the equator.

15. Prove that if the first of two spherical triangles is the polar triangle of the second, then the second is the polar triangle of the first.

16. Show that in two polar triangles each angle of the one is the supplement of the side opposite to it in the other.

17. Show that the sum of the angles of a spherical triangle is greater than two, and less than six, right angles.

18. Discuss the following cases, in which A, a, and b are given in a spherical triangle ABC:

I. $A = 90°$: (1) $b = 90°$; (2) $b < 90°$($a < b$, $a = b$, $a > b$ and $< \pi - b$, $a = \pi - b$, $a > \pi - b$); (3) $b > 90°$($a < \pi - b$, $a = \pi - b$, $a > \pi - b$ and $< b$, $a = b$, $a > b$).

II. $A < 90°$: (1) $b = 90°$($a < A$, $a = A$, $a > A$ and $< b$, $a = b = 90°$, $a > b$); (2) $b < 90°$($a < p$, $a = p$, $a > p$ and $< b$, $a = b$, $a > b$ and $< \pi - b$, $a = \pi - b$, $a > \pi - b$); (3) $b > 90°$($a < p$, $a = p$, $a > p$ and $< \pi - b$, $a = \pi - b$, $a > \pi - b$ and $< b$, $a = b$, $a > b$). [For definition of p, see p. 26.]

III. $A > 90°$: (1) $b = 90°$($a = b$, a between b and $\pi - b$, a between $\pi - b$ and p); (2) $b < 90°$($a > p$, $a = p$, $a < p$ and $> b$, $a = b$, a between b and $\pi - b$); (3) $b > 90°$($a < b$, $a > b$ and $< p$, $a < p$ and $> \pi - b$, a between b and $\pi - b$, $a = b$).

CHAPTER II.

1. Define *spherical angle, spherical triangle, Napier's circular parts, polar triangle, quadrantal triangle, oblique spherical triangle, pole of an arc, spherical excess, spherical polygon.*

2. In a right-angled spherical triangle show that: (a) It is impossible for only one of the three sides to be greater than 90°; (b) The hypotenuse is less than 90° only when both the other sides are in the same quadrant; (c) If another part besides the right angle be right, the triangle is biquadrantal.

3. Prove, by geometry and by trigonometry, that in a right spherical triangle an angle and its opposite side are always in the same quadrant, that is, either both are less or both are greater than 90°.

4. Prove that in a right spherical triangle ABC, $(C = 90°)$. (*a*) $\sin A = \cos B \div \cos b$; (*b*) $\cos c = \cot A \cot B$; (*c*) $\cos c = \cos a \cos b$.

5. (*a*) Mention in order Napier's circular parts, and state the two principal rules for their use. (*b*) State Napier's Rules and write the ten formulas for the right spherical triangle by means of them. (*c*) Prove three of these formulas.

6. What formulas should be used to find B, a, and b of a right spherical triangle ABC $(C = 90°)$ when A and c are given? What formula includes all the required parts?

7. Show how to obtain the formulas for finding a, B, and C of a quadrantal triangle, when A and b are given and $c = 90°$.

8. Given one side and the hypotenuse of a right spherical triangle, write all the formulas for the solution and check, and state how the species of each part will be determined.

9. How many solutions are there for a right spherical triangle ABC, given side b and angle B? Discuss fully.

10. Given A and b of a right spherical triangle ABC $(C = 90°)$: write and derive formulas for computing each of the parts B, a, and c in terms of A and b only; also the check formula.

11. Show how to solve a right spherical triangle, having given (*a*) the sides about the right angle; (*b*) the two oblique angles.

12. (*a*) Show how the solution of a quadrantal triangle may be reduced to that of a right triangle. (*b*) Write the relations between the sides and angles of a quadrantal triangle ABC, in which $c = 90°$.

13. In a spherical triangle ABC, $A = B$: write the relations between the sides and angles of ABC.

14. If A be one of the base angles of an isosceles spherical triangle whose vertical angle is $90°$ and a the opposite side, prove that $\cos a = \cot A$; and determine the limits within which it is necessary that A must lie.

15. Show how oblique spherical triangles can be solved by means of right spherical triangles. (Six cases.)

16. In a right spherical triangle ABC $(C = 90°)$ prove that: (*a*) $\sin^2 B - \cos^2 A = \sin^2 b \sin^2 A$; (*b*) $\sin A \sin 2b = \sin c \sin 2B$; (*c*) $\sin 2A \sin c = \sin 2a \sin B$; (*d*) $\sin 2a \sin 2b = 4 \cos A \cos B \sin^2 c$; (*e*) $\cos^2 A \sin^2 c = \sin^2 c - \sin^2 a$; (*f*) $\sin^2 A \cos^2 c = \sin^2 A - \sin^2 a$.

17. (*a*) In ABC, if $C = 90°$, and $a = b = c$, prove that $\sec A = 1 + \sec a$. (*b*) In ABC $(C = 90°)$ show that if $b = c = \frac{\pi}{2}$, then $\cos a = \cos A$.

18. In a right spherical triangle whose oblique angles are $72° 34'$ and $59° 42'$, find the length of the perpendicular from the right angle upon the base, and the angles which it forms with the sides.

19. Two planes intersecting at right angles are intersected by a third plane making with them angles of 60° and 75° respectively. Find the angles which the three lines of intersection make with each other.

20. Two planes intersect at right angles; from any point of their line of intersection one line is drawn in each plane making the respective angles 60° and 73° with the line of intersection. Find the angle between the two lines thus drawn.

21. A triangle whose sides are 40°, 90°, and 125° respectively, is drawn on the surface of a sphere whose radius is 8 feet. Find in feet the length of each side of this triangle, and also the angles of the polar triangle. Write the formula for finding either angle in terms of functions of the sides.

22. Solve the following spherical triangles given: (1) Right triangle, hypotenuse $= 140°$, one side $= 20°$. (2) Sides 90°, 50°, 50°. (3) Sides 100°, 50°, 60°. (4) Sides each 30° in length. (5) $A = 100°$, $C = 90°$, $a = 112°$. (6) $A = 80°$, $a = 90°$, $b = 37°$. (7) $a = b = 119°$, $C = 85°$. (8) Triangle PQR, $R = 90°$, $P = 63° 42'$, $Q = 123° 18'$. (9) Right triangle, one angle $= 110° 30' 20''$, hypotenuse $= 75° 45'$. (10) $A = 90°$, $b = 21° 30'$, $c = 122° 18'$. (11) $B = 90°$, $C = 79° 40'$, $b = 137° 52'$. (12) $A = 90°$, $a = 108° 23$, $c = 37° 42'$. (13) $B = 90°$, $A = 43° 10'$, $a = 78° 35'$. (14) $B = 90°$, $C = 33° 57'$, $A = 43° 18'$. (15) $A = 87° 40' 20''$, $b = 33° 42' 40''$, $B = 90°$. (16) $A = 33° 42' 40''$, $b = 87° 40' 20''$, $B = 90°$.

CHAPTER III.

1. In a spherical triangle ABC prove that: (*a*) $\sin a : \sin A = \sin b : \sin B = \sin c : \sin C$; (*b*) $\cos a = \cos b \cos c + \sin b \sin c \cos A$; (*c*) $\cos A = -\cos B \cos C + \sin B \sin C \cos a$; (*d*) $\cos \frac{1}{2} A = \sqrt{\sin s \sin (s - a) \div \sin b \sin c}$, where $s = \frac{1}{2}(a + b + c)$; (*e*) $\tan \frac{1}{2} A \cot \frac{1}{2} B = \sin (s - b) \operatorname{cosec} (s - a)$.

2. Give the equations (or proportions) known as Napier's Analogies. Derive them.

3. Derive formulas giving the values of $\sin A$, $\cos A$, $\tan A$, and $\cos c$, in terms of functions of a, b, and c.

4. In a spherical triangle ABC show that: (*a*) If $a = b = c$, then $\sec A = 1 + \sec a$. (*b*) If $b + c = 180°$, then $\sin 2B + \sin 2C = 0$. (*c*) If $C = 90°$, then $\tan \frac{1}{2}(c + a) \tan \frac{1}{2}(c - a) = \tan^2 \frac{b}{2}$.

5. In an equilateral spherical triangle show that: (*a*) $2 \sin \frac{A}{2} \cos \frac{a}{2} = 1$, and hence, that such a triangle can never have its angle less than 60°, nor its side greater than 120°; (*b*) $2 \cos A = 1 - \tan^2 \frac{a}{2}$.

6. Show that: (*a*) If the three angles of spherical triangle ABC are together equal to four right angles, then $\cos^2 \frac{c}{2} = \cot A \cot B$. (*b*) If x is the side of a spherical triangle formed by joining the middle points of the equilateral triangle of side a, then $2 \sin \frac{x}{2} = \tan \frac{a}{2}$.

7. (*a*) In a spherical triangle ABC show that, if $b + c = 90°$, then $\cos a = \sin 2c \cos^2 \frac{A}{2}$. (*b*) If a be the side of an equilateral triangle and a' that of its polar triangle, prove $\cos a \cos a' = \frac{1}{2}$.

8. (*a*) If, in a triangle ABC, l be the length of the arc joining the middle point of the side c to the opposite vertex C, show that $\cos l = (\cos a + \cos b) \div 2 \cos \frac{c}{2}$. (*b*) In a right spherical triangle ABC $(C = 90°)$, if α, β be the arcs drawn from C respectively perpendicular to and bisecting the hypotenuse c, show that $\sin^2 \frac{c}{2}(1 + \sin^2 \alpha) = \sin^2 \beta$.

9. (*a*) Prove that the half sum of two sides of any spherical triangle is in the same quadrant as the half sum of the opposite angles. (*b*) Two sides of a spherical triangle are given: prove that the angle opposite the smaller of them will be greatest when that opposite the larger is a right angle.

10. ABC is a spherical triangle of which each side is a quadrant, and P is a point within it. Prove that $\cos^2 PA + \cos^2 PB + \cos^2 PC = 1$.

11. In a spherical triangle, if $A = 36°$, $B = 60°$, and $C = 90°$, show that $a + b + c = 90°$.

CHAPTER IV.

1. (*a*) Name the six cases for solution of spherical triangles. (*b*) Discuss each case in detail, writing the formulas used in the solution, and deriving these formulas. (*c*) Solve an example under each case. Test the result by (1) solving by right triangles, (2) solving without logarithms, (3) using a check formula.

2. How many solutions are possible for the oblique spherical triangle ABC, given A, B, and a? Discuss in full the question of one solution, two solutions, or no solution. Plan the solution.

3. In a spherical triangle ABC, two sides a and b and the included angle C are given. Write all the formulas used in the solution and check; describe fully the process of solution. Derive the formulas used.

4. Write and deduce the formulas for finding A, B, and C of any spherical triangle when a, b, and c are given.

5. Given A, B, and C. Show how to find the remaining parts, writing the formulas to be used.

6. In an equilateral spherical triangle the side a is given. Find the angle A.

7. Solve the spherical triangle whose sides are 70°, 60°, and 50°. Solve the plane triangle obtained by connecting by straight lines the vertices of this spherical triangle, the sphere on which it is drawn being 2 feet in diameter.

8. In a triangle ABC on the earth's surface (supposed spherical) $a = 483$ miles, $b = 321$ miles, $C = 38° 21'$. Find the length of the side c. [Earth's radius = 3960 miles.]

9. Two planes intersect at an angle of 75°. From any point of their line of intersection one line is drawn in each plane, making the respective angles 55° and 80° with the line of intersection. Find the angle between the lines thus drawn.

10. Two planes intersecting at an angle of 65° are intersected by a third plane, making with them the respective angles 55° and 82°. Find the angles which the three lines of intersection make with one another.

11. A solid angle is contained by three plane angles 62°, 83°, 38°. Find the angle between the planes of the angles 62° and 38°.

12. Two of the three angles which contain a solid angle are 42° and 65° 30′, and their planes are inclined at an angle of 50°. Find the angle of the third plane face and the angles at which this third plane is inclined to the other two planes.

13. A pyramid has each of its slant sides and base an equilateral triangle. Find the angle between any two faces.

14. A pyramid each of whose slant faces is an equilateral triangle has a square base. Find the angle between any two slant faces, also the angle between any slant face and the base.

15. In the following cases ABC is a three-sided spherical figure each of whose sides is an arc of a great circle. Select those which are spherical triangles, and give reasons for so doing. Explain why the other figures cannot be triangles. Solve the triangles and check the results. (Solve some without using logarithms.)

(1) $a = 76°,\ b = 54°,\ c = 36°$. (2) $A = 54° 35' 20'',\ b = 104° 25' 45'',\ c = 92° 10'$. (3) $A = 107° 47' 7'',\ B = 38° 58' 27'',\ c = 51° 41' 14''$. (4) $A = 60°,\ B = 80°,\ C = 100°$. (5) $A = 120°,\ B = 130°,\ C = 80°$. (6) $A = 54° 35',\ b = 104° 24',\ c = 95° 10'$. (7) $A = 61° 37' 53'',\ B = 139° 54' 34'',\ b = 150° 17' 26''$. (8) $a = 72° 18',\ b = 146° 35',\ c = 98° 11'$. (9) $A = 125° 15',\ C = 85° 12',\ b = 100°$. (10) $A = 50°,\ B = 114° 5' 8'',\ b = 50°$. (11) $A = 83° 40',\ b = 73° 45',\ a = 30° 24'$. (12) $A = 83° 40',\ b = 30° 24',\ a = 73° 45'$. (13) $A = 97° 20',\ a = 94° 37',\ b = 36° 17'$. (14) $a = 127° 40',\ b = 143° 50',\ c = 139° 39'$. (15) $A = 40°,\ B = 30°,\ C = 20°$. (16) $A = 40° 35',\ B = 36° 42',\ c = 47° 18'$.

CHAPTER V.

[In the following exercises,

$$n = \sqrt{\sin s \sin (s-a) \sin (s-b) \sin (s-c)},$$

and
$$N = \sqrt{-\cos S \cos (S-A) \cos (S-B) \cos (S-C)};$$

also, r, r_a, r_b r_c, denote the radii of the circles inscribed in the spherical triangle ABC and its three colunar triangles, and R, R_a, R_b, R_c denote the radii of the circumscribing circles of these triangles.]

1. Given a spherical triangle ABC, find (1) the radius of the inscribed circle; (2) the radius of the circumscribing circle; (3) the radii of the inscribed circles of the colunar triangles; (4) the radii of the circumscribing circles of the colunar triangles.

Show that:

2. $\text{Tan}\, r = \dfrac{n}{\sin s}$.

3. $\text{Tan}\, R = \dfrac{N}{\cos (S-A) \cos (S-B) \cos (S-C)}$.

4. (*a*) $\text{Cot}\, R \cot R_a \cot R_b \cot R_c = N^2$;

(*b*) $\text{Tan}\, R \cot R_a \cot R_b \cot R_c = \cos^2 S$.

5. $\text{Tan}\, R = 4 \tan r \dfrac{-\cos S \sin s}{\sin a \sin b \sin c \sin A \sin B \sin C}$.

6. $\text{Tan}\, r_a \tan r_b \tan r_c = \tan r \sin^2 s$.

7. $\text{Tan}\, R + \cot r = \tan R_a + \cot r_a = \tan R_b + \cot r_b$

$$= \tan R_c + \cot r_c = \tfrac{1}{2} (\cot r + \cot r_a + \cot r_b + \cot r_c).$$

8. $\text{Tan}\, R \tan r = -\dfrac{\cos S \sin a}{\sin s \sin A} = -\dfrac{\cos S \sin b}{\sin s \sin B} = \text{etc.}$ Write the other formula of this set.

9. $\text{Tan}^2 R + \tan^2 R_a + \tan^2 R_b + \tan^2 R_c = \cot^2 r + \cot^2 r_a + \cot^2 r_b + \cot^2 r_c$.

10. $\text{Tan}\, r \tan r_a \tan r_b \tan r_c = n^2$; $\cot r \tan r_a \tan r_b \tan r_c = \sin^2 s$.

11. In any equilateral triangle, $\tan R = 2 \tan r$.

12. $\text{Tan}\, R_a = -\dfrac{\tan \frac{1}{2} a}{\cos S} = \dfrac{\cos (S-A)}{N} = \dfrac{\sin \frac{1}{2} a}{\sin A \sin \frac{1}{2} b \sin \frac{1}{2} c}$

$$= \frac{2 \sin \frac{1}{2} a \cos \frac{1}{2} b \cos \frac{1}{2} c}{n} = \frac{1}{2n} [\sin s - \sin (s-a) + \sin (s-b) + \sin (s-c)].$$

Write the corresponding formulas for R_b and R_c.

13. $\text{Cot}\, r_a + \cot r_b + \cot r_c - \cot r = 2 \tan R$.

14. Find the radii of the circles connected with some of the triangles in Ex. 15 of the preceding set.

CHAPTER VI.

1. Define the following terms: *zone* of a sphere, *lune*, *spherical degree*, *spherical excess of a triangle*, *spherical excess of a* (non-re-entrant) *polygon*, *spherical excess* of any figure on a sphere, *spherical measure* and *spherical degree measure* of a solid angle, *spherical pyramid*, *spherical sector*, *spherical segment.*

2. Derive the area of the surface of a sphere.

3. Derive the area of a spherical triangle.

4. Discuss fully the measurement of solid angles.

5. Show how to find the spherical excess of a figure on a sphere when the area of the figure is given (in square units).

6. State and deduce *Roy's Rule* for computing the spherical excess of a triangle of known area on the earth's surface.

7. Derive the volumes of a sphere, a spherical pyramid, a spherical sector, and a spherical segment.

8. The area of an equilateral triangle is one-fourth the area of the sphere: find its sides and angles.

9. If the three sides of a spherical triangle measured on the earth's surface be 12, 16, and 18 miles, find the spherical excess.

10. If $a = b$ and $C = \frac{\pi}{2}$, show that $\tan E° = \frac{\sin^2 a}{2 \cos a}$. (In ABC.)

11. If $a = b = 60°$ and $c = 90°$, show that $E = \cos^{-1} \frac{7}{9}$. (In ABC.)

12. If $C = 90°$ in ABC, then $E = 2 \tan^{-1}(\tan \frac{1}{2} a \tan \frac{1}{2} b)$.

13. In a triangle on the earth's surface (assumed spherical), two sides are 483 and 321 miles, and the angle between them is 38° 21′. Find the area of the triangle in square miles. [Radius of earth = 3960 miles.]

14. The sides of a triangle on the earth's surface (supposed spherical) are 321, 287, and 412 miles; find the area.

15. Prove that in a right triangle ABC ($C = 90°$),

$$\cos \tfrac{1}{2} E = \frac{\cos \frac{1}{2} a \cos \frac{1}{2} b}{\cos \frac{1}{2} c}, \text{ and } \sin \tfrac{1}{2} E = \frac{\sin \frac{1}{2} a \sin \frac{1}{2} b}{\cos \frac{1}{2} c}.$$

16. The spherical excess of a triangle on the earth's surface is 2″.5. Find its area, the radius of the earth being taken as 3960 miles.

17. Find the fraction of the earth's surface (supposed spherical) contained by great-circle arcs joining London, New York, and Paris. Find the spherical degree measure, and the spherical measure of the angle subtended at the centre of the earth by this part of the earth's surface.

18. Find the spherical excess of some of the triangles in Ex. 15, p. 104. Also find their areas in square inches on spheres of radii, say, 4 inches, 10 inches, 12 inches, 20 inches, a inches.

19. Find the spherical measures and the spherical degree measures of the solid angles corresponding to the triangles taken in Ex. 18.

CHAPTER VII.

1. Given the latitude and longitude of each of two places: show how to find the shortest distance between these places, and the direction of one place from the other.

2. Given the latitudes and longitudes of three places on the earth's surface, and also the radius of the earth: show how to find the area of the spherical triangle formed by arcs of great circles passing through them.

3. Given the sun's altitude and declination and the latitude of a place: show clearly how the time of day may be determined.

4. If d represents the sun's declination, what formulas will be required in order to determine the time of sunrise for a place whose latitude is l?

5. Show what formulas must be used to find the length of a degree of longitude on the earth's surface for a place whose latitude is l, r representing the radius of the earth.

6. The shortest distance d between two places and their latitudes l and l' are known; find their difference of longitude.

7. Given the obliquity of the ecliptic ω, and the sun's longitude λ, show that if α and δ denote his right ascension and declination respectively, then $\tan \alpha = \cos \omega \tan \lambda$, and $\sin \delta = \sin \omega \sin \lambda$.

8. The faces of a regular dodecaedron are regular pentagons, three faces meeting at each vertex. Find the diedral angle at the edge of the solid.

9. The ridges of two gable roofs meet at right angles; each roof is inclined to the horizontal at an angle of 65°. Find the diedral angle between the planes of the two roofs, and the angle their line of intersection makes with the ridge of either roof.

10. What is the direction of a wall in latitude 52° 30′ N. which casts no shadow at 6 A.M. on the longest day of the year?

11. Two ports are in the same parallel of latitude, their common latitude being l, and their difference of longitude 2λ. Show that the saving of distance in sailing from one to the other on the great circle instead of sailing due east or west, is

$$2r\{\lambda \cos l - \sin^{-1}(\sin \lambda \cos l)\},$$

λ being expressed in radian measure, and r being the radius of the earth.

12. If a ship sails from New York (40° 28′ N., 74° 8′ W.) starting due east, and continues her course on an arc of a great circle, what will be her latitude when she reaches the meridian of Greenwich, and in what direction will she then be sailing?

13. Find the distance between New York (40° 28′ N., 74° 8′ W.) and Cape Clear (51° 26′ N., 9° 29′ W.), and the bearing of each from the other. [Radius of earth = 3960 miles.]

14. From Victoria, B.C. (48° 25′ N., 123° 23′ W.), a ship sails on an arc of a great circle for 1250 miles, starting in the direction S. 47° 35′ W. Find its latitude and longitude, taking the length of 1° as $69\frac{1}{6}$ miles.

15. Two places are both in latitude 50° N., and the difference of their longitudes is 60°. Find the distance between them (*a*) along the parallel of latitude, (*b*) along a straight line, (*c*) along a great circle. [Earth's radius = 3960 miles.]

16. What will be the first course and the shortest (great circle) distance passed over in sailing from a place in latitude 43° N. to another place 80° east of it and in the same latitude? What is the distance between the two places along the parallel? What is the straight-line distance between them?

17. At what hours will the sun rise in London (51° 30′ 48″ N.) and New York (40° 43′ N.) when its declination is respectively 23° N., 20° N., 15° N., 10° N., 5° N., 5° S., 10° S., 15° S., 20° S., 23° S.?

18. When the sun's declination is 18°, find his right ascension and longitude.

19. What is the altitude of the sun above the horizon when its angular distance from the south point is 75° and from the west point is 60°?

20. The right ascension of Sirius is $6^h\ 38^m\ 37^s.6$, and his declination is 16° 31′ 2″ S.; the right ascension of Aldebaran is $4^h\ 27^m\ 25^s.9$, and his declination is 16° 12′ 27″ N. Find the angular distance between these stars.

21. If the sun's declination be 20° 45′ N. and his altitude be 41° 10′ at 3 P.M., find the observer's latitude.

22. What will be the altitude of the sun at 3.30 P.M. in San Francisco (37° 48′ N.), its declination being 15° S.?

23. In Bombay (18° 54′ N.) the altitude of the sun is observed to be 27° 40′. If the sun's declination is 7° S. and the observation is made in the morning, find the hour of the day.

24. Find the latitude and longitude of a star whose right ascension is $4^h\ 40^m$, and declination 57°.

25. Find the distance in degrees between the sun and moon when their right ascensions are respectively $15^h\ 12'$, $4^h\ 45'$, and their declinations are 21° 30′ S., 5° 30′ N.

26. Find the length of the longest day in the year at the following places (the sun's greatest declination being $23° 27'$ N.): London ($51° 30' 48''$ N.), New York ($40° 43'$ N.), Montreal ($45° 30'$ N.), St. Petersburg ($60°$ N.), Hong Kong ($22° 17'$ N.).

27. Find the length of the shortest day in the year at the places mentioned in Ex. 26. (The sun's declination is then $23° 27'$ S.)

28. At Copenhagen ($55° 40'$ N.), at an afternoon observation, the sun's altitude is $44° 20'$; find the time of day, the sun's declination being $18° 25'$ N.

29. At what time of day will the sun have an altitude of $53° 40'$ for a place in latitude $40° 35'$ N., his declination being $13° 48'$ N.?

30. What will be the sun's altitude at 3.30 P.M. at a place in latitude $44° 40'$ N., his declination being $18°$ N.?

31. What will be the sun's altitude at 10 A.M. at a place in latitude $44° 40'$ N., his declination being $18°$ S.?

32. What is the sun's declination when his altitude at a place in latitude $37° 48'$ N. is $25°$ at 4 P.M.?

NOTE. The Spherical Trigonometries of M'Clelland and Preston, Casey, and Bowser, contain especially good collections of exercises. See Art. 40.

ANSWERS TO THE EXAMPLES.

CHAPTER I.

Art. 24. **I. 4.** $A = 88° 12.2'$, $B = 74° 34.7'$, $C = 43° 8'$; $A = 118° 33.2'$, $B = 113° 11.2'$, $C = 92° 45'$. **II. 4.** $a = 72° 40.6'$, $b = 67° 45.8'$, $c = 51° 43.1'$; $a = 71° 22.7'$, $b = 108° 37.3'$, $c = 104° 56.7'$. **III. 4.** $A = 63° 56'$, $B = 126° 21.2'$, $c = 77° 3'$; $B = 32° 47.1'$, $C = 62° 30.7'$, $a = 84° 29.5'$. **IV. 5.** $b = 70° 5.7'$, $c = 102° 51.3'$, $A = 68° 35.8'$; $a = 46° 1.5'$, $c = 86° 0.7'$, $B = 122° 55.8'$. **VI. 3.** $B = 59° 40.1'$, $C = 114° 55'$, $c = 96° 31.1'$, and $B = 120° 19.9'$, $C = 27° 49.6'$, $c = 30° 45.4'$; $B = 65° 1.8'$, $C = 97° 16.9'$, $c = 100° 26'$; $C = 110° 43.1'$, $b = 33° 8.6'$, $c = 60° 28.8'$; $C = 165° 3.3'$, $b = 125° 1.7'$, $c = 162° 55.7'$, and $C = 119° 47'$, $c = 81° 7'$, $b = 54° 58.3'$.

CHAPTER II.

Art. 27. **4.** $c = 82° 33.9'$, $A = 60° 51.2'$, $B = 76° 56.1'$. **5.** $a = 33° 0.25'$, $b = 36° 29.4'$, $c = 47° 37.8'$.

Art. 31. **5.** (1) $C = 86° 30.9'$, $A = 36° 30.2'$, $B = 87° 25.4'$. (2) $b = 138° 24.4'$, $A = 58° 41.9'$, $B = 129° 43.1'$. (3) $a = 35° 50.6'$, $b = 75° 39.5'$, $B = 81° 29.1'$. (4) $a = 42° 49.8'$, $b = 27° 47.3'$, $c = 49° 33'$. (5) $b = 33° 37.4'$, $c = 79° 2'$, $B = 34° 20.1'$; and $b = 146° 22.6'$, $c = 100° 58'$, $B = 145° 39.9'$. (6) $a = 35° 16.4'$, $c = 51° 10.8'$, $B = 55° 18.6'$.

Art. 32. **1.** (1) $b = 54° 20'$, $A = 32° 0.75'$, $B = 57° 59.25'$, $C = 93° 59.3'$; (2) $b = 66° 29'$, $c = 111° 29.4'$, $B = 50° 17'$, $C = 128° 41.2'$. **2.** (1) $b = 59° 56.2'$, $A = 130°$, $B = 52° 55.5'$. (2) $a = 135° 33'$, $b = 100° 58.6'$, $C = 101° 24.7'$.

CHAPTER III.

Art. 37. **I. 2.** $A = 55° 58.4'$, $B = 74° 14.6'$, $C = 103° 36.6'$. **3.** $A = 43° 58'$, $B = 58° 14.4'$, $C = 108° 4.8'$. **II. 3.** $a = 39° 29.6'$, $b = 35° 36.2'$, $c = 27° 59'$. **4.** $a = 130° 49.6'$, $b = 120° 17.5'$, $c = 54° 56.1'$.

CHAPTER IV.

Art. 42. **2.** $A = 41° 27'$, $B = 66° 26.4'$, $C = 106° 3.2'$. **3.** $A = 144° 26.6'$, $B = 26° 9.1'$, $C = 36° 34.7'$.

Art. 43. **1.** $a = 43° 36'$, $b = 41° 20.9'$, $c = 33° 7.4'$. **2.** $a = 111° 40.2'$, $b = 91° 17.2'$, $c = 71° 7.4'$.

Art. 44. **2.** $A = 101° 24.2'$, $B = 54° 57.9'$, $c = 79° 9.5'$. **3.** $B = 78° 20.6'$, $C = 47° 47'$, $a = 82° 42'$.

Art. 45. **1.** $a = 63° 15.1'$, $b = 43° 53.7'$, $C = 95° 1'$. **2.** $b = 86° 39.5'$, $c = 68° 39.5'$, $A = 59° 44'$.

Art. 46. **2.** $B = 36° 35.5'$, $C = 51° 59.7'$, $c = 42° 38.9'$. **3.** $B = 59° 3.5'$, $C = 97° 38.8'$, $c = 56° 56.9'$; $B = 120° 56.5'$, $C = 28° 5.2'$, $c = 23° 27.8'$.

Art. 47. **1.** $b = 154° 45.1'$, $c = 34° 9.1'$, $C = 70° 17.5'$. **2.** $A = 164° 43.7'$, $a = 162° 37.5'$, $c = 124° 40.6'$; $A = 119° 18.7'$, $a = 81° 18.7'$, $c = 55° 19.4'$.

CHAPTER VI.

Art. 53. **1.** 2827.44 sq. in. **2.** 392.7 sq. in. **3.** 8.25 sq. ft.

Art. 55. **1.** 1.396 sq. ft. **2.** 64.14 sq. ft.

Art. 56. 24° 37′ 47″ (.42986), 33° 56.6′ (.59213), 27° 10.4′ (.47426), 12° (.20944), 86° 20′ (1.5068), etc.

Art. 57. **1.** 42.986 sq. ft., 59.213 sq. ft., 47.426 sq. ft. **2.** 130.9 sq. in., 941.75 sq. in.

Art. 61. **1.** Spherical degree measure = 12, spherical measure = .20944, **2.** Spherical degree measure = 24.63, spherical measure = .42986.

Art. 64. **1.** 143.29 cu. ft., 197.38 cu. ft., 158.09 cu. ft., 1090.8 cu. in., 7847.9 cu. in., etc. **2.** (*a*) 1357.17 cu. in. (*b*) 904.78 cu. ft.

CHAPTER VII.

Art. 66. **2.** 8° 4.3′ S.; course, S. 45° 6′ E. **5.** (*a*) On the equator in long. 18° 56′ E.; course, S. 47° 39′ E. (*b*) Lat. 42° 21′ S., long. 108° 56′ E.; course, E. (*c*) On the equator in long. 161° 4′ W.; course, N. 47° 39′ E. **6.** Distance = (51° 19.8′) = 3547.675 mi.; bearing of New York from Liverpool is N. 71° 6.8′ W., and bearing of Liverpool from New York is N. 48° 5.8′ E.; lat. 51° 44.1′ N.; course, N. 65° 38′ E.

Art. 73. **1.** 8.08 A.M. **2.** 2.33 P.M. **3.** 2.59 P.M. **4.** 4.09 P.M. **5.** 6.09 P.M. **6.** 9.46 A.M.

Art. 74. **1.** (*a*) 5.44; (*b*) 5.06; (*c*) 4.34; (*d*) 5.43; (*e*) 6.21; (*f*) 6.53; (*g*) 7.26. **2.** (*a*) 5.37; (*b*) 4.40; (*c*) 3.51; (*d*) 5.35; (*e*) 6.30; (*f*) 7.19; (*g*) 8.09. **3.** (*a*) 5.29; (*b*) 4.08; (*c*) 2.51; (*d*) 5.25; (*e*) 6.42; (*f*) 7.51. (*g*) 9.09.