ELEMENTS OF QUATERNIONS.

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OF

QUATERNIONS.

BY THE LATE

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I HAVE reserved for the Appendix to this Volume the longer additional and illustrative notes which I have written for the new edition of the "ELEMENTS."

Some of those notes would have been inconveniently long as footnotes; others would have been inconveniently placed. For example, although the Note on Screws relates naturally to Art. 416 and that on the Kinematical Treatment of Curves to Art. 396, I have placed the Note on Screws before the Note on Curves because Hamilton's remarks on screw motion in the earlier Article required some development in order to make the Note on Curves easily intelligible. Accordingly the order of the notes has been arranged with reference to the notes themselves rather than with reference to the text. The selection and treatment of the subjects of these notes have been subordinated to the illustration of quaternion methods. I have not hesitated to sacrifice brevity for suggestiveness, and above all I have tried to render the notation as explicit as possible.

An analysis of the Appendix will be found on pages xlv–xlix.

For greater convenience I have provided an Index to the whole work referring to the pages, the volumes being distinguished by the numbers i and ii.

I take this opportunity of testifying to the extraordinary accuracy both of matter and of printing in the first edition of the "ELEMENTS." Every portion of the work bears evidence of Hamilton's unsparing pains. I cannot recall a single sentence ambiguous in its meaning, or a single case in which a difficulty is not honestly faced. I see no sign of diminished vigour or of relaxed care in those portions of the work written in his failing health. My task as editor has convinced me of the extreme caution with which any endeavour should be made to improve or modify the calculus of Quaternions.

In conclusion, I desire to express my thanks to the College Printer, Mr. George Weldrick, for the great care he has taken in printing this edition for the Board of Trinity College, and for his unvarying courtesy to myself.

CHARLES JASPER JOLY.

THE OBSERVATORY, DUNSINK,

16*th December*, 1900.

TABLE OF CONTENTS.

BOOK III.

Pages

ON QUATERNIONS, CONSIDERED AS PRODUCTS OR POWERS OF VECTORS; AND ON SOME APPLICATIONS OF QUATERNIONS (*continued*), ... 1–358

CHAPTER III.

On some Additional Applications of Quaternions, with some Concluding Remarks.

Section 1.—Remarks Introductory to this Concluding Chapter, ... 1–4

Section 2.—On Tangents and Normal Planes to Curves in Space, . . . 4–10

Section 3.—On Normals and Tangent Planes to Surfaces, ... 11–23

Section 4.—On Osculating Planes, and Absolute Normals to Curves of Double Curvature, ... 24–29

Section 5.—On Geodetic Lines, and Families of Surfaces, ... 29–49

In these Sections, $d\rho$ usually denotes a *tangent to a curve*, and ν a *normal to a surface*. Some of the theorems or constructions may perhaps be new; for instance, those connected with the *cone of parallels* (pp. 6, 26, &c.) to the *tangents to a curve of double curvature*; and possibly the theorem (p. 42), respecting *reciprocal curves in space*: at least, the deductions here given of these results may serve as exemplifications of the Calculus employed. In treating of *Families of Surfaces* by quaternions, a sort of *analogue* (pp. 47, 48) to the formation and integration of *Partial Differential Equations* presents itself; as indeed it had done, on a similar occasion, in the *Lectures* (574).

Section 6.—On Osculating Circles and Spheres, to Curves in Space; with some connected Constructions, ... 50–179

The analysis, however condensed, of this long Section (III. iii. 6), cannot conveniently be performed otherwise than under the heads of the respective *Articles* (389–401) which compose it: each Article being followed by several sub-articles, which form with it a sort of *Series*.*

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* A *Table of initial Pages* of all the *Articles* will be elsewhere given, which will much facilitate *reference*.

Pages

ARTICLE 389.—*Osculating Circle defined*, as the *limit* of a circle, which *touches a given curve* (plane or of double curvature) at a *given point* P, and *cuts* the curve at a *near* point Q (see fig. 77, p. 24). Deduction and interpretation of general expressions for the vector κ of the *centre* K of the centre so defined. The *reciprocal of the radius* KP being called the *vector of curvature*, we have *generally*,

$$\textit{Vector of Curvature} = (\rho - \kappa)^{-1} = \frac{\mathrm{dU}d\rho}{\mathrm{T}d\rho} = \frac{1}{d\rho}\,\mathrm{V}\frac{d^2\rho}{d\rho} = \&c.; \qquad (S)$$

and if the *arc* (s) of the curve be made the independent variable, then

$$\text{Vector of Curvature} = \rho'' = \mathrm{D}_s{}^2\rho = \frac{d^2\rho}{ds^2}. \qquad (S')$$

Examples: curvatures of helix, ellipse, hyperbola, logarithmic spiral; locus of centres of curvature of helix, plane evolute of plane ellipse, ... 50–55

ARTICLE 390.—Abridged general calculations; return from (S') to (S), . . . 55–56

ARTICLE 391.—Centre determined by three scalar equations; *Polar Axis, Polar Developable*, ... 57–58

ARTICLE 392.—*Vector Equation* of osculating circle, ... 58–60

ARTICLE 393.—*Intersection* (or intersections) of a circle with a *plane curve* to which it osculates; example, hyperbola, ... 60–63

ARTICLE 394.—Intersection (or intersections) of a *spherical curve* with a *small circle* osculating thereto; example, *spherical conic; constructions* for the spherical *centre* (or *pole*) of the circle osculating to such a curve, and for the point of *intersection* above mentioned, 63–74

ARTICLE 395.—*Osculating Sphere*, to a curve of double curvature, defined as the *limit* of a sphere, which *contains the osculating circle* to the curve at a *given point* P, and *cuts* the same curve at a *near* point Q (comp. Art. 389). The *centre* S, of the sphere so found, is (as usual) the point in which the *polar axis* (Art. 391) *touches the cusp-edge* of the *polar developable*. Other general construction for the same centre (p. 77, comp. 106). General expressions for the *vector*, σ = OS, and for the *radius*, $R = \overline{\mathrm{SP}}$; R^{-1} is the *spherical curvature* (comp. Art. 397). *Condition of Sphericity* ($S = 1$), and *Coefficient of Non-sphericity* ($S - 1$), for a curve in space. When this last coefficient is *positive* (as it is for the helix), the *curve* lies *outside* the *sphere*, at least in the neighbourhood of the point of osculation, . 74–80

ARTICLE 396.—Notations $\tau, \tau', \ldots$ for $\mathrm{D}_s\rho, \mathrm{D}_s{}^2\rho$, &c.; properties of a curve depending on the *square* (s^2) of its *arc*, measured from a given point P; τ = *unit-tangent*, τ' = *vector of curvature*, $r^{-1} = \mathrm{T}\tau'$ = *curvature* (or *first* curvature, comp. Art. 397), $\nu = \tau\tau'$ = *binormal*; the *three planes*, respectively perpendicular to τ, τ', ν, are the *normal plane*, the *rectifying plane*, and the *osculating plane*; general theory of *emanant lines and planes, vector of rotation, axis of displacement, osculating screw surface; condition of developability* of *surface of emanants*, ... 81–88

ARTICLE 397.—Properties depending on the *cube* (s^3) of the *arc*; *Radius* r (denoted here, for distinction, by a roman letter), and *Vector* $\mathrm{r}^{-1}\tau$, of *Second Curvature; this radius* r may be *either* positive or *negative* (whereas the radius r of *first* curvature is always treated as *positive*), and its reciprocal r^{-1} may be thus expressed (pp. 92, 88),

$$\textit{Second Curvature}^* = \mathrm{r}^{-1} = \mathrm{S}\frac{d^3\rho}{\mathrm{V}d\rho d^2\rho}, \quad (T), \quad \text{or,} \quad \mathrm{r}^{-1} = \mathrm{S}\frac{\tau''}{\tau\tau'}, \qquad (T')$$

the independent variable being the *arc* in (T'), while it is *arbitrary* in (T): but quaternions

* In this Article, or Series, 397, and indeed also in 396 and 398, several references are given to a very interesting Memoir by M. de Saint-Venant, "*Sur les lignes courbes non planes*": in which, however, that able writer *objects* to such known phrases as *second curvature, torsion*, &c., and proposes in their stead a new name "*cambrure*," which it has not been thought necessary here to adopt. (*Journal de l'École Polytechnique*, Cahier xxx.)

Pages

supply a vast variety of *other expressions* for this important scalar (see, for instance, the *Table* in p. 108). We have also (by p. 89, comp. Arts. 389, 395, 396),

$$\textit{Vector of Spherical Curvature} = \mathrm{s}\mathrm{P}^{-1} = (\rho - \sigma)^{-1} = \&\text{c.}, \tag{U}$$

= *projection of vector* (τ') of (*simple* or *first*) *curvature, on radius* (R) *of osculating sphere*: and if p and P denote the *linear* and *angular elevations*, of the centre (s) of this *sphere* above the osculating *plane*, then (by same page 89),

$$p = r \tan P = R \sin P = r'\mathrm{r} = \mathrm{r}\mathrm{D}_s r. \tag{U'}$$

Again (pp. 89, 90), if we write (comp. Art. 396),

$$\lambda = \mathrm{V}\frac{\tau''}{\tau'} = \mathrm{r}^{-1}\tau + \tau\tau' = \textit{Vector of Second Curvature plus Binormal}, \tag{V}$$

this line λ may be called the *Rectifying Vector*; and if H denote the *inclination* (considered first by Lancret), of this *rectifying line* (λ) to the *tangent* (τ) to the curve, then

$$\tan H = r'^{-1} \tan P = r^{-1}\mathrm{r}. \tag{V'}$$

Known right cone with rectifying line for its *axis*, and with H for its *semiangle*, which *osculates* at P to the *developable locus of tangents* to the curve (or by p. 99 to the *cone of parallels* already mentioned); *new right cone*, with a *new semiangle*, C, connected with H by the relation (p. 91),

$$\tan C = \frac{3}{4} \tan H, \tag{V''}$$

which osculates to the *cone of chords*, drawn from the given point P to other points Q of the given curve. *Other* osculating *cones, cylinders, helix*, and *parabola*; this last being (pp. 91, 96) the parabola which *osculates to the projection of the curve, on its own osculating plane*. *Deviation* of *curve*, at any near point Q, from the osculating *circle* at P, decomposed (p. 96) into *two rectangular deviations*, from osculating *helix* and *parabola*. Additional formulæ (p. 109), for the general theory of *emanants* (Art. 396); case of *normally* emanant *lines*, or of *tangentially* emanant *planes*. General *auxiliary spherical curve* (pp. 110–112, comp. p. 28); new proof of the second expression (V') for tan H, and of the theorem that if this *ratio* of *curvatures* be constant, the proposed curve is a *geodetic* on a *cylinder*: new proof that if *each* curvature (r^{-1}, r^{-1}) be *constant*, the cylinder is *right*, and therefore the curve a *helix*, ... 88, 112

ARTICLE 398.—Properties of a curve in space, depending on the *fourth* and *fifth powers* (s^4, s^5) of its *arc* (s), ... 112–156

This *Series* 398 is so much longer than any other in the Volume, and is supposed to contain so much original matter, that it seems necessary here to subdivide the analysis under several separate heads, lettered as (*a*), (*b*), (*c*), &c.

(*a*). Neglecting s^5, we may write (p. 112, comp. Art. 396),

$$\mathrm{OP}_s = \rho_s = \rho + s\tau + \tfrac{1}{2}s^2\tau' + \tfrac{1}{6}s^3\tau'' + \tfrac{1}{24}s^4\tau'''; \tag{W}$$

or (comp. p. 125),

$$\rho_s = \rho + x_s\tau + y_s r\tau' + z_s r\nu, \tag{W'}$$

with expressions (p. 126) for the *coefficients* (or *coordinates*) x_s, y_s, z_s, in terms of r, r', r'', r, r', and s. If s^5 be taken into account, it becomes necessary to add to the expression (W) the term, $\frac{1}{120}s^5\tau^{\mathrm{IV}}$; with corresponding additions to the scalar coefficients in (W'), introducing r''' and r'': the laws for forming which additional terms, and for extending them to higher powers of the arc, are assigned in a subsequent Series (399, pp. 156, 163).

(*b*). Analogous expressions for τ''', ν'', κ'', λ', σ', and p', R', P', H', to serve in questions in which s^5 is neglected, are assigned (in p. 113); τ'', ν', κ', λ, σ, and p, R, P, H, having been previously expressed (in Series 397); while τ^{IV}, ν''', κ''', λ'', σ'', &c. enter into investigations which take account of s^5: the arc s being treated as the independent variable in all *these* derivations.

(c). One of the chief results of the present Series (398), is the introduction (p. 116, &c.) of a *new auxiliary angle*, J, *analogous* in several respects to the *known angle* H (397), but belonging to a *higher order* of theorems, respecting *curves in space*: because the *new* angle J depends on the *fourth* (and lower) powers of the arc s, while Lancret's angle H depends only on s^3 (including s^1 and s^2). In fact, while tan H is represented by the expressions (V'), whereof one is r'^{-1} tan P, tan J admits (with many transformations) of the following analogous expression (p. 116),

$$\tan J = R'^{-1} \tan P; \tag{X}$$

where R' depends* by (b) on s^4, while r' and P depend (397) on no higher power than s^3.

(d). To give a more distinct *geometrical meaning* to this new angle J, than can be easily gathered from such a formula as (X), respecting which it may be observed, in passing, that J is in general more simply defined by expressions for its *cotangent* (pp. 116, 126), than for its tangent, we are to conceive that, at each point P of any proposed curve of double curvature, there is drawn a *tangent plane* to the *sphere*, which *osculates* (395) to the curve at that point; and that then the *envelope* of all these *planes* is determined, which envelope (for reasons afterwards more fully explained) is called here (p. 116) the "*Circumscribed Developable*": being a surface *analogous* to the "*Rectifying Developable*" of Lancret, but belonging (c) to a *higher order* of questions. And then, as the *known* angle H denotes (397) the *inclination*, suitably measured, of the *rectifying line* (λ), which is a *generatrix* of the *rectifying* developable, to the *tangent* (τ) to the *curve*; so the *new* angle J represents the inclination of a *generating line* (ϕ), of what has just been called the *circumscribed* developable, to the *same* tangent (τ), measured likewise in a defined direction (p. 117), but in the tangent plane to the sphere. It may be noted as another analogy (p. 117), that while H is a *right angle* for a *plane curve*, so J is *right* when the curve is *spherical*. For the *helix* (p. 122), the angles H and J are *equal*; and the rectifying and circumscribed *developables coincide*, with each other and with the *right cylinder*, on which the helix is a geodetic line.

(e). If the recent line ϕ be measured from the given point P, in a suitable direction (as contrasted with the opposite), and with a suitable length, it becomes what may be called (comp. 396) the *Vector of Rotation of the Tangent Plane* (d) *to the Osculating Sphere*; and then it satisfies, among others, the equations (pp. 114, 116, comp. (V)),

$$\phi = \mathrm{V}\frac{\nu''}{\nu'}, \quad \mathrm{T}\phi = R^{-1}\operatorname{cosec} J; \tag{X'}$$

this last being an expression for the *velocity of rotation* of the *plane* just mentioned, or of its *normal*, namely the spherical *radius* R, if the *given curve* be conceived to be described by a point moving with a *constant velocity*, assumed = 1. And if we denote by V the point in which the *given radius* R or PS is *nearest* to a *consecutive radius* of the same kind, or to the radius of a *consecutive osculating sphere*, then this point V *divides the line* PS *internally*, into *segments* which may (ultimately) be thus expressed (pp. 115, 116),

$$\overline{\mathrm{PV}} = R \sin^2 J, \quad \overline{\mathrm{VS}} = R\cos^2 J. \tag{X''}$$

But these and other connected results, depending on s^4, have their known *analogues* (with H for J, and r for R), in that earlier theory (c) which introduces only s^3 (besides s^1 and s^2): and they are all *included* in the general theory of *emanant* lines and planes (396, 397), of which some new geometrical illustrations (pp. 117, 120) are here given.

* In other words, the calculation of r' and P introduces no differentials higher than the *third order*; but that of R' requires the *fourth order* of differentials. In the language of modern geometry, the *former* can be determined by the consideration of *four consecutive points of the curve*, or by that of *two consecutive osculating circles*; but the *latter* requires the consideration of *two* consecutive *osculating spheres*, and therefore of *five* consecutive *points* of the curve (supposed to be one of double curvature). *Other* investigations, in the present and immediately following Series (398, 399), especially those connected with what we shall shortly call the *Osculating Twisted Cubic*, will be found to involve the consideration of *six* consecutive points of a curve.

Pages

(*f*). New auxiliary scalar $n (= p^{-1}RR' = \cot J \sec P = \&c.)$, = *velocity of centre* s of *osculating sphere*, if the velocity of the point P of the *given curve* be taken as *unity* (*e*); n vanishes with R', cot J, and (comp. 395) the coefficient $S - 1$ (= $n r \text{T}^{-1}$) of non-sphericity, for the case of a spherical curve (p. 120). Arcs, first and second curvatures, and rectifying planes and lines, of the cusp-edges of the polar and rectifying* developables; these can *all* be expressed without going beyond s^5, and *some* without using any higher power than s^4, or differentials of the orders corresponding; $r_1 = n\text{r}$, and $\text{r}_1 = nr$, are the *scalar radii* of first and second curvature of the *former* cusp-edge, r_1 being *positive* when *that* curve turns its *concavity* at s towards the *given* curve at P: determination of the point R, in which the *latter* cusp-edge is touched by the rectifying line λ to the original curve (pp. 120, 125).

(*g*). Equation with one arbitrary constant (p. 125), of a *cone of the second order*, which has its *vertex* at the given point P, and has *contact* of the *third order* (or *four-side contact*) with the *cone of chords* (397) from that point; equation (p. 128) of a *cylinder* of the second order, which has an arbitrary line PE from P as one *side*, and has contact of the *fourth order* (or *five-point contact*) with the *curve* at P; the constant above mentioned can be so determined, that the right line PE shall be a *side* of the *cone also*, and therefore a *part* of the *intersection* of *cone* and *cylinder*; and then the *remaining* or *curvilinear part*, of the *complete intersection* of those two surfaces of the second order, is (by known principles) a *gauche curve of the third order*, or what is briefly called† a *Twisted Cubic*: and *this last curve*, in virtue of its construction above described, and whatever the assumed direction of the auxiliary line PE may be, has *contact of the fourth order* (or *five-point contact*) *with the given curve* of double curvature at P (pp. 125, 129, comp. pp. 92, 104).

(*h*). Determination (p. 129) of the *constant* in the equation of the *cone* (*g*), so that *this* cone may have *contact* of the *fourth order* (or *five-side contact*) with the *cone of chords* from P; the cone thus found may be called the *Osculating Oblique Cone* (comp. 397), of the second order, to that cone of chords; and the coefficients of its equation involve only r, r, r', r$'$, r'', r$''$, but *not* r''', although this last derivative is of no higher order than r$''$, since each depends only on s^5 (and lower powers), or introduces only *fifth differentials*. Again, the *cylinder* (*g*) will have contact of the *fifth order* (or *six-point contact*) with the given *curve* at P, if the line PE, which is by construction a *side* of that cylinder, and has hitherto had an *arbitrary direction*, be *now* obliged to be a side of a certain *cubic cone*, of which the equation (p. 128) involves as constants not only rrr'r$'r''$r$''$, like that of the *osculating cone* just determined, but also r'''. The *two cones* last mentioned have the tangent (τ) to the given curve for a *common side*,‡ but they have also *three other* common sides, whereof *one*

* The *rectifying plane*, of the cusp-edge of the rectifying developable, is the plane of λ and τ', of which the formula LIV'. in p. 124 is the equation; and the *rectifying line* RH, of the same cusp-edge, *intersects the absolute normal* PK to the given curve, or the *radius* (r) of first curvature, in the point H in which that radius is *nearest* (*e*) to a *consecutive radius* of the same kind. But this last theorem, which is here deduced by quaternions, had been previously arrived at by M. de Saint-Venant (comp. the Note to p. viii), through an entirely different analysis, confirmed by geometrical considerations.

† By Dr. Salmon, in his excellent Treatise on *Analytic Geometry of Three Dimensions* (Dublin, 1862), which is several times cited in the Notes to this final Chapter (III. iii.) of these *Elements*. The gauche curves, above mentioned, have been studied with much success, of late years, by M. Chasles, Sig. Cremona, and other geometers: but their existence, and some of their leading properties, appear to have been first perceived and published by Prof. Möbius (see his *Barycentric Calculus*. Leipzig, 1827, pp. 114–122, especially p. 117).

‡ *This side*, however, *counts as three* (p. 159), in the system of the *six lines of intersection* (real or imaginary) of these *two cones*, which have a *common vertex* P, and are respectively of the *second* and *third orders* (or degrees). Additional light will be thrown on this whole subject, in the following Series (399); in which also it will be shown that there is only *one* osculating twisted cubic, at a given point, to a given curve of double curvature; and that *this cubic curve can be determined, without resolving any cubic* or other *equation*.

Pages

at least is *real*, since they are assigned by a *cubic equation* (p. 129); and by taking *this side* for the line PE in (*g*), there results a *new cylinder* of the *second order*, which *cuts the osculating oblique cone*, *partly* in that *right line* PE itself, and *partly* in a *gauche curve* of the *third order*, which it is proposed to call an *Osculating Twisted Cubic* (comp. again (*g*)), because it has *contact of the fifth order* (or *six-point contact*) *with the given curve* at P (p. 129).

(*i*). *In general*, and independently of any question of *osculation*, a *Twisted Cubic* (*g*), if passing through the *origin* O, may be represented by any one of the *vector equations* (pp. 131, 132),

$$V\alpha\rho + V\rho\phi\rho = 0, \quad (Y); \quad \text{or} \quad (\phi + c)\rho = \alpha, \quad (Y')$$

$$\text{or} \quad \rho = (\phi + c)^{-1}\alpha, \quad (Y''); \quad \text{or} \quad V\alpha\rho + \rho V\gamma\rho + V\rho V\lambda\rho\mu = 0, \quad (Y''')$$

in which α, γ, λ, μ are *real* and *constant vectors*, but c is a *variable scalar*; while $\phi\rho$ denotes (comp. the Section III. ii. 6, or p. xxxii, vol. i., a *linear and vector function*, which is *here* generally *not self-conjugate*, of the *variable vector* ρ of the *cubic curve*. The *number* of the *scalar constants*, in the form (Y'''), or in any other form of the equation, is found to be *ten* (p. 132), with the foregoing supposition that the curve passes *through the origin*, a restriction which it is easy to remove. The curve (Y) is *cut*, as it ought to be, in *three points* (real or imaginary), by an *arbitrary secant plane*; and its three *asymptotes* (real or imaginary) have the *directions* of the *three vector roots* β (see again the last cited Section) of the equation (same p. 131),

$$V\beta\phi\beta = 0: \quad (Z)$$

so that by (P), p. xxxii, vol. i., *these three asymptotes compose a real and rectangular system*, for the *case of self-conjugation* of the function ϕ in (Y).

(*j*). *Deviation* of a near point P_s of the given curve, from the *sphere* (395) which *osculates* at the given point P; this deviation (by p. 132, comp. pp. 79, 120) is

$$\overline{SP_s} - \overline{SP} = \frac{r_1 s^4}{24 r r^2 R} = \frac{R' s^4}{24 r r p} = \frac{n s^4}{24 r r R} = \&c.; \quad (A_1)$$

it is ultimately equal (p. 134) to the *quarter* of the deviation (397) of the same near point P_s from the *osculating circle* at P, multiplied by the *sine* of the *small angle* SPS_s, which the *small arc* SS_s of the *locus* of the *spheric centre* S (or of the *cusp-edge* of the polar developable) *subtends* at the same point P; and it has an *outward* or an *inward* direction, according as this *last arc* is *concave* or *convex* (*f*) at S, towards the *given curve* at P (pp. 122, 134). It is also ultimately equal (p. 136) to the deviation $\overline{PS_s} - \overline{P_sS_s}$, of the *given point* P from the *near sphere*, which *osculates* at the *near point* P_s; and likewise (p. 137) to the *component*, in the direction of SP, of the deviation of that near point from the osculating *circle* at P, measured in a direction parallel to the normal plane at that point, if this *last* deviation be *now* expressed to the accuracy of the *fourth order*: whereas it has *hitherto* been considered sufficient to develope this *deviation from the osculating circle* (397) as far as the *third order* (or third dimension of s); and therefore to treat it as having a direction, *tangential to the osculating* sphere (comp. pp. 97, 133).

(*k*). The deviation (A_1) is also equal to the *third part* (p. 138) of the *deviation* of the *near point* P_s from the *given circle* (which osculates at P), if measured *in the near normal plane* (at P_s), and *decomposed* in the direction of the *radius* R_s of the *near sphere*; or to the *third part* (with direction preserved) of the deviation of the *new near point* in which the given circle is *cut* by the near plane, *from the near sphere*: or finally to the *third part* (as before, and still with an unchanged direction) of the deviation from the *given sphere*, of that *other new point* C, in which the *near circle* (osculating at P_s) is cut by the *given normal plane* (at P), and which is found to satisfy the equation,

$$\overline{SC} = 3\overline{SP_s} - 2\overline{SP}. \quad (B_1)$$

Geometrical connexions (p. 140) between these various results (*j*) (*k*), illustrated by a diagram (fig. 83).

(*l*). The *Surface*, which is the *Locus of the Osculating Circle* to a given curve in space, may be represented rigorously by the *vector expression* (p. 141),

$$\omega_{s,\,u} = \rho_s + r_s\tau_s \sin u + r_s{}^2\tau_s{}' \operatorname{vers} u; \tag{C₁}$$

in which s and u are two independent scalar variables, whereof s is (as before) the *arc* PP_s *of the given curve*, but is *not now treated as small:* and u is the (small or large) *angle subtended at the centre* K_s of the circle, by the *arc of that circle*, measured from its *point of osculation* P_s. But the *same superficial locus* (comp. 392) may be represented also by the *vector equation* (p. 156) involving *apparently* only *one* scalar variable (s),

$$\mathrm{V}\frac{2\tau_s}{\omega - \rho_s} + \nu_s = 0, \tag{D₁}$$

in which $\nu_s = \tau_s\tau'_s$, and $\omega = \omega_{s,\,u} =$ the vector of an arbitrary point of the surface. The general method (p. 11) of the Section III. iii. 3, shows that the *normal* to this *surface* (C_1), at any proposed point thereof, has the direction of $\omega_{s,\,u} - \sigma_s$; that is (p. 141), the direction of the *radius* of the *sphere*, which contains the *circle* through that point, and has the *same point of osculation* P_s to the given curve. The *locus of the osculating circle* is therefore found, by this little calculation with quaternions, to be at the same time the *Envelope of the Osculating Sphere*, as was to be expected from geometrical considerations (comp. the Note to p. 141).

(*m*). The *curvilinear locus of the point* C in (*k*) is *one branch* of the *section of the surface* (*l*), made by the normal plane to the given curve at P; and if D be the projection of C on the tangent at P to this new curve, which tangent PD has a direction *perpendicular* to the radius PS or R of the osculating sphere at P (see again fig. 83, in p. 140), while the ordinate DC is *parallel* to that radius, then (attending only to principal terms), pp. 139, 140) we have the expressions,

$$\mathrm{PD} = \frac{Rs^3}{6r^2\mathrm{r}}\,\mathrm{U}\tau(\sigma - \rho), \quad \mathrm{DC} = \frac{-ns^4}{8r\mathrm{T}R}\,\mathrm{U}(\sigma - \rho), \tag{E₁}$$

and therefore ultimately (p. 141),

$$\frac{\mathrm{DC}^3}{\mathrm{PD}^4} = \frac{81}{32} \cdot \frac{n^3r^5\mathrm{r}(\sigma - \rho)}{R^7} = \text{const.}; \tag{F₁}$$

from which it follows that P is a *singular point* of the *section* here considered, but *not a cusp* of that section, although the *curvature* at P is *infinite:* the *ordinate* DC varying ultimately as the *power* with exponent $\frac{4}{3}$ of the *abscissa* PD. *Contrast* (pp. 141, 142), of this section, with that of the developable *Locus of Tangents*, made by the same normal plane at P to the given curve; the vectors analogous to PD and DC are in *this* case nearly equal to $-\frac{1}{2}s^2\tau'$ and $-\frac{1}{3}s^3\mathrm{r}^{-1}\nu$; so that the latter varies ultimately as the power $\frac{3}{2}$ of the former, and the point P *is* (as it is known to be) a *cusp* of this last section.

(*n*). A given *Curve* of double curvature is therefore *generally* a *Singular Line* (p. 143), although *not a cusp-edge*, upon that *Surface* (*l*), which is at once the *Locus* of its osculating *Circle*, and the *Envelope* of its osculating *Sphere:* and the *new developable surface* (*d*), as being *circumscribed to this superficial locus* (or *envelope*), so as to *touch it along this singular line* (p. 156), may naturally be called, as above, the *Circumscribed Developable* (p. 110).

(*o*). Additional light may be thrown on this whole theory of the *singular line* (*n*), by considering (pp. 143–155) a problem which was discussed by Monge, in two distinct Sections (xxii. xxvi.) of his well-known *Analyse* (comp. the Notes to pp. 144, 145, 153, 154, 155 of these *Elements*); namely, to determine the *envelope of a sphere with varying radius* R, whereof the *centre* S traverses a *given curve in space;* or briefly, to find the

Envelope of a Sphere with One varying Parameter (comp. p. 171): especially for the *Case of Coincidence* (p. 145, &c.), of what are *usually two distinct branches* (p. 144) of a certain *Characteristic Curve* (or *arête de rebroussement*), namely the *curvilinear envelope* (real or imaginary) of all the *circles*, along which the *superficial envelope* of the *spheres* is *touched* by those spheres themselves.

(*p*). Quaternion forms (pp. 145, 146) of the *condition of coincidence* (*o*); one of these can be at once translated into Monge's *equation of condition* (p. 145), or into an equation slightly more general, as leaving the independent variable arbitrary; but a simpler and more easily interpretable form is the following (p. 146),

$$r_1 dr = \pm R dR, \tag{G_1}$$

in which r is the radius of the *circle of contact*, of a sphere with its envelope (*o*), while r_1 is the radius of (first) *curvature* of the *curve* (s), which is the *locus* of the *centre* s of the sphere.

(*q*). The *singular line* into which the *two branches* of the curvilinear envelope are *fused*, when this *condition* is satisfied, is in general an *orthogonal trajectory* (p. 151) to the *osculating planes* of the *curve* (s); *that* curve, which is *now* the *given* one, is therefore (comp. 391, 395) the *cusp-edge* (p. 151) of the *polar developable*, corresponding to the singular line just mentioned, or to what may be called the *curve* (P), which was *formerly* the given curve. In this way there arise many verifications of formulæ (pp. 151, 152); for example, the equation (G_1) is easily shown to be consistent with the results of (*f*).

(*r*). With the geometrical hints thus gained from interpretation of quaternion results, there is now no difficulty in assigning the *Complete and General Integral* of the *Equation of Condition* (*p*), which was presented by Monge under the form (comp. p. 145) of a *non-linear differential equation of the second order*, involving *three variables* (ϕ, ψ, π) considered as *functions of a fourth* (*a*), namely the *coordinates* of the *centre* of the *sphere*, regarded as *varying with the radius*, but which does not appear to have been either *integrated* or *interpreted* by that illustrious analyst. The general integral *here* found presents itself at first in a *quaternion form* (p. 153), but is easily *translated* (p. 154) into the usual language of analysis. A *less general integral* is also assigned, and its geometrical signification exhibited, as answering to a case for which the *singular line* lately considered reduces itself to a *singular point* (p. 155).

(*s*). Among the *verifications* (*q*) of this whole theory, it is shown (pp. 152, 153) that although, when the *two branches* (*o*) of the general *curvilinear envelope* of the *circles* of the system are *real* and *distinct*, *each branch* is a *cusp-edge* (or *arête de rebroussement*, as Monge perceived it to be), upon the *superficial envelope* of the *spheres*, yet in the *case of fusion* (*p*) this *cuspidal character* is *lost* (as was likewise seen by Monge*): and that *then* a *section of the surface*, made by a *normal plane* to the *singular line*, has precisely the *form* (*m*), expressed by the equation (F_1). In short, the result is in many ways confirmed, by calculation and by geometry, that when the *condition of coincidence* (*p*) is satisfied, the *Surface* is, as in (*n*), at once the *Envelope of the osculating Sphere* and the *Locus of the osculating Circle*, to that *Singular Line* on itself, into which by (*q*) the *two branches* (*o*) of its *general cusp-edge* are *fused*.

(*t*). Other applications of preceding formulæ might be given; for instance, the formula for κ'' enables us to assign general expressions (p. 155) for the *centre* and *radius* of the *circle*, which *osculates* at K to the *locus of the centre of the osculating circle*, to a given *curve in space*: with an elementary verification, for the case of the *plane evolute of the plane evolute of a plane curve*. But it is time to conclude this long analysis, which however could scarcely have been much abridged, of the results of Series 398, and to pass to a more brief account of the investigations in the following Series.

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* Compare the first Note to p. 153 of these *Elements*.

Pages

ARTICLE 399.—Additional general investigations, respecting that *gauche curve* of the *third order* (or degree), which has been above called an *Osculating Twisted Cubic* (398, (*h*)), to any proposed curve of double curvature; with applications to the case, where the given curve is a *helix*, ... 156, 167

(*a*). In general (p. 159) the *tangent* PT to the given curve is a *nodal side* of the *cubic cone* (398, (*h*)); *one tangent plane* to that cone (C_3), along that side, being the *osculating plane* (P) to the curve, and therefore touching also, along the same side, the *osculating oblique cone* (C_2) of the *second order*, to the *cone of chords* (397) from P; while the *other* tangent plane to the cubic cone (C_3) *crosses* that *first* plane (P), or the *quadric cone* (C_2), at an angle of which the trigonometric *cotangent* ($\frac{1}{2}$r′) is equal to *half the differential of the radius* (r) *of second curvature, divided by the differential of the arc* (*s*). And the *three common sides*, PE, PE′, PE″, of these *two cones*, which remain when the *tangent* PT is *excluded*, and of which *one* at least must be *real*, are the *parallels* through the given point P to the *three asymptotes* (398, (*i*)) to the gauche curve sought; being also *sides* of *three quadric cylinders*, say (L_2), (L'_2), (L''_2), which contain those asymptotes as *other sides* (or generating lines): and of which *each contains the twisted cubic* sought, and is *cut in it* by the *quadric cone* (C_2).

(*b*). On applying this *First Method* to the case of a given *helix*, it is found (p. 159) that the *general cubic cone* (C_3) breaks up into the system of a *new quadric cone*, (C_2'), and a *new plane* (P'); which latter is the *rectifying plane* (396) of the helix, or the *tangent plane* at P to the *right cylinder*, whereon that given curve is traced. The two quadric cones, (C_2) and (C_2') *touch each other* and the plane (P) along the tangent PT, and have *no other real common side*: whence *two* of the sought *asymptotes*, and *two* of the corresponding *cylinders* (*a*), are in this case *imaginary*, although they can still be used in *calculation* (pp. 159, 160, 162). But the *plane* (P') *cuts* the *cone* (C_2), not only in the tangent PT, but also in a *second real side* PE, to which the *real asymptote* is *parallel* (*a*); and which is at the same time a *side* of a *real quadric cylinder* (L_2) which has that asymptote for *another side* (p. 162), and *contains the twisted cubic*: this gauche curve being thus the *curvilinear part* (p. 161) of the *intersection* of the *real cone* (C_2), with the *real cylinder* (L_2).

(*c*). Transformations and verifications of this result; *fractional expressions* (p. 162), for the coordinates of the twisted cubic; expression (p. 161) for the *deviation* of the *helix* from that *osculating curve*, which deviation is directed *inwards*, and is of the *sixth order*: the *least* distance, between the *tangent* PT and the *real asymptote*, is a right line PB, which is *cut internally* (p. 162) by the *axis* of the *right cylinder* (*b*), in a point A such that PA is to AB as *three* to *seven*.

(*d*). The *First Method* (*a*), which has been established in the preceding Series (398), *succeeds* then for the case of the *helix*, with a facility which arises chiefly from the circumstance (*b*), that for *this case* the *general cubic cone* (C_3) breaks up into *two* separate loci, whereof *one* is a *plane* (P'). But *usually* the foregoing method requires, as in (398, (*h*)), the solution of a *cubic equation*: an inconvenience which is completely avoided, by the employment of a *Second General Method*, as follows.

(*e*). This *Second Method* consists in taking, for a *second locus* of the *gauche osculatrix* sought, a certain *Cubic Surface* (S_3), of which every point is the *vertex** of a quadric *cone*,

* It is known that the *locus of the vertex* of a *quadric cone*, which passes through *six given points of space*, A, B, C, D, E, F, whereof no four are in one plane, is generally a *Surface*, say (S_4), of the *Fourth Degree*: in fact, it is *cut* by the plane of the triangle ABC in a system of *four right lines*, whereof three are the sides of that triangle, and the fourth is the intersection of the two planes, ABC and DEF. If then we investigate the intersection of this surface (S_4) with the *quadric cone*, (A . BCDEF), or say (C_2), which has A for vertex, and passes through the five other given points, we might expect to find (in *some sense*) a curve of the *eighth degree*. But when we set aside the *five right lines*, AB, AC, AD, AE, AF, which are *common* to the two surfaces here considered, we find that the (remaining or) *curvilinear part* of the *complete intersection* is reduced to a *curve of the third degree*,

Pages

having *six-point contact* with the given curve at P: so that this *new surface* is cut by the *plane at infinity*, in the *same cubic curve* as the *cubic cone* (C_3). It is found (p. 166) to be a *Ruled Surface*, with the tangent PT for a *Singular Line*; and when this *right line* is *set aside*, the *remaining* (that is, the *curvilinear*) *part* of the intersection of the *two loci*, (C_2) and (S_3), is the *Osculating Twisted Cubic* sought: which *gauche osculatrix* is thus *completely* and *generally determined*, without any such difficulty or *apparent variety*, as might be supposed to attend the solution of a *cubic equation* (*d*), and with new verifications for the case of the *helix* (p. 167).

ARTICLE 400.—On Involutes and Evolutes in space, ... 167, 173

(*a*). The usual points of Monge's theory are deduced from the two fundamental quaternion equations (p. 168),

$$S(\sigma - \rho)\rho' = 0, \quad V(\sigma - \rho)\sigma' = 0, \qquad (H_1)$$

in which ρ and σ are corresponding vectors of involute and evolute; together with a theorem of Prof. De Morgan (p. 169), respecting the case when the evolute is a spherical curve.

(*b*). An *involute in space* is generally the *only real part* (p. 171) of the *envelope* of a certain variable *sphere* (comp. 398), which has its *centre* on the *evolute*, while its *radius R* is the variable *intercept* between the two curves: but because we have *here* the relation (p. 169, comp. p. 143),

$$R'^2 + \sigma'^2 = 0, \qquad (H_1')$$

the *circles of contact* (398, (*o*)) reduce themselves each to a *point* (or rather to a *pair of imaginary right lines*, intersecting in a real point), and the preceding theory (398), of envelopes of spheres with one varying parameter, undergoes important modifications in its results, the conditions of the applications being different. In particular, the *involute* is indeed, as the equation (H_1) express, an *orthogonal trajectory* to the *tangents* of the *evolute*; but *not* to the *osculating planes* of that curve, as the *singular line* (398, (*q*)) of the former envelope *was*, to those of the curve which was the *locus* of the *centres* of the *spheres* before considered, when a certain *condition of coincidence* or of *fusion*, 398, (*p*)) was satisfied.

(*c*). *Curvature of hodograph of evolute* (p. 173): if P, P_1, P_2, . . and S, S_1, S_2, . . be *corresponding points* of involute and evolute, and if we draw right lines ST_1, ST_2, . . in the directions of S_1P_1, S_2P_2, . . and with a common length = $\overline{SP}$, the *spherical curve* PT_1T_2 . . will have *contact of the second order* at P, *with the involute* PP_1P_2 . . (p. 173).

ARTICLE 401.—Calculations abridged, by the treatment of *quaternion differentials* (which have hitherto been *finite*, comp. p. xxix, vol. i.) as *infinitesimals*;* new deductions of *osculating plane*, *circle*, and *sphere*, with the *vector equation* (392) of the circle; and of the first and second *curvature* of a *curve in space*, ... 173, 179

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which is precisely the *twisted cubic through the six given points*. In applying this general (and perhaps new) method, to the problem of the *osculating* twisted cubic to a *curve*, the osculating *plane* to that curve may be *excluded*, as foreign to the question: and then the *quartic surface* (S_4) is reduced to the *cubic surface* (S_3), above described.

* Although, for the sake of brevity, and even of clearness, some *phrases* have been used in the foregoing analysis of the Series 398 and 399, such as *four-side* or *five-side contact* between *cones*, and *five-point* or *six-point* contact between *curves*, or between a *curve* and a *surface*, which are borrowed from the doctrine of *consecutive points* and *lines*, and therefore from that of *infinitesimals*; with a few other expressions of modern geometry, such as the *plane at infinity*, &c.; yet the *reasonings* in the *text* of these *Elements* have all been rigorously *reduced*, so far, or are all obviously reducible, to the fundamental conception of *Limits*; compare the *definitions* of the *osculating circle* and *sphere*, assigned in Articles 389, 395. The object of Art. 401 is to make it visible how, without abandoning such *ultimate reference to limits*, it is possible to *abridge calculation*, in several cases, by treating (at this stage) the *differential symbols*, $d\rho$, $d^2\rho$, &c., *as if they represented infinitely small differences*, $\Delta\rho$, $\Delta^2\rho$, &c.; without taking the trouble to *write* these latter symbols *first*, as denoting *finite differences*, in the *rigorous statement* of a problem, of which statement it is not always

Pages

SECTION 7.—On Surfaces of the Second Order; and on Curvatures of Surfaces, . . . , ... 179–283

ARTICLE 402.—References to some equations of *Surfaces*, in earlier parts of the volume, ... 179, 180

ARTICLES 403.—Quaternion equations of the *Sphere* ($\rho^2 = -1$, &c.), ... 180, 182

In some of these equations, the Notation N for *norm* is employed (comp. the Section II. i. 6.)

ARTICLE 404.—Quaternion equations of the *Ellipsoid*, ... 183, 185

One of the simplest of these forms is (pp. 325, vol. i., 185) the equation,

$$\mathrm{T}(\iota\rho + \rho\kappa) = \kappa^2 - \iota^2, \qquad (\mathrm{I}_1)$$

in which ι and κ are real and constant vectors, in the directions of the *cyclic normals.* This *form* (I_1) is intimately connected with, and indeed served to suggest, that *Construction of the Ellipsoid* (II. i. 13), by means of a *Diacentric Sphere* and a *Point* (p. 234, vol. i., comp. fig. 53, pp. 234, vol. i., and 184), which was among the *earliest geometrical results* of the Quaternions. The *three semiaxes, a, b, c,* are expressed (comp. p. 238) interms of ι, κ as follows:

$$a = \mathrm{T}\iota + \mathrm{T}\kappa; \quad b = \frac{\kappa^2 - \iota^2}{\mathrm{T}(\iota - \kappa)}; \quad c = \mathrm{T}\iota - \mathrm{T}\kappa; \qquad (\mathrm{I}_1')$$

whence

$$ab^{-1}c = \mathrm{T}(\iota - \kappa). \qquad (\mathrm{I}_1'')$$

ARTICLE 405.—*General Central Surface* of the *Second Order* (or central quadric), $\mathrm{S}\rho\phi\rho = f\rho = 1$, ... 186–189

ARTICLE 406.—*General Cone* of the *Second Order* (or quadric cone), $\mathrm{S}\rho\phi\rho = f\rho = 0$, . 189–196

ARTICLE 407.—*Bifocal Form* of the equation of a *central* but *non-conical surface* of the second order: with some quaternion formulæ, relating to *Confocal Surfaces*, . . . 196–208

(*a*). The *bifocal form* here adopted (comp. the Section III. ii. 6) is the equation,

$$Cf\rho = (\mathrm{S}\alpha\rho)^2 - 2e\mathrm{S}\alpha\rho\mathrm{S}\alpha'\rho + (\mathrm{S}\alpha'\rho)^2 + (1 - e^2)\rho^2 = C, \qquad (\mathrm{J}_1)$$

in which,

$$C = (e^2 - 1)(e + \mathrm{S}\alpha\alpha')l^2. \qquad (\mathrm{J}_1')$$

α, α' are two (real) *focal unit-lines*, common to the whole *system* of confocals; the (real and positive) scalar l is also *constant* for that system: but the scalar *e varies*, in passing from surface to surface, and may be regarded as a *parameter*, of which the value serves to *distinguish* one confocal, say (e), from another (pp. 196, 197).

(*b*). The *squares* (p. 197) of the three *scalar semiaxes* (real or imaginary), arranged in algebraically descending order, are

$$a^2 = (e+1)l^2, \quad b^2 = (e + \mathrm{S}\alpha\alpha')l^2, \quad c^2 = (e-1)l^2; \qquad (\mathrm{K}_1)$$

whence

$$l^2 = \frac{a^2 - c^2}{2}, \quad e = \frac{a^2 + c^2}{a^2 - c^2}; \qquad (\mathrm{L}_1)$$

and the three *vector semiaxes* corresponding are,

$$a\mathrm{U}(\alpha + \alpha'), \quad b\mathrm{UV}\alpha\alpha', \quad c\mathrm{U}(\alpha - \alpha'). \qquad (\mathrm{M}_1)$$

(*c*). *Rectangular, unifocal,* and *cyclic forms* (pp. 197, 203, 205) of the scalar function $f\rho$, to each of which corresponds a form of the vector function $\phi\rho$; deduction, by a new

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easy to assign the proper form, for the case of points, &c., at *finite distances*: and *then* having the additional trouble of *reducing* the complex expressions so found to *simpler forms*, in which *differentials* shall finally appear. In short, it is shown that in *Quaternions*, as in other parts of Analysis, the *rigour of limits* can be combined with the *facility of infinitesimals.*

Pages

analysis, of several known theorems* (pp. 197, 198, 202, 208) respecting *confocal surfaces* and their *focal conics;* the lines α, α' are *asymptotes* to the *focal hyperbola* (p. 202), whatever the *species* of the surface may be: references (in Notes to pp. 203, 204) to the *Lectures,*† for the *focal ellipse* of the *Ellipsoid,* and for several different *generations* of this last surface.

(*d*). *General Exponential Transformation* (p. 206) of the equation of *any central quadric;*

$$\rho = x\alpha + y\mathrm{V}\alpha^t\beta, \quad (\mathrm{N}_1), \quad \text{with } x^2f\alpha + y^2f\mathrm{UV}\alpha\alpha' = 1, \qquad (\mathrm{N}_1')$$

and

$$\beta = \frac{(\alpha' - e\alpha)\,\mathrm{UV}\alpha\alpha'}{e + \mathrm{S}\alpha\alpha'}; \qquad (\mathrm{N}_1'')$$

this *auxiliary vector* β is *constant,* for any *one* confocal (e); the *exponent,* t, in (N_1), is an *arbitrary* or *variable scalar;* and the *coefficients,* x and y, are *two other* scalar variables, which are however *connected* with each other by the relation (N_1').

(*e*). If any *fixed value* be assigned to t, the equation (N_1) then represents the *section* made by a plane through α (p. 207) which section is an *ellipse* if the surface be an *ellipsoid,* but an *hyperbola* for *either hyperboloid;* and the *cutting plane* makes with the *focal plane* of α, α', or with the plane of the focal hyperbola, an angle $= \frac{1}{2}t\pi$.

(*f*). If, on the other hand, we allow t to *vary,* but assign to x and y any *constant values* consistent with (N_1'), the equation (N_1) then represents an *ellipse* (p. 206) whatever the *species* of the *surface* may be; x represents the *distance of its centre* O of the *surface,* measured along the *focal line* α; y is the *radius* of a *right cylinder,* with α for its *axis,* of which the ellipse is a *section,* or the radius of a *circle* in a plane perpendicular to α, into which that ellipse can be orthogonally *projected*: and the angle $\frac{1}{2}t\pi$ is now the *excentric anomaly. Such elliptic sections* of a central quadric may be otherwise obtained from the *unifocal form* (*c*) of the equation of the surface; they are, in some points of view, almost as interesting as the known *circular sections*: and it is proposed (p. 204) to call them *Centro-Focal Ellipses.*

(*g*). And it is obvious that, by interchanging the *two focal lines* α, α' in (*d*) a *Second Exponential Transformation* is obtained, with a *Second System* of *centro-focal ellipses,* whereof the proposed *surface* is the *locus,* as well as of the *first system* (*f*), but which have their *centres* on the line α', and are projected into *circles,* on a plane perpendicular to this *latter line* (p. 203).

(*h*). *Equation of Confocals* (p. 207).

$$\mathrm{V}\nu_{\prime}\phi\nu_{\prime} = \mathrm{V}\nu\phi_{\prime}\nu. \qquad (\mathrm{O}_1)$$

ARTICLE 408.—On Circumscribed Quadric Cones; and on the Umbilics of a central quadric, ... 209–224

(*a*). Equations (p. 209) of *Conjugate Points,* and of *Conjugate Directions,* with respect to the surface $f\rho = 1$,

$$f(\rho, \rho') = 1, \quad (\mathrm{P}_1), \quad \text{and} \quad f(\rho, \rho') = 0; \qquad (\mathrm{P}_1')$$

Condition of Contact, of the same surface with the *right line* PP',

$$(f(\rho, \rho') - 1)^2 = (f\rho - 1)\,(f\rho' - 1); \qquad (\mathrm{Q}_1)$$

this latter is also a form of the equation of the *Cone,* with vertex at P', which is *circumscribed* to the same quadric ($f\rho = 1$).

* For example, it is proved by quaternions (p. 208), that the *focal lines* of the *focal cone,* which has any proposed point P for vertex, and rests on the focal hyperbola, are *generating lines* of the *single-sheeted hyperboloid* (of the given confocal system), which passes through that point: and an *extension* of this result, to the focal lines of *any cone circumscribed to a confocal,* is deduced by a similar analysis, in a subsequent Series (408, p. 213). But such known theorems respecting confocals can only be alluded to, in those Contents.

† *Lectures on Quaternions* (by the present author), Dublin, Hodges and Smith, 1853.

Pages

(*b*). The condition (Q_1) may also be thus transformed (p. 211),

$$F\mathrm{V}\rho\rho' = a^2b^2c^2 f(\rho - \rho'), \qquad (Q_1')$$

F being a scalar function, connected with f by certain relations of *reciprocity* (comp. p. 547, vol. i.); and a simple *geometrical interpretation* may be assigned, for this last equation.

(*c*). The *Reciprocal Cone*, or *Cone of Normals* σ at P'_1 to the circumscribed cone (Q_1) or (Q_1'), may be represented (p. 212) by the very simple equation,

$$F(\sigma : \mathrm{S}\rho'\sigma) = 1; \qquad (Q_1'')$$

which likewise admits of an extremely simple interpretation.

(*d*). A given *right line* (p. 214) is *touched* by *two confocals*, and other known results are easy consequences of the present analysis; for example (pp. 216, 217), the cone circumscribed to any surface of the system, from any point of *either* of the *two real focal curves*, is a *cone of revolution* (real or imaginary): but a similar conclusion holds good, when the *vertex* is on the *third* (or *imaginary*) *focal*, and even more generally (p. 223), when that vertex is *any point* of the (known and imaginary) *developable envelope* of the *confocal system*.

(*e*). A central quadric has in general *Twelve Umbilics* (p. 218), whereof only *four* (at most) can be *real*, and which are its *intersections* with the *three focal curves*: and these *twelve points* are ranged, *three by three*, on *eight imaginary right lines* (p. 222), which *intersect the circle at infinity*, and which it is proposed to call the *Eight Umbilicar Generatrices* of the surface.

(*f*). These (imaginary) *umbilicar generatrices* of a quadric are found to possess several interesting properties, especially in relation to the *lines of curvature*: and their *locus*, for a *confocal system*, is a *developable surface* (p. 222), namely the known *envelope* (*d*) of that system.

ARTICLE 409.—Geodetic Lines on Central Surfaces of the Second Order, . . . 225-229

(*a*). One form of the *general* differential *equation of geodetics* on an *arbitrary surface* being, by III. iii. 5 (p. 29),

$$\mathrm{V}\nu\mathrm{d}^2\rho = 0, \quad (R_1), \quad \text{if} \quad \mathrm{Td}\rho = \text{const.}, \qquad (R_1')$$

this is shown (p. 226) to conduct, for central quadrics, to the first integral,

$$P^{-2}D^{-2} = \mathrm{T}\nu^2 f\mathrm{Ud}\rho = h = \text{const.}; \qquad (S_1)$$

where P is the *perpendicular from the centre* O on the *tangent plane*, and D is the (real or imaginary) *semidiameter* of the surface, which is *parallel to the tangent* ($\mathrm{d}\rho$) to the curve. The known equation of Joachimstal, $P.D = \text{const.}$, is therefore proved anew; this last *constant*, however, being by no means necessarily *real*, if the surface be *not* an *ellipsoid*.

(*b*). Deduction (p. 227) of a theorem of M. Chasles), that the *tangents to a geodetic*, on any one central quadric (*e*) *touch* also a *common confocal* (e_r); and of an integral (p. 228) of the form,

$$e_1 \sin^2 v_1 + e_2 \cos^2 v_1 = e_r = \text{const.}, \qquad (S_1')$$

which agrees with one of M. Liouville.

(*c*). Without the restriction (R_1'), the differential of the scalar h in (S_1) may be thus decomposed into factors (p. 229).

$$\mathrm{d}h = \mathrm{d}\,.\,P^{-2}D^{-2} = 2\mathrm{S}\nu\mathrm{d}\nu\mathrm{d}\rho^{-1}\,.\,\mathrm{S}\nu\mathrm{d}\rho^{-1}\mathrm{d}^2\rho; \qquad (S_1'')$$

but, by the lately cited Section (III. iii. 5, p. 29), the differential equation of the *second order*,

$$\mathrm{S}\nu\mathrm{d}\rho\mathrm{d}^2\rho = 0, \qquad (R_1'')$$

with an arbitrary scalar variable, represents the *geodetic lines* on *any surface*: the theorem (*a*) is therefore in this way reproduced.

Pages

(*d*). But we see, at the same time, by (S_1''), that the quantity h, or $P.D = h^{-\frac{1}{2}}$, is *constant*, *not only* for the *geodetics* on a central quadric, but *also* for a certain *other set* of curves, determined by the differential equation of the *first order*, $S\nu d\nu d\rho = 0$, which will be seen, in the next Series, to represent the *lines of curvature*.

ARTICLE 410.—On Lines of Curvature generally; and in particular on such lines, for the case of a Central Quadric, ... 230–239

(*a*). The differential equation (comp. 409, (*d*)),

$$S\nu d\nu d\rho = 0, \qquad (T_1)$$

represents (p. 229) the *Lines of Curvature* upon an *arbitrary surface*; because it is a *limiting form* of this other equation,

$$S\nu\Delta\nu\Delta\rho = 0, \qquad (T_1')$$

which is the *condition of intersection* (or of parallelism), of the *normals* drawn at the extremities of the two vectors ρ and $\rho + \Delta\rho$.

(*b*). The normal vector ν, in the equation (T_1) may be multiplied (pp. 237, 275) by any constant or variable scalar n, without any real change in that equation; but in this whole theory, of the treatment of *Curvatures of Surfaces* by Quaternions, it is advantageous to consider the expression $S\nu d\rho$ as denoting the *exact differential* of *some scalar function* of ρ; for then (by p. 553, vol. i.) we shall have an equation of the form,

$$d\nu = \phi d\rho = \text{a } \textit{self-conjugate function} \text{ of } d\rho, \qquad (U_1)$$

which usually involves ρ also. For instance, we may write generally (p. 233, comp. (R), p. xxxii, vol. i.),

$$d\nu = gd\rho + V\lambda d\rho\mu; \qquad (U_1')$$

the scalar g, and the vectors λ, μ being *real*, and being *generally** functions of ρ, but not involving $d\rho$.

(*c*). This being understood, the *two*† *directions* of the *tangent* $d\rho$, which satisfy at once the general equation (T_1) of the lines of curvature, and the differential equation $S\nu d\rho = 0$ of the surface, are easily found to be represented by the two vector expressions (p. 233),

$$UV\nu\lambda \pm UV\nu\mu; \qquad (T_1'')$$

they are therefore generally *rectangular* to each other, as they have long been known to be.

(*d*). The *surface itself* remaining still quite *arbitrary*, it is found useful to introduce the conception of an *Auxiliary Surface of the Second Order* (p. 234), of which the variable vector is $\rho + \rho'$, and the equation is,

$$S\rho'\phi\rho' = g\rho'^2 + S\lambda\rho'\mu\rho' = 1, \qquad (U_1'')$$

or more generally = const.; and it is proposed to call *this* surface, of which the *centre* is at the given point P, the *Index Surface*, partly because its *diametral section*, made by the *tangent plane* to the *given* surface at P, is a certain *Index Curve* (p. 231), which may be considered to coincide with the known "*indicatrice*" of Dupin.

(*e*) The expressions (T_1'') show (p. 234), that whatever the *given* surface may be, the *tangents* to the lines of curvature *bisect the angles* formed by the *traces* of the two

* For the case of a *central quadric*, g, λ, μ are *constants*.

† *Generally two*; but in some cases *more*. It will soon be seen, that *three lines of curvature* pass through an *umbilic* of a *quadric*.

Pages

cyclic planes of the *Index Surface* (*d*), on the *tangent plane* to the given surface; these two tangents have also (as was seen by Dupin) the directions of the *axes* of the *Index Curve* (p. 231); and they are distinguished (as he likewise saw) from all *other* tangents to the given surface, at the given point P, by the condition that *each* is *perpendicular to its own conjugate*, with respect to that indicating curve: the *equation* of such *conjugation*, of two tangents τ and τ', being in the present notation (see again p. 232),

$$S\tau\phi\tau' = 0, \quad \text{or} \quad S\tau'\phi\tau = 0. \qquad (U_1''')$$

(*f*). New proof (p. 232) of another theorem of Dupin, namely that *if a developable be circumscribed to any surface*, along *any curve* thereon, its *generating lines* are everywhere *conjugate*, as tangents to the surface, to the corresponding *tangents to the curve*.

(*g*). Case of a central quadric; new proof (p. 235) of still another theorem of Dupin, namely that the *curve of orthogonal intersection* (p. 198) of two *confocal surfaces*, is a *line of curvature* on *each*.

(*h*). The system of the *eight umbilicar generatrices* (408, (*e*)), of a central quadric, is the *imaginary envelope* of the *lines of curvature* on that surface (p. 235); and *each* such *generatrix* is *itself* an *imaginary line of curvature* thereon: so that *through each* of the *twelve umbilics* (see again 408, (*e*)) there pass *three lines of curvature* (comp. p. 242) whereof however only *one*, at most, can be real: namely *two generatrices*, and a *principal section* of the surface. These last results, which are perhaps new, will be illustrated, and otherwise proved, in the following Series (411).

ARTICLE 411.—Additional illustrations and confirmations of the foregoing theory, for the case of a *Central* Quadric*; and especially of the theorem respecting the *Three Lines of Curvature through an Umbilic*, whereof *two* are always *imaginary* and *rectilinear*, . 239–245

(*a*). The general equation of condition (T_1') or $S\nu\Delta\nu\Delta\rho = 0$, for the intersection of two finitely distant normals, may be easily transformed for the case of a quadric, so as to express (p. 240) that *when the normals* at P and P′ *intersect* (or are parallel) the *chord* PP′ is *perpendicular to its own polar*.

(*b*). Under the same conditions, if the point P be given, the *locus* of the *chord* PP′ is *usually* (p. 241) a *quadric cone*, say (*C*); and therefore the locus of the *point* P′ is usually a *quartic curve*, with P for a *double point*, whereat *two branches* of the *curve cut each other* at *right angles*, and *touch* the *two lines* of curvature.

(*c*). If the point P be one of a *principal section* of the given surface, but *not an umbilic*, the *cone* (*C*) breaks up into a *pair of planes*, whereof *one*, say (*P*), is the plane of the section, and the *other*, (*P′*), is perpendicular thereto, and is *not tangential* to the surface; and thus the *quartic* (*b*) breaks up into a *pair of conics* through P, whereof one is the principal section itself, and the other is perpendicular to it.

(*d*). But *if* the given point P be an *umbilic*, the *second plane* (*P′*) becomes a *tangent plane* to the surface; and the *second conic* (*c*) breaks up, at the same time, into a *pair of imaginary*† *right lines*, namely the *two umbilicar generatrices* through P (pp. 242, 245).

(*e*). It follows that the *normal* PN at a *real umbilic* P (of an ellipsoid, or a double-sheeted hyperboloid) is *not intersected by any other real normal*, except those which are in the *same principal section;* but that this real normal PN *is* intersected, in an *imaginary sense*, by *all the normals* P′N′, which are drawn at points P′ of *either* of the two

* Many, indeed most, of the results apply, without modification, to the case of the *Paraboloids;* and the rest can easily be adapted to this latter case, by the consideration of infinitely distant points. We shall therefore often, for conciseness, omit the term *central*, and simply speak of *quadrics*, or surfaces of the *second order*.

† It is well known that the *single-sheeted hyperboloid*, which (alone of central quadrics) has *real generating lines*, has at the same time *no real umbilics* (comp. p. 221).

Pages

imaginary generatrices through the real *umbilic* P; so that *each* of these *imaginary right lines* is seen anew to be a *line* of curvature*, on the surface (comp. 410, (*h*)), because all the normals P'N', at points of this line, are situated in one *common* (*imaginary*) *normal plane* (p. 242): and as before, there are thus *three lines* of curvature *through an umbilic.*

(*f*). These geometrical results are in various ways deducible from calculation with quaternions; for example, a form of the equation of the lines of curvature on a quadric is seen (p. 242) to become an *identity* at an umbilic ($\nu \parallel \lambda$): while the *differential* of that equation breaks up into two *factors*, whereof *one* represents the tangent to the *principal section*, while the *other* ($S\lambda d^2\rho = 0$) assigns the directions of the *two generatrices.*

(*g*). The equation of the *cone* (*C*), which has already presented itself as a certain *locus of chords* (*b*), admits of many quaternion transformations; for instance (see p. 240), it may be written thus,

$$\frac{S\alpha\rho\Delta\rho}{S\alpha\Delta\rho} + \frac{S\alpha'\rho\Delta\rho}{S\alpha'\Delta\rho} = 0, \qquad (V_1)$$

ρ being the vector of the vertex P, and $\rho + \Delta\rho$ that of any other point P' of the cone; while α, α' are still, as in 407, (*a*), two real *focal lines*, of which the *lengths* are *here arbitrary*, but of which the *directions* are *constant*, as before, for a whole *confocal system.*

(*h*). This cone (*C*), or (V_1), is also the *locus* (p. 244) of a system of *three rectangular lines*; and if it be cut by any plane perpendicular to a side, and not passing through the vertex, the *section* is an *equilateral hyperbola.*

(*i*). The same cone (*C*) has, for *three* of its *sides* PP', the *normals* (p. 243) to the *three confocals* (p. 197) of a given system which pass through its vertex P; and therefore also, by 410, (*g*), the *tangents* to the *three lines of curvature* through that point, which are the *intersections* of those three confocals.

(*j*). And because its equation (V_1) does not involve the constant *l*, of 407, (*a*), (*b*), we arrive at the following theorem (p. 243):—*If indefinitely many quadrics, with a common centre* O, *have their asymptotic cones biconfocal, and pass through a common point* P, *their normals at that point have a quadric cone* (*C*) *for their locus.*

ARTICLE 412.—On Centres of Curvature of Surfaces, ... 246–261

(*a*). If σ be the vector of the *centre* S of *curvature* of a normal section of an *arbitrary*

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* It might be natural to suppose, from the known general theory (410, (*c*)) of the *two* rectangular directions, that *each* such generatrix PP' is *crossed perpendicularly*, at every one of its *non-umbilicar points* P', by a *second* (and *distinct*, although *imaginary*) *line of curvature.* But it is an almost equally well known and *received result* of modern geometry, paradoxical as it must at first appear, that *when a right line is directed to the circle at infinity*, as (by 408, (*e*)) the generatrices in question are, then *this imaginary line is everywhere perpendicular to itself.* Compare the Notes to pages 516 vol. i., 236. Quaternions are not at all responsible for the *introduction* of this principle into geometry, but they *recognise* and *employ* it, under the following very simple form: that *if a non-evanescent vector be directed to the circle at infinity*, it is *an imaginary value of the symbol* $0^{\frac{1}{2}}$ (comp. pp. 316, 516 vol. i., 222, 236); and conversely, that *when this last symbol represents a vector which is not null*, the vector thus denoted is *an imaginary line*, which *cuts that circle.* It may be noted here, that such is the case with the *reciprocal polar* of *every chord* of a *quadric*, connecting *any two umbilics* which are *not in one principal plane*; and that thus the *quadratic equation* (XXI., in p. 233) from which the *two directions* (410, (*c*)) can *usually* be derived, becomes an *identity* for *every umbilic*, real *or* imaginary: as it ought to do, for consistency with the foregoing theory of the *three lines* through that umbilic. And as an additional illustration of the *coincidence* of directions of the lines of curvature at any *non-umbilicar point* P' of an umbilicar generatrix, it may be added that the *cone of chords* (*C*), in 411, (*b*), is found to *touch the quadric along that generatrix*, when its *vertex* is at any *such* point P'.

surface, which touches one of the two lines of curvature thereon, at any given point P, we have the two fundamental equations (p. 247),

$$\sigma = \rho + R\mathrm{U}\nu, \quad (\mathrm{W}_1), \quad \text{and} \quad R^{-1}\mathrm{d}\rho + \mathrm{dU}\nu = 0; \qquad (\mathrm{W}_1')$$

whence

$$\mathrm{V}\mathrm{d}\rho\mathrm{dU}\nu = 0, \quad (\mathrm{W}_1''), \quad \text{and} \quad \frac{\mathrm{T}\nu}{R} + \mathrm{S}\frac{\mathrm{d}\nu}{\mathrm{d}\rho} = 0; \qquad (\mathrm{W}_1''')$$

the equation (W_1'') being a *new form* of the *general* differential equation of the *lines of curvature*.

(*b*). Deduction (pp. 248, 249, &c.) of some known theorems from these equations; and of some which introduce the new and general conception of the *Index Surfac* (410, (*d*)), as well as that of the known *Index Curve*.

(*c*). Introducing the auxiliary scalar (p. 251),

$$r = \frac{\mathrm{T}\nu}{R} = -\mathrm{S}\frac{\mathrm{d}\nu}{\mathrm{d}\rho} = -\mathrm{S}\tau^{-1}\phi\tau, \qquad (\mathrm{X}_1)$$

in which τ ($\parallel \mathrm{d}\rho$) is a tangent to a line of curvature, while $\mathrm{d}\nu = \phi\mathrm{d}\rho$, as in ($\mathrm{U}_1$), the two values of r, which answer to the two rectangular directions (T_1'') in 410, (*c*), are given (p. 248) by the expression,

$$r = -g - \mathrm{T}\lambda\mu \, . \cos\left(\angle\frac{\nu}{\lambda} \mp \angle\frac{\nu}{\mu}\right), \qquad (\mathrm{X}_1')$$

in which g, λ, μ are, for any given point P, the constants in the equation (U_1'') of the *index surface*; the *difference* of the *two curvatures* R^{-1} therefore *vanishes at an umbilic* of the *given surface*, whatever the *form* of that surface may be: that is, at a point, where $\nu \parallel \lambda$ or $\parallel \mu$, and where consequently the *index curve* is *a circle*.

(*d*). At any *other* P of the given surface, which is as yet entirely *arbitrary*, the values of r may be thus expressed (p. 249),

$$r_1 = \mathrm{a}_1^{-2}, \quad r_2 = \mathrm{a}_2^{-2}, \qquad (\mathrm{X}_1'')$$

a_1, a_2 being the *scalar semiaxes* (real or imaginary) of the *index curve* (*defined*, comp. 410, (*d*), by the equations $\mathrm{S}\rho'\phi\rho' = 1$, $\mathrm{S}\nu\rho' = 0$).

(*e*). The *quadratic equation*, of which r_1 and r_2, or the *inverse squares* of the two last *semiaxes*, are the *roots*, may be written (p. 252) under the *symbolical form*,

$$\mathrm{S}\nu^{-1}(\phi + r)^{-1}\nu = 0; \qquad (\mathrm{Y}_1)$$

which may be developed (same page) into this other form,

$$r^2 + r\mathrm{S}\nu^{-1}\chi\nu + \mathrm{S}\nu^{-1}\psi\nu = 0, \qquad (\mathrm{Y}_1')$$

the linear and vector functions, ψ and χ, being devived from the function ϕ, on the plan of the Section III. ii. 6 (pp. 489, 494, vol. i).

(*f*). Hence, *generally* the *product* of the *two curvatures* of a *surface* is expressed (p. 253) by the formula

$$R_1^{-1}R_2^{-1} = r_1 r_2 \mathrm{T}\nu^{-2} = -\mathrm{S}\frac{1}{\nu}\psi\frac{1}{\nu}; \qquad (\mathrm{Z}_1)$$

which will be found useful in the following series (413), in connexion with the theory of the *Measure of Curvature*.

(g). The given surface being still quite general, if we write (p. 256),

$$\tau = \mathrm{U}d\rho,\ \tau' = \mathrm{U}(\nu d\rho),\ (\mathrm{A}_2),\ \text{and therefore } \tau\tau' = \mathrm{U}\nu, \qquad (\mathrm{A}_2')$$

so that τ and τ' are *unit tangents* to the lines of curvature, it is easily proved that

$$d\tau' = \tau\,\mathrm{S}\tau' d\tau,\ (\mathrm{B}_2),\quad \text{or that}\quad \mathrm{V}\tau d\tau' = 0\,; \qquad (\mathrm{B}_2')$$

this *general parallelism* of $d\tau'$ to τ being geometrically explained, by obverving that a line of curvature *on any surface* is, at the same time, a line of curvature *on the developable normal surface*, which *rests upon that line*, and to which τ' or $\nu\tau$ is *normal*, if τ be *tangential* to the line.

(h). If the *vector of curvature* (389) of a line of curvature be *projected on the normal* ν to the given surface, the projection (p. 257) is the vector of curvature of the *normal section* of that surface, which has the same tangent τ; but this result, and an analogous one (same page) for the developable normal surface (g), are virtually included in Meusnier's theorem, which will be proved by quaternions in Series 414.

(i). The vector σ of a centre of curvature of the given surface, answering to a given point P thereon, may (by (W_1) and (X_1)) be expressed by the equation,

$$\sigma = \rho + r^{-1}\nu\,; \qquad (\mathrm{C}_2)$$

which may be regarded also as a *general form* of the *Vector Equation* of the *Surface of Centres*, or of the *locus* of the centre S: the variable vector ρ of the point P of the *given* surface being supposed (p. 11) to be expressed as a vector function of two independent and scalar variables, whereof therefore ν, r, and σ become also functions, although the two last involve an ambiguous *sign*, on account of the *Two Sheets* of the surface of centres.

(j). The *normal* at S, to which may be called the *First Sheet*, has the direction of the *tangent* τ to what may (on the same plan) be called the *First Line of Curvature* at P; and the vector υ of the point *corresponding* to S, on the corresponding *sheet* of the *Reciprocal* (comp. pp. 19, 20) *of the Surfaces of Centres*, has (by p. 254) the expression,

$$\upsilon = \tau\,(\mathrm{S}\rho\tau)^{-1}\,; \qquad (\mathrm{D}_2)$$

which may also be considered (comp. (i)) to be a form of the *Vector Equation* of that *Reciprocal Surface*.

(k). The vector υ satisfies generally (p. 254) the *equations of reciprocity*,

$$\mathrm{S}\upsilon\sigma = \mathrm{S}\sigma\upsilon = 1,\quad \mathrm{S}\upsilon\delta\sigma = 0,\quad \mathrm{S}\sigma\delta\upsilon = 0, \qquad (\mathrm{D}_2')$$

$\delta\sigma$, $\delta\upsilon$ denoting any infinitesimal variations of the vectors σ and υ, consistent with the equations of the surface of centres and its reciprocal, or any *linear* and *vector elements* of those two surfaces, at two corresponding points; we have also the relations (p. 255),

$$\mathrm{S}\rho\upsilon = 1,\quad \mathrm{S}\nu\upsilon = 0,\quad \mathrm{S}\nu\upsilon\phi\upsilon = 0. \qquad (\mathrm{D}_2'')$$

(l). The equation $\mathrm{S}\upsilon(\omega - \rho) =$, or more simply,

$$\mathrm{S}\upsilon\omega = 1, \qquad (\mathrm{E}_2)$$

in which ω is a variable vector, represents (p. 254) the *normal plane* to the *first line* (j) of curvature at P; or the *tangent plane* at S to the *first sheet* of the surface of centres: or finally, the tangent plane to that *developable normal surface* (g), which *rests* upon the the *second line* of curvature, and *touches the first sheet* along a certain *curve*, whereof we shall shortly meet with an example. And if υ be regarded, comp. (i), as a vector function of *two* scalar variables, the *envelope of the variable plane* (E_2) is a *sheet of the*

Pages

surface of centres; or rather, on account of the ambiguous *sign* (*i*), it is that surface of centres *itself*; while, in like manner, the *reciprocal surface* (*j*) is the *envelope* of this *other plane*,

$$S\sigma\omega = 1. \qquad (E_2')$$

(*m*). The equations (W_1), (W_1') give (comp. the Note to p. 254),

$$d\sigma = dR.\, U\nu: \qquad (F_2)$$

combining which with (C_2), we see that the equations (H_1) of p. xvi are satisfied, when the derived vectors ρ' and σ' are changed to the corresponding differentials, $d\rho$ and $d\sigma$. The known theorem (of Monge), that each *Line of Curvature* is generally an *involute*, with the corresponding *Curve of Centres* for one of its *evolutes* (400), is therefore in this way reproduced: and the connected theorem (also of Monge), that *this evolute* is a *geodetic on its own sheet* of the surface of centres, follows easily from what precedes.

(*n*). In the foregoing paragraphs of this analysis, the *given* surface has throughout been *arbitrary*, or *general*, as stated in (*d*) and (*g*). But if we now consider specially the *case* of a *central quadric*, several less general but interesting results arise, whereof *many*, but perhaps *not all*, are known; and of which some may be mentioned here.

(*o*). Supposing, then, that not only $d\nu = \phi d\rho$, but also $\nu = \phi\rho$, and $S\rho\nu = f\rho = 1$, the *Index Surface* (410, (*d*)) becomes simply (p. 233) the *given* surface, with its *centre transported* from O to P; whence many simplications follow.

(*p*). For example, the *semiaxes* a_1, a_2 of the *index curve* are now equal (p. 249) to the semiaxes of the *diametral section* of the given surface, made by a plane parallel to the tangent plane; and $T\nu$ is, as in 409, the *reciprocal* P^{-1} of the *perpendicular*, from the centre on this latter plane; whence (by (X_1) and X_1'')) these known expressions for the two* curvatures result:

$$R_1^{-1} = Pa_1^{-2};\quad R_2^{-1} = Pa_2^{-2}. \qquad (G_2)$$

(*q*). Hence, by (*e*), if a *new* surface be *derived* from a *given* central quadric (of *any species*), as the *locus of the extremities of normals* erected at the centre, *to the planes of diametral sections* of the given surface, *each such normal* (when real) having the *length* of *one* of the *semiaxes* of that *section*, the *equation of this new surface*† admits (p. 253) of being written thus:

$$S\rho(\phi - \rho^{-2})^{-1}\rho = 0. \qquad (H_2)$$

(*r*). Under the conditions (*o*), the expression (C_2) for σ gives (p. 254) the two converse forms,

$$\sigma = r^{-1}(\phi + r)\rho, \quad (I_2), \quad \rho = r(\phi + r)^{-1}\sigma; \qquad (I'_2)$$

whence (pp. 254, 260),

$$\nu = r(\phi + r)^{-1}\phi\sigma, \quad (J_2), \quad \sigma = (\phi^{-1} + r^{-1})\,\nu; \qquad (J_2')$$

and therefore (p. 260), by (*d*), (*p*), and by the theory (407) of confocal surfaces,

$$\sigma_1 = \phi_2^{-1}\nu = \phi_2^{-1}\phi\rho, \qquad (K_2)$$

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* Throughout the present series 412, we attend only (comp. (*a*)) to the curvatures of the *two normal sections* of a surface, which have the directions of the *two lines* of curvature: these being in fact what are always regarded as the *two principal curvatures* (or simply as *the* two curvatures) of the *surface*. But, in a shortly subsequent Series (414), the more general case will be considered, of the curvature of *any section*, normal or oblique.

† When the *given* surface is an *ellipsoid* the *derived* surface is the celebrated *Wave Surface* of Fresnel: which thus has (H_2) for a *symbolical form* of its equation. When the given surface is an *hyperboloid*, and a *semiaxis* of a section is *imaginary*, the (scalar and now positive) *square*, of the (imaginary) *normal* erected, is still to be made equal to the *square* of that semiaxis.

if ϕ_2 be formed from ϕ by changing the semiaxes abc to $a_2b_2c_2$; it being understood that the given quadric (abc) is cut by the two confocals ($a_1b_1c_1$) and ($a_2b_2c_2$), in the *first* and *second lines* of curvature through the given point P: and that σ_1 is here the vector of that *first centre* S of curvature, which answers to the *first line* (comp. (j)). Of course, on the same plan, we have the analogous expression,

$$\sigma_2 = \phi_1^{-1}\nu = \phi_1^{-1}\phi\rho, \tag{K$_2'$}$$

for the vector of the *second centre*.

(s). These expressions for σ_1, σ_2 include (p. 260) a theorem of Dr. Salmon, namely that the *centres of curvature* of a given quadric at a given point are the *poles of the tangent plane*, with respect to the two confocals through that point; and *either* of them may be regarded, by an admission of an ambiguous *sign* (comp. (i)), as a *new Vector Form** of the *Equation of the Surface of Centres*, for the *case* (o) of a given *central quadric*.

(t). In connexion with the same expressions for σ_1, σ_2, it may be observed that if r_1, r_2 be the corresponding values of the auxiliary scalar r in (c), and if τ, τ' still denote the unit tangents (g) to the first and second lines of curvature, while abc, $a_1b_1c_1$, and $a_2b_2c_2$ retain their recent significations (r), then (comp. pp. 257, 258, see also p. 208),

$$r_1 = f\tau = f\,\mathrm{U}d\rho = (a^2 - a_2^2)^{-1} = \&c., \tag{L$_2$}$$

and
$$r_2 = f\tau' = f\,\mathrm{U}\nu d\rho = (a^2 - a_1^2)^{-1} = \&c.; \tag{L$_2'$}$$

this association of r_1 and σ_1 with a_2, &c., and of r_2 and σ_2 with a_1, &c., arising from the circumstance that the *tangents* τ and τ' have respectively the directions of the *normals* ν_2 and ν_1, to the two confocal surfaces, ($a_2b_2c_2$) and ($a_1b_1c_1$).

(u). By the properties of such surfaces, the scalar here called r_2 is therefore *constant*, in the whole extent of a *first line* of curvature; and the same *constancy* of r_2, or the equation,

$$d f\,\mathrm{U}\nu d\rho = 0, \tag{M$_2$}$$

may in various ways be proved by quaternions (p. 258).

(v). Writing simply r and r' for r_1 and r_2, so that r' is constant, but r variable, for a *first line* of curvature, while conversely r is constant and r' variable for a *second line*, it is found (pp. 254, 255, 256), that the *scalar equation* of the *surface of centres* (i) may be regarded as the result of the elimination of r^{-1} between the *two* equations,

$$1 = S.\sigma(1 + r^{-1}\phi)^{-2}\phi\sigma, \quad (\mathrm{N}_2), \quad \text{and} \quad 0 = S.\sigma(1 + r^{-1}\phi)^{-3}\phi^2\sigma; \tag{N$_2'$}$$

whereof the latter is the *derivative* of the former with respect to the scalar r^{-1}. It follows (comp. p. 259), that the *First Sheet* of the *Surface of Centres* is *touched by an Auxiliary Quadric* (N$_2$), *along a Quartic Curve* (N$_2$) (N$_2'$), which curve is the *Locus* of the *Centres of First Curvature*, for all the points of a *Line of Second Curvature*; the *same sheet* being also touched (see again p. 259), along the *same curve*, by the *developable normal surface* (l), which *rests* on the *same second line*: with permission to interchange the words, *first* and *second*, throughout the whole of this enunciation.

(w). The given surface being still a central quadric (o), the vectors ρ, σ, ν can be expressed as functions of υ (comp. (j) (k) (l)), and conversely the latter can be expressed as a function of any one of the former; we have, for example, the reciprocal equations (p. 256),

$$\sigma = (1 + r^{-1}\phi)^2\phi^{-1}\upsilon, \quad (\mathrm{O}_2), \quad \text{and} \quad \upsilon = (1 + r^{-1}\phi)^{-2}\phi\sigma; \tag{O$_2'$}$$

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* Dr. Salmon's result, that this surface of centres is of the *twelfth degree*, may be easily deduced from this form.

Pages

from which last the formula (N_2) may be obtained anew, by observing (k) that $S\sigma\upsilon = 1$. Hence also, by (r), we can infer the expressions,*

$$\rho = (\phi^{-1} + r^{-1})\,\upsilon = \phi_2^{-1}\,\upsilon, \quad (P_2), \quad \text{and} \quad \upsilon = \phi_2\,\rho = \nu_2; \qquad (P_2')$$

and in fact it is easy to see otherwise (comp. p. 198), that $\nu_2 \parallel \tau \parallel \upsilon$, and $S\rho\nu_2 = 1 = S\rho\upsilon$, whence $\nu_2 = \upsilon$ as before.

(x). More fully, *the two sheets* of the *reciprocal* (j) of the surface of centres may have their separate *vector equations* written thus,

$$\upsilon_1 = \phi_2\rho = \nu_2, \qquad \upsilon_2 = \phi_1\rho = \nu_1; \qquad (P_2'')$$

and the *scalar equation*† of this *reciprocal surface* itself, considered as including *both* sheets, may (by page 255) be thus written, the functions f and F being related as in 408, (b),

$$\upsilon^4 = (F\upsilon - 1)\,f\upsilon, \qquad (Q_2)$$

with several equivalent forms; one way of obtaining this equation being the elimination of r between the two following (same p. 256):

$$F\upsilon + r^{-1}\,\upsilon^2 = 1, \quad (Q_2'); \quad f\upsilon + r\upsilon^2 = 0. \qquad (Q_2'')$$

(y). The two last equations may also be written thus, for the *first sheet* of the reciprocal surface,

$$F_2\upsilon_1 = 1, \quad (R_2), \quad \text{and} \quad fU\upsilon_1 = r, \qquad (R_2')$$

in which (comp. pp. 255, 260),

$$F_2\upsilon = S\upsilon\phi_2^{-1}\,\upsilon = S\upsilon\,(\phi^{-1} + r^{-1})\upsilon; \qquad (R_2'')$$

and accordingly (comp. pp. 548, vol. i, 199), we have $F_2\nu_2 = F\nu = 1$, and $fU\nu_2 = f\tau = r$.

(z). For a *line of second curvature* on the given surface, the scalar r is *constant*, as before; and then the two equations (Q_2'), (Q_2''), or (R_2), (R_2'), represent jointly (comp. the slightly different enunciation in p. 259) a certain *quartic curve*, in which the *quadric reciprocal* (R_2), of the *second confocal* ($a_2\,b_2\,c_2$), *intersects the first sheet* (y) of the *Reciprocal Surface* (Q_2); this quartic curve, being at the same time the intersection of the *quadric surface* (Q_2') or (R_2), with the *quadric cone* (Q_2'') or (R_2'), which is *biconcyclic* with the *given quadric*, $f\rho = 1$.

ARTICLE 413.—On the Measure of Curvature of a Surface, ... 261–266

The object of this short Series 413 is the deduction by quaternions, somewhat more briefly and perhaps more clearly than in the *Lectures*, of the principal results of Gauss (comp. Note to p. 261), respecting the *Measure of Curvature of a Surface*, and questions therewith connected.

(a). Let P, P₁, P₂ be *any three near points* on a given but *arbitrary surface*, and R, R₁, R₂ the three *corresponding points* (near to *each other*) on the *unit sphere*, which are determined by the *parallelism* of the *radii* OR, OR₁, OR₂ to the *normals* PN, P₁N₁, P₂N₂; then the areas of the *two small triangles* thus formed will bear to each other the *ultimate ratio* (p. 262),

$$\lim. \frac{\Delta RR_1R_2}{\Delta PP_1P_2} = \frac{V\,.\,dU\nu\delta U\nu}{Vd\rho\delta\rho} = -S\frac{1}{\nu}\,\psi\,\frac{1}{\nu}; \qquad (S_2)$$

* The equation $\upsilon = \nu_2$, = the normal to the confocal ($a_2\,b_2\,c_2$) at P, is not actually given in the text of Series 412; but it is easily deduced, as above, from the formulæ and methods of that Series.

† The equation (Q_2) is one of the *fourth degree*; and, when expanded by coordinates, it agrees perfectly with that which was first assigned by Dr. Booth (see a Note to p. 255), for the *Tangential Equation* of the *Surface of Centres* of a quadric, or for the Cartesian equation of the *Reciprocal Surface*.

whence, with Gauss's *definition* of the *measure of curvature*, as the *ultimate ratio of corresponding areas* on surface and sphere, we have, by the formula (Z_1) in 412, (*f*), his *fundamental theorem*,

$$\textit{Measure of Curvature} = R_1^{-1} R_2^{-1}, \qquad (S_2')$$

= *Product of the two Principal Curvatures of Sections.*

(*b*). If the vector ρ of the surface be considered as a function of two scalar variables, t and u, and if derivations with respect to these be denoted by upper and lower accents, this general transformation results (p. 263),

$$\textit{Measure of Curvature} = S\frac{\rho''}{\nu} S\frac{\rho_{,,}}{\nu} - \left(S\frac{\rho'_{,}}{\nu}\right)^2 \qquad (T_2)$$

in which

$$\nu = V\rho'\rho_{,}; \qquad (T_2')$$

with a verification for the notation *pqrst* of Monge.

(*c*). The square of a linear element $ds_{,}$ of the given but arbitrary surface, may be expressed (p. 263) as follows:

$$ds^2 = (Td\rho^2 =) edt^2 + 2fdtdu + gdu^2; \qquad (U_2)$$

and with the recent use (*b*) of accents, the *measure* (T_2) is proved (same page) to be an explicit function of the ten scalars,

$$e, f, g\ ; \quad e', f', g'; \quad e_{,}, f_{,}, g_{,}; \quad \text{and} \quad e_{,,} - 2f'_{,} + g''; \qquad (U_2')$$

the form of this function (p. 264) agreeing, in all its details, with the corresponding expression assigned by Gauss. *

(*d*). Hence follow at once (p. 264) two of the most important results of that great mathematician on this subject; namely, that *every Deformation of a Surface*, consistent with the conception of it as an *infinitely thin* and *flexible* but *inextensible solid*, leaves *unaltered*, Ist, *the Measure of Curvature at any Point, and* IInd, the *Total Curvature of any Area*: this last being the *area* of the *corresponding portion* (*a*) of the *unit-sphere.*

(*e*). By a suitable choice of t and u, as certain *geodetic co-ordinates*, the expression (U_2) may be reduced (p. 264) to the following,

$$ds^2 = dt^2 + n^2du^2; \qquad (U_2'')$$

where t is the *length* of a geodetic arc AP, from a fixed point A to a variable point P of the surface, and u is the *angle* BAP which this variable arc makes with a fixed geodetic AB: so that in the immediate neighbourhood of A, we have $n = t$, and $n' = D_t n = 1$.

(*f*). The general expression (*c*) for the *measure of curvature* takes thus the very simple form (p. 264),

$$R_1^{-1}R_2^{-1} = -n^{-1}n'' = -n^{-1}D_t^2 n; \qquad (V_2)$$

and we have (comp. (*d*)) the equation (p. 265),

$$\textit{Total Curvature of Area } \text{APQ} = \Delta u - \int n'du; \qquad (V_2')$$

this *area* being bounded by *two geodetics*, AP and AQ, which make with each other an angle = Δu, and by an *arc* PQ of an *arbitrary curve* on the given surface, for which t, and therefore n', may be conceived to be a given function of u.

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* References are given, in Notes to pp. 261, &c. of the present Series 413, to the pages of Gauss's beautiful Memoir, "*Disquisitiones generales circa Superficies Curvas,*" as reprinted in the Additions to Liouville's Monge.

Pages

(*g*). If this arc PQ be *itself* a *geodetic*, and if we denote by v the variable angle which it makes at P with AP prolonged, so that $\tan v = ndu : dt$, it is found that $dv = -n'du$; and thus the equation (V_2') conducts (p. 266) to another very remarkable and general theorem of Gauss, for an *arbitrary surface*, which may be thus expressed,

$$\textit{Total Curvature of a Geodetic Triangle } \text{ABC} = \text{A} + \text{B} + \text{C} - \pi, \qquad (V_2'')$$

= what may be called the *Spheroidal Excess* of that triangle, the *total area* (4π) of the *unit-sphere* being represented by *eight right angles*: with *extensions* to *Geodetic Polygons*, and *modifications* for the case of what may on the same plan be called the *Spheroidal Defect*, when the *two curvatures* of the surface are *oppositely directed*.

ARTICLE 414.—On Curvature of Sections (Normal and Oblique) of Surfaces; and on Geodetic Curvatures, ... 266–272

(*a*). The curvatures considered in the two preceding Series having been those of the *principal normal sections* of a surface, the present Series 414 treats briefly the more general case, where the section is made by an *arbitrary plane*, such as the *osculating plane* at P to an *arbitrary curve* upon the surface.

(*b*). The *vector of curvature* (389) of any such curve or section being $(\rho - \kappa)^{-1} = D_s^2\rho$, its *normal* and *tangential components* are found to be (p. 267),

$$(\rho - \sigma)^{-1} = \nu^{-1}S\frac{d\nu}{d\rho} = (\rho - \sigma_1)^{-1}\cos^2 v + (\rho - \sigma_2)^{-1}\sin^2 v, \qquad (W_2)$$

and

$$(\rho - \xi)^{-1} = \nu^{-1}d\rho^{-1}S\nu d\rho^{-1}d^2\rho; \qquad (W_2')$$

the former component being the *Vector of Normal Curvature* of the *Surface*, for the direction of the *tangent* to the curve: and the latter being the *Vector of Geodetic Curvature* of the same *Curve* (or section).

(*c*). In the foregoing expressions, σ and ξ are the vectors of the points S and X, in which the *axis* of the *osculating circle* to the *curve* intersects respectively the *normal* and the *tangent plane* to the *surface* (p. 267); S is also the *centre* of the *sphere*, which *osculates to the surface* in the direction $d\rho$ of the tangent; σ_1, σ_2 are the vectors of the *two centres* S_1, S_2, *of curvature of the surface*, considered in Series 412, which are at the same time the centres of the *two osculating spheres*, of which the curvatures are (algebraically) the *greatest* and *least*: and v is the *angle* at which the curve here considered crosses the *first line of curvature*.

(*d*). The equation (W_2) contains a theorem of Euler, under the form (p. 268),

$$R^{-1} = R_1^{-1}\cos^2 v + R_2^{-1}\sin^2 v; \qquad (W_2'')$$

it contains also Meusnier's theorem (same page), under the form (comp. 412, (*h*)) that the *vector of normal curvature* (*b*) of a *surface*, for any given direction, is the *projection on the normal* ν, of the *vector of oblique curvature*, whatever the *inclination* of the plane of the section to the tangent plane may be.

(*e*). The expression (W_2') for the *vector of geodetic curvature*, admits (p. 271) of various transformations, with corresponding expressions for the *radius* $T(\rho - \xi)$ of geodetic curvature, which is also the *radius of plane curvature* of the *developed curve*, when the developable circumscribed to the given surface along the given curve is *unfolded* into a plane: and when this radius is *constant*, so that the developed curve is a *circle*, or part of one, it is proposed (p. 271), to call the given curve a *Didonia* (as in the *Lectures*), from its possession of a certain *isoperimetrical property*, which was first considered by M. Delaunay, and is represented in quaternions by the formula (p. 271),

$$\int S(U\nu \,.\, d\rho\delta\rho) + c\delta\int Td\rho = 0; \qquad (X_2)$$

or

$$c^{-1}d\rho = V(U\nu \,.\, dUd\rho), \qquad (X'_2)$$

by the rules of what may be called the *Calculus of Variations in Quaternions*: c being a constant, which represents generally (p. 272) the *radius* of the developed circle, and becomes *infinite* for *geodetic lines*, which are thus included as a *case of Didonias*.

Pages
Article 415.—Supplementary Remarks, ... 272–283

(a). Simplified proof (referred to in a Note to p. xxxii, vol. i.), of the general existence of a system of *three real and rectangular directions*, which satisfy the vector equation $V\rho\phi\rho = 0$, (P), when ϕ is a linear, vector, and *self-conjugate* function; and of a system of *three real roots* of the cubic equation $M = 0$ (p. xxxii, vol. i.), under the same condition (pp. 272–274).

(b). It may happen (p. 276) that the *differential equation*,

$$S\nu d\rho = 0, \qquad (Y_2)$$

is *integrable*, or represents a *system of surfaces*, *without* the *expression* $S\nu d\rho$ being an *exact differential*, as it was in 410, (b). In this case, there exists some scalar *factor*, n, such that $Sn\nu d\rho$ is the exact differential of a scalar function of ρ, without the assumption that this vector ρ is *itself* a *function* of a *scalar variable*, t; and then if we write (p. 276, comp. p. xx),

$$d\nu = \phi d\rho, \quad d\,.\,n\nu = \Phi d\rho, \qquad (Y_2')$$

this *new vector function* Φ will be *self-conjugate*, although the function ϕ is *not* such *now*, as it *was* in the equation (U_1).

(c). In this manner it is found (p. 277), that the *Condition* of Integrability* of the equation (Y_2) is expressed by the very simple formula,

$$S\gamma\nu = 0; \qquad (Y_2'')$$

in which γ is a *vector function* of ρ, *not* generally *linear*, and deduced from ϕ on the plan of the Section III. ii. 6 (p. 492, vol. i), by the relation,

$$\phi d\rho - \phi' d\rho = 2V\gamma d\rho; \qquad (Y_2''')$$

ϕ' being the *conjugate* of ϕ, but *not here equal* to it.

(d). Connexions (pp. 278, 279) of the *Mixed Transformations* in the last cited Section, with the known *Modular and Umbilicar Generations* of a surface of the second order.

(e). The equation (p. 279),

$$T(\rho - V\,.\,\beta V\gamma\alpha) = T(\alpha - V\,.\,\gamma V\beta\rho), \qquad (Z_2)$$

in which α, β, γ are *any three vector constants*, represents a *central quadric*, and appears to offer a *new mode of generation*† of such a surface, on which there is not room to enter, at this last stage of the work.

(f). The vector of the *centre* of the quadric, represented by the equation $f\rho - 2S\epsilon\rho = \text{const.}$, with $f\rho = S\rho\phi\rho$, is generally $\kappa = \phi^{-1}\epsilon = m^{-1}\psi\epsilon$ (p. 280); case of *paraboloids*, and of *cylinders*.

(g). The equation (p. 281),

$$Sq\rho q'\rho q''\rho + S\rho\phi\rho + S\gamma\rho + C = 0, \qquad (Z_2')$$

represents the *general surface* of the *third degree*, or briefly the *General Cubic Surface*; C being a constant scalar, γ a constant vector, and q, q', q'' three constant quaternions, while $\phi\rho$ is here again a linear, vector, and self-conjugate function of ρ.

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* It is shown, in a Note to p. 278, that this *monomial equation* (Y_2'') becomes, when expanded, the known *equation of six terms*, which expresses the condition of integrability of the differential equation $pdx + qdy + rdz = 0$.

† In a Note to p. 204 (already mentioned in p. xviii), the reader will find references to the *Lectures*, for several different *generations of the ellipsoid*, derived from quaternion forms of its equation.

Pages

(*h*). The *General Cubic Cone*, with its vertex at the origin, is thus represented in quaternions by the monomial equation (same page),

$$Sq\rho q'\rho q''\rho = 0. \qquad (Z_2'')$$

(*i*). *Screw Surface, Screw Sections* (p. 281); *Skew Centre* of *Skew Arch*, with illustration by a diagram (fig. 85, p. 283).

SECTION 8.—On a few Specimens of Physical Applications of Quaternions, with some Concluding Remarks, 283 to the end.

ARTICLE 416.—On the Statics of a Rigid Body, ... 283–287

(*a*). *Equation of Equilibrium*,

$$V\gamma\Sigma\beta = \Sigma V\alpha\beta; \qquad (A_3)$$

each α is a *vector of application*; β the corresponding *vector of applied force*; γ an *arbitrary* vector; and this *one* quaternion formula (A_3) is equivalent to the system of the *six* usual scalar equations ($X = 0$, $Y = 0$, $Z = 0$, $L = 0$, $M = 0$, $N = 0$).

(*b*.) When

$$S(\Sigma\beta \,.\, \Sigma V\alpha\beta) = 0, \quad (B_3), \quad \text{but } not \quad \Sigma\beta = 0, \qquad (C_3)$$

the applied forces have an *unique resultant* $= \Sigma\beta$, which acts along the line whereof (A_3) is then the equation, with γ for its variable vector.

(*c*). When the condition (C_3) *is* satisfied, the forces compound themselves generally into *one couple*, of which the *axis* $= \Sigma V\alpha\beta$, whatever may be the position of the assumed origin O of vectors.

(*d*). When

$$\Sigma V\alpha\beta = 0, \quad (D_3), \quad \text{with or without } (C_3),$$

the forces have no tendency to turn the body round *that* point O; and when the equation (A_3) holds good, as in (*a*), for an *arbitrary* vector γ, the forces do not tend to produce a rotation* round *any* point C, so that they completely *balance* each other, as before, and *both* the conditions (C_3) and (D_3) are satisfied.

(*e*). In the general case, when *neither* (C_3) nor (D_3) is satisfied, if q be an *auxiliary quaternion*, such that

$$q\Sigma\beta = \Sigma V\alpha\beta, \qquad (E_3)$$

then Vq is the *vector perpendicular* from the origin, on the *central axis* of the system; and if $c = Sq$, then $c\Sigma\beta$ represents, both in quantity and in direction, the *axis of the central couple*.

(*f*). If Q be *another* auxiliary quaternion, such that

$$Q\Sigma\beta = \Sigma\alpha\beta, \qquad (F_3)$$

with $T\Sigma\beta > 0$, then $SQ = c =$ *central moment* divided by *total force*; and VQ is the vector γ of a point C *upon the central axis* which does not vary with the origin O, and which there are reasons for considering as the *Central Point* of the system, or as the *general centre of applied forces*; in fact, for the *case of parallelism*, this point C coincides with what is usually called the centre of parallel forces.

(*g*). Conceptions of the *Total Moment* $\Sigma\alpha\beta$, regarded as being *generally* a *quaternion*; and of the *Total Tension*, $-\Sigma\alpha\beta$, considered as a *scalar* to which that quaternion with its sign changed reduces itself for the case of *equilibrium* (*a*), and of which the value is in that case *independent of the origin* of vectors.

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* It is easy to prove that the *moment* of the *force* β, acting at the end of the vector α from O, and estimated with respect to any unit-line ι from the same origin, or the *energy* with which the force so acting tends to cause the body to *turn* round that line ι, regarded as a *fixed axis*, is represented by the scalar, $-S\iota\alpha\beta$, or $S\iota^{-1}\alpha\beta$; so that when the condition (D_3) is satisfied, the applied forces have no tendency to produce rotation round *any axis through the origin*: which origin becomes an *arbitrary point* C, when the *equation of equilibrium* (A_3) holds good.

Pages

(*h*). *Principle of Virtual Velocities*,

$$\Sigma S\beta\delta\alpha = 0, \qquad (G_3)$$

ARTICLE 417.—On the Dynamics of a rigid body, ... 287–292

(*a*). *General Equation of Dynamics*,

$$\Sigma m S\,(D_t^2\alpha - \xi)\,\delta\alpha = 0\,; \qquad (H_3)$$

the vector ξ representing the accelerating force, or $m\xi$ the moving force, acting on a particle m of which the vector at the time t is α; and $\delta\alpha$ being any infinitesimal variation of this last vector, geometrically compatible with the connexions between the parts of the system, which need not *here* be a rigid one.

(*b*). For the case of a *free system*, we may change each $\delta\alpha$ to $\epsilon + V\iota\alpha$, ϵ and ι being *any two* infinitesimal vectors, which do not change in passing from one particle m to another; and thus the general equation (H_3) furnishes *two* general vector equations, namely,

$$\Sigma m\,(D_t^2\alpha - \xi) = 0, \quad (I_3), \quad \text{and} \quad \Sigma m V\alpha\,(D_t^2\alpha - \xi) = 0\,; \quad (J_3)$$

which contain respectively the law of the *motion of the centre of gravity*, and the law of *description of areas*.

(*c*) If a *body* be supposed to be *rigid*, and to have a *fixed point* O, then only the equation (J_3) need be retained; and we may write,

$$D_t\alpha = V\iota\alpha, \qquad (K_3)$$

ι being *here* a *finite* vector, namely the *Vector Axis of Instantaneous Rotation*: its *versor* $U\iota$ denoting the *direction* of that axis, and its *tensor* $T\iota$ representing the *angular velocity* of that body about it, at the time t.

(*d*) When the forces vanish, or balance each other, or compound themselves into a single force acting at the fixed point, as for the case of a heavy body turning freely about its centre of gravity, then

$$\Sigma m V\alpha\xi = 0, \quad (L_3)\,; \quad \text{and if we write,} \quad \phi\iota = \Sigma m\alpha V\alpha\iota, \quad (M_3)$$

so that ϕ again denotes a linear, vector, and self-conjugate function, we shall have the equations,

$$\phi D_t\iota + V\iota\phi\iota = 0, \quad (N_3)\,; \qquad \phi\iota + \gamma = 0, \quad (O_3)\,; \qquad S\iota\phi\iota = h^2\,; \quad (P_3)$$

whence

$$S\iota\gamma + h^2 = 0, \quad (Q_3); \qquad \text{and} \qquad \phi D_t\iota = V\iota\gamma\,; \quad (R_3)$$

the vector γ being what we may call the *Constant of Areas*, and the scalar h^2 being the *Constant of Living Force*.

(*e*). One of Poinsot's representations of the *motion of a body*, under the circumstances last supposed, is thus reproduced under the form, that the *Ellipsoid of Living Force* (P_3), with its *centre* at the *fixed point* O, *rolls without gliding* on the *fixed plane* (Q_3), which is *parallel to the Plane of Areas* ($S\iota\gamma = 0$); the variable *semidiameter of contact*, ι, being the *vector-axis* (*c*) of *instantaneous rotation* of the body.

(*f*) The *Moment of Inertia*, with respect to *any axis* ι through O, is equal to the *living force* (h^2) divided by the *square* ($T\iota^2$) of the *semidiameter of the ellipsoid* (P_3), which has the direction of that axis; and hence may be derived, with the help of the first *general construction* of an ellipsoid, suggested by quaternions, a simple geometrical representation (p. 290) of the *square root* of the moment of inertia of a body, with respect to *any axis* AD passing through a given point A, as a certain *right line* $\overline{\text{BD}}$, if $\overline{\text{CD}} = \overline{\text{CA}}$, with the help of two other points B and C, which are likewise fixed in the body, but may be chosen in more ways than one.

Pages

(*g*) A *cone* of the second degree,

$$S\iota\nu = 0, \quad (S_3), \quad \text{with} \quad \nu = \gamma^2\phi\iota - h^2\phi^2\iota, \qquad (T_3)$$

is *fixed* in the *body*, but *rolls* in *space* on that *other cone*, which is the *locus* of the instantaneous *axis* ι; and thus a *second representation*, proposed by Poinsot, is found for the *motion of the body*, as the *rolling of one cone on another*.

(*h*) Some of Mac Cullagh's results, respecting the motion here considered, are obtained with equal ease by the same quaternion analysis; for example, the line γ, although *fixed in space*, describes *in the body* an easily assigned *cone of the second degree* (p. 291), which cuts the *reciprocal ellipsoid*,

$$S\gamma\phi^{-1}\gamma = h^2, \qquad (U_3)$$

in a certain *sphero-conic*: and the *cone of normals* to the last mentioned cone (or the locus of the line $\iota + h^2\gamma^{-1}$) *rolls on the plane of areas* $(S\iota\gamma = 0)$.

(*i*). The *Three* (*Principal*) *Axes of Inertia* of the body, for the given point o, have the *directions* (p. 291) of the *three rectangular and vector roots* (comp. (P), p. xxxii, vol. i., and the paragraph 415, (*a*), p. xxx) of the equation

$$V\iota\phi\iota = 0, \quad (V_3), \qquad \text{because, for each, } D\iota\iota = 0; \qquad (V_3')$$

and if A, B, C denote the three *Principal Moments* of inertia corresponding, then the *Symbolical Cubic* in ϕ (comp. the formula (N) in page xxxi, vol. i.) may be thus written,

$$(\phi + A)\ (\phi + B)\ (\phi + C) = 0. \qquad (W_3)$$

(*j*). Passage (p. 292), from moments referred to axes passing through a given point o, to those which correspond to respectively parallel axes, through any other point Ω of the body.

Article 418.—On the motions of a System of Bodies, considered as free particles $m, m', \ldots$ which attract each other according to the law of the Inverse Square, . . 293–298

(*a*). Equation of motion of the system,

$$\Sigma m S D_t^2\alpha\delta\alpha + \delta P = 0, \quad (X_3), \quad \text{if } P = \Sigma mm' T(\alpha - \alpha')^{-1}; \qquad (Y_3)$$

α is the vector, at the time t, of the mass or particle m; P is the *potential* (or *force-function*); and the infinitesimal variations $\delta\alpha$ are arbitrary.

(*b*). Extension of the *notation of derivatives*,

$$\delta P = \Sigma S\,(D_\alpha P \,.\, \delta\alpha). \qquad (Z_3)$$

(*c*). The differential equations of motion of the separate masses $m, \ldots$ become thus,

$$m D_t^2\alpha + D_\alpha P = 0, \ldots; \qquad (A_4)$$

and the laws of the centre of gravity, of areas, and of living force, are obtained under the forms,

$$\Sigma m D_t\alpha = \beta, \quad (B_4); \quad \Sigma m V\alpha D_t\alpha = \gamma: \qquad (C_4)$$

and

$$T = -\tfrac{1}{2}\Sigma m (D_t\alpha)^2 = P + H; \qquad (D_4)$$

β, γ being two vector constants, and H a scalar constant.

(*d*). Writing,

$$F = \int_0^t (P + T)\,dt, \quad (E_4), \quad \text{and} \quad V = \int_0^t 2T dt = F + tH, \qquad (F_4)$$

F may be called the *Principal* Function*, and V the *Characteristic Function*, of the

* References are given to two Essays by the present writer, "*On a General Method in Dynamics*," in the *Philosophical Transactions* for 1834 and 1835, in which the *Action* (V), and a certain *other* function (S), which is here denoted by F, were called, as above, the *Characteristic* and *Principal Functions*. But the *analysis* here used, as being founded on the *Calculus of Quaternions*, is altogether unlike the analysis which was employed in those former Essays.

Pages

motion of the system; *each* depending on the *final vectors* of *position*, $\alpha, \alpha', \ldots$ and on the *initial vectors*, $\alpha_0, \alpha'_0, \ldots$; but F depending *also* (explicitly) on the *time*, t, while V (= the *Action*) depends instead on the *constant* H of *living force*, in addition to those final and initial vectors: the *masses* $m, m', \ldots$ being supposed to be known, or constant.

(*e*). We are led thus to equations of the forms,

$$m\mathrm{D}_t\alpha + \mathrm{D}_\alpha F = 0, \ldots \quad (\mathrm{G}_4); \quad -m\mathrm{D}_0\alpha + \mathrm{D}_{\alpha_0}F = 0, \ldots \quad (\mathrm{H}_4); \quad (\mathrm{D}_tF) = -H, \quad (\mathrm{I}_4)$$

whereof the system (G_4) contains what may be called the *Intermediate Integrals*, while the system (H_4) contains the *Final Integrals*, of the differential *Equations of Motion* (A_4).

(*f*). In like manner we find equations of the forms,

$$\mathrm{D}_\alpha V = -m\mathrm{D}_t\alpha, \ldots \quad (\mathrm{J}_4); \quad \mathrm{D}_{\alpha_0}V = m\mathrm{D}_0\alpha, \ldots \quad (\mathrm{K}_4); \quad \mathrm{D}_H V = t; \quad (\mathrm{L}_4)$$

the *intermediate integrals* (*e*) being here the result of the elimination of H, between the system (J_4) and the equation (L_4); and the *final integrals*, of the same system of differential equations (A_4). being now (theoretically) obtained, by eliminating the same constant H between (K_4) and (L_4).

(*g*). The functions F and V are obliged to satisfy certain *Partial Differential Equations in Quaternions*, of which those relative to the final vectors $\alpha, \alpha', \ldots$ are the following,

$$(\mathrm{D}_tF) - \tfrac{1}{2}\Sigma m^{-1}(\mathrm{D}_\alpha F)^2 = P, \quad (\mathrm{M}_4); \quad \tfrac{1}{2}\Sigma m^{-1}(\mathrm{D}_\alpha V)^2 + P + H = 0; \quad (\mathrm{N}_4)$$

and they are subject to certain geometrical conditions, from which can be deduced, in a new way, and as new verifications, the law of motion of the centre of gravity, and the law of description of areas.

(*h*). General approximate expressions (p. 298) for the functions F and V, and for their derivatives H and t, for the case of a *short motion* of the system.

ARTICLE 419.—On the Relative Motion of a Binary System; and on the Law of the Circular Hodograph, ... 298–320

(*a*). The vector of one body from the other being α, and the distance being r (= $\mathrm{T}\alpha$), while the sum of the masses is M, the differential equation of the relative motion is, with the law of the inverse square,

$$\mathrm{D}^2\alpha = M\alpha^{-1}r^{-1}; \quad (\mathrm{O}_4)$$

D being here used as a characteristic of derivation, with respect to the time t.

(*b*). As a first integral, which holds good also for any *other law* of *central force*, we have

$$\mathrm{V}\alpha\mathrm{D}\alpha = \beta = \text{a constant vector}; \quad (\mathrm{P}_4)$$

which *includes* the *two* usual laws, of the *constant plane* ($\perp \beta$), and of the *constant areal velocity* $\left(\frac{c}{2} = \frac{1}{2}\mathrm{T}\beta\right)$.

(*c*). Writing $\tau = \mathrm{D}\alpha$ = *vector of relative velocity*, and conceiving this new vector τ to be drawn from that one of the two bodies which is here selected for the origin o, the locus of the extremities of the vector τ is (by earlier definitions) the *Hodograph of the Relative Motion*; and this hodograph is proved to be, for the *Law of the Inverse Square*, a *Circle*.

(*d*). In fact, it is shown (p. 302), that for any *law* of central force, the *radius of curvature* of the hodograph is equal to the force, multiplied into the square of the distance, and divided by the doubled areal velocity; or by the *constant parallelogram* c, under the vectors (α and τ) of *position* and *velocity*, or of the *orbit* and the *hodograph*.

(*e*). It follows then, conversely, that the law of the inverse square is the *only law* which renders the hodograph generally a *circle*; so that the law of nature may be characterized, as the *Law of the Circular Hodograph*; from which latter law, however, it is easy to deduce the *form* of the *Orbit*, as a *conic section* with a *focus* at o.

Pages

(*f*). If the *semiparameter* of this *orbit* be denoted, as usual, by p, and if h be the *radius* of the *hodograph*, then (p. 301),

$$h = Mc^{-1} = cp^{-1} = (Mp^{-1})^{\frac{1}{2}}. \quad (Q_4)$$

(*g*). The orbital *excentricity* e is also the hodographic excentricity, in the sense that eh is the distance of the centre H of the hodograph, from the point O which is here treated as the centre of force.

(*h*). The orbit is an *ellipse*, when the point O is *interior* to the hodographic circle ($e < 1$); it is a *parabola*, when O is *on the circumference* of that circle ($e = 1$); and it is an *hyperbola*, when O is an *exterior point* ($e > 1$). And in all these cases, if we write

$$a = p(1 - e^2)^{-1} = ch^{-1}(1 - e^2)^{-1}, \quad (R_4)$$

the constant a will have its usual signification, relatively to the orbit.

(*i*). The quantity Mr^{-1} being here called the *Potential*, and denoted by P, geometrical *constructions* for this quantity P are assigned, with the help of the hodograph (p. 307); and for the *harmonic mean*, $2M(r + r')^{-1}$, between the *two potentials*, P and P', which answer to the extremities T, T′ of any proposed *chord* of that circle: all which constructions are illustrated by a new diagram (fig. 86).

(*j*). If U be the *pole* of the chord TT′; M, M′ the points in which the line OU cuts the circle; L the middle point, and N the pole, of the *new chord* MM′, *one* secant from which last pole is thus the line NTT′; U′ the intersection of *this* secant with the chord MM′, or the *harmonic conjugate* of the point U, with respect to the same chord; and $NT_{,}T'_{,}$ any *near secant* from N, while $U_{,}$ (on the line OU) is the pole of the *near chord* $T_{,}T'_{,}$: then the *two small arcs*, $T_{,}T$ and $T'T'_{,}$, of the hodograph, intercepted *between these two secants*, are proved to be ultimately *proportional* to the *two potentials*, P and P'; or to the *two ordinates* TV, T′V′, namely the perpendiculars let fall from T and T′, on what may here be called the *hodographic axis* LN. Also, the *harmonic mean* between these two ordinates is obviously (by the construction) the line U′L; while UT, UT′, and $U_{,}T_{,}$, $U_{,}T'_{,}$ are *four tangents* to the hodograph, so that *this* circle is *cut orthogonally*, in the *two pairs of points*, T, T′ and $T_{,}$, $T'_{,}$, by *two other circles*, which have the two near points U, $U_{,}$ for their *centres* (pp. 308, 309).

(*k*). In general, for *any motion of a point* (absolute or relative, in one plane or in space, for example, in the motion of the centre of the *moon* about that of the *earth*, under the perturbations produced by the attractions of the sun and planets), with α for the *variable vector* (418) of *position* of the point, the *time* dt which corresponds to any *vector-element* $dD\alpha$ of the *hodograph*, or what may be called the time of *hodographically describing* that element, is the *quotient* obtained by *dividing* the same *element of the hodograph*, by the *vector of acceleration* $D^2\alpha$ in the *orbit*; because we may write *generally* (p. 308),

$$dt = \frac{dD\alpha}{D^2\alpha}, \text{ or } dt = \frac{TdD\alpha}{TD^2\alpha}, \text{ if } dt > 0. \quad (S_4)$$

(*l*). For the law of the *inverse square* (comp. (*a*) and (*i*)), the measure of the *force*, is,

$$TD^2\alpha = Mr^{-2} = M^{-1}P^2; \quad (T_4)$$

the *times* dt, dt', of hodographically describing the small circular *arcs* $T_{,}T$ and $T'T'_{,}$ of the hodograph, being found by multiplying the lengths (*j*) of those two arcs by the mass, and dividing each product by the square of the potential corresponding, are therefore *inversely* as those *two potentials*, P, P', or *directly* as the *distances*, r, r', in the *orbit*: so that we have the proportion,

$$dt : dt' : dt + dt' = r : r' : r + r'. \quad (U_4)$$

(*m*). If we suppose that the *mass*, M, and the *five points* O, L, M, U, U, upon the chord MM′ are *given*, or *constant*, but that the *radius*, h, of the *hodograph*, or the position of the *centre* H on the hodographic *axis* LN, is *altered*, it is found in this way (p. 309) that

although the *two elements of time*, dt, dt', *separately vary*, yet their *sum* remains *unchanged*: from which it follows, that even if the *two circular arcs*, $T_,T$, $T'T_,'$, be *not small*, but still intercepted (j) between *two secants* from the *pole* N of the *fixed chord* MM′, the *sum* (say, $\Delta t + \Delta t'$) of the *two times* is *independent of the radius*, h.

(n). And hence may be deduced (p. 310), by supposing *one* secant to become a *tangent*, this *Theorem of Hodographic Isochronism*, which was communicated without demonstration, several years ago, to the Royal Irish Academy,* and has since been treated as a subject of investigation by several able writers:

If two circular hodographs, having a common chord, which passes through, or tends towards, a common centre of force, be cut perpendicularly by a third circle, the times of hodographically describing the intercepted arcs will be equal.

(o). This *common time* can easily be expressed (p. 310), under the form of the definite integral,

$$\textit{Time of } \mathrm{TMT'} = \frac{2M}{g^3}\int_0^w \frac{dw}{(1-e'\cos w)^2}; \qquad (\mathrm{V}_4)$$

$2g$ being the length of the *fixed chord* MM′; e' the quotient LO : LM, which reduces itself to -1 when O is at M′, that is for the case of a *parabolic orbit*; e' lying *between* ± 1 for an *ellipse*, and *outside* those limits for an *hyperbola*, but being, in all these cases, *constant*; while w is a certain auxiliary angle, of which the sine $= \overline{\mathrm{UT}} : \overline{\mathrm{UL}}$ (p. 312), or $= s(r+r')^{-1}$, if s denote the length $\overline{\mathrm{PP}'}$ of the *chord of the orbit, corresponding* to the *chord* TT′ of the *hodograph*; and w varies from 0 to π, when the whole *periodic time* $2\pi n^{-1}$ for a *closed orbit* is to be computed: with the verification, that the integral (V$_4$) gives, in this last case,

$$M = a^3n^2, \text{ as usual.} \qquad (\mathrm{W}_4)$$

(p). By examining the general composition of the definite integral (V$_4$), or by more purely geometrical considerations, which are illustrated by fig. 87, it is found that, with the law of the inverse square, the *time* t of describing an *arc* PP′ of the *orbit* (closed or unclosed) is a *function* (p. 314) of the *three ratios*,

$$\frac{a^3}{M}, \quad \frac{r+r'}{a}, \quad \frac{s}{r+r'}; \qquad (\mathrm{X}_4)$$

and therefore simply a function of the chord (s, or $\overline{\mathrm{PP}'}$) of the orbit, and of the *sum of the distances* ($r + r'$, or $\overline{\mathrm{OP}} + \overline{\mathrm{OP}'}$) when M and a are given: which is a form of the *Theorem of Lambert*.

(q). The same important theorem may be otherwise deduced, through a quite different analysis, by an employment of *partial derivatives*, and of *partial differential equations in quaternions*, which is analogous to that used in a recent investigation (418), respecting the motions of an *attracting system* of *any number of bodies*, m, m', &c.

(r). Writing now (comp. p. xxxiii) the following expression for the *relative living force*, or for the *mass* ($M = m + m'$), multiplied into the *square* of the *relative velocity* (TDα),

$$2T = -M\mathrm{D}\alpha^2 = 2(P+H) = M(2r^{-1} - a^{-1}); \qquad (\mathrm{Y}_4)$$

introducing the *two new integrals* (p. 314),

$$F = \int_0^t (P+T)dt, \quad (\mathrm{Z}_4), \quad \text{and} \quad V = \int_0^t 2T dt = F + tH, \qquad (\mathrm{A}_5)$$

which have thus (comp. (E$_4$) and (F$_4$)) the *same forms* as before, but with *different* (although *analogous*) *significations*, and may still be called the *Principal* and *Characteristic Functions* of the motion; and denoting by α, α' (instead of α_0, α) the *initial* and *final vectors* of *position*, or of the *orbit*, while r, r' are the two *distances*, and τ, τ' the

* See the *Proceedings* of the 16th of March, 1847. It is understood that the *common centre* O of *force* is occupied by a *common mass*, M.

Pages

two corresponding *vectors of velocity*, or of the *hodograph*: it is found that when M is given, F may be treated as a function of α, α', t, or of r, r', s, t, and V as a function of α, α', a, or of r, r', s, and H; and that their *partial derivatives*, in the first view of these two functions, are (p. 314),

$$D_\alpha F = D_\alpha V = \tau, \quad (B_5); \quad D_{\alpha'} F = D_{\alpha'} V = -\tau'; \qquad (C_5)$$

$$(D_t)F = -H, \quad (D_5); \quad \text{and} \quad D_H V = \frac{2a^2}{M} D_a V = t; \qquad (E_5)$$

while, in the second view of the same functions, they satisfy the *two partial differential equations* (p. 315),

$$D_r F = D_{r'} F, \quad (F_5), \quad \text{and} \quad D_r V = D_{r'} V; \qquad (G_5)$$

along with *two other* equations of the same kind, but of the second degree, for *each* of the functions here considered, which are analogous to those mentioned in p. xxxiv.

(*s*). The equations (F_5) (G_5) express, that the *two distances*, r and r', enter into each of the two functions only by their *sum*; so that, if M be still treated as given, F may be regarded as a function of the *three* quantities, $r + r'$, s, and t; while V, and therefore also t by (E_5), is found in like manner to be a function of the *three* scalars, $r + r'$, s, and a: which last result respecting the *time* agrees with (*p*), and furnishes a *new proof of Lambert's Theorem.*

(*t*). The *three* partial differential equations (*r*) in V conduct, by merely *algebraical* combinations, to expressions for the three partial derivatives, $D_r V$, $D_{r'} V (= D_r V)$, and $D_s V$; and thus, with the help of (E_5), to *two new definite integrals** (p. 317), which express respectively the *Action* and the *Time*, in the *relative motion of a binary system* here considered, namely, the two following:

$$V = \int_{-s}^{s} \left(\frac{M}{r + r' + s} - \frac{M}{4a} \right)^{\frac{1}{2}} ds; \qquad (H_5)$$

$$t = \tfrac{1}{2} \int_{-s}^{s} \left(\frac{4M}{r + r' + s} - \frac{M}{a} \right)^{-\frac{1}{2}} ds; \qquad (I_5)$$

whereof the latter is not to be extended, without modification, beyond the limits within which the radical is finite.

ARTICLE 420.—On the determination of the Distance of a Comet, or new Planet, from the Earth, ... 320

(*a*). The masses of earth and comet being neglected, and the mass of the sun being denoted by M, let r and w denote the distances of earth and comet from sun, and z their distance from each other, while α is the heliocentric vector of the earth ($T\alpha = r$), known by the theory of the sun, and ρ is the unit-vector, determined by observation, which is directed from the earth to the comet. Then it is easily proved by quaternions, that we have the equation (p. 320),

$$\frac{S\rho D\rho D^2\rho}{S\rho D\rho U\alpha} = \frac{r}{z} \left(\frac{M}{r^3} - \frac{M}{w^3} \right), \qquad (J_5)$$

with

$$w^2 = r^2 + z^2 - 2zS\alpha\rho; \qquad (K_5)$$

* References are given to the *First Essay*, &c., by the present writer (comp. the Note to p. xxxiii), in which were assigned integrals, substantially equivalent to (H_5) and (I_5), but deduced by a quite different analysis. It has recently been remarked to him, by his friend Professor Tait of Edinburgh, that while the *area* described, with Newton's Law, about the *full focus* of an orbit, has long been known to be proportional to the *time* corresponding, so the *area about the empty focus* represents (or is proportional to) the *action*.

Pages

eliminating w between these two formulæ, clearing of fractions, and dividing by z, we are therefore conducted in this way to an algebraical *equation of the seventh degree*, whereof *one root* is the *sought distance*, z.

(*b*). The final equation, thus obtained, differs only by its notation, and by the facility of its deduction, from that assigned for the same purpose in the *Mécanique Céleste*; and the *rule* of Laplace there given, for determining, by inspection of a celestial *globe*, *which* of the two bodies (earth and comet) is the *nearer to the sun*, results at sight from the formula (J_5).

ARTICLE 421.—On the Development of the Disturbing Force of the Sun on the Moon; or of one Planet on another, which is nearer than itself to the Sun, ... 320–323

(*a*) Let α, σ be the geocentric vectors of moon and sun; $r(= T\alpha)$, and $s(= T\sigma)$, their geocentric distances; M the sum of the masses of earth and moon; S the mass of the sun; and D (as in recent Series) the mark of derivation with respect to the time: then the *differential equation of the disturbed motion of the moon about the earth* is,

$$D^2\alpha = M\phi\alpha + \eta, \qquad (L_5) \qquad \text{if} \qquad \phi\alpha = \phi(\alpha) = \alpha^{-1}T\alpha^{-1}, \qquad (M_5)$$

and

$$\eta = \textit{Vector of Disturbing Force} = S(\phi\sigma - \phi(\sigma - \alpha)\,; \qquad (N_5)$$

ϕ denoting here a *vector function*, but *not* a *linear* one.

(*b*). If we neglect η, the equation (L_5) reduces itself to the form $D^2\alpha = M\phi\alpha$; which contains (comp. (O_4)) the laws of *undisturbed elliptic motion*.

(*c*). If we develop the disturbing vector η, according to ascending powers of the quotient $r : s$, of the distances of moon and sun from the earth, we obtain an *infinite series* of terms, each representing a *finite group* of *partial disturbing forces*, which may be thus denoted

$$\eta = \eta_1 + \eta_2 + \eta_3 + \&c.\,; \qquad (O_5)$$

$$\eta_1 = \eta_{1,1} + \eta_{1,2}, \quad \eta_2 = \eta_{2,1} + \eta_{2,2} + \eta_{2,3}\ \&c.\,; \qquad (P_5)$$

these partial forces *increasing* in *number*, but *diminishing* in *intensity*, in the passage from any one group to the following; and being connected with each other, *within* any such group, by simple *numerical ratios* and *angular relations*.

(*d*). For example, the *two forces* $\eta_{1,1}$, $\eta_{1,2}$ of the *first group* are, *rigorously*, proportional to the numbers 1 and 3; the *three* forces $\eta_{2,1}$, $\eta_{2,2}$, $\eta_{2,3}$ of the *second* group are as the numbers 1, 2, 5; and the *four* forces of the *third* group are proportional to 5, 9, 15, 35: where the separate *intensities* of the *first forces*, in these *three first group*, have the expressions,

$$T\eta_{1,1} = \frac{Sr}{2s^3}; \qquad T\eta_{2,1} = \frac{3Sr^2}{8s^4}; \qquad T\eta_{3,1} = \frac{5Sr^3}{16s^5}. \qquad (Q_5)$$

(*e*). All these *partial forces* are conceived to *act at the moon*; but their *directions* may be represented by the respectively *parallel unit-lines* $U\eta_{1,1}$, &c., drawn *from the earth*, and terminating on a *great circle* of the *celestial sphere* (supposed here to have its radius equal to unity), which passes *through the geocentric* (or *apparent*) *places*, ☉ and ☽, of the sun and moon *in the heavens*.

(*f*). Denoting then the *geocentric elongation* ☉☽ of *moon* from *sun* (in the *plane of the three bodies*) by $+\theta$; and by ☉$_1$, ☉$_2$, and ☽$_1$, ☽$_2$, ☽$_3$ what may be called *two fictitious suns*, and *three fictitious moons*, of which the corresponding elongations from ☉, in the *same great circle* are $+2\theta$, -2θ, and $-\theta$, $+3\theta$, -3θ, as illustrated by fig. 88 (p. 322); it is found that the *directions* of the *two forces* of the *first group* are represented by the *two radii* of this *unit-circle*, which terminate in ☽ and ☽$_1$; those of the *three* forces of the *second* group, by the three radii to ☉$_1$, ☉, and ☉$_2$; and those of the *four* forces of the third group, by the radii to ☽$_2$, ☽, ☽$_1$, and ☽$_3$; with facilities for *extending* all those results (with the requisite modifications), to the *fourth* and *subsequent groups*, by the *same quaternion analysis*.

Pages

(*g*). And it is important to observe, that *no supposition* is here made respecting any *smallness* of *excentricities* or *inclinations* (p. 323); so that *all the formulæ apply*, with the necessary changes of *geocentric* to *heliocentric* vectors, &c., to the *perturbations of the motion of a comet about the sun*, produced by the *attraction of a planet*, which is (at the time) *more distant* than the comet from the sun.

ARTICLE 422.—On Fresnel's Wave, ... 323-352

(*a*) If ρ and μ be two corresponding vectors of *ray-velocity* and *wave slowness*, or briefly *Ray* and *Index*, in a biaxal crystal, the velocity of light in a vacuum being unity; and if $\delta\rho$ and $\delta\mu$ be any infinitesimal *variations* of these two vectors, consistent with the equations (supposed to be as yet unknown), of the *Wave* (or *wave-surface*), and its *reciprocal*, the *Index-Surface* (or *surface of wave-slowness*): we have then first the fundamental *Equations of Reciprocity* (comp. p. 461, vol. i.),

$$S\mu\rho = -1, \quad (R_5); \qquad S\mu\delta\rho = 0, \quad (S_5); \qquad S\rho\delta\mu = 0, \quad (T_5)$$

which are independent of any hypothesis respecting the *vibrations* of the *ether*.

(*b*). If $\delta\rho$ be next regarded as a *displacement* (or vibration), *tangential* to the *wave*, and if $\delta\epsilon$ denote the *elastic force* resulting, there exists then, on Fresnel's principles, a *relation* between these two small vectors; which relation may (with our notations) be expressed by either of the two following equations,

$$\delta\epsilon = \phi^{-1}\delta\rho, \quad (U_5), \quad \text{or} \quad \delta\rho = \phi\delta\epsilon; \quad (V_5)$$

the function ϕ being of that linear, vector, and *self-conjugate* kind, which has been frequently employed in these Elements.

(*c*). The fundamental connexion, between the functional symbol ϕ, and the optical constants *abc* of the crystal, is expressed (p. 330, comp. the formula (W_3) in p. xxxiii) by the *symbolic and cubic equation*,

$$(\phi + a^{-2})(\phi + b^{-2})(\phi + c^{-2}) = 0; \quad (W_5)$$

of which an extensive use is made in the present Series.

(*d*). The *normal component*, $\mu^{-1}S\mu\delta\epsilon$, of the elastic force $\delta\epsilon$, is *ineffective* in Fresnel's theory, on account of the supposed *incompressibility of the ether*; and the *tangential component*, $\phi^{-1}\delta\rho - \mu^{-1}S\mu\delta\epsilon$, is (in the same theory, and with present notations) to be equated to $\mu^{-2}\delta\rho$, for the propagation of a *rectilinear vibration* (p. 324); we obtain then thus, for such a vibration or *tangential displacement*, $\delta\rho$, the expression,

$$\delta\rho = (\phi^{-1} - \mu^{-2})^{-1}\mu^{-1}S\mu\delta\epsilon; \quad (X_5)$$

and therefore by (S_5) the equation,

$$0 = S\mu^{-1}(\phi^{-1} - \mu^{-2})^{-1}\mu^{-1}, \quad (Y_5)$$

which is a *Symbolical Form* of the scalar *Equation of the Index-Surface*, and may be thus transformed,

$$1 = S\mu(\mu^2 - \phi)^{-1}\mu. \quad (Z_5)$$

(*e*). The *Wave-Surface*, as being the *reciprocal* (*a*) of the *index-surface* (*d*), is easily found (p. 326), to be represented by this *other Symbolical Equation*,

$$0 = S\rho^{-1}(\phi - \rho^{-2})^{-1}\rho^{-1}; \quad (A_6)$$

or

$$1 = S\rho(\rho^2 - \phi^{-1})^{-1}\rho. \quad (B_6)$$

(*f*). In such transitions, from one of these reciprocal surfaces to the other, it is found convenient to introduce *two auxiliary vectors*, υ and ω ($= \phi\upsilon$), namely the lines OU and OW of fig. 89; *both* drawn from the *common centre* O of the two surfaces; but υ terminating (p. 325) on the *tangent plane* to the *wave*, and being *parallel* to the direction of the *elastic force* $\delta\epsilon$; whereas ω terminates (p. 328) on the tangent plane to the *index-surface*, and is parallel to the *displacement* $\delta\rho$.

(*g*). Besides the relation,

$$\omega = \phi\upsilon, \quad \text{or} \quad \upsilon = \phi^{-1}\omega, \qquad (C_6)$$

connecting the two new vectors (*f*) with *each other*, they are connected with ρ and μ by the equations (pp. 325, 328),

$$S\mu\upsilon = -1, \quad (D_6); \quad S\rho\upsilon = 0; \qquad (E_6)$$

$$S\rho\omega = -1, \quad (F_6); \quad S\mu\omega = 0; \qquad (G_6)$$

and generally (p. 328), the following *Rule of the Interchanges* holds good: *In any formula involving* $\rho, \mu, \upsilon, \omega$, *and* ϕ, *or some of them, it is permitted to exchange* ρ *with* μ, υ *with* ω, *and* ϕ *with* ϕ^{-1}; *provided* that we at the same time interchange $\delta\rho$ with $\delta\epsilon$, but *not generally** $\delta\mu$ with $\delta\rho$, when these variations, or any of them occur.

(*h*). We have also the relations (pp. 328, 329),

$$-\rho^{-1} = \upsilon^{-1}V\upsilon\mu = \mu + \upsilon^{-1}; \qquad (H_6)$$

$$-\mu^{-1} = \omega^{-1}V\omega\rho = \rho + \omega^{-1}; \qquad (I_6)$$

with others easily deduced, which may all be illustrated by the above-cited fig. 89.

(*i*). Among such deductions, the following equations (p. 330) may be mentioned,

$$(V\upsilon\phi\upsilon)^2 + S\upsilon\phi\upsilon = 0, \quad (J_6); \quad (V\omega\phi^{-1}\omega)^2 + S\omega\phi^{-1}\omega = 0; \qquad (K_6)$$

which show that the *Locus* of *each* of the *two Auxiliary Points*, U and W, wherein the two vectors υ and ω terminate (*f*), is a *Surface of the Fourth Degree*, or briefly, a *Quartic Surface*; of which *two loci* the *constructions* may be connected (as stated in p. 330) with those of the *two reciprocal ellipsoids*,

$$S\rho\phi\rho = 1, \quad (L_6), \quad \text{and} \quad S\rho\phi^{-1}\rho = 1; \qquad (M_6)$$

ρ denoting, for each, an *arbitrary semidiameter*.

(*j*). It is, however, a much more interesting use of these *two ellipsoids*, of which (by (W_5), &c.) the *scalar semiaxes* are a, b, c for the *first*, and a^{-1}, b^{-1}, c^{-1} for the *second*, to observe that they may be employed (p. 327) for the *Constructions* of the *Wave* and the *Index-Surface*, respectively, by a very simple rule, which (at least for the *first* of these two reciprocal surfaces (*a*)) was assigned by Fresnel himself.

(*k*). In fact, on comparing the *symbolical form* (A_6) of the equation of the *Wave*, with the form (H_2) in p. xxv, or with the equation 412, XLI., in p. 253, we derive at once *Fresnel's Construction*: namely, that if the *ellipsoid* (*abc*) be *cut*, by an arbitrary *plane* through its centre, and if *perpendiculars* to that plane be erected at that central point, which shall have the *lengths* of the *semiaxes* of the *section*, then *the locus of the extremities, of the perpendiculars so erected, will be the sought Wave-Surface.*

(*l*). A precisely similar construction applies, to the derivation of the *Index-Surface* from the ellipsoid ($a^{-1}b^{-1}c^{-1}$): and thus the *two auxiliary surfaces*, (L_6) and (M_6), may be briefly called the *Generating Ellipsoid*, and the *Reciprocal Ellipsoid*.

(*m*). The cubic (W_6) in ϕ enables us easily to express (p. 331) the *inverse function* $(\phi + e)^{-1}$, where e is any scalar; and thus, by changing e to $-\rho^{-2}$, &c., *new forms* of the equation (A_6) of the *wave* are obtained, whereof one is,

$$0 = (\phi^{-1}\rho)^2 + (\rho^2 + a^2 + b^2 + c^2)\,S\rho\phi^{-1}\rho - a^2b^2c^2; \qquad (N_6)$$

with an analogous equation in μ (comp. the *rule* in (*g*)), to represent the *index-surface*: so that each of these two surfaces is of the *fourth degree*, as indeed is otherwise known.

* This apparent *exception* arises (pp. 328, 329) from the circumstance, that $\delta\rho$ and $\delta\epsilon$ have their *directions generally fixed*, in this whole investigation (although subject to a *common reversal* by $\pm$), when ρ and μ are given; whereas $\delta\mu$ continues to be used, as in (*a*), to denote *any infinitesimal vector, tangential to the index-surface* at the end of μ.

(*n*). If *either* $S\rho\phi^{-1}\rho$ or ρ^2 be treated as *constant* in (N_6), the *degree* of that equation is depressed from the *fourth* to the *second*; and therefore the *Wave* is *cut*, by *each* of the *two concentric quadrics*,

$$S\rho\phi^{-1}\rho = h^4, \quad (O_6), \quad \rho^2 + r^2 = 0, \qquad (P_6)$$

in a (real or imaginary) *curve of the fourth degree*: of which *two quartic curves*, answering to all scalar values of the constants h and r, the wave is the *common locus*.

(*o*). The *new ellipsoid* (O_6) is *similar* to the ellipsoid (M_6), and *similarly placed*, while the *sphere* (P_6) has r for *radius*; and *every quartic* of the *second system* (*n*) is a *sphero-conic*, because it is, by the equation (A_6) of the wave, the *intersection* of that *sphere* (P_6) with the *concentric* and *quadric cone*,

$$0 = S\rho(\phi + r^{-2})^{-1}\rho\,; \qquad (Q_6)$$

or, by (B_6), with this *other concentric quadric*,*

$$-1 = S\rho(\phi^{-1} + r^2)^{-1}\rho, \qquad (R_6)$$

whereof the *conjugate* (obtained by changing -1 to $+1$ in the last equation) has [p. 346]

$$a^2 - r^2, \quad b^2 - r^2, \quad c^2 - r^2, \qquad (S_6)$$

for the squares of its scalar semiaxes, and is therefore *confocal with the generating ellipsoid* (L_6).

(*p*). For *any point* P of the *wave*, or at the end of *any ray* ρ, the *tangents* to the *two curves* (*n*) have the directions of ω and $\mu\omega$; so that *these two quartics cross each other at right angles*, and *each* is a *common orthogonal* in *all* the curves of the *other system* [p. 345].

(*q*). But the *vibration* $\delta\rho$ is easily proved to be parallel to ω; hence the curves of the *first system* (*n*) are *Lines of Vibration of the Wave*: and the curves of the *second system* are the *Orthogonal Trajectories*† to those *Lines*.

(*r*). In general, the *vibration* $\delta\rho$ has (on Fresnel's principles) the direction of the *projection* of the *ray* ρ on the *tangent plane* to the *wave*; and the *elastic force* $\delta\epsilon$ has in like manner the direction of the *projection* of the *index-vector* μ on the tangent plane to the *index-surface*: so that the *ray* is thus *perpendicular* to the *elastic force* corresponding.

♀ *August* 25, 1865.

(*s*). When a given or first ray, ρ, prolonged or shortened, becomes a second ray, ρ_1, at the same side of the centre O, so that $U\rho_1 = U\rho$, we can easily derive from LXIII. the expression [p. 349]

$$r_1 = T\rho_1 = abch^{-2}, \qquad (T_6)$$

or

$$r_1^2 = a^2b^2c^2\,(S\rho\phi^{-1}\rho)\,; \qquad (U_6)$$

so that the two quantities, h and r, are constant together or variable together: similarly for the two other quantities, h and r, which are obtained from these by interchanging *sheets*.

(*t*). It follows, then, that one *sheet* of the *cone* (Q_6), which has its surface at the centre of the wave, and rests on a sphero-conic (r_1) traced on the wave-sheet, contains also, or may be considered as likewise *resting upon*, a *line of vibration* (h) on the *other sheet*, and reciprocally; so that *each* of these two curves is *projected* into the other, by rays from O, and one would appear as *superposed* on the other, if we imagine them to be *seen* by an eye placed at that point. As a limiting case, when the *projecting cone* reduces itself to one of the two *principal planes*—for example, to the plane (*a*)—then the ellipse (*a*) in that plane may be represented by the equation $h^2 = bc$, and the circle (*a*) has for equation $r_1 = a$; so that the condition (T_6) is satisfied [p. 350].

* For *real curves* of the *second system* (*n*), this *new quadric* (R_6) is an *hyperboloid*, with *one sheet* or with *two*, according as the constant r lies between a and b, or between b and c; and, of course, the *conjugate hyperboloid* (*o*) has *two sheets* or *one*, in the same two cases respectively.

† In a different theory of light (comp. the next Series, 423), these *sphero-conics* on the wave are *themselves* the *lines of vibration*.

Pages

♑ *August* 26, 1865.

(*u*). In fact the *quadric cone* (Q_6) must cut the *quartic wave* in an *octic curve*, or else in a *system* of curves, of which the product of the dimensions is *eight*; and accordingly we find, as above, that the *complete* intersection, here considered, of the two surfaces, consists of a system of two *quartic curves*, namely, a *sphero-conic* (r_1) on *one sheet*, and a *line of vibration* (h) on the other [p. 349].

(*v*).* [The section of the wave by a principal plane of the generating ellipsoid (L_6) breaks up into a circle and an ellipse (p. 332)

$$\rho^{-2} + a^{-2} = 0, \qquad (V_6); \qquad 1 - b^{-2}c^{-2}S\rho\phi^{-1}\rho = 0, \qquad (W_6)$$

which we may refer to as the circle (a) and the ellipse (a). The intersections of a circle and the corresponding ellipse are nodal points on the wave. Those on the circle (b) and the ellipse (b) alone are real, and may be called by pre-eminence the *Wave-Cusps*. And the vectors ($\pm\rho_0$, $\pm\rho_1$) drawn from the centre o to these four cusps may be termed *Lines of Single Ray-Velocity*, or briefly *Cusp-Rays* (p. 332). At a wave-cusp the vector μ is indeterminate (p. 334) but it is an edge of the cone (p. 336)

$$\mu^2 + S\mu\rho_0 S\mu\mu_0 = 0 \qquad (X_6)$$

where μ_0 corresponds to ρ_0 as terminating on the ellipse (b) (p. 334). *Analogous* cusps lie on the Index Surface at the ends of the vectors ($\pm\nu_0$, $\pm\nu_1$) of *Single Normal Slowness* (p. 335). The tangent cone to a wave-cusp (p. 335) may be thus briefly written (p. 341)

$$(S\mu_0\rho_0\rho)^2 = 4S\rho_0\rho S\mu_0\rho \qquad (Y_6)$$

with various other transformations (pp. 342–4).]

[(*w*). There are four real Circular Ridges on the wave along which it is touched by the four planes

$$S\rho\nu_0 = \pm 1, \quad S\rho\nu_1 = \pm 1 \qquad (Z_6)$$

$\pm\nu_0$ and $\pm\nu_1$ being the vectors thus designated in the last paragraph. The common length of the diameters of these circles is $b^{-1}(a^2 - b^2)^{\frac{1}{2}}(b^2 - c^2)^{\frac{1}{2}}$ and each diameter in the principal plane subtends at the wave centre the angle $\tan^{-1}b^{-2}(a^2 - b^2)^{\frac{1}{2}}(b^2 - c^2)^{\frac{1}{2}}$ (p. 337). In virtue of the law of reciprocity these ridges *correspond* to the conical points on the Index Surface. New determination of a circular ridge by means of its vector equation and without assuming any knowledge of the existence of a wave-cusp. The relation

$$(\phi - \rho^{-2})(\mu + \rho^{-1})^{-1} = \rho^{-1} \qquad (A_7)$$

* [The paragraphs (*s*), (*t*) and (*u*), accidentally omitted in the First Edition, were first printed by the Rev. R. P. Graves in the Appendix to the Third Volume of his *Life of Hamilton* p. 640. They are of peculiar interest as they show that in spite of severe illness Hamilton was occupied in his work until a few days before his death which took place on the 2nd of September. In the manuscript-book nothing follows after (*v*). The Rev. Charles Graves in his Presidential éloge delivered to the Royal Irish Academy referred to Hamilton's labours in the following terms:—"It will be a satisfaction to the members of this Academy to be told that his *Elements of Quaternions*—the work upon which he was engaged with most unceasing activity for the last two years—is all but complete. I have reason to know that at no period of his life—not even when he was in the prime of health and youthful vigour—did he apply himself to his mathematical labours with more devoted diligence. Those who did not actually know how he was employed, or who had formed a false estimate of his character, might imagine him indolently reposing upon his laurels, or pursuing his studies in a desultory way. Such a conception of them would be the very opposite to the true one. His diligence of late was even excessive—interfering with his sleep, his meals, his exercise, his social enjoyments. It was, I believe, fatally injurious to his health."—*Proceedings*, Royal Irish Academy, vol. ix., p. 315, and Graves's *Life of Sir W. R. Hamilton*, vol. iii., p. 224.]

Pages

generally determines the index-vector μ when the ray-vector ρ is given, but when $\phi - \rho^{-2}$ is a binomial the vector μ becomes indeterminate provided ρ is perpendicular to the direction β satisfying $(\phi - \rho^{-2})\beta = 0$. The vector equation of the Index-Ridge is then (p. 338)

$$V\beta(\mu + \rho_0^{-1})^{-1} - V\beta(\mu_0 + \rho_0^{-1})^{-1} = 0 \qquad (B_7)$$

and the vector equation of the Wave-Ridge is (p. 339)

$$V\beta(\rho + \nu_0^{-1})^{-1} - V\beta(\sigma_0 + \nu_0^{-1})^{-1} = 0 \qquad (C_7)$$

where $\sigma_0 = - a^2c^2\phi\nu_0$ (CXIII., p. 337). The existence of the circular ridges may also be manifested (p. 344) by reducing the equation of the wave-surface to

$$(2\rho^2 - (a^2 - c^2)\,S\nu_0\rho S\nu_1\rho + a^2 + c^2)^2 = (a^2 - c^2)^2\,\{1 - (S\nu_0\rho)^2\}\,\{1 - (S\nu_1\rho)^2\} \qquad (D_7).]$$

[(*x*). The laws of the two sets of vibrations, at a cusp and on a ridge, are illustrated by fig. 89 and are intimately connected with the *two Conical Refractions, external* and *internal,* in a biaxial crystal (p. 341). In the first case the vibration is in the tangent plane at the cusp P to the ellipsoid (*b*) (compare W_6) and has the direction of a chord PR of the cone resting upon the index-ridge. In the second case the vibration at R has the direction of the chord RQ' of the wave-ridge through the point on the circle (*b*) (p. 340).]

[(*y*). In addition to the symbolic forms of the equation of the wave (A_6) and (B_6) (paragraph *e*) and (N_6) (paragraph *m*) the cyclic transformation is employed to derive this new equation (p. 332)

$$g\rho^2 = 1 + S\lambda\rho S\lambda'\rho \pm TV\lambda\rho TV\lambda'\rho \qquad (E_7)$$

(the upper sign belonging to one sheet, and the lower to the other sheet) with several other expressions. The bifocal transformation affords the equation (p. 344)

$$(2\rho^2 - (a^2 - c^2)\,S\nu_0\rho S\nu_1\rho + a^2 + c^2)^2 = (a^2 - c^2)^2\,\{1 - (S\nu_0\rho)^2\}\,\{1 - (S\nu_1\rho)^2\} \qquad (F_7)$$

already referred to (*w*), and the equation $T(\iota\rho + \rho\kappa)^2 = (\kappa^2 - \iota^2)^2$ has been selected by Professor Tait as the basis of his paper on "Quaternion Investigations connected with Fresnel's Wave-Surface" (p. 350). Some leading expressions are written down showing the Cartesian equivalents of quaternion forms (p. 352).]

[(*z*). Although the italic letters *i, j, k* are not now much used having been superseded by general signs of operation such as S, V, T, U, K, they may be supposed to be still familiar to the student as links between quaternions and coordinates, p. 351.]

ARTICLE 423.—MacCullagh's Theorem of the Polar Plane, ... 352–358

[(*a*). The vectors ρ, ρ' and ρ'' representing respectively the ray-velocities of light incident on, and refracted and reflected by, a biaxial crystal, and μ' being the index-vector for the refracted light, by all wave theories of light (p. 353)

$$\rho^2 = S\mu'\rho' = \rho''^2 = -1, \quad (G_7); \qquad \rho'' = -\nu\rho\nu^{-1}, \quad (H_7); \qquad \nu = \mu' - \rho, \quad (J_7)$$

where ν is a normal to the face. The corresponding vectors of vibration being τ, τ', τ'', by all theories of tangential vibration

$$S\rho\tau = 0, \quad (K_7); \qquad S\mu'\tau' = 0, \quad (L_7); \qquad S\rho''\tau'' = 0 \quad (M_7).]$$

[(*b*). To these MacCullagh adds I. that the vibration in the crystal is perpendicular to ρ', or

$$S\rho'\tau' = 0; \qquad (N_7)$$

he also assumes II. the Principle of Equivalent Vibrations expressed by

$$\tau - \tau' + \tau'' = 0, \qquad (O_7)$$

III. the Principle of Vis Viva and IV. the Principle of constant Density of the Ether, jointly expressed by

$$S\nu(\rho\tau^2 - \rho'\tau'^2 + \rho''\tau''^2) = 0 \qquad (P_7).]$$

[(c). Eliminating ρ'' and τ'' and solving for τ it is found (p. 354) that

$$2\tau S\rho\nu = V\rho\nu'\tau' \qquad (Q_7) \qquad \text{if} \qquad \nu' = \mu' - \rho' \qquad (R_7)$$

which includes one form of the enunciation of the Theorem of the Polar Plane as expressed by the equation (p. 355)

$$S\nu'\tau\tau' = 0. \qquad (S_7)]$$

[(d). If ω is an arbitrary vector (p. 356) the equations had to

$$V\nu V\{(\rho - \omega)\tau - (\rho' - \omega)\tau' + (\rho'' - \omega)\tau''\} = 0 \qquad (T_7)$$

and this equation combined with the principle of Rectangular Vibrations contained in equations (K_7), (M_7) and (N_7) is sufficient to give the same direction of τ' and the same dependencies of τ and τ'' thereon as those expressed by (O_7), (P_7), (Q_7) and (S_7). Equation (T_7) expresses that three forces τ, $-\tau'$, τ'' applied at the extremities of ρ, ρ', ρ'' would be equivalent to a couple having its axis parallel to ν.]

APPENDIX.

Pages

Note I.—On Quaternion Determinants, ... 361–363

(1.) Convention respecting expansion, p. 361.—(2.) Examples, p. 361.—(3.) Multiplication of determinant by scalar determinant, p. 361.—(4.) Vanishing determinants derived from linear equations, p. 362.—(5.) Linear equations derived from vanishing determinants, p. 362.—(6.) Vanishing determinants with four identical rows, p. 363.

Note II.—Miscellaneous Properties of Two Linear Vector Functions, . 363–365

(1.) Simultaneous reduction of two functions, ϕ and θ, p. 363.—(2.) Symbolic cubics of products unaltered by cyclical transposition, p. 363.—(3.) Mutually conjugate directions for two quadrics, p. 364.—(4.) Expression of a function as a product of two self-conjugate functions Φ_1 and Φ, p. 364.—(5.) Condition for the simultaneous reduction of two functions to $\Phi_1\Phi$ and $\Phi_2\Phi$, p. 364.—(6.) Reduction (1.) fails or is indeterminate according as $\phi^{-1}\theta$ has coincident or indeterminate axes, p. 364.—(7.) Coaxial functions are commutative and conversely, p. 364.

Note III.—The Strain Function, ... 365–373

(1.) Strain with or without perversion, p. 365.—(2.) Ratio of volumes expressed by m, p. 365.—(3.) Criterion as to perversion afforded by sign of m, p. 365.—(4.) Pure strain, strain ellipsoid, p. 366.—(5.) Preliminary to reduction of strain to rotation followed by pure strain, p. 366.—(6.) Determination of square of function producing the pure strain, p. 367.—(7.) Principal square root defined and pure strain found, p. 367.—(8.) Calculation of the rotation, p. 367.—(9.) Reduction to rotation following pure strain, p. 368.—(10.) One direction unaltered in case of strain; quartic cone locus of directions whose inclinations to a given direction are unchanged, p. 368.—(11.) Roots of a strain function as altered by superposed rotation, p. 369.—(12.) Axes may be rendered indeterminate. Hamilton's cyclic form. Decomposition of strain suggested, p. 369.—(13.) Case of displacement parallel to a plane, and to a line, p. 370.—(14.) Case of a shear, p. 370.—(15.) Reduction of strain to dilatation, pair of shears and rotation, p. 370.—(16.) Displacement function, p. 372.—(17.) Small strain, p. 372.—(18.) Superposition of strains, p. 372.—(19.) Examples on Hamilton's forms, p. 373.—(20.) Strain converting one quadric into another, p. 373.

Note IV.—On the Specification of Linear Vector Functions, . . . 374–376

(1.) Function determined by three changes of lengths and three of directions, p. 374.—(2.) Four changes of directions, one of length, p. 374.—(3.) Example on two self-conjugate triads, p. 374.—(4.) Six changes of lengths and one of direction, p. 375.—(5.) Hamilton's *aconic function.* Pascal's theorem, p. 375.—(6.) Five changes of lengths, two of directions, p. 376.

Pages

NOTE V.—Invariants of Linear Vector Functions, ... 376–384

(1.) Geometrical interpretation of vanishing of first invariant m'', or sum of roots, p. 376.—(2.) Of second invariant m', or sum of products of roots, p. 376.—(3.) Of sum of square roots of latent roots, p. 377.—(4.) Of sum of n^{th} roots, p. 377.—(5.) Invariant condition that quadric cone may remain unaltered by strain, p. 377.—(6.) Anharmonic relations, p. 378.—(7.) Invariants depending on equality of roots, p. 378. Condition for indeterminate axes, p. 379.—(8.) Invariants of ϕ depending on roots are also invariants of $\theta\phi\theta^{-1}$, p. 379.—(9.) Invariants of ϕ_1 and ϕ_2 depending on roots of $\phi_2^{-1}\phi_1$. These are unchanged when the functions are multiplied by χ and into θ, p. 379.—(10.) Interpretations, p. 380.—(11.) Case of self-conjugate functions. Latitude of transformation curtailed. Non-projective element, p. 380.—(12.) Vanishing of invariants as depending on rotations not on pure strains, p. 381.—(13.) Covariant functions, p. 381.—(14.) Quaternion invariants as quotients of determinants, p. 382.—(15.) Structure of these invariants, p. 382.—(16.) Deduction of Hamilton's invariants, p. 383.—(17.) Nature of these invariants, p. 383.—(18.) Reduction of their number, p. 383.—(19.) Quotient of determinants of second order, p. 384.

NOTE VI.—On the System of Linear Vector Functions $\phi + t\theta$, . . . 384–387

(1.) Locus of axes. Cubic cone, p. 384.—(2.) Axes of conjugate system. Relations between cones, p. 384.—(3.) Case of self-conjugation, p. 384.—(4.) Envelope of principal planes, p. 384.—(5.) Six functions with coincident axes, p. 385.—(6.) No function has indeterminate axes, p. 385.—(7.) No two functions have a common axis, p. 385.—(8.) Axes form co-residual triads on cubic cone, p. 385.—(9.) Elliptic functions. Sum of parameters constant; half a period for self-conjugate functions. Perspective property, p. 385.—(10.) Extension of results. Locus of axes of $(a_1\phi + b_1\theta + c_1)^{-1}(a_2\phi + b_2\theta + c_2)$, p. 386.—(11.) Generalization of sections (6.) and (7.), p. 386.—(12.) Axes still co-residual, p. 387.

NOTE VII.—On the General Linear Transformation in Space, . . . 387–390

(1.) Formula of relation $\varpi = f\rho$. Inverse relation $\rho = f^{-1}\varpi$, p. 387.—(2.) United points, p. 387.—(3.) Conjugate function f'. Correspondence of planes, p. 388.—(4.) Change of origin, p. 388.—(5.) Anharmonic property, p. 388.—(6.) Self-conjugate function. Application to quadric. Reciprocal quadric. Self-conjugate tetrahedron for two quadrics, p. 389.—(7.) Condition that change of origin should render a function self-conjugate, p. 389.

NOTE VIII.—On the Theory of Screws, ... 390–397

(1.) Twists, twist-velocities and wrenches, p. 390.—(2.) Intensity and amplitude. Resultant. Screw-systems, p. 390.—(3.) Virtual coefficient. Reciprocal screws, p. 390.—(4.) Canonical form for two-system. Cylindroid, p. 391.—(5.) Three-system. Pitch quadrics, p. 391.—(6.) Co-reciprocal screws, p. 392.—(7.) Reciprocal three-system. Reduction of wrench to six co-reciprocals, p. 392.—(8.) Reduction of systems of fourth and fifth order and determination of the reciprocal systems, p. 392.—(9.) Screw invariants as determinants. Application to two-system, p. 393.—(10.) Sexiant of six screws. Linear relation for seven twists or wrenches, p. 393.—(11.) Screw reciprocal to five given screws expressed by determinants, p. 394.—(12.) Chiastic correspondence, p. 394.—(13.) Application to case of rigid body acted on by impulses. Reduced wrenches, p. 394.—(14.) Principal screws for freedom of the n^{th} order. Reality of these screws, p. 395.—(15.) Permanent screws for freedom of the third order, p. 396.—(16.) Principal screws determined and contrasted with permanent screws, p. 396.—(17.) Unique correlation of two three-systems; corresponding impulsive and instantaneous screws. Dynamical constants of a body deduced from the effects of known wrenches, p. 396.

Pages

NOTE IX.—On Finite Displacements, ... 397–403

(1.) Finite displacement of a body, p. 397.—(2.) Translations complementary to rotations about different axes, p. 398.—(3.) Relation connecting the translations and determination of the corresponding rotations, p. 398.—(4.) Central axis, p. 398.—(5.) Displacement effected by two rotations. Relations connecting these rotations, p. 398.—(6.) Successive displacements, p. 399.—(7.) Continuous motion, p. 399.—(8.) Locus of axes of instantaneous screws. Relation of body and space loci, p. 400.—(9.) Case of two degrees of freedom, p. 400.—(10.) Conditions that a carried plane may envelope a surface, the point of contact being fixed in the plane, p. 401.—(11.) Case of all screws having the same pitch. Applicable surfaces, p. 402.—(12.) Formulæ for moving axes, p. 403.

NOTE X.—On the Kinematical Treatment of Curves, ... 403–410

(1.) Explanatory. Angular velocity of vector units. Angles through which system has turned round tangent and binormal, p. 403.—(2.) Formulæ of differentiation, p. 404.—(3.) Development in powers of arc, p. 404.—(4.) Contacts and deviations. Two methods for finding osculating twisted cubic, p. 404.—(5.) Example on integration. Constant ratio of curvature to torsion, p. 405.—(6.) General vector differential equation of curve, p. 405.—(7.) Example of spherical curve, p. 406.—(8.) Locus of centre of spherical curvature, p. 406.—(9.) Deviation from osculating sphere, p. 407.—(10.) System of curves having given locus of spherical curvature, p. 407.—(11.) Conditions for constant inclinations of unit vectors of two curves, p. 407.—(12.) Skew emanant surfaces. Line of striction. Parameter of distribution* or *pitch*, p. 408.—(13.) Anharmonic properties of ruled surface, p. 408.—(14.) Emanant developables. Lines of striction, p. 409.

NOTE XI.—On the Kinematical Treatment of Surfaces, ... 410–417

(1.) Rotations. Formulæ of differentiation, p. 410.—(2.) Translations. Line element, p. 411.—(3.) Differential equations of rotations and translations, p. 411.—(4.) Principal curvatures. Proof that measure of curvature is unchanged by bending, p. 412.—(5.) Lines of curvature; simplified differential equations, p. 412.—(6.) Curvature of arbitrary curve on surface, p. 413.—(7.) Normal component of curvature, p. 413.—(8.) Tangential component or geodesic curvature, p. 414.—(9.) Torsion of curve. Geodesic torsion, p. 414.—(10.) Consequences of Bonnet's relation, p. 415.—(11.) Geodesics as one system of curves of reference. Total curvature, p. 415.—(12.) Case of ruled surfaces, p. 416.—(13.) Determination of orthogonal system for ruled surfaces, p. 417.—(14.) Differential equation of orthogonal system for an arbitrary surface, p. 417.

NOTE XII.—On Systems of Rays, ... 417–431

(1.) Vector equations of regulus, congruency and complex, p. 417.—(2.) Explanation of notation, p. 418.—(3.) Displacement of a ray into a neighbouring position effected by a twist and a translation, p. 418.—(4.) When the translation has a definite direction the twist belongs to a two-system, p. 418.—(5.) Translation zero for congruency. Properties of congruency deduced from properties of cylindroid, p. 419.—(6.) Properties of complex deduced by translating rays of congruency, p. 420.—(7.) Calculation of principal and of extreme virtual

* Hamilton has called this the *coefficient of undevelopability*. (*Theory of Systems of Rays* [45.], *Trans. R.I.A.*, vol. xv.)

foci for congruency, p. 420.—(8.) Inversion of functions, p. 421, and (9.) Explicit forms of equations for the foci. Condition of reality. Central point, p. 421. —(10.) First natural system of vectors of reference. Hamilton's law of the virtual foci, p. 422.—(11.) Second natural system of lines of reference. Equation of cylindroid virtually contained in Hamilton's law, p. 422.—(12.) Foci by projection, p. 423.—(13.) Law of these foci. Ruled surface derivable from cylindroid, p. 423.—(14.) Curvatures of an element of surface orthogonal to a small pencil of rays. Density of the congruency, p. 424.—(15.) Condition for an orthogonal family of surfaces, p. 425.—(16.) Singly infinite system of rays for which the extreme and principal points coincide, p. 425.—(17.) Different class of conditions for coincidence of the extreme points, p. 426.—(18.) Developables of the congruency and other surfaces related to it, p. 426.—(19.) Ray determined by a pair of auxiliary vectors, p. 426.—(20.) Scalar equations of systems of rays, p. 426.—(21.) Duality of forms; systems by reciprocation, p. 427.—(22.) Linear complex. Parameter and central axis, p. 427.—(23.) Transformation between points and planes, p. 427.—(24.) Arrangement of rays, p. 428.—(25.) Illustrated by theory of screws, p. 428.—(26.) General screw system, p. 429.—(27.) Regarded as the assemblage of screws reciprocal to certain correspondents. Assemblage of the reciprocal correspondents, p. 429.—(28.) Screw system includes singly infinite system of complexes, p. 429.—(29.) Nodal cones of complex. Kummer surface common to complex of axes of screws of pitch p and of axes of reciprocal correspondents of pitch p, p. 430.—(30.) Congruency of double edges of cones of complex. Order and class. Scalar and vector equations, p. 430.—(31.) Focal surface of congruency is Kummer surface of complex, p. 431.

Note XIII.—On Hamilton's Operator ∇, ... 432–475

(1.) Properties of the vector ν derived from a scalar function of ρ and defined by $df\rho = -S\nu d\rho$, p. 432.—(2.) Notation $\nabla f\rho = \nu$ introduced to suggest the derivation of the vector ν from the scalar $f\rho$, p. 432.—(3.) Examples of the effect of operating by ∇ on scalars, p. 432.—(4.) Operation on sums and products of scalars, p. 433.—(5.) Operation on a scalar function of several variable vectors, p. 433.—(6.) Differentials of vector and quaternion functions of ρ in terms of ∇, p. 434.—(7.) Symbolic separation of differentiating symbol from operand. First analytic expression for ∇ in terms of arbitrary non-coplanar differentials, p. 434.—(8.) Second method of finding the result of operating by ∇ on a vector, p. 435.—(9.) Examples of operating on vectors by ∇, p. 435.—(10.) Examples of operating on vectors by linear functions of ∇, p. 436.—(11.) The operator ∇ treated as a symbolic vector. Successive operations. First proof that ∇^2 is a scalar operator, p. 436.—(12.) Second more elementary proof depending on reduction of ∇ to its well-known form, p. 436.—(13.) Examples on operations with ∇ in this form, p. 437.—(14.) Interpretation of ∇q by the aid of a parallelepiped, p. 438.—(15.) Results extended to small surface of any form, p. 438.—(16.) Direct and independent method, p. 439.—(17.) Illustration of fluid pressure, p. 440.—(18.) Scalar and vector parts of $\nabla\sigma$, σ being a vector function of ρ, p. 440.—(19.) Illustration of $S\nabla\sigma$ by displacement, force and flux. Equation of continuity, p. 441.—(20.) Small surface taken to be elements of a tube of flow, p. 441.—(21.) Selection of shape of surface to illustrate $V\nabla\sigma$, p. 442.—(22.) Line integrals and surface integrals. Interpretation of $Vd\nu\nabla . q$, p. 443.—(23.) Scalar and vector parts of $Vd\nu\nabla . \sigma$. Interpretation of $V\nabla\sigma$, p. 444.—(24.) Illustration from distribution of force, p. 444.—(25.) If $\phi\alpha = -S\alpha\nabla . \sigma$ and if χ is Hamilton's auxiliary function, $-\chi'\alpha = -V.V\alpha\nabla . \sigma$ represents the mechanical force on a circuit normal to α in which unit current flows, σ being the magnetic induction, p. 444.—(26.) Magnetic illustration of the result of operating by $Vd\nu\nabla$ on a scalar, p. 445.—(27.) On the linear function $\phi\alpha = -S\alpha\nabla . \sigma$, p. 446.—(28.) Second auxiliary function ψ. Expressions for invariants, p. 446.—(29.) Examples on expansion by Taylor's series in terms of ∇, p. 447.—(30.) Conditions for the

Pages

transformation $\int \nabla q dv = \int d\nu q$, p. 448.—(31.) Case of q discontinuous, p. 448.—(32.) Case of q multiple-valued, p. 448.—(33.) Case of q infinite at a point, p. 449.—(34.) Case of multiply-connected region, p. 449.—(35.) Conditions for the transformation $\int d\rho q = \int Vd\nu\nabla . q$, p. 449.—(36.) On the transformation $\int f d\nu = \int f \nabla . dv$; $\int F d\rho = \int F V d\nu\nabla$, p. 450.—(37.) Potential functions, p. 450.—(38.) Interpretation of $p = \nabla^{-2} q$, p. 451.—(39.) Integration throughout limited field, p. 451.—(40.) Quaternion as the result of operating by ∇ on another quaternion, or $q = \nabla Q$, p. 452.—(41.) Relation of Q to q, p. 453.—(42.) Application to vortex motion, p. 453.—(43.) Case of re-entrant vortices, p. 453.—(44.) Equations of motion of a continuous distribution of matter, p. 454.—(45.) Euler's method, p. 454.—(46.) Lagrange's method, p. 455.—(47.) Comparison of the two equations of continuity, p. 455.—(48.) On the forces acting on an element, p. 456.—(49.) Molecular couples. Nature of the stress, p. 456.—(50.) Equation of motion, p. 457.—(51.) Lord Kelvin's theorem on flow, p. 457.—(52.) Case of a perfect fluid, p. 458.—(53.) Interpretation of $D_t V\nabla\sigma$ and $V\nabla D_t\sigma$, p. 458.—(54.) Stress in a viscous fluid, p. 459.—(55.) Equation of motion, p. 459.—(56.) Case of isotropic elastic solid, p. 459.—(57.) The energy equation. Rate of change of energy in fixed region, p. 459.—(58.) Rate of change of energy of portion of the medium, p. 460.—(59.) Rate of storage and of waste of energy, p. 460.—(60.) The dissipation function, p. 461.—(61.) Stress in terms of strain. Stress across directed area ω, $\phi\omega = \theta(\omega, \nabla, \sigma)$ where θ is a linear function of ω, of σ (the displacement) and of ∇ as operating on σ, p. 461.—(62.) Laws of interchange of vectors in the symbol $(\alpha, \beta, \gamma, \delta) = - S\alpha\theta(\beta, \gamma, \delta)$, p. 462.—(63.) When the energy function exists $\theta(\alpha, \beta, \gamma) = \theta(\alpha, \gamma, \beta)$. Enumeration of elastic constants, p. 463.—(64.) In the same case $\theta(\alpha, \beta, \gamma) - \theta(\beta, \alpha, \gamma) = 2V\Theta V\alpha\beta . \gamma$ where Θ is a self-conjugate function, p. 463.—(65.) On the quartic surface $S\rho\theta(\rho, \rho, \rho)$ = const. Planes of symmetry, p. 464.—(66.) Case in which this surface is spherical, p. 464.—(67.) On the quadric $S\rho\Theta\rho$ = const. Natural system of lines of reference afforded by its axes. Planes of symmetry must be principal planes of the quadric, p. 465.—(68.) Case in which the quadric is of revolution. Angles at which planes of symmetry must intersect, p. 465.—(69.) The quadric a sphere. Determination of three new quadrics by means of which planes of symmetry may be detected, p. 466.—(70.) Digression on functions linear in several vectors, p. 467.—(71.) Applications to spherical harmonics. Expansion in terms of these functions, p. 468.—(72.) Fundamental theorems concerning spherical harmonics, p. 469.—(73.) On the analytical structure of ∇, p. 469.—(74.) Expression for ∇^2 in terms of independent differentials, p. 470.—(75.) Case in which the position of a point is determined by three families of surfaces, p. 470.—(76.) First expression for the square of ∇ when thus specified, p. 478.—(77.) Second expression for ∇ in this case, and for ∇^2, p. 471.—(78.) Simplifications where the surfaces are plane or orthogonal, p. 471.—(79.) Case in which a family is isothermal, p. 472.—(80.) Condition that a family should be isothermal, p. 472.—(81.) Example on confocal quadrics, p. 473.—(82.) Condition that a family should be one of three mutually orthogonal systems, p. 474.

Table of Pages for the Figures in Volumes I. and II.

Figure.	First Edition.	New Edition.	Figure.	First Edition.	New Edition.	Figure.	First Edition.	New Edition.
	Page	Page		Page	Page		Page	Page
1	1	3	33 *bis.*	120	122	59	288	302
2	2	3	34	110	113	60	290	304
3	2	4	35	112	115	61	290	305
4	2	4	35 *bis.*	143	144	62	295	311
5	3	4	36	112	115	63	324	348
6	3	4	36 *bis.*	126	128	63 *bis.*	325	348
7	4	6	37	116	119	64	325	349
8	5	6	38	119	121	65	326	349
9	6	7	39	129	131	66	327	351
10	6	8	40	130	132	67	332	358
11	7	8	41	130	132	68	334	360
12	8	9	41 *bis.*	130	132	69	334	361
13	10	11	42	132	133	70	343	371
14	11	12	42 *bis.*	141	141	71	344	373
15	13	13	43	144	144	72	348	378
16	14	14	44	151	151	73	359	391
17	16	16	45	152	151	74	397	436
18	17	17	5 *bis.*	152	152	75	425	471
19	20	19	46	154	153	76	499	8
20	20	19	47	157	157	77	511	24
21	21	20	47 *bis.*	158	158	78	517	31
22	25	23	48	168	169	79	520	36
23	27	25	49	172	174	80	543	65
24	33	30	50	190	192	81	569	100
25	36	33	51	215	220	82	573	106
26	37	33	52	220	227	83	599	140
27	42	37	53	226	234	84	660	219
28	50	44	54	247	257	85	706	283
29	54	48	55	269	281	86	724	308
30	82	78	55 *bis.*	269	281	87	727	311
31	91	89	56	269	281	88	735	322
32	98	97	57	274	287	89	740	329
33	108	111	58	280	293	—	—	—

Table of Initial Pages of Articles.

Article.	First Edition.	New Edition.	Article.	First Edition.	New Edition.	Article.	First Edition.	New Edition.
	Page	Page		Page	Page		Page	Page
1	1	3	34	23	22	67	54	48
2	2	3	35	24	23	68	55	48
3	3	4	36	26	24	69	55	49
4	3	5	37	28	26	70	57	50
5	4	5	38	29	27	71	57	51
6	5	6	39	30	27	72	58	51
7	5	6	40	30	28	73	58	52
8	5	7	41	31	29	74	59	52
9	6	7	42	32	29	75	59	53
10	6	8	43	33	30	76	60	54
11	7	8	44	33	30	77	61	54
12	8	9	45	34	31	78	61	55
13	8	9	46	35	32	79	62	55
14	9	10	47	36	33	80	62	56
15	9	10	48	37	33	81	62	56
16	10	10	49	37	34	82	63	56
17	10	11	50	38	34	83	64	57
18	11	12	51	39	35	84	64	58
19	11	12	52	39	35	85	65	59
20	12	12	53	40	36	86	65	59
21	12	13	54	41	37	87	66	60
22	13	14	55	42	37	88	67	61
23	14	14	56	43	38	89	68	61
24	14	14	57	44	39	90	68	62
25	15	15	58	46	40	91	69	63
26	16	16	59	46	41	92	69	63
27	17	17	60	47	42	93	77	73
28	18	18	61	48	42	94	80	76
29	19	18	62	49	44	95	83	79
30	19	19	63	50	45	96	85	81
31	20	20	64	51	45	97	88	84
32	22	21	65	53	47	98	90	87
33	22	21	66	53	47	99	95	94

Article.	First Edition.	New Edition.	Article.	First Edition.	New Edition.	Article.	First Edition.	New Edition.
	Page	Page		Page	Page		Page	Page
100	98	96	136	121	123	172	150	150
101	103	107	137	121	123	173	150	150
102	104	107	138	122	124	174	151	151
103	104	108	139	122	124	175	151	151
104	105	109	140	123	125	176	152	152
105	105	109	141	123	125	177	153	152
106	106	109	142	124	125	178	153	153
107	106	109	143	124	126	179	154	154
108	106	110	144	125	126	180	155	155
109	107	110	145	126	127	181	157	157
110	108	111	146	129	131	182	158	157
111	108	111	147	130	132	183	159	159
112	109	112	148	130	132	184	161	161
113	110	113	149	131	133	185	162	163
114	111	113	150	132	134	186	163	164
115	111	114	151	133	135	187	166	167
116	111	114	152	133	135	188	167	169
117	112	115	153	134	135	189	168	169
118	112	115	154	134	136	190	169	170
119	113	115	155	135	136	191	170	171
120	113	116	156	135	136	192	171	173
121	114	117	157	136	137	193	173	174
122	114	117	158	137	138	194	174	175
123	115	117	159	138	139	195	175	176
124	116	118	160	139	140	196	176	177
125	116	118	161	140	140	197	183	185
126	116	119	162	142	143	198	184	186
127	117	119	163	143	143	199	185	187
128	117	119	164	144	144	200	187	189
129	117	120	165	144	145	201	190	192
130	118	120	166	145	145	202	190	193
131	118	120	167	146	146	203	192	194
132	119	121	168	147	147	204	193	196
133	120	122	169	148	148	205	200	204
134	120	122	170	149	148	206	202	205
135	121	123	171	149	149	207	203	206

Article.	First Edition.	New Edition.	Article.	First Edition.	New Edition.	Article.	First Edition.	New Edition.
	Page	Page		Page	Page		Page	Page
208	204	207	244	265	277	280	303	323
209	207	210	245	265	277	281	303	323
210	208	211	246	266	278	282	305	326
211	213	218	247	266	278	283	308	329
212	214	219	248	266	278	284	308	329
213	214	220	249	267	278	285	310	331
214	217	223	250	267	279	286	310	331
215	219	225	251	267	279	287	311	332
216	223	230	252	268	280	288	312	332
217	225	232	253	269	281	289	312	333
218	227	235	254	272	284	290	313	333
219	229	238	255	274	287	291	313	334
220	232	240	256	275	288	292	314	335
221	233	242	257	277	290	293	315	335
222	234	243	258	279	293	294	316	337
223	236	245	259	279	293	295	321	344
224	239	249	260	281	295	296	324	347
225	240	250	261	283	298	297	331	356
226	240	250	262	286	301	298	343	372
227	241	251	263	287	302	299	347	377
228	244	254	264	287	302	300	349	379
229	246	256	265	289	304	301	351	382
230	246	256	266	290	305	302	351	382
231	247	257	267	291	306	303	352	383
232	247	257	268	292	306	304	354	385
233	248	258	269	293	308	305	356	387
234	250	260	270	293	308	306	358	390
235	251	262	271	295	310	307	361	394
236	253	264	272	295	310	308	364	396
237	255	266	273	297	313	309	366	399
238	257	268	274	298	313	310	370	404
239	257	268	275	301	321	311	373	407
240	259	270	276	301	321	312	374	408
241	260	271	277	302	322	313	379	414
242	262	274	278	302	322	314	381	417
243	264	275	279	303	322	315	383	420

Article.	First Edition.	New Edition.	Article.	First Edition.	New Edition.	Article.	First Edition.	New Edition.
	Page	Page		Page	Page		Page	Page
316	384	421	352	447	499	388	529	47
317	391	430	353	453	508	389	531	50
318	391	430	354	459	516	390	535	55
319	391	431	355	464	523	391	537	57
320	392	431	356	466	525	392	538	58
321	393	432	357	468	527	393	539	60
322	394	432	358	470	530	394	541	63
323	399	435	359	474	535	395	549	74
324	400	438	360	481	545	396	554	81
325	401	440	361	482	547	397	559	88
326	403	441	362	484	550	398	578	112
327	404	442	363	485	551	399	612	156
328	405	445	364	487	555	400	621	167
329	406	446	365	491	560	401	626	173
330	407	448	366	495	3	402	630	179
331	408	449	367	496	3	403	631	180
332	409	451	368	496	3	404	633	183
333	411	453	369	496	4	405	636	186
334	412	454	370	498	6	406	638	189
335	414	457	371	500	9	407	649	196
336	416	459	372	501	11	408	653	209
337	417	461	373	502	12	409	664	225
338	420	465	374	508	20	410	667	230
339	421	466	375	509	22	411	674	239
340	422	467	376	511	24	412	679	246
341	423	469	377	512	26	413	689	261
342	427	473	378	513	26	414	694	266
343	429	476	379	513	27	415	698	272
344	431	479	380	515	29	416	707	283
345	432	479	381	519	33	417	709	287
346	435	484	382	520	36	418	713	293
347	436	485	383	522	38	419	717	298
348	439	489	384	524	40	420	733	320
449	441	491	385	524	41	421	734	320
350	443	494	386	525	42	422	736	323
351	445	497	387	527	44	423	757	352

BOOK III.

CHAPTER III.

ON SOME ADDITIONAL APPLICATIONS OF QUATERNIONS, WITH SOME CONCLUDING REMARKS

CHAPTER III.

ON SOME ADDITIONAL APPLICATIONS OF QUATERNIONS, WITH SOME CONCLUDING REMARKS.

SECTION 1.

Remarks Introductory to this Concluding Chapter.

366. When the *Third Book* of the present *Elements* was begun, it was hoped (277) that this Book might be made a much shorter one, than either of the two preceding. *That* purpose it was found impossible to accomplish, without injustice to the subject; but at least an intention was expressed (317), at the commencement of the *Second Chapter*, of rendering that Chapter the *last*: while some new *Examples* of *Geometrical Applications*, and some few *Specimens* of *Physical* ones, were promised.

367. The promise, thus referred to, has been perhaps already in part redeemed; for instance, by the investigations (315) respecting certain *tangents*, *normals*, *areas*, *volumes*, and *pressures*, which have served to illustrate certain portions of the theory of *differentials* and *integrals* of quaternions. But it may be admitted, that the six preceding Sections have treated chiefly of that *Theory of Quaternion Differentials*, including of course its *Principles* and *Rules*; and of the connected and scarcely less important *theory* of *Linear* or *Distributive Functions*, of Vectors and Quaternions: *Examples* and *Applications* having thus played hitherto a merely *subordinate* or *illustrative* part, in the progress of the present Volume.

368. Such was, indeed, *designed* from the outset to be, *upon the whole*, the result of the present undertaking: which was rather to *teach*, than to *apply*, the *Calculus of Quaternions*. Yet it still appears to be possible, without quite exceeding suitable limits, and accordingly we shall now endeavour, to condense into a short *Third Chapter* some *Additional Examples*, geometrical and

physical, of the *application* of the *principles* and *rules* of that Calculus, supposed to be already *known*, and even to have become by this time *familiar** to the reader. And then, with a few general remarks, the work may be brought to its close.

SECTION 2.

On Tangents and Normal Planes to Curves in Space.

369. It was shown (100) towards the close of the First Book, that if the *equation* of a *curve in space*, whether *plane* or of *double* curvature, be *given* under the form,

$$\text{I.} \ldots \rho = \phi(t) = \phi t,$$

where t is a scalar variable, and ϕ is a functional sign, then the *derived vector*,

$$\text{II.} \ldots \mathrm{D}\rho = \mathrm{D}\phi t = \phi' t = \rho' = \mathrm{d}\rho : \mathrm{d}t,$$

represents a *line* which is, or is parallel to, the *tangent* to the curve, drawn at the extremity of the variable vector ρ. If then we suppose that T is a point situated upon the tangent thus drawn to a curve PQ at P and that U is a point in the corresponding normal plane, so that the angle TPU is right, and if we denote the vectors OP, OT, OU by ρ, τ, υ, the *equations* of the *tangent line* and *normal plane* at P may now be thus expressed:

$$\text{III.} \ldots \mathrm{V}(\tau - \rho)\rho' = 0; \qquad \text{IV.} \ldots \mathrm{S}(\upsilon - \rho)\rho' = 0;$$

the *vector* τ being treated as the *only variable* in III., and in like manner υ as the only variable in IV., when once the *curve* PQ is *given*, and the *point* P is *selected*.

(1.) It is permitted, however, to express these last equations under *other forms*; for example, we may replace ρ' by $\mathrm{d}\rho$, and thus write, for the same tangent line and normal plane,

$$\text{V.} \ldots \mathrm{V}(\tau - \rho)\mathrm{d}\rho = 0; \qquad \text{VI.} \ldots \mathrm{S}(\upsilon - \rho)\mathrm{d}\rho = 0;$$

where the *vector differential* $\mathrm{d}\rho$ may represent *any line, parallel to the tangent* to the curve at P, and is *not necessarily small* (compare again 100).

(2.) We may also write, as the equation of the tangent,

$$\text{VII.} \ldots \tau = \rho + x\rho', \text{ where } x \text{ is a scalar variable;}$$

---

* Accordingly, even *references* to former Articles will now be supplied more sparingly than before.

and as the equation of the normal plane,

$$\text{VIII.} \ldots d_\rho T(\upsilon - \rho) = 0, \quad \text{or} \quad \text{VIII}'. \ldots dT(\upsilon - \rho) = 0, \quad \text{if} \quad d\upsilon = 0;$$

because this *partial differential* of $T(\upsilon - \rho)$, or of $\overline{\text{PU}}$, is (by 334, XII., &c.),

$$\text{IX.} \ldots dT(\upsilon - \rho) = S(U(\upsilon - \rho) . d\rho).*$$

(3.) For the *circular locus* 314, (1.), or 337, (1.), of which the equation is,

$$\text{X.} \ldots \rho = \alpha^t \beta, \quad \text{with} \quad T\alpha = 1, \quad \text{and} \quad S\alpha\beta = 0,$$

the equation of the tangent is, by VII., and by the value 337, VI. of ρ',

$$\text{XI.} \ldots \tau = \rho + y\alpha\rho, \quad \text{where} \quad y \text{ is a new scalar variable};$$

the *perpendicularity* of the *tangent* to the *radius* being thus put in evidence.

(4.) For the *plane* but *elliptic locus*, 314, (2.), or 337, (2.), for which,

$$\text{XII.} \ldots \rho = V.\alpha^t \beta, \quad \text{with} \quad T\alpha = 1, \quad \text{but } not \quad S\alpha\beta = 0,$$

the value 337, VIII. of ρ' shows that the tangent, at the extremity of any *one* semidiameter ρ, is *parallel* to the *conjugate* semidiameter of the curve; that is, to the one obtained by altering the *excentric anomaly* (314, (2.)), by a *quadrant*: or to the value of ρ which results, when we change t to $t + 1$.

(5.) For the *helix*, 314, (10.), of which the equation is,

$$\text{XIII.} \ldots \rho = ct\alpha + \alpha^t \beta, \quad \text{with} \quad T\alpha = 1, \quad \text{and} \quad S\alpha\beta = 0,$$

c being a scalar constant, we have the derived vector,

$$\text{XIV.} \ldots \rho' = c\alpha + \frac{\pi}{2} \alpha^{t+1}\beta; \quad \text{whence} \quad \text{XV.} \ldots S\alpha^{-1}\rho' = c,$$

$$\text{XVI.} \ldots TV\alpha^{-1}\rho' = \frac{\pi}{2} T\beta, \quad \text{and} \quad \text{XVII.} \ldots (TV : S)\alpha^{-1}\rho' = \frac{\pi T\beta}{2c};$$

the *tangent line* (ρ') to the *helix* is therefore inclined to the *axis* (α) of the

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* [Again we may write, as the equation of the normal plane,

$$(\text{VII.}) \quad \upsilon = \rho + \xi\rho', \text{ where } \xi \text{ is a variable vector at right angles to } \rho';$$

and as the equation of the tangent,

$$(\text{VIII.}) \quad d_\rho U(\tau - \rho) = 0, \quad \text{or} \quad (\text{VIII}'.) \quad dU(\tau - \rho) = 0, \quad \text{if} \quad d\tau = 0.$$

Geometrically, VIII. expresses that the *length* of the line joining a point in the normal plane to the corresponding point on the curve does not vary when we pass to a consecutive point on the curve, and (VIII.) expresses that the *direction* of the line joining a point on the tangent to the corresponding point on the curve does not change when we pass to a consecutive point on the curve.]

cylinder whereon that curve is traced, at a *constant angle* (a), whereof the *trigonometrical tangent* ($\tan a$) is given by this formula XVII.; and accordingly, the numerator $\pi T\beta$ of that formula represents the *semicircumference* of the cylindric *base*; while the denominator $2c$ is an expression for *half* the *interval* between two *successive spires*, measured in a direction parallel to the axis. We may then write,

$$\text{XVIII.} \ldots \pi T\beta = 2c \tan a = 2c \cot b,$$

if a thus denote the *constant inclination* of the helix to the *axis*, while b denotes the constant and complementary inclination of that curve to the *base*, or to the *circles* which it crosses on the cylinder.

(6.) In general, the *parallels* ρ' to the *tangents* to a curve of *double curvature*, which are drawn from a *fixed origin* O, have a certain *cone* for their *locus*; and for the case of the *helix*, the *equation* of this cone is given by the formula XVII., or by any legitimate transformation thereof, such as the following,

$$\text{XIX.} \ldots SU\alpha^{-1}\rho' = \pm \cos a = \pm \sin b;$$

it is therefore, in this case, a *cone of revolution*, with its semiangle $= a$.

(7.) As an example of the determination of a *normal plane* to a curve of double curvature, we may observe that the equation XIII. of the helix gives,

$$\text{XX.} \ldots \rho^2 = \beta^2 - c^2t^2, \quad \text{and therefore} \quad \text{XXI.} \ldots S\rho\rho' = -c^2t;$$

the equation IV. becomes therefore, for the case of this curve,

$$\text{XXII.} \ldots 0 = S\rho'\upsilon + c^2t, \text{ with the value XIV. of } \rho'.$$

(8.) If then it be required to assign the point U in which the normal plane to the helix *meets the axis* of the cylinder, we have only to combine this equation XXII. with the condition $\upsilon \parallel \alpha$, and we find, by XIII. and XIV.,

$$\text{XXIII.} \ldots \text{OU} = \upsilon = -c^2t\alpha : S\alpha\rho' = ct\alpha, \qquad \text{XXIV.} \ldots S\alpha(\upsilon - \rho) = 0;$$

the line PU is therefore *perpendicular* to the axis, being in fact a *normal* to the *cylinder*.

370. *Another view* of *tangents* and *normal planes* may be proposed, which shall connect them in calculation with *Taylor's Series* adapted to quaternions (342), as follows.

(1.) Writing

$$\text{I.} \ldots \rho_t = \rho_0 + u_t t \rho'_0, \quad \text{or briefly,} \quad \text{I}' \ldots \rho_t = \rho + ut\rho',$$

the *coefficient* u_t or u will *generally* be a *quaternion*, but its *limiting value* will be *positive unity*, when t tends to *zero* as its limit; or in symbols,

$$\text{II.} \ldots u_0 = \lim_{t=0} u = 1.$$

(2.) Admitting this, which follows either from Taylor's Series, or (in so simple a case) from the mere *definition* of the *derived vector* ρ', we may conceive that vector ρ' to be *constructed* by *some* given line PT, without *yet* supposing it to be *known* that this *line* is *tangential* at P to the *curve* PQ, of which the *variable vector* is $\text{OQ} = \rho_t$, while $\text{OP} = \rho_0 = \rho$, so that the line $\text{PQ} = ut\rho'$ is a *vector chord* from P, which *diminishes* indefinitely with the *scalar variable*, t, and is *small*, if t be small.

(3.) Conceiving next that $\omega = \text{OR} =$ the vector of some new and arbitrary point R, we may let fall a perpendicular QM on the line PR, and so *decompose* the *chord* PQ into the *two rectangular lines*, PM and MQ; which, when divided by the same chord, give *rigorously* the *two* (generally) *quaternion quotients*,

$$\text{III.} \ldots \frac{\text{PM}}{\text{PQ}} = \frac{\text{S}u\rho'(\omega - \rho)}{u\rho'(\omega - \rho)}, \qquad \text{IV.} \ldots \frac{\text{MQ}}{\text{PQ}} = \frac{\text{V}u\rho'(\omega - \rho)}{u\rho'(\omega - \rho)};$$

the variable t thus disappearing through the divisions, except so far as it enters into u, which *tends* as above to 1.*

(4.) Passing then to the *limits*, we have these *other* rigorous equations,

$$\text{V.} \ldots \lim. \frac{\text{PM}}{\text{PQ}} = \frac{\text{S}\rho'(\omega - \rho)}{\rho'(\omega - \rho)}, \qquad \text{VI.} \ldots \lim. \frac{\text{MQ}}{\text{PQ}} = \frac{\text{V}\rho'(\omega - \rho)}{\rho'(\omega - \rho)};$$

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* [Here $\text{PQ} = \text{PM} + \text{MQ} = \text{PQ} . \text{PR} . \text{PR}^{-1}$, and separately (vol. I. p. 194) $\text{PM} = \text{S}(\text{PQ} . \text{PR}) . \text{PR}^{-1}$ and $\text{MQ} = \text{V}(\text{PQ} . \text{PR}) . \text{PR}^{-1}$. So we have

$$\frac{\text{PM}}{\text{PQ}} = \text{S}(\text{PQ} . \text{PR}) . \text{PR}^{-1}\,\text{PQ}^{-1} = \frac{\text{S}(\text{PQ} . \text{PR})}{\text{PQ} . \text{PR}} \quad \text{and} \quad \frac{\text{MQ}}{\text{PQ}} = \text{V}(\text{PQ} . \text{PR}) . \text{PR}^{-1} . \text{PQ}^{-1} = \frac{\text{V}(\text{PQ} . \text{PR})}{\text{PQ} . \text{PR}}.$$

The formulæ of these sub-articles may be easily deduced from the consideration of the versor

$$\text{U}\frac{\text{PQ}}{\text{PR}} = \pm \frac{\text{U}u\rho'}{\text{U}(\rho - \omega)},$$

or in the limit

$$\lim. \text{U}\frac{\text{PQ}}{\text{PR}} = \pm\, \text{U}\frac{\rho'}{\rho - \omega}.$$

This reduces to a scalar when R is on the tangent, and to a right versor when it is in the normal plane. Observe that $\text{U}tu\rho' = \pm\, \text{U}u\rho'$.]

by comparing which with 369, III. and IV., we see that those two equations represent respectively, as before stated, the *tangent* and the *normal plane* to the proposed curve at P; because, if $V\rho'(\omega-\rho) = 0$, the *chord* PQ *tends*, by V. or VI., to *coincide*, both in *length* and in *direction*, with its *projection* PM on the line PR; while on the other hand, if $S\rho'(\omega-\rho) = 0$, *that* projection tends to *vanish*, even as *compared* with the chord PQ; which chord tends *now* to coincide with its *other* projection MQ, or with the *perpendicular* to the line PR, erected so as to reach the point Q: whence PR must, in this last case, be a *normal* to the *curve* at P.

(5.) We may also investigate an equation for the *normal plane*, by considering it as the *limiting position* of the plane which *perpendicularly bisects the chord*. If R be supposed to be a point of this last plane, then, with the recent notations, the vector $\omega = \text{OR}$ must satisfy the condition,

$$\text{VII.} \ldots \ T(\omega-\rho_t) = T(\omega-\rho_0), \quad \text{or} \quad \text{VIII.} \ldots \ (\omega-\rho-ut\rho')^2 = (\omega-\rho)^2,$$

or

$$\text{IX.} \ldots \ 2Su\rho'(\omega-\rho) = t(u\rho')^2,$$

in which it may be noted that $u\rho'$ is a *vector* (in the direction of the *chord*, PQ), although u itself is generally a *quaternion*, as before: such then is the equation of the *bisecting plane*, with ω for its variable vector, and its *limit* is,

$$\text{X.} \ldots \ S\rho'(\omega-\rho) = 0, \text{ as before.}$$

(6.) The last process may also be presented under the form,

$$\text{XI.} \ldots \ 0 = \lim. t^{-1}\{T(\omega-\rho_t) - T(\omega-\rho_0)\} = D_t T(\omega-\rho_t), \text{ when } t = 0;$$

and thus the equation 369, VIII. may be obtained anew.

(7.) Geometrically, if we set off on RQ a portion RS equal in length to RP, as in the annexed fig. 76, we shall have the limiting equation,

$$\text{XII.} \ldots \ \pm\overline{\text{SQ}} : \overline{\text{PQ}} = (\overline{\text{RQ}} - \overline{\text{RP}}) : \overline{\text{PQ}} = (\textit{ultimately}) - \cos \text{RPT};$$

which agrees with 369, IX.*

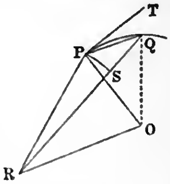

Fig. 76.

(8.) If then the point R be taken *out of* the normal plane at P, this *limit* of the *quotient*, $\overline{\text{RQ}} - \overline{\text{RP}}$ divided by $\overline{\text{PQ}}$, has a *finite value*, positive or negative; and if the *chord* PQ be called *small* of the *first order*, the *difference* of *distances* of its extremities from R may then be said to be *small* of the *same* (first) order. But if R be taken *in* the normal plane at P (and not coincident with that

* [$\overline{\text{SQ}}$ denotes the length of the vector SQ.]

point P *itself*), this difference of distances may *then* be said to be *small*, of an *order higher than the first*: which answers to the *evanescence* of the *first differential* of the *tensor*, $T(\omega - \rho)$ in XI., or $T(\upsilon - \rho)$ in 369, VIII'.

371. A *curve* may occasionally be represented in quaternions, by an equation which is not of the *form*, 369, I., although it must always be conceived capable of *reduction* to that form: for instance, this new equation,

$$\text{I.} \ldots V\alpha\rho \,.\, V\rho\alpha' = (V\alpha\alpha')^2, \quad \text{with} \quad TV\alpha\alpha' > 0,$$

is not *immediately* of the form $\rho = \phi t$, but it is reducible to that form as follows,

$$\text{II.} \ldots \rho = t\alpha + t^{-1}\alpha'.$$

An equation such as I. may therefore have its *differential* or its *derivative* taken, with respect to the scalar variable t on which ρ is thus *conceived* to depend, even if the exact *law* of such dependence be unknown: and $d\rho$, or ρ', may then be changed to the tangential vector $\omega - \rho$ to which it is parallel, in order to form an *equation* of the *tangent*, or a *condition* which the vector ω of a point on that sought line must satisfy.

(1.) To pass from I. to II., we may first operate with the sign V, which gives,

$$\text{III.} \ldots \rho S\alpha\alpha'\rho = 0, \quad \text{or simply,} \quad \text{III'.} \ldots S\alpha\alpha'\rho = 0;$$

whence, t and t' being scalars, we may write,

$$\text{IV.} \ldots \rho = t\alpha + t'\alpha', \quad V\alpha\rho = t'V\alpha\alpha', \quad V\rho\alpha' = tV\alpha\alpha', \quad tt' = 1,$$

and the required reduction is effected: while the *return* from II. to I., or the *elimination* of the scalar t, is an even easier operation.

(2). Under the form II., it is at once seen that ρ is the vector of a *plane hyperbola*, with the origin for *centre*, and the lines α, α' for *asymptotes*; and accordingly all the properties of such a curve may be deduced from the expression II., by the rules of the present Calculus.

(3.) For example, since the derivative of that expression is,

$$\text{V.} \ldots \rho' = \alpha - t^{-2}\alpha',$$

the tangent may (comp. 369, VII.) have its equation thus written:

$$\text{VI.} \ldots \omega = (t + x)\alpha + t^{-2}(t - x)\alpha';$$

it intersects therefore the lines α, α' in the points of which the vectors are $2t\alpha$, $2t^{-1}\alpha'$; so that (as is well known) the *intercept*, upon the tangent, between the *asymptotes*, is *bisected* at the point of *contact*: and the intercepted *area* is *constant*, because $V(t\alpha \,.\, t^{-1}\alpha') = V\alpha\alpha'$, &c.

(4.) But we may also operate *immediately*, as above remarked, on the *form* I.; and thus arrive (by substitution of $\omega - \rho$ for $d\rho$, &c.) at the *equation of conjugation*,

$$\text{VII.} \ldots V\alpha\omega \,.\, V\rho\alpha' + V\alpha\rho \,.\, V\omega\alpha' = 2(V\alpha\alpha')^2,$$

which expresses (comp. 215, (13.), &c.) that if $\rho = \text{OP}$, and $\omega = \text{OR}$, as before, then either R is on the *tangent* to the curve, at the point P, or at least *each* of these two points is situated on the *polar* of the other, with respect to the same hyperbola.

(5.) Again, it is frequently convenient to consider a *curve* as the *intersection* of *two surfaces*; and, in connexion with *this* conception, to represent it by a system of *two scalar equations*, not explicitly involving any *scalar variable*: in which case, *both* equations are to be differentiated, or derivated, with reference to *such* a variable *understood*, and $d\rho$ or ρ' deduced, or replaced by $\omega - \rho$ as before.

(6.) Thus we may substitute, for the equation I., the system of the two following (whereof the first had occurred as III'.):

$$\text{VIII.} \ldots S\alpha\alpha'\rho = 0, \quad \rho^2 S\alpha\alpha' - S\alpha\rho S\alpha'\rho = (V\alpha\alpha')^2;$$

and the derivated equations corresponding are,

$$\text{IX.} \ldots S\alpha\alpha'\rho' = 0, \quad 2S\alpha\alpha' S\rho\rho' - S\alpha\rho' S\alpha'\rho - S\alpha\rho S\alpha'\rho' = 0;$$

or, with the substitution of $\omega - \rho$ for ρ', &c.,

$$\text{X.} \ldots S\alpha\alpha'\omega = 0, \quad 2S\alpha\alpha' S\rho\omega - S\alpha\omega S\alpha'\rho - S\alpha\rho S\alpha'\omega = 2(V\alpha\alpha')^2;$$

the last of which might also have been deduced from VII., by operating with S.

(7.) And it may be remarked that the two equations VIII. represent respectively in general a *plane* and an *hyperboloid*, of which the *intersection* (5.) is the *hyperbola* I. or II.; or a plane and an *hyperbolic cylinder*, if $S\alpha\alpha' = 0$.*

* [If $\rho = \dfrac{\phi(t)}{f(t)}$ where $\phi(t)$ is a rational and integral vector function of degree m in t, and $f(t)$ a rational and integral scalar function of degree n, the degree of the curve is equal to the greater of the two integers m and n. This is evident when we substitute for ρ in the equation of an arbitrary plane, $S\lambda\rho = 1$, for we obtain a scalar equation in t whose roots determine the points in which the curve cuts the plane. Curves of this kind are *unicursal*. In general there is some

SECTION 3.

On Normals and Tangent Planes to Surfaces.

372. It was early shown (100, (9.)), that when a *curved surface* is represented by an equation of the form,

$$\text{I.} \ldots \rho = \phi(x, y),$$

in which ϕ is a functional sign, and x, y are two independent and scalar variables, then either the *two partial differentials*, or the two *partial derivatives*, of the *first order*,

$$\text{II.} \ldots \mathrm{d}_x\rho, \mathrm{d}_y\rho, \quad \text{or}, \quad \text{III.} \ldots \mathrm{D}_x\rho, \mathrm{D}_y\rho,$$

represent *two tangential vectors*, or at least vectors *parallel* to *two tangents* to the surface, drawn at the extremity or *term* P of ρ; so that the *plane* of these two differential vectors, or of lines parallel to them, is (or is parallel to) the *tangent plane* at that point: and the principle has been since exemplified, in 100, (11.) and (12.), and in the sub-articles to 345, &c. It follows that any *vector* ν, which is *perpendicular to both* of two such *non-parallel* differentials, or derivatives, must (comp. 345, (11.)) be a *normal vector* at P, or at least one having the *direction* of the normal to the surface at that point; so that each of the two vectors,

$$\text{IV.} \ldots \mathrm{V} . \mathrm{d}_x\rho \, \mathrm{d}_y\rho, \qquad \text{V.} \ldots \mathrm{V} . \mathrm{D}_x\rho \, \mathrm{D}_y\rho,$$

if *actual*, represents such a normal.

(1.) As an additional example, let us take the case of the *ruled paraboloid*, on which a given *gauche quadrilateral* ABCD is *superscribed*. The expression for the vector ρ of a variable point P of this surface, considered as a function of two independent and scalar variables, x and y, may be thus written (comp. 99, (9.)):

$$\text{VI.} \ldots \rho = xy\alpha + (1-x)y\beta + (1-x)(1-y)\gamma + x(1-y)\delta;$$

where the supposition $y = 1$ places the point P on the line AB; $x = 0$ places it on BC; $y = 0$, on CD; and $x = 1$, on DA.

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irrationality in the functions of t, and the result of substitution in the equation of the plane must be rationalized before the degree of the curve can be determined.

As examples of the equations of curves:—

$\rho = \dfrac{\alpha t^2 + 2\beta t + \gamma}{at^2 + 2bt + c}$ is a conic provided there is no common factor in the numerator or denominator;

$\rho = (\phi + t)^m \alpha$, in which ϕ is a linear vector function and m a constant scalar, represents a right line when $m = 1$, a twisted cubic when $m = -1$, and a twisted quartic when $m = \frac{1}{2}$.]

(2.) We have here, by partial derivations,

$$\text{VII.} \ldots \mathrm{D}_x\rho = y(a-\beta) + (1-y)(\delta-\gamma); \quad \mathrm{D}_y\rho = x(a-\delta) + (1-x)(\beta-\gamma);$$

these then represent the directions of *two* distinct *tangents* to the paraboloid VI., at what may be called *the point* (x, y); whence it is easy to deduce the *tangent plane* and the *normal* at that point, by constructions on which we cannot here delay, except to remark that if (comp. fig. 31, Art. 98) we draw two right lines, QS and RT, through P, so as to cut the sides AB, BC, CD, DA of the quadrilateral in points Q, R, S, T, we shall have by VI. the vectors,

$$\text{VIII.} \ldots \begin{cases} \mathrm{OQ} = xa + (1-x)\beta, & \mathrm{OR} = y\beta + (1-y)\gamma, \\ \mathrm{OS} = x\delta + (1-x)\gamma, & \mathrm{OT} = ya + (1-y)\delta, \end{cases}$$

and therefore, by VII.,

$$\text{IX.} \ldots \mathrm{D}_x\rho = \mathrm{RT}, \quad \mathrm{D}_y\rho = \mathrm{SQ};$$

so that *these* two tangents are simply the *two generating lines* of the surface, which pass through the proposed point P.*

(3.) For example, at the point (1, 1), or A, the *tangents* thus found are the *sides* BA, DA, and the *tangent plane* is that of the *angle* BAD, as indeed is evident from geometry.

(4.) Again, the equation of the *screw surface* (comp. 314, XVI.),

$$\text{X.} \ldots \rho = cxa + ya^x\beta, \quad \text{with} \quad \mathrm{T}a = 1, \quad \text{and} \quad \mathrm{S}a\beta = 0,$$

gives the two tangents,

$$\text{XI.} \ldots \mathrm{D}_x\rho = ca + \frac{\pi}{2}ya^{x+1}\beta, \quad \mathrm{D}_y\rho = a^x\beta,$$

whereof the latter is perpendicular to the former, and to the axis a of the cylinder; so that the corresponding *normal* to the surface X. at the point (x, y) is represented by the product,

$$\text{XII.} \ldots \nu = \mathrm{D}_x\rho \, . \, \mathrm{D}_y\rho = ca^{x+1}\beta + \frac{\pi}{2}y\beta^2 a.$$

373. Whenever a variable vector ρ is thus expressed or even *conceived* to be expressed, as a *function* of *two* scalar variables, x and y (or s and t, &c.),

* [In VIII, Q and S are two variable points dividing homographically AB and DC, and R and T divide BC and AD homographically. The ruled paraboloid is the locus of lines joining corresponding points of the homographic divisions on AB and DC, or on BC and AD, for VI. may be written in either of the forms

$$\rho = y\,\mathrm{OQ} + (1-y)\,\mathrm{OS}, \quad \text{or} \quad \rho = x\,\mathrm{OT} + (1-x)\,\mathrm{OR}.]$$

if we assume any *three* diplanar vectors, such as α, β, γ (or ι, κ, λ, &c.), the *three scalar expressions*, $S\alpha\rho$, $S\beta\rho$, $S\gamma\rho$ (or $S\iota\rho$, $S\kappa\rho$, $S\lambda\rho$, &c.), will then be functions of the same *two* scalar variables; and will therefore be *connected* with each other by some *one* scalar *equation*, of the form,

$$\text{I.} \ldots F(S\alpha\rho, S\beta\rho, S\gamma\rho) = 0,$$

or briefly,

$$\text{II.} \ldots f\rho = C;$$

where C is a scalar constant, introduced (instead of zero) for greater generality of expression; and F, f are used as functional but scalar signs. If then (comp. 361, XIV.) we express the *first differential* of this scalar function $f\rho$ under the form,

$$\text{III.} \ldots df\rho = 2S\nu d\rho,$$

in which ν is a certain *derived vector*, and is *here* considered as being (at least implicitly) a *vector function* (like ρ) of the *two scalar variables* above mentioned, we shall have the two equations,

$$\text{IV.} \ldots S\nu d_x\rho = 0, \quad S\nu d_y\rho = 0,$$

or these two other and corresponding ones,

$$\text{V.} \ldots S\nu D_x\rho = 0, \quad S\nu D_y\rho = 0;$$

from which it follows (by 372) that ν has the direction of the normal to the surface I. or II., at the point P in which the vector ρ terminates. Hence the *equation* of that *normal* (with ω for its variable vector) may, under these conditions, be thus written:

$$\text{VI.} \ldots V\nu(\omega - \rho) = 0;$$

and the corresponding *equation of the tangent plane* at the same point P is,

$$\text{VII.} \ldots S\nu(\omega - \rho) = 0.$$

(1.) For example, if we take the expression 308, XVIII., or 345, XII., namely

$$\text{VIII.} \ldots \rho = rk^t j^s k j^{-s} k^{-t}, \quad \text{in which} \quad kj^{-s} = j^s k, \text{ \&c.},$$

treating the scalar r as constant, but s and t as variable, we have then (comp. 345, XIV.), the equations, a denoting any unit-vector,

$$\text{IX.} \ldots Si\rho = rS\,.\,a^{2t}S\,.\,a^{2s+1}, \quad Sj\rho = rS\,.\,a^{2t-1}S\,.\,a^{2s+1}, \quad Sk\rho = rS\,.\,a^{2s+2};$$

between which s and t can be eliminated, by simply adding their squares, because $(\mathrm{S}a^t)^2 + (\mathrm{S}a^{t-1})^2 = 1$, by 315, V., if $\mathrm{T}a = 1$. In this manner then we arrive at equations of the forms I. and II., namely (comp. 357, VII., and 308, (10.) and (13.)),

$$\text{X.} \ldots (\mathrm{S}i\rho)^2 + (\mathrm{S}j\rho)^2 + (\mathrm{S}k\rho)^2 - r^2 = 0,$$

and

$$\text{XI.} \ldots f\rho = \rho^2 = - r^2 = \text{const.}, \quad \text{or} \quad \text{XI}'. \ldots \mathrm{T}\rho = r;$$

which last results had indeed been otherwise obtained before.*

(2.) With this *form* XI. of $f\rho$, we have the *differential expression* of the first order,

$$\text{XII.} \ldots \mathrm{d}f\rho = 2\mathrm{S}\nu\mathrm{d}\rho = 2\mathrm{S}\rho\mathrm{d}\rho, \quad \text{whence} \quad \text{XIII.} \ldots \nu = \rho;$$

and if we *still conceive* that ρ is, as above, *some vector function* of *two scalar variables*, s and t, although the *particular law* VIII. of its dependence on them may now be supposed to be *unknown* (or to be forgotten), we may write also,

$$\text{XIV.} \ldots \tfrac{1}{2}\mathrm{d}f\rho = \mathrm{S}\nu\mathrm{d}\rho = \mathrm{S}\rho\mathrm{d}\rho = \mathrm{S}\rho(\mathrm{d}_s + \mathrm{d}_t)\rho = \mathrm{S}\rho\mathrm{D}_s\rho \,.\, \mathrm{d}s + \mathrm{S}\rho\mathrm{D}_t\rho \,.\, \mathrm{d}t;$$

if then the *function* $f\rho$ have (as above) a *value*, $= - r^2$, which is *constant*, or is *independent* of *both* the variables, s and t, while *their* differentials are *arbitrary*, and are independent of *each other*, we shall thus have separately (comp. V., and 337, XIII., XVII.),

$$\text{XV.} \ldots \mathrm{S}\rho\mathrm{D}_s\rho = 0, \quad \mathrm{S}\rho\mathrm{D}_t\rho = 0.$$

The *radius* ρ of the *sphere* XI. is therefore in this way seen to have the direction of the *normal* at its own extremity, because it is *perpendicular to two distinct tangents*, $\mathrm{D}_s\rho$ and $\mathrm{D}_t\rho$, at that point; which are indeed, in the present case, perpendicular to *each other* also (337, (8.)).

(3.) Instead of treating the *two* scalar variables, x and y, or s and t, &c., as *both* entirely *arbitrary* and *independent*, we may conceive that *one* is an *arbitrary* (but scalar) *function* of the *other*; and *then* the vector ν, determined by the equation III., will be seen anew to be the *normal* at the extremity P of ρ, because it is *perpendicular* to the *tangent* at P to an *arbitrary curve* upon the surface, which passes through that point: or (otherwise stated)

* [Of course $\rho^2 = r^2k^tj^skj^{-s}k^{-t} \cdot k^tj^skj^{-s}k^{-t} = r^2k^2 = - r^2$.]

because it is a line in an *arbitrary normal plane* at P, if a normal plane to a *curve* on a surface be called (as usual) a normal plane to that *surface* also.

(4.) For example, if we conceive that s in VIII. is thus an arbitrary function of t, the last expression XIV. will take the form,

$$\text{XVI.} \ldots 0 = \tfrac{1}{2}\mathrm{d}f\rho = \mathrm{S}\,.\,\rho(s'\mathrm{D}_s\rho + \mathrm{D}_t\rho)\mathrm{d}t, \quad \text{if} \quad \mathrm{d}s = s'\mathrm{d}t\,;$$

whence, $\mathrm{d}t$ being still arbitrary, we have the *one* scalar equation,

$$\text{XVII.} \ldots \mathrm{S}\,.\,\rho(s'\mathrm{D}_s\rho + \mathrm{D}_t\rho) = 0, \quad \text{or} \quad \text{XVIII.} \ldots \rho \perp s'\mathrm{D}_s\rho + \mathrm{D}_t\rho,$$

and although, on account of the arbitrary coefficient s', this *one* equation XVII. is *equivalent* to the system of the *two* equations XV., yet it *immediately* signifies, as in XVIII., that the *directed radius* ρ, of the sphere XI., is *perpendicular* to the *arbitrary tangent*, $s'\mathrm{D}_s\rho + \mathrm{D}_t\rho$; or to the tangent to an *arbitrary spherical curve* through P, the centre O and tensor $\mathrm{T}\rho$ (or *undirected radius*, r) remaining as before.

(5.) As regards the *logic* of the subject, it may be worth while to read again the *proof* (331), of the validity of the rule for *differentiating a function of a function*; because this *rule* is virtually employed, when after thus reducing, or conceiving as reduced, the scalar function $f\rho$ of a *vector* ρ, to another scalar function such as Ft of a *scalar* t, by treating ρ as equal to some vector function ϕt of this last scalar, we *infer* that

$$\text{XIX.} \ldots \mathrm{d}Ft = \mathrm{d}f\phi t = 2\mathrm{S}\,.\,\nu\mathrm{d}\phi t, \quad \text{if} \quad \mathrm{d}f\rho = 2\mathrm{S}\nu\mathrm{d}\rho, \text{ as before.}$$

(6.) And as regards the *applications* of the formulæ VI. and VII., or of the equations given by them for the *normal* and *tangent plane* to a *surface* generally, the difficulty is only to *select*, out of a multitude of examples which might be given: yet it may not be useless to add a *few* such here, the case of the *sphere* having of course been only taken to illustrate the *theory*, because the normal property of its *radii* was manifest, independently of any calculation.

(7.) Taking then the equation of the *ellipsoid*, under the form,

$$\text{XX.} \ldots \mathrm{T}(\iota\rho + \rho\kappa) = \kappa^2 - \iota^2, \qquad 282, \text{XIX.},$$

of which the first differential may (see the sub-articles to 336) be thus written,

$$\text{XXI.} \ldots 0 = \mathrm{S}\,.\,\{(\iota - \kappa)^2\rho + 2(\iota\mathrm{S}\kappa\rho + \kappa\mathrm{S}\iota\rho)\}\mathrm{d}\rho = \mathrm{S}\nu\mathrm{d}\rho,$$

and introducing an auxiliary vector, ON or ξ, such that

$$\text{XXII.} \ldots \text{ON} = \xi = -2(\iota - \kappa)^{-2}(\iota S\kappa\rho + \kappa S\iota\rho),$$

we have $\nu \parallel \rho - \xi$, and may write, as the equation of the normal at the extremity P of ρ, the following,

$$\text{XXIII.} \ldots V.(\xi - \rho)(\omega - \rho) = 0, \quad \text{or} \quad \text{XXIV.} \ldots \omega = \rho + x(\xi - \rho),$$

in which x is a scalar variable (comp. 369, VII.); making then $x = 1$, we see that ξ is the vector of the point N in which the normal intersects the plane of the two fixed lines ι, κ, supposed to be drawn from the origin, which is here the *centre* of the ellipsoid.

(8.) If we look back on the sub-articles to 216 and 217, we shall see that these lines ι, κ have the directions of the *two real cyclic normals*, or of the normals to the *two* (real) *cyclic planes*; which planes are now represented by the two equations,

$$\text{XXV.} \ldots S\iota\rho = 0, \quad S\kappa\rho = 0.$$

Accordingly the equation XX. of the ellipsoid may be put (comp. 336, 357, 359) under the *cyclic forms*,

$$\text{XXVI.} \ldots S\rho\phi\rho = (\iota^2 + \kappa^2)\rho^2 + 2S\iota\rho\kappa\rho$$
$$= (\iota - \kappa)^2\rho^2 + 4S\iota\rho S\kappa\rho = (\kappa^2 - \iota^2)^2 = \text{const.};$$

hence each of the *two diametral planes* XXV. cuts the surface in a *circle*, the *common radius* of these two *circular sections* being

$$\text{XXVII.} \ldots T\rho = \frac{T\iota^2 - T\kappa^2}{T(\iota - \kappa)} = b,$$

where b denotes, as in 219, (1.), the length of the *mean semiaxis* of the ellipsoid; and in fact, this value of $T\rho$ can be at once obtained from the equation XX., by making either $\iota\rho = -\rho\iota$, or $\rho\kappa = -\kappa\rho$, in virtue of XXV.

(9.) By the sub-article last cited, the *greatest* and *least semiaxes* have for their *lengths*,

$$\text{XXVIII.} \ldots a = T\iota + T\kappa, \quad c = T\iota - T\kappa;$$

and the *construction* in 219, (2.) shows (by fig. 53, annexed to 217, (4.)) that these three semiaxes a, b, c have the respective *directions* of the lines,

$$\text{XXIX.} \ldots \iota T\kappa - \kappa T\iota, \quad V\iota\kappa, \quad \iota T\kappa + \kappa T\iota;$$

all which agrees with the *rectangular transformation*,

$$\text{XXX.}\ldots 1 = \frac{S\rho\phi\rho}{(\kappa^2-\iota^2)^2} = \left(\frac{T(\iota\rho+\rho\kappa)}{\kappa^2-\iota^2}\right)^2$$

$$= \left(\frac{S.\rho U(\iota T\kappa - \kappa T\iota)}{T\iota + T\kappa}\right)^2 + \left(\frac{T(\iota-\kappa)S.\rho UV\iota\kappa}{T\iota^2 - T\kappa^2}\right)^2 + \left(\frac{S.\rho U(\iota T\kappa + \kappa T\iota)}{T\iota - T\kappa}\right)^2,$$

in deducing which (comp. 359, (1.)) from 357, VIII., by means of the formulæ 357, XX. and XXI., we employ the values (comp. XXVI.),

$$\text{XXXI.}\ldots g = \iota^2 + \kappa^2, \quad \lambda = 2\iota, \quad \mu = \kappa.$$

(10.) The *fixed plane* (7.), of the *cyclic normals* ι and κ (8.), is therefore also the plane of the *extreme semiaxes*, a and c (9.), or that which may be called perhaps the *principal plane** of the ellipsoid: namely, the plane of the *generating triangle* (218, (1.)), in that *construction* of the surface (217, (6.) or (7.)) which is illustrated by fig. 53, and was deduced as an *interpretation of the quaternion equation* XX., or of the somewhat less simple form 217, XVI., with the value $T\iota^2 - T\kappa^2$ of t^2.

(11.) Let n denote the length of that portion of the normal, which is intercepted between the surface and the principal plane (10.), so that, by (7.),

$$\text{XXXII.}\ldots n = \overline{\text{NP}} = T(\rho - \xi), \quad n^2 = -(\rho - \xi)^2,$$

with the value XXII. of ξ. Let σ = OS be the vector of a point S on the surface of a new or auxiliary *sphere*, described about the point N as centre, with a radius = n, and therefore *tangential* to the *ellipsoid* at P; and let us inquire in what *curve* or *curves*, real or imaginary, does this sphere *cut* the ellipsoid.

(12.) The *equations* (comp. 371, (5.)) of the sought *intersection* are the two following,

$$\text{XXXIII.}\ldots (\sigma - \xi)^2 + n^2 = 0, \quad \text{and} \quad \text{XXXIV.}\ldots T(\iota\sigma + \sigma\kappa) = \kappa^2 - \iota^2;$$

whereof the first expresses that S is a point of the sphere, and the second that it is a point of the ellipsoid; while ρ or OP enters virtually into XXXIII., through ξ and n, but is *here* treated as a *constant*, the point P being now supposed to be a *given* one.

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* This *plane* may also be said to be the plane of the *principal elliptic section* (219, (9.)); or it may be distinguished (comp. the Note to page 240, vol. i.) as the plane of the *focal hyperbola*, of which important curve we shall soon assign the equation in quaternions.

(13.) We shall *remove* (18.) *the origin* to this point P of the ellipsoid, if we write,

$$\text{XXXV.} \ldots \sigma = \rho + \sigma', \quad \text{or} \quad \text{XXXV}'. \ldots \sigma' = \sigma - \rho = \text{PS};$$

and thus we obtain the new or transformed equations,

$$\text{XXXVI.} \ldots 0 = \sigma'^2 + 2S(\rho - \xi)\sigma', \qquad \text{XXXVII.} \ldots 0 = N(\iota\sigma' + \sigma'\kappa) + 2S\nu\sigma';$$

in which (as in (7.), comp. also 210, XX.),

$$\text{XXXVIII.} \ldots \nu = (\iota - \kappa)^2\rho + 2(\iota S\kappa\rho + \kappa S\iota\rho) = (\iota - \kappa)^2 (\rho - \xi),$$

and

$$\text{XXXIX.} \ldots N(\iota\sigma' + \sigma'\kappa) = (\iota - \kappa)^2\sigma'^2 + 4S\iota\sigma' S\kappa\sigma'.$$

(14.) Eliminating then σ'^2, we obtain from the two equations XXXVI. and XXXVII. this other,

$$\text{XL.} \ldots S\iota\sigma' . S\kappa\sigma' = 0;$$

which like them is of the *second degree* in σ', but breaks up, as we see, into *two linear* and *scalar factors*, representing *two distinct planes*, *parallel* by XXV. to the two *diametral* and *cyclic planes* of the ellipsoid. The sought *intersection* consists then of a *pair* of (real) *circles*, upon that given surface; namely, *two circular* (but *not diametral*) *sections*, which pass *through the given point* P.

(15.) Conversely, because the equations XXXVII. XXXVIII. XXXIX. XL. give XXXVI. and XXXIII., with the foregoing values of ξ and n, it follows that these *two* plane *sections* of the ellipsoid at P are on one *common sphere*, namely that which has N for centre, and n for radius, as above; and thus we might have found, *without differentials*, that the line PN is the *normal* at P; or that this normal *crosses the principal plane* (10.), in the point determined by the formula XXII.

(16.) In general, the *cyclic form* of the equation of *any central surface of the second order*, namely the form (comp. 357, II.),

$$\text{XLI.} \ldots S\rho\phi\rho = g'\rho^2 + 2S\lambda\rho S\mu\rho = C = \text{const.},$$

shows that the *two circles* (real or imaginary) in which that surface is cut by any *two planes*,

$$\text{XLII.} \ldots S\lambda\rho = l, \quad S\mu\rho = m,$$

drawn *parallel* respectively to the two real *cyclic planes*, which are jointly represented (comp. XL., and 216, (7.)) by the one equation,

$$\text{XLIII.} \ldots S\lambda\rho S\mu\rho = 0,$$

are *homospherical*, being *both* on that one sphere of which the equation is,

$$\text{XLIV.} \ldots g'\rho^2 + 2(lS\mu\rho + mS\lambda\rho) = 2lm + C.$$

(17.) But the *centre* (say N) of this *new sphere*, has for its vector (say ξ),

$$\text{XLV.} \ldots \text{ON} = \xi = -g'^{-1}(l\mu + m\lambda);$$

it is therefore situated *in the plane* of the *two real cyclic normals*, λ and μ; and if l and m in XLV. receive the values XLII., then this new ξ is the *vector of intersection* of that *plane*, with the *normal to the surface* at P: because it is (comp. (15.)) the vector of the centre of a sphere which *touches* (though also *cutting*, in the two circular sections) the surface at that point.

(18.) We can therefore thus *infer* (comp. again (15.)), *without the differential calculus*, that the line,

$$\text{XLVI.} \ldots g'(\rho - \xi) = g'\rho + \lambda S\mu\rho + \mu S\lambda\rho = \phi\rho,$$

as having the direction of NP, is the normal at P to the surface XLI.; which agrees with, and may be considered as confirming (if confirmation were required), the conclusion otherwise obtained through the differential expression (361),

$$\text{XLVII.} \ldots dS\rho\phi\rho = 2S\nu d\rho = 2S\phi\rho d\rho;$$

the linear function $\phi\rho$ being here supposed (comp. 361, (3.)) to be self-conjugate.

(19.) Hence, with the notation 362, I., the *equation of the tangent plane* to a central surface of the second order, at the same point P, may by VII. be thus written,

$$\text{XLVIII.} \ldots f(\omega, \rho) = C, \quad \text{if} \quad S\rho\phi\rho = C = \text{const.};$$

in which it is to be remembered, that

$$\text{XLIX.} \ldots f(\omega, \rho) = f(\rho, \omega) = S\omega\phi\rho = S\rho\phi\omega.$$

(20.) And if we choose to *interpret* this equation XLVIII., which is only of the *first degree* (362) with respect to *each* separately of the *two vectors*, ρ and ω, or OP and OR, and involves them *symmetrically*, without requiring that P shall be a point *on the surface*, we may then say (comp. 215, (13.), and 316, (31.)), that the formula in question is an *equation of conjugation*, which expresses that *each* of the two points P and R, is situated in the *polar plane* of the *other*.

(21.) In general, if we suppose that the *length* and *direction* of a line ν are so adjusted as to satisfy the *two* equations (comp. 336, XII. XIII. XIV.),

$$\text{L.} \ldots S\nu\rho = 1, \quad S\nu d\rho = 0, \quad \text{and therefore also} \quad \text{LI.} \ldots S\rho d\nu = 0;$$

then, because the equation VII. of the *tangent plane* to *any curved surface* may now be thus written,

$$\text{LII.} \ldots S\nu(\omega - \nu^{-1}) = 0,$$

it follows that ν^{-1} represents, in length and direction, the *perpendicular from* o *on that tangent plane at* P; so that ν *itself* represents the *reciprocal* of that *perpendicular*, or what may be called (comp. 336, (8.)) the *vector of proximity*, of the tangent plane to the origin. And we see, by LI., that the *two vectors*, ρ and ν, if drawn from a *common origin*, terminate on *two surfaces* which are, in a known and important sense (comp. the sub-arts. to 361), *reciprocals** of one another: the line ρ^{-1}, for instance, being the perpendicular from o on the tangent plane to the *second* surface, at the extremity of the vector ν.

374. In the two preceding Articles, we have treated the symbol $d\rho$ as representing (rigorously) a *tangent to a curve on a given surface*, and therefore also to that surface *itself*; and thus the formula $S\nu d\rho = 0$ has been considered as expressing that ν has the direction of the *normal* to that *surface*, because it is *perpendicular to two tangents* (372), and therefore generally *to every tangent* (373), which can be drawn at a given point P. But without at present introducing any *other*† *signification* for this *symbol* $d\rho$, we may *interpret* in another way, and with a reference to *chords* rather than to *curves*, the *differential equation*,

$$\text{I.} \ldots df\rho = 2S\nu d\rho,$$

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* Compare the Note to page 549, vol. i.

† It is *permitted*, for example, by general principles above explained, to treat the *differential* $d\rho$ as denoting a *chordal vector*, or to substitute it for $\Delta\rho$, and so to represent the *differenced equation* of the surface under the form (comp. 342),

$$0 = \Delta f\rho = (\epsilon^{d} - 1)f\rho = df\rho + \tfrac{1}{2}d^2 f\rho + \&c.;$$

but *with this meaning* of the *symbol* $d\rho$, the *equation* $df\rho = 0$, or $S\nu d\rho = 0$, is *no longer rigorous*, and must (for rigour) be replaced by such an equation as the following,

$$0 = 2S\nu d\rho + Sd\nu d\rho + R, \quad \text{if} \quad df\rho = 2S\nu d\rho, \text{ as before};$$

the *remainder* R *vanishing*, when the *surface* is only of the *second order* (comp. 362, (3.)). Accordingly this last *form* is *useful* in some investigations, especially in those which relate to the *curvatures of normal sections*: but for the present it seems to be clearer to *adhere* to the recent signification of $d\rho$, and therefore to treat it as still denoting a *tangent*, which may or may not be *small*.

supposed still to be a *rigorous* one (in virtue of our *definitions* of differentials, which do not require that $d\rho$ should be *small*); and may *still* deduce from it the *normal property* of the vector ν, but now with the help of *Taylor's Series* adapted to quaternions (comp. 342, 370). In fact, that series gives here a *differenced equation*, of the form,

$$\text{II.} \ldots \Delta f\rho = 2S\nu\Delta\rho + R;$$

where R is a scalar *remainder* (comp. again 342), having the property that

$$\text{III.} \ldots \lim. (R : T\Delta\rho) = 0, \quad \text{if} \quad \lim. T\Delta\rho = 0;$$

whence

$$\text{IV.} \ldots \lim. (\Delta f\rho : T\Delta\rho) = 2 \lim. S\nu U\Delta\rho,$$

whatever the ultimate direction of $\Delta\rho$ *may be.* If then we conceive that $\Delta\rho$ represents a *small* and *indefinitely decreasing chord* PQ of the *surface*, drawn from the extremity P of ρ, so that

$$\text{V.} \ldots \Delta f\rho = f(\rho + \Delta\rho) - f\rho = 0, \quad \text{and} \quad \lim. T\Delta\rho = 0,$$

the equation IV. becomes simply,

$$\text{VI.} \ldots \lim. S\nu U\Delta\rho = 0;$$

and thus proves, in a *new way*, that ν is *normal to the surface at the proposed point* P, by proving that it is *ultimately perpendicular to all the chords* PQ *from that point*, when those chords become *indefinitely small*, or *tend* indefinitely to *vanish*.

(1.) For example, if

$$\text{VII.} \ldots f\rho = \rho^2, \quad \nu = \rho, \quad \text{then} \quad \text{VIII.} \ldots R = \Delta\rho^2, \quad \text{and} \quad R : T\Delta\rho = -T\Delta\rho;$$

thus, for *every point of space*, we have *rigorously*, with this form of $f\rho$,

$$\text{IX.} \ldots \Delta f\rho : T\Delta\rho = 2S\rho U\Delta\rho - T\Delta\rho;$$

and for *every point* Q *of the spheric surface*, $f\rho$ = *const.*, we have *with equal rigour*,

$$\text{X.} \ldots 2S\rho U\Delta\rho = T\Delta\rho, \quad \text{or} \quad \text{XI.} \ldots \overline{PQ} = 2\overline{OP} . \cos OPQ;$$

in fact, either of these two last formulæ expresses simply, that the *projection of a diameter of a sphere, on a conterminous chord*, is *equal to that chord itself*, and of course *diminishes with it*.

(2.) Passing then to the *limit*, or conceiving the point Q of the surface to *approach* indefinitely to P, we derive the limiting equations,

$$\text{XII.} \ldots \lim. \mathrm{S}\rho \mathrm{U}\Delta\rho = 0; \quad \text{XIII.} \ldots \lim. \cos \mathrm{OPQ} = 0;$$

either of which shows, in a new way, that the *radii* of a *sphere* are its *normals*; with the *analogous result* for *other surfaces*, that the *vector* ν in I. has a *normal direction*, as before: *because* its *projection on a chord* PQ tends indefinitely to *diminish with that chord.*

(3.) We may also interpret the differential equation I. as expressing, through II. and III., that the *plane* 373, VII., which is drawn through the point P in a direction perpendicular to ν, is the *tangent plane* to the surface: because the *projection of the chord* $\Delta\rho$ *on the normal* ν *to that plane*, or the *perpendicular distance*,

$$\text{XIV.} \ldots -\mathrm{S}(\mathrm{U}\nu \,.\, \Delta\rho) = \tfrac{1}{2} R \,.\, \mathrm{T}\nu^{-1},$$

of a near point Q *from the plane* thus drawn through P, is *small* of an *order higher than the first* (comp. 370, (8.)), if the *chord* PQ *itself* be considered as *small* of the *first order*.

375. This occasion may be taken (comp. 374, I. II. III.) to give a *new Enunciation of Taylor's Theorem*, in a form *adapted to Quaternions*, which has some advantages over that given (342) in the preceding Chapter. We shall therefore now express that important *Theorem* as follows:—

"*If none of the* $m+1$ *functions*,

$$\text{I.} \ldots fq,\ \mathrm{d}fq,\ \mathrm{d}^2fq \ldots\ \mathrm{d}^m fq, \quad \text{in which} \quad \mathrm{d}^2 q = 0,$$

become infinite in the immediate vicinity of a given quaternion q, *then the quotient*,

$$\text{II.} \ldots Q = \left\{ f(q + \mathrm{d}q) - fq - \mathrm{d}fq - \frac{\mathrm{d}^2 fq}{2} - \frac{\mathrm{d}^3 fq}{2\,.\,3} - \&\text{c.} \right.$$
$$\left. - \frac{\mathrm{d}^m fq}{2\,.\,3 \ldots m} \right\} : \frac{\mathrm{d}q^m}{2\,.\,3 \ldots m},$$

can be made to tend indefinitely to zero, for any ultimate value of the versor U$\mathrm{d}q$, *by indefinitely diminishing the tensor* T$\mathrm{d}q$."

(1.) The *proof* of the theorem, as thus enunciated, can easily be supplied by an attentive reader of Articles 341, 342, and their sub-articles; a few *hints* may however here be given.

(2.) We do not *now* suppose, as in 342, that $d^m fq$ must be *different from zero*; we only assume that it is *not infinite*: and we *add*, to the expression 342, VI. for Fx, the *term*,

$$\text{III.} \ldots \frac{-x^m d^m fq}{2.3\ldots m}.$$

(3.) Hence *each* of the expressions 342, VII., for the successive *derivatives* of Fx, receives an *additional term*; the *last* of them thus becoming,

$$\text{IV.} \ldots D^m Fx = F^{(m)}x = d^m f(q + x dx) - d^m fq;$$

so that we have *now* (comp. 342, X.) the values

$$\text{V.} \ldots F0 = 0, \quad F'0 = 0, \quad F''0 = 0, \ldots \quad F^{(m-1)}0 = 0, \quad F^{(m)}0 = 0.$$

(4.) Assuming therefore now (comp. 342, XII.) the new auxiliary function,

$$\text{VI.} \ldots \psi x = \frac{x^m dq^m}{2.3\ldots m}, \quad \text{with} \quad T dq > 0,$$

which gives,

$$\text{VII.} \ldots \psi 0 = 0, \quad \psi' 0 = 0, \quad \psi'' 0 = 0, \ldots \quad \psi^{(m-1)}0 = 0, \quad \psi^{(m)}0 = dq^m,$$

we find (by 341, (8.), (9.), comp. again 342, XII.) that

$$\text{VIII.} \ldots \lim_{x=0} . (Fx : \psi x) = 0.$$

(5.) But these two new functions, Fx and ψx, are formed from the dividend and the divisor of the quotient Q in II., by changing dq to $x dq$; and (comp. 342, (3.)) instead of thus *multiplying* a *given quaternion differential* dq, by a *small* and indefinitely *decreasing scalar*, x, we may indefinitely *diminish the tensor*, $T dq$, *without changing the versor*, $U dq$.

(6.) And *even if* $U dq$ *be changed*, while the differential dq is thus made to *tend to zero*, we can always conceive that it *tends* to *some limit*; which *limiting* or *ultimate value* of that versor $U dq$ may then be treated *as if* it were a *constant* one, without affecting the *limit* of the quotient Q.

(7.) The *theorem*, as above enunciated, is therefore fully proved; and we are at liberty to *choose*, in any application, between the two forms of statement, 342 and 375, of which one is more convenient at one time, and the other at another.

SECTION 4.

On Osculating Planes, and Absolute Normals, to Curves of Double Curvature.

376. The variable vector ρ_t of a curve in space may in general be thus expressed, with the help of Taylor's Series (comp. 370, (1.)):

$$\text{I.} \ldots \rho_t = \rho + t\rho' + \tfrac{1}{2}t^2 u\rho'', \quad \text{with} \quad u_0 = 1;$$

ρ, ρ', ρ'', u being here abridged symbols for $\rho_0, \rho'_0, \rho''_0, u_t$; and the product $u\rho''$ being a vector, although the factor u is generally a quaternion (comp. 370, (5.)). And the different terms of this expression I. may be thus constructed (compare the annexed fig. 77):

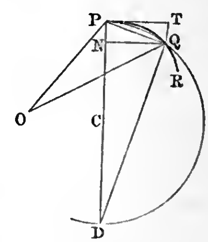

Fig. 77.

$$\text{II.} \ldots \rho = \text{OP}; \quad t\rho' = \text{PT}; \quad \tfrac{1}{2}t^2u\rho'' = \text{TQ};$$

while

$$\text{III.} \ldots \rho_t = \text{OQ}, \quad \text{and} \quad t\rho' + \tfrac{1}{2}t^2u\rho'' = \text{PQ};$$

the line TQ, or the term $\frac{1}{2}t^2u\rho''$, being thus what may be called the *deflexion* of the *curve* PQR, at Q, from its *tangent* PT at P, measured in a *direction* which depends on the *law* according to which ρ_t varies with t, and on the *distance* of Q from P. The *equation of the plane* of the *triangle* PTQ is *rigorously* (by II.) the following, with ω for its variable vector,

$$\text{IV.} \ldots 0 = \text{S}u\rho''\rho'(\omega - \rho);$$

this *plane* IV. then *touches* the curve at P, and (generally) *cuts* it at Q; so that if the point Q be conceived to *approach* indefinitely to P, the resulting formula,

$$\text{V.} \ldots 0 = \text{S}\rho''\rho'(\omega - \rho), \quad \text{or} \quad \text{V}'. \ldots 0 = \text{S}\rho'\rho''(\omega - \rho),$$

is the equation of the plane PTQ in that *limiting position*, in which it is called the *osculating plane*, or is said to *osculate to the curve* PQR, at the point P.

(1.) If the *variable vector* ρ be *immediately* given as a *function* ρ_s of a *variable scalar*, s, which is *itself* a function of the *former* scalar variable t, we shall then have (comp. 331) the expressions,

$$\text{VI.} \ldots \rho'_t = s'\text{D}_s\rho_s, \quad \rho''_t = s''\text{D}_s\rho_s + s'^2\text{D}_s^2\rho_s, \quad \text{with} \quad s' = \text{D}_t s, \quad s'' = \text{D}_t^2 s;$$

thus the *vector* ρ'' *may change*, even in *direction*, when we *change* the *independent scalar variable*; but ρ'' will *always* be a line, either *in* or *parallel* *to the osculating plane*; while ρ' will always represent a *tangent*, whatever scalar variable may be selected.

(2.) As an example, let us take the equation 314, XV., or 369, XIII., of the *helix*. With the independent variable t of that equation, we have (comp. 369, XIV.) the derived expressions,

$$\text{VII.} \ldots \rho' = c\alpha + \frac{\pi}{2}\alpha^{t+1}\beta, \quad \rho'' = -\left(\frac{\pi}{2}\right)^2 \alpha^t\beta = \left(\frac{\pi}{2}\right)^2 (ct\alpha - \rho);$$

ρ'' has therefore here (comp. 369, (8.)) the direction of the *normal* to the cylinder; and consequently, the *osculating plane to the helix* is a *normal plane to the cylinder* of revolution, on which that curve is traced: a result well known, and which will soon be greatly extended.

(3.) When a curve of *double* curvature *degenerates* into a *plane* curve, its *osculating* plane becomes *constant*, and reciprocally. The *condition of planarity* of a *curve in space* may therefore be expressed by the equation,

$$\text{VIII.} \ldots \mathrm{UV}\rho'\rho'' = \pm \text{ a constant unit line};$$

or, by 335, II., and 338, VIII.,

$$\text{IX.} \ldots 0 = \mathrm{V}\frac{\mathrm{V}(\rho'\rho'')'}{\mathrm{V}\rho'\rho''} = \mathrm{V}\frac{\mathrm{V}\rho'\rho'''}{\mathrm{V}\rho'\rho''};$$

or finally,

$$\text{X.} \ldots \mathrm{S}\rho'\rho''\rho''' = 0, \quad \text{or} \quad \text{XI.} \ldots \rho''' \parallel\!\mid \rho', \rho''.$$

(4.) Accordingly, for a *plane curve*, if λ be a given normal to its plane, we have the three equations,

$$\text{XII.} \ldots \mathrm{S}\lambda\rho' = 0, \quad \mathrm{S}\lambda\rho'' = 0, \quad \mathrm{S}\lambda\rho''' = 0;$$

which conduct, by 294, (11.), to X.

(5.) For example, if we had not otherwise known that the equation 337 (2.) represented a *plane ellipse*, we might have perceived that it was the equation of *some* plane curve, because it gives the *three* successive derivatives,

$$\text{XIII.} \ldots \rho' = \frac{\pi}{2}\mathrm{V}\alpha^{t+1}\beta, \quad \rho'' = -\left(\frac{\pi}{2}\right)^2 \mathrm{V}\alpha^t\beta, \quad \rho''' = -\left(\frac{\pi}{2}\right)^3 \mathrm{V}\alpha^{t+1}\beta,$$

which are *complanar* lines, the third having a direction *opposite* to the first.

(6.) And generally, the formula X. enables us to assign, on *any curve* of *double* curvature, for which ρ is expressed as a function of t, the *points** at

* Namely, in a modern phraseology, the places of *four-point contact* with a *plane*. The equation, $\mathrm{V}\rho'\rho'' = 0$, indicates in like manner the places, if any, at which a curve has *three-point contact* with a *right line*. For curves of double curvature, these are also called points of *simple* and *double inflexion*.

which it *most resembles* a *plane* curve, or *approaches most closely* to its own osculating plane.

377. An important and *characteristic property* of the *osculating plane* to a curve of double curvature, is that the *perpendiculars* let fall on it, from points of the curve near to the point of osculation, are *small* of an order *higher* than the *second*, if their *distances* from that *point* be considered as *small* of the *first* order.

(1.) To exhibit this by quaternions, let us begin by considering an *arbitrary plane*,

$$\text{I.} \ldots \mathrm{S}\lambda(\omega - \rho) = 0, \quad \text{with} \quad \mathrm{T}\lambda = 1,$$

drawn through a point P of the curve. Using the expression 376, I., for the vector OQ, or ρ_t, of another point Q of the same curve, we have, for the perpendicular distance of Q from the plane I., this other rigorous expression,

$$\text{II.} \ldots \mathrm{S}\lambda(\rho_t - \rho) = t\mathrm{S}\lambda\rho' + \tfrac{1}{2}t^2\mathrm{S}\lambda u\rho'' ;$$

which represents, *in general*, a small quantity of the *first order*, if t be assumed to be such.

(2.) The expression II. represents however, *generally*, a small quantity of the *second* order, if the *direction* of λ satisfy the condition,

$$\text{III.} \ldots \mathrm{S}\lambda\rho' = 0 ;$$

that is, if the *plane* I. *touch* the *curve*.

(3.) And if the condition,

$$\text{IV.} \ldots \mathrm{S}\lambda\rho'' = 0,$$

be *also* satisfied by λ, *then*, but *not otherwise*, the expression II. *tends* to bear an *evanescent ratio* to t^2, or is *small* of an order *higher* than the *second*.

(4.) But the combination of the *two conditions*, III. and IV., conducts to the expression,

$$\text{V.} \ldots \lambda = \pm \mathrm{UV}\rho'\rho'' ;$$

comparing which with 376, V., we see that the property above stated is one which belongs to the *osculating* plane, and to *no other*.

378. Another remarkable property* of the *osculating plane* to a *curve* is, that it is the *tangent plane* to the *cone of parallels to tangents* (369, (6.)), which has its *vertex* at the *point* of *osculation*.

* The writer does not remember seeing *this* property in print; but of course it is an easy consequence from the doctrine of *infinitesimals*, which doctrine however it has not been thought convenient to adopt, as the *basis* of the present exposition.

(1.) In general, if $\rho = \phi x$ be (comp. 369, I.) the equation of a *curve* in space, the equation of the *cone* which has its vertex at the origin, and passes through this curve, is of the form,

$$\text{I.} \ldots \rho = y\phi x;$$

in which x and y are *two* independent and scalar variables.

(2.) We have thus the two partial derivatives,

$$\text{II.} \ldots \mathrm{D}_x\rho = y\phi' x, \quad \mathrm{D}_y\rho = \phi x;$$

and the tangent plane along the *side* (x) has for equation,

$$\text{III.} \ldots 0 = \mathrm{S}(\omega \,.\, \phi x \,.\, \phi' x); \quad \text{or briefly,} \quad \text{III}' \ldots 0 = \mathrm{S}\omega\phi\phi'.$$

(3.) Changing then x, ϕ, ϕ', ω to t, ρ', ρ'', $\omega - \rho$, we see that the equation 376, V., of the *osculating plane* to the *curve* 376, I., is also that of the *tangent plane* to the *cone of parallels*, &c., as asserted.

379. Among all the *normals* to a *curve*, at any one point, there are *two* which deserve special attention; namely the one which is *in* the osculating plane, and is called the *absolute* (or *principal*) *normal*; and the one which is *perpendicular* to that plane, and which it has been lately proposed to name the *binormal*.* It is easy to assign expressions, by quaternions, for these two normals, as follows.

(1.) The *absolute normal*, as being perpendicular to ρ', but complanar with ρ' and ρ'', has a direction expressed by any one of the following formulæ (comp. 203, 334):

$$\text{I.} \ldots \mathrm{V}\rho''\rho' . \rho'^{-1}; \quad \text{or} \quad \text{II.} \ldots \mathrm{dU}\rho'; \quad \text{or} \quad \text{III.} \ldots \mathrm{dUd}\rho.$$

(2.) There is an extensive class† of cases, for which the following equations hold good:

$$\text{IV.} \ldots \mathrm{T}\rho' = \text{const.}; \qquad \text{V.} \ldots \rho'^2 = \text{const.}; \qquad \text{VI.} \ldots \mathrm{S}\rho'\rho'' = 0;$$

and in all *such* cases, the expression I. reduces itself to ρ'', which is therefore *then* a representative of the absolute normal.

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* By M. de Saint-Venant, as being perpendicular at once to *two* consecutive *elements* of the curve, in the infinitesimal treatment of this subject. See page 261 of the very valuable Treatise on *Analytic Geometry of Three Dimensions* (Hodges and Smith, Dublin), by the Rev. George Salmon, D.D., which has been published in the present year (1862), but not till after the printing of these *Elements of Quaternions* (begun in 1860) had been too far advanced, to allow the writer of them to profit by the study of it, so much as he would otherwise have sought to do.

† Namely, those in which the *arc of the curve*, or that arc multiplied by a scalar constant, is taken as the *independent variable*.

(3.) For example, in the case of the *helix*, with the equation several times before employed, the conditions (2.) are satisfied; and accordingly the absolute normal to that curve coincides with the normal ρ'' to the *cylinder*, on which it is traced: the *locus of the absolute normal* being here that *screw surface* or *Helicoid*, which has been already partially considered (comp. 314, (11.), and 372, (4.)).

(4.) And as regards the *binormal*, it may be sufficient here to remark, that because it is perpendicular to the osculating plane, it has the *direction* expressed by one or other of the two symbols (comp. 377, V.),

$$\text{VII.} \ldots \text{V}\rho'\rho'', \quad \text{or} \quad \text{VII}'. \ldots \text{V}d\rho d^2\rho.$$

(5.) There exists, of course, a system of *three rectangular planes*, the *osculating* plane being *one*, which are connected with the system of the *three rectangular lines*, the tangent, the absolute normal, and the binormal, and of which any one who has studied the Quaternions so far can easily form the expressions.

(6.) And a *construction** for the *absolute normal* may be assigned, analogous to and including that lately given (378) for the *osculating plane*, as an *interpretation* of the expression II. or III., or of the *symbol* $d\text{U}\rho'$ or $d\text{U}d\rho$. From any origin O conceive a system of unit lines ($\text{U}\rho'$ or $\text{U}d\rho$) to be drawn, in the *directions* of the successive *tangents* to the *given curve* of double curvature; these lines will terminate on a certain *spherical curve*; and the *tangent*, say SS′, to this *new* curve, at the point S which *corresponds* to the point P of the *old* one, will have the direction of the absolute normal at that old point.

(7.) At the same time, the *plane* OSS′ of the *great circle*, which *touches* the *new curve* upon the *unit sphere*, being the *tangent plane* to the *cone of parallels* (378), has the direction of the *osculating plane* to the old curve; and the *radius* drawn to its *pole* is parallel to the *binomial*.

(8.) As an *example* of the *auxiliary* (or *spherical*) curve, constructed as in (6.), we may take again the *helix* (369), XIII., &c.) as the *given* curve of double curvature, and observe that the expression 369, XIV., namely,

$$\text{VIII.} \ldots \rho' = c\alpha + \frac{\pi}{2}\alpha^{t+1}\beta, \text{ gives IX.} \ldots \rho'^2 = -c^2 + \frac{\pi^2\beta^2}{4} = \text{const. (comp. (3.))};$$

* *This* construction also has not been met with by the writer in print, so far as he remembers; but it may easily have escaped his notice, even in the books which he has seen.

whence $T\rho'$ is *constant* (as in IV.), and we have the equation (comp. 369, XV. XIX.),

$$\text{X.} \ldots \; SaU\rho' = -c\left(c^2 - \frac{\pi^2\beta^2}{4}\right)^{-\frac{1}{2}} = -\cos a = \text{const.},$$

a being again the inclination of the helix to the axis of its cylinder; which shows that the *new* curve is in *this* case a *plane* one, namely a certain *small circle* of the unit sphere.

(9.) In general, if the *given curve* be conceived to be an *orbit* described by a *point*, which *moves* with a *constant velocity* taken for *unity*, the *auxiliary* or *spherical curve* becomes what we have proposed (100, (5.)) to call the *hodograph* of that *motion*.

(10.) And if the given curve be supposed to be described with a *variable velocity*, the *hodograph* is still *some* curve upon the *cone of parallels* to tangents.

SECTION 5.

On Geodetic Lines, and Families of Surfaces.

380. Adopting as the *definition* of a *geodetic line*, on any proposed curved surface, the property that is one of which the *osculating plane* is always a *normal plane* to that surface, or that the *absolute normal* to the *curve* is also the *normal* to the *surface*, we have *two principal modes* of *expressing* by quaternions this general and *characteristic property*. For we may either write,

$$\text{I.} \ldots \; S\nu\rho'\rho'' = 0, \quad \text{or} \quad \text{II.} \ldots \; S\nu d\rho d^2\rho = 0,$$

to express that the *normal* ν to the *surface* (comp. 373) is *perpendicular* to the *binormal* $V\rho'\rho''$ or $Vd\rho d^2\rho$ to the *curve* (comp. 379, VII. VII'.); or else, at pleasure,

$$\text{III.} \ldots \; V\nu(U\rho')' = 0, \quad \text{or} \quad \text{IV.} \ldots \; V\nu dUd\rho = 0,$$

to express that the *same* normal ν has the *direction* of the *absolute normal* $(U\rho')'$ or $dUd\rho$ (comp. 379, II. III.), to the same *geodetic line*. And thus it becomes easy to deduce the known *relations* of such *lines* (or *curves*) to some important *families of surfaces*, on which they can be traced. Accordingly, after beginning for simplicity with the *sphere*, we shall proceed in the following sub-articles to determine the geodetic lines on *cylindrical* and *conical* surfaces, with *arbitrary bases*; intending afterwards to show how the corresponding lines can be investigated, upon *developable surfaces*, and surfaces of *revolution*.

(1.) On a *sphere*, with centre at the origin, we have $\nu \parallel \rho$, and the *differential equation* IV. admits of an immediate *integration*;* for it here becomes,

$$\text{V.} \ldots 0 = \text{V}\rho d\text{U}d\rho = d\text{V}\rho\text{U}d\rho,$$

whence

$$\text{VI.} \ldots \text{V}\rho\text{U}d\rho = \omega, \quad \text{and} \quad \text{VII.} \ldots \text{S}\omega\rho = 0,$$

ω being some constant vector; the curve is therefore in this case a *great circle*, as being wholly contained in *one diametrical plane*.

(2.) Or we may observe that the equation,

$$\text{VIII.} \ldots \text{S}\rho\rho'\rho'' = 0, \quad \text{or} \quad \text{IX.} \ldots \text{S}\rho d\rho d^2\rho = 0,$$

obtained by changing ν to ρ in I. or II., has *generally* for a *first* integral (comp. 335, (1.)), whether $\text{T}\rho$ be constant or variable,

$$\text{X.} \ldots \text{UV}\rho\rho' = \text{UV}\rho d\rho = \omega = \text{const.};$$

it expresses therefore that ρ is the vector of *some curve* (or line) *in a plane through the origin*; which curve must consequently be here a *great circle*, as before.

(3.) Accordingly, as a verification of X., if we write

$$\text{XI.} \ldots \rho = ax + \beta y, \quad x \text{ and } y \text{ being scalar functions of } t,$$

where t is still some independent scalar variable, and a, β are two vector constants, we shall have the derivatives,

$$\text{XII.} \ldots \rho' = ax' + \beta y', \quad \rho'' = ax'' + \beta y'' \parallel\!\mid \rho, \rho';$$

so that the equation VIII. is satisfied.

(4.) For an *arbitrary cylinder*, with generating lines parallel to a fixed line a, we may write,

$$\text{XIII.} \ldots \text{S}a\nu = 0, \quad \text{XIV.} \ldots \text{S}ad\text{U}d\rho = 0, \quad \text{XV.} \ldots \text{S}a\text{U}d\rho = \text{const.};$$

a *geodetic* on a cylinder *crosses* therefore the *generating lines* at a *constant angle*, and consequently becomes a *right line* when the cylinder is *unfolded*

* We here assume as evident, that the *differential* of a *variable* cannot be *constantly zero* (comp. 335, (7.)); and we employ the principle (comp. 338, (5.)), that $\text{V} \,.\, d\rho\text{U}d\rho = -\text{VT}d\rho = 0$.

into a *plane*: both which known properties are accordingly verified (comp. 369, (5.), and 376, (2.)) for the case of a cylinder of *revolution*, in which case the geodetic is a *helix*.

(5.) For an *arbitrary cone*, with vertex at the origin, we have the equations,

$$\text{XVI.} \ldots \mathrm{S}\nu\rho = 0, \qquad \text{XVII.} \ldots \mathrm{S}\rho d\mathrm{U}d\rho = 0,$$

$$\text{XVIII.} \ldots d\mathrm{S}\rho\mathrm{U}d\rho = \mathrm{S}(d\rho \,.\, \mathrm{U}d\rho) = -\mathrm{T}d\rho;$$

multiplying the last of which equations by $2\mathrm{S}\rho\mathrm{U}d\rho$, and observing that $-2\mathrm{S}\rho d\rho = -d\,.\,\rho^2$, we obtain the transformations,

$$\text{XIX.} \ldots 0 = d\{(\mathrm{S}\rho\mathrm{U}d\rho)^2 + \rho^2\} = d\,.\,(\mathrm{V}\rho\mathrm{U}d\rho)^2, \qquad \text{XX.} \ldots \mathrm{TV}\rho\mathrm{U}d\rho = \text{const.};$$

the *perpendicular from the vertex, on a tangent to any one geodetic* upon a *cone*, has therefore a *constant length*; and all *such* tangents *touch also a concentric sphere*,* or one which has its centre at the vertex of the cone.

(6.) Conceive then that at each point P or P′ of the geodetic a tangent PT or P′T′ is drawn, and that the angles OTP, OT′P′ are right; we shall have, by what has just been shown,

$$\text{XXI.} \ldots \overline{\text{OT}} = \overline{\text{OT}'} = \text{const.} = \text{radius of concentric sphere};$$

and if the *cone* be *developed* (or unfolded) into a *plane*, this constant or *common length*, of the perpendiculars from O on the tangents, will remain *unchanged*, because the length $\overline{\text{OP}}$ and the angle OPT are unaltered by such development; the geodetic becomes therefore *some plane line*, with the *same property* as before; and although *this* property would belong, not only to a *right line*, but also to a *circle* with O for centre (compare the second part of the annexed figure 78), yet we have in this result merely an effect of the *foreign factor* $\mathrm{S}\rho\mathrm{U}d\rho$, which was introduced in (5.), in order to facilitate the integration of the differential equation XVIII., and which (by that very equation) cannot be constantly equal to zero. We are therefore to *exclude* the *curves* in which the *cone* is cut by *spheres* concentric with it: and there remain, as the sought *geodetic lines*, only those of which the *developments* are *rectilinear*, as in (4.).

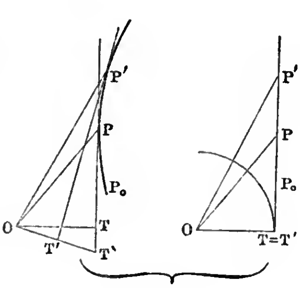

Fig. 78.

* When the *cone* is of the *second order*, this becomes a case of a known theorem respecting *geodetic lines on a surface* of the same *second* order, the *tangents* to any *one* of which *curves* touch *also* a *confocal surface*.

(7.) Another mode of *interpreting*, and at the same time of *integrating*, the equation XVIII., is connected with the interpretation of the *symbol* $\mathrm{Td}\rho$; which can be proved, on the principles of the present Calculus, to represent *rigorously* the *differential* $\mathrm{d}s$ of the *arc* (s) of that *curve*, whatever it may be, of which ρ is the *variable vector*; so that we have the *general* and *rigorous* equation,

$$\text{XXII.} \ldots \mathrm{Td}\rho = \mathrm{d}s, \quad \text{if } s \text{ thus denote the } arc:$$

whether that arc *itself*, or some *other* scalar, t, be taken as the *independent variable*; and whether its differential $\mathrm{d}s$ be *small* or *large*, provided that it be *positive*.

(8.) In fact if we suppose, for the sake of greater generality, that the vector ρ and the scalar s are thus *both* functions, ρ_t and s_t, of some *one* independent and scalar variable, t, our principles direct us first to take, or to conceive as taken, a *submultiple*, $n^{-1}\mathrm{d}t$, of the *finite differential* $\mathrm{d}t$, considered as an assumed and *arbitrary increment* of that *independent variable*, t; to determine next the *vector* $\rho_{t+n^{-1}\mathrm{d}t}$, and the *scalar* $s_{t+n^{-1}\mathrm{d}t}$, which correspond to the *point* $\mathrm{P}_{t+n^{-1}\mathrm{d}t}$ of the *curve* on which ρ_t terminates in P_t, and of which s_t is the *arc*, $\widehat{\mathrm{P}_0\mathrm{P}_t}$, measured to P_t from some fixed point P_0 on the same curve; to take the *differences*,

$$\rho_{t+n^{-1}\mathrm{d}t} - \rho_t, \quad \text{and} \quad s_{t+n^{-1}\mathrm{d}t} - s_t,$$

which represent respectively the *directed chord*, and the *length*, of the *arc* $\mathrm{P}_t\mathrm{P}_{t+n^{-1}\mathrm{d}t}$, which arc will generally be *small*, if the number n be *large*, and will *indefinitely diminish* when that *number* tends to *infinity*; to *multiply* these two decreasing differences, of ρ_t and s_t, by n; and finally to seek the *limits* to which the *products tend*, when n thus tends to ∞: such *limits* being, by our *definitions*, the values of the two sought and *simultaneous differentials*, $\mathrm{d}\rho$ and $\mathrm{d}s$, which answer to the assumed values of t and $\mathrm{d}t$. And because the *small arc*, Δs, and the *length*, $\mathrm{T}\Delta\rho$, of its *small chord*, in the foregoing construction, *tend* indefinitely to a *ratio of equality*, such must be the *rigorous ratio* of $\mathrm{d}s$ and $\mathrm{Td}\rho$, which are (comp. 320) the *limits of their equimultiples*.

(9.) Admitting then the exact *equality* XXII. of $\mathrm{Td}\rho$ and $\mathrm{d}s$, at least when the latter like the former is taken *positively*, we have only to substitute $-\mathrm{d}s$ for $-\mathrm{Td}\rho$ in the equation XVIII., which then becomes immediately integrable, and gives,

$$\text{XXIII.} \ldots s + \mathrm{S}\rho\mathrm{Ud}\rho = s - \mathrm{S}(\rho : \mathrm{Ud}\rho) = \text{const.};$$

where $\mathrm{S}(\rho : \mathrm{Ud}\rho)$ denotes the *projection* $\overline{\mathrm{TP}}$, of the vector ρ or OP, *on the tangent* to the geodetic at P, considered as a *positive scalar* when ρ makes an *acute angle* with $d\rho$, that is, when the *distance* $T\rho$ or $\overline{OP}$ from the vertex is *increasing*; while s denotes, as above, the *length* of the *arc* P_0P of the same curve, measured from *some fixed point* P_0 thereon, and considered as a *scalar* which *changes sign*, when the variable point P passes through the position P_0.

(10.) But the *length* of TP does not change (comp. (6.)), when the cone is *developed*, as before; we have therefore the equations (comp. again fig. 78),

$$\text{XXIV.} \ldots \overset{\frown}{P_0P} - \overline{TP} = \text{const.} = \overset{\frown}{P_0P'} - \overline{T'P'}, \qquad \text{XXV.} \ldots \overset{\frown}{PP'} = \overline{T'P'} - \overline{TP},$$

which must hold good both *before* and *after* the supposed *development* of the conical *surface*; and it is easy to see that this can only be, by the *geodetic* on the cone becoming a *right line*, as before. In fact, if OT′ in the *plane* be supposed to intersect the tangent TP in a point T′, and if P′ be conceived to approach to P, the second member of XXV. bears a limiting ratio of equality to the first member, increased or diminished by $\overline{TT'}$; which latter *line*, and therefore also the *angle* TOT′ between the perpendiculars on the two near tangents, or the angle between those tangents themselves, if existing, must bear an indefinitely decreasing ratio to the *arc* $\overset{\frown}{PP'}$; so that the *radius of curvature* of the *supposed curve* is *infinite*, or T′ *coincides* with T, and the *development* is *rectilinear* as before.

(11.) The important and general equation, $Td\rho = ds$ (XXII.), conducts to many other consequences, and may be put under several other forms. For example, we may write generally,

$$\text{XXVI.} \ldots TD_s\rho = 1, \qquad \text{XXVII.} \ldots (D_s\rho)^2 + 1 = 0;$$

also

$$\text{XXVIII.} \ldots (D_t\rho)^2 + (D_ts)^2 = 0, \quad \text{or} \quad \text{XXIX.} \ldots \rho'^2 + s'^2 = 0,$$

if ρ' and s' be the first derivatives of ρ and s, taken with respect to any independent scalar variable, such as t; whence, by continued derivation,

$$\text{XXX.} \ldots S\rho'\rho'' + s's'' = 0, \quad \text{XXXI.} \ldots S\rho'\rho''' + \rho''^2 + s's''' + s''^2 = 0, \ \&\text{c.}$$

(12.) And if the arc s be *itself* taken as the independent variable, then (comp. 379, (2.)) the equations XXIX., &c., become,

$$\text{XXXII.} \ldots \rho'^2 + 1 = 0, \quad S\rho'\rho'' = 0, \quad S\rho'\rho''' + \rho''^2 = 0, \ \&\text{c.}$$

381. In general, if we conceive (comp. 372, I.) that the *vector* ρ of a *given surface* is expressed as a *given function* of *two scalar variables*, x and y, whereof

one, suppose y, is regarded at first as an *unknown function* of the other, so that we have again,

$$\text{I.} \ldots \rho = \phi(x, y), \quad \text{but now with} \quad \text{II.} \ldots y = fx,$$

where the *form* of ϕ is *known*, but that of f is *sought*; we may then regard ρ as being *implicitly* a function of the *single* (or *independent*) *scalar variable*, x, and may consider the equation,

$$\text{III.} \ldots \rho = \phi(x, fx),$$

as being that of *some curve* on the given surface, to be determined by assigned conditions. Denoting then the *unknown total derivative* $\mathrm{D}\phi(x, fx)$ by ρ', but the *known partial derivatives* of the same first order by $\mathrm{D}_x\phi$ and $\mathrm{D}_y\phi$, with analogous notations for orders higher than the first, we have (comp. 376, VI.) the expressions,

$$\text{IV.} \ldots \rho' = \mathrm{D}_x\phi + y'\mathrm{D}_y\phi, \quad \rho'' = \mathrm{D}_x^2\phi + 2y'\mathrm{D}_x\mathrm{D}_y\phi + y'^2\mathrm{D}_y^2\phi + y''\mathrm{D}_y\phi, \ \&\text{c.};$$

in which $y' = \mathrm{D}_x y = f'x$, $y'' = \mathrm{D}_x^2 y = f''x$, &c. Hence, writing for the *normal* ν to the *surface* the expression,

$$\text{V.} \ldots \nu = \mathrm{V}(\mathrm{D}_x\phi \,.\, \mathrm{D}_y\phi) = \mathrm{V} \,.\, \mathrm{D}_x\phi\mathrm{D}_y\phi \qquad \text{(comp. 372, V.)},$$

or this vector multiplied by any scalar, the *equation* 380, I. of a *geodetic line* takes this *new form*,

$$\text{VI.} \ldots 0 = \mathrm{S}\nu\rho'\rho'' = \mathrm{S}(\mathrm{V} \,.\, \mathrm{D}_x\phi\mathrm{D}_y\phi \,.\, \mathrm{V}\rho'\rho'');$$

or, by a general transformation which has been often employed already (comp. 352, XXXI., &c.),

$$\text{VII.} \ldots 0 = \mathrm{S}\rho'\mathrm{D}_y\phi \,.\, \mathrm{S}\rho''\mathrm{D}_x\phi - \mathrm{S}\rho'\mathrm{D}_x\phi \,.\, \mathrm{S}\rho''\mathrm{D}_y\phi;$$

and thus, by substituting the expressions IV. for ρ' and ρ'', we obtain an *ordinary* (or *scalar*) *differential equation*, of the *second order*, in x and y, which is *satisfied by all the geodetics* on the given *surface*, and of which the *complete integral* (when found) expresses, with *two arbitrary* and *scalar constants*, the *form* of the *scalar function* f in II., or the *law* of the dependence of y on x, for the geodetic *curves* in question.

(1.) As an *example*, let us take the equation,

$$\text{VIII.} \ldots \rho = \phi(x, y) = y\psi x \qquad \text{(comp. 378, I.)},$$

of a *cone* with its vertex at the origin; which cone becomes a *known* one, when the *form* of the vector function ψ is given, that is, when we know a *guiding curve* $\rho = \psi x$, through which the *sides* of the cone all pass. We have here the partial derivatives,

$$\text{IX.} \ldots \mathrm{D}_x\phi = y\mathrm{D}_x\psi x = y\psi', \quad \mathrm{D}_y\phi = \psi x = \psi \qquad \text{(comp. 378, II.)};$$

and

$$\text{X.} \ldots \mathrm{D}_x^2\phi = y\mathrm{D}_x^2\psi x = y\psi'', \quad \mathrm{D}_x\mathrm{D}_y\phi = \psi', \quad \mathrm{D}_y^2\phi = 0;$$

the expressions IV. become, then,

$$\text{XI.} \ldots \rho' = y\psi' + y'\psi, \quad \rho'' = y\psi'' + 2y'\psi' + y''\psi;$$

and since only the *direction* of the normal is important, we may divide V. by $-y$, and write,

$$\text{XII.} \ldots \nu = \mathrm{V}\psi\psi'.$$

(2.) The expressions XI. and XII. give (comp. VI. and VII.) for the *geodetics on the cone* VIII., the differential equation of the second order,

$$\begin{aligned}\text{XIII.} \ldots 0 &= \mathrm{S}(\mathrm{V}\psi\psi'.\mathrm{V}\rho'\rho'') = \mathrm{S}\rho''\psi\mathrm{S}\rho'\psi' - \mathrm{S}\rho''\psi'\mathrm{S}\rho'\psi \\ &= (y\mathrm{S}\psi\psi'' + 2y'\mathrm{S}\psi\psi' + y''\psi^2)\,(y\psi'^2 + y'\mathrm{S}\psi\psi') \\ &\quad - (y\mathrm{S}\psi'\psi'' + 2y'\psi'^2 + y''\mathrm{S}\psi\psi')\,(y\mathrm{S}\psi\psi' + y'\psi^2),\end{aligned}$$

in which ψ^2 and ψ'^2 are abridged symbols for $(\psi x)^2$ and $(\psi' x)^2$; but this equation in x and y may be greatly simplified, by some permitted suppositions.

(3.) Thus, we are allowed to suppose that the *guiding curve* (1.) is the *intersection* of the *cone* with the *concentric unit sphere*, so that

$$\text{XIV.} \ldots \mathrm{T}\psi x = 1, \quad \psi^2 = -1, \quad \mathrm{S}\psi\psi' = 0, \quad \mathrm{S}\psi\psi'' + \psi'^2 = 0;$$

and if we further assume that the *arc* of this *spherical curve* is taken as the *independent variable*, x, we have then, by 380, (12.), combined with the last equation XIV.,

$$\text{XV.} \ldots \mathrm{T}\psi' x = 1, \quad \psi'^2 = -1, \quad \mathrm{S}\psi'\psi'' = 0, \quad \mathrm{S}\psi\psi'' = -\psi'^2 = 1.$$

(4.) With these simplifications, the *differential equation* XIII. becomes,

$$\text{XVI.} \ldots 0 = (y - y'')\,(-y) - (-2y')\,(-y') = yy'' - 2y'^2 - y^2;$$

and its *complete integral* is found *by ordinary methods* to be,

$$\text{XVII.} \ldots y = b \sec (x + c),$$

in which b and c are two arbitrary but scalar constants.

(5.) To *interpret* now this *integrated* and *scalar equation* in x and y, of the *geodetics* on an *arbitrary cone*, we may observe that, by the suppositions (3.), y represents the *distance*, $T\rho$ or $\overline{OP}$, from the vertex O, and $x + c$ represents the *angle* AOP, in the *developed state* of cone and curve, from some *fixed line* OA in the *plane*, to the variable line OP; the *projection* of this *new* OP on that *fixed* line OA is therefore *constant* (being $= b$, by XVII.), and the *developed geodetic* is again found to be a *right line*, as before.

382. Let ABCDE . . . (see the annexed figure 79) be *any given series* of *points in space*. Draw the successive right lines, AB, BC, CD, DE, . . and prolong them to points B′, C′, D′, E′, . . . the lengths of these prolongations being arbitrary; join also B′C′, C′D′, D′E′, ... We shall thus have a *series of plane triangles*, B′BC′, C′CD′, D′DE′, . . . *all* generally *in different planes*; so that BCD′C′B′, CDE′D′C′, . . . are generally *gauche pentagons*, while BCDE′D′C′B′ is a *gauche heptagon*, &c. But we can conceive the *first triangle* B′BC′ to *turn* round its *side* BCC′, till it *comes* into the plane of the *second triangle* C′CD′; which will transform the *first gauche pentagon* into a *plane* one, denoted still by BCD′C′B′. We can then conceive *this* plane figure to turn round its *side* CDD′, till it comes into the plane of the *third triangle*, D′DE′; whereby the *first gauche heptagon* will have *become* a plane one, denoted as before by BCDE′D′C′B′: and so we can proceed indefinitely. Passing then to the *limit*, at which the *points* ABCDE . . . are conceived to be each *indefinitely near* to the one which precedes or follows it in the series, we conclude as usual (comp. 98, (12.)) that the *locus of the tangents* to a *curve of double curvature* is a *developable surface*: or that it admits of being *unfolded* (like a cone or cylinder) *into a plane*, without any breach of continuity. It is now proposed to *translate* these *conceptions* into the *language* of *quaternions*, and to draw from them some of their consequences: especially as regards the determination of the *geodetic lines*, on such a developable surface.

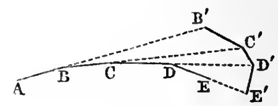

Fig. 79.

(1.) Let ψ_x, or simply ψ, denote the variable vector of a point upon the *curve*, or *cusp-edge*, or *edge of regression* of the developable, to which curve the *generating lines* of that *surface* are thus *tangents*, considered as a *function* ψ of its *arc*, x, measured from some fixed point A upon it; so that while the *equation* of the surface will be of the form (comp. 100, (8.)),

$$\text{I.} \ldots \rho = \phi(x, y) = \psi_x + y\psi'_x = \psi + y\psi',$$

y being a second scalar variable, we shall have the relations (comp. 381, XV.),

$$\text{II.} \ldots T\psi' = 1, \quad \psi'^2 = -1, \quad S\psi'\psi'' = 0, \quad S\psi'\psi''' = -\psi''^2 = z^2, \quad \text{if} \quad z = T\psi''.$$

(2.) Hence $\text{III.} \ldots \mathrm{D}_x\phi = \psi' + y\psi'', \quad \mathrm{D}_y\phi = \psi';$

$$\text{IV.} \ldots \rho' = (1+y')\psi' + y\psi'', \quad \rho'' = y''\psi' + (1+2y')\psi'' + y\psi''';$$

and

$$\text{V.} \ldots \nu = \mathrm{V}\psi'\psi'' = \psi'\psi'', \text{ multiplied by any scalar.}$$

(3.) The differential equation of the geodetics may therefore be thus written (comp. 381, XIII.),

$$\text{VI.} \ldots 0 = \mathrm{S}(\mathrm{V}\psi'\psi'' \,.\, \mathrm{V}\rho'\rho'') = \mathrm{S}\rho'\psi''\mathrm{S}\rho''\psi' - \mathrm{S}\rho''\psi''\mathrm{S}\rho'\psi';$$

in which, by (1.) and (2.),

$$\text{VII.} \ldots \begin{cases} \mathrm{S}\rho'\psi'' = -yz^2, & \mathrm{S}\rho''\psi' = -y'' + yz^2, \\ \mathrm{S}\rho''\psi'' = -(1+2y')z^2 - yzz', & \mathrm{S}\rho'\psi' = -(1+y'); \end{cases}$$

the equation becomes therefore, after division by $-z$,

$$\text{VIII.} \ldots 0 = z\{(1+y')^2 + (yz)^2\} + (1+y')(yz)' - y''yz,$$

or simply,

$$\text{IX.} \ldots z + v' = 0, \quad \text{or} \quad \text{IX}'. . \mathrm{Td}\psi' + \mathrm{d}v = 0, \quad \text{if} \quad \text{X.} \ldots \tan v = \frac{yz}{1+y'} = \frac{y'\mathrm{T}\psi''}{1+y'}.$$

(4.) To *interpret* now this very simple equation IX. or IX'., we may observe that z, or $\mathrm{T}\psi''$, or $\mathrm{Td}\psi' : \mathrm{d}x$, expresses the *limiting ratio*, which the *angle* between *two near tangents* ψ' and $\psi' + \Delta\psi'$, to the *cusp-edge* (1.), bears to the *small arc* Δx of that curve which is intercepted between their points of contact; while v is, by IV., that *other angle*, at which *such* a variable tangent, or *generating line* of the developable, *crosses the geodetic* on that surface; and therefore its *derivative*, v' or $\mathrm{d}v : \mathrm{d}x$, represents the limiting ratio, which the *change* Δv of this last angle, in passing from one generating line to another, bears to the *same small arc* Δx of the curve which those lines *touch*.

(5.) Referring then to fig. 79, in which, instead of *two continuous curves*, there were *two gauche polygons*, or at least *two systems* of *successive right lines*, connected by *prolongations* of the lines of the first system, we see that the recent formula IX. or IX'. is equivalent to this *limiting equation*,

$$\text{XI.} \ldots \lim. \frac{\mathrm{CD'C'} - \mathrm{BC'B'}}{\mathrm{C'CD'}} = -1;$$

but these three *angles* remain *unaltered*, in the *development* of the surface: the *bent line* B'C'D' for *space* becomes therefore *ultimately* a *straight* line in the *plane*, and similarly for *all other* portions of the original polygon, or *twisted line*, B′C′D′E′ . . ., of which B′C′D′ was a *part*.

(6.) Returning then to *curves* and *surfaces* in *space*, the quaternion analysis (3.) is found, by this simple reasoning,* to conduct to an expression for the known and *characteristic property* of the *geodetics on a developable*: namely that they *become right lines*, as those on *cylinders* (380, (4.)), and on *cones* (380, (6.) and (10.), or 381, (5.)), were lately seen to do, when the *surface* on which they are thus traced is *unfolded into a plane*.

383. This known result, respecting *geodetics on developables*, may be very simply verified, by means of a new determination of the *absolute*† *normal* (379) to a curve in space, as follows.

(1.) The arc s of any curve being taken for the independent variable, we may write (comp. 376, I.), by Taylor's Series, the following rigorous expressions,

$$\text{I.} \ldots \rho_{-s} = \rho - s\rho' + \tfrac{1}{2}s^2 u_{-s}\rho'', \quad \rho_0 = \rho, \ \rho_s = \rho + s\rho' + \tfrac{1}{2}s^2 u_s\rho'', \quad \text{with} \quad u_0 = 1,$$

for the vectors of three *near points*, P_{-s} P_0, P_s, on the curve, whereof the second *bisects the arc*, $2s$, intercepted between the first and third.

(2.) If then we conceive the *parallelogram* $\text{P}_{-s}\text{P}_0\text{P}_s\text{R}_s$ to be completed, we shall have, for the *two diagonals* of this new figure these other rigorous expressions,

$$\text{II.} \ldots \text{P}_{-s}\text{P}_s = \rho_s - \rho_{-s} = 2s\rho' + \tfrac{1}{2}s^2(u_s - u_{-s})\rho'';$$

$$\text{III.} \ldots \text{P}_0\text{R}_s = \rho_s + \rho_{-s} - 2\rho_0 = \tfrac{1}{2}s^2(u_s + u_{-s})\rho'';$$

which give the limiting equations,

$$\text{IV.} \ldots \lim_{s=0} s^{-1}\text{P}_{-s}\text{P}_s = 2\rho'; \qquad \text{V.} \ldots \lim_{s=0} s^{-2}\text{P}_0\text{R}_s = \rho''.$$

(3.) But the *length* $\overline{\text{P}_{-s}\text{P}_s}$ of what may be called the *long diagonal*, or the *chord* of the *double arc*, $2s$, is *ultimately equal* to that double arc; we have therefore by IV., the equation,

$$\text{VI.} \ldots \text{T}\rho' = 1, \quad \text{If} \quad \rho' = \text{D}_s\rho, \text{ and if } s \text{ denotes the } arc,$$

considered as the scalar variable on which the vector ρ depends: a result agreeing with what was otherwise found in 380, (12.).

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* In the *Lectures* (page 581), nearly the same *analysis* was employed, for geodetics on a developable; but the *interpretation* of the result was made to depend on an equation which, with the recent significations of ψ and v, may be thus written, as the integral of IX′., $v + \int \text{T}d\psi' = \text{const.}$; where $\int \text{T}d\psi'$ represents the *finite angle* between the *extreme tangents* to the *finite arc* $\int \text{T}d\psi$, or Δx, of the *cusp-edge*, when that curve is *developed* into a *plane* one.

† Called also, and perhaps more usually, the *principal normal*.

(4.) At the same time, since the *ultimate direction* of the same long diagonal is evidently that of the *tangent* at P_0, we see anew that the same *first derived vector* ρ' represents what may be called the *unit tangent** to the curve at that point.

(5.) And because the *lengths* of the *two sides* $P_{-s}P_0$ and P_0P_s, considered as *chords* of the two successive and *equal arcs*, s and s, are *ultimately equal* to them and to each other, it follows that the *parallelogram* (2.) is *ultimately equilateral*, and therefore that its *diagonals* are *ultimately rectangular*; but these diagonals, by IV. and V., have ultimately the directions of ρ' and ρ''; we find therefore anew the equation,

VII. . . $S\rho'\rho'' = 0$, if the arc be the independent variable,

which had been otherwise deduced before, in 380, (12.).†

(6.) But under the same condition, we saw (379, (2.)) that the *second derived vector* ρ'' has the direction of the *absolute normal* to the curve; such then is by V. the *ultimate direction* of what we may call the *short diagonal* P_0R_s, constructed as in (2.); or, *ultimately*, the direction of the *bisector* of the (obtuse) *angle* $P_{-s}P_0P_s$, between the *two near* and *nearly equal chords* from the point P_0: while the *plane* of the *parallelogram* becomes *ultimately* the *osculating plane*.

(7.) All this is quite independent of the consideration of any *surface*, on which the *curve* may be conceived to be *traced*. But if we now conceive that this curve is formed *from a right line* B′C′D′ . . . (comp. fig. 79), by *wrapping* round a *developable* surface a *plane* on which the *line* had been drawn, and if the successive portions B′C′, C′D′, . . of that line be supposed to have been *equal*, then because the *two right lines* C′B′ and C′D′ *originally* made *supplementary angles* with any *other* line C′C in the *plane*, the *two chords* C′B′ and C′D′ of the *curve* on the developable *tend* to make supplementary angles with the *generatrix* C′C of that surface; on which account the *bisector* (6.) of their *angle* B′C′D′ *tends* to be *perpendicular to that generating line* C′C, *as well as* to the *chord* B′D′, or *ultimately* to the *tangent to the curve* at C′, when chords and arcs *diminish* together. The *absolute normal* (6.) to the curve thus formed is therefore *perpendicular to two distinct tangents* to the surface at C′, and is consequently (comp. 372) the *normal* to that *surface* at that point; whence, by the *definition* (380), the *curve* is, as before, a *geodetic on the developable*.

* Compare the first Note to page 152, Vol. I.

† [See note to 396 (19.), p. 88.]

(8.) As regards the asserted *rectangularity* (7.), of the *bisector* of the angle B′C′D′ to the line C′C, when the angles CC′B′ and CC′D′ are supposed to be *supplementary*, but *not in one plane*, a simple proof may be given by conceiving that the right line B′C′ is prolonged to C″, in such a manner that $\overline{C'C''} = \overline{C'D'}$; for then these two *equally long* lines from C′ make *equal angles* with the line C′C, so that the one may be formed from the other by a *rotation* round that line as an *axis*; whence C″D′, which is evidently *parallel* to the bisector of B′C′D′, is also *perpendicular* to C′C.

(9.) In quaternions, if α and ρ be any two vectors, and if t be any scalar, we have the equation,

$$\text{VIII.} \ldots \; \mathrm{S} \, . \, \alpha(\alpha^t\rho\alpha^{-t} - \rho) = 0,$$

which is, by 308, (8.), an expression for the geometrical principle as stated.

384. The recent analysis (382) enables us to deduce with ease, by quaternions, other known and important properties of developable surfaces: for instance, the property that each such surface may be considered as the *envelope* of a *series of planes*, involving only *one* scalar and *arbitrary constant* (or *parameter*) in their *common equation*; and that *each plane* of this series *osculates* to the *cusp-edge* of the *developable.*

(1.) The equation of the developable *surface* being still,

$$\text{I.} \ldots \; \rho = \phi(x, y) = \psi_x + y\psi'_x = \psi + y\psi' \qquad \text{(as in 382, I.)},$$

its *normal* ν is easily found to have, as in 382, V., the *direction* of $\mathrm{V}\psi'\psi''$, whether the scalar variable x be, or be not, the *arc* of the *cusp-edge*, of which curve the equation is,

$$\text{II.} \ldots \; \rho = \psi_x.$$

(2.) Hence, by 373, VII., the equation of the *tangent plane* takes the form,

$$\text{III.} \ldots \; \mathrm{S}\omega\psi'\psi'' = \mathrm{S}\psi\psi'\psi'',$$

from which the *second* scalar variable y thus *disappears*: this *common equation*, of all the tangent planes to the developable, involves therefore, as above stated, only *one* variable and scalar *parameter*, namely x; and the *envelope* of all these *planes* is the developable surface itself.

(3.) The plane III., for any *given value* of this parameter x, that is, for any *given point* of the cusp-edge, *touches the surface* along the *whole extent* of the *generating line*, which is the *tangent* to this last *curve.*

(4.) And by comparing its equation III. with the formula 376, V., we see at once that this plane *osculates* to the same cusp-edge, at the point of *contact* of that curve with the same *generatrix* of the developable.

385. If the *reciprocals* of the *perpendiculars*, let fall from a given origin, on the *tangent planes* to a developable surface, be considered as being themselves vectors from that origin, they terminate on a *curve*, which is connected with the *cusp-edge* of the developable by some interesting relations of *reciprocity* (comp. 373, (21.)): in such a manner that if this *new curve* be made the cusp-edge of a *new developable*, we can *return* from it to the *former* surface, and to *its* cusp-edge, by a *similar process* of construction.

(1.) In general, if ψ_x and χ_x, or briefly ψ and χ, be two vector functions of a scalar variable x, such that χ may be deduced from ψ by the three scalar equations,

$$\text{I.} \ldots \mathrm{S}\psi\chi = c, \quad \mathrm{S}\psi'\chi = 0, \quad \mathrm{S}\psi''\chi = 0,$$

in which $\mathrm{S}\psi\chi$ is written briefly for $\mathrm{S}(\psi_x . \chi_x)$, and c is any scalar constant, we have then this *reciprocal system* of three such equations,

$$\text{II.} \ldots \mathrm{S}\chi\psi = c, \quad \mathrm{S}\chi'\psi = 0, \quad \mathrm{S}\chi''\psi = 0;$$

an intermediate step being the equation,

$$\text{III.} \ldots \mathrm{S}\psi'\chi' = \mathrm{S}\chi'\psi' = 0.$$

(2.) Hence, generally,

$$\text{IV.} \ldots \text{if } \chi = \frac{c\mathrm{V}\psi'\psi''}{\mathrm{S}\psi\psi'\psi''}, \quad \text{then} \quad \text{V.} \ldots \psi = \frac{c\mathrm{V}\chi'\chi''}{\mathrm{S}\chi\chi'\chi''}.$$

(3.) But if ρ be the variable vector of a curve in space, and ρ', ρ'' its first and second derivatives with respect to any scalar variable, then, by the equation 376, V. of the osculating plane to the curve, we have the general expression,

$$\text{VI.} \ldots \frac{\mathrm{S}\rho\rho'\rho''}{\mathrm{V}\rho'\rho''} = \textit{perpendicular from origin on osculating plane};$$

so that if ψ and χ be considered as the vectors of *two curves*, *each* vector is $c \times$ the *reciprocal of the perpendicular*, thus let fall from a common point, on the osculating plane to the *other*.

(4.) We have therefore this *Theorem*:—

If, from any assumed point, O, *there be drawn lines equal to the reciprocals of the perpendiculars from that point, on the osculating planes to a given curve of double curvature, or to those perpendiculars multiplied by any given and constant scalar; then the locus of the extremities of the lines so drawn will be a second* curve, from which we can return to the first curve by a precisely similar process.*

386. The theory of *developable surfaces*, considered as *envelopes of planes* with *one* scalar and *variable parameter* (384), may be additionally illustrated by connecting it with *Taylor's Series*, as follows.

(1.) Let a_t denote *any vector function* of a *scalar* variable t, so that

$$\text{I.} \ldots a_t = a_0 + tu_t a'_0 = a + tua', \quad \text{with} \quad u_0 = 1;$$

or, by another step in the expansion,

$$\text{II.} \ldots a_t = a_0 + ta'_0 + \tfrac{1}{2}t^2 v_t a''_0 = a + ta' + \tfrac{1}{2}t^2 va'', \quad v_0 = 1;$$

where u and v are generally *quaternions*, but ua' and va'' are *vectors*.

(2.) Then, as the *rigorous equation* of the *variable plane*, the reciprocal of the perpendicular on which from the origin is $-a_t$, we have either,

$$\text{III.} \ldots -1 = \text{S}a_t\rho = \text{S}a\rho + t\text{S}ua'\rho,$$

or

$$\text{IV.} \ldots -1 = \text{S}a\rho + t\text{S}a'\rho + \tfrac{1}{2}t^2\text{S}va''\rho,$$

according as we adopt the expression I., or the equally but not more rigorous expression II., for the variable vector a_t.

(3.) Hence, by the form III., the *line of intersection* of the *two planes*, which answer to the *two values* 0 and t of the scalar variable, or *parameter*, t, is *rigorously* represented by the system of the *two scalar equations*,

$$\text{V.} \ldots \text{S}a\rho + 1 = 0, \quad \text{S}ua'\rho = 0.$$

(4.) And the *limiting position* of this *right line* V., which answers to the conceived *indefinite approach* of the *second plane* to the *first*, is given with *equal rigour* by the equations,

$$\text{VI.} \ldots \text{S}a\rho + 1 = 0, \quad \text{S}a'\rho = 0;$$

* The *two curves* may be said to be *polar reciprocals*, with respect to the (real or imaginary) *sphere*, $\rho^2 = c$; and an analogous *relation of reciprocity* exists *generally*, when the *points* of one curve are the *poles* of the *osculating planes* of the other, with respect to *any surface of the second order*: *corresponding tangents* being then *reciprocal polars*. Compare the theory of *developables reciprocal to curves*, given in Salmon's *Analytical Geometry of Three Dimensions*, page 89; see also Chapter XI. (page 224, &c.), of the same excellent work.

whereof it is seen that the *second* may be formed from the *first*, by *derivating* with respect to t, and treating ρ as *constant*: although *no such rule* of *calculation* had been *previously laid down*, for the comparatively *geometrical process* which is here supposed to be adopted.

(5.) The *locus* of all the *lines* VI. is evidently *some ruled surface*; to determine the *normal* ν to which, at the extremity of the vector ρ, we may consider that vector to be a function (372) of *two* independent and scalar variables, whereof one is t, and the other may be called for the moment w; and thus we shall have the two *partial* derivatives,

$$\text{VII.} \ldots \mathrm{S}\alpha\mathrm{D}_t\rho = 0, \quad \mathrm{S}\alpha\mathrm{D}_w\rho = 0, \quad \text{giving} \quad \nu \parallel \alpha.$$

(6.) Hence the *line* α has the direction of the required *normal* ν; the *plane* $\mathrm{S}\alpha\rho + 1 = 0$ *touches* the *surface* (comp. 384, (3.)) along the *whole extent* of the *limiting line* VI.; and the *locus* of all *such lines* is the *envelope* of all the *planes*, of the system recently considered.

(7.) The *line* VI. *cuts* generally the *plane* IV., in a *point* which is rigorously determined by the three equations,

$$\text{VIII.} \ldots \mathrm{S}\alpha\rho + 1 = 0, \quad \mathrm{S}\alpha'\rho = 0, \quad \mathrm{S}v\alpha''\rho = 0;$$

and the *limiting position* of this *intersection* is, with equal rigour, the point determined by this other system of equations,

$$\text{IX.} \ldots \mathrm{S}\alpha\rho + 1 = 0, \quad \mathrm{S}\alpha'\rho = 0, \quad \mathrm{S}\alpha''\rho = 0;$$

in which it may be remarked (comp. (4.)), that the *third* is the *derivative* of the *second*, if ρ be treated as *constant*.

(8.) The *locus* of all these *points* IX. is generally *some curve* upon the *surface* (5.), which is the *locus* of the *lines* VI., and has been seen to be the *envelope* (6.) of the *planes* III. or IV.; and to find the *tangent* to this *curve*, at the point answering to a *given value* of t, or to a *given line* VI., we have by IX. the derived equations,

$$\text{X.} \ldots \mathrm{S}\alpha\rho' = 0, \quad \mathrm{S}\alpha'\rho' = 0, \quad \text{whence} \quad \rho' \parallel \mathrm{V}\alpha\alpha';$$

comparing which with the equations VI. we see that *the lines* VI. *touch the curve*, which is thus *their* common *envelope*.

(9.) We see then, in a new way, that *the envelope of the planes* III., which have *one scalar parameter* (t) in their *common equation*, and may represent *any system* of planes subject to this *condition*, is a *developable surface*:

because it is *in general* (comp. 382) the *locus of the tangents to a curve in space*, although this curve *may* reduce itself to a *point*, as we shall shortly see.

(10.) We may add that if α_t in III. be considered as the vector of a *given curve*, this curve is the *locus of the poles** of the *tangent planes* to the *developable*, taken with respect to the *unit sphere*; and conversely, that the developable surface is the *envelope* of the *polar planes* of the *points* of the same given curve, with respect to the same sphere.

(11.) If then it happen that this *given* curve, with α_t for vector, is a *plane* one, so that we have this new condition,

$$\text{XI.} \ldots \mathrm{S}\beta\alpha_t + 1 = 0, \quad \beta \text{ being any constant vector,}$$

namely the vector of the pole of the supposed plane of the given curve, the *variable plane* III., or $\mathrm{S}\rho\alpha_t + 1 = 0$, of which the surface (5.) is the *envelope*, passes constantly through this *fixed pole*; so that the *developable* becomes in this case a *cone*, with β for the vector of its *vertex*: the equations IX. giving *now* $\rho = \beta$.

(12.) The same *degeneration*, of a developable into a *conical* surface, may also be conceived to take place in another way, by the *cusp-edge* (or at least some finite portion thereof) tending to become *indefinitely small*, while yet the *direction* of its *tangents* does not tend to become *constant*. For example, with recent notations, the *developable* which is the locus of the tangents to the *helix* may have its equation written thus:

$$\text{XII.} \ldots \rho = \phi(x, y) = c\left(x\alpha + \frac{2}{\pi} \tan a \,.\, \alpha^x \mathrm{U}\beta\right) + y\alpha(1 + \tan a \,.\, \alpha^x \mathrm{U}\beta);$$

which when the *quarter interval*, c, between the *spires*, tends to *zero*, without their *inclination* a to the *axis* α being changed, *tends* to become a *cone of revolution* round that axis, with its *semiangle* $= a$.

387. So far, then, we may be said to have considered, in the present section, and in connexion with *geodetic lines*, the four following *families* of *surfaces* (if the first of them may be so called). First, *spherical surfaces*, of which the characteristic *property* is expressed by the equation,

$$\text{I.} \ldots \mathrm{V}\nu(\rho - \alpha) = 0, \quad \text{if } \alpha \text{ be } \textit{vector of centre};$$

second, *cylindrical* surfaces, with the property,

$$\text{II.} \ldots \mathrm{S}\nu\alpha = 0, \quad \text{if } \alpha \text{ be } \textit{parallel to the generating lines};$$

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* Compare the Note to page 42.

third, *conical* surfaces, with the property,

$$\text{III.} \ldots S\nu(\rho - \alpha) = 0, \quad \text{if } \alpha \text{ be } \textit{vector of vertex};$$

and fourth, *developable* surfaces, with the distinguishing property expressed by the more general equation,

$$\text{IV.} \ldots V\nu d\nu = 0, \quad \text{if } d\rho \text{ have the } \textit{direction of a generatrix};$$

ν being in each the *normal vector* to the surface, so that

$$\text{V.} \ldots S\nu d\rho = 0, \quad \text{for } \textit{all tangential directions} \text{ of } d\rho;$$

and the *fourth* family *including* the *third*, which in its turn includes the *second*. A few additional remarks on these equations may be here made.

(1.) The *geometrical signification* of the equation I. (as regards the *radii*) is obvious; but on the side of *calculation* it may be useful to remark, that *elimination* of ν between I. and V. gives, for *spheres*,

$$\text{VI.} \ldots S(\rho - \alpha)d\rho = 0, \quad \text{or} \quad \text{VII.} \ldots T(\rho - \alpha) = \text{const.}$$

(2.) The equations II. and V. show that $d\rho$, and therefore $\Delta\rho$, *may* have the given direction of α; for an *arbitrary cylinder*, then, we have the *vector equation* (372),

$$\text{VIII.} \ldots \rho = \phi(x, y) = \psi_x + y\alpha,$$

where ψ_x is an *arbitrary vector function* of x.*

(3.) From VIII. we can at once infer, that

$$\text{IX.} \ldots S\beta\rho = S\beta\psi_x, \quad S\gamma\rho = S\gamma\psi_x, \quad \text{if} \quad \alpha = V\beta\gamma;$$

the *scalar equation* (373) of a *cylindrical surface* is therefore generally of the *form* (comp. 371, (6.), (7.)),

$$\text{X.} \ldots 0 = F(S\beta\rho, S\gamma\rho);$$

β and γ being two constant vectors, and the generating lines being perpendicular to both.

(4.) The equation III. may be thus written,

$$\text{XI.} \ldots S\nu U\alpha = T\alpha^{-1}S\nu\rho; \quad \text{whence} \quad \text{XII.} \ldots S\nu U\alpha = 0, \quad \text{if} \quad T\alpha = \infty;$$

---

* [In general $d\rho = D_x\rho \,.\, dx + D_y\rho \,.\, dy$, and as one direction of $d\rho$ is parallel to α, we may write without loss of generality, $d\rho = D_x\rho \, dx + \alpha dy$. Moreover, since α is constant, $D_x\rho$ must be a function of x, so on integration $\rho = \psi_x + y\alpha$.]

the equation for *cones* includes therefore that for *cylinders*, as was to be expected, and reduces itself thereto, when the vertex becomes infinitely distant.

(5.) The same equation III., when compared with V., shows that $d\rho$ may have the direction of $\rho - \alpha$, and therefore that $\rho - \alpha$ may be multiplied by any scalar; the *vector equation* of a *conical surface* is therefore of the form,

$$\text{XIII.} \ldots \rho = \alpha + y\psi_x, \quad \psi_x \text{ being an arbitrary vector function.*}$$

(6.) The *scalar equation* of a cone may be said to be the result of the *elimination* of a scalar variable t, between two equations of the forms,

$$\text{XIV.} \ldots \text{S}(\rho - \alpha)\chi_t = 0, \quad \text{S}(\rho - \alpha)\chi'_t = 0,$$

which express that the *cone* is the *envelope* (comp. 386, (11.)) of a *variable plane*, which passes through a *fixed point*, and involves only *one* scalar *parameter* in its equation: with a new reduction to a *cylinder*, in a case on which we need not here delay.

(7.) The equation IV. implies, that for each *point* of the surface there is a *direction* along which we may move, *without changing the tangent plane*; and therefore that the surface is an *envelope of planes*, &c., as in 386, and consequently that it is *developable*, in the sense of Art. 382.†

(8.) The *vector equation* of a *general developable surface* may be written under the form,

$$\text{XV.} \ldots \rho = \phi(x, y) = \psi_x + y\text{U}\psi'_x;$$

the sign of a *versor* being here introduced, for the sake of facilitating the passage, at a certain *limit*, to a *cone* (comp. 386, (12.)).

(9.) And the *scalar equation* of the same *arbitrary developable* may be represented as the result of the elimination of t, between the *two* equations,

$$\text{XVI.} \ldots \text{S}\rho\chi_t + 1 = 0, \quad \text{S}\rho\chi'_t = 0;$$

in which χ_t is an arbitrary vector function of t.

---

* [As in the last note, because one direction of $d\rho$ is parallel to $\rho - \alpha$, we may take

$$d\rho = (\rho - \alpha)y^{-1}dy + \text{D}_x\rho \,.\, dx, \quad \text{or} \quad d \,.\, y^{-1}(\rho - \alpha) = y^{-1}\text{D}_x\rho \,.\, dx = d \,.\, \psi_x.$$

Hence, $\rho = \alpha + y\psi_x$.]

† [The normal at any point of the ruled surface $\rho = \psi_x + y\phi_x$ is parallel to $\text{V}(\psi'_x + y\phi'_x)\phi_x$. If the direction of the normal does not change as we pass along a generator, either $\text{V}\psi'_x\phi_x = 0$, or $\text{V}\phi'_x\phi_x = 0$. The first of these conditions requires the surface to be a developable. The second requires all the generators to be parallel, so that the surface is a cylinder. See Tait's Quaternions, Art. 311.]

(10.) The envelope of a *plane* with *two* arbitrary and scalar parameters, t and u, is generally a *curved* but *undevelopable surface*, which may be represented by the system of the *three scalar equations*,

$$\text{XVII.} \ldots S\rho\chi_{t,u} + 1 = 0, \quad S\rho D_t\chi = 0, \quad S\rho D_u\chi = 0;$$

where $-\chi$ denotes the reciprocal of the perpendicular from the origin on the tangent plane to the surface, at what may be called *the point* (t, u).

388. It remains, on the plan lately stated (380), to consider briefly *surfaces of revolution*, and to investigate the *geodetic lines*, on this additional *family* of surfaces; of which the *equation*, analogous to those marked I. II. III. IV. in 387, for spheres, cylinders, cones, and developables, is of the form,

$$\text{I.} \ldots S\alpha\rho\nu = 0,$$

if α be a given line in the direction of the *axis* of revolution, supposed for simplicity to pass through the origin; but which may also be represented by either of these two other equations, not involving the normal ν,

$$\text{II.} \ldots T\rho = f(S\alpha\rho), \quad \text{or} \quad \text{III.} \ldots TV\alpha\rho = F(S\alpha\rho),$$

where f and F are used as characteristics of two *arbitrary* but *scalar functions*: between which $S\alpha\rho$ may be conceived to be eliminated, and so a *third form* of the same sort obtained.

(1.) In fact, the equation I. expresses that $\nu \parallel\!| \alpha, \rho$, or that the *normal* to the surface *intersects* the *axis*; while II. expresses that the *distance* from a *fixed point* upon that axis is a *function* of its own *projection* on the same *fixed line*, or that the *sections* made by *planes perpendicular* to the axis are *circles*; and the same *circularity* of these sections is otherwise expressed by III., since that equation signifies that the *distance from the axis* depends on the *position* of the *cutting plane*, and is *constant* or *variable* with it: while the two last forms are connected with each other in *calculation*, by means of the general relation (comp. 204, XXI.),

$$\text{IV.} \ldots (T\alpha\rho)^2 = (S\alpha\rho)^2 + (TV\alpha\rho)^2.$$

(2.) The equation I. is *analogous*, in *quaternions*, to a *partial differential equation* of the *first order*, and either of the two other equations, II. and III., is *analogous* to the *integral* of that equation, in the *usual* differential calculus of *scalars*.

(3.) To accomplish the corresponding *integration* in *quaternions*, or to pass from the form I. to II., whence III. can be deduced by IV., we may observe that the equation I. allows us to write (because $S\nu d\rho = 0$),

$$\text{V.} \ldots \nu = x\alpha + y\rho, \qquad \text{VI.} \ldots xS\alpha d\rho + yS\rho d\rho = 0,$$

so that the two scalars $S\alpha\rho$ and $T\rho$ are *together constant*, or *together variable*, and must therefore be *functions* of *each other*.

(4.) Conversely, to *eliminate* the *arbitrary function* from the form II., quaternion *differentiation* gives,

$$\text{VII.} \ldots 0 = S(U\rho \,.\, d\rho) + f'(S\alpha\rho) \,.\, S\alpha d\rho = S \,.\, (U\rho + \alpha f'S\alpha\rho)d\rho\,;$$

hence

$$\text{VIII.} \ldots \nu \parallel U\rho + \alpha f'S\alpha\rho, \quad \text{and} \quad \text{IX.} \ldots \nu \parallel\!| \ \alpha, \rho, \text{ as before};$$

so that we can *return* in this way to the equation I., the *functional sign f disappearing*.

(5.) We have thus the germs of a *Calculus of Partial Differentials in Quaternions*,* *analogous* to that employed by Monge, in his researches respecting *families of surfaces*: but we cannot attempt to pursue the subject farther here.

(6.) But as regards the *geodetic lines* upon a surface of revolution, we have only to substitute for ν, in the recent formula I., by 380, IV., the expression $dUd\rho$, which gives at once the *differential equation*,

$$\text{X.} \ldots 0 = S\alpha\rho dUd\rho = d \,.\, S\alpha\rho Ud\rho \text{ (because } S(\alpha d\rho \,.\, Ud\rho) = - S\alpha Td\rho = 0);$$

whence, by a first integration, c being a scalar constant,

$$\text{XI.} \ldots c = S\alpha\rho Ud\rho = TV\alpha\rho \,.\, SU(V\alpha\rho \,.\, d\rho).$$

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* The same remark was made in page 574 of the *Lectures*, in which also was given the elimination of the arbitrary function from an equation of the recent form III. It was also observed, in page 578, that *geodetics* furnish a very simple example of what may be called the *Calculus of Variations in Quaternions*; since we may write,

$$\delta \textstyle\int ds = \delta \int Td\rho = \int \delta Td\rho = -\int S(Ud\rho \,.\, \delta d\rho)$$

$$= -\textstyle\int S(Ud\rho \,.\, d\delta\rho) = -\Delta S(Ud\rho \,.\, \delta\rho) + \int S(dUd\rho \,.\, \delta\rho),$$

and therefore $dUd\rho \parallel \nu$, or $V\nu dUd\rho = 0$, as in 380, IV., in order that the expression under the last integral sign may vanish for all variations $\delta\rho$ consistent with the *equation of the surface*: while the evanescence of the part which is *outside* that sign $\int$ supplies the *equations of limits*, or shows that the *shortest line between two curves* on a given surface is *perpendicular to both*, as usual.

(7.) The characteristic property of the sought curves is, therefore, that for each of them the *perpendicular distance from the axis of revolution varies inversely as the cosine* of the angle, at which the geodetic crosses a parallel,* or *circular section* of the surface: because, if $T\alpha = 1$, the line $V\alpha\rho$ has the *length* of the *perpendicular* let fall from a point of the curve on the axis, and has the *direction* of a *tangent* to the parallel.

(8.) The equation XI. may also be thus written,

$$\text{XII.} \ldots cT\rho' = S\alpha\rho\rho', \quad \text{where} \quad \rho' = D_t\rho;$$

and if the independent variable t be supposed to denote the *time*, while the *geodetic* is conceived to be a curve described by a *moving point*, then while $T\rho'$ evidently represents the *linear velocity* of that point, as being $= ds : dt$, if s denote the *arc* (comp. 100, (5.), and 380, (7.), (11.)), it is easy to prove that $S\alpha\rho\rho'$ represents the *double areal velocity, projected on a plane perpendicular to the axis*; the *one* of these two *velocities* varies therefore *directly as the other*: and in fact, it is known from mechanics, that *each* velocity would be *constant*,† if the *point* were to describe the *curve*, subject only to the *normal reaction* of the *surface*, and undisturbed by any *other force*.

(9.) As regards the *analysis*, it is to be observed that the *differential equation* X. is satisfied, *not only* by the *geodetics* on the surface of revolution, but *also* by the *parallels* on that surface: which fact of calculation is connected with the obvious geometrical property, that *every normal plane* to such a parallel *contains the axis* of revolution.

(10.) In fact if we draw the normal plane to *any curve* on the surface, at a point where it *crosses* a *parallel*, this *plane* will *intersect* the *axis*, in the point where that axis is met by the *normal* to the *surface*, drawn at the same point of crossing; but *this* construction *fails* to *determine* that normal, if the curve *coincide* with, or even *touch* a parallel, at the point where its normal plane is drawn.

* Unless it happen that this cosine is *constantly zero*, in which case $c = 0$, and the *geodetic* is a *meridian* of the surface.

† This remark is virtually made in page 443 of Professor De Morgan's *Differential and Integral Calculus* (London, 1842), which was alluded to in page 578 of the *Lectures on Quaternions*. [If p is the normal reaction of the surface, the differential equation of motion of the particle is $\rho'' = pU\nu$. From this equation the mechanical properties may be at once deduced.]

SECTION 6.

On Osculating Circles and Spheres, to Curves in Space; with some connected Constructions.

389. Resuming the expression 376, I. for ρ_t, and referring again to fig. 77 [p. 24], we see that if a *circle* PQD be described, so as to *touch* a given curve PQR, or its tangent PT, at a given point P, and to *cut* the curve at a near point Q, and if PN be the *projection* of the *chord* PQ on the *diameter* PD, or on the *radius* CP, then because we have, rigorously,

$$\text{I.} \ldots \text{PQ} = t\rho' + \tfrac{1}{2}t^2 u\rho'', \quad \text{with} \quad u = 1 \quad \text{for} \quad t = 0,$$

we have also

$$\text{II.} \ldots \text{PN} = \tfrac{1}{2}t^2 \text{V}u\rho''\rho' : \rho',$$

and

$$\text{III.} \ldots \frac{1}{\text{PC}} = \frac{2}{\text{PD}} = \frac{2\text{PN}}{\text{PQ}^2} = \frac{\text{V}u\rho''\rho'}{(\rho' + \frac{1}{2}tu\rho'')^2\rho'}.$$

Conceiving then that the near point Q *approaches indefinitely* to the given point P, in which case the *ultimate state* or *limiting position* of the *circle* PQD is said to be that of the *osculating circle* to the curve at that point P, we see that while the *plane* of this last circle is the *osculating plane* (376), the *vector* κ of its *centre* K, or of the *limiting position* of the point C, is rigorously expressed by the formula:

$$\text{IV.} \ldots \kappa = \rho + \frac{\rho'^3}{\text{V}\rho''\rho'};$$

which may however be in many ways *transformed*, by the rules of the present Calculus.

(1.) Thus, we may write, as *transformations* of the expression IV., the following:

$$\text{V.} \ldots \kappa = \rho + \frac{\rho'}{\text{V}\rho''\rho'^{-1}} = \rho - \frac{\text{T}\rho'}{\text{V}\rho''\rho'^{-1}.\text{U}\rho'} = \rho - \frac{\text{T}\rho'}{(\text{U}\rho')'};$$

or introducing *differentials* instead of *derivatives*, but leaving still the independent variable *arbitrary*,

$$\text{VI.} \ldots \kappa = \rho - \frac{\text{d}\rho^3}{\text{V}\text{d}\rho\text{d}^2\rho} = \rho + \frac{\text{d}\rho}{\text{V}\text{d}^2\rho\text{d}\rho^{-1}} = \rho - \frac{\text{T}\text{d}\rho}{\text{d}\text{U}\rho'} = \rho - \frac{\text{d}s}{\text{d}\text{U}\text{d}\rho},$$

if s be the *arc* of the curve; so that the last expression gives this very simple formula, for the *reciprocal of the radius of curvature*, or for the *ultimate value* of 1 : CP,

$$\text{VII.} \ldots (\rho - \kappa)^{-1} = \mathrm{D}_s \mathrm{U}\rho', \quad \text{where} \quad \mathrm{U}\rho' = \mathrm{U}d\rho, \text{ as before.}$$

(2.) To *interpret* this result, we may employ again that *auxiliary* and *spherical curve*, upon the *cone of parallels to tangents*, which has already served us to *construct*, in 379, (6.) and (7.), the *osculating plane*, the *absolute normal*, and the *binormal*, to the *given curve* in space. And thus we see, that while the *semidiameter* PC has ultimately the *direction* of $d\mathrm{U}\rho'$, and therefore that of the *absolute normal* (379, II.) at P, the *length* of the same radius is ultimately equal to the *arc* PQ (or Δs) *of the given curve, divided by the corresponding arc of the auxiliary curve*; or that the *radius of curvature*, or *radius of the osculating circle* at P, is equal to the *ultimate quotient* of the *arc* PQ, divided by the *angle between the tangents*, PT and (say) QU, to that *arc* PQ *itself* at P, and to its *prolongation* QR at Q, *although* these two tangents are *generally* in *different planes*, and have *no common point*, so long as PQ remains *finite*: because we suppose that the *given* curve is *in general* one of *double curvature*, although the *formulæ*, and the *construction*, above given, are applicable to *plane curves also*.

(3.) For the *helix*, the formula IV. gives, by values already assigned for ρ, ρ', ρ'', and a, the expression,

$$\text{VIII.} \ldots \kappa = cta - a^t\beta \cot^2 a, \quad \text{whence} \quad \text{IX.} \ldots \rho - \kappa = a^t\beta \operatorname{cosec}^2 a,$$

a being the inclination of the given helix to the axis; the *locus of the centre* of the osculating circle is therefore in this case a *second helix*, on the *same cylinder*, if $a = \frac{\pi}{4}$, but otherwise on a *co-axal* cylinder, of which the *radius* = the given radius $\mathrm{T}\beta$, multiplied by the square of the cotangent of a; and the *radius of curvature* = $\mathrm{T}(\rho - \kappa) = \mathrm{T}\beta \times \operatorname{cosec}^2 a$, so that *this* radius always *exceeds* the radius of the cylinder, and is *cut* perpendicularly (without being *prolonged*) by the *axis*.

(4.) In general, if $\mathrm{T}\rho' =$ const., and therefore $\mathrm{S}\rho'\rho'' = 0$ (comp. 379, (2.)), the expression IV. becomes,*

$$\text{X.} \ldots \kappa = \rho + \frac{\rho'^2}{\rho''}; \quad \text{whence,} \quad \text{XI.} \ldots \kappa = \rho - \rho''^{-1}, \quad \text{if} \quad \mathrm{T}\rho' = 1,$$

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* The expressions X. XI. may also be easily deduced by limits, from the construction in 383, (2.).

that is, if the *arc* be taken as the *independent variable* (380, (12.)). Under this last condition, then, the formula VII. reduces itself to the following,

$$\text{XII.} \ldots (\rho - \kappa)^{-1} = \rho'' = \mathrm{D}_s^2\rho < \textit{ultimate reciprocal of radius } \mathrm{CP};$$

so that ρ'' (for $\mathrm{T}\rho' = 1$) may be called the *Vector of Curvature*, because its *tensor* $\mathrm{T}\rho''$ is a *numerical measure* for what is *usually* called *the curvature** at the point P, and its *versor* $\mathrm{U}\rho''$ represents the *ultimate direction* of the *semidiameter* PC, of the *circle* constructed as above.

(5.) As an example of the application (2.) of the formula IV. for κ, to a *plane curve*, let us take the *ellipse*,

$$\text{XIII.} \ldots \rho = \mathrm{V}\alpha^t\beta, \quad \mathrm{T}\alpha = 1, \quad \mathrm{S}\alpha\beta \gtrless 0, \qquad 337, (2.),$$

considered as an *oblique section* (314, (4.)) of a *right cylinder*. The expressions 376, (5.) for the derivatives of ρ, combined with the expression XIII. for that vector itself, give here the relations,

$$\text{XIV.} \ldots \mathrm{V}\rho\rho'' = 0, \quad \mathrm{V}\rho'\rho''' = 0;$$

and therefore comp. (338, (5.)),

$$\text{XV.} \ldots \mathrm{V}\rho\rho' = \text{const.}\dagger = \frac{\pi}{2}\beta\gamma, \quad \mathrm{V}\rho'\rho'' = \text{const.} = \left(\frac{\pi}{2}\right)^3\beta\gamma, \quad \text{if} \quad \gamma = \mathrm{V}\alpha\beta;$$

hence for the present curve we have by IV.,

$$\text{XVI.} \ldots \kappa = \rho - \frac{\rho'^3}{\mathrm{V}\rho'\rho''} = \mathrm{V}\alpha^t\beta - (\mathrm{V}\alpha^{t+1}\beta)^3\,(\beta\gamma)^{-1}.$$

(6.) To *interpret* this result, we may write it as follows,

$$\text{XVII.} \ldots \kappa = \rho - \frac{\rho_1^2}{\mathrm{V}\rho\rho' \cdot \rho'^{-1}}, \quad \text{where} \quad \text{XVIII.} \ldots \rho_1 = \frac{2}{\pi}\rho' = \mathrm{V}\alpha^{t+1}\beta;$$

so that ρ_1 is the *conjugate semidiameter* of the ellipse (comp. 369, (4.)), and $\mathrm{V}\rho\rho' : \rho'$ is the *perpendicular from the centre* of that curve *on the tangent*. We recover then, by this simple analysis, the known result, that the *radius of curvature* of an ellipse is equal to the *square* of the *conjugate* semidiameter, *divided* by the *perpendicular*.

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* It may be remarked that the quantity z, or $\mathrm{T}\psi''$, in the investigation (382) respecting *geodetics on a developable*, represents thus the *curvature of the cusp-edge*, for any proposed value of the *arc*, x, of that curve.

† These values XV. might have been obtained *without integrations*, but this seemed to be the readiest way. [The constants may be determined by putting $t = 0$.]

(7.) We may also write the equation XVI. under the form,

$$\text{XIX.} \ldots \kappa = \rho - \frac{\rho_1^{\,3}}{V\rho\rho_1}, \quad \text{where} \quad \text{XX.} \ldots V\rho\rho_1 = \beta\gamma = \text{const.};$$

and may interpret it as expressing, that the radius of curvature is equal to the *cube* of the *conjugate* semidiameter, divided by the *constant parallelogram* under *any two* such conjugates; or by the *rectangle* under the *major* and *minor semiaxes*, which are here the vectors β and γ (comp. 314, (2.)).

(8.) The expression XVI. or XIX. for κ is easily seen to *vanish*, as it ought to do, at the *limit* where the *ellipse* becomes a *circle*, by the *cylinder* being cut *perpendicularly*, or by the condition $S\alpha\beta = 0$ being satisfied; and accordingly if we write,

$$\text{XXI.} \ldots e = SU\alpha\beta = \textit{excentricity of ellipse}, \quad \text{or} \quad \text{XXII.} \ldots \gamma^2 = (1 - e^2)\beta^2,$$

we easily find the expressions,

$$\text{XXIII.} \ldots \rho = \beta S \,.\, \alpha^t + \gamma S \,.\, \alpha^{t-1}, \quad \rho_1 = -\beta S \,.\, \alpha^{t-1} + \gamma S \,.\, \alpha^t;$$

$$\text{XXIV.} \ldots \rho_1^{\,2} = \beta^2(1 - e^2(S \,.\, \alpha^t)^2), \quad \frac{\rho_1}{V\rho\rho_1} = \frac{\rho_1}{\beta\gamma} = \beta^{-2}\left(\beta S \,.\, \alpha^t + \frac{\gamma S \,.\, \alpha^{t-1}}{1 - e^2}\right);$$

so that the formula XIX. becomes,

$$\text{XXV.} \ldots \kappa = e^2\left(\beta(S \,.\, \alpha^t)^3 - \frac{\gamma(S \,.\, \alpha^{t-1})^3}{1 - e^2}\right) = e^2(\beta(S \,.\, \alpha^t)^3 - \frac{\beta^2}{\gamma}(S \,.\, \alpha^{t-1})^3),$$

thus containing e^2 as a factor.

(9.) And it may be remarked in passing, that the expression XVI., or its recent transformation XXV., for κ as a function of t, may be considered as being in quaternions the *vector equation* (comp. 99, I., or 369, I.) *of the evolute** of the ellipse, or the equation of the *locus of centres of curvature* of that plane curve; and that the last form gives, by elimination of t (comp. 315, (1.), and 371, (5.)), the following system of *two scalar equations* for the same evolute,

$$\text{XXVI.} \ldots \left(S\frac{\kappa}{\beta}\right)^{\frac{2}{3}} + \left(S\frac{\gamma\kappa}{\beta^2}\right)^{\frac{2}{3}} = e^{\frac{4}{3}}, \quad S\beta\gamma\kappa = 0;$$

* That is to say, of the *plane evolute*; for we shall soon have occasion to consider briefly those *evolutes of double curvature*, which have been shown by Monge to exist, *even* when the *given* curve is *plane*.

or $$\text{XXVI}' \ldots (S\beta\kappa)^{\frac{2}{3}} + (S\gamma\kappa)^{\frac{2}{3}} = (e\beta)^{\frac{4}{3}}, \text{ \&c.};$$

which will be found to agree with known results.*

(10.) As another example of application to a *plane* curve, we may consider the *hyperbola*,

$$\text{XXVII} \ldots \rho = t\alpha + t^{-1}\beta, \qquad \text{(comp. 371, II.)},$$

with α and β for asymptotes, and with its centre at the origin. In this case the derived vectors are,

$$\text{XXVIII} \ldots \rho' = \alpha - t^{-2}\beta, \quad \rho'' = 2t^{-3}\beta,$$

whence

$$\text{XXIX} \ldots V\rho''\rho' = 2t^{-3}V\beta\alpha = t^{-2}V\rho\rho',$$

and the formula IV. becomes,

$$\text{XXX} \ldots \kappa - \rho = \frac{(T\rho')^2}{V\rho\rho' : \rho'} = \frac{\text{PT}^2}{\text{OV}},$$

where OV is the perpendicular from the centre O on the tangent to the curve at P, and PT is the portion of that tangent, intercepted between the same point P and an asymptote (comp. (6.) and 371, (3.)).

(11.) We may also interpret the denominator in XXX. as denoting the *projection* of the *semidiameter* OP *on the normal*, or as the line NP where N is the foot of the perpendicular from the centre on that normal line; if then K be the sought centre of the osculating circle, we have the *geometrical* equations,

$$\text{XXXI} \ldots \text{NP} \cdot \text{PK} = \text{PT}^2, \qquad \text{XXXII} \ldots \angle\, \text{NTK} = \frac{\pi}{2};$$

whereof the last furnishes evidently an extremely simple *construction* for the *centre of curvature* of an *hyperbola*, which we shall soon find to admit of being extended, with little modification, to a *spherical conic*† and its *cyclic arcs*.

(12.) The *logarithmic spiral* with its *pole* at the *origin*,

$$\text{XXXIII} \ldots \rho = \alpha^t\beta, \quad S\alpha\beta = 0, \quad T\alpha \gtrless 1, \qquad \text{(comp. 314, (5.))}$$

* [The expression $(e\beta)^{\frac{4}{3}}$ is perhaps a little inaccurate. The cube-root of $(e\beta)^4$ is meant.]

† It was in fact for the *spherical* curve that the *geometrical construction* alluded to was *first* perceived by the writer, soon after the invention of the quaternions, and as a consequence of calculation with them: but it has been thought that a sub-article or two might be devoted, as above, to the *plane case*, or *hyperbolic limit*, which may serve at least as a verification.

may be taken as a *third example* of a *plane curve*, for the application of the foregoing formulæ. A first derivation gives, by 333, VII.,

$$\text{XXXIV.} \ldots \rho' = (c+\gamma)\rho = \rho(c-\gamma), \quad \rho'\rho^{-1} = c+\gamma, \text{ if } c = \text{lT}a, \text{ and } \gamma = \frac{\pi}{2}\text{U}a;$$

the *constant quaternion quotient*, $\rho' : \rho$, here showing that the prolonged *vector* OP makes with the *tangent* PT a *constant angle*, n, which is given by the formula,

$$\text{XXXV.} \ldots \tan n = (\text{TV} : \text{S})\,(\rho' : \rho) = c^{-1}\text{T}\gamma, \quad \text{or} \quad \cot n = \frac{2}{\pi}\text{lT}a;*$$

and a second derivation gives next,

$$\text{XXXVI.} \ldots \rho'' = (c+\gamma)^2\rho, \quad \text{V}\rho''\rho' = (c^2-\gamma^2)\rho^2\gamma = \rho'^2\gamma.$$

The formula IV. becomes therefore, in this case,

$$\text{XXXVII.} \ldots \kappa = \rho + \rho'\gamma^{-1} = \rho c\gamma^{-1} = -c\gamma^{-1}\rho = \frac{2\text{lT}a}{\pi\text{T}a} \cdot a^{t+1}\beta;$$

the *evolute* is therefore a *second spiral*, of the same kind as the first, and the *radius of curvature* KP *subtends a right angle* at the *common pole*. But we cannot longer here delay on *applications within the plane*, and must resume the treatment by quaternions of curves of *double curvature*.†

390. When the *logic* by which the expression 389, IV. was obtained, for the vector κ of the centre of the osculating circle, has once been fully *understood*, the *process* may be conveniently and safely *abridged*, as follows. Referring still to fig. 77 [p. 24], we may write briefly, as equations which are all *ultimately true*, or true *at the limit*, in a sense which is supposed to be now distinctly seen :

$$\text{I.} \ldots \text{PT} = d\rho, \quad \text{TQ} = \tfrac{1}{2}d^2\rho, \quad \text{PN} = (\text{part of PQ} \perp \text{PT} =)\ \frac{\text{V}d^2\rho d\rho}{2d\rho},$$

by 203, &c.; whence, ultimately,

$$\text{II.} \ldots \kappa - \rho = \text{PC} = \frac{\text{PQ}^2}{2\text{PN}} = \frac{\text{PT}^2}{2\text{PN}} = \frac{d\rho^3}{\text{V}d^2\rho d\rho},$$

as before: this *last* expression, in which $\text{V}d^2\rho d\rho$ denotes briefly $\text{V}(d^2\rho \,.\, d\rho)$,

---

* If r be radius vector, and θ polar angle, and if we suppose for simplicity that $\text{T}\beta = 1$, the ordinary *polar equation* of the spiral becomes $r = a^\theta$, with $a = \text{T}a^{\frac{2}{\pi}}$, and $\cot n = \text{l}a$, as usual.

† [The differential equations $\rho'' = c\rho$, compare (5.); $\text{V}\beta\rho'' = 0$ (10.); $\rho' = q\rho$ and $\rho'' = q\rho$ (12.) will afford useful exercises in integration and in geometrical and dynamical interpretation.]

being *rigorous*, and permitting the choice of *any scalar*, to be used as the *independent variable*. And then, by writing,

$$\text{III.} \ldots \mathrm{d}\rho = \rho'\mathrm{d}t, \quad \mathrm{d}^2t = 0, \quad \mathrm{d}^2\rho = \rho''\mathrm{d}t^2,$$

the factor $\mathrm{d}t^3$ disappears, and we pass at once to the expression,

$$\text{IV.} \ldots \kappa - \rho = \frac{\rho'^3}{\mathrm{V}\rho''\rho'}, \qquad (389,\ \text{IV.}),$$

which had been otherwise found before.

(1.) When the *arc* of the curve is taken for the independent variable, then (comp. 380, (12.) &c.) the expression II. reduces itself to the following,

$$\text{V.} \ldots \kappa - \rho = \frac{\mathrm{d}\rho^2}{\mathrm{d}^2\rho}, \quad \text{because} \quad \mathrm{S}\mathrm{d}^2\rho\mathrm{d}\rho = 0\,;$$

and accordingly the *angle* PTQ in fig. 77 is then *ultimately right* (comp. 383, (5.)), so that we may at once write, with *this choice* of the scalar variable,

$$\text{VI.} \ldots \kappa - \rho = (ult.)\ \mathrm{PC} = (ult.)\ \frac{\mathrm{PT}^2}{2\mathrm{TQ}} = \frac{\mathrm{d}\rho^2}{\mathrm{d}^2\rho}, \text{ as above.}$$

(2.) Suppose then that we have thus *geometrically* (and *very simply*) *deduced* the expression V. for $\kappa - \rho$, for this *particular choice* of the scalar variable; and let us consider how we might thence *pass*, in *calculation*, to the more *general* formula II., in which that variable is left *arbitrary*. For this purpose, we may write, by principles already stated,

$$\text{VII.} \ldots (\rho - \kappa)^{-1} = \frac{\mathrm{d}^2\rho}{(\mathrm{T}\mathrm{d}\rho)^2} = \frac{1}{\mathrm{T}\mathrm{d}\rho}\,\mathrm{d}\,\frac{\mathrm{d}\rho}{\mathrm{T}\mathrm{d}\rho} = \frac{\mathrm{d}\mathrm{U}\mathrm{d}\rho}{\mathrm{T}\mathrm{d}\rho} = \frac{\mathrm{V}\mathrm{d}^2\rho\mathrm{d}\rho^{-1}.\mathrm{U}\mathrm{d}\rho}{\mathrm{T}\mathrm{d}\rho}$$
$$= -\frac{\mathrm{V}\mathrm{d}^2\rho\mathrm{d}\rho^{-1}}{\mathrm{d}\rho} = \frac{\mathrm{V}\mathrm{d}\rho\mathrm{d}^2\rho}{\mathrm{d}\rho^3}\,;$$

and the required transformation is accomplished.

(3.) And generally, if s denote the *arc* of any curve of which ρ is the variable vector, we may establish the *symbolical equations*,

$$\text{VIII.} \ldots \mathrm{D}_s = \frac{1}{\mathrm{T}\mathrm{d}\rho}\mathrm{d}\,; \quad \mathrm{D}_s^{\,2} = \frac{1}{\mathrm{T}\mathrm{d}\rho}\,\mathrm{d}\,\frac{1}{\mathrm{T}\mathrm{d}\rho}\,\mathrm{d} = \left(\frac{1}{\mathrm{T}\mathrm{d}\rho}\mathrm{d}\right)^2; \ \&\text{c.}$$

(4.) For example (comp. 389, XII.), the *Vector of Curvature*, $\mathrm{D}_s^{\,2}\rho$, admits of being expressed *generally* under any one of the five last forms VII.

391. Instead of determining the vector κ of the centre of the osculating circle by *one vector expression*, such as 389, IV., or any of its transformations, we may determine it by a system of *three scalar equations*, such as the following,

$$\text{I.} \ldots S(\kappa - \rho)\rho' = 0; \qquad \text{II.} \ldots S(\kappa - \rho)\rho'' - \rho'^2 = 0;$$

$$\text{III.} \ldots S(\kappa - \rho)\rho'\rho'' = 0,$$

of which it may be observed that the second is the *derivative* of the first, if κ be treated as constant (comp. 386, (4.)); and of which the first expresses (369, IV.) that the sought *centre* is *in the normal plane* to the curve, while the third expresses (376, V.) that it is *in the osculating plane*; and the second serves to fix its position *on the absolute normal* (379), in which those two planes intersect.

(1.) Using *differentials* instead of derivatives, but leaving still the independent variable arbitrary, we may establish this equivalent system of three equations,

$$\text{IV.} \ldots S(\kappa - \rho)d\rho = 0; \qquad \text{V.} \ldots S(\kappa - \rho)d^2\rho - d\rho^2 = 0;$$

$$\text{VI.} \ldots S(\kappa - \rho)d\rho d^2\rho = 0;$$

of which the second is the differential of the first, if κ be again treated as constant.

(2.) It is also permitted (comp. 369, (2.), 376, (3.), and 380, (2.)), with the same supposition respecting κ, to write these equations under the forms,

$$\text{VII.} \ldots dT(\kappa - \rho) = 0; \qquad \text{VIII.} \ldots d^2T(\kappa - \rho) = 0;$$

$$\text{IX.} \ldots dUV(\kappa - \rho)d\rho = 0;$$

and to connect them with *geometrical interpretations*.

(3.) For instance, we may say that the *centre* of the osculating circle is the point, in which the osculating *plane*, III. or VI. or IX., is *intersected* by the *axis* of that circle; namely, by the *right line* which is drawn through its centre, at right angles to its plane: and which is represented by the *two* scalar equations,

I. and II., or IV. and V., or VII. and VIII.

(4.) And we may observe (comp. 370, (8.)), that whereas for a point R taken arbitrarily *in the normal plane* to a curve at a given point P, we can only say *in general*, that if a *chord* PQ be called *small* of the *first order*, then the *difference* of *distances*, $\overline{RQ} - \overline{RP}$, is small of an order *higher* than the *first*; yet, if the point R be taken *on the axis* (3.) of the osculating circle, then this difference of distance is small, of an order *higher than the second*, in virtue of the equations VII. and VIII.

(5.) The *right line* I. II., or IV. V., or VII. VIII., as being the *locus* of *points* which may be called *poles* of the osculating *circle*, on all possible *spheres* passing through it, is also called the *Polar Axis* of the *curve itself*, corresponding to the given *point of osculation*.

(6.) And because the equation II. is (as above remarked) the *derivative* of I., the known theorem follows (comp. 386), that the *locus* of all such *polar axes* is a *developable surface*, namely that which is called the *Polar Developable*, or the *envelope of the normal planes* to the given curve; of which surface we shall soon have occasion to consider briefly the *cusp-edge*.

392. The following is an entirely different method of investigating, by quaternions, not merely the *radius* or the *centre* of the *osculating circle* to a *curve in space*, but the *vector equation* of that *circle itself*: and in a way which is *applicable alike*, to *plane* curves, and to curves of *double* curvature.

(1.) In general, conceive that OT $= \tau$ is a *given tangent* to a circle, at a given point which is for the moment taken as the origin; and let PP$' = \rho'$ represent a *variable tangent*, drawn at the extremity of the variable chord OP $= \rho$: also let U be the *intersection*, OT · PP$'$, of these two tangents. Then the isosceles triangle OUP, combined with the formula 324, XI. for the differential of a reciprocal, gives easily the equations,

$$\text{I.} \ldots \rho' \parallel \rho\tau^{-1}\rho\,; \qquad \text{II.} \ldots \mathrm{V}\tau\rho^{-1}\rho'\rho^{-1} = -(\mathrm{V}\tau\rho^{-1})' = 0\,;$$

$$\text{III.} \ldots \mathrm{V}\tau\rho^{-1} = \text{const.} = \mathrm{V}\tau\alpha^{-1}, \text{ as in 296, IX.}'',$$

if α be the vector OA of any second *given point* A of the circumference.

(2.) The *vector equation* of the circle PQD (389) is therefore,

$$\text{IV.} \ldots \mathrm{V}\frac{2\rho'}{\omega-\rho} = \mathrm{V}\frac{2\rho'}{\rho_t-\rho} = \frac{2}{t}\mathrm{V}.(1+\tfrac{1}{2}tu\rho''\rho'^{-1})^{-1} = -\mathrm{V}.u\rho''\rho'^{-1}(1+\tfrac{1}{2}tu\rho''\rho'^{-1})^{-1}\,;$$

whence, passing to the *limit* ($t = 0$, $u = 1$), the analogous equation of the *osculating circle* is at once found to be,

$$\text{V.} \ldots \mathrm{V}\frac{2\rho'}{\omega-\rho} = -\mathrm{V}\frac{\rho''}{\rho'}, \quad \text{or} \quad \text{VI.} \ldots \mathrm{V}\left(\frac{2\mathrm{d}\rho}{\omega-\rho} + \frac{\mathrm{d}^2\rho}{\mathrm{d}\rho}\right) = 0\,;$$

with the verification (comp. 296, (9.)), that when we suppose,

$$\text{VII.} \ldots \omega - \rho = 2(\kappa - \rho) \perp \rho',$$

the vector κ of the *centre* is seen to satisfy the equation,

$$\text{VIII.} \ldots \frac{\rho'}{\kappa - \rho} = -\mathrm{V}\frac{\rho''}{\rho'}, \quad \text{or} \quad \text{IX.} \ldots \frac{\mathrm{d}\rho}{\kappa - \rho} + \mathrm{V}\frac{\mathrm{d}^2\rho}{\mathrm{d}\rho} = 0;$$

which agrees with recent results (389, IV., &c.).

(3.) Instead of conceiving that a circle is described (389), so as to *touch* a given curve (fig. 77) at P, and to *cut* it at *one* near point Q, we may conceive that a circle *cuts* the curve in the *given* point P, and *also* in *two* near points, Q and R, unconnected by any given *law*, but *both* tending together to *coincidence* with P: and may inquire what is the *limiting position* (if any) of the *circle* PQR, which thus *intersects* the curve in *three near points*, whereof *one* (P) is *given*.

(4.) In general, if α, β, ρ be *three co-initial chords*, OA, OB, OP, of any one circle, their *reciprocals* α^{-1}, β^{-1}, ρ^{-1}, if still co-initial, are *termino-collinear* (260); applying which principle, we are led to investigate the condition for the three co-initial vectors,

$$\text{X.} \ldots (\omega - \rho)^{-1}, \quad (s\rho' + \tfrac{1}{2}s^2 u_s \rho'')^{-1}, \quad (t\rho' + \tfrac{1}{2}t^2 u_t \rho'')^{-1},$$

with $u_0 = 1$, thus *ultimately* terminating on *one right line*; or for our having ultimately a relation of the form,

$$\text{XI.} \ldots \frac{xs + yt}{\omega - \rho} = \frac{x}{\rho' + \frac{1}{2}s\rho''} + \frac{y}{\rho' + \frac{1}{2}t\rho''};$$

or

$$\text{XII.} \ldots \frac{(xs + yt)\rho'}{\omega - \rho} = \frac{x}{1 + \frac{1}{2}s\rho''\rho'^{-1}} + \frac{y}{1 + \frac{1}{2}t\rho''\rho'^{-1}}$$
$$= x + y - \tfrac{1}{2}(xs + yt)\rho''\rho'^{-1} + \&\text{c.}:$$

in which last equation, both members are generally *quaternions*.*

(5.) The comparison of the *scalar parts* gives here no useful information, on account of the *arbitrary* character of the coefficients x and y; but *these*

* [Observe that $\frac{1}{\alpha + \beta} = \frac{1}{(1 + \beta\alpha^{-1})\alpha} = \frac{1}{\alpha} \cdot \frac{1}{1 + \beta\alpha^{-1}}$ if α and β are any two vectors.]

disappear, with the two *other* scalars, s and t, in the comparison of the *vector parts*, whence follows the *determinate* and *limiting equation*,

$$\text{XIII.} \ldots 2V\rho'(\omega - \rho)^{-1} = -V\rho''\rho'^{-1},$$

which evidently agrees with V.

(6.) It is then found, by this little quaternion calculation, as was of course to be expected,* that the *circle* (3.), through *any three near points* of a curve in space, coincides *ultimately* with the *osculating* circle, if the *latter* be still *defined* (389) with reference to a *given tangent*, and a *near point*, which *tends* to coincide with the *given point* of contact.

393. An osculating circle to a curve of *double* curvature does not generally meet that curve *again*; but it intersects generally a *plane* curve, of the degree n, to which it osculates, in $2n - 3$ points, distinct from the point P of osculation, whereof *one* at least must be *real*, although it may happen to *coincide* with that point P: and such a circle intersects also generally a *spherical* curve of double curvature, and of the degree n, in $n - 3$ other points, namely in those where the osculating *plane* to the curve meets it again. An *example* of each of these two last cases, as treated by quaternions, may be useful.

(1.) In general, if we clear the recent equation, 392, V. or XIII., of fractions, it becomes,

$$\text{I.} \ldots 0 = 2\rho'^2 V\rho'(\omega - \rho) + (\omega - \rho)^2 V\rho''\rho';$$

in which ρ = OP = the vector of the given point of osculation, and ρ', ρ'' are its first and second derivatives, taken with respect to any scalar variable t, and for the particular value (whether zero or not) of that variable, which answers to the *particular point* P; while ω denotes generally the vector of *any point* upon the circle, which osculates to the given curve at that point P.

(2.) Writing then (comp. 389, (10.)),

$$\text{II.} \ldots \rho = t\alpha + t^{-1}\beta, \quad \rho' = \alpha - t^{-2}\beta, \quad \rho'' = 2t^{-3}\beta,$$

and

$$\text{III.} \ldots \omega = \text{OQ} = x\alpha + x^{-1}\beta,$$

to express that we are seeking for the *remaining intersection* Q of a *plane*

* This conclusion is indeed so well-known, and follows so obviously from the doctrine of *infinitesimals*, that it is only deduced here as a *verification* of previous formulæ, and for the sake of *practice* in the present Calculus.

hyperbola with its *osculating circle* at P, the equation I. becomes, after a few easy reductions, including a division by $V\alpha\beta$, the following *biquadratic* in x,

$$\text{IV.} \ldots 0 = (x - t)^3 (t^3\alpha^2 x - \beta^2);$$

in which the *cubic factor* is to be set aside, as answering only to the point P itself.

(3.) Substituting then, in III., the remaining value IV. of x, we find the expression,

$$\text{V.} \ldots \omega = \text{OQ} = \frac{(t\alpha)^2}{t^{-1}\beta} + \frac{(t^{-1}\beta)^2}{t\alpha} = \tfrac{1}{2}\left\{\frac{(2t\alpha)^2}{2t^{-1}\beta} + \frac{(2t^{-1}\beta)^2}{2t\alpha}\right\};$$

comparing which with 371, (3.), we see that if the tangent to the hyperbola at the given point P intersects the asymptotes in the points A, B, then the tangent at the sought point Q meets the same lines OA, OB in points A′, B′, such that

$$\text{VI.} \ldots \text{OA} \,.\, \text{OA}' = \text{OB}^2, \quad \text{OB} \,.\, \text{OB}' = \text{OA}^2;$$

whence Q is at once found, as the bisecting point of the line A′B′.

(4.) A still more simple construction, and one more obviously agreeing with known results, may be derived from the following expression for the *chord* PQ:

$$\text{VII.} \ldots \text{PQ} = \omega - \rho = (t^2\beta^{-2} - t^{-2}\alpha^{-2})\,(t\alpha^2\beta - t^{-1}\alpha\beta^2)$$
$$= (t^3\beta^{-2} - t^{-1}\alpha^{-2})\alpha\rho'\beta \parallel \alpha\rho'^{-1}\beta;$$

whence it follows (comp. 226) that if this chord PQ, both ways prolonged, meets the two asymptotes OB and OA in the points R and S, we have then the *inverse similitude of triangles* (118),

$$\text{VIII.} \ldots \Delta\, \text{ROS} \propto' \text{AOB}.$$

(5.) As regards the *equality* of the *intercepts*, RP and QS, it can be verified *without specifying* the *second point* Q on the hyperbola, or the *second scalar*, x, by observing that the formula III., combined with the first equation II., conducts to the expressions,

$$\text{IX.} \ldots \text{OR} = \frac{x\rho - t\omega}{x - t} = (x^{-1} + t^{-1})\beta, \quad \text{OS} = \frac{t\rho - x\omega}{t - x} = (x + t)\alpha;$$

which give, generally,

$$\text{X.} \ldots \text{RP} = \text{QS} = t\alpha - x^{-1}\beta.$$

(6.) And as regards the *general reduction*, of the determination of the osculating *circle* to a spherical curve of double curvature, to the determination of the osculating *plane*, it is sufficient to observe that when we take the centre of the sphere for the origin, and therefore write (comp. 381, XIV.),

$$\text{XI.} \ldots \rho^2 = \text{const.}, \quad S\rho\rho' = 0, \quad S\rho\rho'' = -\rho'^2,$$

then if we operate on the vector equation I. with the symbol $V.\rho$, and divide by $-\rho'^3$, there results the scalar equation,

$$\text{XII.} \ldots 0 = 2S\rho(\omega - \rho) + (\omega - \rho)^2 = \omega^2 - \rho^2,$$

which expresses that the *circle* is entirely contained on the *same spheric** *surface* as the curve; while the *other* scalar equation,

$$\text{XIII.} \ldots 0 = S\rho''\rho'(\omega - \rho),$$

obtained by operating on I. with $S.\rho''$, expresses (comp. 376, V.) that the same circle is *in the osculating plane* :† so that its *centre* K is the *foot* of the *perpendicular* let fall on that plane from the origin, and we may therefore write (comp. 385, VI.),

$$\text{XIV.} \ldots \text{OK} = \kappa = \frac{S\rho''\rho'\rho}{V\rho''\rho'}, \quad \text{with the relations,} \quad \text{XV.} \ldots S\frac{\omega}{\kappa} = S\frac{\rho}{\kappa} = 1;$$

and with the verification that the expression XIV. agrees with the general formula, 389, IV., because

$$\text{XVI.} \ldots \rho V\rho''\rho' + \rho'^3 = S\rho''\rho'\rho,$$

when the conditions XI. are satisfied.

(7.) And even if the given curve be *not* a *spherical* one, yet if we retain the *general* expression for κ,

$$\text{XVII.} \ldots \kappa = \rho + \frac{\rho'^3}{V\rho''\rho'}, \qquad (389, \text{IV.}),$$

and operate on I. with $S.\rho''$ and $S.\rho''\rho'$, we find again the equation XIII.

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* This conclusion is geometrically evident, but is here drawn as above, for the sake of practice in the quaternions.

† Compare the Note immediately preceding.

of the osculating plane, combined with a new scalar equation, which may after a few reductions be written thus,

$$\text{XVIII.} \ldots (\omega - \kappa)^2 = (\rho - \kappa)^2 ;$$

and which represents a *new sphere*, whereon the osculating circle to the curve is a *great circle*.

394. To give now an *example* of a *spherical curve* of double curvature, with its osculating *circle* and *plane* for any proposed point P, and with a determination of the point Q in which these meet the curve *again* (393), we may consider that *spherical conic*, or *sphero-conic*, of which the equations are (comp. 357, II.),

$$\text{I.} \ldots \rho^2 + r^2 = 0, \qquad \text{II.} \ldots g\rho^2 + S\lambda\rho\mu\rho = 0 ;$$

namely the intersection of the *sphere*, which has its centre at the origin, and its radius $= r$, with a *cone* of the second order, which has the same origin for vertex, and has the given lines λ and μ for its two (real) cyclic normals. And thus we shall be led to some sufficiently simple *spherical constructions*, which *include*, as their *plane limits*, the analogous constructions recently assigned for the case of the common *hyperbola*.

(1.) Since $S\lambda\rho\mu\rho = 2S\lambda\rho S\mu\rho - \rho^2 S\lambda\mu$ (comp. 357, II′.), the equations I. and II. allow us to write, as their first derivatives, or at least as equations consistent therewith,

$$\text{III.} \ldots S\rho\rho' = 0, \quad S\lambda\rho' + S\lambda\rho = 0, \quad S\mu\rho' - S\mu\rho = 0,$$

because the independent variable is here arbitrary, so that we may conceive the first derived vector ρ' to be multiplied by any convenient scalar; in fact, it is only the *direction* of this tangential vector ρ' which is here important, although we must *continue* the derivations consistently, and so must write, as consequences of III., the equations,

$$\text{IV.} \ldots S\rho\rho'' + \rho'^2 = 0, \quad S\lambda\rho'' + S\lambda\rho' = 0, \quad S\mu\rho'' - S\mu\rho' = 0.$$

(2.) Introducing then the auxiliary vectors,

$$\text{V.} \ldots \eta = V\lambda\mu, \quad \sigma = \lambda S\mu\rho + \mu S\lambda\rho, \quad \tau = \rho + \rho', \quad \upsilon = \rho - \rho',$$

whence

$$\text{VI.} \ldots 0 = S\eta\sigma = S\lambda\tau = S\mu\upsilon, \quad S\rho\sigma = 2S\lambda\rho S\mu\rho, \quad S\mu\tau = 2S\mu\rho,$$

$$S\lambda\upsilon = 2S\lambda\rho, \quad \tau^2 = \upsilon^2 = \rho^2 + \rho'^2,$$

and by new derivations,

$$\text{VII.} \ldots \sigma' = \mathrm{V}\eta\rho, \quad \tau' = \rho' + \rho'', \quad \upsilon' = \rho' - \rho'', \quad \mathrm{S}\lambda\tau' = \mathrm{S}\mu\upsilon' = 0,$$
$$\mathrm{S}\mu\tau' = \mathrm{S}\mu\tau, \quad \mathrm{S}\lambda\upsilon' = -\mathrm{S}\lambda\upsilon,$$

we see first that τ and υ are the vectors OT and OU of the points in which the *rectilinear tangent* to the curve at P meets the two *cyclic planes*, perpendicular respectively to λ and μ; and because the *radius* OP is seen to be the *perpendicular bisector* of the *linear intercept* TU between those two planes, so that

$$\text{VIII.} \ldots \rho' = \text{PT} = \text{UP} \perp \text{OP}, \quad \text{we have} \quad \text{IX.} \ldots \text{UOP} = \text{POT},$$

or

$$\text{X.} \ldots \frown \text{AP} = \frown \text{PB},$$

if the *tangent arc* on the sphere, to the same conic at the same point P, meet the two *cyclic arcs* CA and CB in the points A and B: the *intercepted arc* AB being thus *bisected* at its point of *contact* P, which is a well-known property of such a curve.

(3.) Another known property of a *sphero-conic* is, that for any *one* such curve the *sum of the two spherical angles* CAB and ABC, and therefore also the *area* of the *spherical triangle* ABC, is *constant*. We can only here remark, in passing, that quaternions recognise this property, under the form (comp. II.),

$$\text{XI.} \ldots \cos(\text{A} + \text{B}) = -\mathrm{SU}\lambda\rho\mu\rho = -g : \mathrm{T}\lambda\mu = \text{const.}$$

(4.) The scalar equations III. and IV. give immediately the vector expressions,

$$\text{XII.} \ldots \rho' = \frac{\mathrm{V}\rho(\lambda\mathrm{S}\mu\rho + \mu\mathrm{S}\lambda\rho)}{\mathrm{S}\lambda\mu\rho}, \qquad \text{XIII.} \ldots \rho'' = \rho - \frac{(\rho^2 + \rho'^2)\mathrm{V}\lambda\mu}{\mathrm{S}\lambda\mu\rho}$$

or by (2.),

$$\text{XIV.} \ldots \rho' = \frac{\mathrm{V}\rho\sigma}{\mathrm{S}\eta\rho}, \quad \text{and} \quad \text{XV.} \ldots \rho'' = \rho - \xi, \quad \text{if} \quad \text{XVI.} \ldots \xi = \frac{\tau^2\eta}{\mathrm{S}\eta\rho}$$
$$= \tau - \tau' = \upsilon + \upsilon',$$

the new auxiliary vector ξ being thus that of the point X, in which the osculating plane to the conic at P meets the line η of intersection of the cyclic planes: so that we have the geometrical expressions,

$$\text{XVII.} \ldots \rho'' = \text{XP}, \quad \tau' = \text{XT}, \quad -\upsilon' = \text{XU}, \quad \text{if} \quad \xi = \text{OX},$$

and the lines* τ' and υ' are the traces of the osculating plane on those two

* We may also consider the derived vectors τ' and υ', or the lines XT and XU, as *corresponding tangents*, at the points T and U (2.), to the *two sections*, made by the *cyclic planes*, of that *developable surface* which is the *locus of the tangents* TPU to the *spherical conic* in question.

cyclic planes, or of the latter on the former; while σ and σ', as being perpendicular respectively to ρ' and ρ, while each $\perp \eta$, are the traces on the plane $\lambda\mu$ of the two cyclic normals, of the normal plane to the conic at the point P, and of the tangent plane to the sphere at that point: or at least these lines have the *directions* of those *traces*.

(5.) Already, from the expression XVI. for the portion OX of the radius OC (2.), or of that radius prolonged, which is cut off by the *osculating plane* at P, we can derive a simple *construction* for the position of the *spherical centre*, or *pole*, say E, of the *small circle* which osculates at that point P, to the proposed *sphero-conic*. For if we take the radius r for unity, we have the trigonometric expressions,

$$\text{XVIII.} \ldots \sec \text{CE} \cos \text{EP} = (\text{T}\xi = \text{T}\tau^2 : \text{SU}\eta^{-1}\rho =) \sec^2 \text{PB} \sec \text{CP};$$

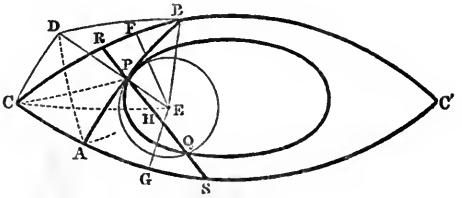

Fig. 80.

or letting fall (comp. fig. 80) the perpendicular CD on the normal arc PE,

$$\text{XIX.} \ldots \cos \text{DE} = \cos \text{DP} \cos \text{PB} \,.\, \cos \text{PB} \cos \text{PE} = \cos \text{DB} \cos \text{BE};$$

or finally, $$\text{XX.} \ldots \text{DBE (or DAE)} = \frac{\pi}{2}.$$

(6.) But although it is a perfectly *legitimate* process to *mix* thus *spherical trigonometry* with *quaternions* (since in fact the latter *include* the former), yet it may be satisfactory to deduce this last result by a more *purely* quaternionic method, which can easily be done as follows. The values (4.) of ρ' and ρ'' give,

$$\text{XXI.} \ldots \text{V}\rho'\rho''\text{S}\eta\rho = \rho\text{S}\sigma\rho'' - \sigma\text{S}\rho\rho'' = \rho\text{S}\rho\sigma + \rho'^2\sigma$$
$$= (\tau - \rho')\text{S}\sigma\tau + \sigma\text{S}\rho'\tau = \tau\text{S}\sigma\tau + \text{V}\tau\rho'\sigma \parallel\parallel \tau, \quad \text{V}\tau\rho'\sigma,$$

in which $\rho'\sigma$ denotes a vector $\perp \rho'$ (because $\text{S}\rho'\sigma = 0$), and $\parallel\parallel \eta, \rho'$ (because $\text{S}\eta\rho'\rho'\sigma = 0$); this line $\rho'\sigma$ has therefore the direction of the *projection* of the line η on a plane perpendicular to ρ', and we are thus led to draw, through the line OC of intersection of the cyclic planes, a *plane* COD perpendicular to the normal plane to the conic at P, or to let fall (as in fig. 80) a perpendicular *arc* CD on the normal arc PD; after which the normal to the sought osculating plane, or the *axis* OE of the osculating circle sought, as being $\parallel V\rho'\rho''$, will be contained *in the plane through the trace* τ, or OT, or OB, which is *perpendicular to the plane of* τ *and* $\rho'\sigma$, or to the plane DOB; and therefore the *spherical angle* DBE (or DAE) will be a *right* angle, as before.

(7.) We may also observe that if K be the *centre* of the osculating circle, considered *in its own plane*, or the *foot of the perpendicular* on that plane from O, then by XXI.,

$$\text{XXII.} \ldots \text{OK} = \kappa = \frac{S\rho\rho'\rho''}{V\rho'\rho''} = \frac{\tau^2 S\rho\sigma}{\rho S\rho\sigma + \rho'^2\sigma}, \quad \text{KP} = \rho - \kappa = \frac{\rho'^2 V\rho\sigma}{\rho S\rho\sigma + \rho'^2\sigma};$$

and therefore

$$\text{XXIII.} \ldots \frac{\text{KP}}{\text{OK}} = \frac{\rho - \kappa}{\kappa} = \frac{\rho'^2}{\tau^2}\frac{V}{S}\rho\sigma, \quad \text{XXIV.} \ldots \tan \text{EP} = \sin^2 \text{PB} \cot \text{PD},$$

which gives again the angular relation XX.; the quotient XXIII. being thus a *vector*, as it ought by 393, XV. to be; and the *trigonometric* formula XXIV. being obtained from its expression, by observing that

$$\text{XXV.} \ldots T\rho'\tau^{-1} = \overline{\text{PT}} : \overline{\text{OT}} = \sin \text{POT} = \sin \text{PB}, \quad \text{and} \quad (V:S)\rho\sigma = U\rho'.\cot \text{PD},$$

because $\sigma \perp \rho'\sigma$, but $\parallel\!\parallel \rho, \rho'\sigma$, or $\rho'\sigma \perp \sigma$, but $\parallel\!\parallel \rho, \sigma$.

(8.) The rectangularity of the planes of τ, κ and τ, $\rho'\sigma$ is also expressed by the equation,

$$\text{XXVI.} \ldots 0 = S(V\kappa\tau . V\rho'\sigma\tau) = S\kappa\tau S\rho'\sigma\tau - \tau^2 S\rho'\sigma\kappa;$$

in proving which we may employ the values,

$$\text{XXVII.} \ldots S\tau\kappa^{-1} = 1, \quad S\rho'\sigma\kappa^{-1} = (-\tau^{-2}\rho'^2 S\eta\rho =) S\rho'\sigma\tau^{-1}.$$

(9.) We may also interpret these equations XXVII., as expressing the system of the two relations,

$$\text{XXVIII.} \ldots \kappa^{-1} - \tau^{-1} \perp \tau, \quad \kappa^{-1} - \tau^{-1} \perp \rho'\sigma;$$

from which it follows that κ^{-1}, and therefore also that κ, is a line in the plane so drawn through τ, as to be perpendicular to the plane through τ and $\rho'\sigma$, as before.

(10.) And the two relations XXVIII. are both included in the following expression,

$$\text{XXIX.} \ldots \kappa^{-1} - \tau^{-1} = \mathrm{V}\tau^{-1}\rho'\sigma : \mathrm{S}\rho\sigma.$$

(11.) We may also easily deduce, from the foregoing *spherical construction*, the following *trigonometric expressions*, for the *arcual radius* $r = $ EP of the *osculating small circle* (5.), and for the *angle* $a = $ PAE $=$ EBP which it subtends at A or at B:

$$\text{XXX.} \ldots \tan r = \sin \frac{c}{2} \tan a; \qquad \text{XXXI.} \ldots \tan a = \tfrac{1}{2}(\cot \mathrm{A} + \cot \mathrm{B});$$

A and B *here* denoting, as in XI., the *base angles* of the triangle ABC with C for vertex, and c denoting as usual the *base* AB, namely the portion of the *arcual tangent* (2.) to the conic, which is intercepted between the cyclic arcs.

(12.) The osculating *plane* and *circle* at P being thus fully and in various ways determined, we may next inquire (393) *in what point* Q do they *meet the conic again*. In symbols, denoting by ω the vector of this point, we have the *three scalar equations*,

$$\text{XXXII.} \ldots \mathrm{S}\kappa\omega = \mathrm{S}\kappa\rho, \quad \mathrm{S}\lambda\omega\mathrm{S}\mu\omega = \mathrm{S}\lambda\rho\mathrm{S}\mu\rho, \quad \omega^2 = \rho^2,$$

which are all evidently *satisfied* by the value $\omega = \rho$, but can in general be satisfied *also* by *one other vector value*, which it is the object of the problem to assign.

(13.) We satisfy the *two first* of these three equations XXXII., by assuming the expression,

$$\text{XXXIII.} \ldots \omega = \xi + \tfrac{1}{2}(x^{-1}\tau' - xv'),$$

in which x is any scalar; in fact we have the relations,

$$\text{XXXIV.} \ldots \mathrm{S}\kappa\xi = \mathrm{S}\kappa\rho, \quad \mathrm{S}\lambda v' = -2\mathrm{S}\lambda\rho, \quad \mathrm{S}\mu\tau' = 2\mathrm{S}\mu\rho,$$
$$0 = \mathrm{S}\lambda\xi = \mathrm{S}\mu\xi = \mathrm{S}\lambda\tau' = \mathrm{S}\mu v' = \mathrm{S}\kappa\tau' = \mathrm{S}\kappa v',$$

whence XXXIII. gives,

$$\text{XXXV.} \ldots \mathrm{S}\lambda\omega = x\mathrm{S}\lambda\rho, \quad \mathrm{S}\mu\omega = x^{-1}\mathrm{S}\mu\rho, \ \&c.$$

And because

$$\text{XXXVI.} \ldots \rho = \xi + \tfrac{1}{2}(\tau' - v'),$$

we shall satisfy also the *third* equation XXXII., if we adopt for x *any root* of that new scalar equation, which is obtained by equating the square of the expression XXXIII. for ω, to what that square becomes when x is changed to 1.

(14.) To facilitate the formation of this new equation, we may observe that the relations,

$$\xi = \rho - \rho'', \quad \tau' = \rho' + \rho'', \quad \upsilon' = \rho' - \rho'', \quad S\rho\rho' = 0, \quad S\rho\rho'' = -\rho'^2,$$

which have all occurred before, give

$$\text{XXXVII.} \ldots 4S\xi\tau' = 3\tau'^2 + \upsilon'^2, \quad 4S\xi\upsilon' = \tau'^2 + 3\upsilon'^2;$$

the resulting equation is therefore, after a few slight reductions, the following *biquadratic in x*,

$$\text{XXXVIII.} \ldots 0 = (x-1)^3 (\upsilon'^2 x - \tau'^2);$$

of which the *cubic factor* is to be rejected (comp. 393, (2.)), as answering only to the point P itself.

(15.) We have then the values,

$$\text{XXXIX.} \ldots x = \tau'^2\upsilon'^{-2}, \quad \text{and} \quad \text{XL.} \ldots \text{OQ} = \omega = \xi + \tfrac{1}{2}\left(\frac{\upsilon'^2}{\tau'} - \frac{\tau'^2}{\upsilon'}\right);$$

comparing which last expression with the formulæ XVII., we see that the required point of intersection Q, of the sphero-conic with its osculating circle, can be *constructed* by the following rule. On the traces (4.), of the osculating plane on the two cyclic planes, determine two points T_1 and U_1, by the conditions,

$$\text{XLI.} \ldots \text{XT} \, . \, \text{XT}_1 = \text{XU}^2, \quad \text{XU} \, . \, \text{XU}_1 = \text{XT}^2; \quad \text{then} \quad \text{XLII.} \ldots \text{T}_1\text{Q} = \text{QU}_1,$$

or in words, *the right line T_1U_1 is bisected by the sought point* Q.

(16.) But a still more simple or more *graphic* construction may be obtained, by investigating (comp. 393, (4.)) the *direction of the chord* PQ. The *vector value* of this *rectilinear chord* is, by XXXVI. and XL.,

$$\text{XLIII.} \ldots \text{PQ} = \omega - \rho = \tfrac{1}{2}(\upsilon'^2 - \tau'^2)(\upsilon'^{-1} + \tau'^{-1}) = \tfrac{1}{2}(\tau'^{-2} - \upsilon'^{-2})\,\tau'(\tau' + \upsilon')\upsilon'$$

$$= \left(\frac{\rho'^2}{\tau'^2} - \frac{\rho'^2}{\upsilon'^2}\right)\tau'\rho'^{-1}\upsilon', \quad \text{because} \quad \rho' = \tfrac{1}{2}(\tau' + \upsilon');$$

the chord PQ has therefore the direction (or its opposite) of the *fourth proportional* (226) to the *three vectors*, ρ', τ', and $-\upsilon'$, or PT, XT, and XU; if then we conceive this chord or its prolongations to meet the traces XT, XU in two new points T_2, U_2, we shall have (comp. 393, VIII.) the two *inversely similar triangles* (118),

$$\text{XLIV.} \ldots \Delta\, T_2XU_2 \propto' UXT.$$

(17.) To deduce hence a *spherical construction* for Q, we may conceive *four planes*, through the *axis* OKE, *perpendicular* respectively to the *four* following *right lines* in the *osculating plane*:

$$\text{XLV.} \ldots \tau', -\upsilon', \rho', \omega - \rho, \quad \text{or} \quad XT,\ XU,\ PT,\ PQ;$$

which planes will cut the *sphere* in *four great circles*, whereof the four *arcs*,

$$\text{XLVI.} \ldots EF,\ EG,\ EP,\ EH,$$

are *parts*, if F, G, H (see again fig. 80) be the feet of the three *arcual perpendiculars* from the *pole* E of the osculating circle on the two cyclic arcs CB, CA, and on the arcual chord PQ.

(18.) *These four arcs* XLVI. are therefore connected by the *same angular relation* as the *four lines* XLV.; and we have thus the very simple formula,

$$\text{XLVII.} \ldots GEH = PEF,$$

expressing an equality between *two spherical angles* at the pole E, which serves to determine the *direction* of the *arc* EH, and therefore also the *positions* of the *points* H and Q, by means of the relations,

$$\text{XLVIII.} \ldots PHE = \frac{\pi}{2}, \quad \frown PH = \frown HQ.$$

(19.) If the arcual chord PQ, both ways prolonged, or any chord of the conic, cut the cyclic arcs CB and CA in the points R and S (fig. 80), it is well-known that there exists the *equality of intercepts* (comp. 270, (2.)),

$$\text{XLIX.} \ldots \frown RP = \frown QS;$$

and conversely this equation, combined with the formulæ (11.), or with the trigonometric expression,

$$\text{L.} \ldots \tan PE = \tan r = \tfrac{1}{2} \sin \frac{c}{2} (\cot A + \cot B),$$

for the tangent of the *arcual radius* of the osculating circle, enables us to determine what may be called perhaps the arcual *chord of osculation* PQ, by determining the spherical angle RPB, or simply P, from principles of *spherical trigonometry alone*, in a way which may serve as a verification of the results above deduced from *quaternions*.

(20.) Denoting by t the semitransversal RH = HS, and by s the semichord PH = HQ, the oblique-angled triangles RPB, SPA give the equations,

$$\text{LI.} \ldots \begin{cases} \cot(t-s)\sin\frac{c}{2} = \cos \text{P} \cos\frac{c}{2} + \sin \text{P} \cot \text{B}, \\ \cot(t+s)\sin\frac{c}{2} = \cos \text{P} \cos\frac{c}{2} - \sin \text{P} \cot \text{A}; \end{cases}$$

while the right-angled triangle PHE gives,

$$\text{LII.} \ldots \tan s = \sin \text{P} \tan r.$$

Equating then the values of cot $2s$, deduced from LI. and LII., we eliminate s and t, and obtain a quadratic in tan P, of which one root is zero, when tan r has the value L.; such then might in this new way be inferred to be the tangent of the arcual radius of curvature of the conic, and the remaining root of the equation is then,

$$\text{LIII.} \ldots \tan \text{P} = \frac{\cos\frac{c}{2}(\cot \text{B} - \cot \text{A})}{\cot \text{A} \cot \text{B} + \cos^2\frac{c}{2} - \tan^2 r};$$

a formula which ought to determine the inclination P, or RPB, or QPA, of the chord PQ to the tangent PA, but which does not appear at first sight to admit of any simple *interpretation*.*

(21.) On the other hand, the *construction* (17.) (18.), to which the *quaternion analysis* led us, gives

$$\text{LIV.} \ldots \text{HEP} = \text{GEP} - \text{GEH} = \text{GEP} - \text{PEF} = \text{FEB} + \text{GEA},$$

---

* We might however at once see from this formula, that P = A − B at the *plane limit*; which agrees with the known construction 393, (4.), for the corresponding chord PQ in the case of the plane *hyperbola*.

and therefore, by the four right-angled triangles, PHE, BFE, AGE, and BPE or EPA, conducts to this other formula,

$$\text{LV.} \ldots \cot^{-1}(\cos r \cot \text{P}) = \cot^{-1}\left(\cos r \cos \frac{c}{2} \tan (\text{B} + a)\right)$$
$$- \cot^{-1}\left(\cos r \cos \frac{c}{2} \tan (\text{A} + a)\right),$$

in which a is the same auxiliary angle as in XXXI.; we ought therefore to find, as the proposed *verification* (19.), that this last equation LV. expresses virtually the *same relation* between A, B, c, and P, as the formula LIII., although there seems at first to be no connexion between them; and such agreement can accordingly be proved to exist, by a chain of ordinary *trigonometric transformations*, which it may be left to the reader to investigate.

(22.) A *geometrical proof* of the validity of the construction (17.) (18.) may be derived in the following way. The *product* of the *sines* of the *arcual perpendiculars*, from a point of a given *sphero-conic* on its two *cyclic arcs*, is well-known to be *constant*; hence also the *rectangle* under the *distances* of the same variable point from the two *cyclic planes* is constant, and the curve is therefore the *intersection* of the *sphere* with an *hyperbolic cylinder*, to which those planes are *asymptotic*. It may then be considered to be thus geometrically evident, that the *circle* which *osculates* to the spherical curve, at any given point P, osculates *also* to the *hyperbola*, which is the *section* of that *cylinder*, made by the *osculating plane* at this point; and that the point Q, of recent investigations, is the point in which this hyperbola is met *again*, by *its own* osculating circle at P. But the determination 393, (4.) of *such* a point of intersection, although above deduced (for practice) by quaternions, is a *plane problem* of which the solution was *known*; we may then be considered to have *reduced*, to this known and plane problem, the corresponding *spherical problem* (12.); and thus the *inverse similarity* of the *two plane triangles* XLIV., although *found* by the quaternion analysis, may be said to be *geometrically explained*, or accounted for: the *traces* XT and XU, or τ' and $-v'$, of the osculating plane to the conic on the two cyclic planes (4.), being evidently the *asymptotes* of the hyperbola in question.

(23.) In quaternions, the constant *product of sines*, &c., is expressed by this form of the equation II. of the *cone*,

$$\text{LVI.} \ldots \text{SU}\lambda\rho \,.\, \text{SU}\mu\rho = (g - \text{S}\lambda\mu) : 2\text{T}\lambda\mu = \text{const.};$$

and the scalar equation of the hyperbolic *cylinder*, obtained by eliminating ρ^2 between I. and II., after the first substitution (1.), is

$$\text{LVII.} \ldots \mathrm{S}\lambda\rho\mathrm{S}\mu\rho = \tfrac{1}{2}r^2(g - \mathrm{S}\lambda\mu) = \text{const.};$$

while the expression XXXIII. for ω may be considered as the vector equation of the *hyperbola*, of which the intersection Q with the *circle*, or with the *sphere*, is determined by combining that equation with the condition $\omega^2 = \rho^2 (= - r^2)$.

(24.) In the foregoing investigation, we have treated a *sphero-conic* in connexion with its *cyclic arcs* (2.); but it would have been about equally easy to have treated the same curve, with reference to its *focal points*: or to the *focal lines* of the *cone*, of which it is the *intersection* with a concentric *sphere*. (Compare what has been called the *bifocal transformation*, in 360, (2.).)

(25.) We can however only state generally here the *result* of such an application of quaternions, as regards the construction of the osculating small circle to a spherical conic, considered relatively to its *foci*: which *construction** can indeed be also *geometrically* deduced, as a certain *polar reciprocal* of the one given above. Two focal points (not mutually opposite) being called F and G, let PN be the *normal arc* at P, which is thus *equally inclined*, by a well-known principle, to the two *vector arcs*, FP, GP; so that if the focus G be suitably distinguished from its own opposite, the spherical angle FPG is *bisected* by the arc PN, which is here supposed to *terminate* on the *given arc* FG. *At* N *erect an arc* QNR, *perpendicular to* PN, *and terminating in* Q *and* R *on the two vector arcs. Perpendiculars*, QE, RE, *to these last arcs, will meet on the normal arc* PN, *in the sought pole* (or *spherical centre*) E, *of the sought small circle, which osculates to the conic at the given point* P.

(26.) The two *focal* and arcual *chords of curvature* from P, which pass through F and G, and terminate on the osculating circle, are evidently *bisected* at Q and R, in virtue of the foregoing *construction*, which may therefore be thus enunciated:—

The great circle QR, *which is the common bisector of the two focal and arcual chords of curvature from a given point* P, *intersects the normal arc* PN *on the fixed arc* FG, *connecting the two foci*; that is, *on the arcual major axis of the conic.*

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* The reader can easily draw the figure for himself. As regards the *known rule*, lately alluded to (in 393, (4.), and 394, (22.)), for determining the *chord of intersection* of a *plane conic* with its *osculating circle*, it will be found (for instance) in page 194 of *Hamilton's Conic Sections* (in Latin, London, 1758). The *two spherical constructions*, for the *small circle* osculating to a *spherical conic*, were early deduced and published by the present writer, as consequences of quaternion calculations. Compare the second Note to page 54.

(27.) The construction (5.) fails to determine the position of the auxiliary point D in fig. 80, for the case when the given point P is *on the minor axis* of the conic; and in fact the expressions (4.) for ρ' and ρ'' become infinite, when the denominator $S\lambda\mu\rho$ is zero. But it is easy to see that the auxiliary vector σ, which represents *generally* the *trace* of the *normal plane* to the curve on the plane of the two *cyclic normals*, becomes at the *limit* here considered the required *axis* of the osculating circle; and accordingly, if we assume simply (comp. (1.) and (2.)),

$$\text{LVIII.} \ldots \rho' = V\rho\sigma, \quad \text{and therefore} \quad \rho'' = V\rho'\sigma + V\rho\sigma',$$

we have

$$\text{LIX.} \ldots \sigma' = 0, \quad \text{and} \quad V\rho'\rho'' \parallel \sigma, \quad \text{when} \quad S\lambda\mu\rho = 0.$$

(28.) In general, if we determine three points L, M, S in the plane of $\lambda\mu$, by the formulæ (comp. again (2.)),

$$\text{LX.} \ldots \text{OL} = \frac{\lambda\rho^2}{S\lambda\rho}, \quad \text{OM} = \frac{\mu\rho^2}{S\mu\rho}, \quad \text{OS} = \frac{\sigma\rho^2}{S\sigma\rho} = \tfrac{1}{2}(\text{OL} + \text{OM}),$$

then L and M will be the intersections of the cyclic normals λ, μ with the tangent plane to the sphere at P, and the normal plane to the curve at the same point will bisect the right line LM in the point S; we shall also have this proportion of sines,

$$\text{LXI.} \ldots \sin \text{LOS} : \sin \text{SOM} = SU\lambda\rho : SU\mu\rho$$
$$= \cos \text{LOP} : \cos \text{POM} = \sin \text{PP}_1 : \sin \text{PP}_2, \qquad \text{(comp. (23.))},$$

if PP_1, PP_2 be the arcual perpendiculars from the point P of the conic on the two cyclic arcs; and this *general rule* for determining the position of the line OS, or σ, applies even to the *limiting case* (27.), when that variable line becomes the *axis* of the osculating circle, at a *minor summit* of the curve.

(29.) As an *example*, let us suppose that the constants g, λ, μ in the equation II. are connected by the relation,

$$\text{LXII.} \ldots g = -S\lambda\mu, \quad \text{whence} \quad \text{LXIII.} \ldots S(V\lambda\rho \,.\, V\mu\rho) = 0;$$

the *cyclic normals* are therefore in *this* case *sides* of the cone, and the *two planes* which connect them with *any third side* are mutually *rectangular*; so that the *conic* is now the *locus of the vertex* of a *right-angled spherical triangle*, of which the *hypotenuse* is *given*. And by applying either the formula LXI., or the construction (28.) which it represents, we find that the trigonometric *tangent* of the *arcual radius* of the osculating small circle to *such* a conic, at either end of the given hypotenuse, is equal to *half* the tangent* of that *hypotenuse itself.*

(30.) It is obvious that every determination, of an *osculating circle* to a *spherical curve*, is at the same time the determination of what may be (and is) called an *osculating right cone* (or cone of *revolution*), to the *cone* which *rests* upon that curve, and has its *vertex* at the *centre* of the sphere. Applying this remark to the last example (29.), we arrive at the following theorem, which can however be otherwise deduced :—

If a cone be cut in a circle by a plane perpendicular to a side, the axis of the right cone which osculates to it along that side passes through the centre of the section.

395. When a given curve of double curvature is *not* a *spherical curve*, we may propose to investigate the *spheric surface* which approaches to it *most closely*, at any assigned point. An *osculating circle* has been defined (389) to be the *limit* of a circle, which *touches* a given curve, or its tangent PT, at a *given point* P, and *cuts* the same curve at a *near* point Q ; while the *tangent* PT itself had been regarded (100) as the *limit* of a rectilinear *secant*, or as the *ultimate position* of the *small chord* PQ. It is natural then to *define* the *osculating sphere*, as being the *limit of a spheric surface*, which passes *through* the osculating *circle*, at a given point P of a curve, and also *cuts* that *curve* in a point Q, which is supposed to *approach* indefinitely to P, and *ultimately* to *coincide* with it. Accordingly we shall find that this *definition* conducts by quaternions to *formulæ* sufficiently simple ; and that their geometrical *interpretations* are consistent with known results : for example, the *centre of spherical curvature*, or the *centre of the osculating sphere*, will thus be shown to be, as usual, the point in which the *polar axis* (391, (5.)) *touches the cusp-edge* of the *polar developable* (391, (6.)). It will also be seen, that whereas *in general*, if R be a point *in the normal plane* (370, (8.)) to a given curve at P, we can only say that the *difference of distances*, $\overline{RQ} - \overline{RP}$, is small of an order *higher than the first*, if the *chord* PR be small of the *first order* ; and whereas, even if R be *on the polar axis* (391, (4.)), we can only say generally that this difference of distances is small, of an order *higher than the second* ; *yet*, if R be placed *at the centre* S *of spherical curvature*, the difference $\overline{SQ} - \overline{SP}$ is small,

* This may also be inferred by limits from the formulæ (11.) ; in which r and a were used, provisionally, to denote a certain spherical *arc* and *angle*.

of an order *higher than the third*: so that *the distance of a near point* Q, *from the osculating sphere at the given point* P, *is generally small of the fourth order*, the *chord* being still small of the *first*.

(1.) Operating with S.λ, where λ is an arbitrary line, on the vector equation 392, V. of the osculating circle, we obtain the scalar equation of a sphere through that circle under the form,

$$\text{I.} \ldots 0 = 2\text{S}\,\frac{\lambda\rho'}{\omega - \rho} + \text{S}\,\frac{\lambda\rho''}{\rho'};$$

which may however, by 393, (7.), be brought to this other form, better suited to our present purpose,

$$\text{II.} \ldots (\omega - \kappa)^2 = (\rho - \kappa)^2 + 2c\text{S}\rho''\rho'(\omega - \rho);$$

c being any scalar constant, while κ is still the vector of the centre K of the circle: and the vector σ of the centre S of the sphere is given by the formula,

$$\text{III.} \ldots \sigma = \kappa + c\text{V}\rho''\rho',$$

which evidently expresses that this last centre is on the polar axis.

(2.) To express now that this sphere cuts the curve in a near point Q, we are to substitute for ω the expression,

$$\text{IV.} \ldots \omega = \rho_t = \rho + t\rho' + \tfrac{1}{2}t^2\rho'' + \tfrac{1}{6}t^3 u_t \rho''', \quad \text{with} \quad u_0 = 1;$$

but κ has been seen (in 391) to satisfy the three equations,

$$\text{V.} \ldots 0 = \text{S}\rho'(\kappa - \rho), \quad 0 = \text{S}\rho''(\kappa - \rho) - \rho'^2, \quad 0 = \text{S}\rho''\rho'(\kappa - \rho);$$

reducing then, dividing by $\tfrac{1}{3}t^3$, and passing to the limit, we find for the *osculating sphere* the condition,

$$\text{VI.} \ldots \text{S}\rho'''(\rho - \kappa) + 3\text{S}\rho'\rho'' = c\text{S}\rho'''\rho''\rho';$$

so that finally the vector σ satisfies the three scalar equations,

$$\text{VII.} \ldots 0 = \text{S}\rho'(\sigma - \rho), \quad 0 = \text{S}\rho''(\sigma - \rho) - \rho'^2, \quad 0 = \text{S}\rho'''(\sigma - \rho) - 3\text{S}\rho'\rho'',$$

by which it is completely determined, and of which the two last are seen to be the successive derivatives of the first, while that first is the equation of the normal plane: whence the *centre* S of this *sphere* is (by the sub-arts. to 386, comp. 391, (6.)) the point where the *polar axis* KS *touches the cusp-edge* of the polar developable.*

(3.) Differentials may be substituted for derivatives in the equations VII., which may also be thus written (comp. 391, (4.)),

$$\text{VIII.} \ldots 0 = d\text{T}(\rho - \sigma), \quad 0 = d^2\text{T}(\rho - \sigma), \quad 0 = d^3\text{T}(\rho - \sigma), \quad \text{if} \quad d\sigma = 0;$$

the *distance* of a near point Q of the given curve from the osculating *sphere* is therefore *small* (as above said), of an order *higher than the third*, if the *chord* PQ be small of the *first* order.

(4.) The two first equations VII., combined with V., give also

$$\text{IX.} \ldots 0 = \text{S}\rho'(\sigma - \kappa), \quad 0 = \text{S}\rho''(\sigma - \kappa), \quad 0 = \text{S}(\kappa - \rho)(\sigma - \kappa);$$

which express that the line KS is perpendicular to the osculating plane and absolute normal at P, as it ought to be, because it is part of the polar axis.

(5.) Conceiving the *three points* P, K, S, or their *vectors* ρ, κ, σ, to *vary together*, the equations V. and VII., combined with their own derivatives, give among other results the following:

$$\text{X.} \ldots 0 = \text{S}\kappa'\rho' = \text{S}\sigma'\rho' = \text{S}\sigma'\rho'' = \text{S}\sigma'(\kappa - \rho) = \text{S}\sigma''\rho';$$

of which the geometrical interpretations are easily perceived.

(6.) Another easy combination is the following,

$$\text{XI.} \ldots 0 = \text{S}\kappa'(\sigma + \rho - 2\kappa),$$

as appears by derivating the last equation IX., with attention to other relations; but $2\kappa - \rho$ is the vector of the extremity, say M, of the *diameter* of

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* [The equation of the osculating sphere may be obtained in a manner analogous to the instructive method of 392, (3.), p. 59. Let ρ, ρ_1, ρ_2, and ρ_3 be the vectors to any four points on the curve, and let ω be the vector to a variable point on the sphere which passes through these four points, then for certain scalars x, y, and z,

$$\frac{x + y + z}{\omega - \rho} = \frac{x}{\rho_1 - \rho} + \frac{y}{\rho_2 - \rho} + \frac{z}{\rho_3 - \rho},$$

because the coinitial vectors reciprocal to four coinitial chords of a sphere are termino-complanar.

Let $$\rho_1 = \rho + t_1\rho' + \tfrac{1}{2}t_1^2\rho'' + \tfrac{1}{6}t_1^3 u_{t_1}\rho''', \text{ \&c.},$$

and the relation becomes

$$(x + y + z)\frac{\rho'}{\omega - \rho} = \frac{x t_1^{-1}}{1 + \frac{1}{2}t_1\rho''\rho'^{-1} + \frac{1}{6}t_1^2 u_{t_1}\rho'''\rho'^{-1}} + \text{\&c.}$$

the osculating *circle*, drawn from the given point P: we have therefore this construction :—

On the tangent KK′ *to the locus of the centre of the osculating circle, let fall a perpendicular from the extremity* M *of the diameter drawn from the given point* P; *this perpendicular prolonged will intersect the polar axis, in the centre* S *of the osculating sphere to the given curve at* P.

(7.) In general, the three scalar equations VII. conduct to the vector expression,

$$\text{XII.} \ldots \sigma = \rho + \frac{3\mathrm{V}\rho'\rho''\mathrm{S}\rho'\rho'' + \rho'^2\mathrm{V}\rho'''\rho'}{\mathrm{S}\rho'\rho''\rho'''};$$

or with differentials,

$$\text{XIII.} \ldots \sigma = \rho + \frac{3\mathrm{V}d\rho d^2\rho\mathrm{S}d\rho d^2\rho + d\rho^2\mathrm{V}d^3\rho d\rho}{\mathrm{S}d\rho d^2\rho d^3\rho};$$

the scalar variable being still left arbitrary.

(8.) And if, as an *example*, we introduce the values for the *helix*,

$$\text{XIV.} \ldots \rho = cta + a^t\beta, \quad \rho' = ca + \frac{\pi}{2}a^{t+1}\beta, \quad \rho'' = -\left(\frac{\pi}{2}\right)^2 a^t\beta,$$

$$\rho''' = -\left(\frac{\pi}{2}\right)^3 a^{t+1}\beta,$$

whereof the three first occurred before, we find after some slight reductions the expression, in which a denotes again the constant inclination of the curve to the axis of the cylinder,

$$\text{XV.} \ldots \sigma = \rho - a^t\beta \operatorname{cosec}^2 a = cta - a^t\beta \cot^2 a;$$

but this is precisely what we found for κ, in 389, VIII.; *for the helix*, then, the *two centres*, K and S, of *absolute and spherical curvature, coincide.*

Ultimately, when the four points on the curve approach indefinitely, this reduces to

$$(x+y+z)\frac{\rho'}{\omega-\rho} = \frac{xt_1^{-1}}{1+\frac{1}{2}t_1\rho''\rho'^{-1}+\frac{1}{6}t_1^2\rho'''\rho'^{-1}} + \&c.$$

$$= xt_1^{-1}(1-\tfrac{1}{2}t_1\rho''\rho'^{-1}+\tfrac{1}{4}t_1^2(\rho''\rho'^{-1})^2-\tfrac{1}{6}t_1^2\rho'''\rho'^{-1}+\&c.)+\&c.$$

$$= (xt_1^{-1}+yt_2^{-1}+zt_3^{-1}) - \tfrac{1}{2}(x+y+z)\rho''\rho'^{-1}$$

$$+ \tfrac{1}{6}(xt_1+yt_2+zt_3)\left(\tfrac{1}{2}(\rho''\rho'^{-1})^2 - z\rho'''\rho'^{-1}\right) + \&c.$$

Taking the vector part,

$$(x+y+z)\,\mathrm{V}\left(\frac{\rho'}{\omega-\rho} - \tfrac{1}{2}\rho''\rho'^{-1}\right) = \tfrac{1}{6}(xt_1+yt_2+zt_3)\,(\mathrm{V}\rho''\rho'^{-1}\mathrm{S}\rho''\rho'^{-1} - 3\mathrm{V}\rho'''\rho'^{-1});$$

and hence

$$\mathrm{S}\left(\frac{\rho'}{\omega-\rho} - \tfrac{1}{2}\rho''\rho'^{-1}\right)(\rho''\mathrm{S}\rho''\rho'^{-1} - 3\rho''') = 0,$$

which is the equation required.]

(9.) This known result is a consequence, and may serve as an illustration, of the general *construction* (6.); because it is easy to infer, from what was shown in 389, (3.), respecting the *locus* of the *centre* K of the *osculating circle* to the helix, as being *another helix* on a *co-axal cylinder*, that the *tangent* KK′ to *this* locus is perpendicular to the radius of curvature KP, while the same tangent (KK′ or κ') is *always* perpendicular (X.) to the tangent (PP′ or ρ') to the *curve*; KK′ is therefore *here* at right angles to the *osculating plane* of the *given* helix, or coincides with its *polar axis*: so that the perpendicular on it from the extremity M of the *diameter of curvature* falls at the point K *itself*, with which consequently the point S in the present case *coincides*, as found by calculation in (8.).

(10.) In general, if we introduce the expressions 376, VI., or the following,

$$\text{XVI.} \ldots \rho' = s'\mathrm{D}_s\rho, \quad \rho'' = s'^2\mathrm{D}_s{}^2\rho + s''\mathrm{D}_s\rho, \quad \rho''' = s'^3\mathrm{D}_s{}^3\rho + 3s's''\mathrm{D}_s{}^2\rho + s'''\mathrm{D}_s\rho,$$

in which s denotes the *arc* of the curve, but the accents still indicate derivations with respect to an *arbitrary* scalar t; and if we observe (comp. 380, (12.)) that the relations,

$$\text{XVII.} \ldots \mathrm{D}_s\rho^2 = -1, \quad \mathrm{S}\,.\,\mathrm{D}_s\rho\mathrm{D}_s{}^2\rho = 0, \quad \mathrm{S}\,.\,\mathrm{D}_s\rho\mathrm{D}_s{}^3\rho + \mathrm{D}_s{}^2\rho^2 = 0,$$

in which $\mathrm{D}_s\rho^2$ and $\mathrm{D}_s{}^2\rho^2$ denote the squares of $\mathrm{D}_s\rho$ and $\mathrm{D}_s{}^2\rho$, and $\mathrm{S}\,.\,\mathrm{D}_s\rho\mathrm{D}_s{}^2\rho$ denotes $\mathrm{S}(\mathrm{D}_s\rho\,.\,\mathrm{D}_s{}^2\rho)$, &c., exist independently of the *form* of the curve; we find that s'' and s''' disappear from the numerator and denominator of the expression XII. for $\sigma - \rho$, and that they have s'^6 for a common factor: setting aside which, we have thus the simpler formulæ,

$$\text{XVIII.} \ldots \sigma - \rho = \frac{\mathrm{V}\,.\,\mathrm{D}_s\rho\mathrm{D}_s{}^3\rho}{\mathrm{S}\,.\,\mathrm{D}_s\rho\mathrm{D}_s{}^2\rho\mathrm{D}_s{}^3\rho} = \frac{\mathrm{D}_s\,.\,\mathrm{D}_s\rho\mathrm{D}_s{}^2\rho}{\mathrm{S}\,.\,\mathrm{D}_s\rho\mathrm{D}_s{}^2\rho\mathrm{D}_s{}^3\rho}.$$

And accordingly the three scalar equations VII., which determine the centre of the osculating sphere, may now be written thus,

$$\text{XIX.} \ldots \mathrm{S}(\sigma - \rho)\mathrm{D}_s\rho = 0, \quad \mathrm{S}(\sigma - \rho)\mathrm{D}_s{}^2\rho + 1 = 0, \quad \mathrm{S}(\sigma - \rho)\mathrm{D}_s{}^3\rho = 0.$$

(11.) Conversely, when we have any formula involving thus the successive derivatives of the vector ρ taken with respect to the *arc*, s, we can always and easily *generalize* the *expression*, and introduce an *arbitrary* variable t, by inverting the equations XVI.; or by writing (comp. 390, VIII.),

$$\text{XX.} \ldots \mathrm{D}_s\rho = s'^{-1}\rho', \quad \mathrm{D}_s{}^2\rho = s'^{-1}(s'^{-1}\rho')' = s'^{-2}\rho'' - s'^{-3}s''\rho', \text{ \&c.}$$

(12.) It may happen (comp. 379, (2.)) that the independent variable t is only *proportional* to s, without being *equal* thereto; but as we have the general relation,

$$\text{XXI.} \ldots \mathrm{D}_t{}^n\rho = s'^n\mathrm{D}_s{}^n\rho, \quad \text{if} \quad s' = \mathrm{D}_t s = \mathrm{T}\rho' = \text{const.},$$

it is nearly or quite as easy to effect the transformations (10.) and (11.) in the case here supposed, or to pass from t to s and reciprocally, as if we had $s' = 1$.

(13.) If the vector σ be treated as *constant* in the derivations, or if we consider for a moment the *centre* S of the *sphere* as a *fixed point*, and attend only to the *variations of distance* of a point on the *curve* from *it*, then (remembering that $\mathrm{T}(\rho - \sigma)^2 = -(\rho - \sigma)^2$) we not only easily put (comp. VIII.) the three equations XIX. under the forms,

$$\text{XXII.} \ldots 0 = \mathrm{D}_s\mathrm{T}(\rho - \sigma) = \mathrm{D}_s{}^2\mathrm{T}(\rho - \sigma) = \mathrm{D}_s{}^3\mathrm{T}(\rho - \sigma),$$

but also obtain by XVII. this *fourth* equation,

$$\text{XXIII.} \ldots \mathrm{T}(\rho - \sigma)\, \mathrm{D}_s{}^4\mathrm{T}(\rho - \sigma) = \mathrm{S}.(\sigma - \rho)\, \mathrm{D}_s{}^4\rho + \mathrm{D}_s{}^2\rho^2.$$

(14.) If then we write, for abridgment,

$$\text{XXIV.} \ldots r = \mathrm{T}(\kappa - \rho) = \mathrm{T}\mathrm{D}_s{}^2\rho^{-1} = \textit{radius of osculating circle};$$

$$\text{XXV.} \ldots R = \mathrm{T}(\sigma - \rho) = \textit{radius of osculating sphere};$$

and

$$\text{XXVI.} \ldots S = \frac{\mathrm{S}(\sigma - \rho)\mathrm{D}_s{}^4\rho}{-\mathrm{D}_s{}^2\rho^2} = \frac{\mathrm{S}.\mathrm{D}_s\rho^3\mathrm{D}_s{}^3\rho\mathrm{D}_s{}^4\rho}{\mathrm{S}.\mathrm{D}_s\rho\mathrm{D}_s{}^2\rho^3\mathrm{D}_s{}^3\rho},$$

we see that *this scalar*, S, *must be constantly equal to unity*, for *every spherical curve*; but that for a curve which is *non-spherical*, the *distance* $\overline{\mathrm{SQ}}$ of a *near point* Q, from the *centre* S of the *osculating sphere* at P, is generally given by an expression of the form,

$$\text{XXVII.} \ldots \overline{\mathrm{SQ}} = R + \frac{(S-1)u_s s^4}{24r^2R}, \quad \text{with} \quad u_0 = 1;$$

so that, at least for *near points* Q, on *each side* of the *given point* P, the *curve* lies *without* or *within* the *sphere* which *osculates* at that given point, according as the *scalar*, S, determined as above, is *greater* or *less* than *unity*.

(15.) In the case (12.), the formula XXVI. may be thus written,

$$\text{XXVIII.} \ldots S = \frac{S \,.\, \rho'^3 \rho''' \rho^{\text{IV}}}{S \,.\, \rho' \rho''^3 \rho'''};$$

whence, by carrying the derivations one step farther than in (8.), we find for the *helix*,

$$\text{XXIX.} \ldots S = \operatorname{cosec}^2 a > 1, \quad \text{or} \quad \text{XXIX}'. \ldots S - 1 = \cot^2 a > 0;$$

and accordingly it is easy to prove that *this* curve lies *wholly without* its osculating sphere, except at the point of osculation.

(16.) In general, the *scalar* $S - 1$, which vanishes (14.) for *all spherical curves*, and which enters as a *coefficient* into the expression XXVII. for the *deviation* $\overline{\text{SQ}} - \overline{\text{SP}}$ of a *near point* of any *other* curve from its own osculating sphere, may be called the *Coefficient of Non-Sphericity*; and if QT be the *perpendicular* from that near point Q *on the tangent* PT to the curve at the given point P, we have then this *limiting equation*, by which the value of that coefficient may be expressed,

$$\text{XXX.} \ldots S - 1 = \lim. 3 \left(\frac{\overline{\text{SQ}}^2 - \overline{\text{SP}}^2}{\overline{\text{QT}}^2} \right).$$

(17.) Besides the forms XVIII., other transformations of the expressions XII. XIII. for the vector σ of the centre of an osculating sphere might be assigned; but it seems sufficient here to suggest that some useful practice may be had, in proving that those expressions for σ reduce themselves generally to zero, when the condition,

$$\text{XXXI.} \ldots T\rho = \text{const.}$$

is satisfied.

(18.) It may just be remarked, that as r^{-1} is often called (comp. 389, (4.)) the *absolute curvature*, or simply *the curvature*, of the curve in space which is considered, so R^{-1} is sometimes called the *spherical curvature* of that curve: while r and R are called the *radii** of those two curvatures respectively.

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* We shall soon have occasion to consider *another scalar radius*, which we propose to denote by the small roman letter r, of what is not uncommonly called the *torsion*, or the *second curvature*, of the same curve in space.

396. When the *arc* (s) of the curve is made the independent variable, the *calculations* (as we have seen) become considerably simplified, while no essential *generality* is lost, because the transformations requisite for the introduction of an *arbitrary* scalar variable (t) follow a simple and uniform *law* (395, (11.), &c.). Adopting then the expression (comp. 395, IV.),

$$\text{I.} \ldots \rho_s = \rho + s\tau + \tfrac{1}{2}s^2\tau' + \tfrac{1}{6}s^3 u_s \tau'', \quad \text{with} \quad u_0 = 1,$$

in which

$$\text{II.} \ldots \tau = \mathrm{D}_s\rho, \quad \tau' = \mathrm{D}_s^{\,2}\rho, \quad \tau'' = \mathrm{D}_s^{\,3}\rho,$$

and therefore

$$\text{III.} \ldots \tau^2 + 1 = 0, \quad \mathrm{S}\tau\tau' = 0, \quad \mathrm{S}\tau\tau'' + \tau'^2 = 0,$$

we shall proceed to deduce some *other affections* of the *curve*, besides its *spherical curvature* (395, (18.)), which do not involve the consideration of the *fourth power* of the *arc* (or chord). In particular, we shall determine expressions for that known *Second Curvature* (or *torsion*), which depends on the *change of the osculating plane*, and is measured by the *ultimate ratio* of that change, expressed as an *angle*, to the *arc* of the curve itself; and shall assign the quaternion equations of the known *Rectifying Plane*, and *Rectifying Line*, which are respectively the *tangent plane*, and the *generating line*, of that known *Rectifying Developable*, whereon the proposed curve is a *geodetic* (382): so that it would *become* a *right line*, by the *unfolding* of this last *surface* into a *plane*. But first it may be well to express, in this new notation, the principal affections or properties of the curve, which depend only on the *three first terms* of the expansion I., or on the *three initial vectors* ρ, τ, τ', or rather on the *two last* of these; and which *include*, as we shall see, the *rectifying plane*, but *not* the rectifying *line*: *nor* what has been called above the *second** *curvature*.

(1.) Using then first, instead of I., this less expanded but still rigorous expression (comp. 376, I.),

$$\text{IV.} \ldots \rho_s = \rho + s\tau + \tfrac{1}{2}s^2 u_s \tau', \quad \text{with} \quad u_0 = 1,$$

and with the relations II. and III., we have at once the following system of

* In a Note to a very able and interesting Memoir, "*Sur les lignes courbes non planes*" (referred to by Dr. Salmon in the Note to page 277 of his already cited Treatise, and published in *Cahier* XXX. of the *Journal de l'École Polytechnique*), M. de Saint-Venant brings forward several objections to the use of this appellation, and also to the phrases *torsion*, *flexion*, &c., instead of which he proposes to introduce the new name, "*cambrure*": but the expression "*second curvature*" may serve us for the present, as being at least not unusual, and appearing to be sufficiently suggestive.

three rectangular lines, which are conceived to be all drawn from the given point P of the curve:

$$\text{V.} \ldots \tau = \textit{unit tangent}; \qquad \text{VI.} \ldots \tau' = \textit{vector of curvature}\ (389, (4));$$

and

$$\text{VII.} \ldots \nu = \tau\tau' = -\tau'\tau = \tau'\tau^{-1} = \textit{binormal}\ (\text{comp. } 379, (4.));$$

τ being a line drawn in the direction of a conceived *motion* along the curve, in virtue of which the *arc* (s) *increases*; while τ' is directed *towards the centre of curvature*, or of the *osculating circle*, of which centre K the vector is now,

$$\text{VIII.} \ldots \text{OK} = \kappa = \rho - \tau'^{-1} = \rho + r^2\tau' = \rho + r\text{U}\tau',$$

if

$$\text{IX.} \ldots r^{-1} = \text{T}\tau' = \textit{curvature at } \text{P}, \quad \text{or} \quad \text{IX}'. \ldots r = \text{T}\tau'^{-1} = \textit{radius of curvature};$$

and the *third* line ν (which is *normal* at P to the *surface of tangents* to the curve) has the *same length* ($\text{T}\nu = r^{-1}$) as τ', and is *directed* so that the rotation round it from τ to τ' is *positive*.

(2.) At the same time, we have evidently a system of *three rectangular vector units* from the same point P, which may be called respectively the *tangent unit*, the *normal unit*, and the *binormal unit*, namely the three lines,

$$\text{X.} \ldots \text{U}\tau = \tau, \quad \text{U}\tau' = r\tau', \quad \text{U}\nu = r\tau\tau';$$

the *normal unit* being thus directed (like τ') *towards the centre of curvature*.

(3.) The *vector equation* (comp. 392, (2.)) of the *circle of curvature* takes now the form,

$$\text{XI.} \ldots \text{V}\frac{2\tau}{\omega - \rho} = -\nu;$$

with the verification that it is satisfied by the value,

$$\text{XII.} \ldots \omega = \mu = 2\kappa - \rho = \rho - 2\tau'^{-1},$$

in which μ (comp. 395, (6.)) is the vector OM of the extremity of the *diameter of curvature* PM.

(4.) The *normal plane*, the *rectifying plane*, and the *osculating plane*, to the curve at the given point, form a *rectangular system of planes* (comp. 379, (5.)), *perpendicular* respectively to the *three lines* (1.); so that their scalar equations are, in the present notation,

$$\text{XIII.} \ldots \text{S}\tau(\omega - \rho) = 0; \quad \text{XIV.} \ldots \text{S}\tau'(\omega - \rho) = 0; \quad \text{XV.} \ldots \text{S}\nu(\omega - \rho) = 0;$$

by *pairing* which we can represent the *tangent*, *normal*, and *binormal* to the curve, regarded *as indefinite right lines*; or by the three vector equations,

$$\text{XVI.}\ldots \mathrm{V}\tau(\omega-\rho)=0;\quad \text{XVII.}\ldots \mathrm{V}\tau'(\omega-\rho)=0;\quad \text{XVIII.}\ldots \mathrm{V}\nu(\omega-\rho)=0.$$

(5.) In general, if the two vector equations,

$$\text{XIX.}\ldots \mathrm{V}\eta(\omega-\rho)=0,\quad\text{and}\quad \text{XIX}'.\ldots \mathrm{V}\eta_s(\omega_s-\rho_s)=0,$$

represent *two right lines*, PH and P_sH_s, which are conceived to *emanate* according to *any given law* from *any given curve* in space, the *identical formula*,*

$$\text{XX.}\ldots \rho_s-\rho+\mathrm{V}\left(\mathrm{V}\eta\eta_s.\mathrm{V}\frac{\rho_s-\rho}{\mathrm{V}\eta\eta_s}\right)=\frac{\mathrm{S}\eta\eta_s(\rho_s-\rho)}{\mathrm{V}\eta\eta_s},$$

shows that the *common perpendicular* to these *two emanants*, which *as a vector* is represented by *either member* of this formula XX., *intersects* the *two lines* in the *two points* of which the vectors are,

$$\text{XXI.}\ldots \omega=\rho+\eta\mathrm{S}\frac{(\rho_s-\rho)\eta_s}{\mathrm{V}\eta\eta_s};\quad \text{XXI}'.\ldots \omega_s=\rho_s+\eta_s\mathrm{S}\frac{(\rho_s-\rho)\eta}{\mathrm{V}\eta\eta_s}.$$

(6.) In *general* also, the *passage of a right line* from any *one* given position in space to any *other* may be conceived to be accomplished by a sort of *screw motion*, with the *common perpendicular* for the *axis* of the screw, and with *two proportional velocities*, of *translation along*, and of *rotation round* that axis: the *locus* of the *two given* and of *all* the *intermediate positions* of the *line* (when *thus interpolated*) being a *Screw Surface*, such as that of which the vector equation was assigned in 314, (11.), and was used in 372, (4.).

(7.) Again, for *any quaternion*, q, we have (by 316, XX. and XXIII.†) the two equations,

$$\text{XXII.}\ldots \mathrm{lU}q=\angle q.\mathrm{UV}q,\qquad \text{XXII}'.\ldots \mathrm{VU}q=\sin\angle q.\mathrm{UV}q;$$

comparing which we see that

$$\text{XXIII.}\ldots \mathrm{VU}q:\mathrm{lU}q=\sin\angle q:\angle q=(\text{very nearly})\ 1,$$

---

* It is obvious that we have thus an easy quaternion solution of the problem, *to draw a common perpendicular to any two right lines in space.*

† Although the expression XXII'. for $\mathrm{VU}q$ is here deduced from 316, XXIII., yet it might have been introduced at a much earlier stage of these *Elements*; for instance, in connexion with the formula 204, XIX., namely $\mathrm{TVU}q=\sin\angle q$.

if the *angle* of the quaternion be *small*; so that the *logarithm* and the *vector* of the *versor* of a *small-angled quaternion* are very nearly *equal* to each other, and we may write the following general *approximate formula* for such a *versor*:

$$\text{XXIV.} \ldots \mathrm{U}q = (\varepsilon^{\mathrm{IU}q} =)\ \varepsilon^{\mathrm{VU}q}, \textit{ nearly}, \text{ if } \angle\, q \text{ be } \textit{small};$$

the *error* of this last formula being in fact small of the *third order*, if the *angle* be small of the *first*.

(8.) And thus or otherwise (comp. 334, XIII. and XV.), we may perceive that if the quaternion q have the form (comp. (5.)),

$$\text{XXV.} \ldots q = \eta_s \eta^{-1}, \quad \text{with} \quad \text{XXVI.} \ldots \eta_s = \eta + s\eta' + \ldots,$$

and if we write for abridgment,

$$\text{XXVII.} \ldots \theta = \mathrm{V}\frac{\eta'}{\eta}, \quad \text{and} \quad \text{XXVIII.} \ldots h = \mathrm{S}\frac{\eta'}{\eta},$$

we shall then have nearly, if s be small, the expressions,

$$\text{XXIX.} \ldots \mathrm{U}q = \mathrm{U}\frac{\eta_s}{\eta} = \varepsilon^{s\theta}, \quad \text{and} \quad \text{XXX.} \ldots \mathrm{T}q = \mathrm{T}\frac{\eta_s}{\eta} = 1 + sh;$$

or, neglecting s^2,

$$\text{XXXI.} \ldots \eta_s = (1 + sh)\,\varepsilon^{s\theta}\eta = \varepsilon^{s\theta}\eta + sh\eta,$$

in which last binomial, the *first* (or *exponential*) *term* alone influences the *direction* of the *near emanant line* (5.).

(9.) At the same time, by supposing s to tend to 0, the formula XXI. gives, as a limit,

$$\text{XXXII.} \ldots \mathrm{OH} = \omega_0 = \rho + \eta\mathrm{S}\frac{\tau\eta}{\mathrm{V}\eta\eta'} = \rho - \eta\mathrm{S}\frac{\tau}{\theta\eta},$$

for the vector of the *point*, say H, *on the given emanant* PH, in which that given line is *ultimately intersected* by the *common perpendicular* (5.), or by the *axis* of the *screw rotation* (6.); but the *direction* of that axis is represented by the *versor* $\mathrm{U}\theta$, and the *angular velocity* of that rotation is represented by the *tensor* $\mathrm{T}\theta$, if the velocity of *motion* (1.) along the *given curve* be taken as *unity*: we may therefore say that the *vector* θ *itself*, or the *factor* which *multiplies the arc*, s, in the *exponential term* XXXI., if set off *from the point*

H determined by XXXII., is the *Vector of Rotation of the Emanant*, whatever the *law* (5.) of the *emanation* may be.

(10.) And as regards the *screw translation* (6.), its *linear velocity* is in like manner represented, in length and in direction, by the following expression (obtained by limits from XX.),

$$\text{XXXIII.} \ldots \iota = \theta S \frac{\tau}{\theta} \text{ (set off from H)} = \textit{Vector of Translation of Emanant},$$

$$= \textit{projection of unit-tangent on screw-axis} \text{ (or of } \tau \text{ on } \theta).$$

And the *indefinite right line* through the point H, of which this line ι is a *part*, may be called the *Axis of Displacement of the Emanant*.

(11.) It is easy in this manner to assign what may be called the *Osculating Screw Surface* to the (*generally gauche*) *Surface of Emanants*, or indeed to any proposed *skew surface*; namely, the *screw* surface which has the *given emanant* (or other) *line* for *one* of its *generatrices*, and *touches the skew surface* in the *whole extent* of that right line.

(12.) It is however more important here to observe, that in the *case* when the surface of emanants is *developable*, the *vector* ι of *translation vanishes*; and that conversely this vector ι cannot be *constantly zero*, if that surface be *undevelopable*. The *Condition of Developability* of the *Surface of Emanants* is therefore expressed by the equation,

$$\text{XXXIV.} \ldots \iota = 0, \quad \text{or} \quad S\tau\theta = 0, \quad \text{or} \quad \text{XXXIV}'. \ldots S\eta\eta'\tau = 0;$$

and accordingly this condition is *satisfied* (as was to be expected) when $\eta = \tau$, that is, for the *surface of tangents*.

(13.) In the same case, of $\eta =$ or $\parallel \tau$, the vector θ of *rotation* becomes equal (by XXVII. and VII.) to the *binormal* ν; and the expression XXXII., for the vector ω_0 of the foot H of the *axis* reduces itself to ρ; and thus we might be led to see (what indeed is otherwise evident), that the *passage from a given tangent to a near one* may be approximately made, by a *rotation round the binormal, through the small angle*, $sT\nu = sr^{-1} =$ *arc divided by radius of curvature*.

(14.) Instead of emanating *lines*, we may consider a system of *emanating planes*, which are respectively *perpendicular* to those lines, and pass through the *same points* of the given curve. It may be sufficient here to remark, that the *passage* from one to another of two such near emanant planes, represented by the equations,

$$\text{XXXV.} \ldots S\eta(\omega - \rho) = 0, \qquad \text{XXXV}'. \ldots S\eta_s(\omega - \rho) = 0,$$

may be conceived to be made by a *rotation through an angle* = $s\mathrm{T}\theta$, *round the right line*,

$$\text{XXXVI.} \ldots \mathrm{S}\eta(\omega - \rho) = 0, \quad \mathrm{S}\eta'(\omega - \rho) - \mathrm{S}\eta\tau = 0,$$

or

$$\text{XXXVI}'. \ldots \mathrm{V}\theta(\omega - \rho) + \eta^{-1}\mathrm{S}\eta\tau = 0,$$

in which the *plane* XXXV. *touches its developable envelope*, and which is *parallel* to the recent vector θ, or to the *vector of rotation* (9.) of the *emanant line*; so that if an *equal vector* be set off *on this new line* XXXVI., it may be said to be *the Vector Axis of Rotation of the Emanant Plane*.

(15.) For example, if we again make $\eta = \tau$, so that the equation XXXV. represents now the *normal plane* to the curve, we are led to combine the equation XIII. of that plane with its *derived* equation, and so to form the system of the *two* scalar equations,

$$\text{XXXVII.} \ldots \mathrm{S}\tau(\omega - \rho) = 0, \quad \mathrm{S}\tau'(\omega - \rho) + 1 = 0,$$

whereof the second represents a plane *parallel* to the *rectifying plane* XIV., and *drawn through the centre of curvature* VIII.; and which jointly represent the *polar axis* (391, (5.)), considered as an *indefinite* right line, which is represented otherwise by the *one* vector equation,

$$\text{XXXVIII.} \ldots \mathrm{V}\nu(\omega - \kappa) = 0, \quad \text{or} \quad \text{XXXVIII}'. \ldots \mathrm{V}\nu(\omega - \rho) = -\tau.$$

(16.) And if, *on this indefinite line*, we set off a portion *equal to the binormal* ν, such *portion* (which may conveniently be measured *from the centre* κ) may be said, by (14.), to be *the Vector Axis of Rotation of the Normal Plane*; or briefly, *the Polar Axis*, considered as representing not only the *direction* but also the *velocity* of that rotation, which velocity = $\mathrm{T}\nu = r^{-1}$ = the *curvature* (IX.) of the given curve: while *another portion* = $\mathrm{U}\nu$ = the *binormal unit* (2.), set off on the *same axis* from the *same centre* of curvature, may be called the *Polar Unit*.

(17.) This suggests a *new way* of representing the *osculating circle* by a *vector equation* (comp. (3.), and 316), as follows:

$$\begin{aligned}\text{XXXIX.} \ldots \omega_s &= \kappa + \epsilon^{s\nu}(\rho - \kappa) = \rho + (\epsilon^{s\nu} - 1)\tau'^{-1} \\ &= \rho + s\tau + (\epsilon^{s\nu} - 1 - s\nu)\tau'^{-1} \\ &= \rho + s\tau + \tfrac{1}{2}s^2\tau' + (\epsilon^{s\nu} - 1 - s\nu - \tfrac{1}{2}s^2\nu^2)\tau'^{-1};\end{aligned}$$

which agrees, as we see, with the expression I. or IV., if s^3 be neglected; and of which, when the expansion is continued, the *next term* is,

$$\text{XL.} \ldots \tfrac{1}{6}s^3\nu^3\tau'^{-1} = \tfrac{1}{6}s^3\nu\tau' = -\frac{s^3\tau}{6r^2}.$$

(18.) The *complete expansion* of the *exponential form* XXXIX., for the variable *vector of the osculating circle*, may be briefly summed up in the following *trigonometric* (but *vector*) *expression*:

$$\text{XLI.} \ldots \omega_s = \kappa + \left(\cos\frac{s}{r} + \mathrm{U}\nu . \sin\frac{s}{r}\right)(\rho - \kappa),$$

in which,

$$\text{XLII.} \ldots \rho - \kappa = -r^2\tau', \quad \text{and} \quad \mathrm{U}\nu . (\rho - \kappa) = r\nu\tau'^{-1} = r\tau;$$

so that we may also write, *neglecting no power of s*,

$$\text{XLIII.} \ldots \omega_s = \rho + r\tau \sin\frac{s}{r} + r^2\tau' \operatorname{vers}\frac{s}{r};$$

and if *this* be subtracted from the *full* expression for the vector ρ_s, the *remainder* may be called the *deviation of the given curve in space, from its own circle of curvature*: which *deviation*, as we already see, is *small* of the *third order*, and will soon be decomposed into its *two principal parts*, or *terms*, of *that* order, in the directions of the *normal* and the *binormal* respectively.

(19.) Meantime we may remark, that if we only neglect terms of the *fourth order*, the expansion I. gives, by III. and IX., for the *length* of a *small chord* PP_s, the formula:

$$\begin{aligned} \text{XLIV.} \ldots \overline{\text{PP}_s} &= \mathrm{T}(\rho_s - \rho) = \mathrm{T}(s\tau + \tfrac{1}{2}s^2\tau' + \tfrac{1}{6}s^3\tau'') \\ &= \sqrt{\{-(s\tau + \tfrac{1}{2}s^2\tau' + \tfrac{1}{6}s^3\tau'')^2\}} \\ &= \sqrt{\{s^2 + s^4\tau'^2(\tfrac{1}{3} - \tfrac{1}{4})\}} \\ &= \sqrt{\left(s^2 - \frac{s^4}{12r^2}\right)} = s - \frac{s^3}{24r^2} = 2r\sin\frac{s}{2r}; \end{aligned}$$

this *length* then is the *same* (to this degree of approximation), as that of the *chord of an equally long arc* of the *osculating circle*: and although the *chord* of even a *small arc* of a *curve* is *always shorter* than that *arc itself*, yet we see that the *difference* is *generally* a small quantity of the *third** *order*, if the arc be small of the *first*.

397. Resuming now the expression 396, I., but suppressing here the coefficient u_s, of which the limit is unity, and therefore writing simply,

$$\text{I.} \ldots \rho_s = \rho + s\tau + \tfrac{1}{2}s^2\tau' + \tfrac{1}{6}s^3\tau'',$$

with the relations,

$$\text{II.} \ldots \tau^2 = -1, \quad S\tau\tau' = 0, \quad S\tau\tau'' = -\tau'^2 = r^{-2}, \quad S\tau'\tau'' = r^{-3}r',$$

if $s = arc$, and $r^{-1} = T\tau' = curvature$,† as before, or $r = radius$ of curvature (> 0), while $r' = D_s r$; and introducing the *new scalar*,

$$\text{III.} \ldots \mathrm{r}^{-1} = S\frac{\tau''}{\tau\tau'} = r^{-1}V\frac{\nu'}{\nu} = \textit{Second‡ Curvature},$$

with $\nu = \tau\tau' = binormal$, or the *new vector*,

$$\text{IV.} \ldots \mathrm{r}^{-1}\tau = \tau S\frac{\tau''}{\tau\tau'} = V\frac{\nu'}{\nu} = \textit{Vector of Second Curvature},$$

supposed to be set off tangentially from the given point P of the curve, or finally this *other* new scalar (> or < 0),

$$\text{V.} \ldots \mathrm{r} = \left(S\frac{\tau''}{\tau\tau'}\right)^{-1} = \textit{Radius of Second Curvature},$$

which gives the expression,

$$\text{VI.} \ldots \tau'' = -r^{-2}\tau - r^{-1}r'\tau' + \mathrm{r}^{-1}\tau\tau' = -r^{-2}U\tau + (r^{-1})'U\tau' + (r\mathrm{r})^{-1}U\nu\,;$$

we proceed to deduce some of the chief affections of a curve in space, which depend on the *third power* of the *arc* or chord. In doing this, although everything *new* can be *ultimately* reduced to a dependence on the *two new scalars*, r' and r, or on the *one* new *vector* τ'', or even on $\nu' = V\tau\tau''$, yet some

* This ought to have been expressly stated in the reasoning of 383, (5.), for which it was not sufficient to observe that the *arc* and *chord tend* to bear to *each other* a *ratio of equality*, without showing (or at least mentioning) that their *difference tends* to *vanish*, *even* as *compared* with a line which is ultimately of the same order as the *square* of either.

† Whenever this word *curvature* is thus used, without any qualifying adjective, it is always to be understood as denoting the *absolute* (or *first*) curvature of the curve in space.

‡ Compare the Note to page 81.

auxiliary symbols will be found useful, and almost necessary. Retaining then the symbols ν, κ, σ, R, as well as τ, τ', r, and therefore writing as before (comp. 396, VIII.),

$$\text{VII.}\ldots \text{OK} = \kappa = \rho - \tau'^{-1} = \rho + r\text{U}\tau' = \rho + r^2\tau',$$

$$\text{VIII.}\ldots (\rho - \kappa)^{-1} = r^{-1}\text{U}(\kappa - \rho) = \tau' = \text{D}_s^2\rho = \textit{Vector of Curvature},$$

we may now write also, by 395, XVIII.,

$$\text{IX.}\ldots \text{OS} = \sigma = \rho - \frac{\nu'}{\text{S}\tau'\nu'} = \kappa + rr'\text{r}\nu = \kappa + r'\text{rU}\nu,$$

and

$$\text{X.}\ldots (\rho - \sigma)^{-1} = R^{-1}\text{U}(\sigma - \rho) = \nu'^{-1}\text{S}\tau'\nu' = \textit{Vector of Spherical Curvature},$$
$$= \textit{projection of vector } (\tau') \textit{ of curvature on radius } (R) \textit{ of osculating sphere};$$

because we have now, by VI.,

$$\text{XI.}\ldots \nu' = (\tau\tau')' = \text{V}\tau\tau'' = -\text{r}^{-1}\tau' - r^{-1}r'\nu,$$

or

$$\text{XI}'.\ldots (\text{U}\nu)' = (r\nu)' = -r\text{r}^{-1}\tau' = -\text{r}^{-1}\text{U}\tau',$$

and

$$\text{XII.}\ldots \text{S}\tau'\nu' = -\text{S}\tau\tau'\tau'' = -\text{r}^{-1}\tau'^2 = r^{-2}\text{r}^{-1}.$$

If then we denote by p and P the *linear* and *angular elevations*, of the *centre* S of the osculating *sphere* above the osculating *plane*, we shall have these *two* new auxiliary *scalars*, which are positive or negative together, according as the linear *height* KS has the direction of $+\nu$ or of $-\nu$:

$$\text{XIII.}\ldots p = \frac{\sigma - \kappa}{\text{U}\nu} = r'\text{r}; \quad \text{XIV.}\ldots P = \text{KPS} = \tan^{-1}\frac{p}{r} = \sin^{-1}\frac{p}{R} = \cos^{-1}\frac{r}{R};$$

while

$$\text{XV.}\ldots R = \text{T}(\sigma - \rho) = \sqrt{(r^2 + p^2)} = \sqrt{(r^2 + r'^2\text{r}^2)};$$

the *angle* P being treated as generally *acute*. Another important *line*, and an accompanying *angle* of elevation, are given by the formulæ,

$$\text{XVI.}\ldots \lambda = \text{V}\frac{\tau''}{\tau'} = r^2\text{V}\tau'\tau'' = \text{r}^{-1}\tau + \tau\tau' = \text{r}^{-1}\text{U}\tau + r^{-1}\text{U}\nu$$
$$= \text{V}\nu'\nu^{-1} + \nu = \textit{Rectifying Vector} \text{ (set off from given point P)},$$
$$= \textit{Vector of Second Curvature plus Binormal};$$

$$\text{XVII.}\ldots H = \angle\frac{\lambda}{\tau} = \tan^{-1}\frac{\text{r}}{r} = \textit{Elevation of Rectifying Line } (> 0, < \pi),$$
$$= \text{the } \textit{angle} \text{ (acute or obtuse, but here regarded as positive)},$$

which that known and important *line* (396) makes with the *tangent* to the curve; so that (by XIII., XIV.) these *two auxiliary angles*,* H and P, from which (instead of deducing them from r' and r) all the affections of the curve depending on s^3 can be deduced, are connected with each other and with r' by the relation,

$$\text{XVIII.} \ldots \tan P = r' \tan H.$$

Many other combinations of the symbols offer themselves easily, by the rules of the present calculus; for instance, the vector σ may be determined by the three scalar equations (comp. 395, XIX.),

$$\text{XIX.} \ldots \mathrm{S}\tau(\sigma - \rho) = 0, \quad \mathrm{S}\tau'(\sigma - \rho) = -1, \quad \mathrm{S}\tau''(\sigma - \rho) = 0,$$

whence, by XVI.,

$$\text{XX.} \ldots r^2\tau'' = r^2\mathrm{V}(\mathrm{V}\tau'\tau''.(\sigma - \rho)) = \mathrm{V}\lambda(\sigma - \rho),$$

a result which also follows from the expressions,

$$\text{XXI.} \ldots \tau'' = \left(\mathrm{V}\frac{\tau''}{\tau'} + \mathrm{S}\frac{\tau''}{\tau'}\right)\tau' = (\lambda - r^{-1}r')\tau',$$

and

$$\text{XXII.} \ldots \sigma - \rho = r^2\tau' + rp\nu = r\mathrm{U}\tau' + p\mathrm{U}\nu,$$

because

$$\text{XXIII.} \ldots rp\mathrm{V}\lambda\nu = -rp\mathrm{r}^{-1}\tau' = -r\mathrm{r}'\tau';$$

we may therefore replace the formula I. for the vector of the curve by the following, which is true to the same order of approximation,†

$$\text{XXIV.} \ldots \rho_s = \rho + s\tau + \frac{s^2}{2r^2}(\kappa - \rho) + \frac{s^3}{6r^2}\mathrm{V}\lambda(\sigma - \rho):$$

and may thus exhibit, even to the eye, the dependence of all affections connected with s^3, on the *two new lines*, λ and $\sigma - \rho$, which were not required when s^3 was neglected, but can now be determined by the *two scalars* r and p

* The angle H appears to have been first considered by Lancret, in connexion with his theory of rectifying lines, planes, and surfaces: but the angle here called P was virtually included in the earlier results of Monge.

† As regards the *homogeneity* of such expressions, if we treat the four vectors ρ_s, ρ, κ, and σ, and the five scalars s, r, R, p, and r, as being each of the *first dimension*, we are then to regard the dimensions of τ, r', κ', H, and P as being each *zero*; those of τ', ν, and λ as each equal to -1; and that of either τ'' or ν' as being $= -2$.

(or r and r', or H and P as before). The *geometrical signification* of the scalar p is evident from what precedes, namely, the *height* (κs) of the *centre* of the osculating *sphere* above that of the osculating *circle*, divided by the *binormal unit* ($U\nu$) ; and as regards what has been called the *radius* r of *second curvature* (V.), we shall see that this is in fact the *geometrical radius* of a *second circle*, which *osculates*, at the extremity of the tangential vector $r\tau$, to the *principal normal section* of the developable *Surface of Tangents* ; and thereby determines an *osculating oblique cone* to that important surface, and *also* an *osculating right cone** thereto, of which *latter* cone the *semiangle* is H, and the *rectifying line* λ is the *axis of revolution* : being also a *side* of an *osculating right cylinder*, on which is traced what is called the *osculating helix*. We shall assign the quaternion equations of these two cones, and of this cylinder, and helix ; and shall show that although the *helix* has *not* generally *complete contact* of the *third order* with the *given* curve, yet it approaches *more nearly* to that curve (supposed to be of *double* curvature), than does the osculating *circle*. But an *osculating parabola* will also be assigned, namely, the parabola which osculates to the *projection* of the curve, *on its own osculating plane* : and it will be shown that this *parabola* represents or *constructs one* of the *two* principal and *rectangular components* (396, (18.)), of the *deviation* of the *curve* from its osculating *circle*, in a direction which is (ultimately) *tangential to the osculating sphere*, while the *helix constructs* the *other component*. An *osculating right cone* to the *cone of chords*, drawn *from a given point* of the curve, will also be assigned by quaternions : and will be shown to have in general a *smaller acute semiangle* C (or $\pi - C$), than the acute semiangle H (or $\pi - H$), of the osculating right cone (above mentioned) to the *surface of tangents*, or (as will be seen) to the *cone of parallels* to tangents (369, (6.), &c.) : the *relation* between these *two semiangles*, of *two* osculating *right cones*, being rigorously expressed by the formula,

$$\text{XXV.} \ldots \tan C = \tfrac{3}{4} \tan H.$$

A *new oblique cone* of the second order will be assigned, which has contact of the *same order* with the *cone of chords*, as the *second right cone* (C), while the latter *osculates* to *both* of them ; and also an *osculating parabolic cylinder*, which rests upon the *osculating parabola*, and is *cut perpendicularly* in that

* *These two osculating cones*, oblique and right, to the *surface of tangents*, appear to have been first assigned, in the Memoir already cited, by M. de Saint Venant : the osculating (circular) *helix*, and the osculating (circular) *cylinder*, having been previously considered by M. Olivier.

auxiliary curve by the *osculating plane* to the given curve. And the *intersection* of these *two last surfaces* of the second order (oblique cone and parabolic cylinder) will be found to consist partly of the *binormal* at the given point, and partly of a certain *twisted cubic** (or *gauche curve* of the *third degree*), which latter *curve* has *complete contact* of the *third order* with the *given* curve in space. *Constructions* (comp. 395, (6.)) will be assigned, which will connect, more closely than before, the *tangent* to the *locus of centres of curvature*, with *other* properties or affections of that given curve. And finally we shall prove, by a very simple quaternion analysis, as a consequence of the formula XI'., the known theorem,† that *when the ratio of the two curvatures is constant, the curve is a geodetic on a cylinder.*

(1.) The scalar expression III., for the *second curvature* of a curve in space, as *defined* in 396, may be deduced from the formulæ (396, (5.), &c.) of the recent theory of *emanants*, which give,

$$\text{XXVI.} \ldots \theta = \mathrm{V}\nu'\nu^{-1} = \mathrm{r}^{-1}\tau, \quad \omega_0 = \rho, \quad \iota = \tau, \quad \text{if} \quad \eta = \nu,$$

while the *line of contact* (396, (14.)), of the emanant *plane* with its envelope, coincides in position with the *tangent* to the *curve*; in passing, then, from the given point P to the *near* point P_s, the *binormal* (ν) and the *osculating plane* ($\perp \nu$) have (nearly) *revolved together, round that tangent* (τ) as a *common axis*, through a *small angle* $= \mathrm{r}^{-1}s$, and therefore with a *velocity* $= \mathrm{r}^{-1}$, *if this symbol have the value* assigned by III., or by the following extended expression, in which the *scalar variable* (t) is *arbitrary* (comp. 395, (11.), &c.),

$$\text{XXVII.} \ldots \mathrm{r}^{-1} = \mathrm{S}\frac{\rho'''}{\mathrm{V}\rho'\rho''} = \mathrm{S}\frac{\mathrm{d}^3\rho}{\mathrm{Vd}\rho\mathrm{d}^2\rho} = \textit{Second Curvature}:$$

while the *binormal* has at the same time been *translated* (nearly), in a direction perpendicular to the tangent τ, through the *small interval* $\iota s = s\tau$, which (in the present order of approximation) represents the *small chord* PP_s.

(2.) As an *example*, if we take this *new form* of the equation of the *helix*,

$$\text{XXVIII.} \ldots \rho_t = b(at \cot a + \varepsilon^{a^t}\beta), \quad \text{with} \quad \mathrm{T}a = \mathrm{T}\beta = 1, \quad \text{and} \quad \mathrm{S}a\beta = 0,$$

* This convenient appellation (of *twisted cubic*) has been proposed by Dr. Salmon, for a curve of the kind here considered: see pages 241, &c., of his already cited Treatise. The *osculating twisted cubic* will be considered somewhat later.

† This theorem was established, on sufficient grounds, in the cited Memoir of M. de Saint Venant (page 26); but it has also been otherwise deduced by M. Serret, in the Additions to M. Liouville's Edition of Monge (Paris, 1850, page 561, &c.).

which gives the derived vectors,

$$\text{XXIX.} \ldots \rho_t' = ba(\cot a + \varepsilon^{at}\beta), \quad \rho_t'' = - b\varepsilon^{at}\beta, \quad \rho_t''' = a\rho_t'',$$

and this expression for the *arc* s (supposed to begin with t),

$$\text{XXX.} \ldots s = s't, \quad \text{where} \quad s' = \mathrm{T}\rho' = b \operatorname{cosec} a = \text{const.},$$

we easily find (after a few reductions) the following values for the *two curvatures* :

$$\text{XXXI.} \ldots r^{-1} = b^{-1} \sin^2 a, \quad \mathrm{r}^{-1} = b^{-1} \sin a \cos a\,;$$

while the *common centre* (395), of the osculating *circle* and *sphere*, has now for its vector (comp. 389, (3.)),

$$\text{XXXII.} \ldots \kappa = \sigma = \rho_t - b\varepsilon^{at}\beta \operatorname{cosec}^2 a = b \cot a(at - \varepsilon^{at}\beta \cot a)\,;$$

b being here the *radius* of the *cylinder*, but a denoting still the constant *inclination* of the *tangent* (ρ') to the *axis* (α).

(3.) The *rectifying line* (396), considered merely as to its *position*, being the *line of contact* of the *rectifying plane* (396, XIV.) with its own envelope, is represented by the equations,

$$\text{XXXIII.} \ldots 0 = \mathrm{S}\tau'(\omega - \rho) = \mathrm{S}\tau''(\omega - \rho), \quad \text{or} \quad \text{XXXIII}'. \ldots 0 = \mathrm{V}\lambda(\omega - \rho),$$

with the signification XVI. of λ ; and accordingly, if we treat the rectifying planes as *emanants*, or change η to τ', we find the value $\theta = \mathrm{V}\tau''\tau'^{-1} = \lambda$, which shows also that in the passage from P to P_s the *rectifying plane turns* (nearly) *round the rectifying line, through a small angle* $= s\mathrm{T}\lambda$, or with a *velocity* of rotation represented by the tensor,

$$\text{XXXIV.} \ldots \mathrm{T}\lambda = \sqrt{(r^{-2} + \mathrm{r}^{-2})} = r^{-1} \operatorname{cosec} H = \mathrm{r}^{-1} \sec H\,;$$

so that what we have called the *rectifying vector*, λ, coincides in fact (by the general theory of emanants) with the *vector axis* (396, (14.)) of this *rotation of the rectifying plane* : as the *vector of second curvature* ($\mathrm{r}^{-1}\tau$) has been seen to be, in the *same full sense* (comp. (1.)), the *vector axis* of rotation of the *osculating plane*, when *velocity*, *direction*, and *position* are *all* taken into account.

(4.) When the derivative s' of the *arc* is only *constant*, without being equal to *unity* (comp. 395, (12.)), the expression XVI. may be put under this slightly more general form,

$$\text{XXXV.} \ldots \lambda = \mathrm{V}\frac{\rho'''}{s'\rho''} = \mathrm{V}\frac{\mathrm{d}^3\rho}{\mathrm{d}s\mathrm{d}^2\rho} = \textit{Rectifying Vector};$$

and accordingly for the *helix* (2.) we have thus the values,

$$\text{XXXVI.} \ldots \lambda = as'^{-1} = ab^{-1}\sin a = ar^{-1}\operatorname{cosec} a, \quad \mathrm{U}\lambda = a;$$

the *rectifying line* is therefore, for this curve, *parallel* to the *axis*, and *coincides* with the *generating line* of the *cylinder*, as is otherwise evident from geometry. The value, $\mathrm{T}\lambda = b^{-1}\sin a$, of the *velocity of rotation* of the *rectifying plane*, which is here the *tangent plane* to the cylinder, when compared with a conceived velocity of *motion along the curve*, is also easily interpreted; and the formulæ XVII., XVIII. give, for the same helix (by XXXI.), the values,

$$\text{XXXVII.} \ldots r' = 0, \quad H = a, \quad P = 0.$$

(5.) The *normal* (or the radius of curvature), as being *perpendicular* to the rectifying plane, revolves with the *same velocity*, and round a *parallel line*; to determine the *position* of which new line, or the *point* H in which it *cuts* the normal, we have only to change η to τ' in the formula 396, XXXII., which then becomes,

$$\begin{aligned}\text{XXXVIII.} \ldots \mathrm{OH} = \omega_0 &= \rho - \tau'\mathrm{S}\frac{\tau}{\lambda\tau'} = \rho - \lambda^{-2}\tau' \\ &= \rho + \frac{r^{-2}(\kappa - \rho)}{r^{-2} + \mathrm{r}^{-2}} = \frac{r^2\rho + \mathrm{r}^2\kappa}{r^2 + \mathrm{r}^2} \\ &= \rho\cos^2 H + \kappa\sin^2 H;\end{aligned}$$

the *vector of rotation* (396, (9.)) of the *normal* is therefore a line $\parallel$ and $= \lambda$, which *divides* (internally) the *radius* (r) of *curvature* into the *two segments*,*

$$\text{XXXIX.} \ldots \overline{\mathrm{PH}} = r\sin^2 H, \quad \overline{\mathrm{HK}} = r\cos^2 H;$$

namely, into segments which are *proportional to the squares* (r^{-2} and r^{-2}) of the *first* and *second curvatures*.

---

* This law of *division* of a *radius* of curvature into segments, by the *common perpendicular* to that radius and to its *consecutive*, has been otherwise deduced by M. de Saint Venant, in the Memoir already referred to.

(6.) At the same time, what we have called generally the *vector of translation* of an emanant line becomes, for the *normal* (by 396, (10.), changing θ to λ), the line

$$\text{XL.} \ldots \iota = \lambda S\frac{\tau}{\lambda} = U\lambda \cos H = -r^{-1}\lambda^{-1}, \text{ set off from the same point H};$$

and the *indefinite right line*, or *axis*, through that point H,

$$\text{XLI.} \ldots 0 = V\lambda(\omega - \omega_0), \quad \text{or} \quad \text{XLI}'. \ldots 0 = V\lambda(\omega - \rho \cos^2 H - \kappa \sin^2 H),$$

along which axis the normal *moves*, through the *small line* $s\iota$, while it *turns round* the *same* axis (as before) through the *small angle* $sT\lambda$, may be called (comp. again 396, (10.)) the *Axis of Displacement of the Normal* (or of the radius of curvature).

(7.) As a verification, for the *helix* (2.) we have thus the values,

$$\text{XLII.} \ldots \overline{\text{PH}} = b, \quad \omega_0 = \rho_t - b\epsilon^{at}\beta = bat \cot a, \quad \iota = \alpha \cos a;$$

so that the *axis* of *displacement* (6.) coincides with the *axis* (α) of the *cylinder*, as was of course to be expected.

(8.) When the given curve is *not* a helix, the values VI., XVI., XXXVIII., and XL., of τ'', λ, ω_0, and ι, enable us to put the expression I. for ρ_s under the form,

$$\text{XLIII.} \ldots \rho_s = \omega_0 + s\iota + \epsilon^{s\lambda}(\rho - \omega_0) - \frac{s^3 r'\tau'}{6r};$$

the *curve* therefore generally *deviates*, by this last *small vector* of the *third order*, namely by that *part* of the *term* $\frac{1}{6}s^3\tau''$ which has the direction of the *normal* τ', or of $-\tau'$, and which depends on r', *from the osculating helix*,

$$\text{XLIV.} \ldots \omega_s = \omega_0 + s\iota + \epsilon^{s\lambda}(\rho - \omega_0),$$

and from the *osculating right cylinder*,

$$\text{XLV.} \ldots TV\lambda(\omega - \omega_0) = \sin H,$$

whereon that *helix* is traced, and of which the *rectifying line* (XXXIII.) is a *side*, while its *axis of revolution* (comp. (7.)) is the *axis of displacement* (XLI.) of the normal.

(9.) *Another general transformation*, of the expression I. for the vector of the curve, is had by the substitution,

$$\text{XLVI.} \ldots s = t + \frac{t^2 r'}{6r} + \frac{t^3}{6r^2},$$

in which t is a new scalar variable; for this gives the new form,

$$\text{XLVII.} \ldots \rho_t = \rho + t\tau + \tfrac{1}{2}t^2\left(\tau' + \frac{r'\tau}{3r}\right) + \tfrac{1}{6}t^3\text{r}^{-1}\nu,$$

and therefore shows that the *curve deviates*, by this *other small vector* of the *third order*,

$$\text{XLVIII.} \ldots \tfrac{1}{6}t^3\text{r}^{-1}\nu = \tfrac{1}{6}s^3\text{r}^{-1}\tau\tau',$$

that is, by the *part* of the *term* $\frac{1}{6}s^3\tau''$ which has the direction of the *binormal* ν, and which depends on r, from what we propose to call the *Osculating Parabola*, namely that new auxiliary curve of which the equation is,

$$\text{XLIX.} \ldots \omega_t = \rho + t\tau + \tfrac{1}{2}t^2\left(\tau' + \frac{r'\tau}{3r}\right):$$

or from the *parabola* which *osculates* at the given point P, to the *projection* of the given curve *on its own osculating plane*.

(10.) And because the small *deviation* XLVIII. of the *curve* from the *parabola* is also the deviation of the same curve from this last *plane*, if we conceive that a *near point* Q of the curve is *projected* into *three* new points Q_1, Q_2, Q_3, on the *tangent*, *normal*, and *binormal* respectively, we shall have the limiting equation,

$$\text{L.} \ldots \lim. \frac{3\text{PQ}_3}{\text{PQ}_1 . \text{PQ}_2} = \text{r}^{-1} = \textit{Second Curvature};$$

the *sign* of this scalar *quotient* being determined by the rules of quaternions.

(11.) But we may also (comp. 396, (17.), (18.)) employ this *third general transformation* of I., *analogous* to the forms XLIII. and XLVII.,

$$\text{LI.} \ldots \rho_s = \kappa + \varepsilon^{s\nu}(\rho - \kappa) + \frac{s^3}{6}\nu'\tau,$$

with the value XI. of ν'; in which the *sum* of the *two first terms* gives the vector of the point of the *osculating circle*, which is distant from the given point PP_s by an *arc* of that *circle* equal to the arc s of the given *curve*; and the *third term*,

$$\text{LII.} \ldots \tfrac{1}{6}s^3\nu'\tau = \tfrac{1}{6}s^3(\tau'' + r^{-2}\tau) = -\tfrac{1}{6}s^3r^{-1}r'\tau' + \tfrac{1}{6}s^3r^{-1}\nu,$$

which represents the *deviation* from the *same circle*, measured in a direction (comp. IX. or X.) *tangential* to the osculating *sphere*, is (as we see) the *vector sum* of *two rectangular components*, which represent respectively the deviations of the curve, from the osculating *helix* (8.), and from the osculating *parabola* (9.).

(12.) It follows, then, that although *neither helix nor parabola* has in general *complete contact* of the *third order* with a given *curve in space*, since the deviation from *each* is generally a small vector of *that* (third) *order*, yet *each* of these *two auxiliary curves*, one on a *right cylinder* XLV., and the other on the osculating *plane*, approaches in general *more closely* to the *given* curve, than does the *osculating circle*: while circle, helix, and parabola have, *all three*, complete contact of the *second** *order* with the curve, and with each other.

(13.) As regards the *geometrical signification* of the *new variable scalar*, t, in the equation XLIX. of the parabola, that equation gives,

$$\text{LIII.} \ldots \mathrm{T}\omega'_t = \mathrm{T}\left\{\left(1 + \frac{r't}{3r}\right)\tau + t\tau'\right\} = 1 + \frac{r't}{3r} + \frac{t^2}{2r^2} \cdot \cdot,$$

and therefore (to the present order of approximation),

$$\text{LIV.} \ldots \textit{Arc of Osculating Parabola (from } \omega_0 \text{ to } \omega_t)$$
$$= \int_0^t \mathrm{T}\omega'_t dt = t + \frac{r't^2}{6r} + \frac{t^3}{6r^2} = s \text{ (by XLVI.)}$$
$$= \textit{Arc of Curve in Space (from } \rho_0 \text{ to } \rho_s);$$

if then an *arc* = s be thus set off *upon the parabola*, with the same initial point P, and the same initial direction, and if this *parabolic arc*, or its *chord* $\omega_t - \omega_0$, be *obliquely projected* on the *initial tangent* τ, by drawing a *diameter* of the

* It appears then that we may say that the *helix* and *parabola* have each a *contact* with the *curve in space*, which is *intermediate between the second and third orders*: or that the *exponent* of the *order* of *each* contact is the *fractional index*, $2\frac{1}{2}$. But it must be left to mathematicians to judge, whether this phraseology can properly be adopted.

parabola through its final point, the *oblique tangential projection* so obtained will be $= t\tau$ by XLIX.; and its *length*, or the *ordinate to that diameter*, will be the scalar t.

(14.) And as regards the *direction* of the *diameter* of the osculating parabola, drawn as we may suppose from P, if we denote for a moment by D its inclination to the normal $+ \tau'$, regarded as positive when towards the tangent $+ \tau$, we have (by XLIX. and XVIII.) the formula,

$$\text{LV.} \ldots \tan D = \frac{r'}{3} = \tfrac{1}{3} \tan P \cot H :$$

which is an instance of the reducibility, above mentioned, of *all affections* of the curve depending on s^3, to a dependence on the *two angles*, H and P.

(15.) *Some* of these affections, besides the *direction* of the *rectifying line* λ, can be deduced from the angle H *alone*. As an example, we may observe that the vector equation of the *surface of tangents* is of the form,

$$\text{LVI.} \ldots \omega_{s,\,t} = \rho_s + t\rho'_s = \rho_s + t\tau_s,$$

in which s and t are *two* independent and scalar variables, and

$$\text{LVII.} \ldots \tau_s = \tau + s\tau' + \frac{s^2}{2}\tau'',$$

+ terms depending on s^4 in ρ_s. If then we *cut* this *developable* LVI. by the *plane*,

$$\text{LVIII.} \ldots \mathrm{S}\tau(\omega - \rho) = -c = \text{any given scalar constant},$$

which is, relatively to the *surface*, a *normal plane* at the extremity of the tangential vector $c\tau$ from P, while this *tangent* is also a *generating line*, we get thus a *principal** *normal section*, of which the variable vector has for its approximate expression,

$$\text{LIX.} \ldots \omega_s = (\rho + c\tau) + (cs + \ldots)\,\tau' + (\tfrac{1}{2}cs^2 r^{-1} + \ldots)\nu\,;$$

the terms suppressed being of higher orders than the terms retained, and having no influence on the *curvature* of the section. We find then thus,

* Some *general* acquaintance with the known theory of *sections of surfaces* is here supposed, although that subject will soon be briefly treated by quaternions.

that the *vector of the centre* of the *osculating circle* to this *normal section of the surface of tangents* to the given curve is, *rigorously*,

$$\text{LX.} \ldots \rho + c\tau + \frac{(cs\tau')^2}{cs^2\mathrm{r}^{-1}\nu} = \rho + c(\tau + \mathrm{r}\nu) = \rho + c\mathrm{r}\lambda\,;$$

so that the *locus* of all *such centres* is the *rectifying line* XXXIII′. And if, in particular, we make $c = r$, or cut the developable at the extremity of the tangential vector $r\tau$, the expression LX. becomes then $\rho + r\tau + \mathrm{r}\mathrm{U}\nu$; which expresses that the *radius* of the *circle of curvature* of *this* normal section of the surface is precisely what has been called the *Radius* (r) *of Second Curvature*, of the given curve in space. But *this* radius ($\mathrm{r} = r \tan H$) depends only on the angle H, when the radius (r) of (absolute) curvature is given, or has been previously determined.

(16.) The *cone of the second order*, represented by the quaternion equation,

$$\text{LXI.} \ldots 0 = 2\mathrm{r}\mathrm{S}\tau(\omega - \rho)\, \mathrm{S}\nu(\omega - \rho) + (\mathrm{V}\tau(\omega - \rho))^2,$$

has its *vertex* at the given point P, and *rests* upon the *circle* last determined; it is then the *locus* of *all* the circles lately mentioned (15.), and is therefore (in a known sense) an *osculating oblique cone* to the developable *surface of tangents*: its *cyclic normals* (comp. 357, &c.) being τ and $\tau + 2\mathrm{r}\nu$, or τ and $r\tau + 2\mathrm{r}\mathrm{U}\nu$. But, by 394, (30.), the osculating *right cone* to *this* cone LXI., and therefore also (in a sense likewise known) to the *surface* of tangents *itself*, is one which has the recent *locus of centres* (15.), namely the *rectifying line* (λ), for its *axis of revolution*, while the *tangent* (τ) to the curve is one of its *sides*: its *semiangle* is therefore $= H$, and a form of the quaternion equation of this *osculating right cone* is the following (comp. XLV.),

$$\text{LXII.} \ldots \mathrm{TVU}\lambda(\omega - \rho) = \sin H.$$

(17.) The *right cone* LXII., which thus osculates to the developable *surface of tangents* LVI., along the given tangent τ, osculates *also* along that tangential line to the *cone of parallels to tangents*, which has its vertex at the given point P; as is at once seen (comp. 394, (30.)), by changing ρ' and ρ'' to τ' and τ'', in the general expression $\mathrm{V}\rho'\rho''$ (393, (6.), or 394, (6.)), for a line in the direction of the *axis* of the osculating *circle* to a curve upon a *sphere*. And the *axis* of the right *cone* thus determined, namely (again) the *rectifying line* (λ), *intersects* the *plane* of the *great circle* of the *osculating* *sphere*, which is *parallel* to the osculating plane, in a point L of which the vector is,

$$\text{LXIII.} \ldots \text{OL} = \rho + rp\lambda = \rho + rr'\tau + rp\nu.$$

(18.) We have thus, *in general*, a *gauche quadrilateral*, PKSL, right-angled except at L, with the help of which *one* figure *all* affections of the curve, not depending on s^4, can be geometrically represented or *constructed*: although it must be observed that when $r' = 0$, which happens for the *helix* (XXXVII.), the *osculating* circle is then *itself* a *great* circle of the osculating *sphere*, and the points P and L, like the points K and S, *coincide*.

(19.) In the general case, it may assist the conceptions to suppose lines set off, from the given point P, on the tangent and binormal, as follows:

$$\text{LXIV.} \ldots \text{PT} = \text{BL} = rr'\tau\,; \quad \text{PB} = \text{TL} = \text{KS} = rp\nu\,;$$

for thus we shall have a *right triangular prism*, with the two right-angled triangles, TPK and LBS, in the osculating plane and in the parallel plane (17.), for two of its faces, while the three others are the rectangles, PKSB, PBLT, KSLT, whereof the two first are situated respectively in the normal and rectifying planes.

(20.) *All scalar properties* of this auxiliary *prism* may be deduced, by our general methods, from the *three scalars*, r, r, r', or r, H, P; and *all vector properties* of the same prism can in like manner be deduced from the *three vectors*, τ, τ', τ'', or from τ, ν, ν', which (as we have seen) are not entirely arbitrary, but are subject to certain conditions.

(21.) As an example of such deduction (compare the annexed figure 81), the *equation of the diagonal plane* SPL, which contains the radius (R) of spherical curvature and the rectifying line (λ), and the *equation of the trace*, say PU, of that plane on the osculating plane, which *trace* is evidently *parallel* (by the construction) to the *edges* LS, TK of the *prism* are in the recent notations (comp. XX.),

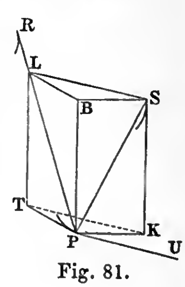

Fig. 81.

$$\text{LXV.} \ldots 0 = S\tau''(\omega - \rho)\,; \quad \text{LXVI.} \ldots 0 = V(r^{-1}\tau)'\,(\omega - \rho)\,;$$

with the verification that $rS\tau'\tau'' = r'S\tau\tau'' = r^{-2}r'$, by II.

(22.) In general, by 204, (22.), if a and β be any two vectors, we have the expressions,

$$\text{LXVII.} \ldots \tan \angle \frac{\beta}{a} = \tan \angle \frac{a}{\beta} = -\tan \angle \beta a = -\tan \angle a\beta$$

$$= TV\frac{\beta}{a} : S\frac{\beta}{a} = \frac{TV}{S} \cdot \frac{\beta}{a} = -(TV : S)\, a\beta,$$

the *angles of quaternions* here considered being supposed as usual (comp. 130) to be generally > 0, but < π; for example, we have thus,

$$\text{LXVIII.} \ldots \tan H = \tan \angle \frac{\lambda}{\tau} = (\mathrm{TV:S})\, \lambda\tau^{-1} = (\mathrm{TV:S})\,(\mathrm{r}^{-1} - \tau') = \mathrm{r}\mathrm{T}\tau' = \mathrm{r}r^{-1},$$

as in XVII.; and in like manner we have generally, by principles already explained (comp. 196, XVI.),

$$\text{LXIX.} \ldots \cos \angle \frac{\beta}{a} = \cos \angle \frac{a}{\beta} = -\cos \angle \beta a = -\cos \angle a\beta$$
$$= \mathrm{S}\frac{\beta}{a} : \mathrm{T}\frac{\beta}{a} = \mathrm{SU}\frac{\beta}{a} = -\mathrm{SU}a\beta.$$

(23.) Applying these principles to investigate the inclinations of the vector τ'', which is perpendicular to the diagonal plane LXV. of the prism, to the three rectangular lines τ, τ', ν, or the inclinations of that diagonal plane itself to the normal, rectifying, and osculating planes, with the help of the expressions deduced from VI. for the three products,* $\tau\tau''$, $\tau'\tau''$, $\nu\tau''$, we arrive easily at the following results:

$$\text{LXX.} \ldots \cos \angle \frac{\tau''}{\tau} = \frac{-r^{-2}}{\mathrm{T}\tau''}; \quad \cos \angle \frac{\tau''}{\tau'} = -\frac{r^{-2}r'}{\mathrm{T}\tau''}; \quad \cos \angle \frac{\tau''}{\nu} = \frac{r^{-1}\mathrm{r}^{-1}}{\mathrm{T}\tau''};$$

with the verification, that the sum of the squares of these three cosines is unity, because

$$\text{LXXI.} \ldots r^2\mathrm{T}\tau'' = \sqrt{(1 + \mathrm{r}^{-2}R^2)} = \sqrt{(1 + r'^2 + r^2\mathrm{r}^{-2})};$$

or

$$\text{LXXI}'. \ldots r\mathrm{T}\tau'' = \sqrt{(r^{-2}r'^2 + \mathrm{T}\lambda^2)}, \quad \mathrm{T}\tau'' = \sqrt{(r^{-4} + \mathrm{T}\nu'^2)}.$$

(24.) Or we may write, on the same general plan,

$$\text{LXXII.} \ldots \tan \angle \frac{\tau''}{\tau} = \frac{-R}{\mathrm{Tr}}; \quad \tan \angle \frac{\tau''}{\tau'} = \frac{-r\mathrm{T}\lambda}{r'}; \quad \tan \angle \frac{\tau''}{\nu} = \frac{\mathrm{r}}{r}\sqrt{(1 + r'^2)};$$

or

$$\text{LXXIII.} \ldots \tan \angle \tau\tau'' = R\mathrm{Tr}^{-1}; \ \tan \angle \tau'\tau'' = rr'^{-1}\mathrm{T}\lambda; \ \tan \angle \nu\tau'' = -\mathrm{r}r^{-1}\sqrt{(1 + r'^2)};$$

* A student, who should be inclined to pursue this subject, might find it useful to form for himself a *table* of all the *binary products* of the nine vectors,

$$\tau, \ \tau', \ \tau'', \ \nu, \ \nu', \ \lambda, \ \sigma - \rho, \ \sigma - \mu, \text{ and } \kappa',$$

considered as so many *quaternions*, and reduced to the *common quadrinomial form*, $a + b\tau + c\tau' + e\nu$, in which a, b, c, e are *scalars*, whereof some may vanish, but which are generally functions of r, r, and r'.

and may modify the expressions, by introducing the auxiliary angles H and P, with which may be combined, if we think fit, the following *angle of the prism*,

$$\text{LXXIV.} \ldots \text{PKT} = \text{BSL} = \tan^{-1} r'.$$

(25.) Instead of thus comparing the *plane* SPL with the three rectangular planes (379, (5.)) of the construction, we may inquire what is the value of the *angle* SPL, which the radius (R) of spherical curvature makes with the rectifying line (λ); and we find, on the same plan, by quaternions, the following very simple expression for the cosine of this angle, which may however be deduced by spherical trigonometry also,

$$\text{LXXV.} \ldots \cos \text{SPL} = -\text{SU}\lambda(\sigma - \rho) = \frac{pr^{-1}}{R\text{T}\lambda} = \sin P \sin H;$$

or

$$\text{LXXV}'. \ldots \cos \text{SPL} = \cos \text{SPB} \cos \text{BPL}.$$

(26.) In general, it is easy to form, by methods already explained, the quaternion equation of a *cone* which has a *given vertex*, and *rests* on a *given curve* in space; and also to determine the *right cone* which *osculates* (394, (30.)) to this *general cone*, along any *given side* of it.

(27.) But if we merely wish to assign the *osculating right cone* to the *cone of chords* from P, or to the *locus* of the *line* PP_s, we may imitate a recent process: and may observe that if this *new cone* be *cut* by the *normal plane* LVIII., the *vector* of the *section* has the following approximate expression, analogous to LIX., and like it sufficient for our purpose,

$$\text{LXXVI.} \ldots \omega_s = \rho + c\tau + \tfrac{1}{2}cs\tau' + \tfrac{1}{6}cs^2\text{r}^{-1}\nu;$$

from which it may be inferred (comp. (15.), (16.)), that the *axis of revolution* of the *new right cone* has for equation,

$$\text{LXXVII.} \ldots 0 = \text{V}(\text{r}^{-1}\tau + \tfrac{3}{4}\nu)(\omega - \rho).$$

This *axis* is therefore situated *in the rectifying plane, between the rectifying line* (λ or $\text{r}^{-1}\tau + \nu$), *and the tangential vector* (IV.) *of second curvature* ($\text{r}^{-1}\tau$): while the *semiangle* C of the same new cone (measured like H from $+\tau$ towards $+\nu$) has the value already assigned by anticipation in the formula XXV., and is therefore *less* than the semiangle H if both be *acute*, but *greater* than H if both be *obtuse*; so that, *in each case*, the *new right cone* (C) is *sharper* than the *old* right cone (H).

(28.) The same result may be otherwise obtained, by observing that an unit-vector in the direction of the chord PP_s has (by 396, XLIV., and 397, I.) the approximate expression,

$$\text{LXXVIII.} \ldots \chi_s = \text{U}(\rho_s - \rho) = \left(1 + \frac{s^2}{24r^2}\right)\left(\tau + \frac{s\tau'}{2} + \frac{s^2\tau''}{6}\right)$$
$$= \tau + \frac{s\tau'}{2} + \frac{s^2}{6}\left(\tau'' + \frac{r^{-2}\tau}{4}\right);$$

whence the axis of the osculating right cone to the cone of chords (27.) has rigorously the direction of the line $\text{V}\chi'\chi''$ (for $s = 0$), or of the vector,

$$\text{LXXIX.} \ldots \xi = \text{V}\tau'(r^2\tau'' + \tfrac{1}{4}\tau) = \lambda - \tfrac{1}{4}\nu = \text{r}^{-1}\tau + \tfrac{3}{4}\nu, \text{ as before.}$$

(29.) *This axis* ξ makes (if we neglect s^3) the *same angle* C, with the *chord* PP_s, as with the *tangent* τ; whereas the *former axis* λ makes *unequal angles* with those two lines, within the same *order* (or degree) of approximation: for our methods conduct to the expression,

$$\text{LXXX.} \ldots \angle \frac{\rho_s - \rho}{\lambda} = H - \frac{s^2}{24r\text{r}},$$

from which the *relation* XXV., between the *two right cones*, may easily be deduced anew.

(30.) Neglecting only s^4, and employing the substitution XLVI., the expression XLVII. for the vector of the given curve becomes,

$$\text{LXXXI.} \ldots \rho_t = \rho + t\tau + \tfrac{1}{2}t^2\upsilon + \tfrac{1}{6}t^3\text{r}^{-1}\nu, \quad \text{if} \quad \text{LXXXII.} \ldots \upsilon = \tau' + \frac{r'\tau}{3r};$$

where the *variable scalar* t denotes, by (13.), the *ordinate of the osculating parabola*, and the *constant vector* υ has the direction, by (14.), of the *diameter* of that parabola.

(31.) In the *present order of approximation*, then, the proposed *curve in space* may be considered to be the *common intersection* of the *three* following *surfaces of the second order*, all passing through the given point P.

$$\text{LXXXIII.} \ldots 2(\text{S}\tau'(\omega - \rho))^2 = 3\text{rS}\nu(\omega - \rho)\text{S}\upsilon\nu(\omega - \rho);$$

$$\text{LXXXIV.} \ldots 2\text{S}\tau'(\omega - \rho) = - r^2(\text{S}\upsilon\nu(\omega - \rho))^2;$$

$$\text{LXXXV.} \ldots 3\text{rS}\nu(\omega - \rho) = - r^2\text{S}\tau'(\omega - \rho)\text{S}\upsilon\nu(\omega - \rho);$$

whereof the *first* represents a *new osculating oblique cone*, which has a *contact* of the *same* (*second*) *order* with the *cone of chords*, as the *osculating right cone* (27.); the *second* represents an *osculating parabolic cylinder*, which is *cut perpendicularly* in the *osculating parabola* (9.), by the *osculating plane* to the curve; and the *third* represents a certain *osculating hyperbolic* (or *ruled*) *paraboloid*, whereof the *tangent* (τ) is *one* of the *generating lines*, while the *diameter* (υ) of the osculating *parabola* is *another*.

(32.) *Each* of these *three surfaces* (31.) has in fact generally a *contact of the third order* with the *given curve*; or has its *equation satisfied*, not only (as is obvious on inspection) by the *point* P *itself*, but also when we *derivate* successively with respect to the scalar variable t, and then substitute the values (comp. LXXXI.),

$$\text{LXXXVI.} \ldots \omega = \rho_0 = \rho, \quad \omega' = \rho_0' = \tau, \quad \omega'' = \rho_0'' = \upsilon, \quad \omega''' = \rho_0''' = \mathrm{r}^{-1}\nu;$$

r, r, ρ, τ', ν, and υ being treated as *constants* of the *equation*, or of the *surface*, in each of *these* derivations.

(33.) The *cone* LXXXIII., and the *cylinder* LXXXIV., have a *common generatrix*, namely the *binormal** (ν); and in like manner, *another* generating line of the *same cone*, namely the *tangent* (τ) to the curve, has just been seen (31.) to be a line *on the paraboloid* LXXXV.: and although the *cylinder and paraboloid* have *no finitely distant right line common*, yet *each* may be said to contain the *line at infinity, in the diametral plane* of the cylinder, namely in the plane of ν and υ, of which plane the quaternion equation is (comp. (14.)),

$$\text{LXXXVII.} \ldots 0 = S\nu\upsilon(\omega - \rho), \quad \text{or} \quad \text{LXXXVII}'. \ldots 0 = S(rr'\tau' - 3\tau)(\omega - \rho);$$

or the line in which this *diametral* meets the parallel *axial plane*.

(34.) On the whole, then, it is clear, from the known theory of intersections of surfaces of the second order having a common generating line, that *the given curve of double curvature* (whatever it may be) *has contact of the third order with the twisted cubic*,† or *gauche curve of the third degree*, which is

* The *geometrical reason*, for the *osculating cone* LXXXIII. to the *cone of chords* containing the *binormal* (ν), is that *if* the expression LXXXI. for ρ_t were *rigorous*, and if the variable t were supposed to *increase indefinitely*, the *ultimate direction* of the *chord* PP_t would be *perpendicular to the osculating plane*. And the same binormal is a generating line of the *parabolic cylinder* also, because that cylinder passes through P, and *all* its generating lines are perpendicular to the last mentioned plane. It is sufficient however to observe, on the side of calculation, that the *equations* LXXXIII. and LXXXIV. are *satisfied*, when we suppose $\omega - \rho \parallel \nu$.

† Compare again page 241, already cited, of Dr. Salmon's Treatise; also Art. 285, in page 225 of the same work.

represented without ambiguity by the system of the *two scalar equations*,

$$\text{LXXXVIII.} \ldots y = x^2, \quad z = x^3,$$

if we write for abridgment,

$$\text{LXXXIX.} \ldots \begin{cases} x = (t =) - r^2\text{S}\upsilon\nu(\omega - \rho), \\ y = (t^2 =) - 2r^2\text{S}\tau'(\omega - \rho), \\ z = (t^3 =) - 6r^2\text{r}\text{S}\nu(\omega - \rho). \end{cases}$$

(35.) As another geometrical connexion between the elements of the present theory, it may be observed that while the *osculating plane* to the *curve*, of which plane the equation is,

$$\text{XC.} \ldots \text{S}\nu(\omega - \rho) = 0, \text{ as in 396, XV.,}$$

touches the oblique cone LXXXIII., *along the tangent* τ to the same curve, the *diametral plane* LXXXVII. *touches the same cone along the binormal* ν, which was lately seen (33.) to be, as well as τ, a *side* of that oblique cone; but these *two sides of contact*, τ and ν, are both *in the rectifying plane* (396, XIV.), and the *two tangent planes* corresponding *intersect* in the *diameter* υ of the *parabola* (9.); we have therefore this *theorem*:—

The diameter of the osculating parabola to a curve of double curvature is the polar of the rectifying plane, with respect to the osculating oblique cone LXXXIII.; that is, with respect to a certain cone of the *second order*, which has been above deduced from the expression LXXXI. for the vector ρ_t of the curve, as one naturally suggested thereby, and as having a *contact of the third order with the curve* at P, and therefore also a contact of the *second order* with the *cone of chords* from that point.

(36.) Conversely, *this particular cone* LXXXIII. is *geometrically distinguished* from all *other** cones of the *same* (*second*) *order*, which have their *vertices* at the *given point* P, and have each a *contact* of the same *second* order,

* The cone of *this system* (36.), which is *touched* along the *binormal* by the *normal plane*, and which therefore *intersects* the *parabolic cylinder* LXXXIV. in a *new twisted cubic* (comp. (34.)), having *also* contact of the *third order* with the *curve*, is easily found to have, for its quaternion equation, the following:

$$2r^2(\text{S}\tau'(\omega - \rho))^2 = 3\text{r}\text{S}\tau(\omega - \rho)\text{S}\nu(\omega - \rho);$$

and with respect to *this cone* (comp. (35.)), the *polar of the rectifying plane* is the (*absolute*) *normal* (τ') *to the curve*.

with the *given cone* of *chords* from that point, or of the *third* order with the given *curve*, by the *condition* that it is *touched* (as above), *along the binormal* (ν), by the *diametral plane* ($\nu\nu$) of the *osculating parabolic cylinder* LXXXIV.

(37.) We have already considered, in 395, (5.), the simultaneous variations of the points P and K, or of the vectors ρ and κ. With recent notations, including the expression $\mu = 2\kappa - \rho$, we have the following among other transformations, for the *first derivative* of the latter vector, and therefore for the *tangent* KK′ to the *locus of centres of curvature*, of a given curve in space:

$$\text{XCI.} \ldots \mathrm{KK}' = \mathrm{D}_s\kappa = \kappa' = (\rho - \tau'^{-1})' = \tau + \tau'^{-1}\tau''\tau'^{-1}$$
$$= (\rho + r^2\tau')' = \tau + r^2\tau'' + 2rr'\tau'$$
$$= rr'\tau' + r^2\mathrm{r}^{-1}\nu = rr'(\tau' + p^{-1}r\nu) = r\mathrm{r}^{-1}(p\tau' + r\nu)$$
$$= \frac{rr'}{\rho - \kappa} - \frac{rr'}{\sigma - \kappa} = \frac{rr'(\sigma - \mu)}{(\sigma - \kappa)(\kappa - \rho)} = \mathrm{r}^{-1}(\sigma - \mu)\tau$$
$$= \cot H(\mathrm{U}\tau' \tan P + \mathrm{U}\nu) = \mathrm{r}^{-1}R(\mathrm{U}\tau' \sin P + \mathrm{U}\nu \cos P)$$
$$= r^4\nu\nu'\tau' = r^4\tau'\nu'\nu = \nu^{-1}\nu'\tau'^{-1} = \tau'^{-1}\nu'\nu^{-1}$$
$$= \mathrm{r}^{-1}\nu(\rho - \sigma)(\kappa - \rho) = \mathrm{r}^{-1}(\kappa - \rho)(\rho - \sigma)\nu$$
$$= \mathrm{r}^{-1}R\mathrm{U}(\nu(\rho - \sigma)(\kappa - \rho)) = \&c.;$$

if then we draw the *diameter of curvature* PM, and let fall a perpendicular KN from the centre K of the osculating circle on the new radius SM of the osculating sphere (as in the annexed figure 82), *this perpendicular will touch* the locus of the centre* K, a result which agrees with the construction in 395, (6.); and we see, at the same time, that the *length* of the line KK′, or the *tensor* Tκ', may be expressed (comp. LXXIII.) as follows,

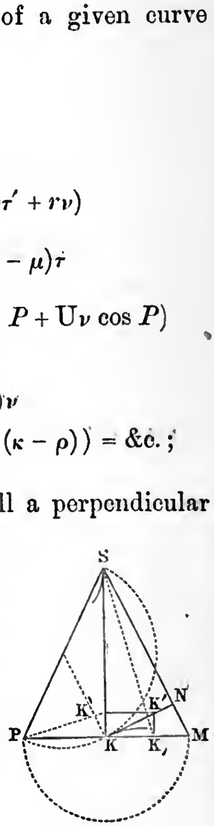

Fig. 82.

$$\text{XCII.} \ldots \overline{\mathrm{KK}'} = \mathrm{T}\kappa' = R\mathrm{Tr}^{-1} = r^2\mathrm{T}\nu' = \tan \angle \tau\tau''.$$

(38.) If we *project* the tangent KK′, into its two rectangular components, KK, and KK′, on the diameter of curvature and

* Geometrically, and by infinitesimals, if we conceive K′ to be an infinitely near point of the locus of K, and therefore in the normal plane at P, the angle PK′S (like PKS) will be right, and the point K′ will be *on the semicircle* PKS; but the *radius* of this semicircle drawn to K (comp. fig. 82) is *parallel* to the line SM, to which line the *tangent* KK′ is therefore *perpendicular*, as above.

the polar axis, we shall have by XCI. the expressions:

$$\text{XCIII.} \ldots \kappa\kappa_{,} = rr'\tau' = r'\mathrm{U}\tau' = \frac{rr'}{\rho - \kappa} = \&c.;$$

$$\text{XCIV.} \ldots \kappa\kappa_{\backprime} = r^2\mathrm{r}^{-1}\nu = r\mathrm{r}^{-1}\mathrm{U}\nu = \frac{-rr'}{\sigma - \kappa} = \&c.;$$

these two projections then, or the *vector-tangent* KK′ itself, would *suffice* to determine r and r', or H and P, and thereby *all* the affections of the curve which depend on s^3, but *not* on s^4.

(39.) We have also the *similar triangles* (see again fig. 82),

$$\text{XCV.} \ldots \Delta\, \mathrm{K_{,}K'K} \propto \mathrm{K_{\backprime}KK'} \propto \mathrm{KMS};$$

and the *vector equations*,

$$\text{XCVI.} \ldots \mathrm{KK'} : \mathrm{SM} = \mathrm{KK_{,}} : \mathrm{SK} = \mathrm{KK_{\backprime}} : \mathrm{KM} = \mathrm{KK_{\backprime}} : \mathrm{PK}$$
$$= \mathrm{r}^{-1}\tau = \textit{Vector of second curvature (IV.)};$$

whence also result the *scalar expressions*,

$$\text{XCVII.} \ldots \tan \mathrm{KSK_{,}} = \tan \mathrm{KPK_{\backprime}} = \mathrm{r}^{-1} = \textit{Second* Curvature (III.)}:$$

this last scalar being *positive* or *negative*, according as the *rotation* KSK, (or KPK‵) *appears* to be positive or negative, when seen from *that side* of the *normal plane*, towards which the conceived *motion* (396, (1.)) along the given *curve*, or the *unit tangent* $+\tau$, is directed.†

(40.) Besides the *seven* expressions, III., XXVII., L., and XCVII., this important scalar r^{-1} admits of many others, of which the following, numbered for reference as 8, 9, &c., and deduced from formulæ and principles already laid down, are examples: and may serve as *exercises in transformation*, according to the rules of the present Calculus, while some of them may also be found useful, in future *geometrical applications*.

* In illustration it may be observed, that if ds be treated as *infinitely small*, and if the line KK′ be supposed to represent (not the *derivative* κ', but) the *differential* vector $d\kappa = \kappa' ds$, then the *projections* KK, and KK‵ become dr and $r\mathrm{r}^{-1}ds$ (comp. XCIII. and XCIV.); while KPK‵ (in fig. 82) represents the *infinitesimal angle* $\mathrm{r}^{-1}ds$, through which the *osculating plane* (comp. (1.)) *revolves*, round the tangent τ to the curve during the change ds of the *arc*.

† This direction of $+\tau$ is to be conceived (comp. fig. 81, [p. 100]) to be towards the *back* of fig. 82, as drawn, if the scalars r' and r (and therefore also p) be *positive*.

(41.) We have then (among others) the transformations:

XCVIII. . . *Second Curvature* = r^{-1} (= *seven preceding expressions*)

$$= p^{-1}r' = r^{-1}\cot H = \mathrm{T}\lambda\cos H = r^{-1}r'\cot P \qquad (8, 9, 10, 11)$$

$$= r^2\mathrm{S}\nu'\tau' = -\,\mathrm{S}\nu'\tau'^{-1} = -\,r^2\mathrm{S}\tau\tau'\tau'' = \mathrm{S}\tau\tau'^{-1}\tau'' \qquad (12, 13, 14, 15)$$

$$= -\,r^2\mathrm{S}\nu\tau'' = \mathrm{S}\nu^{-1}\tau'' = -\,\mathrm{S}\nu\kappa' = \mathrm{S}\tau\kappa'\tau' \qquad (16, 17, 18, 19)$$

$$= \tau\kappa'(\sigma-\mu)^{-1} = \mathrm{S}\lambda\tau^{-1} = (\kappa-\rho)\mathrm{V}\lambda\nu = -\,\tau'^{-1}\mathrm{V}\lambda\nu \qquad (20, 21, 22, 23)$$

$$= r^2\tau'\mathrm{V}\lambda\nu = r^2\mathrm{S}\lambda\nu\tau' = \mathrm{S}\lambda\tau'\nu^{-1} = \mathrm{S}\lambda\tau'^{-1}\nu \qquad (24, 25, 26, 27)$$

$$= r^2\mathrm{S}\nu'\lambda\tau = r^2\mathrm{S}\nu'\nu\tau = \mathrm{S}\tau\nu^{-1}\nu' = r^2\mathrm{S}\nu'\nu^{-1}\tau'' \qquad (28, 29, 30, 31)$$

$$= r^4\mathrm{S}\nu\nu'\tau'' = \tau''^{-1}\mathrm{V}\nu'\lambda = r^3r'^{-1}\mathrm{S}\nu'\lambda\tau' = r^3r'^{-1}\mathrm{S}\nu\lambda\tau'' \qquad (32, 33, 34, 35)$$

$$= \mathrm{S}\nu'\lambda\tau''^{-1} = \mathrm{T}\tau''^{-2}\mathrm{S}\lambda\nu'\tau'' = \frac{-(r\nu)'}{r\tau'} = \frac{-r^2\nu'}{\sigma-\rho} \qquad (36, 37, 38, 39)$$

$$= \frac{-r\nu'}{r\tau'+p\nu} = \frac{r^2\tau''+\tau}{\tau(\sigma-\rho)} = R^{-1}\tan\angle\,\mathrm{r}\tau\tau'' = R^{-1}\tan\angle\frac{\mathrm{V}\lambda\nu'}{\tau} \qquad (40, 41, 42, 43)$$

$$= \frac{rr'\nu}{\sigma-\kappa} = \frac{rr'\tau'}{(\sigma-\kappa)\tau} = \frac{r'}{r}\cdot\frac{\tau(\kappa-\rho)}{\sigma-\kappa} = \frac{rr'\tau}{(\sigma-\kappa)(\rho-\kappa)} \qquad (44, 45, 46, 47)$$

$$= \mathrm{S}\frac{rp\lambda}{(\sigma-\kappa)(\rho-\kappa)} = \mathrm{S}\frac{\rho+rp\lambda-\kappa}{(\sigma-\kappa)(\rho-\kappa)} = \mathrm{S}\frac{\mathrm{KL}}{\mathrm{KS}\,.\,\mathrm{KP}} \qquad (48, 49, 50)$$

$$= \mathrm{S}\frac{\mathrm{SL}}{\mathrm{PK}\,.\,\mathrm{KS}} = \frac{-(\mathrm{S}\alpha r\nu)'}{r(\mathrm{S}\alpha\tau)'} = \frac{-\mathrm{d}\cos\angle\frac{\nu}{\alpha}}{r\mathrm{d}\cos\angle\frac{\tau}{\alpha}}; \qquad (51, 52, 53)$$

PKSL, in the forms 50 and 51, being points of the same *gauche quadrilateral* as in (18.); and α, in 52 and 53,* denoting *any constant vector*: while several other varieties of form may be deduced from the foregoing by very simple processes, such as the substitution of $\mathrm{U}\nu$ for $r\nu$, &c., which gives for instance (comp. XI′.), from the form 38, these others,

$$\text{XCVIII}'.\ .\ .\ \mathrm{r}^{-1} = \frac{-(\mathrm{U}\nu)'}{r\tau'} = \frac{-(\mathrm{U}\nu)'}{\mathrm{U}\tau'} = \frac{-\mathrm{dU}\nu}{r\mathrm{d}\tau}. \qquad (54, 55, 56)$$

We may also write, with the significations (10.) of Q_1 and Q_3, the following expression analogous to L.,

$$\text{XCVIII}''.\ .\ .\ \mathrm{r}^{-1} = 6\mathrm{KP}\,.\,\lim.\,\frac{\mathrm{PQ}_3}{\mathrm{PQ}_1{}^3}, \qquad (57)$$

* This last form 53 corresponds to and contains a theorem of M. Serret, alluded to in the second Note to page 92.

which contains the law of the *inflexion* of the *plane curve*, into which the proposed curve of *double curvature* is *projected*, on its own *rectifying plane*; the *sign* of the *scalar*, to which this last expression ultimately reduces itself, being determined by the rules of quaternions.

(42.) And besides the various expressions for the positive scalar r^{-2}, which are immediately obtained by *squaring* the foregoing forms, the following are a few others:

XCIX. . . *Square of Second Curvature* $= \mathrm{r}^{-2} = \mathrm{Tr}^{-2}$

$$= \mathrm{T}\lambda^2 - r^{-2} = r^2\mathrm{S}\tau''\tau'\lambda - r^{-2} = r^2\mathrm{T}\nu'^2 - r^{-2}r'^2 \qquad (1, 2, 3)$$

$$= r^2\mathrm{S}\tau\nu'\tau'' - r^{-2}r'^2 = r^2\mathrm{T}\tau''^2 - r^{-2} - r^{-2}r'^2 = R^{-2}(r^4\mathrm{T}\tau''^2 - 1) \qquad (4, 5, 6)$$

$$= R^{-2}r^4\mathrm{T}\nu'^2 = R^{-2}\mathrm{T}\kappa'^2 = R^{-2}\tan^2\angle\,\tau\tau''; \qquad (7, 8, 9)$$

while the important vector τ'', besides its two original forms VI., admits of the following among other expressions (comp. XX. XXI.):

C. . . $\tau'' = \mathrm{D}_s{}^3\rho$ (= the two expressions VI.)

$$= r^{-2}\mathrm{V}\lambda(\sigma - \rho) = \lambda\tau' - r^{-1}r'\tau' = \nu'\tau - r^{-2}\tau \qquad (3, 4, 5)$$

$$= \mathrm{r}\mathrm{V}\nu'\lambda = r^{-2}\mathrm{r}^{-1}\tau(\sigma - \rho - \mathrm{r}) = r^{-3}p + r^{-2}\lambda(\sigma - \rho) \qquad (6, 7, 8)$$

$$= ((\rho - \kappa)^{-1})' = \tau'(\kappa' - \tau)\tau' = -r^{-2}\tau - \frac{r^{-1}r'}{\rho - \kappa} - \frac{r^{-1}r'}{\sigma - \kappa}. \qquad (9, 10, 11)$$

(43.) As regards the general theory (396, (5.), &c.) of *emanant lines* (η) from curves, it might have been observed that if we write,

CI. . . $\zeta = \mathrm{V}\dfrac{\tau}{\theta}$, with CII. . . $\theta = \mathrm{V}\dfrac{\eta'}{\eta}$, as in 396, XXVII.,

the equation 396, XXXII. takes the simplified form,

CIII. . . $\mathrm{PH} = \omega_0 - \rho = \eta\mathrm{S}\eta^{-1}\zeta$ = *projection of vector* ζ *on emanant* η;

for example, when $\eta = \nu$, then $\theta = \mathrm{r}^{-1}\tau$, and $\zeta = 0$, $\mathrm{PH} = 0$, or $\omega_0 = \rho$, as in (1.); and when $\eta = \tau$, then $\theta = \nu$, $\zeta = r^2\tau' \perp \eta$, so that the *projection* PH again vanishes, as in 396, (13.).

(44.) In an extensive class of applications, the *emanant lines* are *perpendicular to the given curve* ($\eta \perp \tau$); and since we have, by (43.),

CIV. . . $\zeta = \dfrac{\mathrm{V}\tau\mathrm{V}\eta'\eta}{\eta^2\theta^2} = \eta^{-1}\theta^{-2}\mathrm{S}\tau\eta' = \dfrac{\eta^{-1}\mathrm{S}\eta\tau'}{\mathrm{T}\theta^2}$, if $\mathrm{S}\tau\eta = 0$, we may write, for this *case of normal emanation*, the formula,

$$\text{CV.} \ldots \text{PH} = \zeta = \frac{\textit{projection of vector of curvature } (\tau') \textit{ on emanant line } (\eta)}{\textit{square of velocity } (\mathrm{T}\theta) \textit{ of rotation of that emanant}};$$

for example, when the emanant (η) coincides with the *absolute normal* (τ'), we have then $\theta = \lambda$, as in (3.), and the recent formula CV. becomes,

$$\text{CVI.} \ldots \text{PH} = \omega_0 - \rho = \zeta = \tau'\mathrm{T}\lambda^{-2} = r^2\tau' \sin^2 H = (\kappa - \rho) \sin^2 H,$$

which agrees with the expression XXXVIII.

(45.) And in the corresponding case of *tangential emanant planes*, by making $\mathrm{S}\tau\eta = 0$ in the second equation 396, XXXVI., and passing to a second derived equation, we find for the *intercept* between the point P of the *curve*, and the point, say R, in which the *line of contact* of the *plane* with its own *envelope* touches the *cusp-edge* of that *developable surface*, the expression,

$$\text{CVII.} \ldots \text{PR} = \frac{-\mathrm{V}\eta\eta'\mathrm{S}\eta\tau'}{\mathrm{S}\eta\eta'\eta''} = \frac{-\mathrm{S}\eta\tau'(\text{or } + \mathrm{S}\tau\eta')}{\textit{projection of } \eta'' \textit{ on } \theta};$$

which accordingly vanishes, as it ought to do, when $\eta = \nu$, that is, when the *emanant plane* $\mathrm{S}\eta(\omega - \rho) = 0$ coincides with the *osculating plane* XC.

(46.) Some additional light may be thrown on this whole theory, of the *affections in a curve in space* depending on the *third power of the arc*, and even on those affections which depend on *higher powers* of s, by that *conception* of an *auxiliary spherical curve*, which was employed in 379, (6.) and (7.), to supply *constructions* (or geometrical representations) for the *directions*, not only of the *tangent* (ρ') to the *given* curve, to which indeed the *unit-vector* (τ) of the *new* curve is *parallel*, but also of the *absolute normal*, the *binormal*, and the *osculating plane*; while the *same* auxiliary curve served also, in 389, (2.), to furnish a *measure* of the *curvature* of the original curve, which is in fact the *velocity** of motion in the *new* or *spherical* curve, if that in the *old* or *given* one be supposed to be *constant*, and be taken for *unity*.

(47.) We might for instance have observed, that while the *normal plane* to the *curve in space* is represented (in direction) by the *tangent plane* to the *sphere*, the *rectifying plane* (as being perpendicular to the absolute normal) is represented similarly by the *normal* plane to the *spherical curve*: and it is not

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* Accordingly the *vector of velocity* τ', of this *conceived motion* in the *auxiliary curve*, is precisely what we have called (389, (4.), comp. 396, VI.) the *vector of curvature* of the proposed *curve in space*: and its *tensor* ($\mathrm{T}\tau'$) is equal to the *reciprocal* of the *radius* (r) of that curvature.

difficult to prove that the *rectifying line* has the direction of that *new radius* of the sphere, which is drawn to the *point* (say L) where the *normal arc* to the auxiliary curve *touches its own envelope*.

(48.) The *point* L thus determined is the *common spherical centre* (comp. 394, (5.)) *of curvature*, of the *auxiliary* curve *itself*, and of that *reciprocal** *curve* on the same sphere, of which the *radii* have the directions (comp. 379, (7.)) of the *binormals* to the original curve; the *trigonometric tangent* of the *arcual radius of curvature* of the auxiliary curve is therefore ultimately equal to a *small arc* of that curve, *divided by the corresponding arc* of the reciprocal curve (or rather by the latter arc with its *direction reversed*, if the point L fall *between* the two curves upon the sphere); and therefore to the *first curvature* (r^{-1}) of the *given* curve, *divided by the second curvature* (r^{-1}) : and thus we have not only a simple *geometrical interpretation* of the *quaternion equation* XI'., but also a *geometrical proof* (which may be said to require *no calculation*), of the important but known relation XVII., which connects the *ratio* ($\mathrm{r} : r$) *of the two curvatures*, with the *angle* (H) between the *tangent* (τ) and the *rectifying line* (λ), for *any curve in space*.

(49.) In whatever manner this known relation ($\tan H = \mathrm{r} : r$) has once been established, it is geometrically evident, that *if the ratio of the two curvatures be constant*, then, because the *curve crosses* the *generating lines* of its own *rectifying developable* (396) under a *constant angle* (H), that *developable surface* must be *cylindrical*: or in other words, the proposed *curve of double curvature* must, in the case supposed, be a *geodetic*† *on a cylinder* (comp. 380, (4.)). Accordingly the point L, in the two last sub-articles, becomes then a *fixed point* upon the *sphere*, and is the *common pole* of *two complementary small circles*, to which the *auxiliary spherical curve* (46.), and the *reciprocal curve* (48.), in the case here considered, reduce themselves; so that the *tangent* and

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* The *reciprocity* here spoken of, between these *two spherical curves*, is of that known kind, in which each *point* of *one* is a *pole* of the *great-circle tangent*, at the *corresponding point* of the *other*: and accordingly, with our recent symbols, we have not only $\nu = \mathrm{V}\tau\tau'$, but also, $\mathrm{V}\nu\nu' = r^{-2}\mathrm{V}\nu'\nu^{-1} = r^{-2}\mathrm{r}^{-1}\tau \parallel \tau$.

† The writer has not happened to meet with the *geometrical proof* of this known theorem, which is attributed to M. Bertrand by M. Liouville, in page 558 of the already cited Additions to Monge; but the deduction of it as above, from the fundamental property (396) of the *rectifying line*, is sufficiently obvious, and appears to have suggested the method employed by M. de Saint-Venant, in the part (p. 26) of his Memoir *sur les lignes courbes non planes*, &c., before referred to, in which the result is enunciated. *Another*, and perhaps even a *simpler* method, *suggested by quaternions*, of *geometrically* establishing the same theorem, will be sketched in the present sub-article (49.); and in the following sub-article (50.), a proof by the *quaternion analysis* will be given, which seems to leave nothing to be desired on the side of simplicity of *calculation*.

the *binormal* to the *curve in space* make (in the same case) *constant angles*, with the *fixed radius* drawn to that point: and the *curve itself* is therefore (as before) a *geodetic line*, on *some* cylindrical surface.

(50.) By quaternions, when the *two curvatures* have thus a *constant ratio*, the equations XI′. and XVI. give,

$$\text{CVIII.} \ldots (r\lambda)' = (\mathrm{U}\nu + r\mathrm{r}^{-1}\tau)' = (r\mathrm{r}^{-1})'\tau = 0,$$

or

$$\text{CIX.} \ldots r\lambda = \textit{a constant vector};$$

the *tangent* (τ) makes therefore, in this case, a *constant angle* (H) with a *constant line* ($r\lambda$): and the *curve* is thus seen again, by this very simple *analysis*, to be a *geodetic on a cylinder*. And because it is easy to prove (comp. XXXI.), that we have in the same case the expression,

$$\text{CX.} \ldots r \sin^2 H = \textit{radius of curvature of base},$$

or of the *section* of the cylinder made by a plane perpendicular to the generating lines, this *other* known theorem results, with which we shall conclude the present series of sub-articles: *When both the curvatures are constant, the curve is a geodetic on a right circular cylinder* (or *cylinder of revolution*); or it is what has been called above, for simplicity and by eminence, a *helix*.*

398. When the *fourth power* (s^4) of the *arc* is taken into account, the expansion of the vector ρ_s involves *another term*, and takes the form (comp. 397, I.),

$$\text{I.} \ldots \rho_s = \rho + s\tau + \tfrac{1}{2}s^2\tau' + \tfrac{1}{6}s^3\tau'' + \tfrac{1}{24}s^4\tau''',$$

in which

$$\text{II.} \ldots \tau''' = \mathrm{D}_s{}^4\rho, \quad \text{and} \quad \text{III.} \ldots \mathrm{S}\tau\tau''' = -3\mathrm{S}\tau'\tau'' = -3r^{-3}r';$$

so that the *new affections* of the curve, thus introduced, depend only on *two new scalars*, such as r' and r'', or r' and R', or H' and P', &c. We must be

* In general, the expression XLIV. for the vector ω_s of the *osculating helix*, in which $\iota = -\mathrm{r}^{-1}\lambda^{-1} = \tau - \lambda^{-1}\tau'$, and $\rho - \omega_0 = \lambda^{-2}\tau'$, gives $\mathrm{T}\omega'_s = 1$; so that the *deviation* (8.) may be considered (comp. (13.)) to be measured from the extremity of an *arc of the helix*, which is *equal in length* to the arc s of the *curve*, and is set off from the same initial point P, with the same initial direction: while ω_0 does not *here* denote the value of ω_s answering to $s = 0$, but has a special signification assigned by the formula XXXVIII. It may also be noted that the *conception*, referred to in (46.), of an *auxiliary spherical curve*, corresponds to the ideal substitution of the *motion of a point* with a *varying velocity* upon a *sphere*, for a motion with an *uniform velocity* in *space*, in the investigation of the *general properties* of *curves of double curvature*: and that thus it is intimately connected (comp. 379, (9.)) with the general theory of *hodographs*.

content to offer here a very few remarks on the theory of *such* affections, and on the manner in which it may be extended by the introduction of derivatives of *higher orders*.

(1.) The new vector τ''', on which everything here depends, is easily reduced to the following forms,* analogous to the expressions 397, VI. for τ'':

$$\text{IV.} \ldots \tau''' = \frac{r(r^{-3})'}{\tau} + \frac{(r^{-3}r')' - \tau''^2}{\tau'} - \frac{(r^{-2}\mathrm{r}^{-1})'}{\nu}$$
$$= 3r^{-3}r'\tau + (r(r^{-1})'' + \lambda^2)\tau' + (r^{-2}\mathrm{r}^{-1})'r^2\nu.$$

(2.) The first derivatives of the four vectors, ν', κ', λ, σ, taken in like manner with respect to the arc s of the curve, are the following:

$$\text{V.} \ldots \nu'' = (\mathrm{V}\tau\tau'')' = \mathrm{V}\tau\tau''' + r^{-2}\lambda$$
$$= r^{-2}\mathrm{r}^{-1}\tau + (r^{-2}\mathrm{r}^{-1})'\tau'^{-1} + (r(r^{-1})'' - \mathrm{r}^{-2})\nu;$$

$$\text{VI.} \ldots \kappa'' = -r^{-1}r'\tau + (rr'' - r^2\mathrm{r}^{-2})\tau' + (r^2\mathrm{r}^{-1})'\nu;$$

$$\text{VII.} \ldots \lambda' = (\mathrm{r}^{-1})'\tau + (r^{-1})'r\nu, \text{ or VII'.} \ldots (r\lambda)' = (r\mathrm{r}^{-1})'\tau \text{ (comp. 397, CVIII.)};$$

$$\text{VIII.} \ldots \sigma' = (\kappa + pr\nu)' = (p' + r\mathrm{r}^{-1})r\nu = RR'p^{-1}r\nu;$$

in which last the scalar derivatives p' and R' are determined, in terms of r'' and r', by the equations,

$$\text{IX.} \ldots p' = (r'\mathrm{r})' = r''\mathrm{r} + r'\mathrm{r}',$$

and

$$\text{X.} \ldots R' = R^{-1}(pp' + rr') = p' \sin P + r' \cos P = (p' + \cot H) \sin P.$$

We have also the derivatives,

$$\text{XI.} \ldots H' = \frac{r\mathrm{r}' - r'\mathrm{r}}{r^2 + \mathrm{r}^2} = \frac{r^{-1}r' - \mathrm{r}^{-1}\mathrm{r}'}{r\mathrm{r}\lambda^2},$$

$$\text{XII.} \ldots P' = \frac{rp' - r'p}{r^2 + p^2} = \frac{(rr'' - r'^2)\mathrm{r} + rr'\mathrm{r}'}{R^2};$$

and the relations,

$$\text{XIII.} \ldots \mathrm{S}\tau\tau'\tau''' = \mathrm{S}\nu\tau''' = -(r^{-2}\mathrm{r}^{-1})';$$

$$\text{XIV.} \ldots \mathrm{S}\tau\tau''\tau''' = \mathrm{S}\nu'\tau''' = -r^{-3}\mathrm{r}^{-2}(p' - r\mathrm{r}\lambda^2);$$

$$\text{XV.} \ldots \mathrm{S}\tau'\tau''\tau''' = r^{-2}\mathrm{S}\lambda\tau''' = -r^{-5}(r\mathrm{r}^{-1})';$$

* In these new expressions, on the plan of the second Note to page 90, the scalars r', p', R', and the vector σ', are to be regarded as of the dimension *zero*; r'', H', P', and κ'' of the dimension -1; λ' of the dimension -2; and ν'' and τ''', as being each of the dimension -3.

which may be proved in various ways, and by the two first (or the two last) of which, the derivatives r' and p', and therefore also H' and P', can be *separately* calculated, as *scalar functions* of the *four vectors* $\tau, \tau', \tau'', \tau'''$, or of some three of them, including the *new* vector τ'''.

(3.) We may also deduce, from either V. or VIII., the following *vector expressions*, of which the *geometrical signification* is evident from the recent theory (396, 397) of *emanant lines and planes*:

$$\text{XVI.} \ldots \textit{Vector of Rotation of Radius } (R) \textit{ of Spherical Curvature}$$
$$= \textit{Vector of Rotation of Tangent Plane to Osculating Sphere}$$
$$= \text{(say) } \phi = \mathrm{V}\frac{\nu''}{\nu'} = \mathrm{V}\frac{\sigma' - \tau}{\sigma - \rho} = R^{-2}\tau(\nu^{-1}\sigma' + \sigma - \rho) \qquad (1, 2, 3)$$
$$= \frac{\tau}{R}\left(\frac{rR'}{p} + \frac{\sigma - \rho}{R}\right) = \frac{r\tau}{R^2}\left(p' + \frac{r}{\mathrm{r}} + r\tau' + p\nu\right) = R^{-2}r(r\lambda + p'\tau - p\tau')\,; \qquad (4, 5, 6)$$

whence follows this *tensor value* for the *common angular velocity* of these two connected *rotations*, compared still with the velocity of *motion* along the *curve*,

$$\text{XVII.} \ldots \textit{Velocity of Rotation of Radius } (R), \textit{ or of Tangent Plane to Sphere},$$
$$= \mathrm{T}\phi = \mathrm{TV}\frac{\nu''}{\nu'} = R^{-1}\surd(1 + R'^2 \cot^2 P) = R^{-1}\surd\{1 + (p' + \cot H)^2 \cos^2 P\}\,;$$

with the verifications, for the case of the *helix*, for which $p = 0$, $p' = 0$, $P = 0$, and $R = r$, that these expressions XVI. and XVII. become,

$$\text{XVI}' \ldots \phi = \lambda, \quad \text{and} \quad \text{XVII}' \ldots \mathrm{T}\phi = \mathrm{T}\lambda = r^{-1} \operatorname{cosec} H,$$

which agree with those found before, for the vector and velocity of rotation of the *radius* (r) of *absolute curvature*.

(4.) As another verification, we have $R' = 0$ for every *spherical curve*, and the general expressions take then the forms,

$$\text{XVI}'' \ldots \phi = \frac{-\tau}{\sigma - \rho}, \quad \text{and} \quad \text{XVII}'' \ldots \mathrm{T}\phi = R^{-1},$$

of which the interpretation is easy.

(5.) In general, the formula XVII. may also be thus written,

$$\text{XVIII.} \ldots R^2\phi^2 + 1 = -R'^2 \cot^2 P = R'^2 - p^{-2}R^2R'^2 = R'^2 + \sigma'^2 = \sigma'^2 \cos^2 P\,;$$

or thus,

$$\text{XIX.} \ldots R\mathrm{T}\phi = \surd(1 + \mathrm{T}\sigma'^2 \cos^2 P) = \surd(1 + \mathrm{T}\sigma'^2 - R'^2)\,;$$

or finally,

$$\text{XX.} \ldots R^2 \text{T}\phi = \sqrt{(R^2 - r^2\sigma'^2)} = \sqrt{(R^2 + r^2\text{T}\sigma'^2)};$$

so that the *small angle*, $s\text{T}\phi$, between the *two near radii* of *spherical curvature*, R and R_s, is ultimately equal to the *square root of the sum of the squares* of the *two small angles*, in *two rectangular planes*, sR^{-1} and $rsR^{-2}\text{T}\sigma'$, or PSP_s and SPS_s, which are subtended, respectively, *at the centre* S *of the osculating sphere* by the *small arc* s of the *given curve*, and *at the given point* P by the *small corresponding arc* $s\text{T}\sigma'$ of the *locus of centre* S of *spherical curvature*, or of the *cusp-edge* (395, (2.)) of the *polar developable*; exactly* as the *small angle* $s\text{T}\lambda$, between *two near radii* (397, (5.)) of *absolute curvature*, r and r_s, is ultimately the square root of the sum of the squares of the *two other small angles*, sr^{-1} and $s\text{r}^{-1}$, or PKP_s and KPK_s, which are likewise situated in two *rectangular planes*, and are subtended at the centre K of the *osculating circle* by the small arc s of the *curve*, and at the given point P by the corresponding arc $s\text{T}\kappa'$ of the *locus* of the *centre* K (comp. 397, XXXIV., XCIV.).

(6.) The *point*, say V, in which the *radius* R of the osculating *sphere* at P approaches *most nearly* to the *near radius* R_s from P_s, is *ultimately* determined (comp. 397, CV. and X.) by the formula,

$$\text{XXI.} \ldots \text{PV} = \zeta = \frac{\textit{Vector of Spherical Curvature}}{\textit{Square of Angular Velocity of Radius } (R)}$$

$$= (\rho - \sigma)^{-1}\text{T}\phi^{-2} = \frac{\sigma - \rho}{1 + R'^2 \cot^2 P} = \frac{\sigma - \rho}{1 + p^{-2}r^2R'^2};$$

the *vector* of this point V (in its ultimate position) is therefore

$$\text{XXII.} \ldots \text{OV} = \rho + \zeta = \frac{r^2R'^2\rho + p^2\sigma}{r^2R'^2 + p^2} = \frac{r^2R'^2\rho + \text{r}^2r'^2\sigma}{r^2R'^2 + \text{r}^2r'^2};$$

with the verification, that (by X., comp. XVII.) the scalar $p^{-1}rR'$ or $R' \cot P$ reduces itself to $\cot H$, or to $r\text{r}^{-1}$, for the case $p = 0$, $p' = 0$, $P = 0$ (comp. (3.)): and that thus the expression 397, XXXVIII., for the vector OH of the *point* of *nearest approach*, of a *radius* (r) *of absolute curvature* to a *consecutive*† *radius* of the same kind, is reproduced.

* It will soon be seen that these two results, and others connected with them, depend geometrically on one *common principle*, which extends to all systems of *normal emanants* (397, (44.)).

† This usual expression, *consecutive*, is obviously borrowed here from the *language* of *infinitesimals*, but is supposed to be *interpreted*, like those used in other parts of the present series of Articles, by a reference to the conception of *limits*.

(7.) In general, if we introduce a *new auxiliary angle*, J, determined by the formula,

$$\text{XXIII.} \ldots \cot J = p^{-1}rR' = R' \cot P = (p' + \cot H) \cos P = R(\mathrm{r}^{-1} + P'),$$

the expression XXII. takes the simplified form (comp. again 397, XXXVIII.),

$$\text{XXIV.} \ldots \mathrm{OV} = \rho + \zeta = \rho \cos^2 J + \sigma \sin^2 J\,;$$

and the *segments*, into which the point V divides (internally) the radius R of the *sphere*, have the values (comp. 397, XXXIX.),

$$\text{XXV.} \ldots \overline{\mathrm{PV}} = R \sin^2 J, \quad \overline{\mathrm{VS}} = R \cos^2 J.$$

(8.) A *geometrical signification* may be assigned for this *new angle* J, which is *analogous* to the known signification of the angle H (397, XVII.). In fact, the *tangent plane* to the *osculating sphere* at P *touches* its own *developable envelope* along a *new right line*, of which the scalar equations are,

$$\text{XXVI.} \ldots \mathrm{S}(\sigma - \rho)(\omega - \rho) = 0, \quad \mathrm{S}(\sigma' - \tau)(\omega - \rho) = 0\,;$$

and because the *developable locus* of all such *lines* can be shown to be *circumscribed*, along the *given curve*, to the *locus of the osculating circle*, which is at the same time the *envelope of the osculating sphere*, we shall briefly call this *locus of the line* XXVI. the *Circumscribed Developable*. And the *inclination* of the *generatrix* of this *new developable surface*, to the *tangent* to the given *curve* at P, if suitably measured *in the tangent plane* to the *sphere*, is precisely the *angle* which has been above denoted by J.

(9.) To render this conception more completely clear, let us suppose that a *finite right line* PJ is set off from the given point P, *on the indefinite line* XXVI., so as to represent, by its *length* and *direction*, the *velocity* of the *rotation of the tangent plane* to the osculating sphere; and so to be, in the phraseology (396, (14.)) of the general theory of *emanants*, the *vector-axis* of that *rotation*. We shall then have the values,

$$\text{XXVII.} \ldots \mathrm{PJ} = \phi (= \textit{the six expressions } \text{XVI.})$$
$$= R^{-1}\tau(\cot J + \mathrm{U}(\sigma - \rho)) = R^{-1} \operatorname{cosec} J(\tau \cos J + \tau \mathrm{U}(\sigma - \rho) \sin J)\,; \quad (7, 8)$$

the angle J being determined by the formula XXIII., and a new expression, $\mathrm{T}\phi = R^{-1} \operatorname{cosec} J$, being thus obtained for the velocity XVII.

(10.) Hence the new angle J, if conceived to be included (like H) between the limits 0 and π, may be considered to be measured *from* τ *to* ϕ, or *from* the *unit-tangent* to the curve at P, *to the generating line* PJ of the *circumscribed developable* (8.), in the *direction* from τ to $\tau(\sigma - \rho)$: which *last* tangent to the osculating sphere makes generally, like the tangent ϕ or PJ itself, an *acute* angle with the positive binormal ν, as appears from the common *sign* of the scalar *coefficients* of that vector, in their developed expressions.

(11.) It may also be remarked, as an additional point of analogy, and as serving to verify some formulæ, that while the older angle H becomes *right*, when the given curve is *plane*, so the *new* angle $J = \frac{\pi}{2}$, for every *spherical curve*.

(12.) As *another geometrical illustration* of the properties of the angle J, and of some other results of recent sub-articles, which may serve to connect them, still more closely, with the general theory of *normal emanants* from curves (397, (44.)), let us conceive that AB, BC, CD are three successive right lines, perpendicular each to each; let us denote by a and b the angles BCA and CBD, and by c the inclination of the line AD to BC: and let us suppose that these two lines are intersected by their common perpendicular in the points G and H respectively.

(13.) Then, by completing the rectangle BCDE, and letting fall the perpendicular BF on the hypotenuse of the right-angled triangle ABE, we obtain the projections, AE and FB, of the two lines AD and GH, on the plane through B perpendicular to BC; and hence, by elementary reasonings, we can infer the relations:

$$\text{XXVIII.} \ldots \tan^2 c = \tan^2 \text{ADE} = \tan^2 a + \tan^2 b;$$

and

$$\text{XXIX.} \ldots \frac{\text{BH}}{\text{BC}} = \frac{\text{AG}}{\text{AD}} = \frac{\text{AF}}{\text{AE}} = \frac{\text{AB}^2}{\text{AE}^2} = \sin^2 \text{AEB},$$

or

$$\text{XXIX}'. \ldots \text{BH} = \text{BC} \sin^2 j, \quad \text{if} \quad \tan j = \tan a \cot b;$$

nothing here being supposed to be *small*. It may also be observed, that the *two rectilinear angles*, BCA and CBD, or a and b, represent respectively the inclinations of the *plane* ACD to the *plane* BCD, and of the plane ABD to the plane ABC.

(14.) Conceive next that PQ and P_sQ_s are two near normal emanants, touching the polar developable in the points Q and Q_s, whereof Q is thus on the given polar axis KS, and Q_s is on the near polar axis K_sQ_s; and let the second emanant be cut, in the points P′ and Q′, by planes through P and Q, perpendicular to the first emanant PQ. The line PP′ will then be very nearly tangential to the given curve at P; and the line QQ′ will be very nearly situated in the corresponding normal plane to that curve: so that these two new lines will be very nearly perpendicular to each other, and the gauche quadrilateral P′PQQ′ will ultimately have the properties of the recently considered quadrilateral ABCD.

(15.) This being perceived, if we denote by e the length of the emanant line PQ, the small angle a is very nearly $= e^{-1}s$; and if the small angle b be put under the form $b's$, then the new coefficient b' is ultimately equal (by XXIX′.) to $e^{-1}\cot j$: where j is an auxiliary angle, not generally small, and is such that we have ultimately $\text{PH} = \text{PQ} \,.\, \sin^2 j$, if H be the point in which the given normal emanant PQ approaches most closely to the consecutive emanant P_sQ_s.

(16.) We have then the ultimate equation,

$$\text{XXX.} \ldots \cot j = eb' = \overline{\text{PQ}} \times \lim.(s^{-1} \,.\, \text{QPQ}_s)$$
$$= \textit{length of emanant line } (\text{PQ})$$
$$\times \textit{ angular velocity of the tangential plane } (\text{P}'\text{PQ}) \textit{ containing it};$$

this latter plane being here conceived as *turning*, for a moment, round the tangent to the given curve at P, and the velocity of *motion* along that curve being still taken for unity.

(17.) Accordingly, when we change e to r, b' to r^{-1}, and j to H, we recover in this way the fundamental value $\cot H = r\text{r}^{-1}$ (397, XVII.), for the cotangent of the *older* angle H; and when, on the other hand, we treat the radius of *spherical* curvature as the normal emanant, supposing Q to coincide with S, and therefore changing e to R, and b' to $\text{r}^{-1} + P'$, we recover the last of the expressions XXIII. for the cotangent of the *new* but analogous angle J, namely $\cot J = R(\text{r}^{-1} + P')$, together with an *interpretation*, which may not have at first seemed obvious: although that expression *itself* was deducible, in the following among other ways, from equations previously established,

$$\text{XXXI.} \ldots R^{-1}\cot J - \text{r}^{-1} = \frac{rR'}{pR} - \frac{r'}{p} = -\frac{R}{p}\left(\frac{r}{R}\right)' = -\frac{(\cos P)'}{\sin P} = P'.$$

(18.) As regards the *angular velocity*, say v, of the emanant *line* PQ, or the ultimate quotient of the *angle* between two such near lines, divided by the small *arc* s of the given curve, we see by XXVIII. (comp. (5.)) that this *small angle* vs is ultimately equal to the *square root of the sum of the squares* of the *two other small angles*, above denoted by a and b, and found to be equal, nearly, to $e^{-1}s$ and $e^{-1}s \cot j$ respectively: we may then establish the general formula,

$$\text{XXXII.} \ldots \textit{Angular Velocity of Normal Emanant} = v = e^{-1} \operatorname{cosec} j;$$

which reproduces the values, $r^{-1} \operatorname{cosec} H$, and $R^{-1} \operatorname{cosec} J$, already found for the angular velocities of the two radii, r and R.

(19.) And if we observe that the *projection of the vector of curvature*, KP^{-1}, *on the emanant* PQ, is easily proved to be $= \text{QP}^{-1} = e^{-2} \,.\, \text{PQ}$, we see by XXXII. that if this *projection* be *divided by the square of the angular velocity* (v) of the line PQ, the *quotient* is the line $\text{PQ} \,.\, \sin^2 j$, or PH (15.): which reproduces the general result, 397, CV., for all *systems of normal emanants*, together with a geometrical *interpretation*.

(20.) As still *another geometrical illustration* of the properties of the new angle J, we may observe that in the construction (12.) and (13.) the corresponding auxiliary angle j was equal to AEB, or to ABF, and that the line BF (= HG) was *perpendicular* to *both* BC and AD, although not *intersecting* the latter. Substituting then, as in (14.), the quadrilateral P′PQQ′ for ABCD, and passing to the limit, we may say that if a new line PJ be a *common perpendicular*, at the given point P, to *two consecutive** *normal emanants*, PQ and P′Q′, the *general auxiliary angle* j is simply the *inclination* P′PJ, of that *common perpendicular* PJ, to the *tangent* PP′ to the *curve*.

(21.) And if, instead of *normally emanating lines* PQ, we consider a system of *tangential emanant planes* (as in 397, (45.)), to which those *lines* are *perpendicular*, we may then (comp. 396, (14.)) consider the recent line PJ as being a *generating line* of the *developable surface*, which is the *envelope* of all the *planes* of the system; the *auxiliary angle*,† j, is therefore *generally* by (20.) the *inclination* of this *generatrix* to the *tangent*: a result which agrees with, and includes, the known and fundamental property (397, XVII.) of the angle H, in connexion with the *Rectifying Developable* (396); and also

* Compare the second Note to page 115.

† In these *geometrical illustrations*, the angle j has been treated, for simplicity, as being both *positive* and *acute*; although the *general formulæ*, which involve the corresponding angles H and J, permit and require that we should occasionally attribute to them *obtuse* (but still positive) values: while those angles may also become *right*, in some particular cases (comp. (11.)).

the *analogous* property of the newer angle J, connected (8.) with what it has been above proposed to call the *Circumscribed Developable.*

(22.) We shall soon return briefly on the theory of that *new developable surface* (8.), and of the *new locus* (of the osculating *circle*, or *envelope* of the osculating *sphere*) to which it has been said to be *circumscribed* : but may here observe, that if we write for abridgment (comp. VIII. and XXIII.),

$$\text{XXXIII.} \ldots n = \frac{\sigma'}{r\nu} = \frac{RR'}{p} = p' + \cot H = \cot J \sec P,$$

then what has been called the *coefficient of non-sphericity* (comp. 395, (14.) and (16.)) is easily seen to have by XIV. the values,

$$\text{XXXIV.} \ldots S - 1 = \frac{S\tau^3\tau''\tau'''}{S\tau\tau'^3\tau''} - 1 = -r^4\text{r}S\nu'\tau''' - 1 \qquad (1, 2)$$

$$= \frac{r}{\text{r}}(p' - r\text{r}\lambda^2) - 1 = \frac{r}{\text{r}}\left(p' + \frac{r}{\text{r}}\right) = nr\text{r}^{-1} \qquad (3, 4, 5)$$

$$= \frac{\sigma'}{\text{r}\nu} = \cot H \cot J \sec P = \frac{rRR'}{p\text{r}}; \qquad (6, 7, 8)$$

whence also the *deviation* of a near point P_s of the curve, from the osculating sphere at P, is ultimately (by 395, XXVII.).

$$\text{XXXV.} \ldots \overline{\text{SP}_s} - \overline{\text{SP}} = \frac{(S-1)s^4}{24r^2R} = \frac{ns^4}{24r\text{r}R} = \frac{R's^4}{24r\text{r}p};$$

and accordingly, the square of the vector $\rho_s - \sigma$ is given now (comp. I.) by the expression,

$$(\rho_s - \sigma)^2 = (\rho - \sigma)^2 - \frac{s^4}{12r^2}\{r^2S(\sigma - \rho)\tau''' - 1\},$$

in which

$$r^2S(\sigma - \rho)\tau''' = S = 1 + nr\text{r}^{-1} = \&c., \text{ as above.}$$

(23.) The same auxiliary scalar n enters into the following expressions for the *arc*, and for the *scalar radii* of the *first* and *second curvatures*, of the *locus of the centre* S of the osculating *sphere*, or of the *cusp-edge* of the *polar developable* (comp. 391, (6.), and 395, (2.)) :

XXXVI. . . $\pm\int n ds$ = *Arc of that Cusp-Edge* (or of *locus of* S) ;

XXXVI′. . . $r_1 = n\text{r} = r + p'\text{r} = \dfrac{RR'}{r'}$ = (*Scalar*) *Radius of Curvature of same edge* ;

XXXVI″. . . $\text{r}_1 = nr = \sigma'\nu^{-1}$ = (*Scalar*) *Radius of Second Curvature of same curve* ;

these two latter being here called *scalar* radii, because the *first* as well as the *second* (comp. 397, V.) is conceived to have an algebraic *sign*. In fact, if we denote by κ_1 the *centre of the osculating circle* to the *cusp-edge* in question, its *vector* is (by the general formula 389, IV.),

$$\text{XXXVII.} \ldots \text{OK}_1 = \kappa_1 = \sigma + \frac{\sigma'^3}{V\sigma''\sigma'} = \sigma - nr\mathrm{r}\tau' = \rho - p'\mathrm{r}r\tau' + pr\nu = \sigma - r_1 r\tau',$$

with the signification XXXVI'. of r_1; because by XXXIII. (comp. 397, XI'.),

$$\text{XXXVIII.} \ldots \sigma' = nr\nu, \quad \sigma'' = n'r\nu + n(r\nu)' = n'r\nu - nr\mathrm{r}^{-1}\tau',$$

and therefore

$$\text{XXXIX.} \ldots \sigma'^2 = -n^2, \quad V\sigma'\sigma'' = n^2\mathrm{r}^{-1}\tau.$$

We may also observe that the relation $\sigma' \parallel \nu$ gives (by 397, IV.),

$$\text{XL.} \ldots V\frac{\sigma''}{\sigma'} = V\frac{\nu'}{\nu} = \mathrm{r}^{-1}\tau = \textit{Vector of Second Curvature of given curve};$$

and that we have the equation,

$$\text{XLI.} \ldots \frac{\text{K}_1\text{S}}{\text{PK}} = \frac{\sigma - \kappa_1}{\kappa - \rho} = \frac{r_1}{r}, \quad \text{with} \quad r > 0, \quad \text{but} \quad r_1 > \text{or} < 0,$$

according as the *cusp-edge* turns its *concavity* or its *convexity* towards the *given curve* at P.

(24.) The *radius* of (*first*) *curvature* of that cusp-edge, when regarded as a *positive* quantity, is therefore represented by the *tensor*,

$$\text{XLII.} \ldots \sqrt{r_1^2} = \pm r_1 = \text{T}r_1 = R\text{T}\frac{R'}{r'} = \pm\frac{R\text{d}R}{\text{d}r}(> 0);$$

and as regards the *scalar radius* XXXVI''. *of second curvature* of the same cusp-edge, its expression follows by XXXVIII. from the general formula 397, XXVII., which gives here,

$$\text{XLIII.} \ldots \mathrm{r}_1^{-1} = S\frac{\sigma'''}{V\sigma'\sigma''} = \frac{1}{nr}S\frac{\nu''}{V\nu\nu'} = n^{-1}r^{-1}, \text{ because XLIII'.} \ldots S\frac{\nu''}{V\nu\nu'} = 1;$$

the *two scalar derivatives*, n' and n'', which would have introduced the derived vectors τ^{IV} and τ^{V}, or $D_s^5\rho$ and $D_s^6\rho$, of the *fifth* and *sixth orders*, thus *disappearing from the expressions of the two curvatures* of the *locus of the centre* S of the *osculating sphere*, as was to be expected from geometrical* considerations.

(25.) For the *helix*, the formula XXXVII. gives $\kappa_1 = \rho$, or $K_1 = P$; we have then thus, as a verification, the known result, that the *given point* P of *this* curve is *itself* the *centre of curvature* K_1 of that *other helix* (comp. 389, (3.), and 395, (8.)), which is in this case the *common locus* of the *two coincident centres*, K and S. It is scarcely necessary to observe that for the helix we have also $J = H$.

(26.) In general, the *rectifying plane* of the *locus of* S is *parallel* to the rectifying plane of the *given curve*, because the *radii* of their *osculating circles* are parallel; the *rectifying lines* for these two curves are therefore not only *parallel* but *equal*; and accordingly we have here the formula,

$$\text{XLIV.} \ldots \lambda_1 = \mathrm{V}\frac{\tau_1''}{\tau_1'} = \mathrm{V}\frac{\tau''}{\tau'} = \lambda \text{ (by 397, XVI.)},$$

which will be found to agree with this other expression (comp. 397, XVII.),

$$\text{XLV.} \ldots \tan H_1 = \frac{\mathrm{r}_1}{\mathrm{T}r_1} = \frac{r}{\mathrm{r}}\mathrm{U}r_1 = \pm \cot H,$$

the upper or lower sign being taken, according as the *new curve* is *concave* (as in figs. 81, 82 [pp. 100, 106]), or is *convex* at S (comp. (23.)), towards the *old* (or given) curve at P: and the *new angle* H_1 being measured in the *new rectifying plane*, *from* the *new tangent* σ' or $nr\nu$, *to* the *new rectifying line* λ_1, and in the *direction* from that new tangent to the *new binormal* ν_1, or (comp. XL.) to a line from S which is equal to the *vector of second curvature* $\mathrm{r}^{-1}\tau$ of the *given curve*, multiplied by a *positive scalar*, namely by $\mathrm{T}n^{-1}$, or by the coefficient n^{-1} taken positively.

(27.) The *former rectifying line* λ *touches* the *cusp-edge* of the *rectifying developable* (396) of the given curve, in a *new point* R (comp. fig. 81), of which by 397, (45.), and by XV., the vector *from the given point* is, generally,

$$\text{XLVI.} \ldots \mathrm{PR} = -\frac{\mathrm{V}\tau'^3\tau''}{\mathrm{S}\tau'\tau''\tau'''} = \frac{r^{-2}\lambda}{\mathrm{S}\lambda\tau'''} = -\frac{r\lambda}{(r\mathrm{r}^{-1})'} = \frac{\mathrm{U}\lambda \sin H}{H'};$$

* In fact, n represents here the velocity of motion of the point S along its own locus, while r^{-1} and r^{-1} represent respectively the velocities of rotation of the tangent and binormal to that curve: so that $n\mathrm{r}$ and nr must be, as above, the radii of its two curvatures.

with the verification that this expression becomes infinite (comp. 397, (49.), (50.)), when the *curve* is a *geodetic on the cylinder*.

(28.) In general, the vector OR of the *point of contact* R, which vector we shall here denote by υ, may be thus expressed,

$$\text{XLVII.} \ldots \upsilon = \text{OR} = \rho + l\text{U}\lambda, \quad \text{if} \quad \text{XLVIII.} \ldots l = \frac{\sin H}{H'} = \frac{-r\text{T}\lambda}{(r\text{r}^{-1})'};$$

and because $(r\lambda)' = (r\text{r}^{-1})'\tau$, by VII′., its first derivative is,

$$\text{XLIX.} \ldots \upsilon' = r\lambda\left(\frac{\upsilon - \rho}{r\lambda}\right)' = \text{U}\lambda \operatorname{cosec} H(l \sin H)' = \text{U}\lambda(l' + \cos H);$$

in which however the new derived scalar l' involves H'', and so depends on τ^{IV}: while the *scalar coefficient* l itself represents the *portion* ($\pm\ \overline{\text{PR}}$) of *the rectifying line*, intercepted *between* the *given curve*, and the *cusp-edge* (27.) of the rectifying developable, and considered as *positive* when the *direction* of this *intercept* PR coincides with that of the line $+\lambda$, but as negative in the contrary case.

(29.) For abridgment of discourse, the cusp-edge *last* considered, namely that of the *rectifying developable*, as being the locus of a point which we have denoted by the letter R, may be called simply "*the curve* (R)"; while the *former* cusp-edge (23.), or that of the *polar* developable, may be called in like manner "*the curve* (S)"; the *locus* of the *centre* K of (absolute) *curvature* may be called "*the curve* (K)": and the *given curve* itself (comp. again figs. 81, 82) may be called, on the same plan, "*the curve* (P.)."

(30.) The *arc* RR_s, of the *curve* (R), is (by XLIX., comp. XXXVI.),

$$\text{L.} \ldots \pm\int_0^s \text{T}\upsilon' ds = l_s - l + \int_0^s \cos H ds;$$

this arc being treated as *positive*, when the direction of *motion* along it coincides with that of $+\lambda$.

(31.) The expression VII. for λ', combined with the former expression 397, XVI. for λ, gives easily by the general formula 389, IV.,

LI. . . *Vector of Centre of Curvature of the Curve* (R)

$$= \upsilon + \frac{\upsilon'}{\text{V}\upsilon''\upsilon'^{-1}} = \upsilon + \frac{\upsilon'}{\text{V}\lambda'\lambda^{-1}} = \upsilon + \frac{\upsilon'}{H'}\text{U}\tau';$$

whence

$$\text{LII.} \ldots \textit{Radius of Curvature of Curve } (\text{R}) = \mathrm{T}\frac{\upsilon'}{H'} = \mathrm{T}\frac{\mathrm{d}\upsilon}{\mathrm{d}H'},$$

the scalar variable being here arbitrary.

(32.) We see, at the same time, that the *angular velocity of the rectifying line* λ, or of the *tangent* to this *curve* (R), is represented by $\pm H'$; or that the *small angle** between *two such near lines*, λ and λ_s, is nearly equal to sH', or to $H_s - H$: while the *vector axis* $(\mathrm{V}\lambda'\lambda^{-1})$ *of rotation of the rectifying line*, set off from the point R, has $-H'\mathrm{U}\tau'$, or $-H'r\tau'$, for its expression.

(33.) As regards the *second curvature* of the same curve (R), we may observe that the expression (comp. VII. and LI.),

$$\text{LIII.} \ldots \lambda'' = (\mathrm{r}^{-1})''\tau + (r^{-1})''r\nu + r^{-1}(r\mathrm{r}^{-1})'\tau' = (\mathrm{r}^{-1})''\tau + (r^{-1})''r\nu + \mathrm{V}\lambda\lambda',$$

combined with the parallelism (XLIX.) of υ' to λ, gives, by the general formula 397, XXVII.,

LIV. . . *Radius of Second Curvature of Curve* (R)

$$= \left(\mathrm{S}\frac{\upsilon'''}{\mathrm{V}\upsilon'\upsilon''}\right)^{-1} = \frac{\upsilon'}{\lambda}\left(\mathrm{S}\frac{\lambda''}{\mathrm{V}\lambda\lambda'}\right)^{-1} = \frac{\upsilon'}{\lambda} = \frac{l' + \cos H}{\mathrm{T}\lambda};$$

with the verification, that while $l' + \cos H$ represents, by (30.), the velocity of *motion along this curve* (R), $\mathrm{T}\lambda$ represents, by 397, (3.), the velocity of *rotation of its osculating plane*, namely the *rectifying plane* of the *given curve* (P): and it is worth observing, that although *each* of these *two radii* of curvature, LII. and LIV., depends on τ^{IV} through l' (28.), yet *neither* of them depends on τ^{V} (comp. (24.)). As another verification, it can be shown that the *plane of the two lines* λ and τ' from P, namely the plane,

$$\text{LIV}'. \ldots \mathrm{S}\tau'\lambda(\omega - \rho) = 0,$$

which is the *normal plane to the rectifying developable* along the rectifying line, and contains the absolute normal to the given curve (P), *touches its own developable envelope along the line* RH, if H be the point determined by

* A result substantially equivalent to this is deduced, by an entirely different analysis, in the above cited Memoir of M. de Saint-Venant, and is illustrated by geometrical considerations: which also lead to expressions for the two curvatures (or, as he calls them, the *courbure* and *cambrure*), of the cusp-edge of the rectifying developable; and to a determination of the *rectifying line* of that *cusp-edge*.

the formula 397, XXXVIII., or the *point of nearest approach* of a *radius of curvature* (r) of that *given* curve *to its consecutive* (comp. (6.)); this line RH must therefore be the *rectifying line of the curve* (R); and accordingly (comp. 397, XVII.), the trigonometric tangent of its *inclination* to the *tangent* RP to this last curve has for expression (abstracting from sign),

$$\text{LIV}''. \ldots \tan \text{PRH} = \overline{\text{PH}} : \overline{\text{PR}} = \pm\, l^{-1} r \sin^2 H = \pm\, rH' \sin H = T\lambda^{-1} H'$$

$$= \frac{\textit{Radius (LIV.) of Second Curvature of Curve (R)}}{\textit{Radius (LII.) of First Curvature of same Curve}}.$$

(34.) Without even introducing τ^{IV}, we can assign as follows a *twisted cubic* (comp. 397, (34.)), which shall have *contact of the fourth order* with the given curve at P; or rather an *indefinite variety* of *such cubics*, or *gauche curves* of the *third degree*. Writing, for abridgment,

$$\text{LV.} \ldots x = -\,S\tau(\omega - \rho), \quad y = -\,Sr\tau'(\omega - \rho), \quad z = -\,Sr\nu(\omega - \rho),$$

so that

$$\text{LVI.} \ldots \omega = \rho + x\tau + yr\tau' + zr\nu,$$

the scalar equation,

$$\text{LVII.} \ldots \left(\frac{2\text{r}y}{r}\right)^2 = 6\left(\frac{\text{r}}{r}\right)^3 xz + \left(\frac{\text{r}^3}{r^2}\right)' yz + ez^2,$$

in which e is an arbitrary but scalar constant, represents evidently, by its *form*, a *cone of the second order*, with its *vertex* at the given point P; and *this cone* can be proved to have *contact of the fourth order* with the *curve** at that point: or of the *third order* with the *cone of chords* from it (comp. 397, (31.), (32.)). In fact the coefficients will be found to have been so determined, that the difference of the two members of this equation LVII. contains s^6 as a factor, when we change ω to ρ_s, as given by the formula I., or when we substitute for xyz their approximate values for the curve, as functions of the

* In the language of infinitesimals, the cone LVII. contains *five consecutive points* of the *curve*, or has *five-point contact* therewith: but it contains only *four consecutive sides* of the *cone of chords* from the given point, or has only *four-side* contact with *that cone*, except for one particular *value* of the *constant*, e, which we shall presently assign. It may be observed that xyz form *here* a (scalar) system of *three rectangular coordinates*, of the usual kind, with their *origin* at the point P of the curve, and with their *positive semiaxes* in the directions of the *tangent* τ, the *vector of curvature* τ', and the *binormal* ν.

arc s; namely, by the expressions IV. for τ''', and 397, VI. for τ'',

$$\text{LVIII.} \ldots \begin{cases} x_s = s - \dfrac{s^3}{6r^2} + \dfrac{r's^4}{8r^3}, \\ y_s = \dfrac{s^2}{2r} - \dfrac{r's^3}{6r^2} - \dfrac{s^4}{24}((r^{-2}r')' + r^{-3} + r^{-1}\mathrm{r}^{-2}), \\ z_s = \dfrac{s^3}{6r\mathrm{r}} + \dfrac{rs^4}{24}(r^{-2}\mathrm{r}^{-1})'; \end{cases}$$

where the terms set down are more than sufficient for the purpose of the proof. It may be added that the coefficient of $\frac{-s^4}{24}$ in y_s, which is the only one at all complex here, may be transformed as follows:

$$\text{LVIII}'. \ldots \mathrm{S}r\tau'\tau''' = -(r^{-1})'' - r^{-1}\lambda^2 = r^{-3}S + p(r^{-2}\mathrm{r}^{-1})';$$

S being that scalar for which (or more immediately for its excess over unity) several expressions* have lately been assigned (22.), and which had occurred in an earlier investigation (395, (14.), &c.).

(35.) With the same significations LV. of the three scalars xyz, this other equation,

$$\text{LIX.} \ldots 18ry - (3x - r'y)^2 = (9 + r'^2 - 3rr'' - 3r^2\mathrm{r}^{-2})y^2,$$

or

$$\text{LIX}'. \ldots 2ry - (x - \tfrac{1}{3}r'y)^2 = (1 - \tfrac{1}{2}r^{\frac{4}{3}}(r^{\frac{2}{3}})'' - \tfrac{1}{3}r^2\mathrm{r}^{-2})y^2,$$

will be found to be satisfied when we substitute for x and y the values LVIII. of x_s and y_s, and neglect or suppress s^5; it therefore represents an *elliptic* (or *hyperbolic*) *cylinder*, which is *cut perpendicularly*, by the *osculating plane* to the given curve at P, in an *ellipse* (or *hyperbola*), having *contact* of the *fourth order* with the *projection* (397, (9.)) of that given *curve* upon that osculating *plane*: and the *cylinder itself* has *contact* of the *same* (*fourth*) *order* with the *curve in space*, at the same given point P, so that we may call it (comp. 397, (31.)) the *Osculating Elliptic* (or *Hyperbolic*) *Cylinder*, *perpendicular to the osculating plane*.

(36.) As a verification, if we suppress the second member of either LIX. or LIX'., we obtain, under a new form, the equation of what has

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* It might have been observed, in addition to the eight forms XXXIV., that we have also,

$$\text{XXXIV}'. \ldots S - 1 = R\mathrm{r}^{-1}\cot J = n \cot H. \qquad (9, 10)$$

been already called the *Osculating Parabolic Cylinder* (397, LXXXIV.); and as another verification, the *coefficient of* y^2 in that second member *vanishes*, as it ought to do, when the given curve is supposed to be a *parabola*: *that plane curve*, in fact, satisfying the *differential equation of the second order*,

$$\text{LX.} \ldots 3rr'' - r'^2 = 9, \quad \text{or} \quad \text{LX}'. \ldots r^{\frac{4}{3}}(r^{\frac{2}{3}})'' = 2,$$

or

$$\text{LX}''. \ldots r^{-\frac{2}{3}}\left(\left(\frac{dr}{3ds}\right)^2 + 1\right) = \text{const.} = p^{-\frac{2}{3}},$$

if r be still the *radius of curvature*, considered as a function of the *arc*, s, while p is here the *semiparameter*.

(37.) The *binormal* ν is, by the construction, a *generating line* of the cylinder LIX.; and although this line is *not generally* a *side* of the *cone* LVII., yet we can *make* it such, by assigning the particular value *zero* to the *arbitrary constant*, e, in its equation, or by suppressing the *term*, ez^2. And when this is done, the cone LVII. will *intersect* the *cylinder* LIX., not only in this *common side* ν (comp. 397, (33.)), but also in a certain *twisted cubic*, which will have *contact* of the *fourth order* with the *given curve* at P, as stated at the commencement of (34.).

(38.) But, as was also stated there, *indefinitely many* such *cubics* can be described, which shall have contact of the *same* (*fourth*) *order*, with the *same curve*, at the *same point*. For we may *assume any point* E of space, or *any vector* (comp. LVI.),

$$\text{LXI.} \ldots \text{OE} = \epsilon = \rho + a\tau + br\tau' + cr\nu,$$

in which a, b, c are *any three scalar constants*; and then the vector equation,

$$\text{LXII.} \ldots \omega = \rho_s + t(\epsilon - \rho),$$

in which t is a *new scalar variable*, will represent a *cylindric surface*, not *generally* of the *second order*, but passing *through the given curve*, and having the *line* PE for a *generatrix*. We can then *cut* (generally) this new cylinder by the *osculating plane* to the curve at P, and so obtain (generally) a *new* and *oblique projection* of the *curve* upon that *plane*; the x and y of which *new projected curve* will depend on the arc s of the original curve by the relations,

$$\text{LXIII.} \ldots x = x_s - ac^{-1}z_s, \quad y = y_s - bc^{-1}z_s;$$

with the approximate expressions LVIII. for $x_s y_s z_s$. And if we then determine *two new scalar constants*, B and C, by the condition that the substitution of these last expressions LXIII. for x and y shall satisfy this new equation,

$$\text{LXIV.} \ldots 2ry = x^2 + 2Bxy + Cy^2,$$

if only s^5 be neglected (comp. (35.)), or by *equating the coefficients of* s^3 *and* s^4, in the result of such substitution, then, on *restoring* the significations LV. of xyz, and writing for abridgment,

$$\text{LXV.} \ldots X = x - ac^{-1}z, \quad Y = y - bc^{-1}z,$$

the *equation of the second degree*,

$$\text{LXVI.} \ldots 2rY = X^2 + 2BXY + CY^2,$$

will represent generally an *oblique osculating elliptic* (or hyperbolic) *cylinder*, which has contact of the *fourth order* with the given curve at P, and contains the assumed line PE. If then we determine finally the constant e in LVII., by the result of the substitution of abc for xyz, or by the condition,

$$\text{LXVII.} \ldots \left(\frac{2\mathrm{r}b}{r}\right)^2 = 6\left(\frac{\mathrm{r}}{r}\right)^3 ac + \left(\frac{\mathrm{r}^3}{r^2}\right)' bc + ec^2,$$

the *cone* LVII., and the *cylinder* LXVI., will have that line PE for a *common side*; and will *intersect* each other, not only in *that line*, but *also* (as before) in a *twisted cubic*, although now a *new* one, which will have the *required* (*fourth*) *order of contact*, with the given curve at the given point.

(39.) If, after the substitution (38.) in LXIV., we equate the coefficients of the *three powers*, s^3, s^4, s^5, and then eliminate B and C, we are conducted to an *equation of condition*, which is found to be of the *form*,

$$\text{LXVIII.} \ldots \mathrm{a}b^3 + \mathrm{b}b^2c + \mathrm{c}bc^2 + \mathrm{e}c^3 = ac(b\mathrm{g} + c\mathrm{h});$$

in which the ratios of abc still serve to determine the direction of the generating line PE, while the coefficients, a, b, c, e, g, h are *assignable functions* of r, r, r', r$'$, r'', r$''$, and r''', depending on the vector r^{IV}: and when this *condition* LXVIII. is satisfied, the *cylinder* LXVI. has *contact* of the *fifth order* with the given *curve* at P.

(40.) Again, if we *improve* the approximate expressions LVIII. for the three scalars x_s, y_s, z_s, by taking account of s^5, or by introducing the *new term* $\frac{s^5\rho^{\text{IV}}}{120}$ (comp. I.) of ρ_s, and if we substitute the expressions so improved, instead of x, y, z, in the equation of the *cone* LVII. and then equate to zero (comp. (34.)) the coefficient of s^6 in the difference of the two members of that equation, we obtain a *definite expression* for the *constant*, e, which had been *arbitrary* before, but becomes now a *given function* of $r\mathrm{r}r'\mathrm{r}'r''$ and r'' (*not involving* r'''), namely the following,

$$\text{LXIX.} \ldots e = \frac{\mathrm{r}^4}{5}\left(\frac{9}{r^4} - \frac{21}{r^2\mathrm{r}^2} + \frac{r'^2}{r^4} - \frac{3r''}{r^3} + \frac{3r'\mathrm{r}'}{r^3\mathrm{r}} - \frac{27\mathrm{r}'^2}{4r^2\mathrm{r}^2} + \frac{9\mathrm{r}''}{r^2\mathrm{r}}\right);$$

and when the constant e receives *this value*,* the *cone* has contact of the *fifth order* with the *curve* at the given point.

(41.) Finally, if we multiply the equation LXVII. by $bg + ch$, we can at once eliminate a by LXVIII., and so obtain a *cubic equation* in $b : c$, which has *at least one real root*, answering to a *real system of ratios* a, b, c, and therefore to a *real direction* of the *line* PE in (38.). It is therefore possible to assign *at least one real cylinder* of the second order (39.), which shall have *contact* of the *fifth order* with the curve at P, and shall at the same time have *one side* PE *common* with the *cone* of the second order (40.), which has contact of the *same* (*fifth*) *order* with the *curve* (or of the *fourth* order with the *cone of chords*): and consequently it is possible in this way to assign, as the *intersection* of this cylinder with this cone, at least *one real twisted cubic*, which has *contact of the fifth*† *order with the given curve of double curvature*, at the given point thereof. And *such* a cubic curve may be called, by eminence, an *Osculating*‡ *Twisted Cubic*.

(42.) Not intending to return, in these *Elements*, on the subject of such *cubic curves*, we may take this occasion to remark, that the very simple *vector equation*,§

$$\text{LXX.} \ldots \mathrm{V}\alpha\rho = \rho\mathrm{V}\beta\rho,$$

represents a *curve of this kind*, if α and β be any two constant and non-parallel

* Compare the Note to page 125.

† Accordingly it is known (see page 242 of Dr. Salmon's Treatise, already cited), that a *twisted cubic* can generally be described through any *six given points*; and also (page 248), that *three quadric cylinders* (or cylinders of the second order or degree) can be described, containing a *given cubic curve*, their *edges* being parallel to the *three* (real or imaginary) *asymptotes*.

‡ Compare the first Note to page 92.

§ This example was given in pages 679, &c., of the *Lectures*, with some connected transformations, the equation having been found as a certain *condition* for the *inscription of a gauche quadrilateral*, or vectors. In fact, if we operate on this equation by the symbol S . λ, in which λ is an *arbitrary* but constant vector, the *scalar equation* so obtained, namely,

$$\text{LXXI.} \ldots \text{S}\lambda\alpha\rho = \text{S}\lambda\rho\text{S}\beta\rho - \rho^2\text{S}\beta\lambda,$$

represents a *surface of the second order*, on which the *curve* is wholly contained; making then successively $\lambda = \alpha$ and $\lambda = \beta$, we get, in particular, the *two* equations,

$$\text{LXXII.} \ldots \text{S}(\text{V}\alpha\rho . \text{V}\beta\rho) = 0, \quad \text{and} \quad \text{LXXIII.} \ldots (\text{V}\beta\rho)^2 + \text{S}\alpha\beta\rho = 0,$$

representing respectively a *cone* and *cylinder* of that order, with the vector β from the origin as a *common side*: and the *remaining part* of the *intersection* of these two surfaces, is precisely the *curve* LXX., which therefore is a *twisted cubic*, in the known sense already referred to.

(43.) *Other surfaces* of the same order, containing the same curve, would be obtained by assigning other values to λ; for example (comp. 397, (31.)), we should get generally an *hyperbolic paraboloid* from the form LXXI., by taking $\lambda \perp \beta$. But it may be more important here to observe, that *without* supposing *any acquaintance* with the theory of *curved surfaces*, the *vector equation* LXX. can be shown, *by quaternions*, to represent a *curve of the third degree*, in the sense that it is *cut*, by an *arbitrary plane*, in *three points* (real or imaginary). In fact, we may write the equation as follows,

$$\text{LXXIV.} \ldots \text{V}q\rho = -\alpha, \quad \text{if} \quad \text{LXXV.} \ldots q = g + \beta,$$

q being here a *quaternion*, of which the *vector part* β is *given*, but the *scalar part* g is *arbitrary*; and then, by *resolving* (comp. 347) this *linear equation* LXXIV., we may still further transform it as follows,

$$\text{LXXVI.} \ldots g(g^2 - \beta^2)\rho = \beta\text{S}\beta\alpha + g\text{V}\beta\alpha - g^2\alpha,$$

which conducts to a *cubic equation* in g, when combined with the equation

$$\text{LXXVII.} \ldots \text{S}\varepsilon\rho = e,$$

of any proposed *secant plane*.

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other *even-sided polygon*, in a *given spheric surface* (comp. the sub-articles to 296): the $2n$ successive *sides* of the figure being obliged to pass through the same *even number* of *given points of space*. It was shown that the *curve* might be said to *intersect* the *unit-sphere* ($\rho^2 = -1$) in *two imaginary points at infinity*, and *also* in *two real* and *two imaginary points*, situated on *two real right lines*, which were *reciprocal polars* relatively to the sphere, and might be called *chords of solution*, with respect to the proposed *problem of inscription* of the polygon; and that *analogous results* existed for *even-sided polygons in ellipsoids*, and *other surfaces* of the *second order*: whereas the corresponding problem, of the inscription of an *odd-sided polygon* in such a *surface*, conducted only to the assignment of a *single chord of solution*, as happens in the known and analogous theory of *polygons in conics*, whether the number of sides be (in *that* theory) *even* or *odd*. But we cannot here pursue the subject, which has been treated at some length in the *Lectures*, and in the *Appendices* to them.

(44.) The vector equation LXX., however, is *not sufficiently general*, to represent an *arbitrary twisted cubic*, through an *assumed point* taken as *origin*; for which purpose, *ten scalar constants* ought to be disposable, in order to allow of the curve being made to pass through *five* other arbitrary points*: whereas the equation referred to involves *only five* such *constants*, namely the *four* included in $U\alpha$ and $U\beta$, and the *one* quotient of tensors $T\beta : T\alpha$ (comp. 358).

(45.) It is easy, however, to accomplish the *generalization* thus required, with the help of that theory of *linear and vector functions* $(\phi\rho)$ *of vectors*, which was assigned in the Sixth Section of the preceding Chapter (Arts. 347, &c.). We have only to write, instead of the equation LXX., this other but analogous *form* which *includes* it,

$$\text{LXXVIII.} \ldots V\alpha\rho + V\rho\phi\rho = 0, \quad \text{or} \quad \text{LXXVIII}'. \ldots \phi\rho + c\rho = \alpha,$$

and which gives, by principles and methods already explained (comp. 354, (1.)), the transformation

$$\text{LXXIX.} \ldots \rho = (\phi + c)^{-1}\alpha = \frac{\psi\alpha + c\chi\alpha + c^2\alpha}{m + m'c + m''c^2 + c^3};$$

α, $\psi\alpha$, and $\chi\alpha$ being here *fixed vectors*, and m, m', m'' being *fixed scalars*, but c being an *arbitrary* and *variable scalar*, which may receive *any value*, without the expression LXXIX. ceasing to satisfy the equation LXXVIII.

(46.) The *curve* LXXVIII. is therefore *cut* (comp. (43.)) by the *plane* LXXVII. in *three points* (real or imaginary), answering to and determined by the *three roots* of the *cubic in c*, which is formed by substituting the expression LXXIX. for ρ in the equation of that secant plane; and consequently it is a curve of the *third degree*, the *three* (real or imaginary) *asymptotes* to which have directions corresponding to the *three values* of c, obtained by equating to zero the *denominator* of that expression LXXIX., or by making $M = 0$, in a notation formerly employed: so that they have the directions of the *three lines* β, which satisfy this *other* vector equation (comp. 354, I.),

$$\text{LXXX.} \ldots V\beta\phi\beta = 0.$$

* Compare the second Note to page 129. In general, when a *curve in space* is supposed to be represented (comp. 371, (5.)) by *two scalar equations*, each *new arbitrary point*, through which it is required to *pass*, introduces a necessity for *two new disposable constants*, of the *scalar* kind: and accordingly *each new order*, say the n^{th}, of *contact* with such a *curve*, has been seen to introduce a *new vector*, $D_s{}^n\rho$, or $\tau^{(n-1)}$, subject to a *condition* resulting from the general equation $TD_s\rho = 1$, or $\tau^2 = -1$ (comp. 380, XXVI., and 396, III.), but involving virtually *two new scalar constants*. Thus, besides the *four* such constants, which enter through τ and τ' into the determination of the *directions* of the *rectangular system* of lines, *tangent*, *normal*, and *binormal* (comp. 379, (5.), or 396, (2.)), and of the *length* of the *radius* of (*first*) *curvature*, r, the *three successive derivatives*, r', r'', r''', of *that* radius, and the radius r of *second curvature*, with its *two* first derivatives, r′ and r″, have been seen to enter, through the *three other vectors*, τ'', τ''', τ^{IV}, into the determination (41.) of the *osculating twisted cubic*.

(47.) Accordingly, if β be *such* a line, and if γ be *any vector* in the *plane* of α and β, the curve LXXVIII. is a *part* of the *intersection* of the *two surfaces* of the *second order*,

$$\text{LXXXI.} \ldots S\alpha\rho\phi\rho = 0, \quad \text{and} \quad \text{LXXXII.} \ldots S\gamma\alpha\rho + S\gamma\rho\phi\rho = 0,$$

whereof the first is a cone, and which have the *line* β from the origin for a *common side* (comp. (42.)): the *curve* is therefore found *anew* to be a *twisted cubic.*

(48.) And as regards the *number* of the *scalar constants*, which are to be conceived as entering into its *vector equation* LXXVIII., when we take for $\phi\rho$ the *form* $Vq_0\rho + V\lambda\rho\mu$ assigned in 357, I., in which q_0 is an arbitrary but *constant quaternion*, such as $g + \gamma$, and λ, μ are *constant vectors*, the *term* $g\rho$ of $\phi\rho$ *disappears* under the symbol of operation $V . \rho$, and the equation (45.) of the curve becomes,

$$\text{LXXXIII.} \ldots V\alpha\rho + \rho V\gamma\rho + V\rho V\lambda\rho\mu = 0;$$

in which the *four versors*, $U\alpha$, $U\gamma$, $U\lambda$, $U\mu$, introduce each *two* scalar constants, while the *two tensor quotients*, $T\gamma : T\alpha$ and $T\lambda\mu : T\alpha$, count as *two others*: so that the required number of *ten* such *constants* (44.) is exactly made up, the *curve* being still supposed to pass through an assumed *origin*, and therefore to have *one point given*. It is scarcely worth observing, that we can at once *remove* this last *restriction*, by merely *adding* a *new constant vector* to ρ, in the last equation, LXXXIII.

(49.) Although, for the determination of the *osculating twisted cubic* (41.) to a given *curve of double curvature*, it was necessary (comp. (40)) to employ the vector τ^{IV} or $D_s^5\rho$, or to take account of s^5 in the vector ρ_s, or in the connected scalars $x_sy_sz_s$ of (34.), and therefore to *improve* the expressions LVIII., by carrying in each of them (or at least in the two latter), the approximation *one step farther*, yet there are many *other problems* relating to *curves in space*, besides some that have been already considered, for which those *scalar expressions* LVIII. are *sufficiently approximate*: or for which the *vector expression* I. suffices.

(50.) Resuming, for instance, the questions considered in (22.) and (23.), we may throw some additional light on the *law of the deviation* of a *near point* P_s of the *curve*, from the *osculating sphere* at P, as follows. Eliminating n by XXXVI′. from XXXV., we find this *new expression*,

$$\text{LXXXIV.} \ldots \overline{SP_s} - \overline{SP} = \frac{r_1 s^4}{24rr'^2R};$$

the *direction* of this *deviation* from the *sphere* (R) depends therefore on the *sign* of the *scalar radius* r_1 (23.) of *curvature of the cusp-edge* (s) of the *polar developable*: and it is *outward* or *inward* (comp. 395, (14.)), according as that cusp-edge turns its *concavity* (comp. XLI.) or its *convexity*, at the centre s of the osculating sphere, towards the point P of the *given curve*, that is, *towards the point of osculation.*

(51.) Again, if we only take account of s^3, the deviation of P_s *from the osculating circle* at P has been seen to be a vector *tangential to the osculating sphere*, which may be thus expressed (comp. 397, IX., LII.),

$$\text{LXXXV.} \ldots C_sP_s = \frac{s^3}{6}\nu'\tau = \frac{s^3\tau(\sigma-\rho)}{6r^2\text{r}},$$

if C_s be the point *on the circle*, which is distant from the given point P by an *arc of that circle* = s, with the *same initial direction* of motion, or of departure from P, represented by the *common unit tangent* τ; the *quantity* of *this* deviation is therefore expressed by the *scalar* $\frac{s^3R}{6r^2\text{r}}$: that is, by the deviation $\frac{s^3}{6r\text{r}}$ (comp. 397, (9.), (10.)) *from the osculating plane** at P, multiplied by the *secant* $(r^{-1}R)$ of the *inclination* (P) of the radius (R) of *spherical* curvature, to the radius (r) of *absolute curvature*, and *positive* when this *last* deviation has the direction of the *binormal* ν.

(52.) On the other hand (comp. (5.)) the *small angle*, which the *small arc* SS_s of the *cusp-edge* (s) of the *polar developable* subtends at the point P, is ultimately expressed by the *scalar*,

$$\text{LXXXVI.} \ldots SPS_s = (\overline{PS_s} - \overline{PS}) \,.\, R^{-1}\cot P = \frac{rR's}{pR} = \frac{nrs}{R^2} \text{ (by XXXIII.)},$$

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* Besides the nine expressions in 397, (42.) for the *square* r^{-2} of the *second curvature*, the following may be remarked, as containing the *law of the regression* of the *projection of a curve* of double curvature *on its own normal plane*:

$$\text{r}^{-2} = \frac{9}{2\text{KP}} \,.\, \lim. \frac{\text{PQ}_3{}^2}{\text{PQ}_2{}^3}, \qquad 397, \text{ XCIX., (10.)}$$

K being still the centre of the osculating circle, and Q_1, Q_2, Q_3 being still (as in 397, (10.)) the projections of a *near point* Q (or P_s), on the *tangent*, the absolute *normal* (or inward radius of curvature PK), and the *binormal* at P. In fact, the *principal terms* of the *three vector projections* corresponding, of the *small chord* PQ (or PP_s), are (comp. LVIII.):

$$\text{PQ}_1 = s\tau; \quad \text{PQ}_2 = (\tfrac{1}{2}s^2\tau' =) \frac{s^2}{2r}\text{U}\tau'; \quad \text{PQ}_3 = (\tfrac{1}{6}s^3\text{r}^{-1}\nu =) \frac{s^3}{6r\text{r}}\text{U}\nu;$$

whence, ultimately,

$$\frac{9}{2}\cdot\frac{\text{PQ}_3{}^2}{\text{PQ}_2{}^3} = -\text{r}^{-2}r\text{U}\tau' = \text{r}^{-2}\,.\,\text{KP}.$$

this *angle* being treated as *positive*, when the corresponding *rotation** round $+\tau$ from PS to PS$_s$ is positive: and if we multiply *this* scalar, by that which has just been assigned (51.), as an expression for the deviation C$_s$P$_s$ from the osculating *circle*, we get, by XXXV., the product,

$$\text{LXXXVII.} \ldots \frac{s^3R}{6r^2\mathrm{r}} \cdot \frac{rR's}{pR} = \frac{R's^4}{6r\mathrm{r}p} = 4\,(\overline{\mathrm{SP}_s} - \overline{\mathrm{SP}}).$$

(53.) Combining then the recent results (50.), (51.), (52.), we arrive at the following *Theorem*:

The deviation of a near point P$_s$ *of a curve in space, from the osculating sphere at the given point* P, *is ultimately equal to the quarter of the deviation of the same near point from the osculating circle at* P, *multiplied by the sine of the small angle which the arc* SS$_s$, *of the locus of centres of spherical curvature* (S), or of the *cusp-edge* of the *polar developable, subtends at the same point* P; and *this deviation* $(\overline{\mathrm{SP}_s} - \overline{\mathrm{SP}})$ *from the sphere has an outward or an inward direction, according as the same arc* SS$_s$ *is concave or convex towards the same given point.*

(54.) The vector of the centre S$_s$, of the *near osculating sphere* at P$_s$, is (in the same order of approximation, comp. I.),

$$\text{LXXXVIII.} \ldots \mathrm{OS}_s = \sigma_s = \sigma + s\sigma' + \tfrac{1}{2}s^2\sigma'' + \tfrac{1}{6}s^3\sigma''' + \tfrac{1}{24}s^4\sigma^{\mathrm{IV}};$$

and although $\sigma - \rho$ is already a function (by 397, IX., &c.) of τ, τ', τ'', so that σ' is (as in (2.) or (22.)) a function of τ', τ'', τ''', and $\sigma'', \sigma''', \sigma^{\mathrm{IV}}$ introduce respectively the new derived vectors $\tau^{\mathrm{IV}}, \tau^{\mathrm{V}}, \tau^{\mathrm{VI}}$, or $\mathrm{D}_s{}^5\rho$, $\mathrm{D}_s{}^6\rho$, $\mathrm{D}_s{}^7\rho$, which we are not at present employing (49.), yet we have seen, in (23.) and (24.), that some useful *combinations* of σ'', and σ''' can be expressed *without* $\tau^{\mathrm{IV}}, \tau^{\mathrm{V}}$: and the following is another remarkable example of the same species of *reduction*, involving not only σ'' and σ''' but also σ^{IV}, but still admitting, like the former, of a simple geometrical *interpretation*.

(55.) Remembering (comp. (22.), and 397, XV.) that

$$\text{LXXXIX.} \ldots (\sigma - \rho)^2 + R^2 = 0, \text{ and } \text{XC.} \ldots \mathrm{S}\tau'''(\sigma - \rho) = r^{-2}S = r^{-2} + nr^{-1}\mathrm{r}^{-1},$$

and reducing the successive derivatives of LXXXIX. with the help of the equations 397, XIX., and of their derivatives, we are conducted easily to the

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* Considered as a *rotation*, this small angle may be represented by the *small vector*, $rp^{-1}R'R^{-1}s\tau$; and if the *vector deviation* LXXXV. from the osculating *circle* be multiplied *by this*, the *quarter* of the *product is* (comp. XXXV.) the *vector deviation* from the osculating *sphere*, under the form,

$$\frac{s^4(\rho - \sigma)}{24R} \cdot \frac{R'}{r\mathrm{r}p}.$$

following system of equations, into which the derived vectors τ, τ', &c. do not expressly enter, but which involve σ', σ'', σ''', σ^{IV}, and R' R'', R''', R^{IV}:

$$\text{XCI.} \ldots \text{S}\sigma'(\sigma - \rho) + RR' = 0; \qquad \text{XCII.} \ldots \text{S}\sigma'\sigma''(\sigma - \rho) = 0;$$

$$\text{XCIII.} \ldots \text{S}\sigma''(\sigma - \rho) + \sigma'^2 + (RR')' = 0;$$

$$\text{XCIV.} \ldots \text{S}\sigma'''(\sigma - \rho) + 3\text{S}\sigma'\sigma'' + (RR')'' = 0;$$

$$\text{XCV.} \ldots \text{S}\sigma^{\text{IV}}(\sigma - \rho) + 4\text{S}\sigma'\sigma''' + 3\sigma''^2 + (RR')''' = -\frac{RR'}{r\text{r}p} = -\frac{n}{r\text{r}};$$

auxiliary equations being,

$$\text{XCVI.} \ldots \text{S}\sigma'\tau = 0, \quad \text{S}\sigma'\tau' = 0, \quad \text{S}\sigma''\tau = 0, \qquad \text{(comp. 395, X.)}$$

and
$$\text{XCVII.} \ldots \text{S}\sigma'''\tau = -\text{S}\sigma''\tau' = \text{S}\sigma'\tau'' = \text{S}\tau\tau'' - \text{S}(\sigma - \rho)\tau'''$$
$$= -r^{-2}(S - 1) = -nr^{-1}\text{r}^{-1}.$$

(56.) But, if R_s denote the radius of the near sphere, and if we still neglect s^5, we have,

$$\text{XCVIII.} \ldots \overline{\text{P}_s\text{S}_s}^2 = -(\sigma_s - \rho_s)^2 = R_s^2$$
$$= R^2 + 2sRR' + s^2(RR')' + \frac{s^3}{3}(RR')'' + \frac{s^4}{12}(RR')''';$$

whence follows, by LXXXVIII., and by the recent equations, this very simple expression, from which (comp. (24.)) *everything depending on* τ^{IV}, τ^{V}, τ^{VI} *has disappeared*,

$$\text{XCIX.} \ldots (\sigma_s - \rho)^2 + R_s^2 = \frac{-RR's^4}{12r\text{r}p};$$

and which gives (within the same order of approximation, attending to XXXV.) the *geometrical relation*,

$$\text{C.} \ldots \overline{\text{PS}_s} - \overline{\text{P}_s\text{S}_s} = \text{T}(\sigma_s - \rho) - R_s = \frac{R's^4}{24r\text{r}p} = \frac{ns^4}{24r\text{r}R} = \overline{\text{SP}_s} - \overline{\text{SP}};$$

or
$$\text{C}'. \ldots \overline{\text{S}_s\text{P}} - \overline{\text{SP}_s} = \overline{\text{S}_s\text{P}_s} - \overline{\text{SP}} = R_s - R.$$

(57.) This result might have been *foreseen*, from the following very simple consideration. When the *coefficient* $S - 1$ *of non-sphericity* (395, (16.)), or of the deviation of a curve from a sphere, is *positive*, so that a *near point* P_s of the *curve* is *exterior* to (what we may call) the *given sphere*, which *osculates* to that curve at P, by an amount which is ultimately proportional to the *fourth power* of the *arc*, s, of the curve, then the *given point* P must be, for the same reason, *exterior to the near sphere*, which osculates at the point P_s; and the *two* *deviations*, $\overline{PS_s} - \overline{P_sS_s}$ and $\overline{SP_s} - \overline{SP}$, which have been found by *calculation* to be *equal* (C.), if s^5 be neglected, must in fact bear to each other an *ultimate ratio of equality*, because the *two arcs*, $+s$ and $-s$, from P to P_s, and from P_s *back* to P, are *equally long*, although *oppositely directed*; or because $(+s)^4 = (-s)^4$. And precisely the same reasoning applies, when the coefficient $S - 1$ is *negative*, so that the *deviations*, equated in the formula C., are *both inwards*.

(58.) As regards the deviation (51.) of the near point P_s of the curve *from the osculating circle* at P, we may generalize and render more exact the expression LXXXV., by considering a point C_t of that circle, which is distant *by a circular arc* $= t$ from the given point P; and of which the vector is, *rigorously*, by 396, (18.),

$$\text{CI.} \ldots \text{OC}_t = \omega_t = \rho + r\tau \sin\frac{t}{r} + r^2\tau' \text{ vers}\frac{t}{r};$$

or if we only neglect t^5,

$$\text{CII.} \ldots \text{OC}_t = \omega_t = \rho + \tau\left(t - \frac{t^3}{6r^2}\right) + r\tau'\left(\frac{t^2}{2r} - \frac{t^4}{24r^3}\right).$$

(59.) In this way we shall have (comp. (34.)) the *vector deviation*,

$$\text{CIII.} \ldots \text{C}_t\text{P}_s = \rho_s - \omega_t = X\tau + Yr\tau' + Zr\nu,$$

with the *scalar coefficients*,

$$\text{CIV.} \ldots X = x_s - r\sin\frac{t}{r}, \quad Y = y_s - r \text{ vers}\frac{t}{r}, \quad Z = z_s;$$

or, neglecting s^5 and t^5, and attending to the expressions LVIII. and LVIII'.,

$$\text{CV.} \ldots \begin{cases} X = s - t - \dfrac{s^3 - t^3}{6r^2} + \dfrac{r's^4}{8r^3}; \\[2ex] Y = \dfrac{s^2 - t^2}{2r} - \dfrac{p}{r}Z - \dfrac{s^4 - t^4}{24r^3} - \dfrac{ns^4}{24r^2\text{r}}; \\[2ex] Z = \dfrac{s^3}{6r\text{r}} + \dfrac{rs^4}{24}(r^{-2}\text{r}^{-1})'; \end{cases}$$

in which r, r' r, p, and n have the same significations as before.

(60.) Assuming then for the *circular arc* t the value,

$$\text{CVI.} \ldots t = s + \frac{r's^4}{8r^3},$$

which differs (as we see) by only a quantity of the *fourth order* from the arc s of the *curve*, we shall have, to the same order of approximation, the expressions,

$$\text{CVII.} \ldots X = 0, \quad Y = \frac{-p}{r}Z - \frac{ns^4}{24r^2\text{r}}, \quad Z = z_s = \&\text{c., as before,}$$

the *deviation* at P_s from the *circle* being here measured in a direction *parallel to the normal plane* at P; and if s^4 be neglected (although the expressions enable us to take account of it), this deviation is also *parallel* (as before) to the *tangent* $\tau(\sigma - \rho)$ *to the osculating sphere* in that plane: while it is represented in quantity by $Rr^{-1}z_s$, which agrees with the result in (51).

(61.) The expressions CVII. give also, *without neglecting* s^4,

$$\text{CVIII.} \ldots \frac{rY + pZ}{R} = -\frac{ns^4}{24r\mathrm{T}R} = \overline{\mathrm{SP}} - \overline{\mathrm{SP}}_s;$$

such then is the *component* of the *deviation* from the osculating *circle*, which is *parallel to the normal* PS *to the sphere* at P; and we see that it only differs in *sign* (because it is *positive* when its *direction* is that of the *inward normal*, or *inward radius* PS), from the expression XXXV. (comp. C.), for the *outward deviation* $\overline{\mathrm{SP}}_s - \overline{\mathrm{SP}}$ of the near point P_s, from the same osculating *sphere* at the given point P.

(62.) This *latter* component (61.) is *small*, even as *compared* with the *former* small component (60.); and the *small quotient*, of the latter divided by the former, is ultimately (by LXXXVI.),

$$\text{CIX.} \ldots \frac{rY + pZ}{rZ - pY} = \frac{-nrs}{4R^2} = -\tfrac{1}{4}\mathrm{SPS}_s;$$

where the *small angle* SPS_s is positive or negative, according to the rule stated in (52.), and may be replaced by its sine, or by its tangent.

(63.) Instead of cutting the given osculating circle, as in (60.), by a plane which is *parallel to the given normal plane* at P, we may propose to *cut that circle by the near normal plane* at P_s, or to satisfy this *new condition*,

$$\text{CX.} \ldots 0 = S\tau_s(\rho_s - \omega_t), \quad \text{or} \quad \text{CX}'. \ldots 0 = XS\tau\tau_s + YSr\tau'\tau_s + ZSr\nu\tau_s;$$

which is easily found to give by CV. the values (s and t being still supposed to be small, and s^5 being still neglected):

$$\text{CXI.} \ldots t = s - \frac{r's^4}{24r^3}, \quad \text{and} \quad \text{CXII.} \ldots X = \frac{r's^4}{6r^3}, \; Y = \&\text{c.}, \; Z = \&\text{c.}, \text{ as in CVII.};$$

so that in passing to this *new near point* C_t of the *circle*, we only change X from *zero* to a small quantity of the *fourth order*, and make *no change* in the values of Y and Z.

(64.) The *new deviation* C_tP_s from the *given circle* may be decomposed into *two partial deviations*, in the *near normal plane*, of which *one* has the direction of the *unit-tangent* $R_s^{-1}\tau_s(\sigma_s - \rho_s)$ to the *near sphere* at P_s, and the other has that of the *unit-normal* $R_s^{-1}(\sigma_s - \rho_s)$ to the same sphere at the same point (or the opposites of these two directions); and the *scalar coefficients* of these *two vector units*, if we attend only to *principal terms*, are easily found to be,

$$\text{CXIII.} \ldots \frac{rZ - pY}{R} = \frac{Rs^3}{6r^2\text{r}}, \quad \text{and} \quad \text{CXIV.} \ldots \frac{rY + (p + ns)Z}{R} = \frac{ns^4}{8r\text{r}R}.$$

(65.) We may then write:

CXV. . . *Deviation of near point* P_s *from given osculating circle, measured in the near normal plane to the curve at* P_s,

$$= \text{new } C_tP_s = \frac{Rs^3}{6r^2\text{r}}\,\text{U}\tau_s(\sigma_s - \rho_s) + \frac{ns^4}{8r\text{r}R}\,\text{U}(\sigma_s - \rho_s);$$

in which it may be observed, that the *second scalar coefficient* is equal to *three times* the *scalar deviation* $\overline{SP_s} - \overline{SP}$ (XXXV. or C.), of the *near point* P_s of the *curve*, from the *given osculating sphere* (at P).

(66.) But we may also interpret the *new coefficient* last mentioned, as representing a *new deviation*; namely, that of the *point* C_t of the *given circle*, *from the near osculating sphere* at P_s, considered as *positive* when that *new point* C_t is *exterior* to that near sphere; or as denoting the *difference of distances* $\overline{S_sC_t} - \overline{S_sP_s}$. We have therefore (comp. (56.)) this *new geometrical relation*, of an extremely simple kind:

$$\text{CXVI.} \ldots \overline{S_sC_t} - \overline{S_sP_s} = 3(\overline{SP_s} - \overline{SP}) = 3(\overline{S_sP} - \overline{S_sP_s});$$

or

$$\text{CXVI}'. \ldots \overline{S_sC_t} = 3\overline{S_sP} - 2\overline{S_sP_s}.$$

(67.) Supposing, then, at first, that the *coefficient of non-sphericity* $S - 1$ is *positive* (comp. 395, (16.)), if we conceive a point to move *backwards, upon the curve*, from P_s to P, and then *forwards, upon the circle* which *osculates at* P, to the *new point* C_t (63.), we see that it will *first* attain (at P) a position *exterior to the sphere* which *osculates at* P_s, or will have an amount, determined in (56.), of *outward deviation*, with respect to that *near osculating sphere*; and that it will *afterwards* attain (at the new point C_t) a *deviation of the same* *character* (namely *outwards*, if $S > 1$), from the *same near sphere*, but one of which the *amount* will be *threefold* the former : this last *relation* holding also when $S < 1$, or when *both* deviations are *inwards*.

(68.) It is easy also to infer from (65.), (comp. (57.)), that if we go *back from* P_s, *on the near circle* which *osculates at that near point*, through an *arc* (t) *of that circle*, which will only *differ* by a small quantity of the *fourth order* (comp. (60.)) from the *arc* (s) *of the curve*, so as to arrive at a point, which for the moment we shall simply denote by C, and in which (as well as in *another point of section*, not necessary here to be considered) the *near osculating circle* is *cut* by the *given normal plane* at P, the *vector deviation* of this *new point* C of the *new circle*, *from the given point* P of the *curve*, must be, nearly :

$$\text{CXVII.} \ldots \text{PC} = \frac{Rs^3}{6r^2\text{r}} \text{U}\tau(\sigma - \rho) - \frac{ns^4}{8r\text{r}R} \text{U}(\sigma - \rho) ;$$

the coefficients being formed from those of the formula CXV., by first changing s to $-s$, and then changing the signs of the results : while the relation CXVI. or CXVI′. takes now the form,

$$\text{CXVIII.} \ldots \overline{\text{SC}} - \overline{\text{SP}} = 3(\overline{\text{SP}}_s - \overline{\text{SP}}), \quad \text{or} \quad \text{CXVIII}'. \ldots \overline{\text{SC}} = 3\overline{\text{SP}}_s - 2\overline{\text{SP}}.$$

(69.) Accordingly if, after going from P to P_s along the *curve*, we go forward or backward, through *any positive or negative arc*, t, of the *circle*, which *osculates* at that point P_s, we shall arrive at a point which we may here denote by $C_{s,t}$; and the *vector* (comp. again 396, (18.)) of this *near point* (*more general* than any of those hitherto considered) will be *rigorously*,

$$\text{CXIX.} \ldots \omega_{s,t} = \text{O}C_{s,t} = \rho_s + r_s\tau_s \sin\frac{t}{r_s} + r_s^2\tau'_s \text{ vers } \frac{t}{r_s} \cdot$$

And if we *develop* this new expression to the accuracy of the *fourth order* inclusive, we find that we satisfy the *new condition* (comp. (63.)),

$$\text{CXX.} \ldots \text{S}\tau(\omega_{s,t} - \rho) = 0, \quad \text{when} \quad \text{CXXI.} \ldots t = -s - \frac{r's^4}{24r^3} ;$$

and that then the expression CXIX. agrees with CXVII., within the order of approximation here considered.

(70.) A *geometrical connexion* can be shown to exist, between the *two equivalents* which have been found above, one for the *quadruple* (LXXXVII., comp. (53.)), and the other for the *triple* (CXVIII.), of the *deviation* $\overline{\text{SP}_s} - \overline{\text{SP}}$ of a *near point* P_s of the *curve*, from the *sphere* which *osculates* at the *given point* P: in such a manner that if *either* of those two expressions be regarded as *known*, the *other* can be *inferred* from it.

(71.) In fact if we draw, in the normal plane, perpendiculars PD and PE to the lines PS and PS_s, and determine points D and E upon them by drawing a parallel to PS through the point C of (68.), letting fall also a perpendicular CF on PS_s, the *two small lines* PD and DC will ultimately represent the *two terms* or components CXVII. of PC; and the *small angle* DPC will ultimately be equal to *three quarters* of the small angle SPS_s, and will correspond to the *same direction* of *rotation* round τ, because

$$\text{CXXII.} \ldots \frac{\text{DC}}{\text{PD}} = \tfrac{3}{4} \cdot \frac{nrs\tau}{R^2} = \tfrac{3}{4}\, \text{V} \frac{\sigma' s}{\sigma - \rho},$$

or

$$\text{CXXIII.} \ldots \text{DPC} = \tfrac{3}{4}\text{SPS}_s = \tfrac{3}{4}\text{DPE};$$

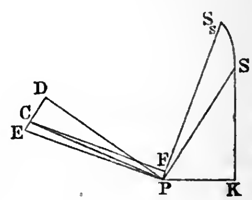

Fig. 83.

so that we shall have the *ultimate ratios* (comp. the annexed fig. 83*):

$$\text{CXXIV.} \ldots \text{DC} : \text{DE} : \text{CE} \text{ (or FP)} = 3 : 4 : 1.$$

But the line CF is ultimately the *trace*, on the given normal plane, of the tangent plane at C to the near osculating sphere; the small line FP (or CE) represents therefore the *deviation* $\overline{\text{S}_s\text{P}} - \overline{\text{S}_s\text{P}_s}$ of the given point P from that near sphere, or the *equal* deviation (57.), $\overline{\text{SP}_s} - \overline{\text{SP}}$; its ultimate *quadruple*, DE, represents the *product* mentioned in (52.); and the ultimate *triple*, DC, of the same small line CE, is a geometrical representation of that *other* deviation $\overline{\text{SC}} - \overline{\text{SP}}$, which has been more recently considered.

(72.) When the *two scalars*, s and t, are supposed capable of receiving *any values*, the *point* $\text{C}_{s,t}$ in (69.) may be *any point of the Locus* (8.) *of the Osculating Circle* to the given curve of double curvature; and if we seek the direction of the *normal* to this superficial *locus*, at this point, on the plan of Art. 372, writing first the equation of the surface under the

* In figs. 81, 82, the little arc near S is to be conceived as *terminating there*, or as being a *preceding arc* of the curve which is the locus of S, if r', r, n, and therefore also p and r_1, be *positive* (comp. the second Note to page 107). In the new figure 83, the triangle PDE is to be conceived as being in fact *much smaller* than PKS, though *magnified* to exhibit angular and other relations.

slightly simplified, but equally rigorous form,

$$\text{CXXV.} \ldots \omega_{s,u} = \rho_s + r_s\tau_s \sin u + r_s^2\tau'_s \operatorname{vers} u,$$

with

$$\text{CXXVI.} \ldots u = r_s^{-1}t = \text{P}_s\text{K}_s\text{C}_{s,t},$$

so that u is here a new scalar variable, representing the *angle subtended at the centre* K_s, of the osculating circle at P_s, *by the arc*, t, of that *circle*, we are led, after a few reductions, to the expression,

$$\text{CXXVII.} \ldots \text{V}(\text{D}_u\omega_{s,u} \,.\, \text{D}_s\omega_{s,u}) = r_s\text{r}_s^{-1}(\omega_{s,u} - \sigma_s) \operatorname{vers} u\,;$$

which proves, *by quaternions*, what was to be expected from *geometrical* considerations*, that the *locus of the osculating circle* is also (as stated in (8.) and (22.)) the *Envelope of the Osculating Sphere*.

(73.) The *normal* to this *locus*, at any proposed point $\text{C}_{s,t}$ of any *one* osculating circle, is thus the *radius* of the *sphere* to which *that circle* belongs, or which has the *same point of osculation* P_s with the *given curve*, whether the *arc* (s) of that *curve*, and the arc (t) of the *circle*, be *small* or *large*. We must therefore consider the *tangent plane* to the *locus*, at the given point P of the *curve*, as *coinciding* with the tangent plane to the *osculating sphere* at that point; and in fact, while this latter plane ($\perp$ PS) *contains* the *tangent* τ to the *curve*, which is at the same time a *tangent* to the *locus*, it contains *also* the tangent $\tau(\sigma - \rho)$ to the *sphere*, which is by CXVII. *another tangent* to the locus, as being the tangent at P to the *section* of that surface, which is made by the *normal plane* to the curve.

(74.) But when we come to examine, with the help of the same equation CXVII., what is the *law of the deviation* DC (comp. fig. 83) of that *normal section* of the *locus*, considered as a *new curve* (C), *from its own tangent* PD, we find that *this law* is ultimately expressed (comp. (71.)) by the formula,

$$\text{CXXVIII.} \ldots \frac{\text{DC}^3}{\text{PD}^4} = \frac{81}{32} \cdot \frac{n^3r^5\text{r}(\sigma - \rho)}{R^8} = \text{const.}\,;$$

hence $\overline{\text{DC}}$ varies ultimately as the *power* of $\overline{\text{PD}}$, which has the *fraction* $\frac{4}{3}$ for its exponent; the *limit* of $\overline{\text{PD}}^2 : \overline{\text{DC}}$ is therefore *null*, and the *curvature of the section* is *infinite at* P.

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* In the language of infinitesimals, *two consecutive osculating spheres*, to any curve in space, *intersect* each other in an *osculating circle* to that curve.

(75.) It follows that this point P is a *singular point* of the *curve* (C), in which the *locus* (8.) is *cut* (73.), by the normal plane to the given curve at that point; but it is *not a cusp* on that *section*, because the *tangential component* PD of the *vector chord* PC is ultimately proportional to an *odd power* (namely to the *cube*, by CXVII., comp. (71.)) of the *scalar variable*, s, and therefore has its *direction reversed*, when that *variable changes sign*: whereas the *normal component* DC of the same *chord* PC is proportional to an *even power* (namely the *fourth*, by the same equation CXVII.) of the same *arc*, s, of the *given curve*, and therefore retains its *direction unchanged*, when we pass from a near point P_s, on *one side* of the given point P, to a near point P_{-s} on the *other side* of it.

(76.) To illustrate this by a contrasted case, let G be the point in which the *tangent* to the given curve at P_s is cut by the *normal plane* at P; or a point of the *section*, by that *plane*, of the *developable surface of tangents*. We shall then have the sufficiently approximate expressions,

$$\text{CXXIX.} \ldots \text{PG} = \rho_s - \rho - \left(s + \frac{s^3}{3r^2}\right)\tau_s = \frac{-s^2\tau'}{2} - \frac{s^3\nu}{3\text{r}} = -\text{PQ}_2 - 2\text{PQ}_3,$$

with the significations 397, (10.) of Q_2 and Q_3; hence the point P of the *curve* is (as is well known) a *cusp* of the *section* (G) of the developable *surface of tangents* (comp. 397, (15.)), because the *tangential component* ($-\text{PQ}_2$) of the *vector chord* (PG) has *here* a *fixed direction*, namely that of the *outward radius* (KP prolonged) of the *circle of curvature* at P: while it is *now* the *normal component* (-2PQ_3) which *changes direction*, when the arc s of the *curve* changes *sign*. At the same time we see* that the *equation* of this *last* section (G) may *ultimately* be thus expressed:

$$\text{CXXX.} \ldots \frac{(-2\text{PQ}_3)^2}{(-\text{PQ}_2)^3} = \frac{8\text{PK}}{9\text{r}^2} = \text{const.};$$

comparing which with the equation CXXVIII., we see that although, *in each case*, the *curvature of the section* is *infinite*, at the point P of the *curve*, yet the normal component (or coordinate) varies (ultimately) as the *power* $\frac{3}{2}$ of the *tangential* component, for the section (G) of the *Surface of Tangents*: whereas the former component varies by (74.) as the power $\frac{4}{3}$ of the latter, for the corresponding section (C) of the *Locus of the Osculating Circle*.

* Compare the Note to page 133.

(77.) It follows also that the *curve* (P) *itself*, although it is *not a cusp-edge* of the last-mentioned *locus* (8.), while it *is* such on the *surface of tangents*, is yet a *Singular Line* upon *that locus likewise*: the nature and origin of which *line* will perhaps be seen more clearly, by reverting to the *view* (8.), (22.), (72.), according to which that *Locus of a Circle* is at the same time the *Envelope of a Sphere*.

(78.) In general, if we suppose that σ and R are *any two real functions*, of the vector and scalar kinds, *of any one real* and *scalar variable*, t, and that σ', R', and σ'', R'', &c. denote their successive *derivatives*, taken with respect to it, then σ may be conceived to be the *variable vector* of a *point* S of a *curve in space*, and R to be the *variable radius* of a *sphere*, which has its *centre* at that point S, but *alters* generally its *magnitude*, at the same time that it alters its *position*, by the *motion* of its centre *along the curve* (S).

(79.) Passing from *one* such sphere, with centre S and radius R, considered as *given*, and represented by the scalar equation,*

$$(\sigma - \rho)^2 + R^2 = 0, \qquad \text{LXXXIX.},$$

in which ρ is *now* conceived to be the vector of a *variable point* P *upon its surface*, to a *near sphere* of the *same system*, for which σ, S, and R are replaced by σ_t, S_t, and R_t, where t is supposed to be *small*, we easily infer (comp. 386, (4.)) that the equation,

$$S\sigma'(\sigma - \rho) + RR' = 0, \qquad \text{XCI.},$$

which is formed from LXXXIX. by once derivating σ and R with respect to t, but treating ρ as constant, represents the *real plane* (comp. 282, (12.)) of the (*real or imaginary*) *circle*, which is the *ultimate intersection* of the near sphere with the given one; the *radius* of this *circle*, which we shall call r, being found by the following formula,

$$\text{CXXXI.} \ldots r^2\sigma'^2 = R^2(R'^2 + \sigma'^2), \quad \text{or} \quad \text{CXXXI}'. \ldots r^2T\sigma'^2 = R^2(T\sigma'^2 - R'^2),$$

and being therefore *real* when

$$\text{CXXXII.} \ldots R'^2 + \sigma'^2 < 0, \quad \text{or} \quad \text{CXXXII}'. \ldots R'^2 < T\sigma'^2;$$

* This equation, and a few others which we shall require, occurred before in this series, but in a connexion so different, that it appears convenient to repeat them here.

while the *centre*, say K, of the circle is *always real*, and its *vector* is,

$$\text{CXXXI}''. \ldots \text{OK} = \kappa = \sigma + RR'\sigma'^{-1};$$

and the *plane* XCI. of the same circle is *parallel* to the *normal plane* of the *curve* (S).

(80.) With the *condition* CXXXII., the *two scalar equations*, LXXXIX. and XCI., represent then *jointly* a *real circle*; and the *locus* of all such *circles* (comp. 386, (6.)) is easily proved to be also the *envelope* of all the *spheres*, of which one is represented by the equation LXXXIX. *alone*; *each such sphere touching this locus*, in the *whole extent* of the corresponding *circle* of the system.

(81.) The *plane* XCI., considered as *varying* with t, has a *developable surface* for its *envelope*; and the *real right line*, or *generatrix*, along which *one* touches the *other*, is represented (comp. again 386, (6.)) by the system of the *two* scalar equations, XCI. and

$$S\sigma''(\sigma - \rho) + \sigma'^2 + (RR')' = 0, \qquad \text{XCIII.};$$

where ρ is now the *variable vector of the line of contact*, although it has been *treated as constant* (comp. 386, (4.)), in the process by which we are here conceived to pass, by a *second derivation*, from LXXXIX. through XCI. to XCIII.

(82.) This real *right line* (81.) *meets* generally the *sphere*, and also the *circle* (as being in its plane), in *two* (real or imaginary) *points*, say P_1, P_2; and the *curvilinear locus* of all *such points* forms generally a species of *singular line*,* upon the *superficial locus* (or *envelope*) recently considered (80.); or rather it forms in general *two branches* (real or imaginary) of *such a line*: which *generally two-branched line* (or *curve*) is the (real or imaginary) *envelope* (comp. 386, (8.)), of all the *circles* of the system.

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* Called by Monge an *arête de rebroussement*, except in the *case* to which we shall next proceed, when its *two branches coincide*. The *envelope* (80.) of a *varying sphere* has been considered in two distinct Sections, § XXII. and § XXVI., of the *Application de l'Analyse à la Géométrie*; but the author of that great work does not appear to have perceived the *interpretation* which will soon be pointed out, of the *condition* of such *coincidence*. Meantime it may be mentioned, in passing, that quaternions are found to confirm the geometrical result, that when the *two branches* (P_1) (P_2) are *distinct*, then *each* is a *cusp-edge* of the *surface*; but that when they are *coincident*, the *singular line* (P) in which they merge has then a *different character*.

(83.) The equation

$$S\sigma'\sigma''(\sigma - \rho) = 0, \qquad \text{XCII.},$$

which now represents (comp. 376, V.) the *osculating plane to the curve* (s), shows that this *plane through the centre* s of the *sphere* is *perpendicular to the right line* (81.), and consequently *contains* the perpendicular let fall *from that centre on that line*: the *foot* P of this last *perpendicular* is therefore found by combining the *three* linear and scalar equations, XCI., XCII., XCIII., and its *vector* is,

$$\text{CXXXIII.} \ldots \text{OP} = \rho = \sigma + \frac{g\sigma' + RR'\sigma''}{V\sigma'\sigma''},$$

if

$$\text{CXXXIV.} \ldots g = -\sigma'^2 - R'^2 - RR'' = T\sigma'^2 - (RR')'.$$

(84.) The *condition of contact* of the *right line* (81.) with the *sphere* (78.), or with the *circle* (79.), or the condition of contact between *two consecutive** *circles* of the system (80.), or finally the *condition of coincidence* of the *two branches* (82.) of that *singular line* upon the surface which is *touched by all those circles*, is at the same time the *condition of coexistence* of the *four scalar equations*, LXXXIX., XCI., XCII., XCIII.; it is therefore expressed by the equation (comp. CXXXIII.),

$$\text{CXXXV.} \ldots R^2(V\sigma'\sigma'')^2 = (g\sigma' + RR'\sigma'')^2;$$

which may also be thus written,†

$$\text{CXXXVI.} \ldots (RS\sigma'\sigma'' - R'g)^2 = (R'^2 + \sigma'^2)(R^2\sigma''^2 + g^2),$$

or thus,

$$\text{CXXXVII.} \ldots R^2(R'^2 + \sigma'^2)(V\sigma'\sigma'')^2 = (g\sigma'^2 + RR'S\sigma'\sigma'')^2;$$

the *scalar variable* t (78.), with respect to which the derivations are performed,

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* Compare the second Note to page 115.

† In page 372 of Liouville's Edition already cited, or in page 325 of the Fourth Edition (Paris, 1809), of the *Application de l'Analyse*, &c., it will be found that this condition is assigned by Monge, as that of the *evanescence* of a certain *radical*, under the form (an accidentally omitted exponent of π'' in the second part of the first member being here restored):

$$[a(\phi'\phi'' + \psi'\psi'' + \pi'\pi'') - h^2]^2 + h^2[a^2(\phi''^2 + \psi''^2 + \pi''^2) - h^4] = 0;$$

in which he writes, for abridgment,

$$h^2 = 1 - \phi'^2 - \psi'^2 - \pi'^2,$$

and ϕ, ψ, π are the three rectangular coordinates of the centre of a moving sphere, considered as remaining still entirely *arbitrary*, but the *point* P, which is determined by the formula CXXXIII., being *now* situated on both the *sphere* and the *circle*: and its *curvilinear locus*, which we may call the *curve* (P), being now the *singular line itself*, in its *reduced* and *one-branched* state. And the *last form* CXXXVII. shows, what was to be expected from geometry, that when this *condition of coincidence* is satisfied, the earlier *condition of reality* CXXXII. is satisfied also: together with this *other inequality*,

$$\text{CXXXVIII.} \ldots R^2\sigma''^2 + g^2 < 0,$$

which then results from the form CXXXVI.

(85.) The equations CXXXI., CXXXIV., and the general formula 389, IV., give the expressions,

$$\text{CXXXIX.} \ldots \frac{rr'}{RR'} = \frac{g\sigma'^2 + RR'\text{S}\sigma'\sigma''}{-\sigma'^4}; \quad \text{CXL.} \ldots r_1^{-2} = \frac{(\text{V}\sigma'\sigma'')^2}{\sigma'^6};$$

where r is still the *radius* of the *circle of contact* of the *sphere* with its *envelope*, and r_1 is the *radius of curvature* of the *locus of the centre* S of the same variable sphere; whence it is easy to infer, that the *condition* CXXXV. may be reduced to the following very simple form (comp. XXXVI'. and XLII.):

$$\text{CXLI.} \ldots (r'r_1)^2 = (RR')^2; \quad \text{or} \quad \text{CXLI'.} \ldots r_1 \text{d}r = \pm R\text{d}R;$$

the independent variable being still *arbitrary*.

(86.) If the *arc* of the *curve* (S) be taken as that variable t, the form CXXXVI. of the same condition is easily reduced to the following,

$$\text{CXLII.} \ldots R^2 = (RR')^2 + g^2r_1^2, \quad \text{with} \quad \text{CXLIII.} \ldots g = 1 - (RR')';$$

derivating then, and dividing by $2g$, we have this *new differential equation*,

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functions of its radius a. Accordingly, if we change R to a, and σ to $i\phi + j\psi + k\pi$, supposing also that $R' = a' = 1$, and $R'' = a'' = 0$, whereby g is changed to $-h^2$, and $R'^2 + \sigma'^2$ to h^2, in the condition CXXXVI., that condition takes, by the rules of quaternions, the exact form of the equation cited in this Note: which, for the sake of reference, we shall call, for the present, the *Equation of Monge*, although it does not appear to have been either *interpreted* or *integrated* by that illustrious author. Indeed, if Monge had not hastened over this *case of coincident branches*, on which he seems to have designed to *return* in a subsequent Memoir (unhappily not written, or not published), he would scarcely have chosen such a symbol as h^2 (instead of $-h^2$), to denote a quantity which is *essentially negative*, whenever (as here) the *envelope* of the *sphere* is *real*.

which is *of linear form with respect to RR′*, whereas the *condition itself* may be considered as a differential equation of the *second degree*, as well as of the *second order*,*

$$\text{CXLIV.} \ldots RR' = r_1(gr_1)'; \quad \text{or} \quad \text{CXLV.} \ldots r_1^2u'' + r_1r_1'(u' - 1) + u = 0,$$

if

$$\text{CXLVI.} \ldots u = RR' = R\mathrm{D}_tR, \quad \text{and therefore} \quad \text{CXLVII.} \ldots u^2 = R^2 - r^2,$$

by CXXXI. or CXXXI′., because we have now,

$$\text{CXLVIII.} \ldots \sigma'^2 = -1, \quad \text{or} \quad \mathrm{T}\sigma' = 1, \quad \text{or} \quad \mathrm{d}t = \mathrm{Td}\sigma:$$

so that the *new scalar variable*, RR', or u, with respect to which the *linear equation* CXLIV. or CXLV. is only of the *second order*, represents the perpendicular *height*† of the *centre* s of the *sphere*, above the *plane* of the *circle*, considered as a *function of the arc* (t) of the *curve* (s), and as *positive* when the *radius* R of the *sphere increases*, for *positive motion* along that curve, or for an *increasing value* of its *arc*.

(87.) If the curve (s) be *given*, or even if we only know the *law* according to which its *radius of curvature* (r_1) depends on its *arc* (t), the *coefficients* of the *linear equation* CXLV. are *known*; and if we succeed in *integrating* that equation, so as to find an expression for the *perpendicular* u as a function of that *arc* t, we shall then be able to express also, as functions of the *same* arc, the *radii* R and r of the *sphere* and *circle*, by the formulæ,

$$\text{CXLIX.} \ldots \pm r = gr_1 = r_1(1 - u'), \quad \text{and} \quad \text{CL.} \ldots R^2 = 2\int u\mathrm{d}t = u^2 + r_1^2(1 - u')^2;$$

the *third scalar constant*, which the integral $2\int u\mathrm{d}t$ would otherwise introduce into the expression for R^2, being in this manner *determined*, by means of the *other two*, which arise from the integration of the *equation* above mentioned.

(88.) For example, it may happen that the locus of the centre s of the sphere has a *constant curvature*, or that r_1 = const.; and then the *complete integral* of the linear equation CXLV. is at once seen to be of the form,

$$\text{CLI.} \ldots u = a \sin(r_1^{-1}t + b),$$

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* We shall soon assign the *complete integral* of the *differential equation in quaternions* (84.), and also that of the corresponding *Equation of Monge*, cited in the preceding Note.

† It will be found that this new scalar u, if we abstract from *sign*, corresponds precisely to the p of earlier sub-articles, although presenting itself in a *differential connexion*: for the *sphere* (78.), and the *circle* (79.), under the *condition* (84.), will soon be shown to be the *osculating* sphere and circle to the recent *curve* (P), or to the *singular line* (84.) upon the *surface* at present considered, that is, on the *locus* or *envelope* (80.).

a and b being *two arbitrary* (but scalar) constants; after which we may write, by (87.),

$$\text{CLII.} \ldots \pm r = r_1 - a \cos (r_1^{-1}t + b);$$

$$\text{CLIII.} \ldots R^2 = r_1^2 - 2ar_1 \cos (r_1^{-1}t + b) + a^2;$$

so that, *in this case*, *both* the *radii*, r and R, of *circle* and *sphere*, are *periodical functions* of the *arc* of the curve (s).

(89.) In general, if that *curve* (s) be *completely given*, so that the *vector* σ is a *known function* of a *scalar variable*, and if an expression have been *found* (or *given*) for the *scalar* R which satisfies any one of the forms of the *condition* (84.), we can then determine *also* the *vector* ρ, by the formula CXXXIII., as a function of the *same* variable; and so can assign the *point* P of the *singular line* (84.), which *corresponds* to any *given position* of the *centre* S of the *sphere*. For this purpose we have, when the arc of the curve (s) is taken, as in (86.), for the independent variable t, the formula,

$$\text{CLIV.} \ldots \rho = \sigma - u\sigma' - (1 - u')\sigma''^{-1} = \kappa_1 - u\sigma' - r_1^2 u'\sigma'',$$

if κ_1 be the *vector* of the *centre*, say K_1, of the *osculating circle* at S *to that given curve*, so that (comp. 389, XI.) it has the value,

$$\text{CLV.} \ldots \text{OK}_1 = \kappa_1 = \sigma - \sigma''^{-1} = \sigma + r_1^2\sigma'', \quad \text{with} \quad \text{CLV}'. \ldots \sigma''^2 + r_1^{-2} = 0.$$

If then we denote by v the *distance* of the point P from *this* centre K_1, and attend to the *linear equation* CXLV., we see that

$$\text{CLVI.} \ldots v = \overline{\text{K}_1\text{P}} = \text{T}(\rho - \kappa_1) = \sqrt{(u^2 + r_1^2 u'^2)},$$

and

$$\text{CLVI}'. \ldots vv' = r_1 r_1' u_1', \quad \text{with} \quad \text{T}\sigma' = 1;$$

or more generally,

$$\text{CLVII.} \ldots vv's_1' = r_1 r_1' u',$$

if

$$\text{CLVII}'. \ldots u = RR's_1'^{-1}, \quad \text{and} \quad \text{CLVII}''. \ldots s_1 = \int \text{T}d\sigma,$$

while

$$\text{CLVI}''. \ldots v^2 = u^2 + r_1^2 u'^2 s_1'^{-2};$$

so that s_1 denotes the *arc* of the *curve* (s), when the independent variable t is again left arbitrary. This *distance*, v, is therefore *constant* (= a) in the *case* (88.), namely when the *radius of curvature* r_1 of that curve is *itself* a constant quantity.

(90.) When $s_1' = T\sigma' = 1$, as in CXLVIII., the part $\sigma - u\sigma'$ of the first expression CLIV. for ρ becomes $= \kappa$, by CXXXI″. and CXLVI.; attending then to CLV., we have the *scalar quotient*,

$$\text{CLVIII.} \ldots \frac{\kappa - \rho}{\sigma - \kappa_1} = 1 - u';$$

whence generally,

$$\text{CLVIII}'. \ldots \frac{\kappa - \rho}{\sigma - \kappa_1} = 1 - \frac{1}{s_1'}\left(\frac{RR'}{s_1'}\right)' = 1 - \left(\frac{\mathrm{d}}{\mathrm{d}s_1}\right)^2\left(\frac{R^2}{2}\right),$$

the independent variable t being again arbitrary. Accordingly, if we combine the general expression CXXXIII. for ρ, with the expression CXXXI″. for κ, and with the following for κ_1 (comp. 389, IV.),

$$\text{CLIX.} \ldots \kappa_1 = \sigma + \frac{\sigma'^3}{V\sigma''\sigma'}, \text{ for an } \textit{arbitrary} \text{ scalar variable,}$$

we easily deduce this new form of the scalar quotient,

$$\text{CLIX}'. \ldots \frac{\kappa - \rho}{\sigma - \kappa_1} = 1 + ((RR')' - RR'S\sigma'^{-1}\sigma'')\sigma'^{-2};$$

which agrees with CLVIII′., because $-\sigma'^2 = s_1'^2$, and $S\frac{\sigma''}{\sigma'} = \frac{s_1''}{s_1'}$.

(91.) It has then been fully shown, how to *determine the vector* ρ as a *function of the scalar* t, when σ and R are *two known functions* of that variable, which satisfy any one of the forms of the *condition* (84.). It must then be possible to determine also the *derived vectors*, ρ', ρ'', &c., as functions of the same variable; and accordingly this can be done, by *derivating* any *three* of the *four* scalar equations, LXXXIX. XCI. XCII. XCIII., of which that condition (84.) expresses the *coexistence*. Now if we derivate a first time the two first of these, and then reduce by the second and fourth, we get the equations,

$$\text{CLX.} \ldots S\rho'(\sigma - \rho) = 0, \quad S\rho'\sigma' = 0, \quad \text{whence} \quad \text{CLX}'. \ldots \rho' \parallel V\sigma'(\sigma - \rho);$$

and although this last formula only determines the *direction* of the *tangent* to the *singular line* at P, namely that of the *common tangent* at that point to *two consecutive circles* (84.), yet it enables us to infer, by the remaining equation XCII., that

$$\text{CLXI.} \ldots \rho' \perp \sigma'', \quad \rho' \parallel V\sigma'\sigma'', \quad \text{and} \quad \text{CLXI}'. \ldots S\rho'\sigma'' = 0;$$

reducing by which the derivative of XCIII., we find,

$$S\sigma'''(\sigma - \rho) + 3S\sigma'\sigma'' + (RR')'' = 0, \qquad \text{XCIV.},$$

the scalar variable being still arbitrary. And *conversely*, the *system** of the *four equations* LXXXIX. XCI. XCIII. XCIV. gives the *three* equations CLX. CLXI′., and so conducts to the equation XCII., and thence to the *condition* (84.); *unless we suppose that* ρ *is a constant vector* α, or that the *variable sphere* passes *through a fixed point* A, a case which we do not here consider, because in it the singular *line* (P) would reduce itself to that one *point*.

(92.) Derivating the two equations CLX., and reducing with the help of CLXI′., we find these new equations,

$$\text{CLXII.} \ldots S\rho''(\sigma - \rho) - \rho'^2 = 0, \quad S\rho''\sigma' = 0;$$

whence

$$\text{CLXIII.} \ldots S\rho'''(\sigma - \rho) - 3S\rho'\rho'' = 0.$$

We are led then, by *elimination of the derivatives of* σ, to the system of the three equations 395, VII.; and we conclude, that *the point* S *is the centre*, and the *radius* R *is the radius*, *of the osculating sphere*† *to the singular line* (P): whence it is easy to infer also, that the *plane of contact* (79.) of the *sphere* with its *envelope* is the *osculating plane*, and that the *circle of contact* (80.) is the *osculating circle* (comp. (72.)), to the same *curve* (P), at the *point* where *two consecutive circles touch one another* (84.).

(93.) *In general*, and even *without* the condition (84.), the *tangent* to a *branch* (82.) of the *curvilinear envelope* of the *circles* of the system, at any point P_1 of that branch, has the direction represented by the vector $V\sigma'(\sigma - \rho_1)$, of the tangent to the circle at that point; but when that *condition* is satisfied,

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* In the language of infinitesimals, this system of equations expresses that *four consecutive spheres intersect*, in one *common point* P. When that point happens to be a *fixed* one, the *condition* (84.) requires that we should have the relation $S\sigma'\sigma''(\sigma - \alpha) = 0$; or geometrically, that the *curve* (S) should be *in a plane through a fixed point*, which is then a *singular point* of the envelope.

† In the language of infinitesimals (comp. the preceding Note), if every *four consecutive spheres* of a system *intersect* in *one point of a curve*, then *each sphere* passes *through four consecutive points* of that curve. Simple as this geometrical reasoning is, the writer is not aware that it has been anticipated; and indeed he is at present led to suppose that this whole theory, of the *Locus of the Osculating Circle*, as the *Envelope of the Osculating Sphere*, is new. Monge had however considered, but rejected (page 374 of Liouville's Edition), the case of a *system of circles* having each a *simple contact* with a *curve in space*.

so that the *two branches* of the singular line *coincide*, the *point* P of that line is *in the osculating plane* (83.) *to the curve* (S): and *then* the equation XCII. shows that the *tangent* ρ', or $V\sigma'(\sigma - \rho)$, to the *line*, is perpendicular to σ'', or parallel to $V\sigma'\sigma''$ (comp. CLXI.), and therefore that *the singular line crosses that plane at right angles*.

(94.) It follows that, with the *condition* (84.), the singular line (P) is an *orthogonal trajectory* to the *system of osculating planes* to the curve (S); and whereas, when this last curve is given, there ought to be *one* such trajectory for *every point* of a *given* osculating plane, this circumstance is analytically represented, in our recent calculations, by the *biordinal form* of the *differential equation* CXLV., of which the *complete integral* must be conceived (87.) to involve generally, as in the case (88.), *two arbitrary constants*.

(95.) It follows also that, with the same condition of *coincidence of branches*, the *singular line* (P) must have the *curve* (S) for the *cusp-edge* of its *polar developable*; or that the *sphere*, with S for centre, and with R for radius, must be the *osculating sphere* to the *curve* (P), as otherwise found by calculation in (92.): while the *circle* (80.) must be, as before, the *osculating circle* to that curve.

(96.) Accordingly, *all equations*, and *inequalities*, which have been stated in the recent sub-articles (79.), &c., respecting the envelope of a moving sphere with variable radius, under that *condition* (84.), and without any special *selection* of the independent variable, admit of being *verified*, by means of the *earlier* formulæ for the osculating circle and sphere to a curve (P) treated as a *given* one, when the *arc* (s) of *that* curve is taken as such a variable.

(97.) For example, we had lately the *two inequalities*, $R'^2 + \sigma'^2 < 0$, CXXXII., and $R^2\sigma''^2 + g^2 < 0$, CXXXVIII. And accordingly the earlier sub-articles (22.), (23.) give, for those two combinations, the *essentially negative values*,

$$\text{CLXIV.} \ldots R'^2 + \sigma'^2 = -p^{-2}r^2R'^2; \qquad \text{CLXV.} \ldots R^2\sigma''^2 + g^2 = -((nr)')^2;$$

in obtaining which last, the following transformations have been employed

$$\text{CLXVI.} \ldots \sigma''^2 = -n'^2 - n^2\text{r}^{-2}; \qquad \text{CLXVII.} \ldots g = -n'p + nr\text{r}^{-1}.$$

(98.) As regards the verification of the *equations*, it may be sufficient to give one example; and we shall take for it the *last general form* CLVII of the differential *equation of condition* (84.). For this purpose we may now write, by (22.) and (23.),

$$\text{CLXVIII.} \ldots s_1' = \pm n, \quad u = \pm p, \quad u' = \pm p', \quad r_1 u_1' s_1'^{-1} = p' r_1 n^{-1} = p' \mathrm{r};$$

and have only to observe that

$$\text{CLXIX.} \ldots \tfrac{1}{2}(p^2 + p'^2 \mathrm{r}^2)' = p'\mathrm{r}(r + p'\mathrm{r})', \quad \text{because} \quad p = r'\mathrm{r}.$$

(99.) If we denote by c_1, c_2, c_3 the first members of the equations XCI., XCIII., XCIV., then besides the equation LXXXIX., which may be regarded as a mere *definition* of the radius R, we have $c_1 = 0$ for the *whole* of the superficial *locus* or *envelope* (80.); but we have not *also* $c_2 = 0$, except for a point on one or other of the *two* (generally distinct) *branches* of the *singular line* (82.) upon that locus. And if, at any *other* and *ordinary point*, we *cut* the surface by a *plane* perpendicular to the *circle* at that point, we find, by a process of the same kind as some which have been already employed, expressions for the *tangential* and *normal components* of the vector chord, whereof the *principal terms* involve the scalar c_2 as a *factor*, while the latter varies (ultimately) as the *square* of the former, so that the *curvature of the section* is *finite* and known, but *tends* to become *infinite* when c_2 tends to *zero*.

(100.) If the *condition of coincidence* (84.) be *not satisfied*, so that the two branches of the singular line (82.) remain *distinct*, and that thus $c_2 = 0$, but *not* $c_3 = 0$ (comp. (91.)), for any ordinary point on *one* of those two branches, then if we *cut* the surface at that point by a *plane* perpendicular to the *branch*, or to the *circle* which *touches* it there, we find an ultimate expression for the vector chord which involves the scalar c_3 as a factor, and of which the *normal* component varies as the *sesquiplicate power* of the *tangential* one: so that we have here the case of a *semicubical cusp*, and *each branch* of the *singular line* is a *cusp-edge** of the surface, exactly in the *same known sense* (comp. (76.)) as that in which a *curve* of double curvature is *generally such*, on the *developable locus* of its *tangents*.

(101.) But when the condition (84.) *is* satisfied, so that the two branches *coincide*, and that thus (comp. again (91.)) we have at once the *three* equations,

$$\text{CLXX.} \ldots c_1 = 0, \quad c_2 = 0, \quad c_3 = 0,$$

then the *terms*, which were lately the *principal* ones (100.), *disappear*: and a

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* Compare the Note to page 144.

new expression arises, for the vector chord of a section of the surface, made by a plane perpendicular to the singular line, which (when we take $t = s$, as in (96.)) is found to admit of being identified with the formula CXVII., and of course conducts to precisely the same system of consequences; the *tangential* component *now* varying ultimately as the *cube*, and the *normal* component as the *fourth power* of a small variable, so that the *cuspidal property* of the point P of the *section* no longer exists, although the *curvature* at that point is still *infinite*, as in (74.): and the *Singular Line*, reduced now to a *single branch*, to which all the *circles* of the system *osculate*, (92.), (95.), is *not a cusp-edge* of the *Surface*, as had been otherwise found before (77.), but a line of a *different character*,* which may thus be regarded, with reference to a *more general Envelope* (80.), as the result of a *Fusion* (84.) of *Two Cusp-Edges*.

(102.) The *condition* of such *fusion* (or coincidence) has been seen (84.) to be expressible by the *differential equation* of the *second order*, and *second degree*,

$$(RS\sigma'\sigma'' - R'g)^2 = (R'^2 + \sigma'^2)(R^2\sigma''^2 + g^2), \qquad \text{CXXXVI.}$$

with

$$g = -\sigma'^2 - (RR')', \qquad \text{CXXXIV.}$$

and with the independent variable arbitrary. And we are now prepared to assign the *complete general integral*† of this differential equation; namely the system of the *two* following equations (comp. 395, (7.) and (14.)), of the *vector* and *scalar* kinds,

$$\text{CLXXI.} \ldots \sigma = \rho + \frac{3V\rho'\rho''S\rho'\rho'' + V\rho'''\rho'^3}{S\rho'\rho''\rho'''}, \quad \text{and} \quad \text{CLXXII.} \ldots R = T(\sigma - \rho);$$

in which ρ is an *arbitrary vector function* of *any scalar variable*, t, and which express, when geometrically interpreted, that σ is the *variable vector* of the

* Compare the Note to page 144. Monge (in page 372 of Liouville's Edition) has the remark, that (when a certain radical vanishes) "les deux branches de la courbe touchée par toutes les caractéristiques se confondent en une seule: et cette courbe, sans cesser d'être une ligne singulière de la surface, n'est plus une arête de rebroussement, elle est une ligne de striction." The propriety of this last *name*, "line of striction," appears to the present writer questionable: although he has confirmed, as above, by calculations with quaternions, the result that, in the case referred to, the *singular line* is *not a cusp-edge*. Monge does not seem to have perceived that, in the same *case of fusion*, the curved *line* in question is not merely *touched*, but *osculated*, by all the *circles* of the system.

† Compare the first Note to page 147. We say here, *general integral*, because a *less* general one, although involving *one arbitrary function* (of the scalar kind), will soon be pointed out.

centre S, and that R is the *variable radius*, of the *osculating sphere*, to an *arbitrary curve* (P), of which the variable vector of a point P is ρ.

(103.) In fact, if we *met* the cited *equation of condition* CXXXVI., g representing therein the expression CXXXIV., without any previous knowledge of its meaning or origin, we might first, by the *rules of quaternions*, and as a mere affair of *calculation*, *transform* it to the equation CXXXV.; which would evidently allow the assumption of the formula CXXXIII., ρ being treated as an *auxiliary vector*, which satisfies (in virtue of the supposed *condition*) the system of the *four scalar equations*, LXXXIX., XCI., XCII., XCIII.; whence derivating and combining, as in (91.) and (92.), we are led to a *new system** of *four* scalar equations, whereof *one* is again the equation LXXXIX., and may be written under the form CLXXII.; while the *three others* are those formerly numbered as 395, VII., and conduct (except in a particular *case* which we shall presently consider) to the *vector expression* CLXXI., which conversely is *sufficient* to represent them, all *derivatives* of σ and of R being thus *eliminated*.

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* The *Equation of Monge* (comp. the second Note to page 145) may be considered as the *condition of coexistence* of the *four* following equations, in which ϕ, ψ, π are supposed to be functions of a, and to be differentiated or derivated as such:

$$(1) \ldots (x-\phi)^2 + (y-\psi)^2 + (z-\pi)^2 = a^2;$$

$$(2) \ldots (x-\phi)\phi' + (y-\psi)\psi' + (z-\pi)\pi' + a = 0;$$

$$(3) \ldots (x-\phi)\phi'' + (y-\psi)\psi'' + (z-\pi)\pi'' + 1 - \phi'^2 - \psi'^2 - \pi'^2 = 0;$$

$$(4) \ldots (x-\phi)(\psi'\pi'' - \pi'\psi'') + (y-\psi)(\pi'\phi'' - \phi'\pi'') + (z-\pi)(\phi'\psi'' - \psi'\phi'') = 0;$$

whereof the first *three* have been employed by Monge himself, but the *fourth* does not seem to have been perceived by him, the condition of *evanescence of a radical* having been used in its stead. And by a *translation* of quaternion results, above deduced, into the usual language of analysis, it is found that the *complete* and *general integral*, of the *non-linear differential equation* of the *second order*, which is obtained by the elimination of x, y, z between these four, is expressed by a *new system* of four equations, the equation (1) being *one* of them; and the *three others*, in which x, y, z are now treated as *arbitrary functions* of a, and are derivated as such, being the following:

$$(5) \ldots (x-\phi)x' + (y-\psi)y' + (z-\pi)z' = 0;$$

$$(6) \ldots (x-\phi)x'' + (y-\psi)y'' + (z-\pi)z'' + x'^2 + y'^2 + z'^2 = 0;$$

$$(7) \ldots (x-\phi)x''' + (y-\psi)y''' + (z-\pi)z''' + 3(x'x'' + y'y'' + z'z'') = 0.$$

By treating a as a function of some *other* independent variable, t, the terms $+a$ and $+1$, in (2) and (3), come to be replaced by $+aa'$ and $+aa'' + a'^2$; and the slightly *more general form*, which Monge's Equation thus assumes, has *still* its complete *general integral* assigned by the system (1) (5) (6) (7), if x, y, z (as well as a) be now regarded as arbitrary functions of the *new* variable t, in the place of which it is permitted (for instance) to take x, and so to write $x' = 1$, $x'' = 0$: only *two arbitrary functions* thus entering, in the last analysis, into the *general solution*, as was to be expected from the form of the equation.

(104.) The *case* just now alluded to, in which the *general integral* (102.) is replaced by a *less general form*, is the case (91.) when the *variable sphere* passes through a *fixed point* A, to which *point*, in that case, the *singular line* reduces itself. And the *integral equations*,* which then replace CLXXI. and CLXXII. may be thus written:

$$\text{CLXXIII.} \ldots \sigma = \alpha + t\beta + u\gamma, \quad \text{with} \quad u = F(t),$$

and

$$\text{CLXXIV.} \ldots R = \mathrm{T}(t\beta + u\gamma);$$

the *second scalar coefficient*, u, being here an *arbitrary function* of the *first* scalar coefficient, or of the independent variable t, and α, β, γ being *three arbitrary* but *constant vectors*: so that the *curve* (s) is now obliged to lie *in some one plane*† through the *fixed point* A, but remains in other respects *arbitrary*. Accordingly it will be found that this *last integral system*, although *less general* than the former system (102.), and not properly *included* in it, *satisfies* the differential equation CXXXVI.; whereof the two members acquire, by the substitutions indicated, this *common value*,

$$\text{CLXXV.} \ldots (R\mathrm{S}\sigma'\sigma'' - R'g)^2 = \&\text{c.} = R^{-2}t^2(tu' - u)^2u''^2(\mathrm{V}\beta\gamma)^4.$$

(105.) Other problems might be proposed and resolved, with the help of formulæ‡ already given, respecting the properties or affections of curves

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* The *particular integral* corresponding, of the *Equation of Monge*, is expressed by the following system:

$$\phi = a + et + lu, \quad \psi = b + ft + mu, \quad \varpi = c + gt + nu,$$

$$(et + lu)^2 + (ft + mu)^2 + (gt + nu)^2 = a^2;$$

abcefglmn being *nine arbitrary constants*, while t and u are *two functions* of a, whereof *one* is *arbitrary*, but the *other* is algebraically *deduced* from it, by means of the fourth equation. The writer is not aware that either of these integrals has been assigned before.

† Compare the first Note to page 150.

‡ We might for example employ the formula VI. for κ'', in conjunction with one of the expressions 397, XCI. for κ', to determine, by the general formula 389, IV., the *vector* (say ξ) of the *centre of curvature* of the *curve* (κ), and therefore also the *radius of curvature* of that curve, which is the *locus of the centres of curvature* of the *given curve* (P), supposed to be *in general* one of *double* curvature. After a few reductions, with the help of XII., we should thus find the equations,

$$\text{CLXXVII.} \ldots \mathrm{V}\frac{\kappa''}{\kappa'} = \frac{-r'\tau}{r\kappa'} + (r^{-1} - P')\tau,$$

$$\text{CLXXVIII.} \ldots \xi = \kappa + \frac{\kappa'}{\mathrm{V}\frac{\kappa''}{\kappa'}} = \kappa + \frac{\sigma - 2\kappa + \rho}{1 - \frac{r\,dP}{ds} + \frac{p\,ds}{r\,d\kappa}},$$

in space which depend on the *fourth power* (s^4) of the *arc*, or on the *fourth derivative* $D_s^4\rho$ or τ''' of the *vector* ρ_s; but it is time to conclude this series of sub-articles, which has extended to a much greater length than was designed, by observing that, in virtue of the *vector form* 396, XI. for the equation of a *circle of curvature*, the *Locus* (8.) *of the Osculating Circle* may be concisely but sufficiently represented by the *Vector Equation*,

$$\text{CLXXVI.} \ldots \mathrm{V}\,\frac{2\tau_s}{\omega - \rho_s} + \nu_s = 0,$$

which *apparently* involves only *one scalar variable*, s, namely, the *arc of the curve* (P), the *other* scalar variable, such as t, which corresponds (69.) to the *arc of the circle*, *disappearing* under the *sign* V: and that the *surface*, which was called in (8.) the *Circumscribed Developable*, is now seen to be in fact *circumscribed* to that *Locus*, or *Envelope*, in a certain *singular* (or eminent) *sense*, as *touching it along its Singular Line*.

399. When we take account of the *fifth power* (s^5) of the *arc*, the expression for ρ_s receives a *new term*, and becomes (comp. 398, I.),

$$\text{I.} \ldots \rho_s = \rho + s\tau + \tfrac{1}{2}s^2\tau' + \tfrac{1}{6}s^3\tau'' + \tfrac{1}{24}s^4\tau''' + \tfrac{1}{120}s^5\tau^{\mathrm{IV}};$$

and although some of the consequences of such an expression have been already considered, especially as regards the general determination of what has been above called the *Osculating Twisted Cubic* to a curve of double curvature, or the *gauche curve of the third degree* which has *contact of the fifth order* with a given *curve in space*, yet, without repeating any calculations already made, some additional light may be thrown on the subject as follows.

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in which last the denominator is a quaternion, and the scalar variable is arbitrary: whence also,

CLXXIX. . . *Radius of curvature of curve* (K),
or of *locus of centres of osculating circles to a given curve* (P) *in space*,

$$= \mathrm{T}(\xi - \kappa) = R\left\{\left(1 - \frac{r\,\mathrm{d}P}{\mathrm{d}s}\right)^2 + \left(\frac{pr}{Rr}\right)^2\right\}^{-\frac{1}{2}}$$

$$= \pm\frac{R\mathrm{d}r}{p\mathrm{d}s}\left\{\left(\frac{1}{r} - \frac{\mathrm{d}P}{\mathrm{d}s}\right)^2 + \left(\frac{p}{Rr}\right)^2\right\}^{-\frac{1}{2}};$$

with the verification, that for the case of a *plane curve* (P), for which therefore $\frac{R}{p} = 1$, and $\frac{1}{r} = 0 = \frac{\mathrm{d}P}{\mathrm{d}s}$, we have thus the elementary expression,

$$\text{CLXXX.} \ldots \textit{Radius of Curvature of Plane Evolute} = \pm\frac{r\mathrm{d}r}{\mathrm{d}s},$$

r being still the radius of curvature, and s the *arc*, of the *given curve*.

(1.) As regards the successive deduction of the derived vectors in the formula I., it may be remarked that if we write (comp. 398, LVI., LXI.),

$$\text{II.} \ldots \mathrm{D}^{n+1}_s \rho = \tau^{(n)} = a_n \tau + b_n r \tau' + c_n r \nu,$$

we shall have, generally,

$$\text{III.} \ldots a_{n+1} = a'_n - r^{-1} b_n, \quad b_{n+1} = b'_n + r^{-1} a_n - \mathrm{r}^{-1} c_n, \quad c_{n+1} = c'_n + \mathrm{r}^{-1} b_n,$$

with the initial values,

$$\text{IV.} \ldots a_0 = 1, \quad b_0 = 0, \quad c_0 = 0, \quad \text{or} \quad \text{IV}'. \ldots a_1 = 0, \quad b_1 = r^{-1}, \quad c_1 = 0;$$

whence $$\text{V.} \ldots \begin{cases} a_2 = -r^{-2}, & b_2 = (r^{-1})', \quad c_2 = r^{-1}\mathrm{r}^{-1}, \\ a_3 = 3r^{-3}r', & b_3 = (r^{-1})'' - r^{-3} - r^{-1}\mathrm{r}^{-2}, \quad c_3 = r(r^{-2}\mathrm{r}^{-1})', \end{cases}$$

as in the expressions 397, VI. for τ'', and 398, IV. for τ'''; the corresponding coefficients of τ^{IV} being in like manner found to be,

$$\text{VI.} \ldots \begin{cases} a_4 = -2(r^{-2})'' + ((r^{-1})')^2 + r^{-2}(r^{-2} + \mathrm{r}^{-2}); \\ b_4 = (r^{-1})''' - 2(r^{-3})' - 3(r^{-1}\mathrm{r}^{-1})'\mathrm{r}^{-1}; \\ c_4 = r^{-1}(\mathrm{r}^{-1})'' + 3((r^{-1})'\mathrm{r}^{-1})' - r^{-1}\mathrm{r}^{-1}(r^{-2} + \mathrm{r}^{-2}); \end{cases}$$

and being sufficient for the investigation of all affections or properties of a curve in space, which depend only on the *fifth power* of the arc s.

(2.) For the *helix* the two curvatures are *constant*, so that all the derivatives of the two radii r and r vanish; the expressions become therefore greatly simplified, and a *law* is easily perceived, allowing us to *sum* the infinite *series* for ρ_s, and so to obtain the following *rigorous expressions* for the *coordinates** x_s, y_s, z_s of this particular curve, instead of those which were

* We have here, and in this whole investigation, an instance of the facility with which *quaternions* can be combined with *coordinates*, whenever the *geometrical nature* of a question may render it convenient so to combine them, by offering to our notice any obvious planes of reference. If it be thought useful to pass to a system connected more immediately with the *right cylinder* than with the *helix*, we may write,

$$\text{VII}'. \ldots \begin{cases} \mathrm{x}_s = l(r^{-1}x_s - \mathrm{r}^{-1}z_s) = l^2 r^{-1} \sin t, \\ \mathrm{y}_s = l^2 r^{-1} - y_s \qquad\quad = l^2 r^{-1} \cos t, \\ \mathrm{z}_s = l(\mathrm{r}^{-1}x_s + r^{-1}z_s) = l^2 \mathrm{r}^{-1} t, \end{cases}$$

where $l^2 r^{-1} = r \sin^2 H$ is the radius of the cylinder, with converse formulæ easily assigned.

developed *generally* in 398, LVIII., but only as far as s^4 inclusive:

$$\text{VII.} \ldots x_s = l^3(\mathrm{r}^{-2}t + r^{-2}\sin t); \quad y_s = l^2 r^{-1} \operatorname{vers} t; \quad z_s = l^3 r^{-1}\mathrm{r}^{-1}(t - \sin t);$$

where l and t are an auxiliary constant and variable, namely,

$$\text{VIII.} \ldots l = (r^{-2} + \mathrm{r}^{-2})^{-\frac{1}{2}} = r \sin H, \quad t = l^{-1}s,$$

l being thus what was denoted in earlier formulæ by $T\lambda^{-1}$, and t being the angle between two axial planes; while the origin is still placed at the point P of the curve, and the tangent, normal, and binormal are still made the axes of xyz.

(3.) The *cone* of the second order, 398, (40.), which has *generally* a *contact of the fifth order* with a proposed *curve in space*, at a point P taken for vertex, has in this case of the *helix* the equation (comp. 398, LVII. and LXIX.),

$$\text{IX.} \ldots y^2 = \frac{3}{2}\frac{\mathrm{r}}{r}\left\{x + \left(\frac{3}{10}\frac{\mathrm{r}}{r} - \frac{7}{10}\frac{r}{\mathrm{r}}\right)z\right\}z.$$

Accordingly it can be shown, by elementary methods, that if we write, for a moment,

$$\text{X.} \ldots f(t) = 3(t - \sin t)(3t + 7\sin t) - 20 \operatorname{vers}^2 t,$$

we have the *eight* evanescent values,

$$\text{XI.} \ldots f0 = f'0 = f''0 = f'''0 = f^{\text{IV}}0 = f^{\text{V}}0 = f^{\text{VI}}0 = f^{\text{VII}}0 = 0;$$

whence it is easy to infer that this *cone* IX. has (in the present example, although not generally) a contact as high as the *sixth order** with the *curve*, of which the coordinates have here the expressions VII.; and consequently that the cone in question must wholly *contain* the *osculating twisted cubic* to that curve.

(4.) In general, to find a *second locus* for such a *cubic curve*, the method of recent sub-articles (398, (38.) &c.) leads us to form the equation (398,

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* Or in modern language, *seven-point contact*, in the sense that the cone passes, *in this case*, through *seven consecutive points* of the curve. It may be remarked that the *gauche curve* of the *fourth degree*, or the *quartic curve*, in which this *cone cuts* the *cylinder of revolution* whereon the helix is traced (cutting also in it a certain *other* cylinder of the second order), and which has the point P for a *double point*, *crosses the helix* by *one* of its *two branches* at that point, while it has *seven-point contact* with the same helix by its *other branch*: and that thus the fact of calculation, expressed by the formula XI., is geometrically accounted for.

LXVI.) of a *cylinder* of the *second order*, or briefly of a *quadric* cylinder*, which like the *quadric cone* (3.) shall have *contact* of the *fifth order* with the proposed *curve* in space, at the given point P; the *ratios* of *abc*, which determine the *direction* of a generating line PE, being obliged for this purpose to satisfy a certain *equation of condition* (398, LXVIII.), of which the *form* indicates that the *locus* of this *line* PE is *generally* a certain *cubic cone*, having the *tangent* (say PT) to the *curve* for a *nodal side*: along which side it is *touched*, not only (like the quadric cone) by the *osculating plane* ($z = 0$) to that given curve, but *also* by a *second plane*, whereof the equation ($gy + hz = 0$, or after reductions $y - \frac{1}{2}r'z = 0$ shows that the *second branch* of the cubic cone *crosses the first branch*, or the quadric cone, or the osculating plane to the curve, at an *angle* of which the trigonometric *cotangent* is equal to *half the differential of the radius* (r) *of second curvature, divided by the differential of the arc* (s); so that this *second tangent plane* to the *cone* coincides with the *rectifying plane* to the *curve*, when the second curvature happens to be *constant*. The *tangent* PT therefore *counts as three* of the *six common sides* of the *two cones* with P for vertex: and the *three other* common sides, for the assigning of which it has been shown (in 398, (41.)) how to form a *cubic equation* in $b : c$, are the *parallels* from that point P to the three real or imaginary *asymptotes*† of the *twisted cubic*, and are *generating lines* PE of *three quadric cylinders*, whereof *one* at least is necessarily *real*, and *contains*, as a *second locus*, that sought *osculating gauche curve* of the *third degree*.

(5.) In applying this *general method* to the case of the *helix*, it is found that the cubic cone breaks up, in this example, into a system of a *new quadric cone*, which *touches* the *former* quadric cone IX. along the *tangent* PT to the curve (the *two other* common sides of *these* two cones being *imaginary*), and of a *plane* ($y = 0$), namely the *rectifying plane* (comp. (4.)) of the helix, or the *tangent plane* to the *cylinder of revolution* on which that given curve is traced: and that this last plane cuts the *first* quadric cone in *two real right lines*, the *tangent* being again *one* of them, and the *other* having the *sought direction* of a *real asymptote* to the sought *osculating twisted cubic*. Without entering here into details of calculation, the resulting equation of the *real*‡ *quadric cylinder*,

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* So called by Dr. Salmon, in his Treatise already cited. Compare the second Note to page 129 of these *Elements*.

† Compare again the Note last referred to.

‡ As regards the *two imaginary quadric cylinders*, their equations can be formed by the same general method, employing as generating lines the *two imaginary common sides* (5.), of the *cone* IX.,

on which that sought gauche curve is situated, may be at once stated to be (with the present system of coordinates),

$$\text{XII.}\ldots 2ry = \left\{x + \left(\frac{3}{10}\frac{\mathrm{r}}{r} - \frac{7}{10}\frac{r}{\mathrm{r}}\right)z\right\}^2 + \frac{3}{5}\left(1 + \frac{r^2}{\mathrm{r}^2}\right)y^2;$$

in such a manner that *if we set aside the right line*,

$$\text{XIII.}\ldots y = 0, \quad x + \left(\frac{3}{10}\frac{\mathrm{r}}{r} - \frac{7}{10}\frac{r}{\mathrm{r}}\right)z = 0,$$

which is a *common side* of the *cone* IX. and of the *cylinder* XII., the *curve*, which is the *remaining part* of their *complete* intersection, is the *twisted cubic sought*. As an elementary verification of the fact, that *this gauche curve* of intersection IX. XII. has *contact of the fifth order* with the *helix* at the point P, it may be observed that if we change the coordinates xyz in XII. to the expressions VII., and write for abridgment,

$$\text{XIV.}\ldots F(t) = (3t + 7t\sin t)^2 - 200\,\text{vers}\,t + 60\,\text{vers}^2\,t,$$

we have then (comp. X. XI.) the *six* evanescent values,

$$\text{XV.}\ldots F0 = F'0 = F''0 = F'''0 = F^{\text{IV}}0 = F^{\text{V}}0 = 0.$$

(6.) As another *verification*, which is at the same time a sufficient *proof*, of the *à posteriori* kind, that the *gauche curve* IX. XII. has in fact *contact* of the *fifth order* with the *helix*, it can be shown that while the coordinates y_s and z_s of the *latter* may (by VII., writing simply x for x_s, and neglecting x^7) be thus developed,

$$\text{XVI.}\ldots \begin{cases} y_s = \dfrac{x^2}{2r} + \dfrac{x^4}{24r}\left(\dfrac{3}{r^2} - \dfrac{1}{\mathrm{r}^2}\right) + \dfrac{x^6}{720r}\left(\dfrac{45}{r^4} - \dfrac{24}{r^2\mathrm{r}^2} + \dfrac{1}{r^4}\right), \\ z_s = \dfrac{x^3}{6r\mathrm{r}} + \dfrac{x^5}{120r\mathrm{r}}\left(\dfrac{9}{r^2} - \dfrac{1}{\mathrm{r}^2}\right), \end{cases}$$

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and of that *other quadric cone* above referred to, which is *here* a *separable part* of the *general cubic locus*, and has for equation,

$$\text{IX}'\ldots \frac{20}{9}y^2 = 5\frac{\mathrm{r}}{r}xz + \left(3\frac{\mathrm{r}^2}{r^2} - 2\right)z^2.$$

It seems sufficient here to remark, that by taking the sum and difference of the equations of those two imaginary cylinders, *two new real quadric surfaces* are obtained, which *also contain* the *osculating twisted cubic*, and *intersect* each other in that gauche curve: namely *two hyperbolic paraboloids*, which have a *common side at infinity*, and of which the equations can be *otherwise deduced* (by way of verification), *without imaginaries*, through easy algebraical combinations of the two real equations IX. and XII.

the corresponding coordinates y and z of the *former*, that is, of the *curvilinear part* of the *intersection* of the cone IX. with the cylinder XII., have (in the same order of approximation) developments which may be thus abridged,

$$\text{XVII.} \ldots y = y_s - \frac{(r^{-2} + \text{r}^{-2})^2 x^6}{800r}, \quad z = z_s.$$

(7.) The *deviation* of the helix from the gauche curve IX. XII. is therefore of the *sixth order* (with respect to x, or s), and it has an *inward direction*, or in other words, the *osculating twisted cubic* deviates *outwardly* from the *helix*, with respect to the right cylinder; the *ultimate* (or initial) *amount* of this deviation, or the *law* according to which it *tends* to vary, being represented by the formula,

$$\text{XVII}'. \ldots y_s - y = \frac{(r^{-2} + \text{r}^{-2})^2 s^6}{800r} = \frac{t^4 y_s}{400};$$

where t denotes as in (2.) the *angle*, which a *plane* drawn through a near point P_s, and through the axis of the *right cylinder*,*

$$\text{XVIII.} \ldots 2ry = \left(x - \frac{r}{\text{r}} z\right)^2 + \left(1 + \frac{r^2}{\text{r}^2}\right) y^2,$$

whereon the helix is traced, makes with the plane drawn through the same *axis of revolution*, or through the right line,

$$\text{XIX.} \ldots x = \frac{r}{\text{r}} z, \quad y = r^{-1}(r^{-2} + \text{r}^{-2})^{-1} = l^2 r^{-1},$$

and through the given point P: while y_s is still the (inward) distance of the same near point P_s, from the tangent plane to the same cylinder at the same given point P.

(8.) If we cut the cone IX., and the cylinder XII., by any plane,

$$\text{XX.} \ldots 2ry = w \left\{x + \left(\frac{3}{10}\frac{\text{r}}{r} - \frac{7}{10}\frac{r}{\text{r}}\right) z\right\},$$

drawn through their common side XIII., we obtain two other sides, one for

* With the coordinates VII'. of a recent Note (to page 157), the equation of this cylinder would be,

$$\text{XVIII}'. \ldots \text{x}^2 + \text{y}^2 = l^4 r^{-2}.$$

each of these two quadric surfaces; and these two new right lines, in this plane XX., intersect each other in a new point,* of which the coordinates xyz are given, as functions of the new variable w, by the *three fractional expressions*,†

$$\text{XXI.} \ldots x = \frac{w + \left(\frac{7}{\text{r}^2} - \frac{3}{r^2}\right)\frac{w^3}{60}}{1 + \frac{3}{20}\frac{w^2}{l^2}}; \quad 2ry = \frac{w^2}{1 + \frac{3}{20}\frac{w^2}{l^2}}; \quad 6r\text{r}z = \frac{w^3}{1 + \frac{3}{20}\frac{w^2}{l^2}};$$

while the *twisted cubic*, which *osculates* (as above) to the helix at P, is the *locus* of all the *points of intersection* thus determined. Accordingly, if we develop xyz by XXI., in ascending powers of w, neglecting w^7 (or x^7), we are conducted, by elimination of w, to expressions for y and z in terms of x, which agree with those found in (6.), and thereby establish in a new way the existence of the required contact of the *fifth order*, between the two curves of double curvature.

(9.) The *real asymptote* to the cubic curve is found by supposing the auxiliary variable w to tend to infinity in the expressions XXI.; it is therefore the right line (comp. XX.),

$$\text{XXII.} \ldots y = \frac{10}{3}\frac{l^2}{r}, \quad x + \left(\frac{3}{10}\frac{\text{r}}{r} - \frac{7}{10}\frac{r}{\text{r}}\right)z = 0,$$

namely the *second side* in which the *elliptic cylinder* XII. is cut by a normal plane through the side XIII.; and by comparing the value of its y with the equation XIX., we see that the *least distance between the real asymptote to the osculating twisted cubic, and the axis of revolution of the cylinder* on which the *helix* is traced, is equal to *seven-thirds of the radius of that right cylinder*.

(10.) As regards the *two imaginary asymptotes*, they correspond to the two imaginary values of w, which cause the *common denominator* of the expressions

* The plane XX., as containing the line XIII., is parallel to an asymptote, and therefore meets the cubic at infinity; it also passes through the given point P: and therefore it can only cut the twisted cubic in *one other point*, of which the position is expressed by the equations XXI.

† *Quaternions* suggest such fractional expressions, through the formula 398, LXXIX. for the *vector* $(\phi + c)^{-1}a$; but it is proper to state that expressions of *fractional form*, for the coordinates of a *curve in space* of the *third order* (or degree) were given by Möbius, who appears to have been the first to discover the existence of *such* gauche curves, and who published several of their principal properties in his *Barycentric Calculus* (der barycentrische Calcul, Leipzig, 1827).

XXI. to vanish; but it may be sufficient here to observe, that because those expressions give, generally,

$$\text{XXIII.} \ldots x + \left(\frac{6}{5}\frac{\mathrm{r}}{r} + \frac{1}{5}\frac{r'}{\mathrm{r}}\right)z = w,$$

the two imaginary lines in question are to be considered as being contained in *two imaginary planes*, which are both *parallel to the real plane** *through* P,

$$\text{XXIV.} \ldots x + \left(\frac{6}{5}\frac{\mathrm{r}}{r} + \frac{1}{5}\frac{r'}{\mathrm{r}}\right)z = 0;$$

namely to a certain *common normal plane* to the *two real cylinders* XII. and XVIII., or to the *elliptic* and *right cylinders* already mentioned.

(11.) *In general*, instead of seeking to determine, as above, a *cylinder* of the second order, which shall have *contact* of the *fifth order* with *any given curve* of double curvature, at a given point P, we may propose to find a *second cone* of the same (second) order, which shall have *such* contact with that curve at that point, its *vertex* being at *some other point* of space (abc). Writing (comp. 398, LXVI.) the equation of such a *cone* under the form,

$$\text{XXV.} \ldots 2r(cy - bz)(c - z) = (cx - az)^2 + 2B(cx - az)(cy - bz) + C(cy - bz)^2;$$

substituting for xyz the coordinates $x_s y_s z_s$ of the curve, under the forms (comp. 398, LVIII.),

$$\text{XXVI.} \ldots \begin{cases} x_s = s - \dfrac{s^3}{6r^2} + \dfrac{a_3 s^4}{24} + \dfrac{a_4 s^5}{120}, \\[2ex] y_s = \dfrac{s^2}{2r} - \dfrac{r' s^3}{6r^2} + \dfrac{b_3 s^4}{24} + \dfrac{b_4 s^5}{120}, \\[2ex] z_s = \dfrac{s^3}{6r\mathrm{r}} + \dfrac{c_3 s^4}{24} + \dfrac{c_4 s^5}{120}, \end{cases}$$

in which the coefficients $a_3 b_3 c_3$ and $a_4 b_4 c_4$ have the values assigned in (1.); developing according to powers of s, neglecting s^6, and comparing coefficients of s^3, s^4, s^5; we find first the expressions,

$$\text{XXVII.} \ldots B = \frac{-1}{3}\left(r' + \frac{b}{c}\frac{r}{\mathrm{r}}\right), \quad C = -\frac{4}{9}\left(r' + \frac{b}{c}\frac{r}{\mathrm{r}}\right)^2 + \frac{4}{3}\left(1 + \frac{a}{c}\frac{r}{\mathrm{r}}\right) + \frac{r^3}{3}\left(b_3 - \frac{b}{c}c_3\right),$$

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* The *right line at infinity*, in this plane XXIV., is the *common side* of the *two hyperbolic paraboloids* mentioned in the third Note to page 159, as each containing the whole twisted cubic.

which are the *same* for *cone* as for cylinder: and then are led to the *new equation of condition*,

$$\text{XXVIII.} \ldots \frac{r}{5}\left(b_4 - \frac{b}{c}c_4\right) = a_3 - \frac{a}{c}c_3 + \frac{2}{cr\mathrm{r}}$$
$$+ B\left(b_3 - \frac{b}{c}c_3 - \frac{2}{r^3} - \frac{2a}{cr^2\mathrm{r}}\right) - 2C\left(\frac{r'}{r^3} + \frac{b}{cr^2\mathrm{r}}\right),$$

which *differs* from the corresponding equation for the determination of a *cylinder* having the *same* (fifth) *order of contact* with the curve, but only by the *one term* $\frac{2}{cr\mathrm{r}}$ in the second member, which term *vanishes* when the coordinate c of the vertex is *infinite*.

(12.) Eliminating B and C, and substituting for $a_3b_3c_3$ and $a_4b_4c_4$ their values V. and VI., we find that the condition XXVIII. may be thus expressed (comp. 398, LXVIII.):

$$\text{XXIX.} \ldots ac\left(b - \frac{\mathrm{r}'}{2}c\right) - \mathrm{r}c^2 = \mathrm{a}b^3 + \mathrm{b}b^2c + \mathrm{c}bc^2 + \mathrm{e}c^3;$$

in which we have written, for abridgment,

$$\text{XXX.} \ldots \begin{cases} \mathrm{a} = \dfrac{4}{9}\dfrac{r}{\mathrm{r}}; \quad \mathrm{b} = \dfrac{r'}{3} - \dfrac{r}{\mathrm{r}}\dfrac{\mathrm{r}'}{2}; \\ \mathrm{c} = \dfrac{1}{30}(6r''\mathrm{r} - 3r\mathrm{r}'' - 2r^{-1}r'^2\mathrm{r} - 6r'\mathrm{r}' + 6r\mathrm{r}^{-1}\mathrm{r}'^2 - 18r^{-1}\mathrm{r} + 12r\mathrm{r}^{-1}); \\ \mathrm{e} = \dfrac{1}{90}(9r'''\mathrm{r}^2 - 9r^{-1}r'r''\mathrm{r}^2 + 4r^{-2}r'^3\mathrm{r}^2 + 36r^{-2}r'\mathrm{r}^2 + 18r' - 27r\mathrm{r}^{-1}\mathrm{r}'). \end{cases}$$

The *locus* of the *vertex* of the sought *quadric cone* XXV. is therefore that *cubic surface*, or surface of the *third order*, which is represented by the equation XXIX. in abc; this surface, then, is a *second locus* (comp. (4.)) for the *osculating twisted cubic*, whatever the *given curve* in space may be: a *first locus* for that *cubic curve* being still the *quadric cone* (comp. (3.)), of which the equation in abc is (by 398, LXVII. and LXIX.),

$$\text{XXXI.} \ldots 4\left(\frac{\mathrm{r}}{r}\right)^2 b^2 = 6\left(\frac{\mathrm{r}}{r}\right)^3 ac + \left(\frac{\mathrm{r}^3}{r^2}\right)' bc$$
$$+ \frac{\mathrm{r}^4}{5}\left(\frac{9}{r^4} - \frac{21}{r^2\mathrm{r}^2} + \frac{r'^2}{r^4} - \frac{3r''}{r^3} + \frac{3r'\mathrm{r}'}{r^3\mathrm{r}} - \frac{27\mathrm{r}'^2}{4r^2\mathrm{r}^2} + \frac{9\mathrm{r}''}{r^2\mathrm{r}}\right)c^2,$$

and which has contact of the *fifth order* with the *curve*, while its *vertex* is *at the given point* P of osculation.*

(13.) Instead of thus introducing, as *data*, the *derivatives* of the two *radii of curvature*, r and r, taken with respect to the *arc*, s, it may be more convenient in many applications to treat the two coordinates y and z of the curve as functions of the third coordinate x, assumed as the independent variable: and so to write (comp. (6.)) these new developments,

$$\text{XXXII.}\ldots\ y_x = \frac{x^2}{2r} + \frac{y'''x^3}{6} + \frac{y^{\text{IV}}x^4}{24} + \frac{y^{\text{V}}x^5}{120}, \qquad z_x = \frac{x^3}{6r\mathrm{r}} + \frac{z^{\text{IV}}x^4}{24} + \frac{z^{\text{V}}x^5}{120};$$

and then the equation of the quadric cone XXXI. will be found to become (in xyz),

$$\text{XXXIII.}\ldots\ y^2 = \frac{3}{2}\frac{r}{\mathrm{r}}\,xz + 2gyz + hz^2,$$

with the coefficients,

$$\text{XXXIV.}\ldots\ g = r\mathrm{r}\left(y''' - \frac{3}{8}\mathrm{r}z^{\text{IV}}\right), \quad h = \frac{3}{2}r\mathrm{r}^2\left(y^{\text{IV}} - \frac{3}{10}\mathrm{r}z^{\text{V}}\right)$$
$$- r^2\mathrm{r}^2\left(y'''^2 + \frac{3}{4}\mathrm{r}z^{\text{IV}}y''' - \frac{9}{16}\mathrm{r}^2z^{\text{IV}2}\right);$$

while the cubic surface XXIX. will also come to be represented by an equation of the *same form* as before, namely (in xyz) by the following,

$$\text{XXXV.}\ldots\ xz\,(y + \mathrm{h}z) - \mathrm{r}z^2 = \mathrm{a}y^3 + \mathrm{b}y^2z + \mathrm{c}yz^2 + \mathrm{e}z^3,$$

in which the coefficients are,

$$\text{XXXVI.}\ldots\begin{cases} \mathrm{a} = \dfrac{4}{9}\dfrac{r}{\mathrm{r}}\ \text{(as before)}; \quad \mathrm{b} = -\dfrac{4}{3}r^2y''' + \dfrac{r^2\mathrm{r}}{2}z^{\text{IV}}; \quad \mathrm{h} = -r\mathrm{r}y''' + \tfrac{1}{2}r\mathrm{r}^2z^{\text{IV}}; \\[2ex] \mathrm{c} = \dfrac{4}{9}r^3\mathrm{r}y'''^2 - \tfrac{1}{2}r^3\mathrm{r}^2y'''z^{\text{IV}} - \tfrac{1}{2}r^2\mathrm{r}y^{\text{IV}} + \tfrac{1}{10}r^2\mathrm{r}^2z^{\text{V}}; \\[2ex] \mathrm{e} = -\dfrac{4}{9}r^4\mathrm{r}^2y'''^3 + \tfrac{1}{2}r^3\mathrm{r}^2y'''y^{\text{IV}} - \tfrac{1}{10}r^2\mathrm{r}^2y^{\text{V}}. \end{cases}$$

(14.) Whichever set of expressions for the *coefficients* we may adopt, some general consequences may be drawn from the mere *forms* of the *equations*,

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* The quadric cone XXXI. may be said to have *five-side contact* with the *cone of chords* of the given curve (compare the Note to page 125).

XXXI. and XXIX., or XXXIII. and XXXV., of the *quadric cone* and *cubic surface*, considered as *two loci* (12.) of the *osculating twisted cubic* to a given curve of double curvature. Thus, if we eliminate ac (comp. 398, (41.)) from XXIX. by XXXI., or xz by XXXIII. from XXXV., we get an equation between b, c, or between y, z, which rises *no higher* than the *third degree*, and is of the *form*,

$$\text{XXXVII.} \ldots 2rz^2 = ay^3 + b_{\prime}y^2z + c_{\prime}yz^2 + e_{\prime}z^3,$$

with the same value of a as before; such then is the equation of the *projection of the twisted cubic, on the normal plane* to the curve; and we see that, as was to be expected, the *plane cubic* thus obtained has a *cusp* at the given point P, which (when we neglect s^7 or x^7) *coincides* with the *corresponding cusp** of the projection of the given curve of double curvature *itself*, on the same normal plane.

(15.) The equation XXXVII. may also be considered as representing a *cubic cylinder*, which is a *third locus* of the twisted cubic; and on which the *tangent* PT to the curve is a *cusp-edge*, in such a manner that an arbitrary *plane through this line*, suppose the plane

$$\text{XXXVIII.} \ldots 3rz = vy,$$

where v is any assumed constant, *cuts* the cylinder *in that line twice*, and a *third* time in a *real* and parallel *right line*, which intersects the *quadric cone* in a point at infinity (because the tangent PT is a *side* of that cone), and in *another real point*, which is *on the twisted cubic*, and may be made to be *any point* of that sought curve, by a suitable value of v: in fact, the plane XXXVIII. *touches both curves* at P, and therefore *intersects* the *cubic* curve in *one* other real point. And thus may *fractional expressions* (comp. (8.)) for the coordinates of the *osculating cubic* be found *generally*, which we shall not here delay to write down.

(16.) Without introducing the *cubic cylinder* XXXVII., it is easy to see that *any plane*, such as XXXVIII., which is *tangential* to the given curve at P, *cuts* the *cubic surface* XXXV. in a section which may be said to consist of the *tangent twice taken*, and of a certain *other right line*, which varies with the direction of this secant plane, so that the *locus* XXXV. or XXIX. is a *Ruled*

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* Compare the first formula of the Note to page 133.

Cubic Surface, with the given tangent PT for a *singular* line*, which is *intersected* by *all the other right lines* on that surface, determined as above: and if we *set aside this line*, the *remaining part* of the *complete intersection* of that *cubic surface* with the *quadric cone* XXXIII. or XXXI. is the *twisted cubic sought*. We may then consider ourselves to have *completely* and *generally determined* the *Osculating Twisted Cubic* to a curve of double curvature, *without requiring* (as in 398, (41.)), the *solution* of any *cubic* or other *equation*.†

(17.) As illustrations and verifications, it may be added that the general *ruled cubic surface*, and *cubic cylinder*, lately considered, take for the case of the *helix* (2.), the particular forms,‡

$$\text{XXXIX.}\ldots xyz - \mathrm{r}z^2 = \frac{4}{9}\frac{r}{\mathrm{r}}y^3 + \left(\frac{2}{5}\frac{r}{\mathrm{r}} - \frac{3}{5}\frac{\mathrm{r}}{r}\right)yz^2,$$

and

$$\text{XL.}\ldots \mathrm{r}z^2 = \frac{2}{9}\frac{r}{\mathrm{r}}y^3 + \frac{3}{10}\left(\frac{r}{\mathrm{r}} + \frac{\mathrm{r}}{r}\right)yz^2;$$

and that accordingly these two last equations are satisfied, independently of w, when the *fractional expressions* XXI. are substitued for xyz.

400. The general theory§ of *evolutes of curves in space* may be briefly treated by quaternions, as follows: a *second curve* (in space, or in one plane) being *defined* to bear to a *first curve* the relation of *evolute* to *involute*, when the *first cuts the tangents to the second at right angles.*

(1.) Let ρ and σ be corresponding vectors, OP and OS, of involute and evolute, and let $\rho', \sigma', \rho'', \sigma''$ denote their first and second derivatives, taken

* If the *cubic surface* be cut by a plane *perpendicular* to the tangent PT, at any point T *distinct* from the point P itself, the *section* is a *plane cubic*, which has T for a *double point*; and this point counts for *three* of the *six common points*, or points of *intersection*, of the plane cubic just mentioned with the *plane conic* in which the *quadric cone* is cut by the same secant plane, because *one branch*, or *one tangent*, of the plane cubic at T *touches* the plane conic at that point, in the osculating plane to the given curve at P, while the *other branch*, or the *other tangent*, *cuts* that plane conic there.

† It may be remarked that, by equating the second member of XXXVII. to zero, and changing y, z to b, c, we obtain generally the *cubic equation*, referred to in 398, (41.); and that by suppressing the term $-\mathrm{r}c^2$ in XXIX., or the term $-\mathrm{r}z^2$ in XXXV., we pass, in like manner *generally*, from the *cubic surface* of recent sub-articles, to the earlier *cubic cone* (4.).

‡ By suppressing the term $-\mathrm{r}z^2$, dividing by $\frac{ry}{5\mathrm{r}}$, and transposing, we pass for the case of the *helix* from the equation XXXIX. of the *cubic locus*, to the equation IX'. in the last Note to page 159; namely to the equation of that *quadric cone* which forms (in this example) a *separable part* of the general *cubic cone*, the *other part* being here the *tangent plane* ($y = 0$) to the *right cylinder*.

§ Invented by Monge.

with respect to a scalar variable t, on which they are both conceived to depend. Then the two fundamental equations, which express the relation between the two curves, as above defined, are the following:

$$\text{I.} \ldots S(\sigma - \rho)\rho' = 0; \qquad \text{II.} \ldots V(\sigma - \rho)\sigma' = 0;$$

which express, respectively, that the point S is *in the normal plane* to the *involute* at P, and that the latter point is *on the tangent* to the *evolute* at S: so that the *locus of* P (the involute) is a *rectangular trajectory* to all such *tangents to the locus of* S (the evolute).

(2.) Eliminating $\sigma - \rho$ between the two preceding equations, and taking their derivatives, we find,

$$\text{III.} \ldots S\rho'\sigma' = 0, \quad \text{IV.} \ldots S(\sigma - \rho)\rho'' - \rho'^2 = 0, \quad \text{V.} \ldots V(\sigma - \rho)\sigma'' - V\rho'\sigma' = 0;$$

whence also, $$\text{VI.} \ldots S\rho'\sigma'\sigma'' = 0.$$

(3.) Interpreting these results, we see *first*, by IV. combined with I. (comp. 391, (5.)), that the *point* S of the evolute is *on the polar axis* of the involute at P, and therefore that the *evolute itself* is *some curve on the polar developable* of the *involute*; and *second*, by VI. (comp. 380, I.), that this curve is a *geodetic line* on that *polar surface*, because the *osculating plane to the evolute* at S contains the *tangent to the involute* at P, and therefore also the (parallel) *normal to the locus of evolutes.*

(4.) The *locus of centres of curvature* (395, (6.)) of a *curve in space* is *not* generally an *evolute* of that curve, because the *tangents** KK′ to that *locus* do not generally *intersect* the *curve* at all; but a *given plane involute* has always the locus just mentioned for *one* of its evolutes; and has, besides, indefinitely many *others*,† which are all *geodetics on the cylinder* which rests perpendicularly on that one *plane evolute* as its *base.*

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* It might have been remarked, in connexion with a recent series of sub-articles (397), that this tangent KK′ or κ' is inclined to the rectifying line λ, at an angle of which the cosine is,

$$- SU\kappa'\lambda = \pm R^{-1}T\lambda^{-1} = \pm \sin H \cos P;$$

upper or lower signs being taken, according as the second curvature r^{-1} is positive or negative, because $S\kappa'\lambda = - r^{-1}$.

† Compare the Note to page 53; from the formulæ of which page it *now* appears, that if the *involute* be an *ellipse*, with β = OB and γ = OC for its major and minor semiaxes, and therefore with the scalar equations,

$$(S\beta^{-1}\rho)^2 + (S\gamma^{-1}\rho)^2 = 1, \quad S\beta\gamma\rho = 0,$$

the *evolutes* are geodetics on the *cylinder* of which the corresponding equation is,

$$(S\beta\sigma)^{\frac{2}{3}} + (S\gamma\sigma)^{\frac{2}{3}} = (\beta^2 - \gamma^2)^{\frac{2}{3}}.$$

(5.) An easy combination of the foregoing equations gives,

$$\text{VII.} \ldots (\mathrm{T}(\sigma - \rho))' = -\mathrm{S}(\mathrm{U}(\sigma - \rho) \,.\, (\sigma' - \rho')) = \mp \mathrm{S}\sigma'\mathrm{U}\sigma' = \pm \mathrm{T}\sigma',$$

or with differentials,

$$\text{VIII.} \ldots \mathrm{dT}(\sigma - \rho) = \pm \mathrm{Td}\sigma\,;$$

whence by an immediate integration (comp. 380, XXII. and 397, LIV.),

$$\text{IX.} \ldots \Delta\mathrm{T}(\sigma - \rho) = \pm \int \mathrm{Td}\sigma = \pm \textit{ arc of the evolute}:$$

this arc then, *between two points* such as S and S_1 of the latter curve, is equal to the *difference between the lengths* of the *two lines*, PS and P_1S_1, intercepted *between the two curves* themselves.

(6.) Another quaternion combination of the same equations gives, after a few steps of reduction, the differential formula (comp. 335, VI.),

$$\text{X.} \ldots \mathrm{d} \cos \mathrm{OPS} = -\mathrm{dSU}\frac{\sigma - \rho}{\rho} = \frac{\mathrm{dT}\rho}{\mathrm{T}(\sigma - \rho)} \,.\, \mathrm{S}\frac{\sigma}{\rho}\,;$$

if then the *involute* be a curve *on a given sphere*, with its centre at the origin O, so that the *evolute* is a *geodetic on a concentric cone*, this differential X. *vanishes*, and we have the integrated equation,

$$\text{XI.} \ldots \cos \mathrm{OPS} = \text{const.}, \quad \text{or simply,} \quad \text{XI}'. \ldots \mathrm{OPS} = \text{const.}\,;$$

the *tangents* PS to the evolute being thus *inclined* (in the case here considered) at a *constant angle*,* to the *radii* OP of the *sphere*.

(7.) In general, if we denote by R the *interval* $\overline{\mathrm{PS}}$ between two *corresponding points* of involute and evolute, we shall have the equation,

$$\text{XII.} \ldots (\sigma - \rho)^2 + R^2 = 0, \quad \text{or} \quad \text{XII}'. \ldots \mathrm{T}(\sigma - \rho) = R\,;$$

and the formula VII. may be replaced by the following,

$$\text{XIII.} \ldots R'^2 + \sigma'^2 = 0, \quad \text{or} \quad \text{XIII}'. \ldots \mathrm{D}_t R = \pm \mathrm{TD}_t\sigma,$$

in which the independent variable t is still left *arbitrary*.

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* This property of the *evolutes of a spherical curve* was deduced by Professor De Morgan, in a Paper *On the Connexion of Involute and Evolute in Space* (Cambridge and Dublin Mathematical Journal for November, 1851); in which also a *definition* of involute and evolute was proposed, substantially the same as that above adopted.

(8.) But if we take for that variable the *arc* s_0s_t of the *evolute*, measured from some fixed point of that curve, we may then write,

$$\text{XIV.} \ldots t = \int \mathrm{T}d\sigma, \quad \text{XV.} \ldots dR_t = \pm\, dt, \quad \text{XVI.} \ldots D_t R_t = \pm 1;$$

whence

$$\text{XVII.} \ldots D_t(R_t \mp t) = 0, \quad \text{and} \quad \text{XVIII.} \ldots R_t \mp t = \text{const.} = R_0,$$

the integral IX. being thus under a new form reproduced.

(9.) In this *last mode* of obtaining the result,

$$\text{XIX.} \ldots \Delta\overline{\text{PS}} = R_t - R_0 = \pm\, t = \pm\, \textit{arc}\ \overset{\frown}{s_0 s_t}\ \textit{of evolute},$$

no use is made of *infinitesimals*,* or even of *small differentials*. We only infer, as in XVIII. (comp. 380, (9.)), that the quantity $R_t \mp t$ is *constant*,† because its *derivative* is *null*: it having been previously proved (380, (8.)), as a consequence of our *definition of differentials* (320, 324) that if s be the arc and ρ the vector of *any curve*, then the equation $ds = \mathrm{T}d\rho$ (380, XXII.) is *rigorously satisfied*, whatever the *independent variable* t may be, and whether the two connected and *simultaneous differentials* be *small* or *large*.

(10.) But when we employ the *notation of integrals*, and introduce, as above, the symbol $\int \mathrm{T}d\sigma$, we are then led to *interpret* that *symbol* as denoting the *limit of a sum* (comp. 345, (12.)); or to write, generally,

$$\text{XX.} \ldots \int \mathrm{T}d\rho = \lim.\ \Sigma \mathrm{T}\Delta\rho, \quad \text{if} \quad \lim.\ \Delta\rho = 0,$$

with analogous formulæ for other cases of *integration in quaternions*. Geometrically, the equation,

$$\text{XXI.} \ldots \int \mathrm{T}d\rho = \Delta s, \quad \text{or} \quad \text{XXI}'. \ldots \int \mathrm{T}d\sigma = \Delta t,$$

if s and t denote *arcs* of curves of which ρ and σ are *vectors*, comes thus to be interpreted as an expression of the well-known principle, that the *perimeter of any curve* (or of any *part* thereof) is the *limit of the perimeter* of an *inscribed polygon* (or of the corresponding *portion* of that polygon), when the *number*

* In general, it may have been observed that we have hitherto *abstained*, at least in the *text* of this whole Chapter of *Applications*, from making any use of *infinitesimals*, although they have been often referred to in these *Notes*, and employed therein to assist the *geometrical investigation* or *enunciation* of results. But as regards the *mechanism* of *calculation*, it is at least as easy to use *infinitesimals in quaternions* as in any other system: as will perhaps be shown by a few examples, farther on.

† Compare the Note to page 30.

of the *sides* is indefinitely *increased*, and when their *lengths* are *diminished* indefinitely.

(11.) The equations I. and XII. give,

$$\text{XXII.} \ldots S\sigma'(\sigma - \rho) + RR' = 0,$$

the independent variable t being again arbitrary; but these equations XII. and XXII. coincide with the formulæ 398, LXXXIX. and XCI.; we may then, by 398, (79.) and (80.), consider the *locus of the point* P as the *envelope of a variable sphere*, namely of the sphere which has S for centre and R for radius, and is represented by the recent equation XII., if $\rho = $ OP be the vector of a *variable point* thereon.

(12.) But whereas *such* an envelope has been seen to be *generally* a *surface*, which is *real* or *imaginary* (398, (79.)) according as $R'^2 + \sigma'^2 <$ or > 0, we have *here* by XIII. the *intermediate* or *limiting case* (comp. 398, CXXXI.), for which the *circles* of the system become *points*, and the surface itself degenerates into a *curve*, which is here the *involute* (P) above considered. The *involutes of a given curve* (S) are therefore *included*, as a *limit*, in that general *system of envelopes* which was considered in the lately cited sub-articles, and in others immediately following.

(13.) The *equation of condition*, 398, CXXXVI., is in this case satisfied by XIII., both members vanishing; but we cannot now put it under the form 398, CXLI., because in the passage to that form, in 398, (85.), there was tacitly effected a *division by* r^2, which is not now allowed, the radius r of the circle on the envelope being in the present case equal to zero. For a similar reason, we cannot now *divide* by g, as was done in 398, (86.); and because, in virtue of II., the *two* equations 398, CLX. reduce themselves to *one*, they no longer conduct to the formulæ 398, CLX′. CLXI. CLXI′. CLXIII. XCIV.; nor to the second equation 398, CLXII.

(14.) The general geometrical relations of the curves (P) and (S), which were investigated in the sub-articles to 398 for the case when the *condition**

* If, without thinking of *evolutes*, we merely suppose that the *condition* 398, CXXXVI. is satisfied, as lately in (13.), by our having the relation $R'^2 + \sigma'^2 = 0$, it will be found (comp. the symbolical expression 274, XX. for $0^{\frac{1}{2}}$, and the imaginary solution in 353, (18.) of the system $S\gamma\rho = 0$, $\rho^2 = 0$), that the *envelope* of the *sphere* $(\sigma - \rho)^2 + R^2 = 0$, or the *locus* of the (null) *circles* in which such spheres are (conceived to be) *cut* by the (tangent) *planes*, $S\sigma'(\sigma - \rho) + RR' = 0$, may be said to be *generally* the system of all those *imaginary points*, of which the vectors (or the *bivectors*, comp. 214, (6.)) are assigned by the formula,

$$\rho = \sigma - RR'^{-1}\sigma' + (U\sigma' + \sqrt{-1})\,V\sigma'\mu;$$

above referred to is satisfied, are therefore only very *partially* applicable to a system of *involute* and *evolute* in space: at least if we still consider the *former* curve (the involute) as being a *rectangular trajectory* to the *tangents* to the *latter* (the evolute), instead of being, like the curve (P) previously considered, a rectangular trajectory (398, (94.)) to the *osculating planes** of the curve (S).

(15.) If the *arc* of the *evolute* be again taken for the independent variable t, and if the positive direction of motion along that arc be always *towards* the *involute*, we may write,

$$\text{XXIII.} \ldots \rho = \sigma + R\sigma', \quad R' = -1, \quad \sigma'^2 = -1, \text{ \&c.};$$

whence

$$\text{XXIV.} \ldots \rho' = R\sigma'', \quad \rho'' = R\sigma''' - \sigma'', \quad \mathrm{V}\rho''\rho' = R^2\mathrm{V}\sigma'''\sigma'';$$

if then $\kappa = \mathrm{OK}$ be the vector of the centre K of the circle which osculates to the involute at P, the general formula 389, IV. gives, after a few reductions,† the expression (comp. 397, XVI. XXXIV., and XCVIII. (15.)),

$$\text{XXV.} \ldots \kappa = \rho + \frac{\rho'^3}{\mathrm{V}\rho''\rho'} = \sigma + R\left(\sigma' + \frac{\sigma''^3}{\mathrm{V}\sigma'''\sigma''}\right)$$

$$= \sigma + \frac{R\mathrm{S}\sigma'\sigma'''\sigma''}{\mathrm{V}\sigma'''\sigma''} = \sigma - \frac{R\mathrm{S}\sigma'\sigma''^{-1}\sigma'''}{\mathrm{V}\sigma'''\sigma''^{-1}}$$

$$= \sigma - R\mathrm{r}_1^{-1}\lambda_1^{-1} = \sigma + \mathrm{U}\lambda_1 \,.\, R\cos H_1,$$

if r_1, H_1, and λ_1 be what r, H, and λ in 397 become, when we pass from the curve (P) to the curve (S), with the present relations between those two curves; this *centre of curvature* K is therefore the *foot of the perpendicular* let fall *from the point* P *of the involute, on the rectifying line* λ_1 *of*

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where μ is an *arbitrary vector*, and $\sqrt{-1}$ is the *old imaginary* of algebra. By making $\mu = 0$ we reduce this expression for ρ to the *real vector form*,

$$\rho = \sigma - RR'^{-1}\sigma' = \sigma + RR'\sigma'^{-1},$$

= the κ of 398, CXXXI.''; and thus the *curve* (P), which is here the *locus of the centres of the null circles of contact*, and coincides with the *involute* in the present series of sub-articles, may *still* be called a *Singular Line* upon the *Envelope of the Sphere* (with *One Variable Parameter*), as being in the present case the *only real part* of that *elsewhere imaginary surface*.

* The curve to the *osculating planes* of which another curve is thus an *orthogonal trajectory*, and which is therefore (398, (95.)) the *cusp-edge* of the *polar developable* of the latter curve, was called by Lancret its *evolute by the plane* (developpée par le plan); whereas the curve (S) of the *present series* (400) of sub-articles, to whose *tangents* the corresponding curve (P) is an orthogonal trajectory, has been called by way of distinction the *evolute by the thread* (developpée par le fil) of this last curve. It would be improper to delay here on subjects so well known to geometers: but the student may be invited to read again, in connexion with them, the sub-articles (88.) and (89.) to Art. 398.

† Especially by observing that $\mathrm{V}\sigma'\mathrm{V}\sigma'''\sigma'' = -\sigma''^3$, because $\mathrm{S}\sigma'\sigma'' = 0$, and $\mathrm{S}\sigma'\sigma''' = -\sigma''^2$.

the evolute: as indeed is evident from geometrical considerations, because by (3.) this *rectifying line* of the curve (S) is the *polar axis* of the curve (P).

(16.) If we conceive (comp. 389, (2.)) an *auxiliary spherical curve* to be described, of which the variable unit-vector shall be,

$$\text{XXVI.} \ldots \text{OT} = \tau = \sigma' = \mathrm{U}(\rho - \sigma) = R^{-1}(\rho - \sigma),$$

and suppose that υ is the vector OU of the centre of curvature of this *new curve*, at the point T which corresponds to the point S of the evolute, we shall then have by XXV. the expression,

$$\text{XXVII.} \ldots \text{TU} = \upsilon - \tau = \frac{\tau'^3}{\mathrm{V}\tau''\tau'} = \frac{\sigma''^3}{\mathrm{V}\sigma'''\sigma''} = \frac{\kappa - \rho}{R} = \text{PK} : \overline{\text{PS}};$$

we have therefore this *theorem*, that the *inward radius of curvature of the hodograph of the evolute* (conceived to be an *orbit* described, as in 379, (9.), with a *constant velocity* taken for *unity*) is equal to the *inward radius of curvature of the involute, divided by the interval R* between the two curves (P) and (S): and that these *two radii of curvature*, TU and PK, have one *common direction*, at least if the direction of *motion on the evolute* be supposed, as in (15.), to be *towards the involute*.

(17.) The following is perhaps a simpler enunciation of the theorem* just stated:—*If* $\text{P}, \text{P}_1, \text{P}_2, \ldots$ *and* $\text{S}, \text{S}_1, \text{S}_2, \ldots$ *be corresponding points of involute and evolute, and if we draw lines* $\text{ST}_1 \parallel \text{S}_1\text{P}_1$, $\text{ST}_2 \parallel \text{S}_2\text{P}_2, \ldots$ *with a common length* $= \overline{\text{SP}}$, *the spherical curve* $\text{PT}_1\text{T}_2 \ldots$ *will then have contact of the second order with the curve* $\text{PP}_1\text{P}_2 \ldots$, that is with the involute at P.

401. The fundamental formula 389, IV., for the vector of the centre of the osculating circle to a curve in space, namely the formula,

$$\text{I.} \ldots \kappa = \rho + \frac{\rho'^3}{\mathrm{V}\rho''\rho'}, \quad \text{or} \quad \text{II.} \ldots \kappa = \rho + \frac{\mathrm{d}\rho^3}{\mathrm{V}\mathrm{d}^2\rho\mathrm{d}\rho},$$

which has been so extensively employed throughout the present Section, has hitherto been established and used in connexion with *derivatives* and

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* Some additional light may be thrown on this theorem, by comparing it with the *construction* in 397, (48.); and by observing that the equations 397, XVI. XXXIV. give generally, in the notations of the Article referred to, for the vector of the *centre of curvature* of the *hodograph of any curve*, the transformations,

$$\tau + \frac{\tau'}{\mathrm{V}\tau''\tau'^{-1}} = \tau - \frac{\tau'}{\lambda} = -r^{-1}\lambda^{-1} = \mathrm{U}\lambda \,.\, \cos H.$$

differentials of vectors, rather than with *differences*, great or small. We may however establish, in another way, an essentially *equivalent* formula, into which *differences* enter by their *limits* (or rather by their *limiting relations*), namely, the following,

$$\text{III.} \ldots \kappa = \rho + \lim. \frac{\Delta\rho^3}{\mathrm{V}\Delta^2\rho\Delta\rho}, \quad \text{if} \quad \lim. \Delta\rho = 0, \quad \text{and} \quad \lim. \frac{\Delta^2\rho}{\Delta\rho} = 0,$$

the denominator $\mathrm{V}\Delta^2\rho\Delta\rho$ being understood to signify the same thing as $\mathrm{V}(\Delta^2\rho \,.\, \Delta\rho)$; and *then* may, if we think fit, *interpret* the *differential expression* II. as if $d\rho$ and $d^2\rho$ in it denoted *infinitesimals*,* of the *first and second orders*: with similar interpretations in other but analogous investigations.

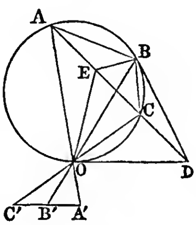

Fig. 58, *bis*.

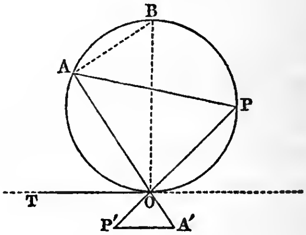

Fig. 64, *bis*.

(1.) If in the second expression 316, L.,† for the perpendicular from O on the line AB, we change α and β to their reciprocals (compare figures 58, 64, pp. 293, 349, vol. i.) and then take the reciprocal of the result, we obtain this new expression [but with the letters C and D referring to points not marked in fig. 58],

$$\text{IV.} \ldots \text{OD} = \delta = \frac{\alpha^{-1} - \beta^{-1}}{\mathrm{V}\beta^{-1}\alpha^{-1}} = \frac{\alpha(\beta - \alpha)\beta}{\mathrm{V}\beta\alpha} = \frac{\text{OA} \,.\, \text{AB} \,.\, \text{OB}}{\mathrm{V}(\text{OB} \,.\, \text{OA})},$$

in the denominator of which, OB may be replaced by AB, or by AO + AB, for the *diameter* OD of the *circle* OAB; so that if C be the *centre* of this circle, its vector $\gamma = \text{OC} = \frac{1}{2}\text{OD} = \frac{1}{2}\delta = \&\text{c}.$ Supposing then that P, Q, R are *any three points* of *any given curve in space*, while O is as usual an *arbitrary origin*, and writing

$$\text{V.} \ldots \text{OP} = \rho, \quad \text{OQ} = \rho + \Delta\rho, \quad \text{OR} = \rho + 2\Delta\rho + \Delta^2\rho,$$

and therefore

$$\text{VI.} \ldots \text{PQ} = \Delta\rho, \quad \text{QR} = \Delta\rho + \Delta^2\rho, \quad \tfrac{1}{2}\text{PR} = \Delta\rho + \tfrac{1}{2}\Delta^2\rho,$$

* Compare 345, (17.), and the first Note to page 170.

† [Namely $\rho = \dfrac{\mathrm{V}\beta\alpha}{\alpha - \beta}$ on page 427, vol. i.]

the *centre* C of the circle PQR has the following *rigorous expression* for its vector:

$$\text{VII.} \ldots \text{OC} = \gamma = \rho + \frac{\Delta\rho(\Delta\rho + \Delta^2\rho)(\Delta\rho + \frac{1}{2}\Delta^2\rho)}{\text{V}(\Delta^2\rho \, . \, \Delta\rho)};$$

whence passing to the *limit*, we obtain successively the expressions III. and II. for the vector κ of the *centre of curvature* to the curve PQR at P; the two other points, Q and R, being *both* supposed to *approach indefinitely* to the given point P, according to *any law* (comp. 392, (6.)), which allows the two successive vector chords, PQ and QR, to bear to each other an *ultimate ratio of equality*.

(2.) Instead of thus *first* forming a *rigorous* expression, such as VII., involving the *differences* $\Delta\rho$ and $\Delta^2\rho$; then *simplifying* the formula so found, by the rejection of *terms*, which become *indefinitely small*, with respect to the terms retained; and finally changing differences to *differentials* (comp. 344, (2.)), namely $\Delta\rho$ to $\text{d}\rho$, and $\Delta^2\rho$ to $\text{d}^2\rho$, in the *homogeneous* expression which results, and of which the *limit* is to be taken: we may *abridge* the *calculation*, by *at once writing* the *differential symbols*, in place of *differences*, and *at once suppressing* any terms, of which we *foresee* that they must *disappear from the final result*. Thus, in the recent example, when we have perceived, by quaternions, that if K be the centre of the circle PQR, the equation

$$\text{VIII.} \ldots \text{PK} = \frac{\text{PQ} \, . \, \text{QR} \, . \, \frac{1}{2}(\text{PQ} + \text{QR})}{\text{V}\{(\text{QR} - \text{PQ})\text{PQ}\}}$$

is *rigorous*, we may *at once* change *each* of the *three factors* of the *numerator* to $\text{d}\rho$, while the factor QR − PQ in the *denominator* is to be changed to $\text{d}^2\rho$; and thus the *differential expression* II., for the *inward vector-radius of curvature* $\kappa - \rho$, is at once obtained.

(3.) It is scarcely necessary to observe, that this expression for that *radius*, as a *vector*, agrees with and *includes* the known expressions for the same radius of curvature of a curve in space, considered as a (positive) *scalar*, which has been denoted in the present Section by the italic letter r (because the more usual symbol ρ would have *here* caused confusion). Thus, while the formula II. gives *immediately* (because $\text{Td}\rho = \text{d}s$) the equation,

$$\text{IX.} \ldots r^{-1}\text{d}s^3 = \text{TVd}\rho\text{d}^2\rho,$$

it gives also (because $\text{d}\rho^2 = -\text{d}s^2$, and $\text{Sd}\rho\text{d}^2\rho = -\text{d}s\text{d}^2s$) the transformed equation,

$$\text{X.} \ldots r^{-1}\text{d}s^2 = \sqrt{(\text{Td}^2\rho^2 - \text{d}^2s^2)};$$

and it conducts (by 389, VI.) to this still simpler formula (comp. the equation $r^{-1} = T\tau'$, 396, IX.),

$$\text{XI.} \ldots r^{-1}ds = TdUd\rho.$$

(4.) Accordingly, if we employ the *standard trinomial form* (295, I.) for a *vector*,

$$\text{XII.} \ldots \rho = ix + jy + kz,$$

which gives, by the laws of the symbols ijk (182, 183),

$$\text{XIII.} \ldots \begin{cases} d\rho = idx + jdy + kdz, & ds = Td\rho = \sqrt{(dx^2 + dy^2 + dz^2)}, \\ d^2\rho = id^2x + jd^2y + kd^2z, & Td^2\rho = \sqrt{(d^2x^2 + d^2y^2 + d^2z^2)}, \\ Vd\rho d^2\rho = i(dyd^2z - dzd^2y) + j(dzd^2x - dxd^2z) + k(dxd^2y - dyd^2x), & \\ Ud\rho = i\dfrac{dx}{ds} + j\dfrac{dy}{ds} + k\dfrac{dz}{ds}, & dUd\rho = id\dfrac{dx}{ds} + \ldots, \end{cases}$$

the recent equations IX. X. XI. take these *known forms*:

$$\text{IX}'. \ldots r^{-1}ds^3 = \sqrt{((dyd^2z - dzd^2y)^2 + \ldots)};$$

$$\text{X}'. \ldots r^{-1}ds^2 = \sqrt{(d^2x^2 + d^2y^2 + d^2z^2 - d^2s^2)};$$

$$\text{XI}'. \ldots r^{-1}ds = \sqrt{\left(\left(d\frac{dx}{ds}\right)^2 + \left(d\frac{dy}{ds}\right)^2 + \left(d\frac{dz}{ds}\right)^2\right)}.$$

(5.) The formula IV., which lately served us to determine a diameter of a circle through three given points, may be more symmetrically written as follows. *If* AD *be a diameter of the circle* ABC, *then*

$$\text{XIV.} \ldots \text{AD}\,.\,V(\text{AB}\,.\,\text{BC}) = \text{AB}\,.\,\text{BC}\,.\,\text{CA};$$

an equation* in which V(AB . BC) may be changed to V(AB . AC), &c., and in

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* A student might find it useful practice to verify, that if we write in like manner,

$$\text{XIV}'. \ldots \text{BE}\,.V(\text{BC}\,.\,\text{CA}) = \text{BC}\,.\,\text{CA}\,.\,\text{AB},$$

so that BE is a *second diameter*, then AB = ED, or ABDE is a *parallelogram*. He may employ the principles, that $\alpha\beta\gamma = \gamma\beta\alpha$, if $S\alpha\beta\gamma = 0$, and that $\beta\gamma - \gamma\beta = 2V\beta\gamma$; in virtue of which, after subtracting XIV′. from XIV., and dividing by V(BC . CA), or by its equal V(AB . BC), the equation AD − BE = 2AB is obtained, and proves the relation mentioned. It is easy also to prove that

$$\text{XIV}''. \ldots \text{BD}\,.V(\text{BC}\,.\,\text{CA}) = \text{AB}\,.\,S(\text{BC}\,.\,\text{CA}),$$

and therefore that ABDE is a *rectangle*.

which it may be remarked that each member is an expression (comp. 296, V.) for a *vector* AT, which *touches at* A *the segment* ABC: while its *length* is at once a representation of the *product of the lengths of the sides of the triangle* ABC, and also of the *double area of that triangle* (comp. 281, XIII.), *multiplied by the diameter of the circumscribed circle.*

(6.) In general, if PQRS be *any four concircular points*, they satisfy (by 260, IX., comp. 296, (3.)) the *condition of concircularity*,

$$\text{XV.} \ldots \mathrm{V}\left(\frac{\text{PS}}{\text{SQ}} \cdot \frac{\text{QR}}{\text{RP}}\right) = 0,$$

which may be thus transformed:*

$$\text{XVI.} \ldots \mathrm{V}\left(\frac{\text{PQ}}{\text{PS}} + \frac{\text{QP} + \text{QR}}{\text{PR}}\right) = \mathrm{V}\left(\frac{1}{\text{PS}} \cdot \text{PQ} \cdot \frac{\text{QP} + \text{QR}}{\text{PR}}\right).$$

Writing then (comp. VI., and the remarks in (2.)),

$$\text{XVII.} \ldots \text{PS} = \omega - \rho, \quad \text{PQ} = \mathrm{d}\rho, \quad \text{PR} = 2\mathrm{d}\rho + \mathrm{d}^2\rho, \quad \text{QP} + \text{QR} = \mathrm{d}^2\rho,$$

the second member is seen to be, *on the present plan*, an *infinitesimal* of the *second order*, which is therefore to be *suppressed*, because the first member is only of the *first order*; and thus we obtain at once the following *vector equation of the osculating circle* to the curve PQR at P,

$$\text{XVIII.} \ldots \mathrm{V}\left(\frac{\mathrm{d}\rho}{\omega - \rho} + \frac{\mathrm{d}^2\rho}{2\mathrm{d}\rho}\right) = 0;$$

which agrees with the equation 392, VI., although deduced in a quite different manner, and conducts anew to the expression II. for $\kappa - \rho$, under the form,

$$\text{XIX.} \ldots \frac{\mathrm{d}\rho}{\kappa - \rho} + \mathrm{V}\frac{\mathrm{d}^2\rho}{\mathrm{d}\rho}, \text{ as in 392, VIII.}$$

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* Without having recourse to this *transformation* XVI., we might treat the *condition* XV. by *infinitesimals*, as follows:

$$\text{XVII}'. \ldots \begin{cases} \dfrac{\text{PS}}{\text{QS}} = 1 + \dfrac{\text{PQ}}{\text{QS}} = 1 + \dfrac{\mathrm{d}\rho}{\omega - \rho - \mathrm{d}\rho} = 1 + \dfrac{\mathrm{d}\rho}{\omega - \rho}; \\[2ex] \dfrac{2\text{QR}}{\text{PR}} = 1 + \dfrac{\text{QP} + \text{QR}}{\text{PR}} = 1 + \dfrac{\mathrm{d}^2\rho}{2\mathrm{d}\rho + \mathrm{d}^2\rho} = 1 + \dfrac{\mathrm{d}^2\rho}{2\mathrm{d}\rho} \end{cases}$$

equating then to zero the vector part of the product of these two expressions, and suppressing the infinitesimal of the second order, the equation XVIII. of the osculating circle is obtained anew.

(7.) Again, if OD = δ be the *diameter* from the origin, of any *sphere* through that point O, which passes also through any *three other given points* A, B, C, with OA = α, &c., we have by 296, XXVI. the formula,

$$\text{XX.} \ldots \delta S\alpha\beta\gamma = V\alpha(\beta - \alpha)(\gamma - \beta)\gamma\,;$$

writing then (comp. XVII.),

$$\text{XXI.} \ldots \alpha = d\rho, \quad \beta - \alpha = d\rho + d^2\rho, \quad \gamma - \beta = d\rho + 2d^2\rho + d^3\rho,$$

and

$$\text{XXII.} \ldots \delta = 2\text{PS} = 2(\sigma - \rho),$$

where σ is (as in 395, &c.) the vector OS (from an *arbitrary origin* O) of the *centre* S of the *osculating sphere* to a curve of double curvature at P, we have by *infinitesimals*, suppressing terms which are of the *seventh* and higher *orders*, because the first member is only of the *sixth* order, and reducing* by the rules of quaternions,

$$\text{XXIII.} \ldots (\sigma - \rho)Sd\rho d^2\rho d^3\rho = \tfrac{1}{2}Vd\rho(d\rho + d^2\rho)(d\rho + 2d^2\rho + d^3\rho)$$
$$(3d\rho + 3d^2\rho + d^3\rho) = 3Vd\rho d^2\rho Sd\rho d^2\rho + d\rho^2 Vd^3\rho d\rho\,;$$

which agrees precisely with the formula 395, XIII., although obtained by a process so different.

(8.) Finally as regards the *osculating plane*, and the *second curvature*, of a curve in space, *infinitesimals* give *at once* for that *plane* the equation,

$$\text{XXIV.} \ldots S(\omega - \rho)d\rho d^2\rho = 0, \text{ agreeing with 376, V.};$$

and if *three consecutive elements* of the curve be represented (comp. XXI.) by the differential expressions,

$$\text{XXV.} \ldots \text{PQ} = d\rho, \quad \text{QR} = d\rho + d^2\rho, \quad \text{RS} = d\rho + 2d^2\rho + d^3\rho,$$

the second curvature r^{-1}, defined as in 396, is easily seen to be connected as

* Of the eighteen terms which would follow the sign of operation $\frac{1}{2}V$, if the second member of XXIII. were fully developed, *one* is of the *fourth order*, but is a *scalar*; *three* are of the *fifth* order, but have a *scalar sum*; *nine* are of orders *higher* than the sixth; and *two* terms of the *sixth* order are *scalars*, so that there remain only *three terms* of that order to be considered. In this manner it is found that the second member in question reduces itself to the sum of the *two vector parts*,

$$\tfrac{3}{2}V.(d\rho d^2\rho)^2 = 3Vd\rho d^2\rho\,.\,Sd\rho d^2\rho,$$

and

$$\tfrac{1}{2}d\rho^2V(d\rho d^3\rho + 3d^3\rho d\rho) = d\rho^2Vd^3\rho d\rho\,;$$

and thus the third member of XXIII. is obtained.

follows with the *angle* of a certain *auxiliary quaternion* q, which *differs infinitely little from unity*:

$$\text{XXVI.} \ldots \mathrm{r}^{-1}\mathrm{d}s = \angle\, q, \quad \text{if} \quad \text{XXVII.} \ldots q = \frac{\mathrm{V}(\mathrm{QR}\,.\,\mathrm{RS})}{\mathrm{V}(\mathrm{PQ}\,.\,\mathrm{QR})} = 1 + \frac{\mathrm{V}\mathrm{d}\rho\mathrm{d}^3\rho}{\mathrm{V}\mathrm{d}\rho\mathrm{d}^2\rho};$$

we have then the expression,

$$\text{XXVIII.} \ldots \textit{Second Curvature} = \mathrm{r}^{-1} = \frac{\mathrm{V}q}{\mathrm{d}\rho} = \mathrm{S}\frac{\mathrm{d}^3\rho}{\mathrm{V}\mathrm{d}\rho\mathrm{d}^2\rho},$$

which agrees with the formula 397, XXVII., and has been illustrated, in the sub-articles to 397 and 398, by numerous geometrical applications.

(9.) On the whole, then, it appears that although the *logic of derived vectors*, and of *differentials of vectors* considered as *finite lines, proportional* to such *derivatives*, is perhaps *a little clearer* than that of *infinitesimals*, because it shows *more evidently* (especially when combined with *Taylor's Series adapted to Quaternions*, 342, 375) that *nothing is neglected*, yet it is perfectly *possible* to *combine** quaternions, *in practice*, with methods founded on the *more usual notion* of *Differentials*, as *infinitely small differences*: and that when this combination is judiciously made, *abridgments of calculation* arise, *without any ultimate error*.

SECTION 7.

On Surfaces of the Second Order; and on Curvatures of Surfaces.

402. As early as in the First Book of these *Elements*, some specimens were given of the treatment or expression of *Surfaces of the Second Order* by *Vectors*; or by *Anharmonic Equations* which were *derived* from the theory of *vectors*, without any introduction, at that stage, of *Quaternions* properly so called. Thus it was shown, in the sub-articles to 98, that a very simple *anharmonic equation* ($xz = yw$) might represent either a *ruled paraboloid*, or a *ruled hyperboloid*, according as a certain *condition* ($ac = bd$) was or was not satisfied, by the *constants* of the surface. Again, in the

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* Compare the first Note to page 170. It will however be of course necessary, in any *future applications* of quaternions, to specify in *which* of these *two senses*, as a *finite differential*, or as an *infinitesimal*, such a *symbol* as $\mathrm{d}\rho$ is employed.

sub-articles to 99, *two* examples were given, of *vector expressions* for *cones* of the *second order* (and *one* such expression for a *cone* of the *third* order, with a *conjugate ray* (99, (5.)); while an expression of the same sort, namely,

$$\text{I.} \ldots \rho = x\alpha + y\beta + z\gamma, \quad \text{with} \quad x^2 + y^2 + z^2 = 1,$$

was assigned (99, (2.)) as representing *generally* an *ellipsoid*,* with α, β, γ, or OA, OB, OC, for three *conjugate semidiameters*. And finally, in the sub-articles (11.) and (12.) to Art. 100, an instance was furnished of the determination of a *tangential plane to a cone*, by means of *partial derived vectors*.

403. In the Second Book, a much greater *range of expression* was attained, in consequence of the introduction of the *peculiar symbols*, or *characteristics of operation*, which belong to the present Calculus; but still with that *limitation* which was caused, by the *conception* and *notation* of a *Quaternion* being confined, in that Book, to *Quotients of Vectors* (112, 116, comp. 307, (5.)), without yet admitting *Products* or *Powers* of *Directed Lines in Space*: although *versors*, *tensors*, and even *norms*† of such *vectors* were already introduced (156, 185, 273).

(1.) The *Sphere*,‡ for instance, which has its *centre* at the *origin*, and has the *vector* OA, or α, with a *length* $T\alpha = a$, for one of its *radii*, admitted of being represented, not only (comp. 402, I.) by the *vector expression*,

$$\text{I.} \ldots \rho = x\alpha + y\beta + z\gamma, \quad x^2 + y^2 + z^2 = 1,$$

with

$$\text{I}'. \ldots T\alpha = T\beta = T\gamma = a, \quad \text{and} \quad \text{I}''. \ldots S\frac{\beta}{\alpha} = S\frac{\gamma}{\alpha} = S\frac{\gamma}{\beta} = 0,$$

---

* In like manner the expression,

$$\text{II.} \ldots \rho = x\alpha + y\beta + z\gamma, \quad \text{with} \quad x^2 + y^2 - z^2 = 1, \quad \text{or} \quad = -1,$$

represents a *general hyperboloid*, of *one sheet*, or of *two*, with $\alpha\beta\gamma$ for conjugate semi-diameters: while, with the *scalar equation* $x^2 + y^2 - z^2 = 0$, the *same vector expression* represents their *common asymptotic cone* (not generally of *revolution*).

† The *notation* $N\alpha$, for $(T\alpha)^2$, although not formally introduced before Art. 273, had been used by anticipation in 200, (3.), page 191, vol. i.

‡ That is to say, the *spheric surface* through A, with O for centre. Compare the Note to page 199, vol. i.

but also by any one of the following *equations*, in which it is permitted to change α to $-\alpha$:

$$\text{II.} \ldots \frac{\alpha}{\rho} = K\frac{\rho}{\alpha}; \qquad \text{III.} \ldots \frac{\rho}{\alpha} K \frac{\rho}{\alpha} = 1;$$

$$\text{IV.} \ldots N\frac{\rho}{\alpha} = 1; \quad 145, (8.), (12.) \qquad \text{V.} \ldots T\rho = a;$$

$$\text{VI.} \ldots T\rho = T\alpha; \qquad \text{VII.} \ldots T\frac{\rho}{\alpha} = 1; \ 186, (2.), 187, (1.)$$

$$\text{VIII.} \ldots S\frac{\rho - \alpha}{\rho + \alpha} = 0; \qquad \text{IX.} \ldots N\frac{\rho}{\varepsilon} = N\frac{\alpha}{\varepsilon};$$

$$\text{X.} \ldots N\rho = N\alpha; \quad 200, (11.), \ 215, (10.), \ 273, (1.) \qquad \text{XI.} \ldots \left(S\frac{\rho}{\alpha}\right)^2 - \left(V\frac{\rho}{\alpha}\right)^2 = 1;$$

$$\text{XII.} \ldots NS\frac{\rho}{\alpha} + NV\frac{\rho}{\alpha} = 1; \ 204, (6.), \text{XXV., XXVI.}$$

$$\text{XIII.} \ldots N\left(S\frac{\rho}{\alpha} + V\frac{\rho}{\alpha}\right) = 1; \qquad \text{XIV.} \ldots T\left(S\frac{\rho}{\alpha} + V\frac{\rho}{\alpha}\right) = 1; 204, (9.)$$

or by the *system of equations*,

$$\text{XV.} \ldots S\frac{\rho}{\alpha} = x, \quad \left(V\frac{\rho}{\alpha}\right)^2 = x^2 - 1 \ (\leqq 0), \qquad 204, (4.)$$

representing a *system of circles*, with the *spheric surface* for their *locus*.

(2.) *Other forms* of equation, for the same spheric surface, may on the same principles be assigned; for example we may write,

$$\text{XVI.} \ldots \frac{\rho}{\alpha} = K\frac{\alpha}{\rho}; \qquad \text{XVII.} \ldots N\frac{\alpha}{\rho} = 1; \qquad \text{XVIII.} \ldots T\frac{\alpha}{\rho} = 1;$$

$$\text{XIX.} \ldots \angle\frac{\rho - \alpha}{\rho + \alpha} = \frac{\pi}{2}; \qquad \text{XX.} \ldots S\frac{2\alpha}{\rho + \alpha} = 1; \qquad \text{XXI.} \ldots S\frac{2\rho}{\rho + \alpha} = 1;$$

or (comp. 186, (5.), and 200, (3.)),

$$\text{XXII.} \ldots T(\rho - c\alpha) = T(c\rho - \alpha), \quad c^2 \gtrless 1;$$

under which *last form*, the *sphere* may be considered to be generated by the *revolution* of the *circle*, which has been already spoken of as the *Apollonian* Locus.*

(3.) And from *any one* to *any other*, of all these various *forms*, it is possible, and easy to *pass*, by general *Rules of Transformation*,† which were established in the Second Book: while *each* of them is capable of receiving, on the principles of the same Book, a *Geometrical Interpretation.*

(4.) But we could not, on the principles of the Second Book *alone*, advance to such subsequent *equations* of the same *sphere*, as

$$\text{XXIII.} \ldots \rho^2 = a^2, \quad \text{or} \quad \text{XXIV.} \ldots \rho^2 + a^2 = 0, \qquad 282, \text{VII. XIII.}$$

whereof the latter includes (282, (9.)) the important equation $\rho^2 + 1 = 0$, or $\rho^2 = -1$, of what we have called the *Unit-Sphere* (128); nor to such an *exponential expression* for the *variable vector* ρ of the *same spheric surface*, as

$$\text{XXV.} \ldots \rho = ak^t j^s k j^{-s} k^{-t}, \qquad 308, \text{XVIII.}$$

in which j and k belong to the fundamental system ijk of *three rectangular unit-lines* (295), connected by the fundamental Formula A of Art. 183, namely,

$$i^2 = j^2 = k^2 = ijk = -1, \qquad (A)$$

while s and t are *two arbitrary* and *scalar variables*, with simple *geometrical*‡ *significations*: because we were not *then* prepared to introduce any *symbol*, such as ρ^2, or k^t, which should represent a *square* or other *power* of a *vector*.§ And similar remarks apply to the representation, by quaternions, of *other surfaces of the second order.*

* Compare the first Note to page 130, vol. i.

† This *richness of transformation*, of *quaternion expressions* or equations, has been noticed, by some friendly critics, as a *characteristic* of the present *Calculus*. In the preceding parts of this work, the reader may compare pages 130, 141, 185, vol. i., and pages 106, 108, 109, vol. ii.; in the two last of which, the *variety* of the expressions for the *second curvature* (r^{-1}) of a *curve in space* may be considered worthy of remark. On the other hand, it may be thought remarkable that, in this Calculus, a *single expression*, such as that given by the first formula (389, IV.) of page 50, vol. ii., adapts itself with *equal ease* to the determination of the vector (κ) of the *centre* of the *osculating circle*, to a *plane curve*, and to a curve of *double curvature*, as has been sufficiently exemplified in the foregoing Section.

‡ Compare the second Note to page 398, vol. i.

§ It is true that the formula A was established in the course of the Second Book (page 160, vol. i.); but it is to be remembered that the symbols ijk were *there* treated as denoting a system of *three right versors*, in *three mutually rectangular planes* (181): although it has *since* been found possible and useful, in this Third Book, to *identify* those right *versors* with their own *indices* or *axes* (295), and so to treat them as a system of *three rectangular lines*, as above.

404. A brief review, or *recapitulation*, of some of the chief expressions connected with the *Ellipsoid*, for example, which have been already established in these *Elements*, with *references* to a few others, may not be useless here.

(1.) Besides the *vector expression* $\rho = x\alpha + y\beta + z\gamma$, with the *scalar relation* $x^2 + y^2 + z^2 = 1$, and with *arbitrary vector values* of the constants α, β, γ, which was lately cited (402) from the First Book, or the equations 403, I., *without* the *conditions* 403, I′., II′. which are peculiar to the *sphere*, there were given in the Second Book (204, (13.), (14.)) equations which differed from those lately numbered as 403, XI. XII. XIII. XIV. XV., only by the substitution of $\mathrm{V}\frac{\rho}{\beta}$ for $\mathrm{V}\frac{\rho}{\alpha}$; for instance, there was the equation,

$$\text{I.} \ldots \left(\mathrm{S}\frac{\rho}{\alpha}\right)^2 - \left(\mathrm{V}\frac{\rho}{\beta}\right)^2 = 1, \qquad 204, (14.)$$

analogous to 403, XI., and representing generally* an *ellipsoid*, regarded as the *locus* of a certain system of *ellipses*, which were thus substituted for the *circles*† (403, XV.) of the *sphere*, by a species of geometrical *deformation*, which led to the establishment of certain *homologies* (developed in the sub-articles to 274).

(2.) Employing still *only quotients* of *vectors*, but introducing *two other pairs* of *vector-constants*, γ, δ and ι, κ, instead of the pair α, β in the equation I., which were however connected with that pair and with each other by certain assigned *relations*, that equation was transformed successively to

$$\text{II.} \ldots \mathrm{T}\left(\frac{\rho}{\gamma} + \mathrm{K}\frac{\rho}{\delta}\right) = 1, \qquad 216, \text{X.}$$

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* In the *case of parallelism* of the two vector constants ($\beta \parallel \alpha$), the equation I. represents generally a *Spheroid of revolution*, with its *axis* in the direction of α; while in the contrary *case of perpendicularity* ($\beta \perp \alpha$), the same equation I. represents an *elliptic Cylinder*, with its *generating lines* in the direction of β. Compare 204, (10.), (11.), and the Note to page 231, vol. i.

† The equation I. might also have been thus written, on the principles of the Second Book,

$$\text{I}'. \ldots \left(\mathrm{S}\frac{\rho}{\alpha} + \mathrm{S}\frac{\rho}{\beta}\right)\left(\mathrm{S}\frac{\rho}{\alpha} - \mathrm{S}\frac{\rho}{\beta}\right) + \left(\mathrm{T}\frac{\rho}{\beta}\right)^2 = 1;$$

whence it would have followed at once (comp. 216, (7.)), that the *ellipsoid* I. is *cut in two circles*, with a common radius = $\mathrm{T}\beta$, by the *two diametral planes*,

$$\text{I}''. \ldots \mathrm{S}\frac{\rho}{\alpha} + \mathrm{S}\frac{\rho}{\beta} = 0, \quad \mathrm{S}\frac{\rho}{\alpha} - \mathrm{S}\frac{\rho}{\beta} = 0.$$

In fact, this equation I′. is what was called in 359 a *cyclic form*, while I. itself is what was there called a *focal form*, of the equation of the surface; the lines $\alpha^{-1} \pm \beta^{-1}$ being, by the Third Book, the *two* (real) *cyclic normals*, while β is *one* of the two (real) *focal lines* of the (imaginary) *asymptotic cone*. Compare the Note to page 535, vol. i.

and to a form which may be written thus (comp. 217, (5.)),

$$\text{III.} \ldots \mathrm{T}\left(\iota + \mathrm{K}\frac{\kappa}{\rho} \cdot \rho\right)\mathrm{T}\rho = \mathrm{T}\iota^2 - \mathrm{T}\kappa^2; \qquad 217, \text{XVI.}$$

and this last *form* was *interpreted*, so as to lead to a *Rule of Construction** (217, (6.), (7.)), which was illustrated by a *Diagram* (fig. 53), and from which many *geometrical properties* of that surface were deduced (218, 219) in a very simple manner, and were confirmed by calculation with quaternions: the equation and construction being also modified afterwards, by the introduction (220) of a *new pair* of vector-constants, ι' and κ', which were shown to admit of being substituted for ι and κ, in the recent form III.

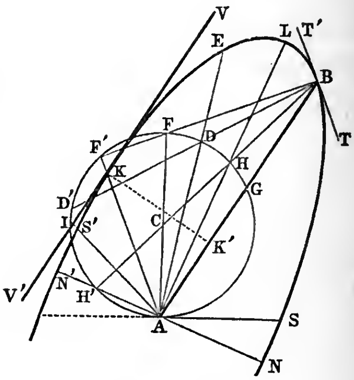

Fig. 53, *bis*.

(3.) And although the *Equation of Conjugation*

$$\text{IV.} \ldots \mathrm{S}\frac{\lambda}{\alpha}\mathrm{S}\frac{\mu}{\alpha} - \mathrm{S}\left(\mathrm{V}\frac{\lambda}{\beta}.\mathrm{V}\frac{\mu}{\beta}\right) = 1, \quad 316, \text{LXIII.}$$

which connects the vectors λ, μ of any two points L, M, whereof *one* is on the *polar plane* of the other, with respect to the ellipsoid I., was not *assigned* till near the end of the First Chapter of the present Book, yet it was there *deduced* by principles and processes of the Second Book *alone*: which thus were *adequate*, although not in the most practically convenient way, to the treatment of questions respecting *tangent planes* and *normals* to an *ellipsoid*, and similarly for *other surfaces*† of the same second order.

* This *Construction of the Ellipsoid*, by means of a *Generating Triangle* and a *Diacentric Sphere* (page 234, vol. i.), is believed to have been new, when it was deduced by the writer in 1846, and was in that year stated to the Royal Irish Academy (see its *Proceedings*, vol. iii., pp. 288, 289), as a result of the *Method of Quaternions*, which had been previously communicated by him to that Academy (in the year 1843).

† The following are a few other references, on this subject, to the Second Book. Expressions for a *Right Cone* (or for a single *sheet* of such a cone) have been given in vol. i. in pages 121, 180, 226, 227. In page 181 the equation $\mathrm{S}\frac{\rho}{\alpha}\mathrm{S}\frac{\beta}{\rho} = 1$, has been assigned, with a transformation in page 182, to represent generally a *Cyclic Cone*, or a cone of the *second order*, with its vertex at the origin; and to exhibit its *cyclic planes*, and *subcontrary sections* (pp. 182, 184). *Right Cylinders* have occurred in pages 195, 199, 201, 202, 223. A case of an *Elliptic Cylinder* has been already mentioned (the case when $\beta \perp \alpha$ in I.); and a transformation of the equation III. of the *Ellipsoid*, by means of *reciprocals* and *norms* of *vectors*, was assigned in page 314. And several expressions (comp. 403), for a *Sphere* of which the *origin* was *not* the *centre*, occurred in pages 165, 180, 192, vol. i., and perhaps elsewhere, without any employment of *products of vectors*.

(4.) But in this Third Book we have been able to write the equation III. under the simpler form,*

$$\text{V.} \ldots \mathrm{T}(\iota\rho + \rho\kappa) = \kappa^2 - \iota^2, \qquad 282, \text{XXIX.}$$

which has again admitted of numerous transformations; for instance, of all those which are obtained by equating $(\kappa^2 - \iota^2)^2$ to any one of the expressions 336, (5.), for the *square* of this last *tensor* in V., or for the *norm* of the *quaternion* $\iota\rho + \rho\kappa$; *cyclic forms*† of equation thus arising, which are easily converted into focal forms (359); while a *rectangular transformation* (373, XXX.) has subsequently been assigned, whereby the *lengths* (*abc*), and also the *directions*, of the *three semiaxes* of the surface, are expressed in terms of the *two vector-constants*, ι, κ: the results thus obtained by *calculation* being found to agree with those previously deduced, from the geometrical *construction* (2.) in the Second Book.

(5.) The equation V. has also been *differentiated* (336), and a *normal vector* $\nu = \phi\rho$ has thus been deduced, such that, for the ellipsoid in question,

$$\text{VI.} \ldots \mathrm{S}\nu d\rho = 0, \quad \text{and} \quad \text{VII.} \ldots \mathrm{S}\nu\rho = 1;$$

a process which has since been *extended* (361), and appears to furnish one of the best *general methods* of treating *surfaces*‡ *of the second order* by quaternions: especially when combined with that theory of *linear and vector functions* ($\phi\rho$) *of vectors*, which was developed in the Sixth Section§ of the Second Chapter of the present Book.

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* Mentioned by anticipation in the Note to page 241, vol. i.

† Compare the second Note to page 183. The vectors ι and κ are here the *cyclic normals*, and $\iota - \kappa$ is *one* of the *focal lines*; the *other* being the line $\iota' - \kappa'$ of the page 241, vol. i.

‡ The following are a few additional references to preceding parts of this Third Book, which has extended to a much greater length than was designed (page 322, vol. i.). In the First Chapter, the reader may consult pages 325, 326, 327, 328, vol. i., for some other forms of equation of the ellipsoid and the sphere. In the Second Chapter, pages 460, 461, vol. i., contain some useful practice, above alluded to, in the differentiation and transformation of the equation $r^2 = \mathrm{T}(\iota\rho + \rho\kappa)$. As regards the Sixth Section of that Chapter, which we are about to *use* (405), as one supposed to be familiar to the reader, it may be sufficient here to mention Arts. 357–362, and the Notes (or some of them) to pages 523, 525, 527, 535, 546, 549, vol. i. In this Third Chapter, the sub-articles (7.)–(21.) to 373 (pages 15, &c.) might be re-perused; and perhaps the investigations respecting *cones* and *sphero-conics*, in 394, and its sub-articles (pages 63, &c.), including remarks on an *hyperbolic cylinder*, and its *asymptotic planes* (in page 72). Finally, in a few longer and later series of sub-articles, to Arts. 397, &c., a certain degree of *familiarity* with some of the chief properties of surfaces of the second order has been assumed; as in pages 103, 126, 129, and generally in the recent investigations respecting the *osculating twisted cubic* (pages 129, 166, &c.), to a *helix*, or *other curve in space*.

§ It appears that this *Section* may be conveniently *referred to*, as III. ii. 6; and similarly in other cases.

405. Dismissing then, at least for the present, the *special* consideration of the *ellipsoid*, but still confining ourselves, for the moment, to *Central Surfaces* of the *Second Order*, and using freely the principles of this Third Book, but especially those of the Section (III. ii. 6) last referred to, we may denote any such *central* and *non-conical surface* by the scalar equation (comp. 361, [p. 547, vol. i.]),

$$\text{I.} \ldots f\rho = S\rho\phi\rho = 1\,;$$

the *asymptotic cone* (real or imaginary) being represented by the connected equation,

$$\text{II.} \ldots f\rho = S\rho\phi\rho = 0\,;$$

and the *equation of conjugation*, between the vectors ρ, ρ' of any *two points* P, P′, which are *conjugate* relatively to this surface I. (comp. 362, and 404, (3.)), see also 373, (20.)), being,

$$\text{III.} \ldots f(\rho, \rho') = f(\rho', \rho) = S\rho\phi\rho' = S\rho'\phi\rho = 1\,;$$

while the *differential equation* of the surface is of the form (361),

$$\text{IV.} \ldots 0 = \mathrm{d}f\rho = 2S\nu\mathrm{d}\rho, \quad \text{with} \quad \text{V.} \ldots \nu = \phi\rho\,;$$

this *vector-function* $\phi\rho$, which represents the *normal* ν to the surface, being at once *linear* and *self-conjugate* (361, (3.)); and the *surface itself* being the *locus* of all the *points* P which are *conjugate to themselves*, so that its *equation* I. may be thus written,

$$\text{I}'. \ldots f(\rho, \rho) = 1, \quad \text{because} \quad f(\rho, \rho) = f\rho, \qquad \text{by 362, IV.}$$

(1.) Such being the *form* of $\phi\rho$, it has been seen that there are always *three real* and *rectangular unit-lines*, α_1, α_2, α_3, and *three real scalars*, c_1, c_2, c_3, such as to satisfy (comp. 357, III.) the *three vector equations*,

$$\text{VI.} \ldots \phi\alpha_1 = -c_1\alpha_1, \quad \phi\alpha_2 = -c_2\alpha_2, \quad \phi\alpha_3 = -c_3\alpha_3\,;$$

whence also these *three scalar equations* are satisfied,

$$\text{VII.} \ldots f\alpha_1 = c_1, \quad f\alpha_2 = c_2, \quad f\alpha_3 = c_3\,;$$

and therefore (comp. 362, VII.),

$$\text{VIII.} \ldots f(c_1^{-\frac{1}{2}}\alpha_1) = f(c_2^{-\frac{1}{2}}\alpha_2) = f(c_3^{-\frac{1}{2}}\alpha_3) = 1.$$

(2.) It follows then that the *three* (real or imaginary) *rectangular lines*,

$$\text{IX.} \ldots \beta_1 = c_1^{-\frac{1}{2}}\alpha_1, \quad \beta_2 = c_2^{-\frac{1}{2}}\alpha_2, \quad \beta_3 = c_3^{-\frac{1}{2}}\alpha_3,$$

are the *three* (real or imaginary) *vector semiaxes* of the surface I.; and that the *three* (positive or negative) *scalars*, c_1, c_2, c_3, namely the *three roots* of the *scalar* and *cubic equation** $M = 0$ (comp. 357, (1.)), are the (always real) *inverse squares* of the three (real or imaginary) *scalar semiaxes*, of the same central surface of the second order.

(3.) For the *reality* of that *surface* I., it is necessary and sufficient that *one at least* of the three scalars c_1, c_2, c_3 should be *positive*; if *all* be such, the surface is an *ellipsoid*; if *two*, but not the third, it is a *single-sheeted hyperboloid*; and if only *one*, it is a *double-sheeted hyperboloid*: those scalars being here supposed to be each *finite*, and different from *zero*.

(4.) We have already seen (357, (2.)) how to obtain the *rectangular transformation*,

$$\text{X.} \ldots f\rho = c_1(S\alpha_1\rho)^2 + c_2(S\alpha_2\rho)^2 + c_3(S\alpha_3\rho)^2,$$

which may now, by IX., be thus written,

$$\text{XI.} \ldots f\rho = (S\beta_1^{-1}\rho)^2 + (S\beta_2^{-1}\rho)^2 + (S\beta_3^{-1}\rho)^2;$$

but it is to be remembered that, by (2.) and (3.), *one* or even *two* of these three *vectors* $\beta_1\beta_2\beta_3$ may become *imaginary*, without the *surface* ceasing to be *real*.

(5.) We had also the *cyclic transformation* (357, II. II'.),

$$\text{XII.} \ldots f\rho = g\rho^2 + S\lambda\rho\mu\rho = \rho^2(g - S\lambda\mu) + 2S\lambda\rho S\mu\rho,$$

in which the scalar g and the vector λ, μ are *real*, and the latter have the directions of the two (real) cyclic normals;† in fact it is obvious on inspection, that the surface is *cut in circles*, by *planes perpendicular* to these *two last lines*.

* It is unnecessary *here* to write $M_0 = 0$, as in page 520, vol. i., &c., because the function ϕ is here supposed to be *self-conjugate*; its *constants* being also *real*.

† Compare the Note to page 527, vol. i., see also the proof by quaternions, in 373, (16.), &c., of the known theorem, that any two *subcontrary* circular sections are *homospherical*, with the *equation* (373, XLIV.) of their *common sphere*, which is found to have its *centre* in the *diametral plane* of the *two cyclic normals* λ, μ.

(6.) It has been proved that the *four real scalars*, $c_1c_2c_3g$, and the *five real vectors*, $\alpha_1\alpha_2\alpha_3\lambda\mu$, are connected by the relations* (357, XX. and XXI.),

$$\text{XIII.} \ldots c_1 = -g - \mathrm{T}\lambda\mu, \quad c_2 = -g + \mathrm{S}\lambda\mu, \quad c_3 = -g + \mathrm{T}\lambda\mu;$$

$$\text{XIV.} \ldots \alpha_1 = \mathrm{U}(\lambda\mathrm{T}\mu - \mu\mathrm{T}\lambda), \quad \alpha_2 = \mathrm{UV}\lambda\mu, \quad \alpha_3 = \mathrm{U}(\lambda\mathrm{T}\mu + \mu\mathrm{T}\lambda);$$

at least if the *three roots* $c_1c_2c_3$ of the cubic $M = 0$ be arranged in *algebraically ascending order* (357, IX.), so that $c_1 < c_2 < c_3$.

(7.) It may happen (comp. (3.)), that *one* of these three roots *vanishes*; and in that case (comp. (2.)), *one* of the three *semiaxes* becomes *infinite*, and the *surface* I. becomes a *cylinder*.

(8.) Thus, in particular, if $c_1 = 0$, or $g = -\mathrm{T}\lambda\mu$, so that the *two other* roots are both *positive*, the equation takes (by XII., comp. 357, XXII.) a form which may be thus written,

$$\text{XV.} \ldots (\mathrm{S}\lambda\mu\rho)^2 + (\mathrm{S}\lambda\rho\mathrm{T}\mu + \mathrm{S}\mu\rho\mathrm{T}\lambda)^2 = \mathrm{T}\lambda\mu - \mathrm{S}\lambda\mu > 0;$$

and it represents an *elliptic cylinder*.†

(9.) Again, if $c_2 = 0$, or $g = \mathrm{S}\lambda\mu$, the equation becomes,

$$\text{XVI.} \ldots 2\mathrm{S}\lambda\rho\mathrm{S}\mu\rho = 1,$$

and represents an *hyperbolic cylinder*; the root c_1 being in this case *negative*, while the remaining root c_3 is *positive*.

(10.) But if we suppose that $c_3 = 0$, or $g = \mathrm{T}\lambda\mu$, so that c_1 and c_2 are *both negative*, the equation may (by 357, XXIII.) be reduced to the form,

$$\text{XVII.} \ldots (\mathrm{S}\lambda\mu\rho)^2 + (\mathrm{S}\lambda\rho\mathrm{T}\mu - \mathrm{S}\mu\rho\mathrm{T}\lambda)^2 = -\mathrm{T}\lambda\mu - \mathrm{S}\lambda\mu < 0;$$

it represents therefore, in *this* case, *nothing real*, although it may be *said* to be, in the same case, the equation of an *imaginary*‡ *elliptic cylinder*.

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* These relations and a few others mentioned are so useful that, although they occurred in an earlier part of the work, it seems convenient to restate them here.

† [XV. and XVII. may be directly obtained by means of the identity $\rho = (\mathrm{V}\lambda\mu)^{-1}(\mathrm{S}\lambda\mu\rho + \mathrm{VV}\lambda\mu . \rho)$.]

‡ In the Section (III. ii. 6) above referred to, many symbolical results have been established, respecting *imaginary cyclic normals*, or *focal lines*, &c., on which it is unnecessary to return. But it it may be remarked that as, when the *scalar function* $f\rho$ admits of *changing sign*, for a *change of direction* of the *real vector* ρ, so as to be *positive* for *some* such directions, and *negative* for *others*, although $f(-\rho) = f(+\rho)$, the two equations, $f\rho = +1$, $f\rho = -1$, represent then *two real* and *conjugate hyperboloids*, of *different species*: so, when the function $f\rho$ is either *essentially positive*, or else *essentially negative*, for *real* values of ρ, the equation $f\rho = 1$ and $f\rho = -1$ may *then* be said to represent *two conjugate ellipsoids*, *one real*, and the *other imaginary*.

(11.) It is scarcely worth while to remark, that we have here supposed *each* of the *two vectors* λ and μ to be not only *real* but *actual* (Art. 1); for if *either* of them were to *vanish*, the equation of the surface would take by XII. the form,

$$\text{XVIII.} \ldots \rho^2 = g^{-1}, \quad \text{or} \quad \text{XVIII}'. \ldots \mathrm{T}\rho = (-g)^{-\frac{1}{2}},$$

and would represent a *real* or *imaginary sphere*, according as the scalar constant g was *negative* or *positive*: λ and μ have also *distinct directions*, except in the case of *surfaces of revolution*.

(12.) In general, it results from the relations (6.), that the *plane* of the *two* (real) *cyclic normals*, λ, μ, is *perpendicular* to the (real) *direction* of that (real or imaginary) *semiaxis*, of which, when considered as a *scalar* (2.), the *inverse square* c_2 is *algebraically intermediate* between the inverse squares c_1, c_3 of the *other two*; or that the *two diametral and cyclic planes* ($\mathrm{S}\lambda\rho = 0$, $\mathrm{S}\mu\rho = 0$) *intersect* in that *real line* ($\mathrm{V}\lambda\mu$) which has the direction of the *real unit-vector* α_2 (1.), corresponding to the *mean root* c_2 of the cubic equation $M = 0$: all which agrees with known results, respecting the *circular sections* of the (real) *ellipsoid*, and of the two *hyperboloids*.

406. Some additional light may be thrown on the theory of the *central surface* 405, I., by the consideration of its *asymptotic cone* 405, II.; of which *cone*, by 405, XII., the equation may be thus written,

$$\text{I.} \ldots f\rho = g\rho^2 + \mathrm{S}\lambda\rho\mu\rho = \rho^2(g - \mathrm{S}\lambda\mu) + 2\mathrm{S}\lambda\rho\mathrm{S}\mu\rho = 0;$$

and which is *real* or *imaginary*, according as we have the inequality,

$$\text{II.} \ldots g^2 < \lambda^2\mu^2, \quad \text{or} \quad \text{III.} \ldots g^2 > \lambda^2\mu^2;$$

that is, by 405, (6.), according as the *product* c_1c_3 of the *extreme roots* of the *cubic* $M = 0$ is *negative* or *positive*; or finally, according as the *surface* $f\rho = 1$ is a (real) *hyperboloid*, or an *ellipsoid* (real or imaginary*).

(1.) As regards the asserted *reality* of the cone I., when the *condition* II. is satisfied, it may suffice to observe that if we cut the cone by the *plane*,

$$\text{IV.} \ldots \mathrm{S}\lambda(\rho - \mu) = -g,$$

the *section* is a *circle* of the *real* and *diacentric sphere*,

$$\text{V.} \ldots \rho^2 = 2\mathrm{S}\mu\rho, \quad \text{or} \quad \text{V}'. \ldots (\rho - \mu)^2 = \mu^2;$$

* Compare the Note immediately preceding; also the second Note to page 535, vol. i.

and a *real circle*, because it is on the *real cylinder of revolution*,

$$\text{VI.} \ldots \mathrm{TV}(\rho - \mu)\mathrm{U}\lambda = (\mathrm{T}\mu^2 - g^2\mathrm{T}\lambda^{-2})^{\frac{1}{2}},$$

so that its *radius* is equal to this last *real radical*.

(2.) For example, the cone

$$\text{VII.} \ldots \mathrm{S}\frac{\rho}{\alpha}\mathrm{S}\frac{\beta}{\rho} = 1, \quad \text{or} \quad \text{VII}'. \ldots 2(\mathrm{S}\alpha\rho\mathrm{S}\beta\rho - \alpha^2\rho^2) = 0,$$

which under the form VII. occurred as early as 196, (8.), and for which $\lambda = \alpha$, $\mu = \beta$, $g = \mathrm{S}\alpha\beta - 2\alpha^2$, and therefore $\mathrm{T}\lambda\mu + g > 0$, the condition II. reduces itself to $\mathrm{T}\lambda\mu - g > 0$; or after division by $2\mathrm{T}\alpha^2$, &c., to the form (comp. 199, XII.),

$$\text{VIII.} \ldots \tfrac{1}{2}(\mathrm{T} + \mathrm{S})\frac{\beta}{\alpha} > 1, \quad \text{or} \quad \text{VIII}'. \ldots \mathrm{S}\sqrt{\frac{\beta}{\alpha}} > 1;$$

and accordingly, when either of these two last inequalities exists, it will be found that the *sphere* $\mathrm{S}\frac{\beta}{\rho} = 1$ is *cut* by the *plane* $\mathrm{S}\frac{\rho}{\alpha} = 1$ in a *real circle*, the base of a *real cone* VII.

(3.) As an example of the *variety of processes* by which problems in this Calculus may be treated, we might propose to determine, by the general formula 389, IV., the vector κ of the *centre* of the *osculating circle* to the *curve* IV. V., considered merely as an *intersection* of two surfaces. The first derivatives of the equations would allow us to assume $\rho' = \mathrm{V}\lambda(\rho - \mu)$, and therefore $\rho'' = \lambda\rho'$; whence, by the formula, we have

$$\text{IX.} \ldots \kappa = \rho + \frac{\rho'^3}{\mathrm{V}\rho''\rho'} = \rho + \frac{\rho'}{\lambda} = \frac{\mathrm{S}\rho\lambda + \mathrm{V}\mu\lambda}{\lambda} = \mu - g\lambda^{-1};$$

the *section* is therefore a *circle*, because its *centre of curvature* is *constant*; and its *radius* is,

$$\text{X.} \ldots r = \mathrm{T}(\rho - \kappa) = \mathrm{T}(\rho - \mu + g\lambda^{-1}) = (\mathrm{T}\mu^2 - g^2\mathrm{T}\lambda^{-2})^{\frac{1}{2}},$$

= the radius of the cylinder VI.

(4.) When the *opposite inequality* III. exists, the *radius* X., the *cylinder* VI., the *circle* IV. V., and the *cone* I., become all four *imaginary*; the *plane* IV. being then wholly *external* to the *sphere* V., as happens, for instance, with the plane and sphere in (2.), when the condition VIII. or VIII′. is *reversed*.

(5.) In the *intermediate case*, when

$$\text{XI.} \ldots g^2 = \lambda^2\mu^2, \quad \text{or} \quad \text{XI}'. \ldots g = \mp \mathrm{T}\lambda\mu,$$

the radius r in X. *vanishes*; the right cylinder VI. reduces itself to its *axis*; and the circle IV. V. becomes a *point*, in which the sphere is *touched* by the plane. In this case, then, the cone I. is reduced to a *single* (*real**) *right line*, which has (compare the equations of the *elliptic cylinders*, 405, XV. XVII.) the direction of $\lambda\mathrm{T}\mu - \mu\mathrm{T}\lambda$, if $g = -\mathrm{T}\lambda\mu$, but the perpendicular direction of $\lambda\mathrm{T}\mu + \mu\mathrm{T}\lambda$, if $g = +\mathrm{T}\lambda\mu$.

(6.) In general (comp. 405, X.), the equation of the cone I. admits of the *rectangular transformation*,

$$\text{XII.} \ldots f\rho = c_1(\mathrm{S}\alpha_1\rho)^2 + c_2(\mathrm{S}\alpha_2\rho)^2 + c_3(\mathrm{S}\alpha_3\rho)^2 = 0;$$

and the two *sub-cases* last considered (5.) correspond respectively (by 405, (6.)) to the *evanescence* of the *roots* c_1, c_3 of the cubic $M = 0$, with the resulting *directions* α_1, α_3 of the only *real side* of the cone. An analogous but intermediate case (comp. 405, (9.)) is that when $c_2 = 0$, or $g = \mathrm{S}\lambda\mu$; in which case, the *cone* I. reduces itself to the *pair of* (*real*) *planes*,

$$\text{XIII.} \ldots \mathrm{S}\lambda\rho \,.\, \mathrm{S}\mu\rho = 0,$$

namely to the *asymptotic planes* of the *hyperbolic cylinder* 405, XVI., or to those which are usually the *two cyclic*† *planes* of the cone.

(7.) The *case* (comp. 394, (29.)),

$$\text{XIV.} \ldots g = -\mathrm{S}\lambda\mu, \quad \text{or} \quad \text{XIV}'. \ldots c_1 - c_2 + c_3 = 0,$$

for which the equation I. of the cone becomes,

$$\text{XV.} \ldots 0 = f\rho = 2(\mathrm{S}\lambda\rho\mathrm{S}\mu\rho - \rho^2\mathrm{S}\lambda\mu) = 2\mathrm{S}(\mathrm{V}\lambda\rho \,.\, \mathrm{V}\mu\rho),$$

may deserve a moment's attention. In this case, the *two planes*, of $\lambda\rho$ and $\mu\rho$, which connect the *two cyclic normals* λ and μ with an *arbitrary side* ρ of the cone, are always *rectangular* to each other; and these two *normals* to the cyclic planes are at the same time *sides* of the cone, which thus is *cut in*

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* It may however be said, that in this case the *cone* consists of a *pair of imaginary planes*, which *intersect* in a *real right line*.

† The cones and surfaces which have a common centre, and common values of the vectors λ and μ, but different values of the scalar g, may thus be said, in a known phraseology, to be *biconcyclic*.

circles, by planes *perpendicular to those two sides.* And because the equation of the cone may (in the same case) be thus written,

$$\text{XVI.} \ldots \mathrm{TV}(\lambda + \mu)\rho = \mathrm{TV}(\lambda - \mu)\rho,$$

while the *lengths* of λ and μ may *vary*, if their *product* $\mathrm{T}\lambda\mu$ be left unchanged, so that $\lambda + \mu$ and $\lambda - \mu$ may represent *any two lines* from the vertex, in the *plane of the two cyclic normals*, and *harmonically conjugate with respect to them*, it follows that, *for this cone* XV., the *sines of the inclinations of an arbitrary side* ρ, *to these two new lines*, have a *constant ratio* to each other.

(8.) *In general*, the second form I. of $f\rho$ shows (comp. 394, (23.)), that the *constant product of the sines of the inclinations*, of a *side* ρ of the *cone* to the two *cyclic planes*, has for expression,

$$\text{XVII.} \ldots \cos \angle \frac{\rho}{\lambda} \cdot \cos \angle \frac{\rho}{\mu} = \tfrac{1}{2}\left(\frac{g}{\mathrm{T}\lambda\mu} + \cos \angle \frac{\mu}{\lambda}\right);$$

while the first form I. of the same function $f\rho$ reproduces the *condition of reality* II., by showing that $g : \mathrm{T}\lambda\mu$ is (for a *real cone*) the *cosine* of a *real angle*, namely, that of the quaternion product $\lambda\rho\mu\rho$, since it gives the relation,

$$\text{XVIII.} \ldots \frac{g}{\mathrm{T}\lambda\mu} = \mathrm{SU}\lambda\rho\mu\rho = \cos \angle\, \lambda\rho\mu\rho = \cos \angle \frac{\rho\mu^{-1}\rho}{\lambda}.$$

(9.) We may also observe that in the *case of reality* II., with exclusion of the sub-case (6.), if α_3 have the direction of the *internal axis* of the cone, so that

$$\text{XIX.} \ldots c_1 < 0, \quad c_2 < 0, \quad c_3 > 0, \quad \text{or} \quad \text{XIX}'. \ldots g > \mathrm{S}\lambda\mu, \quad g < \mathrm{T}\lambda\mu,$$

the *two sides* (of one sheet) in the plane of $\lambda\mu$ have the directions,

$$\text{XX.} \ldots \rho_1 = c_3^{-\frac{1}{2}}\alpha_3 + (-c_1)^{-\frac{1}{2}}\alpha_1, \quad \rho_2 = c_3^{-\frac{1}{2}}\alpha_3 - (-c_1)^{-\frac{1}{2}}\alpha_1;$$

if then their mutual *inclination*, or the *angle of the cone* in the *plane of the cyclic normals*, be denoted by 2b, we have the values,

$$\text{XXI.} \ldots \tan^2 \mathrm{b} = \frac{c_3}{-c_1}, \quad \text{XXI}'. \ldots \cos 2\mathrm{b} = \frac{-c_1 - c_3}{-c_1 + c_3} = \frac{g}{\mathrm{T}\lambda\mu};$$

the *angle of the quaternion* $\lambda\rho\mu\rho$ is therefore (by XVIII.), equal to this angle 2b, namely to the *arcual minor axis* of the *sphero-conic*, in which the cone is cut by the concentric unit-sphere.

(10.) The same condition of reality II. may be obtained in a quite different way, as that of the reality of the *reciprocal cone*, which is the *locus of the normal vector*,

$$\text{XXII.} \ldots \nu = \phi\rho = g\rho + \text{V}\lambda\rho\mu.$$

Inverting this *linear function* ϕ, by the method of the Section III. ii. 6, we find first the expression (comp. 354, (12.), and 361, (6.)),

$$\text{XXIII.} \ldots m\rho = \psi\nu = \mu^2\lambda\text{S}\lambda\nu + \lambda^2\mu\text{S}\mu\nu - g(\lambda\text{S}\mu\nu + \mu\text{S}\lambda\nu) + (g^2 - \lambda^2\mu^2)\nu,$$

in which $$\text{XXIV.} \ldots m = (g - \text{S}\lambda\mu)(g^2 - \lambda^2\mu^2) = -c_1c_2c_3;$$

and next the *reciprocal equation* (comp. 361, XXVII.),

$$\text{XXV.} \ldots 0 = \text{S}\nu\psi\nu = \mu^2(\text{S}\lambda\nu)^2 + \lambda^2(\text{S}\mu\nu)^2 - 2g\text{S}\lambda\nu\text{S}\mu\nu + (g^2 - \lambda^2\mu^2)\nu^2,$$

which may be put under the form,

$$\text{XXVI.} \ldots \cos\left(\angle\frac{\nu}{\lambda} + \angle\frac{\nu}{\mu}\right) = \frac{-g}{\text{T}\lambda\mu};$$

the *quotient* $g : \text{T}\lambda\mu$ thus presenting itself anew as a *cosine*, namely as that of the *supplement of the sum of the inclinations of the normal* ν (to the cone I.), *to the two cyclic normals* λ, μ (of *that* cone); or as the cosine* of $\pi - \text{A} - \text{B}$, if A and B denote (comp. fig. 80 [vol. ii., p. 65] the *two spherical angles*, which the *tangent arc* to the *sphero-conic* (9.) makes with the two *cyclic arcs*: so that by comparison of XXI′. and XXVI. we have the relation,

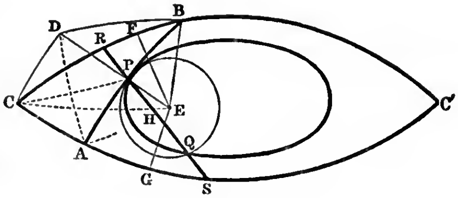

Fig. 80, *bis*.

$$\text{XXVII.} \ldots \text{A} + \text{B} = \angle\frac{\nu}{\lambda} + \angle\frac{\nu}{\mu} = \pi - 2\text{b}.$$

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* This relation was mentioned by anticipation in 394, (3.); and the relation in XXVII. may easily be verified, by conceiving the point of contact P in fig. 80 (vol. ii., page 65) to tend towards a minor summit of the conic, or the tangent arc APB to tend to pass through the two points C, C′, in which the cyclic arcs intersect.

(11.) Comparing the expression XXI'. for cos 2b, with the last expression XVIII. for $g : \mathrm{T}\lambda\mu$, we derive the following *construction* for a *sphero-conic*, which may easily be verified by geometry :*

Having assumed two points (L, M) *on a sphere, and having described a small circle round one of them* (say L), *bisect the arcs* (MQ) *which are drawn to its circumference from the other point* ; *the locus of the bisecting points* (P) *will be a sphero-conic, with the two fixed points for its two cyclic poles* (or for the poles of its cyclic arcs), *and with an arcual minor axis* (2b) *equal to the arcual radius of the small circle.*†

(12.) As regards the *arcual major axis* (say 2a) of the same sphero-conic, it is (with the conditions XIX.) the angle between the two sides (comp. XX.),

$$\text{XXVIII.} \ldots \rho_3 = c_3^{-\frac{1}{2}}a_3 + (-c_2)^{-\frac{1}{2}}a_2, \quad \rho_4 = c_3^{-\frac{1}{2}}a_3 - (-c_2)^{-\frac{1}{2}}a_2\,;$$

whence (comp. XXI.),

$$\text{XXIX.} \ldots \tan^2 \mathrm{a} = \frac{c_3}{-c_2}, \quad \text{or} \quad \text{XXIX}'. \ldots \cos 2\mathrm{a} = \frac{-c_2 - c_3}{-c_2 + c_3} = (\text{say})\ e,$$

and therefore, a few easy reductions being made,

$$\text{XXX.} \ldots \frac{\sin \mathrm{b}}{\sin \mathrm{a}} = \sqrt{\left\{\tfrac{1}{2}\left(1 + \mathrm{SU}\frac{\mu}{\lambda}\right)\right\}} = \cos \tfrac{1}{2} \angle \frac{\mu}{\lambda}\,;$$

from which we can at once infer, that if a *focus* of the conic be determined, by drawing from a minor summit to the major axis an arc equal to the major semiaxis a, *the minor axis subtends at this focus* (or at the other) *a spherical angle equal to the angle between the two cyclic arcs.*

(13.) For the *two real unifocal transformations* of the equation of the *cone*, or the *forms*,

$$\text{XXXI.} \ldots a(\mathrm{V}\alpha\rho)^2 + b(\mathrm{S}\beta\rho)^2 = 0, \quad \text{and} \quad \text{XXXI}'. \ldots a(\mathrm{V}\alpha'\rho)^2 + b(\mathrm{S}\beta'\rho)^2 = 0,$$

with one *common* set of real values of the scalar *coefficients*, a and b, but with *two real focal unit lines* α, α', and *two real directive normals* β, β' corresponding, it may be sufficient here to refer to the sub-articles to 358 ; except that it should be noticed, that if the cone be *real*, and if the line a_3 have the direction

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* In fact, the bisecting radii OP are parallel to the supplementary chords M'Q, if MM' be a diameter of the sphere ; and the locus of all such chords is a cyclic cone, resting on the small circle as its base.

† [By quaternions, if $\mathrm{OQ} = \kappa$, $\mathrm{U}\kappa\mu^{-1} = \mathrm{U}(\rho\mu^{-1})^2$ or $\mathrm{U}\kappa = -\mathrm{U}\rho\mu\rho$, &c.]

of its *internal axis*, so that the inequalities XIX. are satisfied, and therefore also (by 405, (6.)),

$$\text{XXXII.} \ldots c_3^{-1} > 0 > c_1^{-1} > c_2^{-1},$$

instead of the inequalities 358, III., or 359, XXXVII., we are now to change, in the earlier formulæ referred to, the symbols $c_1c_2c_3a_1a_2a_3$ to $c_3c_1c_2a_3a_1a_2$, so that we have now the values,

$$\text{XXXIII.} \ldots a = -c_1, \quad b = c_3 - c_1 + c_2, \quad \text{if} \quad T\beta = T\beta' = 1.$$

(14.) And as regards the *interpretation* of the *unifocal form* XXXI., with these last values, it is evidently contained in this other equation,

$$\text{XXXIV.} \ldots \sin \angle \frac{\rho}{\alpha} \cdot \sec \angle \frac{\rho}{\beta} = \frac{TV\alpha\rho}{-S\beta\rho} = \left(\frac{c_3 - c_1 + c_2}{-c_1}\right)^{\frac{1}{2}} = \text{const.};$$

the *sines of the inclinations* of an arbitrary *side* (ρ) of the *cone*, to a *focal line* (α), and to the corresponding *director plane* ($\perp \beta$), thus bearing to each other (as is well known) a *constant ratio*, which remains unchanged when we pass to the *other* (real) *focal line* (α'), and at the same time to the *other* (real) *director plane* ($\perp \beta'$) : and the *focal plane* of these *two lines* (α, α') being *perpendicular* to *that one* of the *three axes*, which corresponds to the *root* (here c_1 by XXXII.) of the *cubic*, of which the *reciprocal* is algebraically *intermediate* between the reciprocals of the other two.

(15.) It is, however, more *symmetric* to employ the *bifocal transformation* (comp. 360, VI.*),

$$\text{XXXV.} \ldots 0 = (S\alpha\rho)^2 - 2eS\alpha\rho S\alpha'\rho + (S\alpha'\rho)^2 + (1 - e^2)\rho^2;$$

in which the scalar constant e has the value (comp. XXIX'.),

$$\text{XXXVI.} \ldots e = \cos 2\text{a};$$

and α, α' are the *two*† real and *focal unit lines*, recently considered (13.).

* It is to be remembered that, in the formula here cited, the symbols α, α' did *not* denote unit-vectors.

† When those two vectors α, α' remain *constant*, but the scalar e *changes*, there arises a system of *biconfocal cones*: or, by their intersections with a concentric sphere, a system of *biconfocal sphero-conics*. Compare the second Note to page 191.

(16.) The equation XXXV., for the case of a *real cone*, may be thus written (comp. XXVI. XXXVI.),

$$\text{XXXVII.} \ldots \angle\frac{\rho}{a} + \angle\frac{\rho}{a'} = \cos^{-1} e = 2\text{a};$$

the *sum** of the *inclinations* of the *side* ρ to the *two* focal lines a, a' being thus *constant*, and equal (as is well known) to the *major axis* of the *spherical conic*: and although, when $e > 1$, the *cone* becomes *imaginary*, yet it is then asymptotic to a *real ellipsoid*, as we shall shortly see.

407. The *bifocal form* (406, XXXV.) of the equation of a *cone* may suggest the corresponding *form*,

$$\text{I.} \ldots C = Cf\rho = (\text{S}a\rho)^2 - 2e\text{S}a\rho\text{S}a'\rho + (\text{S}a'\rho)^2 + (1 - e^2)\rho^2,$$

in which a and a' are given and generally non-parallel unit-lines, while e and C are scalar constants, as capable of representing *generally* (comp. 360, (2.), (3.)) a *central* but *non-conical surface* ($f\rho = 1$) of the *second order*. And we shall find that if, in passing from *one* such surface to *another*, we suppose a and a' to remain *unchanged*, but e and C to *vary together*, so as to be always connected by the relation,

$$\text{II.} \ldots C = (e^2 - 1)(e + \text{S}aa')l^2,$$

in which l is some real, positive, and *given scalar*, then all the surfaces I. so deduced, or in other words the surfaces represented by the common equation,

$$\text{III.} \ldots l^2 = l^2 f\rho = \frac{(\text{S}a\rho)^2 - 2e\text{S}a\rho\text{S}a'\rho + (\text{S}a'\rho)^2 + (1 - e^2)\rho^2}{(e^2 - 1)(e + \text{S}aa')},$$

with e for the *only variable parameter*, compose a *Confocal System*.

(1.) The *scalar form* III. of $f\rho$ gives the connected *vector form*,

$$\text{IV.} \ldots l^2\nu = l^2\phi\rho = \frac{a\text{S}(a - ea')\rho + a'\text{S}(a' - ea)\rho + (1 - e^2)\rho}{(e^2 - 1)(e + \text{S}aa')},$$

* Or the *difference*, according to the choice between two opposite directions, for *one* of the two focal lines. The *angular transformation* XXXVII. may be accomplished, by *resolving* the equation XXXV. as a *quadratic* in e, and then interpreting the result.

which may also be thus written, with the value II. of C,

$$\text{V.} \ldots C\nu = C\phi\rho = (\alpha - e\alpha')S\alpha\rho + (\alpha' - e\alpha)S\alpha'\rho + (1 - e^2)\rho,$$

so that the function ϕ is *self-conjugate*, as it ought to be.

(2.) And because we have thus,

$$\text{VI.} \ldots (e^2 - 1)\,l^2\phi\alpha = \alpha' - e\alpha, \quad (e^2 - 1)\,l^2\phi\alpha' = \alpha - e\alpha',$$

if we write, for abridgment,

$$\text{VII.} \ldots a^2 = (e + 1)\,l^2, \quad b^2 = (e + S\alpha\alpha')l^2, \quad c^2 = (e - 1)l^2,$$

we shall have the values,

$$\text{VIII.} \ldots \begin{cases} \phi(\alpha + \alpha') = -a^{-2}(\alpha + \alpha'), \\ \phi V\alpha\alpha' \quad = -b^{-2}V\alpha\alpha', \\ \phi(\alpha - \alpha') = -c^{-2}(\alpha - \alpha')\,; \end{cases}$$

comparing which with 405, (1.), (2.), we see that the three (real or imaginary) lines,

$$\text{IX.} \ldots aU(\alpha + \alpha'), \quad bUV\alpha\alpha', \quad cU(\alpha - \alpha'),$$

of any one of which the direction may be reversed, are the *three vector semiaxes* of the *surfaces* $f\rho = 1$; and therefore, by VII., that the *system* III. is one of *confocals*, as asserted.

(3.) The *rectangular transformations*, scalar and vector, are now (comp. 405, X., and 357, V. VIII.):

$$\text{X.} \ldots l^2 = l^2 f\rho = \frac{(S\rho U(\alpha + \alpha'))^2}{e + 1} + \frac{(S\rho UV\alpha\alpha')^2}{e + S\alpha\alpha'} + \frac{(S\rho U(\alpha - \alpha'))^2}{e - 1}\,;$$

$$\text{XI.} \ldots l^2\nu = l^2\phi\rho = \frac{U(\alpha + \alpha').S\rho U(\alpha + \alpha')}{e + 1} + \frac{UV\alpha\alpha'.S\rho UV\alpha\alpha'}{e + S\alpha\alpha'} + \frac{U(\alpha - \alpha').S\rho U(\alpha - \alpha')}{e - 1}\,;$$

which can both be established, by the rules of the present Calculus, in several other ways, and from the first of which it follows that (as is well known) *through any proposed point* P of space there can in general be drawn *three confocal surfaces*, of a *given system* III.; *one* an *ellipsoid*, for which $e > 1$, and therefore $a^2 > b^2 > c^2 > 0$; *another* a *single-sheeted hyperboloid*, for which $e < 1$, $e > -\mathrm{S}\alpha\alpha'$, $a^2 > b^2 > 0 > c^2$; and the *third* a *double-sheeted hyperboloid*, for which $e < -\mathrm{S}\alpha\alpha'$, $e > -1$, $a^2 > 0 > b^2 > c^2$.

(4.) From the *other* rectangular transformation XI. it follows, that if we denote by $\nu_1 = \phi_1\rho$ what the normal vector $\nu = \phi\rho$ becomes, when ρ remains the same, but e is changed to a *second root* e_1 of the equation III. or X. of the *surface*, considered as a *cubic* in e, then

$$\text{XII.} \ldots \frac{\nu_1 - \nu}{e_1 - e} = l^2\phi\nu_1 = l^2\phi_1\nu = l^2\phi_1\phi\rho = l^2\phi\phi_1\rho;$$

but

$$\text{XIII.} \ldots \mathrm{S}\rho\nu_1 = \mathrm{S}\rho\nu = f_1\rho = f\rho = 1,$$

$f_1\rho$ being formed from $f\rho$, by the substitution of e_1 for e; therefore,

$$\text{XIV.} \ldots 0 = \mathrm{S}\rho\phi\nu_1 = \mathrm{S}\nu_1\phi\rho = \mathrm{S}\nu_1\nu,$$

and the known theorem results, that *confocal surfaces cut each other orthogonally*.*

(5.) It follows, from V. and VI., that the *inverse function* $\phi^{-1}\rho$ can be expressed as follows:

$$\text{XV.} \ldots \phi^{-1}\rho = l^2(\alpha\mathrm{S}\alpha'\rho + \alpha'\mathrm{S}\alpha\rho) - b^2\rho;$$

or that ρ may be deduced from ν by the formula,

$$\text{XVI.} \ldots \rho = \phi^{-1}\nu = l^2(\alpha\mathrm{S}\alpha'\nu + \alpha'\mathrm{S}\alpha\nu) - b^2\nu,$$

which can easily be otherwise established. Hence (comp. 361, (4.)), the equation of the surface *reciprocal* to the surface I. or III., or of that *new surface* which has ν (instead of ρ) for its *variable vector*, is

$$\text{XVII.} \ldots 1 = F\nu = \mathrm{S}\nu\phi^{-1}\nu = 2l^2\mathrm{S}\alpha\nu\mathrm{S}\alpha'\nu - b^2\nu^2;$$

the *fixed focal lines* α, α' of the *confocal system* III., or of the corresponding system of the *asymptotic cones*, becoming thus (in agreement with known results) the *fixed cyclic normals* (or *cyclic lines*, comp. 361, (6.)) of the *reciprocal system* XVII.

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* We shall soon see that the *same formula* XII., by expressing that ν, ν_1, and $\phi\nu_1$ or $\phi_1\nu$ are *complanar*, contains this *other part* of the known theorem referred to, that the *intersection* is a *line of curvature*, on *each* of the two *confocals*. [Compare 410, (12.).]

(6.) In thus deducing the equation XVII. from III., *no use* has been made of the *rectangular transformations* X. XI., of the functions $f\rho$ and $\phi\rho$. *Without* the transformations last referred to, we could therefore have inferred, by a slight modification of the form XVII., that the *reciprocal surface* ($F\nu = 1$) with ν for its variable vector, which has the *same rectangular system* of *directions* for its three semiaxes as the *original surface* ($f\rho = 1$), but with *inverse squares* (the roots of *its* cubic) equal to the *direct* squares of the *original* semiaxes, has for equation (comp. 405, XII.),

$$\text{XVIII.} \ldots 1 = F\nu = l^2(\text{S}\alpha\nu\alpha'\nu - e\nu^2) = \text{S}\lambda\nu\mu\nu + g\nu^2,$$

if

$$\text{XIX.} \ldots \lambda = l\alpha, \quad \mu = l\alpha', \quad g = -el^2 = -e\text{T}\lambda\mu;$$

the values VII. of a^2, b^2, c^2 being thus deduced *anew*, but by a process quite different from that employed in (2.), under the forms (comp. 405, XIII.),

$$\text{XX.} \ldots a^2 = c_3 = -g + \text{T}\lambda\mu; \quad b^2 = c_2 = -g + \text{S}\lambda\mu; \quad c^2 = c_1 = -g - \text{T}\lambda\mu;$$

while the *directions* IX. of the corresponding semiaxes may be deduced as those of α_3, α_2, α_1, from the formulæ 405, XIV.

(7.) If the symbol $\omega(\nu)$, or simply $\omega\nu$, be used to denote a *new* linear and self-conjugate vector function of ν, defined by the equation,

$$\text{XXI.} \ldots \omega\nu = \rho\text{S}\rho\nu - l^2(\alpha\text{S}\alpha'\nu + \alpha'\text{S}\alpha\nu),$$

with ρ *here* treated as a vector constant, then (because $\text{S}\rho\nu = 1$) the equation XVI. may be thus written (comp. 354, &c.),

$$\text{XXII.} \ldots (\omega + b^2)\nu = 0;$$

the *three rectangular directions*, of the *three normals* ν, ν_1, ν_2 to the *three confocals* through P, are therefore those which satisfy (comp. again 354) the *vector quadratic* equation,

$$\text{XXIII.} \ldots \text{V}\nu\omega\nu = 0;$$

and they are the *directions* of the *axes* of this *new surface* of the second order (comp. 357, &c.),

$$\text{XXIV.} \ldots \text{S}\nu\omega\nu = (\text{S}\rho\nu)^2 - 2l^2\text{S}\alpha\nu\text{S}\alpha'\nu = 1,$$

in which ρ is still treated as a *constant* vector, but ν as a *variable* one.

(8.) The *inverse squares* of the *scalar semiaxes* of this *new* surface ($\text{S}\nu\omega\nu = 1$) are the *direct* squares b^2, b_1^2, b_2^2 of what may be called the *mean semiaxes* of the *three confocals*; these *latter* squares must therefore be the *roots* of this *new cubic*,

$$\text{XXV.} \ldots 0 = m + m'b^2 + m''(b^2)^2 + (b^2)^3,$$

in which the coefficients m, m', m'', deduced here from the *new function* ω, as they were deduced from ϕ in the Section III. ii. 6, have the values,

$$\text{XXVI.} \ldots \begin{cases} m \;\, = l^4(\mathrm{S}\alpha\alpha'\rho)^2; \\ m' \;= l^4(\mathrm{V}\alpha\alpha')^2 + 2l^2\mathrm{S}(\mathrm{V}\alpha\rho\,.\mathrm{V}\alpha'\rho); \\ m'' = \rho^2 - 2l^2\mathrm{S}\alpha\alpha'. \end{cases}$$

Accordingly, if we observe that (because $\mathrm{T}\alpha = \mathrm{T}\alpha' = 1$) we have among others the transformation,

$$\text{XXVII.} \ldots (\mathrm{S}\alpha\alpha'\rho)^2 = \rho^2(\mathrm{V}\alpha\alpha')^2 - (\mathrm{S}\alpha\rho)^2 - 2\mathrm{S}\alpha\alpha'\mathrm{S}\alpha\rho\mathrm{S}\alpha'\rho - (\mathrm{S}\alpha'\rho)^2,$$

we can express this last cubic equation XXV., with these values XXVI. of its coefficients, under the form,

$$\begin{aligned} \text{XXVIII.} \ldots 0 = (b^2 + \rho^2)\,\{(b^2 - l^2\mathrm{S}\alpha\alpha')^2 - l^4\} \\ + 2l^2(b^2 - l^2\mathrm{S}\alpha\alpha')\,\mathrm{S}\alpha\rho\mathrm{S}\alpha'\rho - l^4((\mathrm{S}\alpha\rho)^2 + (\mathrm{S}\alpha'\rho)^2); \end{aligned}$$

which, when we change b^2 by VII. to its value $l^2(e + \mathrm{S}\alpha\alpha')$, and divide by l^4, becomes the *cubic* in e, or the equation III. under the form,

$$\text{XXIX.} \ldots 0 = (e^2 - 1)\,\{l^2(e + \mathrm{S}\alpha\alpha') + \rho^2\} + 2e\mathrm{S}\alpha\rho\mathrm{S}\alpha'\rho - (\mathrm{S}\alpha\rho)^2 - (\mathrm{S}\alpha'\rho)^2.$$

(9.) As an additional test of the *consistency* of this whole theory and method, the *directions* of the *three axes* of the *new surface* XXIV., or those of the *three normals* (7.) to the confocals, or the *three vector roots* (354) of the equation XXIII., ought to admit of being assigned by three expressions of the forms,

$$\text{XXX.} \ldots \begin{cases} n\nu \;\;\, = \psi\sigma \;\, + b^2\chi\sigma \;\, + b^4\sigma, \\ n_1\nu_1 = \psi\sigma_1 + b_1^{\,2}\chi\sigma_1 + b_1^{\,4}\sigma_1, \\ n_2\nu_2 = \psi\sigma_2 + b_2^{\,2}\chi\sigma_2 + b_2^{\,4}\sigma_2; \end{cases}$$

in which b^2, $b_1^{\,2}$, $b_2^{\,2}$ are the *three scalar roots* of the cubic XXV. or XXVIII., while σ, σ_1, σ_2 are *three arbitrary vectors*; n, n_1, n_2 are *three scalar coefficients*, which can be determined by the conditions $\mathrm{S}\rho\nu = \mathrm{S}\rho\nu_1 = \mathrm{S}\rho\nu_2 = 1$ (comp. XIII.); and ψ, χ are two new *auxiliary linear* and *vector functions*, to be deduced here from the function ω, in the same manner as they were deduced from ϕ in the Section lately referred to.

(10.) Accordingly, by the method of that Section, taking for convenience the given* vector ρ (instead of the arbitrary vectors σ, σ_1, σ_2) as the subject of the operations ψ and χ, we find the expressions,

$$\text{XXXI.} \ldots \psi\rho = l^4 \mathrm{V}\alpha\alpha'\mathrm{S}\alpha\alpha'\rho, \quad \chi\rho = l^2(\alpha\mathrm{S}\alpha'\rho + \alpha'\mathrm{S}\alpha\rho - 2\rho\mathrm{S}\alpha\alpha');$$

whence, after a few reductions, with elimination of n by the relation $\mathrm{S}\rho\nu = 1$, and by the cubic in b^2, the first equation XXX. becomes:

$$\text{XXXII.} \ldots 0 = (b^2\nu + \rho)\{(b^2 - l^2\mathrm{S}\alpha\alpha')^2 - l^4\}$$
$$+ l^2(b^2 - l^2\mathrm{S}\alpha\alpha')(\alpha\mathrm{S}\alpha'\rho + \alpha'\mathrm{S}\alpha\rho) - l^4(\alpha\mathrm{S}\alpha\rho + \alpha'\mathrm{S}\alpha'\rho);$$

which is in fact a *form* of the relation between ν and ρ, for any *one* of the confocals, as appears (for instance) by again changing b^2 to $l^2(e + \mathrm{S}\alpha\alpha')$, and comparing with the equation IV.

(11.) Another and a more interesting *auxiliary surface*, of which the axes have still the directions of the normals ν, is found by *inverting* the new linear function ω, or by forming from XXII. the *inverse equation*,

$$\text{XXXIII.} \ldots (\omega^{-1} + b^{-2})\nu = 0;$$

in which,

$$\text{XXXIV.} \ldots \omega^{-1}\nu \,.\, (\mathrm{S}\alpha\alpha'\rho)^2 = \mathrm{V}\alpha\alpha'\mathrm{S}\alpha\alpha'\nu + l^{-2}(\mathrm{V}\alpha\rho\mathrm{S}\alpha'\rho\nu + \mathrm{V}\alpha'\rho\mathrm{S}\alpha\rho\nu);$$

and from which it follows that the *normals* ν to the *confocals* through P have the directions of the *axes* of this *new cone*,

$$\text{XXXV.} \ldots \mathrm{S}\nu\omega^{-1}\nu = 0, \quad \text{or} \quad \text{XXXVI.} \ldots 0 = l^2(\mathrm{S}\alpha\alpha'\nu)^2 + 2\mathrm{S}\alpha\rho\nu\mathrm{S}\alpha'\rho\nu,$$

with ρ treated as a constant, as before.

(12.) The *vertex* of this auxiliary cone being placed at the given point P, of intersection of the three confocals, we may inquire *in what curve* is the *cone cut*, by the *plane* of the given *focal lines*, α, α', drawn through the *common centre* O of all the surfaces III. Denoting by $\sigma = t\alpha + t'\alpha'$ the vector of a point S of this sought *section*, and writing

$$\text{XXXVII.} \ldots \nu = \sigma - \rho = t\alpha + t'\alpha' - \rho,$$

the equation XXXVI. gives the relation,

$$\text{XXXVIII.} \ldots tt' = \frac{l^2}{2} = \frac{a^2 - c^2}{4} = \text{const.};$$

* The general expressions for $\psi\sigma$ and $\chi\sigma$ include terms, which vanish when $\sigma = \rho$.

the section is therefore an *hyperbola*, which is *independent of the point* P, and has the focal lines of the *system* for its *asymptotes*. And because its *vector equation* may be thus written (comp. 371, II.),

$$\text{XXXIX.} \ldots \sigma = ta + \tfrac{1}{2}l^2t^{-1}a',$$

or what may be called its *quaternion equation* as follows (comp. 371, I.),

$$\text{XL.} \ldots 2\mathrm{V}a\sigma.\mathrm{V}\sigma a' = l^2(\mathrm{V}aa')^2,$$

it satisfies the *two scalar equations*,

$$\text{XLI.} \ldots m = 0, \quad m' = 0,$$

with the significations XXVI. of m and m'; it is therefore that important curve, which is known by the name of the *Focal Hyperbola*:* namely the *limit* to which the *section* of the *confocal surface* by the *plane* of its *extreme*† *axes* tends, when the *mean axis* ($2b$) tends to *vanish*. We are then led thus to the known theorem, that *if, with any assumed point* P *for vertex, and with the focal hyperbola*‡ *for base, a cone be constructed, the axes of this focal cone have the directions of the normals to the confocals through* P.

(13.) As regards the *Focal Ellipse*, its two scalar equations may be deduced from the rectangular form X., by equating to zero both the numerator and the denominator of its last term; they are therefore,

$$\text{XLII.} \ldots \mathrm{S}(a - a')\rho = 0, \quad 2l^2 = (\mathrm{S}\rho\mathrm{U}(a + a'))^2 + \left(\frac{\mathrm{S}\rho\mathrm{UV}aa'}{\mathrm{S}\sqrt{aa'}}\right)^2;$$

the curve being thus given as a *perpendicular section* of an *elliptic cylinder*, with $l\sqrt{2}$ and $l\sqrt{(1 + \mathrm{S}aa')}$, or $(a^2 - c^2)^{\frac{1}{2}}$ and $(b^2 - c^2)^{\frac{1}{2}}$, for the semiaxes of its base, or of the ellipse itself.

(14.) The same curve may also be represented by the equations,

$$\text{XLIII.} \ldots \mathrm{S}a\rho = \mathrm{S}a'\rho, \quad \mathrm{TV}a\rho = (b^2 - c^2)^{\frac{1}{2}},$$

or

$$\text{XLIII'.} \ldots \mathrm{S}a'\rho = \mathrm{S}a\rho, \quad \mathrm{TV}a'\rho = (b^2 - c^2)^{\frac{1}{2}};$$

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* Compare the Notes to pages 240, vol. i., and 17, vol. ii.

† Namely, those two of which the *squares* algebraically include between them that of the third; this latter being, for the same reason, considered here as the *mean*.

‡ We shall soon see that quaternions give, with equal ease, a more general known theorem, in which this is included as a limit. [Compare 408, (13.), page 214.]

which express that it is the *common intersection* of its own *plane* ($\perp \alpha - \alpha'$) with *two right cylinders*,* which have the *two focal lines* α, α' of the system for their *axes of revolution*, and have *equal radii*, denoted each by the radical last written.

(15.) In general, the *unifocal form* (comp. 406, (13.)) of the equation III., namely,

$$\text{XLIV.} \ldots 0 = (1 - e^2)\,((\text{V}\alpha\rho)^2 + b^2) + (\text{S}(\alpha' - e\alpha)\rho)^2,$$

in which α and α' may be interchanged, shows that the *two equal right cylinders*,

$$\text{XLV.} \ldots (\text{V}\alpha\rho)^2 + b^2 = 0, \qquad \text{XLV}'. \ldots (\text{V}\alpha'\rho)^2 + b^2 = 0,$$

or

$$\text{XLVI.} \ldots \text{TV}\alpha\rho = b, \qquad \text{XLVI}'. \ldots \text{TV}\alpha'\rho = b,$$

which are *real* if their common *radius* b be such, that is, if the confocal (e) be either an *ellipsoid* (supposed to be *real*), or else a *single-sheeted hyperboloid*, and which have the *focal lines* α, α' of the system for their *axes* of revolution, *envelope*† *that confocal surface*; the *planes* of the two *ellipses of contact* (which again are *real curves*, if b be real) being given by the equations,

$$\text{XLVII.} \ldots \text{S}(\alpha' - e\alpha)\rho = 0, \qquad \text{XLVII}'. \ldots \text{S}(\alpha - e\alpha')\rho = 0;$$

so that they pass *through the centre* O of the surface (or of the system), and are the (real) *director planes* (comp. 406, (14.)) of the *asymptotic cone* (real or imaginary), to the particular confocal (e).

(16.) Whether the mean *semiaxis* (b) be real or imaginary, the *surface* III. (supposed to be itself *real*) is always, by the form XLIV. of its equation, the *locus* of a system of *real ellipses* (comp. 404, (1.)), in planes *parallel* to the *director plane* XLVII., which have their *centres on the focal line* α, and are orthogonally *projected into circles* on a plane *perpendicular to that line*.

(17.) The *same surface* is also the locus of a *second system* of such ellipses, related similarly to the second focal line α', and to the second director

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* The reader may consult page 513 of the *Lectures*, for the case of this theorem which answers to a given *ellipsoid*. The *focal ellipse* may also be represented generally by the expression (comp. page 417, vol. i., of these *Elements*),

$$\rho = (a^2 - c^2)^{\frac{1}{2}}\,\text{V}.\alpha^t\text{U}(\alpha + \alpha');$$

or by the same expression, with α and α' interchanged.

† Compare pages 202, 236, 241, 315, vol. i.

plane XLVII′.; and it appears that *these two systems* of *elliptic sections* of a surface of the second order, which from some points of view are nearly as interesting as the *circular sections*, may conveniently be called its *Centro-Focal Ellipses.*

(18.) For example, when the *first quaternion form* (204, (14.), or 404, I.) of the equation of the *ellipsoid* is employed, *one* system of such ellipses coincides with the system (204, (13.)) of which, in the *first generation** of the surface, the ellipsoid was treated as the *locus*; and an *analogous generation* of the *two hyperboloids*, by geometrical *deformation* of two corresponding surfaces of *revolution*, with certain resulting *homologies* (comp. sub-arts. to 274), through substitution of (*centro-focal*) *ellipses* for *circles*, conducts to *equations* of those hyperboloids of the same *unifocal form*; namely, if α and β have significations analogous to those in the cited equation of the ellipsoid (so that β and *not* α is *here* a *focal line*),

$$\text{XLVIII.} \ldots \left(\mathrm{S}\frac{\rho}{\alpha}\right)^2 + \left(\mathrm{V}\frac{\rho}{\beta}\right)^2 = \mp 1;$$

the upper or the lower *sign* being taken, according as the surface consists of *one sheet* or of *two*.

(19.) It may also be remarked that as, by changing β to α in the corresponding equation of the *ellipsoid*, we could return (comp. 404, (1.)) to a form (403, XI.) of the equation of the *sphere*, so the same change in

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* Besides that *first generation* (I) of the Ellipsoid, which was a *double* one, in the sense that a *second system* (17.) of *generating ellipses* might be employed, and which served to connect the surface with a *concentric sphere*, by certain relations of *homology* (274); and the *second double generation* or *construction* (II), by means of either of *two diacentric spheres* (217, (4.), (6.), (7.), and 220, (3.)), which was illustrated by fig. 53 (page 234, vol. i., and page 184, vol. ii.): several *other* generations of the same important surface were deduced from quaternions in the *Lectures* to which it is only possible here to *refer*. A reader, then, who happens to have a copy of that earlier work, may consult page 499 for a *generation* (III) of a system of *two reciprocal ellipsoids*, with a *common mean axis* ($2b$), by means of a *moving sphere*, of which the *radius* ($= b$) is *given*, but of which the *centre* has the *original* ellipsoid for its *locus*; while the *corresponding point* on the *reciprocal surface*, and also the *normals* at the two points, are easily deduced from the construction. In page 502, he will find another and perhaps a *simpler generation* (IV), of the *same pair* of *reciprocal ellipsoids*, by means of *quadrilaterals inscribed in a fixed sphere* (the *common mean sphere*, comp. 216, (10.)); the *directions* of the *four sides* of such a quadrilateral being *given*, and *one pair* of *opposite sides intersecting* in a point of *one surface*, while the *other pair* have for their intersection the *corresponding point* of the *other* (or reciprocal) ellipsoid. In the page last cited, and in the following page, there is given a *new double generation* (V) of any *one* ellipsoid; its *circular sections* (of either system) being constructed as *intersections of two equal spheres* (or spheric surfaces), of which the *line of centres* retains a *fixed direction*, while the *spheres slide* within *two equal and right cylinders*, whose *axes* intersect each other

XLVIII. conducts to equations of the *equilateral hyperboloids of revolution*, of *one* sheet and of *two*, under the very simple forms* (comp. 210, XI.),

$$\text{XLIX.} \ldots \mathrm{S}\left(\frac{\rho}{\alpha}\right)^2 = -1, \quad \text{and} \quad \text{L.} \ldots \mathrm{S}\left(\frac{\rho}{\alpha}\right)^2 = +1;$$

in which it seems unnecessary to insert *points* after the signs S, and of which the geometrical *interpretations* become obvious when then they are written thus (comp. 199, V.),

$$\text{LI.} \ldots \mathrm{T}\frac{\rho}{\alpha} = \sqrt{\sec 2\left(\frac{\pi}{2} - \angle\frac{\rho}{\alpha}\right)}, \quad \text{LII.} \ldots \mathrm{T}\frac{\rho}{\alpha} = \sqrt{\sec 2\angle\frac{\rho}{\alpha}};$$

where $\mathrm{T}\frac{\rho}{\alpha} = \overline{\text{OP}} : \overline{\text{OA}}$, while $\angle\frac{\rho}{\alpha}$ is the inclination AOP of the *semidiameter* OP to the *axis* of revolution OA, and $\frac{\pi}{2} - \angle\frac{\rho}{\alpha}$ is the inclination of the same semidiameter to a *plane* perpendicular to that axis.

(20.) The *real cyclic forms* of the equation of the surface III. might be deduced from the *unifocal form* XLIV., by the general method of the sub-articles to 359; but since we have ready the *rectangular form* X., it is simpler to obtain them from that form, with the help of the identity,

$$\text{LIII.} \ldots -\rho^2 = (\mathrm{S}\rho\mathrm{U}(\alpha + \alpha'))^2 + (\mathrm{S}\rho\mathrm{U}\mathrm{V}\alpha\alpha')^2 + (\mathrm{S}\rho\mathrm{U}(\alpha - \alpha'))^2,$$

by eliminating the *first* of these *three terms* for the case of a *single-sheeted*

(in the *centre* of the generated surface), and of which the *common radius* is the *mean semiaxis* (b). Finally, in page 699 of the same volume, there will be found a *new generation* (VI) of the *original ellipsoid* (abc), *analogous* to the generation (IV) by the *fixed* (*mean*) *sphere*, but with *new directions* of the *sides* of the *quadrilaterals*, which are also (in this *last* generation) *inscribed* in the *circles* of a certain *mean ellipsoid* (or *prolate spheroid*) of *revolution*, which has the *mean axis* ($2b$) for *its major axis*; and has *two medial foci* on that axis, whose *common distance* from the *centre* is represented by the expression,

$$\frac{\sqrt{(a^2 - b^2)}\sqrt{(b^2 - c^2)}}{\sqrt{(a^2 - b^2 + c^2)}};$$

the *common tangent planes*, to this *mean* (or *medial*) ellipsoid, and to the *given* (or *generated*) ellipsoid (abc), which are *parallel* to their *common axis* ($2b$), being parallel also to the *two umbilicar diameters* of the latter surface.

* The *same forms*, but with σ for ρ, and β for α, may be deduced from XLVIII. on the plan of 274, (2.), (4.), by assuming an auxiliary vector σ such that $\mathrm{S}\frac{\sigma}{\beta} = \pm\,\mathrm{S}\frac{\rho}{\alpha}$, and $\mathrm{V}\frac{\sigma}{\beta} = \mathrm{V}\frac{\rho}{\beta}$; the *homologies*, above alluded to, between the *general hyperboloid* of either species, and the *equilateral hyperboloid of revolution* of the same species, admitting also thus of being easily exhibited.

hyperboloid (for which $b^{-2} > a^{-2} > 0 > c^{-2}$); the *second* for an *ellipsoid* ($c^{-2} > b^{-2} > a^{-2} > 0$); and the *third* for a *double-sheeted hyperboloid* ($a^{-2} > 0 > c^{-2} > b^{-2}$).

(21.) Whatever the *species* of the surface III. may be, we can always derive from the unifocal form XLIV. of its equation what may be called an *Exponential Transformation*; namely the vector expression,

$$\text{LIV.} \ldots \rho = x\alpha + y\text{V}\alpha^t\beta, \quad \text{with} \quad \text{LV.} \ldots x^2 f\alpha + y^2 f\text{UV}\alpha\alpha' = 1;$$

the *scalar exponent*, t, remaining *arbitrary*, but the *two scalar coefficients*, x and y, being *connected* by this last equation of the second degree: provided that the *new constant vector* β be derived from α, α', and e, by the formula,

$$\text{LVI.} \ldots \beta = \frac{(\alpha' - e\alpha)\text{UV}\alpha\alpha'}{e + \text{S}\alpha\alpha'},$$

which gives after a few reductions (comp. the expression 315, III. for α^t, when $\text{T}\alpha = 1$),

$$\text{LVII.} \ldots \text{V}\alpha\beta = \text{UV}\alpha\alpha', \quad \text{S}(\alpha' - e\alpha)\beta = 0, \quad \text{S}\alpha\alpha'\beta = 0;$$

$$\text{LVIII.} \ldots \text{V}\alpha^t\beta = \beta\text{S}\,.\,\alpha^t + \text{UV}\alpha\alpha'.\,\text{S}\,.\,\alpha^{t-1};$$

$$\text{LIX.} \ldots \text{V}.\,\dot{\alpha}\text{V}\alpha^t\beta = \alpha^t\text{UV}\alpha\alpha' = \text{T}^{-1}1;$$

$$\text{LX.} \ldots \text{S}(\alpha' - e\alpha)\rho = x(e + \text{S}\alpha\alpha'), \quad \text{V}\alpha\rho = y\alpha^t\text{UV}\alpha\alpha';$$

while

$$\text{LXI.} \ldots f\alpha = a^{-2}b^2c^{-2}, \quad \text{and} \quad \text{LXII.} \ldots f\beta = f\text{UV}\alpha\alpha' = b^{-2}.$$

(22.) If we treat the *exponent*, t, as the *only variable* in the expression LIV. for ρ, then (comp. 314, (2.)) that *exponential expression* represents what we have called (17.) a *centro-focal ellipse*; the *distance* of *its centre* (or of its *plane*) from the centre of the *surface*, measured *along the focal line* α, being represented by the coefficient x; and the *radius* of the *right cylinder*, of which the ellipse is a *section*, or the radius of the *circle* (16.) into which that ellipse is *projected*, on a plane $\perp \alpha$, being represented by the *other* coefficient, y: while $\frac{1}{2}t\pi$ is the *excentric anomaly*.

(23.) If, on the contrary, we treat the *exponent* t as *given*, but the *coefficients* x and y as *varying together*, so as to satisfy the equation LV. of the second degree, the expression LIV. then represents a *different section* of the *surface* III., made by a *plane through the line* α, which makes with the *focal plane* (of α, α') an angle $= \frac{t\pi}{2}$; this latter section (like the former) being *always real*, if the surface *itself* be such: but being an *ellipse* for an *ellipsoid*, and an *hyperbola* for *either hyperboloid*, because

$$\text{LXIII.} \ldots f\alpha \,.\, f\text{UV}\alpha\alpha' = a^{-2}c^{-2} \text{ by LXI. and LXII.}$$

(24.) And it is scarcely necessary to remark, that by interchanging α and α' we obtain a *Second Exponential Transformation*, connected with the *second system* (17.) of *centro-focal ellipses*, as the *first* exponential transformation LIV. is connected with the *first system* (16.).

(25.) The *asymptotic cone* $f\rho = 0$ has likewise its *two systems* of *centro-focal ellipses*, and its equation admits in like manner of *two exponential transformations*, of the form LIV.; the only difference being, that the equation LV. is replaced by the following,

$$\text{LXIV.} \ldots x^2 f\alpha + y^2 f\text{UV}\alpha\alpha' = 0,$$

in which, for a *real cone*, the coefficients of x^2 and y^2 have *opposite signs* by (23.).

(26.) Finally, as regards the *confocal relation* of the surfaces III., which may represent *any confocal system* of surfaces of the second order, it may be perceived from (4.) that an *essential character* of such a relation is expressed by the equation,

$$\text{LXV.} \ldots \text{V}\nu_{\prime}\phi\nu_{\prime} = \text{V}\nu\phi_{\prime}\nu;$$

which may perhaps be called, on that account, the *Equation of Confocals*.

(27.) It is understood that the *two confocal surfaces* here considered, are represented by the two scalar equations,

$$\text{LXVI.} \ldots \text{S}\rho\phi\rho = 1, \quad \text{S}\rho\phi_{\prime}\rho = 1, \quad \text{or} \quad \text{LXVI}'. \ldots f\rho = 1, \quad f_{\prime}\rho = 1;$$

and that the *two linear and vector functions*, ν and $\nu_{\prime}$, of an *arbitrary vector* ρ, which represent *normals* to the two *concentric* and *similar* and *similarly posited surfaces*,

$$\text{LXVII.} \ldots f\rho = \text{const.}, \quad f_{\prime}\rho = \text{const.},$$

passing through any proposed point P, are expressed as follows,

$$\text{LXVIII.} \ldots \nu = \phi\rho, \quad \nu_{\prime} = \phi_{\prime}\rho.$$

(28.) It is understood also, that the two surfaces LXVI. or LXVI'. are not only *concentric*, as their equations show, but also *coaxal*, so far as the *directions* of their *axes* are concerned: or that the *two vector quadratics* (comp. 354),

$$\text{LXIX.} \ldots \mathrm{V}\rho\phi\rho = 0, \quad \text{and} \quad \text{LXX.} \ldots \mathrm{V}\rho\phi_{\prime}\rho = 0,$$

are satisfied by one *common system* of *three rectangular unit lines*. And with these understandings, it will be found that the equation LXV., which has been called above the *Equation of Confocals*, is not only *necessary* but *sufficient*, for the establishment of the relation required.

(29.) It is worth while however to observe, before closing the present series of sub-articles, that the equations XII., and those formed from them by introducing e_2 and ν_2, give the following among other relations:

$$\text{LXXI.} \ldots f\mathrm{U}\nu_1 = (b^2 - b_1^2)^{-1} = -f_1\mathrm{U}\nu; \quad f_1\mathrm{U}\nu_2 = (b_1^2 - b_2^2)^{-1} = -f_2\mathrm{U}\nu_1; \ \&\text{c.};$$

and

$$\text{LXXII.} \ldots f(\nu_1, \nu_2) = f_1(\nu_2, \nu) = f_2(\nu, \nu_1) = 0;$$

and therefore,

$$\text{LXXIII.} \ldots f_1\{(b_2^2 - b_1^2)^{\frac{1}{2}}\mathrm{U}\nu_2 \pm (b_1^2 - b^2)^{\frac{1}{2}}\mathrm{U}\nu\} = 0;$$

whence it is easy to see that the *two vectors* under the functional sign f_1 in this last expression have the directions of *generating lines* of the *single-sheeted hyperboloid* (e_1) through P, if we suppose that $b_2^2 > b_1^2 > 0 > b^2$, so that the confocal (e_2) is *here* an *ellipsoid*, and (e) a *double-sheeted hyperboloid.*

(30.) But if σ be taken to denote the variable vector of the *auxiliary surface* XXIV., the equation of that surface may by (7.) and (8.) be brought to the following rectangular form, with the meaning XXI. of ω,

$$\text{LXXIV.} \ldots 1 = \mathrm{S}\sigma\omega\sigma = (\mathrm{S}\rho\sigma)^2 - 2l^2\mathrm{S}\alpha\sigma\mathrm{S}\alpha'\sigma = b^2(\mathrm{S}\sigma\mathrm{U}\nu)^2 + b_1^2(\mathrm{S}\sigma\mathrm{U}\nu_1)^2 + b_2^2(\mathrm{S}\sigma\mathrm{U}\nu_2)^2;$$

hence, with the inequalities (29.), its *cyclic normals*, or those of its *asymptotic cone* $\mathrm{S}\sigma\omega\sigma = 0$, or the *focal lines* of the *reciprocal cone* $\mathrm{S}\sigma\omega^{-1}\sigma = 0$, that is of the cone XXXVI., or finally the *focal lines of the focal* cone* (12.), which *rests on the focal hyperbola*, have the directions of the lines LXXIII.; *those focal lines* are therefore (by what has just been seen) *the generating lines of the hyperboloid* (e_1), which passes through the given point P.

(31.) And for an *arbitrary* σ we have the transformation,

$$\text{LXXV.} \ldots l^{-2}(\mathrm{S}\rho\sigma)^2 - \mathrm{S}\alpha\sigma\alpha'\sigma = e(\mathrm{S}\sigma\mathrm{U}\nu)^2 + e_1(\mathrm{S}\sigma\mathrm{U}\nu_1)^2 + e_2(\mathrm{S}\sigma\mathrm{U}\nu_2)^2.$$

* A more general known theorem, including this, will soon be proved by quaternions [page 213].

408. The general *equation** *of conjugation*,

$$\text{I.} \ldots f(\rho, \rho') = 1, \qquad 405, \text{III.}$$

connecting the vectors ρ, ρ' of any two *points* P, P′ which are conjugate with respect to the *central* but *non-conical surface* $f\rho = 1$, may be called for that reason the *Equation of Conjugate Points*; while the analogous equation,

$$\text{II.} \ldots f(\rho, \rho') = 0,$$

which replaces the former for the case of the *asymptotic cone* $f\rho = 0$, may be called by contrast the *Equation of Conjugate Directions*: in fact, it is satisfied by any two *conjugate semidiameters*, as may be at once inferred from the *differential equation* $f(\rho, d\rho) = 0$ of the surface $f\rho$ = const. (comp. 362). Each of these two formulæ admits of numerous applications, among which we shall here consider the deduction, and some of the transformations, of the *Equation of a Circumscribed Cone*,

$$\text{III.} \ldots (f(\rho, \rho') - 1)^2 = (f\rho - 1)(f\rho' - 1);$$

which may also be considered as the *Condition of Contact*, of the *right line* PP′ with the *surface* $f\rho = 1$.

(1.) In this last view, the equation III. may be at once deduced, as the condition of *equal roots* in the scalar and *quadratic* equation (comp. 216, (2.), and 316, (30.)),

$$\text{IV.} \ldots 0 = f(x\rho + x'\rho') - (x + x')^2,$$

or

$$\text{V.} \ldots 0 = x^2(f\rho - 1) + 2xx'(f(\rho, \rho') - 1) + x'^2(f\rho' - 1);$$

which gives in general the *two vectors of intersection*, as the two values of the expression $\dfrac{x\rho + x'\rho'}{x + x'}$.

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* For the notation used, Art. 362 may be again referred to. [On page 550, vol. i., are printed the formulæ

$$f(\rho, \rho') = f(\rho', \rho) = S\rho\phi\rho' = S\rho'\phi\rho,$$

and

$$f(\rho, \rho) = f\rho,$$

which sufficiently explain the notation employed.]

(2.) If we treat the *point* P′ as *given*, and denote the *two secants* drawn from it in any *given direction* τ by $t_1^{-1}\tau$ and $t_2^{-1}\tau$, then t_1 and t_2 are the roots of this other quadratic, $f(\rho' + t^{-1}\tau) = 1$, or

$$\text{VI.} \ldots 0 = f(t\rho' + \tau) - t^2 = t^2(f\rho' - 1) + 2tf(\rho', \tau) + f\tau;$$

denoting then by $t_0^{-1}\tau$ the *harmonic mean* of these two secants, so that $2t_0 = t_1 + t_2$, and writing $\rho = \rho' + t_0^{-1}\tau$, we have

$$\text{VII.} \ldots t_0(1 - f\rho') = f(\rho', \tau), \quad f(\rho, \rho') = 1;$$

we are then led in this way to the formula I., as the *Equation of the Polar Plane* of the point P′, if that plane be here supposed to be *defined* by its well-known *harmonic property* (comp. 215, (16.), and 316, (31.), (32.)).

(3.) At the same time we obtain this *other form* of the *condition of contact* III., as that of equal roots in VI.,

$$\text{VIII.} \ldots f(\rho', \tau)^2 = f\tau \,.\, (f\rho' - 1),$$

the first member being an abridgment of $(f(\rho', \tau))^2$: and because this last equation VIII. is *homogeneous with respect to* τ, it represents a cone, namely the *Cone of Tangents* (τ) to the given surface $f\rho = 1$, from the given point P′. Accordingly it is easy to prove that the equation III. may be thus written,

$$\text{IX.} \ldots f(\rho', \rho - \rho')^2 = f(\rho - \rho') \,.\, (f\rho' - 1),$$

under which last form it is seen to be homogeneous with respect to $\rho - \rho'$.

(4.) Without expressly introducing τ, the transformation IX. shows that the equation III. represents *some cone*, with the given point P′ for its vertex; and because the *intersection* of this *cone* with the given *surface* is expressed by the *square* of the equation I. of the *polar plane* of that point, the cone must be (as above stated) *circumscribed* to the surface $f\rho = 1$, touching it *along the curve* (real or imaginary) in which that surface is *cut* by that *plane* I.

(5.) Another important transformation, or set of transformations, of the equation III. may be obtained as follows. In general, for *any two vectors* ρ and ρ', if the scalar constant m, the vector function ψ, and the scalar function F, be derived from the linear and vector function ϕ, which is here *self-conjugate* (405), by the method of the Section III. ii. 6, we have successively,

$$\text{X.} \ldots f(\rho, \rho')^2 - f\rho \,.\, f\rho' = \text{S}\rho\phi\rho' \,.\, \text{S}\rho'\phi\rho - \text{S}\rho\phi\rho \,.\, \text{S}\rho'\phi\rho' = \text{S}\,(\text{V}\rho\rho' \,.\, \text{V}\phi\rho\phi\rho')$$
$$= \text{S} \,.\, \rho\rho'\psi\text{V}\rho\rho' = m\text{S} \,.\, \rho\rho'\phi^{-1}\text{V}\rho\rho' = mF\text{V}\rho\rho';$$

and thus the equation III. of the *circumscribed cone* becomes,

$$\text{XI.} \ldots mF\text{V}\rho\rho' + f(\rho - \rho') = 0, \quad \text{or} \quad \text{XII.} \ldots mF\text{V}\tau\rho' + f\tau = 0,$$

if $\tau = \rho - \rho'$ be a *tangent* from P'. Or because $\phi\psi = m$, and $m = - c_1c_2c_3 = - a^{-2}b^{-2}c^{-2}$, by 406, XXIV., we may write (with $\tau = \rho - \rho'$) either

$$\text{XIII.} \ldots 0 = \text{S}\tau\psi^{-1}\tau + \text{S}\nu\phi^{-1}\nu, \quad \text{if} \quad \nu = \text{V}\tau\rho' = \text{V}\rho\rho',$$

or

$$\text{XIV.} \ldots F\text{V}\rho\rho' = a^2b^2c^2f(\rho - \rho'),$$

as the *condition of contact* of the *line* PP' with the *surface* $f\rho = 1$.*

(6.) A *geometrical interpretation*, of this last *form* XIV. of that *condition*, can easily be assigned as follows. Supposing at first for simplicity that the surface is an ellipsoid, let P be the point of contact, so that $f\rho = 1, f(\rho, \tau) = 0$; and let the tangent PP' be taken equal to the parallel semidiameter OT, so that $f\tau = f(\rho - \rho') = 1$. Then, with the signification XIII. of ν, the equation XIV. becomes,

$$\text{XV.} \ldots \sqrt{F\nu} = \text{T}\nu \,.\, \sqrt{F\text{U}\nu} = abc;$$

in which the factor $\text{T}\nu$ represents the area of the parallelogram under the conjugate semidiameters OP, OT of the given surface $f\rho = 1$; while the other factor $\sqrt{F\text{U}\nu}$ represents the reciprocal of the semidiameter of the reciprocal surface $F\nu = 1$, which is perpendicular to their plane POT; or the perpendicular distance between that plane, and a parallel plane which touches the given ellipsoid: so that their product $\sqrt{F\nu}$ is equal, by elementary principles, to the product of the three semiaxes, as stated in the formula XV. And the result may easily be extended by squaring, to other central surfaces.

(7.) It may be remarked in passing, that if ρ, σ, τ be *any three conjugate semidiameters* of any central surface $f\rho = 1$, so that

$$\text{XVI.} \ldots f\rho = f\sigma = f\tau = 1, \quad \text{and} \quad \text{XVII.} \ldots f(\rho, \sigma) = f(\sigma, \tau) = f(\tau, \rho) = 0,$$

* [The constituents of these auxiliary vectors ν and τ correspond to Plücker's *six coordinates* of a right line. A scalar equation of the type $f(\nu, \tau) = 0$ represents a *complex* of right lines provided the relation is independent of the absolute magnitudes of the tensors of ν and τ. The lines of the complex which pass through the extremity of a given vector ρ' lie on the cone $f(\text{V}\tau\rho', \tau) = 0$, τ being variable. Moreover, if $\text{S}\lambda\rho = 1$, $\text{S}\lambda'\rho = 1$ are the equations of any pair of planes through the right line (ν, τ), and if we take the new auxiliary vectors $\tau_1 = \lambda - \lambda'$ and $\nu_1 = \text{V}\lambda\lambda'$, it is easy to prove that $\tau_1 = x\nu$ and $\nu_1 = x\tau$, x being a scalar. Thus we may replace ν and τ by τ_1 and ν_1 respectively in the equation of the complex, and we have $f(\tau_1, \nu_1) = 0$ or $f(\tau_1, \text{V}\tau_1\lambda') = 0$. The second of these equations when λ' is regarded as known, and τ_1 as variable represents the reciprocal of the cone whose vertex is at the origin and which is touched by the lines of the complex which lie in the arbitrary plane $\text{S}\lambda'\rho = 1$.]

and if $x\rho + y\sigma + z\tau$ be *any other* semidiameter of the same surface, we have then the scalar equation,

$$\text{XVIII.} \ldots f(x\rho + y\sigma + z\tau) = x^2 + y^2 + z^2 = 1\,;$$

a relation between the coefficients, x, y, z, which has been already noticed for the *ellipsoid* in 99, (2.), and in 402, I., and is indeed deducible for *that* surface, from principles of *real scalars* and *real vectors* alone: but in extending which to the *hyperboloids*, one at least of those three *coefficients* becomes *imaginary*, as well as one at least of the three *vectors* ρ, σ, τ.

(8.) Under the same conditions XVI. XVII., we have also,

$$\text{XIX.} \ldots \mathrm{V}\rho\sigma = \pm\, abc\phi\tau = \pm\,(-m)^{-\frac{1}{2}}\phi\tau\,;$$

$$\text{XX.} \ldots \tau = \pm\,(-m)^{\frac{1}{2}}\phi^{-1}\mathrm{V}\rho\sigma = \mp\,(-m)^{-\frac{1}{2}}\mathrm{V}\phi\rho\phi\sigma\,;$$

$$\text{XXI.} \ldots \mathrm{S}\rho\sigma\tau = \pm\, abc = \pm\,(-m)^{-\frac{1}{2}}\,;$$

together with this very simple relation,

$$\text{XXII.} \ldots \mathrm{S}\rho\sigma\tau\,.\,\mathrm{S}\phi\rho\phi\sigma\phi\tau = -\,1.$$

(9.) Under the same conditions, if $x\rho + y\sigma + z\tau$ and $x'\rho + y'\sigma + z'\tau$ have only *conjugate directions*, that is, if they have the *directions* of any two *conjugate semidiameters*, the six scalar coefficients must satisfy (comp. II.) the equation,

$$\text{XXIII.} \ldots xx' + yy' + zz' = 0.$$

(10.) The equation VIII., with ρ for ρ', may be written under the form,

$$\text{XXIV.} \ldots 0 = \mathrm{S}\sigma\tau = \mathrm{S}\tau\omega\tau, \quad \text{if} \quad \text{XXV.} \ldots \sigma = \omega\tau = \phi\rho\mathrm{S}\rho\phi\tau + \phi\tau(1 - f\rho),$$

= a *new* linear and vector function, which represents a *normal to the cone of tangents* from P, to the surface $f\rho = 1$. *Inverting* this last function, we find

$$\text{XXVI.} \ldots \tau = \omega^{-1}\sigma = \frac{\phi^{-1}\sigma - \rho\mathrm{S}\rho\sigma}{1 - f\rho}\,;$$

the equation in σ of the *reciprocal cone*, or of the *cone of normals* to the *circumscribed cone* from P, is therefore,

$$\text{XXVII.} \ldots \mathrm{S}\sigma\omega^{-1}\sigma = 0, \quad \text{or} \quad \text{XXVIII.} \ldots F\sigma = (\mathrm{S}\rho\sigma)^2,$$

or finally

$$\text{XXVIII}'. \ldots F(\sigma : \mathrm{S}\rho\sigma) = 1\,;$$

a remarkably *simple form*, which admits also of a simple *interpretation*. In fact, the line $\sigma : S\rho\sigma$ is the *reciprocal of the perpendicular*, from the centre o, *on a tangent plane to the cone*, which is also a tangent plane *to the surface*; it is therefore *one of the values of the vector* ν (comp. (6.), and 373, (21.)), and consequently it is a *semidiameter of the reciprocal surface* $F\nu = 1$.

(11.) As an application of the equation XXVIII., let the surface be the *confocal* (e), represented by the equation 407, III. or X., of which the *reciprocal* is represented by 407, XVII. or XVIII. Substituting for $F\sigma$ its value thus deduced, the equation of the *reciprocal cone* (10.), with σ for a *side*, becomes,*

$$\text{XXIX.} \ldots 2l^2 S\alpha\sigma S\alpha'\sigma - (S\rho\sigma)^2 = b^2\sigma^2,$$

or

$$\text{XXIX}'. \ldots S\alpha\sigma\alpha'\sigma - l^{-2}(S\rho\sigma)^2 = e\sigma^2;$$

if then the *vertex* P be *fixed*, but the *confocal vary*, by a change of e, or of b^2 which varies with it, the *cone* XXIX. will *also vary*, but will belong to a *biconcyclic system*; whence it follows that the (*direct* or) *circumscribed cones from a given point* are all *biconfocal*: and also, by 407, (30.), that their *common focal lines* are the *generating lines* of the *confocal hyperboloid*† of one sheet, which passes through their *common vertex*.

(12.) Changing e to $e_{,}$ in XXIX′., and using the transformation 407, LXXV., with the identity (comp. 407, LIII.),

$$-\sigma^2 = (S\sigma U\nu)^2 + (S\sigma U\nu_1) + (S\sigma U\nu_2)^2,$$

we find that if σ be a *normal* to the *cone of tangents* from P to ($e_{,}$), it satisfies the equation,

$$\text{XXX.} \ldots 0 = (e - e_{,})\,(S\sigma U\nu)^2 + (e_1 - e_{,})\,(S\sigma U\nu_1)^2 + (e_2 - e_{,})\,(S\sigma U\nu_2)^2;$$

and therefore that if τ be a *tangent* from the same point P, to the same confocal ($e_{,}$), it satisfies this other condition,

$$\text{XXXI.} \ldots 0 = (e - e_{,})^{-1}(S\tau U\nu)^2 + (e_1 - e_{,})^{-1}(S\tau U\nu_1)^2 + (e_2 - e_{,})^{-1}(S\tau U\nu_2)^2,$$

which thus is a form of the equation of the *circumscribed cone* to ($e_{,}$), with its *vertex* at a *given point* P: the *confocal character* (11.) of all *such* cones being hereby exhibited anew.

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* It may be observed that, when $b = 0$, this equation XXIX. represents the *ymptotic cone* the *auxiliary surface* 407, XXIV.; and at the same time the *reciprocal* of that *focal cone*, 407, XXXVI., which rests on the *focal hyperbola*.

† This theorem (which includes that of 407, (30.)) is cited from Jacobi, and is proved, in page 143 of Dr. Salmon's Treatise, referred to in several former Notes.

(13.) It follows also from XXXI., that the *axes* of every *cone* thus *circumscribed* have the directions of the *normals* ν, ν_1, ν_2 to the *three confocals through* P; and this known theorem* may be otherwise deduced, from the *Equation of Confocals* (407, LXV.), by our general method, as follows. That equation gives

$$\nu_{,} - \nu \parallel \phi_{,}\nu \,(\text{because } \phi\nu_{,} = \phi_{,}\nu), \text{ and therefore,}$$

$$\text{XXXII.} \ldots (\nu_{,} - \nu)\,S\nu\nu_{,} = \phi_{,}\nu\,(f_{,}\rho - 1), \quad V\nu\nu_{,}S\nu\nu_{,} + V\nu\phi_{,}\nu\,(1 - f_{,}\rho) = 0;$$

changing then V to S, and ν to τ, we see that ν, ν_1, ν_2, as being the *roots* (354) of this last *vector quadratic* XXXII., have the *directions* of the axes of the cone, with τ for side,

$$\text{XXXIII.} \ldots f_{,}(\rho, \tau)^2 + f_{,}\tau\,.\,(1 - f_{,}\rho) = 0;$$

that is, by VIII., the directions of the *axes of the cone of tangents*, from P to $(e_{,})$.

(14.) As an application of the formula XIV., with the abridged symbols τ and v of (5.) for $\rho - \rho'$ and $V\rho\rho'$, the *condition of contact* of the *line* PP′ with the *confocal* (e) becomes, by the expressions 407, III., XVIII., and VII. for the functions f, F, and the squares a^2, b^2, c^2, the following *quadratic* in e:

$$\text{XXXIV.} \ldots (S\alpha\tau)^2 - 2eS\alpha\tau S\alpha'\tau + (S\alpha'\tau)^2 + (1 - e^2)\tau^2 = t^{-2}(S\alpha v\alpha' v - ev^2);$$

there are therefore in general (as is known) *two confocals*, say (e) and $(e_{,})$, of a *given system*, which touch *a given right line*; and their *parameters*,† e and $e_{,}$, are the *two roots* of the last equation: for instance, their *sum* is given by the formula,

$$\text{XXXV.} \ldots (e + e_{,})\tau^2 = t^{-2}v^2 - 2S\alpha\tau S\alpha'\tau.$$

(15.) Conceive then that ρ is a *given semidiameter* of a *given confocal* (e), and that $d\rho$ is a *tangent, given in direction*, at its extremity; the equation XXXIV. will then of course be satisfied,‡ if we change τ to $d\rho$, and v to $V\rho d\rho$, retaining the *given value* of e; but it will *also* be satisfied, for the *same*

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* Compare the third Note to page 202.

† This name of *parameter* is here given, as in 407, to the arbitrary constant $e = \frac{a^2 + c^2}{a^2 - c^2}$, of which the value distinguishes *one* confocal (e) of a system from another.

‡ In fact it follows easily from the transformations (5.), that

$$f\rho\,.\,fd\rho - a^{-2}b^{-2}c^{-2}FV\rho d\rho = f(\rho, d\rho)^2.$$

ρ and $d\rho$ (or for the same τ and υ), when we change e to this *new parameter*,

$$\text{XXXVI.} \ldots e_{,} = -e + 2S\alpha Ud\rho \,.\, S\alpha' Ud\rho - l^{-2}(V\rho Ud\rho)^2;$$

that is to say, the *new confocal* ($e_{,}$), with a parameter determined by this last formula, will *touch the given tangent* to the *given* confocal (e).

(16.) If we *at once* make $l^2 = 0$ in the equation 407, III. of a *Confocal System* of *Central Surfaces*, leaving the parameter *e finite*, we fall back on the system 406, XXXV. of *Biconfocal Cones*; but if we conceive that l^2 only *tends* to *zero*, and that e at the same time tends to *positive infinity*, in such a manner that their *product* tends to a *finite limit*, r^2, or that

$$\text{XXXVII.} \ldots \lim . l = 0, \quad \lim . e = \infty, \quad \lim . el^2 = r^2,$$

then the equation of the surface (e) tends to this *limiting form*,

$$\text{XXXVIII.} \ldots \rho^2 + r^2 = 0, \quad \text{or} \quad \text{XXXVIII}'. \ldots T\rho = r;$$

a *system of biconfocal cones* is therefore to be *combined* with a system of *concentric spheres*, in order to make up a *complete confocal system*.

(17.) Accordingly, any *given right line* PP′ is in general *touched* by only *one cone* of the system just referred to, namely by *that* particular *cone* (e), for which (comp. XXXIV.) we have the value,

$$\text{XXXIX.} \ldots e = S\alpha\upsilon\alpha'\upsilon^{-1}, \quad \text{or} \quad \text{XXXIX}'. \ldots e + S\alpha\alpha' = 2S\alpha\upsilon S\alpha'\upsilon^{-1},$$

with $\upsilon = V\rho\rho'$, as before, so that υ is *perpendicular to the given plane* OPP′, which contains the *vertex* and the *line*; in fact, the *reciprocals* of the *biconfocal cones* 406, XXXV., when α, α' are treated as *given unit lines*, but e as a *variable parameter*, compose the *biconcyclic* system* (comp. 407, XVIII.),

$$\text{XL.} \ldots S\alpha\upsilon\alpha'\upsilon = e\upsilon^2.$$

But, besides the *tangent cone* thus found, there is a *tangent sphere* with the same centre O; of which, by passing to the limits XXXVII., the radius r may be found from the same formula XXXIV. to be,

$$\text{XLI.} \ldots r = T\frac{\upsilon}{\tau} = T\frac{V\rho\rho'}{\rho - \rho'};$$

and such is in fact an expression (comp. 316, L.) for the length of the perpendicular from the origin on the given line PP′.

* The *bifocal form* o tne equation of this *reciprocal system* of cones XL. was given in 406, XXV., but with *other constants* (λ, μ, g), connected with the *cyclic form* (406, I.) of the equation of the *given system*.

(18.) In general, the equation XXXIV. is a form of the *equation of the cone*, with ρ for its variable vector, which has a *given vertex* P′, and is *circumscribed* to a *given confocal* (e). Accordingly, by making $e = -Saa'$ in that formula, we are led (after a few reductions, comp. 407, XXVII.) to an equation which may be thus written,

$$\text{XLII.} \ldots 0 = l^2(Saa'\tau)^2 + 2Sa\rho'\tau Sa'\rho'\tau,$$

with the variable side $\tau = \rho - \rho'$, as before; and which differs only by the substitution of ρ' and τ for ρ and ν, from the equation 407, XXXVI. for that *focal cone*, which rests on the *focal hyperbola*. The *other* (real) *focal cone* which has the same arbitrary vertex P′, but rests on the *focal ellipse*, has for equation,

$$\text{XLIII.} \ldots l^2(S(a - a')\tau)^2 = Sa\upsilon a'\upsilon - \upsilon^2,$$

as is found by changing e to 1 in the same formula XXXIV.

(19.) It is however simpler, or at least it gives more symmetric results, to change $e_{,}$ in XXXI. to $-Saa'$ for the focal hyperbola, and to $+1$ for the focal ellipse, in order to obtain the *two real focal cones* with P for vertex, which rest on those two curves; while that *third* and wholly *imaginary focal cone*, which has the same vertex, but rests on the known *imaginary focal curve*, in the plane of b and c, is found by changing $e_{,}$ to -1. This imaginary focal cone, and the two real ones which rest as above on the hyperbola and ellipse respectively, may thus be represented by the three equations,

$$\text{XLIV.} \ldots 0 = a^{-2}(S\tau U\nu)^2 + a_1^{-2}(S\tau U\nu_1)^2 + a_2^{-2}(S\tau U\nu_2)^2;$$

$$\text{XLV.} \ldots 0 = b^{-2}(S\tau U\nu)^2 + b_1^{-2}(S\tau U\nu_1)^2 + b_2^{-2}(S\tau U\nu_2)^2;$$

$$\text{XLVI.} \ldots 0 = c^{-2}(S\tau U\nu)^2 + c_1^{-2}(S\tau U\nu_1)^2 + c_2^{-2}(S\tau U\nu_2)^2;$$

τ being in each case a side of the cone, and ν, ν_1, ν_2 having the same significations as before.

(20.) On the other hand, if we place the *vertex* of a circumscribed cone at a point P of a *focal curve*, real or imaginary, the *enveloped surface* being the *confocal* $(e_{,})$, we find first, by XXX., for the *reciprocal* cones, or *cones of normals* σ, with the same order of succession as in (19.), the three equations,

$$\text{XLVII.} \ldots a^2(SU\nu\sigma)^2 = a_{,}^2;$$

$$\text{XLVIII.} \ldots b^2(SU\nu\sigma)^2 = b_{,}^2;$$

$$\text{XLIX.} \ldots c^2(SU\nu\sigma)^2 = c_{,}^2;$$

and next, for the *circumscribed cones* themselves, or cones of *tangents* τ, the connected equations:

$$\text{L.} \ldots a^2(\text{VU}\nu\tau)^2 + a_{,}^2 = 0;$$

$$\text{LI.} \ldots b^2(\text{VU}\nu\tau)^2 + b_{,}^2 = 0;$$

$$\text{LII.} \ldots c^2(\text{VU}\nu\tau)^2 + c_{,}^2 = 0;$$

all which have the *forms* of equations of *cones of revolution*, but on the geometrical *meanings* of the three last of which it may be worth while to say a few words.

(21.) The cone L. has an *imaginary vertex*, and is always *itself* imaginary; but the *two other cones*, LI. and LII., have *each* a *real vertex* P, with $b^2 > 0$ for the first, and $c^2 < 0$ for the second; b being the mean semiaxis of the *ellipsoid*, which passes through a given point of the *focal hyperbola*, and c^2 being the negative and algebraically least square of a scalar semiaxis of the *double-sheeted hyperboloid*, which passes through a given point of the *focal ellipse*: while, *in each case*, ν has the direction of the *normal* to the *surface*, which is also the *tangent* to the *curve* at that point, and is at the same time the *axis* of revolution of the *cone*.

(22.) The *semiangles* of the two last cones, LI. and LII., have for their respective *sines* the two quotients,

$$\text{LIII.} \ldots b_{,} : b, \quad \text{and} \quad \text{LIV.} \ldots (-c_{,}^2)^{\frac{1}{2}} : (-c^2)^{\frac{1}{2}};$$

each of those *two cones* is therefore *real*, if circumscribed to a *single-sheeted hyperboloid*, because, for *such* an enveloped surface $(e_{,})$, $b_{,}$ is *real*, and *less* than the b of *any confocal ellipsoid*, while $c_{,}$ is imaginary, and its *square* is algebraically *greater* (or nearer to zero) than the square of the imaginary semiaxis c of *every double-sheeted hyperboloid*, of the same confocal system; but the *cone* LI. is *imaginary*, if the *enveloped surface* $(e_{,})$ be either an hyperboloid of *two* sheets ($b_{,}$ imaginary), or an *exterior ellipsoid* ($b_{,} > b$); and the *other* cone LII. is imaginary, if the surface $(e_{,})$ be either *any ellipsoid* ($c_{,}$ real), or else an *exterior* and *double*-sheeted hyperboloid ($a_{,}^2 < a^2$, $c_{,}^2 < c^2$, $-c_{,}^2 > -c^2$). Accordingly it is known that the focal *hyperbola*, which is the *locus of the vertex* of the cone LI., lies entirely *inside every double*-sheeted hyperboloid of the system; while the focal *ellipse*, which is in like manner the *locus* of the *vertex* of the cone LII., is *interior* to *every ellipsoid*: and *real tangents* to a *single*-sheeted hyperboloid can be drawn, from *every real point* of space.

(23.) The *twelve points* (whereof only *four* at most can be *real*), in which a *surface* (*e*) or (*abc*) is cut by the *three focal curves*, are called the *Umbilics* of that surface; the vectors, say ω, $\omega_{,}$, $\omega_{,,}$, of *three* such umbilics, in the respective planes of *ca*, *ab*, *bc*, are:

$$\text{LV.} \ldots \omega = \frac{a}{2}(\alpha + \alpha') + \frac{c}{2}(\alpha - \alpha');$$

$$\text{LVI.} \ldots \omega_{,} = \frac{a(\alpha + \alpha')}{1 - S\alpha\alpha'} + \frac{\sqrt{-1}\, b V\alpha\alpha'}{1 - S\alpha\alpha'};$$

$$\text{LVII.} \ldots \omega_{,,} = \frac{c(\alpha - \alpha')}{1 + S\alpha\alpha'} - \frac{\sqrt{-1}\, b V\alpha\alpha'}{1 + S\alpha\alpha'};$$

and the others can be formed from these, by changing the signs of the terms, or of some of them. The four *real* umbilics of an *ellipsoid* are given by the formula LV., and those of a *double-sheeted hyperboloid* by LVI., with the changes of sign just mentioned.

(24.) In transforming expressions of this sort, it is useful to observe that the expressions for the squares of the semiaxes,

$$a^2 = l^2(e + 1), \quad b^2 = l^2(e + S\alpha\alpha'), \quad c^2 = l^2(e - 1), \qquad 407, \text{VII.}$$

combined with $T\alpha = T\alpha' = 1$, give not only $a^2 - c^2 = 2l^2$, but also,

$$\text{LVIII.} \ldots T\frac{\alpha + \alpha'}{2} = \sqrt{\frac{1 - S\alpha\alpha'}{2}} = \cos \tfrac{1}{2} \angle \frac{\alpha'}{\alpha} = \left(\frac{a^2 - b^2}{a^2 - c^2}\right)^{\frac{1}{2}};$$

$$\text{LIX.} \ldots T\frac{\alpha - \alpha'}{2} = \sqrt{\frac{1 + S\alpha\alpha'}{2}} = \sin \tfrac{1}{2} \angle \frac{\alpha'}{\alpha} = \left(\frac{b^2 - c^2}{a^2 - c^2}\right)^{\frac{1}{2}};$$

and

$$\text{LX.} \ldots TV\alpha\alpha' = \sqrt{(1 - (S\alpha\alpha')^2)} = \sin \angle \frac{\alpha'}{\alpha} = l^{-2}(a^2 - b^2)^{\frac{1}{2}}(b^2 - c^2)^{\frac{1}{2}},$$

with the verification, that because

$$\text{LXI.} \ldots (\alpha - \alpha')(\alpha + \alpha') = 2V\alpha\alpha',$$

therefore

$$\text{LXI}'. \ldots T(\alpha - \alpha') \,.\, T(\alpha + \alpha') = 2TV\alpha\alpha'.$$

We have also the relations,

$$\text{LXII.} \ldots T(\alpha + \alpha')^{-2} + T(\alpha - \alpha)^{-2} = (TV\alpha\alpha')^{-2};$$

$$\text{LXIII.} \ldots T(\alpha + \alpha')^{-2} - T(\alpha - \alpha')^{-2} = S\alpha\alpha' \,.\, (TV\alpha\alpha')^{-2};$$

with others easily deduced.

(25.) The expression LV. conducts to the following among other consequences, which all admit of elementary verifications,* and may be illustrated by the annexed fig. 84. Let U, U′ be the two real points in which an ellipsoid (abc) is cut by one branch of the focal hyperbola, with H for summit, and with F for its interior focus; the adjacent major summit of the surface being E, and R, R′ being (as in the figure) the adjacent points of intersection of the same surface with the focal lines α, α', that is, with the asymptotes to the hyperbola. Let also V, T be the points in which the same asymptotes α, α' meet the tangent to the hyperbola at U, or the normal to the ellipsoid at that real umbilic, of which we may suppose that the vector OU is the ω of the formula LV.; and let S be the foot of the perpendicular on this normal to the surface, or tangent TV to the curve, let fall from the centre O. Then, besides the obvious values,

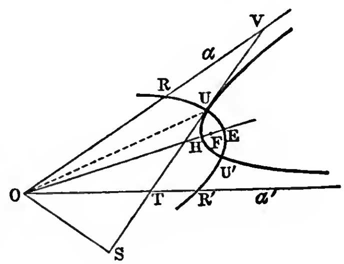

Fig. 84.

$$\text{LXIV.} \ldots \overline{\text{OE}} = a, \quad \overline{\text{OF}} = (a^2 - c^2)^{\frac{1}{2}}, \quad \overline{\text{OH}} = (a^2 - b^2)^{\frac{1}{2}},$$

and the obvious relations, that the intercept TV is bisected at U, and that the point F is at once a summit of the focal ellipse, and a focus of that other ellipse in which the surface is cut by the plane (ac) of the figure, we shall have these vector expressions (comp. 371, (3.), and 407, VIII. LXI.):

$$\text{LXV.} \ldots \text{OV} = (a + c)\alpha, \quad \text{OT} = (a - c)\alpha', \quad \text{TV} = a(\alpha - \alpha') + c(\alpha + \alpha');$$

$$\text{LXVI.} \ldots \text{SU}^{-1} = \phi\omega = -\frac{a^{-1}}{2}(\alpha + \alpha') - \frac{c^{-1}}{2}(\alpha - \alpha'), \quad \text{SU} = -ac : \text{TU};$$

$$\text{LXVII.} \ldots \text{OR} = \frac{\alpha}{\sqrt{f\alpha}} = ab^{-1}c\alpha, \quad \text{OR}' = \frac{\alpha'}{\sqrt{f\alpha'}}, = ab^{-1}c\alpha';$$

whence follow by (24.) these other values,

$$\text{LXVIII.} \ldots \overline{\text{OV}} = a + c, \quad \overline{\text{OT}} = a - c, \quad \overline{\text{TV}} = 2b;$$

$$\text{LXIX.} \ldots \overline{\text{TU}} = \overline{\text{UV}} = b, \quad \overline{\text{SU}} = \overline{\text{OR}} = \overline{\text{OR}'} = ab^{-1}c;$$

$$\text{LXX.} \ldots \overline{\text{OU}} = \text{T}\omega = (a^2 - b^2 + c^2)^{\frac{1}{2}};$$

$$\text{LXXI.} \ldots \overline{\text{OS}} = (a^2 - b^2 + c^2 - a^2b^{-2}c^2)^{\frac{1}{2}} = b^{-1}(a^2 - b^2)^{\frac{1}{2}}(b^2 - c^2)^{\frac{1}{2}}.$$

* Some such verifications were given in the *Lectures*, pages 691, 692, in connexion with fig. 102 of that former volume, which answered in several respects to the present fig. 84.

(26.) It follows that the lengths of the *sides* OV, OT, TV of the *umbilicar triangle* TOV are equal to the *sum* and *difference* $(a \pm c)$ of the *extreme semiaxes*, and to the *mean axis* $(2b)$ of the *ellipsoid*; while the *area* of that triangle $= \overline{\text{OS}} . \overline{\text{TU}} = (a^2 - b^2)^{\frac{1}{2}} (b^2 - c^2)^{\frac{1}{2}}$ = the *rectangle* under the *two semiaxes* of the *hyperbola*, if *both* be treated as *real*. The length $(\text{T}\phi\omega)^{-1}$, or $\overline{\text{SU}}$, or the *perpendicular* from the *centre* O, on the *tangent plane* at an umbilic U, is $ab^{-1}c$; and the *sphere* concentric with the ellipsoid, which *touches the four umbilicar tangent planes*, passes *through the points* R, R′ of *intersection* of that ellipsoid with the *focal lines* α, α', that is, as before, with the *asymptotes* to the *hyperbola*; or, by (21.) (22.), with the *axes* of the *two circumscribed right cylinders*.* And finally the length, say u, of the *umbilicar semidiameter* OU, is given by the formula,

$$\text{LXII.} \ldots u^2 = a^2 - b^2 + c^2;$$

all which agrees (25.) with known results.

(27.) An *umbilic* of a surface of the second order may be otherwise *defined* (comp. (23.)), as a real or imaginary point at which the *tangent plane* is *parallel* to a *cyclic plane*; and accordingly it is easy to prove (comp. 407, (20.)) that the *umbilicar normal* $\phi\omega$ in LXVI. has the direction of a *cyclic normal*. To employ this known property in verification of the recent expressions (25.), (26.), for the lengths of OU and SU, it is only necessary to observe that the *common radius* of the diametral and *circular sections* of the ellipsoid is the *mean semiaxis* b (comp. 216, (7.) (9.), &c.); and that, by a slight extension of the analysis in (7.), (8.), (9.), it can be shown that if ρ, σ, τ and ρ', σ', τ' be *any two systems* of *three conjugate semidiameters of any central surface*, $f\rho = 1$, then

$$\text{LXXIII.} \ldots \rho'^2 + \sigma'^2 + \tau'^2 = \rho^2 + \sigma^2 + \tau^2,$$

and

$$\text{LXXIV.} \ldots (\text{S}\rho'\sigma'\tau')^2 = (\text{S}\rho\sigma\tau)^2.$$

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* Compare 218, (5.), and 220, (4.); in which the points B, B′ (comp. also fig. 53, p. 234, vol. i. [and p. 184, vol. ii.]) may now be conceived to coincide with the points R, R′ of the new figure 84. It is obvious that the theory of *circumscribed cylinders* is included in that of circumscribed *cones*; so that the cylinder circumscribed to the confocal (e), with its generating lines parallel to a given (real or imaginary) semidiameter γ of that surface $(f\gamma = 1)$, may be represented (comp. III. XIV.) by the equation,

$$\text{III}' \ldots f(\rho, \gamma)^2 = f\rho - 1; \quad \text{or} \quad \text{XIV}' \ldots F\text{V}\gamma\rho = a^2b^2c^2;$$

with interpretations easily deduced, from principles already established.

(28.) A less elementary verification of the value LXXII. of u^2, but one which is useful for other purposes, may be obtained from either the cubic in b^2, or that in e, assigned in 407, (8.). For if b_0^2, b_1^2, b_2^2 be the roots of the former cubic, and e_0, e_1, e_2 the roots of the latter, inspection of those equations shows at once that we have *generally*,

$$\text{LXXV.} \ldots -\rho^2 = b_0^2 + b_1^2 + b_2^2 - 2l^2 S\alpha\alpha' = l^2(e_0 + e_1 + e_2 + S\alpha\alpha');$$

or

$$\text{LXXVI.} \ldots \overline{\text{OP}}^2 = T\rho^2 = a_0^2 + b_1^2 + c_2^2 = b_0^2 + c_1^2 + a_2^2 = \&\text{c.},$$

where the semiaxes a_0, b_1, c_2 belong to the three confocals through *any* proposed point P. Making then,

$$\text{LXXVII.} \ldots a_0^2 = a^2, \quad b_1^2 = 0, \quad c_2^2 = c^2 - b^2,$$

we recover the expression assigned above, for the square of the length u of an *umbilicar semidiameter* of an *ellipsoid*.

(29.) For *any* central surface, the principle (27.) shows that if λ, μ be, as in 405, (5.), &c., the *two real cyclic normals*, and if g be the *real scalar* associated with them as before, then the vectors of the *four real umbilics* (if such exist) must admit of being thus expressed:

$$\text{LXXVIII.} \ldots \pm \phi^{-1}\lambda : \sqrt{F\lambda} = \pm\, abc\,(gU\lambda + \mu T\lambda);$$

$$\text{LXXIX.} \ldots \pm \phi^{-1}\mu : \sqrt{F\mu} = \pm\, abc\,(gU\mu + \lambda T\mu);$$

and thus we see anew, that an *hyperboloid* with *one sheet* has (as is well known) *no real umbilic*, because for *that* surface the product abc of the semiaxes is imaginary; or because it has *no real tangent plane parallel* to either of its two real planes of *circular section*.

(30.) Of whatever *species* the surface may be, the *three umbilicar vectors* (23.), of which only *one* at most can be *real*, with the particular *signs* there given, but which have the *forms* of lines *in the three principal planes*, must be conceived, in virtue of their *expressions* LV. LVI. LVII., to *terminate on an imaginary right line*, of which the vector equation is,

$$\text{LXXX.} \ldots \rho = \frac{-a(e'+1)}{\alpha+\alpha'} - \sqrt{-1}\,\frac{b(e'+S\alpha\alpha')}{V\alpha\alpha'} + \frac{c(e'-1)}{\alpha-\alpha'};$$

e' being a scalar variable, which receives the three values, $-S\alpha\alpha'$, $+1$, and -1, when ρ comes to coincide with ω, $\omega_{,}$, and $\omega_{,,}$ respectively. And *such* an imaginary *right line*, which is easily proved to *satisfy*, for *all values* of the variable e', both the *rectangular* and the *bifocal forms* of the *equation of the surface* (e), or to be (in an imaginary sense) *wholly contained* upon that surface, may be called an *Umbilicar Generatrix*.

(31.) There are in general *eight such generatrices* of *any central surface* of the second order, whereof *each* connects *three umbilics*, in the *three principal planes*, *two* passing through *each* of the *twelve* umbilicar points (23.); and because e'^2 disappears from the *square* of the expression LXXX. for ρ, which square reduces itself to the following,

$$\text{LXXXI.} \ldots \rho^2 = -l^2(2e' + e + \mathrm{S}\alpha\alpha') = -b^2 - 2l^2e',$$

they may be said to be the *eight generating lines* through the *four imaginary points*, in which the *surface meets* the *circle at infinity*.

(32.) In general, from the cubics in e and in b^2, or from either of them, it may be without difficulty inferred (comp. (28.)), that the *eight intersections* (real or imaginary) of *any three confocals* (e_0) (e_1) (e_2) have their vectors ρ represented by the formula:

$$\text{LXXXII.} \ldots \rho = \frac{\pm a_0a_1a_2}{l^2(\alpha + \alpha')} \pm \frac{\sqrt{-1}\, b_0b_1b_2}{l^2\mathrm{V}\alpha\alpha'} \pm \frac{c_0c_1c_2}{l^2(\alpha - \alpha')};$$

comparing which with the vector expression LXXX., we see that the three confocals, through the point determined by that former expression, for any given value of e', are (e), (e'), and (e') *again*; and therefore that *two* of the *three* confocal surfaces through *any point* of an *umbilicar generatrix* (30.) *coincide*: a result which gives in a new way (comp. LXXV.) the expression LXXXI. for ρ^2.

(33.) The *locus* of all *such* generatrices, for *all* the confocals (e) of the system, is a certain *ruled surface*, of which the doubly variable *vector* may be thus expressed, as a function of the *two* scalar variables, e and e':

$$\text{LXXXIII.} \ldots \rho_{e,e'} = \frac{\pm l(e+1)^{\frac{1}{2}}(e'+1)}{\alpha + \alpha'} \pm \frac{\sqrt{-1}\, l(e + \mathrm{S}\alpha\alpha')^{\frac{1}{2}}(e' + \mathrm{S}\alpha\alpha')}{\mathrm{V}\alpha\alpha'} \pm \frac{l(e-1)^{\frac{1}{2}}(e'-1)}{\alpha - \alpha'};$$

and because we have thus, for *any one set of signs*, the *differential relation*,

$$\text{LXXXIV.} \ldots \mathrm{D}_e\rho_{e,e'} = \tfrac{3}{2}\mathrm{D}_{e'}\rho_{e,e'},$$

it follows that *this ruled locus* is a *Developable Surface*: its *edge of regression* being that wholly *imaginary curve*, of which the *vector* is $\rho_{e,\,e}$, and which is therefore by (32.) the *locus* of all the *imaginary points*, through each of which pass *three coincident confocals*.

(34.) The *only real part* of this *imaginary developable* consists of the *two real focal curves*, which are *double lines* upon it, as are also the *imaginary focal*, and the *circle at infinity* (31.) ; and the *scalar equation* of the same *imaginary surface*, obtained by elimination of the two arbitrary scalars e and e', is found to be of the *eighth degree*, namely the following:

$$\text{LXXXV.}\ldots\begin{cases} 0 = \Sigma m^2x^8 + 2\Sigma m(m-n)x^6y^2 + \Sigma(p^2 - 6mn)x^4y^4 \\ \quad + 2\Sigma(3m^2 - np)x^4y^2z^2 + 2\Sigma m^2(n-p)x^6 + 2\Sigma m(mp - 3n^2)x^4y^2 \\ \quad + 2(m-n)\,(n-p)\,(p-m)x^2y^2z^2 + \Sigma m^2(m^2 - 6np)x^4 \\ \quad + 2\,\Sigma mn(mn - 3p^2)x^2y^2 + 2\Sigma m^2np(p-n)x^2 + m^2n^2p^2\,; \end{cases}$$

in which we have written, for abridgment,

$$\text{LXXXVI.}\ldots x = -\,\mathrm{S}\rho\mathrm{U}(a + a'), \quad y = -\,\mathrm{S}\rho\mathrm{UV}aa', \quad z = -\,\mathrm{S}\rho\mathrm{U}(a - a'),$$

and

$$\text{LXXXVII.}\ldots m = b^2 - c^2, \quad n = c^2 - a^2, \quad p = a^2 - b^2,$$

so that

$$\text{LXXXVIII.}\ldots m + n + p = 0\,;$$

while each sign Σ indicates a sum of three or of six terms, obtained by cyclical or binary* interchanges.

(35.) From the manner in which the equation of this *imaginary surface* (33.) or (34.) has been deduced, we easily see by (32.) that it has the double property: I.st of being (comp. (20.)) the *locus* of the *vertices* of all the (real or imaginary) *right cones*, which can be *circumscribed* to the *confocals* of the system; and II.nd of being at the same time the *common envelope* of all those confocals: which *envelope* accordingly is known to be a *developable*† *surface*.

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* When xyz and abc are *cyclically* changed to yzx and bca, then mnp are similarly changed to npm; but when, for instance, retaining x and a unchanged, we make only *binary interchanges* of y, z, and of b, c, we then change m, n, and p, to $-m$, $-p$, and $-n$ respectively.

† This theorem is given, for instance, in page 157 [Art. 221] of the several times already cited Treatise by Dr. Salmon, who also mentions the *double lines* &c. upon the surface; but the present writer does not yet know whether the theory above given, of the *eight umbilicar generatrices*, has been anticipated: the *locus* (33.) of which *imaginary right lines* (30.) is *here* represented by the *vector equation* LXXXIII., from which the *scalar equation* LXXXV. has been above deduced (34.), and ought to be found to agree (notation excepted) with the known coordinate equation of the *developable envelope* (35.) of a *confocal system*.

(36.) The *eight imaginary lines* (31.) will come to be mentioned again, in connexion with the *lines of curvature* of a surface of the second order* ; and before closing the present series of sub-articles, it may be remarked that the equation in (15.), for the determination of the *second confocal* $(e_{,})$ which *touches a given tangent*, $d\rho$ or PP′, to a *given surface* (e) of the same system, will soon appear under a new form, in connexion with that theory of *geodetic lines*, on surfaces of the second order, to which we next proceed.†

* [Compare the sub-articles to 410, page 235.]

† [Although repetition is unavoidable, it seems well to supplement Arts. 407 and 408 by a few examples on the use of the general equation of confocals $S\rho(\Phi + u)^{-1}\rho = -1$, in which $-(\Phi + u)$ replaces the ϕ^{-1} of 407, XV., so that $u - b^2$ is constant. The vector ϖ to the pole of the plane $S\lambda\rho = 1$, with respect to the quadric u is given by $(\Phi + u)^{-1}\varpi = -\lambda$ or $\varpi = -(\Phi + u)\lambda$. The locus of poles of the plane is thus a right line normal to the plane, and the distance between any pair of poles is $T(\varpi - \varpi') = (u' - u)T\lambda = (b'^2 - b^2)p^{-1}$, p being the central perpendicular on the plane, and b and b' the mean semiaxes of the quadrics. The plane touches one quadric of the system whose parameter u_0 is given by $S\lambda(\Phi + u_0)\lambda = -1$, this being the condition that the corresponding pole should lie in the plane. The vector to the point of contact is $\varpi = -\lambda^{-1}(S + V)(\Phi + u_0)\lambda$, or, by the condition, $\varpi = \lambda^{-1} - \lambda^{-1}V\lambda\Phi\lambda$.

If in this equation we replace λ by $x\lambda$ where x is a variable scalar, we see at once that the locus of points of contact of a system of parallel planes is a rectangular hyperboloid, and if we replace λ by $(\lambda + x\lambda')(1 + x)^{-1}$, we find the locus of the points of contact of planes through a given line to be a twisted cubic. In this case also the locus of poles of the planes is a hyperbolic paraboloid $\rho = -(\Phi + u)(\lambda + x\lambda')(1 + x)^{-1}$, since the form of the equation shows that it is the locus of lines dividing the line loci for any two of the planes in the same ratio.

If u_1, u_2, and u_3 are the parameters of the three confocals which pass through the extremity of a given vector α, and ν_1, ν_2, and ν_3 the corresponding vectors of proximity, $-\alpha = (\Phi + u_1)\nu_1 = (\Phi + u_1)\nu_2 = (\Phi + u_3)\nu_3$, and $S\alpha\nu_1 = S\alpha\nu_2 = S\alpha\nu_3 = 1$. Combining the three expressions for α, we deduce $u_1S\nu_2\nu_1 + S\nu_2\Phi\nu_1 = S\nu_1\Phi\nu_2 + u_2S\nu_1\nu_2$, so $S\nu_1\nu_2 = 0$, since u_1 is not equal to u_2, or the surfaces cut at right angles. Again $(\Phi + u_1)(\nu_1 - \nu_2) = (u_2 - u_1)\nu_2$, and on inversion $\nu_1 - \nu_2 = (u_2 - u_1)(\Phi + u_1)^{-1}\nu_2$. Operating on this by $S\nu_3$, we see that ν_2 and ν_3 as well as being at right angles are conjugate with respect to the quadric u_1, and therefore parallel to the principal axes of the section of that quadric made by $S\rho\nu_1 = 0$; operating by $S\nu_2^{-1}$, we find $-1 = -(u_1 - u_2)S\nu_2^{-1}(\Phi + u_1)^{-1}\nu_2 = +(u_1 - u_2)SU\nu_2(\Phi + u_1)^{-1}U\nu_2$, so the lengths of these semiaxes are $(u_1 - u_2)^{\frac{1}{2}}$ and $(u_1 - u_3)^{\frac{1}{2}}$, respectively.

Introducing a new linear vector function analogous to that of 407, (7.), and defined by the equation $\Theta\rho = \Phi\rho + \alpha S\alpha\rho$, we see on referring to the relations between α, ν_1, ν_2, and ν_3 that

$$(\Theta + u_1)\nu_1 = (\Theta + u_2)\nu_2 = (\Theta + u_3)\nu_3 = 0,$$

so the vectors of proximity at α are the solutions of this new function and the parameters of the surfaces are the corresponding roots. This again proves the surfaces cut at right angles, for Θ is self-conjugate, and its solutions are consequently mutually rectangular.

If $S\lambda\rho = 1$ is any plane through the extremity of α, the equation $\varpi = -(\Phi + u)\lambda$ which determines its pole with respect to the quadric u, may be replaced by $\varpi - \alpha = -(\Theta + u)\lambda$, because $S\lambda\alpha = 1$. If the pole is in the plane, $\varpi - \alpha$ is at right angles to λ, and we determine at once the parameter of the touched quadric u and the point of contact by operating on λ by $-\Theta$, and then resolving the vector obtained in and normal to the given plane. Setting off from α the component in the plane we get the point of contact, while the parameter of the quadric is the ratio which the component normal to the plane bears to the vector λ.

In this case also we have $S(\varpi - \alpha)(\Theta + u)^{-1}(\varpi - \alpha) = 0$ for the equation of the tangent cone from the point α to the surface u, and the form of the equation shows that the tangent cones are confocal, so that the quadrics appear to cut at right angles as well as actually doing so. Also the

409. A general theory of *geodetic lines*, as treated by quaternions, was given in the Fifth Section (III. iii. 5) of the present Chapter; and was illustrated by applications to several different *families of surfaces*. We can only here spare room for applying the same theory to the deduction, in a new way, of a few known but *principal properties* of geodetics on *central surfaces of the second order*; the differential

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common principal axes of the system are along the normals at α for these are the solutions of $V\nu\Theta\nu = 0$. Replacing $\varpi - \alpha$ by a given vector τ, we have $S\tau(\Theta + u)^{-1}\tau = 0$ to determine the two quadrics which the line touches. If λ and λ' are the vectors of proximity to the points of contact as before, we have the vectors from α to the points of contact given by $\tau = -(\Theta + u)\lambda$, and $\tau' = -(\Theta + u')\lambda'$, u and u' being roots of the quadratics, and τ being parallel to τ'. But λ and λ' are normal to the corresponding cones, hence we see $S\lambda\Theta\lambda' = 0$ as well as $S\lambda\lambda' = 0$. We may also write $\tau = -\lambda^{-1}V\lambda\Theta\lambda$, and this, coupled with the condition $S\lambda\alpha = 1$, determines the locus of the points of contact, λ being supposed to vary consistently with the condition.

Another method of treatment is often useful. Any quadric may be derived from a sphere by operating on its vector radii by a self-conjugate linear vector function which is however real only when the quadric is an ellipsoid (Tait's *Quaternions*, Third Edition, page 207.)

It is obvious that if we can determine a self-conjugate linear vector function θ so that $\theta^2 = \phi$, we may write $S\rho\phi\rho = -1$ in the form $(\theta\rho)^2 = -1$ or $\theta\rho = \eta$, where $T\eta = 1$. Even in the more general case when ϕ is not self-conjugate but expressible in the form $\phi\rho = (a\alpha S\beta\gamma\rho + b\beta S\gamma\alpha\rho + c\gamma S\alpha\beta\rho)(S\alpha\beta\gamma)^{-1}$, any one of the eight functions given by $\theta\rho = (\pm\sqrt{a}\alpha S\beta\gamma\rho \pm \sqrt{b}\beta S\gamma\alpha\rho \pm \sqrt{c}\gamma S\alpha\beta\rho)(S\alpha\beta\gamma)^{-1}$, satisfies the condition $\theta^2 = \phi$. It is evident that all functions of this type or of the type $(\phi + u)^{\frac{1}{2}}$ are commutative in order of operation. More generally it can be shown that two functions are commutative in order of operation when and only when their vector solutions are parallel, a condition obviously true for the functions to be considered. We may consequently use the vector equation $\rho = (\Phi + u)^{\frac{1}{2}}\eta$, where $T\eta = 1$ as the equation of a confocal system, for $\eta^2 = ((\Phi + u)^{-\frac{1}{2}}\rho)^2 = S\rho(\Phi + u)^{-1}\rho = -1$. Points on two confocals derived from the same point on a unit sphere are called corresponding points, and it is easy to show in this notation if P and Q on one confocal correspond respectively to P′ and Q′ on another that $\overline{PQ'} = \overline{QP'}$.

Now three confocals pass through a given point. We have thus three different expressions for a vector $\rho = (\Phi + u_1)^{\frac{1}{2}}\eta_1 = (\Phi + u_2)^{\frac{1}{2}}\eta_2 = (\Phi + u_3)^{\frac{1}{2}}\eta_3$, $\eta_1\eta_2\eta_3$ being certain unit vectors, and u_1, u_2, and u_3 being the parameters of the confocals through ρ. The form of these equations suggests the new expression

$$\rho = [(\Phi + u_1)(\Phi + u_2)(\Phi + u_3)]^{\frac{1}{2}}\epsilon,$$

and substituting this for ρ in $S\rho(\Phi + u_1)^{-1}\rho = -1$, the result is $S\epsilon(\Phi + u_2)(\Phi + u_3)\epsilon = -1$. This must be satisfied for all values of u_2 and u_3, so we see ϵ is one of eight imaginary vectors constant for the whole system, and satisfying $\epsilon^2 = 0$, $S\epsilon\Phi\epsilon = 0$, and $S\epsilon\Phi^2\epsilon = -1$. For a value of ϵ satisfying these equations, and for suitable choice of the three parameters ρ may be made the vector to any point in space; if one parameter is given ρ describes the corresponding quadric, and if two of the parameters are assigned, ρ describes the curve of intersection of the quadrics determined by them.

This notation is suitable for investigating the properties of the umbilical generators. When $u_3 = u_2$, we have $\rho = (\Phi + u_2)(\Phi + u_1)^{\frac{1}{2}}\epsilon$ which represents a right line of a simply infinite system when u_1 is given and u_2 variable. If for the moment $\tau = (\Phi + u_1)^{\frac{1}{2}}\epsilon$, we deduce from the properties of ϵ, $\tau^2 = S\tau(\Phi + u_1)^{-1}\tau = 0$, and $S\tau\Phi\tau = -1$, and from these it appears that the line is a generator of the quadric passing through one of the points in which the asymptotic cone intersects the circle at infinity (408, (30.)). Again (33.) if $t = \frac{2}{3}(u_2 - u_1)$ the equation of one of these lines becomes $\rho = \left(1 + t\dfrac{d}{du_1}\right)(\Phi + u_1)^{\frac{3}{2}}\epsilon$, showing that they belong to a developable whose cuspidal edge is $\rho = (\Phi + u)^{\frac{3}{2}}\epsilon$, the locus of points through each of which pass three coincident confocals.]

equation employed being one of those formerly used, namely (comp. 380, IV.),

$$\text{I.} \ldots \mathrm{V}\nu\mathrm{d}^2\rho = 0, \quad \text{if} \quad \text{II.} \ldots \mathrm{Td}\rho = \text{const.};$$

that is, if the *arc* of the geodetic be made the independent variable.

(1.) In general, for *any surface*, of which ν is a normal vector, so that the *first* differential equation of the surface is $\mathrm{S}\nu\mathrm{d}\rho = 0$, the *second differential* equation $\mathrm{dS}\nu\mathrm{d}\rho = 0$ gives, by I., for a *geodetic* on that surface, the expression,

$$\text{III.} \ldots \mathrm{d}^2\rho = -\nu^{-1}\mathrm{Sd}\nu\mathrm{d}\rho.$$

(2.) Again, the surface $f\rho = $ const. being still quite *general*, if we write (comp. 363, X'., 373, III., &c.),

$$\text{IV.} \ldots \mathrm{d}f\rho = 2\mathrm{S}\nu\mathrm{d}\rho = 2\mathrm{S}\phi\rho\mathrm{d}\rho, \quad \text{we shall have} \quad \text{V.} \ldots \mathrm{d}f\mathrm{d}\rho = 2\mathrm{S}(\phi\mathrm{d}\rho\,.\,\mathrm{d}^2\rho);$$

and therefore, by III., for a *geodetic*,

$$\text{VI.} \ldots \frac{\mathrm{d}f\mathrm{d}\rho}{\mathrm{Sd}\rho\mathrm{d}\phi\rho} + 2\mathrm{S}\frac{\phi\mathrm{d}\rho}{\phi\rho} = 0.$$

(3.) For a *central surface* of the *second order*, $\phi\rho$ is a *linear function*, and we may write (comp. 361, IV.),

$$\text{VII.} \ldots \phi\mathrm{d}\rho = \mathrm{d}\phi\rho = \mathrm{d}\nu, \quad \mathrm{Sd}\rho\mathrm{d}\phi\rho = \mathrm{Sd}\rho\phi\mathrm{d}\rho = f\mathrm{d}\rho;$$

the general differential equation VI. becomes therefore here,

$$\text{VIII.} \ldots \frac{\mathrm{d}f\mathrm{d}\rho}{f\mathrm{d}\rho} + 2\mathrm{S}\frac{\mathrm{d}\nu}{\nu} = 0;$$

and gives, by a first integration, with the condition II.,

$$\text{IX.} \ldots \nu^2 f\mathrm{d}\rho = h\mathrm{d}\rho^2, \quad \text{or} \quad \text{IX'.} \ldots \mathrm{T}\nu^2 f\mathrm{Ud}\rho = h = \text{const.};$$

or

$$\text{X.} \ldots P^{-2}D^{-2} = h, \quad \text{or} \quad \text{X'.} \ldots P\,.\,D = h^{-\frac{1}{2}} = \text{const.};$$

where

$$P = \mathrm{T}\nu^{-1} = \textit{perpendicular from centre on tangent plane},$$

and

$$D = (f\mathrm{Ud}\rho)^{-\frac{1}{2}} = \textit{semidiameter parallel to tangent};$$

these two last quantities being treated as scalars, whereof the latter may be real or imaginary,* together with the last scalar constant $h^{-\frac{1}{2}}$.

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* For the case of the *ellipsoid*, for which the product $P\,.\,D$ is necessarily *real*, the foregoing deduction, by quaternions, of Joachimstal's celebrated first integral, $P\,.\,D = $ const., was given (in substance) in page 580 of the *Lectures*.

(4.) The following is a quite different way of accomplishing a first integration, which conducts to another known result of not less interest, although rather of a *graphic* than of a *metric* kind. Operating on the equation 407, XVI. by $S.d\rho$, and remembering that $S\rho\nu = 1$, and $S\nu d\rho = 0$, we obtain the differential equation,

$$\text{XI.} \ldots S\rho\nu S\rho d\rho = l^2(S\alpha'\nu S\alpha d\rho + S\alpha\nu S\alpha' d\rho);$$

that is, by I. and II.,

$$\text{XII.} \ldots S\rho d\rho . S\rho d^2\rho - \rho^2 S d\rho d^2\rho = l^2 d(S\alpha d\rho . S\alpha' d\rho),$$

in which the first member, like the second, is an exact differential, because

$$\text{XIII.} \ldots S(V\rho d\rho . V\rho d^2\rho) = \tfrac{1}{2} d(V\rho d\rho)^2;$$

hence, for the geodetic,

$$\text{XIV.} \ldots l^{-2}(V\rho d\rho)^2 - 2S\alpha d\rho S\alpha' d\rho = h' d\rho^2,$$

or

$$\text{XV.} \ldots 2S\alpha U d\rho . S\alpha' U d\rho - l^{-2}(V\rho U d\rho)^2 = h',$$

h' being a new scalar constant.

(5.) Comparing this last equation with the formula 408, XXXVI., we find that the new constant h' is the *sum*, $e + e_{,}$, of what have been above called the *parameters*,* of the *given surface* (e) on which the geodetic is traced, and of the *confocal* $(e_{,})$ which *touches* a given *tangent* to that curve: whence follows the known† theorem, that *the tangents to a geodetic, on any central surface of the second order, all touch one common confocal.*‡

(6.) The new constant $e_{,}(= h' - e)$ may, by 407, LXXV. and 408, LXXV. (with e for e_0), be thus transformed:

$$\text{XVI.} \ldots e_{,} = e_1(TVU\nu_1 d\rho)^2 + e_2(TVU\nu_2 d\rho)^2$$
$$= e_1(SU\nu_2 d\rho)^2 + e_2(SU\nu_1 d\rho)^2 = \text{const.};$$

where e_1, e_2 are the parameters of the two confocals through the point P of the geodetic on (e), and ν_1, ν_2 are as before the normals at that point, to those two surfaces (e_1), (e_2).

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* Compare the second Note to page 214.

† Discovered by M. Chasles.

‡ This touched confocal becomes a *sphere*, when the given confocal is a *cone*. Compare 380, (5.), and 408, (16.), (17.); also the Note to page 31.

(7.) In fact, the two equations last cited give the *general* transformation,

$$\text{XVII.} \ldots l^{-2}(\mathrm{V}\rho\sigma)^2 - 2\mathrm{S}\alpha\sigma\mathrm{S}\alpha'\sigma = e(\mathrm{V}\sigma\mathrm{U}\nu)^2 + e_1(\mathrm{V}\sigma\mathrm{U}\nu_1)^2 + e_2(\mathrm{V}\sigma\mathrm{U}\nu_2)^2;$$

σ being an *arbitrary* vector, which may for instance be replaced by $d\rho$. Equating then this last expression to $(e + e_{,})\sigma^2$, or to $e(\mathrm{V}\sigma\mathrm{U}\nu)^2 - e_{,}\mathrm{T}\sigma^2$, since $\mathrm{S}\nu\sigma = 0$, we obtain the first and therefore also the second transformation XVI., because the three normals $\nu\nu_1\nu_2$ compose a rectangular system (comp. 407, (4.), &c.).

(8.) It is, however, simpler to deduce the second expression XVI. from the equation 408, XXXI. of the cone of tangents from P to $(e_{,})$, by changing τ to $\mathrm{U}d\rho$; and then if we write

$$\text{XVIII.} \ldots v_1 = \angle \frac{d\rho}{\nu_1},$$

so that v_1 denotes the angle at which the geodetic crosses the normal ν_1 to (e_1), considered as a tangent to the given surface (e), the first integral XVI. takes the form,*

$$\text{XIX.} \ldots e_{,} = e_1 \sin^2 v_1 + e_2 \cos^2 v_1,$$

or

$$\text{XX.} \ldots a_{,}^2 = a_1^2 \sin^2 v_1 + a_2^2 \cos^2 v_1, \text{ \&c.};$$

in which the constant $a_{,}$ is the primary semiaxis of the touched confocal (5.).

(9.) Without supposing that $\mathrm{T}d\rho$ is constant, we may investigate as follows the differential of the real scalar h in IX. or X., or of the product $P^{-2} . D^{-2}$, for *any curve* on a central surface of the second order. Leaving *at first* the *surface arbitrary*, as in (1.) and (2.), and resolving $d^2\rho$ in the three rectangular directions of ν, $d\rho$, and $\nu d\rho$, we get the *general* expression,

$$\text{XXI.} \ldots d^2\rho = -\nu^{-1}\mathrm{S}d\nu d\rho + d\rho^{-1}\mathrm{S}d\rho d^2\rho + (\nu d\rho)^{-1}\mathrm{S}\nu d\rho d^2\rho;$$

of which, under the conditions I. and II., the two last terms vanish, as in III. Without assuming those conditions, if we *now* introduce the relations

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* Under this form XX., the integral is easily seen to coincide with that of M. Liouville,

$$\mu^2 \cos^2 i + \nu^2 \sin^2 i = \mu'^2 = \text{const.},$$

cited in page 290 of Dr. Salmon's Treatise.

VII. which belong to a central surface of the second order, we have by V. and IX. the expression,*

$$\text{XXII.} \ldots \tfrac{1}{2}\mathrm{d}h \,.\, \mathrm{d}\rho^2 = \nu^2 \mathrm{Sd}\nu\mathrm{d}^2\rho + \mathrm{S}\nu\mathrm{d}\nu\mathrm{Sd}\nu\mathrm{d}\rho$$
$$- h\mathrm{Sd}\rho\mathrm{d}^2\rho = \mathrm{S}\nu\mathrm{d}\nu\mathrm{d}\rho^{-1} \,.\, \mathrm{S}\nu\mathrm{d}\rho\mathrm{d}^2\rho,$$

or

$$\text{XXIII.} \ldots \mathrm{d}h = \mathrm{d} \,.\, \nu^2\mathrm{Sd}\nu\mathrm{d}\rho^{-1} = \mathrm{d} \,.\, P^{-2}D^{-2} = 2\mathrm{S}\nu\mathrm{d}\nu\mathrm{d}\rho^{-1}\mathrm{S}\nu\mathrm{d}\rho^{-1}\mathrm{d}^2\rho\,;$$

or finally,

$$\text{XXIV.} \ldots \mathrm{d}h \,.\, \mathrm{d}\rho^4 = 2\mathrm{S}\nu\mathrm{d}\nu\mathrm{d}\rho \,.\, \mathrm{S}\nu\mathrm{d}\rho\mathrm{d}^2\rho,$$

the scalar variable with respect to which the differentiations are performed being here entirely arbitrary.

(10.) For a *geodetic line* on *any surface*, referred thus to *any scalar variable*, we have by 380, II. the differential equation,

$$\text{XXV.} \ldots \mathrm{S}\nu\mathrm{d}\rho\mathrm{d}^2\rho = 0\,;$$

and therefore by XXIV., for *such* a line on a central surface of the *second order*, we have *again*, as in (3.),

$$\text{XXVI.} \ldots \mathrm{d}h = 0, \quad \text{or} \quad \text{XXVI}'. \ldots h = \text{const.},$$

with $h = P^{-2}D^{-2}$ as in X.

(11.) But we now see, by XXIV., that for such a surface the condition XXVI. is satisfied, not only by this *differential equation* of the *second order* XXV. but also by this *other* differential equation,

$$\text{XXVII.} \ldots \mathrm{S}\nu\mathrm{d}\nu\mathrm{d}\rho = 0\,;$$

the product $P^{-2}D^{-2}$ (or PD itself) is therefore *constant*, *not only* as in (3.) for *every geodetic* on the surface, but *also* for *every curve* of *another set*,† represented by this *last* equation XXVII., which is only of the *first order*, and the geometrical meaning of which we next propose to consider.

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* In deducing this expression, it is to be remembered that

$$\mathrm{dSd}\nu\mathrm{d}\rho = \mathrm{d}f\mathrm{d}\rho = 2\mathrm{Sd}\nu\mathrm{d}^2\rho\,;$$

in fact, the linear and self-conjugate form of $\nu = \phi\rho$ gives,

$$\mathrm{Sd}\rho\mathrm{d}^2\nu = f(\mathrm{d}\rho,\, \mathrm{d}^2\rho) = \mathrm{Sd}\nu\mathrm{d}^2\rho.$$

[The second part of the transformation in XXII. may be effected by replacing $\mathrm{d}^2\rho$ in the term $\nu^2\mathrm{Sd}\nu\mathrm{d}^2\rho$ by the value given in XXI.]

† Namely, the *lines of curvature*, as is known, and as will presently be proved by quaternions.

410. In general, if ν and $\nu + \Delta\nu$ have the directions of the normals to *any surface*, at the extremities of the vectors ρ and $\rho + \Delta\rho$, the *condition of intersection* (or parallelism) of these two normals is, rigorously,

$$\text{I.} \ldots S\nu\Delta\nu\Delta\rho = 0\,;$$

the *differential equation** of what are called the *Lines of Curvature*, on an *arbitrary surface*, is therefore (comp. 409, XXVII.),

$$\text{II.} \ldots S\nu d\nu d\rho = 0\,;$$

from which we are now to deduce a few *general* consequences, together with some that are peculiar to surfaces of the *second order*.

(1.) The differential equation of the *surface* being, as usual,

$$\text{III.} \ldots S\nu d\rho = 0,$$

the normal vector ν is generally *some* function of ρ, although *not* generally *linear*, because the surface is as yet *arbitrary*: its differential $d\nu$ is therefore generally some function of ρ and $d\rho$, which is linear relatively to the *latter*. And if, attending only to the dependence of $d\nu$ on $d\rho$, we write

$$\text{IV.} \ldots d\nu = \phi d\rho,$$

it results from what has been already proved (363), that *this linear* and *vector function* ϕ is at the same time *self-conjugate*.

(2.) Denoting then by τ *a tangent*† PT to a line of curvature, drawn at the given extremity P of ρ, we see that the vector τ must satisfy the *two* following scalar equations, in which ν is supposed to be given,

$$\text{V.} \ldots S\nu\tau = 0, \quad \text{and} \quad \text{VI.} \ldots S\nu\tau\phi\tau = 0\,;$$

this tangent τ admits therefore (355) of *two real and rectangular directions*, but *not* in general of *more*: *opposite* directions being *not here* counted as

* In this equation II., $d\rho$ and $d\nu$ are *two simultaneous differentials*, which may (according to the theory of the present Chapter, and of the one preceding it) be at pleasure regarded, either as *two finite right lines*, whereof $d\rho$ is (rigorously) *tangential* to the surface, and to the line of curvature; or else as *two infinitely small vectors*, $d\rho$ being, on this latter plan, an *infinitesimal chord* $\Delta\rho$. (Compare pages 97, 431, vol. i., and pages 4, 174, and the Notes to pages 170, 179, vol. ii.) The treatment of the *equations* is the *same*, in these *two views*, whereof one may appear clearer to some readers, and the other view to others.

† This symbol τ is used here partly for abridgment, and partly that the reader may not be *obliged* to interpret $d\rho$ as denoting a *finite tangent*, although the principles of this work *allow* him so to interpret it.

distinct. Hence, as is indeed well known, *through each point of any surface there pass generally two lines of curvature*: and *these two curves intersect each other at right angles.*

(3.) A *construction* for the two rectangular directions of τ can easily be assigned as follows. Assuming, as we may, that the *length* of the tangent τ varies with its *direction*, according to the law,

$$\text{VII.} \ldots S\tau\phi\tau = 1,$$

which gives

$$\text{VIII.} \ldots S(\phi\tau \,.\, d\tau) = 0,$$

or briefly

$$\text{VIII}'. \ldots S\phi\tau d\tau = 0,$$

by the properties above-mentioned of ϕ; and remembering that ν is treated as a constant in V., so that we may write,

$$\text{IX.} \ldots S\nu d\tau = 0, \quad \text{and therefore (by VI.),} \quad \text{X.} \ldots S\tau d\tau = 0;\text{*}$$

we see that, under the condition of the question, the above-mentioned *length* $T\tau$, of this tangential vector τ, is a *maximum* or *minimum*: and therefore that the *two directions* sought are those of the *two axes* of the *plane conic* V. VII., which has its *centre at the given point* P of the surface, and is *in the tangent plane* at that point.

(4.) This plane conic V. VII. may be called the *Index Curve*, for the given surface at the given point P; in fact it is easily proved to coincide, if we abstract from mere dimensions, with the known *indicatrix* (la courbe indicatrice) of Dupin,† who first pointed out the coincidence (3.) of the directions of its *axes*, with those of the lines of curvature; and also established a more general relation of *conjugation* between *two tangents* to a surface at one point, which exists when they have the directions of any two *conjugate semidiameters* of that curve: so that the lines of curvature are distinguished by this *characteristic property*, that the *tangent to each* is *perpendicular to its conjugate.*

(5.) In our notations, this relation of conjugation between two tangents τ, τ', which satisfy as such the equations,

$$\text{V.} \ldots S\nu\tau = 0, \quad \text{and} \quad \text{V}'. \ldots S\nu\tau' = 0,$$

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* [Since $d\tau \parallel V\phi\tau\nu$ by VIII. and IX.]

† *Développements de Géométrie* (Paris, 1813), pages 48, 145, &c.

is expressed by the formula,

$$\text{XI.} \ldots \mathrm{S}\tau\phi\tau' = 0, \quad \text{or} \quad \text{XI}'. \ldots \mathrm{S}\tau'\phi\tau = 0;$$

we have therefore the parallelisms,*

$$\text{XII.} \ldots \tau \parallel \mathrm{V}\nu\phi\tau', \qquad \text{XII}'. \ldots \tau' \parallel \mathrm{V}\nu\phi\tau;$$

so that the equation VI. may be written under the very simple form,

$$\text{XIII.} \ldots \mathrm{S}\tau\tau' = 0,$$

which gives at once the *rectangularity* lately mentioned.

(6.) The parallelism XII′. may be otherwise expressed by saying (comp. (4.)) that

$$\text{XIV.} \ldots d\rho \quad \text{and} \quad \mathrm{V}\nu d\nu$$

have the directions of *conjugate tangents*; or that the two vectors,

$$\text{XV.} \ldots \Delta\rho \quad \text{and} \quad \mathrm{V}\nu\Delta\nu,$$

have *ultimately* such directions, when $\mathrm{T}\Delta\rho$ diminishes indefinitely. But whatever may be this *length* of the *chord* $\Delta\rho$, the vector $\mathrm{V}\nu\Delta\nu$ has the *direction* of the *line of intersection* of the *two tangent planes* to the surface, drawn at its two extremities: another theorem of Dupin† is therefore reproduced, namely, that *if a developable be circumscribed to any surface, along any proposed curve thereon, the generating lines of this developable are everywhere conjugate, as tangents to the surface, to the corresponding tangents to the curve*, with the recent definition (4.) of such conjugation.

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* The *conjugate character* of these two parallelisms, or the relation,

$$\mathrm{V}.\nu\phi\mathrm{V}\nu\phi\tau \parallel \tau, \quad \text{if} \quad \mathrm{S}\nu\tau = 0,$$

may easily be deduced from the *self-conjugate* property of ϕ, with the help of the formula 348, VII., in page 490, vol. i. [The equation cited becomes for present purposes $\phi\mathrm{V}\nu\phi\tau = \mathrm{V}\psi\nu\tau$.]

† Dupin proved *first* (*Dév. de Géométrie*, pp. 43, 44, &c.), that two such tangents as are described in the text have a relation of *reciprocity* to each other, on which account he *called* them "*tangentes conjuguées*": and *afterwards* he gave a sort of *image*, or *construction*, of this relation and of others connected with it, by means of the *curve* which he named "*l'indicatrice*" (in his already cited page 48, &c.).

(7.) The following is a very simple mode of proving by quaternions, that *if* a tangent τ satisfies the equation VI., then the *rectangular tangent*,

$$\text{XVI.} \ldots \tau' = \nu\tau,$$

satisfies the same equation. For this purpose we have only to observe, that the *self-conjugate* property of ϕ gives, by VI. and XVI.,

$$\text{XVII.} \ldots 0 = \text{S}\tau'\phi\tau = \text{S}\tau\phi\tau' = \nu^{-2}\text{S}\nu\tau'\phi\tau'.$$

(8.) Another way of exhibiting, by quaternions, the mutual rectangularity of the lines of curvature, is by employing (comp. 357, I.) the self-conjugate *form*,

$$\text{XVIII.} \ldots \phi\tau = g\tau + \text{V}\lambda\tau\mu;$$

in which the vectors λ, μ, and the scalar g, depend only on the surface and the point, and are independent of the direction of the tangent. The equation VI. then becomes by V.,

$$\text{XIX.} \ldots 0 = \text{S}\nu\tau\lambda\tau\mu = \text{S}\nu\tau\lambda\text{S}\mu\tau + \text{S}\nu\tau\mu\text{S}\lambda\tau;$$

assuming then the expression,

$$\text{XX.} \ldots \tau = x\text{V}\nu\lambda + y\text{V}\nu\mu,$$

we easily find that

$$\text{XXI.} \ldots y^2(\text{V}\nu\mu)^2 = x^2(\text{V}\nu\lambda)^2,$$

or

$$\text{XXI}'. \ldots y\text{TV}\nu\mu = \pm x\text{TV}\nu\lambda;$$

the *two directions* of τ are therefore those of the two lines,

$$\text{XXII.} \ldots \text{UV}\nu\lambda \pm \text{UV}\nu\mu,$$

which are evidently perpendicular* to each other.

(9.) An *interpretation*, of some interest, may be given to this last expression XXII., by the introduction of a certain *auxiliary surface* of the *second order*, which may be called the *Index Surface*, because the *index curve* (4.) is the *diametral section* of this *new* surface, made by the *tangent plane* to the *given* one. With the recent signification of ϕ, this *index surface* is represented by the equation VII., if τ be *now* supposed (comp. (2.)) to

* *This mode*, however, of determining *generally* the directions of the lines of curvature, gives only an illusory result, when the normal ν has the direction of either λ or μ, which happens at an *umbilic* of the surface. Compare 408, (27.), (29.), and the first Note to page 525, vol. i.

represent a line PT drawn *in any direction* from the given point P, and therefore *not now obliged* to satisfy the *condition* V. of *tangency*. Or if, for greater clearness, we denote by $\rho + \rho'$ the vector from the origin O to a point of the index surface, the equation to be satisfied is, by the form XVIII. of ϕ (comp. 357, II.),

$$\text{XXIII.} \ldots 1 = S\rho'\phi\rho' = g\rho'^2 + S\lambda\rho'\mu\rho';$$

the *centre* of this auxiliary surface being thus at P, and its two (real) *cyclic normals* being the lines λ and μ: so that $V\nu\lambda$ and $V\nu\mu$ have the directions of the *traces of its two cyclic planes, on that diametral plane* ($S\nu\rho' = 0$) which *touches the given surface.* We have therefore, by XXII., this general *theorem,* that *the bisectors of the angle formed by those two traces are the tangents to the two lines of curvature, whatever the form of the given surface may be.*

(10.) Supposing *now* that the *given surface* is *itself* one of the *second order,* and that its *centre* is at the *origin* O, so that it may be represented (comp. 405, XII.) by the equation,

$$\text{XXIV.} \ldots 1 = S\rho\phi\rho = g\rho^2 + S\lambda\rho\mu\rho,$$

with *constant values* of λ, μ, and g, which will reproduce *with those values* the *form* XVIII. of ϕ, we see that the *index surface* (9.) becomes in this case simply that *given* one, with its *centre transported* from O to P; and therefore with a *tangent plane at the origin,* which is *parallel to the given tangent plane.* And thus the *traces* (9.), of the *cyclic planes* on the *diametral plane* of the *index surface,* become *here* the *tangents* to the *circular sections* of the *given surface.* We recover then, as a *case* of the general theorem in (9.), this *known* but *less general theorem*: that *the angles formed by the two circular sections, at any point of a surface of the second order, are bisected by the lines of curvature,* which pass through the same point.

(11.) And because the *tangents* to these latter lines *coincide generally,* by (3.) (4.) (9.), with the *axes* of the *diametral section* of the *index surface,* made by the *tangent plane* to the *given surface,* they are *parallel,* in the *case* (10.), as indeed is well known, *to the axes* of the *parallel section* of a given surface of the *second order.*

(12.) And if we now look back to the *Equation of Confocals* in 407, (26.), and to the earlier formulæ of 407, (4.), we shall see that because the vector ν_1, in the last cited sub-article, represents a *tangent* to the given surface $S\rho\phi\rho = 1$, *complanar** with the *normal* ν and the *derived vector* $\phi\nu_1$, so that it satisfies

* Compare the Note to page 198.

(comp. 407, XII. XIV., and the recent formulæ V. VI.) the two scalar equations,

$$\text{XXV.}\ldots S\nu\nu_1 = 0, \quad \text{and} \quad \text{XXVI.}\ldots S\nu\nu_1\phi\nu_1 = 0,$$

which are likewise satisfied (comp. (7.)) when we change ν_1 to the *rectangular tangent* ν_2, it follows that *these two vectors*, ν_1 and ν_2, which are the *normals to the two confocals* to (*e*) *through* P, are also the *tangents to the two lines of curvature* on that *given surface* of the second order at that point: whence follows this other theorem* of Dupin, that *the curve of orthogonal intersection* (407, (4.)), *of two confocal surfaces, is a line of curvature on each.*

(13.) And by combining this known theorem, with what was lately shown respecting the *umbilicar generatrices* (in 408, (30.), (32.), comp. also (35.), (36.)), we may see that while, on the one hand, the *lines of curvature* on a central surface of the *second order* have *no real envelope*, yet on the other hand, *in an imaginary sense*, they have for their *common envelope*† the *system* of the *eight imaginary right lines* (408, (31.)), which *connect the twelve* (real or imaginary) *umbilics* of the surface, *three by three*, and are at once *generating lines* of the *surface itself*, and also of the known *developable envelope* of the *confocal system.*

(14.) It may be added, as another curious property of these *eight imaginary right lines*, that *each* is, in an imaginary sense, *itself a line of curvature* upon the surface: or rather, each represents *two coincident lines* of that kind. In fact, if we denote the variable vector 408, LXXX. of such a generatrix by the expression,

$$\text{XXVII.}\ldots \rho = c'\sigma + \sigma',$$

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* *Dév. de Géométrie*, page 271, &c.

† The writer is not aware that this theorem, to which he was conducted by quaternions, has been enunciated before; but it has evidently an intimate connexion with a result of Professor Michael Roberts, cited in page 290 of Dr. Salmon's Treatise, respecting the *imaginary geodetic tangents to a line of curvature*, drawn *from an umbilicar point*, which are *analogous* to the *imaginary tangents* to a *plane conic*, drawn from a *focus* of that curve. An illustration, which is almost a *visible representation*, of the theorem (13.) is supplied by Plate II. to Liouville's Monge (and by the corresponding plate in an earlier edition), in which the *prolonged* and *dotted parts* of certain *ellipses*, answering to the *real projections* of *imaginary portions* of the *lines of curvature* of the *ellipsoid*, are *seen* to touch a system of *four real right lines*, namely the *projections* (on the same plane of the greatest and least axes), of the *four real umbilicar tangent planes*, and therefore also of what have been above called (408, (30.), (31.)) the *eight* (*imaginary*) *umbilicar generatrices* of the surface. Accordingly Monge observes (page 150 of Liouville's edition), that "toutes les ellipses, projections des lignes de courbure, seront inscrites dans ce parallélogramme dont chacune d'elles touchera les quatre côtés": with a similar remark in his explanation of the corresponding figure (page 160).

in which e' is a *variable scalar*, but σ, σ' are two *given* or *constant* but *imaginary vectors*, such that

$$\text{XXVIII.} \ldots \sigma^2 = 0, \quad S\sigma\sigma' = -l^2, \quad \sigma'^2 = -b^2,$$

and

$$\text{XXIX.} \ldots f\sigma = S\sigma\phi\sigma = 0, \quad f(\sigma, \sigma') = S\sigma'\phi\sigma = 0, \quad f\sigma' = 1,$$

we have the *imaginary normal* ν, with (for the case of a *real umbilic*) a *real tensor*,

$$\text{XXX.} \ldots \nu = e'\phi\sigma + \phi\sigma' \perp \sigma, \quad \text{XXXI.} \ldots T\nu = \pm \frac{(e - e')l^2}{abc};$$

and we find, after reductions, the imaginary expression,

$$\text{XXXII.} \ldots \nu\sigma = \pm \sqrt{-1}\, \sigma T\nu,$$

whence

$$\text{XXXIII.} \ldots S\nu\sigma = 0, \quad S\nu\sigma\phi\sigma = 0.$$

The *differential equations* V. VI. of a line of curvature are therefore *symbolically satisfied*, when we substitute, for the tangential vector τ, either the imaginary line σ *itself*, or the apparently *perpendicular* but in an imaginary sense *coincident** vector $\nu\sigma$; and the recent assertions are justified.

(15.) As regards the *real* lines of curvature, on a central surface of the second order, we see by comparing the *general* differential equation II. with the expression 409, XXIII. for the differential of h, or of $P^{-2}D^{-2}$, that this latter product, or the product $P \,.\, D$ itself, is *constant*† for a *line of curvature*, as well as for a *geodetic line*, on such a surface, as indeed it is well known to be: although this *last* constant $(P \,.\, D)$ *may* become *imaginary*, for the case of a *single-sheeted*‡ *hyperboloid*, and *must* be such for a line of curvature on an hyperboloid of *two sheets*.

* As regards the paradox, of the *imaginary vector* σ being thus apparently *perpendicular to itself*, a similar one had occurred before, in the investigation 353, (17.), (18.), (19.); and it is explained, on the principles of modern geometry, by observing that this imaginary vector is *directed to the circle at infinity*. Compare 408, (31.), and the Note to page 516, vol. i.

† Compare the second Note to page 229.

‡ Although the writer has been content to employ, in the present work, some of these usual but rather long appellations, he feels the elegance of Dupin's phraseology, adopted also by Möbius, and by some other authors, according to which the two central hyperboloids are distinguished, as *elliptic* (for the case of *two* sheets), and *hyperbolic* (for the case of *one*). The phrase "*quadric*," for the *general* surface of the *second order* (or *second degree*), employed by Dr. Salmon and Mr. Cayley, is also very convenient. It may be here remarked, that Dupin was perfectly aware of, or rather appears to have first discovered, the existence of what have since his time come to be called the *focal conics*; which important curves were considered by him, as being at once *limits of confocal surfaces*, and also *loci of umbilics*. Comp. *Dév. de Géométrie*, pages 270, 277, 278, 279; see also page 390 of the *Aperçu Historique*, &c., by M. Chasles (Brussels, 1837).

(16.) And as regards the *general theory* of the *index surface* (9.), it is to be observed that this auxiliary surface depends *primarily* on the *scalar function f*, in the equation $f\rho = 1$, or generally $f\rho =$ const., of the *given surface*; and that it is *not entirely determined* by means of *that surface alone*. For if we write, for instance,

$$\text{XXXIV.} \ldots \mathrm{f}f\rho = \mathrm{f}1, \quad \text{with} \quad \mathrm{d}f\rho = 2S\nu \mathrm{d}\rho \text{ as before,}$$

we shall have, as the new first differential equation of the same given surface, instead of III.,

$$\text{XXXV.} \ldots 0 = \mathrm{df}f\rho = 2Sn\nu \mathrm{d}\rho,$$

with

$$\text{XXXVI.} \ldots n = \mathrm{f}'f\rho;$$

and if we then write, by analogy to IV.,

$$\text{XXXVII.} \ldots \mathrm{d} \,.\, n\nu = \Phi \mathrm{d}\rho = n\phi \mathrm{d}\rho + n'\nu S\nu \mathrm{d}\rho,$$

with

$$\text{XXXVIII.} \ldots n' = 2\mathrm{f}''f\rho,$$

the *new index surface*, constructed on the plan (9.), will have for its equation, analogous to XXIII., the following:

$$\text{XXXIX.} \ldots S\rho'\Phi\rho' = nS\rho'\phi\rho' + n'(S\nu\rho')^2 = \text{const.}$$

(17.) But if we take this last constant $= n$, the *two index surfaces*, XXIII. and XXXIX., will have a *common diametral section*, made by the *given tangent plane*, namely the *index curve* (4.); and they will *touch each other*, in the *whole extent of that curve*. And it will be found that the *construction* (9.), for the *directions* of the *lines of curvature*, applies *equally well* to the *one* as to the *other*, of these *two* auxiliary surfaces: in fact, it is evident that the *differential equation* II., namely $S\nu \mathrm{d}\nu \mathrm{d}\rho = 0$, receives *no real alteration*, when ν is multiplied by *any scalar*, n, even if that scalar should be variable.

(18.) And instead of supposing that the *variable vector* ρ is thus obliged, as in 373, to satisfy a *given scalar equation*, of the form*

$$f\rho = \text{const.},$$

* If $\rho = ix + jy + kz$, and $v = f\rho = F(x, y, z)$, and if we write,

$$\mathrm{d}v = p\mathrm{d}x + q\mathrm{d}y + r\mathrm{d}z, \qquad \mathrm{d}p = p'\mathrm{d}x + r''\mathrm{d}y + q''\mathrm{d}z,$$
$$\mathrm{d}q = q'\mathrm{d}y + p''\mathrm{d}z + r''\mathrm{d}x, \qquad \mathrm{d}r = r'\mathrm{d}z + q''\mathrm{d}x + p''\mathrm{d}y,$$

we may suppose, as in 372, that ρ is a *given vector function* of *two scalar variables*, x and y, between which there will then arise, by the same fundamental formula II., a *differential equation* of the *first order* and *second degree*, to be integrated (when possible) by known methods. For example, if we write,

$$\text{XL.} \ldots \rho = ix + jy + kz, \quad \mathrm{d}z = p\mathrm{d}x + q\mathrm{d}y,$$

---

we may then write also, on the present plan, which gives $\mathrm{d}f\rho = 2\mathrm{S}\nu\mathrm{d}\rho$,

$$\mathrm{d}\rho = i\mathrm{d}x + j\mathrm{d}y + k\mathrm{d}z, \qquad \nu = -\tfrac{1}{2}(ip + jq + kr),$$

$$\mathrm{d}\nu = -\tfrac{1}{2}(i\mathrm{d}p + j\mathrm{d}q + k\mathrm{d}r), \qquad \mathrm{Sd}\rho\mathrm{d}\nu = \tfrac{1}{2}(\mathrm{d}x\mathrm{d}p + \mathrm{d}y\mathrm{d}q + \mathrm{d}z\mathrm{d}r);$$

and the *index surface*, constructed as in (9.), and with ρ' changed to $\Delta\rho = i\Delta x + j\Delta y + k\Delta z$, will thus have the equation,

$$(\text{a}). \ldots \tfrac{1}{2}p'\Delta x^2 + \tfrac{1}{2}q'\Delta y^2 + \tfrac{1}{2}r'\Delta z^2 + p''\Delta y\Delta z + q''\Delta z\Delta x + r''\Delta x\Delta y = 1,$$

or more generally = const.; so that it may be made in this way to depend upon, and be entirely determined by, the *six partial differential coefficients of the second order*, $p' \ldots p'' \ldots$, of the function v or $f\rho$, taken with respect to the *three rectangular coordinates*, xyz. And by comparing this equation (a) with the following equation of the same auxiliary surface, which results more directly from the principles employed in the text (comp. XVIII. XXIII.),

$$(\text{b}). \ldots \mathrm{S}\Delta\rho\phi\Delta\rho = g\Delta\rho^2 + \mathrm{S}\lambda\Delta\rho\mu\Delta\rho = 1,$$

we can easily deduce *expressions* for those six partial *coefficients*, in terms of g, λ, μ. Thus, for example,

$$\tfrac{1}{2}\mathrm{D}_x{}^2v = \tfrac{1}{2}p' = -g + \mathrm{S}\lambda i\mu i = \mathrm{S}\lambda\mu - g + 2\mathrm{S}i\lambda\mathrm{S}i\mu;$$

but $$\mathrm{S}i\lambda\mathrm{S}i\mu + \mathrm{S}j\lambda\mathrm{S}j\mu + \mathrm{S}k\lambda\mathrm{S}k\mu = -\mathrm{S}\lambda\mu; \text{ therefore,}$$

$$(\text{c}). \ldots \tfrac{1}{2}(\mathrm{D}_x{}^2v + \mathrm{D}_y{}^2v + \mathrm{D}_z{}^2v) = \mathrm{S}\lambda\mu - 3g = c_1 + c_2 + c_3 = -m'',$$

if c_1, c_2, c_3 be the roots and m'' a coefficient of a certain cubic (354, III.), deduced from the linear and vector function $\mathrm{d}\nu = \phi\mathrm{d}\rho$, on a plan already explained. If then the function v satisfy, as in several physical questions, the *partial differential equation*,

$$(\text{d}). \ldots \mathrm{D}_x{}^2v + \mathrm{D}_y{}^2v + \mathrm{D}_z{}^2v = 0,$$

the *sum* of these *three roots*, c_1, c_2, c_3, will *vanish*: and consequently, the *asymptotic cone* to the *index surface*, found by changing 1 to 0 in the second member of (a), is *real*, and has (comp. 406, XXI., XXIX.) the property that

$$(\text{e}). \ldots \cot^2 \mathrm{a} + \cot^2 \mathrm{b} = 1,$$

if a, b denote its two extreme semiangles. An entirely different method of transforming, by quaternions, the well known equation (d), occurred early to the present writer, and will be briefly mentioned somewhat farther on. In the mean time it may be remarked, that because $m'' = 0$ by (c), when the equation (d) is satisfied, we have then, by the general theory III. ii. 6 of linear and vector functions, and especially by the sub-articles to 350, remembering that ϕ is here self-conjugate, the formulæ,

$$(\text{f}). \ldots \mathrm{d}\nu + \chi\mathrm{d}\rho = 0, \quad \text{and} \quad (\text{g}). \ldots \psi\sigma - \phi^2\sigma = m'\sigma,$$

χ, ψ being auxiliary functions, and m' another coefficient of the cubic, while σ is an arbitrary vector. For the same reason, and under the same condition (d), the function ϕ itself has the properties expressed by the equations,

$$(\text{h}). \ldots \phi\mathrm{V}\iota\kappa = \kappa\phi\iota - \iota\phi\kappa, \quad \text{and} \quad (\text{i}). \ldots \phi^2\mathrm{V}\iota\kappa = \mathrm{V}\phi\iota\phi\kappa - m'\mathrm{V}\iota\kappa;$$

in which the *two vectors* ι, κ are *arbitrary*, and m' is the same *scalar coefficient* as before.

we shall satisfy the equation III. by assuming (with a constant factor understood),

$$\text{XLI.} \ldots \nu = ip + jq - k, \quad \text{whence} \quad \text{XLII.} \ldots \mathrm{d}\nu = i\mathrm{d}p + j\mathrm{d}q\,;$$

and thus the *general equation* II., for the lines of curvature on an arbitrary surface, receives (by the laws of ijk) the form,

$$\text{XLIII.} \ldots \mathrm{d}p\,(\mathrm{d}y + q\mathrm{d}z) = \mathrm{d}q\,(\mathrm{d}x + p\mathrm{d}z)\,;$$

which last form has accordingly been assigned, and in several important questions employed by Monge*: but which is now seen to be *included* in the still more concise (and more easily deduced and interpreted) *quaternion equation*,

$$\mathrm{S}\nu\mathrm{d}\nu\mathrm{d}\rho = 0.$$

411. For a *central surface* of the *second order*, we have as usual $\nu = \phi\rho$, $\Delta\nu = \phi\Delta\rho$, and therefore (by 347, 348, and by the self-conjugate form of ϕ),

$$\text{I.} \ldots \mathrm{V}\nu\Delta\nu = \mathrm{V}\phi\rho\phi\Delta\rho = \psi\mathrm{V}\rho\Delta\rho = m\phi^{-1}\mathrm{V}\rho\Delta\rho\,;$$

the *general condition of intersection* 410, I. of *two normals*, at the extremities of a *finite chord* $\Delta\rho$, and the *general differential equation* 410, II. of the *lines of curvature*, may therefore for *such* a surface receive these *new* and *special forms*:

$$\text{II.} \ldots \mathrm{S}\Delta\rho\phi^{-1}\mathrm{V}\rho\Delta\rho = 0, \quad \text{or} \quad \text{II}'. \ldots \mathrm{S}\rho\Delta\rho\phi^{-1}\Delta\rho = 0\,;$$

$$\text{III.} \ldots \mathrm{S}\mathrm{d}\rho\phi^{-1}\mathrm{V}\rho\mathrm{d}\rho = 0, \quad \text{or} \quad \text{III}'. \ldots \mathrm{S}\rho\mathrm{d}\rho\phi^{-1}\mathrm{d}\rho = 0\,;$$

which admit of geometrical interpretations, and conduct to some new theorems, especially when they are transformed as follows:

$$\text{IV.} \ldots \mathrm{S}\lambda\Delta\rho\,.\,\mathrm{S}\rho\Delta\rho\phi^{-1}\mu + \mathrm{S}\mu\Delta\rho\,.\,\mathrm{S}\rho\Delta\rho\phi^{-1}\lambda = 0,$$

$$\text{V.} \ldots \mathrm{S}\lambda\mathrm{d}\rho\,.\,\mathrm{S}\rho\mathrm{d}\rho\phi^{-1}\mu + \mathrm{S}\mu\mathrm{d}\rho\,.\,\mathrm{S}\rho\mathrm{d}\rho\phi^{-1}\lambda = 0,$$

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* See the enunciation of the formula here numbered as XLIII., in page 133 of Liouville's Monge: compare also the applications of it, in pages 274, 303, 305, 357. (The corresponding pages of the Fourth Edition are, 115, 240, 265, 267, 312.) The quaternion equation, $\mathrm{S}\nu\mathrm{d}\nu\mathrm{d}\rho = 0$, was published by the present writer, in a communication to the Philosophical Magazine, for the month of October, 1847 (page 289). See also the Supplement to the same Volume xxxi. (Third Series); and the Proceedings of the Royal Irish Academy for July, 1846.

λ and μ being (as in 405, (5.), &c.) the *two real cyclic normals* of the surface: while the same equations may also be written under the still more simple forms,

$$\text{VI.} \ldots \mathrm{S}\alpha\Delta\rho \,.\, \mathrm{S}\alpha'\rho\Delta\rho + \mathrm{S}\alpha'\Delta\rho \,.\, \mathrm{S}\alpha\rho\Delta\rho = 0,$$

$$\text{VII.} \ldots \mathrm{S}\alpha d\rho \,.\, \mathrm{S}\alpha'\rho d\rho + \mathrm{S}\alpha' d\rho \,.\, \mathrm{S}\alpha\rho d\rho = 0,$$

α, α' being, as in several recent investigations, the *two real focal unit lines*, which are common to a whole *confocal system*.

(1.) The vector $\phi^{-1}\mathrm{V}\rho\Delta\rho$ in II. has by I. the direction of $\mathrm{V}\nu\Delta\nu$; whence, by 410, (6.), the interpretation of the recent equation II., or (for the present purpose) of the more general equation 410, I., is that *the chord* PP′ *is perpendicular to its own polar, if the normals at its extremities intersect.* Accordingly, if their point of intersection be called N, the polar of PP′ is perpendicular at once to PN and P′N, and therefore to PP′ itself.

(2.) The equation II′. may be interpreted as expressing, that *when the normals* at P and P′ thus *intersect in a point* N, *there exists a point* P″ *in the diametral plane* OPP′, *at which the normal* P″N″ *is parallel to the chord* PP′: a result which may be otherwise deduced, from elementary principles of the geometry of surfaces of the second order.

(3.) It is unnecessary to dwell on the *converse* propositions, that when *either* of these conditions is satisfied, there *is intersection* (or parallelism) of the *two normals* at P and P′: or on the corresponding but *limiting* results, expressed by the equations III. and III′.

(4.) In order, however, to make any use in *calculation* of these new forms II., III., we must select some suitable expression for the self-conjugate function ϕ, and deduce a corresponding expression for the inverse function ϕ^{-1}. The *form*,*

$$\text{VIII.} \ldots \phi\rho = g\rho + \mathrm{V}\lambda\rho\mu,$$

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* The *vector form* VIII. occurred, for instance, in pages 520, 529, 535, 549, vol. i., and 193, 233, vol. ii.; and the connected *scalar form*,

$$f\rho = g\rho^2 + \mathrm{S}\lambda\rho\mu\rho, \qquad 357,\ \text{II}.$$

has likewise been frequently employed.

which has already several times occurred, has also been more than once inverted: but the following *new inverse* form*,

$$\text{IX.} \ldots (g - S\lambda\mu) \cdot \phi^{-1}\rho = \rho - \lambda S\rho\phi^{-1}\mu - \mu S\rho\phi^{-1}\lambda,$$

has an advantage, for our present purpose, over those assigned before. In fact, this form IX. gives at once the equation,

$$\text{X.} \ldots (g - S\lambda\mu) \cdot \phi^{-1}V\rho\Delta\rho = V\rho\Delta\rho - \lambda S\rho\Delta\rho\phi^{-1}\mu - \mu S\rho\Delta\rho\phi^{-1}\lambda\,;$$

and so conducts immediately from II. to IV., or from III. to V. as a limit.

(5.) The equation IV. expresses *generally*, that the *chord* $\Delta\rho$, or PP′, is a *side* of a certain *cone* of the second order, which has its vertex at the point P of the given surface, and passes through all the points P′ for which the normals to that surface *intersect* the *given normal* at P; and the equation V. expresses *generally*, that the *two sides* of this last cone, in which it is *cut* by the *given tangent plane* at the same point P, are the *tangents to the line of curvature.*

(6.) But if the surface be an *ellipsoid*, or a *double*-sheeted hyperboloid, then (comp. 408, (29.)) the *always real vectors*,† $\phi^{-1}\lambda$ and $\phi^{-1}\mu$, have the directions of *semidiameters* drawn to *two of the four real umbilics*; supposing then that ρ is *such* a semidiameter, and that it has the direction of $\pm\,\phi^{-1}\lambda$, the second term of the first member of the equation IV. vanishes, and the *cone* IV. breaks up into a *pair of planes*, of which the equations in ρ' are,

$$\text{XI.} \ldots S\lambda(\rho' - \rho) = 0, \quad \text{and} \quad \text{XII.} \ldots S\rho'\phi^{-1}\lambda\phi^{-1}\mu = 0\,;$$

whereof the *former* represents the *tangent plane at the umbilic* P, and the *latter* represents the *plane of the four real umbilics.*

(7.) It follows, then, that *the normal at the real umbilic* P *is not intersected by any real normal* to the surface, *except those which are drawn at points* P′ *of that principal section, on which all the real umbilics are situated*: *but* that *the*

* *Inverse forms*, for $\phi^{-1}\rho$ or $m^{-1}\psi\rho$, have occurred in pages 521, 549, vol. i., and 193, vol. ii. In comparing these with the form IX., it will easily be seen (comp. page 221) that

$$\phi^{-1}\lambda = \frac{g\lambda - \lambda^2\mu}{g^2 - \lambda^2\mu^2}, \quad \phi^{-1}\mu = \frac{g\mu - \mu^2\lambda}{g^2 - \lambda^2\mu^2}.$$

† Compare the Note immediately preceding.

same real umbilicar normal PN *is, in an imaginary sense, intersected by all the imaginary normals, which are drawn from the imaginary points* P′ *of either of the two imaginary generatrices through* P.

(8.) In fact, the locus of the point P′, under the *condition of intersection* of its normal P′N′ with a *given normal* PN, is *generally* a *quartic curve*, namely the intersection of the *given surface* with the *cone* IV.; but when this cone *breaks up*, as in (6.), into *two planes*, whereof *one* is *normal*, and the *other tangential* to the surface, the general quartic is likewise *decomposed*, and becomes a system of a *real conic*, namely the *principal section* (7.) and *a pair of imaginary right lines*, namely the *two umbilicar generatrices* at P.

(9.) We see, at the same time, in a new way (comp. 410, (14.)), that *each such generatrix* is (in an imaginary sense) *a line of curvature*: because the (imaginary) *normals to the surface*, at all the points of that generatrix, are situated by (7.) in one *common* (imaginary) *normal plane*.

(10.) Hence *through a real umbilic*, on a surface of the second order there pass *three lines of curvature*: whereof *one* is a *real conic* (8.), and the *two others* are *imaginary right lines*, namely, the *umbilicar generatrices* as before.

(11.) If we prefer *differentials* to *differences*, and therefore use the equation V. of the lines of curvature, we find that this equation takes the form $0 = 0$, if the point P be an umbilic; and that if the normal at that point be parallel to λ, the *differential of the equation* V. breaks up into *two factors*, namely,

$$\text{XIII.} \ldots S\lambda d^2\rho = 0, \quad \text{and} \quad \text{XIV.} \ldots Sd\rho\phi^{-1}\lambda\phi^{-1}\mu = 0\,;$$

whereof the former gives *to imaginary directions*, and the latter gives *one real direction*, coinciding precisely with the *three directions* (10.).

(12.) And if ρ, instead of being the vector of an *umbilic*, be only the vector of a point *on a generatrix* corresponding, we shall still *satisfy the differential equation* V., by supposing that $d\rho$ belongs to the *same imaginary right line*: because we shall then have, as at the umbilic itself,

$$\text{XV.} \ldots S\lambda d\rho = 0, \quad S\rho d\rho\phi^{-1}\lambda = 0.$$

An *umbilicar generatrix* is therefore *proved anew* (comp. (9.)) to be, *in its whole extent*, a *line of curvature*.

(13.) The recent reasonings and calculations apply (6.), not only to an ellipsoid, but also to a double-sheeted hyperboloid, four umbilics for *each* of these two surfaces being *real*. But if for a moment we now consider specially the case of an *ellipsoid*, and if we denote for abridgment the real quotient $\frac{a-c}{a+c}$ by h, we may then substitute in IV. and V. for λ, μ, $\phi^{-1}\lambda$, $\phi^{-1}\mu$ the expressions,

$$\text{XVI.} \ldots \alpha - h\alpha' = \frac{2b\text{U}\lambda}{a+c}; \qquad h\alpha - \alpha' = \frac{2b\text{U}\mu}{a+c};$$

$$\text{XVII.} \ldots \alpha + h\alpha' = \frac{-2b\phi^{-1}\text{U}\lambda}{ac(a+c)}; \quad -h\alpha - \alpha' = \frac{-2b\phi^{-1}\text{U}\mu}{ac(a+c)};$$

and then, after division by $h^2 - 1$, there remain only the two vector constants α, α', the equation IV. reducing itself to VI., and V. to VII.

(14.) The simplified equations thus obtained are not however peculiar to *ellipsoids*, but extend to a whole *confocal system*. To prove this, we have only to combine the equations II. and III. with the *inverse form*,

$$\text{XVIII.} \ldots t^{-2}\phi^{-1}\rho = \alpha\text{S}\alpha'\rho + \alpha'\text{S}\alpha\rho - \rho(e + \text{S}\alpha\alpha'),$$

which follows from 407, XV., and gives at once the equations VI. and VII., whatever the *species* of the surface may be.

(15.) The differential equation VII. must then be satisfied by the *three rectangular directions* of $d\rho$, or of a *tangent to a line of curvature*, which answer to the *orthogonal intersections* (410, (12.)) of the *three confocals* through a given point P; it ought therefore, as a verification, to be satisfied *also*, when we substitute ν for $d\rho$, ν being a *normal to a confocal* through that point: that is, we ought to have the equation,

$$\text{XIX.} \ldots \text{S}\alpha\nu\text{S}\alpha'\rho\nu + \text{S}\alpha'\nu\text{S}\alpha\rho\nu = 0.$$

And accordingly this is at once obtained from 407, XVI., by operating with $\text{S}.\rho\nu$; so that the *three normals* ν are all *sides of this cone* XIX., or of the cone VII. with $d\rho$ for a side, with which the cone V. is found to coincide (13.).

(16.) And because this last equation XIX., like VI. and VII., involves *only* the two *focal lines* α, α' as its *constants*, we may infer from it this *theorem*: "*If indefinitely many surfaces of the second order have only their asymptotic cones biconfocal,* and pass through a given point, their normals at*

* That is, if the surfaces (supposed to have a common centre) be cut by the plane at infinity in biconfocal conics, real or imaginary.

that point have a cone of the second order for their locus"; which latter *cone* is *also* the *locus of the tangents*, at the same point, *to all the lines of curvature which pass through it*, when different values are successively assigned to the scalar constant $a^2 - c^2$ (or $2l^2$): that is, when the *asymptotes* α, α' to the *focal hyperbola* remain *unchanged in position*, but the *semiaxes* $(a^2 - b^2)^{\frac{1}{2}}$, $(b^2 - c^2)^{\frac{1}{2}}$ of that curve (here treated as both real) *vary together.*

(17.) The equation VI. of the *cone of chords* (5.) introduces the *fixed focal lines* α, α' by their *directions* only. But if we suppose that the *lengths* of those two lines are *equal*, without being *here* obliged to assume that each of those lengths is *unity*, we shall then have (comp. 407, (2.), (3.)), the following *rectangular system of unit lines*, in the directions of the *axes of the system*,

$$\text{XX.} \ldots \mathrm{U}(\alpha + \alpha'), \quad \mathrm{UV}\alpha\alpha', \quad \mathrm{U}(\alpha - \alpha'),$$

which obey in all respects the laws of ijk, and may often be conveniently *denoted* by those symbols, in investigations such as the present. And then, by decomposing the semidiameter ρ, and the chord $\Delta\rho$, in these three directions XX., we easily find the following *rectangular transformation** of the foregoing equation VI.,

$$\text{XXI.} \ldots \frac{\mathrm{S}(\alpha + \alpha')^{-1}\rho}{\mathrm{S}(\alpha + \alpha')\Delta\rho} + \frac{\mathrm{S}(\alpha - \alpha')^{-1}\rho}{\mathrm{S}(\alpha - \alpha')\Delta\rho} = \frac{\mathrm{S}\,.\,(\mathrm{V}\alpha\alpha')^{-1}\rho}{\mathrm{S}\,.\,\mathrm{U}\alpha\alpha'\Delta\rho};$$

in which it is permitted to change $\Delta\rho$ to $d\rho$, in order to obtain a *new form* of the differential equation of the *lines of curvature*; or else at pleasure to ν, and so to find, in a new way, a *condition* satisfied by the *three normals*, to the *three confocals* through P.

(18.) The *cone*, VI. or XXI., is *generally* the *locus of a system of three rectangular lines*; each *plane through the vertex*, which is *perpendicular to any real side*, cutting it in a *real pair of mutually rectangular sides*: while, for the

* The corresponding *form*, in rectangular coordinates, of the *condition of intersection*, of *normals at two points* (xyz) and $(x'y'z')$, to the surface,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

is the equation (probably a known one, although the writer has not happened to meet with it),

$$\frac{(b^2 - c^2)x'}{x - x'} + \frac{(c^2 - a^2)y'}{y - y'} + \frac{(a^2 - b^2)z'}{z - z'} = 0;$$

in which it is evident that xyz and $x'y'z'$ may be interchanged.

same reason, the *section* of the same cone, by any plane which does *not* pass through its *vertex* P, but cuts any *side* perpendicularly, is generally an *equilateral hyperbola*.

(19.) If, however, the *point* P be situated *in any one* of the *three principal planes*, perpendicular to the three lines XX., then the *cone* XXI. (as its equation shows) *breaks up* (comp. (6.)) into a *pair of planes*, of which *one* is that principal plane itself, while the *other* is *perpendicular* thereto. And while the *former* plane *cuts* the surface in a *principal section*, which is *always* a *line of curvature* through P, the *latter* plane *usually* cuts the surface in *another conic*, which *crosses* the former section at *right angles*, and gives the *direction* of the second line of curvature.

(20.) *But* if we further purpose, as in (6.), that the point P is an *umbilic*, then (as has been seen) the *second plane* is a *tangent plane*; and the *second conic* (19.) is itself *decomposed*, into a *pair of imaginary right lines*: namely, as before, the *two umbilicar generatrices* through the point, which have been shown to be, in an imaginary sense, both *lines of curvature themselves*, and also a *portion* of the *envelope of all the others*.

(21.) We shall only here add, as *another transformation* of the general equation VI. of the *cone of chords*, which does not *even* assume $T\alpha = T\alpha'$, the following:

$$\text{XXII.} \ldots \; S(\alpha + \alpha')\Delta\rho \,.\, S(\alpha + \alpha')\rho\Delta\rho = S(\alpha - \alpha')\Delta\rho \,.\, S(\alpha - \alpha')\rho\Delta\rho\,;$$

where the *directions* of the *two new lines*, $\alpha + \alpha'$ and $\alpha - \alpha'$, are only obliged to be *harmonically conjugate* with respect to the directions of the *fixed focal lines* of the system: or in other words, are those of any two *conjugate semidiameters* of the *focal hyperbola*.*

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* [In order to obtain additional illustrations of the remark made at the beginning of this Article that $S\rho\Delta\rho\phi^{-1}\Delta\rho = 0$, and the equivalent equations lead to geometrical theorems relating to a system of quadrics having the same pair of focal lines, we see in the first place if ω and ω' are any two vectors terminating on the chord, that the equation may be written in the form $S\omega\omega'\phi^{-1}(\omega - \omega') = 0$. This is equivalent to the vector equation $(\phi^{-1} + h')\omega = (\phi^{-1} + h)\omega'$. Operating on this by $(\phi^{-1} + g)^{-1}$, we easily find $\varpi = \omega + (h' - g)(\phi^{-1} + g)^{-1}\omega = \omega' + (h - g)(\phi^{-1} + g)^{-1}\omega'$. It is obvious from the form of these relations that the normal at ω to the quadric $S\rho(\phi^{-1} + g)^{-1}\rho = S\omega(\phi^{-1} + g)^{-1}\omega$ intersects the normal at ω' to the similar quadric $S\rho(\phi^{-1} + g)^{-1}\rho = S\omega'(\phi^{-1} + g)^{-1}\omega'$; and that ϖ is the vector to the point of intersection. In particular, if ω and ω' happen to lie on the same quadric, the normals still intersect. Returning to the general case and allowing the arbitrarily assumed scalar g to vary, it is obvious that the point of intersection of the normals describes a twisted cubic if we remember the results of p. 131.

The relation between ω, ω', h, and h' suggests the use of an auxiliary vector τ in terms of which we may write $\omega = (\phi^{-1} + h)\tau$ and $\omega' = (\phi^{-1} + h')\tau$. Thus τ is parallel to the chord, and the equation of the chord is $\rho = \phi^{-1}\tau + x\tau$. In terms of this vector, the vector to the point of intersection of

412. The subject of *Lines of Curvature* receives of course an additional illustration, when it is combined with the known conception of the corresponding *Centres of Curvature*. Without yet entering on the *general* theory of the *curvatures of sections* of an arbitrary surface, we may at least consider here the curvatures of those *normal sections*, which *touch* at any given point the *lines* of curvature. Denoting then by σ the vector of the *centre* s of

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normals becomes $\varpi = (\phi^{-1}+g)^{-1}(\phi^{-1}+h)(\phi^{-1}+h')\tau$. Regarding ω as fixed, we have $\varpi = (\phi^{-1}+g)^{-1}(\phi^{-1}+h')\omega$ as the vector equation of the locus of intersections of the normals at ω with the corresponding normals at the variable point $\omega' = (\phi^{-1}+h')\tau$. This surface locus which consists of right lines and twisted cubics is easily seen to be the quadric $S\varpi\omega\phi^{-1}(\varpi-\omega) = 0$. But we obtain a second interpretation for this locus since $\omega = (\phi^{-1}+g)(\phi^{-1}+h')^{-1}\varpi = \varpi + (g-h')(\phi^{-1}+h')^{-1}\varpi$ expresses that the normal at ϖ to the quadric $S\rho(\phi^{-1}+h')^{-1}\rho = S\varpi(\phi^{-1}+h')^{-1}\varpi$ passes through the fixed point ω. So we may say that the quadric is the locus of points whose normals with respect to the doubly infinite system of quadrics $S\rho(\phi^{-1}+h')^{-1}\rho = C$ pass through the extremity of the given vector ω. Returning to the vector equation of the locus, we see that the locus of points whose normals pass through a fixed point is a twisted cubic when h' is constant, or when we have to do only with a system of similar and similarly placed quadrics. If, on the other hand, we confine our attention to a system of confocal quadrics so that C is constant but h' variable, we have $S\varpi(\phi^{-1}+h')^{-1}\varpi = C$ or $S\omega(\phi^{-1}+g)^{-2}(\phi^{-1}+h')\omega = C$, giving h' in terms of g. From this we deduce $(h'-g)((\phi^{-1}+g)^{-1}\omega)^2 = C - S\omega(\phi^{-1}+g)^{-1}\omega$, and the vector equation of the locus of points on the system of confocals, whose normals pass through the extremity of ω, becomes

$$\varpi = \omega + (C - S\omega(\phi^{-1}+g)^{-1}\omega)\,((\phi^{-1}+g)^{-1}\omega)^{-1},$$

or

$$\varpi = (C + V\omega(\phi^{-1}+g)^{-1}\omega)\,((\phi^{-1}+g)^{-1}\omega)^{-1}.$$

We cannot delay on this curve except to state that it is a twisted quintic and unicursal, and that, being a quintic, it meets any quadric of the system in ten points, four of which must be foreign to the present inquiry as only six normals can be drawn to a quadric from a point.

Returning to the equation $\varpi = (\phi^{-1}+g)^{-1}(\phi^{-1}+h)(\phi^{-1}+h')\tau$, we shall express that the two points ω and ω' lie on the *same* quadric $S\rho(\phi^{-1}+g)^{-1}\rho = C$. In terms of τ and h, we see that ω lies on this quadric if $S\tau(\phi^{-1}+h)^2(\phi^{-1}+g)^{-1}\tau = C$, and if ω' likewise lies on it, h' must be the second root of this quadratic in h. Expanding in terms of $h-g$ for convenience, we have

$$(h-g)^2 S\tau(\phi^{-1}+g)^{-1}\tau + 2(h-g)\tau^2 + S\tau(\phi^{-1}+g)\tau = C,$$

and using this equation to eliminate h and h' from the expression for ϖ, we find

$$\varpi = (\phi^{-1}+g)\tau - \frac{2\tau^3}{S\tau(\phi^{-1}+g)^{-1}\tau} + (\phi^{-1}+g)^{-1}\tau\,\frac{S\tau(\phi^{-1}+g)\tau - C}{S\tau(\phi^{-1}+g)^{-1}\tau}.$$

This may be reduced to simpler forms, one being

$$\varpi = \frac{V(\phi^{-1}+g)\tau V\tau(\phi^{-1}+g)^{-1}\tau - (\phi^{-1}+g)V\tau V\tau(\phi^{-1}+g)^{-1}\tau - C(\phi^{-1}+g)^{-1}\tau}{S\tau(\phi^{-1}+g)^{-1}\tau}.$$

It is obvious when C alone varies that the locus is a right line; it is easily seen when g alone varies that the locus is a conic section, and when both vary, it may be proved that the locus is a ruled quartic having the line $\rho = \phi^{-1}\tau + x\tau$ for a triple line.

Finally, it easily follows from the equations of this note, that every line of the triply infinite system obtained by assigning all possible values to τ in the equation $\rho = \phi^{-1}\tau + x\tau$ is at every point normal to some one quadric, and at every point touches two quadrics of the doubly infinite system $S\rho(\phi^{-1}+g)^{-1}\rho = C$ along lines of curvature.]

curvature of *such* a section, and by R the *radius* PS, considered as a *scalar* which is positive when it has the direction of $+\nu$, it is easy to see that we have the *two fundamental equations*:

$$\text{I.} \ldots \sigma = \rho + R\mathrm{U}\nu;$$

$$\text{II.} \ldots R^{-1}\mathrm{d}\rho + \mathrm{dU}\nu = 0;$$

whence follows this *new form* of the general differential equation 410, II. of the *lines of curvature*,

$$\text{III.} \ldots \mathrm{V}\mathrm{d}\rho\mathrm{dU}\nu = 0;$$

with several other combinations or transformations,* among which the following may be noticed here:

$$\text{IV.} \ldots \frac{\mathrm{T}\nu}{R} + \mathrm{S}\frac{\mathrm{d}\nu}{\mathrm{d}\rho} = 0.$$

(1.) The equation I. requires no proof; and from it the equation II. is obtained by merely differentiating† as if σ and R were constant: after which the formula III. follows at once, and IV. is easily deduced.

(2.) To obtain from this last equation a more developed expression for R, we may assume for $\mathrm{d}\nu$, considered as a linear and self-conjugate function of $\mathrm{d}\rho$ (410, (1.)), the general form (comp. 410, XVIII.),

$$\text{V.} \ldots \mathrm{d}\nu = g\mathrm{d}\rho + \mathrm{V}\lambda\mathrm{d}\rho\mu,$$

in which g, λ, μ are independent of $\mathrm{d}\rho$; and then, while the *tangent* $\mathrm{d}\rho$ has (by 410, XXII.) one or other of the *two directions*,

$$\text{VI.} \ldots \mathrm{d}\rho \parallel \mathrm{UV}\nu\lambda \pm \mathrm{UV}\nu\mu,$$

the *curvature* R^{-1} receives one or other of the *two values* corresponding,

$$\text{VII.} \ldots R^{-1} = -\mathrm{T}\nu^{-1}(g + \mathrm{S}\lambda\mathrm{U}\nu \,.\, \mathrm{S}\mu\mathrm{U}\nu \pm \mathrm{TV}\lambda\mathrm{U}\nu \,.\, \mathrm{TV}\mu\mathrm{U}\nu).$$

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* [The expression $R^{-1}d\rho + \mathrm{T}\nu^{-1}\mathrm{d}\nu = x\nu$ is at times a useful transformation of II. The value of the scalar x need not generally be considered, though it is $-\mathrm{dT}\nu^{-1}$.]

† To students who are accustomed to *infinitesimals*, the *easiest* way is here to conceive the *differentials* to be such. But it has already been abundantly shown, that this *view* of the latter is by no means *necessary*, in the treatment of them by quaternions. (Compare the first Note to page 230.)

(3.) One mode of arriving at this last transformation, or of showing that if (comp. again 410, XXII.) we assume,

$$\text{VIII.} \ldots \tau = (\text{or } \|)\text{UV}\lambda\nu \pm \text{UV}\mu\nu,$$

then

$$\text{IX.} \ldots \text{S}\lambda\tau\mu\tau^{-1} = \text{S}\lambda\text{U}\nu \,.\, \text{S}\mu\text{U}\nu \pm \text{TV}\lambda\text{U}\nu \,.\, \text{TV}\mu\text{U}\nu,$$

or

$$\text{X.} \ldots 2\text{S}\lambda\tau \,.\, \text{S}\mu\tau^{-1} = \text{S}(\text{V}\lambda\text{U}\nu \,.\, \text{V}\mu\text{U}\nu) \pm \text{TV}\lambda\text{U}\nu \,.\, \text{TV}\mu\text{U}\nu,$$

or finally,

$$\text{XI.} \ldots 2\text{SU}\lambda\tau \,.\, \text{SU}\mu\tau^{-1} = \text{S}(\text{VU}\lambda\nu \,.\, \text{VU}\mu\nu) \pm \text{TVU}\lambda\nu \,.\, \text{TVU}\mu\nu,$$

is to introduce the *auxiliary quaternion*,

$$\text{XII.} \ldots q = \text{VU}\lambda\nu \,.\, \text{VU}\mu\nu;$$

and to prove that, with the value (or direction) VIII. of τ, we have thus the equation (in which $\text{V}q^2$, as usual, represents the square of $\text{V}q$),

$$\text{XIII.} \ldots 2\text{SU}\lambda\tau \,.\, \text{SU}\mu\tau^{-1} = \text{S}q \pm \text{T}q = \frac{\text{V}q^2}{\text{S}q \mp \text{T}q}.$$

(4.) And this may be done, by simply observing that we have thus (with the value VIII.) the expressions,

$$\text{XIV.} \ldots \text{S}\tau\text{U}\lambda = \frac{\pm \text{SU}\lambda\mu\nu}{\text{TVU}\mu\nu}, \quad \text{S}\tau\text{U}\mu = \frac{-\text{SU}\lambda\mu\nu}{\text{TVU}\lambda\nu};$$

$$\text{XV.} \ldots \text{S}\tau\text{U}\lambda \,.\, \text{S}\tau\text{U}\mu = \frac{\mp (\text{SU}\lambda\mu\nu)^2}{\text{TVU}\lambda\nu \,.\, \text{TVU}\mu\nu} = \frac{\pm \text{V}q^2}{\text{T}q},$$

because

$$\text{XVI.} \ldots \text{V}q = -\text{U}\nu \,.\, \text{SU}\lambda\mu\nu;$$

and

$$\text{XVII.} \ldots \tau^2 = -2 \pm 2\text{SU}q = \pm \frac{2(\text{S}q \mp \text{T}q)}{\text{T}q}.$$

(5.) Admitting then the expression VII., for the curvature R^{-1}, we easily see that it may be thus transformed:

$$\text{XVIII.} \ldots R^{-1} = -\text{T}\nu^{-1}\left(g + \text{T}\lambda\mu \,.\, \cos\left(\angle\frac{\nu}{\lambda} \mp \angle\frac{\nu}{\mu}\right)\right);$$

and that the *difference* of the *two* (principal) *curvatures*, of normal *sections* of an *arbitrary surface*, answering *generally* to the *two* (rectangular) *directions* of the *lines* of curvature through the particular point considered, *vanishes* when the *normal* ν has the *direction* of either of the two *cyclic* normals, λ, μ, of the *index surface* (410, (9.)); that is, when the *index curve* (410, (4.)), considered as a *section* of that index surface, is a *circle*: or finally, when the point in question is, in a received sense, an *umbilic** of the given surface.

(6.) That surface, although considered to be a *given* one, has hitherto (in these last sub-articles) been treated as quite *general*. But if we now suppose it to be a central surface of the *second order*, and to be represented by the equation,

$$\text{XIX.} \ldots f\rho = g\rho^2 + S\lambda\rho\mu\rho = 1,$$

which has already several times occurred, we see at once, from the formula VII. or XVIII. (comp. 410, (10.)), that the *difference of curvatures*, of the two principal normal sections of any such surface, *varies* proportionally to the *perpendicular* ($T\nu^{-1}$ or P) from the centre on the tangent plane, multiplied by the *product of the sines of the inclinations* of that plane, to the two *cyclic planes* of the surface.

(7.) In general (comp. 409, (3.)), it is easy to see that

$$\text{XX.} \ldots S\frac{d\nu}{d\rho} = S\tau^{-1}\phi\tau = -D^{-2},$$

if D denote the (scalar) *semidiameter* of the *index surface*, in the direction of $d\rho$ or of τ; but for the two directions of the *lines of curvature*, these semidiameters become (410, (3.), (4.)) the *semiaxes* of the *index curve*. Denoting then by a_1 and a_2 these last semiaxes, the *two principal radii of curvature* of *any surface* come by IV. to be thus expressed:

$$\text{XXI.} \ldots R_1 = a_1^2 T\nu; \quad R_2 = a_2^2 T\nu.$$

And if the surface be a central one, of the *second order*, then a_1, a_2 are the semiaxes of the *diametral section*, parallel to the *tangent plane*; while $T\nu$ is (comp. again 409, (3.)) the *reciprocal* P^{-1} of the *perpendicular*, let fall on that plane from the centre. Accordingly (comp. (6.), and 219, (4.)), it is known that the *difference* of the *inverse squares* of those *semiaxes* varies proportionally to the *product* of the *sines* of the *inclinations*, of the plane of the section to the two cyclic planes.†

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* Compare the Note to page 233.

† [The expressions of this sub-article enable us to deduce the equation of a system of quadrics having at a given point on an arbitrary surface the same elements of lines of curvature as the arbitrary surface, and the same values of the principal curvatures.

We know that the lines of curvature at a point on a quadric are parallel to the principal axes of the central section parallel to the tangent plane. If τ_1 and τ_2 are unit vectors touching the lines of

(8.) And as regards the *squares themselves*, it follows from 407, LXXI., that they may be thus expressed, in terms of the *principal semiaxes* of the *confocal surfaces*, and in agreement with known results:

$$\text{XXII.} \ldots \mathrm{a}_1^2 = a^2 - a_1^2; \quad \mathrm{a}_2^2 = a^2 - a_2^2;$$

being thus *both positive* for the case of an *ellipsoid*; *both negative*, for that of a *double-sheeted hyperboloid*; and *one* positive, but the *other* negative, for the case of an hyperboloid of *one sheet* (comp. 410, (15.)).

(9.) In *all* these cases, the *normal* $+ \nu$ is drawn towards the *same side* of the *tangent plane*, as that on which the *centre* o of the *surface* is situated (because $S\nu\rho = 1$); hence (by I. and XXI.) *both* the *radii* of curvature R_1, R_2 are drawn in *this direction*, or towards *this side*, for the *ellipsoid*; but *one* such radius for the *single*-sheeted *hyperboloid*, and *both* radii for the hyperboloid of *two* sheets, are directed towards the *opposite side*, as indeed is evident from the forms of these surfaces.

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curvature, and if ρ is the vector from the centre to the point, the vectors $\mathrm{a}_1\tau_1$, $\mathrm{a}_2\tau_2$ and ρ compose a system of mutually conjugate radii of the quadric. It is easy to prove (see below) that

$$(S\omega\beta\gamma)^2 + (S\omega\gamma\alpha)^2 + (S\omega\alpha\beta)^2 = (S\alpha\beta\gamma)^2$$

is the equation of a quadric of which α, β, and γ are conjugate radii. In particular

$$\mathrm{a}_2^2(S\omega\tau_2\rho)^2 + \mathrm{a}_1^2(S\omega\tau_1\rho)^2 + \mathrm{a}_1^2\mathrm{a}_2^2(S\omega\tau_1\tau_2)^2 = \mathrm{a}_1^2\mathrm{a}_2^2(S\rho\tau_1\tau_2)^2$$

is the equation of a quadric having its centre at the origin and arbitrarily assumed directions for the lines of curvature at the extremity of ρ. Now the central perpendicular on the tangent plane at ρ has its length equal to $P = S\rho\tau_1\tau_2 = S\rho U\nu$. So, by XXI., we have

$$R_2(S\omega\tau_2\rho)^2 + R_1(S\omega\tau_1\rho)^2 + R_1R_2(S\omega U\nu)^2 S\rho U\nu = R_1R_2(S\rho U\nu)^3$$

for the equation of a quadric with its centre at the origin, having at an assumed point arbitrarily assumed directions for the lines of curvatures and arbitrarily assumed values for the curvatures. By varying the position of the centre, we can thus determine a system of quadrics having contact of the high order described with any surface at a given point.

We cannot delay discussing this system of quadrics except to state that when the centre lies on a certain line, the lines of curvature of the quadric have four point contact with those of the surface. We can, moreover, only suggest as an exercise on the notation given in the Note to page 225, the investigation of the locus of points on a quadric or on a confocal system at which one or both of the principal curvatures are given. It seems, however, to be worth while to prove the expression for a quadric in terms of the conjugate radii. If the equation of the quadric is $S\omega\phi\omega = 1$, and if α, β, and γ are conjugate radii, among the conditions are $S\alpha\phi\alpha = 1$ and $S\beta\phi\alpha = S\gamma\phi\alpha = 0$. Thus

$$\phi\alpha = V\beta\gamma(S\alpha\beta\gamma)^{-1} \quad \text{and because} \quad \omega S\alpha\beta\gamma = \alpha S\beta\gamma\omega + \beta S\gamma\alpha\omega + \gamma S\alpha\beta\omega,$$

we have

$$\phi\omega(S\alpha\beta\gamma)^2 = V\beta\gamma S\beta\gamma\omega + V\gamma\alpha S\gamma\alpha\omega + V\alpha\beta S\alpha\beta\omega.$$

The forms of the invariants of this function afford proofs of certain well-known theorems. We see also easily that $\phi^{-1}\omega = \alpha S\alpha\omega + \beta S\beta\omega + \gamma S\gamma\omega$ from which known theorems may be derived, and this function ϕ^{-1} may be used with advantage in certain questions relating to confocals. Again to find a set of directions $U\alpha$, $U\beta$, and $U\gamma$ conjugate to two quadrics depending on two functions ϕ and ϕ_1, we have to solve $V\phi_1\omega\phi\omega = 0$ or $V\omega\phi_1^{-1}\phi\omega = 0$.]

(10.) The following is another method of deducing *generally* the two principal curvatures of a surface, from the *self-conjugate function*,*

$$\text{XXIII.} \ldots \mathrm{d}\nu = \phi \mathrm{d}\rho, \qquad 410, \text{IV}.$$

which affords some good practice in the processes of the present Calculus. Writing, for abridgment,

$$\text{XXIV.} \ldots r = \frac{\nu}{\sigma - \rho} = R^{-1}\mathrm{T}\nu = -\mathrm{S}\frac{\mathrm{d}\nu}{\mathrm{d}\rho} = -\mathrm{S}\tau^{-1}\phi\tau,$$

where τ is still a tangent to a line of curvature, the equation II. is easily brought to the form,

$$\text{XXV.} \ldots - r\tau = \nu^{-1}\mathrm{V}\nu\phi\tau = \phi\tau - \nu^{-1}\mathrm{S}\tau\phi\nu = \Phi\tau,$$

where Φ denotes a *new linear and vector function*, which however is *not* in general *self-conjugate*, because we have not generally $\phi\nu \parallel \nu$. Treating then this *new function* on the plan of the Section III. ii. 6, we derive from it a *new cubic equation*, of the form,

$$\text{XXVI.} \ldots 0 = M + M'r + M''r^2 + r^3,$$

and with the coefficients,

$$\text{XXVII.} \ldots M = 0, \quad M' = \mathrm{S}\nu^{-1}\psi\nu, \quad M'' = m'' - \mathrm{S}\nu^{-1}\phi\nu;$$

ψ being a certain *auxiliary function* $(= m\phi^{-1})$, and m'' being the *coefficient*†

* [Compare the Note to p. 554, vol. i., by which it appears that this function is self-conjugate only when n in the equation $\mathrm{d}f\rho = n\mathrm{S}\nu\mathrm{d}\rho$ is a constant or a function of $f\rho$ (see also 410, (16.)). As an example, if we take $n = \mathrm{T}\nu$ and write $\mathrm{dU}\nu = \theta\mathrm{d}\rho$, equation II. of the present article becomes $R^{-1}\mathrm{d}\rho + \theta\mathrm{d}\rho = 0$. Thus the principal curvatures are two roots of the cubic of θ, and the tangents to the lines of curvature are two of the solutions of $\mathrm{Vd}\rho\theta\mathrm{d}\rho = 0$. We can see that the third root is zero because $\mathrm{SU}\nu\delta\mathrm{U}\nu = 0$ or for any value of $\delta\rho$, $\mathrm{S}\delta\rho\theta'\mathrm{U}\nu = 0$. So $\theta'\mathrm{U}\nu = 0$, and therefore a root of the conjugate θ is also zero. If then the symbolic cubic of θ is $\theta^3 - N''\theta^2 + N'\theta = 0$, we have the following expressions:

$$R_1^{-1} + R_2^{-1} = -N'' \quad \text{and} \quad R_1^{-1}R_2^{-1} = N'.$$

We may also write $\mathrm{dU}\nu = -\mathrm{S}\mathrm{d}\rho\nabla\,.\,\mathrm{U}\nu = \theta\mathrm{d}\rho$ where ∇ is Hamilton's operator, and from the properties of this operator it is not hard to see that

$$R_1^{-1} + R_2^{-1} = \mathrm{S}\nabla\mathrm{U}\nu \quad \text{and} \quad R_1^{-1}R_2^{-1} = \tfrac{1}{2}\mathrm{SV}\nabla\nabla'\mathrm{VU}\nu\mathrm{U}\nu',$$

where the accents are to be omitted when the operations indicated have been performed. The function Φ introduced in this sub-article is closely analogous to the function θ of this Note.]

† Compare the Note to page 237, continued in page 238. The reason of the evanescence of the coefficient M, or of the occurrence of a null root of the cubic, is that we have here $\Phi\phi^{-1}\nu = 0$, so that the symbol $\Phi^{-1}0$ may represent an actual vector (comp. 351). Geometrically, this corresponds to the circumstance that when we pass, along a semidiameter prolonged, from a surface of the second order to another surface of the same kind, concentric, similar, and similarly placed, the direction of the normal does not change.

analogous to M'', in the cubic derived from the function ϕ *itself*. The root $r = 0$ is foreign to the present inquiry; but the *two curvatures*, R_1^{-1}, R_2^{-1}, are the *two roots* of the following *quadratic* in R^{-1}, obtained from the equation XXVI. by the rejection of that foreign root:

$$\text{XXVIII.} \ldots 0 = (R^{-1}\mathrm{T}\nu)^2 + M''R^{-1}\mathrm{T}\nu + M'.$$

(11.) As a first application of this general equation XXVIII., let $\phi\tau$ have again, as in V., the form $g\tau + \mathrm{V}\lambda\tau\mu$; we shall then have the values,

$$\text{XXIX.} \ldots M'' = 2(g + \mathrm{S}\lambda\mathrm{U}\nu \,.\, \mathrm{S}\mu\mathrm{U}\nu),$$

and

$$\text{XXX.} \ldots M' = (g + \mathrm{S}\lambda\mathrm{U}\nu \,.\, \mathrm{S}\mu\mathrm{U}\nu)^2 - (\mathrm{V}\lambda\mathrm{U}\nu)^2\,(\mathrm{V}\mu\mathrm{U}\nu)^2,$$

= a great variety of transformed expressions; and the two resulting curvatures agree with those assigned by VII.

(12.) As a second application, let the surface be central of the second order, with abc for its scalar semiaxes (real or imaginary); then the *symbolical cubic* (350) in ϕ becomes,

$$\text{XXXI.} \ldots 0 = \phi^3 - m''\phi^2 + m'\phi - m = (\phi + a^{-2})\,(\phi + b^{-2})\,(\phi + c^{-2});$$

and the coefficients of the quadratic XXVIII. in R^{-1} take the values, in which N denotes the semidiameter of the surface in the direction of the normal:

$$\text{XXXII.} \ldots R_1^{-1} + R_2^{-1} = -M''\mathrm{T}\nu^{-1} = -(m'' + f\mathrm{U}\nu)P = (a^{-2} + b^{-2} + c^{-2} - N^2)P;$$

$$\text{XXXIII.} \ldots R_1^{-1}R_2^{-1} = M'\mathrm{T}\nu^{-2} = -m\nu^{-4} = a^{-2}b^{-2}c^{-2}P^4;$$

both of which agree with known results, and admit of elementary verifications.*

(13.) *In general*, if we observe that $m'' - \phi = \chi$ (350, XVI.), we shall see that the quadratic XXVIII. in r (or in $R^{-1}\mathrm{T}\nu$) may be thus written:

$$\text{XXXIV.} \ldots 0 = \mathrm{S}\nu^{-1}(r^2\nu + r\chi\nu + \psi\nu);$$

or thus more briefly (comp. 398, LXXIX.),

$$\text{XXXV.} \ldots 0 = \mathrm{S}\nu^{-1}(\phi + r)^{-1}\nu.$$

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* As an easy verification by quaternions of the expression XXXII., it may be remarked (comp. 408, (27.)), that if α, β, γ be *any three rectangular unit lines*, then

$$f\alpha + f\beta + f\gamma = \text{const.} = c_1 + c_2 + c_3 = a^{-2} + b^{-2} + c^{-2}.$$

(14.) Accordingly, the formula XXV. gives the expression,

$$\text{XXXVI.} \ldots \nu^2\tau = (\phi + r)^{-1}\nu \,.\, \mathrm{S}\tau\phi\nu\,;$$

from which, under the condition $\mathrm{S}\nu\tau = 0$, the equation XXXV. follows at once.

(15.) We have therefore *generally*, for the *product* of the two principal *curvatures of sections* of *any surface* at any point, the expression:

$$\text{XXXVII.} \ldots R_1^{-1}R_2^{-1} = r_1r_2\mathrm{T}\nu^{-2} = -\nu^{-4}\mathrm{S}\nu\psi\nu = -\mathrm{S}\frac{1}{\nu}\psi\frac{1}{\nu}\,;$$

which contains an important theorem of Gauss, whereto we shall presently proceed.

(16.) Meanwhile we may remark that the recent analysis shows, that the squares a_1^2, a_2^2 (7.) of the semiaxes of the index-curve are *generally* the roots of the following equation,

$$\text{XXXVIII.} \ldots 0 = \mathrm{S}\nu(\phi + a^{-2})^{-1}\nu,$$

when developed as a quadratic in a^2.

(17.) And that the same quadratic assigns the squares of the semiaxes of a diametral section, made by a plane $\perp \nu$, of the central surface of the second order which has $\mathrm{S}\rho\phi\rho = 1$ for its equation.

(18.) Accordingly, $\mathrm{V}\rho\phi\rho$ has the direction of a tangent to this surface, which is perpendicular to ρ at its extremity; and therefore the vector,

$$\text{XXXIX.} \ldots \sigma = \rho^{-1}\mathrm{V}\rho\phi\rho = \phi\rho - \rho^{-1} = (\phi - \rho^{-2})\rho,$$

is perpendicular to the plane of the diametral section, which has the *semi-diameter* ρ for a *semiaxis*: so that it is perpendicular also to ρ itself. The equation,

$$\text{XL.} \ldots \mathrm{S}\sigma(\phi - \rho^{-2})^{-1}\sigma = 0,$$

assigns therefore the values of the squares $(-\rho^2)$ of the scalar semiaxes of the central section $\perp \sigma$; which agrees with the formula XXXVIII.

(19.) If then a *surface* be *derived* from a *given* central surface of the *second order*, as the *locus of the extremities of normals* (erected at the centre) *to the diametral sections* of the given surface, *each such normal* (when real) having the *length* of *one of the semiaxes* of that section, the *equation of this new surface** (or locus) will admit of being written thus:

$$\text{XLI.} \ldots \mathrm{S}\rho(\phi - \rho^{-2})^{-1}\rho = 0.$$

* When the given surface is an *ellipsoid*, this *derived* surface XLI. is therefore the celebrated *Wave Surface* of Fresnel, which will be briefly mentioned somewhat farther on.

(20.) The first of the values XXIV., for the auxiliary scalar r, gives the expression (if $\nu = \phi\rho$, as it is for a central surface of the second order),

$$\text{XLII.} \ldots \sigma = \rho + r^{-1}\nu = (1 + r^{-1}\phi)\rho = r^{-1}(\phi + r)\rho\,;$$

whence, by inversion, and operation with ϕ,

$$\text{XLIII.} \ldots \rho = r(\phi + r)^{-1}\sigma\,; \quad \text{XLIV.} \ldots \nu = r(\phi + r)^{-1}\phi\sigma\,;$$

and therefore, because $S\rho\nu = 1$,

$$\text{XLV.} \ldots r^{-2} = S((\phi + r)^{-1}\sigma \,.\, (\phi + r)^{-1}\phi\sigma) = S \,.\, \sigma(\phi + r)^{-2}\phi\sigma.$$

(21.) The following is a quite different way of arriving at this result, which is also useful for other purposes. Considering σ as the vector OS of a point S on the *Surface of Centres*, that is, on the *locus* of all the centres of curvature of principal normal sections, the vector (say υ) of the *Reciprocal Surface* is connected with σ (comp. 373, (21.)) by the *equations of reciprocity*,*

$$\text{XLVI.} \ldots S\sigma\upsilon = S\upsilon\sigma = 1\,; \quad \text{XLVII.} \ldots S\upsilon d\sigma = 0\,; \quad \text{XLVIII.} \ldots S\sigma d\upsilon = 0\,;$$

which are all satisfied by the vector expression,

$$\text{XLIX.} \ldots \upsilon = \frac{\tau}{S\rho\tau},$$

where τ is, as before, a tangent to the line of curvature: so that, if ω denote the variable vector of the normal plane to this last curve, the equation of that plane (comp. 369, IV.) may be thus written,

$$\text{L.} \ldots S\upsilon(\omega - \rho) = 0.$$

This *normal plane*, to the *line of curvature* at P, is therefore at the same time the *tangent plane* to the *surface of centres* at S, as indeed it is known to be, from simple geometrical considerations, independently of the *form* of the *given* surface, which remains here entirely arbitrary.

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* It is understood that $d\sigma$ and $d\upsilon$, in the differential equations XLVII., XLVIII., are in general only obliged to have directions *tangential* to the surface of centres, and to its reciprocal, at corresponding points: so that the equations might be in some respects more clearly written thus, $S\upsilon\delta\sigma = 0$, $S\sigma\delta\upsilon = 0$, the mark d being reserved to indicate changes which arise from motion along a *given line* of curvature, while δ should have a more general signification. Accordingly if, in particular, we write $\delta\rho = \nu d\rho$, for a variation answering to motion along the *other line*, and denote the two radii of curvature for the two directions $d\rho$ and $\delta\rho$ by R_1 and R_2, we shall have by II., $R_1^{-1}d\rho + dU\nu = 0$, $R_2^{-1}\delta\rho + \delta U\nu = 0$, and therefore by I.,

$$d\sigma = dR_1 \,.\, U\nu, \quad \delta\sigma = \delta\rho + \delta(R_1 U\nu) = (1 - R_1 R_2^{-1})\nu d\rho + \delta R_1 \,.\, U\nu\,;$$

so that we have *both* $Sd\rho d\sigma = 0$, and $Sd\rho\delta\sigma = 0$, and therefore the *tangent* $d\rho$ or τ to the *given line* of curvature has the direction of the *normal* υ to the corresponding *sheet* of the surface of centres, as is otherwise visible from geometry. And when we have thus found an equation of the form $t\upsilon = \tau$, operation with $S \,.\, \sigma$ gives by XLVI. the value $t = S\rho\tau$, as in XLIX., because $\sigma - \rho \parallel \nu \perp \tau$.

(22.) The expression XLIX. for υ gives *generally* the relation,

$$\text{LI.} \ldots \mathrm{S}\rho\nu = 1\,;$$

giving also, by 410, V. and VI., these two other equations,

$$\text{LII.} \ldots \mathrm{S}\nu\upsilon = 0, \quad \text{and} \quad \text{LIII.} \ldots \mathrm{S}\nu\upsilon\phi\upsilon = 0,$$

which are still independent of the *form* of the given surface.

(23.) But if that surface be a *central quadric*,* then the equation LI. may be thus written,

$$\text{LIV.} \ldots 1 = \mathrm{S}\upsilon\phi^{-1}\nu = \mathrm{S}\nu\phi^{-1}\upsilon\,;$$

combining which with LII. and LIII., we derive the expressions:

$$\text{LV.} \ldots \nu = \frac{\upsilon^2\phi\upsilon - \upsilon f\upsilon}{\upsilon^4 - f\upsilon\,.\,F\upsilon}; \qquad \text{LVI.} \ldots \rho = \phi^{-1}\nu = \frac{\upsilon^3 - \phi^{-1}\upsilon f\upsilon}{\upsilon^4 - f\upsilon\,.\,F\upsilon};$$

wherein $f\upsilon = \mathrm{S}\upsilon\phi\upsilon$, and $F\upsilon = \mathrm{S}\upsilon\phi^{-1}\upsilon$, as usual.

(24.) Operating with $\mathrm{S}\,.\,\nu$ on this last expression for ρ, and attending to LII. and LIV., we find the following *quaternion forms* of the *Equation of the Reciprocal of the Surface of Centres*:

$$\text{LVII.} \ldots 1 = (\mathrm{S}\nu\rho =) \frac{-f\upsilon}{\upsilon^4 - f\upsilon\,.\,F\upsilon} \quad \text{or} \quad \text{LVIII.} \ldots \upsilon^4 = (F\upsilon - 1)f\upsilon\,;$$

or

$$\text{LIX.} \ldots 1 = (F\upsilon - 1)f\frac{1}{\upsilon}; \quad \text{or} \quad \text{LX.} \ldots F\upsilon - \frac{1}{f\frac{1}{\upsilon}} = 1\,; \ \&\text{c.},$$

whereof the second, when translated into coordinates, is found to agree perfectly with a known† equation of the same reciprocal surface.

(25.) Differentiating the form LX., and observing that

$$\text{LXI.} \ldots \left(f\frac{1}{\upsilon}\right)^{-1} = \frac{\upsilon^4}{f\upsilon}, \quad \mathrm{d}\,.\,\upsilon^4 = 4\mathrm{S}\upsilon^3\mathrm{d}\upsilon, \quad \mathrm{d}f\upsilon = 2\mathrm{S}\phi\upsilon\mathrm{d}\upsilon, \quad \mathrm{d}F\upsilon = 2\mathrm{S}\phi^{-1}\upsilon\mathrm{d}\upsilon,$$

we find, by comparison with XLVI. and XLVIII., the expression:

$$\text{LXII.} \ldots \sigma = \phi^{-1}\upsilon - \frac{2\upsilon^3}{f\upsilon} + \frac{\upsilon^4\phi\upsilon}{(f\upsilon)^2}; \quad \text{or} \quad \text{LXIII.} \ldots \sigma = \phi^{-1}\upsilon + \frac{2\upsilon}{f\mathrm{U}\upsilon} + \frac{\phi\upsilon}{(f\mathrm{U}\upsilon)^2};$$

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* Compare the last Note to page 236; see also the use made of this known name "quadric," for a surface of the second order (or degree), in the sub-articles to 399 (pages 159, &c.).

† The equation alluded to, which is one of the *fourth degree*, appears to have been first assigned by Dr. Booth, in a Tract on *Tangential Coordinates* (1840), cited in page 163 of Dr. Salmon's Treatise. See also the Abstract of a Paper by Dr. Booth, in the Proceedings of the Royal Society for April, 1858.

or finally by XLIX., with the recent signification XXIV. of r,

$$\text{LXIV.} \ldots \sigma = r^{-2}(\phi + r)^2\phi^{-1}\upsilon, \quad \text{because} \quad \text{LXV.} \ldots r = f\mathrm{U}\tau = f\mathrm{U}\upsilon:$$

and, for the same reason, the equation LX. of the *reciprocal surface* may be thus briefly written,

$$\text{LXVI.} \ldots F\upsilon + r^{-1}\upsilon^2 = 1, \quad \text{while} \quad \text{LXVI}'. \ldots f\upsilon + r\upsilon^2 = 0.$$

(26.) Inverting the last form for σ, and using again the relation XLVI., we first find for υ the expression,

$$\text{LXVII.} \ldots \upsilon = r^2(\phi + r)^{-2}\phi\sigma;$$

and then are conducted anew to the equation XLV., or to the following,

$$\text{LXVIII.} \ldots 1 = \mathrm{S}\,.\,\sigma(1 + r^{-1}\phi)^{-2}\phi\sigma.$$

(27.) This last equation may also be thus written,

$$\text{LXIX.} \ldots 1 = \mathrm{S}\,.\,\sigma(1 + r^{-1}\phi)^{-3}(\phi + r^{-1}\phi^2)\sigma;$$

but by combining XLIII. LI. LXVII. we have,

$$\text{LXX.} \ldots 1 = (\mathrm{S}\rho\upsilon =)\mathrm{S}\,.\,\sigma(1 + r^{-1}\phi)^{-3}\phi\sigma;$$

hence

$$\text{LXXI.} \ldots 0 = \mathrm{S}\,.\,\sigma(1 + r^{-1}\phi)^{-3}\phi^2\sigma,$$

a result which may be otherwise and more directly deduced, under the form $\mathrm{S}\nu\upsilon = 0$ (LII.), from the expressions XLIV. LXVII. for ν and υ.

(28.) If we write,

$$\text{LXXII.} \ldots \tau = \mathrm{Ud}\rho, \quad \tau' = \mathrm{U}(\nu\mathrm{d}\rho), \quad \text{and therefore} \quad \text{LXXIII.} \ldots \tau\tau' = \mathrm{U}\nu,$$

τ and τ' being thus *unit-tangents* to the lines of curvature, the equation III. gives, generally,

$$\text{LXXIV.} \ldots 0 = \mathrm{V}\tau\mathrm{d}(\tau\tau') = -\mathrm{d}\tau' + \tau\mathrm{S}\tau'\mathrm{d}\tau, \quad \text{whence} \quad \text{LXXIV}'. \ldots \mathrm{d}\tau' \parallel \tau;$$

of which *general parallelism* of $\mathrm{d}\tau'$ to τ, the geometrical reason is (comp. again III.) that a *line of curvature* on an *arbitrary surface* is, at the same time, a line of curvature on the *developable normal surface* which *rests* upon that line, and *to which* the vectors τ' or $\nu\mathrm{d}\rho$ are *normals*.

(29.) The same substitution LXXIII. for $U\nu$ gives by II., if we denote by s the *arc* of a line of curvature, measured from any fixed point thereof, so that (by 380, (7.), &c.),

$$\text{LXXV.} \ldots \mathrm{Td}\rho = \mathrm{d}s, \quad \mathrm{d}\rho = \tau \mathrm{d}s, \quad \mathrm{D}_s\rho = \tau,$$

the following *general expression* for the curvature of the given surface, in the direction τ of the given line, which by LXXIV'. is also that of $\mathrm{d}\tau'$:

$$\text{LXXVI.} \ldots R^{-1} = \mathrm{S} \,.\, \tau \mathrm{D}_s(\tau\tau') = -\,\mathrm{S} \,.\, \tau\tau'\mathrm{D}_s\tau = \mathrm{S}(\mathrm{U}\nu^{-1} \,.\, \mathrm{D}_s^2\rho)\,;$$

but $\mathrm{D}_s^2\rho$ is (by 389, (4.)) what we have called the *vector of curvature* of the line of curvature, considered as a *curve in space*, and $R^{-1}\mathrm{U}\nu$ is the corresponding vector of curvature of the *normal section* of the given surface, which has the same tangent τ at the given point: hence *the latter vector of curvature is* (generally) *the projection of the former, on the normal* ν *to the given surface.*

(30.) In like manner, if we denote for a moment by $R_{\prime}^{-1}$ the curvature of the developable normal surface (28.), for the same direction τ, the general formula II. gives, by LXXIV.,

$$\text{LXXVII.} \ldots R_{\prime}^{-1} = \tau \mathrm{D}_s\tau' = -\,\mathrm{S}\tau'\mathrm{D}_s\tau = \mathrm{S} \,.\, \tau'^{-1}\mathrm{D}_s^2\rho\,;$$

the *vector* $R_{\prime}^{-1}\tau'$ of this *new curvature* is therefore the *projection on the new normal* τ', of the *vector of curvature* $\mathrm{D}_s^2\rho$ of the *given line* of curvature. But we shall soon see that these two last results are included in one more general,* respecting *all plane sections* of an arbitrary surface.

(31.) The general parallelism LXXIV'. conducts easily, for the case of a central quadric, to a known and important theorem, which may be thus investigated. Writing, for such a surface,

$$\text{LXXVIII.} \ldots r = f\tau, \quad r' = f\tau',$$

so that r retains here its recent signification LXV., and r' is the analogous scalar for the other direction of curvature, we have by LXXIV. the differential,

$$\text{LXXIX.} \ldots \mathrm{d}r' = 2\mathrm{S}\phi\tau'\mathrm{d}\tau' = 2\mathrm{S}\tau\phi\tau'\mathrm{S}\tau'\mathrm{d}\tau = 0,$$

because $\mathrm{S}\tau\phi\tau' = 0$, by 410, XI.

* Namely in Meusnier's Theorem, which can be proved *generally* by quaternions with about the same ease as the two foregoing *cases* of it.

(32.) We have then the relation,

$$\text{LXXX.} \ldots fU(\nu d\rho) = f\tau' = r' = \text{const.};$$

that is to say, the *square* (r'^{-1}) of the scalar *semidiameter* (D') of the surface, which is *parallel to the second tangent* (τ'), is *constant for any one line of curvature* (τ); and accordingly (comp. XXII., and the expression 407, LXXI. for $fU\nu_1$), the value of this square is,

$$\text{LXXXI.} \ldots (fU\nu d\rho)^{-1} = r'^{-1} = a^2 - a'^2 = b^2 - b'^2 = c^2 - c'^2,$$

if a', b', c' be the scalar semiaxes of the confocal, which cuts the given quadric (abc) along the line of curvature, whereof the variable tangent is τ.

(33.) This constancy of $fU\nu d\rho$ may be proved in other ways; for instance, the general equation $S\nu d\nu d\rho = 0$ gives, for a line of curvature on an *arbitrary surface*,

$$\text{LXXXII.} \ldots d\nu = \nu S\nu^{-1}d\nu + d\rho S\frac{d\nu}{d\rho}; \qquad \text{LXXXIII.} \ldots Vd\nu d\rho = \nu d\rho S\nu^{-1}d\nu;$$

and

$$\text{LXXXIV.} \ldots S.d\rho\phi(\nu d\rho) = 0, \quad \text{because} \quad d\nu = \phi d\rho;$$

while for a *central quadric* ($f\rho = 1$, $\phi\rho = \nu$) it is easy to show that we have also,

$$\text{LXXXV.} \ldots \phi(\nu d\rho) = V\rho d\rho f(\nu U d\rho);$$

hence, for such a surface, if we suppose for simplicity that ds or $Td\rho$ is constant, which gives $V\nu d^2\rho \parallel d\rho$, we have,

$$\text{LXXXVI.} \ldots df(\nu d\rho) = 2S(\phi(\nu d\rho).d(\nu d\rho)) = 2S\nu^{-1}d\nu.f(\nu d\rho),$$

a differential equation of the *second order*, of which a *first integral* is evidently,

$$\text{LXXXVII.} \ldots f(\nu d\rho) = C\nu^2 d\rho^2, \quad \text{or} \quad \text{LXXXVII}'. \ldots fU(\nu d\rho) = C = \text{const.}$$

(34.) But we see that the *lines of curvature* on a central quadric are thus *included* in a *more general system* of curves on the same surface, represented by the differential equation LXXXVI., of which the *complete integral* would involve *two constants*: and which expresses that the *semidiameters* parallel to those *tangents* to the *surface*, which *cross* any *one* such curve at *right angles*, have a *common square*, and therefore (if real) a *common length*, so that (in this case) they *terminate on a sphero-conic*.*

* Compare the sub-articles (6.) (7.) (8.) to 219, in page 240, vol. i.

(35.) Admitting however, as a *case* of this property, the *constancy* LXXX. of the scalar lately called r', namely the *second root* of the quadratic XXXIV. or XXXV., of which the coefficients and the first root r *vary*, in passing from one point to another of what we may call for the moment a *line of first curvature*, we have only to conceive r and ν to be accented in the equations LXVI. LXVI'., in order to perceive this *theorem*, which perhaps is new:

The Curve on the Reciprocal* (24) *of the Surface of Centres* of curvature of a central quadric, which answers to the *second curvature* of that given surface for all the points of a *given line* of *first* curvature, or which is *itself* in a known sense the *reciprocal* (with respect to the given centre) of the *developable normal surface* (28.) which *rests* upon that *line*, is the *intersection* of *two quadrics*; whereof *one* (LXVI'.) is a *cone, concyclic with the given surface* ($f\rho = 1$); while the *other* (LXVI.) is a surface *concyclic with the reciprocal* of that given quadric ($F\nu = 1$).

(36.) Again, the scalar *Equation of the Surface of Centres* (21.) may be said to be the result of the *elimination* of r^{-1} between the equations LXVIII. and LXXI., whereof the latter is the *derivative*† of the former with respect to that scalar; we have therefore this theorem:

An Auxiliary Quadric (LXVIII. or XLV.) *touches the Second Sheet of the Surface of Centres* of a given quadric, *along a Quartic Curve*, which is the *locus of the centres of Second Curvature* for all the points of a *Line of First Curvature* (35.); and (for the same reason) the same auxiliary quadric is *circumscribed*, along the same quartic, by the *Developable Normal Surface* (28), which *rests* on that *first line*: with permission, of course, to interchange the words *first* and *second*, in this enunciation.

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* The *variable vector* of this *curve* is easily seen (comp. XLIX.) to be,

$$\nu' = \frac{\tau'}{S\tau'\rho} = \frac{\nu\tau}{S\nu\tau\rho};$$

and the *reciprocal surface* (21.) or (24.) is by (25.) the *locus of this quartic* (35.).

† The analogous relation, between the coordinate forms of the equations, was perhaps thought too obvious to be mentioned, in page 161 of Dr. Salmon's Treatise; or possibly it may have escaped notice, since the quartic curve (36.) is only mentioned there as an *intersection of two quadrics*, which is *on the surface of centres*, and answers to points of a *line of curvature* upon the given surface. But as regards the *possible novelty*, even *in part*, of any such *geometrical deductions* as those given in the text from the *quaternion analysis* employed, the writer wishes to be understood as expressing himself with the utmost diffidence, and as most willing to be corrected, if necessary. The power of *derivating* (or differentiating) any *symbolical expression* of the form LXVIII., or of any analogous *form*, with respect to *any scalar* which it involves *explicitly*, as if the expression were *algebraical*, is an important but an easy consequence from the principles of the Section III. ii. 6, which has been so often referred to.

(37.) When the arbitrary constant r is thus allowed to take successively all values, corresponding to *both systems* of lines of curvature, the *Surface of Centres* is therefore at once the *Envelope* of the Auxiliary Quadric* LXVIII., and the *Locus of the Quartic Curve* (36), in which one or other of its *two sheets* is *touched*, by that auxiliary quadric in one of its successive states, and also by one of the developable surfaces of normals to the given surface.

(38.) To obtain the *vector equation* of that envelope or locus we may proceed as follows, using a new expression for σ, in terms of ν or of ρ, which may then be transformed into a function of two independent and scalar variables. Denoting (comp. (32.)) by a_1, b_1, c_1 the semiaxes of the confocal which cuts the given surface in the given line of curvature, and by a_2, b_2, c_2 those of the other confocal, so that the *normals* ν_1, ν_2 to these two confocals have the directions of the *tangents* τ', τ lately considered, we have not only the expressions LXXXI. for r'^{-1}, with $a'b'c'$ changed to a_1, b_1, c_1, but also the analogous expressions (comp. 407, LXXI.),

$$\text{LXXXVIII.} \ldots r^{-1} = a^2 - a_2^2 = b^2 - b_2^2 = c^2 - c_2^2.$$

We have therefore by XLII., combined with 407, XVI., this very simple expression for σ:

$$\text{LXXIX.} \ldots \sigma = (\phi^{-1} + r^{-1})\nu = \phi_2^{-1}\nu = \phi_2^{-1}\phi\rho\ ;$$

containing, in the present notation, and as a result of the present analysis, a known and interesting theorem,† on which however we cannot here delay.

(39.) It follows from this last value of σ, combined with the expression 408, LXXXII. for ρ, that we may write,

$$\text{XC.} \ldots \sigma = l^{-2}\left(\frac{a^{-1}a_1a_2^3}{a + a'} + \frac{\sqrt{-1}b^{-1}b_1b_2^3}{\text{V}aa'} + \frac{c^{-1}c_1c_2^3}{a - a'}\right),$$

as the sought *Vector Equation of the Surface of Centres* of curvature of a given quadric (abc); ambiguous signs being virtually included in these three terms,

* Compare the Note immediately preceding.

† Namely Dr. Salmon's theorem (page 161 of his Treatise), that the *centres of curvature* of a given quadric at a given point are the *poles of the tangent plane*, with respect to the two confocals. The connected theorem (page 136), respecting the *rectilinear locus* of the *poles* of a *given plane*, with respect to the surfaces of a *confocal system*, is at once deducible from the quaternion expression 407, XVI. for $\phi^{-1}\nu$, although the theorem did not happen to be known to the present writer, or at least remembered by him, when he investigated that *formula of inversion* for other applications, of which some have been already given.

because in the subsequent eliminations* the semiaxes enter only by their squares: while l, a, a' are *constants*, as in 407, &c., for the whole confocal *system*, and abc are also constant *here*, but $a^2 - a_1^2$ and $a^2 - a_2^2$, or r'^{-1} and r^{-1} (38.), are *variable*, and may be considered to be the *two independent scalars* of which σ is a *vector function*.†

413. Some brief remarks may here be made, on the connexion of the general formula,

$$\text{I.} \ldots S\nu^{-1}(\phi + r)^{-1}\nu = 0, \qquad 412, \text{XXXV.}$$

in which $r = R^{-1}T\nu$ (412, XXIV.), and which when developed by the rules of the Section III. ii. 6 takes (comp. 398, LXXIX.) the form of the quadratic,

$$\text{II.} \ldots r^2 + rS\nu^{-1}\chi\nu + S\nu^{-1}\psi\nu = 0, \qquad 412, \text{XXXIV.}$$

with Gauss's‡ theory of the *Measure of Curvature of a Surface*; and especially with his fundamental result, that this *measure* is equal to the *product of the two principal curvatures of sections* of that surface: a relation which, in our notations, may be thus expressed,

$$\text{III.} \ldots V.dU\nu\delta U\nu = R_1^{-1}R_2^{-1}Vd\rho\delta\rho.$$

(1.) As regards the deduction, by quaternions, of the equation III., in which d and δ may be regarded as two§ distinct symbols of differentiation,

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* The corresponding elimination in coordinates was first effected by Dr. Salmon, who thus determined the equation of the surface of centres of curvature of a quadric to be one of the *twelfth degree*. (Compare pages 161, 162 of his already cited Treatise.)

† [In the notation of the Note to page 225, the vector to the centre of curvature of the quadric u_1 along its intersection with u_2 is $\sigma = \rho + x(\Phi + u_1)^{-1}\rho$, the value of the scalar x being found by expressing that σ does not change while u_3 in the expression $\rho = \{(\Phi + u_1)(\Phi + u_2)(\Phi + u_3)\}^{\frac{1}{2}}\epsilon$ receives a small increment. This gives at once $\frac{1}{2}(\Phi + u_1 + x)\rho du_3 + (\Phi + u_3)\rho dx = 0$, and therefore $x = u_3 - u_1$. Hence $\sigma = (\Phi + u_1)^{-1}(\Phi + u_3)\rho$, or in terms of ϵ the vector equation of the surface of centres is when u_2 and u_3 are variable

$$\sigma = (\Phi + u_1)^{-\frac{1}{2}}(\Phi + u_2)^{\frac{1}{2}}(\Phi + u_3)^{\frac{3}{2}}\epsilon.$$

It may also be shown in various ways that the vector equation of the reciprocal of this surface is

$$\nu = -(\Phi + u_1)^{\frac{1}{2}}(\Phi + u_2)^{\frac{1}{2}}(\Phi + u_3)^{-\frac{1}{2}}\epsilon.]$$

‡ The reader is referred to the Additions to Liouville's Monge (pages 505, &c.), in which the beautiful Memoir by Gauss, entitled: *Disquisitiones generales circa superficies curvas*, is with great good taste reprinted in the Latin, from the *Commentationes recentiores* of the Royal Society of Göttingen. He is also supposed to look back, if necessary, to the Section III. ii. 6 of these *Elements* (pages 484, vol. i., &c.), and especially to the deduction in page 486, vol. i., of ψ from ϕ, remembering that the latter function (and therefore also the former) is here self-conjugate.

§ Compare page 553, vol. i., and the Note to page 254.

performed with respect to two independent scalar variables, we may observe that, by principles and rules already established,

$$\text{IV.} \ldots \mathrm{d}\mathrm{U}\nu = \mathrm{V}\frac{\mathrm{d}\nu}{\nu}\,.\,\mathrm{U}\nu, \quad \delta\mathrm{U}\nu = \mathrm{V}\frac{\delta\nu}{\nu}\,.\,\mathrm{U}\nu = -\,\mathrm{U}\nu\,.\,\mathrm{V}\frac{\delta\nu}{\nu}\,;$$

and that therefore the first member of III. may be thus transformed:

$$\text{V.} \ldots \mathrm{V}\,.\,\mathrm{d}\mathrm{U}\nu\delta\mathrm{U}\nu = \mathrm{V}\left(\mathrm{V}\frac{\mathrm{d}\nu}{\nu}\,.\,\mathrm{V}\frac{\delta\nu}{\nu}\right) = -\,\nu^{-1}\mathrm{S}\nu^{-1}\mathrm{d}\nu\delta\nu.$$

(2.) Again, since we have $\mathrm{d}\nu = \phi\mathrm{d}\rho$ (410, IV., &c.), and in like manner $\delta\nu = \phi\delta\rho$, the relations $\mathrm{S}\nu\mathrm{d}\rho = 0$, $\mathrm{S}\nu\delta\rho = 0$, and the self-conjugate property of ϕ, allow us to write,

$$\text{VI.} \ldots \mathrm{V}\mathrm{d}\nu\delta\nu = \psi\mathrm{V}\mathrm{d}\rho\delta\rho, \quad \text{and} \quad \text{VII.} \ldots \mathrm{V}\mathrm{d}\rho\delta\rho = \nu^{-1}\mathrm{S}\nu\mathrm{d}\rho\delta\rho\,;$$

whence follows at once by V. the formula III., if we remember the general expression, deduced from the quadratic II.,

$$\text{VIII.} \ldots R_1^{-1}R_2^{\ 1} = -\,\nu^{-2}r_1r_2 = -\,\mathrm{S}\frac{1}{\nu}\,\psi\,\frac{1}{\nu}. \qquad 412,\ \text{XXXVII.}$$

(3.) If then we suppose that P, P_1, P_2 are *any three near points* on an *arbitrary surface*, and that R, R_1, R_2 are three near and *corresponding points* on the *unit sphere*, determined by the condition of parallelism of the *radii* OR, OR_1, OR_2 to the *normals* PN, $\mathrm{P}_1\mathrm{N}_1$, $\mathrm{P}_2\mathrm{N}_2$, the *two small triangles* thus formed will bear to each other the *ultimate ratio*,

$$\text{IX.} \ldots \lim.\ \frac{\Delta\mathrm{RR}_1\mathrm{R}_2}{\Delta\mathrm{PP}_1\mathrm{P}_2} = R_1^{-1}R_2^{-1}\,;$$

a result which justifies (although by an entirely new analysis) the adoption by Gauss of this *product** of curvatures of *sections*, as the *measure* of the curvature of the *surface*, with his signification of the phrase.

(4.) As another form of this important product or measure, if we conceive that the vector ρ of the surface is expressed as a function (372) of two independent scalars, t and u, and if we write for abridgment,

$$\text{X.} \ldots \mathrm{D}_t\rho = \rho', \quad \mathrm{D}_u\rho = \rho_{,}, \quad \mathrm{D}^2_t\rho = \rho'', \quad \mathrm{D}_t\mathrm{D}_u\rho = \rho'_{,}, \quad \mathrm{D}_u^2\rho = \rho_{,,},$$

* If it be supposed to be in any manner known that a *limit* such as IX. *exists*, or that the *quotient* of the two *vector areas* in III. is a scalar *independent of the directions* of PP_1, PP_2, or of $\mathrm{d}\rho$, $\delta\rho$, we have only to assume that these are the directions of the *lines of curvature*, in order to obtain *at once*, by 412, II. [page 247], the *product* $R_1^{-1}R_2^{-1}$ as the *value* of this quotient or limit.

which will allow us (comp. 372, V.) to assume for the normal vector ν the expression,

$$\text{XI.} \ldots \nu = \text{V}\rho'\rho_{,},$$

it is easy to prove* that we have generally,

$$\text{XII.} \ldots R_1^{-1}R_2^{-1} = \text{S}\frac{\rho''}{\nu}\,\text{S}\frac{\rho_{,,}}{\nu} - \left(\text{S}\frac{\rho_{,}'}{\nu}\right)^2;$$

which takes as a verification the well-known form,

$$\text{XIII.} \ldots R_1^{-1}R_2^{-1} = \frac{rt - s^2}{(1 + p^2 + q^2)^2},$$

when we write (comp. 410, (18.)),

$$\text{XIV.} \ldots \rho = ix + jy + kz, \quad \rho' = \text{D}_x\rho = i + kp, \quad \rho_{,} = \text{D}_y\rho = j + kq;$$

$$\text{XV.} \ldots \nu = \text{V}\rho'\rho_{,} = k - ip - jq, \quad \rho'' = kr, \quad \rho_{,}' = ks, \quad \rho_{,,} = kt.$$

(5.) In general, the equation XII. may be thus transformed,

$$\text{XVI.} \ldots \nu^4 R_1^{-1}R_2^{-1} = \text{S}(\text{V}\nu\rho''.\text{V}\nu\rho_{,,}) - (\text{V}\nu\rho_{,}')^2 + \nu^2(\text{S}\rho''\rho_{,,} - \rho_{,}'^2);$$

also

$$\text{XVII.} \ldots \text{Td}\rho^2 = e\text{d}t^2 + 2f\text{d}t\text{d}u + g\text{d}u^2,$$

if

$$\text{XVIII.} \ldots e = -\rho'^2, \quad f = -\text{S}\rho'\rho_{,}, \quad g = -\rho_{,}^2,$$

whence

$$\text{XIX.} \ldots \nu^2 = f^2 - eg,$$

and if we still denote, as in X., derivations relatively to t and u by upper and lower accents, we may substitute in the quadruple of the equation XVI. the values,

$$\text{XX.} \ldots 2\text{V}\nu\rho'' = (e_{,} - 2f')\rho' + e'\rho_{,}, \quad 2\text{V}\nu\rho_{,}' = -g'\rho' + e_{,}\rho_{,},$$
$$2\text{V}\nu\rho_{,,} = -g_{,}\rho' + (2f_{,} - g')\rho_{,},$$

and

$$\text{XXI.} \ldots 2(\text{S}\rho''\rho_{,,} - \rho_{,}'^2) = e_{,,} - 2f_{,}' + g'';$$

hence *the measure of curvature is an explicit function of the ten scalars*,

$$\text{XXII.} \ldots e, f, g; \quad e', f', g'; \quad e_{,}, f_{,}, g_{,}; \quad \text{and} \quad e_{,,} - 2f_{,}' + g'':$$

* The quadratic in R^{-1} may be formed by operating on 412, II. with $\text{S}\,.\,\rho'$ and $\text{S}\,.\,\rho_{,}$, and then eliminating $\text{d}t : \text{d}u$.

and therefore, as was otherwise proved by Gauss, *this measure depends only* on the expression* (XVII.) *of the square of a linear element*, in terms of two independent scalars (t, u), and of their differentials (dt, du).

(6.) Hence follow also these two other theorems† of Gauss:—

If a *surface* be considered as an *infinitely thin solid*, and supposed to be *flexible* but *inextensible*, then every *deformation* of it, as such, will leave *unaltered*, Ist, the *Measure of Curvature at any Point*, and IInd, the *Total Curvature of any Area*; that is, the area of the corresponding portion of the *unit sphere*, determined as in (3.) by radii parallel to normals.‡

(7.) Supposing now that t and u are *geodetic coordinates*, whereof the former represents the *length* of a geodetic AP from a *fixed point* A of the surface, and the latter represents the *angle* BAP which this variable geodetic makes at A with a *fixed geodetic* AB, it is easy to see that the general expression XVII. takes the shorter form,

$$\text{XXIII.} \ldots \mathrm{Td}\rho^2 = dt^2 + n^2 du^2, \quad \text{in which} \quad \text{XXIV.} \ldots n = \mathrm{T}\rho_{,} = \mathrm{T}\nu;$$

so that we have now the values,

$$\text{XXV.} \ldots e = 1, \quad f = 0, \quad g = n^2, \quad g' = 2nn', \quad g'' = 2nn'' + 2n'^2,$$

and the derivatives of e and f all vanish. And thus the general expression XII. for the *measure of curvature* reduces itself by (5.) to the very simple form,

$$\text{XXVI.} \ldots R_1^{-1}R_2^{-1} = -n^{-1}n'' = -n^{-1}\mathrm{D}_t^2 n;$$

in which n is generally a function of both t and u, although here twice derivated with respect to the former only.

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* The proof by quaternions, above given, of this exclusive dependence, is perhaps as simple as the subject will allow, and is somewhat shorter than the corresponding proof in the *Lectures*; in page 605 of which is given however the equation,

$$4(eg - f^2)^2 R_1^{-1}R_2^{-1} = e(g'^2 - 2g_{,}f' + g_{,}e_{,}) + f(e'g_{,} - e_{,}g' - 2e_{,}f_{,} - 2g'f' + 4f'f_{,}) \\ + g(e_{,}^2 - 2e'f_{,} + e'g') - 2(eg - f^2)(e_{,,} - 2f_{,}' + g''),$$

which may now be deduced *at sight* from XVI., by the substitutions XIX. XX. XXI., and differs only in notation from the equation of Gauss (Liouville's Monge, page 523, or Salmon, page 309).

† See page 524 of Liouville's Monge.

‡ [If q is a quaternion or versor function of the two scalars t and u, and if $d\varpi = q\,d\rho\,q^{-1}$ is the differential of a vector function of t and u, the squares of the linear elements $d\varpi$ and $d\rho$ are identical. The surfaces described by ρ and ϖ correspond point to point, and the measure of curvature at any point on one surface is equal to that at the corresponding point on the other. Under these circumstances the surfaces are *applicable*. To find the condition to be satisfied by q, we express that $d\varpi$ is a differential of a function of t and u by equating $\mathrm{D}_t\mathrm{D}_u\varpi = \mathrm{D}_u\mathrm{D}_t\varpi$. This gives in the notation of the text a partial differential equation for q

$$q'\rho_{,}q^{-1} - q\rho_{,}q^{-1}q'q^{-1} = q_{,}\rho'q^{-1} - q\rho'q^{-1}q_{,}q^{-1}, \quad \text{or} \quad \mathrm{V.V}q^{-1}q'.\rho_{,} = \mathrm{V.V}q^{-1}q_{,}.\rho'.]$$

(8.) The point P being denoted by the symbol (t, u), and any other point P′ of the surface by $(t + \Delta t, u + \Delta u)$, we may consider the two connected points P_1, P_2, of which the corresponding symbols are $(t + \Delta t, u)$ and $(t, u + \Delta u)$; and then the *quadrilateral* $PP_1P'P_2$, bounded by two portions PP_1, P_2P' of *geodetic lines* from A, and (as we may suppose) by two arcs PP_2, P_1P' of *geodetic circles* round the same fixed point, will have its *area* ultimately $= n\Delta t\Delta u$ (by XXIII.), and therefore (by XXVI., comp. (3.), (6.)) its *total curvature* ultimately $= - n''\Delta t\Delta u$, or $= - \Delta_t n' . \Delta u$, when Δt and Δu diminish together, by an approach of P′ to P.

(9.) Again, in the immediate neighbourhood of A, we have $n = t$, $n' = 1$; changing then $- \Delta_t n'$ to $- d_t n'$, and integrating with respect to t from $t = 0$, we obtain $1 - n'$ as the coefficient of Δu in the result, and are thus conducted to the expression:

XXVII. . . *Total Curvature of Triangle* APP′ $= (1 - n')\Delta u$, ultimately,

if AP, AP′ be *any two geodetic lines*, making with each other a *small angle* $= \Delta u$, and if PP′ be *any small arc* (geodetic or not) on the same surface.

(10.) Conceive then that PQ is a *finite arc* of *any curve* upon the surface, for which therefore t, and consequently n', may be conceived to be a function of u; we shall have this other expression of the same kind,

XXVIII. . . *Total Curvature of Area* APQ $= \int (1 - n')du = \Delta u - \int n'du$;

the *area* here considered being bounded by the *two geodetic lines* AP, AQ, which make with each other the finite angle Δu, and by the *arc* PQ of the *arbitrary curve*.

(11.) If this curve be *itself* a *geodetic*, and if we treat its coordinates t, u, and its vector ρ, as functions of its arc, s, then the second differential of ρ, namely,

$$\text{XXIX.} \ldots d^2\rho = \rho' d^2 t + \rho_, d^2 u + \rho'' dt^2 + 2\rho_,' dt du + \rho_{,,} du^2,$$

must be normal to the surface at P, and consequently perpendicular to ρ' and $\rho_,$. Operating* therefore with $S . \rho'$, and attending to the relations XVIII. and XXV., which give

$$\text{XXX.} \ldots \rho'^2 = -1, \quad S\rho'\rho_, = S\rho'\rho'' = S\rho'\rho_,' = 0, \quad S\rho'\rho_{,,} = - S\rho_,\rho_,' = nn',$$

* To operate with $S . \rho_,$ would give a result not quite so simple, but reducible to the form XXXI., with the help of $d^2 s = 0$.

we obtain the differential equation,

$$\text{XXXI.} \ldots \mathrm{d}^2t = nn'\mathrm{d}u^2, \quad \text{or} \quad \text{XXXII.} \ldots \mathrm{d}v = -n'\mathrm{d}u,$$

if we observe that we may write,

$$\text{XXXIII.} \ldots \mathrm{d}t = \cos v\mathrm{d}s, \quad n\mathrm{d}u = \sin v\mathrm{d}s,$$

because

$$\text{XXXIV.} \ldots \mathrm{d}t^2 + n^2\mathrm{d}u^2 = \mathrm{d}s^2\,;$$

v being here the variable angle, which the geodetic PQ makes at P with AP prolonged.

(12.) Substituting then for $-n'\mathrm{d}u$, in XXVIII., its value $\mathrm{d}v$ given by XXXII., the integration becomes possible, and the result is $\Delta u + \Delta v$; where Δu is still the angle at A, and $\pi + \Delta v = (\pi - v) + (v + \Delta v)$ is the sum of the angles at P and Q, in the *geodetic triangle* APQ.

(13.) Writing then B and C instead of P and Q, we thus arrive at another most remarkable Theorem* of Gauss, which may be expressed by the formula:

$$\text{XXXV.} \ldots \textit{Total Curvature of a Geodetic Triangle } \mathrm{ABC} = \mathrm{A} + \mathrm{B} + \mathrm{C} - \pi,$$

= what may be called the *Spheroidal Excess*; A, B, C, in the second member, being used to denote the *three angles* of the triangle: and the *total surface* of the *unit sphere* ($= 4\pi$) being represented by 720°, when the *part* corresponding to the *geodetic triangle* is thus represented by the *angular excess*, $\mathrm{A} + \mathrm{B} + \mathrm{C} - 180°$.

(14.) And it is easy to perceive, on the one hand, how this theorem admits of being *extended*, as it was by Gauss, to *all geodetic polygons*: and on the other hand, how it may require to be *modified*, as it was by the same eminent geometer, so as to give what would on the same plan be called a *spheroidal defect*, when the *measure of curvature* is *negative*, as it is for surfaces (or parts of surfaces) of which the principal sections have their curvatures *oppositely directed*.

414. The only sections of a surface, of which the curvatures have been above determined, are the *two principal normal sections* at any proposed point; but the general expressions of III. iii. 6 may be applied to find the curvature of *any plane section*, normal or oblique, and therefore also of *any curve* on a

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* The enunciation of this theorem, respecting which its illustrious discoverer justly says, "Hoc theorema, quod, ni fallimur, ad elegantissima in theoria superficierum curvarum referendum esse videtur," . . . is given in page 533 of the Additions to Liouville's Monge. A proof by quaternions was published in the *Lectures* (pages 606–609, see also the few preceding pages), but the writer conceives that the one given above will be found to be not only shorter, but more clear.

given surface, when only its *osculating plane* is known. Denoting (as in 389, &c.) by ρ and κ the vectors of the given point P, and of the *centre* K of the *osculating circle* at that point, and by s the *arc* of the curve, we have generally (by 389, XII. and VI.),

$$\text{I.} \ldots \textit{Vector of Curvature of Curve} = \text{KP}^{-1} = (\rho - \kappa)^{-1} = \mathrm{D}_s^2\rho = \frac{1}{\mathrm{d}\rho} \mathrm{V} \frac{\mathrm{d}^2\rho}{\mathrm{d}\rho}\,;$$

the independent variable in the last expression being arbitrary. And if we denote by σ and ξ the vectors of the points S and X, in which the *axis* of the osculating circle meets respectively the *normal* and the *tangent plane* to the given surface, we shall have also, by the right-angled triangles, the general decomposition, $\text{KP}^{-1} = \text{SP}^{-1} + \text{XP}^{-1}$ (as vectors), or

$$\text{II.} \ldots \mathrm{D}_s^2\rho = (\rho - \kappa)^{-1} = (\rho - \sigma)^{-1} + (\rho - \xi)^{-1}\,;$$

where the two components admit of being transformed as follows:

III. . . *Normal Component of Vector of Curvature of Curve* (*or Section*)

$$= (\rho - \sigma)^{-1} = \nu^{-1}\mathrm{S}\frac{\mathrm{d}\nu}{\mathrm{d}\rho} = (\rho - \sigma_1)^{-1}\cos^2 v + (\rho - \sigma_2)^{-1}\sin^2 v$$

= *Vector of Normal Curvature of Surface* for the direction of the given tangent;

σ_1, σ_2 being the vectors of the *centres* S_1, S_2 (comp. 412) of the *two principal curvatures,* and v being the *angle* at which the curve (or its tangent $\mathrm{d}\rho$) crosses the *first line of curvature* (or its tangent τ_1), while σ is the vector of the centre S of the *sphere* which is said to *osculate* to the *surface,* in the *given direction* (of $\mathrm{d}\rho$); and

IV. . . *Tangential Component of Vector of Curvature*

$$= (\rho - \xi)^{-1} = \nu^{-1}\mathrm{d}\rho^{-1}\mathrm{S}\nu\mathrm{d}\rho^{-1}\mathrm{d}^2\rho$$

= *Vector of Geodetic Curvature of Curve* (*or Section*);

this latter *vector* being here so called, because in fact its *tensor* represents what is known by the name of the *geodetic** *curvature* of a *curve upon a surface*: the independent variable being still arbitrary.

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* The name, "*courbure géodésique,*" was introduced by M. Liouville, and has been adopted by several other mathematical writers. Compare pages 568, 575, &c. of his *Additions* to Monge.

(1.) As regards the decomposition II., if α, β be any two rectangular vectors OA, OB, and if γ = OC = the perpendicular from O on AB, then (comp. 316, L., and 408, XLI.),

$$\text{V.} \ldots \gamma^{-1} = \frac{\beta}{\mathrm{V}\alpha\beta} + \frac{\alpha}{\mathrm{V}\beta\alpha} = \alpha^{-1} + \beta^{-1}.$$

(2.) To prove the first transformation III., we have, by I. and II., observing that $\mathrm{dS}\nu\mathrm{d}\rho = 0$,

$$\text{VI.} \ldots \frac{\nu}{\rho - \sigma} = \mathrm{S}\frac{\nu}{\rho - \kappa} = \mathrm{S}\,.\,\frac{\nu}{\mathrm{d}\rho}\mathrm{V}\frac{\mathrm{d}^2\rho}{\mathrm{d}\rho} = \frac{-\mathrm{S}\nu\mathrm{d}^2\rho}{\mathrm{d}\rho^2} = \frac{\mathrm{Sd}\nu\mathrm{d}\rho}{\mathrm{d}\rho^2} = \mathrm{S}\frac{\mathrm{d}\nu}{\mathrm{d}\rho}.$$

(3.) Hence, by 412, (7.), if we denote the vector III. of normal curvature by $R^{-1}\mathrm{U}\nu$, we have the general expressions (comp. 412, I. XXI.),

$$\text{VII.} \ldots \sigma = \rho + R\mathrm{U}\nu, \quad R = D^2\,.\,\mathrm{T}\nu, \quad \text{with} \quad \text{VIII.} \ldots \mathrm{T}\nu = P^{-1},$$

for the case of a *central quadric*; D being *generally* the *semidiameter* of the *index surface* (410, (9.), &c.), or for a quadric the semidiameter of that surface *itself*, which has the direction of the *tangent* (or of $\mathrm{d}\rho$): and P being, for the latter surface, the *perpendicular* from the centre on the *tangent plane*, as in some earlier formulæ.

(4.) To deduce the second transformation III., which contains a theorem of Euler, let τ, τ_1, τ_2 denote unit tangents to the section and the two lines of curvature, so that

$$\text{IX.} \ldots \tau = \tau_1 \cos v + \tau_2 \sin v, \quad \text{and} \quad \tau^2 = \tau_1^2 = \tau_2^2 = -1;$$

we may then write generally (comp. 412, IV.),

$$\text{X.} \ldots R^{-1}\mathrm{T}\nu = \frac{\nu}{\sigma - \rho} = -\mathrm{S}\frac{\mathrm{d}\nu}{\mathrm{d}\rho} = -\mathrm{S}\tau^{-1}\phi\tau = \mathrm{S}\tau\phi\tau,$$

and shall have the values (comp. 410, XI.),

$$\text{XI.} \ldots \mathrm{S}\tau_1\phi\tau_1 = R_1^{-1}\mathrm{T}\nu, \quad \mathrm{S}\tau_2\phi\tau_2 = R_2^{-1}\mathrm{T}\nu, \quad \mathrm{S}\tau_1\phi\tau_2 = \mathrm{S}\tau_2\phi\tau_1 = 0;$$

whence

$$\text{XII.} \ldots R^{-1} = R_1^{-1}\cos^2 v + R_2^{-1}\sin^2 v,$$

and the required transformation is accomplished.

(5.) The theorem of Meusnier may be considered to be a result of the *elimination* (2.) of $\mathrm{d}^2\rho$ from the expressions for the *normal component* III. of what we may call the *Vector* $D_s^2\rho$ *of Oblique Curvature*; and it may be expressed by the equation,

$$\text{XIII.} \ldots S\frac{\rho - \sigma}{\rho - \kappa} = 1, \quad \text{or} \quad \text{XIII}'. \ldots S\frac{\sigma - \kappa}{\rho - \kappa} = 0,$$

which gives

$$\text{XIII}''. \ldots \text{PKS} = \frac{\pi}{2},$$

if it be now understood that the point S, of which σ is the vector, is the *centre* of the *circle* which *osculates* to the *normal section*; or of the *sphere* which osculates in the *same direction* to the *surface*, as will be more clearly seen by what follows.

(6.) In general, if $\rho + \Delta\rho$ be the vector of *any second point* P′ of the given surface, the equation

$$\text{XIV.} \ldots S\frac{\nu}{\omega - \rho} = S\frac{\nu}{\Delta\rho}, \text{ with } \omega \text{ for a variable vector,}$$

represents rigorously the sphere which *touches* the surface at the given point P, and passes *through* the second point P′; conceiving then that the latter point *approaches* to the former, and observing that the development* by Taylor's Series of the equation $f\rho$ = const. gives (if $df\rho = 2S\nu d\rho$, and $d\nu = \phi d\rho$),

$$\text{XV.} \ldots 0 = \Delta\rho^{-2}\Delta f\rho = 2S\frac{\nu}{\Delta\rho} + S\frac{\phi\Delta\rho}{\Delta\rho} + \text{terms which vanish } \textit{generally} \text{ with } \Delta\rho,$$

even if they be not *always null*, we are conducted in a new way, by the known conception of the *Osculating Sphere* for a given *direction* to a *surface*, to the same *centre* S, and *radius* R, as before: the equation of *this* sphere being,

$$\text{XVI.} \ldots S\frac{2\nu}{\omega - \rho} = \left(\lim. S\frac{2\nu}{\Delta\rho} = -\lim. S\frac{\phi\Delta\rho}{\Delta\rho} = \right) - S\frac{d\nu}{d\rho}.$$

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* Compare Art. 374, and the second Note to page 20. The occasional use, there mentioned, of the differential symbol $d\rho$ as signifying a finite and *chordal vector*, in the development of $f(\rho + d\rho)$, has appeared obscure, in the *Lectures*, to some friends of the writer; and he has therefore aimed, for the sake of clearness, in at least the *text* of these *Elements*, and especially in the geometrical applications, to confine that symbol to its *first* signification (100, 369, 373, &c.), as denoting a *tangential vector* (finite or infinitely small, and to a curve or surface): ρ itself being generally regarded as a *vector function*, and *not* as an independent variable (comp. 362, (3.)).

(7.) Conversely, if we *assume* a *radius* R, such that R^{-1} is algebraically *intermediate* between R_1^{-1} and R_2^{-1}, the *tangent sphere*,

$$\text{XVII.} \ldots \mathrm{S}\frac{2\nu}{\omega-\rho} = \frac{\mathrm{T}\nu}{R}, \quad \text{or} \quad \text{XVII}'. \ldots \mathrm{S}\frac{2\mathrm{U}\nu}{\omega-\rho} = R^{-1},$$

will *cut the surface* in *two directions of osculation*, assigned by the formula XII.; but if R^{-1} be *outside* those *limits*, there will be *only contact*, and *not* any (real) *intersection*, at least in the vicinity of P.

(8.) If P′ be again, as in (6.), *any second point* of the surface, and if we denote for a moment by (Π) and (Σ) the normal *plane* PNP′ and the normal *section* corresponding, we may suppose that N is the point in which the normals *to the plane curve* (Σ) at P and P′ intersect; and if we then erect a perpendicular at N to the plane (Π), it will be crossed by *every* perpendicular at P′ to the tangent P′T′ to the section, and therefore in particular by the *normal* at P′ to the *surface*, in a point which we may call N′: so that the line P′N is the *projection*, on the plane PP′N, of this *second normal* P′N′ to the surface. Conceiving then the *plane* (Π) to be *fixed*, but the *point* P′ to *approach* indefinitely to P, we see that the *centre* S of *curvature* of the *normal section* (Σ), which is also by (6.) the centre of the *osculating sphere* to the surface for the *same direction*, is the *limiting position* of the point N, in which the *given normal* at P is *intersected by the projection** *of the near normal* P′N′, *on the given normal plane.*

(9.) The two components III. and IV are included in the *binomial expression*,

XVIII. . . *Vector of Oblique Curvature* (or of *Curvature of Oblique Section*)

$$= (\rho - \kappa)^{-1} = \nu^{-1}\mathrm{S}d\nu d\rho^{-1} + \nu^{-1}d\rho^{-1}\mathrm{S}\nu d\rho^{-1}d^2\rho,$$

which is obtained by substituting in I. the general equivalent 409, XXI. for $d^2\rho$, and in which (as before) the independent variable is arbitrary; and the *tangential component* IV. may be otherwise found by observing that, by I. and II.,

$$\text{XIX.} \ldots \frac{\nu d\rho}{\rho-\xi} = \mathrm{S}\frac{\nu d\rho}{\rho-\kappa} = \mathrm{S}\frac{\nu d^2\rho}{d\rho} = -\mathrm{S}\nu d\rho^{-1}d^2\rho,$$

and that

$$-(\nu d\rho)^{-1} = \nu^{-1}d\rho^{-1}, \quad \text{because} \quad \mathrm{S}\nu d\rho = 0.$$

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* The reader may compare the calculations and constructions, in pages 600, 601 of the *Lectures*. In the language of infinitesimals, an *infinitely near normal* P′N′ *intersects the axis of the osculating circle*, to the given normal section.

(10.) Another way of deducing the same component IV., is to resolve the following system of three scalar equations, which by the geometrical definition of the point x the vector ξ must satisfy:

$$\text{XX.} \ldots S(\xi - \rho)\nu = 0; \quad S(\xi - \rho)d\rho = 0; \quad S(\xi - \rho)d^2\rho = d\rho^2;$$

and which give,

$$\text{XXI.} \ldots \xi - \rho = \frac{\nu d\rho^3}{S\nu d\rho d^2\rho} = \frac{\nu d\rho}{S\nu d\rho^{-1}d^2\rho},$$

or $(\rho - \xi)^{-1} = \&c.$, as before. We have also the transformations,

$$\text{XXII.} \ldots \textit{Vector of Geodetic Curvature} = (\rho - \xi)^{-1}$$
$$= (\nu d\rho)^{-1} S(\nu Ud\rho . dUd\rho) = -\nu d\rho \, S\frac{d\rho^{-2}d^2\rho}{\nu d\rho} = \&c.$$

(11.) The definition of the point x shows also easily, that *if a developable surface* (D) *be circumscribed to a given surface* (S), *along a given curve* (C), *and if, in the unfolding of the former surface, the point* x *be carried with the tangent plane, originally drawn to the latter surface at* P, *it will become the centre of curvature, at the new point* (P), *to the new or plane curve* (C′) obtained by this development: so that the *radius* (PX) *of geodetic curvature* is equal, as indeed it is known* to be, to the *radius of plane curvature* of the *developed curve*.

(12.) This *plane* curve (C′) is therefore a *circle*† (or part of one) if the condition,

$$\text{XXIII.} \ldots \overline{PX} = T(\xi - \rho) = \text{const.},$$

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* Compare page 576 of the *Additions* to Liouville's Monge.

† The *curves* on any given surface, which thus *become circles* by development, have also the *isoperimetrical property* expressed in quaternions (comp. the Note to page 48) by the formula,

$$\text{XXVI.} \ldots \int S(U\nu . d\rho\delta\rho) + c\delta \int Td\rho = 0,$$

which conducts to the differential equation,

$$\text{XXVII.} \ldots c^{-1}d\rho = V.U\nu dUd\rho \text{ (comp. 380, IV. [page 29])},$$

and in which the scalar constant c can be shown to have the value,

$$\text{XXVIII.} \ldots c = (\xi - \rho)U.\nu d\rho = \pm T(\xi - \rho) = \textit{Radius of Geodetic Curvature},$$

= radius of developed circle; and each such curve includes, by XXVI., on the given surface, a *maximum area* with a *given perimeter*: on which account, and in allusion to a well-known classical story, the writer ventured to propose, in page 582 of the *Lectures*, the *name* "Didonia" for a curve of this kind, while acknowledging that the *curves themselves* had been discovered and discussed by M. Delaunay.

be satisfied; but it degenerates into a *right line*, if this radius of *geodetic* curvature be *infinite*, that is, if

$$\text{XXIV.} \ldots \mathrm{T}(\rho - \xi)^{-1} = 0, \quad \text{or} \quad \text{XXV.} \ldots \mathrm{S}\nu d\rho d^2\rho = 0,$$

or finally (by 380, II., comp. 409, XXV.), if the original curve (c) be a *geodetic line* on the *given surface* (s), and therefore also on the *developable* (D): which agrees with the fundamental property (382, 383) of geodetics on a developable surface.

(13.) Accordingly it may be here observed that the general formula IV., combined with the notations and calculations of 382, conducts to the expression $(z + v')\,\mathrm{T}\rho'^{-1}$, or $\frac{z dx + dv}{ds}$, for the geodetic curvature of *any curve* on a developable surface, whereof the element ds crosses a generating line at the variable angle v, while $z dx$ is the angle between two such consecutive lines: a result easily confirmed by geometrical considerations, and agreeing with the differential equation $z + v' = 0$ (382, IX.) of *geodetics* on a developable.

415. We shall conclude the present Section with a few supplementary remarks, including a new and simplified proof of an important *theorem* (354), which we have had frequent occasion to employ for purposes of *geometry*, and which presents itself often in *physical* applications of quaternions also: namely, that *if the linear and vector function* ϕ *be self-conjugate*, then the *Vector Quadratic*,

$$\text{I.} \ldots \mathrm{V}\rho\phi\rho = 0, \qquad 354, \text{I.}$$

represents generally a *System of Three Real and Rectangular Directions*; and that these (comp. 405, (1.), (2.), &c.) are the directions of the *Axes of the Central Surfaces of the Second Order*, which are represented by the scalar equation,

$$\text{II.} \ldots \mathrm{S}\rho\phi\rho = \text{const.};$$

or more generally,

$$\text{III.} \ldots \mathrm{S}\rho\phi\rho = C\rho^2 + C', \text{ where } C \text{ and } C' \text{ are any two scalar constants.}$$

(1.) It is an easy consequence of the theory (350) of *the symbolic and cubic equation* in ϕ, that if c be a root of the derived *algebraical cubic* $M = 0$ (354), and if we write $\Phi = \phi + c$ (as in that Article), the *new* linear and vector function $\Phi\rho$ must be reducible to the *binomial form* (351),

$$\text{IV.} \ldots \Phi\rho = \phi\rho + c\rho = \beta\mathrm{S}\alpha\rho + \beta'\mathrm{S}\alpha'\rho, \quad \text{with} \quad \text{V.} \ldots \mathrm{V}\beta\alpha + \mathrm{V}\beta'\alpha' = 0,$$

as the *condition* (353, XXXVI.) of *self-conjugation.* With this condition we may then write,

$$\text{VI.} \ldots \beta = A\alpha + B\alpha', \quad \beta' = A'\alpha' + B\alpha;$$

and it is easy to see that no essential generality is lost, by supposing that α and α' are two rectangular vector units, which may be turned about in their own plane, if β and β' be suitably modified: so that we may assume,

$$\text{VII.} \ldots \alpha^2 = \alpha'^2 = -1, \quad S\alpha\alpha' = 0; \quad \text{whence} \quad \text{VIII.} \ldots \Phi\alpha = -\beta, \quad \Phi\alpha' = -\beta',$$

and $$\text{IX.} \ldots V\beta'\alpha' = B\alpha\alpha' = -V\beta\alpha, \quad V\beta\alpha' = A\alpha\alpha', \quad V\beta'\alpha = -A'\alpha\alpha'.$$

(2.) The equation I., under the form,

$$\text{X.} \ldots V\rho\Phi\rho = 0, \quad \text{is satisfied by} \quad \text{XI.} \ldots \phi\rho = 0, \quad \text{or} \quad \text{XII.} \ldots V\alpha\alpha'\rho = 0;$$

and it cannot be satisfied otherwise, unless we suppose,

$$\text{XIII.} \ldots \rho = x\alpha + x'\alpha', \quad \text{and} \quad \text{XIV.} \ldots V(x\beta + x'\beta')(x\alpha + x'\alpha') = 0;$$

that is, by IX.,

$$\text{XV.} \ldots B(x'^2 - x^2) + (A - A')xx' = 0:$$

while conversely the expression XIII. will satisfy I., under this condition XV. But this quadratic in $x' : x$, of which the coefficients B and $A - A'$ do not *generally* vanish, has necessarily *two real roots*, with a product $= -1$; hence there *always exists*, as asserted, a system of *three real and rectangular directions*, such as the following,

$$\text{XVI.} \ldots x\alpha + x'\alpha', \quad x'\alpha - x\alpha', \quad \text{and} \quad \alpha\alpha' \text{ (or } V\alpha\alpha'),$$

which satisfy the equation I.; and this system is *generally definite*; which proves the *first part* of the Theorem.

(3.) The lines α, α' may be made by (1) to *turn* in their own plane, till they coincide with the two first directions XVI.; which will give,

$$\text{XVII.} \ldots B = 0, \quad \beta = A\alpha, \quad \beta' = A'\alpha',$$

and therefore,

$$\text{XVIII.} \ldots \phi\rho = -c\rho + A\alpha S\alpha\rho + A'\alpha' S\alpha'\rho$$
$$= (c + A)\,\alpha S\alpha\rho + (c + A')\,\alpha' S\alpha'\rho + c\alpha\alpha' S\alpha\alpha'\rho;$$

and thus the scalar equation II. will take the form,

$$\text{XIX.} \ldots S\rho\phi\rho = (c + A)\,(S\alpha\rho)^2 + (c + A')\,(S\alpha'\rho)^2 + c\,(S\alpha\alpha'\rho)^2 = \text{const.},$$

which represents generally a *central surface* of the *second order*, with its *three axes* in the *three directions* α, α', $\alpha\alpha'$ of ρ; and does not cease to represent such a surface, and with such axes, when for $S\rho\phi\rho$ we substitute, as in III., this new expression :

$$\text{XX.} \ldots S\rho\phi\rho - C\rho^2 = S\rho\phi\rho + C\,((S\alpha\rho)^2 + (S\alpha'\rho)^2 + (S\alpha\alpha'\rho)^2) = C' = \text{const.};$$

the *second surface* being in fact *concyclic* (or having the same cyclic planes) with the first, and the *new term*, $- C\rho$, in $\phi\rho$, disappearing under the sign $V.\rho$: so that the *second part* of the Theorem is proved anew.

(4.) It would be useless to dwell here on the *cases*, in which the *surfaces* XIX., XX. come to be of *revolution*, or even to be *spheres*, and when consequently the *directions* of their *axes*, or of ρ in I., become partially or even wholly *indeterminate*. But as an example of the *reduction* of an equation in quaternions to the *form* I., without its *at first* presenting itself under that form, we may take the very simple equation,

$$\text{XXI.} \ldots \rho\iota\rho\kappa = \iota\rho\kappa\rho, \text{ with } \kappa \text{ not } \parallel \iota,$$

which may be reduced (comp. 354, (12.)) to

$$\text{XXII.} \ldots V.\rho V\iota\rho\kappa = 0;$$

and which is accordingly satisfied (comp. 373, XXIX.) by the *three rectangular directions*,

$$\text{XXIII.} \ldots U\iota - U\kappa, \quad V\iota\kappa, \quad U\iota + U\kappa,$$

of the *axes* (abc) of the *ellipsoid*,

$$\text{XXIV.} \ldots T(\iota\rho + \rho\kappa) = \kappa^2 - \iota^2, \qquad 282, \text{XIX.}$$

which is *one* of the surfaces of the *concyclic system* (comp. III.),

$$\text{XXV.} \ldots S\iota\rho\kappa\rho = C\rho^2 + C',$$

as appears from the transformations 336, XI., &c.

(5.) In applying the theorem thus recently proved anew, we have on several occasions used the expression,

$$\text{XXVI.} \ldots d\nu = \phi d\rho, \qquad 410, \text{IV.}$$

in which ν is a vector normal to a surface whereof ρ is the variable vector, and the function ϕ is treated as *self-conjugate* (363).

(6.) It is, however, important to remark that, in order to justify the assertion of this last property, the following *expression of integral form*,

$$\text{XXVII.} \ldots \int S\nu d\rho,$$

must admit of being equated to *some scalar function* of ρ, such as $\frac{1}{2}f\rho$ + const., *without* its being assumed that ρ *itself* is a *function*, of any determinate form, of a scalar variable, t. The *self-conjugation* of the linear and vector function ϕ in XXVI., is the *condition of the existence of the integral* XXVII., considered as representing *such a scalar function* (comp. again 363).

(7.) There are indeed several investigations, in which it is sufficient to regard ν as denoting *some normal vector*, of which only the *direction* is important, and which may therefore be multiplied by an *arbitrary scalar coefficient*, constant *or* variable, without any change in the results (comp. the calculations respecting *geodetic lines*, in the Section III. iii. 5, and many others which have already occurred).

(8.) And there have been other general investigations, such as those regarding the *lines of curvature* on an arbitrary surface, in which $d\nu$ was treated as a self-conjugate function of $d\rho$, while yet (comp. 410, (17.)) the fundamental differential equation $S\nu d\nu d\rho = 0$ was not affected by any such multiplication of ν by n.

(9.) But there are questions in which a factor of this sort may be introduced, with advantage for *some* purposes, while yet it is inconsistent with the *self-conjugation* above mentioned, unless the multiplier n be such as to render the *new expression* $Sn\nu d\rho$ (comp. XXVII.) an *exact differential* of some scalar function of ρ.

(10.) For example, in the theory of *Reciprocal Surfaces* (comp. 412, (21.)), it is convenient to employ the system of the *three* connected equations,

$$\text{XXVIII.} \ldots S\nu\rho = 1, \quad S\nu d\rho = 0, \quad S\rho d\nu = 0; \qquad 373, \text{L. LI.}$$

but when the *length* of ν is determined so as to satisfy the *first* of these equations, ν^{-1} being then the *vector perpendicular* from the origin *on the tangent plane* to the *given* but *arbitrary surface* of which ρ is the vector, while ρ^{-1} is the *corresponding perpendicular* for the *reciprocal surface* with ν for vector, the *differential* $d\nu$ *loses generally its self-conjugate character*, as a linear and vector function of $d\rho$: although it *retains* that character if the scalar function $f\rho$ be *homogeneous*, in the equation $f\rho$ = const. of the original surface, as it is for the case of a *central quadric,** for which $\nu = \phi\rho$, $d\nu = \phi d\rho$, &c., as in former Articles.

(11.) In fact, the introduction of the first equation XXVIII. is equivalent to the multiplication of ν by the factor $n = (S\nu\rho)^{-1}$; and if we write (comp. 410, (16.)),

$$\text{XXIX.} \ldots df\rho = 2S\nu d\rho, \quad d\nu = \phi d\rho, \quad dn = S\sigma d\rho,$$

we shall have this new pair of conjugate linear and vector functions,

$$\text{XXX.} \ldots d \,.\, n\nu = \Phi d\rho = n\phi d\rho + \nu S\sigma d\rho, \quad \text{XXXI.} \ldots \Phi' d\rho = n\phi d\rho + \sigma S\nu d\rho;$$

and these will not be *equal generally*, because we shall not in general have $\sigma \parallel \nu$. But this last parallelism exists in the *case of homogeneity* (10.), because we have then the relations,

$$\text{XXXII.} \ldots 2S\nu\rho = rf\rho, \quad d \,.\, n^{-1} = dS\nu\rho = rS\nu d\rho,$$

if r be the number which represents the *dimension* of $f\rho$ (supposed to be *whole*).

(12.) On the other hand it may happen, that the *differential equation* $S\nu d\rho = 0$ represents a *surface*, or rather a set of surfaces, without the *expression* $S\nu d\rho$ being an *exact differential*, as in (6.); and *then* there necessarily exists a scalar *factor*, or multiplier, n, which *renders* it such a differential.

(13.) For example the differential equation,

$$\text{XXXIII.} \ldots S\gamma\rho d\rho = S\nu d\rho = 0, \text{ with } \text{XXXIV.} \ldots \nu = V\gamma\rho, \quad d\nu = V\gamma d\rho = \phi d\rho,$$

represents an *arbitrary plane* (or a *set of planes*), drawn *through a given line* γ; but the *expression* $S\gamma\rho d\rho$ *itself* is *not* an exact differential, and the *integral* XXVII. represents *no scalar function of* ρ, with the present form of ν, of which the differential $d\nu$ is accordingly a linear function $\phi d\rho$, which is *not conjugate to itself, but to its opposite* (comp. 349, (4.)), so that we have *here* $\phi' d\rho = - \phi d\rho$.

(14.) But if we multiply ν by the *factor*,

$$\text{XXXV.} \ldots n = \nu^{-2} = (V\gamma\rho)^{-2}, \text{ which gives } \text{XXXVI.} \ldots dn = S\sigma d\rho, \quad \sigma = 2n^{-2}\gamma V\gamma\rho,$$

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* It was for this reason that the symbol $T\nu$ was not interpreted *generally* as denoting the reciprocal, P^{-1}, of the length of the perpendicular from the origin on the tangent plane, in the formulæ of 410, 412, 414: although, in several of those formulæ, as in an equation of 409, (3.), that symbol *was* so interpreted, for the *case* of a central surface of the second order.

and therefore $S\gamma\sigma = 0$, $S\rho\sigma = -2n$, then the *new normal vector* $n\nu$, or ν^{-1}, is found to have the *self-conjugate differential*,

$$\text{XXXVII.} \ldots \mathrm{d} \,.\, n\nu = \mathrm{d} \,.\, \nu^{-1} = -\nu^{-1}V\gamma \mathrm{d}\rho \,.\, \nu^{-1} = \Phi \mathrm{d}\rho = \Phi' \mathrm{d}\rho\,;$$

and accordingly the *new expression*,

$$\text{XXXVIII.} \ldots Sn\nu \mathrm{d}\rho = S\nu^{-1}\mathrm{d}\rho = S\frac{\mathrm{d}\rho}{V\gamma\rho}, \text{ with } \gamma \text{ constant,}$$

is easily seen to be an *exact differential*, namely (if $T\gamma = 1$), that of the *angle* which the *plane* of γ and ρ makes with a *fixed plane* through γ: so that, when ν is thus changed to $n\nu$, the *integral* in XXVII. acquires a *geometrical signification*, which is often useful in *physical applications*, since it then represents the *change* of this *angle*, in passing from one position of ρ to another; or the angle through which the *variable plane* of $\gamma\rho$ has *revolved*.

(15.) In fact, the general formula 335, XV. for the *differential of the angle of a quaternion* gives, if we write

$$\text{XXXIX.} \ldots q = \frac{V\gamma\rho}{V\gamma\rho_0}, \quad \gamma = \text{const.}, \quad \rho_0 = \text{const.}, \quad T\gamma = 1,$$

the two connected expressions:

$$\text{XL.} \ldots \mathrm{d}\angle q = \pm S\frac{\mathrm{d}\rho}{V\gamma\rho}; \qquad \text{XLI.} \ldots \int S\frac{\mathrm{d}\rho}{V\gamma\rho} = \pm\Delta\angle(V\gamma\rho : V\gamma\rho_0)\,;$$

which contain the above-stated result, and can easily be otherwise established.

(16.) *In general*, if the linear and vector function $\mathrm{d}\nu = \phi \mathrm{d}\rho$ be *not* self-conjugate, and if the function $\mathrm{d}\,.\,n\nu = \Phi \mathrm{d}\rho$ be formed from it as in (11.), it results from that sub-article, and from 349, (4.), that we may write,

$$\text{XLII.} \ldots (\phi - \phi')\mathrm{d}\rho = 2V\gamma \mathrm{d}\rho, \quad (\Phi - \Phi')\mathrm{d}\rho = 2V\gamma_{,}\mathrm{d}\rho,$$

with the relation,

$$\text{XLIII.} \ldots 2\gamma_{,} = 2n\gamma + V\nu\sigma\,;$$

where γ, $\gamma_{,}$ are *independent of* $\mathrm{d}\rho$, although they *may* depend on ρ *itself*. If then the *new* linear function $\Phi \mathrm{d}\rho$ is to be *self-conjugate*, so that $\gamma_{,} = 0$, we must have

$$\text{XLIV.} \ldots 2n\gamma + V\nu\sigma = 0, \quad \text{and therefore} \quad \text{XLV.} \ldots S\gamma\nu = 0\,;$$

which latter very simple equation, not involving either n or σ, is thus a form, in quaternions, of the *Condition of Integrability** *of the differential equation* $S\nu d\rho = 0$, if the vector γ be deduced from ν as above.

(17.) The *Bifocal Transformation* of $S\rho\phi\rho$, in 360, (2.), has been sufficiently considered in the present Section (III. iii. 7); but it may be useful to remark here, that the *Three Mixed Transformations* of the same scalar function $f\rho$, in the same series of sub-articles, include virtually the whole known theory of the *Modular and Umbilicar Generations* of *Surfaces of the Second Order*.†

(18.) Thus, in the formulæ of 360, (4), if we make $e = 1$, ϵ is the vector of an *Umbilicar Focus* of the surface $f\rho = 1$, and ζ is the vector of a point on the *Umbilicar Directrix* corresponding; whence the *umbilicar focal conic* and *dirigent cylinder* (real or imaginary) can be deduced, as the *loci* of this *point* and *line*.

(19.) Again, by making e_1 and e_3 each $= 1$, in the formulæ of 360, (6.), we obtain *Two Modular Transformations* of the equation of the same surface; ϵ_1, ϵ_3 being vectors of *Modular Foci*, in two distinct planes, and ζ_1, ζ_3 being vectors of points upon the *Modular Directrices* corresponding: whence the *modular focal conics*, and *dirigent cylinders* (real or imaginary), are found by easy eliminations.

(20.) Thus, by assuming that either

$$\text{XLVI.} \ldots S\lambda(\rho - \zeta_1) = 0, \quad S\lambda(\rho - \zeta_3) = 0,$$

or

$$\text{XLVII.} \ldots S\mu(\rho - \zeta_1) = 0, \quad S\mu(\rho - \zeta_3) = 0,$$

---

* If the proposed equation be

$$S\nu d\rho = pdx + qdy + rdz = 0, \quad \text{so that} \quad \nu = -(ip + jq + kr),$$

we easily find that $2\gamma = iP + jQ + kR$, where

$$P = D_z q - D_y r, \quad Q = D_x r - D_z p, \quad R = D_y p - D_x q;$$

the condition of integrability XLV. becomes therefore here,

$$pP + qQ + rR = 0, \text{ which agrees with known results.}$$

[In terms of the operator ∇, the condition is $S\nu\nabla\nu = 0$. For if $nS\nu d\rho = df\rho = -Sd\rho\nabla \cdot f\rho$ is true for all differentials $d\rho$, we must have $\nabla f\rho = -n\nu$. Operating on this by ∇ and remembering that ∇^2 is scalar, $\nabla^2 f\rho = -n\nabla\nu - \nabla n \cdot \nu$ gives, on operating by $S\nu$, the condition as stated above.]

† [The formula of the three mixed transformations are

$$S\rho\phi\rho = g(\rho - \epsilon)^2 + 2S\lambda(\rho - \zeta)S\mu(\rho - \zeta) + e, \qquad \text{360, VII.}$$

$$S\rho\phi\rho = g_1(\rho - \epsilon_1)^2 + (S\lambda_1(\rho - \zeta_1))^2 + (S\mu_1(\rho - \zeta_1))^2 + e_1, \qquad \text{XVI.}$$

and

$$S\rho\phi\rho = g_3(\rho - \epsilon_3)^2 - (S\lambda_3(\rho - \zeta_3))^2 - (S\mu_3(\rho - \zeta_3)^2 + e_3, \qquad \text{XVII.}$$

with obvious conditions for homogeneity in ρ. See pages 545, 546, vol. i.]

the equations 360, XVI., XVII. may be brought to the forms,

$$\text{XLVIII.} \ldots (\rho - \varepsilon_1)^2 = m_1^2 (\rho - \zeta_1)^2, \qquad \text{XLIX.} \ldots (\rho - \varepsilon_3)^2 = m_3^2 (\rho - \zeta_3)^2,$$

with the values,

$$\text{L.} \ldots m_1^2 = 1 - \frac{c^2}{c_1}, \quad \text{and} \quad \text{LI.} \ldots m_3^2 = 1 - \frac{c^2}{c_3};$$

in which c_1, c_2, c_3 are the *three roots* of a certain *cubic* ($M = 0$), or the *inverse squares* of the three scalar *semiaxes* (real or imaginary) of the surface, arranged in algebraically ascending order (357, IX., XX.; 405, (6.), &c.): and m_1, m_3 are the *two* (real or imaginary) *Moduli*, or represent the *modular ratios*, in the *two modes of Modular Generation** corresponding.

(21.) It is obvious that an equation of the form,

$$\text{LII.} \ldots T\phi\rho = C = \text{const.}$$

represents a *central quadric*, if $\phi\rho$ be *any linear*† and vector function of ρ, of the kind considered in the Section III. ii. 6, whether self-conjugate or not; but it requires a little more attention to perceive, that an equation of this *other form*,

$$\text{LIII.} \ldots T(\rho - V.\beta V\gamma\alpha) = T(\alpha - V.\gamma V\beta\rho),$$

represent *such* a surface, whatever the *three vector constants* α, β, γ may be. The discussion of this last *form* would present some circumstances of interest, and might be considered to supply a *new mode of generation*, on which however we cannot enter here.

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* MacCullagh's rule of modular generation, which includes both those modes, was expressed in page 437 of the *Lectures* by an equation of the form,

$$T(\rho - \alpha) = TV.\gamma V\beta\rho;$$

in which the origin is on a directrix, β is the vector of another point of that right line, α is the vector of the corresponding focus, γ is perpendicular to a directive (that is, generally, to a cyclic) plane, ρ is the vector of any point P of the surface, and $\pm S\beta\gamma$ is the constant modular ratio, of the distance $\overline{\text{AP}}$ of P from the focus, to the distance of the same point P from the directrix OB, measured parallel to the directive plane. The new forms (360), above referred to, are however much better adapted to the working out of the various consequences of the construction; but it cannot be necessary, at this stage, to enter into any details of the quaternion transformations: still less need we here pause to give references on a subject so interesting, but by this time so well known to geometers, as that of the modular and umbilicar generations of surfaces of the second order. But it may just be noted, in order to facilitate the applications of the formulæ L. and LI., that if we write, as usual, for *all* the central quadrics, $a^2 > b^2 > c^2$, whether b^2 and c^2 be positive or negative, then the roots c_1, c_2, c_3 coincide, for the *ellipsoid*, with a^{-2}, b^{-2}, c^{-2}; for the *single*-sheeted *hyperboloid*, with c^{-2}, a^{-2}, b^{-2}; and for the *double*-sheeted hyperboloid with b^{-2}, c^{-2}, a^{-2}, (comp. page 206).

† In page 226 the notation,

$$df\rho = 2S\nu d\rho = 2S\phi\rho d\rho, \qquad 409, \text{IV.}$$

was employed for an *arbitrary surface*; but with the understanding that *this function* $\phi\rho$ (comp. 363)

(22.) The surfaces of the second order, considered hitherto in the present Section, have all had the *origin* for *centre*. But if, retaining the significations of ϕ, f, and F, we compare the two equations,

$$\text{LIV.} \ldots f(\rho - \kappa) = C, \quad \text{and} \quad \text{LV.} \ldots f\rho - 2\text{S}\epsilon\rho = C',$$

we shall see (by 362, &c.) that the constants are connected by the two relations,

$$\text{LVI.} \ldots \epsilon = \phi\kappa, \quad C' = C - f\kappa = C - \text{S}\epsilon\kappa = C - F\epsilon;$$

so that the equation,

$$\text{LVII.} \ldots f\rho - 2\text{S}\epsilon\rho = f(\rho - \phi^{-1}\epsilon) - F\epsilon,$$

is an *identity*.

(23.) If then we meet an equation of the form LV., in which (as has been usual) we have still $f\rho = \text{S}\rho\phi\rho$ = a scalar and *homogeneous* function of ρ, of the *second* dimension, we shall know that it represents *generally* a surface of *that* order, with the expression (comp. 347, IX., &c.),

$$\text{LVIII.} \ldots \kappa = \phi^{-1}\epsilon = m^{-1}\psi\epsilon = \textit{Vector of Centre}.$$

(24.) It may happen, however, that the two relations,

$$\text{LIX.} \ldots m = 0, \quad \text{T}\psi\epsilon > 0,$$

exist together; and *then* the *centre* may be said to be at an *infinite distance*, but in a *definite direction*: and the surface becomes a *Paraboloid*, elliptic or hyperbolic, according to conditions which are easy consequences from what has been already shown.

(25.) On the other hand it may happen that the two equations,

$$\text{LX.} \ldots m = 0, \quad \psi\epsilon = 0,$$

are satisfied together; and then the vector κ of the centre acquires, by LVIII., an *indeterminate value*, and the surface becomes a *Cylinder*, as has been already sufficiently exemplified.

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was *generally non-linear*. It may be better, however, as a general rule, to *avoid* writing $\nu = \phi\rho$, *except* for central quadrics; and to confine ourselves to the notation $d\nu = \phi d\rho$, as in some recent and several earlier sub-articles, when we wish, for the sake of *association* with other investigations and results, to treat the *function* ϕ as *linear* (or *distributive*); because we shall thus be at liberty to treat the *surface* as *general*, notwithstanding this property of ϕ. As regards the methods of *generating* a quadric, it may be worth while to look back at the Note to page 204, respecting the *Six Generations of the Ellipsoid*, which were given by the writer in the *Lectures*, with suggestions of a few others, as interpretations of quaternion equations.

(26.) It would be tedious to dwell here on such details; but it may be worth while to observe, that the *general equation of a Surface of the Third Degree* may be thus written:

$$\text{LXI.} \ldots \mathrm{S}q\rho q'\rho q''\rho + \mathrm{S}\rho\phi\rho + \mathrm{S}\gamma\rho + C = 0;$$

C and γ being any scalar and vector constants; $\phi\rho$ any linear, vector, and self-conjugate function; and q, q', q'' any three constant quaternions: while ρ is, as usual, the variable vector of the surface.

(27.) In fact, besides the *one scalar constant*, C, *three* are included in the *vector* γ, and *six others* in the function ϕ (comp. 358); and of the *ten* which remain to be introduced, for the expression of a scalar and *homogeneous* function of ρ, of the *third degree*, the *three versors* $\mathrm{U}q$, $\mathrm{U}q'$, $\mathrm{U}q''$ supply *nine* (comp. 312), and the *tensor* $\mathrm{T}.qq'q''$ is the *tenth*.

(28.) And for the same reason the *monomial equation*,

$$\text{LXII.} \ldots \mathrm{S}q\rho q'\rho q''\rho = 0,$$

with the same significations of q, q', q'', represents the *general Cone of the Third Degree*, or *Cubic Cone*, which has its vertex at the origin of vectors.

(29.) If then we combine this last equation with that of a *secant plane*, such as $\mathrm{S}\epsilon\rho + 1 = 0$, we shall get a quaternion expression for a *Plane Cubic*, or *plane curve of the third degree*: and if we combine it with the equation $\rho^2 + 1 = 0$ of the *unit-sphere*, we shall obtain a corresponding expression for a *Spherical Cubic*,* or for a curve upon a spheric surface, which is cut by an arbitrary great circle in *three pairs* of *opposite points*, real or imaginary.

(30.) Finally, as an example of *sections of surfaces*, represented by *transcendental equations*, let us consider the *Screw Surface*, or *Helicoid*,† of which the vector equation may be thus written (comp. the sub-arts. to 314):

$$\text{LXIII.} \ldots \rho = c(x + a)\alpha + y\alpha^x\gamma, \quad \text{with} \quad \mathrm{T}\alpha - 1, \quad \gamma = \mathrm{V}\alpha\beta, \quad \text{and} \quad y > 0;$$

* Compare the Note to page 38, vol. i.; see also the *theorem* in that page, which contains perhaps a new mode of *generation* of cubic curves in a given plane; or, by an easy modification, of the corresponding curves upon a sphere.

† Already mentioned in pages 419, vol. i., 12, 28, 85. The condition $y > 0$ answers to the supposition that, in the generation of the surface, the perpendiculars from a given helix on the axis of the cylinder are not prolonged beyond that axis.

α being the *unit axis*, while β, γ are two other constant vectors, a, c two scalar constants, and x, y two variable scalars.

(31.) Cutting this surface by the plane of $\beta\gamma$, or supposing that

$$\text{LXIV.} \ldots 0 = \mathrm{S}\gamma\beta\rho = \beta^2\mathrm{S}\alpha\rho - \mathrm{S}\alpha\beta\mathrm{S}\beta\rho, \quad \text{and writing} \quad \text{LXV.} \ldots c = b\mathrm{S}\alpha\beta,$$

we easily find that the scalar and vector equations of what we may call the *Screw Section* may be thus written :

$$\text{LXVI.} \ldots b(x + a) = y\mathrm{S}\,.\,\alpha^{x-1}\,; \qquad \text{LXVII.} \ldots \rho = y(\gamma\mathrm{S}\,.\,\alpha^x - \beta\mathrm{S}\,.\,\alpha^{x-1}).$$

(32.) Derivating these with respect to x, and eliminating β and y', we arrive at the equation,

$$\text{LXVIII.} \ldots \rho = (x + a)\rho' + z\gamma, \quad \text{if} \quad \text{LXIX.} \ldots 2bz = \pi y^2\,;$$

but $z\gamma$ in LXVIII. is the vector of the point, say G, in which the *tangent to the section* at the point (x, y), or P, *intersects the given line* γ, namely the line in the plane of that section which is *perpendicular to the axis* α : we see then, by LXIX., that *this point of intersection depends only on the constant, b, and on the variable, y,* being *independent of the constant, a, and of the variable, x.*

(33.) To interpret this result of calculation, which might have been otherwise found with the help of the expression 372, XII. (with β changed to γ) for the *normal* ν to a screw-surface, we may observe, first, that the equation LXVII., which may be written as follows,

$$\text{LXX.} \ldots \rho = y\mathrm{V}\,.\,\alpha^{x+1}\beta, \quad \text{and gives} \quad \text{LXXI.} \ldots \mathrm{TV}\alpha\rho = y\mathrm{T}\gamma,$$

would represent an *ellipse*, if the coefficient y were treated as *constant*; namely, the section of the right *cylinder* LXXI. by the *plane* LXIV.; the *vector semiaxes* (major and minor) of this ellipse being $y\beta$ and $y\gamma$ (comp. 314, (2.)).

(34.) By assigning a new value to the constant a, we pass to a *new screw surface* (30.), which differs only in *position* from the former, and may be conceived to be formed from it by *sliding* along the *axis* α ; while the value of x, corresponding to a *given* y, will *vary* by LXVI., and thus we shall have a *new screw section* (31.), which will *cross the ellipse* (33.) in a *new point* Q : but the *tangent* to the *section* at this point will *intersect* by (32.) the *minor axis* of the ellipse in the *same point* G as before.

(35.) We shall thus have a *Figure** such as the following (fig. 85.); in which if F be a *focus* of the ellipse BC, and G (as above) the *point of convergence* of the *tangents* to the *screw sections* at the points P, Q, &c., of that ellipse, it is easy to prove, by pursuing the same analysis a little farther, Ist, that the *angle* (g), subtended at this focus F by the minor semiaxis OC, which is also a *radius* (r) of the *cylinder* LXXI., is equal to the *inclination* of the *axis* (a) of that cylinder to the *plane* of the ellipse, as may indeed be inferred from elementary principles; and IInd, what is less obvious, that the *other angle* (h), subtended at the same focus (F) by the interval OG, or by what may be called (with reference to the present construction, in which it is supposed that $b < 0$, or that the angles made by $D_x\rho$ and β with α are either *both acute*, or *both obtuse*) the *Depression* (s) *of the Skew Centre* (G), is equal to the inclination of the same axis (a) to the *helix* on the same cylinder, which is obtained (comp. 314, (10.)) by treating y as constant, in the equation LXIII. of the *Screw Surface*.

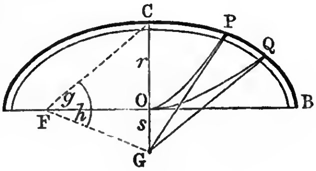

Fig. 85.

SECTION 8.

On a few Specimens of Physical Application of Quaternions, with some Concluding Remarks.

416. It remains to give, according to promise (368), before concluding this work, some examples† of *physical applications* of the present Calculus: and as a first specimen, we shall take the *Statics of a Rigid Body*.

(1.) Let $\alpha_1, \ldots \alpha_n$ be n *Vectors of Application*, and let $\beta_1, \ldots \beta_n$ be n corresponding *Vectors of Force*, in the sense that n forces are applied at the points $A_1, \ldots A_n$ of a *free* but *rigid system*, and are represented as usual by so many right lines from those points, to which *lines* the *vectors* $OB_1, \ldots OB_n$ are *equal*, though drawn from a common origin; and let γ (= OC) be the vector

* Those who are acquainted, even slightly, with the theory of *Oblique Arches* (or *skew bridges*), will at once see that this fig. 85 may be taken as representing rudely such an arch: and it will be found that the *construction* above deduced agrees with the celebrated *Rule of the Focal Excentricity*, discovered practically by the late Mr. Buck. This application of Quaternions was alluded to, in page 620 of the *Lectures*.

† The reader may compare the remarks on *hydrostatic pressure*, in pages 483, 484, vol. i.

of an *arbitrary point* c of space. Then the *Equation* of Equilibrium* of the system or body, under the action of these n applied forces, may be thus written :

$$\text{I.} \ldots \Sigma V(\alpha - \gamma)\beta = 0; \quad \text{or thus,} \quad \text{I}'. \ldots V\gamma\Sigma\beta = \Sigma V\alpha\beta.$$

(2.) The supposed *arbitrariness* (1.) of γ enables us to break up the formula I. or I'., into the *two* vector equations:

$$\text{II.} \ldots \Sigma\beta = 0; \qquad \text{III.} \ldots \Sigma V\alpha\beta = 0;$$

of *each* of which it is easy to assign, as follows, the *physical signification.*

(3.) The equation II. expresses that if the forces, which are applied at the points A_1 . . of the body, were all *transported* to the origin o, their *statical resultant*, or *vector sum*, would be *zero.*

(4.) The equation III. expresses that the resultant of all the *couples*, produced in the usual way by such a transference of the applied forces to the assumed origin, is *null.*

(5.) And the equation I., which as above includes *both* II. and III., expresses that if all the given forces be transported to *any common point* c, the *couples* hence arising will *balance each other* : which is a sufficient condition of equilibrium of the system.

(6.) When we have only the *relation*,

$$\text{IV.} \ldots S(\Sigma\beta \,.\, \Sigma V\alpha\beta) = 0,$$

without $\Sigma\beta$ vanishing, the applied forces have then an *Unique Resultant* = $\Sigma\beta$, acting along the line of which I. or I'. is the equation, with γ for its variable vector.

(7.) And the *physical interpretation* of this condition IV. is, that when the forces are transported to o, as in (3.) and (4.) the *resultant force* is *in the plane* of the *resultant couple.*

(8.) When the equation II., but not III., is satisfied, the applied forces compound themselves into *One Couple,* of which the *Axis* = $\Sigma V\alpha\beta$, whatever may be the position of the *origin.*

(9.) When *neither* II. nor III. is satisfied, we may still propose so to place the *auxiliary point* c, that when the given forces are transferred *to it,*

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* We say here, "*equation*": because the *single quaternion formula*, I. or I'., contains virtually the *six usual scalar equations*, or conditions, of the equilibrium at present considered.

as in (5.), the *resultant force* $\Sigma\beta$ may have the *direction* of the *axis* $\Sigma V(\alpha-\gamma)\beta$ of the *resultant couple*, or else the *opposite* of that direction; so that, in each case, the condition,*

$$\text{V.} \ldots \quad V\frac{\Sigma V(\alpha-\gamma)\beta}{\Sigma\beta} = 0,$$

shall be satisfied by a suitable limitation of the auxiliary vector γ.

(10.) This last equation V. represents therefore the *Central Axis* of the given system of applied forces, with γ for the variable vector of that right line: or the *axis* of the *screw-motion* which those forces *tend* to produce, when they are *not in balance*, as in (1.), and neither tend to produce *translation alone*, as in (6.), nor *rotation alone*, as in (8.).†

(11.) In general, if q be an *auxiliary quaternion*, such that

$$\text{VI.} \ldots \quad q\Sigma\beta = \Sigma V\alpha\beta,$$

its *vector part*, Vq, is equal by (V.) to the *Vector-Perpendicular*, let fall from the origin *on the central axis*; while its *scalar part*, Sq, is easily proved to be the *quotient*,‡ of what may be called the *Central Moment*, divided by the *Total Force*: so that $Vq = 0$ when the central axis passes *through the origin*, and $Sq = 0$ when there exists an *unique resultant*.

(12.) When the total force $\Sigma\beta$ does not vanish, let Q be a *new auxiliary quaternion*, such that

$$\text{VII.} \ldots \quad Q = \frac{\Sigma\alpha\beta}{\Sigma\beta} = q + \frac{\Sigma S\alpha\beta}{\Sigma\beta},$$

with

$$\text{VIII.} \ldots \quad c = SQ = Sq, \quad \text{and} \quad \text{IX.} \ldots \quad \gamma = \text{OC} = VQ,$$

for its scalar and vector parts; then $c\Sigma\beta$ represents, both in quantity and in direction, the *Axis of the Central Couple* (9.), and γ is the vector of a point C which is *on the central axis* (10.), considered as a right line having *situation in space*: while the *position* of this *point* on this *line* depends only on the *given system* of applied forces, and does not vary with the assumed *origin* O.

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* The equation V. may also be obtained from the condition,

$$\text{V}'. \ldots \quad T\Sigma V(\alpha-\gamma)\beta = \text{a } minimum,$$

when γ is treated as the only variable vector; which answers to a known property of the *Central Moment*.

† [In the expressive language of Sir Robert S. Ball the forces constitute a *wrench* upon a screw.]

‡ [This scalar has been aptly termed by Sir Robert Stawell Ball the *pitch* of the screw.]

(13.) Under the same conditions, we have the transformations,

$$\text{X.} \ldots \Sigma\alpha\beta = (c + \gamma)\Sigma\beta; \qquad \text{XI.} \ldots \mathrm{T}\Sigma\alpha\beta = (c^2 - \gamma^2)^{\frac{1}{2}}\mathrm{T}\Sigma\beta;$$

$$\text{XII.} \ldots \Sigma\mathrm{V}\alpha\beta = c\Sigma\beta + \mathrm{V}\gamma\Sigma\beta; \qquad \text{XIII.} \ldots (\Sigma\mathrm{V}\alpha\beta)^2 = c^2(\Sigma\beta)^2 + (\mathrm{V}\gamma\Sigma\beta)^2;$$

whereof XII. contains the known *law*, according to which the *axis* of the *couple* (4.), obtained by transferring all the forces to an assumed point O, *varies* generally in *quantity* and in *direction* with the *position* of that point: while XIII. expresses the known corollary from that law, in virtue of which the *quantity alone*, or the *energy* ($\mathrm{T}\Sigma\mathrm{V}\alpha\beta$) of the couple here considered, is the *same* for all the points O of any one *right cylinder*, which has the *central axis* of the system for its *axis of revolution*.

(14.) If we agree to call the *quaternion product* PA . AA′ the *quaternion moment*, or simply the *Moment*, of the *applied force* AA′ at A, *with respect to the Point* P, the *quaternion sum* $\Sigma\alpha\beta$ in X. may then be said to be the *Total Moment* of the given system of forces, with respect to the assumed *origin* O; and the formula XI. expresses that the *tensor* of this *sum*, or what may be called the *quantity* of this total moment, is *constant* for all points O which are situated *on any one spheric surface*, with the point C determined in (12.) for its *centre*: being also a *minimum* when O is placed at that point C *itself*, and being then equal to what has been already called the *central moment*, or the energy of the central couple.

(15.) For these and other reasons, it appears not improper to call *generally* the point C, above determined, the *Central Point*, or simply the *Centre*, of the given system of applied forces, when the total force does not vanish; and accordingly in the particular but important case, when all those forces are *parallel*, without their *sum* being *zero*, so that we may write,

$$\text{XIV.} \ldots \beta_1 = b_1\beta, \ldots \beta_n = b_n\beta, \quad \mathrm{T}\Sigma\beta > 0,$$

the scalar c in (12.) vanishes, and the vector γ becomes (comp. Art. 97 on *bary-centres*),

$$\text{XV.} \ldots \mathrm{OC} = \gamma = \frac{b_1\alpha_1 + \ldots + b_n\alpha_n}{b_1 + \ldots + b_n} = \frac{\Sigma b\alpha}{\Sigma b};$$

so that the *point* C, thus determined, is *independent of the common direction* β, and coincides with what is usually called the *Centre of Parallel Forces*.

(16.) The conditions of equilibrium (1.), which have been already expressed by the formula I., may also be included in this other quaternion equation,

$$\text{XVI.} \ldots \textit{Total Moment} = \Sigma\alpha\beta = \textit{a scalar constant},$$

of which the *value* is *independent of the origin*; and which, with its sign changed represents what may perhaps be called the *Total Tension* of the system.*

(17.) *Any infinitely small change*, in the *position of a rigid body*, is equivalent to the alteration of each of its vectors α to another of the form,

$$\text{XVII.} \ldots \alpha + \delta\alpha = \alpha + \epsilon + \mathrm{V}\iota\alpha,$$

ϵ and ι being *two* arbitrary but infinitesimal vectors, which do not vary in the passage from one point A of the body to another†: and thus the *conditions of equilibrium* (1.) may be expressed by this other formula,

$$\text{XVIII.} \ldots \Sigma\mathrm{S}\beta\delta\alpha = 0,$$

which contains, for the case here considered, the *Principle of Virtual Velocities*, and admits of being extended easily to other cases of Statics.

417. The general *Equation of Dynamics* may be thus written,

$$\text{I.} \ldots \Sigma m\mathrm{S}\,(\mathrm{D}_t^2\alpha - \xi)\delta\alpha = 0,$$

with significations of the symbols which will soon be stated; but as we only propose (416) to give here some *specimens* of physical application, we shall aim chiefly, in the following sub-articles, at the deduction of a few formulæ and theorems, respecting *Axes* and *Moments* of *Inertia*, and subjects there-with connected.

(1.) In the formula I., α is the vector of position, at the time t, of an element m of the system; $\delta\alpha$ is any variation of that vector, geometrically compatible with the mutual connexions between the parts of that system;

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* [This of course is what Clausius has since called the *virial.*]

† [Compare pages 83–85, and observe that the transformation

$$\epsilon = \epsilon\iota^{-1} \,.\, \iota = (\mathrm{S}\epsilon\iota^{-1} + \mathrm{V}\epsilon\iota^{-1})\iota = (p + \varpi)\iota, \; (p = \mathrm{S}\epsilon\iota^{-1}, \; \varpi = \mathrm{V}\epsilon\iota^{-1})$$

shows that the displacement of the body may be accomplished by a rotation round the axis whose equation is $\rho = \varpi + x\iota$, accompanied by a proportional translation along that axis. This screw translation is called a *twist* by Sir Robert Ball. In the same way a moving body is said to have a twist-velocity on an instantaneous screw.]

the vector $m\xi$ represents a moving force, or ξ an accelerating force, which acts on the element m of mass; D and S are marks, as usual, of derivating and taking the scalar; and the summation denoted by Σ extends to all the elements, and is generally equivalent to a triple integration, or to an addition of triple integrals in space. And the formula is obtained (comp. 416, (17.)), by a combination of D'Alembert's principle with the principle of virtual velocities, which is analogous to that employed in the *Mécanique Analytique* by Lagrange.

(2.) For the case of a *free* but *rigid body*, we may substitute for δa the expression $\varepsilon + V\iota a$, assigned by 416, XVII.; and then, on account of the arbitrariness of the two infinitesimal vectors ε and ι, the formula I. breaks up into the two following,

$$\text{II.} \ldots \Sigma m(D_t^2 a - \xi) = 0; \qquad \text{III.} \ldots \Sigma m V a(D_t^2 a - \xi) = 0;$$

which correspond to the two statical equations 416, II. and III., and contain respectively the law of motion of the centre of gravity, and the law of description of areas.

(3.) If the body have a *fixed point*, which we may take for the origin o, we eliminate the reaction at that point, by attending only to the equation III.; and may then express the connexions between the elements m by the formula,

$$\text{IV.} \ldots D_t a = V\iota a, \quad \text{whence} \quad \text{V.} \ldots D_t^2 a = \iota V\iota a - V a D_t \iota;$$

ι being the *Vector-Axis of instantaneous Rotation* of the body, in the sense that its *versor* $U\iota$ represents the *direction* of the *axis*, and that its *tensor* $T\iota$ represents the *angular velocity*, of such rotation at the time t.

(4.) By V., the equation III. becomes,

$$\text{VI.} \ldots \Sigma m a V a D_t \iota = \Sigma m(V\iota a S\iota a - V a \xi);$$

and other easy combinations give the laws of areas and living force, under the forms,

$$\text{VII.} \ldots \Sigma m a D_t a - \Sigma m V \textstyle\int a\xi dt = \gamma = \text{a constant vector};$$

$$\text{VIII.} \ldots \tfrac{1}{2}\Sigma m(D_t a)^2 - \Sigma m S \textstyle\int \iota a \xi dt = c = \text{a constant scalar}.$$

(5.) When the applied forces vanish, or balance each other, or more generally when they compound themselves into a single force acting at the fixed point, so that in each case the condition

$$\text{IX.} \ldots \Sigma m V a \xi = 0$$

is satisfied, the equations (4.) are simplified; and if we introduce a linear, vector, and self-conjugate function ϕ, such that

$$\text{X.} \ldots \phi\iota = \Sigma m a \mathrm{V} a\iota = \iota \Sigma m a^2 - \Sigma m a \mathrm{S} a\iota,$$

and write h^2 for $-2c$, they take the forms,

$$\text{XI.} \ldots \phi \mathrm{D}_t\iota + \mathrm{V}\iota\phi\iota = 0; \quad \text{XII.} \ldots \phi\iota + \gamma = 0; \quad \text{XIII.} \ldots \mathrm{S}\iota\phi\iota = h^2;$$

γ and h being two real constants, of the vector and scalar kinds, connected with each other and with ι by the relation,

$$\text{XIV.} \ldots \mathrm{S}\iota\gamma + h^2 = 0; \quad \text{also} \quad \text{XV.} \ldots \phi \mathrm{D}_t\iota = \mathrm{V}\iota\gamma.$$

It may be added that γ is now the *vector sum* of the doubled *areal velocities* of all the elements of the body, multiplied each by the mass m of that element, and each represented by a *right line* $a\mathrm{D}_t a$ perpendicular to the plane of the area described round the fixed point o in the time dt; and that h^2 is the *living force*, or *vis viva* of the body, namely the *positive sum* of all the products obtained by multiplying each element m by the *square* of its *linear velocity*, regarded as a *scalar* ($\mathrm{TD}_t a$).*

* [The following elegant method of dealing with a body rotating about a fixed point is due to Clifford (*Dynamic*, vol. ii., page 75). If ρ is the vector to any moving point, $\mathrm{D}_t\rho$ its velocity, and $\mathrm{d}_t\rho$ its velocity relative to the body, it is geometrically evident that

$$\mathrm{D}_t\rho = \mathrm{d}_t\rho + \mathrm{V}\iota\rho.$$

This may be regarded as a formula of differentiation connecting D_t and d_t, as ρ may be any vector whatever. In particular, replacing ρ by ι,

$$\mathrm{D}_t\iota = \mathrm{d}_t\iota,$$

or the rate of change of the angular velocity is the same whether referred to fixed axes or to axes moving in the body. (Compare Routh's *Rigid Dynamics*, Part I., Arts. 249, 250.)

Again, from fundamental principles the rate of change of angular momentum of the body about the fixed point is equal to the impressed couple about that point. If then η is the couple, and if $-\phi\iota$ is the angular momentum (retaining Hamilton's notation) we have, on replacing ρ by $-\phi\iota$, the dynamical equation

$$-\eta = \mathrm{D}_t\phi\iota = \mathrm{d}_t\phi\iota + \mathrm{V}\iota\phi\iota. \quad \text{But} \quad \mathrm{d}_t\phi\iota = \phi\mathrm{d}_t\iota,$$

because the function ϕ does not change relatively to the body, so Euler's equations are contained in

$$\phi\mathrm{d}_t\iota + \mathrm{V}\iota\phi\iota = -\eta.$$

As another example, on replacing ρ by $\mathrm{D}_t\rho$ or by its equivalent $\mathrm{d}_t\rho + \mathrm{V}\iota\rho$, we deduce the formula of acceleration

$$\begin{aligned} \mathrm{D}_t^2\rho &= \mathrm{d}_t(\mathrm{d}_t\rho + \mathrm{V}\iota\rho) + \mathrm{V}\iota(\mathrm{d}_t\rho + \mathrm{V}\iota\rho) \\ &= \mathrm{d}_t^2\rho + 2\mathrm{V}\iota\mathrm{d}_t\rho + \mathrm{V}\mathrm{d}_t\iota\rho + \mathrm{V}\iota\mathrm{V}\iota\rho. \end{aligned}$$

If ρ is the vector to a point fixed in the body, this becomes $\mathrm{D}_t^2\rho = \mathrm{V}\mathrm{d}_t\iota\rho + \mathrm{V}\iota\mathrm{V}\iota\rho$, and on taking moments and summing for the various elements of the body, the dynamical equations may be easily derived anew.]

(6.) When ι is regarded as a variable vector, the equation XIII. represents an *ellipsoid*, which is *fixed in the body*, but *moveable with it*; and the equation XIV. represents a *tangent plane* to this ellipsoid, which plane is *fixed in space*, but *changes* in general its position relatively to the *body*. And thus the *motion* of that body may generally be conceived, as was shown by Poinsot, to be performed by the *rolling* (*without gliding*) of an *ellipsoid upon a plane*; the former *carrying the body* with it, while its *centre* O remains *fixed*: and the *semidiameter* (ι) *of contact* being the *vector-axis* (3.) of *instantaneous rotation*.

(7.) The ellipsoid XIII. may be called, perhaps, the *Ellipsoid of Living Force*, on account of the signification (5.) of the constant h^2 in its equation; and the fixed plane XIV., on which it rolls, is parallel to what may be called the *Plane of Areas* ($S\iota\gamma = 0$): no use whatever having hitherto been made, in this investigation, of any *axes* or *moments* of *inertia*. But if we here admit the usual definition of such a moment, we may say that the *Moment of Inertia of the body*, with respect to *any axis* ι through the fixed point, is equal to the *living force* h^2 *divided by the square** *of the semidiameter* $T\iota$ of the ellipsoid XIII.; because this moment is,

$$\text{XVI.} \ldots \Sigma m(TV\alpha U\iota)^2 = \iota^{-2}\Sigma m(V\iota\alpha)^2 = -S\iota^{-1}\phi\iota = h^2 T\iota^{-2}.$$

(8.) The equations XII. and XIII. give,

$$\text{XVII.} \ldots 0 = \gamma^2 S\iota\phi\iota - h^2(\phi\iota)^2 = S\iota\nu, \quad \text{if} \quad \text{XVIII.} \ldots \nu = \gamma^2\phi\iota - h^2\phi^2\iota;$$

and this equation XVII. represents a *cone* of the second degree, fixed in the body (comp. (6.)), but moveable with it, of which the axis ι is always a *side*, and to which the *normal*, at any point of that side, has the direction of the line ν. But it follows from XI., or from XII. XV., and from the properties of the function ϕ, that $D_t\iota$ is perpendicular to both $\phi\iota$ and $\phi^2\iota$, and therefore also by XVIII. to ν; the cone XVII. is therefore *touched*, along the side ι, by that *other cone*, which is the *locus in space* of the *instantaneous axis* of rotation. We are then led by this simple quaternion analysis, to a *second*

* Hence it may easily be inferred, with the help of the general *construction of an ellipsoid* (217, (6.)), illustrated by fig. 53 in page 234, vol. i., and page 184, that for *any solid body*, and *any given point* A thereof, there can always be found (indeed in more ways than one) *two other points*, B and C, which are likewise *fixed in the body*, and are such that the *square-root of the moment of inertia*, round *any axis* AD, is geometrically *constructed by the line* BD, if the point D be determined on the axis, by the condition that A and D shall be *equally distant* from C. This theorem, with some others here reproduced, was given in the Abstract of a Paper read before the Royal Irish Academy on the 10th of January, 1848, and was published in the *Proceedings* of that date.

representation of the *motion of the body,* which also was proposed by Poinsot: namely, as the *rolling of one cone on another.*

(9.) To treat briefly by quaternions some of MacCullagh's results on this subject, it may be noted that the *line* γ, though *fixed in space,* describes *in the body* a *cone* of the second degree, of which the equation is, by what precedes,

$$\text{XIX.} \ldots g^2 S\gamma\phi^{-1}\gamma + h^2\gamma^2 = 0, \quad \text{if} \quad \text{XX.} \ldots g = T\gamma, \quad \text{or} \quad \text{XXI.} \ldots \gamma^2 + g^2 = 0;$$

while, if we write $\gamma = \text{OC}$, the *point* C is indeed fixed in *space,* but describes a *sphero-conic* in the *body,* which is part of the common intersection of the *cone* XIX., the *sphere* XXI., and the *reciprocal ellipsoid* (comp. XIII.),

$$\text{XXII.} \ldots S\gamma\phi^{-1}\gamma = h^2.$$

(10.) Also, the *normal* to the *new cone* (9.), at any point of the side γ, has the direction of $g^2\phi^{-1}\gamma + h^2\gamma$, or of $\iota + h^2\gamma^{-1}$ (comp. XIV.); and if a line in this direction be drawn through the fixed point O, it will be the *side of contact* of the *plane of areas* (7.), with the *cone of normals* at O to the cone XIX.; which *last* (or *reciprocal*) *cone rolls on that plane of areas.*

(11.) As regards the *Axes of Inertia,* it may be sufficient here to observe that if the body revolve round a *permanent axis,* and with a *constant velocity,* the *vector axis* ι is constant; and must therefore satisfy the equation,

$$\text{XXIII.} \ldots V\iota\phi\iota = 0, \quad \text{because} \quad \text{XXIV.} \ldots D_t\iota = 0;$$

it has therefore in general (comp. 415) one or other of *Three Real and Rectangular Directions,* determined by the condition XXIII.: namely, those of the *Axes of Figure* of either of the two *Reciprocal Ellipsoids,* XIII. XXII.

(12.) And the *Three Principal Moments,* say A, B, C, corresponding to those *three principal axes,* are by XVI. the three scalar values of $-\iota^{-1}\phi\iota$; so that the *symbolical cubic* (350) in ϕ may be thus written,

$$\text{XXV.} \ldots (\phi + A)(\phi + B)(\phi + C) = 0.$$

(13.) Forming then this symbolical cubic by the general method of the Section III. ii. 6, we find that the *three moments* A, B, C, are the *three roots* (always *real,* by this analysis) of the *algebraic* and *cubic equation,*

$$\text{XXVI.} \ldots A^3 - 2n^2A^2 + (n^4 + n'^2)A - (n^2n'^2 - n''^2) = 0;$$

in which, n^2, n'^2, n''^2 are three positive scalars, namely,

$$\text{XXVII.} \ldots n^2 = -\Sigma m\alpha^2; \quad n'^2 = -\Sigma mm'(\mathrm{V}\alpha\alpha')^2; \quad n''^2 = \Sigma mm'm''(\mathrm{S}\alpha\alpha'\alpha'')^2;$$

and the combination $n^2n'^2 - n''^2$ is another positive scalar, of which the value may be thus expressed,

$$\text{XXVIII.} \ldots ABC = n^2n'^2 - n''^2 = \Sigma m^2m'\alpha^2(\mathrm{V}\alpha\alpha')^2 \\ + 2\Sigma mm'm''(\mathrm{T}\alpha\alpha'\mathrm{T}\alpha'\alpha''\mathrm{T}\alpha''\alpha + \mathrm{S}\alpha\alpha'\mathrm{S}\alpha'\alpha''\mathrm{S}\alpha''\alpha),$$

if α, α', α'', &c. be the vectors of the mass-elements m, m', m'', &c.

(14.) And because the equation XXV. gives this other symbolical result,

$$\text{XXIX.} \ldots -ABC\phi^{-1} = \phi^2 + (A + B + C)\phi + BC + CA + AB,$$

it follows that

$$\text{XXX.} \ldots \phi^{-1}0 = 0;$$

and therefore, by XV., &c., that if a body, with a fixed point, &c., *begin* to revolve round one of its three principal axes of inertia, it will *continue* to revolve round that axis, with an *unchanged velocity* of rotation.

(15.) It has hitherto been supposed, that all the moments of inertia are referred to axes passing through *one point* o of the body; but it is easy to remove this restriction. For example, if we denote the moment XVI., by I_0, and if I_ω be the corresponding moment for an axis *parallel* to ι, but drawn through a *new point* Ω, of which the vector is ω, then

$$\text{XXXI.} \ldots I_\omega = \iota^{-2}\Sigma m(\mathrm{V}\iota(\alpha - \omega))^2 = I_0 + 2\Sigma m \,.\, \mathrm{S}(\omega\iota^{-1}\mathrm{V}\iota\kappa) + p^2\Sigma m,$$

if

$$\text{XXXII.} \ldots \kappa\Sigma m = \Sigma m\alpha, \quad \text{and} \quad \text{XXXIII.} \ldots p = \mathrm{TV}\omega\mathrm{U}\iota,$$

so that κ is the vector of the *centre of inertia* (or of gravity) of the body, and p is the *distance* between the two parallel axes.

(16.) If then we suppose that the condition

$$\text{XXXIV.} \ldots \mathrm{V}\iota\kappa = 0$$

is satisfied, that is, if the axis ι pass through the centre of inertia, we shall have the very simple relation,

$$\text{XXXV.} \ldots I_\omega = I_0 + p^2\Sigma m;$$

which agrees with known results.*

* [In like manner, if

$$\phi_\omega\iota = \Sigma m(\alpha - \omega)\,\mathrm{V}(\alpha - \omega)\iota,$$

we find

$$\phi_\omega\iota = \phi_0\iota - \mathrm{V}(\kappa\mathrm{V}\omega\iota + \omega\mathrm{V}\kappa\iota)\Sigma m + \omega\mathrm{V}\omega\iota\Sigma m.$$

418. As a *third specimen* of physical applications of quaternions, we propose to consider briefly the motions of a *System of Bodies*, $m, m', m'', \ldots$ regarded as free material points, of which the variable vectors are $\alpha, \alpha', \alpha'', \ldots$ and which are supposed to attract each other according to the law of the inverse square: the fundamental formula employed being the following,

$$\text{I.} \ldots \Sigma m \text{S} D_t^2\alpha\delta\alpha + \delta P = 0, \quad \text{if} \quad \text{II.} \ldots P = \Sigma \frac{mm'}{\text{T}(\alpha - \alpha')}:$$

P thus denoting the *Potential* (or *force-function*) of the system, and the variations $\delta\alpha, \delta\alpha', \ldots$ being infinitesimal, but otherwise arbitrary.

(1.) To deduce the formula I., with the signification II. of P, from the general equation 417, I. of dynamics, we have first, for the case of two bodies, the following expressions for the accelerating forces,

$$\text{III.} \ldots \xi = \frac{m'}{(\alpha - \alpha')r}, \quad \xi' = \frac{m}{(\alpha' - \alpha)r}, \quad \text{if} \quad r = \text{T}(\alpha - \alpha');$$

whence follows the transformation,*

$$\text{IV.} \ldots -\text{S}(m\xi\delta\alpha + m'\xi'\delta\alpha') = \frac{-mm'}{r}\text{S}\frac{\delta(\alpha - \alpha')}{\alpha - \alpha'} = \delta\frac{mm'}{r};$$

a result easily extended, as above. If the *law* of attraction were supposed different, there would be no difficulty in modifying the expression for the potential accordingly.

(2.) *In general*, when a *scalar*, f (as here P), is a *function* of one or more *vectors*, $\alpha, \alpha', \ldots$ its *variation* (or differential) can be expressed as a *linear and scalar function* of *their* variations (or differentials), of the form $\text{S}\beta\delta\alpha + \text{S}\beta'\delta\alpha' + \ldots$ (or $\Sigma\text{S}\beta d\alpha$); in which $\beta, \beta' \ldots$ are certain *new* and *finite vectors*, and are themselves generally *functions* of $\alpha, \alpha', \ldots$, *derived*

When the point o is at the centre of inertia, so that κ is zero, this takes the simple form

$$\phi_\omega\iota = \phi_0\iota + M\omega\text{V}\omega\iota = (\phi_0 - M\omega\text{S}\omega)\iota + M\omega^2\iota,$$

M being the mass of the body. It is evident that the linear functions ϕ_ω and $\phi_0 - M\omega\text{S}\omega$ have the same principal directions, and comparing XXI., page 199, and the Note to page 224, it appears that these directions are the normals to the three confocals

$$\text{S}\omega(M^{-1}\phi_0 + h)^{-1}\omega = 1$$

which pass through the point Ω (Binet's theorem). The distribution of the assemblage of these principal axes has virtually been considered in the Note to page 245.]

* It may not be useless here to compare the expression in page 461, vol. i., for the *differential of a proximity*.

from the *given* scalar function f. And we shall find it convenient to *extend* the *Notation** *of Derivatives*, so as to denote these *derived vectors* β, β', &c., by the *symbols*, $D_\alpha f$, $D_{\alpha'} f$, &c. In this manner we shall be able to write,

$$\text{V.} \ldots \delta P = \Sigma S(D_\alpha P \, . \, \delta\alpha);$$

and the differential equations of motion of the bodies m, m', m'', .. will take by I. the forms:

$$\text{VI.} \ldots m D_t^2 \alpha + D_\alpha P = 0, \quad m' D_t^2 \alpha' + D_{\alpha'} P = 0, \text{ \&c.};$$

or more fully,

$$\text{VII.} \ldots D_t^2 \alpha = \frac{m'}{(\alpha - \alpha') T(\alpha - \alpha')} + \frac{m''}{(\alpha - \alpha'') T(\alpha - \alpha'')} + \ldots; \text{ \&c.}$$

(3.) The laws of the centre of gravity, of areas, and of living force, result immediately from these equations, under the forms,

$$\text{VIII.} \ldots \Sigma m D_t \alpha = \beta; \qquad \text{IX.} \ldots \Sigma m V \alpha D_t \alpha = \gamma;$$

and

$$\text{X.} \ldots T = -\tfrac{1}{2} \Sigma m (D_t \alpha)^2 = P + H;$$

in which β, γ are constant vectors, H is a constant scalar, and $2T$ is the living force of the system (comp. 417, (5.)).

(4.) One mode (comp. 417, (2.)) of deducing the three equations, of which these are the first integrals, is the following. To obtain VIII., change every variation $\delta\alpha$ in I. to one *common* but *arbitrary* infinitesimal vector, ϵ. For IX., change $\delta\alpha$ to $V\iota\alpha$, $\delta\alpha'$ to $V\iota\alpha'$, &c.; ι being *another* arbitrary and infinitesimal vector. Finally, to arrive at X., change *variations* to *differentials* ($\delta\alpha$ to $d\alpha$, &c.), and integrate once, as for the two former equations, with respect to the time t.

(5.) The formula I. admits of being *integrated by parts*, without any restriction on the *variations* $\delta\alpha$, by means of the general transformation,

$$\text{XI.} \ldots S(D_t^2 \alpha \, . \, \delta\alpha) = D_t S(D_t \alpha \, . \, \delta\alpha) - \tfrac{1}{2} \delta \, . \, (D_t \alpha)^2,$$

combined with the introduction of the following definite integral (comp. X.),

$$\text{XII.} \ldots F = \int_0^t (P + T) dt.$$

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* In this extended notation, such a formula as $df\rho = 2S\nu d\rho$ would give, $\nu = \tfrac{1}{2} D_\rho f\rho$.

(6.) In fact, if we denote by a_0, a'_0, . . the *initial values* of the vectors a, a', . . or their values when $t = 0$, and by $D_0 a$, $D_0 a'$, . . the corresponding values of $D_t a$, $D_t a'$, . ., we shall thus have, as a first integral of the equation I., the formula,

$$\text{XIII.} \ldots \Sigma m S(D_t a \cdot \delta a - D_0 a \cdot \delta a_0) + \delta F = 0;$$

in which no variation δt is assigned to t, and which conducts to important consequences.

(7.) To draw from it some of these, we may observe that if the masses m, m', . . be treated as constant and known, the complete integrals of the equations VI. or VII. must be conceived to give what may be called the *final vectors* of *position* a, a', . . and of *velocity* $D_t a$, $D_t a'$, . . in terms of the *initial vectors* a_0, a'_0, . . $D_0 a$, $D_0 a'$, . . and of the *time*, t: whence, conversely, we may conceive the initial vectors of velocity to be expressible as functions of the initial and final vectors of position, and of the time. In this way, then, we are led to consider P, T, and F as being *scalar functions* (whether we are or are not prepared to *express* them as such), of a, a', . . a_0, a'_0, . . and t; and thus, by (2.), the recent formula XIII. breaks up into the two following systems of equations:

$$\text{XIV.} \ldots mD_t a + D_a F = 0, \quad m'D_t a' + D_{a'} F = 0, \text{ \&c.};$$

and

$$\text{XV.} \ldots -mD_0 a + D_{a_0} F = 0, \quad -m'D_0 a' + D_{a'_0} F = 0, \text{ \&c.};$$

whereof the *former* may be said to be *intermediate integrals*, and the *latter* to be *final integrals*, of the *differential equations of motion* of the system, which are included in the formula I.

(8.) In fact, the equations XIV. do not involve the final *vectors of acceleration* $D_t^2 a$, . . as the differential equations VI. or VII. had done; and the equations XV. express, at least theoretically, the dependence of the *final* vectors of *position* a, . . on the *time*, t, and on the *initial* vectors of position a_0, . . and of *velocity* $D_0 a$, . . as by (7.) the *complete* integrals ought to do. And on account of these and other important properties, the function here denoted by F may be called the *Principal* Function of Motion of the System.*

* This function was in fact so called, in two Essays by the present writer, "On a General Method in Dynamics," published in the *Philosophical Transactions* (London), for the years 1834 and 1835; although of course *coordinates*, and not *quaternions*, were *then* employed, the latter not having been discovered until 1843: and the notation S, since adopted for *scalar*, was then used instead of F.

(9.) If the initial vectors α_0, . . and $D_0\alpha$, . . be *given*, that is, if we consider the actual *progress* in space of the mutually attracting system of masses m, . . from one set of positions to another, then the function F depends upon the *time* alone; and by its definition XII., its *rate* or *velocity* of increase, or its *total derivative* with respect to t, is thus expressed,

$$\text{XVI.} \ldots D_tF = P + T.$$

(10.) But we may inquire what is the *partial derivative*, say (D_tF), of the same definite integral F, when regarded (7.) as a function of the final and initial vectors of position α, . . α_0, . . which involves *also* the *time explicitly*, and is *now* to be derivated with respect *only* to *that* variable t, *as if* the final vectors α, . . were *constant*: whereas in fact those vectors *alter* with the time, in the course of any actual *motions* of the system.

(11.) For this purpose, it is sufficient to observe that the *part* of the *total derivative* D_tF, which arises from the last-mentioned changes of α, . . is (by XIV. and X.),

$$\text{XVII.} \ldots \Sigma S(D_\alpha F \,.\, D_t\alpha) = 2T;$$

and therefore (by XVI. and X.), that the *remaining part* must be,

$$\text{XVIII.} \ldots (D_tF) = P - T = -H.$$

(12.) The *complete variation* of the function F is therefore (comp. XIII.), when t as well as α, . . and α_0, . . is treated as *varying*,

$$\text{XIX.} \ldots \delta F = -H\delta t - \Sigma m S D_t\alpha\delta\alpha + \Sigma m S D_0\alpha\delta\alpha_0.$$

(13.) And hence, with the help of the equations X. XIV. XV., it is easy to infer that the principal function F must satisfy the two following *Partial Differential Equations in Quaternions*:

$$\text{XX.} \ldots (D_tF) - \tfrac{1}{2}\Sigma m^{-1}(D_\alpha F)^2 = P;$$

$$\text{XXI.} \ldots (D_tF) - \tfrac{1}{2}\Sigma m^{-1}(D_{\alpha_0} F)^2 = P_0;$$

in which P_0 denotes the initial value of the potential P.

(14.) If we write

$$\text{XXII.} \ldots V = \int_0^t 2T\mathrm{d}t,$$

so that V represents what is called the *Action*, or the accumulated living force, of the system during the time t, then by X. and XII. the two definite integrals F and V are connected by the very simple relation,

$$\text{XXIII.} \ldots V = F + tH;$$

whence by XIX. the *complete variation* of V, considered as a function of the final and initial vectors of position, and of the constant H of living force, which does not explicitly involve the time, may be thus expressed,

$$\text{XXIV.} \ldots \delta V = t\delta H - \Sigma m \mathrm{SD}_t\alpha\delta\alpha + \Sigma m\mathrm{SD}_0\alpha\delta\alpha_0.$$

(15.) The *partial derivatives* of this new function V, which is for *some* purposes more useful than F, and may be called, by way of distinction from it, the *Characteristic* Function* of the motion of the system, are therefore,

$$\text{XXV.} \ldots \mathrm{D}_\alpha V = -m\mathrm{D}_t\alpha, \text{ \&c.};$$

$$\text{XXVI.} \ldots \mathrm{D}_{\alpha_0} V = +m\mathrm{D}_0\alpha, \text{ \&c.};$$

and

$$\text{XXVII.} \ldots \mathrm{D}\ \ V = t.$$

(16.) The *intermediate integrals* (7.) of the differential equations of motion, which were before expressed by the formulæ XIV., may now, somewhat less simply, be regarded as the result of the elimination of H between the formulæ XXV. XXVII.; and the *final integrals* of those equations VI. or VII., which were expressed by XV., are now to be obtained by eliminating the same constant H between the recent equations XXVI. XXVII.

(17.) The *Characteristic Function*, V, is obliged (comp. (13.)) to satisfy the two following *partial differential equations*,

$$\text{XXVIII.} \ldots \tfrac{1}{2}\Sigma m^{-1}(\mathrm{D}_\alpha V)^2 + P + H = 0;$$

$$\text{XXIX.} \ldots \tfrac{1}{2}\Sigma m^{-1}(\mathrm{D}_{\alpha_0} V)^2 + P_0 + H = 0;$$

it vanishes, like F, when $t = 0$, at which epoch $\alpha = \alpha_0$, $\alpha' = \alpha'_0$, &c.; *each* of these two functions, F and V, depends *symmetrically* on the initial and final vectors of position: and each does so, only by depending on the mutual *configuration* of *all* those initial and final *positions*.

* The *Action*, V, was in fact so called, in the two Essays mentioned in the preceding Note. The properties of this *Characteristic Function* had been perceived by the writer, before those of that which he came afterwards to call the *Principal Function*, as above.

(18.) It follows (comp. (4.), see also 416, (17.), and 417, (2.)), that the function F must satisfy the two conditions,

$$\text{XXX.} \ldots \Sigma(\mathrm{D}_\alpha F + \mathrm{D}_{\alpha_0} F) = 0; \quad \text{XXXI.} \ldots \Sigma \mathrm{V}(\alpha \mathrm{D}_\alpha F + \alpha_0 \mathrm{D}_{\alpha_0} F) = 0;$$

which accordingly are forms, by XIV. XV., of the equations VIII. and IX., and therefore are expressions for the law of motion of the centre of gravity, and the law of description of areas. And, in like manner, the function V is obliged to satisfy these two analogous conditions,

$$\text{XXXII.} \ldots \Sigma(\mathrm{D}_\alpha V + \mathrm{D}_{\alpha_0} V) = 0; \quad \text{XXXIII.} \ldots \Sigma \mathrm{V}(\alpha \mathrm{D}_\alpha V + \alpha_0 \mathrm{D}_{\alpha_0} V) = 0;$$

which accordingly, by XXV. XXVI., are new forms of the same equations VIII. IX., and consequently are new expressions of the same two laws.

(19.) All the foregoing conditions are satisfied when t is *small*, that is, when the *time* of motion of the system is *short*, by the following *approximate expressions* for the functions F and V, with the respectively derived and mutually connected expressions for H and t:

$$\text{XXXIV.} \ldots F = \frac{t}{2}(P + P_0) + \frac{s^2}{2t};$$

$$\text{XXXV.} \ldots V = s(P + P_0 + 2H)^{\frac{1}{2}};$$

$$\text{XXXVI.} \ldots H = -(\mathrm{D}_t F) = -\tfrac{1}{2}(P + P_0) + \frac{s^2}{2t^2};$$

$$\text{XXXVII.} \ldots t = \mathrm{D}_H V = s(P + P_0 + 2H)^{-\frac{1}{2}};$$

in which s denotes a real and positive scalar, such that

$$\text{XXXVIII.} \ldots s^2 = -\Sigma m(\alpha - \alpha_0)^2, \quad \text{or} \quad \text{XXXIX.} \ldots s = \sqrt{\Sigma m \mathrm{T}(\alpha - \alpha_0)^2}.$$

419. As a *fourth specimen*, we shall take the case of a free point or particle, attracted to a fixed centre* o, from which its variable vector is α, with an accelerating force $= Mr^{-2}$, if $r = \mathrm{T}\alpha$ = the distance of the point from the

* When *two free masses*, m and m', with variable vectors α and α', attract each other according to the law of the inverse square, the differential equation of the *relative motion* of m about m' is, by 418, VII.,

$$\text{I}' \ldots \mathrm{D}^2(\alpha - \alpha') = (m + m')(\alpha - \alpha')^{-1} r^{-1}, \quad \text{if} \quad r = \mathrm{T}(\alpha - \alpha');$$

and this equation I′. reduces itself to I., when we write α for $\alpha - \alpha'$, and M for $m + m'$.

centre, while M is the attracting mass: the differential equation of the motion being,

$$\text{I.} \ldots \mathrm{D}^2\alpha = M\alpha^{-1}r^{-1},$$

if D (abridged from D_t) be the sign of derivation, with respect to the time t.

(1.) Operating on I. with $\mathrm{V}.\alpha$, and integrating, we obtain immediately the equation (comp. 338, (5.)),

$$\text{II.} \ldots \mathrm{V}\alpha\mathrm{D}\alpha = \beta = \text{const.};$$

which expresses at once that the *orbit is plane*, and also that the *area* described in it is *proportional to the time*; $\mathrm{U}\beta$ being the fixed *unit-normal* to the *plane*, round which the point, in its angular motion, *revolves positively*; and $\mathrm{T}\beta$ representing in quantity the *double areal velocity*, which is often denoted by c.

(2.) And it is important to remark, that these conclusions (1.) would have been obtained by the same analysis, if r^{-1} in I. had been replaced by *any other scalar function*, $f(r)$, of the *distance*; that is, for *any other law* of *central force*, instead of the law of the *inverse square*.

(3.) *In general*, we have the transformation,

$$\text{III.} \ldots \alpha^{-1}\mathrm{T}\alpha^{-1} = d\mathrm{U}\alpha : \mathrm{V}\alpha d\alpha,$$

because, by 334, XV., &c., we have,

$$\text{IV.} \ldots d\mathrm{U}\alpha = \mathrm{V}(d\alpha\,.\,\alpha^{-1})\,.\,\mathrm{U}\alpha = \alpha^{-2}\mathrm{U}\alpha\,.\,\mathrm{V}\alpha d\alpha = \alpha^{-1}\mathrm{T}\alpha^{-1}\,.\,\mathrm{V}\alpha d\alpha;$$

the equation I. may therefore by II. be transformed as follows,

$$\text{V.} \ldots \mathrm{D}^2\alpha = \gamma\mathrm{D}\mathrm{U}\alpha, \quad \text{if} \quad \text{VI.} \ldots \gamma = -M\beta^{-1};$$

and thus it gives, by an immediate integration,

$$\text{VII.} \ldots \mathrm{D}\alpha = \gamma(\mathrm{U}\alpha - \varepsilon), \quad \text{or} \quad \text{VII}'. \ldots \mathrm{D}\alpha = (\varepsilon - \mathrm{U}\alpha)\gamma,$$

ε being a *new constant vector*, but one situated *in the plane* of the orbit, to which plane β and γ are *perpendicular*.

(4.) But α, $\mathrm{D}\alpha$, $\mathrm{D}^2\alpha$ are here (comp. 100, (5.) (6.) (7.)) the vectors of *position*, *velocity*, and *acceleration* of the moving point; and it has been defined (100, (5.)) that if, for *any motion* of a point, the *vectors of velocity* be set off from any *common origin*, the *curve* on which they *terminate* is the *Hodograph** of that motion.

(5.) Hence α and $D\alpha$, if the latter like the former be drawn from the fixed point O, are the vectors of *corresponding points* of *orbit* and *hodogragh*; and because the formula VII. gives,

$$\text{VIII.} \ldots S\gamma D\alpha = 0, \quad \text{and} \quad \text{IX.} \ldots (D\alpha + \gamma\varepsilon)^2 = \gamma^2,$$

it follows that the hodograph is, in the present question, a *Circle*, in the plane of the orbit, with $-\gamma\varepsilon$ (or $+\varepsilon\gamma$) for the vector of its *centre*, and with $T\gamma = MT\beta^{-1}$ for its *radius*, which radius we shall also denote by h.

(6.) The *Law of the Circular*† *Hodograph* is therefore a mathematical consequence of the *Law of the Inverse Square*; and conversely it will soon be proved, that *no other law* of *central force* would allow generally the *hodograph* to be a *circle*.

(7.) For the law of nature, the *Radius* (h) *of the Hodograph* is equal, by (1.) and (5.), to the quotient of the *attracting mass* (M), *divided by the double areal velocity* ($T\beta$ or c) in the *orbit*; and if we write

$$\text{X.} \ldots e = T\varepsilon,$$

this positive scalar e may be called the *Excentricity* of the hodograph, regarded as a *circle excentrically situated*, with respect to the *fixed centre of force*, O.

(8.) Thus, if $e < 1$, the fixed point O is *interior* to the hodograph circle; if $e = 1$, the point O is *on the circumference*; and if $e > 1$, the centre O of force is then *exterior* to the hodograph, being however, in *all* these cases, situated *in its plane*.

(9.) The equation VII. gives,

$$\text{XI.} \ldots \varepsilon - U\alpha = -\gamma^{-1}D\alpha = D\alpha \cdot \gamma^{-1};$$

operating then on this with $S \cdot \alpha$, and writing for abridgment,

$$\text{XII.} \ldots p = \beta\gamma^{-1} = M^{-1}T\beta^2 = c^2M^{-1}, \quad \text{and} \quad \text{XIII.} \ldots SU\alpha\varepsilon = \cos v,$$

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* Compare fig. 32, p. 97, vol. i. [and p. 302]; see also pages 99, vol. i., 29, 112, from the two latter of which it may be perceived, that the *conception* of the *hodograph* admits of some purely *geometrical* applications.

† This *law of the circular hodograph* was deduced *geometrically*, in a paper read before the Royal Irish Academy, by the present author, on the 14th of December, 1846; but it was virtually contained in a *quaternion formula*, equivalent to the recent equation VII., which had formed part of an earlier communication, in July, 1845. (See the *Proceedings* for those dates; and especially pages 345, 347, and xxxix, xlix, of vol. iii.)

so that p is a constant and positive scalar, while v is the inclination of α to $-\varepsilon$, we find,

$$\text{XIV.} \ldots r + \mathrm{S}\alpha\varepsilon = p; \quad \text{or} \quad \text{XV.} \ldots r = \frac{p}{1 + e \cos v};$$

the *orbit* is therefore a *plane conic*, with the centre of force o for a *focus*, having e for its *excentricity*, and p for its *semiparameter*.

(10.) And we see, by XII., that if this semiparameter p be *multiplied* by the attracting mass M, the *product* is the *square* of the double areal velocity c; so that this constant c may be denoted by $(Mp)^{\frac{1}{2}}$, which agrees with known results.

(11.) If, on the other hand, we *divide* the mass (M) by the semiparameter (p), the *quotient* is by XII. the *square* of the *radius* ($M\mathrm{T}\beta^{-1}$ or h) of the *hodograph*.

(12.) And if we *multiply* the same semiparameter p by this radius $M\mathrm{T}\beta^{-1}$ of the *hodograph*, the *product* is then, by the same formula XII., the constant $\mathrm{T}\beta$ or c of double areal velocity in the *orbit*, so that $h = Mc^{-1} = cp^{-1}$.

(13.) If we had operated with $\mathrm{V}.\alpha$ on VII′., we should have found,

$$\text{XVI.} \ldots \beta = \mathrm{V}.\alpha(\varepsilon - \mathrm{U}\alpha)\gamma = (\mathrm{S}\alpha\varepsilon + r)\gamma;$$

which would have conducted to the same equations XIV. XV. as before.

(14.) If we operate on VII. with $\mathrm{S}.\alpha$, we find this other equation,

$$\text{XVII.} \ldots -r\mathrm{D}r = \mathrm{S}\alpha\mathrm{D}\alpha = \gamma\mathrm{V}\alpha\varepsilon;$$

but

$$\text{XVIII.} \ldots -\gamma^2 = h^2 = \frac{M}{p} \text{ (by VI. and XII., comp. (11.))},$$

and

$$\text{XIX.} \ldots -(\mathrm{V}\alpha\varepsilon)^2 = e^2r^2 - (p - r)^2 = p(2r - p - r^2a^{-1}),$$

if we write

$$\text{XX.} \ldots a = \frac{p}{1 - e^2};$$

hence squaring XVII., and dividing by r^2, we obtain the equation,

$$\text{XXI.} \ldots \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = M\left(\frac{2}{r} - \frac{1}{a} - \frac{p}{r^2}\right).$$

(15.) It is obvious that this last equation, XXI., connects the *distance*, r, with the *time*, t, as the formula XV. connects the same distance r with the *true anomaly*, v; that is, with the *angular elongation* in the orbit, from the *position of least distance.* But it would be improper here to delay on any of the elementary consequences of these two known equations: although it seemed useful to show, as above, how the equations themselves might easily be deduced by *quaternions*, and be connected with the theory of the *hodograph.*

(16.) The equation II. may be interpreted as expressing, that the *parallelogram* (comp. fig. 32, p. 97, vol. i.) under the vectors a and Da of position and velocity, or under any two *corresponding vectors* (5.) of the *orbit* and *hodograph*, has a *constant plane and area*, represented by the constant vector β, which is *perpendicular* (1.) to that plane. But it is to be observed that, by (2.), these *constancies*, and this *representation*, are *not peculiar* to the law of the *inverse square*, but exist for *all other laws* of *central force.*

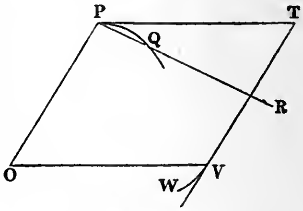

Fig. 32, *bis.*

(17.) In general, if any scalar function R (instead of Mr^{-2}) represent the accelerating force of attraction, at the distance r from the fixed centre O, the differential equation of motion will be (instead of I.),

$$\text{XXII.} \ldots D^2 a = Rra^{-1} = -RUa;$$

and if we still write $VaDa = \beta$, as in II., the formula IV. will give,

$$\text{XXIII.} \ldots D^3 a = -DR \,.\, Ua - Rr^{-2}\beta Ua, \quad \text{and} \quad \text{XXIV.} \ldots V\frac{D^3 a}{D^2 a} = r^{-2}\beta;$$

in which $\beta = cU\beta$, if $c = T\beta$, as before.

(18.) Applying then the general formula 414, I., we have, for *any law** *of force*, the expressions,

$$\text{XXV.} \ldots \textit{Vector of Curvature of Hodograph} = \frac{1}{D^2 a} V \frac{D^3 a}{D^2 a} = \frac{c}{Rr^2} Ua\beta;$$

$$\text{XXVI.} \ldots \textit{Radius } (h) \textit{ of Curvature of Hodograph} = Rr^2 c^{-1}$$

$$= \frac{\textit{Force} \times \textit{Square of Distance}}{\textit{Double Areal Velocity in Orbit}};$$

of which the last not only conducts, in a new way, for the *law of nature*, to the constant value (7.), $h = Mc^{-1}$, but also proves, as stated in (6.), that for

* The *general value* XXVI., of the *radius of curvature of the hodograph*, was *geometrically* deduced in the Paper of 1846, referred to in a recent Note.

any other law of central force *the hodograph cannot be a circle*, unless indeed the *orbit* happens to be such, and to have moreover the centre of force at *its* centre.

(19.) Confining ourselves however at present to the law of the inverse square, and writing for abridgment (comp. (5.)),

$$\text{XXVII.} \ldots \kappa = \text{OH} = \varepsilon\gamma = \textit{Vector of Centre H of Hodograph},$$

which gives, by (5.) and (7.),

$$\text{XXVIII.} \ldots T\kappa = eh,$$

the *origin* O of vectors being still the centre of *force*, we see by the properties of the *circle*, that the *product* of any two *opposite velocities* in the *orbit* is *constant*; and that this constant product* may be expressed as follows,

$$\text{XXIX.} \ldots (e-1)hU\kappa \,.\, (e+1)hU\kappa = h^2(1-e^2) = Ma^{-1},$$

by XVIII. and XX.

(20.) The expression XXIX. may be otherwise written as $\kappa^2 - \gamma^2$; and if υ be the vector of any point U *external* to the circle, but in its plane, and u the length of a *tangent* UT from that point, we have the analogous formula,

$$\text{XXX.} \ldots u^2 = \gamma^2 - (\upsilon - \kappa)^2 = T(\upsilon - \kappa)^2 - h^2.$$

(21.) Let τ and τ' be the vectors OT, OT′ of the *two* points of contact of tangents thus drawn to the hodograph, from an external point U in its plane; then *each* must satisfy the system of the *three* following scalar equations,

$$\text{XXXI.} \ldots S\gamma\tau = 0;$$

$$\text{XXXII.} \ldots (\tau - \kappa)^2 = \gamma^2;$$

$$\text{XXXIII.} \ldots S(\tau - \kappa)(\upsilon - \kappa) = \gamma^2;$$

* In strictness, it is only for a *closed orbit*, that is, for the case (8.) of the centre of force being *interior* to the *hodograph* ($e < 1$), that two velocities *can* be *opposite*; their *vectors* having then, by the fundamental rules of quaternions, a *scalar* and *positive product*, which is here found to be $= Ma^{-1}$, by XXIX., in consistency with the known theory of *elliptic motion*. The result however admits of an *interpretation*, in other cases also. It is obvious that when the centre O of force is *exterior* to the hodograph, the *polar* of that point divides the *circle* into *two parts*, whereof one is *concave*, and the other *convex*, towards O; and there is no difficulty in seeing, that the *former* part corresponds to the *branch* of an *hyperbolic orbit*, which can be described under the influence of an *attracting force*: while the *latter* part answers to that *other branch* of the same complete hyperbola, whereof the description would require the force to be *repulsive*.

whereof the *first alone* represents the *plane*; the *two first* jointly represent (comp. (5.)) the *circle*; and the *third* expresses the *condition of conjugation* of the points T and U, and may be regarded as the *scalar equation of the polar* of the latter point. It is understood that $S\gamma\upsilon = 0$, as well as $S\gamma\kappa = 0$, &c., because γ is perpendicular (3.) to the plane.

(22.) Solving this system of equations (21.), we find the two expressions,

$$\text{XXXIV.} \ldots \tau = \kappa + \gamma(\gamma + u)(\upsilon - \kappa)^{-1};$$

$$\text{XXXIV}'. \ldots \tau' = \kappa + \gamma(\gamma - u)(\upsilon - \kappa)^{-1};$$

in which the scalar u has the same value as in (20.). As a verification, these expressions give, by what precedes,

$$\text{XXXV.} \ldots S(\tau - \kappa)(\tau - \upsilon) = 0; \quad \text{XXXV}'. \ldots S(\tau' - \kappa)(\tau' - \upsilon) = 0;$$

and

$$\text{XXXVI.} \ldots (\tau - \upsilon)^2 = (\tau' - \upsilon)^2 = -u^2.$$

In fact it is found that

$$\text{XXXVII.} \ldots \tau - \upsilon = u(u + \gamma)(\upsilon - \kappa)^{-1};$$

$$\text{XXXVIII.} \ldots T(u + \gamma) = T(\upsilon - \kappa);$$

and

$$\text{XXXIX.} \ldots (\tau - \upsilon)(\tau - \kappa) = u\gamma;$$

$u + \gamma$ being here a quaternion.

(23.) If υ' be the vector OU′ of *any point* U′, on the *polar* of the point U with respect to the circle, then changing τ to υ', and u to z, in XXXIV., we find this *vector form* (comp. (21.)) of the *equation of that polar*,

$$\text{XL.} \ldots \upsilon' = \kappa + \gamma(\gamma + z)(\upsilon - \kappa)^{-1},$$

or, by an easy transformation,

$$\text{XLI.} \ldots (h^2 + u^2)\upsilon' = h^2\upsilon + u^2\kappa + z\gamma(\kappa - \upsilon),$$

in which z is an arbitrary scalar.

(24.) If then we suppose that U′ is the *intersection* of the chord TT′ with the right line OU, the condition

$$\text{XLII.} \ldots V\upsilon'\upsilon = 0 \quad \text{will give} \quad \text{XLIII.} \ldots z\gamma = \frac{u^2 V\kappa\upsilon}{\upsilon^2 - S\kappa\upsilon};$$

but

$$\text{XLIV.} \ldots V\kappa\upsilon \,.\, (\kappa - \upsilon) = \kappa S(\kappa\upsilon - \upsilon^2) + \upsilon S(\kappa\upsilon - \kappa^2);$$

the coefficient then of κ, in the expanded expression for υ', disappears as it ought to do: and we find, after a few reductions,

$$\text{XLV.} \ldots \upsilon' = \upsilon\left(1 + \frac{u^2}{\upsilon^2 - S\kappa\upsilon}\right) = \frac{\gamma^2 - \kappa^2 + S\kappa\upsilon}{\upsilon - \upsilon^{-1}S\kappa\upsilon},$$

a result which might have been otherwise obtained, by eliminating a new scalar y between the two equations,

$$\text{XLVI.} \ldots \upsilon' = y\upsilon, \quad S(y\upsilon - \kappa)(\upsilon - \kappa) = \gamma^2.$$

(25.) Introducing then two auxiliary vectors, λ, μ, such that

$$\text{XLVII.} \ldots \lambda = \upsilon^{-1}S\kappa\upsilon, \quad \text{or} \quad S\kappa\upsilon = \upsilon\lambda = \lambda\upsilon,$$

and therefore

$$\text{XLVII}'. \ldots \lambda - \kappa = \upsilon^{-1}V\kappa\upsilon, \quad S\kappa\lambda = \lambda^2, \quad (\lambda - \kappa)^2 = \kappa^2 - \lambda^2,$$

and

$$\text{XLVIII.} \ldots \mu = \lambda\left(1 + \left(1 + \frac{\gamma^2 - \kappa^2}{\lambda^2}\right)^{\frac{1}{2}}\right), \quad \text{whence} \quad \mu \parallel \lambda, \quad (\mu - \kappa)^2 = \gamma^2,$$

we have the very simple relation,

$$\text{XLIX.} \ldots (\upsilon - \lambda)(\upsilon' - \lambda) = (\mu - \lambda)^2, \quad \text{or} \quad \text{L.} \ldots \text{LU} . \text{LU}' = \text{LM}^2,$$

if $\lambda = \text{OL}$, and $\mu = \text{OM}$. Accordingly, the point L is the foot of the perpendicular let fall from the centre H on the right line OU, while M is one of the two points M, M′ of intersection of that line with the circle; so that the equation L. expresses, that the points U, U′ are *harmonically conjugate*, with respect to the *chord* MM′, of which L is the middle point, as is otherwise evident from geometry.

(26.) The vector α of the *orbit* (or of *position*), which *corresponds* to the vector $\tau (= D\alpha)$ of the *hodograph* (or of *velocity*), and of which the length is $T\alpha = r =$ the *distance*, may be deduced from τ by the equations,

$$\text{LI.} \ldots \alpha = r(\kappa - \tau)\gamma^{-1}, \quad \text{and} \quad \text{LII.} \ldots V\tau\alpha = -\beta = M\gamma^{-1};$$

whence follow the expressions,

$$\text{LIII.} \ldots \textit{Potential} = Mr^{-1} = (\text{say})\ P = S\tau(\kappa - \tau) = S\upsilon(\kappa - \tau);$$

the second expression for P being deduced from the first, by means of the relation XXXV.

(27.) The *first* expression LIII. for P shows that the potential is equal, Ist, to the *rectangle* under the *radius* of the hodograph, and the *perpendicular* from the centre O of force, on the tangent at T to that circle; and IInd, to the *square* of the *tangent* from the same point T of the hodograph, to what may be called the *Circle of Excentricity*, namely to that new circle which has OH for a diameter. And the first of these values of the potential may be otherwise deduced from the equality (7.) of the mass M, to the product hc of the radius h of the hodograph, multiplied by the constant c of double areal velocity, or by the *constant parallelogram* (16.) under any two corresponding vectors.

(28.) The *second* expression LIII. for the potential P, corresponding to the point T of the hodograph, may (by XXXIV., &c.) be thus transformed, with the help of a few reductions of the same kind as those recently employed:

$$\text{LIV.} \ldots P = \frac{M}{r} = \frac{h^2 \mathrm{S}q + u\gamma \mathrm{V}q}{h^2 + u^2}, \quad \text{if} \quad \text{LV.} \ldots q = \upsilon(\kappa - \upsilon),$$

q being thus an auxiliary quaternion; and in like manner, for the other point T′ lately considered, we have the analogous value,

$$\text{LVI.} \ldots P' = \frac{M}{r'} = \frac{h^2 \mathrm{S}q - u\gamma \mathrm{V}q}{h^2 + u^2};$$

whence

$$\text{LVII.} \ldots P \,.\, P' = \frac{h^2(\mathrm{S}q^2 + u^2\upsilon^2)}{h^2 + u^2};$$

and therefore,

$$\text{LVIII.} \ldots \frac{r}{M} = P^{-1} = \frac{\mathrm{S}q + u\gamma^{-1}\mathrm{V}q}{\mathrm{S}q^2 + u^2\upsilon^2};$$

$$\text{LIX.} \ldots \frac{r'}{M} = P'^{-1} = \frac{\mathrm{S}q - u\gamma^{-1}\mathrm{V}q}{\mathrm{S}q^2 + u^2\upsilon^2};$$

and finally,

$$\text{LX.} \ldots \frac{2M}{r + r'} = \frac{2PP'}{P + P'} = \mathrm{S}q + \frac{u^2\upsilon^2}{\mathrm{S}q} = \upsilon(\lambda - \upsilon') = \mathrm{OU} \,.\, \mathrm{U'L}.$$

(29.) In fact, the same second expression LIII. shows, that if V and V′ be the feet of perpendiculars from T and T′ on HL, then the potentials are,

$$\text{LXI.} \ldots P = \text{OU} \,.\, \text{TV}, \quad \text{and} \quad P' = \text{OU} \,.\, \text{T}'\text{V}';$$

and it is easy to prove, geometrically, that the *segment* U′L is the *harmonic mean* between what may be called the *ordinates*, TV, T′V′, to the *hodographic axis* HL.

(30.) If we suppose the point U to take any new but near position $\text{U}_,$ in the plane, the polar chord TT′, and (in general) the length u of the tangent UT, will change; and we shall have the differential relations:

$$\text{LXII.} \ldots \mathrm{d}\tau = (\tau - \upsilon)^{-1}\mathrm{S}(\tau - \kappa)\,\mathrm{d}\upsilon;$$

$$\text{LXII}'. \ldots \mathrm{d}\tau' = (\tau' - \upsilon)^{-1}\mathrm{S}(\tau' - \kappa)\,\mathrm{d}\upsilon;$$

and

$$\text{LXIII.} \ldots \mathrm{d}u = u^{-1}\mathrm{S}(\kappa - \upsilon)\,\mathrm{d}\upsilon.$$

(31.) Conceiving next that U moves along the line OU, or LU, so that we may write,

$$\text{LXIV.} \ldots \upsilon = (x - e')\,(\mu - \lambda), \quad \text{if} \quad x = \frac{\text{LU}}{\text{LM}} = \frac{\text{LM}}{\text{LU}'}, \quad \text{and} \quad e' = \frac{\text{LO}}{\text{LM}},$$

we shall have,

$$\text{LXV.} \ldots \mathrm{d}\upsilon = (\mu - \lambda)\,\mathrm{d}x = \upsilon\,(x - e')^{-1}\mathrm{d}x, \quad \text{with} \quad x > 1 > e',$$

if U be on LM prolonged, and if O be on the concave side of the arc TMT′; and thus, by LIII., the differential expressions (30.) become,

$$\text{LXVI.} \ldots \mathrm{d}\tau = (\upsilon - \tau)^{-1}P\,(x - e')^{-1}\mathrm{d}x; \quad \mathrm{d}\tau' = (\upsilon - \tau')^{-1}P'\,(x - e')^{-1}\mathrm{d}x;$$

and

$$\text{LXVII.} \ldots \mathrm{d}u = u^{-1}\mathrm{S}q \,.\, (x - e')^{-1}\mathrm{d}x, \quad \text{with} \quad \mathrm{S}q = \upsilon\,(\lambda - \upsilon);$$

so that

$$\text{LXVIII.} \ldots \mathrm{T}\mathrm{d}\tau = \frac{P\mathrm{d}x}{u\,(x - e')}, \quad \mathrm{T}\mathrm{d}\tau' = \frac{P'\mathrm{d}x}{u\,(x - e')}, \quad \text{if} \quad \mathrm{d}x > 0.$$

Such then are the *lengths* of the two *elementary arcs* $\text{TT}_,$ and $\text{T}'\text{T}'_,$ of the hodograph, intercepted between two near secants NTT′ and $\text{NT}_,\text{T}'_,$ drawn from the pole N of the chord MM′, and having U and $\text{U}_,$ for their own poles; and we see that these *arcs* are proportional to the *potentials*, P and P', or by LXI. to the *ordinates*, TV, T′V′, or finally to the lines NT, NT′: and accordingly we have the *inverse similarity* (comp. 118), of the two small triangles with N for vertex,

$$\text{LXIX.} \ldots \Delta \mathrm{NTT}_{,} \propto' \mathrm{NT}'_{,}\mathrm{T}',$$

as appears on inspection of the annexed figure 86.

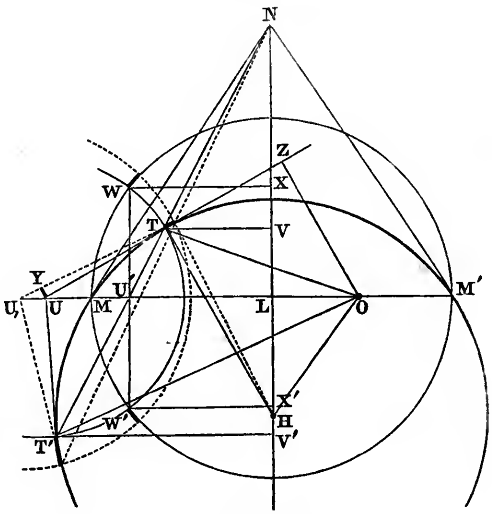

Fig. 86.

(32.) For *any motion of a point*, however complex, the *element* dt of *time* which corresponds to a given element dDα of the *hodograph*, is found by *dividing* the latter element by the *vector* $\mathrm{D}^2\alpha$ of *accelerating force* ; if then we denote by dt and dt' the times corresponding to the elements dτ and dτ' (31.), we have the expressions,

$$\text{LXX.} \ldots \mathrm{d}t = M \,.\, P^{-2} \,.\, \mathrm{Td}\tau = \frac{M\mathrm{d}x}{Pu(x-e')} = \frac{r\mathrm{d}x}{u(x-e')},$$

$$\text{LXX}'. \ldots \mathrm{d}t' = M \,.\, P'^{-2} \,.\, \mathrm{Td}\tau' = \frac{M\mathrm{d}x}{P'u(x-e')} = \frac{r'\mathrm{d}x}{u(x-e')},$$

because, for the motion here considered, the measure or quantity of the force is, by I. and LIII.,

$$\text{LXXI.} \ldots \mathrm{TD}^2\alpha = Mr^{-2} = M^{-1}P^2.$$

(33.) The *times of hodographically describing* the *two small circular arcs*, T,T and T′T,′, are therefore *inversely* proportional to the *potentials*, or *directly* proportional to the *distances* in the *orbit*; and their *sum* is,

$$\text{LXXII.} \ldots \mathrm{d}t + \mathrm{d}t' = \left(\frac{M}{P} + \frac{M}{P'}\right) \frac{u^{-1}\mathrm{d}x}{x - e'} = \frac{(r + r')\,\mathrm{d}x}{u\,(x - e')};$$

that is, by LX. and LXIV.,

$$\text{LXXIII.} \ldots \mathrm{d}t + \mathrm{d}t' = \frac{2Mx\mathrm{d}x}{u\,(x - e')^2 g^2}, \quad \text{if} \quad \text{LXXIV.} \ldots g = \mathrm{T}(\mu - \lambda) = \overline{\mathrm{LM}}.$$

(34.) We have also the relations,

$$\text{LXXV.} \ldots u = (x^2 - 1)^{\frac{1}{2}} g, \quad \text{and} \quad \text{LXXVI.} \ldots \frac{M}{a} = (1 - e'^2) g^2;$$

so that the sum of the two small times may be thus expressed,

$$\text{LXXVII.} \ldots \mathrm{d}t + \mathrm{d}t' = \frac{2(a(1 - e'^2))^{\frac{3}{2}}}{M^{\frac{1}{2}}} \cdot \frac{(1 - e'x^{-1})^{-2}\mathrm{d}x}{x(x^2 - 1)^{\frac{1}{2}}},$$

or finally,

$$\text{LXXVIII.} \ldots \mathrm{d}t + \mathrm{d}t' = 2\left(\frac{a^3(1 - e'^2)^3}{M}\right)^{\frac{1}{2}} \cdot \frac{\mathrm{d}w}{(1 - e'\cos w)^2},$$

if

$$\text{LXXIX.} \ldots x = \sec w, \quad \text{or} \quad w = \angle\,\mathrm{MLW} \text{ in fig. 86},$$

in which figure U′W is an ordinate of a semicircle, with the chord MM′ of the hodograph for its diameter.

(35.) The two near secants (31.), from the pole N of that chord, have been here supposed to cut the half chord LM *itself*, as in the cited figure 86; but if they were to cut the *other* half chord LM′, it is easy to prove that the formulæ LXXVIII. LXXIX. would still hold good, the only difference being that the angle w, or MLW, would be now *obtuse*, and its secant $x < -1$.

(36.) A *circle*, with U for centre, and u for radius, *cuts* the hodograph *orthogonally* in the points T and T′; and in like manner a *near circle*, with U, for centre, and $u + \mathrm{d}u$ for radius, is *another orthogonal*, cutting the same hodograph in the near points T, and T,′ (31.). And by conceiving a *series* of such orthogonals, and observing that the differential expression LXXVIII. depends only on the *four scalars*, $M^{-1}a^3$, e', w, and $\mathrm{d}w$, which are all known when the *mass* M and the *five points* O, L, M, U, U, are given, so that they do not change when we retain that mass and those points, but alter the radius h of the hodograph, or the perpendicular HL let fall from its centre H on the fixed chord MM′, we see that the *sum of the times* (comp. (33.)), of *hodographically describing any two circular arcs*, such as T,T and T′T,′, even if they be *not small*, but intercepted between *any two secants* from the *pole* N of the *fixed chord*, is *independent of the radius* (h), or of the *height* HL of the *centre* H of the hodograph.

(37.) *If then two circular hodographs*, such as the two in fig. 86, having a *common chord* MM′, which passes through, or tends towards, a *common centre of force* O, with a *common mass* M there situated, be *cut by any two common orthogonals*, the *sum of the two times* of *hodographically describing* (33.) the *two intercepted arcs* (small or large) will be the *same* for those *two* hodographs.

(38.) And as a *case* of this *general* result, we have the following *Theorem* of Hodographic Isochronism* (or *Synchronism*):

"*If two circular hodographs, having a common chord, which passes through, or tends towards, a common centre of force, be cut perpendicularly by a third circle, the times of hodographically describing the intercepted arcs will be equal.*"

For example, in fig. 86, we have the equation,

$$\text{LXXX.} \ldots \textit{Time of } \mathrm{TMT'} = \textit{time of } \mathrm{WMW'}.$$

(39.) The time of thus describing the arc TMT′ (fig. 86), if this arc be throughout *concave*† towards O (so that $x > 1 > e'$, as in LXV.), is expressed (comp. LXXVIII.) by the definite integral,

$$\text{LXXXI.} \ldots \textit{Time of } \mathrm{TMT'} = 2\left(\frac{a^3(1-e'^2)^3}{M}\right)^{\frac{1}{2}} \int_0^w \frac{dw}{(1-e'\cos w)^2};$$

and the time of describing the *remainder* of the hodographic circle, if *this* remaining arc T′M′T be throughout concave towards the centre O of force, is expressed by this other integral,

$$\text{LXXXII.} \ldots \textit{Time of } \mathrm{T'M'T} = 2\left(\frac{a^3(1-e'^2)^3}{M}\right)^{\frac{1}{2}} \int_w^\pi \frac{dw}{(1-e'\cos w)^2}.$$

(40.) Hence, for the *case* of a *closed orbit* ($e'^2 < 1$, $e < 1$, $a > 0$), if n denote the *mean angular velocity*, we have the formula,

$$\text{LXXXIII.} \ldots \textit{Periodic Time} = \frac{2\pi}{n} = 2\left(\frac{a^3}{M}\right)^{\frac{1}{2}}(1-e'^2)^{\frac{3}{2}} \int_0^\pi \frac{dw}{(1-e'\cos w)^2} = 2\pi\left(\frac{a^3}{M}\right)^{\frac{1}{2}};$$

or

$$\text{LXXXIV.} \ldots M = a^3 n^2, \text{ as usual.}$$

* This Theorem, in which it is understood that the *common centre* of force (O) is occupied by a *common mass* (M), was communicated to the Royal Irish Academy on the 16th of March, 1847. (See the *Proceedings* of that date, vol. iii., page 417.) It has since been treated as a subject of investigation by several able writers, to whom the author cannot hope to do justice on this subject, within the very short space which now remains at his disposal.

† Compare the Note to page 303.

The same result, for the same case of *elliptic motion*, may be more rapidly obtained, by conceiving the chord MM′ through O to be perpendicular to OH; for, in this position of that chord, its middle point L coincides with O, and $e' = 0$ by LXIV.

(41.) In general, by LXXVI., we are at liberty to make the substitution,

$$\text{LXXXV.} \ldots \left(\frac{a^3(1-e'^2)^3}{M}\right)^{\frac{1}{2}} = \frac{M}{g^3}, \text{ with } g = \textit{half chord of the hodograph};$$

supposing then that $e' = -1$, or placing O at the extremity M′ of the chord, we have by LXXXI.,

$$\text{LXXXVI.} \ldots \textit{Parabolic time of } \text{TMT}' = \frac{2M}{g^3}\int_0^w \frac{dw}{(1+\cos w)^2};$$

for, when the *centre of force* is thus situated *on the circumference* of the hodographic circle, we have by (8.) the *excentricity* $e = 1$, and the *orbit* becomes by XV. a *parabola*. For *hyperbolic motion* ($e'^2 > 1$, $e > 1$, $a < 0$), the formula LXXXI. (with or without the substitution LXXXV.) is to be employed if $e' < -1$, that is, if O be on LM′ prolonged; and the formula LXXXII., if $e' > 1$, $e' < \sec w$, that is, if O be situated between M and U.

(42.) For *any law of central force*, if P, P′ be the points of the *orbit* which correspond to the points T, T′ of the *hodograph*, and if Q be the point of meeting of the tangents to the orbit at P, P′, as in the annexed figure 87, while the tangents to the hodograph at T, T′ meet as before in U, we shall have the parallelisms,

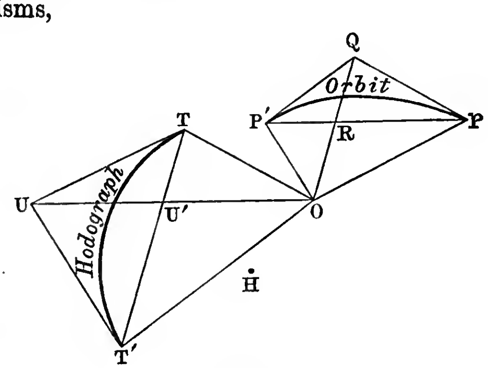

Fig. 87.

$$\text{LXXXVII.} \ldots \text{OP} \parallel \text{UT}, \quad \text{OP}' \parallel \text{T}'\text{U}, \quad \text{PQ} \parallel \text{OT}, \quad \text{QP}' \parallel \text{OT}';$$

writing then,

$$\text{LXXXVIII.} \ldots \text{OP} = a, \ \text{OP}' = a', \ \text{OT} = \text{D}a = \tau, \ \text{OT}' = \text{D}a' = \tau', \ \text{OU} = v, \ \text{OQ} = \omega,$$

most of which notations have occurred before, we have the equations,

$$\text{LXXXIX.}\ldots 0 = \mathrm{V}a(\tau - \upsilon) = \mathrm{V}a'(\upsilon - \tau') = \mathrm{V}\tau(\omega - a) = \mathrm{V}\tau'(a' - \omega);$$

thus

$$\text{XC.}\ldots \mathrm{V}a\upsilon = \mathrm{V}a\tau = \beta = \mathrm{V}a'\tau' = \mathrm{V}a'\upsilon, \quad a' - a \parallel \upsilon, \quad \mathrm{PP}' \parallel \mathrm{OU},$$

and

$$\text{XCI.}\ldots \mathrm{V}\tau\omega = \mathrm{V}\tau a = -\beta = \mathrm{V}\tau' a' = \mathrm{V}\tau'\omega', \quad \tau - \tau' \parallel \omega, \quad \mathrm{T'T} \parallel \mathrm{OQ}.$$

Geometrically, the *constant parallelogram* (16.) under OP, OT, or under OP′, OT′, is equal, by LXXXVII., to each of the four following parallelograms: I. under OP, OU; II. under OP′, OU; III. under OQ, OT; and IV. under OQ, OT′; whence PP′ ∥ OU, and T′T ∥ OQ, as before.

(43.) The *parallelism* XC. may be otherwise deduced for the law of the *inverse square*, with recent notations, from the quaternion formulæ,

$$\text{XCII.}\ldots \frac{a' - a}{r + r'} = \frac{u}{\lambda - \upsilon} = \frac{\upsilon - \upsilon'}{u}, \quad \text{in which,} \quad \text{XCII}'.\ldots \upsilon' = \frac{r\tau + r'\tau'}{r + r'},$$

and which may be obtained in various ways; whence it may also be inferred, that if s denote the *length* $\mathrm{T}(a' - a)$ of the *chord* PP′ of the *orbit*, then (comp. fig. 86.),

$$\text{XCIII.}\ldots \frac{s}{r + r'} = \frac{u}{\mathrm{T}(\lambda - \upsilon)} = \overline{\mathrm{UT}} : \overline{\mathrm{UL}} = \&\text{c.} = \sin w;$$

w being the same auxiliary angle as in (34.), &c.

(44.) It is easy to prove that

$$\text{XCIV.}\ldots \lambda - \tau = \left(1 + \frac{u}{\gamma}\right)\frac{P}{\upsilon}, \quad \lambda - \tau' = \left(1 - \frac{u}{\gamma}\right)\frac{P'}{\upsilon},$$

whence

$$\text{XCV.}\ldots \mathrm{T}\frac{\tau' - \lambda}{\tau - \lambda} = \frac{P'}{P} = \frac{r}{r'}, \quad \text{and} \quad \text{XCVI.}\ldots P'^{-1}(\tau' - \lambda)\upsilon = \mathrm{K}\,.\,P^{-1}(\tau - \lambda)\upsilon;$$

the lines LT, LT′ are therefore in *length* proportional to the *potentials*, P, P'; and their *directions* are *equally inclined* to that of OU, but at *opposite sides* of it, so that the line LU *bisects the angle* TLT′. Accordingly (see fig. 86), the three points T, L, T′ are on the *circle* (not drawn in the figure) which has HU for diameter; so that the angles ULT′, TLU are equal to each other, as being respectively equal to the angles UTT′, TT′U, which the chord TT′ of the hodograph makes with the tangents at its extremities: the triangles TLV, T′LV′ are therefore similar, and $\overline{\mathrm{LT}}$ is to $\overline{\mathrm{LT}'}$ as TV to T′V′, that is, by LXI., as P to P', or as r' to r.

(45.) Again, calculation with quaternions gives,

$$\text{XCVII.}\ldots\frac{(\upsilon-\tau)(\lambda-\tau)}{\upsilon'-\tau}=\frac{(\upsilon-\tau')(\lambda-\tau')}{\upsilon'-\tau'}=(\upsilon-\kappa)(\upsilon-\lambda)(\upsilon-\kappa)^{-1},$$

whence

$$\text{XCVIII.}\ldots\text{T}\frac{\upsilon'-\tau}{\lambda-\tau}=\text{T}\frac{\upsilon'-\tau'}{\lambda-\tau'}=\text{T}\frac{\tau-\upsilon}{\lambda-\upsilon}=\overline{\text{UT}}:\overline{\text{UL}}=\sin w\,;$$

such then is the *common ratio*, of the *segments* $\overline{\text{TU}'}$, $\overline{\text{U}'\text{T}'}$ of the *base* TT′ of the triangle TLT′, to the adjacent *sides* $\overline{\text{LT}}$, $\overline{\text{LT}'}$, which are to *each other* as r' to r (44.); and because this ratio is also that of s to $r + r'$, by (43.), we have the proportion,

$$\text{XCIX.}\ldots\overline{\text{OP}}:\overline{\text{OP}'}:\overline{\text{PP}'}=r:r':s=\overline{\text{LT}'}:\overline{\text{LT}}:\overline{\text{TT}'},$$

and the formula of inverse similarity (118),

$$\text{C.}\ldots\Delta\,\text{LT}'\text{T}\propto'\text{OPP}'.$$

Accordingly (comp. the two last figures), the base angles OPP′, OP′P of the second triangle are respectively equal, by the parallelisms (42.), to the angles TUL, T′UL, and therefore, by the circle (44.), to the base angles TT′L, T′TL, of the first triangle: but the two rotations, round O from P to P′, and round L from T′ to T, are oppositely directed.

(46.) The investigations of the three last sub-articles have not assumed any knowledge of the *form* of the *orbit* (as *elliptic*, &c.), but only the *law* of *attraction* according to the *inverse square*, or by (6.) the *Law of the Circular Hodograph*. And the same general principles give not only the expression LXXVI. for the constant Ma^{-1}, but also (by LX. LXIV. LXXIV. LXXIX.) this other expression,

$$\text{CI.}\ldots\frac{2M}{r+r'}=(1-e'\cos w)\,g^2;\quad\text{whence}\quad\text{CII.}\ldots\frac{r+r'}{2a}=\frac{1-e'^2}{1-e'\cos w},$$

which last may be considered as a quadratic in e', assigning two values (real or imaginary) for that scalar, when the first member of CII. and the angle w are given; the sine of this latter angle being already expressed by XCIII.

(47.) Abstracting, then, from any *ambiguity** of solution, we see, by the

* That there *ought* to be some such ambiguity is evident from the consideration, that when a *focus* O, and *two points* P, P′ of an *elliptic orbit* are *given*, it is still permitted to conceive the motion to be performed along *either* of the *two elliptic arcs*, PP′, P′P, which together make up the whole periphery. But into details of this kind we cannot enter here.

definite integrals in (39.), that the *time of describing an arc* PP′ *of an orbit*, with the law of the *inverse square*, is a *function* (comp. (36.)) *of the three ratios*,

$$\text{CIII.} \ldots \frac{a^3}{M}, \quad \frac{r+r'}{a}, \quad \frac{s}{r+r'};$$

which is a form *of Lambert's Theorem*, but presents itself *here* as *deduced* from the recently stated *Theorem of Hodographic Isochronism* (38.), *without the employment of any property of conic sections*.

(48.) The differential equation I. of the present relative motion may be thus written (comp. 418, I., and generally the preceding Series 418):

$$\text{CIV.} \ldots \text{S}\,.\,\text{D}^2\alpha\delta\alpha + \delta P = 0, \quad \text{whence} \quad \text{CV.} \ldots T = P + H,$$

as in 418, X., if we now write,

$$\text{CVI.} \ldots T = -\tfrac{1}{2}\text{D}\alpha^2 = -\tfrac{1}{2}\tau^2, \quad \text{and} \quad \text{CVII.} \ldots H = \frac{-M}{2a};$$

in fact (by LIII., comp. (20.) (21.)),

$$\text{CVIII.} \ldots -2H = 2(P - T) = 2P + \tau^2 = \kappa^2 - \gamma^2 = \frac{M}{a}.$$

(49.) Integrating CIV. by parts, &c., and writing (as in 418, XII. XXII.),

$$\text{CIX.} \ldots F = \int_0^t (T + P)\,\text{d}t, \quad \text{and} \quad \text{CX.} \ldots V = \int_0^t 2T\text{d}t,$$

so that F may again be called the *Principal Function* and V the *Characteristic Function* of the motion, we have the variations,

$$\text{CXI.} \ldots \delta F = \text{S}\tau\delta\alpha - \text{S}\tau'\delta\alpha' - H\delta t; \quad \text{CXII.} \ldots \delta V = \text{S}\tau\delta\alpha - \text{S}\tau'\delta\alpha' + t\delta H;$$

in which α, α' (instead of α_0, α) denote now what may be called the *initial* and *final vectors* (OP, OP′) of the *orbit*; whence follow the *partial derivatives*,

$$\text{CXIII.} \ldots \text{D}_\alpha F = \text{D}_\alpha V = \tau; \quad \text{CXIII}'. \ldots \text{D}_{\alpha'} F = \text{D}_{\alpha'} V = -\tau';$$

$$\text{CXIV.} \ldots (\text{D}_t F) = -H; \quad \text{and} \quad \text{CXV.} \ldots \text{D}_H V = t;$$

F being here a scalar function of α, α', t, while V is a scalar function of α, α', H, if M be treated as given.

(50.) The two *vectors* α, α' can enter into these two *scalar functions*, only through their dependent scalars r, r', s (comp. 418, (17.)); but

$$\text{CXVI.} \ldots \delta r = -r^{-1}S\alpha\delta\alpha, \quad \delta r' = -r'^{-1}S\alpha'\delta\alpha', \quad \delta s = -s^{-1}S(\alpha'-\alpha)(\delta\alpha'-\delta\alpha);$$

confining ourselves then, for the moment, to the function V, and observing that we have by CXII. the formula,

$$\text{CXVII.} \ldots S(\tau\delta\alpha - \tau'\delta\alpha') = D_r V.\delta r + D_{r'}V.\delta r' + D_s V.\delta s,$$

in which the variations $\delta\alpha, \delta\alpha'$ are arbitrary, we find the expressions,

$$\text{CXVIII.} \ldots \tau = -\alpha r^{-1}D_r V + (\alpha'-\alpha)s^{-1}D_s V;$$

$$\text{CXVIII}'. \ldots \tau' = +\alpha' r'^{-1}D_{r'}V + (\alpha'-\alpha)s^{-1}D_s V;$$

which give these others,

$$\text{CXIX.} \ldots D_r V = rV(\alpha'-\alpha)\tau : V\alpha\alpha';$$

$$\text{CXIX}'. \ldots D_{r'}V = r'V(\alpha-\alpha')\tau' : V\alpha\alpha';$$

and

$$\text{CXX.} \ldots D_s V = s\beta : V\alpha\alpha',$$

because

$$V\alpha\tau = V\alpha'\tau' = \beta.$$

(51.) But, by XCII'.,

$$\text{CXXI.} \ldots r\tau + r'\tau' = (r+r')\upsilon' \parallel \upsilon \parallel \alpha' - \alpha,$$

the chord TT' of the hodograph, in figs. 86, 87, being divided at U' into segments TU', U'T', which are inversely as the distances r, r', or as the lines OP, OP' in the orbit; we have therefore the partial differential equation,

$$\text{CXXII.} \ldots D_r V = D_{r'}V, \quad \text{and similarly,} \quad \text{CXXIII.} \ldots D_r F = D_{r'}F;$$

so that *each* of the two functions, F and V, depends on the *distances* r, r', only by depending on their *sum*, $r + r'$.

(52.) Hence, if for greater generality we now treat M as *variable*, the *Principal Function* F, and therefore by CXIV. its partial derivative $H = -(D_t F)$, are functions of the *four* scalars,

$$\text{CXXIV.} \ldots r + r', \quad s, \quad t, \quad \text{and} \quad M.$$

(53.) And in like manner, the *Characteristic Function* (or *Action-Function*) V, and *its* partial derivative (by CXV.) the *Time*, $t = \mathrm{D}_H V$, may be considered as functions of this *other system* of four scalars (comp. (47.)),

$$\text{CXXV.} \ldots r + r', \quad s, \quad H, \quad \text{and} \quad M;$$

no knowledge whatever being here assumed, of the form or properties of the *orbit*, but only of the *law* of attraction.

(54.) But this dependence of the *time*, t, on the four scalars CXXV., is a new form of *Lambert's Theorem* (47.); which celebrated theorem is thus obtained in a new way, by the foregoing *quaternion analysis*.

(55.) Squaring the equations CXVIII. CXVIII'., attending to the relation CXXII., and changing signs, we get these new partial differential equations,

$$\text{CXXVI.} \ldots 2P + 2H = (\mathrm{D}_r V)^2 + (\mathrm{D}_s V)^2 + \frac{r^2 - r'^2 + s^2}{rs} \mathrm{D}_r V . \mathrm{D}_s V;$$

$$\text{CXXVI'.} \ldots 2P' + 2H = (\mathrm{D}_{r'} V)^2 + (\mathrm{D}_s V)^2 + \frac{r'^2 - r^2 + s^2}{r's} \mathrm{D}_{r'} V . \mathrm{D}_s V;$$

because

$$\text{CXXVII.} \ldots \alpha^2 = -r^2, \quad \alpha'^2 = -r'^2, \quad (\alpha' - \alpha)^2 = -s^2.$$

Hence, by merely algebraical combinations (because $P = Mr^{-1}$, and $P' = Mr'^{-1}$), we find:

$$\text{CXXVIII.} \ldots \tfrac{1}{2}((\mathrm{D}_r V)^2 + (\mathrm{D}_s V)^2) = H + \frac{M}{r + r' + s} + \frac{M}{r + r' - s};$$

$$\text{CXXIX.} \ldots \mathrm{D}_r V . \mathrm{D}_s V = \frac{M}{r + r' + s} - \frac{M}{r + r' - s};$$

$$\text{CXXX.} \ldots (\mathrm{D}_r V + \mathrm{D}_s V)^2 = 2H + \frac{4M}{r + r' + s} = M\left(\frac{4}{r + r' + s} - \frac{1}{a}\right);$$

$$\text{CXXX'.} \ldots (\mathrm{D}_r V - \mathrm{D}_s V)^2 = 2H + \frac{4M}{r + r' - s} = M\left(\frac{4}{r + r' - s} - \frac{1}{a}\right).$$

(56.) But, by CXII. CXVII. CXXII., we have the variation,

$$\text{CXXXI.} \ldots \delta V - t\delta H = \tfrac{1}{2}(\mathrm{D}_r V + \mathrm{D}_s V)\,\delta\,(r + r' + s)$$
$$+ \tfrac{1}{2}(\mathrm{D}_r V - \mathrm{D}_s V)\,\delta\,(r + r' - s);$$

and the function V vanishes with t, and therefore with s, at least at the commencement of the motion; whence it is easy to infer the expressions,*

$$\text{CXXXII.} \ldots V = \int_{-s}^{s} \left(\frac{M}{r + r' + s} + \frac{H}{2} \right)^{\frac{1}{2}} ds = \int_{-s}^{s} \left(\frac{M}{r + r' + s} - \frac{M}{4a} \right)^{\frac{1}{2}} ds;$$

$$\text{CXXXIII.} \ldots t = \tfrac{1}{4} \int_{-s}^{s} \left(\frac{M}{r + r' + s} + \frac{H}{2} \right)^{-\frac{1}{2}} ds = \tfrac{1}{2} \int_{-s}^{s} \left(\frac{4M}{r + r' + s} - \frac{M}{a} \right)^{-\frac{1}{2}} ds.$$

As a verification,† when t and s are small, and therefore r' nearly $= r$, we have thus the approximate values,

$$\text{CXXXIV.} \ldots V = (2P + 2H)^{\frac{1}{2}} s = (2T)^{\frac{1}{2}} s = 2Tt;$$

$$\text{CXXXV.} \ldots t = (2P + 2H)^{-\frac{1}{2}} s = (2T)^{-\frac{1}{2}} s;$$

in which s may be considered to be a *small arc* of the *orbit*, and $(2T)^{\frac{1}{2}}$ the *velocity* with which that arc is described.

(57.) Some not inelegant constructions, deduced from the theory of the hodograph, might be assigned for the case of a *closed orbit*, to represent the *excentric* and *mean anomalies*; but whether the orbit be closed or *not*, the *arc* TMT′ of the *hodographic circle*, in fig. 86, represents the arc of *true anomaly* described: for it subtends at the hodographic *centre* H an *angle* THT′, which is equal to the *angular motion* POP′ in the orbit.

(58.) We may add that, whatever the *special form* of the orbit may be, the equations CXVIII. CXVIII′. give, by CXXII.,

$$\text{CXXXVI.} \ldots \tau' - \tau = (\mathrm{U}\alpha' + \mathrm{U}\alpha)\, \mathrm{D}_r V;$$

from which it follows that the *chord* TT′ of the hodograph is *parallel* to the *bisector* of the *angle* POP′ in the orbit: and therefore, by XCI., that this angle is bisected by OQ in fig. 87, so that the *segments* PR, RP′, in that figure, of the *chord* PP′ *of the orbit*, are *inversely proportional* to the segments TU′, U′T′ of the *chord* TT′ *of the hodograph.*

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* Expressions by definite integrals equivalent to these, for the *action* and *time* in the relative motion of a binary system, were deduced by the present writer, but by an entirely different analysis, in the *First Essay*, &c., already cited, and will be found in the *Phil. Trans.* for 1834, Part ii., pages 285, 286. It is supposed that the radical in CXXXIII. does not become infinite within the extent of the integration; if it did so become, transformations would be required, on which we cannot enter here.

† An analogous verification may be applied to the definite integral LXXXI.; in which however it is to be observed that *both* $r + r'$ and s *vary*, along with the variable w: whereas, in the recent integrals CXXXII. CXXXIII., $r + r'$ is treated as *constant*.

(59.) We arrive then thus, in a new way, and as a new verification, at this known theorem: that *if two tangents* (QP, QP′) *to a conic section be drawn from any common point* (Q), *they subtend equal angles at a focus* (O), whatever the special form of the conic may be.

(60.) And although, in several of the preceding sub-articles, *geometrical constructions* have been used only to *illustrate* (and so to *confirm*, if confirmation were needed) *results* derived through *calculation with quaternions*; yet the eminently *suggestive* nature of the present Calculus enables us, in this as in many other questions, to *dispense with its own processes*, when once they have indicated a definite train of *geometrical investigation*, to serve as their substitute.

(61.) Thus, after having in any manner been led to perceive that, for the motion above considered, the *hodograph* is a *circle** (5.), of which the *radius* HT is equal (7.) to the attracting *mass M*, divided by the constant *parallelogram* (16.) under the vectors OP, OT of position and velocity, in the recent figures 86 and 87, which parallelogram is equal to the *rectangle* under the distance OP in the orbit, and the *perpendicular* OZ let fall from the centre O of force on the tangent UT to the hodograph, we see *geometrically* that the *potential P*, or the mass divided by the distance, for the point P of the orbit corresponding to the point T of the hodograph, is equal (as in (27.)) to the rectangle under HT and OZ, and therefore, by the similar triangles HTV, UOZ, to the rectangle under OU and TV (as in (29.)).

(62.) In like manner, the three potentials corresponding to the second point T′ of the first hodograph, and to the points W and W′ of the second hodograph, in fig. 86, are respectively equal to the rectangles under the same line OU, and the three other perpendiculars T′V′, WX, W′X′, on what we have called (29.) the *hodographic axis*, HL; so that, for these *two pairs of points*, in which the *two circular hodographs*, with a *common chord* MM′, are cut by a *common orthogonal* with U for centre, the *four potentials* are directly proportional to the *four hodographic ordinates* (29.).

(63.) And because the force (Mr^{-2}) is equal to the *square* of the *potential* (Mr^{-1}), divided by the *mass* (M), the *four forces* are directly as the *squares* of the *four ordinates* corresponding; each force, when divided by the square of

* This follows, among other ways, from the general value XXVI. for the *radius of curvature* of the hodograph, with *any law* of *central force*; which value was *geometrically deduced*, as stated in the Note to page 302, compare the Note to page 300, by the present writer, in a Paper read before the Royal Irish Academy in 1846, and published in their *Proceedings*. In fact, that *general expression* for the radius of hodographic curvature may be obtained with great facility, by dividing the element fdt of the hodograph (in which f denotes the force), by the corresponding element $cr^{-2}dt$ of angular motion in the orbit.

the corresponding hodographic ordinate, giving the constant or *common quotient*,

$$\text{CXXXVII.} \ldots \overline{\text{OU}}^2 : M.$$

(64.) It has been already seen (31.) to be a geometrical consequence of the two pairs of similar triangles, NTT$_,$, NT$_,$T$'$, and NTV, NT$'$V$'$, that the *two small arcs* of the *first hodograph*, near T and T$'$, intercepted between two near secants from the pole N of the *fixed chord* MM$'$, or between two near orthogonal circles, with U and U$_,$ for centres, are proportional to the *two ordinates*, TV, T$'$V$'$.

(65.) Accordingly, if we draw, as in fig. 86, the *near radius* (represented by a dotted line from H) of the first hodograph, and also the *small perpendicular* UY, erected at the centre U of the first orthogonal to the tangent UT, and terminated in Y by the tangent from the near centre U$_,$, the two new pairs of similar triangles, THT$_,$, UTY, and THV, UU$_,$Y, give the proportion,

$$\text{CXXXVIII.} \ldots \overline{\text{TT}_,} : \overline{\text{TV}} = \overline{\text{UU}_,} : \overline{\text{UT}};$$

which not merely confirms what has just been stated (64.), for the case of the *first* hodograph, but proves that the *four small arcs*, of the *two circular hodographs* in fig. 86, intercepted between the two near orthogonals, are directly proportional to the *four ordinates* already mentioned.

(66.) But the *time* of describing any small hodographic arc is the quotient (32.) of that *arc* divided by the *force*; and therefore, by (63.), (65.), the *four small times* are *inversely proportional* to the *four ordinates*. And the *harmonic mean* U$'$L between the *two* ordinates TV, T$'$V$'$ of the *first* hodograph, does not vary when we pass to the *second*, or to *any other hodograph*, with the *same fixed chord* MM$'$, and the *same orthogonal circles*; it follows then, *geometrically*, that the *sum* (33.) of the *two small times* is the *same*, in any *one* hodograph as in any *other*, under the conditions above supposed: and that this sum is equal to the expression,

$$\text{CXXXIX.} \ldots \frac{2M . \overline{\text{UU}'}}{\overline{\text{OU}}^2 . \overline{\text{UT}} . \overline{\text{U}'\text{L}}} = \frac{2M . \overline{\text{UU}'} . \overline{\text{UL}}}{\overline{\text{OU}}^2 . \overline{\text{LM}}^2 . \overline{\text{UT}}},$$

which agrees with the formula LXXIII.

(67.) On the whole, then, it is found that the *Theorem of Hodographic Isochronism* (38.) admits of being *geometrically** *proved*, although by processes

* It appears from an unprinted memorandum, to have been nearly thus that the author orally deduced the theorem, in his communication of March, 1847, to the Royal Irish Academy; although, as usually happens in cases of invention, his own previous processes of investigation had involved principles and methods, of a much less simple character.

suggested (60.) by quaternions: and sufficient *hints* have been already given, in connexion with fig. 87, as regards the *geometrical passage* from *that* theorem to the well-known *Theorem of Lambert*, without necessarily employing any property of *conic sections*.

420. As a *fifth specimen*, we shall deduce by quaternions an equation, which is adapted to assist in the determination of the *distance* of a *comet*, or *new planet*, from the *earth*.

(1.) Let M be the mass of the sun, or (somewhat more exactly) the sum of the masses of sun and earth; and let α and ω be the heliocentric vectors of earth and comet. Write also,

$$\text{I.} \ldots \mathrm{T}\alpha = r, \quad \mathrm{T}\omega = w, \quad \mathrm{T}(\omega - \alpha) = z, \quad \mathrm{U}(\omega - \alpha) = \rho,$$

so that r and w are the distances of earth and comet from the sun, while z is their distance from each other, and ρ is the unit-vector, directed from earth to comet. Then (comp. 419, I.),

$$\text{II.} \ldots \mathrm{D}^2\alpha = -Mr^{-3}\alpha, \quad \mathrm{D}^2\omega = -Mw^{-3}\omega,$$

and

$$\text{III.} \ldots \mathrm{D}^2 \,.\, z\rho = \mathrm{D}^2(\omega - \alpha) = M(r^{-3} - w^{-3})\alpha - Mzw^{-3}\rho,$$

with

$$\text{IV.} \ldots w^2 = -(\alpha + z\rho)^2 = r^2 + z^2 - 2z\mathrm{S}\alpha\rho.$$

(2.) The vector α, with its tensor r, and the mass M, are given by the theory of the earth (or sun); and ρ, $\mathrm{D}\rho$, $\mathrm{D}^2\rho$ are deduced from three (or more) near observations of the comet; operating then on III. with $\mathrm{S} \,.\, \rho\mathrm{D}\rho$, we arrive at the formula,

$$\text{V.} \ldots \frac{\mathrm{S}\rho\mathrm{D}\rho\mathrm{D}^2\rho}{\mathrm{S}\rho\mathrm{D}\rho\mathrm{U}\alpha} = \frac{r}{z}\left(\frac{M}{r^3} - \frac{M}{w^3}\right);$$

which becomes by IV., when cleared of fractions and radicals, and divided by z, an algebraical equation of the seventh degree, whereof one root is the sought distance* z, of the comet, (or planet) from the earth.

421. As a *sixth specimen*, we shall indicate a method, suggested by quaternions, of developing and geometrically decomposing the disturbing force of the sun on the moon, or of a relatively superior on a relatively inferior planet.

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* Compare the equation in the *Mécanique Céleste* (Tom. I., p. 241, new edition, Paris, 1843). Laplace's rule for determining, by inspection of a globe, which of the two bodies is the nearer to the sun, results at once from the formula V.

(1.) Let α, σ be the geocentric vectors of moon and sun; r, s their geocentric distances ($r = \mathrm{T}\alpha$, $s = \mathrm{T}\sigma$); M the sum of the masses of earth and moon; and S the mass of the sun; then the differential equation of motion of the moon about the earth may be thus written (comp. 418, 419),

$$\text{I.} \ldots \mathrm{D}^2\alpha = M \,.\, \phi\alpha + S \,.\, (\phi\sigma - \phi(\sigma - \alpha)),$$

if D be still the mark of derivation relatively to the time, and

$$\text{II.} \ldots \phi\alpha = \phi(\alpha) = \alpha^{-1}\mathrm{T}\alpha^{-1};$$

so that $\phi\alpha$ is here a *vector-function* of α, but *not* a *linear* one.

(2.) If we confine ourselves to the *term* $M\phi\alpha$, in the second member of I., we fall back on the equation 419, I., and so are conducted anew to the laws of *undisturbed relative elliptic motion.*

(3.) If we denote the *remainder* of that second member by η, then η may be called the *Vector of Disturbing Force*; and we propose now to *develope* this vector, according to *descending powers* of $\mathrm{T}(\sigma : \alpha)$, or according to *ascending powers* of the *quotient* $r : s$, of the *distances* of moon and sun from the earth.

(4.) The expression for that vector may be thus transformed :*

$$\text{III.} \ldots \textit{Vector of Disturbing Force} = \eta = \mathrm{D}^2\alpha - M\phi\alpha$$
$$= Ss^{-1}\sigma^{-1}\{1 - (1 - \alpha\sigma^{-1})^{-1}\mathrm{T}(1 - \alpha\sigma^{-1})^{-1}\}$$
$$= Ss^{-1}\sigma^{-1}\{1 - (1 - \alpha\sigma^{-1})^{-\frac{3}{2}}\ (1 - \sigma^{-1}\alpha)^{-\frac{1}{2}}\}$$
$$= Ss^{-1}\sigma^{-1}\left\{1 - \left(1 + \frac{3}{2}\alpha\sigma^{-1} + \frac{3\,.\,5}{2\,.\,4}(\alpha\sigma^{-1})^2 + \ldots\right)\left(1 + \tfrac{1}{2}\sigma^{-1}\alpha + \frac{1\,.\,3}{2\,.\,4}(\sigma^{-1}\alpha)^2 + \ldots\right)\right\};$$

that is,

$$\text{IV.} \ldots \eta = \eta_1 + \eta_2 + \eta_3 + \&c.,$$

if

$$\text{V.} \ldots \eta_1 = -\ Ss^{-1}\sigma^{-1}(\tfrac{1}{2}\sigma^{-1}\alpha + \tfrac{3}{2}\alpha\sigma^{-1}) = \frac{S}{2s^3}(\alpha + 3\sigma\alpha\sigma^{-1}) = \eta_{1,\,1} + \eta_{1,\,2};$$

$$\text{VI.} \ldots \eta_2 = \frac{3Sr^2}{8s^5}(\alpha\sigma\alpha^{-1} + 2\sigma + 5\sigma\alpha\sigma\alpha^{-1}\sigma^{-1}) = \eta_{2,\,1} + \eta_{2,\,2} + \eta_{2,\,3};\ \&c.$$

the *general term*† of this development being easily assigned.

* [Observe that $(\sigma - \alpha)^{-1} = \{(1 - \alpha\sigma^{-1})\sigma\}^{-1} = \sigma^{-1}(1 - \alpha\sigma^{-1})^{-1}$.]

† Such a general term was in fact assigned and interpreted in a communication of June 14, 1847, to the Royal Irish Academy (*Proceedings*, vol. iii., p. 514); and in the *Lectures*, page 616. The development may also be obtained, although less easily, by *Taylor's Series* adapted to quaternions. Compare pages 473, 475, 477, 478, vol. i. of the present work; and see page 358, vol. i., &c., for the interpretation of such symbols as $\sigma\alpha\sigma^{-1}$, $\alpha\sigma\alpha^{-1}$.

(5.) We have thus a *first group* of *two* component and disturbing forces, which are of the same order as $\frac{Sr}{s^3}$; a *second group* of *three* such forces, of the same order as $\frac{Sr^2}{s^4}$; a *third* group of *four* forces, and so on.

(6.) The *first component* of the *first group* has the following tensor and versor,

$$\text{VII.} \ldots \mathrm{T}\eta_{1,1} = \frac{Sr}{2s^3}, \quad \mathrm{U}\eta_{1,1} = \mathrm{U}\alpha;$$

it is therefore a purely *ablatitious force* MN, acting along the moon's geocentric vector EM prolonged, as in the annexed figure 88.

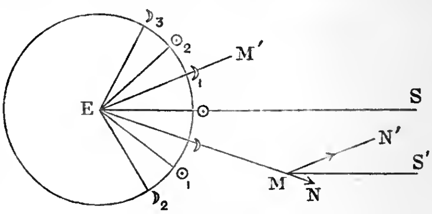

Fig. 88.

(7.) The *second component* MN′, of the same first group, has an exactly *triple intensity* $\overline{\mathrm{MN}'} = 3\overline{\mathrm{MN}}$; and its *direction* is such that the *angle* NMN′, between these two forces of the first group, is *bisected* by a line MS′ from the moon, which is *parallel to the sun's geocentric vector* ES.

(8.) If then we conceive a line EM′ from the earth, having the same direction as the last force MN′, this new line will meet the heavens in what may be called for the moment a *fictitious moon* ☽$_1$, such that the *arc* ☽☽$_1$ of a *great circle*, connecting it with the *true moon* ☽ in the heavens, shall be *bisected by the sun* $\odot$, as represented in fig. 88.

(9.) Proceeding to the *second group* (5.), we have by VI. for the *first component* of this group,

$$\text{VIII.} \ldots \mathrm{T}\eta_{2,1} = \frac{3Sr^2}{8s^4}, \quad \mathrm{U}\eta_{2,1} = \mathrm{U}\alpha\sigma\alpha^{-1} = \frac{\alpha\mathrm{U}\sigma}{\alpha};$$

a line from the earth, parallel to this new force, meets therefore the heavens in what may be called a *first fictitious sun*, $\odot_1$, such that the arc of a great circle, $\odot\odot_1$, connecting it with the true sun, is *bisected by the moon* ☽, as in the same fig. 88.

(10.) The *second* component force, of the same second group, has an *intensity* exactly *double* that of the *first* ($\mathrm{T}\eta_{2,2} = 2\mathrm{T}\eta_{2,1}$); and in *direction* it is parallel to the sun's geocentric vector ES, so that a line drawn in its direction from the earth would meet the heavens in the place of the sun $\odot$.

(11.) The *third* component of the present group has an intensity which is precisely *five-fold* that of the *first* component ($\mathrm{T}\eta_{2,3} = 5\mathrm{T}\eta_{2,1}$); and a line drawn in its direction from the earth meets the heavens in a *second fictitious sun* $\odot_2$, such that the arc $\odot_1\ \odot_2$, connecting these *two* fictitious suns, is *bisected by the true sun* $\odot$.

(12.) There is no difficulty in extending this analysis, and this interpretation, to *subsequent groups* of component disturbing forces, which forces *increase in number*, and *diminish in intensity*, in passing from any one group to the next; their *intensities*, for each *separate* group, bearing *numerical ratios* to each other, and their *directions* being connected by simple *angular relations*.

(13.) For example, the *third group* consists (5.) of *four small forces*, $\eta_{3,1} \ldots \eta_{3,4}$, of which the *intensities* are represented by $\frac{Sr^3}{16s^5}$, multiplied respectively by the four whole numbers, 5, 9, 15, and 35; and which have *directions* respectively parallel to lines drawn from the earth, towards a second fictitious moon ☽$_2$, the true moon, the first fictitious moon ☽$_1$ (8.), and a third fictitious moon ☽$_3$; these *three* fictitious *moons*, like the *two* fictitious *suns* lately considered, being all situated in the *momentary plane* of the *three bodies* E, M, S: and the *three* celestial *arcs*, ☽$_2$☽, ☽☽$_1$, ☽$_1$☽$_3$, being each equal to double the arc ☽$\odot$ of apparent *elongation* of sun from moon in the heavens, as indicated in the above cited fig. 88.

(14.) An exactly similar method may be employed to develop or decompose the disturbing force of one *planet* on another, which is nearer than it to the sun; and it is important to observe that no supposition is here made, respecting any *smallness* of *excentricities* or *inclinations*.

422. As a *seventh specimen* of the physical application of quaternions, we shall investigate briefly the construction and some of the properties of *Fresnel's Wave Surface*, as deductions from his principles or hypotheses* respecting light.

(1.) Let ρ be a *Vector of Ray-Velocity*, and μ the corresponding *Vector of Wave-Slowness* (or *Index-Vector*), for propagation of light from an origin O, within a biaxal crystal; so that

$$\text{I.} \ldots S\mu\rho = -1; \quad \text{II.} \ldots S\mu\delta\rho = 0; \quad \text{and therefore} \quad \text{III.} \ldots S\rho\delta\mu = 0,$$

* The present writer desires to be understood as not expressing any opinion of his own, respecting these or any rival hypotheses. In the next Series (423), as an *eighth specimen* of application, he proposes to deduce, from a *quite different set of physical principles respecting light*, expressed however still in the language of the present Calculus, MacCullagh's Theorem of the *Polar Plane*; intending then, as a *ninth* and *final specimen*, to give briefly a quaternion transformation of a celebrated equation in partial differential coefficients, of the first order and second degree, which occurs in the theory of *heat*, and in that of the *attraction of spheroids*.

if $\delta\rho$ and $\delta\mu$ be any infinitesimal variations of the vectors ρ and μ, consistent with the scalar equations (supposed to be as yet unknown), of the *Wave-Surface* and its *Reciprocal* (with respect to the unit-sphere round o), namely the *Surface of Wave-Slowness*, or (as it has been otherwise called) the *Index**-*Surface*: the velocity of light in a vacuum being here represented by unity.

(2.) The variation $\delta\rho$ being next conceived to represent a small *displacement*, tangential to the wave, of a particle of ether in the crystal, it was supposed by Fresnel that such a displacement $\delta\rho$ gave rise to an *elastic force*, say $\delta\epsilon$, *not generally* in a direction exactly *opposite* to that displacement, but still a *function* thereof, which function is of the kind called by us (in the Sections III. ii. 6, and III. iii. 7) *linear*, *vector*, and *self-conjugate*; and which there will be a convenience (on account of its connexion with certain *optical constants*, *a*, *b*, *c*) in denoting here by $\phi^{-1}\delta\rho$ (instead of $\phi\delta\rho$): so that we shall have the two converse formulæ,

$$\text{IV.} \ldots \delta\rho = \phi\delta\varepsilon; \qquad \text{V.} \ldots \delta\varepsilon = \phi^{-1}\delta\rho.$$

(3.) The *ether* being treated as *incompressible*, in the theory here considered, so that the *normal component* $\mu^{-1}\mathrm{S}\mu\delta\varepsilon$ of the elastic force may be neglected, or rather suppressed, there remains only the *tangential component*,

$$\text{VI.} \ldots \mu^{-1}\mathrm{V}\mu\delta\varepsilon = \delta\varepsilon - \mu^{-1}\mathrm{S}\mu\delta\varepsilon,$$

as regulating the *motion*, tangential to the wave, of a disturbed and *vibrating particle*.

(4.) If then it be admitted that, for the propagation of a *rectilinear vibration*, tangential to a wave of which the velocity is $\mathrm{T}\mu^{-1}$, the *tangential force* (3.) must be *exactly opposite* in direction to the *displacement* $\delta\rho$, and *equal* in quantity to that displacement multiplied by the *square* ($\mathrm{T}\mu^{-2}$) of the *wave-velocity*, we have, by V. and VI., the equation,

$$\text{VII.} \ldots \phi^{-1}\delta\rho - \mu^{-1}\mathrm{S}\mu\delta\varepsilon = \mu^{-2}\delta\rho,$$

or

$$\text{VIII.} \ldots \delta\rho = (\phi^{-1} - \mu^{-2})^{-1}\mu^{-1}\mathrm{S}\mu\delta\varepsilon;$$

* This brief and expressive name was proposed by the late Prof. MacCullagh (*Transactions* R. I. A., vol. xviii., part i., page 38), for that *reciprocal* of the wave-surface which the present writer had previously called the *Surface of Components of Wave-Slowness*, and had employed for various purposes: for instance, to pass from the *conical cusps* to the *circular ridges* of the *Wave*, and so to establish a geometrical connexion between the theories of the *two conical refractions*, *internal* and *external*, to which his own methods had conducted him (*Transactions* R. I. A., vol. xvii., part i., pages 125–144). He afterwards found that the same Surface had been otherwise employed by M. Cauchy (*Exercises de Mathématiques*, 1830, page 36), who did not seem however to have perceived its *reciprocal relation* to the Wave.

combining which with II., we obtain at once this *Symbolical Form* of the scalar equation of the *Index Surface*,

$$\text{IX.} \ldots 0 = S\mu^{-1}(\phi^{-1} - \mu^{-2})^{-1}\mu^{-1};$$

or by an easy transformation,

$$\text{X.} \ldots 1 = S\mu\phi^{-1}(\phi^{-1} - \mu^{-2})^{-1}\mu^{-1};$$

or finally,

$$\text{XI.} \ldots 1 = S\mu(\mu^2 - \phi)^{-1}\mu;$$

while the *direction* of the vibration $\delta\rho$, for any given tangent plane to the wave, is determined *generally* by the formula VIII.

(5.) That formula for the displacement, combined with the expression V. for the elastic force resulting, gives

$$\text{XII.} \ldots \delta\rho = -\phi\upsilon S\mu\delta\epsilon, \quad \text{and} \quad \text{XIII.} \ldots \delta\epsilon = -\upsilon S\mu\delta\epsilon,$$

if

$$\text{XIV.} \ldots (\phi - \mu^2)\upsilon = \mu, \quad \text{or} \quad \text{XV.} \ldots \upsilon = (\phi - \mu^2)^{-1}\mu,$$

υ being thus an auxiliary vector; and because the equation XI. of the index surface gives,

$$\text{XVI.} \ldots S\mu\upsilon = -1, \quad \text{while} \quad \text{XVII.} \ldots V\upsilon\delta\epsilon = 0, \qquad \text{by XIII.},$$

it follows that the vector υ, if drawn like ρ and μ from o, *terminates on the tangent plane to the wave*, and is *parallel to the direction of the elastic force*.

(6.) The equations XIV. XVI. give,

$$\text{XVIII.} \ldots \mu^2\upsilon^2 - S\upsilon\phi\upsilon = 1,$$

whence

$$\text{XIX.} \ldots \upsilon^2 S\mu\delta\mu = S\mu\delta\upsilon = -S\upsilon\delta\mu,$$

because $\delta S\mu\upsilon = 0$, by XVI., and $\delta S\upsilon\phi\upsilon = 2S(\phi\upsilon \,.\, \delta\upsilon)$, by the self-conjugate property of ϕ; comparing then XIX. with III., we see that $\pm\rho$ (as being $\perp$ every $\delta\mu$) has the direction of $\mu + \upsilon^{-1}$, and therefore, by I. and XVI., that we may write,

$$\text{XX.} \ldots \rho^{-1} = -\mu - \upsilon^{-1}; \quad \text{XXI.} \ldots \rho^{-2} = \mu^2 - \upsilon^{-2}; \quad \text{XXII.} \ldots S\rho\upsilon = 0;$$

which last equation shows, by (5.), that *the ray is perpendicular* (on Fresnel's principles) *to the elastic force* $\delta\epsilon$, produced by the displacement $\delta\rho$.

(7.) The equations XX. and XXI. show by XIV. that

$$\text{XXIII.} \ldots (\rho^{-2} - \phi)\upsilon = \rho^{-1}, \quad \text{whence} \quad \text{XXIV.} \ldots \upsilon = (\rho^{-2} - \phi)^{-1}\rho^{-1};$$

we have therefore, by XXII., the following *Symbolical Form* (comp. (4.)) *of the Equation of the Wave Surface,*

$$\text{XXV.} \ldots 0 = S\rho^{-1}(\phi - \rho^{-2})^{-1}\rho^{-1};$$

or, by transformations analogous to X. and XI.,

$$\text{XXVI.} \ldots 1 = S\rho\phi(\phi - \rho^{-2})^{-1}\rho^{-1}; \quad \text{XXVII.} \ldots 1 = S\rho(\rho^2 - \phi^{-1})^{-1}\rho;$$

and we see that we can *return* from each *equation of the wave*, to the corresponding *equation of the index surface*, by merely changing ρ to μ, and ϕ to ϕ^{-1}: but this result will soon be seen to be included in one more general, which may be called the *Rule of the Interchanges*.*

(8.) The equation XXV. may also be thus written,

$$\text{XXVIII.} \ldots S\rho(\phi - \rho^{-2})^{-1}\rho = 0;$$

---

* [Tait finds the envelope of the plane $S\mu\rho = -1$ subject to the condition XI., $1 = S\mu(\mu^2 - \phi)^{-1}\mu$, and thus obtains the equation of the wave surface. If we differentiate XI. and introduce the auxiliary vector υ of equation XIV., the result becomes $Sd\mu(\upsilon^{-1} + d\mu) = 0$. Also $S\rho d\mu = 0$, and as $d\mu$ is otherwise arbitrary

$$x\rho = \upsilon^{-1} + \mu.$$

Squaring this relation we find $x^2\rho^2 = \mu^2 - \upsilon^{-2}$, and operating by $S\mu$ we have $x = \mu^2 - \upsilon^{-2} = x^2\rho^2$. Thus $x = \rho^{-2}$ and we recover XX., whence the result follows as in the text.

The equation of the electro-magnetic wave surface has been obtained by Tait on the following lines. (*Proceedings R. S. E.* April 2, 1894, or *Scientific Papers*, vol. ii., pages 390–1.)

A system of plane waves running with normal velocity $v\alpha = -\mu^{-1}$, $(T\alpha = 1)$ is defined by

$$\theta_1 = \epsilon f(vt + S\alpha\rho), \quad \theta_2 = \eta f(vt + S\alpha\rho). \qquad \text{(i)}$$

These equations satisfy

$$\phi_1\dot{\theta}_1 = V\nabla\theta_2, \quad \phi_2\dot{\theta}_2 = -V\nabla\theta_1, \qquad \text{(ii)}$$

the quaternion equivalents of Clerk Maxwell's electro-magnetic equations provided

$$\phi_1\epsilon = V\mu\eta, \quad \phi_2\eta = -V\mu\epsilon. \qquad \text{(iii)}$$

Assuming the linear functions ϕ_1 and ϕ_2 to be both self-conjugate, we find on elimination of η,

$$\phi_1\epsilon = -V\mu\phi_2^{-1}V\mu\epsilon = -m_2^{-1}V\mu V\phi_2\mu\phi_2\epsilon,$$

if m_2 is the third invariant of ϕ_2. An easy step shows that

$$m_2\phi_1\epsilon + S\mu\phi_2\mu \,.\, \phi_2\epsilon = \phi_2\mu S\mu\phi_2\epsilon,$$

so that

$$\epsilon = (m_2\phi_1 + S\mu\phi_2\mu \,.\, \phi_2)^{-1}\phi_2\mu S\mu\phi_2\epsilon. \qquad \text{(iv)}$$

Operating on (iv) by $S\phi_2\mu$, we have the equation of the index surface

$$1 = S\mu\phi_2(m_2\phi_1 + S\mu\phi_2\mu \,.\, \phi_2)^{-1}\phi_2\mu. \qquad \text{(v)}$$

but under this last form it coincides with the equation 412, XLI.; hence, by 412, (19.) the *Wave Surface* may be *derived* from the *auxiliary* or *Generating Ellipsoid*,

$$\text{XXIX.} \ldots S\rho\phi\rho = 1,$$

by the following *Construction*, which was in fact assigned by Fresnel* himself, but as the result of far more complex calculations :—*Cut the ellipsoid (abc) by an arbitrary plane through its centre, and at that centre erect perpendiculars to that plane, which shall have the lengths of the semiaxes of the section; the locus of the extremities of the perpendiculars so erected will be the sought wave surface.*

(9.) And we see, by IX., that the *Index Surface* may be *derived*, by an exactly similar construction, from that *Reciprocal Ellipsoid*, of which the equation is, on the same plan,

$$\text{XXX.} \ldots S\rho\phi^{-1}\rho = 1.$$

(10.) If the scalar equations, XXVII. and XI., of the wave and index surface, be denoted by the abridged forms,

$$\text{XXXI.} \ldots f\rho = 1, \quad \text{and} \quad \text{XXXII.} \ldots F\mu = 1,$$

Comparing this equation with XI., we are led to assume $\mu' = \phi_2^{\frac{1}{2}}\mu$, and substitution in (v) affords the equation

$$1 = S\mu'(m_2\phi_2^{-\frac{1}{2}}\phi_1\phi_2^{-\frac{1}{2}} + \mu'^2)^{-1}\mu'. \qquad \text{(vi)}$$

The equation of the tangent plane to the wave surface is

$$S\mu\rho = -1, \quad \text{or} \quad S\mu'\phi_2^{-\frac{1}{2}}\rho = -1, \quad \text{or} \quad S\mu'\rho' = -1, \qquad \text{(vii)}$$

if $\rho' = \phi_2^{-\frac{1}{2}}\rho$. Comparing these results with I. and XI., we see that ρ', μ', and $-m_2\phi_2^{-\frac{1}{2}}\phi_1\phi_2^{-\frac{1}{2}}$ correspond respectively to Hamilton's ρ, μ, and ϕ, and we deduce the equation

$$1 = S\rho'(\rho'^2 + m_2^{-1}\phi_2^{\frac{1}{2}}\phi_1^{-1}\phi_2^{\frac{1}{2}})^{-1}\rho' \qquad \text{(viii)}$$

analogous to XXVII. It only remains to replace ρ' in terms of ρ by a transformation the converse of that from (v) to (vi), and we obtain the equation of the wave surface

$$1 = S\rho\phi_2^{-1}(S\rho\phi_2^{-1}\rho \,.\, \phi_2^{-1} + m_2^{-1}\phi_1^{-1})^{-1}\phi_2^{-1}\rho, \qquad \text{(ix)}$$

or by a transformation like that from XXVII. to XXVIII.

$$S\rho(\phi_2 + m_2 S\rho\phi_2^{-1}\rho \,.\, \phi_1)^{-1}\rho = 0. \qquad \text{(x)}$$

It will be noticed that the electro-magnetic wave-surface (ix) is produced from the Fresnel surface (viii) by the transformation or pure strain $\rho = \phi_2^{\frac{1}{2}}\rho'$, so that many of the theorems of these sub-articles can be extended to this more general case.]

* See Sir John F. W. Herschel's Treatise on Light, in the *Encyclopædia Metropolitana*, page 545, Art. 1017.

then the relations I. II. III. enable us to infer the expressions (comp. the notation in 418, (2.) [page 294]),

$$\text{XXXIII.} \ldots \mu = \frac{-D_\rho f\rho}{S\rho D_\rho f\rho}; \quad \text{XXXIV.} \ldots \rho = \frac{-D_\mu F\mu}{S\mu D_\mu F\mu};$$

in which (comp. 412, (36.), and the Note that sub-article [page 259]),

$$\text{XXXV.} \ldots \tfrac{1}{2}D_\rho f\rho = (\rho^2 - \phi^{-1})^{-1}\rho - \rho S\rho(\rho^2 - \phi^{-1})^{-2}\rho = -\omega - \omega^2\rho,$$

and

$$\text{XXXVI.} \ldots \tfrac{1}{2}D_\mu F\mu = (\mu^2 - \phi)^{-1}\mu - \mu S\mu(\mu^2 - \phi)^{-2}\mu = -\upsilon - \upsilon^2\mu;$$

υ being the same auxiliary vector XV. as before, and ω being a new auxiliary vector, such that (by XXIV. XXVII. and IX. XV.),

$$\text{XXXVII.} \ldots \omega = (\phi^{-1} - \rho^2)^{-1}\rho = \phi\upsilon; \quad \text{XXXVIII.} \ldots S\rho\omega = -1;$$
$$\text{XXXIX.} \ldots S\mu\omega = 0;$$

whence also $\omega \parallel \delta\rho$ by XII., so that (comp. (5.)) if ω be drawn (like ρ, μ, and υ) from the point O, *this new vector terminates on the tangent plane to the index surface*, and is *parallel to the displacement on the wave*; also $\delta\rho : \delta\epsilon = \omega : \upsilon$.

(11.) Hence, by XXXIII. XXXV. XXXVIII.,

$$\text{XL.} \ldots \mu = \frac{\omega + \omega^2\rho}{1 - \omega^2\rho^2} = \frac{\omega^{-1} + \rho}{\omega^{-2} - \rho^2} = -(\omega^{-1} + \rho)^{-1}, \quad \text{or} \quad \text{XLI.} \ldots -\mu^{-1} = \rho + \omega^{-1};$$

and similarly, by XXXIV. XXXVI. and XVI.,

$$\text{XLII.} \ldots \rho = \frac{\upsilon + \upsilon^2\mu}{1 - \upsilon^2\mu^2} = \frac{\upsilon^{-1} + \mu}{\upsilon^{-2} - \mu^2} = -(\upsilon^{-1} + \mu)^{-1}, \quad \text{or} \quad -\rho^{-1} = \mu + \upsilon^{-1}, \text{ as in XX.};$$

so that, with the help of the expressions XV. and XXXVII. for υ and ω, the *ray-vector* ρ and the *index-vector* μ are expressed as *functions of each other*: which functions are *generally definite*, although we shall soon see *cases*, in which one or other becomes partially *indeterminate*.

(12.) It is easy now to enunciate the *rule of the interchanges*, alluded to in (7.), as follows:—*In any formula involving the vectors*, ρ, μ, υ, ω, and the *functional symbol* ϕ, or some of them, it is *permitted to exchange* ρ *with* μ, υ *with* ω, *and* ϕ *with* ϕ^{-1}; *provided* that we at the same time interchange $\delta\rho$ with $\delta\epsilon$ (but *not** generally with $\delta\mu$), when either $\delta\rho$ or $\delta\epsilon$ occurs.

* It is true that, in passing from II. to III. (instead of passing to XLIII.), we may be said to have exchanged not only ρ with μ, but *also* $\delta\rho$ and $\delta\mu$. But *usually*, in the present investigation, $\delta\rho$ represents a small *displacement* (2.), which is conceived to have a *definite direction*, tangential to the wave; whereas $\delta\mu$ continues, as in (1.) to represent *any infinitesimal tangent* to the *index surface*, while $\delta\epsilon$ still denotes the *elastic force* (2.), resulting from the displacement $\delta\rho$.

For example, we pass thus from XX. to XLI., and conversely from the latter to the former; from II. we pass by the same rule, to the formula,

$$\text{XLIII.} \ldots S\rho\delta\epsilon = 0, \quad \text{which agrees by XVII. with XXII.};$$

and, as other verifications, the following equations may be noticed,

$$\text{XLIV.} \ldots \delta\rho = \mu V\mu\delta\epsilon; \quad \text{XLV.} \ldots \delta\epsilon = \rho V\rho\delta\rho; \quad \text{XLVI.} \ldots S\mu\delta\epsilon = S\rho\delta\rho.$$

(13.) The relations between the vectors may be illustrated by the annexed figure 89; in which,

$$\text{XLVII.} \ldots \text{OP} = \rho, \ \text{OQ} = \mu, \ \text{OU} = \upsilon, \ \text{OW} = \omega,$$

and

$$\text{XLVIII.} \ldots \text{OP}' = -\rho^{-1}, \quad \text{OQ}' = -\mu^{-1}, \quad \text{OU}' = -\upsilon^{-1}, \quad \text{OW}' = -\omega^{-1};$$

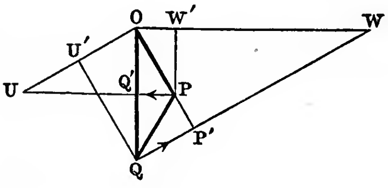

Fig. 89.

in fact it is evident on inspection, that

$$\text{XLIX.} \ldots \text{OP}.\text{OP}' = \text{OQ}.\text{OQ}' = \text{OU}.\text{OU}' = \text{OW}.\text{OW}';$$

and the common value of these four scalar products is here taken as negative unity.

(14.) As examples of such illustration, the equation XX. becomes $\text{P}'\text{O} = \text{QU}'$; XLI. becomes $\text{OQ}' = \text{W}'\text{P}$; XXIII. may be written as $\omega + \rho^{-1} = \rho^{-2}\upsilon$, or as $\text{P}'\text{W} : \text{OU} = \text{P}'\text{O} : \text{OP}$; &c. And because the lines PQ′U and QP′W are sections of the tangent planes, to the wave at the extremity P of the ray, and to the index surface at the extremity Q of the index vector, made by the plane of those two vectors ρ and μ, while $\delta\rho$ and $\delta\epsilon$ (as being parallel to ω and υ) have the directions of PQ′ and QP′; we see that the *displacement* (or vibration) has *generally*, in Fresnel's theory, the direction of the *projection of the ray on the tangent plane to the wave*; and that the *elastic force* resulting has the direction of the *projection of the index vector on the tangent plane to the index surface*: results which might however have been otherwise deduced, from the formulæ alone.

(15.) It may be added, as regards the reciprocal deduction of the two vectors μ and ρ from each other, that (by XLI. XXXVIII., and XX. XVI.) we have the expressions,

$$\text{L.} \ldots -\mu^{-1} = \omega^{-1}V\omega\rho, \quad \text{and} \quad \text{LI.} \ldots -\rho^{-1} = \upsilon^{-1}V\upsilon\mu;$$

which answer in fig. 89 to the relations, that OQ′ is the part (or component) of OP, perpendicular to OW; and that OP′ is, in like manner, the part of OQ ⊥ OU.

(16.) We have also the expressions,

$$\text{LII.} \ldots -\mu^{-1} = \omega^{-1}\mathrm{V}\omega\upsilon, \quad \text{and} \quad \text{LIII.} \ldots -\rho^{-1} = \upsilon^{-1}\mathrm{V}\upsilon\omega,$$

which may be similarly interpreted; and which conduct to the relations,

$$\text{LIV.} \ldots -(\mathrm{V}\upsilon\omega)^2 = \upsilon^2\rho^{-2} = \omega^2\mu^{-2} = \mathrm{S}\upsilon\omega.$$

Hence, *the Locus of each of the two Auxiliary Points* U and W, in fig. 89, *is a Surface of the Fourth Degree*; the scalar equations of these two loci being,

$$\text{LV.} \ldots (\mathrm{V}\upsilon\phi\upsilon)^2 + \mathrm{S}\upsilon\phi\upsilon = 0, \quad \text{and} \quad \text{LVI.} \ldots (\mathrm{V}\omega\phi^{-1}\omega)^2 + \mathrm{S}\omega\phi^{-1}\omega = 0;$$

from which it would be easy to deduce *constructions* for those surfaces, with the help of the two *reciprocal ellipsoids*, XXIX. and XXX.

(17.) The equations XII. XXII., combined with the self-conjugate property of ϕ, give

$$\text{LVII.} \ldots 0 = \mathrm{S}(\phi^{-1}\rho \,.\, \delta\rho), \quad \text{or} \quad \text{LVIII.} \ldots 0 = \delta\mathrm{S}\rho\phi^{-1}\rho;$$

hence (between suitable limits of the constant), *every ellipsoid* of the form,

$$\text{LIX.} \ldots \mathrm{S}\rho\phi^{-1}\rho = h^4 = \text{const.},$$

which is thus *concentric and coaxal with the reciprocal ellipsoid* XXX., being also *similar* to it, and *similarly placed*, *contains* upon its surface what may be called a *Line of Vibration* on the Wave*; the *intersection* of this *new* ellipsoid LIX. with the wave surface being generally such, that the *tangent* at each point of that line (or curve) has the *direction* of Fresnel's vibration.

(18.) The fundamental connexion (2.) of the *function* ϕ with the *optical constants*, a, b, c, of the crystal, is expressed by the *symbolical cubic* (comp. 350, I., and 417, XXV.),

$$\text{LX.} \ldots (\phi + a^{-2})(\phi + b^{-2})(\phi + c^{-2}) = 0;$$

from which it is easy to infer, by methods already explained, that if e be *any scalar*, and if we write,

$$\text{LXI.} \ldots E = (e - a^{-2})(e - b^{-2})(e - c^{-2}),$$

---

* Such *lines of vibration* were discussed by the present writer, but by means of a quite different analysis, in his Memoir of 1832 (*Third Supplement on Systems of Rays*), which was published in the following year, in the *Transactions* of the Royal Irish Academy. See reference in the Note to p. 324.

we have then this *formula of inversion*,

$$\text{LXII.} \ldots E(\phi + e)^{-1} = e^2 - e(\phi + a^{-2} + b^{-2} + c^{-2}) - a^{-2}b^{-2}c^{-2}\phi^{-1}.$$

(19.) Changing then e to $-\rho^{-2}$, the equation XXVIII. of the wave becomes,

$$\text{LXIII.} \ldots 0 = \rho^{-2} + a^{-2} + b^{-2} + c^{-2} + \mathrm{S}\rho^{-1}\phi\rho - a^{-2}b^{-2}c^{-2}\mathrm{S}\rho\phi^{-1}\rho :$$

the *Wave* is therefore (as is otherwise known) a *Surface of the Fourth Degree*: and (as is likewise well known), the *Index Surface* is of the *same* degree, its equation (found by changing ρ, ϕ, a, b, c to μ, ϕ^{-1}, a^{-1}, b^{-1}, c^{-1}) being, on the same plan,

$$\text{LXIV.} \ldots 0 = \mu^{-2} + a^2 + b^2 + c^2 + \mathrm{S}\mu^{-1}\phi^{-1}\mu - a^2b^2c^2\mathrm{S}\mu\phi\mu.$$

(20.) These equations may be variously transformed, with the help of the cubic LX. in ϕ, which gives the analogous cubic in ϕ^{-1},

$$\text{LXV.} \ldots (\phi^{-1} + a^2)(\phi^{-1} + b^2)(\phi^{-1} + c^2) = 0 ;$$

for instance, another form of the equation of the wave is,

$$\text{LXVI.} \ldots 0 = \mathrm{S}\rho\phi^{-2}\rho + (\rho^2 + a^2 + b^2 + c^2)\mathrm{S}\rho\phi^{-1}\rho - a^2b^2c^2 ;$$

in which it may be remarked that $\mathrm{S}\rho\phi^{-2}\rho = (\phi^{-1}\rho)^2 < 0$, whereas $\mathrm{S}\rho\phi^{-1}\rho > 0$.

(21.) Substituting then, for $\mathrm{S}\rho\phi^{-1}\rho$ in LXVI., its value h^4 from (17.), we find that this *second variable ellipsoid*, with h for an arbitrary constant or parameter,

$$\text{LXVII.} \ldots 0 = (\phi^{-1}\rho)^2 + h^4(\rho^2 + a^2 + b^2 + c^2) - a^2b^2c^2,$$

contains upon its surface the *same line of vibration* as the first variable ellipsoid LIX., which involves the same arbitrary constant h; and therefore that the *line* in question is a *quartic curve*, or *Curve of the Fourth Degree*, as being the intersection of these *two* variable but connected *ellipsoids*: and that the *wave* itself is the *locus* of all such quartic curves.

(22.) The *Generating Ellipsoid* ($\mathrm{S}\rho\phi\rho = 1$) has a, b, c for its semiaxes ($a > b > c > 0$); and for *any vector* ρ, in the *plane* of bc, we have the *symbolical quadratic* (comp. 353, (9.)),

$$\text{LXVIII.} \ldots (\phi + b^{-2})(\phi + c^{-2}) = 0,$$

or

$$\text{LXIX.} \ldots - b^{-2}c^{-2}\phi^{-1} = \phi + b^{-2} + c^{-2} ;$$

making then this last substitution for $\phi + b^{-2} + c^{-2}$ in LXIII., we find, for the *section* of the wave by this principal plane of the ellipsoid XXIX., an equation which breaks up into the *two factors*,

$$\text{LXX.} \ldots \rho^{-2} + a^{-2} = 0, \quad \text{and} \quad \text{LXXI.} \ldots 1 - b^{-2}c^{-2}S\rho\phi^{-1}\rho = 0;$$

whereof the *first* represents (the *plane* being understood) a *circle*, with *radius* = a, which we may call briefly *the circle* (a); while the *second* represents (with the same understanding) an *ellipse*, which may by analogy be called here *the ellipse* (a): its two semiaxes having the *lengths* of c and b, but in the *directions* of b and c, for which directions $\phi + b^{-2} = 0$ and $\phi + c^{-2} = 0$, respectively, so that *this* ellipse (a) is merely the *elliptic section* (bc) of the *ellipsoid* (abc), *turned through a right angle* in its own plane, as by the *construction* (8.) it evidently ought to be. And an exactly similar analysis shows, what indeed is otherwise known, that the plane of ca cuts the wave in the system of a circle (b), and an ellipse (b); and that the plane of ab cuts the same wave surface, in a *circle* (c), and an *ellipse* (c).

(23.) The circle (a) is entirely *exterior* to the ellipse (a); and the circle (c) is wholly *interior* to the ellipse (c); but the circle (b) *cuts* the ellipse (b), in *four real points*, which are therefore (in a sense to be soon more fully examined) *cusps* (or *nodal points*) on the wave surface, or briefly *Wave-Cusps*; and the *vectors* ρ, say $\pm\rho_0$ and $\pm\rho_1$, which are drawn from the centre o to these *four cusps*, may be called *Lines of Single Ray-Velocity*, or briefly *Cusp-Rays*.

(24.) It is clear, from the construction (8.), that these lines or rays must have the directions of the *cyclic normals* of the ellipsoid (abc); which suggests our using here the *cyclic forms*,

$$\text{LXXII.} \ldots \phi\rho = g\rho + V\lambda\rho\lambda',$$

and

$$\text{LXXIII.} \ldots S\rho\phi\rho = g\rho^2 + S\lambda\rho\lambda'\rho = 1,$$

for the function ϕ, and the generating ellipsoid (8.); λ' being written, to avoid confusion, instead of the μ of 357, &c., to represent the second cyclic normal.

(25.) Changing then μ to λ', ν to ρ, and g to $g - \rho^{-2}$, in the expression 361, XXVII.* for $F\nu$ or $S\nu\phi^{-1}\nu$; equating the result to zero, and resolving the equation so obtained, as a quadratic in g; we find this *new form* of the *Equation* XXVIII. *of the Wave*,

$$\text{LXXIV.} \ldots g\rho^2 = 1 + S\lambda\rho S\lambda'\rho \pm TV\lambda\rho TV\lambda'\rho;$$

the upper sign belonging to *one sheet*, and the lower sign to the *other sheet*,

* [This equation which occurs on page 549, vol. i., is

$$mF\nu = (g^2 - \lambda^2\mu^2)\nu^2 + \lambda^2(S\mu\nu)^2 + \mu^2(S\nu\lambda)^2 - 2gS\lambda\nu S\mu\nu.]$$

of that wave surface. The new equation may also be thus written, as an expression for the *inverse square* of the *ray-velocity* $T\rho$, or of the *radius-vector*, say r, of the wave,

$$\text{LXXV.} \ldots r^{-2} = T\rho^{-2} = \frac{a^{-2}+c^{-2}}{2} + \frac{a^{-2}-c^{-2}}{2}\cos\left(\angle\frac{\rho}{\lambda} \mp \angle\frac{\rho}{\lambda'}\right),$$

because, by 405, (2.), (6.), &c.,

$$\text{LXXVI.} \ldots a^{-2} = -g - T\lambda\lambda', \quad b^{-2} = -g + S\lambda\lambda', \quad c^{-2} = -g + T\lambda\lambda';$$

and we have the verification, for a *cusp-ray* (23.), that

$$\text{LXXVII.} \ldots r^{-2} = b^{-2}, \quad \text{or} \quad r = T\rho = b, \quad \text{if} \quad \rho \parallel \lambda \text{ or } \lambda'.$$

(26.) If we write (comp. XXXI.),

$$\text{LXXVIII.} \ldots f\rho = -\rho^{-2}(1 + S\rho\phi\rho) + a^{-2}b^{-2}c^{-2}S\rho\phi^{-1}\rho,$$

the equation LXIII. of the wave takes the form,

$$\text{LXXIX.} \ldots f\rho = a^{-2} + b^{-2} + c^{-2} = \text{const.};$$

and we have the partial derivative (comp. XXXV.),

$$\text{LXXX.} \ldots \tfrac{1}{2}D_\rho f\rho = \rho^{-3}(1 + S\rho\phi\rho) - \rho^{-2}\phi\rho + a^{-2}b^{-2}c^{-2}\phi^{-1}\rho$$
$$= \rho^{-3}(1 - V\rho\phi\rho) + a^{-2}b^{-2}c^{-2}\phi^{-1}\rho;$$

which gives by XXXIII. the expression,

$$\text{LXXXI.} \ldots \mu = \frac{\rho^{-3}(V\rho\phi\rho - 1) - a^{-2}b^{-2}c^{-2}\phi^{-1}\rho}{\rho^{-2} + a^{-2}b^{-2}c^{-2}S\rho\phi^{-1}\rho},$$

and therefore a *generally definite value* (comp. (11.)) for the *index vector* μ, when the *ray* ρ is given.

(27.) If the ray be *in the plane* of ac, then (comp. LXIX.),

$$\text{LXXXII.} \ldots \phi\rho + (a^{-2} + c^{-2})\rho + a^{-2}c^{-2}\phi^{-1}\rho = 0,$$

whence

$$\text{LXXXIII.} \ldots V\rho\phi\rho = -a^{-2}c^{-2}V\rho\phi^{-1}\rho = a^{-2}c^{-2}(S\rho\phi^{-1}\rho - \rho\phi^{-1}\rho);$$

and therefore by LXXXI.,

$$\text{LXXXIV.} \ldots \mu = \frac{\rho^{-3}(S\rho\phi^{-1}\rho - a^2c^2) - (\rho^{-2} + b^{-2})\phi^{-1}\rho}{b^{-2}(S\rho\phi^{-1}\rho - a^2c^2) + (\rho^{-2} + b^{-2})a^2c^2};$$

an expression which gives, *definitely*,

$$\text{LXXXV.} \ldots \mu = -\rho^{-1}, \quad \text{if} \quad \text{LXXXVI.} \ldots \rho^{-2} + b^{-2} = 0,$$

but *not*

$$\text{LXXXVII.} \ldots S\rho\phi^{-1}\rho = a^2c^2,$$

that is (comp. (22.)), if the ray terminate on the *circle* (*b*), at any point which is *not also on the ellipse* (*b*); and with equal *definiteness*,

$$\text{LXXXVIII.} \ldots \mu = -a^{-2}c^{-2}\phi^{-1}\rho, \text{ if LXXXVII. but } \textit{not} \text{ LXXXVI. hold good,}$$

that is, if the ray terminate on the *ellipse* (*b*), at any point which is *not also* on the *circle*.

(28.) The *normal* then to the *wave*, in each of the two cases last mentioned, *coincides* with the normal to the *section*, made by the plane of *ac*; and if we abstract for a moment from the *cusps* (23.), we see that the *wave* is *touched*, *along the circle* (*b*), by the concentric *sphere* LXXXVI. with radius = *b*, which we may call *the sphere* (*b*); and *along the ellipse* (*b*) by the concentric *ellipsoid* LXXXVII. which may on the same plan be called *the ellipsoid* (*b*).

(29.) An exactly similar analysis shows that the wave is touched along the circles (*a*) and (*c*), by two other concentric spheres, with radii *a* and *c*, which may be briefly called *the spheres* (*a*) *and* (*c*); and along the ellipses (*a*) and (*c*) by two other concentric and similar ellipsoids, which may by analogy be called *the ellipsoids* (*a*) *and* (*c*). And by comparing the equation LXXXVII. of the ellipsoid (*b*) with the form LIX., we see that the *three elliptic sections* (*a*) (*b*) (*c*) of the wave, made by the *three principal planes* of the generating ellipsoid (*abc*), are *lines of vibration* (17.); the constant h^4 receiving the three values, b^2c^2, c^2a^2, a^2b^2, for these three ellipses respectively.

(30.) But *at a cusp* the *two* equations LXXXVI. and LXXXVII. *coexist*, and the expression LXXXIV. for μ takes the *indeterminate form* $\frac{0}{0}$; in fact, there is in this case no reason for preferring *either* to the *other* of the *two values*, *within the plane* of *ac*,

$$\text{LXXXIX.} \ldots \mu = -\rho_0^{-1}, \qquad \text{XC.} \ldots \mu = \mu_0, \quad \text{if} \quad \text{XCI.} \ldots \mu_0 = -a^{-2}c^{-2}\phi^{-1}\rho_0;$$

in which ρ_0 is the *cusp-ray* (23.), and the *first* value of μ corresponds to the *circle*, but the *second* to the *ellipse* (*b*).

(31.) The *indetermination* of μ, at a *wave-cusp*, is however even *greater* than this. For, if we observe that the equations LXXIX. and LXXX. give, for this case, by LXXXIII. LXXXVI. LXXXVII.,

$$\text{XCII.} \ldots \mathrm{f}\rho_0 = a^{-2} + b^{-2} + c^{-2}, \quad \text{and} \quad \text{XCIII.} \ldots \mathrm{D}_\rho \, \mathrm{f}\rho = 0, \quad \text{for} \quad \rho = \rho_0,$$

we shall see that if ρ be changed to $\rho_0 + \rho'$ in the expression LXXVIII. for $\mathrm{f}\rho$, and only terms which are of the *second dimension* in ρ' retained, the result equated to zero will represent a *cone of tangents* ρ', or a *Tangent Cone to the Wave at the Cusp*: which cone is of the *second degree*, and *every normal* μ to which, if limited by the condition I., is here to be considered as *one value of the vector* μ, corresponding to the value ρ_0 of ρ.

(32.) And it is evident, by the law (12.) of *transition* from the wave to the index surface, that if $\pm \nu_0$, $\pm \nu_1$ be the *Lines of Single Normal Slowness*, or the four values of μ which are *analogous** to the *four cusp-rays* $\pm \rho_0$, $\pm \rho_1$ (23.), then, at the end of each such new line, there must be a *Conical Cusp on the Index Surface*, analogous to the *Conical Cusp* (31.) *on the Wave*, which is in like manner one of *four* such cusps.

(33.) In forming and applying the equation above indicated (31.), of the tangent cone to the wave at a cusp, the following transformations are useful:

$$\begin{aligned}\text{XCIV.} \ldots -(\rho + \rho')^{-2} &= -\rho^{-2}(1 + \rho^{-1}\rho')^{-1}(1 + \rho'\rho^{-1})^{-1} \\ &= -\rho^{-2} + 2\rho^{-2}\mathrm{S}\rho'\rho^{-1} + \rho^{-4}\rho'^2 - 4\rho^{-6}(\mathrm{S}\rho\rho')^2 + \&\text{c.}\end{aligned}$$

the terms not written being of the third and higher dimensions in ρ', and ρ, ρ' being *any two vectors* such that $\mathrm{T}\rho' < \mathrm{T}\rho$ (comp. 421, (4.)); also, without neglecting *any terms* the self-conjugate property of ϕ gives (comp. 362),

$$\text{XCV.} \ldots \mathrm{S}(\rho + \rho')\,\phi\,(\rho + \rho') = \mathrm{S}\rho\phi\rho + 2\mathrm{S}\rho'\phi\rho + \mathrm{S}\rho'\phi\rho',$$

with an analogous transformation for the corresponding expression in ϕ^{-1}; while the cubic LX. in ϕ, or LXV. in ϕ^{-1}, gives for an *arbitrary* ρ,

$$\text{XCVI.} \ldots \phi\,(\phi + a^{-2})\,(\phi + c^{-2})\rho = -\,b^{-2}\,(\phi + a^{-2})\,(\phi + c^{-2})\rho,$$

$$\text{XCVII.} \ldots \phi^{-1}(\phi + a^{-2})\,(\phi + c^{-2})\rho = -\,b^{2}(\phi + a^{-2})\,(\phi + c^{-2})\rho\,;$$

and therefore, among other transformations of the same kind,

$$\text{XCVIII.} \ldots (\phi + a^{-2})^2\,(\phi + c^{-2})^2\rho = (a^{-2} - b^{-2})\,(c^{-2} - b^{-2})\,(\phi + a^{-2})\,(\phi + b^{-2})\rho.$$

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* This word "analogous" is here more proper than "corresponding"; in fact, the *cusps* on *each* of the two surfaces will soon be seen to *correspond* to *circles* on the *other*, in virtue of the *law of reciprocity*.

We have also for a *cusp*, the values,

$$\text{XCIX.} \ldots \phi\rho_0 = \mu_0 - (a^{-2} + c^{-2})\,\rho_0; \quad \text{XCIX}'. \ldots 1 + S\rho_0\phi\rho_0 = (a^{-2} + c^{-2})\,b^2,$$

$$\text{C.} \ldots \mu_0^2 = a^{-4}c^{-4}S\rho_0\phi^{-2}\rho_0 = a^{-2}b^2c^{-2} - (a^{-2} + c^{-2}).$$

(34.) In this way *the equation of the tangent cone* is easily found to take the form,

$$\text{CI.} \ldots 0 = b^4 S\rho'(\phi + a^{-2})(\phi + c^{-2})\,\rho' - 4S\rho'\rho_0 S\rho'\mu_0$$

and to give, by operating with $D_{\rho'}$ (comp. (10.) (26.) (31.)),

$$\text{CII.} \ldots x\mu = b^4(\phi + a^{-2})(\phi + c^{-2})\,\rho' - 2\rho_0 S\rho'\mu_0 - 2\mu_0 S\rho'\rho_0,$$

the scalar coefficient x being determined, for each direction of the tangent ρ' to the wave at the cusp, by the condition I., which here becomes (31.)),

$$\text{CIII.} \ldots S\mu\rho_0 = S\mu_0\rho_0 = -1;$$

also, by CII., &c., we have, after some slight reductions,

$$\text{CIV.} \ldots xS\mu\rho_0 = 2(b^2 S\rho'\mu_0 + S\rho'\rho_0);$$

$$\text{CV.} \ldots xS\mu\mu_0 = 2(S\rho'\mu_0 - \mu_0^2 S\rho'\rho_0);$$

$$\begin{aligned}\text{CVI.} \ldots x^2\mu^2 &= 4(b^2\mu_0^2 + 1)\,S\rho'\rho_0 S\rho'\mu_0 + 4(\rho_0 S\rho'\mu_0 + \mu_0 S\rho'\rho_0)^2 \\ &= -4b^2(S\rho'\mu_0)^2 + 4(b^2\mu_0^2 - 1)\,S\rho'\rho_0 S\rho'\mu_0 + 4\mu_0^2(S\rho'\rho_0)^2;\end{aligned}$$

but this last expression is equal, by CIV. CV., to $-x^2 S\mu\rho_0 S\mu\mu_0$; the equation of the *cone of perpendiculars*, let fall from the *wave-centre* O on the *tangent planes at the cusp*, takes then this very simple form,

$$\text{CVII.} \ldots \mu^2 + S\mu\rho_0 S\mu\mu_0 = 0;$$

so that *this cone* of the second degree has the two vectors ρ_0 and μ_0 at once for *sides* and *cyclic normals* (comp. 406, (7.)); and it is *cut*, by the *plane* CIII., in a *circle*, of which the *diameter* is,

$$\text{CVIII.} \ldots T(\mu_0 + \rho_0^{-1}) = (T\mu_0^2 - b^{-2})^{\frac{1}{2}} = b\,(b^{-2} - a^{-2})^{\frac{1}{2}}(c^{-2} - b^{-2})^{\frac{1}{2}};$$

and therefore *subtends*, at the centre O, and in the plane of ac, the *angle*,

$$\text{CIX.} \ldots \angle\frac{\mu_0}{\rho_0} = \tan^{-1}.\, b^2(b^{-2} - a^{-2})^{\frac{1}{2}}(c^{-2} - b^{-2})^{\frac{1}{2}}.$$

(35.) And by combining the equations CIII. CVII., we see that this circle (34.) is a *small circle of the sphere*,

$$\text{CX.} \ldots \mu^2 = S\mu\mu_0, \quad \text{or} \quad \text{CX}' \ldots S\mu^{-1}\mu_0 = 1,$$

which passes through the wave-centre, and has the vector μ_0 for a diameter, passing also through the extremity of the vector $-\rho_0^{-1}$.

(36.) This circle is, by III., a *curve of contact* of the *plane* CIII. with the *surface* of which μ is the vector, because *every vector* μ *of the curve corresponds*, by (31.), to the *one vector* ρ_0 of the *wave*; it is therefore one of *Four Circular Ridges on the Index Surface*, the three others having *equal diameters*, and corresponding to the three remaining *cusp-rays*, $-\rho_0$, ρ_1, $-\rho_1$ (23.); and there are, in like manner, *Four Circular Ridges on the Wave*, along which it is *touched by the four planes*,

$$\text{CXI.} \ldots S\rho\nu_0 = -1, \quad S\rho\nu_0 = +1, \quad S\rho\nu_1 = -1, \quad S\rho\nu_1 = +1,$$

$\pm\nu_0$, $\pm\nu_1$ being the four lines introduced in (32.); also the *common length* of the *diameters*, of these four *circles on the wave*, is (comp. CVIII.),

$$\text{CXII.} \ldots T(\sigma_0 + \nu_0^{-1}) = (T\sigma_0^2 - b^2)^{\frac{1}{2}} = b^{-1}(a^2 - b^2)^{\frac{1}{2}}(b^2 - c^2)^{\frac{1}{2}},$$

where

$$\text{CXIII.} \ldots \sigma_0 = -a^2c^2\phi\nu_0, \quad \text{CXIV.} \ldots T\nu_0 = b^{-1}, \ \text{and} \ \text{CXV.} \ldots S\nu_0\sigma_0 = -1;$$

finally, $-\nu_0^{-1}$ and σ_0 are the two values* of ρ, in the plane of ac, for the first of the four new circles: and the *angle* between these two vectors, or the angle which the diameter of the circle, in the same plane, *subtends at the wave-centre*, is (comp. CIX.),

$$\text{CXVI.} \ldots \angle\frac{\sigma_0}{\nu_0} = \tan^{-1} . b^{-2}(a^2 - b^2)^{\frac{1}{2}}(b^2 - c^2)^{\frac{1}{2}}.$$

(37.) In the recent calculations (33.) (34.), the *circle of contact* (36.) on the index surface was deduced *from the tangent cone* at a wave-cusp, as a *section* of a certain *cone of normals* CVII. to that tangent cone CI., made by the plane CIII.; but the following is a simpler, and perhaps more elegant, method of deducing and representing the *same circle* by means of its *vector equation* (comp. 392, IX. &c.), and *without* assuming *any previous knowledge* of the character, or even the *existence*, of that *conical wave-cusp*.

* It is not difficult to show that these are the vectors of two points, in which the circle and ellipse (b), wherein the wave is cut by the plane of ac, are touched by a *common tangent*.

(38.) In general, by eliminating the auxiliary vector υ between XX. and XXIII., we arrive at the following equation,

$$\text{CXVII.} \ldots (\phi - \rho^{-2})(\mu + \rho^{-1})^{-1} = \rho^{-1};$$

which holds good for *every pair of corresponding vectors* ρ and μ, of the wave and index surface. And, *in general*, this relation is *sufficient*, to determine the index-vector μ, when the ray-vector ρ is given: because $(\phi + e)^{-1}0$ is generally $= 0$.

(39.) But when e is a root of the equation $E = 0$, with the signification LXI. of E, then, by the formula of inversion LXII., the symbol $(\phi + e)^{-1}0$ takes the indeterminate form $\frac{0}{0}$ and therefore, for *every point* of each of the *three circles* (*a*) (*b*) (*c*) of the wave, the *formula* CXVII. *fails to determine* μ: although it is *only* at a *cusp* (23.), that the *value* of μ becomes in fact *indeterminate* (comp. (27.) (28.) (29.) (30.) (31.)).

(40.) At such a cusp ($\rho = \rho_0$), the equation CXVII. takes the symbolical form,

$$\text{CXVIII.} \ldots (\mu + \rho_0^{-1})^{-1} = (\phi + b^{-2})^{-1}\rho_0^{-1} = (\mu_0 + \rho_0^{-1})^{-1} + (\phi + b^{-2})^{-1}0;$$

μ_0 retaining its recent signification XCI., and the symbol $(\phi + b^{-2})^{-1}0$ denoting *any vector* of the *form* $y\beta$, if β be the *mean vector semiaxis* of the *generating ellipsoid* XXIX., so that,

$$\text{CXIX.} \ldots S\beta\phi\beta = 1, \quad (\phi + b^{-2})\beta = 0, \quad T\beta = b.$$

(41.) Writing then for abridgment (comp. XX.),

$$\text{CXX.} \ldots \upsilon_0 = -(\mu_0 + \rho_0^{-1})^{-1},$$

the *Vector Equation* of the *Index Ridge* (36.) is obtained under the sufficiently simple form,

$$\text{CXXI.} \ldots V\beta(\mu + \rho_0^{-1})^{-1} + V\beta\upsilon_0 = 0;$$

and this equation does in fact represent a *Circle* (comp. 296, (7.)), which is easily proved to be the *same* as the *circular section* (34.), of the *cone* CVII. by the *plane* CIII.; its *diameter* CVIII. being thus found anew under the form,

$$\text{CXXII.} \ldots T\upsilon_0^{-1} = bTV\lambda\lambda' = b(b^{-2} - a^{-2})^{\frac{1}{2}}(c^{-2} - b^{-2})^{\frac{1}{2}},$$

with the significations (24.) (25.) of λ, λ'; in fact we have now the expressions,

$$\text{CXXIII.} \ldots \rho_0 = b\text{U}\lambda, \quad \nu_0 = \rho_0^{-1}(\text{V}\lambda\lambda')^{-1},$$

with the verification, that

$$\text{CXXIV.} \ldots (\phi + b^{-2})\,\nu_0 = \lambda\text{S}\lambda'\nu_0 + \lambda'\text{S}\lambda\nu_0 = b^{-1}\text{U}\lambda = -\rho_0^{-1}.$$

(42.) And by a precisely similar analysis, we have first the *new general relation* (comp. CXVII.), for *any two corresponding vectors*, ρ and μ,

$$\text{CXXV.} \ldots (\phi^{-1} - \mu^{-2})(\rho + \mu^{-1})^{-1} = \mu^{-1};$$

and then in particular (comp. CXVIII.), for $\mu = \nu_0$,

$$\text{CXXVI.} \ldots (\rho + \nu_0^{-1})^{-1} = (\phi^{-1} + b^2)^{-1}\nu_0^{-1} = (\sigma_0 + \nu_0^{-1})^{-1} + (\phi^{-1} + b^2)^{-1}0;$$

so that finally, if we write for abridgment (comp. XLI. CXX.),

$$\text{CXXVII.} \ldots \omega_0 = -(\sigma_0 + \nu_0^{-1})^{-1},$$

the *Vector Equation of a Wave-Ridge* is found (comp. CXXI.) to be,

$$\text{CXXVIII.} \ldots \text{V}\beta(\rho + \nu_0^{-1})^{-1} + \text{V}\beta\omega_0 = 0,$$

β being still (as in CXIX.) the *mean vector semiaxis* of the *generating ellipsoid* ($\text{S}\rho\phi\rho = 1$): and the *diameter* CXII., of this *circle of contact* of the *wave* with the first plane CXI., is thus found *anew* (comp. CXXII.) *without any reference to cusps* (37.), as the value of $\text{T}\omega_0^{-1}$.

(43.) Several of the foregoing results may be illustrated, by a new use of the last diagram (13.). Thus if we suppose, in that fig. 89, that we have the values,

$$\text{CXXIX.} \ldots \text{OP} = \rho_0, \ \text{OQ} = \mu_0, \ \text{OU} = \nu_0, \quad \text{whence} \quad \text{CXXX.} \ldots \text{OP}' = -\rho_0^{-1}, \ \&\text{c.},$$

then the *index-ridge* (36.), corresponding to the *wave-cusp* P (23.), will be the *circle* which has P′Q for diameter, in a *plane* perpendicular to the plane of the figure, which is here the plane of *ac*; the *cone of normals* μ (34.), to the *tangent cone* to the *wave* at P, has the *wave-centre* O for its *vertex*, and *rests* on the last-mentioned circle, having also for a subcontrary *section* that *second circle* which has PQ′ for diameter, and has its plane in like manner at right angles to the plane of POQ; also, if R and S be any two points on the second and first circles, such that ORS is a right line, namely, a *side* μ of the cone here considered, then the *chord* PR of the second circle is *perpendicular* to this last line, and has the *direction* of the *vibration* $\delta\rho$, which answers here to the *two* vectors $\rho (= \rho_0)$ and μ: because (comp. (14.)) this chord is perpendicular to μ, but complanar with ρ and μ.

(44.) Again, to illustrate the theory of the *wave-ridge* (36.), which *corresponds* to a *cusp* (32.) on the *index-surface*, we may suppose that *this* cusp is at the point Q in fig. 89, writing *now* (instead of CXXIX. CXXX.),

$$\text{CXXXI.} \ldots \text{OQ} = \nu_0, \quad \text{OP} = \sigma_0, \quad \text{OW} = \omega_0, \quad \text{OQ}' = -\nu_0^{-1}, \ \&\text{c.};$$

for then the *ridge* (or *circle of contact*) on the *wave* will coincide with the *second circle* (43.), and the *cone of rays* ρ from O, which rests upon *this* circle, will have the *first* circle (43.) for a sub-contrary section: also the *vibration*, at any point R of the wave-ridge, will have the direction of the *chord* RQ', for reasons of the same kind as before.

(45.) Let K and K' denote the *bisecting points* of the lines PQ' and QP', in the same fig. 89; then K' is the *centre of the index-ridge*, in the case (43.); while, in the case (44.), K is the *centre of the wave-ridge.*

(46.) In the *first* of these two cases, the point K is *not* the *centre* of *any ridge*, on *either* wave or index-surface; but it *is* the centre of a certain *sub-contrary* and *circular section* (43.), of the *cone* with O for vertex which *rests* upon an index-ridge; and *each* of its *chords* PR has the *direction* (43.), of a *vibration* $\delta\rho_0$, at the *wave-cusp* P corresponding: so that this *cusp-vibration revolves*, in the *plane* of the *circle* last mentioned, with exactly *half the angular velocity* of the *revolving radius* KR.

(47.) And *every one* of those *cusp-vibrations* $\delta\rho_0$, which (as we have seen) are all situated *in one plane*, namely, in the *tangent plane* at the *cusp* P to the *ellipsoid* (*b*) of (28.), has (as by (14.) it ought to have) the direction of the *projection* of the *cusp-ray* ρ_0, on *some tangent plane* to the *tangent cone* to the *wave*, at that point P: to the determination of which *last cone*, by some new methods, we purpose shortly to return.

(48.) In the *second* of the two cases (45.), namely, in the case (44.), PQ' is a *diameter* of a *wave-ridge*, with K for the *centre* of that circle, and with a *plane* (perpendicular to that of the figure) which *touches* the *wave* at *every point* of the same circular ridge; and the *vibration*, at any such point R, has been seen to have the direction of the *chord* RQ', which is in fact the *projection* (14.) of the *ray* OR upon the *tangent plane* at R to the wave.

(49.) And we see that, in passing from one point to another of this *wave-ridge*, the *vibration* RQ′ *revolves* (comp. (46.)) round the *fixed point* Q′ of that circle, namely, round the *foot* of the *perpendicular from* O on the *ridge-plane*, with (again) *half the angular velocity* of the *revolving radius* KR.

(50.) These *laws* of the *two sets of vibrations*, at a *cusp* and at a *ridge* upon the *wave*, are intimately connected with the *two conical polarizations*, which accompany the *two conical refractions*,* *external* and *internal*, in a *biaxal crystal*; because, on the one hand, the *theoretical deduction* of those two *refractions* is associated with, and was in fact accomplished by, the consideration of those *cusps* and *ridges*: while, on the other hand, in the theory of Fresnel, the *vibration* is always *perpendicular* to the *plane of polarization*. But into the details of such investigations, we cannot enter here.

(51.) It is not difficult to show, by decomposing ρ' into two other vectors, ρ'_1 and ρ'_2, perpendicular and parallel to the plane of ac, that we have the *general transformation*, for *any* vector ρ',

$$\text{CXXXII.} \ldots b^4 S\rho'(\phi + a^{-2})(\phi + c^{-2})\rho' = (S\mu_0\rho_0\rho')^2;$$

the equation CI. of the *tangent cone* at a *wave-cusp* may therefore be thus more briefly written,

$$\text{CXXXIII.} \ldots (S\mu_0\rho_0\rho')^2 = 4S\rho_0\rho' S\mu_0\rho';$$

and under this form, the *cone* in question is easily proved to be the *locus* of the *normals* from the *cusp*, to that *other cone* CVII., which has μ for a side, and the *wave-centre* O for its vertex: while the same cone CVII. is now seen, more easily than in (34.), to be reciprocally the locus of the perpendiculars from O on the tangent planes to the wave at the cusp, in virtue of the new equation CXXXIII., of the tangent cone at that point.

(52.) Another form of the equation of the cusp-cone may be obtained as follows. The equation LXXIV. of the *wave* may be thus modified (comp.

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* The writer's anticipation, from theory, of the two Conical Refractions, was announced at a general meeting of the Royal Irish Academy, on the 22nd of October, 1832, in the course of a final reading of that *Third Supplement on Systems of Rays*, which has been cited in a former Note (p. 324). The very elegant experiments, by which his friend, the Rev. Humphrey Lloyd, succeeded shortly afterwards in exhibiting the expected results, are detailed in a Paper *On the Phenomena presented by Light, in its passage along the Axes of Biaxal Crystals*, which was read before the same Academy on the 28th of January, 1833, and is published in the same first part of vol. xvii. of their *Transactions*. Dr. Lloyd has also given an account of the same phenomena, in a separate work since published, under the title of an *Elementary Treatise on the Wave Theory of Light* (London, Longmans and Co., 1857, Chapter XI.).

LXXVI.), by the introduction of the two non-opposite cusp-rays, $\rho_0 = b\mathrm{U}\lambda$ (CXXIII.), and $\rho_1 = b\mathrm{U}\lambda'$:

$$\text{CXXXIV.} \ldots 2a^2b^2c^2 + (a^2 + c^2)\, b^2\rho^2 + (a^2 - c^2)\, \mathrm{S}\rho_0\rho \,.\, \mathrm{S}\rho_1\rho = \mp (a^2 - c^2)\, \mathrm{TV}\rho_0\rho \,.\, \mathrm{TV}\rho_1\rho\,;$$

where it will be found that the first member vanishes, as well as the second, at the cusp for which $\rho = \rho_0$.

(53.) Changing then ρ to $\rho_0 + \rho'$, and retaining only terms of *first* dimension in ρ' (comp. (31.)), we find an equation of *unifocal form* (comp. 359, &c.),

$$\text{CXXXV.} \ldots \mathrm{S}\beta_0\rho' = \mp \mathrm{TV}\alpha_0\rho', \quad \text{or} \quad \text{CXXXV}'. \ldots (\mathrm{V}\alpha_0\rho')^2 + (\mathrm{S}\beta_0\rho')^2 = 0\,;$$

with the two constant vectors,

$$\text{CXXXVI.} \ldots \alpha_0 = (b^{-2} - a^{-2})^{\frac{1}{2}} (c^{-2} - b^{-2})^{\frac{1}{2}} \rho_0\,; \quad \text{CXXXVI}'. \ldots \beta_0 = \mu_0 - \rho_0^{-1}\,;$$

and this equation CXXXV. or CXXXV′. represents the *tangent cone*, with ρ' for side, $\mathrm{S}\beta_0\rho'$ being positive for one sheet, but negative for the other.

(54.) As regards the calculations which conduct to the recent expressions for α_0, β_0, it may be sufficient here to observe that those expressions are found to give the equations,

$$\text{CXXXVII.} \ldots 2a^2b^2c^2\alpha_0 = (a^2 - c^2)\, \rho_0 \mathrm{TV}\rho_0\rho_1\,;$$

$$\text{CXXXVII}'. \ldots 2a^2b^2c^2\beta_0 = 2\,(a^2 + c^2)\, b^2\rho_0 + (a^2 - c^2)\, (\rho_0 \mathrm{S}\rho_0\rho_1 - b^2\rho_1)\,;$$

and that, in deducing these, we employ the values,

$$\text{CXXXVIII.} \ldots \mathrm{S}\rho_0\rho_1 = \frac{b^2\mathrm{S}\lambda\lambda'}{\mathrm{T}\lambda\lambda'}, \quad \mathrm{TV}\rho_0\rho_1 = \frac{b^2\mathrm{TV}\lambda\lambda'}{\mathrm{T}\lambda\lambda'}\,;$$

together with the formula XCIX., and the following,

$$\text{CXXXIX.} \ldots \phi\,(\rho_0 - \rho_1) = -\,a^{-2}\,(\rho_0 - \rho_1)\,; \quad \phi\,(\rho_0 + \rho_1) = -\,c^{-2}\,(\rho_0 + \rho_1).$$

(55.) It is not difficult to show that the equation CXXXV. or CXXXV′., of the tangent cone at a cusp, can be transformed into the equation CXXXIII.; but it may be more interesting to assign here a *geometrical interpretation*, or *construction*, of the *unifocal form* last found (53.).

(56.) Retaining then, for a moment, the use made in (43.) of fig. 89, as serving to illustrate the case of a wave-cusp at P, with the signification (45.) of the new point K′ as bisecting the line P′Q, or as being the centre of the index-ridge; and conceiving a *parallel cone*, with O instead of P for *vertex*, and with a variable *side* OT = ρ'; then the *cusp-ray* OP (= $\rho_0 \parallel \alpha_0$) is a *focal line* of the new cone, and the line OK′ (= $\frac{1}{2}(\mu_0 - \rho_0^{-1}) = \frac{1}{2}\beta_0$) is the *directive normal*, or the normal to the *director plane* corresponding; and the formula CXXXV. is found to conduct to the following,

$$\text{CXL.} \ldots \cos \text{K}'\text{OT} = \sin \text{POK}' \sin \text{POT},$$

which may be called a *Geometrical Equation of the Cusp-Cone*: or (more immediately) of that *Parallel Cone*, which has (as above) its vertex removed to the wave-centre O.

(57.) Verifications of CXL. may be obtained, by supposing the side OT to be one of the two right lines, ρ'_1, ρ'_2, in which the cone is *cut* by the *plane* of the figure (or of *ac*); that is, by assuming either

$$\text{CXLI.} \ldots \text{OT} = \rho'_1 = \mu_0 + \rho_0^{-1} \parallel \text{OU}, \quad \text{or} \quad \text{CXLI}'. \ldots \text{OT} = \rho'_2 = \rho_0 + \mu_0^{-1} \parallel \text{OW};$$

and it is easy to show, not only that these *two sides*, OU, OW, make (as in fig. 89) an *obtuse angle* with *each other*, but also that they belong to one *common sheet*, of the cone here considered, because *each* makes an *acute angle* with the directive normal OK′.

(58.) Another way of arriving at this result, is to observe that the equation CXXXIII. takes easily the *rectangular form*,

$$\text{CXLII.} \ldots (S\rho'(U\mu_0 + U\rho_0))^2 = (S\rho'(U\mu_0 - U\rho_0))^2 + T\mu_0\rho_0(S\rho'U\mu_0\rho_0)^2;$$

the *internal axis* of the *cusp-cone* has therefore the direction of $U\mu_0 + U\rho_0$, that is, of the *internal bisector* of the angle POQ, while the *external bisector* of the same angle is *one* of the two *external axes*, and the *third* axis is perpendicular to the *plane* of ρ_0, μ_0; but $S\rho'(U\mu_0 + U\rho_0) < 0$, whether $\rho' = \rho'_1$, or $= \rho'_2$: and therefore these *two sides*, ρ'_1 and ρ'_2, belong (as above) to *one sheet*, because *each* is inclined at an acute angle to the internal *axis* $U\mu_0 + U\rho_0$.

(59.) It is easy to see that the *second focal line* of the *parallel cone* (56.) is μ_0 or OQ; and that the *second directive normal* corresponding is the line OK (45.), in the same fig. 89; whence may be derived (comp. CXL.) this *second geometrical equation* of the cone at O,

$$\text{CXLIII.} \ldots \cos \text{KOT} = \sin \text{KOQ} \sin \text{QOT}; \quad \text{with} \quad \text{KOQ} = \text{POK}'.$$

(60.) And finally, as a *bifocal* but still *geometrical form* of the equation of the *cusp-cone*, with its vertex thus *transferred* to O, we may write,

$$\text{CXLIV.} \ldots \angle \text{POT} + \angle \text{QOT} = \text{const.} = \angle \text{WOU}.$$

(61.) *Any legitimate form* of any *one* of the *four functions* $\phi\rho$, $\phi^{-1}\rho$, $S\rho\phi\rho$, $S\rho\phi^{-1}\rho$, when treated by rules of the present Calculus which have been already stated and exemplified, not only conducts to the connected forms of the *three other* functions of the group, but also gives the corresponding forms of equation, of the *Wave* and the *Index-Surface*.

(62.) For instance, with the significations (32.) of ν_0 and ν_1, the scalar function $S\rho\phi^{-1}\rho$, which is $= 1$ in the equation XXX. of the *Reciprocal Ellipsoid* (9.), may be expressed by the following *cyclic form*, with ν_0, ν_1 for the *cyclic normals* of that ellipsoid,

$$\text{CXLV.} \ldots S\rho\phi^{-1}\rho = -b^2\rho^2 + (a^2 - c^2)b^2 S\nu_0\rho S\nu_1\rho;$$

reciprocating which (comp. 361), we are led to a *bifocal form* of the function $S\rho\phi\rho$, which function was made $= 1$ in the equation XXIX. of the *Generating Ellipsoid* (8.), and is now expressed by this other equation (comp. 360, 407),

$$\text{CXLVI.} \ldots \frac{4a^2c^2}{(a^2 - c^2)^2}(S\rho\phi\rho + b^{-2}\rho^2) = (S\nu_0\rho)^2 + (S\nu_1\rho)^2 - 2\frac{a^2 + c^2}{a^2 - c^2} S\nu_0\rho S\nu_1\rho;$$

ν_0, ν_1 being here the two (real) *focal lines* of the same ellipsoid (8.), or of its (imaginary) asymptotic cone.

(63.) Substituting then these forms (62.), of $S\rho\phi\rho$ and $S\rho\phi^{-1}\rho$, in the equation LXIII., we find (after a few reductions) this *new form* of the *Equation of the Wave*:

$$\text{CXLVII.} \ldots (2\rho^2 - (a^2 - c^2)S\nu_0\rho S\nu_1\rho + a^2 + c^2)^2 = (a^2 - c^2)^2 \{1 - (S\nu_0\rho)^2\}\{1 - (S\nu_1\rho)^2\};$$

whence it follows at once, that *each of the four planes* CXI. *touches the wave, along the circle in which it cuts the quadric*, with ν_0, ν_1 for cyclic normals, which is found by equating to zero the expression squared in the first member of CXLVII. For example, the *first* plane CXI. touches the wave along that *circle*, or *wave-ridge*, of which on this plan the equations are,

$$\text{CXLVIII.} \ldots S\nu_0\rho + 1 = 0, \quad 2\rho^2 + (a^2 - c^2)S\nu_1\rho - (a^2 + c^2)S\nu_0\rho = 0;$$

and because

$$\text{CXLIX.} \ldots \phi(\nu_0 + \nu_1) = -a^{-2}(\nu_0 + \nu_1), \quad \phi(\nu_0 - \nu_1) = -c^{-2}(\nu_0 - \nu_1),$$

and therefore, with the value CXIII. of σ_0,

$$\text{CL.} \ldots \sigma_0 = - a^2c^2\phi\nu_0 = \tfrac{1}{2}((a^2 + c^2)\nu_0 - (a^2 - c^2)\nu_1),$$

the second equation CXLVIII. represents (comp. CX.) the *diacentric sphere*,

$$\text{CLI.} \ldots \rho^2 = \text{S}\sigma_0\rho, \quad \text{or} \quad \text{CLI}'. \ldots \text{S}\sigma_0\rho^{-1} = 1,$$

which passes through the *wave-centre* o, and of which the *ridge* here considered is a *section*. The *diameter* of that ridge may thus be shown again to have the value CXII.; and it may be observed that the circle is a section also of the *cone*,

$$\text{CLII.} \ldots \text{S}\nu_0\rho\text{S}\sigma_0\rho = - \rho^2, \quad \text{or} \quad \text{CLII}'. \ldots \text{S}\nu_0\rho\text{S}\sigma_0\rho^{-1} = - 1.$$

(64.) It was shown in (17.) that the *vibration* $\delta\rho$, at *any point* of the wave-surface, or at the end of *any ray* ρ, is perpendicular to $\phi^{-1}\rho$, as well as to μ by II.; and is therefore *tangential* to the variable *ellipsoid* LIX., as well as to the *wave* itself. Hence it is easy to infer, that this vibration must have generally the direction of the auxiliary vector ω, because not only $\text{S}\mu\omega = 0$, by XXXIX., but also $\text{S}\omega\phi^{-1}\rho = \text{S}\rho\phi^{-1}\omega = \text{S}\rho\upsilon = 0$, by XXII. and XXXVII. Indeed, this parallelism of $\delta\rho$ to ω results at once by XXXVII. from XII.

(65.) If then we denote by $\delta'\rho$ an infinitesimal vector, such as $\mu\delta\rho$, which is *tangential to the wave*, but *perpendicular to the vibration* $\delta\rho$, the parallelism $\delta\rho \parallel \omega$ will give,

$$\text{CLIII.} \ldots \delta'\rho = \mu\delta\rho \parallel \mu\omega \perp \rho, \quad \text{because} \quad \text{CLIII}'. \ldots \text{S}\rho\mu\omega = 0;$$

whence

$$\text{CLIV.} \ldots \text{S}\rho\delta'\rho = 0, \quad \delta'\text{T}\rho = 0, \quad \text{or} \quad \text{CLV.} \ldots \text{T}\rho = r = \text{const.},$$

for this *new direction* $\delta'\rho$ of motion upon the wave.

(66.) And thus (or otherwise) it may be shown, that the *Orthogonal Trajectories to the Lines of Vibration* (17.) are the curves in which the *Wave* is *cut* by *Concentric Spheres*, such as CLV.; that is by the spheres $\rho^2 + r^2 = 0$, in which the radius r is *constant* for any *one*, but *varies* in passing from one to *another*.

(67.) The *spherical curves* (r), which are thus *orthogonal* to what we have called the *lines* (h) of vibration, are *sphero-conics* on the wave; either because each such curve (r) is, by XXVIII., situated on a *concentric and quadric cone*, namely,

$$\text{CLVI.} \ldots 0 = S\rho(\phi + r^{-2})^{-1}\rho;$$

or because, by XXVII., it is on this *other concentric quadric*,

$$\text{CLVII.} \ldots -1 = S\rho(\phi^{-1} + r^{2})^{-1}\rho.$$

(68.) It is easy to prove (comp. LXXV.) that, for any *real* point of the wave, r^2 cannot be less than c^2, nor greater than a^2; and that the squares of the scalar semiaxes of the new quadric CLVII. are, in algebraically ascending order, $r^2 - a^2$, $r^2 - b^2$, $r^2 - c^2$; so that this surface is generally an *hyperboloid*, with *one sheet* or with *two*, according as $r >$ or $< b$.

(69.) And we see, at the same time, that the *conjugate hyperboloid*,

$$\text{CLVIII.} \ldots +1 = S\rho(\phi^{-1} + r^{2})^{-1}\rho,$$

which has *two sheets* or *one*, in the same two cases, $r > b$, $r < b$, and has (in descending order) the values,

$$\text{CLIX.} \ldots a^2 - r^2, \quad b^2 - r^2, \quad c^2 - r^2,$$

for the squares of *its* scalar semiaxes, is *confocal* with the *generating ellipsoid* XXIX.; so that the quadric CLVII. itself is the *conjugate of such a confocal.*

(70.) To form a distinct conception (comp. (67.)) of the *course* of a curve (r) upon the wave, it may be convenient to distinguish the *five* following *cases*:

$$\text{CLX.} \ldots (\alpha) \ldots r = a; \quad (\beta) \ldots r < a, > b; \quad (\gamma) \ldots r = b;$$
$$(\delta) \ldots r < b, > c; \quad (\varepsilon) \ldots r = c.$$

(71.) In each of the *three* cases (α) (γ) (ε), the *conic* (r) becomes a *circle*, in one or other of the three principal planes: namely the circle (a), for the case (α); (b) for (γ); and (c) for (ε).

(72.) In the case (β), the *curve* (r) is one of *double curvature*, and consists of *two closed ovals*, *opposite* to each other on the *wave*, and *separated* by the *plane* (a), which plane is *not* (really) *met*, in *any point*, by the complete *spherconic* (r); and *each* separate oval *crosses the plane* (b) *perpendicularly*, in *two* (real) *points* of the *ellipse* (b), which are *external* to the *circle* (b): while the *same oval* crosses *also* the *plane* (c) at *right angles*, in some *two* real points of the *ellipse* (c).

(73.) Finally, in the remaining case (δ), the ovals are separated by the plane (*c*), and each crosses the plane (*b*) at right angles, in *two* points of the ellipse (*b*), which are *interior* to the circle (*b*); crossing also perpendicularly the plane (*a*), in two points of the ellipse (*a*).

(74.) Analogous remarks apply to the *lines of vibration* (*h*); which are either the *ellipses* (*a*) (*b*) (*c*), or else *orthogonals* to the *circles* (*a*) (*b*) (*c*), and generally to the *sphero-conics* (*r*), as appears easily from foregoing results.

(75.) It may be here observed, that when we only know the *direction* ($U\mu$), but not the *length* ($T\mu$), of an *index-vector* μ, so that we have *two parallel tangent planes* to the *wave*, at one *common side* of the *centre*, the *directions* of the *vibrations* $\delta\rho$ *differ* generally for these *two planes*, according to a *law* which it is easy to assign as follows.

(76.) The *second values* of μ and $\delta\rho$ being denoted by $\mu_,$ and $\delta\rho_,$, we have, by the equation IX. of the index-surface, these two other equations:

$$\text{CLXI.} \ldots 0 = S\mu(\phi^{-1} - \mu^{-2})^{-1}\mu; \quad \text{CLXI}'. \ldots 0 = S\mu(\phi^{-1} - \mu_,^{-2})^{-1}\mu;$$

of which the difference gives, suppressing the factor $\mu_,^{-2} - \mu^{-2}$,

$$\text{CLXII.} \ldots 0 = S\mu(\phi^{-1} - \mu_,^{-2})^{-1}(\phi^{-1} - \mu^{-2})^{-1}\mu;$$

or

$$\text{CLXII}'. \ldots 0 = S(\phi^{-1} - \mu^{-2})^{-1}\mu(\phi^{-1} - \mu_,^{-2})^{-1}\mu,$$

because $(\phi^{-1} - \mu_,^{-2})^{-1}$, as a functional operator, is *self-conjugate*, so that μ may be transferred from one side of it to the other; just as, if $\nu = \phi\rho$ be such a self-conjugate function of ρ, then $\nu^2 = S\nu\phi\rho = S\rho\phi\nu = S\rho\phi^2\rho$, &c.

(77.) But, by VIII., we have the parallelisms,

$$\text{CLXIII.} \ldots \delta\rho \parallel (\phi^{-1} - \mu^{-2})^{-1}\mu; \quad \text{CLXIII}'. \ldots \delta\rho_, \parallel (\phi^{-1} - \mu_,^{-2})^{-1}\mu;$$

hence, by CLXII′., we have the very simple relation,

$$\text{CLXIV.} \ldots S\,\delta\rho\,\delta\rho_, = 0;$$

that is, *the two vibrations*, in the *two parallel planes*, are *mutually rectangular*.

(78.) The following quite different method has however the advantage of not only proving anew this *known relation* of *rectangularity*, but also of assigning *quaternion expressions* for the *two directions separately*: and, at the same time, that of leading easily to what appears to be a *new* and elegant *Geometrical Construction*, simpler in some respects than the *known* one, which can indeed be deduced from it.

(79.) By the first principles of Fresnel's theory (comp. (3.)), the *vibration* ($\delta\rho$), on any *one* tangent plane to the *wave*, is situated in the *normal plane* (through μ), which contains the direction ($\delta\varepsilon$) of the *elastic force*; that is to say, we have the *Equation of Complanarity*,

$$\text{CLXV.} \ldots \ S\mu\delta\rho\delta\varepsilon = 0.$$

(80.) We have then, by II. and V., the system of the two equations,

$$\text{CLXVI.} \ldots \ S\mu\delta\rho = 0, \quad S\mu\delta\rho\phi^{-1}\delta\rho = 0;$$

comparing which with the equations of the same form,

$$S\nu\tau = 0, \quad S\nu\tau\phi\tau = 0, \qquad 410, \text{V. VI.}$$

we derive at once the following *Construction, which may* also be expressed as a *Theorem* :—

"*At either of the two points* Q *of the Reciprocal Ellipsoid* XXX., *the tangent plane at which is parallel to that at the given point* P *of the Wave, the tangents to the Lines of Curvature on the Ellipsoid are parallel to the tangents to the Lines of Vibration on the Wave*"; namely, to *one* at that given point P *itself*, and to *another* at the *other point* P′, on the *same side* of the *centre*, at which the tangent plane is parallel to each of the two others above mentioned.

(81.) Thus, for *each* of the two points P, P′ the line of *vibration* is parallel to *one* of the lines of *curvature* at Q; and it is evident, from what precedes, that the *other* of these last lines has the direction of the corresponding *Orthogonal* (66.) at P or P′: nor is there any danger of confusion.

(82.) As regards *quaternion expressions*, for the *two vibrations* on a *given wave-front*, the sub-article, 410, (8.), with notations suitably modified, shows by its formulæ XIX. XXII. that we have here the equations,

$$\text{CLXVII.} \ldots \ 0 = S\mu\delta\rho\nu_0\delta\rho\nu_1 = S\mu\delta\rho\nu_0 S\nu_1\delta\rho + S\mu\delta\rho\nu_1 S\nu_0\delta\rho,$$

and

$$\text{CXVIII.} \ldots \ \delta\rho \parallel UV\mu\nu_0 \pm UV\mu\nu_1,$$

if ν_0, ν_1 be, as in earlier formulæ of the present Series 422, the *cyclic normals* of the *reciprocal ellipsoid*, which are often called the *Optic Axes* of the *Crystal*.

(83.) And hence may be deduced the *known* construction, namely, that "for any *given direction of wave-front*, the *two planes of polarization*, perpendicular respectively to the two vibrations in Fresnel's theory, *bisect* the *two supplementary* and *diedral angles*, which the *two optic axes subtend at the normal* to the front": or that these planes of polarization *bisect, internally and externally*, the angle between the *two planes*, $\mu\nu_0$ and $\mu\nu_1$.

(84.) It may not be irrelevant here to remark, that if μ and $\mu_,$ be any two index-vectors, which have (as in (76.)) the *same direction*, but *not* the *same length*, the equation LXIV. enables us to establish the two converse relations:

$$\text{CLXIX.} \ldots abc\text{T}\mu_, = (\text{S}\mu\phi\mu)^{-\frac{1}{2}}; \qquad \text{CLXIX}' \ldots abc\text{T}\mu = (\text{S}\mu_,\phi\mu_,)^{-\frac{1}{2}}.$$

(85.) Either by changing a, b, c, ϕ, μ to $a^{-2}, b^{-2}, c^{-2}, \phi^{-1}, \rho$, or by treating the *form* LXIII., in (19.), of the *Equation of the Wave*, as we have just treated the form LXIV., of the equation of *Index-Surface*, in the same sub-article (19.), we see that if ρ and $\rho_,$ be *any two condirectional rays* ($\text{U}\rho_, = \text{U}\rho$), then,

$$\text{CLXX.} \ldots (abc)^{-1}\text{T}\rho_, = (\text{S}\rho\phi^{-1}\rho)^{-\frac{1}{2}}, \quad \text{or,} \quad abc\text{T}\rho_,^{-1} = (\text{S}\rho\phi^{-1}\rho)^{\frac{1}{2}};$$

and

$$\text{CLXX}' \ldots (abc)^{-1}\text{T}\rho = \text{S}\,(\rho_,\phi^{-1}\rho_,)^{-\frac{1}{2}}, \text{ or,} \quad abc\text{T}\rho^{-1} = (\text{S}\rho_,\phi^{-1}\rho_,)^{\frac{1}{2}}.$$

(86.) A somewhat interesting geometrical consequence may be deduced from these last formulæ, when combined with the equation LIX. of that *variable ellipsoid*, $\text{S}\rho\phi^{-1}\rho = h^4$, which *cuts the wave* in a *line of vibration* (h). For if we introduce this symbol h^4 for $\text{S}\rho\phi^{-1}\rho$, and write $r_,$, instead of $\text{T}\rho_,$ to denote the *length* of the *second ray* $\rho_,$, the first equation CLXX. will take this simple form,

$$\text{CLXXI.} \ldots r_, = abch^{-2},$$

which shows at once that $r_,$ and h are *together constant*, or *together variable*; and therefore, that "*a Line of Vibration on one Sheet of the Wave is projected into an Orthogonal Trajectory to all such Lines on the other Sheet, and conversely the latter into the former, by the Vectors* ρ *of the Wave*": so that *one* of these *two curves* would appear to be *superposed* upon the *other*, to an eye placed at the *Wave-Centre* o.

(87.) The *visual cone*, here conceived, is represented by the equation CLVI., with some constant value of r; and as being a surface of the *second* degree, it ought to *cut the wave*, which is one of the *fourth*, in some *curve* of the *eighth degree*; or in some *system* of curves, which have the *product* of their dimensions equal to *eight*. Accordingly we now see that the *complete intersection*, of the *cone* CLVI. with the *wave*, consists of *two curves*, *each* of the *fourth degree*; *one* of these being, as in (67.), a *complete sphero-conic* (r), and the *other* a complete *line of vibration* (h): a new geometrical *connexion* being thus established between these *two quartic curves*.

(88.) As additional verifications, we may regard the *three principal planes*, as *limits of the cutting cones*; for then, in the *plane* (*a*) for instance, the *circle* (*a*) and the *ellipse* (*a*), which are (in a sense) *projections* of each other, and of which the *latter* has been seen to be a *line of vibration*, are represented respectively by the two equations,

$$\text{CLXXII.} \ldots r = a, \quad \text{and} \quad \text{CLXXII}'. \ldots bc = h^2,$$

in agreement with CLXXI.; and similarly for the two other planes.

(89.) It was an early result of the quaternions, that an ellipsoid with its centre at the origin might be adequately represented by the equation (comp. 281, XXIX., or 282, XIX.),

$$\text{CLXXIII.} \ldots \mathrm{T}(\iota\rho + \rho\kappa) = \kappa^2 - \iota^2, \quad \text{if} \quad \mathrm{T}\iota > \mathrm{T}\kappa;$$

or, without *any* restriction on the *two* vector constants, ι, κ, by this *other* equation,*

$$\text{CLXXIII}'. \ldots \mathrm{T}(\iota\rho + \rho\kappa)^2 = (\kappa^2 - \iota^2)^2.$$

(90.) Comparing this with $\mathrm{S}\rho\phi\rho = 1$, as the equation XXIX. of the *Generating Ellipsoid*, we see that we are to satisfy, *independently of* ρ, or as an *identity*, the relation (comp. 336):

$$\text{CLXXIV.} \ldots (\kappa^2 - \iota^2)^2\mathrm{S}\rho\phi\rho = (\iota\rho + \rho\kappa)(\rho\iota + \kappa\rho) = (\iota^2 + \kappa^2)\rho^2 + 2\mathrm{S}\iota\rho\kappa\rho;$$

which is done by assuming (comp. again 336) this *cyclic form* for ϕ,

$$\text{CLXXV.} \ldots (\kappa^2 - \iota^2)^2\phi\rho = (\iota^2 + \kappa^2)\rho + 2\mathrm{V}\kappa\rho\iota = (\iota - \kappa)^2\rho + 2\iota\mathrm{S}\kappa\rho + 2\kappa\mathrm{S}\iota\rho;$$

or as in (24.) comp. 359, III. IV.,

$$\phi\rho = g\rho + \mathrm{V}\lambda\rho\lambda', \quad \mathrm{S}\rho\phi\rho = g\rho^2 + \mathrm{S}\lambda\rho\lambda'\rho = 1; \quad \text{LXXII. LXXIII.}$$

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* This equation, CLXXIII'. or CLXXII., which had been assigned by the author as a form of the equation of an ellipsoid, has been selected by his friend Professor Peter Guthrie Tait, now of Edinburgh, as the basis of an admirable Paper, entitled: "Quaternion Investigations connected with Fresnel's Wave-Surface," which appeared in the May number for 1865, of the *Quarterly Journal of Pure and Applied Mathematics*; and which the present writer can strongly recommend to the careful perusal of all quaternion students. Indeed, Professor Tait, who has already published tracts on *other* applications of Quaternions, mathematical and physical, including some on Electro-Dynamics, appears to the writer eminently fitted to carry on, happily and usefully, this new branch of mathematical science: and likely to become in it, if the expression may be allowed, one of the chief successors to its inventor.

with expressions for the constants g, λ, λ', which give, by LXXVI., the following values for the scalar semiaxes,*

$$\text{CLXXVI.} \ldots a = T\iota + T\kappa; \quad b = \frac{\kappa^2 - \iota^2}{T(\iota - \kappa)}; \quad c = T\iota - T\kappa;$$

whence conversely,

$$\text{CLXXVII.} \ldots T\iota = \frac{a+c}{2}; \quad T\kappa = \frac{a-c}{2}; \quad T(\iota - \kappa) = \frac{ac}{b}; \ \&c.$$

(91.) Knowing thus the form CLXXV. of the function ϕ, which answers in the present case to the given equation CLXXIII. of the generating ellipsoid, there would be no difficulty in carrying on the calculations, so as to reproduce, in connexion with the *two* constants ι, κ, all the preceding theorems and formulæ of the present Series, respecting the Wave and the Index-Surface. But it may be more useful to show briefly, before we conclude the Series, how we can *pass* from *Quaternions* to *Cartesian Coordinates*, in any question or formula, of the kind lately considered.

(92.) The *three italic letters*, ijk, conceived to be connected by the *four fundamental relations*,

$$i^2 = j^2 = k^2 = ijk = -1, \qquad \text{(A), 183,}$$

were *originally* the *only peculiar symbols* of the present Calculus; and although they are not *now* so much used, as in the *early practice* of quaternions, because certain general *signs of operation*, such as S, V, T, U, K, have since been introduced, yet they (the symbols ijk) may be supposed to be *still familiar* to a student, as *links* between *quaternions* and *coordinates*.

(93.) We shall therefore merely write down here some leading expressions, of which the meaning and utility seem likely to be at once perceived, especially after the Calculations above performed in this Series.

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* The reader, at this stage, might perhaps usefully turn back to that *Construction of the Ellipsoid*, illustrated by fig. 53 (page 234, vol. i., and page 184), with the Remarks thereon, which were given in the few last Series of the Section II. i. 13, pages 230, 242, vol. i. It will be seen there that the *three vectors*, ι, κ, $\iota - \kappa$, of which the lengths are expressed by CLXXVII., are the *three sides* CB, CA, AB, of what may be called the *Generating Triangle* ABC in the figure; and that the deduction CLXXVI., of the *three semiaxes*, *abc*, from the *two vector constants*, ι, κ, with many connected results, can be very simply exhibited by *Geometry*. The whole subject, of the equation $T(\iota\rho + \rho\kappa) = \kappa^2 - \iota^2$ of the ellipsoid, was very fully treated in the *Lectures*; and the calculations may be made more general, by the transformations assigned in the long but important Section III. ii. 6 of the present *Elements*, so that it seems unnecessary to dwell more on it in this place.

(94.) The vector semiaxes of the generating ellipsoid being called α, β, γ (comp. (40.) (42.)), we may write,

$$\text{CLXXVIII.} \ldots \alpha = ia, \quad \beta = jb, \quad \gamma = kc;$$

$$\text{CLXXIX.} \ldots \phi\rho = \alpha^{-1}S\alpha^{-1}\rho + \beta^{-1}S\beta^{-1}\rho + \gamma^{-1}S\gamma^{-1}\rho = \Sigma\alpha^{-1}S\alpha^{-1}\rho = -\Sigma ia^{-2}x;$$

$$\text{CLXXX.} \ldots S\rho\phi\rho = \Sigma(S\alpha^{-1}\rho)^2 = \Sigma a^{-2}x^2; \quad \text{CLXXXI.} \ldots S\rho\phi^{-1}\rho = \Sigma a^2x^2;$$

$$\text{CLXXXII.} \ldots (\phi + e)\rho = \Sigma\alpha(\alpha^{-2} + e)S\alpha^{-1}\rho;$$

$$\text{CLXXXIII.} \ldots (\phi + e)^{-1}\rho = \Sigma\alpha(\alpha^{-2} + e)^{-1}S\alpha^{-1}\rho;$$

$$\text{CLXXXIV.} \ldots \text{if } r^2 = T\rho^2 = \Sigma x^2, \text{ then } \upsilon = r^{-2}(\phi + r^{-2})^{-1}\rho$$
$$= r^{-2}\Sigma\frac{\alpha S\alpha^{-1}\rho}{r^{-2} - a^{-2}} = -\Sigma\frac{ia^2x}{r^2 - a^2};$$

$$\text{CLXXXV.} \ldots \text{for } \textit{Wave}, \ 0 = S\rho\upsilon = \Sigma\frac{a^2x^2}{r^2 - a^2} = \frac{a^2x^2}{r^2 - a^2} + \frac{b^2y^2}{r^2 - b^2} + \frac{c^2z^2}{r^2 - c^2};$$

or

$$\text{CLXXXVI.} \ldots 1 = -S\rho\omega = -S\rho\phi\upsilon = -S\upsilon\phi\rho$$
$$= \Sigma\frac{x^2}{r^2 - a^2} = \frac{x^2}{r^2 - a^2} + \frac{y^2}{r^2 - b^2} + \frac{z^2}{r^2 - c^2};$$

and the *Index-Surface* may be treated similarly, or obtained from the *Wave* by changing abc to their reciprocals.

423. As an *eighth specimen* of physical application we shall investigate, by quaternions, Mac Cullagh's *Theorem of the Polar Plane*,* and some things therewith connected, for an important case of incidence of polarized light on a biaxal crystal: namely, for what was called by him the case of *uniradial vibrations*.

(1.) Let homogeneous light in air (or in a vacuum), with a velocity† taken for unity, fall on a plane face of a doubly refracting crystal, with such a polarization that only *one* refracted ray shall result; let ρ, ρ', ρ'' denote the *vectors of ray-velocity* of the incident, refracted, and reflected lights respectively, ρ having the direction of the *incident ray*, prolonged *within the crystal*, but ρ''

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* See pages 39, 40 of the Paper by that great mathematical and physical philosopher, "*On the Laws of Crystalline Reflexion and Refraction*," already referred to in the Note to page 324 (*Transactions* R. I. A., vol. xviii., part i.).

† Of course, by a suitable choice of the *units* of time and space, the *velocities* and *slownesses*, here spoken of, may be represented by *lines* as *short* as may be thought convenient.

that of the *reflected ray outside*; and let μ' be the *vector of wave-slowness*, or the *index-vector* (comp. 422, (1.)), for the refracted light: these *four* vectors being all drawn from a given point of incidence o, and μ', like ρ', being *within* the crystal.

(2.) Then, by *all** *wave theories of light*, translated into the present notation, we have the equations,

$$\text{I.} \ldots \rho^2 = S\mu'\rho' = \rho''^2 = -1;$$

$$\text{II.} \ldots \rho'' = -\nu\rho\nu^{-1}, \quad \text{with} \quad \text{II}'. \ldots \nu = \mu' - \rho,$$

where ν is a *normal to the face*; whence also,

$$\text{III.} \ldots \rho'' = \rho S\frac{\mu' + \rho}{\mu' - \rho} - 2\mu' S\frac{\rho}{\mu' - \rho};$$

$$\text{IV.} \ldots \rho'' + \rho = 2\iota, \quad \text{if} \quad \text{IV}'. \ldots \iota = \nu^{-1}V\mu'\rho = \nu^{-1}V\nu\rho;$$

and

$$\text{V.} \ldots \rho'' - \rho = -2\nu S\rho\nu^{-1} = -2\nu^{-1}S\rho\nu;$$

so that the *three vectors*, ρ, μ', ρ'', terminate *on one right line*, which is *perpendicular to the face of the crystal*: and the *bisector* of the *angle* between the *first* and *third* of them, or between the incident and reflected rays, is the *intersection* ι of the *plane of incidence* with the same plane face.

(3.) Let τ, τ', τ'' be the *vectors of vibration* for the *three rays* ρ, ρ', ρ'', conceived to be drawn from their respective extremities; then, by *all*† theories of *tangential* vibration, we have the equations,

$$\text{VI.} \ldots S\rho\tau = 0; \qquad \text{VII.} \ldots S\mu'\tau' = 0; \qquad \text{VIII.} \ldots S\rho''\tau'' = 0;$$

to which Mac Cullagh *adds* the supposition (*a*), that the vibration *in the crystal* is *perpendicular to the refracted ray*: or, with the present symbols, that

$$\text{IX.} \ldots S\rho'\tau' = 0; \quad \text{whence} \quad \text{X.} \ldots \tau' \parallel V\mu'\rho',$$

the *direction* of the *refracted vibration* τ' being thus in general *determined*, when those of the vectors ρ' and μ' are given.

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* These equations may be deduced, for example, from the principles of Huyghens, as stated in his *Tractatus de Lumine* (Opera reliqua, Amst., 1728).

† The equations VI. VII. VIII. hold good, for instance, on Fresnel's principles; but Fresnel's tangential vibration in the crystal has a direction *perpendicular* to that adopted by Mac Cullagh.

(4.) To deduce from τ' the *two other* vibrations, τ and τ'', Mac Cullagh assumes, (*b*), the *Principle of Equivalent Vibrations*, expressed here by the formula,

$$\text{XI.} \ldots \tau - \tau' + \tau'' = 0,$$

in virtue of which the *three* vibrations are *parallel* to one *common plane*, and the refracted vibration is the *vector sum* (or *resultant*) of the other two; (*c*), the *Principle of the Vis Viva*, by which the *reflected* and *refracted lights* are *together equal* to the *incident light*, which is conceived to have *caused* them; and (*d*), the *Principle of Constant Density of the Ether*, whereby the *masses of ether*, disturbed by the *three lights*, are simply *proportional* to their *volumes*: the *two last* hypotheses* being here jointly expressed by the equation,

$$\text{XII.} \ldots \text{S}\nu\,(\rho\tau^2 - \rho'\tau'^2 + \rho''\tau''^2) = 0.$$

(5.) Eliminating ρ'' and τ'' from XII. by V. and XI., τ^2 goes off; and we find, with the help of I. and II'., the following *linear equation* in τ,

$$\text{XIII.} \ldots 2\text{S}\frac{\tau}{\tau'} = 1 + \frac{\text{S}\nu\rho'}{\text{S}\nu\rho} = \frac{\text{S}\rho\nu'}{\text{S}\rho\nu}, \quad \text{if} \quad \text{XIII'.} \ldots \nu' = \mu' - \rho';$$

a *second* such equation is obtained by eliminating ρ'' and τ'' by III. and XI. from VIII., and attending to I. VI. VII., namely,

$$\text{XIV.} \ldots 2\text{S}\rho\nu\text{S}\mu'\tau = (\rho^2 - \mu'^2)\,\text{S}\rho\tau' = -\,\text{S}\mu'\nu'\text{S}\rho\tau';$$

and a *third* linear equation in τ is given immediately by VI.

(6.) Solving then for τ, by the rules of the present Calculus, this system of the three linear and scalar equations VI. XIII. XIV., we find for the *incident vibration* the following *vector expression*,†

$$\text{XV.} \ldots \tau = \frac{\text{V}\rho\nu'\tau'}{2\text{S}\rho\nu}; \quad \text{or} \quad \text{XV'.} \ldots 2\tau\text{S}\rho\nu = \tau'\text{S}\rho\nu' - \nu'\text{S}\rho\tau';$$

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* In the concluding Note (page 74) to this Paper, Professor MacCullagh refers to an elaborate Memoir by Professor Neumann, published in 1837 (in the Berlin Transactions for 1835), as containing precisely the *same system* of hypothetical principles respecting Light. But there was evidently a complete mutual independence, in the researches of those two eminent men. Some remarks on this subject will be found in the *Proceedings* of the R. I. A., vol. i., pages 232, 374, and vol. ii., page 96.

† The expressions XV. XVI. enable us to determine, not only the *directions* $\text{U}\tau$, $\text{U}\tau''$ of the *incident* and *reflected vibrations*, but also their *amplitudes* $\text{T}\tau$, $\text{T}\tau''$, or the *intensities* $\text{T}\tau^2$, $\text{T}\tau''^2$ of the incident and reflected *lights*, for any given or assumed amplitude $\text{T}\tau'$ of the *refracted vibration*, or intensity $\text{T}\tau'^2$ of the *refracted light*, after having determined the *direction* $\text{U}\tau'$ of the refracted vibration by means of the formula X.

and accordingly it may be verified by mere inspection, with the help of VII. and IX., that this vector value of τ satisfies the three scalar equations (5.). And when the *incident* vibration has been thus deduced from the *refracted* vibration τ', the *reflected* vibration τ'' is at once given by the formula XI., or by the expression,

$$\text{XVI.} \ldots \tau'' = \tau' - \tau;$$

7.) The relation XV'. gives at once the *equation of complanarity*,

$$\text{XVII.} \ldots S\nu'\tau\tau' = 0, \quad \text{or the formula} \quad \text{XVIII.} \ldots \mu' - \rho' \,|||\, \tau, \tau';$$

if then a *plane* be anywhere so drawn, as to be *parallel* (4.) to the *three vibrations* τ, τ', τ'', it will be parallel *also* to the line $\mu' - \rho'$, which *connects two corresponding points*, on the *wave* and *index-surface* in the crystal: but this is one form of enunciation of Professor Mac Cullagh's *Theorem of the Polar Plane*, which theorem is thus deduced with great simplicity by quaternions, from the principles above supposed.

(8.) For example, if we suppose that OP and OQ, in fig. 89, represent the *refracted ray* ρ', and the *index vector* μ' corresponding, and if we draw through the line PQ a plane perpendicular to the plane of the figure, then the plane so drawn will *contain* (on the principles here considered) the *refracted vibration* τ', and will be *parallel* to *both* the *incident vibration* τ and the *reflected vibration* τ''; whence the *directions* of the two latter vibrations may be in general determined, as being also *perpendicular* respectively to the *incident* and *reflected rays*, ρ and ρ'': and then the *relative intensities* ($T\tau^2$, $T\tau'^2$, $T\tau''^2$) of the *three lights* may be deduced from the *relative amplitudes* ($T\tau$, $T\tau'$, $T\tau''$) of the *three vibrations*, which may themselves be found from the *three complanar directions*, by a simple *resolution* of one line τ' into two others, of which it is the *vector sum, as if* the vibrations were *forces*.

(9.) The equations II'. IV'. V. and XIII'. enable us to express the four vectors, $\mu' (= \rho + \nu)$, $\iota (= \rho - \nu^{-1}S\nu\rho)$, $\rho'' (= \rho - 2\nu^{-1}S\nu\rho)$, and $\rho' (= \rho + \tau - \nu')$, in terms of the three vectors ρ, ν, ν', which are connected with each other by the relation,

$$\text{XIX.} \ldots \iota (= \rho - \nu^{-1}S\nu\rho), \quad \rho'' (= \rho - 2\nu^{-1}S\nu\rho), \quad \text{and} \quad \rho' (= \rho + \nu - \nu'),$$

$$\text{XIX.} \ldots \nu^2 + 2S\nu\rho = S\nu'(\rho + \nu), \quad \text{because} \quad \text{XIX}'. \ldots S\nu\rho' = S(\nu' - \nu)\rho,$$

as in XIII., or because $\mu'^2 - \rho^2 = S\mu'\nu'$ by I. and XIII'.; and with which τ' is connected (VII. and IX.), by the two equations,

$$\text{XX.} \ldots S(\rho + \nu)\tau' = 0, \quad \text{and} \quad \text{XXI.} \ldots S\nu'\tau' = 0;$$

while τ and τ'' are connected with the same three vectors, and with τ', by the relations VI. VIII. XI. XIII., which conduct, by elimination of τ'', to the following system (comp. (5.)) of three linear and scalar equations in τ,

$$\text{XXII.} \ldots S\rho\tau = 0; \quad 2S\nu\rho S\nu\tau = S\nu'(\rho + \nu) S\nu\tau'; \quad 2S\nu\rho S\tau'^{-1}\tau = S\nu'\rho;$$

and therefore to the vector expression,

$$2\tau S\nu\rho = V\rho\nu'\tau', \text{ as in XV.}$$

(10.) By these or other transformations, there is no difficulty in deducing this new equation, in which ω may be any vector,

$$\text{XXIII.} \ldots V\nu V\{(\rho - \omega)\tau - (\rho' - \omega)\tau' + (\rho'' - \omega)\tau''\}\tau' = 0;$$

and conversely, when ω is thus treated as *arbitrary*, the formula XXIII., with the relations (9.) between the vectors ρ, ρ', ρ'', ν, ν', μ', but *without* any restriction (*except itself*) on τ, τ', τ'', is *sufficient* to give the *two* vector equations,

$$\text{XI.} \ldots \tau - \tau' + \tau'' = 0, \quad \text{and} \quad \text{XXIV.} \ldots \rho\tau - \rho'\tau' + \rho''\tau'' = x\nu^{-1} + y,$$

in which

$$\text{XXV.} \ldots x = S\nu(\rho\tau - \rho'\tau' + \rho''\tau'') = S\nu\nu'\tau', \text{ and XXVI.} \ldots y = S(\rho\tau - \rho'\tau'\nu + \rho''\tau'');$$

and which conduct to the *two* scalar equations (among others),

$$\text{XXVII.} \ldots S\kappa(\rho\tau - \rho'\tau' + \rho''\tau'') = 0, \quad \text{if} \quad \text{XXVII}'. \ldots S\kappa\nu = 0,$$

and

$$\text{XXVIII.} \ldots S\nu\rho(S\rho\tau - S\rho''\tau'') = S\nu\rho' S\mu'\tau';$$

so that if we *now* suppose the equations VI. VIII. IX. to be *given*, the equation VII. will *follow*, by XXVIII.; while, as a *case* of XXVII., and with the signification IV. or IV'. of ι, we have the equation,

$$\text{XXIX.} \ldots S\iota(\rho\tau - \rho'\tau' + \rho''\tau'') = 0.$$

(11.) And thus (or otherwise) it may be shown, that the *three* scalar equations VI. VIII. IX., combined with the *one* vector formula XXIII., which (on account of the arbitrary ω) is equivalent to *five* scalar equations, are sufficient to give the *same direction* of τ', and the *same dependencies* of τ and τ'' thereon, as those expressed by the equations X. XV. XVI.; and therefore (among other consequences), to the formulæ XII. and XVII.

(12.) But the equations VI. VIII. IX. contain what may be called the *Principle of Rectangular Vibrations* (or of *vibrations rectangular to rays*); and the formula XXIII. is easily interpreted (416.), as expressing what may be termed the *Principle of the Resultant Couple*: namely, the theorem, that *if the three vibrations* (or displacements), τ, τ', τ'', *be regarded as three forces,* RT, R′T′, R″T″, *acting at the ends of the three rays,* ρ, ρ', ρ'', or OR, OR′, OR″ (drawn in the directions (1.) from the point of incidence O), *then this other system of three forces,* RT, − R′T′, R″T″ (conceived as applied to a solid body), *is equivalent to a single couple, of which the plane is parallel* (or the axis perpendicular) *to the face of the crystal.*

(13.) It follows then, by (10.) and (11.), that from these two principles,* (I.) and (II.), we can infer *all* the following:

(III.) the *Principle of Tangential Vibrations* (or of vibrations *tangential to the waves*);

(IV.) the *Principle of Equivalent Vibrations* (4.);

(V.) the *Principle of the Vis Viva*, as expressed (in conjunction with that of the *Constant Density of the Ether*) by the equation XII.;

(VI.) the *Principle* (or *Theorem*) *of the Polar Plane*;

And (VII.) what may be called the *Principle of Equivalent Moments,*† namely, theorem that *the Moment of the Refracted Vibration* (R′T′) *is equal to*

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* The word "Principle" is here employed with the usual latitude, as representing *either* an *hypothesis assumed*, or a *theorem deduced*, but made a *ground* of *subsequent* deduction. The *principle* (I.) of *rectangular vibrations coincides*, for the case of an *ordinary medium*, with the principle (III.) of *tangential* vibrations; but, for an *extraordinary* medium, except for the case (not here considered) of *ordinary rays* in an *uniaxal crystal*, these two principles are *distinct*, although *both* were assumed by MacCullagh and Neumann. The present writer has already disclaimed (in the Note to page 323) any responsibility for the *physical hypotheses*; so that the results given above are offered merely as instances of *mathematical deduction* and *generalization* attained through the Calculus of Quaternions.

† In a very clear and able Memoir, by Arthur Cayley, Esq. (now Professor Cayley), "On Professor MacCullagh's Theorem of the Polar Plane," which was read before the Royal Irish Academy on the 23rd of February, 1857, and has been printed in vol. vi. of the *Proceedings* of that Academy (pages 481–491), this *name* "principle of equivalent moments," is given to a statement (page 489), that "the moment of $R't'$ round the axis AH, is equal to the sum of the moments of Rt and $R''t''$ round the same axis"; the line AH being (page 487) the intersection of the plane of incidence with the plane of separation of the two media, that is, with the *face* of the crystal; while Rt, $R't'$, $R''t''$ are lines representing (page 488) the *three vibrations* (incident, refracted, and reflected), at the ends of the *three rays* AR, AR', AR'', which are drawn from the point of incidence A, so as to lie, *all three* (page 487), *within the crystal*. And in fact, if this statement be modified, either by *changing the sign* of the moment of $R''t''$ (page 491), or by drawing the *reflected ray* AR'', like the line OR″ of the present investigation *in the air* (or *in vacuo*), *instead of prolonging it backwards within* the biaxal crystal, it *agrees* with the *case* XXIX. of the more general formula XXVII., which is itself *included* in what has been called above the *Principle of the Resultant Couple*. In venturing thus to point out, as the subject obliged him to do, what seemed to him to be a slight inadvertence in a Paper of such interest and value, the present writer hopes that he will not be supposed to

the Sum of the Moments of the Incident and Reflected Vibrations (RT and R″T″), *with respect to any line, which is on, or parallel to, the Face of the Crystal.*

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[It appears by the Table of Initial Pages (as printed in the First Edition), that the Author had intended to complete the work by the addition of Seven Articles.]

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be deficient in the admiration (long since publicly expressed by him), which is due to the vast attainments of a mathematician so eminent as Professor Cayley.

Since the preceding Series 423, including its Notes (so far), was copied and sent to the printers, the writer's attention has been drawn to a *later* Paper by MacCullagh (read December 9th, 1839, and published in vol. xxi., part i., of the *Transactions* of the Royal Irish Academy, pages 17–50), entitled "*An Essay towards a Dynamical Theory of crystalline Reflexion and Refraction*"; in which there is given at page 43) a *theorem* essentially equivalent to the above-stated "Principle of the Resultant Couple," but expressed so as to include the case where the vibrations are *not uniradial*, so that the *double* refraction of the crystal is allowed to manifest itself. MacCullagh speaks, in his enunciation of the theorem, of measuring each ray, *in the direction of propagation*: which agrees with, but of course anticipates, the *direction of the reflected ray*, adopted in the preceding investigation. The writer believes that subsequent experiments, by Jamin and others, are considered to diminish much the *physical value* of the theory above discussed.

APPENDIX.

NOTES.

I.—ON QUATERNION DETERMINANTS.

(1.) Quaternion determinants were first investigated by Cayley (*Phil. Mag.* xxvi., 1845, pages 141–145). Because quaternion multiplication is not commutative, a determinant whose constituents are quaternions is unmeaning until some additional convention is adopted concerning its expansion. If it be agreed that the order of the constituents in the expansion shall follow the order of the rows, all indefiniteness is removed.

(2.) On this supposition

$$\begin{vmatrix} p & q \\ p' & q' \end{vmatrix} = pq' - qp', \quad \text{but not} \quad pq' - p'q\,; \tag{i}$$

$$\begin{vmatrix} p & q \\ p & q \end{vmatrix} = pq - qp = 2\mathrm{V}.\mathrm{V}p\mathrm{V}q, \quad \text{and} \quad \begin{vmatrix} p & p \\ q & q \end{vmatrix} = pq - pq = 0. \tag{ii}$$

It is also obvious that if x is any scalar

$$\begin{vmatrix} p & q \\ p' & q' \end{vmatrix} = \begin{vmatrix} p & xp+q \\ p' & xp'+q' \end{vmatrix}, \quad \text{but not} = \begin{vmatrix} p & q \\ xp+p' & xq+q' \end{vmatrix}. \tag{iii}$$

(3.) Thus the columns may be treated as in ordinary determinants with scalar constituents; but it is not lawful to treat the rows in this manner. The former of these processes is consistent with the convention that the order of the constituents shall follow the order of the rows; the latter violates this convention.

The following example illustrates multiplication of a quaternion determinant by a scalar determinant:—

$$\begin{vmatrix} p & q \\ p' & q' \end{vmatrix} . \begin{vmatrix} x & y \\ x' & y' \end{vmatrix} = \begin{vmatrix} px+qy & px'+qy' \\ p'x+q'y & p'x'+q'y' \end{vmatrix} = \begin{vmatrix} px+qx' & py+qy' \\ p'x+q'x' & p'y+q'y' \end{vmatrix} \tag{iv}$$

if the x and y are scalars, and the p and q determinants. This method is applicable for any order.

(4.) Again, when we have equations of the type

$$p_1x + q_1y + r_1z = 0, \quad p_2x + q_2y + r_2z = 0, \quad p_3x + q_3y + r_3z = 0, \qquad \text{(v)}$$

in which x, y, and z are scalars, every determinant obtained by interchanging the rows in

$$\begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix} \qquad \text{(vi)}$$

must vanish. There are six of these. Further every determinant deducible from

$$\begin{vmatrix} p_1 & q_1 & r_1 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} p_1 & q_1 & r_1 \\ p_1 & q_1 & r_1 \\ p_1 & q_1 & r_1 \end{vmatrix} \qquad \text{(vii)}$$

by interchange of rows and by alteration of the suffixes must be zero. For by (3.) the columns may be multiplied by x, y, and z and added together, and thus one column may be reduced to zero when equations (v) subsist.

These results may be extended to a system of linear equations of any order.

(5.) The determinants of the third order of the last section are not all independent. If the determinant (vii) with identical rows vanishes, we have by (ii)

$$p_1 V.Vq_1Vr_1 + q_1V.Vr_1Vp_1 + r_1V.Vp_1Vq_1 = 0. \qquad \text{(viii)}$$

Taking the scalar part, we see that the three vectors are coplanar, so that we may write

$$xVp_1 + yVq_1 + zVr_1 = 0.$$

Hence, it appears by operating on this by $V.Vp_1$ and $V.Vq_1$, that (viii) may be replaced by

$$xp_1 + yq_1 + zr_1 = 0. \qquad \text{(ix)}$$

From this it immediately follows that the vanishing of the first determinant (vii) is equivalent to

$$xp_2 + yq_2 + zr_2 = 0.$$

If in this determinant the suffix 2 is replaced by 3, and if the new determinant vanishes, equations (v) are reproduced and all the other determinants will vanish.

In a similar manner for determinants of the second order, if we suppose that the four quaternions p_1, q_1, p_2, and q_2 are not all coplanar, three of the equations

$$\begin{vmatrix} p_1 & q_1 \\ p_1 & q_1 \end{vmatrix} = 0, \quad \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} = 0, \quad \begin{vmatrix} p_2 & q_2 \\ p_1 & q_1 \end{vmatrix} = 0, \quad \begin{vmatrix} p_2 & q_2 \\ p_2 & q_2 \end{vmatrix} = 0, \qquad \text{(x)}$$

imply the fourth, and require

$$xp_1 + yq_1 = 0, \qquad xp_2 + yq_2 = 0. \qquad \text{(xi)}$$

(6.) The determinant of the fourth order, whose rows are identical, vanishes. For if p, q, r, and s are any four quaternions, we can find scalars x, y, z, and w so that

$$xp + y\dot{q} + zr + ws = \mathrm{V}^{-1}0 = \text{a scalar}.$$

One column can thus be reduced to the same scalar repeated four times, and, when we expand by the minors of the second order, every product of minors will involve a vanishing minor. By means of this result many identities may be obtained.*

II.—MISCELLANEOUS PROPERTIES OF TWO LINEAR VECTOR FUNCTIONS.

(1.) In general a pair of linear and vector functions may be simultaneously expressed in the form

$$\phi\rho = \lambda \mathrm{S}\alpha\rho + \mu\mathrm{S}\beta\rho + \nu\mathrm{S}\gamma\rho\,; \quad \theta\rho = a\lambda\mathrm{S}\alpha\rho + b\mu\mathrm{S}\beta\rho + c\nu\mathrm{S}\gamma\rho. \qquad \text{(i)}$$

Assuming the possibility of the reduction, it is clear that

$$\theta\mathrm{V}\beta\gamma = a\phi\mathrm{V}\beta\gamma, \quad \theta\mathrm{V}\gamma\alpha = b\phi\mathrm{V}\gamma\alpha, \quad \theta\mathrm{V}\alpha\beta = c\phi\mathrm{V}\alpha\beta, \qquad \text{(ii)}$$

and consequently $\mathrm{V}\beta\gamma$, $\mathrm{V}\gamma\alpha$, and $\mathrm{V}\alpha\beta$ are the axes, and a, b, and c the roots of the function $\phi^{-1}\theta$. The vectors α, β, and γ having being found, λ, μ, and ν are determined by three equations of the type

$$\lambda = \phi\mathrm{V}\beta\gamma(\mathrm{S}\alpha\beta\gamma)^{-1}. \qquad \text{(iii)}$$

Otherwise α, β, and γ may be determined directly as the axes of the conjugate of $\phi^{-1}\theta$, that is of $\theta'\phi'^{-1}$. Combining (iii) and (ii), we see that λ, μ, and ν are the axes, and a, b, and c the roots of the new function $\theta\phi^{-1}$.

(2.) Thus it is proved that $\phi^{-1}\theta$ and $\theta\phi^{-1}$ have the same latent roots, and consequently the same symbolic cubic. More generally, all functions expressed as products of others and derivable from one another by *cyclical* transposition of the factors have the same cubic; for example, θ and $\phi\theta\phi^{-1}$.

The same thing is evident when the cubic

$$(\phi\theta)^3 - M''(\phi\theta)^2 + M'\phi\theta - M = 0 \qquad \text{(iv)}$$

is multiplied by θ and into θ^{-1}, for it becomes

$$(\theta\phi)^3 - M''(\theta\phi)^2 + M'\theta\phi - M = 0. \qquad \text{(v)}$$

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* Applications and examples will be given in the Note on Invariants, and in the Note on Screws (Note V., Section 14, p. 382, and Note VIII., Section 9, page 393).

(3.) When both functions are self-conjugate $V\beta\gamma$, $V\gamma\alpha$, and $V\alpha\beta$ are mutually conjugate with respect to the quadrics, $S\rho\theta\rho = const.$, $S\rho\phi\rho = const.$, at least if a, b, and c are unequal; for by (ii)

$$aSV\gamma\alpha\phi V\beta\gamma = SV\gamma\alpha\theta V\beta\gamma = SV\beta\gamma\theta V\gamma\alpha = bSV\beta\gamma\phi V\gamma\alpha,$$

so
$$a = b, \quad \text{or} \quad SV\gamma\alpha\phi V\beta\gamma = 0. \tag{vi}$$

From (vi) and similar relations coupled with (iii), we find in this case where the roots are unequal

$$\lambda \parallel \alpha, \quad \mu \parallel \beta, \quad \nu \parallel \gamma. \tag{vii}$$

(4.) Hence we can see how to reduce an arbitrary function ϕ to the product $\Phi_1\Phi$ of two self-conjugate functions. For the axes of ϕ must be mutually conjugate to Φ and to Φ_1^{-1}, and therefore if x, y, and z are arbitrary scalars

$$\Phi\rho = xV\beta\gamma S\beta\gamma\rho + yV\gamma\alpha S\gamma\alpha\rho + zV\alpha\beta S\alpha\beta\rho,$$

$$\Phi_1\rho = (ax^{-1}\alpha S\alpha\rho + by^{-1}\beta S\beta\rho + cz^{-1}\gamma S\gamma\rho)\,(S\alpha\beta\gamma)^{-2}, \tag{viii}$$

are the necessary forms, α, β, and γ being the axes, and a, b, and c the roots of ϕ. Even if ϕ has a pair of imaginary roots (b, c), and axes (β, γ), the functions Φ and Φ_1 are real, provided y and z are conjugate imaginaries.*

(5.) If two functions ϕ and θ can be reduced simultaneously to the forms $\Phi_1\Phi$ and $\Phi_2\Phi$, the axes of ϕ and θ must be edges of a quadric cone. Let $S\rho\Theta\rho = 0$ be the cone through the axes α, β, and γ of ϕ and two of the axes α' and β' of θ. Then because the axes of each function are mutually conjugate to Φ,

$$\Phi\alpha \parallel V\beta\gamma, \text{ \&c.,} \quad \Phi\alpha' \parallel V\beta'\gamma', \text{ \&c.;}$$

and $S\alpha\Theta\alpha = 0$ is equivalent to $S\alpha\Theta\Phi^{-1}V\beta\gamma = 0$. Hence the first invariant of the function $\Theta\Phi^{-1}$ vanishes, or

$$S\alpha\Theta\Phi^{-1}V\beta\gamma + S\beta\Theta\Phi^{-1}V\gamma\alpha + S\gamma\Theta\Phi^{-1}V\alpha\beta = 0. \tag{ix}$$

Replacing α, β, γ in this invariant by α', β', γ', the first and second terms vanish and the third must be zero likewise. Thus γ' is also an edge of the cone.

(6.) In the case of simple equality among the roots of $\phi^{-1}\theta$, two of its axes coincide, and the reduction (i) becomes impossible. When equality among the roots carries with it indeterminateness of the axes of $\phi^{-1}\theta$, the reduction likewise becomes indeterminate instead of being unique as in the general case.

(7.) Two functions are commutative in order of operation if, and only if, their axes coincide. The first part of the proposition is evident, and, to prove the second,

* Tait shows that if the roots of ϕ are real and positive, so also are the roots of Φ and Φ_1. *Proc. R. S. E.*, May 18 and June 1, 1896, or *Scientific Papers*, vol. ii., p. 407.

when the reduction (i) is possible, it is sufficient to observe that if $\phi\theta = \theta\phi$ or $\theta\phi^{-1} = \phi^{-1}\theta$, the vectors λ, μ, and ν must be parallel, respectively, to $V\beta\gamma$, $V\gamma\alpha$, and $V\alpha\beta$. These vectors are, in this case, axes of both the functions ϕ and θ. More generally, and without postulating the possibility of the reduction, if

$$\phi\xi = g\xi, \quad \text{then} \quad \phi\theta\xi = \theta\phi\xi = g\theta\xi. \tag{x}$$

Thus, ξ and $\theta\xi$ are both axes of ϕ, and correspond to the same root, and this requires $\theta\xi \parallel \xi$ (so that ξ is also an axis of θ) or else, ϕ must have indeterminate axes. When the second alternative is admitted, if η is any second vector in the plane of the indeterminate axes of ϕ, $\theta\eta$ lies also in this plane, and the four vectors ξ, η, $\theta\xi$, $\theta\eta$ are coplanar. It is always possible to find two other vectors ξ' and η' in this plane, so that $\theta\xi' \parallel \xi'$ and $\theta\eta' \parallel \eta'$ and these vectors are axes of ϕ as well as of θ.

III.—THE STRAIN FUNCTION.

(1.) The application of the linear vector function to the theory of strain has been admirably developed by Professor Tait in the Tenth Chapter of Kelland and Tait's *Introduction to Quaternions.* From this source a large portion of this Note has been adapted.

When a linear vector function operates on every vector of a system, vectors originally equal remain equal after the operation; consequently, all equal similar and similarly placed figures transform into figures equal similar and similarly placed. There are two classes of this kind of transformation when the function ϕ is real. In the first rotation from $\phi\alpha$ to $\phi\beta$ to $\phi\gamma$ has the same sense as that from α to β to γ, whatever vectors α, β, and γ may be. In the second class the sense of rotation is reversed. The first class of transformation is identical with a homogeneous strain; the second is equivalent to a homogeneous strain accompanied by a reflection as in a plane mirror, or to a homogeneous strain accompanied by reversal of every vector. In fact, reversal of every vector is equivalent to rotation through two right angles about some axis through the origin and reflection with respect to the plane through the origin at right angles to the axis.

(2.) Hamilton's third invariant of the function ϕ

$$m = S\phi\alpha\phi\beta\phi\gamma\,(S\alpha\beta\gamma)^{-1} \tag{i}$$

is the ratio which the volume of the parallelepiped, transformed from that whose edges are α, β, γ, bears to the volume of the original. It is quite independent of any particular set of vectors α, β, γ, and is, therefore, the ratio in which any volume is altered.

(3.) The sign of m affords the criterion concerning the class of the transformation (1.). If m is positive, the sense of rotation from $\phi\alpha$ to $\phi\beta$ to $\phi\gamma$ remains the same as that from α to β to γ. The contrary is the case when m is negative.

(4.) In the case of a pure strain, three mutually rectangular lines preserve their directions. If unit vectors along these are i, j, and k, and if the unit vectors are strained into $e_1 i$, $e_2 j$, and $e_3 k$, where e_1, e_2, and e_3 are three positive scalars, any other vector is strained into

$$\Phi\rho = -e_1 i S i\rho - e_2 j S j\rho - e_3 k S k\rho. \tag{ii}$$

This is a particular case of the general theorem, that a linear vector function is determinate when the results of operating by it, on three known vectors, are given. In fact, given α, β, γ, $\phi\alpha$, $\phi\beta$, and $\phi\gamma$, we have, in general,

$$\phi\rho = (\phi\alpha S\beta\gamma\rho + \phi\beta S\gamma\alpha\rho + \phi\gamma S\alpha\beta\rho)(S\alpha\beta\gamma)^{-1}.$$

The function Φ, defined by (ii), is self-conjugate, and its latent roots e_1, e_2, and e_3 are all positive. A function of this nature may be said to be *ellipsoidal*.

The sphere $T\rho = r$ is changed by the transformation $\varpi = \phi\rho$ into the quadric—the strain ellipsoid—determined by the equation

$$T\phi^{-1}\varpi = r, \quad \text{or} \quad S\varpi\phi'^{-1}\phi^{-1}\varpi = -r^2 \quad \text{or} \quad S\varpi(\phi\phi')^{-1}\varpi = -r^2, \tag{iii}$$

since $\phi'^{-1}\phi^{-1} = (\phi\phi')^{-1}$. It is, in general, an ellipsoid, for ϖ cannot be infinite, while ρ is finite. When the strain is pure, the equation of the strain ellipsoid is more simply

$$T\Phi^{-1}\varpi = r, \quad \text{or} \quad S\varpi\Phi^{-2}\varpi = -r^2,$$

or again, in terms of i, j, and k, e_1, e_2, and e_3,

$$\frac{(Si\rho)^2}{e_1^2} + \frac{(Sj\rho)^2}{e_2^2} + \frac{(Sk\rho)^2}{e_3^2} = r^2.$$

Thus, i, j, and k are unit vectors along its axes, and e_1, e_2, e_3 are the ratios of the semi-axes to the radius of the sphere. In general, the axes of the ellipsoid are parallel to the axes of the self-conjugate function $\phi\phi'$.

(5.) We shall now prove that the transformation produced by any linear function ϕ is equivalent to a rotation followed by a pure strain, and accompanied in the case where m is negative by a reversal of every vector.

Assuming generally for all vectors ρ

$$\phi\rho = \pm \Phi q\rho q^{-1}, \tag{iv}$$

where Φ is an ellipsoidal function, the third invariant of ϕ (i)

$$\begin{aligned} m &= \pm S\Phi q\alpha q^{-1}\Phi q\beta q^{-1}\Phi q\gamma q^{-1}(S\alpha\beta\gamma)^{-1} \\ &= \pm S\Phi q\alpha q^{-1}\Phi q\beta q^{-1}\Phi q\gamma q^{-1}(Sq\alpha q^{-1}q\beta q^{-1}q\gamma q^{-1}) = \pm e_1 e_2 e_3, \end{aligned} \tag{v}$$

if e_1, e_2, and e_3 are the positive roots of Φ. Hence, if m is positive, the plus sign is to be taken; and, if m is negative, the minus sign.

(6.) Again, taking the conjugate of ϕ

$$\phi'\rho = \pm\, q^{-1}\Phi\rho q, \qquad \text{(vi)}$$

and therefore

$$\phi\phi'\rho = \Phi^2\rho \quad \text{or} \quad \phi\phi' = \Phi^2. \qquad \text{(vii)}$$

This equation requires $\phi\phi'$ to be ellipsoidal, and it must be so if ϕ, and therefore ϕ', is real for

$$\mathrm{T}\,(\phi'\rho)^2 = -\,\mathrm{S}\rho\phi\phi'\rho \qquad \text{(viii)}$$

cannot be finite for any infinite value of ρ.

The latent roots of the self-conjugate function $\phi\phi'$ being all positive may be denoted by e_1^2, e_2^2, and e_3^2, and if i, j, k are the axes

$$\phi\phi'\rho = \Phi^2\rho = -\,e_1^2 i\mathrm{S}i\rho - e_2^2 j\mathrm{S}j\rho - e_3^2 k\mathrm{S}k\rho. \qquad \text{(ix)}$$

(7.) We are now at liberty to define Φ by the equation

$$(\phi\phi')^{\frac{1}{2}}\rho = \Phi\rho = -\,e_1 i\mathrm{S}i\rho - e_2 j\mathrm{S}j\rho - e_3 k\mathrm{S}k\rho, \qquad \text{(x)}$$

the roots e_1, e_2, and e_3 being all positive. In general a function has eight square roots, and the eight square roots of $\phi\phi'$ correspond to the various combinations of signs attributable to the radicals in

$$-\sqrt{e_1^2}\,i\mathrm{S}i\rho - \sqrt{e_2^2}\,j\mathrm{S}j\rho - \sqrt{e_3^2}\,k\mathrm{S}k\rho. \qquad \text{(xi)}$$

We may speak of the function Φ as the *principal* square root as in this case positive signs are chosen throughout.

(8.) In order to justify the assumption made in equation (iv), it is necessary to prove that $\mathrm{U}q$ is determinate. Writing equation (iv) in the form

$$\phi\rho = \Phi\chi\rho, \qquad \text{(xii)}$$

where χ is a linear vector function to be determined, the conjugate of $\chi = \Phi^{-1}\phi$ is likewise its inverse for

$$\chi' = \phi'\Phi^{-1}, \quad \text{and} \quad \Phi^{-1}\phi\,.\,\phi'\Phi^{-1} = \Phi^{-1}\Phi^2\Phi^{-1} = 1.$$

So χ satisfies the equation

$$\chi\chi' = 1. \qquad \text{(xiii)}$$

Now this equation shows that whatever vector ρ may be its tensor is equal to that of $\chi\rho$, and therefore all figures remain equal after the transformation represented by χ. The transformation must therefore be equivalent to a rotation, or to a rotation accompanied by a reflection or the reversal of every vector. The assumption made in (iv) is thus completely verified.

Supposing m positive, and writing (vi) in the form

$$q\phi'\rho - \Phi\rho q = 0,$$

the scalar and vector parts furnish the equations

$$\mathrm{SV}q\,(\phi' - \Phi)\rho = 0, \quad (\phi' - \Phi)\rho\mathrm{S}q + \mathrm{V.V}q(\phi' + \Phi)\rho = 0,$$

and the second of these virtually includes the first. In terms of two arbitrary vectors ρ and ρ', we find, without difficulty,

$$\mathrm{V}q = x\mathrm{V}(\phi' - \Phi)\rho\,(\phi' \quad \Phi)\rho'; \quad \mathrm{S}q = -\,x\mathrm{S}(\phi' + \Phi)\rho\,(\phi' - \Phi)\rho'. \qquad \text{(xiv)}$$

Also $\mathrm{V}q$ satisfies the equation

$$(\phi - \Phi)\mathrm{V}q = 0,$$

because otherwise $\mathrm{S}(\phi - \Phi)\mathrm{V}q\rho = 0$ could not be satisfied for every value of ρ.

The symbolic cubic of χ must be of the form

$$(\chi \mp 1)\,(\chi^2 + n\chi + 1) = 0, \qquad \text{(xv)}$$

for the symbolic cubic of a function is also satisfied by its conjugate, and in this case the conjugate is the inverse. The upper sign corresponds to the positive value of m.

(9.) Similarly if the rotation follows the pure strain, the assumption

$$\phi\rho = \pm\, p\Psi\rho p^{-1}, \quad \text{or} \quad \phi'\rho = \pm\, \Psi p^{-1}\rho p \qquad \text{(xvi)}$$

may be justified by an analogous train of reasoning. Here $\phi'\phi\rho = \Psi^2\rho$ and Ψ is the ellipsoidal principal square root of the ellipsoidal function $\phi'\phi$. The latent roots of Ψ and Φ are identical (compare Note II., (2.), p. 363).

(10.) In every homogeneous strain one direction at least remains unchanged. When m is positive, one latent root of the function ϕ must be positive. This is obvious when the roots are all real; and when two of the roots are imaginary, $a + \sqrt{-1}\,b$ and $a - \sqrt{-1}\,b$, their product $a^2 + b^2$ is always positive, and therefore the remaining and real root is positive. The axis of ϕ, corresponding to the real positive root, retains its direction. It is evident by superposing a rotation upon a pure strain that any selected direction may be preserved unaltered.

If two directions $\mathrm{U}\alpha$ and $\mathrm{U}\beta$ remain unchanged, they are connected by the relation

$$\mathrm{SU}\phi\alpha\phi\beta = \mathrm{SU}\alpha\beta, \quad \text{or} \quad \mathrm{SU}\Psi\alpha\Psi\beta = \mathrm{SU}\alpha\beta, \qquad \text{(xvii)}$$

where $\phi\rho = p\Psi\rho p^{-1}$ (xvi). Either of these equations expresses that the cosine of the angle of inclination of the strained lines is equal to that of the unstrained lines. Rationalizing the second of equations (xvii), it appears if α is given that the locus of β is one sheet of the quartic cone

$$\alpha^2\beta^2\,(\mathrm{S}\alpha\Psi^2\beta)^2 = (\mathrm{S}\alpha\beta)^2\mathrm{S}\alpha\Psi^2\alpha\mathrm{S}\beta\Psi^2\beta. \qquad \text{(xviii)}$$

If in this we substitute $\beta = \alpha + t\alpha'$ where $\mathrm{S}\alpha\alpha' = 0$, we find that α is a double edge of the cone, and discarding the factor t^2 we obtain a quadratic in t to determine the edges of the cone in the plane of α and α'. One solution only is appropriate as (xviii) includes both the conditions

$$\mathrm{SU}\Psi\alpha\Psi\beta = \pm\,\mathrm{SU}\alpha\beta.$$

It is easy to see that the roots of the quadratic are always real since Ψ is ellipsoidal. If two directions are unaltered, a third is likewise unaltered.

(11.) The roots of the function ϕ, g_1, g_2, and g_3 may have any values (within certain limits) subject to the single condition

$$g_1 g_2 g_3 = e_1 e_2 e_3, \tag{xix}$$

which expresses that no change of volume is produced by the rotation.

If we assume g_1, g_2, and g_3 subject to (xix), and try to satisfy the equations

$$\phi\alpha = p\Psi\alpha p^{-1} = g_1\alpha, \quad \phi\beta = p\Psi\beta p^{-1} = g_2\beta, \quad \phi\gamma = p\Psi\gamma p^{-1} = g_3\gamma, \tag{xx}$$

we see that that the axes α, β, and γ must be edges, respectively, of the cones

$$\mathrm{T}\Psi\mathrm{U}\rho = \mathrm{T}g_1, \quad \mathrm{T}\Psi\mathrm{U}\rho = \mathrm{T}g_2, \quad \mathrm{T}\Psi\mathrm{U}\rho = \mathrm{T}g_3, \tag{xxi}$$

where $\mathrm{T}g_1$ is the positive value of the scalar g_1 irrespective of its sign. These cones are the loci of vectors whose lengths are altered in a given ratio. Selecting, at pleasure, any vector α on the first cone, β is determined on the second cone by the aid of the relation

$$\mathrm{S}\Psi\alpha\Psi\beta = g_1 g_2 \mathrm{S}\alpha\beta \tag{xxii}$$

implied in (xx) and equivalent to (xvii). Up, and therefore the rotation, may be found by combining the first and second of equations (xx). Hence, ϕ is determined, and the third vector γ is the result of operating by $(\phi - g_1)(\phi - g_2)$ on an arbitrary vector.

Again, by (xxi), the magnitudes of g_1, g_2, and g_3 must lie between the greatest and least values of $\mathrm{T}\Psi\mathrm{U}\rho$, that is, between the greatest and least of the scalars e_1, e_2, e_3. In fact the magnitudes of the roots are inversely proportional to the radii of the ellipsoid

$$\mathrm{T}\Psi\rho = 1 \quad \text{or} \quad \mathrm{S}\rho\Psi^2\rho = -1, \tag{xxiii}$$

which are parallel to the corresponding axes. This ellipsoid is converted by the strain into a sphere of unit radius.

(12.) It is possible to superpose a rotation upon a pure strain, so that the function ϕ may have indeterminate axes. These axes evidently must lie in one or other of the cyclic planes of the ellipsoid (xxiii). Expressing Ψ^2 in Hamilton's cyclic form

$$\Psi^2\rho = e_2^2\rho + \lambda\mathrm{S}\mu\rho + \mu\mathrm{S}\lambda\rho \tag{xxiv}$$

has one root equal to e_2^2, and the other roots are

$$e_1^2 = e_2^2 + \mathrm{S}\lambda\mu + \mathrm{T}\lambda\mu, \quad e_3^2 = e_2^2 + \mathrm{S}\lambda\mu - \mathrm{T}\lambda\mu. \tag{xxv}$$

Assuming

$$\phi\rho = e_2\rho + \nu\mathrm{S}\lambda\rho; \tag{xxvi}$$

this function has indeterminate axes in the plane $\mathrm{S}\lambda\rho = 0$, and it appears without difficulty that $\phi'\phi = \Psi^2$ if ν satisfies the equation

$$e_2\nu + \tfrac{1}{2}\nu^2\lambda = \mu. \tag{xxvii}$$

Also the third invariant of ϕ being equal to that of Ψ,

$$e_2^2(e_2 + S\nu\lambda) = e_1e_2e_3. \qquad \text{(xxviii)}$$

When we operate on (xxvii) by $S\lambda$ and use (xxv) and (xxviii), we find,

$$\nu^2\lambda^2 = (e_1 - e_3)^2 \quad \text{and} \quad e_2\nu = \mu - \tfrac{1}{2}(e_1 - e_3)^2\lambda^{-1}. \qquad \text{(xxix)}$$

Thus ν is completely determined, and the assumption made as to the form of ϕ (xxvi) is justified. Also we see, by the form of the function ϕ, that the most general strain may be effected in three stages, by displacing in one direction ($U\nu$) a system of planes perpendicular to another direction ($U\lambda$) by amounts proportional to the distances of the planes from the origin; by uniformly altering the linear dimensions (in the ratio e_2 to unity); and by rotating the body as a whole.

(13.) When unity is included between the limits $e_1 > e_2 > e_3$, that is, when elongation and contraction both occur, a rotation may be applied to a pure strain, so that one root of ϕ is unity. In this case one root of $\phi - 1$ is zero, or this function is a binomial reducing every vector to a fixed plane. But $\phi\rho - \rho$ or $(\phi - 1)\rho$ is the *displacement* due to the strain, and accordingly under the above conditions a rotation may be superposed upon the strain so as to render the resultant displacement of every point parallel to a plane.

Again, by (xxvi), if e_2 is unity, a suitable rotation will render the displacement of every point parallel to a line. In this case the pure part of the strain is plane, for when one root of Ψ is unity, the strain is completely specified by that in the plane at right angles to the corresponding axis.

(14.) In the case of a plane strain when there is no dilatation the intermediate root e_2 is evidently unity. The condition for no dilatation is now $e_1e_3 = 1$, and this, coupled with $e_2 = 1$, shows that (xxvi) and (xxviii) are equivalent to

$$\phi\rho = \rho + \nu S\lambda\rho, \quad S\nu\lambda = 0. \qquad \text{(xxx)}$$

The strain represented by (xxx) is a simple shear, the system of planes normal to λ being displaced parallel to themselves and proportionately to their distances from the origin. In general a plane strain without dilatation is equivalent to a shear and a rotation.

It also appears from (xxviii) that

$$e_2^2 = e_1e_3 = 1 \qquad \text{(xxxi)}$$

are the conditions that a strain should be equivalent to a rotation and a shear.

(15.) We shall investigate the reduction of the general strain to a dilatation, a pair of shears and a rotation. If this is possible the general linear vector function must be expressible in the form

$$\phi\rho = m^{\frac{1}{3}}p(1 + t'\alpha'S\beta')(1 + t\alpha S\beta)\rho \,.\, p^{-1}, \qquad \text{(xxxii)}$$

where $S\alpha\beta = S\alpha'\beta' = 0$. For convenience we take α, β, α', β' to be unit vectors.

Observing that

$$(1 - ta\mathrm{S}\beta)(1 + ta\mathrm{S}\beta) = 1$$

(xxxii) is equivalent to

$$\phi(1 - ta\mathrm{S}\beta)\rho = m^{\frac{1}{3}}p(1 + t'a'\mathrm{S}\beta')\rho p^{-1}, \qquad \text{(xxxiii)}$$

and taking conjugates

$$(1 - t\beta\mathrm{S}a)\phi'\rho = m^{\frac{1}{3}}(1 + t'\beta'\mathrm{S}a')p^{-1}\rho p. \qquad \text{(xxxiv)}$$

Hence eliminating p, we find

$$m^{-\frac{2}{3}}(1 - t\beta\mathrm{S}a)\phi'\phi(1 - ta\mathrm{S}\beta) = (1 + t'\beta'\mathrm{S}a')(1 + t'a'\mathrm{S}\beta'). \qquad \text{(xxxv)}$$

We shall now calculate the roots of the function on the left. If we can arrange so that one root is unity, the pure part of the strain $m^{-\frac{1}{3}}\phi(1 - ta\mathrm{S}\beta)$ will be plane, and if it is plane it must be a shear for neither $m^{-\frac{1}{3}}\phi$ nor $(1 - ta\mathrm{S}\beta)$ produces any dilatation (14.). Obviously the function on the right has one root unity, the corresponding axis being $\gamma' = a'\beta'$.

For brevity, replacing $m^{-\frac{2}{3}}\phi'\phi$ by Θ, the roots s of the function on the right are given by

$$\mathrm{S}[(1 - t\beta\mathrm{S}a)\Theta a - sa][(1 - t\beta\mathrm{S}a)\Theta(\beta + ta) - s\beta][(1 - t\beta\mathrm{S}a)\Theta\gamma - s\gamma] = 0,$$

if $\gamma = a\beta$. This equation is equivalent to

$$\mathrm{S}[\Theta a - s(a - t\beta)][\Theta(\beta + ta) - s\beta](\Theta\gamma - s\gamma) = 0,$$

deduced from it by operating on every vector by $1 + t\beta\mathrm{S}a$.

Observing that the third invariant of Θ is unity, so that

$$\mathrm{V}\Theta\beta\Theta\gamma = \Theta^{-1}a, \quad \mathrm{V}\Theta\gamma\Theta a = \Theta^{-1}\beta, \quad \mathrm{V}\Theta a\Theta\beta = \Theta^{-1}\gamma,$$

the equation reduces to

$$\mathrm{S}a\beta\gamma - s\mathrm{S}[(a - t\beta)\Theta^{-1}(a - t\beta) + \beta\Theta^{-1}\beta + \gamma\Theta^{-1}\gamma]$$
$$+ s^2\mathrm{S}[a\Theta a + (\beta + ta)\Theta(\beta + ta) + \gamma\Theta\gamma] - s^3\mathrm{S}a\beta\gamma = 0.$$

Finally, the equation of the cubic takes the form

$$1 - s\mathrm{N}' + s^2\mathrm{N}'' - s^3 = 0, \qquad \text{(xxxvi)}$$

where

$$\left.\begin{aligned} \mathrm{N}' &= M' + \mathrm{S}a\Theta^{-1}a - \mathrm{S}(a - t\beta)\Theta^{-1}(a - t\beta), \\ \mathrm{N}'' &= M'' + \mathrm{S}\beta\Theta\beta - \mathrm{S}(\beta + ta)\Theta(\beta + ta), \end{aligned}\right\} \qquad \text{(xxxvii)}$$

and

the first and second invariants of Θ being M'' and M'.

The condition that one root should be unity is

$$\mathrm{N}' = \mathrm{N}'', \qquad \text{(xxxviii)}$$

and observing that N′ and N″ are quadratic in the scalar t which specifies the amount of the shear, it appears that we may arbitrarily select the vectors a and β (that is the plane and direction of the shear), and that its amount is then given by a root of the quadratic (xxxviii) in t. The determination of the complementary shear $1 + t'\alpha'S\beta'$, and of the rotation presents no difficulty.

(16.) It is sometimes convenient, especially in dealing with small strains, to replace ϕ by $1+\theta$. In this notation $\theta\rho$ is the displacement of the extremity of ρ, the origin of vectors being supposed to be kept fixed.* Resolving the displacement along and at right angles to ρ, we have

$$\theta\rho = \theta\rho\rho^{-1}.\rho = (e+\eta)\rho, \qquad \text{(xxxix)}$$

if $$e = S\theta\rho\rho^{-1}, \quad \text{and} \quad \eta = V\theta\rho\rho^{-1}. \qquad \text{(xl)}$$

The scalar e is called the *elongation*. It is equal to the inverse square of the corresponding radius of the elongation quadric

$$S\rho\theta\rho = -1. \qquad \text{(xli)}$$

When the strain is pure so that θ is self-conjugate, the vectors η and ρ are parallel to the principal axes of a central section of this quadric. Thus $\eta\rho$ the component of the displacement at right angles to ρ is normal to that plane section of (xli) of which ρ is a principal axis. Also the magnitude of η is equal to the area of the triangle formed by lines along the corresponding radius and central perpendicular on the tangent plane of the quadric equal in length to the reciprocals of the radius and the perpendicular.

(17.) If the cubic of θ is

$$\theta^3 - n''\theta^2 + n'\theta - n = 0, \qquad \text{(xlii)}$$

the ratio of alteration of volume (i) is

$$m = S(1+\theta)\alpha\,(1+\theta)\beta\,(1+\theta)\gamma\,(S\alpha\beta\gamma)^{-1} = 1 + n'' + n' + n. \qquad \text{(xliii)}$$

If the strain is so small that terms involving the square and cube of the small function θ may be neglected, m is approximately equal to $1+n''$; n'' is the dilatation.

The ratio of lines is $T(1+\theta)U\alpha$ or approximately $1 + S\alpha^{-1}\theta\alpha$ (compare xl). The ratio of areas is $TV(1+\theta)\alpha\,(1+\theta)\beta\,T(V\alpha\beta)^{-1}$, or $Tm(1+\theta')^{-1}U\lambda$ if $U\lambda = UV\alpha\beta$. Now, for a small function θ,

$$(1+\theta)(1-\theta) = 1 \quad \text{or} \quad 1-\theta = (1+\theta)^{-1}, \qquad \text{(xliv)}$$

so the ratio of areas is approximately

$$T(1+n''-\theta')U\lambda \quad \text{or} \quad 1 + n'' - S\lambda^{-1}\theta\lambda.$$

(18.) The result of superposing the strain $1+\theta_2$ upon $1+\theta_1$ is $1+\theta_1+\theta_2+\theta_2\theta_1$, and this is generally distinct from $1+\theta_2+\theta_1+\theta_1\theta_2$ due to the strain $1+\theta_1$ following $1+\theta_2$. However, when both strains are small, so that $\theta_1\theta_2$ and $\theta_2\theta_1$ are negligible, the order

* Compare the Note on *Hamilton's Operator*, Section (27.), where the case of non-homogeneous small strain is considered, page 446.

in which the strains are effected is indifferent, and the displacement, due to the resultant strain, is the resultant of the displacement due to each strain separately. In particular, a small rotation changes ρ into $(1 + V\epsilon)\rho$. If this is followed by a small pure strain $1 + \theta_0$, after the double operation ρ becomes

$$\rho + \theta_0\rho + V\epsilon\rho \quad \text{or} \quad \rho + \theta\rho \tag{xlv}$$

if ϵ is the *spin-vector* of θ. Hence the origin of the name spin-vector. Again, for small strains

$$(1+\theta)(1+\theta') = (1+\theta')(1+\theta) = 1+\theta+\theta' = 1+2\theta_0 = (1+\theta_0)^2,$$

and the functions Φ and Ψ or $(\phi\phi')^{\frac{1}{2}}$ and $(\phi'\phi)^{\frac{1}{2}}$ are identical with $\frac{1}{2}(\phi+\phi')$. Also, the equation of the strain quadric (iii) becomes

$$T(1-\theta)\rho = r \quad \text{or} \quad \rho^2 - 2S\rho\theta_0\rho + r^2 = 0. \tag{xlvi}$$

(19.) In (12) we have given an example of the application of one of Hamilton's forms for the linear vector function. They all admit of simple interpretation. Take, for instance, the focal form

$$\theta\rho = a\alpha V\alpha\rho + b\beta S\beta\rho, \tag{xlvii}$$

and we see that the most general pure strain may be compounded of a contraction $(a\alpha V\alpha\rho)$ round one line $(U\alpha)$, and of an elongation $(b\beta S\beta\rho)$ parallel to another $(U\beta)$. (See Minchin, *Treatise on Statics*, Art. 379.)

The form

$$\theta\rho = g\rho + \lambda S\mu\rho + \mu S\lambda\rho \tag{xlviii}$$

shows that the pure strain may be resolved into shifting planes normal to μ in a direction parallel to λ, and planes normal to λ in a direction parallel to μ, and by superposing a general dilatation $3g$.

(20.) Reference has been made in the Note to page 225 to the strain which converts a quadric into a sphere. More generally if the strain ϕ converts any quadric

$$S\rho\Phi\rho = -1 \quad \text{into} \quad S\rho\Psi\rho = -1,$$

the function ϕ must satisfy the equation

$$\phi'\Psi\phi = \Phi. \tag{xlix}$$

In order to simplify this, assume

$$\phi = \Psi^{-\frac{1}{2}}\chi\Phi^{\frac{1}{2}}, \quad \text{or} \quad \phi' = \Phi^{\frac{1}{2}}\chi'\Psi^{-\frac{1}{2}}, \tag{l}$$

and it appears that χ must be a solution of the equation

$$\chi'\chi = 1.$$

This has been considered and solved in (8.); χ must represent a rotation or a rotation combined with a reflection. We are instructed therefore by the form of the function ϕ (xlix) to strain the first quadric into a sphere; to rotate the sphere with or without reflection; and to strain the sphere into the second quadric.

IV.—ON THE SPECIFICATION OF LINEAR VECTOR FUNCTIONS.

(1.) A linear function is determinate given the vectors derived by it from three known vectors*. Given the directions† β_1, β_2, β_3, into which three known directions α_1, α_2, and α_3 are changed, and the ratios a_4, a_5, a_6, in which the lengths of α_4, α_5, and α_6 are altered, we have

$$\phi\rho = (x\beta_1 S\alpha_2\alpha_3\rho + y\beta_2 S\alpha_3\alpha_1\rho + z\beta_3 S\alpha_1\alpha_2\rho)(S\alpha_1\alpha_2\alpha_3)^{-1}, \qquad \text{(i)}$$

where the scalars x, y, z satisfy

$$TS\alpha_1\alpha_2\alpha_3 = Ta_4^{-1}\Sigma x\beta_1 S\alpha_2\alpha_3\alpha_4 = Ta_5^{-1}\Sigma x\beta_1 S\alpha_2\alpha_3\alpha_5 = Ta_6^{-1}\Sigma x\beta_1 S\alpha_2\alpha_3\alpha_6. \qquad \text{(ii)}$$

Rationalizing and solving these equations for x, y, and z, eight systems of values are obtained, and, corresponding to these, eight functions ϕ may be found. Four of these functions are simply the negatives of the remaining four, and, in general, the eight functions correspond to the eight arrangements of sign attributable to the scalars a.

(2.) Given four directions derived from four others (compare Note V., Section (6.))

$$A\phi\rho = \frac{\beta_1 S\alpha_2\alpha_3\rho S\beta_2\beta_3\beta_4}{S\alpha_2\alpha_3\alpha_1 S\alpha_2\alpha_3\alpha_4} + \frac{\beta_2 S\alpha_3\alpha_1\rho S\beta_3\beta_1\beta_4}{S\alpha_3\alpha_1\alpha_2 S\alpha_3\alpha_1\alpha_4} + \frac{\beta_3 S\alpha_1\alpha_2\rho S\beta_1\beta_2\beta_4}{S\alpha_1\alpha_2\alpha_3 S\alpha_1\alpha_2\alpha_4} \qquad \text{(iii)}$$

where A is an arbitrary scalar. Given the ratio in which a fifth line is altered, A is determined.

(3.) This method of representing a linear vector function leads to some remarkable expressions. For instance, if α_1, α_2, α_3 and α_4, α_5, α_6 are unit vectors along two sets of mutually conjugate radii of a quadric $S\rho\Phi\rho = 1$, we have

$$A_4\Phi\rho = V\alpha_2\alpha_3 \frac{S\alpha_2\alpha_3\rho S\alpha_1\alpha_5\alpha_6}{S\alpha_2\alpha_3\alpha_1 S\alpha_2\alpha_3\alpha_4} + V\alpha_3\alpha_1 \frac{S\alpha_3\alpha_1\rho S\alpha_2\alpha_5\alpha_6}{S\alpha_3\alpha_1\alpha_2 S\alpha_3\alpha_1\alpha_4} + V\alpha_1\alpha_2 \frac{S\alpha_1\alpha_2\rho S\alpha_3\alpha_5\alpha_6}{S\alpha_1\alpha_2\alpha_3 S\alpha_1\alpha_2\alpha_4}. \qquad \text{(iv)}$$

Hence, if for brevity $S\alpha_1\alpha_2\alpha_3$ is denoted by (123), and $S\alpha_1\Phi\alpha_1$ by a_1^{-2}, we obtain

$$\frac{A_4}{(123)} = a_1^2\frac{(156)}{(234)} = a_2^2\frac{(256)}{(314)} = a_3^2\frac{(356)}{(124)} = a_4^2\frac{(456)}{(123)}, \qquad \text{(v)}$$

together with other relations, which can easily be supplied, connecting the signs of the solid angles (α_1, α_2, α_3) with the radii a_1, a_2, . . . a_6 of the quadric. These relations are due to Sir Robert Ball, and are of importance in the theory of co-reciprocal screws. Again, by (iv), we see that the shape and orientation of a quadric are determined, given the directions of three mutually conjugate diameters, and the direction of a fourth line (α_4) conjugate to the plane normal to a given direction $V\alpha_5\alpha_6$.

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* Compare Note III., Section (4.), p. 366.

† We suppose, for convenience, that the vectors are all of unit length.

(4.) If we seek to determine as far as possible a linear vector function by expressing that the lengths of given vectors are to be altered in given ratios, we shall find that we may assign six directions and six ratios, and that the function remains indeterminate to the extent of an arbitrary rotation which may be superposed upon it.* For, given the centre of a quadric, six conditions determine it, and if $\phi\alpha_1 = a_1\beta_1$, &c., the ratios (a_1) are inversely proportional to the radii of the quadric $T\phi\rho = 1$ parallel to the corresponding directions (α_1). In this way we can find the self-conjugate function $\phi'\phi$. Taking its square root, and superposing an arbitrary rotation, we have the general function satisfying the conditions. Or, given six ratios and one direction which a seventh vector must assume, the function is determinate.

(5.) In terms of Hamilton's *Aconic Function*† we can write down the relation between the seven ratios in which the lengths of seven vectors are altered by a strain.

The aconic function of six vectors is

$$[7] = SV.V\alpha_1\alpha_2V\alpha_4\alpha_5V.V\alpha_2\alpha_3V\alpha_5\alpha_6V.V\alpha_3\alpha_4V\alpha_6\alpha_1. \qquad \text{(vi)}$$

If it vanishes, the six vectors lie on a cone, and the form of the expression contains a direct proof of Pascal's theorem, for it shows that the lines of intersection of the planes α_1, α_2; α_4, α_5, and α_2, α_3; α_5, α_6; and α_3, α_4; α_6, α_1 are coplanar.

To fix the signs appropriate to the seven aconic functions formed by omitting one of seven vectors, mark seven points 1, 2, 3, 4, 5, 6, 7 on a circle, and go round it in this order, starting always from the point 1, and omitting one point.‡ Then the relation between the seven unit vectors and the seven ratios is

$$[1]a_1^2 - [2]a_2^2 + [3]a_3^2 - [4]a_4^2 + [5]a_5^2 - [6]a_6^2 + [7]a_7^2 = 0. \qquad \text{(vii)}$$

In fact, allowing α_7 and a_7 to vary, this is the equation of a quadric concentric with the origin whose radii are inversely proportional to a_7, and which passes through the extremities of the six vectors $\alpha_1 a_1^{-1}$, &c. To prove this, it is only necessary to show that the sign of [7] is changed whenever any two vectors in it are transposed;§ for, when $\alpha_7 = \alpha_6$, the function [6] becomes [7], and all the others vanish. When $\alpha_7 = \alpha_5$ all vanish except [5], which becomes $-$ [7], with one interchange of vectors. If the six vectors happen to lie on a quadric cone [7] is zero, and the ratio a_7 is not determined. The equation (vii) (omitting the last term) must then be satisfied for every possible direction α_7, and the six ratios cannot be arbitrarily chosen.

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* Compare Note III., Section (9.), p. 368.

† Lectures on Quaternions, Art. 442.

‡ Thus, for example, we may also write

$$[3] = \{124567\}.$$

§ The most direct way of doing this seems to be, to express α_4, α_5, and α_6 in terms of α_1, α_2, and α_3.

The equation is then equivalent to that of the sphero-conic determined by the five vectors $a_1\alpha_1^{-1}$, . . . $a_5\alpha_5^{-1}$, and expresses that $a_6\alpha_6^{-1}$ terminates on this curve. More fully draw any quadric $S\rho\theta\rho = 1$ through the extremities of the five vectors and having its centre at the origin. Let $S\rho\chi\rho = 0$ be the cone containing the five vectors. The sixth must terminate on the curve common to the system

$$S\rho(\theta + t\chi)\rho = 1. \qquad \text{(viii)}$$

(6.) Hence we can see how to determine a linear vector function given five ratios and two directions. For let (viii) (compare (4.)) be the quadric whose corresponding radii (α_1, . . . α_5) are inversely proportional to the ratios (a_1 . . . a_5), and let β_6 and β_7 be the directions into which α_6 and α_7 are to be changed by the function ϕ. Then, if we determine t from the relation

$$SU(\theta + t\chi)\alpha_6 U(\theta + t\chi)\alpha_7 = S\beta_6\beta_7, \qquad \text{(ix)}$$

we can superpose a rotation upon $\theta + t\chi$, so as to render the vectors derived from α_6 and α_7 parallel to β_6 and β_7.

V.—INVARIANTS OF LINEAR VECTOR FUNCTIONS.

Before touching on the general theory of quaternion invariants of linear vector functions, it seems to be desirable to point out a few consequences of relations connecting the roots of a single function ϕ. The signification of the geometrical interpretations will, in due course, be greatly extended, and we shall come to regard the invariants of the earlier sections of this note as invariants of two linear functions ϕ and unity (compare Section 9).

(1.) Writing the symbolic cubic of ϕ in the form

$$\phi^3 - m''\phi^2 + m'\phi - m = 0, \quad \text{or} \quad (\phi - g_1)(\phi - g_2)(\phi - g_3) = 0, \qquad \text{(i)}$$

we know that every triad of vectors α, β, and γ satisfies the equation

$$S\beta\gamma\phi\alpha + S\gamma\alpha\phi\beta + S\alpha\beta\phi\gamma = 0, \qquad \text{(ii)}$$

when

$$m'' = 0, \quad \text{or} \quad g_1 + g_2 + g_3 = 0. \qquad \text{(iii)}$$

Thus assuming at pleasure two vectors α and β, and determining a third vector γ by the equations $S\beta\gamma\phi\alpha = S\gamma\alpha\phi\beta = 0$, the third equation $S\alpha\beta\phi\gamma = 0$ must be true when $m'' = 0$. In other words, in this case it is possible to determine an infinite number of triads of vectors α, β, and γ, so that each vector of the derived triad $\phi\alpha$, $\phi\beta$, $\phi\gamma$ is coplanar with a pair of vectors of the original. Or we may say briefly the *edges* of the derived lie on the corresponding *faces* of the original triad. Conversely, if this arrangement is possible in any one case, it is possible in an infinite number of cases.

(2.) Similarly when $m' = 0$, triads may be determined so that the faces of the derived triads contain the corresponding edges of the original, and the converse is also true.

(3.) Further, if for any arrangement of signs

$$\pm\sqrt{g_1} \pm \sqrt{g_2} \pm \sqrt{g_3} = 0, \tag{iv}$$

the sum of the roots of the corresponding square root of ϕ is zero (compare the Note on Strain, Section 7, page 367).

We can then determine triads α, β, γ, whose faces contain the edges of the triads $\sqrt{\phi}\alpha$, $\sqrt{\phi}\beta$, $\sqrt{\phi}\gamma$, and we shall show that the faces of these derived triads contain the edges of the triads $\phi\alpha$, $\phi\beta$, $\phi\gamma$. For if $S\beta\gamma\sqrt{\phi}\alpha = 0$, we obtain, on multiplying by the third invariant of $\sqrt{\phi}$, this other equation $S\sqrt{\phi}\beta\ \sqrt{\phi}\gamma\ \ \phi\alpha = 0$; this proves the theorem. In other words when (iv) is satisfied, it is possible to determine in an indefinite number of ways a triad α, β, γ so related to the derived triad $\phi\alpha$, $\phi\beta$, $\phi\gamma$ that, in every case, an intermediate triad can be inscribed to the first and circumscribed to the second. On rationalizing (iv), the condition takes the form

$$(g_1 + g_2 + g_3)^2 - 4(g_2 g_3 + g_3 g_1 + g_1 g_2) = 0, \quad \text{or} \quad m''^2 = 4m'. \tag{v}$$

(4.) The converse of this property is true, and the theorem admits of considerable extension. If

$$\sqrt[n]{g_1} + \sqrt[n]{g_2} + \sqrt[n]{g_3} = 0, \tag{vi}$$

n being an integer, triads α, β, γ and $\phi\alpha$, $\phi\beta$, $\phi\gamma$ can be found connected by a series of inscribed and circumscribed triads derived from the original by successive applications of the operator $\sqrt[n]{\phi}$. Still more generally an interpretation can be assigned for the case in which n is a fraction.

(5.) Otherwise we may deduce invariants by proposing suitable geometrical conditions instead of interpreting geometrically the meaning of the vanishing of assumed invariants. For instance, we may inquire into the conditions that a linear vector transformation may leave a given quadric cone unaltered. The vectors $\phi\rho$ derived from edges of the cone $S\rho\Phi\rho = 0$ are edges of the new cone $S\phi^{-1}\rho\Phi\phi^{-1}\rho = 0$, or $S\rho\phi'^{-1}\Phi\phi^{-1}\rho = 0$. If these cones are identical, ϕ must satisfy the equation

$$\phi'^{-1}\Phi\phi^{-1} = m^{-\frac{2}{3}}\Phi, \quad \text{or} \quad \phi'\Phi\phi = m^{\frac{2}{3}}\Phi, \tag{vii}$$

the factor $m^{\frac{2}{3}}$ being introduced so as to render equal the third invariants of the functions in each number of the equations. A similar equation has occurred in the Note on Strain (Note III., Section (20.)), and, as in the place cited, the general relation between ϕ and Φ is of the form

$$\phi = \pm\, m^{\frac{1}{3}}\Phi^{-\frac{1}{2}}\chi\Phi^{\frac{1}{2}}, \quad \text{where} \quad \chi\chi' = 1, \tag{viii}$$

and the function χ produces a rotation or a rotation and a reflection. Now (Note II., Section 2) the symbolic cubics of $m^{-\frac{1}{3}}\phi$ and of χ must be identical, but the cubic of χ is reciprocal, and so therefore must be that of $m^{-\frac{1}{3}}\phi$, or we must have the invariant relation

$$mm''^3 - m'^3 = 0. \tag{ix}$$

As a rotation leaves unchanged every right cone having its axis coincident with that of the rotation, we are led to infer and can verify at once that the whole system of cones

$$S\rho\Phi\rho + u(S\kappa\rho)^2 = 0, \quad \text{where} \quad \phi'\kappa = m^{\frac{1}{3}}\kappa, \tag{x}$$

transforms into itself when ρ is changed to $\phi\rho$, provided the invariant (ix) vanishes, and provided ϕ is a solution of equation (viii).

(6.) It will be noticed that the foregoing interpretations depend simply on the directions of the vectors involved. If a function changes the directions of α, β, γ into the directions of λ, μ, ν, it must be of the type

$$\phi\rho = u\lambda S\beta\gamma\rho + v\mu S\gamma\alpha\rho + w\nu S\alpha\beta\rho, \tag{xi}$$

the scalars u, v, and w being arbitrary. If, in addition, the direction of δ is changed into that of ϖ,

$$\phi\rho = \lambda S\mu\nu\varpi\frac{S\beta\gamma\rho}{S\beta\gamma\delta} + \mu S\nu\lambda\varpi\frac{S\gamma\alpha\rho}{S\gamma\alpha\delta} + \nu S\lambda\mu\varpi\frac{S\alpha\beta\rho}{S\alpha\beta\delta}, \tag{xii}$$

and in this there is nothing arbitrary except the tensor of the product $\lambda\mu\nu\varpi\delta^{-1}$. (Compare Note IV., Section (2), p. 374.)

From this point of view we can see the connexion with the theory of anharmonic coordinates in a plane (pp. 23–29, vol. i.). For if $\delta = a\alpha + b\beta + c\gamma$, $\varpi = a'\lambda + b'\mu + c'\nu$, and $\rho = xa\alpha + yb\beta + zc\gamma$, we can verify at once that $\phi\rho = (xa'\lambda + yb'\mu + zc'\nu)S\lambda\mu\nu$. Also (compare p. 25, vol. i.),

$$(\text{OA . BDCP}) = \frac{y}{z}, \quad (\text{OB . CDAP}) = \frac{z}{x}, \quad (\text{OC . ADBP}) = \frac{x}{y}, \tag{xiii}$$

where (OA . BDCP) is the anharmonic of the four planes (α, β), (α, δ), (α, γ), and (α, ρ), respectively. The equations (xiii) remain true when A, B, C, D, P are changed to A′, B′, C′, D′, P′, where generally OP′ ∥ ϕOP. Thus, OP′ can be found by linear constructions when OP is given as the tensors of the vectors α, β, γ, δ, λ, μ, ν, ϖ, and ρ may be chosen so that the extremities of these nine vectors may lie in an assumed plane.

(7.) As the axes of ϕ are the vector solutions of the equation

$$V\rho\phi\rho = 0, \tag{xiv}$$

the cone

$$S\alpha\rho\phi\rho = 0 \tag{xv}$$

contains three fixed lines which are quite independent of the vector α. This quadric cone is the locus of a line, so that it and its derived are coplanar with a fixed line (α). For various values of this vector, we obtain a doubly infinite system of cones having three common edges. If two of the solutions of (xiv) coincide, the cones touch one another; if all three solutions coincide they osculate, and they break up into pairs of planes, one fixed plane being common to every pair, if the solutions of (xiv) become indeterminate in a certain plane. The conditions for contact and osculation can be expressed at once in terms of the invariants m, m', m'', being merely the conditions that the root cubic should have two roots equal or should be a perfect cube. The condition for indeterminate axes is of a different kind. Here there must be a double root g, and $\phi - g$ must destroy every vector in the plane of the indeterminate axes. $\phi - g$ is, therefore, a monomial $\lambda S\mu\rho$; its cubic is *depressed* to a quadratic, or, what is equivalent, Hamilton's function

$$\psi_g = 0, \quad \text{or} \quad \phi^2 - m''\phi + m' + g(\phi - m'') + g^2 = 0. \tag{xvi}$$

This, then, is the condition for indeterminate axes. (Compare 352 (20.), p. 504, vol. i., and the remaining sub-articles.)

It is easy to show that the cone (xv) cannot degrade into a pair of planes unless α is coplanar with a pair of axes of ϕ. If the cone is a pair of planes, and if ϖ is the vector of intersection, $S\alpha\rho\phi\varpi + S\alpha\varpi\phi\rho$ must vanish for every vector ρ. Hence

$$V\phi\varpi\alpha + \phi' V\alpha\varpi = 0. \tag{xvii}$$

Now, as $S\alpha\varpi\phi\varpi = 0$, we may write $\phi\varpi = u\varpi + v\alpha$; and substitution in (xvii) shows that

$$\phi' V\alpha\varpi = uV\alpha\varpi, \tag{xviii}$$

or u must be a root, and $V\alpha\varpi$ the corresponding axis of the conjugate ϕ'. But the axes of ϕ' are the normals to the planes containing pairs of axes of ϕ; hence, α must be coplanar with a pair of axes of ϕ, as it is at right angles to an axis of ϕ'.

In the case of indeterminate axes, ϕ must be of the form

$$\phi\rho = g\rho + \lambda S\mu\rho, \tag{xix}$$

and the cones (xv) all break up into pairs of planes

$$S\alpha\lambda\rho S\mu\rho = 0. \tag{xx}$$

(8.) We have seen (Note II. (2.) page 363) that the roots of $\theta\phi\theta^{-1}$ and of ϕ are identical. Consequently, the theorems proved up to the present in this note are also true for $\theta\phi\theta^{-1}$ as well as for ϕ.

(9.) Again, if we write $\phi = \phi_2^{-1}\phi_1$, and

$$S(\phi_1 - g\phi_2)\alpha(\phi_1 - g\phi_2)\beta(\phi_1 - g\phi_2)\gamma = (m_1 - gl_1 + g^2 l_2 - g^3 m_2) S\alpha\beta\gamma, \tag{xxi}$$

where m_1 and m_2 are the third invariants of ϕ_1 and ϕ_2, respectively, and l_1 and l_2 are two new invariants, we obviously have the relations

$$m_2 m = m_1; \quad m_2 m' = l_1; \quad m_2 m'' = l_2, \tag{xxii}$$

since the left-hand side of (xxi) may be replaced by

$$m_2 S(\phi_2^{-1}\phi_1 - g)\alpha(\phi_2^{-1}\phi_1 - g)\beta(\phi_2^{-1}\phi_1 - g)\gamma = m_2(m - gm' + g^2 m'' - g^3) S\alpha\beta\gamma. \tag{xxiii}$$

Furthermore, the values of the ratios

$$m_1 : l_1 : l_2 : m_2$$

are unchanged when ϕ_1 and ϕ_2 are replaced by $\chi\phi_1\theta$ and $\chi\phi_2\theta$, respectively, χ and θ being two arbitrary vector functions, for

$$(\chi\phi_2\theta)^{-1}(\chi\phi_1\theta) = \theta^{-1}\phi_2^{-1}\phi_1\theta = \theta^{-1}\phi\theta, \qquad \text{(xxiv)}$$

and the functions ϕ and $\theta^{-1}\phi\theta$ have the same roots.

Thus, the invariants depending solely on the roots of $\phi_2^{-1}\phi_1$ are invariantal in a very wide sense. Not only may the vectors α, β, γ be changed in any way, but the functions ϕ_1 and ϕ_2 may also be transformed within very wide limits.

It is well to bear in mind that $\phi_2^{-1}\phi_1$, $\phi_1\phi_2^{-1}$ and their conjugates $\phi'_1\phi'^{-1}_2$ and $\phi'^{-1}_2\phi'_1$ have the same roots.

(10.) Hence, by (1),

$$l_2 = 0 \qquad \text{(xxv)}$$

is the condition that ϕ_1 and ϕ_2 (or, more generally, that $\chi\phi_1\theta$ and $\chi\phi_2\theta$) should be capable of producing from a triad of vectors α, β, γ two new triads so related that the faces of the second contain the edges of the first; also, by (2),

$$l_1 = 0, \qquad \text{(xxvi)}$$

if the faces of the first can contain the edges of the second; and by (3) and (v)

$$l_2^2 = 4m_2l_1 \quad \text{or} \quad l_1^2 = 4m_1l_2 \qquad \text{(xxvii)}$$

if an intermediate triad can be inscribed to the first (or second), and circumscribed to the second (or first). Further, by (5),

$$m_1l_2^3 = m_2l_1^3 \qquad \text{(xxviii)}$$

if the transformation represented by ϕ_2^{-1} can restore a system of cones transformed by ϕ_1 into their original state.

It would be tedious, and cannot be necessary, to elaborate this subject any further.

(11.) In the particular case in which ϕ_1 and ϕ_2 are self-conjugate, we may fall back on the invariants of a pair of cones or conics. For instance, if a triad of vectors satisfies $S\phi_2\alpha\phi_1\beta\phi_1\gamma = 0$, and two similar equations derived from this by cyclical interchange, we may replace the equations by three of the type $S\phi_2\alpha\phi_1^{-1}V\beta\gamma = 0$. The form suggests $\phi_1^{-1}V\beta\gamma \parallel \alpha$ with the condition $S\alpha\phi_2\alpha = 0$, &c., and the invariant l_1 vanishes if a triad can be found upon the cone $S\rho\phi_2\rho = 0$ self-conjugate to the cone $S\rho\phi_1\rho = 0$.

In the general case also in which the functions are not self-conjugate, the invariants of their self-conjugate parts (which are of course invariants of the functions themselves) may be regarded as invariants of cones. But there is an important distinction between the two classes of invariants. We have seen (9) that the invariants expressible in terms of the roots of the quotient $\phi_2^{-1}\phi_1$ are merely multiplied by a factor when ϕ_1 and ϕ_2 are replaced by $\chi\phi_1\theta$ and $\chi\phi_2\theta$. This transformation, on the other hand, completely modifies the invariants of the self-conjugate parts; in fact they cease to be invariantal for this transformation. The self-conjugate parts Φ_1 and Φ_2 cease to be self-conjugate where multiplied by χ and into θ. When we restrict the range of the transformation by supposing χ and θ to be conjugates or $\chi = \theta'$, the function ϕ_1, its conjugate ϕ'_1, and its self-conjugate part Φ_1 all undergo the same transformation; and $\phi_2^{-1}\phi_1$, $\phi'^{-1}_2\phi'_1$, and $\Phi_2^{-1}\Phi_1$ transform into $\theta^{-1}\phi_2^{-1}\phi_1\theta$, $\theta^{-1}\phi'^{-1}_2\phi'_1\theta$, and $\theta^{-1}\Phi_2^{-1}\Phi_1\theta$, so the roots of $\Phi_2^{-1}\Phi_1$ (upon which the cone invariants depend) are unaltered as well as those of $\phi_2^{-1}\phi_1$.

We can see a reason for this. The natural interpretation of the equation

$$S\rho\phi_1\rho = 0, \quad \text{or} \quad S\rho\Phi_1\rho = 0, \quad \text{or} \quad S\rho\phi'_1\rho = 0, \tag{xxix}$$

is that it is the locus of lines at *right angles* to their derived lines. Here a non-projective element is introduced, and the latitude of transformation consistent with invariance is restricted.

(12.) The vanishing of invariants depending on the roots of $\phi_2^{-1}\phi_1$ does not, in general, imply any peculiarity in the pure parts of the strains represented by ϕ_1 and ϕ_2. For we have seen in Note III., Section (11.), that a rotation χ can be determined which shall render the roots of $\chi\phi$ equal to any values assignable within certain limits subject to the single condition that their product shall be constant. The magnitudes of the roots may be selected so as to render any function of them zero, and a corresponding rotation χ can be found which will annul any invariant of ϕ_1 and $\phi_2\chi^{-1}$.

It may be shown also that a rotation χ can be found, so that within certain limits one at least of the roots of $\chi\phi_2^{-1}\chi^{-1}\phi_1$ may acquire a selected magnitude. This applies to the invariants of self-conjugate functions (compare the last Section).

(13.) A function ϕ compounded from two or more given functions may be said to be covariant with them, provided ϕ changes into $\chi\phi\theta$ when each of the component functions is multiplied by χ and into θ. Thus $\phi_1\phi_2^{-1}\phi_1$ is covariant with ϕ_1 and ϕ_2, but $\phi_1\phi_2$ and $\phi_1\phi_2^{-1}$ are not.

Again, if ϕ_1, ϕ_2, ϕ_3, &c. are covariant, and if

$$V\Sigma t_1\phi_1\lambda\Sigma t_1\phi_1\mu = \Sigma\Sigma t_1t_2\psi'_{12}V\lambda\mu, \tag{xxx}$$

it appears that the functions ψ' transform into $\chi'^{-1}\psi'\theta'^{-1}$ multiplied by the third invariants of θ and of χ when the functions ϕ are changed to $\chi\phi\theta$. Thus the functions ψ' are covariant among themselves, and of course their conjugates ψ which transform into $\theta^{-1}\psi\chi^{-1}$ are likewise covariant among themselves. In like manner, from the functions ψ' we obtain new functions ϕ_{12} by the equation

$$V\Sigma t_1\psi'_1\lambda\Sigma t_1\psi'_1\mu = \Sigma\Sigma t_1t_2\phi_{12}V\lambda\mu, \tag{xxxi}$$

which are covariant with the original functions ϕ if we disregard a scalar factor depending on the third invariants of χ and θ.

(14.) The principles explained in the Note on Quaternion Determinants (Note I., Sections (2.) and (3.)) enable us to write down the quaternion invariants of a system of linear vector functions. By actual transformation of the vectors α, β, γ it may be shown that the quotient

$$q(\phi_1, \phi_2, \phi_3) = \begin{vmatrix} \phi_1\alpha & \phi_1\beta & \phi_1\gamma \\ \phi_2\alpha & \phi_2\beta & \phi_2\gamma \\ \phi_3\alpha & \phi_3\beta & \phi_3\gamma \end{vmatrix} \div \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha & \beta & \gamma \\ \alpha & \beta & \gamma \end{vmatrix} \qquad \text{(xxxii)}$$

is quite independent of these vectors, and is therefore an invariant of the three functions ϕ_1, ϕ_2, ϕ_3. For if

$$\alpha = x\lambda + y\mu + z\nu, \quad \beta = x'\lambda + y'\mu + z'\nu, \quad \gamma = x''\lambda + y''\mu + z''\nu,$$

we find

$$\begin{vmatrix} \phi_1\alpha & \phi_1\beta & \phi_1\gamma \\ \phi_2\alpha & \phi_2\beta & \phi_2\gamma \\ \phi_3\alpha & \phi_3\beta & \phi_3\gamma \end{vmatrix} = \begin{vmatrix} \phi_1\lambda & \phi_1\mu & \phi_1\nu \\ \phi_2\lambda & \phi_2\mu & \phi_2\nu \\ \phi_3\lambda & \phi_3\mu & \phi_3\nu \end{vmatrix} \cdot \begin{vmatrix} x & y & z \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix}, \qquad \text{(xxxiii)}$$

and in forming the quotient (xxxii) the scalar determinant $(xy'z'')$ cancels.

(15.) By direct expansion of the determinant, we find

$$6q(\phi_1, \phi_2, \phi_3)S\alpha\beta\gamma = \Sigma\phi_1\alpha(\phi_2\beta\phi_3\gamma - \phi_2\gamma\phi_3\beta), \qquad \text{(xxxiv)}$$

where the sign Σ indicates summation for cyclical transposition of the vectors α, β, γ. The scalar part of the quaternion is

$$l_{123} = Sq(\phi_1, \phi_2, \phi_3) = \tfrac{1}{6}\Sigma S\phi_1\alpha(\phi_2\beta\phi_3\gamma + \phi_3\beta\phi_2\gamma)\,(S\alpha\beta\gamma)^{-1}; \qquad \text{(xxxv)}$$

and the vector part reduces without difficulty to

$$\begin{aligned} Vq(\phi_1, \phi_2, \phi_3) = \tfrac{1}{6}[&\Sigma\phi_1\alpha S(\phi_2\beta\phi_3\gamma - \phi_3\beta\phi_2\gamma) \\ &- \Sigma\phi_2\alpha S(\phi_3\beta\phi_1\gamma - \phi_1\beta\phi_3\gamma) \\ &+ \Sigma\phi_3\alpha S(\phi_1\beta\phi_2\gamma - \phi_2\beta\phi_1\gamma)]\,(S\alpha\beta\gamma)^{-1}. \end{aligned} \qquad \text{(xxxvi)}$$

Now if η_{32} is the spin vector of $\phi'_3\phi_2$,

$$S(\phi_2\beta\phi_3\gamma - \phi_3\beta\phi_2\gamma) = S(\phi'_3\phi_2 - \phi'_2\phi_3)\beta\gamma = 2S\eta_{32}\beta\gamma; \qquad \text{(xxxvii)}$$

and the quaternion invariant reduces to

$$q(\phi_1, \phi_2, \phi_3) = l_{123} - \tfrac{1}{3}(\phi_1\eta_{23} - \phi_2\eta_{31} + \phi_3\eta_{12}): \qquad \text{(xxxviii)}$$

for it must be observed that the spin vectors satisfy the equations

$$\eta_{12} + \eta_{21} = 0, \quad \eta_{11} = 0. \qquad \text{(xxxix)}$$

It is evident that the scalar part l_{123} is unchanged when the functions are interchanged in any way. We see by (xxxviii) that briefly

$$q_{321} = Kq_{123}, \quad \text{and} \quad \tfrac{2}{3}\phi_1\eta_{23} = q_{132} - q_{123}. \qquad \text{(xl)}$$

The effect of interchange of rows in determinants of the third order is thus exhibited, and we see that the six quaternion invariants obtained by every possible interchange of ϕ_1, ϕ_2, and ϕ_3 are equivalent to one scalar and three vector invariants, l_{123} and $\phi_1\eta_{23}$, $\phi_2\eta_{31}$, and $\phi_3\eta_{12}$.

(16.) For a single function ϕ we obtain Hamilton's three invariants as particular cases of (xxxviii) in the forms

$$q(\phi, 1, 1) = \tfrac{1}{3}(m'' + 2\epsilon)\,; \quad q(\phi, \phi, 1) = \tfrac{1}{3}(m' + 2\phi\epsilon)\,; \quad q(\phi, \phi, \phi) = m. \qquad \text{(xli)}$$

For the first of these $\epsilon = \eta_{21} = \eta_{31}$.

(17.) Manifestly these vector invariants are totally different in character from the scalar invariants of the earlier sections of this Note. It is easy to see, when we multiply the three functions (xxxii) into θ, that the quaternion is merely multiplied by the third invariant of θ; in fact, the determinant quotient is multiplied by the quotient $q(\theta, \theta, \theta)$. It may be proved, without difficulty directly, that η_{23} becomes $n\theta'^{-1}\eta_{23}$ in this case. But, when the functions are multiplied by a common function χ, although the scalar part is merely multiplied by the third invariant, the vector part is, in general, completely changed. $\phi'_2\phi_3$ becomes $\phi'_2\chi'\chi\phi_3$, and the spin vector of the transformed function is quite distinct from that of the original, except in the case in which $\chi'\chi$ is a scalar. (Compare also (11.).)

(18.) It would take too long to investigate the reduction of the number of independent quaternion invariants of two or three functions. The functions may be combined in any way, such as in products $\phi_1\phi_2$, $\phi_2\phi_1$, &c., and the various invariants may be obtained by substituting those combinations for the simpler functions in (xxxii). It must suffice to remark that, in addition to obvious reductions obtainable by means of symbolic cubics, a simple relation connects the spin vectors of $\phi_1\phi_2$ and of $\phi'_1\phi_2$. For, if ϵ_{12} is the spin vector of $\phi_1\phi_2$, and, as before, if η_{12} is that of $\phi'_1\phi_2$,

$$\begin{aligned} 2V\epsilon_{12}\rho &= (\phi_1\phi_2 - \phi'_2\phi'_1)\rho = ((\phi_1 - \phi'_1) + \phi'_1)\phi_2\rho + \phi'_2((\phi_1 - \phi'_1) - \phi_1)\rho \\ &= 2V\epsilon_1\phi_2\rho + 2\phi'_2V\epsilon_1\rho + 2V\eta_{12}\rho. \end{aligned}$$

This affords the relation

$$\epsilon_{12} = \eta_{12} + (m''_2 - \phi_2)\epsilon_1, \qquad \text{(xlii)}$$

in virtue of the fundamental equation

$$V\phi_2\lambda\mu + V\lambda\phi_2\mu + \phi'_2V\lambda\mu = m''_2V\lambda\mu.$$

From symmetry we may write down, in like manner,

$$\epsilon_{21} = \eta_{21} + (m''_1 - \phi_1)\epsilon_2. \qquad \text{(xliii)}$$

In the same way η'_{12}, the spin vector of $\phi_1\phi'_2$, can be expressed in terms of ϵ_{12} and simpler vector invariants. Thus the spin vectors of $\phi_1\phi_2$, $\phi_2\phi_1$, $\phi'_1\phi_2$, $\phi'_2\phi_1$, $\phi_1\phi'_2$, $\phi_2\phi'_1$, $\phi'_1\phi'_2$, $\phi'_2\phi'_1$ are all expressible in terms of η_{12}, ϵ_1, ϵ_2, and the results of operating on ϵ_1 and ϵ_2 by the functions ϕ_1, ϕ_2 and their conjugates.

By repeated application of these formulæ the spin vectors of all functions derived from a given product of functions, by cyclically transposing the functions and altering them to their conjugates, can be reduced to the spin vector of one of the products and the results of operating on simpler invariants.

19. We may also notice that the quotient

$$\begin{vmatrix} \phi_1\alpha & \phi_1\beta \\ \phi_2\alpha & \phi_2\beta \end{vmatrix} \div \begin{vmatrix} \alpha & \beta \\ \alpha & \beta \end{vmatrix} = \frac{\psi'_{12}V\alpha\beta - 2S\eta_{12}V\alpha\beta}{2V\alpha\beta} \qquad \text{(xliv)}$$

is unchanged when α and β are replaced by any other vectors in their plane (compare (xxx)).

VI.—ON THE SYSTEM OF LINEAR VECTOR FUNCTIONS $\phi + t\theta$.

(1.) When t is eliminated from the equation

$$V\rho(\phi + t\theta)\rho = 0, \qquad \text{(i)}$$

the locus of axes of the system $\phi + t\theta$ is found to be the cubic cone represented by

$$f = S\rho\phi\rho\theta\rho = 0. \qquad \text{(ii)}$$

This is also the locus of a vector coplanar with the vectors derived from it by any two functions of the system.

(2.) The cone

$$f' = S\rho\phi'\rho\theta'\rho = 0 \qquad \text{(iii)}$$

is the locus of axes of the conjugate system. Bearing in mind that the axes of a function are perpendicular to the corresponding planes containing pairs of axes of its conjugate, we see that every edge ρ_1 of the cone f corresponding to the root g_1 of the function $\phi + t\theta$ is perpendicular to ρ'_2 and ρ'_3 edges of the cone f' and axes of the conjugate function. The third edge of the cone f' may, without difficulty, be shown to be $V\rho_1\phi\rho_1$, or $V\rho_1\theta\rho_1$.

(3.) In particular, when both functions are self-conjugate the cones f and f' coincide, and every edge is at right angles to two others. Also since

$$Vi\phi i + Vj\phi j + Vk\phi k = 0, \qquad \text{(iv)}$$

when ϕ is self-conjugate, i, j, and k being any mutually rectangular system of unit vectors, it appears that in this case the planes containing pairs of axes of any function cut the cone again in lines which lie in a plane.

(4.) In general the reciprocal of the cone f is the envelope of the principal planes of the system, and, as this is of the third class, three principal planes are parallel to any line.

(5.) The root cubic of the function $\phi + t\theta$ is of the form

$$g^3 - M_1 g^2 + M_2 g - M_3 = 0, \qquad \text{(v)}$$

and the coefficients contain t in the order indicated by their suffixes. On equating the discriminant to zero, the result is a sextic in t whose roots determine six functions having double roots and pairs of coincident axes.

(6.) No function of the system can have, in general, indeterminate axes; for if $\phi + t\theta$ were such a function, it could be reduced to the form $g\rho + \lambda S\mu\rho$; and the equation of the cone f might then be written in the form

$$S\rho\phi\rho\lambda S\mu\rho = 0. \qquad \text{(vi)}$$

In this case, therefore, the cone breaks up into a quadric cone and a plane.

(7.) Moreover, if two functions have a common axis, it must be common to the whole system, and the cubic cone must have a double edge. For if in (i) two values of t correspond to a given vector ρ, we must have separately

$$V\rho\phi\rho = 0, \quad \text{and} \quad V\rho\theta\rho = 0. \qquad \text{(vii)}$$

Hence ρ is an axis of every function, and also a double edge of the cone.

(8.) The quadric cone

$$S\alpha\rho(\phi + t\theta)\rho = 0 \qquad \text{(viii)}$$

contains the axes of the function defined by t. Or if we regard t as arbitrary and α given, (viii) represents a singly infinite system, a particular cone being determined by the condition that it shall contain any assumed line. This singly infinite system passes through four fixed lines which may be found by combining the equations

$$S\alpha\rho\phi\rho = 0, \quad S\alpha\rho\theta\rho = 0. \qquad \text{(ix)}$$

From these we obtain the equation

$$x\alpha = VV\rho\phi\rho V\rho\theta\rho = -\rho S\rho\phi\rho\theta\rho, \qquad \text{(x)}$$

which must be satisfied by the four fixed vectors. One solution is obviously $\rho \parallel \alpha$, and for the remaining lines we must have $x = 0$, and $S\rho\phi\rho\theta\rho = 0$. Thus three of the lines are on the cubic cone $f = 0$. Hence the axes of all functions of the system compose co-residual triads upon the cubic cone for a quadric cone can be drawn through any set of axes to meet the cubic again in three fixed lines.

(9.) In the notation of elliptic functions, three edges of a cubic cone lie in a plane if the sum of their elliptic parameters is zero. A quadric cone intersects a cubic in six lines, and the sum of the corresponding elliptic parameters is zero. Hence the sum of the parameters of the axes of any function of the system is constant, and the value of this sum is a characteristic of the system.

If the sum is half a *period*, the axes of any pair of functions lie upon a quadric cone. This is the case when the functions are self-conjugate, and more generally when the axes of one function (ϕ) are in perspective with the vectors derived from them by the operation of another function θ. This condition is expressed by the equation

$$SV\rho_1\theta\rho_1V\rho_2\theta\rho_2V\rho_3\theta\rho_3 = 0\ ; \qquad \text{(xi)}$$

and this is also the condition that the axes of ϕ and θ should lie upon a quadric cone. In fact, dropping the suffix 3, (xi) is the equation of a cone through the three axes of θ and two of the axes ρ_1 and ρ_2 of ϕ, so, if (xi) is satisfied, it contains the third axis ρ_3 likewise.

The co-residual property shows that, if the perspective property is true for any pair of functions of the system, it is true for every pair.

The condition (xi) may be expressed by the vanishing of the invariant of the functions

$$\Sigma S\theta\phi\alpha\phi\theta\beta\phi\theta\gamma - \Sigma S\phi\theta\alpha\theta\phi\beta\theta\phi\gamma = 0, \qquad \text{(xii)}$$

in which α, β, and γ may be any three vectors.

This may be proved (compare *Trans.*, R.I.A., vol. xxx., p. 723, and *Proc.*, R.I.A., 3rd series, vol. iv., p. 13) by replacing α, β, and γ by ρ_1, ρ_2, and ρ_3, and by writing

$$\theta\rho_1 = a_{11}\rho_1 + a_{12}\rho_2 + a_{13}\rho_3,\ \theta\rho_2 = a_{21}\rho_1 + a_{22}\rho_2 + a_{23}\rho_3,\ \theta\rho_3 = a_{31}\rho_1 + a_{32}\rho_2 + a_{33}\rho_3, \quad \text{(xiii)}$$

and substituting in (xi) and in (xii). The result in both cases is proportional to

$$a_{12}a_{23}a_{31} - a_{21}a_{32}a_{13}. \qquad \text{(xiv)}$$

(10.) The results already obtained admit of very considerable extension. The equation of the cone (ii) may be replaced by

$$S\chi_1\rho\chi_2\rho\chi_3\rho = 0, \quad \text{where} \quad \chi_1 = a_1\phi + b_1\theta + c_1, \ \&\text{c}. \qquad \text{(xv)}$$

or by

$$f_1 = S\rho\phi_1\rho\theta_1\rho = 0, \quad \text{where} \quad \phi_1 = \chi_1^{-1}\chi_2,\ \theta_1 = \chi_1^{-1}\chi_3. \qquad \text{(xvi)}$$

Thus, the cone is the locus of axes of all functions of the type

$$(a_1\phi + b_1\theta + c_1)^{-1}\,(a_2\phi + b_2\theta + c_2), \qquad \text{(xvii)}$$

a_1, b_1, c_1, a_2, b_2, c_2 being arbitrary scalars. The cone of the axes of the conjugates

$$f'_1 = S\rho\phi'_1\rho\theta'_1\rho = 0, \quad \text{where} \quad \phi'_1 = \chi'_2\chi'^{-1}_1,\ \theta'_1 = \chi'_3\chi'^{-1}_1 \qquad \text{(xviii)}$$

is not the same as the cone f'. In fact, the equation (iii) of that cone may be replaced by

$$f' = S\rho\chi'^{-1}_1\chi'_2\rho\chi'^{-1}_1\chi'_3\rho = 0,$$

and this is not the same as (xviii), because the functions χ'_1, χ'_2, and χ'_3 are not commutative.

(11.) It appears, from (6), that, in general, no function of the type (xvii) can have indeterminate axes; and, by (7), no pair of functions $\phi_1 = \chi_1^{-1}\chi_2$, $\theta_1 = \chi_1^{-1}\chi_3$ can have a common axis.

(12.) We shall now extend the theorem of section (8), and show that the axes of all functions of the type (xvii) form co-residual triads. The equations (ix) are equivalent to

$$S\alpha\rho\chi_1\rho = 0, \quad S\alpha\rho\chi_2\rho = 0, \tag{xix}$$

and these equations require

$$x V\alpha\rho = V\chi_1\rho\chi_2\rho \tag{xx}$$

if ρ is a common edge. Hence, we obtain the new quadric cone

$$S\alpha\chi_1\rho\chi_2\rho = 0, \quad \text{or} \quad S\chi_1^{-1}\alpha\rho\chi_1^{-1}\chi_2\rho = 0 \tag{xxi}$$

which contains the three residual lines. But the second form of its equation shows that it contains also the axes of $\chi_1^{-1}\chi_2$, and these axes, therefore, are residual to the three intersections of the cones (xix) or (ix), and consequently co-residual with the axes of every function of the type (xvii).

VII.—ON THE GENERAL LINEAR TRANSFORMATION IN SPACE.

(1.) If ϖ and ρ are the vectors from an assumed origin to a pair of corresponding points, the relation between the vectors may be written in the form

$$\varpi = f\rho = \frac{\phi\rho + \alpha}{S\beta\rho + 1}, \tag{i}$$

where α and β are constant vectors, and ϕ is a constant linear vector function.

There is no difficulty in verifying that

$$\rho = f^{-1}\varpi = \frac{\phi^{-1}\varpi(S\beta\phi^{-1}\alpha - 1) - \phi^{-1}\alpha(S\beta\phi^{-1}\varpi - 1)}{S\beta\phi^{-1}\varpi - 1}. \tag{ii}$$

(2.) The united points of the transformation are the extremities of vectors satisfying the equation

$$f\rho = \rho, \tag{iii}$$

or

$$\phi\rho + \alpha = t\rho, \quad \text{if} \quad t = S\beta\rho + 1. \tag{iv}$$

Eliminating ρ between these two equations, the result

$$t - 1 + S\beta(\phi - t)^{-1}\alpha = 0 \tag{v}$$

is equivalent to a quartic equation, and the united points correspond to the roots of this quartic and lie upon the twisted cubic

$$\rho = -(\phi - t)^{-1}\alpha. \tag{vi}$$

(3.) If the plane $S\lambda\rho + 1 = 0$ transforms into $S\mu\varpi + 1 = 0$, and if we write symbolically

$$S\mu f\rho = S\rho f'\mu, \qquad \text{(vii)}$$

it appears that

$$\lambda = f'\mu = \frac{\phi'\mu + \beta}{S\alpha\mu + 1}. \qquad \text{(viii)}$$

It is needless to write down the equations corresponding to (ii), (iii), (iv), and (v). They are obtained by replacing ϖ, ρ, α, β, ϕ by λ, μ, β, α, ϕ', respectively; and it will be noticed that the quartic (v) is unaltered by this interchange.

(4.) For a change of origin to the extremity of the vector ϵ, the new symbols are connected with the old by the relations

$$\phi_{\prime} = \frac{\phi - \epsilon S\beta}{S\beta\epsilon + 1}; \quad \alpha_{\prime} = \frac{\alpha + \phi\epsilon - \epsilon(S\beta\epsilon + 1)}{S\beta\epsilon + 1};$$

$$\beta_{\prime} = \frac{\beta}{S\beta\epsilon + 1}; \quad \lambda_{\prime} = \frac{\lambda}{S\lambda\epsilon + 1}; \quad \rho_{\prime} = \rho - \epsilon, \qquad \text{(ix)}$$

and a root $t_{\prime}$ of the quartic transformed from (v) is simply proportional to the corresponding root t of the original, the connexion being

$$t_{\prime} = \frac{t}{S\beta\epsilon + 1}. \qquad \text{(x)}$$

The ratios of the roots of (v) are therefore independent of the origin.

(5.) When we express an arbitrary vector ρ in terms of the vectors ρ_1, ρ_2, ρ_3, and ρ_4 to the four united points ABCD by the equation (compare Art. 79, page 55, vol. i)

$$\text{OP} = \rho = \frac{x_1\rho_1 + x_2\rho_2 + x_3\rho_3 + x_4\rho_4}{x_1 + x_2 + x_3 + x_4}, \qquad \text{(xi)}$$

the derived vector $f\rho$ can easily be seen to be expressible in the form*

$$\text{OQ} = f\rho = \frac{x_1t_1\rho_1 + x_2t_2\rho_2 + x_3t_3\rho_3 + x_4t_4\rho_4}{x_1t_1 + x_2t_2 + x_3t_3 + x_4t_4}. \qquad \text{(xii)}$$

Thus (compare Art. 83, page 58, vol. i),

$$(\text{BC . APDQ}) = \frac{t_1}{t_4}, \ (\text{CA . BPDQ}) = \frac{t_2}{t_4}, \ (\text{AB . CPDQ}) = \frac{t_3}{t_4}, \qquad \text{(xiii)}$$

where (BC . APDQ) is the anharmonic of the four planes through the line BC and the points A, P, D, and Q. Or again, the ratio of the volumes of the pyramids, whose bases are a face of the tetrahedron formed by the united points, and whose vertices are Q and P, respectively, is proportional to the corresponding value of t.

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* Compare Note V., Section (6.), p. 378.

(6.) If $f = f' = F$, so that the function f may be said to be self-conjugate,

$$F\rho = \frac{\Phi\rho + \alpha}{S\alpha\rho + 1} \tag{xiv}$$

where Φ is a self-conjugate function. The general equation of a quadric, any point being origin, may be expressed in the form,

$$S\rho F\rho + 1 = 0. \tag{xv}$$

The extremities of ω and ω' are conjugate with respect to this quadric, if

$$S\omega F\omega' = S\omega' F\omega = -1. \tag{xvi}$$

Hence $F\omega$ is the vector to the reciprocal of the polar plane of the extremity of ω with respect to the quadric, the centre of reciprocation being the origin of vectors, and the radius of reciprocation being unity. The reciprocal quadric is

$$S\nu F^{-1}\nu + 1 = 0. \tag{xvii}$$

To determine the tetradedron self-conjugate to a pair of quadrics,

$$S\rho F\rho + 1 = 0 \quad \text{and} \quad S\rho G\rho + 1 = 0,$$

it is necessary to solve

$$F\omega = G\omega, \quad \text{or} \quad G^{-1}F\omega = \omega, \quad \text{or} \quad F^{-1}G\omega = \omega; \tag{xviii}$$

for the first equation expresses that the extremity of ω has identical polar planes with respect to the two quadrics.

The first equation may be solved directly. In fact, if Ψ and β in G correspond to Φ and α in F,

$$(\Phi - t\Psi)\,\omega = -(\alpha - t\beta) \quad \text{if} \quad S\alpha\omega + 1 = t(S\beta\omega + 1), \tag{xix}$$

and, therefore,

$$\omega = -(\Phi - t\Psi)^{-1}(\alpha - t\beta), \tag{xx}$$

where

$$t - 1 + S(\alpha - t\beta)(\Phi - t\Psi)^{-1}(\alpha - t\beta) = 0. \tag{xxi}$$

We cannot delay on this subject, except to remark that the twisted cubic (xx) is the locus of the vertices of tetrahedra self-conjugate to any pair of quadrics of the doubly infinite system

$$S\omega(\Phi + u\Psi)\,\omega + 2S(\alpha + u\beta)\,\omega + v = 0.$$

(7.) It appears, from (ix), that on change of origin a self-conjugate function will, in general, cease to be self-conjugate. Under what conditions can the origin be selected so that a function may be self-conjugate?

If η is the spin-vector of ϕ, change of origin to the extremity of ϵ will render $\phi_,$ self-conjugate, and $\alpha_, = \beta$, (ix), if the equations

$$2\eta = V\epsilon\beta, \quad \alpha + \phi\epsilon - \epsilon(S\beta\epsilon + 1) = \beta \tag{xxii}$$

can be satisfied. The second equation gives, by the process already employed,

$$\epsilon = (\phi - t)^{-1} (\beta - \alpha) \text{ if } t = S\beta\epsilon + 1, \text{ or } 1 - t + S\beta (\phi - t)^{-1} (\beta - \alpha) = 0. \quad \text{(xxiii)}$$

Thus, four points can be found for which $\alpha_{,} = \beta_{,,}$ and consequently η must be equal to one or other of four determinate vectors.

VIII.—ON THE THEORY OF SCREWS.*

(1.) If ϵ is the translation and ι the rotation, the origin being taken as base-point, for any small displacement of a body, the transformation†

$$\epsilon = \epsilon\iota^{-1} . \iota = (S\epsilon\iota^{-1} + V\epsilon\iota^{-1}) \iota = (p + \varpi) \iota, \quad (p = S\epsilon\iota^{-1}, \quad \varpi = V\epsilon\iota^{-1}) \qquad \text{(i)}$$

shows that the displacement may be accomplished by a rotation round the axis whose equation is $\rho = \varpi + x\iota$, accompanied by a proportional translation along that axis. This screw displacement is called a *twist* by Sir Robert Ball. In the same way a moving body is said to have a *twist-velocity* on an instantaneous screw. In the following brief applications of quaternions to the admirable Theory of Screws of Sir Robert Ball, what is said of wrenches will be seen to be equally true of twist-velocities and of small twists.

(2.) If μ represents the resultant couple at the origin of vectors arising from any distribution of forces and couples, and if λ represents the resultant force, the equivalent wrench may be represented by the symbol (μ, λ). The intensity of a wrench (or the amplitude of a twist) is measured by the tensor of the vector λ; thus $(t\mu, t\lambda)$ or $t(\mu, \lambda)$ is a wrench having the same axis and the same pitch as (μ, λ), but t-fold its intensity. It is obvious that the resultant of any number of wrenches $t_1(\mu_1, \lambda_1)$, $t_2(\mu_2, \lambda_2)$, &c., may be represented by the wrench $(t_1\mu_1 + t_2\mu_2 + \&c., \ t_1\lambda_1 + t_2\lambda_2 + \&c.)$, and by the principle of superposition of small motions this is equally true for twists provided they are small. Every wrench compounded in this manner from n independent wrenches is said to belong to an n-system, and any particular wrench of the system is determined by the values of the scalars t.

(3.) When a body, acted on by a wrench (μ, λ), receives a small twist (μ', λ'), the work done by the wrench is

$$- S(\mu\lambda' + \mu'\lambda), \qquad \text{(ii)}$$

remembering that μ' represents a translation, and λ' a rotation. The symmetry of this expression shows that the same amount of work would have been done by (μ', λ') considered as a wrench, had the body received the twist represented by (μ, λ). When the work done is zero, the screws are said to be *reciprocal*. It is obvious from the linear character of the condition of reciprocity that a screw reciprocal to n screws is reciprocal to every screw that can be compounded from them.

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* Sir Robert Stawell Ball. *A Treatise on the Theory of Screws.* Cambridge, 1900.

† Compare pages 83–85, and 285–287.

Again, in terms of the vector perpendiculars ϖ and the pitches p, the expression for the work becomes

$$-(p+p')\,S\lambda\lambda' - S(\varpi - \varpi')\,\lambda\lambda' \quad \text{or} \quad (p+p')\cos A + d\sin A \qquad \text{(iii)}$$

into the product of the tensors of λ and λ', if A is the angle and d the shortest distance between the axes. Hence, if the axes of reciprocal screws intersect, they cut at right angles, or else the sum of the pitches is zero; the converse is also true.

(4.) Two screws of the *two-system* $(\mu + t\mu',\ \lambda + t\lambda')$ are reciprocal if

$$S(\mu + t\mu')(\lambda + t'\lambda') + S(\mu + t'\mu')(\lambda + t\lambda') = 0, \qquad \text{(iv)}$$

and the axes cut at right angles if $S(\lambda + t\lambda')(\lambda + t'\lambda') = 0$. These equations lead to a quadratic in t, whose roots determine the pair of screws. Their axes intersect, and, if the origin is taken at the point of intersection, the screws may be represented by (ai, i) and (bj, j), a and b being the pitches. Any screw of the system can be represented by $(ai\cos\theta + bj\sin\theta,\ i\cos\theta + j\sin\theta)$, and from the relation $ai\cos\theta + bj\sin\theta = (p + \varpi)(i\cos\theta + j\sin\theta)$, we find at once*

$$p = a\cos^2\theta + b\sin^2\theta, \quad \text{and} \quad \varpi = (b-a)\,k\sin\theta\cos\theta. \qquad \text{(v)}$$

Hence, the equation of the cylindroid, the locus of the axes, is seen to be

$$\rho = (b-a)\,k\sin\theta\cos\theta + s(i\cos\theta + j\sin\theta) \quad \text{or} \quad z(x^2+y^2) = (b-a)\,xy. \qquad \text{(vi)}$$

(5.) Let (μ_1, λ_1), (μ_2, λ_2), and (μ_3, λ_3) be any three wrenches, and let θ be the linear vector function determined by the three equations

$$\mu_1 = \theta\lambda_1, \quad \mu_2 = \theta\lambda_2, \quad \text{and} \quad \mu_3 = \theta\lambda_3. \qquad \text{(vii)}$$

Every wrench that can be compounded from the given wrenches may be represented by $(\theta\alpha, \alpha)$, the vector α being an abbreviation for $t_1\lambda_1 + t_2\lambda_2 + t_3\lambda_3$. If ρ is the vector to any point on the central axis of the wrench $(\theta\alpha, \alpha)$,

$$\theta\alpha = p\alpha + V\rho\alpha, \quad \text{or} \quad (\theta_0 - V(\rho - \epsilon))\,\alpha = p\alpha, \qquad \text{(viii)}$$

where θ_0 is self-conjugate, and $\theta = \theta_0 + V\epsilon$. Thus p is a root, and α is an axis of the linear function $\theta_0 - V(\rho - \epsilon)$. The cubic determining the roots of this function is

$$p^3 - m''p^2 + (m' - (\rho - \epsilon)^2)\,p - (m - S(\rho - \epsilon)\,\theta_0\,(\rho - \epsilon)) = 0, \qquad \text{(ix)}$$

if $\theta_0^3 - m''\theta_0^2 + m'\theta_0 - m = 0$ is the symbolic cubic of θ_0 (Note, p. 520, vol. i.). Hence, the locus of axes of screws of the system, having a given pitch p, is the quadric (ix), one of a concentric system. It is also evident by (viii) and (ix) that three axes pass through an assumed point, and that the sum of the corresponding pitches is constant. Again, the pitch and the vector perpendicular are, respectively,†

$$p = S\theta\alpha\alpha^{-1}, \quad \text{and} \quad \varpi = V\theta\alpha\alpha^{-1}. \qquad \text{(x)}$$

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* Compare the Note on Systems of Rays, Section 11, p. 422.

† Compare Note III., (16.), p. 372.

On comparison with (ix), it appears that the pitch of any screw is inversely proportional to the square of the parallel radius of the zero pitch quadric

$$m = S(\rho - \epsilon)\,\theta_0\,(\rho - \epsilon). \qquad \text{(xi)}$$

We cannot delay upon the locus of the feet of the vector-perpendiculars except to state that it is a Steiner's quartic with three double lines intersecting at the origin, and that the form of the equation $\varpi = V\theta(\alpha_1 + t\alpha_2)(\alpha_1 + t\alpha_2)^{-1}$ shows that the locus for axes parallel to a plane is an ellipse, t alone being variable.

(6.) The wrenches $(\theta\alpha_1, \alpha_1)$ and $(\theta\alpha_2, \alpha_2)$ are reciprocal if

$$S(\alpha_1\theta\alpha_2 + \alpha_2\theta\alpha_1) = 0, \quad \text{or if} \quad S\alpha_1\theta_0\alpha_2 = 0; \qquad \text{(xii)}$$

that is, if the directions of their axes are conjugate with respect to the zero-pitch quadric (xi). Corresponding to three mutually conjugate directions α_1, α_2, and α_3 are three mutually reciprocal or co-reciprocal screws.

(7.) If (μ', λ') is reciprocal to the whole system $(\theta\alpha, \alpha)$, the equation $S(\mu' + \theta'\lambda')\alpha = 0$ must be satisfied for every possible vector α. Hence $\mu' + \theta'\lambda' = 0$, and further, in general, $(-\theta'\alpha', \alpha')$ belongs to a three-system reciprocal to the given three-system. From these considerations it is easy to refer any wrench to six co-reciprocals.

If we assume

$$\mu = \theta\alpha - \theta'\alpha', \quad \text{and} \quad \lambda = \alpha + \alpha', \qquad \text{(xiii)}$$

where θ is any vector function whatever, and (μ, λ) any given wrench, we see that the auxiliary vectors α and α' are in general determinate, being in fact

$$\alpha = (\theta + \theta')^{-1}(\mu + \theta'\lambda), \quad \text{and} \quad \alpha' = -(\theta + \theta')^{-1}(\mu - \theta\lambda).$$

Selecting then any two triads $\alpha_1\alpha_2\alpha_3$ and $\alpha_4\alpha_5\alpha_6$ of mutually conjugate directions with respect to the quadric (xi), and referring α and α' to these, so that

$$\alpha = t_1\alpha_1 + t_2\alpha_2 + t_3\alpha_3, \quad \text{and} \quad \alpha' = t_4\alpha_4 + t_5\alpha_5 + t_6\alpha_6, \qquad \text{(xiv)}$$

it appears that the given wrench can be resolved into component wrenches on six arbitrary co-reciprocals. The six scalars t are proportional to the intensities of the components, and play the part of coordinates of the wrench.*

(8.) To refer a four-system to a set of co-reciprocals, determine the vector function θ from three wrenches of the system as in (vii), and reduce any fourth wrench as in (xiii). Thus α_4 is found, and for every wrench of the system t_5 and t_6 are zero. The two-system $(-\theta'(t_5\alpha_5 + t_6\alpha_6), t_5\alpha_5 + t_6\alpha_6)$ is reciprocal to the four-system. In like manner for a five-system we find α' to be of the form $t_4\alpha_4 + t_5\alpha_5$, and the single screw reciprocal to the system is $(-\theta'\alpha_6, \alpha_6)$. Similarly any wrench can be resolved into two components, one belonging to a screw-system, the other to the reciprocal system.

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* Compare Note IV., Section 3, p. 374.

(9.) The principles explained in the Note on Quaternion Determinants furnish us with a means of writing down a number of invariants for the various screw-systems. For instance, the ratios

$$\begin{vmatrix} \mu_1 & \mu_2 \\ \mu_1 & \mu_2 \end{vmatrix} : 2 \begin{vmatrix} \mu_1 & \mu_2 \\ \lambda_1 & \lambda_2 \end{vmatrix} : \begin{vmatrix} \lambda_1 & \lambda_2 \\ \lambda_1 & \lambda_2 \end{vmatrix} \qquad \text{(xv)}$$

are quite independent of any particular screw of the two-system $(\mu_1 + t\mu_2, \lambda_1 + t\lambda_2)$.

In terms of the vector (ϵ) to the centre of the cylindroid and the screws of reference (4.) these ratios reduce to

$$[abk - \epsilon S\epsilon k - (aiS\epsilon i + bjS\epsilon j) \, . \, k] : [(a + b)\,k + 2S\epsilon k + V\epsilon k] : k, \qquad \text{(xvi)}$$

so that if we write, for brevity, (xv) in the form $p : q : r$,

$$\epsilon = V \, . \, (Vq + \tfrac{1}{2}Sq)r^{-1}; \; a + b = S \, . \, qr^{-1}; \; ab = S \, . pr^{-1} + \tfrac{1}{4}r^{-2}(Sq)^2.$$

On solution of the equations in a and b we can determine everything in terms of p, q, and r.

Again, for a two-system, every determinant composed of rows μ_1, μ_2, μ_3, followed by rows λ_1, λ_2, λ_3 vanishes.

(10.) It is more interesting, however, to consider the relations for systems of higher orders. Write down a determinant, formed by three identical rows of six μ's, followed by three identical rows of six λ's. This is the *sexiant* of the six screws, $(\mu_1, \lambda_1) \ldots (\mu_6, \lambda_6)$. If it vanishes, the screws belong to a five-system. Write down four identical rows of seven μ's, followed by three rows of seven λ's. The result vanishes identically, for a determinant with four identical vector rows vanishes (Note I., (6.)), but we may expand it in the form

$$\mu_1(1) + \mu_2(2) + \mu_3(3) + \mu_4(4) + \mu_5(5) + \mu_6(6) + \mu_7(7) = 0, \qquad \text{(xvii)}$$

where (1) is the sexiant of the screws omitting the first.* Again, four identical rows (λ), followed by three identical rows (μ), form a vanishing determinant expanding into

$$\lambda_1(1) + \lambda_2(2) + \lambda_3(3) + \lambda_4(4) + \lambda_5(5) + \lambda_6(6) + \lambda_7(7) = 0, \qquad \text{(xviii)}$$

the same symbols denoting the sexiants as before. We see thus how to express an arbitrary screw in terms of six given screws.†

* It is simplest to expand a sexiant in terms of the minors of the third order when it is seen to be $\Sigma S\mu_1\mu_2\mu_3 S\lambda_4\lambda_5\lambda_6$.

† Indeed, from this point of view, the theory of screws is equivalent to the theory of vector pairs (μ, λ), every pair denoting an entity. There is a corresponding theory of vector triplets (ν, μ, λ), &c. Writing down four identical rows of ten ν's, followed by three of μ's and three of λ's, we see how to express an arbitrary triplet in terms of nine given triplets by means of functions of nine which may be called noniants in analogy to the sexiants.

(11.) Again, write down the determinant* of three rows of five μ's, followed by two of five λ's, and call it μ'. Similarly, if $-\lambda'$ is the determinant of two rows of μ's, followed by three of λ's (the same as before), the screw (μ', λ') may be easily seen to be the reciprocal of the five given screws. If, however, the fifth screw (μ_5, λ_5) is quite arbitrary, the variable screw (μ', λ') obtained in this way generates the two-system reciprocal to the given four-system.†

(12.) If a free rigid body, acted on by any system of forces, receives a small twist from a position of stable equilibrium the forces no longer equilibrate, and a certain wrench corresponding to the twist acts on the body. We shall consider the important case in which the wrench (μ, λ) is linearly expressible in terms of the twist (σ, ω), that is, when the one-to-one relation between twist and wrench can be expressed by equations of the type

$$\mu = \phi\omega + \chi\sigma, \quad \lambda = \theta\omega + \psi\sigma, \tag{xix}$$

ϕ, χ, θ, and ψ being four linear and vector functions. As the twist changes from (σ, ω) to $(\sigma + d\sigma, \omega + d\omega)$, the work done by the forces is

$$-S(\mu d\omega + \lambda d\sigma) = -S(\phi\omega + \chi\sigma)d\omega - S(\theta\omega + \psi\sigma)d\sigma. \tag{xx.}$$

This is a perfect differential, or the forces are conservative, if, and only if, ϕ and ψ are self-conjugate, and if χ and θ are conjugate. The truth of this property is apparent when we differentiate an expression such as $-\frac{1}{2}S\omega\Phi\omega - S\sigma\Theta\omega - \frac{1}{2}S\sigma\Psi\sigma$ and compare results on assigning arbitrary values to the four vectors σ, ω, $d\sigma$, and $d\omega$. In what follows we shall limit ourselves to the case of conservative forces, so that we may take ϕ and ψ to be self-conjugate, and

$$\mu = \phi\omega + \chi\sigma, \quad \lambda = \chi'\omega + \psi\sigma. \tag{xxi}$$

This type of relation has been called *Chiastic* by Sir Robert Ball because of the cross-connexion expressed by the equations

$$\begin{aligned} S(\mu\omega' + \lambda\sigma') &= S(\phi\omega + \chi\sigma)\omega' + S(\chi'\omega + \psi\sigma)\sigma' \qquad \text{(xxii)} \\ &= S(\phi\omega' + \chi\sigma')\omega + S(\chi'\omega' + \psi\sigma')\sigma = S(\mu'\omega + \lambda'\sigma), \end{aligned}$$

which show that if (μ, λ) is reciprocal to (σ', ω'), then is (μ', λ') reciprocal to (σ, ω).

(13.) A free rigid body is acted on by an impulsive wrench, and begins, in consequence, to twist about an instantaneous screw. Taking the centre of inertia as base-point, it appears that the wrench and twist-velocity are chiastically related, for the dynamical equations are

$$\mu = \phi\omega, \qquad \lambda = M\sigma, \tag{xxiii}$$

if $\phi\omega$ is the linear vector function of the angular velocity ω which represents the angular momentum, and if M is the mass of the body and σ the velocity of translation of the centre of inertia. Here, as before, the chiastic conditions are satisfied, for ϕ

* This determinant is a vector as appears on expansion by minors of μ's and minors of λ's.

† A system of screws of the most general type is partially considered in Note XII., Sections 26–31.

is self-conjugate; also χ and θ are zero, and ψ is a scalar M. We proceed to consider the general case of chiastic relation since from this the properties of impulsive and instantaneous screws are at once deducible. We suppose ϕ, χ, and ψ to be known. When the body is not perfectly free we resolve the wrench (μ, λ) into two components, one (η, ξ) belonging to the screw system of the freedom, and the other (η', ξ') belonging to the reciprocal screw-system. The wrench (η, ξ) is the *reduced wrench*. Thus

$$\mu = \eta + \eta' = \phi\omega + \chi\sigma, \quad \text{and} \quad \lambda = \xi + \xi' = \chi'\omega + \psi\sigma. \qquad \text{(xxiv)}$$

Obviously, when (σ, ω) is given, (η, ξ) and (η', ξ') are at once determinate. Again, when (η, ξ) is given, (η', ξ') and (σ, ω) are still determinate. For if (σ_1, ω_1), (σ_2, ω_2), &c., are n given twists determining the freedom, we may express the known wrench (η, ξ) in the form $(\Sigma a_1\sigma_1, \Sigma a_1\omega_1)$, and the unknown twist (σ, ω) in the form $(\Sigma x_1\sigma_1, \Sigma x_1\omega_1)$, and remembering that (η', ξ') is reciprocal to all the twists of the freedom, we obtain n equations such as

$$S(\eta\omega_1 + \xi\sigma_1) = S\omega_1(\phi\omega + \chi\sigma) + S\sigma_1(\chi'\omega + \psi\sigma),$$

which afford the n unknown scalars $x_1, x_2, \ldots x_n$.

(14.) Again, for freedom of the n^{th} order there exist n principal screws upon which the reduced wrench and the corresponding twist are situated. For if we replace η and ξ by $t\sigma$ and $t\omega$ respectively in the n equations we find on elimination of the scalars x an equation of the n^{th} degree in t, and every root of this determines a principal screw. These screws form a co-reciprocal system. Let (σ_1, ω_1) and (σ_2, ω_2) be two principal twists corresponding to t_1 and t_2 respectively. Thus if we write

$$[12] = S\omega_1\phi\omega_2 + S\omega_1\chi\sigma_2 + S\omega_2\chi\sigma_1 + S\sigma_1\psi\sigma_2 = [21], \qquad \text{(xxv)}$$

we see that

$$[12] = S(\eta_1\omega_2 + \xi_1\sigma_2) = t_1 S(\sigma_1\omega_2 + \sigma_2\omega_1),$$

and also

$$[12] = S(\eta_2\omega_1 + \xi_2\sigma_1) = t_2 S(\sigma_1\omega_2 + \sigma_2\omega_1); \qquad \text{(xxvi)}$$

hence, if t_1 is not equal to t_2, we must have $S(\sigma_1\omega_2 + \sigma_2\omega_1) = 0$, and the screws are reciprocal; and also we have $[12] = 0$, and the screws are said to be *conjugate* screws of the potential. We shall now examine the conditions of reality of the principal screws. They are evidently real if all the roots t are real. If, however, $t_1 = t + ht'$, and if $t_2 = t - ht'$, where $h = \sqrt{-1}$, it appears that the corresponding twists must be of the form

$$(\sigma_1, \omega_1) = (\sigma + h\sigma', \omega + h\omega'), \quad \text{and} \quad (\sigma_2, \omega_2) = (\sigma - h\sigma', \omega - h\omega').$$

If these are conjugate we must have

$$[12] = (S\omega\phi\omega + 2S\omega\chi\sigma + S\sigma\psi\sigma) + (S\omega'\phi\omega' + 2S\omega'\chi\sigma' + S\sigma'\psi\sigma') = 0. \qquad \text{(xxvii)}$$

But this cannot be the case when the potential function $-\frac{1}{2}S\omega\phi\omega - S\omega\chi\sigma - \frac{1}{2}S\sigma\psi\sigma$ is essentially one-signed. Under this condition therefore the principal screws are real.

(15.) On Sir Robert Ball's suggestion I append the quaternion treatment of two important parts of the theory of screws. In general, the twist velocity of a body acted on by constraints alone is constantly changing. Under certain conditions, however, the twist velocity remains for a moment unchanged. The instantaneous screw is then said to be *permanent*. Permanent and principal screws are in general quite distinct, though they are identical in the case of a body having one point fixed. We may write the dynamical equations in the form

$$\Sigma m\rho = \xi', \quad \Sigma m V\rho\ddot{\rho} = \eta', \tag{xxviii}$$

m being the element of mass at the extremity of the vector ρ, and (η', ξ') being the wrench arising from the constraints referred to the origin of vectors as base point. If (σ, ω) is the twist velocity, we have

$$\dot{\rho} = \sigma + V\omega\rho, \quad \text{and} \quad \ddot{\rho} = \dot{\sigma} + V\dot{\omega}\rho + V\omega\dot{\rho} = \dot{\sigma} + V\dot{\omega}\rho + V\omega(\sigma + V\omega\rho). \tag{xxix}$$

For a permanent screw $\dot{\sigma}$ and $\dot{\omega}$ vanish, and if the origin of vectors is taken at the centre of mass, we find, on summation,

$$\Sigma m\ddot{\rho} = MV\omega\sigma = \xi', \quad \Sigma m V\rho\dot{\rho} = V\omega\phi\omega = \eta', \tag{xxx}$$

where $\phi\omega = \Sigma m V\rho V\omega\rho$ is the angular momentum of the body (compare 417, X.). In particular for three degrees of freedom if $\sigma = \theta\omega$, then $\eta' = -\theta'\xi'$, because the wrenches arising from the constraints are reciprocal to the twist velocities. Hence

$$V\omega\phi\omega = \eta' = -\theta'\xi' = -M\theta' V\omega\sigma = -M\theta' V\omega\theta\omega,$$

or, if n is the third invariant of θ,

$$V\omega\phi\omega = Mn^{-1}V\omega\theta^{-1}\omega: \tag{xxxi}$$

so the permanent screws have their axes parallel to the axes of the function $\phi - Mn^{-1}\theta^{-1}$.

(16.) In the same case of freedom of the third order the principal screws are given by

$$\phi\omega = t\sigma + \eta', \quad M\sigma = t\omega + \xi', \tag{xxxii}$$

and from these as $\eta' = -\theta'\xi'$, and $\sigma = \theta\omega$,

$$\phi\omega = t(\theta + \theta')\omega - M\theta'\theta\omega, \tag{xxxiii}$$

so that the principal screws are parallel to the axes of the function $(\theta + \theta')^{-1}(\phi + M\theta'\theta)$.

(17.) The second point suggested by Sir Robert Ball is a proof of the theorem that "two three-systems can in general be *in one way* correlated so that each screw in one regarded as an impulsive screw, has a corresponding screw in the other regarded as an instantaneous screw" (Theory of Screws, Art. 318). This theorem arises from the determination of the dynamical constants of a free body by administering three known impulsive wrenches, and by observing the twist velocities produced. The dynamical equations are three pairs of the type

$$\phi\omega = \mu + V\lambda\rho, \quad M(\sigma + V\omega\rho) = \lambda, \tag{xxxiv}$$

where (μ, λ) is a known wrench, (σ, ω) a known twist velocity, and where ϕ, ρ, and M, the vector function, the vector to the centre of mass and the mass of the body are unknown. The three wrenches produce a three-system $\mu = \theta_1\lambda$, the three twist velocities another $\sigma = \theta_2\omega$, and in terms of θ_1 and θ_2 the equations (xxxiv) become

$$\phi\omega = (\theta_1 - \mathrm{V}\rho)\lambda, \quad M(\theta_2 - \mathrm{V}\rho)\omega = \lambda. \qquad \text{(xxxv)}$$

Hence, for three and therefore for all vectors ω,

$$\phi\omega = M(\theta_1 - \mathrm{V}\rho)(\theta_2 - \mathrm{V}\rho)\omega. \qquad \text{(xxxvi)}$$

Now ϕ is a self-conjugate, and therefore if n''_1 and n''_2 are the first invariants of θ_1 and θ_2 respectively, by a well-known property of Hamilton's function χ,

$$\begin{aligned}(\theta_1\theta_2 - \theta'_2\theta'_1)\omega &= \mathrm{V}\rho\theta_2\omega + \theta'_2\mathrm{V}\rho\omega + \theta_1\mathrm{V}\rho\omega + \mathrm{V}\rho\theta'_1\omega \\ &= \mathrm{V}(n''_2 - \theta_2)\rho\omega + \mathrm{V}(n''_1 - \theta'_1)\rho\omega = 2\mathrm{V}\epsilon_{12}\omega,\end{aligned}$$

if ϵ_{12} is the spin-vector of $\theta_1\theta_2$. Thus ρ is uniquely determined by the equation

$$(n''_1 + n''_2 - \theta_2 - \theta'_1)\rho = 2\epsilon_{12}, \qquad \text{(xxxvii)}$$

which is a necessary consequence of the self-conjugate property of ϕ. The vector ρ being known, $M^{-1}\phi$ is determined by (xxxvi), and M by the second equation (xxxiv). Thus ϕ, ρ, and M are uniquely found. Also the unique correlation between the three-systems is established by (xxxv). It is very instructive to investigate step by step the amount of information afforded as to the dynamical nature of the body by observing the twist velocities produced by known wrenches.*

IX.—ON FINITE DISPLACEMENTS.

(1.) It has been shown that the operator $q(\quad)q^{-1}$ produces a conical rotation of a system of vectors about their common origin, the axis of the rotation having the direction of the quaternion or versor q, and the angle of the rotation being double the angle q. Any displacement of a rigid body may be effected by rotating the body about the origin of vectors until lines in the body are parallel to the positions they will ultimately occupy, and by then translating the body until one point (and therefore all points) attains its final position. Thus if τ is equal and parallel to the translation, the vectors ϖ and ρ to a point in the body, in its initial and final positions, are connected by the relation

$$\rho = \tau + q\varpi q^{-1}. \qquad \text{(i)}$$

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* Additional illustrations of the Theory of Screws will be found in the Notes of this Appendix IX. to XII. inclusive.

(2.) We may write this equation in the form

$$\rho = \tau' + \epsilon + q(\varpi - \epsilon)q^{-1}, \quad \text{if} \quad \tau' = \tau + q\epsilon q^{-1} - \epsilon, \tag{ii}$$

with the interpretations following. The same rotation about the extremity of ϵ changes ϖ into $\epsilon + q(\varpi - \epsilon)q^{-1}$, and the translation τ' completes the displacement. Or a translation $-\epsilon$, followed by the rotation about the origin, and then by the translation $\tau' + \epsilon$, is equivalent to the displacement. For example, writing (i) in the form $\rho = q(q^{-1}\tau q + \varpi)q^{-1}$, we see that the translation $q^{-1}\tau q$, followed by the rotation about the origin, effects the displacement.

(3.) The relation (ii) connecting the translations τ' and τ, which must follow rotations about different points in any given displacement, shows that the difference of these translations $(\tau' - \tau)$ is equal to the displacement one point would receive were the body rotated about the other. The components of τ' and τ along the axis of rotation are consequently equal. Or multiplying the relation (ii) into q, we have

$$(\tau' - \tau)q = q\epsilon - \epsilon q = 2\mathrm{V}.\mathrm{V}q\epsilon; \tag{iii}$$

whence

$$\epsilon = -\tfrac{1}{2}(\tau' - \tau)q(\mathrm{V}q)^{-1} + x\mathrm{V}q; \quad \mathrm{S}(\tau' - \tau)\mathrm{V}q = 0. \tag{iv}$$

Thus given τ', the locus of the extremity of ϵ is a right line parallel to the axis of rotation.

(4). The equation of the central axis is found by expressing that the translation $\tau' = \tau_0$ is parallel to the axis of rotation. By (iv) the equation of the central axis, and the value of τ_0, are found to be

$$\epsilon = \tfrac{1}{2}\mathrm{V}\tau q(\mathrm{V}q)^{-1} + x\mathrm{V}q; \tag{v}$$

$$\tau_0 = (\mathrm{V}q)^{-1}\mathrm{S}\tau\mathrm{V}q. \tag{vi}$$

(5). The general decomposition of a displacement into a pair of displacements results from comparison of (i) with

$$\rho = \tau_2 + q_2(\tau_1 + q_1\varpi q_1^{-1})q_2^{-1}, \tag{vii}$$

and the conditions evidently are

$$q_2 q_1 = q; \qquad \tau = \tau_2 + q_2\tau_1 q_2^{-1}. \tag{viii}$$

If these displacements are a pair of rotations effected about the points E and F fixed in space, or the points E and F′ fixed in the body,

$$\tau_1 = \epsilon - q_1\epsilon q_1^{-1}; \quad \tau_2 = \epsilon + \eta - q_2(\epsilon + \eta)q_2^{-1}; \quad \eta = q_1\eta' q_1^{-1} \tag{ix}$$

if ϵ = OE, η = EF, η' = EF′. Thus (viii) becomes

$$\tau = \epsilon - q\epsilon q^{-1} + \eta - q_2\eta q_2^{-1} = \epsilon - q\epsilon q^{-1} + q_1\eta' q_1^{-1} - q\eta' q^{-1}. \tag{x}$$

Writing for abridgment

$$\text{ET}' = \tau' = \tau - \epsilon + q\epsilon q^{-1}, \qquad \text{(xi)}$$

if E is arbitrarily selected the point T′ is determined, and by (x) the second rotation changes EF into a line equal and parallel to T′F. The locus of F is, therefore, the plane bisecting ET′ at right angles, and the second rotation may be made about any line in this plane. The first rotation may be found by (viii) when q_2 is suitably selected. In like manner if $\text{T}''\text{E} = q^{-1}\text{ET}'q$, we find by (x) $q_2^{-1}\text{EF}'q_2 = \text{T}''\text{F}'$, and the locus in the body is the plane bisecting ET″ at right angles.

Again by (x) if q_1 or q_2 is arbitrarily selected, E and F must lie respectively in the planes determined by

$$S(\tau - \epsilon + q\epsilon q^{-1})Vq_2 = 0 \quad \text{and} \quad S(\tau - \eta + q_2\eta q_2^{-1})Vq = 0. \qquad \text{(xii)}$$

Selecting any point E in the first of these planes, T′ is determined by (xi) and the axis of the second rotation is the intersection of the second plane with that bisecting ET′ at right angles.

(6.) When the body is in the position (i) defined by τ and q, a rotation about the point E′, followed by a translation, carries it to the position given by

$$\rho' = \tau' + \epsilon' + q'(\tau - \epsilon')q'^{-1} + q'q\varpi q^{-1}q'^{-1},$$

while the same translation, followed by the same rotation about E″, carries it to the position

$$\rho'' = \epsilon'' + q'(\tau - \epsilon'' + \tau')q'^{-1} + q'q\varpi q^{-1}q'^{-1}.$$

The difference in the positions is equivalent to a translation

$$\rho' - \rho'' = \tau' + \epsilon' - \epsilon'' - q'(\tau' + \epsilon' - \epsilon'')q'^{-1} = 2V \, . \, (\tau' + \epsilon' - \epsilon'')Vq' \, . \, q'^{-1}, \qquad \text{(xiii)}$$

which is small and of the second order if $\tau' + \epsilon' - \epsilon''$ and $Vq' \, . \, q'^{-1}$ are small and of the first order. Under these conditions the order in which the additional translation and rotation are effected is in the limit immaterial. If the additional rotation is made about the origin of the vectors ϖ, which is a point fixed in the body and at the extremity of τ, the new position is given by

$$\rho' = \tau' + \tau + q'q\varpi q^{-1}q'^{-1}. \qquad \text{(xiv)}$$

(7.) When q and τ are functions of a single parameter, the equation (i) contains full particulars of the path of the body, and if the parameter is a known function of the time, the velocity of the displacement may be completely determined. If

$$\rho + d\rho = \tau + d\tau + (q + dq)\varpi(q + dq)^{-1},$$

the two following expressions for $d\rho$ supposed infinitesimal,

$$d\rho = d\tau + Vd\omega q\varpi q^{-1} \quad \text{where} \quad d\omega = 2V \, . \, dqq^{-1}, \qquad \text{(xv)}$$

$$d\rho = d\tau + qVd\iota\varpi q^{-1} \quad \text{where} \quad d\iota = 2V \, . \, q^{-1}dq,\text{*} \qquad \text{(xvi)}$$

* Evidently $q^{-1}d\omega q = 2V \, . \, q^{-1}(dqq^{-1})q = d\iota$.

easily follow from the consideration that if a is any small quaternion and ξ any vector

$$(1 + a)\xi(1 + a)^{-1} - \xi = (1 + a)\xi(1 - a) - \xi = a\xi - \xi a = 2\mathrm{VV}a\xi.$$

The first of these expressions shows that the additional displacement is due to the translation $d\tau$ and to the rotation $d\omega$ about the body origin applied after the rotation $q(\quad)q^{-1}$. The second shows that had the body originally received the rotation about the body origin represented by $d\iota$, then the rotation $q(\quad)q^{-1}$, and finally the translation $\tau + d\tau$, the same position would have been attained.

(8.) If ϵ is the vector from the fixed origin to any point on the axis of the instantaneous screw, and if p is its pitch,

$$d\tau = (p + \mathrm{V}(\epsilon - \tau))d\omega, \qquad \text{(xvii)}$$

because $\epsilon - \tau$ is the vector from the body origin, the base point to which $d\tau$ and $d\omega$ are referred, to a point on the axis. The pitch and the equation of the axis are, by (xv),

$$p = \tfrac{1}{2}\mathrm{S}d\tau(\mathrm{V}\,.\,dqq^{-1})^{-1}; \qquad \epsilon = \tau + \tfrac{1}{2}\mathrm{V}(d\tau + x)(\mathrm{V}\,.\,dqq^{-1})^{-1}, \qquad \text{(xviii)}$$

x being a variable scalar. Allowing the parameter in τ and q, as well as x, to vary, the vector equation represents the locus in space of the axes of the instantaneous screws.

The body locus of the axes may be obtained by substituting $\epsilon - \tau = q\eta q^{-1}$ in (xvii) or (xviii), when we find

$$q^{-1}d\tau q = (p + \mathrm{V}\eta)d\iota \qquad \text{(xix)}$$

and

$$\eta = \tfrac{1}{2}\mathrm{V}(q^{-1}d\tau q + x)(\mathrm{V}\,.\,q^{-1}dq)^{-1}. \qquad \text{(xx)}$$

If we suppose these two surfaces to be constructed and fitted with guiding threads or projections and depressions of suitable pitch, then when the body locus is suitably placed on the space locus with corresponding generators in contact and rolled over it subject to the constraint of the guides which will cause gliding along the line of contact, the body will traverse the path prescribed.

(9.) When q and τ are functions of two parameters, u and v, the body has two degrees of freedom. Given ϖ, equation (i) represents a surface which would be generated by a point fixed in the body were the body to describe every possible path. The equation (xvii) is linear in the ratio $du : dv$, and represents a singly infinite system of screws whose axes lie upon a cylindroid represented by the vector equation (xviii), when the ratios $x : du : dv$ vary arbitrarily. On account of the linearity of (xvii) it appears that if two screws of the same pitch intersect, all the screws lie in a plane, and pass through a common point. When the expression for p is rendered integral by multiplying by $\mathrm{T}(\mathrm{V}^{-1}dqq^{-1})^2$, a quadratic in $du : dv$ results which determines two, and, in general, only two instantaneous motions of given pitch. If three screws have the same pitch, then all have the same pitch, and it is not hard to see, by (xvii), without assuming any property of the cylindroid, that all the axes lie in a plane, and pass through a common point.

In general, for two degrees of freedom, every point of the body on one or other of two lines, will describe an element of a line, not of a surface, as the body receives every possible small displacement from a given position. The lines are the axes of the two screws of zero pitch. For every small displacement may be compounded from rotations about these lines, and a point on one line suffers displacement only on account of the rotation about the other. The normal to the element of surface described by any point in the body must intersect these two lines, for the normal is at right angles to every possible displacement of the point, and, in particular, to that due to a rotation about either of the lines.*

(10.) We arrive at the particular case of which Darboux has made an extensive use in his kinematical treatment of surfaces by expressing that a plane carried by the body constantly touches the surface described by a point fixed in the plane. Comparing (xv) and (xvi) the condition is simply

$$S d\rho q\gamma q^{-1} = 0, \quad \text{or} \quad S(d\tau q\gamma q^{-1} + d\omega q\varpi\gamma q^{-1}) = 0, \quad \text{or} \quad S(d\sigma\gamma + d\iota\varpi\gamma) = 0, \quad \text{(xxi)}$$

if γ is the direction fixed in the body of the normal to the plane, and if, for brevity, $d\sigma = q^{-1}d\tau q$. This condition must be satisfied for every possible displacement, so that if we write

$$d\sigma = \sigma_1 du + \sigma_2 dv, \quad d\iota = \iota_1 du + \iota_2 dv, \quad \text{(xxii)}$$

we must have separately

$$S(\sigma_1\gamma + \iota_1\varpi\gamma) = 0, \quad S(\sigma_2\gamma + \iota_2\varpi\gamma) = 0. \quad \text{(xxiii)}$$

And there is no difficulty in seeing that the same results would have followed had we expressed that a line fixed in the body constantly intersects the two axes of zero pitch (9.).

It is evident that $d\iota$ may be regarded as representing the elementary angular rotation $d\omega$ referred to directions fixed in the body; indeed, it has been shown that $d\iota = q^{-1}d\omega q$. In like manner, $d\sigma$ is the small displacement in space of the body origin referred to fixed directions in the body. It must be carefully remembered that $d\sigma$, $d\omega$, and $d\iota$ are not like $d\tau$ or dq differentials of vectors σ, ω, and ι. In fact, by (7),†

$$\frac{\partial\sigma_1}{\partial v} = 2V.q^{-1}\frac{\partial\tau}{\partial u}qVq^{-1}\frac{\partial q}{\partial v} + q^{-1}\frac{\partial^2\eta}{\partial u\partial v}q,$$

and

$$\frac{\partial\sigma_2}{\partial u} = 2V.q^{-1}\frac{\partial\tau}{\partial v}qVq^{-1}\frac{\partial q}{\partial u} + q^{-1}\frac{\partial^2\eta}{\partial u\partial v}q,$$

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* Compare Darboux, *Leçons sur la Théorie Générale des Surfaces*, Art. 58.

† Compare Darboux, *loc. cit.*, Arts. 55 and 40.

so that, on subtraction,

$$\frac{\partial\sigma_1}{\partial v} - \frac{\partial\sigma_2}{\partial u} = \mathrm{V}(\sigma_1\iota_2 - \sigma_2\iota_1)\,; \quad \text{also} \quad \frac{\partial\iota_1}{\partial v} - \frac{\partial\iota_2}{\partial u} = \mathrm{V}\iota_1\iota_2\,; \qquad \text{(xxiv)}$$

because

$$2\frac{\partial}{\partial v}\mathrm{V}q^{-1}\frac{\partial q}{\partial u} - 2\frac{\partial}{\partial u}\mathrm{V}q^{-1}\frac{\partial q}{\partial v} = -\,2\mathrm{V}q^{-1}\frac{\partial q}{\partial v}q^{-1}\frac{\partial q}{\partial u} + 2\mathrm{V}q^{-1}\frac{\partial q}{\partial u}q^{-1}\frac{\partial q}{\partial v}$$

$$= 4\mathrm{VV}q^{-1}\frac{\partial q}{\partial u}\mathrm{V}q^{-1}\frac{\partial q}{\partial v}.$$

Returning to equations (xxiii), if these are always satisfied for constant vectors γ and ϖ, we can derive four new equations by differentiating each with respect to u and v, and equating the results to zero. Thus, six equations are obtained which lead to differential equations in τ, q, u, and v, when γ and ϖ are eliminated. Observing that ϖ occurs in the equations only in the combination $\mathrm{V}\varpi\gamma$, it is evident that every point on the line in the body through the extremity of ϖ parallel to γ will describe a surface constantly touched by the plane through the point and at right angles to γ.

(11.) We have noticed in Section 9 the conditions under which a point is common to the axes of all the screws corresponding to small motions with two degrees of freedom. Replacing $d\tau$ and $d\omega$ by τ_1, ω_1 and τ_2, ω_2 successively in (xvii), we deduce, from the resulting equations, the expression for the vectors to the point of intersection,

$$\epsilon = \tau - \frac{\mathrm{V}(\tau_1 - p\omega_1)(\tau_2 - p\omega_2)}{\mathrm{S}(\tau_1 - p\omega_1)\omega_2}, \quad \text{and} \quad 2p\mathrm{S}\omega_1\omega_2 = \mathrm{S}(\omega_1\tau_2 + \omega_2\tau_1). \qquad \text{(xxv)}$$

If the pitches are everywhere equal and constant, the space and body loci of the common intersection of the axes are

$$\rho = \epsilon, \quad \text{and} \quad \varpi = q^{-1}(\epsilon - \tau)q\,; \qquad \text{(xxvi)}$$

and corresponding elements of these surfaces are

$$d\rho = d\epsilon, \quad \text{and} \quad d\varpi = q^{-1}(d\epsilon - p\,d\omega)q\,; \qquad \text{(xxvii)}$$

the second expression being reduced by (xvii) from

$$d\varpi = q^{-1}(d\epsilon - d\tau)q + 2q^{-1}(\mathrm{V}(\epsilon - \tau)\mathrm{V}dqq^{-1})q.$$

In the case in which p is zero, the lengths of corresponding elements are equal, and we deduce the elegant theorem of M. Ribaucour,* the loci not only touch, but roll on one another without sliding. The surfaces are *applicable*.

* Darboux, *loc. cit.*, Art. 58; Ribaucour, *Sur la déformation des surfaces* (*Comptes rendus*, t. lxx., p. 330).

(12.) The usual formulæ with respect to moving axes may be deduced from the following equations, in which the quaternion and vector, q and σ are variable in any manner. The vectors ρ and $q\varpi q^{-1}$ being regarded as terminating at a common point, while the former originates at a point fixed in space, and the latter at the extremity of the vector $q\sigma q^{-1}$,

$$\rho = q(\sigma + \varpi)q^{-1}, \quad \text{and} \quad d\rho = q(d\sigma + d\varpi)q^{-1} + qVd\iota(\sigma + \varpi)q^{-1}. \qquad \text{(xxviii)}$$

Hence

$$D\rho = q^{-1}d\rho q = d\sigma + d\varpi + Vd\iota(\sigma + \varpi), \qquad \text{(xxix)}$$

if we write simply as a matter of notation, $D\rho = q^{-1}d\rho q$. This formula includes the usual formulæ of small displacement, or velocity, with respect to moving axes. Differentiating ρ a second time by the characteristic d, we may write

$$D^2\rho = q^{-1}d^2\rho q = d^2\sigma + d^2\varpi + Vd^2\iota(\sigma + \varpi) + 2Vd\iota(d\sigma + d\varpi) + Vd\iota Vd\iota(\sigma + \varpi), \qquad \text{(xxx)}$$

which includes the formulæ of acceleration. Of course if ϖ is fixed with respect to the moving axes, the terms in $d\varpi$ and $d^2\varpi$ disappear.

X.—ON THE KINEMATICAL TREATMENT OF CURVES.

(1.) To extend the kinematical method employed in Art. 396, imagine a point travelling with unit velocity along the curve and carrying with it three mutually rectangular unit vectors α, β, and γ, so that α continually touches the curve, while the plane of α and β preserves closest contact with it, or in other words, osculates it. If we choose we may select β, so that in its initial position it has the direction of the principal normal. Having made a selection once for all, there will be no confusion, provided the motion is continuous.

It is geometrically obvious that the angular velocity (ω) of the system may be resolved into two components round α and γ; thus we may write

$$\omega = \alpha a_1 + \gamma c_1. \qquad \text{(i)}$$

We may regard a_1 and c_1 as the deriveds with respect to the arc s of two angles a and c. Of these a is the total angle through which the system has turned about α, starting from any initial point on the curve; in like manner c is the total angle through which the system has turned about γ; and it is convenient to suppose that at the initial point $s = 0$. We shall use a_2, c_3, &c., to denote the second and third deriveds of a and c.

It is obvious that a_1 is the torsion, and c_1 the curvature at the point considered.

(2.) If, as in 396 (5.), η is any emanant vector drawn from the moving point, we have by the general formula given in the note to p. 293 the relation

$$D_s\eta = \frac{d\eta}{ds} + V\omega\eta, \tag{ii}$$

where in passing along the curve $D_s\eta$ is the absolute rate of change of the vector η, and $\frac{d\eta}{ds}$ its rate of change relative to the moving system. If η is fixed relatively to the system, (ii) reduces to

$$D_s\eta = V\omega\eta, \tag{iii}$$

and this equation by (i) gives, if $\alpha\beta = +\gamma$,

$$D_s\alpha = \beta c_1; \quad D_s\beta = \gamma a_1 - \alpha c_1; \quad D_s\gamma = -\beta a_1. \tag{iv}$$

(3.) Again, if ρ is the vector to the moving point

$$D_s\rho = \alpha, \tag{v}$$

and we may differentiate successively and obtain by (iv)

$$\left.\begin{aligned} D_s^2\rho &= \beta c_1, \\ D_s^3\rho &= \beta c_2 + (\gamma a_1 - \alpha c_1)c_1, \\ D_s^4\rho &= \beta c_3 + 2(\gamma a_1 - \alpha c_1)c_2 + (\gamma a_2 - \alpha c_2)c_1 - \beta(a_1^2 + c_1^2)c_1. \end{aligned}\right\} \tag{vi}$$

In general, if $$D_s^n\rho = A_n\alpha + B_n\beta + C_n\gamma, \tag{vii}$$

the coefficients A_n, B_n, C_n are assignable functions of the scalars a_1 and c_1 and of their successive deriveds. It is evident by (vi) that the deriveds of highest order occur in the term

$$\beta c_{n-1} + \gamma a_{n-2} c_1. \tag{viii}$$

(4.) This method lends itself readily to the consideration of the contacts of curves. Writing down a few terms in the development in powers of s,

$$\rho = \rho_0 + s\alpha + \tfrac{1}{2}s^2\beta c_1 + \tfrac{1}{6}s^3(\beta c_2 + (\gamma a_1 - \alpha c_1)c_1) + \&c., \tag{ix}$$

we see that the deviation from an osculating curve is $\frac{1}{6}s^3(\beta c_2 + \gamma a_1 c_1)$ for a circle; $\frac{1}{6}s^3\gamma a_1 c_1$ for a parabola or conic; and $\frac{1}{6}s^3\beta c_2$ for a helix (p. 97); because a_1 and c_2 are zero for the circle, a_1 for the conic and c_2 for the helix. Generally by (viii) the deviation between two curves is ultimately

$$\frac{s^n}{\underline{|n}}\left[\beta(c_{n-1} - c'_{n-1}) + \gamma(a_{n-2} - a'_{n-2})c_1\right]. \tag{x}$$

Though the work is necessarily long, there is no difficulty in finding the equation of Hamilton's twisted cubic by assuming as its equation $V(\rho - \rho_0)\phi(\rho - \rho_0) + V\lambda(\rho - \rho_0) = 0$, and differentiating until a sufficient number of equations are obtained to determine λ and ϕ. Perhaps even more briefly the result may be obtained by assuming

$$\rho = \rho_0 + (\phi - t)^{-1}\lambda \quad \text{where} \quad t = s^{-1} + t_0 + t_1 s + \&c., \qquad \text{(xi)}$$

and determining λ, ϕ and the coefficients t_0, t_1, &c., so that λ may equal $(\phi - t)(\rho - \rho_0)$, neglecting only powers of the sixth order in s.

(5.) To illustrate integration of equations (iv), assume that the ratio of curvature to torsion is constant. If $c_1 = a_1 \tan H$, the first and third equations give immediately

$$D_s(\alpha \cos H + \gamma \sin H) = 0, \quad \text{or} \quad \alpha \cos H + \gamma \sin H = k, \qquad \text{(xii)}$$

k being a constant and unit vector of integration. The principal normal β is thus at right angles to a fixed direction, and the curve is traced on a cylinder whose generators are parallel to k. It is a geodesic because β is normal to the cylinder or because the curve cuts the generators at a constant angle H.

We now assume $\beta = i \cos l + j \sin l$ when the second equation (iv) shows that

$$D_s\beta = k\beta \frac{dl}{ds} = (\gamma \cos H - \alpha \sin H)\frac{dl}{ds} = \gamma a_1 - \alpha c_1,$$

and requires

$$dl = \operatorname{cosec} H dc = \sec H da = \sqrt{a_1^2 + c_1^2}\, ds.$$

Hence if a_1 or c_1 is known in terms of s, l is determined; otherwise l is an arbitrary function of s. Since $\gamma = \alpha\beta$, we have by (xii)

$$D_s\rho = \alpha = k(\cos H - (i \cos l + j \sin l) \sin H),$$

so that if ρ_0 is a new constant of integration

$$\rho = \rho_0 + \int_0^s \alpha ds = \rho_0 + ks \cos H - k \sin H \int_0^s (i \cos l + j \sin l)\, dl.$$

In particular for the helix $l = \sqrt{a_1^2 + c_1^2}\, s$, and the integration can be effected.

(6.) The vectors α, β, and γ may be expressed in terms of the deriveds of ρ either directly or by aid of (v) and (vi) in the forms

$$\alpha = D_s\rho;\quad \beta = c_1^{-1}D_s^2\rho;\quad \gamma = a_1^{-1}D_s(c_1^{-1}D_s^2\rho) + a_1^{-1}c_1 D_s\rho\,; \qquad \text{(xiii)}$$

and by differentiating the expression for γ we find a differential equation of the fourth order

$$D_s(a_1^{-1}D_s(c_1^{-1}D_s^2\rho) + a_1^{-1}c_1 D_s\rho) + a_1 c_1^{-1} D_s^2\rho = 0. \qquad \text{(xiv)}$$

This can be integrated in certain cases.

(7.) If the curve lies upon the sphere

$$T(\rho - \sigma) = R, \qquad \text{(xv)}$$

successive differentiations with respect to s afford the equations

$$S\alpha(\rho - \sigma) = 0; \;\; S\beta(\rho - \sigma) = \frac{1}{c_1}; \;\; S\gamma(\rho - \sigma) = \frac{1}{a_1}\frac{d}{ds}\left(\frac{1}{c_1}\right) = \frac{d}{da}\left(\frac{1}{c_1}\right). \qquad \text{(xvi)}$$

Hence

$$\sigma = \rho + \beta\frac{1}{c_1} + \gamma\frac{d}{da}\left(\frac{1}{c_1}\right) \quad \text{and} \quad R^2 = \frac{1}{c_1^2} + \left(\frac{d}{da}\frac{1}{c_1}\right)^2, \qquad \text{(xvii)}$$

and these equations in general determine the osculating sphere. If the curve is spherical so that σ and R are constant differentiation of either of the equations (xvii), or of the third of (xvi) gives

$$E = \frac{d^2}{da^2}\frac{1}{c_1} + \frac{1}{c_1} = 0; \quad \text{hence} \quad \frac{1}{c_1} = R\cos(a - a_0), \qquad \text{(xviii)}$$

a_0 being a constant of integration.

(8.) In the general case, instead of (xviii) differentiation of (xvii) shows that

$$d\sigma = \gamma E da; \;\; RdR = Ed\frac{1}{c_1}, \qquad \text{(xix)}$$

so if dotted letters refer to the locus of centres of spherical curvature, we may write

$$d\sigma = \alpha' ds' \quad \text{and} \quad \alpha' = \gamma \quad \text{if} \quad ds' = + E da. \qquad \text{(xx)}$$

Having selected a sign (we must have $\alpha' = \pm\gamma$) we differentiate again by the formulæ (iv) and

$$\beta' c_1' ds' = - \beta a_1 ds. \qquad \text{(xxi)}$$

Here again there is a latitude in the choice of sign, but if we select

$$\beta' = - \beta, \quad \text{then} \quad c_1' = E^{-1}; \; dc' = da, \quad \text{or} \quad c' = a; \qquad \text{(xxii)}$$

if the angles c' and a are measured from corresponding initial points.

Since $\alpha'\beta' = -\gamma\beta$, it follows of necessity that

$$\gamma' = \alpha, \quad \text{whence} \quad -\beta' a'_1 ds' = \beta c_1 ds \quad \text{and} \quad a'_1 = c_1 a_1^{-1} E^{-1} \quad \text{and} \quad a' = c. \qquad \text{(xxiii)}$$

Finally we may remark that as the moving point travels with unit velocity along the curve, the centre of spherical curvature travels with the velocity Ea_1 and at right angles to the osculating plane; moreover the angular velocities of the two systems have the same direction but that of the derived system is $(Ea_1)^{-1}$ times that of the original.

(9.) To determine the sign of E geometrically, we must calculate the deviation from the osculating sphere. This may be readily done by assuming it to be

$$\frac{1}{\lfloor 4} s^4 x (\rho - \sigma),$$

or by expressing that the curve

$$\rho - \frac{1}{\lfloor 4} s^4 x (\rho - \sigma)$$

passes through five consecutive points on the sphere. The result is

$$x = a_1^2 c_1 R^{-2} E. \qquad \text{(xxiv)}$$

(10.) On the other hand, given the locus of centres, the relation between its arc ds' and that of the sought curve ds affords the differential equation

$$\frac{ds'}{dc'} = \frac{1}{c_1} + \frac{d^2}{dc'^2}\frac{1}{c_1}; \qquad \text{(xxv)}$$

and the solution is

$$\frac{1}{c_1} = \sin c' \int \frac{ds'}{dc'} \cos c' dc' - \cos c' \int \frac{ds'}{dc'} \sin c' dc', \qquad \text{(xxvi)}$$

arbitrary constants of integration being understood. Hence if c_1^{-1} is any particular integral and x and y constants of integration,

$$\rho = \sigma + \beta' \left(\frac{1}{c_1} + x \cos c' + y \sin c' \right) - a' \frac{d}{dc'} \left(\frac{1}{c_1} + x \cos c' + y \sin c' \right). \qquad \text{(xxvii)}$$

All these curves have of course corresponding elements parallel. The ratio of the element of the arc of the curve x, y to that for which x and y are zero is

$$1 + c_1 (x \cos c' + y \sin c'). \qquad \text{(xxviii)}$$

(11.) We may inquire under what conditions the unit vectors at corresponding points on two curves can remain constantly inclined to one another. This condition, of course, is satisfied by the curves of the last Section.

Assuming

$$a' = l a + m \beta + n \gamma, \qquad \text{(xxix)}$$

where l, m, and n are constant, we find on differentiation

$$\beta' c'_1 ds' = - a m c_1 ds + \beta (l c_1 - n a_1) ds + \gamma m a_1 ds. \qquad \text{(xxx)}$$

If this vector is constantly inclined to a, β, γ we must have either the ratio $a_1 : c_1$ constant, or $m = 0$ and $ln = 0$. The conditions in (8.) are $m = l = 0$. For $m = n = 0$ corresponding elements of the curves are parallel. In every other case both curves must be geodesics on cylinders (5.).

(12.) For the emanating lines and emanant surfaces generated by them, two different notations suggest themselves. One is already given (ii), and the motion of the line is referred to the moving system. In the second notation we may suppose the line at any instant to be twisting about a certain screw (396, (10.)) of pitch p with an angular velocity θ. This vector θ is, of course, at right angles to the unit vector η, and we have (compare (ii))

$$D_s\eta = \theta\eta = \frac{d\eta}{ds} + V\omega\eta. \qquad \text{(xxxi)}$$

The shortest distance between neighbouring positions of the emanant is obviously $p\theta ds$, and for neighbouring positions we have

$$\rho + t_0\eta + p\theta ds = \rho + D\rho + (t_0 + x ds)(\eta + D\eta), \qquad \text{(xxxii)}$$

in which $\varpi = \rho + t\eta$ is the equation of the emanant line in one position, t_0 the value of t for the point of closest approach to the next position, and x some scalar. Replacing $D\rho$ by αds and $D\eta$ by $\theta\eta ds$, and retaining only terms of the first order, (xxxii) reduces to

$$p\theta = \alpha + t_0\theta\eta + x\eta. \qquad \text{(xxxiii)}$$

From this, $$p = S\alpha\theta^{-1}; \quad t_0 = S\alpha\eta\theta^{-1}; \quad x = S\alpha\eta. \qquad \text{(xxxiv)}$$

The line of *striction* of the surface is the curve whose equation is

$$\varpi = \rho + t_0\eta = \rho + \eta S\alpha\eta\theta^{-1} = \rho - \eta S\alpha(D_s\eta)^{-1}, \qquad \text{(xxxv)}$$

or it is the locus of points of closest approach of consecutive generators.

The scalar p is called the *parameter of distribution*, and is usually defined as the ratio borne by the distance between two close generators to the angle between them. It vanishes for a developable.

(13.) When s and t both vary in the equation

$$\varpi = \rho + t\eta \qquad \text{(xxxvi)}$$

the element of any arc on the surface is

$$D\varpi = (\alpha + t\theta\eta)\, ds + \eta dt = (p\theta + (t - t_0)\,\theta\eta - x\eta)\, ds + \eta dt \qquad \text{(xxxvii)}$$

by (xxxiii); and the normal vector is, consequently,

$$\nu = V\alpha\eta - t\theta = \theta\,(p\eta + t_0 - t) = (t_0 - t - p\eta)\,\theta. \qquad \text{(xxxviii)}$$

The anharmonic of four normals at points on a common generator is

$$\frac{\nu_1 - \nu_2}{\nu_2 - \nu_3}\cdot\frac{\nu_3 - \nu_4}{\nu_4 - \nu_1} = \frac{t_1 - t_2}{t_2 - t_3}\cdot\frac{t_3 - t_4}{t_4 - t_1}; \qquad \text{(xxxix)}$$

and in this equation, which expresses that the anharmonic of the vectors is the anharmonic of the points on the generator, we may replace the vectors ν by any linear and distributive function of them, for instance, by $S\nu(\varpi - \rho)$. Thus, the ratio is also the anharmonic of the four tangent planes.

If we express that two of these normals are perpendicular, by (xxxviii) the condition reduces to

$$S(p\eta + t_0 - t)(t_0 - t' - p\eta) = 0, \quad \text{or} \quad (t - t_0)(t' - t_0) + p^2 = 0. \qquad \text{(xl)}$$

The corresponding points form a system in involution having its centre on the line of striction and having imaginary foci. Moreover, as

$$U\frac{\nu}{\nu_0} = U\frac{\theta(p\eta + t_0 - t)}{\theta p\eta} = U\frac{p\eta + t - t_0}{p\eta} = U\left(1 + \frac{t_0 - t}{p}\eta\right), \qquad \text{(xli)}$$

the tangent of the angle A between the tangent plane at t and that at t_0 is

$$\tan A = \frac{t_0 - t}{p}. \qquad \text{(xlii)}$$

It may be proved that the measure of curvature, or the product of the principal curvatures K_1 and K_2, can be expressed by the very simple equation

$$K_1K_2 = -\frac{p^2}{(p^2 + (t - t_0)^2)^2} = -\frac{\cos^4 A}{p^2}, \qquad \text{(xliii)}$$

but we cannot delay on this.*

(14.) We can obtain a more explicit form of the condition

$$p = 0, \quad \text{or} \quad S\alpha\theta = 0, \quad \text{or} \quad S\alpha\eta D_s\eta = 0,$$

that the emanant surface should be a developable by assuming

$$\eta = \alpha \cos l + \beta \sin l \cos m + \gamma \sin l \sin m, \qquad \text{(xliv)}$$

and substituting in the third form of the condition. This gives

$$\sin l \, d(a + m) - \cos l \sin m \, dc = 0, \quad \text{or} \quad \sin l = 0. \qquad \text{(xlv)}$$

Hence, the only developables generated in the plane of α and γ ($m = \frac{1}{2}\pi$) are the tangent line developable ($l = 0$), and the rectifying developable ($\cot l = \tan H$). No line except α in the plane of α and β can generate a developable. Any line whatever in the plane of β and γ is capable of generating a developable provided $d(m + a) = 0$, or $m = a_0 - a$, a_0 being a constant.

We thus obtain the system of developables

$$\varpi = \rho + t(\beta \cos(a - a_0) - \gamma \sin(a - a_0)), \qquad \text{(xlvi)}$$

whose cuspidal edges are

$$\varpi = \rho + \frac{\beta}{c_1} - \frac{\gamma}{c_1}\tan(a - a_0). \qquad \text{(xlvii)}$$

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* Compare the Note immediately following (section (12.)), p. 416.

These curves all lie on the polar developable generated by the normal planes to the curve, and they are the evolutes of the curve.

Again, by (xlv), except for a geodesic on a cylinder, no developable can be generated by a line fixed relatively to α, β, γ, except the tangent-line developable.

In general, it may be shown that the cuspidal edge of any developable is determined by

$$t_0 = \frac{\sin l}{l_1 + c_1 \cos m}. \tag{xlviii}$$

Also, if the line of striction of any emanant surface coincides with the original curve, $S\alpha D_s\eta = 0$ gives

$$dl + \cos m\, dc = 0, \quad \text{or} \quad \sin l = 0, \tag{xlix}$$

and, in this case, the pitch of the surface is

$$p = -\frac{\sin l}{c_1 \cos l \sin m - (a_1 + m_1) \sin l}. \tag{l}$$

XI.—ON THE KINEMATICAL TREATMENT OF SURFACES.

(1.) For the kinematical treatment of surfaces we may conceive two systems of curves, determined by two parameters u and v, to be traced upon any surface so that the curves of one system are orthogonal to those of the other. At any point on the surface, let α be a unit vector tangent to the curve *u variable*, and β tangent to the curve *v variable*. Then $\alpha\beta = \gamma$ is a unit vector normal to the surface. These three vectors may be supposed connected with three fixed vectors i, j, and k by the equations

$$\alpha = qiq^{-1}, \quad \beta = qjq^{-1}, \quad \gamma = qkq^{-1}, \tag{i}$$

where q is a quaternion function of u and v.

Generally (compare Note IX., Sections 7 and 12, pages 399 and 403)

if $\quad \xi = q\eta q^{-1}, \quad$ then $\quad d\xi = q(d\eta + Vd\iota\eta)q^{-1}, \quad$ where $\quad d\iota = 2Vq^{-1}dq. \qquad$ (ii)

And, in particular, if η is invariably connected with i, j, and k, the relation is simplified into $d\xi = qVd\iota\eta q^{-1}$ because $d\eta = 0$. We shall use the notation

$$d\iota = ida + jdb + kdc = (ia_1 + jb_1 + kc_1)du + (ia_2 + jb_2 + kc_2)dv, \tag{iii}$$

but it must be observed that da is not the differential of a scalar function (a) of u and v, because $d\iota$ is not the differential of a vector. In this notation we find at once, by (i) and (iii),

$$d\alpha = \beta dc - \gamma db, \quad d\beta = \gamma da - \alpha dc, \quad d\gamma = \alpha db - \beta da. \tag{iv}$$

(2.) Now, if ρ is the vector to a point on the surface,

$$d\rho = \alpha A du + \beta B dv, \qquad \text{(v)}$$

where A and B are functions of u and v, and the square of the linear element is

$$ds^2 = T d\rho^2 = A^2 du^2 + B^2 dv^2 \qquad \text{(vi)}$$

and elements of the orthogonal curves are Adu and Bdv, respectively.

We shall also write generally

$$d\rho = (\alpha \cos l + \beta \sin l) ds, \qquad \text{(vii)}$$

l being the angle any curve on the surface makes with the curve *u variable*.

(3.) We shall now show that the eight scalar functions of u and v, A, B, a_1, b_1, c_1, a_2, b_2, and c_2 are not all independent. Since ρ is a function of u and v, the condition

$$\frac{\partial^2 \rho}{\partial v \partial u} = \frac{\partial}{\partial v}(A\alpha) = \frac{\partial}{\partial u}(B\beta) = \frac{\partial^2 \rho}{\partial u \partial v} \qquad \text{(viii)}$$

must be satisfied. By (iv) this becomes

$$\alpha \frac{\partial A}{\partial v} + (\beta c_2 - \gamma b_2) A = \beta \frac{\partial B}{\partial u} + (\gamma a_1 - \alpha c_1) B,$$

so the scalars are connected by the equations

$$\frac{\partial A}{\partial v} = - c_1 B; \quad \frac{\partial B}{\partial u} = c_2 A; \quad a_1 B + b_2 A = 0. \qquad \text{(ix)}$$

Moreover, we can show that

$$\frac{\partial \iota_1}{\partial v} - \frac{\partial \iota_2}{\partial u} = V \iota_1 \iota_2. \qquad \text{(x)}$$

For remembering that $dq^{-1} = - q^{-1} dq q^{-1}$, we have

$$\frac{\partial \iota_1}{\partial v} = 2 \frac{\partial}{\partial v} V q^{-1} \frac{\partial q}{\partial u} = 2 V q^{-1} \frac{\partial^2 q}{\partial v \partial u} - 2 V q^{-1} \frac{\partial q}{\partial v} q^{-1} \frac{\partial q}{\partial u}$$

and

$$\frac{\partial \iota_2}{\partial u} = 2 \frac{\partial}{\partial u} V q^{-1} \frac{\partial q}{\partial v} = 2 V q^{-1} \frac{\partial^2 q}{\partial u \partial v} - 2 V q^{-1} \frac{\partial q}{\partial u} q^{-1} \frac{\partial q}{\partial v};$$

remembering also that q is a function of u and v (1.) so that the term involving the second differential of q cancels on subtraction, and bearing in mind that $pq - qp = 2V.VpVq$, if p and q are any two quaternions, there is no difficulty in establishing equation (x).

Hence we obtain three additional equations equivalent to (x),

$$\frac{\partial a_1}{\partial v} - \frac{\partial a_2}{\partial u} = b_1c_2 - b_2c_1\,; \quad \frac{\partial b_1}{\partial v} - \frac{\partial b_2}{\partial u} = c_1a_2 - c_2a_1\,; \quad \frac{\partial c_1}{\partial v} - \frac{\partial c_2}{\partial u} = a_1b_2 - a_2b_1. \qquad \text{(xi)}$$

The same equations (xi) would have been found, though somewhat less simply, had we employed this other vector $d\omega = 2Vdqq^{-1} = qd\iota q^{-1}$.

The vectors ω_1 and ω_2, analogous to ι_1 and ι_2, satisfy

$$\frac{\partial \omega_1}{\partial v} - \frac{\partial \omega_2}{\partial u} = -V\omega_1\omega_2.$$

(4.) If R is a principal radius of curvature the usual equation

$$d\rho + R\,dU\nu = 0$$

becomes in this notation

$$\alpha A du + \beta B dv + R(\alpha db - \beta da) = 0, \qquad \text{(xii)}$$

which affords the two scalar equations

$$(A + Rb_1)\,du + Rb_2 dv = 0, \qquad Ra_1 du - (B - Ra_2)\,dv = 0. \qquad \text{(xiii)}$$

From these, on elimination of the ratio $du : dv$, we obtain the quadratic

$$R^2(a_1b_2 - a_2b_1) + R(b_1B - a_2A) + AB = 0, \qquad \text{(xiv)}$$

whose roots are the principal radii. This equation may be modified by (ix) and (xi) so as to exhibit Gauss's remarkable theorem on the measure of curvature. In fact

$$R_1^{-1}R_2^{-1} = A^{-1}B^{-1}(a_1b_2 - a_2b_1) = A^{-1}B^{-1}\left(\frac{\partial c_1}{\partial v} - \frac{\partial c_2}{\partial u}\right) \text{ by (xi)};$$

and by (ix) this reduces to

$$R_1^{-1}R_2^{-1} = -A^{-1}B^{-1}\left(\frac{\partial}{\partial v}\left(B^{-1}\frac{\partial A}{\partial v}\right) + \frac{\partial}{\partial u}\left(A^{-1}\frac{\partial B}{\partial u}\right)\right). \qquad \text{(xv)}$$

The measure of curvature thus depends solely on the linear element (vi), and is unaltered when the surface is bent or twisted in any manner without altering the length of any arc.

(5.) Eliminating R from (xiii), the directions of the lines of curvature are given by

$$A du da + B dv db = 0 \quad \text{or} \quad Aa_1du^2 + (Aa_2 + Bb_1)du dv + Bb_2 dv^2 = 0. \qquad \text{(xvi)}$$

Hence, by (ix), we can see that the lines of curvature cut at right angles, and if we take these lines for the orthogonal systems (1.), we must have

$$a_1 = 0, \qquad b_2 = 0, \qquad \text{(xvii)}$$

whence by (xiii)

$$b_1 = -AR_1^{-1}, \quad a_2 = BR_2^{-1},$$

or more conveniently, if K_1 and K_2 are the principal curvatures by (vii)

$$da = BK_2 dv = K_2 \sin l\, ds, \quad db = -AK_1\,du = -K_1 \cos l\, ds. \qquad \text{(xviii)}$$

The relations (ix) and (xi) are equivalent to

$$c_1 = -\frac{1}{B}\frac{\partial A}{\partial v}; \quad c_2 = \frac{1}{A}\frac{\partial B}{\partial u}; \quad \frac{\partial K_2}{\partial u} = \frac{K_1 - K_2}{B}\frac{\partial B}{\partial u}; \quad \frac{\partial K_1}{\partial v} = \frac{K_2 - K_1}{A}\frac{\partial A}{\partial v}; \quad \text{(xix)}$$

together with (xv).

(6.) For any curve on the surface by (v) and (vii) a unit tangent vector is

$$\rho' = \mathrm{U}(\alpha A du + \beta B dv) = \alpha \cos l + \beta \sin l, \quad \text{(xx)}$$

the accent denoting, as usual, that the vector ρ has been differentiated with respect to the arc s.

Taking the differential of the second of these expressions by (iv)

$$d\rho' = \rho'' ds = (\beta dc - \gamma db)\cos l + (\gamma da - \alpha dc)\sin l + (-\alpha \sin l + \beta \cos l)\, dl,$$

or more simply

$$\rho'' ds = \gamma\rho'(dc + dl) + \gamma(\sin l da - \cos l db). \quad \text{(xxi)}$$

From this equation may be deduced all the properties of a curve traced upon the surface depending on differentials of the second order.

(7.) The projection of the vector of curvature (ρ'') on the normal to the surface (γ), or the component of the curvature in the plane containing the normal to the surface and the tangent to the curve is

$$K = \frac{\sin l da - \cos l db}{ds} = \frac{B dv da - A du db}{A^2 du^2 + B^2 dv^2}$$

$$= \sin l\left(\frac{a_1}{A}\cos l + \frac{a_2}{B}\sin l\right) - \cos l\left(\frac{b_1}{A}\cos l + \frac{b_2}{B}\sin l\right), \quad \text{(xxii)}$$

these transformations being effected by the relations

$$\cos l ds = A du, \quad \sin l ds = B dv.$$

As K does not involve the differential of l, or the second differentials of u or v, it is the same for all curves having a common tangent and lying on the surface. In fact, K is the curvature of the normal section of the surface, and Meusnier's theorem is incidentally proved. Also Euler's theorem follows by (xviii) as we may write

$$K = K_1 \cos^2 l + K_2 \sin^2 l, \quad \text{(xxiii)}$$

when we take the lines of curvatures as the curves of reference.

The curvatures of the normal sections through the curves of reference are $-A^{-1}b_1$ and $B^{-1}a_2$.

(8.) The component of the curvature in the tangent plane is, in like manner,

$$K' = (dc + dl)\,ds^{-1}. \qquad \text{(xxiv)}$$

This is the geodesic curvature of the curve. It vanishes if the curve is a geodesic; and in this case the curve projects into a curve in the tangent plane inflexionally touching the tangent. Hence

$$dc + dl = 0, \quad \text{or} \quad \frac{\partial A}{\partial v}\frac{du}{B} - \frac{\partial B}{\partial u}\frac{dv}{A} - d\tan^{-1}\frac{Bdv}{Adu} = 0 \qquad \text{(xxv)}$$

is the equation of a geodesic, the transformation being made by the aid of (ix). As this equation involves only A and B, the coefficients of the line element (vi), geodesics remain geodesics when the surface is deformed without stretching. This, of course, is otherwise obvious.

The geodesic curvatures of the curves of reference are $A^{-1}c_1$ and $B^{-1}c_2$.

(9.) Instead of proceeding directly to a third differentiation, it is simpler to modify the results already obtained by writing in accordance with the notation used in the kinematical treatment of curves (Note X., page 404),

$$\rho' = \alpha', \quad \rho'' = \beta' c'_1, \quad \text{and} \quad d\beta' = (\gamma' a'_1 - \alpha' c'_1)\,ds,$$

and also by introducing a new angle m suggested by (xxi), and defined by the equations

$$c'_1 \cos m\, ds = dc + dl, \quad c'_1 \sin m\, ds = \sin l\, da - \cos l\, db. \qquad \text{(xxvi)}$$

In this notation, the relation (xxi) affords

$$\beta' = \gamma\alpha' \cos m + \gamma \sin m \quad \text{and} \quad \gamma' = \gamma \cos m - \gamma\alpha' \sin m; \qquad \text{(xxvii)}$$

whence,

$$\gamma = \beta' \sin m + \gamma' \cos m, \quad \gamma\alpha' = \beta' \cos m - \gamma' \sin m. \qquad \text{(xxviii)}$$

We may observe that m is zero for an asymptotic curve and a right angle for a geodesic. It is, in general, the angle between the normal to the surface, and the binormal to the curve.

Thus prepared, when we differentiate γ expressed in terms of m, β', and γ' by (xxviii), we have

$$\alpha db - \beta da = (\beta' \cos m - \gamma' \sin m)(dm - a'_1 ds) - \alpha' c'_1 \sin m\, ds. \qquad \text{(xxix)}$$

From this we recover the second of (xxvi), as well as the new equation,

$$da' - dm = \cos l\, da + \sin l\, db. \qquad \text{(xxx)}$$

This equation may be reduced to

$$da' - dm = (K_2 - K_1) \sin l \cos l\, ds, \qquad \text{(xxxi)}$$

when, without loss of generality, we take the lines of curvature as the systems of reference (xviii).

(10.) Thus the difference between the torsion of a curve traced upon a surface, and the rate at which the angle between its osculating plane and the tangent plane varies, is equal to half the difference of the principal curvatures multiplied by the sine of double the angle between the curve and a line of curvature. This theorem has many consequences. In the first place $da' - dm$ is the same for all curves having a common tangent; it vanishes for a line of curvature; when a surface is cut by a plane, the rate of the variation of the angle between the plane and the tangent plane at any point of the section equals half the difference of the curvatures at the point multiplied by the sine of twice the angle between the trace and a line of curvature; when a line of curvature is plane, the surface cuts the plane at a constant angle; and when a surface cuts a plane at a constant angle, the intersection is a line of curvature or the surface is a sphere; the torsion of a geodesic is

$$(K_2 - K_1) \sin l \cos l\,;$$

and this has been called the geodesic torsion by M. O. Bonnet, to whom the important and elegant relation (xxxi) is due.*

Also for the intersection of two surfaces,

$$(K_2 - K_1) \sin l \cos l - (K'_2 - K'_1) \sin l' \cos l' = \frac{dM}{ds} \qquad \text{(xxxii)}$$

gives the rate of change of the angle at which the surfaces cut. Hence, if two surfaces cut at a constant angle along a line of curvature on one, the intersection is also a line of curvature on the other.

It is well to remark that we have now exhausted all the relations which are not obtainable by direct differentiation from those already found. We have seen (Note X.) that all the affections of a curve can be expressed in terms of the unit vector α', β', γ' of the curve, and in terms of the curvature and torsion and their deriveds. But we have found the curvature and the torsion, and have expressed α', β', γ' in terms of α, β, γ, l, and m.

(11.) If we take the curves u *variable* to be geodesics, we have by (xxv) $c_1 = 0$. Hence by (ix) A is a function of u. Changing the variable u to $\int A du$, the new variable is simply the arc of the geodesics. Then A becomes unity, equation (xv) reduces to

$$K_1 K_2 = -\frac{1}{B}\frac{\partial^2 B}{\partial u^2}, \qquad \text{(xxxiii)}$$

and the geodesic curvature of any curve (xxiv) is

$$K' = \frac{\sin l}{B}\frac{\partial B}{\partial u} + \frac{\partial l}{\partial s}, \quad \text{because } c_2 = \frac{\partial B}{\partial u}, \quad dv = \frac{\sin l ds}{B}. \qquad \text{(xxxiv)}$$

* Compare Darboux, *loc. cit.*, Art. 505.

Hence the total curvature of any portion of the surface is

$$\iint \mathrm{K}_1\mathrm{K}_2 dS = -\iint \frac{\partial^2 B}{\partial u^2}\,du dv = -\int \frac{\partial B}{\partial u}\,dv = \int \left(\frac{\partial l}{\partial s} - \mathrm{K}'\right) ds, \qquad \text{(xxxv)}$$

the single integrals being taken over the bounding curve, using Stokes's theorem in the transformation.

If the bounding curve is made up of geodesics, K' is zero; and the integral is 2π minus the sum of the angles through which the direction of a point travelling round the boundary suddenly turns at the points of intersection of the bounding geodesics.

We may also notice the relation

$$\mathrm{K}_1\mathrm{K}_2 = -\frac{\partial \mathrm{K}'_2}{\partial u} - \mathrm{K}'^2_2, \qquad \text{(xxxvi)}$$

where K'_2 is the geodesic curvature of the orthogonal curve v variable.

(12.) For ruled surfaces, when we take the generators to be the curves *u variable*, a is independent of u, and $b_1 = c_1 = 0$. The conditions (ix) and (xi) reduce to

$$\frac{\partial B}{\partial u} = c_2;\quad a_1 B + b_2 = 0;\quad \frac{\partial a_1}{\partial v} - \frac{\partial a_2}{\partial u} = 0;\quad \frac{\partial b_2}{\partial u} = c_2 a_1;\quad \frac{\partial c_2}{\partial u} = -a_1 b_2. \qquad \text{(xxxvii)}$$

These give on combination if $G = (-\mathrm{K}_1\mathrm{K}_2)^{\frac{1}{2}}$ (compare (xxxiii))

$$c_2 = \frac{\partial B}{\partial u};\quad a_1 = G;\quad b_2 = -BG;\quad \frac{\partial a_2}{\partial u} = \frac{\partial G}{\partial v};\quad \frac{\partial}{\partial u}(B^2 G) = 0. \qquad \text{(xxxviii)}$$

The last of these expresses that B^2 is quadratic in u. Hence a_1, b_2, and c_2 are unchanged when the surface is deformed without stretching.

The second equation shows how to find the measure of curvature of a ruled surface in an excessively simple manner. By (iv)

$$a_1 = S\beta\frac{\partial \gamma}{\partial u} = S\gamma\alpha\frac{\partial \gamma}{\partial u},$$

so if α is the direction of a generator, and ν the normal vector at any point

$$G = S\alpha\nu^{-1}\frac{\partial \nu}{\partial u} = (-\mathrm{K}_1\mathrm{K}_2)^{\frac{1}{2}}. \qquad \text{(xxxix)}$$

Or if $\rho = \phi + t\alpha$ is the surface (compare Note X., (xxxiv), (xxxviii), and (xliii)),

$$G = \frac{S\alpha\alpha'\phi'}{V\alpha(\phi' + t\alpha')^2} = \frac{-p}{p^2 + (t - t_0)^2}, \qquad \text{(xl)}$$

where t_0 corresponds to the point on the line of striction.

(13.) We may also modify (xxxvii) of the Note just cited by replacing η, θ, s, and ϖ by α, $\alpha\alpha'$, v, and ρ, respectively, when we obtain

$$\mathrm{D}\rho = (p\alpha + (t - t_0))\,\alpha'\,\mathrm{d}v + \alpha\,(\mathrm{d}t - x\mathrm{d}v). \qquad \text{(xli)}$$

If we now take as a new variable

$$\mathrm{d}u = \mathrm{d}t - x\mathrm{d}v, \quad \text{or} \quad u = t - \int \mathrm{S}\phi'\alpha\,\mathrm{d}v, \qquad \text{(xlii)}$$

we find from (xli)

$$\mathrm{D}\rho = (p\alpha + (u - u_0))\alpha'\,\mathrm{d}v + \alpha\mathrm{d}u\,; \quad \mathrm{TD}\rho^2 = \mathrm{T}\,.\,(p\alpha + u - u_0)^2\,\alpha'^2\,\mathrm{d}v^2 + \mathrm{d}u^2. \qquad \text{(xliii)}$$

Thus for a ruled surface

$$B = \mathrm{T}\alpha'\mathrm{T}\,(p\alpha + u - u_0), \qquad \text{(xliv)}$$

and in particular for a developable $B = \mathrm{T}\alpha'\,(u - u_0)$. Obviously $\mathrm{T}\alpha'$ is the angular velocity of the generator if v is taken to be the time. It is also for the developable the curvature of the cuspidal edge if v is the arc of that edge.

For a developable γ does not vary with u, hence $a_1 = 0$ and all the scalars vanish except c_2 and a_2, the curvature and torsion of the cuspidal edge when v is its arc.

(14.) In the last section the change of variable is introduced artificially. To determine the orthogonal system directly for the surface

$$\rho = \phi + t\alpha, \quad \mathrm{T}\alpha = 1, \qquad \text{(xlv)}$$

assume $t = f(v) + u$ where f is some function of v. The direction of the tangent to the curve $\rho = \phi + \alpha f$ is $\mathrm{d}\rho = (\phi' + \alpha' f + \alpha f')\,\mathrm{d}v$ and this is at right angles to α if

$$\mathrm{S}\alpha\phi' - f' = 0 \quad \text{so} \quad f = \int \mathrm{S}\alpha\phi'\mathrm{d}v = \int \mathrm{S}\alpha\mathrm{d}\phi. \qquad \text{(xlvi)}$$

Thus the orthogonal system is found.

In like manner to determine the system of curves orthogonal to the system v *constant* on the surface

$$\rho = \phi\,(t, v), \qquad \text{(xlvii)}$$

assume $t = f(u, v)$ and we see that f must be a solution of the differential equation of the first order

$$\frac{\partial f}{\partial v} + \mathrm{S}\frac{\partial \phi}{\partial v}\left(\frac{\partial \phi}{\partial f}\right)^{-1} = 0. \qquad \text{(xlviii)}$$

XII.—SYSTEMS OF RAYS.

(1.) If the vector β is a given function of a variable vector α the equation

$$\rho = \beta + t\alpha \qquad \text{(i)}$$

represents a regulus, a congruency or a complex of lines according as one, two or three scalar variables are involved in the constitution of α, or in other words according as the vectors α when coinitial terminate upon a curve or upon a surface or are wholly arbitrary.

A regulus of lines composes a ruled surface. We shall not consider those surfaces here as they have been dealt with in another note.*

For a congruency the simplest form of equation (i) appears to be that in which the vectors α are of constant length. They may then be considered to involve two angular parameters, and the most general congruency can be represented by an equation of this kind.

(2.) In general we shall write

$$d\beta = \phi d\alpha = (\Phi + V\epsilon)\, d\alpha, \tag{ii}$$

ϕ being a linear vector function having Φ for its self-conjugate part and ϵ for its spin-vector. We shall also use the notation

$$d\omega = V\frac{d\alpha}{\alpha} = \frac{dU\alpha}{U\alpha}, \tag{iii}$$

so that $d\omega$ is the rotation which applied to the ray represented by α renders it parallel to the ray represented by $\alpha + d\alpha$.

(3.) The simplest mode from a kinematical point of view in which a ray of the complex can be displaced into the position of a neighbouring ray is to twist it about a certain screw. We shall however find it more convenient to suppose the displacement effected by a translation combined with a twist about a screw. If $d\tau$ is the translation, and p the pitch of the screw, the resultant translation is $d\tau + pd\omega$. Expressing that this translation applied to the ray (α) makes it intersect the ray ($\alpha + d\alpha$) we have the equation

$$\beta + t\alpha + d\tau + pd\omega = \beta + \phi d\alpha + (t + dx)(\alpha + d\alpha), \tag{iv}$$

where dx is some small scalar, and where t determines the point on the ray (α) brought to intersection with the ray ($\alpha + d\alpha$).

(4.) Now, we notice that $d\omega$ depends only on the component of $d\alpha$ at right angles to α. This suggests that we should consider separately the two components of $d\alpha$; so we write in general

$$d\alpha = Vd\alpha\alpha^{-1}.\alpha + Sd\alpha\alpha^{-1}.\alpha = d\omega.\alpha + dy.\alpha, \tag{v}$$

where dy is the scalar $Sd\alpha\alpha^{-1}$. Hence neglecting the small term of the second order $dxd\alpha$ the relation (iv) reduces to

$$d\tau + pd\omega = \phi(d\omega\alpha) + td\omega\alpha + dy\phi\alpha + (dx + tdy)\,\alpha. \tag{vi}$$

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* Note X., sections (12.) and (13.), page 408.

We are at liberty to select $d\tau$ in any way we please. The simplest selection is

$$d\tau = \alpha^{-1}V\alpha\phi\alpha dy, \tag{vii}$$

and then (vi) reduces to

$$(p + t\alpha)\,d\omega = \alpha^{-1}V\alpha\phi\,(d\omega\alpha), \tag{viii}$$

because

$$dx + (t + S\alpha^{-1}\phi\alpha)\,dy + S\alpha^{-1}\phi\,(d\omega\alpha) = 0,$$

by (vi) when $d\tau$ is at right angles to α (vii).

(5.) A slight knowledge of the properties of the cylindroid will now give us the key to an extensive view of the arrangement of the rays of a complex or congruency in the neighbourhood of a given ray. Equation (viii) may be regarded as determining a two-system of screws, for $d\omega$ ($= Vd\alpha\alpha^{-1}$) can be resolved into two components having fixed directions normal to α and dy has completely disappeared from this equation. We commence then by twisting the ray (α) about any screw of this system. The position it occupies after an infinitesimal twist is that of a ray of the congruency determined by the condition $dy = 0$ or $dT\alpha = 0$. In this way by twisting the initial ray about all the screws of the cylindroid we obtain the whole assemblage of rays of the congruency in the neighbourhood.

The ray through any point on the cylindroid near the axis may be constructed by drawing a perpendicular to the generator of the cylindroid through that point inclined to the axis at a small angle whose circular measure is the quotient of the intercept on the generator by the pitch appropriate to that generator.

Now two screws of the system have in general zero pitch. Any small twist on any screw of the system may be resolved into rotations about the axes of these screws of zero pitch. When the initial ray receives a small rotation about one of these axes its point of intersection with the second describes a small arc of a circle normal to $U\alpha$ and to the axis of rotation. A small rotation about the second axis will cause this point to deviate from the arc by a small distance of the second order of magnitudes. So to the first order all the rays of the congruency intersect two fixed lines both of which are at right angles to α and each intersects one axis of zero pitch and is at right angles to the other. In particular two rays intersect the initial ray. These have been rotated about one axis only. The axes of zero pitch are the focal lines and their points of intersection with the initial ray are the principal foci of that ray.

Again the point at which the initial ray is closest to a neighbour is called by Hamilton a *virtual focus*. We see that the closest points on the neighbours lie on the cylindroid generated by the shortest distances. Hence as the cylindroid is contained between two planes normal to the initial ray the virtual foci are limited to a certain range on that ray.*

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* *Trans.* Roy. Ir. Acad. vol. xvi. p. 52.

(6.) We now turn to equation (vii) which shows that all the rays of a complex close to a given ray may be constructed by successively displacing the rays of a certain congruency by translations in a fixed direction normal to the initial ray and of varying but small amounts. Inasmuch as the rays of the congruency intersect two elements of right lines, the rays of the complex pass through two small parallelograms situated in parallel planes each having a pair of sides equal and parallel to the translation. All the rays are parallel which intersect a line in the plane of one parallelogram parallel to one of these sides.

(7.) To verify the conclusions of section (5.), and to calculate the positions of the various lines, we re-write equation (viii) in the form

$$(\phi + t) \,.\, d\omega\alpha = p d\omega + \alpha dz \qquad \text{(ix)}$$

whence

$$p = S\frac{\phi \,.\, d\omega\alpha}{d\omega}; \quad t = -S\frac{\phi \,.\, d\omega\alpha}{d\omega\alpha}; \quad dz = S\frac{\phi \,.\, d\omega\alpha}{\alpha}. \qquad \text{(x)}$$

At a principal focus two rays intersect and p is zero; therefore

$$d\omega\alpha = (\phi + t)^{-1}\alpha dz. \qquad \text{(xi)}$$

Operating on this equation by $S\alpha$ we find

$$S\alpha(\phi + t)^{-1}\alpha = 0, \qquad \text{(xii)}$$

which is equivalent to a quadratic in t whose roots determine the foci.

The extreme points are given by the analogous equation*

$$S\alpha(\Phi + t)^{-1}\alpha = 0 \qquad \text{(xiii)}$$

in which the self-conjugate part Φ replaces ϕ. For we see by (x) that t is the inverse square of a radius of the conic†

$$S\rho\Phi\rho = 1, \quad S\alpha\rho = 0, \qquad \text{(xiv)}$$

and its greatest and least values are the inverse squares of the axes of the conic.

If the line $\rho = \beta + s\alpha + y\tau$ (y variable) meets all the neighbouring rays

$$S(\phi \,.\, d\omega\alpha - s\alpha)(\alpha + d\omega\alpha)\tau = 0$$

or

$$Sd\omega\alpha(\phi' + s)V\tau\alpha = 0 \qquad \text{(xv)}$$

when terms of the second order are rejected. If this is true for all vectors $d\omega\alpha$ that is for all vectors at right angles to α, it is equivalent to

$$(\phi' + s)V\tau\alpha \parallel \alpha \quad \text{or} \quad V\tau\alpha \parallel (\phi' + s)^{-1}\alpha. \qquad \text{(xvi)}$$

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* This equation and the last are given by Molenbroek, *Anwendung der Quaternionen auf die Geometrie*, pp. 236–238.

† Compare p. 253.

Operating on this by $S\alpha$ we see that $s = s_1$ one of the roots of (xii) and that the line passes through a principal focus. If $\tau = \alpha^{-1}(\phi' + s_1)^{-1}\alpha$ the conditions are satisfied and it is very easy to show that $\alpha^{-1}(\phi + s_2)^{-1}\alpha$, which (xi) is the axis of the second screw of zero pitch, is at right angles to τ.

(8.) We shall now invert the functions in (xii) and (xiii) and exhibit the relations connecting the roots of the two equations.

If for a moment we replace α by $V\lambda\mu$ and $\phi + t$ by ϕ_t by Hamilton's fundamental theorem of inversion (xii) is equivalent to

$$SV\lambda\mu V\phi_t'\lambda\phi_t'\mu = SV\lambda\mu V(\Phi_t - V\epsilon)\lambda(\Phi_t - V\epsilon)\mu = 0. \qquad \text{(xvii)}$$

A slight expansion shows that the part linear in ϵ disappears since Φ_t is self-conjugate (ii), while the part quadratic in ϵ is $-(S\epsilon V\lambda\mu)^2$.

(9.) We have therefore two forms for (xii)

$$t^2\alpha^2 + 2tS\alpha(m'' - \phi)\alpha + S\alpha\psi\alpha = 0;\ t^2\alpha^2 + 2tS\alpha(m'' - \Phi)\alpha + S\alpha\Psi\alpha - (S\epsilon\alpha)^2 = 0; \quad \text{(xviii)}$$

and similarly two forms for (xiii)

$$t^2\alpha^2 + 2tS\alpha(m'' - \Phi)\alpha + S\alpha\Psi\alpha = 0;\ t^2\alpha^2 + 2tS\alpha(m'' - \phi)\alpha + S\alpha\psi\alpha + (S\epsilon\alpha)^2 = 0, \quad \text{(xix)}$$

remembering that by Hamilton's formula of inversion

$$\psi_t = m_t\phi_t^{-1} = \psi + t(m'' - \phi) + t^2$$

and that the first invariants (m'') of a function (ϕ) and of its self-conjugate part (Φ) are the same.

Hence, if s_1 and s_2 are the roots of (xii) or (xviii), and t_1 and t_2 those of (xiii) or (xix)

$$2t_0 = s_1 + s_2 = t_1 + t_2 = SU\alpha(m'' - \Phi)U\alpha = SU\alpha(m'' - \phi)U\alpha \qquad \text{(xx)}$$

and

$$s_1s_2 = t_1t_2 + (S\epsilon U\alpha)^2 = -SU\alpha\psi U\alpha;\ t_1t_2 = s_1s_2 - (S\epsilon U\alpha)^2 = -SU\alpha\Psi U\alpha. \quad \text{(xxi)}$$

We may now write for the focal points and for the extreme points in accordance with (xx)

$$s_1 = t_0 - f,\ s_2 = t_0 + f;\quad t_1 = t_0 - e,\ t_2 = t_0 + e. \qquad \text{(xxii)}$$

Thus the four points are symmetrically situated with respect to the *central* point.

Again (xxi) affords the relation

$$e^2 - f^2 = (S\epsilon U\alpha)^2, \qquad \text{(xxiii)}$$

which shows that the focal points are real if $e^2 > (S\epsilon U\alpha)^2$. The extreme points are always real for t_1 and t_2 are the inverse squares of the axes of a conic (xiv) and these are real whether the conic is real or not. The reality of these points is also a geometrical consequence of section (5.).*

* The symmetrical arrangement of the four points, principal and extreme virtual foci, with respect to the central point is the only element in the arrangement of the rays which cannot be deduced from the properties of the cylindroid. This arrangement depends upon the distribution of pitch.

(10.) The directions i and j of the principal axes of the conic (xiv) afford a first natural system of lines of reference coupled with $U\alpha = k$. As we are now dealing only with a congruency we may suppose $T\alpha = 1$ without loss of generality and we may regard

$$\alpha = U\alpha = k$$

unless the contrary is stated.

These vectors obey the laws (compare again (xiv))

$$t_1 = Si\phi i; \quad t_2 = Sj\phi j; \quad Si\Phi j = 0; \quad Si\phi j = -Sj\phi i = -S\epsilon k. \qquad \text{(xxiv)}$$

If we introduce an angle u so that

$$Ud\omega = i \cos u + j \sin u \qquad \text{(xxv)}$$

and a new scalar w the relation (ix) becomes

$$(\phi + t)(i \sin u - j \cos u) = p(i \cos u + j \sin u) + wk. \qquad \text{(xxvi)}$$

Solving this for p and t by operating by $SUd\omega$ and $SUd\omega k$ we find on reference to (xxiv) and (xxii)

$$p = -S\epsilon k + (t_2 - t_1) \sin u \cos u = p_0 + e \sin 2u \qquad \text{(xxvii)}$$

and

$$t = t_1 \sin^2 u + t_2 \cos^2 u = t_0 + e \cos 2u \qquad \text{(xxviii)}$$

where $p_0 = -S\epsilon k$ is the pitch corresponding to the extreme points $u = 0, \frac{\pi}{2}$.*

The focal points are given by

$$p_0 + e \sin 2u = 0 \qquad \text{(xxix)}$$

and are real if $e^2 > p_0^2$.

Again eliminating u from (xxvii) and (xxviii) we find t and p connected by

$$(t - t_0)^2 + (p - p_0)^2 = e^2 \text{ or } (t - t_1)(t - t_2) + (p - p_0)^2 = 0, \qquad \text{(xxx)}$$

which includes as a particular case

$$f^2 + p_0^2 = e^2.$$

(11.) A second natural system of lines of reference is formed by $U\alpha$ and the bisectors of the angles between i and j. Now the angle $u = 0$ at the point t_2 and $u = \frac{\pi}{2}$ at t_1 so if we take $v = u - \frac{\pi}{2}$ so that t_2 and t_1 correspond to $v = -\frac{\pi}{4}$ and $v = +\frac{\pi}{4}$, we have

$$p = p_0 + e \cos 2v; \quad t = t_0 - e \sin 2v. \qquad \text{(xxxi)}$$

If we take these three lines as Cartesian axes and put

$$z = (t - t_0) \quad \text{and} \quad \tan v = yx^{-1}$$

the equation of the cylindroid follows from (xxxi) in its canonical form

$$z(x^2 + y^2) = -2exy. \qquad \text{(xxxii)}$$

* Compare Hamilton, "Supplement to an Essay on the Theory of Systems of Rays," *Trans. R.I.A.* vol. xvi. p. 54, where (xxviii) is obtained in the form

$$r = r_1 (\cos w)^2 + r_2 (\sin w)^2.$$

(12.) There is yet a third kind of focus which Hamilton calls* a "focus by projection." The vector drawn from the point t on the initial ray perpendicular to it and terminating on a neighbouring ray is

$$d\varpi = \alpha V\alpha^{-1}(\phi + t) \, . \, d\omega\alpha \qquad \text{(xxxiii)}$$

as may be verified without any trouble. If the perpendicular $d\varpi'$ at the point t' terminating on the same near ray is at right angles to this the projection of the ray on the plane of α and $d\varpi$ cuts the initial ray at the point t'. This point is a focus by projection.

(13.) To investigate the properties of this new class of foci we shall use the first natural system of lines of reference, the vectors i, j, k of section (10.). We shall also replace α by $U\alpha$ or by k, and we shall write

$$d\varpi = (i \cos w + j \sin w) \, Td\varpi, \qquad \text{(xxxiv)}$$

retaining the previous notation (xxv) for $Ud\omega$.

The angle w is the angle between the plane upon which the projection is made (or briefly the plane of projection) and a plane of extreme virtual foci.

If then

$$PTd\omega = Td\varpi \qquad \text{(xxxv)}$$

equation (xxxiii) becomes

$$P(i \cos w + j \sin w) = kVk(\phi + t)(j \cos u - i \sin u). \qquad \text{(xxxvi)}$$

Remembering the laws of the units (xxiv) and that $p_0 = Si\phi j = -Sj\phi i$, we find

$$P \cos w = p_0 \cos u + (t - t_1) \sin u = p_0 \cos u + (e + g) \sin u,$$
$$P \sin w = p_0 \sin u + (t_2 - t) \cos u = p_0 \sin u + (e - g) \cos u, \qquad \text{(xxxvii)}$$

the symbol e being given by (xxii), and the new symbol g being equal $t - t_0$.

If $g' = g + h$ determines the focus by projection, namely the point at which w has increased by a right angle while u remains constant,

$$\tan u = \frac{p_0 \tan w - (e - g)}{p_0 - (e + g) \tan w} = -\frac{p_0 + (e - g - h) \tan w}{p_0 \tan w + (e + g + h)}, \qquad \text{(xxxviii)}$$

and solving for h in terms of w, we obtain the equivalent of Hamilton's remarkable formula† containing the law of the focus by projection (compare (xxx.))

$$\frac{1}{h} = \frac{(e - g) \cos^2 w - (e + g) \sin^2 w}{g^2 - f^2}. \qquad \text{(xxxix)}$$

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* Hamilton, *Trans.* R.I.A. vol. xvi. p. 47.

† Hamilton's equation printed on p. 50, *loc. cit.*, is

$$\frac{1}{p} = \frac{1}{p_1}(\cos . \Pi)^2 + \frac{1}{p_2}(\sin . \Pi)^2.$$

From this we see that the foci by projection are excluded from a finite portion of the line contained within the extreme points of projection determined by

$$h_1 = -\frac{g^2 - f^2}{e + g},\ h_2 = \frac{g^2 - f^2}{e - g} \text{ or by } g_1 = \frac{ge + f^2}{e + g},\ g_2 = \frac{eg - f^2}{e - g}; \qquad \text{(xl)}$$

while

$$h_0 = g\frac{g^2 - f^2}{e^2 - g^2} \text{ or } g_0 = g\frac{e^2 - f^2}{e^2 - g^2} \text{ and } h_2 - h_1 = 2e\frac{g^2 - f^2}{e^2 - g^2} = g_2 - g_1 \qquad \text{(xli)}$$

give the central point of the excluded portion and its length. The planes of extreme projection are parallel to the planes of the extreme virtual foci no matter where the point g may be.

We cannot delay to consider the cubic ruled surface

$$\frac{x^2 + y^2}{z} = \frac{x^2}{h_1} + \frac{y^2}{h_2} \qquad \text{(xlii)}$$

generated by the perpendiculars to the initial ray in the planes of projection and through the corresponding foci (compare (xxxix)) except to state that the initial ray is a double line; that the surface consists of two sheets wholly exterior to the planes of extreme projection; and that it may be derived from a cylindroid by drawing lines parallel to the generators of the cylindroid from points on the axis whose distances from a fixed point (also on the axis) are inversely proportional to the corresponding distances of the generators of the cylindroid from the same point. Nor can we consider the scalars P associated with the generators of this surface corresponding to the pitch (p) associated with each generator of a cylindroid.

(14.) In order to study more closely the arrangement of the rays near a given point (t) on the initial ray, we shall show how to find a function of the variable vector α so that the surface

$$\rho = \beta + \alpha f(\alpha) \qquad \text{(xliii)}$$

may pass through the given point and that its *element* at the point may be normal to all the contiguous rays. Differentiating (xliii)

$$d\rho = \phi\,.\,d\omega\alpha + d\omega\alpha f + \alpha df \qquad \text{(xliv)}$$

and we see that the condition is satisfied neglecting the second order of small quantities if when $\alpha = k$,

$$f(k) = t \quad \text{and} \quad (df) = Sk\phi\,.\,d\omega k \qquad \text{(xlv)}$$

where (df) denotes that k has been substituted for α in the differential of $f(\alpha)$.

We shall now find the principal radii of curvature of a surface satisfying this condition at the point under consideration. Using the formula $d\rho + RdU\nu = 0$, we obtain at once

$$(\phi + t + R)d\omega k + k(df) = 0 \qquad \text{(xlvi)}$$

so that (compare (xii)) R is given by the quadratic

$$S k(\phi + t + R)^{-1} k = 0. \qquad \text{(xlvii)}$$

Hence the centres of curvature coincide with the principal foci and in the notations of (xxiv) and (xxxvii) the quadratic determining the radii of curvature is

$$(R + t - s_1)(R + t - s_2) = 0, \quad \text{or} \quad (R + g + f)(R + g - f) = 0. \qquad \text{(xlviii)}$$

The measure of curvature of the orthogonal element of surface is the density* of the congruency being the ratio which the area traced on a unit sphere by the rays through a small normal circuit bears to the area of the circuit. This is equal to the inverse of the product of the distances of the point from the foci. We may also speak of the sum of the curvatures of the orthogonal element as the concentration of the congruency.†

(15.) It is not possible in general to draw a surface through an arbitrary point orthogonal to all the lines of the congruency. The condition (xlv) is equivalent to

$$\mathrm{d}f = S a \mathrm{d}\beta, \qquad \text{(xlix)}$$

and if this holds continuously over a surface and not merely at a point we can differentiate again and write

$$\mathrm{d}'\mathrm{d}f = S a \mathrm{d}'\mathrm{d}\beta + S \mathrm{d}'a \mathrm{d}\beta = \mathrm{d}\mathrm{d}'f = S a \mathrm{d}\mathrm{d}'\beta + S \mathrm{d}a \mathrm{d}'\beta$$

provided the differentiations are independent.

Hence the condition is

$$S \mathrm{d}'a \mathrm{d}\beta = S \mathrm{d}a \mathrm{d}'\beta \quad \text{or} \quad S \mathrm{d}'a \phi \mathrm{d}a = S \mathrm{d}a \phi \mathrm{d}'a$$

or again (compare (ii))

$$S \epsilon \mathrm{d}a \mathrm{d}'a = 0 \quad \text{or} \quad S \epsilon a = 0 \qquad \text{(l)}$$

because $V \mathrm{d}a \mathrm{d}'a$ is parallel to a. Referring back to sections (9.) and (10.) we see that in this case the focal and the extreme points coincide and that $p_0 = 0$ and $e = f$.

Also an infinite number of surfaces can be drawn orthogonal to the rays because an arbitrary constant may be added to $f(a)$. For rays of light these are the wave-surfaces when the medium is isotropic.

(16.) From any congruency it is possible to select a singly infinite system of rays on which the focal and extreme points coincide. The system may be defined by the equations

$$\rho = \beta + t a, \quad S \epsilon a = 0 \qquad \text{(li)}$$

but the second equation is not an identity as in the last section. These rays have certain other peculiarities especially in connexion with the foci by projection (compare (xli)).

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* Hamilton *loc. cit.* used the word *condensation* in a similar sense.

† Royal Irish Academy *Transactions*, p. 377, vol. xxxi., 1900.

(17.) We do not determine a singly infinite system by equating to zero the discriminant of (xiii) or (xix) and thus expressing that the two extreme points coincide. For if we consider the mode in which this equation was arrived at, we see that for equal roots α must be normal to a cyclic plane of a certain quadric. Two conditions must therefore be satisfied and only a limited number of rays can possess the property in question.

Nor can the principal foci coincide except under special conditions. For the two axes of zero pitch on the cylindroid would then intersect. From this it follows that the cylindroid must reduce to a plane, and the extreme points must likewise coincide.*

(18.) Important surfaces connected with the congruency are the focal surface, the locus of the extreme points and the locus of the centres; of this last the equation is

$$\rho = \beta - \tfrac{1}{2}\alpha S\alpha^{-1}(m'' - \phi)\alpha. \qquad \text{(lii)}$$

We may moreover write down the differential equations of families of ruled surfaces composed of rays. For instance (compare (x))

$$p = S\frac{\phi d\alpha}{\alpha d\alpha} = const., \quad S\alpha d\alpha = 0 \qquad \text{(liii)}$$

lead to a relation in α which coupled with $\rho = \beta + t\alpha$ determines a family of ruled surfaces for which the parameter of distribution, or the pitch p, is constant. In particular for $p = 0$ we have the developables of the congruency. Geometrically, selecting any ray we can choose one of the rays into which it can be screwed with pitch p and from that another and so on and thus construct a surface included in the integral of (liii).

(19.) There is another and very useful method for the treatment of systems of rays. If ρ_1 and ρ_2 are the vectors to any two points, and if

$$\sigma = V\rho_2\rho_1 \quad \text{and} \quad \tau = \rho_1 - \rho_2 \qquad \text{(liv)}$$

the vectors σ, τ or any equimultiples determine the line through the two points. Its equation is

$$\rho = \sigma\tau^{-1} + x\tau, \qquad \text{(lv)}$$

and the ratio of the tensors only is important. The constituents of these vectors are equivalent to Plücker's six coordinates of a line. Thus given any pair of vectors σ and τ satisfying

$$S\sigma\tau = 0 \qquad \text{(lvi)}$$

a definite line is determinate.

(20.) A scalar relation between σ and τ, homogeneous in the tensors, may be regarded as the equation of a complex; one restriction is imposed on the generality of the lines. Two scalar equations of this kind represent a congruency, three a regulus of lines constituting a ruled surface, and four a finite number of lines.

* Compare Sir Robert Ball, *Theory of Screws*, Chap. II., Cambridge 1900.

(21.) Again a line may be determined by means of two planes intersecting in the line. If these are

$$S\lambda_1\rho + 1 = 0, \qquad S\lambda_2\rho + 1 = 0, \qquad \text{(lvii)}$$

it is evident or may at once be verified that λ_1, λ_2 are connected with σ, τ by the equations

$$\tau = xV\lambda_2\lambda_1, \qquad \sigma = y(\lambda_1 - \lambda_2).$$

Also $$\sigma = V\rho_1\tau = xV\rho_1 V\lambda_2\lambda_1 = -x(\lambda_1 - \lambda_2) \text{ by (lvii)}$$

so $$\rho_1 - \rho_2 = \tau = xV\lambda_2\lambda_1, \qquad V\rho_2\rho_1 = \sigma = -x(\lambda_1 - \lambda_2) \qquad \text{(lviii)}$$

and therefore any function homogeneous in the tensors may be exhibited in three forms

$$f(\sigma, \tau) = 0; \quad f(V\rho_2\rho_1, \rho_1 - \rho_2) = 0; \quad f(\lambda_1 - \lambda_2, -V\lambda_2\lambda_1) = 0. \qquad \text{(lix)}$$

The third equation may also be regarded as that of the reciprocal complex formed by reciprocation with respect to the unit sphere $\rho^2 + 1 = 0$.

It is important to observe that change of origin is without effect on τ, but alters σ into $\sigma + V\epsilon\tau$.

(22.) The general linear and scalar relation

$$S\gamma\sigma + S\delta\tau = 0 \qquad \text{(lx)}$$

reduces on change of origin to

$$S\gamma(\sigma + p\tau) = 0, \qquad \text{(lxi)}$$

if $$p\gamma = V\gamma\epsilon + \delta \quad \text{or} \quad \epsilon = V\delta\gamma^{-1} + x\gamma, \quad p = S\delta\gamma^{-1}. \qquad \text{(lxii)}$$

The equation (lx) represents the general linear complex; (lxi) is the reduced form of this equation when the origin is taken on the central axis determined by the second equation (lxii) and p is the parameter of the complex.

(23.) If ρ is the vector to any point on a ray through a given point, the extremity of α, $\sigma = V\alpha\tau = V\alpha\rho$, and by (lxi) the lines lie in the plane

$$S\gamma(\alpha + p)(\rho - \alpha) = 0. \qquad \text{(lxiii)}$$

Identifying this equation with $S\lambda\rho + 1 = 0$ we see that

$$\gamma(\alpha + p) + (p\lambda - 1)S\gamma\alpha = 0 \quad \text{whence} \quad \gamma^2 + S\gamma\lambda S\gamma\alpha = 0,$$

or more symmetrically the equations

$$\frac{p\lambda - 1}{\alpha + p} = -\frac{\gamma}{S\gamma\alpha} = \frac{S\gamma\lambda}{\gamma} \qquad \text{(lxiv)}$$

give λ without ambiguity in terms of α and α in terms of λ so that the lines in a plane also pass through a point.

These equations lead to an important transformation. The equations

$$f(a) = 0, \; f\frac{V\gamma(p\lambda - 1)}{S\gamma\lambda} = 0 \qquad \text{(lxv)}$$

represent respectively the locus of a point and the tangential equation of the transformed locus. For instance a line transforms into the intersection of the planes corresponding to two points on it; a surface of degree n transforms into a surface of class n.

(24.) If P is the shortest distance between the central axis and a ray, on replacing σ by $V(x\gamma + PUV\gamma\tau)\tau$ in the equation of the complex (lxi) we find

$$P = -p\frac{S\gamma\tau}{TV\gamma\tau} = p \tan l \qquad \text{(lxvi)}$$

if l is the angle the ray makes with the plane normal to the axis. The rays therefore envelope helices coaxial with the complex and having the tangent of their inclination directly proportional to the radius (P) of the containing cylinder.

(25.) The theory of screws affords a vivid illustration of the arrangement of the rays of a linear complex. If a body is attached to a nut fitting a screw of pitch p and axis γ on which the origin of vectors is situated, the point in the body at the extremity of the vector ρ can only move in the direction of the vector

$$p\gamma + V\gamma\rho. \qquad \text{(lxvii)}$$

Applying a force τ to this point no motion is produced if τ is at right angles to this direction or if (compare (lxi))

$$pS\gamma\tau + S\gamma\sigma = 0, \quad \text{where} \quad \sigma = V\rho\tau. \qquad \text{(lxviii)}$$

Again, any point of the body is free to describe a helix whose tangent of inclination is *inversely* proportional to the radius of its cylinder. The direction of any force whose line of action touches this cylinder, and which does not disturb equilibrium must be at right angles to the helix of motion through its point of contact. The tangent of inclination of the force is consequently *directly* proportional to the radius of the cylinder.

We see thus that the linear complex is a very particular case of the general relation*

$$S(\sigma_1\omega + \sigma\omega_1) = 0, \qquad \text{(lxix)}$$

which expresses that the screw (σ, ω) is reciprocal to the screw (σ_1, ω_1) when we do not suppose $S\sigma\omega$ to be zero. This being so and as linear systems of screws are

* Compare the Note on Screws, section (3.), p. 390.

discussed in the Note on Screws, we shall not here consider systems of linear complexes. Moreover in the following sections we shall consider the general complex as a particular case of the general system of screws satisfying a single condition.

(26.) The equation of such a system is of the form

$$f(\sigma, \omega) = 0 \qquad \text{(lxx)}$$

homogeneous in the vectors (σ, ω) or in other words independent of the absolute magnitude of their tensors. If we write the differential as

$$df(\sigma, \omega) = S(\omega_1 d\sigma + \sigma_1 d\omega) \qquad \text{(lxxi)}$$

we may replace the equation of the system by

$$S(\omega_1\sigma + \sigma_1\omega) = 0 \qquad \text{(lxxii)}$$

because the function f is homogeneous.

(27.) In the language of the theory of screws* we may say that the screw (σ_1, ω_1) is reciprocal to (σ, ω). Moreover by (lxxi), σ_1 and ω_1 are determinate functions of σ and ω, or

$$\sigma_1 = \theta(\sigma, \omega), \quad \omega_1 = \chi(\sigma, \omega). \qquad \text{(lxxiii)}$$

Thus we may regard (lxxiii) as establishing a correspondence between a pair of screws (σ_1, ω_1) and (σ, ω), and (lxx) or (lxxii) as representing the assemblage of screws reciprocal to their correspondents.

Further (lxxiii) implies relations

$$\sigma = \theta_1(\sigma_1, \omega_1), \quad \omega = \chi_1(\sigma_1, \omega_1) \qquad \text{(lxxiv)}$$

and the first and third of the equations

$$f(\sigma, \omega) = 0, \quad S(\omega_1\sigma + \sigma_1\omega) = 0, \quad f_1(\sigma_1, \omega_1) = 0 \qquad \text{(lxxv)}$$

represent the assemblage of screws and the assemblage of their reciprocal correspondents while the condition of reciprocity is expressed by the second. Or again the second equation may be regarded as determining either of the assemblages having regard to (lxxiii) or (lxxiv).

(28.) If p is the pitch and α the vector to a point on the axis of the screw (σ, ω),

$$\sigma = (p + V\alpha)\,\omega. \qquad \text{(lxxvi)}$$

Substitution in (lxx) affords the equation

$$f((p + V\alpha)\,\omega, \omega) = 0 \qquad \text{(lxxvii)}$$

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* Compare again the Note on Screws, section (3.); and for the correspondence (lxxiii) compare the particular case of linear correspondence of section (12.) of the note cited, p. 394.

which admits of the following interpretations:—

I. Given p it is the equation of the complex of axes of screws of given pitch belonging to the assemblage.

II. It represents a singly infinite system of complexes depending on the parameter p.

III. It represents the cone of axes of screws of given pitch p which pass through a given point (α).

IV. It is equivalent to a scalar equation determining the pitches of the screws of the assemblage whose axes coincide with a given line ($V\alpha\omega$ and ω given).

V. By (lix) if we suppose

$$V\alpha\omega = \lambda_1 - \lambda_2 = \mu, \quad \omega = -V\lambda_2\lambda_1 = V\mu\lambda_1 \qquad \text{(lxxviii)}$$

we see that the rays of the complex (p given) which lie in the plane $S\lambda_1\rho + 1 = 0$ envelope the curve in which the plane cuts the envelope of the variable plane $S\mu\rho = 0$ where

$$f(V\mu(p\lambda_1 + 1), V\mu\lambda_1) = 0. \qquad \text{(lxxix)}$$

Evidently the order of the cone III., the degree of the equation IV. and the class of the curve V. are all equal to the order in which the vectors (σ, ω) jointly occur in (lxx).

(29.) If the cone III. (lxxvii) has a double edge (ω) the differential vanishes no matter what vector $d\omega$ may be; so in the notation of (lxxi),

$$S(\omega_1(p + V\alpha)d\omega + \sigma_1 d\omega) = 0 \quad \text{or} \quad S(\sigma_1 - (-p + V\alpha)\omega_1)d\omega = 0. \qquad \text{(lxxx)}$$

Hence as $d\omega$ is quite arbitrary

$$\sigma_1 = (-p + V\alpha)\omega_1. \qquad \text{(lxxxi)}$$

Comparing (lxxvi) we see that in this case the axes of the reciprocal correspondents (σ_1, ω_1), (σ, ω) intersect and their pitches are equal and opposite. These two consequences are of course not independent; the latter implies the former. The symmetry of these relations shows that the locus of vertices of cones with double edges which are composed of axes of screws of pitch p, is likewise the locus of vertices of nodal cones composed of axes of the reciprocal correspondents of pitch $-p$.

The locus of the vertices of the nodal cones of a complex is the Kummer surface. Consequently the Kummer surfaces of the two complexes just described are identical.

(30.) The double edges of the cones of the complex form a congruency specified by the three equations

$$p = S\sigma\omega^{-1}; \quad S(\sigma\omega_1 + \omega\sigma_1) = 0; \quad p = -S\sigma_1\omega_1^{-1} \qquad \text{(lxxxii)}$$

If n is the order of the original equation in σ and ω, $n-1$ is the order of σ_1 and ω_1

in the same vectors and $2(n-1)$ is that of $p\omega_1^2 = -S\sigma_1\omega_1$. The first and third equations (lxxxii) determine a complex of order $2(n-1)$, and the rays common to this and to the complex of order n determined by the first and second (lxxxii) compose a congruency whose order and class are both equal to $2(n-1)n$. For the order is the number of rays through a point or the number of common edges of two cones of degree n and $2(n-1)$; and the class is the number of rays in a plane, or the number of common tangents of two curves of class n and $2(n-1)$.

The congruency is likewise specified by the vector equations in which α is the vector to a point on the Kummer surface

$$\sigma = (p + V\alpha)\omega; \quad \sigma_1 = (-p + V\alpha)\omega_1, \qquad \text{(lxxxiii)}$$

it being understood that σ_1 and ω_1 are given functions of σ and ω.

(31.) It is easy to see that the rays of the congruency touch the Kummer surface of the complex and from this property it will follow that the Kummer surface is part at least of the focal surface of the congruency.

Using the equations (lxxxiii) we have

$$d\sigma = (p + V\alpha)d\omega + Vd\alpha\omega; \quad d\sigma_1 = (-p + V\alpha)d\omega_1 + Vd\alpha\omega_1 \qquad \text{(lxxxiv)}$$

for the consecutive screws of pitches $\pm p$ whose axes intersect at a consecutive point $(\alpha + d\alpha)$ on the Kummer surface. Operating on the first by $S\omega_1$ and attending to (lxxxiii) and (lxxi) we find

$$df(\sigma, \omega) = S(\omega_1 d\sigma + \sigma_1 d\omega) = Sd\alpha\omega\omega_1 = 0 \qquad \text{(lxxxv)}$$

because the screw $(\sigma + d\sigma, \omega + d\omega)$ belongs to the assemblage $f(\sigma, \omega) = 0$. In like manner exactly the same equation is found by operating on the second by $S\omega$. Hence $V\omega\omega_1$ is at right angles to all tangential vectors $(d\alpha)$ to the surface and in particular the axes of the screws (ω and ω_1) touch the surface.

Now if the lines of a congruency touch a surface that surface is part at least of the focal surface. For take any ray touching the surface at α and having the direction $\delta\alpha$. The consecutive ray touching the surface at $\alpha + \delta\alpha$ intersects this ray and the point of intersection is a principal focus on both. The surface therefore is part of the focal surface. If $d\alpha$ is the conjugate direction to $\delta\alpha$, the second ray which intersects $\rho = \alpha + t\delta\alpha$ touches the surface at $\alpha + d\alpha$ and the point of intersection of these two rays lies on the other part of the focal surface. In fact if two rays intersect at ρ,

$$d\alpha + dt\delta\alpha + td\delta\alpha = 0 \quad \text{whence} \quad Sd\alpha\delta\alpha d\delta\alpha = 0.$$

XIII.—ON THE OPERATOR ∇.

(1.) If $f\rho$ is any scalar function of a vector ρ, corresponding differentials are connected by a relation of the form

$$df\rho = -S\nu d\rho, \qquad \text{(i)}$$

in which ν is a vector derived from $f\rho$ depending merely on the function f and on the value of the variable vector ρ but not at all on the differential $d\rho$.

Regarding ρ as a vector of position, the rate of change of the function of position $f\rho$ along any direction $Ud\rho$ is evidently $-S\nu Ud\rho$, in other words it is equal to the projection of the vector ν upon that direction. This rate of change is greatest along the direction $U\nu$ being then equal to $T\nu$. In any other direction it is equal to $T\nu$ multiplied by the cosine of the angle between the assumed direction and that of ν.

(2.) It is convenient to use a special notation to suggest the dependence of the vector ν on the scalar function $f\rho$. For this purpose Hamilton* introduced the symbol Nabla or ∇ and connected ν with $f\rho$ by the symbolical equation

$$\nu = \nabla f\rho, \qquad \text{(ii)}$$

in which ν is conceived to be the result of a certain operation performed on $f\rho$.

(3.) We shall now illustrate by a few examples the effect of operating by ∇ on scalar functions. It must be observed however that these are merely translations into the new notation of results already obtained in the course of this work Thus:—

(*a*) $\nabla S\lambda\rho = -\lambda$ because $dS\lambda\rho = S\lambda d\rho$.

(*b*) $\nabla\rho^2 = -2\rho$,, $d\rho^2 = 2S\rho d\rho$.

(*c*) $\nabla S\rho\Phi\rho = -2\Phi\rho$,, $dS\rho\Phi\rho = 2S\Phi\rho d\rho$.

(*d*) $\nabla T\rho = +U\rho$,, $dT\rho = -SU\rho d\rho$.

(*e*) $\nabla TV\lambda\rho = +UV\lambda\rho\,.\,\lambda$,, $dTV\lambda\rho = -SUV\lambda\rho\, V\lambda d\rho$.

(*f*) $\nabla fT(\rho-\lambda) = U(\rho-\lambda)f'T(\rho-\lambda)$,, $dfT(\rho-\lambda) = -f'T(\rho-\lambda)SU(\rho-\lambda)d\rho$.

(*g*) $\nabla T(\rho-\lambda)^{-1} = -U(\rho-\lambda)\,T(\rho-\lambda)^{-2}$,, $dT(\rho-\lambda)^{-1} = -T(\rho-\lambda)^{-2}dT(\rho-\lambda)$.

(*h*) $\nabla T\phi\rho = \phi'U\phi\rho$,, $dT\phi\rho = -S\phi'U\phi\rho\,.\,d\rho$.

All these expressions are consequences of the equations

$$\nabla f\rho = \nu, \qquad df\rho = -S\nu d\rho,$$

which may be regarded (compare (i) and (ii)) as a definition of ∇, the vector λ and the vector functions Φ and ϕ being supposed constant in the examples.

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* *Proceedings* Royal Irish Academy, vol. iii., p. 291. See note, p. 548, vol. i.

Example (g) is of fundamental importance in the theory of attractions for it shows that $\nabla T(\rho-\lambda)^{-1}$ represents in magnitude and direction the attraction at the extremity of ρ due to a unit mass at the extremity of λ.

(4.) Again if f and g are any two *scalar* functions of ρ

$$\nabla(f+g)=\nabla f+\nabla g \quad \text{and} \quad \nabla(fg)=g\nabla f+f\nabla g \qquad \text{(iii)}$$

because

$$d(f+g)=df+dg \quad \text{and} \quad d(fg)=g\,df+f\,dg.$$

Generally as a matter of convenience it is desirable if possible to place the operand immediately to the right of the operator ∇. This can be effected in the second equation (iii) because f and g are scalars and therefore commutative with vectors and quaternions. We shall soon see however that we can assign a definite meaning to the result of operating on a vector or quaternion by ∇. But since we must regard ∇ as a symbolical vector or at least as possessing certain characteristics of a vector (for by definition it produces a vector from a scalar), we are not at liberty to write $p\nabla q=\nabla q\,.\,p$ when p is a quaternion, nor *à fortiori* when p and q are both quaternions. Hence it is not in general possible to place the operand immediately to the right. We are therefore obliged to have recourse to brackets or accents or some temporary mark in order to distinguish the operand. For instance we may write $\nabla(f)g$ to denote that f is excluded from the operation of ∇; or we may accent ∇ and g and $\nabla' fg'$ will then sufficiently indicate that g and not f is the subject of operation.

(5.) Consider in the next place a scalar function of several independent variable vectors ρ, ρ_1, ρ_2, &c. We may in an obvious notation write (compare V, p. 294),

$$\left.\begin{aligned} d\,.\,F(\rho,\rho_1,\rho_2\ldots) &= -\;Sd\rho\nu - Sd\rho_1\nu_1 - Sd\rho_2\nu_2 - \&c. \\ &= -\;Sd\rho\nabla F - Sd\rho_1\nabla_1 F - Sd\rho_2\nabla_2 F - \&c. \end{aligned}\right\} \quad \text{(iv)}$$

where ∇_1 operates on F as if ρ_1 were the only variable. In fact $-\nabla$, $-\nabla_1$, &c. correspond precisely to Hamilton's D_ρ, $D_{\rho'}$ (or rather D_α, D_α') of the formula just cited. It may sometimes be even clearer to distinguish the corresponding operator by a sub-index of the vector operated on, thus (compare section (3.)) we have

$$(a') \qquad \nabla_\rho S\lambda\rho=-\lambda;\quad \nabla_\lambda S\lambda\rho=-\rho$$

$$(e') \qquad \nabla_\rho TV\lambda\rho=-UV\lambda\rho\,.\,\lambda;\quad \nabla_\lambda TV\lambda\rho=-UV\rho\lambda\,.\,\rho$$

$$(g') \qquad \nabla_\rho T(\rho-\lambda)^{-1}=-U(\rho-\lambda)\,T(\rho-\lambda)^{-2}=-\nabla_\lambda T(\rho-\lambda)^{-1}.$$

Also for any function of λ and ρ

$$\text{I. } V\lambda\nabla_\rho f=0, \qquad \text{II. } S\lambda\nabla_\rho f=0, \qquad \text{III. } \nabla_\rho f+\nabla_\lambda f=0 \qquad \text{(v)}$$

if f is a function of I. $S\lambda\rho$, II. $V\lambda\rho$, III. $\rho-\lambda$.

Again (compare (3.) (g) and (4.) (iii)) if P is the potential at the extremity of the vector ω of a system of attracting particles m_1, m_2, &c. whose position vectors are ρ_1, ρ_2, &c., the law of attraction being the law of nature, the force on a unit mass at ω is

$$\nabla_\omega P = \nabla_\omega \Sigma m \mathrm{T}(\omega - \rho)^{-1} = - \Sigma m \mathrm{U}(\omega - \rho)\ \mathrm{T}(\omega - \rho)^{-2} = - \Sigma m \nabla_\rho \mathrm{T}(\omega - \rho)^{-1}. \quad \text{(vi)}$$

Or if we have to do with a continuous distribution of mattter the force is given by

$$\nabla_\omega P = \nabla_\omega \int \frac{\mathrm{d}m}{\mathrm{T}(\omega - \rho)} = - \int \frac{\mathrm{U}(\omega - \rho)\mathrm{d}m}{\mathrm{T}(\omega - \rho)^2} = - \int \mathrm{d}m \nabla_\rho \frac{1}{\mathrm{T}(\omega - \rho)}. \quad \text{(vii)}$$

(6.) Again if α, β, γ are any constant vectors and X, Y and Z any scalar functions of ρ (compare the first of (iii)),

$$\left.\begin{aligned} \mathrm{d}(\alpha X + \beta Y + \gamma Z) &= \mathrm{d}X . \alpha + \mathrm{d}Y . \beta + \mathrm{d}Z . \gamma \\ &= - (\mathrm{Sd}\rho\nabla . X)\alpha - (\mathrm{Sd}\rho\nabla . Y)\beta - (\mathrm{Sd}\rho\nabla . Z)\gamma \\ &= - \mathrm{Sd}\rho\nabla . (X\alpha + Y\beta + Z\gamma). \end{aligned}\right\} \quad \text{(viii)}$$

Thus for any vector function of ρ we may write generally

$$\mathrm{d}\sigma = - \mathrm{Sd}\rho\nabla . \sigma \quad \text{(ix)}$$

for we may always resolve the two vectors σ and $\mathrm{d}\sigma$ along three given and fixed directions. In this equation (ix) we may suppose σ replaced by any quaternion function of ρ for by the distributive property if $\sigma = \mathrm{V}q$ we may add to (ix) the equation $\mathrm{dS}q = - \mathrm{Sd}\rho\nabla . \mathrm{S}q$ so that

$$\mathrm{d}q = - \mathrm{Sd}\rho\nabla . q. \quad \text{(x)}$$

It must be carefully observed that in these equations we regard σ and q as functions of ρ alone. For instance if q involves the time t as well as ρ the *total* differential is*

$$\mathrm{d}q = \frac{\partial q}{\partial t}\mathrm{d}t - \mathrm{Sd}\rho\nabla . q \quad \text{(xi)}$$

where the first term on the right refers to t as occurring in q but not in ρ.

(7.) We have now shown that the general formula (x) is true whether q be quaternion, vector, or scalar so that we may write generally and symbolically

$$\mathrm{d} = - \mathrm{Sd}\rho\nabla \quad \text{(xii)}$$

or what is equivalent

$$\nabla = - \frac{\mathrm{V d}'\rho \mathrm{d}''\rho . \mathrm{d} + \mathrm{Vd}''\rho\mathrm{d}\rho . \mathrm{d}' + \mathrm{Vd}\rho\mathrm{d}'\rho . \mathrm{d}''}{\mathrm{Sd}\rho\mathrm{d}'\rho\mathrm{d}''\rho} \quad \text{(xiii)}$$

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* We must particularly distinguish between

$$\mathrm{Sd}\rho\nabla . q \quad \text{and} \quad \mathrm{S} . \mathrm{d}\rho\nabla q.$$

where $d\rho$, $d'\rho$ and $d''\rho$ are any three non-coplanar differentials of ρ, and where d, d′ and d″ are the corresponding symbols of differentiation. In fact this equation (xiii) is equivalent to the three

$$d = -Sd\rho\nabla, \quad d' = -Sd'\rho\nabla, \quad d'' = -Sd''\rho\nabla$$

as appears in various ways.*

(8.) As an independent method we have (compare (6.))

$$\nabla(\alpha X + \beta Y + \gamma Z) = \nabla X . \alpha + \nabla Y . \beta + \nabla Z . \gamma \qquad \text{(xiv)}$$

where we employ merely the distributive principle (iii) and the commutative property of a scalar (X) with a vector (α). But we already know how to calculate the effect of ∇ on a scalar, so we can determine its effect on a quaternion or vector by referring the vector part or vector to any three fixed directions.

To trace the relation between these two methods we have

$$dX = -Sd\rho\nabla . X, \ \&c.,$$

whence without employing symbolical equations

$$\nabla X = -\frac{Vd'\rho d''\rho . dX + Vd''\rho d'\rho . d'X + Vd\rho d'\rho . d''X}{Sd\rho d'\rho d''\rho}.$$

Multiplying into α and forming similar expressions in ∇Y and ∇Z we find on addition if $\sigma = \alpha X + \beta Y + \gamma Z$

$$\nabla\sigma = -\frac{Vd'\rho d''\rho . d\sigma + Vd''\rho d'\rho . d'\sigma + Vd\rho d'\rho . d''\sigma}{Sd\rho d'\rho d''\rho} \qquad \text{(xv)}$$

which agrees with (xiii).

(9.) We give a few examples of operating on vectors with hints for verification

(a) $\nabla\rho = -3$; $Vd'\rho d''\rho . d\rho + Vd''\rho d\rho . d'\rho + Vd\rho d'\rho . d''\rho = 3Sd\rho d'\rho d''\rho$.

(b) $\nabla V\lambda\rho = 2\lambda = -\nabla(\rho\lambda - S\rho\lambda)$ or $= -\lambda S\nabla\rho + S\lambda\nabla . \rho$.

(c) $\nabla\phi\rho = 2\epsilon - m''$; $V\beta\gamma . \phi\alpha + V\gamma\alpha . \phi\beta + V\alpha\beta . \phi\gamma = (m'' - 2\epsilon)S\alpha\beta\gamma$.

(d) $\nabla U\rho = -2T\rho^{-1} = \nabla(\rho T\rho_0^{-1} + T\rho^{-1} . \rho_0)$.

(e) $\nabla\rho^{-1} = -\rho^{-2}$; $\nabla(\rho^{-1}\rho) = 0$.

(f) $\nabla^2 T(\rho - \lambda)^{-1} = 0$ if ρ is not equal λ.

(g) $\nabla^2 fT\rho = -f''T\rho - 2T\rho^{-1}f'T\rho$.

(h) $\nabla^2 TV\lambda\rho = -T(V\lambda^{-1}\rho)^{-1}$.

(i) $\nabla^2 \log TV\lambda\rho = 0$.

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* Either by verification or by multiplying these three equations by

$$Vd'\rho d''\rho, \quad Vd''\rho d\rho, \quad Vd\rho d'\rho$$

and adding. See also the next section.

For one unfamiliar with the subject it is however far better to employ no short cuts except an intelligent selection of the differentials of ρ if he uses the formula (xiii). For instance in (*b*) he may take these to be λ, ρ and $V\lambda\rho$. He must however be careful if he employs variable differentials to operate on these in subsequent operations* involving ∇. As explained in the last section the results may all be obtained by resolving the vectors along fixed directions.

(10.) To the examples of the last article we may add the following :—

$$(a) \quad V\lambda\nabla \,.\, \rho = -2\lambda.$$

$$(b) \quad V\lambda\nabla \,.\, \phi\rho = (\phi' - m'')\lambda + 2S\epsilon\lambda.$$

$$(c) \quad \phi\nabla \,.\, \rho = -(m'' + 2\epsilon).$$

$$(d) \quad \phi\nabla \,.\, T\rho = \phi U\rho.$$

$$(e) \quad \phi\nabla \,.\, U\rho = -(m'' + 2\epsilon)T\rho^{-1} + \phi U\rho \,.\, \rho^{-1}.$$

$$(f) \quad \phi\nabla \,.\, V\lambda\rho = (m'' - \phi')\lambda + 2S\epsilon\lambda.$$

And simpler examples may be obtained by selecting special forms of ϕ.

(11.) To anyone acquainted with the Calculus of Operations† it is manifest immediately the form (xiii) is obtained that ∇ may be combined with vectors and quaternions just as if it were an ordinary vector. In fact we may regard the symbols of differentiation d, d′, d″ as mere scalars and manipulate our formulæ in any way until we see fit to operate. Of course when successive operators ∇ occur in the same equation they must in general be distinguished by suitable marks and treated as independent vectors. This implies that the symbols d, d′, d″ of each operator must also be ear-marked whenever necessary.

We infer among other deductions that the operator ∇^2 or $\nabla \,.\, \nabla$ is a scalar because the square of a vector is a scalar. In the next section we shall verify this result from an elementary point of view.

(12.) It has been abundantly shown that ∇ is totally independent of any particular coordinates, parameters or differentials. We therefore take the case most familiar and choose our differentials so that

$$d\rho = i dx, \quad d'\rho = j dy, \quad d''\rho = k dz \qquad \text{(xvi)}$$

and therefore

$$d = dx \frac{\partial}{\partial x}, \quad d' = dy \frac{\partial}{\partial y}, \quad d'' = dz \frac{\partial}{\partial z} \qquad \text{(xvii)}$$

writing $\rho = ix + jy + kz$, i, j and k being *constant*.

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* Compare section (74.).

† See Boole, "*Differential Equations*," chap. xvii., or Forsyth, "*Differential Equations*," chap. iii., or indeed any work on this subject which treats of symbolical methods.

The relation (xiii) reduces at once to the well-known form

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \qquad \text{(xviii)}$$

which shows us that the operation performed by ∇ is equivalent to taking the partial differential *coefficients* of any function with respect to the three scalars x, y and z and multiplying these respectively by i, j and k and then adding the results.

Now when we operate twice by ∇ we have

$$\left.\begin{aligned}\nabla'\nabla &= \left(i\frac{\partial'}{\partial x} + j\frac{\partial'}{\partial y} + k\frac{\partial'}{\partial z}\right)\left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \\ &= -\left(\frac{\partial'}{\partial x}\frac{\partial}{\partial x} + \frac{\partial'}{\partial y}\frac{\partial}{\partial y} + \frac{\partial'}{\partial z}\frac{\partial}{\partial z}\right) + i\left(\frac{\partial'}{\partial y}\frac{\partial}{\partial z} - \frac{\partial'}{\partial z}\frac{\partial}{\partial y}\right) \\ &\qquad + j\left(\frac{\partial'}{\partial z}\frac{\partial}{\partial x} - \frac{\partial'}{\partial x}\frac{\partial}{\partial z}\right) + k\left(\frac{\partial'}{\partial x}\frac{\partial}{\partial y} - \frac{\partial'}{\partial y}\frac{\partial}{\partial x}\right)\end{aligned}\right\} \qquad \text{(xix)}$$

because the vectors i, j, k are constant. Suppressing the ear-marking accents as no longer necessary when we operate on a single function,

$$\nabla^2 = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \qquad \text{(xx)}$$

because the order in which the partial differentiation is effected with respect to y and z is indifferent. As we have stated at the beginning of this section ∇ and therefore ∇^2 is quite independent of any particular analytical representation, and thus apart from any *à priori* inferences arising from the form (xiii) we have proved that ∇^2 is a scalar operator; it is in fact with sign changed Laplace's most important operator.

The fuller discussion of the analytical forms attributable to ∇ is postponed to a later section.*

(13.) It may be as well to print here the equation

$$\begin{aligned}\nabla q = -\frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y} - \frac{\partial Z}{\partial z} + i\left(\frac{\partial W}{\partial x} + \frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}\right) + j\left(\frac{\partial W}{\partial y} + \frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x}\right) \\ + k\left(\frac{\partial W}{\partial z} + \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right)\end{aligned} \qquad \text{(xxi)}$$

which is obtained by operating by ∇ in the form (xviii) on q in the form

$$q = W + iX + jY + kZ. \qquad \text{(xxii)}$$

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* See section (73.).

This gives at once expressions for $S\nabla q$, $V\nabla q$, ∇Sq, $S\nabla Vq$, and $V\nabla Vq$. We observe that

$$S\nabla q = S\nabla Vq \quad \text{and} \quad V\nabla q = \nabla Sq + V\nabla Vq \tag{xxiii}$$

and we notice that the form of these equations is precisely the same as if ∇ were an ordinary vector—a verification of the *à priori* inference drawn in (11.).

It may be instructive to the student to find expressions for

$$\nabla Uq, \quad \nabla Tq, \quad T.\nabla q, \quad \nabla Kq, \quad K\nabla q, \quad K\nabla Kq, \quad q\nabla,$$

and other combinations of the symbols q, ∇ and the characteristics S, V, T, U, and K.

(14.) We can at once assign an interpretation to

$$\nabla q = -\frac{Vd'\rho d''\rho \,.\, dq + Vd''\rho d\rho \,.\, d'q + Vd\rho d'\rho \,.\, d''q}{Sd\rho d'\rho d''\rho} \tag{xxiv}$$

(compare (xiii)) by considering the parallelepiped whose centre is at the extremity of ρ and whose small* vector edges are $d\rho$, $d'\rho$ and $d''\rho$. The vectors from the centre to the centres of the faces are $\pm \frac{1}{2}d\rho$, $\pm \frac{1}{2}d'\rho$, $\pm \frac{1}{2}d''\rho$ and the outwardly directed areas of these faces are $\pm Vd'\rho d''\rho$, $\pm Vd''\rho d\rho$, $\pm Vd\rho d'\rho$, the signs corresponding if $Sd\rho d'\rho d''\rho$ is negative.

Now the mean value of q over the face $+ Vd'\rho d''\rho$ may be taken as its value at the centre of the face or ultimately as $q + \frac{1}{2}dq$, q being the value at the centre of the parallelepiped. But

$$-\frac{Vd'\rho d''\rho \,.\, dq}{Sd\rho d'\rho d''\rho} = \frac{+Vd'\rho d''\rho}{-Sd\rho d'\rho d''\rho} \,.\, (q + \tfrac{1}{2}dq) + \frac{-Vd'\rho d''\rho}{-Sd\rho d'\rho d''\rho} (q - \tfrac{1}{2}dq)\,; \tag{xxv}$$

that is it is equal to the sum of the mean value of q over each face multiplied by the directed area of that face and divided by the volume $(- Sd\rho d'\rho d''\rho)$ of the parallelepiped.

Adding up we see that ∇q equals the sum of the products of the directed elements of the surface into the corresponding values of q divided by the volume included.

(15.) We shall extend this result so as to be able to write for any small closed surface surrounding the extremity of ρ;

$$\nabla q = lim. \frac{1}{v} \int d\nu q \tag{xxvi}$$

where v is the volume included by the surface and $d\nu$ a directed element of area, the normal being outwardly drawn. Conceive the region enclosed by the surface divided arbitrarily into an infinite number of small parallelepipeds. For each of these

$$\nabla q dv = \int d\nu q$$

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* It is not necessary in (xiii) that the differentials should be small.

where ∇q on the left refers to the centre of the parallelepiped. On summation

$$\Sigma\nabla q dv = \Sigma \int d\nu q.$$

But over a common interface $d\nu$ regarded as referring to one parallelepiped is opposite to $d\nu$ referring to the other. Hence if there is no discontinuity in q the interfaces contribute nothing and $\Sigma \int d\nu q$ is due simply to the bounding faces of the extreme parallelepipeds, so that in the limit as these become indefinitely small

$$\int \nabla q dv = \int d\nu q \qquad \text{(xxvii)}$$

where the integral on the left is taken throughout the volume and that on the right over the surface. Conceive now the surface to shrink indefinitely and we find in the limit the required result (xxvi).

(16.) On account of the importance of this result and also as an exercise we shall calculate directly the integral taken over any small closed surface including the extremity of the vector ρ. Let τ be the variable vector drawn from this point and terminating on the surface.

Since τ is small we may put for the value of q at its extremity

$$q_\tau = q - S\tau\nabla \,.\, q \qquad \text{(xxviii)}$$

q being the value at the extremity of ρ. Here we assume as in the last section that the function q is continuous. If $d\nu$ or $Vd\tau d'\tau$ is an outwardly directed element of the surface

$$\int d\nu q_\tau = \int d\nu(q - S\tau\nabla \,.\, q) = -\int d\nu \,.\, S\tau\nabla \,.\, q \qquad \text{(xxix)}$$

because the surface is closed so that $\int d\nu = 0$ or $\int d\nu q = 0$.

In this if we choose we may regard

$$-\int d\nu \,.\, S\tau\nabla \qquad \text{(xxx)}$$

as an operator acting on q since we have to deal with τ only so far as integration is concerned, or indeed we may take ∇ outside the sign of integration and regard

$$-(\int d\nu S\tau)\nabla \qquad \text{(xxxi)}$$

as a linear and vector function of ∇.

In any case we have for any vectors τ, $d\tau$, and $d'\tau$

$$Vd\tau d'\tau S\tau\nabla + Vd'\tau\tau Sd\tau\nabla + V\tau d\tau Sd'\tau\nabla = S\tau d\tau d'\tau \,.\, \nabla \qquad \text{(xxxii)}$$

and also identically when d and d′ operate on τ and its differentials alone

$$\left.\begin{aligned} d(Vd'\tau\tau S\tau\nabla) &= dVd'\tau\tau \,.\, S\tau\nabla + Vd'\tau\tau \,.\, Sd\tau\nabla \\ d'(V\tau d\tau S\tau\nabla) &= d'V\tau d\tau \,.\, S\tau\nabla + V\tau d\tau \,.\, Sd'\tau\nabla \end{aligned}\right\} \qquad \text{(xxxiii)}$$

and

But

$$dVd'\tau\tau + d'V\tau d\tau = 2Vd'\tau d\tau + Vdd'\tau\tau + V\tau d'd\tau \qquad \text{(xxxiv)}$$

and the second and third term on the right cancel if we choose $d\tau$ and $d'\tau$ to be *independent* differentials so that $dd'\tau = d'd\tau$. In this case by means of (xxxiii) and (xxxiv) the equation (xxxii) reduces to

$$3Vd\tau d'\tau \,.\, S\tau\nabla + d(Vd'\tau\tau S\tau\nabla) + d'(V\tau d\tau S\tau\nabla) = S\tau d\tau d'\tau \,.\, \nabla. \qquad \text{(xxxv)}$$

Integrating over the surface we find

$$3\int Vd\tau d'\tau\nabla \,.\, S\tau \,.\, q = \int S\tau d\tau d'\tau \,.\, \nabla q \qquad \text{(xxxvi)}$$

since $\int d(Vd'\tau\tau S\tau\nabla) \,.\, q$ vanishes* if we suppose as we may that $d\tau$ (and also $d'\tau$) is an element of a closed curve drawn on the closed surface. Now $\frac{1}{3}S\tau d\tau d'\tau$ is the *negative* volume of the pyramid whose vertex is at the origin of vectors τ and whose base is the *outwardly* directed element of area $Vd\tau d'\tau$ or $d\nu$. Hence if v is the volume of the closed surface by (xxxvi) and (xxix) we find

$$\int d\nu q_\tau = v\nabla q.$$

Or finally dropping the sub-index τ as not now necessary we have rigorously

$$\nabla q = lim.\, \frac{1}{v}\int d\nu q\,; \qquad \text{(xxxvii)}$$

or the value of ∇q at any point is the limit of the integral of the outwardly directed elements of any small closed surface surrounding the point multiplied into the corresponding quaternion q and divided by the volume enclosed by the surface.

(17.) We proceed at once to the interpretation of the results of the last two sections.

The case of q a scalar is aptly illustrated by a hydrostatic pressure p. As $d\nu$ has been supposed measured outwardly, $-d\nu p$ is the pressure in direction and magnitude on the directed element, and $-\int d\nu p = -v\nabla p$ is the resultant pressure over the surface. This urges the element in the direction $-U\nabla p$, that is in the direction in which p diminishes most rapidly for we have seen (1.) that $+U\nabla p$ is the direction in which p increases most rapidly.

(18.) In the case of q a vector (σ) unlike the former case, the integral consists of a scalar as well as a vector part. We notice that the scalar part depends merely on the components of the vectors σ normal to the surface and the vector part on the tangential components. For

$$\nabla\sigma = v^{-1}(S + V)\int d\nu\sigma = v^{-1}\int Sd\nu\sigma + v^{-1}\int Vd\nu\sigma \qquad \text{(xxxviii)}$$

because S and V are distributive, the scalar of a sum for instance being the sum of the scalars.

* We repeat that q is quite independent of τ being in fact the value of q at the origin of the vectors τ.

We shall consider the scalar and vector integrals separately, that is the integral of the *inwardly** directed normal components + $Sd\nu\sigma$; and the integral of the tangential components turned through a right angle in the tangent plane for

$$Vd\nu\sigma = d\nu(d\nu)^{-1}Vd\nu\sigma.$$

(19.) Taking the scalar part first we have to interpret

$$S\nabla\sigma = \frac{1}{v}\int Sd\nu\sigma. \qquad \text{(xxxix)}$$

In the first place let σ represent the displacement of a point in a body—the extremity of the vector ρ—deformable in any way. The integral then represents the sum of the inward components of displacement of the elements of the small surface; in other words it is the diminution of volume. The ratio of this to the volume is the *condensation*. To put this in a clearer light we resort to the suffix τ (xxviii) and (compare (xxix)) we write $\int Sd\nu\sigma_\tau = \int Sd\nu(\sigma_\tau - \sigma)$ so that we only have to consider the displacement relative to the origin of vectors τ.

Secondly let σ denote any distribution of force. The integral represents the *total* normal force over the surface.

Thirdly if σ represents the *flux* of a fluid the integral measures the rate at which the inflow into the little region exceeds the outflow. The quotient of this by the volume is the rate at which the fluid accumulates in unit volume or the rate of increase of density at the point. Otherwise if σ is the *velocity* and c the density $S\nabla(c\sigma)$ is the rate of increase of density or

$$\frac{\partial c}{\partial t} = S\nabla(c\sigma). \qquad \text{(xl)}$$

For these reasons Clerk Maxwell called $S\nabla\sigma$ the convergence of the vector σ.†

(20.) Now we may choose the small surface to be any surface we please. We shall take it to be a portion of a tube of flow according to the hydrodynamical analogy, or we shall suppose that the vectors σ are tangential to its sides and normal to its ends. The integral vanishes consequently except over the ends and

$$S\nabla\sigma = -\frac{1}{v}\int Td\nu T\sigma \qquad \text{(xli)}$$

the integral being taken over the two ends.

The areas of the ends being small and the distance between them small we have ultimately

$$S\nabla\sigma = -\,lim.\,\frac{AdT\sigma + dAT\sigma}{Adl} = -\,lim.\left(\frac{dT\sigma}{dl} + \frac{T\sigma}{A}\frac{dA}{dl}\right) = -\frac{1}{T\sigma}\,lim.\,\frac{d\log AT\sigma}{dl} \qquad \text{(xlii)}$$

* $d\nu$ being outwards, $Sd\nu\sigma$ is $-Td\nu T\sigma\cos\theta$ if θ is the angle between the normal $d\nu$ and σ.

† *Electricity and Magnetism*, Art. 25.

A being the mean area of a normal section and dl the length of the tube. $S\nabla\sigma$ is thus equal to the rate of diminution of $T\sigma$ along a line of flow together with the rate of contraction of the normal cross-section multiplied by $T\sigma$. This is the interpretation of the transformation

$$S\nabla\sigma = SU\sigma_0\nabla \,.\, T\sigma + T\sigma_0 S\nabla U\sigma \qquad \text{(xliii)}$$

in which the suffixes denote that the marked symbol is not to be operated on by ∇. We notice moreover that if a is any constant vector

$$S\nabla\sigma = SU(\sigma_0 - a)\nabla \,.\, T(\sigma - a) + T(\sigma_0 - a)S\nabla U(\sigma - a). \qquad \text{(xliv)}$$

The property (xxxix) remains true if any constant velocity is added to the velocities existing.

(21.) As regards the vector part of $\nabla\sigma$ we have seen that

$$V\nabla\sigma = lim. \frac{1}{v}\int V d\nu\sigma \qquad \text{(xlv)}$$

depends only on the tangential components of σ turned through a right angle round the normal. We may indeed find interpretations of this surface integral taken over an arbitrary surface, but none are satisfactory until we choose a surface presenting a definite direction upon which to fix the attention. For instance for a sphere we find

$$V\nabla\sigma = lim. \frac{1}{vT\tau}\int V\tau\sigma T d\nu \qquad \text{(xlvi)}$$

showing that the vector is the integrated moment of σ about the centre divided by the product of radius and volume. But when we select a small portion of a cylinder whose sides have a fixed direction a and whose ends are normal, we obtain results easily interpretable. Let dA be an element of the small cross-section, dl an element of a generator, $d\tau$ a tangential vector on the curved boundary forming with a and $d\nu$ a mutually rectangular system so that $d\tau$, a, $d\nu$ are in positive order.* Then over the curved boundary $d\nu = dlVd\tau a$ and over the plane faces $d\nu = \pm\, a dA$. Thus

$$V\nabla\sigma = lim. \frac{1}{Al}\left(\int dlV.Vd\tau a \,.\, \sigma + \int Va(\sigma_2 - \sigma_1)dA\right). \qquad \text{(xlvii)}$$

Taking l so small that it may be integrated by itself in the first integral, we may replace $\sigma_2 - \sigma_1$ in the second by $-\,lSa\nabla \,.\, \sigma$.

The expression (xliv) reduces consequently to

$$V\nabla\sigma = lim. \frac{1}{A}\left(\int V.Vd\tau a \,.\, \sigma - \int Va \,.\, Sa\nabla \,.\, \sigma dA\right). \qquad \text{(xlviii)}$$

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* That is rotation round $d\tau$ from a to $d\nu$ is positive.

It may be further transformed since

$$V.Vd\tau\alpha.\sigma = -\alpha Sd\tau\sigma + d\tau S\alpha\sigma = -\alpha Sd\tau\sigma - V\alpha Vd\tau\sigma$$

because α is at right angles to every $d\tau$; thus

$$V\nabla\sigma = -lim.\frac{1}{A}(\alpha\int Sd\tau\sigma + V\alpha(\int Vd\tau\sigma + \int S\alpha\nabla.\sigma dA)) \qquad \text{(xlix)}$$

which gives separately

$$S\alpha V\nabla\sigma = lim.\frac{1}{A}\int Sd\tau\sigma;\quad V\alpha V\nabla\sigma = -lim.\frac{1}{A}\alpha V\alpha(\int Vd\tau\sigma + \int S\alpha\nabla.\sigma dA). \qquad \text{(l)}$$

As we give an independent and superior method of obtaining analogous results in the next section we shall not consider the interpretation of these until section (23.).

(22.) The transformation of the last section suggests the investigation of line integrals $\int d\rho q$. Take a small parallelogram, centre at ρ and edges $d\rho$, $d'\rho$, and circuit it in the order from $d\rho$ to $d'\rho$. In this order the vector sides are $+d\rho$, $+d'\rho$, $-d\rho$, $-d'\rho$ and the corresponding vectors from the centre to their middle points are $-\frac{1}{2}d'\rho$, $+\frac{1}{2}d\rho$, $+\frac{1}{2}d'\rho$, $-\frac{1}{2}d\rho$, so the four sides contribute in order

$$+d\rho(q + \tfrac{1}{2}Sd'\rho\nabla.q),\quad +d'\rho(q - \tfrac{1}{2}Sd\rho\nabla.q),$$
$$-d\rho(q - \tfrac{1}{2}Sd'\rho\nabla.q),\quad -d'\rho(q + \tfrac{1}{2}Sd\rho\nabla.q) \qquad \text{(li)}$$

and the sum of these, which we may write

$$\int d\rho q = Vd\nu\nabla.q; \qquad \text{(lii)}$$

where $d\nu = Vd\rho d'\rho$ is the directed area of the parallelogram, because

$$d\rho Sd'\rho\nabla. - d'\rho Sd\rho\nabla. = V.Vd\rho d'\rho.\nabla.$$

Also rotation round $d\nu$ in the direction of circuiting is positive, viz. from $d\rho$ to $d'\rho$.

We shall prove that the same relation (lii) is true whatever be the shape of the small plane circuit. Conceive the small area divided arbitrarily into small parallelograms, and let each be circuited in the same direction and the sum taken. Any side common to two is traversed twice in opposite directions. If there is no discontinuity in q such a side contributes nothing for $d\rho q + (-d\rho)q = 0$. Hence only the bounding sides contribute and in the limit when these approach coincidence with the curve

$$\int d\rho q = \int Vd\nu\nabla.q$$

the first integral being taken over the bounding curve and the second over the plane area. From this in the limit we recover (lii).

(23.) When q is a vector (lii) affords the two equations true for any small plane circuit (compare (l))

$$\int Sd\rho\sigma = Sd\nu V\nabla\sigma \tag{liii}$$

and

$$\int Vd\rho\sigma = V.Vd\nu\nabla . \sigma. \tag{liv}$$

The first shows that what we may call the circulation in any small circuit ($-\int Sd\rho\sigma$) is equal to the product of the area into the component of $V\nabla\sigma$ along the positive normal.* We shall see in the case of fluid motion that $V\nabla\sigma$ is twice the angular velocity of an element. Just as the rate of change of a scalar function in any direction is the component of ∇P in that direction so the circulation in any unit plane circuit is the component of $V\nabla\sigma$ along its positive normal. The circuit normal to $UV\nabla\sigma$ may be called the principal circuit, the circulation therein being a maximum.

(24.) If σ represents a distribution of force, by carrying a small unit mass round a circuit we gain from the forces an amount of work represented by $-\int S\sigma d\rho$ or $-Sd\nu V\nabla\sigma$. Hence the condition that the forces should be conservative, or that no work could be gained in carrying a small mass round any complete small circuit, is $V\nabla\sigma = 0$; or what is equivalent this is the condition that $\int Sd\rho\sigma$ should be integrable without a factor, the integral being taken between arbitrary limits. In fact the integral must be a function of the vectors ρ at the limits. We may therefore write

$$P = -\int_{\rho_0}^{\rho} S\sigma d\rho = f(\rho, \rho_0). \tag{lv}$$

Whence $\nabla P = \sigma$, because as P is a function of ρ

$$dP = -Sd\rho\nabla P = -S\sigma d\rho$$

for all vectors $d\rho$. Thus the equation

$$V\nabla\sigma = 0 \quad \text{implies} \quad \sigma = \nabla P \tag{lvi}$$

just as the latter implies the former (compare (xx)). A distribution of vectors satisfying this condition is said to be irrotational.

(25.) Introducing the symbol χ' to denote a linear and vector function, we write equation (liv) in the form

$$-\chi' d\nu = V.Vd\nu\nabla . \sigma = \int Vd\rho\sigma. \tag{lvii}$$

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* For brevity let the normal to the circuit about which the positive rotation is the same as that of the circuit be called the *positive* normal.

This function χ' and its conjugate may be expressed* by

$$\chi'\alpha = -\,\mathrm{V.V}\alpha\nabla\,.\,\sigma = -\,\alpha\mathrm{S}\nabla\sigma + \nabla\mathrm{S}\alpha\sigma\,;\quad \chi\alpha = -\,\mathrm{V}\nabla\mathrm{V}\sigma\alpha = -\,\alpha\mathrm{S}\nabla\sigma + \sigma\mathrm{S}\alpha\nabla. \quad \text{(lviii)}$$

In fact χ is Hamilton's auxiliary function for ϕ or

$$\chi\alpha = (m'' - \phi)\alpha \quad \text{where} \quad \phi\alpha = -\,\mathrm{S}\alpha\nabla\,.\,\sigma \qquad \text{(lix)}$$

since $m'' = -\,\mathrm{S}\nabla\sigma$ (compare (27.)).

When a unit electric current flows in the small circuit $-\,\chi' d\nu$ is the resultant mechanical force acting on the circuit provided σ is the magnetic induction due to extraneous causes.† We shall therefore in the most general case briefly term $-\,\chi'\alpha$ the force on the circuit α.

The force on the circuit is normal or tangential to its plane according as α satisfies

$$\mathrm{V}\alpha\chi'\alpha = 0 \quad \text{or} \quad \mathrm{S}\alpha\chi'\alpha = 0. \qquad \text{(lx)}$$

The force on the circuit α has $\mathrm{S}\beta\chi'\alpha$ for its component along β and this is generally different from the component along α of the force on the circuit β because χ' is not self-conjugate. The spin-vector of χ is easily seen to be $-\,\frac{1}{2}\mathrm{V}\nabla\sigma$ and whenever this vanishes the force on α has the same component along β as the force on β has along α.

In a steady magnetic field

$$\nabla\sigma = 0 \quad \text{or} \quad \sigma = -\,\nabla\Omega \qquad \text{(lxi)}$$

where Ω is the magnetic potential and (lviii)

$$\chi'\alpha = \chi\alpha = \nabla\mathrm{S}\alpha\sigma = \mathrm{S}\alpha\nabla\,.\,\sigma \qquad \text{(lxii)}$$

or the force is the rate of change of the induction (σ) along the normal.

(26.) As the last particular case we suppose q to be a scalar P, then for all small circuits

$$\int d\rho P = \mathrm{V}d\nu\nabla\,.\,P. \qquad \text{(lxiii)}$$

The most direct illustration of this formula seems to be to suppose P the magnetic potential of the field. The expression on the right with sign changed represents the couple on a small magnet whose magnetic moment is $d\nu$. As this can be expressed as a line integral round a circuit whose directed area is $d\nu$, the equation suggests the equivalence of the magnetic action due to a unit current in the circuit and that due to the magnet. It shows moreover that the couple acting on the circuit is the negative of the integral of its elements multiplied by the corresponding potentials P.

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* There can be no possible objection to placing ∇ after the operand σ in an equation of this kind (lviii) as no confusion is likely to arise.

† Clerk Maxwell, *Electricity and Magnetism*, Art. 490.

(27.) We shall now consider the linear vector function and its conjugate

$$\phi\alpha = -\,S\alpha\nabla\,.\,\sigma \quad\text{and}\quad \phi'\alpha = -\,\nabla S\alpha\sigma \qquad \text{(lxiv)}$$

of which an auxiliary function has occurred in (25.). It is only necessary to find expressions for its invariants and for the second auxiliary function ψ in terms of ∇ for its meaning has been fully investigated. In fact if σ is the strain-displacement of the extremity of ρ, the displacement of a near point $(\rho + d\rho)$ is $\sigma + \phi d\rho$, so that $\phi d\rho$ is the displacement of this near point with respect to the point (ρ). The strain being supposed small we have seen* that the molecular rotation of the element at (ρ) is $\epsilon = \frac{1}{2}V\nabla\sigma$; also the dilatation is given by the first invariant $m'' = -\,S\nabla\sigma$. The pure part of the strain is due to

$$\Phi\alpha = -\,\tfrac{1}{2}S\alpha\nabla\,.\,\sigma - \tfrac{1}{2}\nabla S\alpha\sigma = \tfrac{1}{2}(\phi + \phi')\alpha. \qquad \text{(lxv)}$$

Of course now, in contrast to the case treated in the Note cited, the strain is not homogeneous.

On account of the great importance of this function we shall prove these expressions for m'' and ϵ. For three arbitrary vectors

$$V\beta\gamma\,.\,\phi\alpha + V\gamma\alpha\,.\,\phi\beta + V\alpha\beta\,.\,\phi\gamma = -\,(V\beta\gamma S\alpha\nabla + V\gamma\alpha S\beta\nabla + V\alpha\beta S\gamma\nabla)\sigma.$$

Hence by (xiii) and the well-known expression for the invariant of ϕ

$$m'' - 2\epsilon = -\,\nabla\sigma. \qquad \text{(lxvi)}$$

(28.) In forming the function ψ it is necessary to use temporary marks to distinguish the corresponding operator and operand.

We write therefore

$$\psi V\alpha\beta = V\phi'\alpha\phi'\beta = V\nabla S\alpha\sigma\,.\,\nabla' S\beta\sigma' = V\nabla\nabla' S\alpha\sigma S\beta\sigma'.$$

Now we may write equally well

$$\psi V\alpha\beta = V\nabla' S\alpha\sigma'\,.\,\nabla S\beta\sigma = -\,V\nabla\nabla' S\alpha\sigma' S\beta\sigma.$$

So that treating ∇, ∇', σ, and σ' as four distinct vectors we obtain on addition of these two forms

$$\psi\gamma = -\,\tfrac{1}{2}V\nabla\nabla' SV\sigma\sigma'\gamma. \qquad \text{(lxvii)}$$

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* Note on Strain, sections (16.) and (17.), p. 372.

The accents may be removed when but not till when the operations indicated have been performed.* Just as in (lxvi)

$$m' - 2\phi\epsilon = -\tfrac{1}{2}\mathrm{V}\nabla\nabla' \,.\, \mathrm{V}\sigma\sigma' \qquad \text{(lxviii)}$$

and this result should be compared with the former and the expressions for the vectors verified. It is also a useful exercise to verify that the third invariant is

$$m = \tfrac{1}{6}\mathrm{S}\nabla\nabla'\nabla''\mathrm{S}\sigma\sigma'\sigma''. \qquad \text{(lxix)}$$

(29.) Instead of retaining only the first term in the expansion we may, for the particular case in which q is a function of ρ, write Hamilton's expression for Taylor's series in the form†

$$q_a = e^{\mathrm{d}}q = e^{-\mathrm{S}a\nabla}q = q - \mathrm{S}a\nabla \,.\, q + \tfrac{1}{2}(\mathrm{S}a\nabla)^2 q - \&\mathrm{c}. \qquad \text{(lxx)}$$

Here as there is no danger of confusion we need not accent or distinguish the several operators there being but one operand.

If the quaternion q_a is associated with each element of mass $\mathrm{d}m$ of a body

$$\int q_a \mathrm{d}m = qM - \mathrm{S}a_0\nabla \,.\, qM - \tfrac{1}{2}(\tfrac{1}{2}(A + B + C)\nabla^2 - \mathrm{S}\nabla\Phi\nabla) \,.\, q + \&\mathrm{c}. \qquad \text{(lxxi)}$$

where a_0 is the vector to the centre of mass; A, B, and C the principal moments and Φ the momentum function of the body with respect to the origin.‡ To prove this it is only necessary to observe that $(\mathrm{S}a\nabla)^2 = a^2\nabla^2 + (\mathrm{V}a\nabla)^2$ and to employ the notation explained in the note on page 291.

In like manner

$$\int a q_a \mathrm{d}m = a_0 qM + \tfrac{1}{2}(A + B + C)\nabla q - \Phi\nabla \,.\, q. \qquad \text{(lxxii)}$$

From (lxxi) we obtain Clerk Maxwell's expression for the mean value of q throughout a sphere when we put $\Phi = A = B = C$ and $a_0 = 0$.

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* A device precisely similar is used in Aronhold's symbolic method of denoting a quantic by $a_x{}^n = 0$, $b_x{}^n = 0$, &c. The Hessian of a quantic is represented by

$$\Delta = n^2(n-1)^2 a_1 b_2 (ab) a_x{}^{n-2} b_x{}^{n-2} = \tfrac{1}{2}n^2(n-1)^2(ab)^2 a_x{}^{n-2} b_x{}^{n-2}$$

where $(ab) = a_1b_2 - a_2b_1$. (Compare Clebsch, *Vorlesungen über Geometrie*, p. 191, Leipzig, 1876.)

† See p. 473, vol. i., and the second Note to p. 20 in the present volume. It is undoubtedly strange that Hamilton has deliberately avoided the employment of the symbol ∇ in the *Elements*. We have seen several times in the course of this Note that our results are merely translations into this notation of investigations in which ∇ was not explicitly employed (comp. sections (2.), (5.), (27.)). He even introduces a new notation (see p. 294 and section (5.)) when ∇ was ready to his hand. The key to this neglect of ∇ seems to be contained in Art. 422, (92.), p. 351.

‡ $+\Phi\omega$ is the angular momentum of the body (comp. p. 291) spinning with angular velocity ω. Hamilton uses a negative sign.

If the body is subject to the attraction of matter having a potential P we find for the force and couple at the centre of mass

$$\lambda = M\nabla P + \tfrac{1}{2}S\nabla\Phi\nabla \, . \, \nabla P, \quad \mu = - V\Phi\nabla \, . \, \nabla P. \qquad \text{(lxxiii)}$$

Hence it is not hard to deduce, using the examples in section (10.), when $P = \int T(\alpha - \alpha')^{-1}dm'$ that

$$\lambda = - M\int \beta r^{-3}dm' - 3\int((A + B + C)\beta + 2\Phi\beta - 5\beta S\beta^{-1}\Phi\beta)r^{-5}dm'$$

and

$$\mu = - 3\int V\beta^{-1}\Phi\beta r^{-3}dm' \qquad \text{(lxxiv)}$$

where for brevity $\beta = \alpha - \alpha'$ and $r = T\beta$.

(30.) It is not necessary to examine in any detail the extension of the integrations of sections (14.) and (15.) to finite regions because the method is almost precisely the same as in the case of scalar integrals. If (I.) there is no discontinuity in the quaternion q, if (II.) it is single-valued and (III.) does not become infinite at any point of the region, if moreover (IV.) the region is simply-connected, we can fill it with small parallelepipeds in any way we please and since over an interface the aspects of the corresponding directed elements of the adjoining parallelepipeds are opposed the interfaces contribute nothing. In the limit therefore when the conditions I., II., III., and IV. are satisfied, the volume integral equals the surface integral or

$$\int \nabla q d\mathrm{v} = \int d\nu q. \qquad \text{(lxxv)}$$

(31.) I. When there is a surface of discontinuity suppose the region divided into two by that surface and apply the equation (lxxv) separately to each region and add. Then

$$\int \nabla q d\mathrm{v} = \int d\nu q + \int d\nu_{12}(q_1 - q_2) \qquad \text{(lxxvi)}$$

when over the surface of discontinuity an element affords the parts

$$d\nu_{12}q_1 \quad \text{and} \quad d\nu_{21}q_2 \quad \text{or} \quad d\nu_{12}(q_1 - q_2).$$

(32.) II. If q is not single-valued by reasoning almost precisely similar to that of Clerk Maxwell* we can see when infinite values of ∇q are excluded from the region that assuming the value of q at any one point its value at every other point is determinate. In fact starting from a point A with a given value of q we can return to it with a different value only if we thread some circuit along which q is indeterminate; and if q is indeterminate anywhere in the region its corresponding deriveds must be infinite. In the case in which a circuit locus of indeterminate values of q exists in the region, we may enclose it in a tube but the region then becomes multiply-connected (IV.).

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* *Electricity and Magnetism*, Art. 96 (*b*).

(33.) III. If q becomes infinite at any point, we exclude that point by a small sphere and include the surface integral over the sphere in the result. Taking for the moment the origin at the point and writing $T\rho = r$ let

$$q = q_0 + \frac{f_1 U\rho}{r} + \frac{f_2 U\rho}{r^2} + \frac{f_3 U\rho}{r^3} + \&c. \qquad \text{(lxxvii)}$$

Then if $d\Omega$ is an element of solid angle the integral over the sphere is

$$-\int dvq = \int d\Omega \,.\, U\rho r^2 \left(q_0 + \frac{f_1 U\rho}{r} + \frac{f_2 U\rho}{r^2} + \&c. \right).$$

This in general is ultimately infinite or indeterminate if $f_3 U\rho$, &c. are not zero. Excluding these cases, in the limit

$$-\int dvq = \int d\Omega U\rho f_2 U\rho.$$

We need only consider the case in which $f_2 U\rho$ is a linear function* of $U\rho$, and we may take it to be

$$f_2 U\rho = S\eta' U\rho + \Sigma\lambda S\mu U\rho = S\eta' U\rho + \phi U\rho. \qquad \text{(lxxviii)}$$

It is easy to see in various ways (compare for instance (29.)) that

$$\int d\Omega U\rho S\alpha U\rho = -\tfrac{4}{3}\pi\alpha \quad \text{and} \quad \int d\Omega U\rho f_2 U\rho = -\tfrac{4}{3}\pi(\eta' + \Sigma\mu\lambda).$$

Hence (lxxv) becomes modified by the infinite point into

$$\int \nabla q dv = \int dvq + \tfrac{4}{3}\pi(\eta' + m'' - 2\epsilon) \qquad \text{(lxxix)}$$

if a term $r^{-2} f U\rho$ occurs in q, the part of $fU\rho$ linear in $U\rho$ being $S\eta' U\rho + \phi U\rho$.

(34.) IV. If the region is multiply-connected we render it simply connected by drawing diaphragms when we fall back on case I. if q is many valued. A diaphragm corresponds to a surface of discontinuity and $q_1 - q_2 = np$ where p is the cyclic increment of q and n an integer.

(35.) In order to extend the integrations of section (22.) to any closed curve directly we must be able to connect all points of the curve by a continuous net of small parallelograms for each of which q must be (I.) continuous, (II.) single-valued and (III.) without infinite differentials. Then because a common side is traversed in opposite directions

$$\int d\rho q = \int V d\nu \nabla \,.\, q \qquad \text{(lxxx)}$$

where the line integral is over the curve and the surface with which the net ultimately coincides. Under these conditions the surface integral extended to a closed surface is always zero.

* By an application of a well-known theorem in spherical harmonics (72.).

(I.) In the case of discontinuity as in (31.) we take an arbitrary curve on the surface of discontinuity and when this curve is *specified* we have on adding the results for the two circuits

$$\int d\rho q + \int d\rho_{12}(q_1 - q_2) = \int V d\nu\nabla \,.\, q \tag{lxxxi}$$

where the second integral on the left is taken over the specified curve on the surface of discontinuity. Let this curve be ACB terminating on the given circuit at A and B. Draw any other curve ADB, then letting the accented line integral refer to this curve and the second surface integral to the portion DBCA of the surface of discontinuity

$$\int d\rho q + \int' d\rho_{12}(q_1 - q_2) = \int V d\nu\nabla \,.\, q + \int V d\nu_{12}(q_1 - q_2) \tag{lxxxii}$$

provided the portion of the surface of discontinuity can be covered with a continuous net. Applying (lxxx) to the surface of discontinuity it is evident we get the same value for $\int d\rho q$ in both cases.

(II.) If q is not single-valued over the continuous net, its value is definite if a definite value is chosen at some one point of the net, or else q is indeterminate at a point of the net and as a consequence its differential may become infinite (III.). This point may be surrounded by a small curve joined by a barrier to the given circuit, and the barrier must then be treated as a line of discontinuity and the value of the integral round the closed curve must be taken account of.

(36.) We shall not delay to prove the more general relations

$$\int f d\nu = \int f \nabla d\nu \,; \quad \int F d\rho = \int F V d\nu\nabla \tag{lxxxiii}$$

where f and F are linear functions and where ∇ operates on them *in situ* in the two expressions $f\nabla$ and $FVd\nu\nabla$. They may be proved exactly as in the simpler case when we have to do only with a quaternion multiplier; in fact $fVd'\rho d''\rho$ at the extremity of ρ becomes

$$(1 - \tfrac{1}{2}Sd\rho\nabla)fVd'\rho d''\rho \quad \text{or} \quad f(1 - \tfrac{1}{2}Sd\rho\nabla)\,.\,Vd'\rho d''\rho$$

at the extremity of $\rho + \frac{1}{2}d\rho$ it being understood that ∇ operates on the constituents of f alone. We may remark that the symbol of taking the conjugate K may be applied to the integrals (lxxx) or (lxxv).

(37.) Let the quaternion p or p_ω be the value of the integral

$$p = \int^\infty T(\rho - \omega)^{-1} q d\nu \tag{lxxxiv}$$

at the extremity of the vector ω, ρ being now the vector variable in the integration which is extended throughout all space or at least everywhere that q is not zero. We suppose q is never infinite and has never infinite differentials corresponding to finite differentials of ρ. Considering separately the parts of the integral inside and outside a small sphere, centre ω, we have on operating by ∇_ω^2

$$\nabla_\omega^2 p = \nabla_\omega^2 \int' T(\rho - \omega)^{-1} q dv \qquad \text{(lxxxv)}$$

where the accent denotes that the integration is confined to the interior of the sphere for by 9(f), $\nabla_\omega^2 T(\rho - \omega)^{-1} = 0$ wherever ω does not coincide with ρ. The sphere may be taken so small that q is sensibly constant within it. q may thus be removed outside the sign of integration and

$$\nabla_\omega^2 p = \nabla_\omega^2 \int' T\,(\rho - \omega)^{-1} dv \,.\, q_\omega = 4\pi q_\omega \qquad \text{(lxxxvi)}$$

because by Poisson's theorem (compare (xx))

$$\nabla_\omega^2 \int' T(\rho - \omega)^{-1} dv = 4\pi.$$

(38.) From these results we infer conversely if two quaternions p and q are connected by the equation

$$q = \nabla^2 p \qquad \text{(lxxxvii)}$$

that

$$p = p_\omega = (4\pi)^{-1} \int^\infty T(\rho - \omega)^{-1} q dv \qquad \text{(lxxxviii)}$$

the integration being extended throughout all space or wherever q is not zero, and we may regard this expression as the equivalent of the inverse operation in the equation

$$p = \nabla^{-2} q. \qquad \text{(lxxxix)}$$

On this supposition the operator ∇^{-2} presents no ambiguity.

(39.) The difference between $\nabla^{-2} q$ or the integral (lxxxviii) taken over an unlimited field and the same integral taken throughout a circumscribed region may by Green's theorem be expressed as a surface integral over the boundary of the region. The extremity of ω being within this region we have by (33.) when the volume integral is taken in the limited region outside the small sphere, the first surface integral over the boundary and the accented integral over the surface of the sphere,

$$\int \nabla \,.\, (T(\rho - \omega)^{-1} \nabla p) dv = \int d\nu T(\rho - \omega)^{-1} \,.\, \nabla p + \int' d\nu T(\rho - \omega)^{-1} \,.\, \nabla p \qquad \text{(xc)}$$

and also by (36.)

$$\int \nabla T(\rho - \omega)^{-1} (\nabla) p dv = \int \nabla T(\rho - \omega)^{-1} \,.\, d\nu p + \int' \nabla T(\rho - \omega)^{-1} \,.\, d\nu p. \qquad \text{(xci)}$$

In the volume integral (xc) the first operator ∇ operates on all that follows it (except dv) and in the second the bracketed (∇) operates *in situ* upon p and also upon $\nabla T(\rho - \omega)^{-1}$.

The surface integral over the sphere in (xc) vanishes; that in (xci) (comp. (3. (g)) and (33.)) becomes*

$-\int' d\Omega p = -4\pi p$ because $d\nu = -U(\rho-\omega)r^2 d\Omega$ and $\nabla T(\rho-\omega)^{-1} = -U(\rho-\omega)\,T(\rho-\omega)^{-2}$.

Also the term in the first volume integral is

$$\nabla\,.\,(T(\rho-\omega)^{-1}\nabla p) = \nabla T(\rho-\omega)^{-1}\,.\,\nabla p + T(\rho-\omega)^{-1}\,.\,\nabla^2 p \qquad \text{(xcii)}$$

and that in the second is

$$\nabla T(\rho-\omega)^{-1}\,(\nabla) p = \nabla T(\rho-\omega)^{-1}\,.\,\nabla p + \nabla^2 T(\rho-\omega)^{-1}\,.\,p, \qquad \text{(xciii)}$$

for it is easy to prove† that $\nabla T(\rho-\omega)^{-1}\nabla = \nabla^2 T(\rho-\omega)^{-1}$. Moreover this part vanishes since ω is not included in the limited field.

By these considerations (xc) and (xci) reduce to

$$\int(\nabla T(\rho-\omega)^{-1}\,.\,\nabla p + T(\rho-\omega)^{-1}\,.\,\nabla^2 p)dv = \int d\nu T(\rho-\omega)^{-1}\,.\,\nabla p \qquad \text{(xciv)}$$

$$\int \nabla T(\rho-\omega)^{-1}\,.\,\nabla p dv = \int \nabla T(\rho-\omega)^{-1}\,.\,d\nu p - 4\pi p \qquad \text{(xcv)}$$

so that on subtraction

$$\int T(\rho-\omega)^{-1}\,.\,\nabla^2 p dv - 4\pi p = \int d\nu T(\rho-\omega)^{-1}\,.\,\nabla p - \int \nabla T(\rho-\omega)^{-1}\,.\,d\nu p. \qquad \text{(xcvi)}$$

Or if we suppose p and q connected by the equation (lxxxvii) or (lxxxix)

$$\frac{1}{4\pi}\int T(\rho-\omega)^{-1} q dv = \nabla^{-2} q + \frac{1}{4\pi}\int d\nu T(\rho-\omega)^{-1}\,.\,\nabla.\nabla^{-2} q - \int \nabla T(\rho-\omega)^{-1}\,.\,d\nu\,.\,\nabla^{-2} q. \qquad \text{(xcvii)}$$

Thus the difference of the integral over a limited and unlimited field (lxxxviii) has been expressed as a surface integral over the boundary of the former.

(40.) We have seen (lxxxv) that when we operate with ∇_ω^2 on a potential function it is only necessary to take account of the element at which ρ and ω coincide. Provided therefore we introduce surface integrals wherever necessary we may limit the field of integration and write generally for all points within that field

$$q = \nabla_\omega^2 \int \frac{q dv}{4\pi T(\rho-\omega)}. \qquad \text{(xcviii)}$$

By the associative principle we deduce

$$4\pi q = \nabla_\omega\,.\,\nabla_\omega \int T(\rho-\omega)^{-1} q dv = \nabla_\omega S\nabla_\omega \int T(\rho-\omega)^{-1} q dv + \nabla_\omega V\nabla_\omega \int T(\rho-\omega)^{-1} q dv. \qquad \text{(xcix)}$$

Hence any quaternion may be expressed as the result of operating by ∇ on another

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* Hence another proof of (lxxxvi). † In fact $\rho\nabla = \nabla\rho = -3$.

quaternion (Q) or as the sum of the results of operating by ∇ on a scalar and on a vector; or generally

$$q = \nabla Q. \qquad \text{(c)}$$

(41.) We shall transform this new quaternion Q so as to exhibit more clearly its relation to q. Integrating through the limited field and excluding the small sphere round (ω),

$$\int \nabla T(\rho - \omega)^{-1} \, . \, q dv + \int T(\rho - \omega)^{-1} \, . \, \nabla q dv = \int \nabla \, . \, T(\rho - \omega)^{-1} q dv = \int d\nu T(\rho - \omega)^{-1} q \qquad \text{(ci)}$$

the surface integral being taken over the boundary of the field (40.) and the surface integral over the small sphere being omitted as it ultimately vanishes (33.). Now $(\nabla + \nabla_\omega) T(\rho - \omega)^{-1} = 0$,

so $$4\pi Q = \nabla_\omega \int T(\rho - \omega)^{-1} q dv = \int T(\rho - \omega)^{-1} \, . \, \nabla q dv - \int T(\rho - \omega)^{-1} d\nu q. \qquad \text{(cii)}$$

The surface integral here disappears when the field of integration is unlimited.

(42.) This transformation is of importance in vortex motion for example.

Considering more particularly the vector part of the volume integral (cii), we have by section (20.) (xlii),

$$\nabla^2 S q = S \nabla V \nabla q = -\frac{1}{TV\nabla q} \, . \, lim. \frac{d \log (dA \, . \, TV\nabla q)}{dl} \qquad \text{(ciii)}$$

dA being the small area of a cross-section of a tube formed by the vectors $V\nabla q$ and dl an element of the length of the tube. Using the relation

$$T\xi \, . \, dA = dm \quad \text{where} \quad \xi = V\nabla V q \qquad \text{(civ)}$$

and where dm is the *strength** of the tube of vectors ξ we have

$$\int T(\rho - \omega)^{-1} \, . \, \xi dv = \int T(\rho - \omega)^{-1} \, . \, U\xi dl dm = \int T(\rho - \omega)^{-1} d\rho dm \qquad \text{(cv)}$$

if $d\rho = U\xi \, . \, dl$ is a directed element along the tube because $dv = dAdl$.

(43.) For the case in which the tubes (ξ) are re-entrant and included within the limits of integration the integral on the right may be regarded as the sum of a number of integrals taken round closed curves. If then we describe any surface through one of these curves so that it does not pass through the extremity of ω, by (lxxxiii)

$$\int \nabla T(\rho - \omega)^{-1} \, . \, d\rho = \int \nabla T(\rho - \omega)^{-1} \, . \, V d\nu \nabla = \int \nabla S d\nu \nabla T(\rho - \omega)^{-1} \qquad \text{(cvi)}$$

* Lamb, *Hydrodynamics*, p. 223, Cambridge, 1895.

because as $\nabla^2 T(\rho - \omega)^{-1} = 0$, the term involving ∇^2 in the expansion $\nabla . V d\nu \nabla = \nabla S d\nu \nabla - d\nu \nabla^2$ disappears. Again we may replace the equation (cvi) by

$$\nabla_\omega \int T(\rho - \omega)^{-1} . d\rho = - \int \nabla S d\nu \nabla T(\rho - \omega)^{-1} = - \nabla_\omega \int S d\nu U(\rho - \omega) T(\rho - \omega)^{-2} \quad \text{(cvii)}$$

and if we suppose the surface built up of elementary cones through the extremity of the vector ω, it is evident that the cross-sections of these cones alone contribute so that we may replace $d\nu$ by $U(\rho - \omega) T(\rho - \omega)^2 d\Omega$ and finally*

$$\nabla_\omega \int T(\rho - \omega)^{-1} . d\rho = \nabla_\omega \int d\Omega = \nabla_\omega \Omega \quad \text{(cviii)}$$

where Ω is the solid angle subtended by the re-entrant tube at the extremity of ω. Thus if none of the circuits pass through the extremity of ω

$$\nabla_\omega \int T(\rho - \omega)^{-1} . \xi dv = \nabla_\omega \int \Omega dm. \quad \text{(cix)}$$

(44.) To illustrate the use of the operator we shall briefly consider the equations of motion of a continuous distribution of matter. Directing the attention to any selected portion its momentum is

$$M\sigma = \int \dot{\rho} dm \quad \text{(cx)}$$

σ being the velocity of the centre of mass, dm an element moving with velocity $\dot{\rho}$ and M the mass of the portion. If λ is the resultant force acting on the mass it is equal to the rate of change of momentum, or

$$D_t \sigma = M^{-1} \lambda. \quad \text{(cxi)}$$

We may evidently suppose the selected portion of such a size that the velocity of its centre of mass approaches indefinitely the velocity of the matter about that point. Again taking moments about the centre of mass we may write

$$D_t \int V(\rho - \rho_0)(\dot{\rho} - \dot{\rho}_0) dm = \mu + \int V(\rho - \rho_0) d\lambda \quad \text{(cxii)}$$

where μ is the resultant couple arising from other causes than the force-couple $\int V(\rho - \rho_0) d\lambda$.

(45.) We shall now consider the transformations of the vector of acceleration $D_t \sigma$ (cxi). If we regard† σ as a function of ρ and t we have (xi) its total differential expressed by

$$D\sigma = \frac{\partial \sigma}{\partial t} dt - S d\rho \nabla . \sigma \quad \text{(cxiii)}$$

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* Hence the result of operating by ∇ on a vector of a certain kind is equivalent to the result of operating on a scalar.

† Of course on the supposition made in the last section the vectors σ and $\dot{\rho}$ are identical.

the partial derived with respect to the time being $\frac{\partial\sigma}{\partial t}$, or in other words $\frac{\partial\sigma}{\partial t}$ being the rate at which the vector σ corresponding to a point fixed in space is changing. But $D_t\sigma$ is the rate at which the vector σ as corresponding to a definite portion of the matter is changing. So when we follow the motion of the matter, $d\rho = \sigma dt$ and

$$D_t\sigma = \frac{\partial\sigma}{\partial t} - S\sigma\nabla \,.\, \sigma = M^{-1}\lambda \tag{cxiv}$$

in which ∇ of course operates only on the σ to the right. In this case the appropriate form of the equation of continuity is (xl) if c is the density

$$\frac{\partial c}{\partial t} = S\nabla(c\sigma). \tag{cxv}$$

(46.) On the other hand if in Lagrange's method we suppose ρ to be a function of t and of three parameters u, v, and w which individualize any element of matter the velocity and acceleration of the centre of mass may be represented simply by $\dot{\rho}$ and $\ddot{\rho}$, the partial deriveds of ρ with respect to the time, and the equation of motion is

$$\ddot{\rho} = M^{-1}\lambda. \tag{cxvi}$$

Also the appropriate form of the equation of continuity is

$$cS\rho_1\rho_2\rho_3 = const. = -C \tag{cxvii}$$

which expresses that the mass $\pm cS\rho_1\rho_2\rho_3 du dv dw$ of a small definite parallelepiped of the matter does not vary, ρ_1, ρ_2, and ρ_3 being the deriveds of ρ with respect to u, v, and w.

(47.) It is easy to derive (cxvii) from (cxv) for remembering the meaning of the fluxional notation*

$$\dot{c} = \frac{\partial c}{\partial t} - S\sigma\nabla . c = cS\nabla\sigma = cS\nabla\dot{\rho}. \tag{cxviii}$$

But exactly as in section (12.), when ρ is expressed in terms of three parameters u, v, and w, the appropriate form of ∇ derived from (xiii) by taking

$$d\rho = \rho_1 du, \quad d'\rho = \rho_2 dv, \quad \text{and} \quad d''\rho = \rho_3 dw \tag{cxix}$$

is

$$\nabla = -\frac{1}{S\rho_1\rho_2\rho_3} \,.\, \left(V\rho_2\rho_3\frac{\partial}{\partial u} + V\rho_3\rho_1\frac{\partial}{\partial v} + V\rho_1\rho_2\frac{\partial}{\partial w}\right). \tag{cxx}$$

Hence evidently

$$\frac{d}{dt}\log c = S\nabla\dot{\rho} = -\frac{d}{dt}\log S\rho_1\rho_2\rho_3 \tag{cxxi}$$

* As an exercise one may verify that $S\dot{\nabla}\rho = -S\nabla\dot{\rho}$.

where the differentiations have the same meaning as the fluxional notation which is not here convenient for printing.

(48.) As regards the forces acting on the element, we have in the first place bodily or external forces ξ acting at each point and generally specified with respect to unit of volume. These contribute the volume integral $\int c\xi dv$.

In the second place there are the forces due to the interaction of the parts of the substance. Their resultant is suitably represented by a surface integral $\int \Phi d\nu$ where $\Phi d\nu + \Phi(-d\nu) = 0$ because the interaction across a directed element from one side is balanced by that on the other, and where $\Phi d\nu = \Phi U d\nu . T d\nu$ because the force is ultimately proportional to the area. Thus (cxi) becomes

$$D_t\sigma = M^{-1}\int c\xi dv + M^{-1}\int \Phi d\nu = M^{-1}\int c\xi dv + M^{-1}\int \Phi_0 d\nu - M^{-1}\int S\tau\nabla . \Phi_0 d\nu \quad \text{(cxxii)}$$

Φ_0 being what the function Φ becomes at the origin of the small vectors τ (16.) which may for convenience be taken centrally within the element. The integral $\int \Phi_0 d\nu$ must vanish as it is only of the second order in the linear dimensions of the element while the others are of the third order. Hence Φ (or Φ_0) must be a linear and distributive function for $\Sigma\Phi_0 d\nu = 0$ whenever $\Sigma d\nu = 0$.* And therefore by an application of the integration theorem (lxxxiii) because Φ is distributive and linear

$$\int \Phi d\nu = \int \Phi\nabla . dv. \quad \text{(cxxiii)}$$

From this (cxxii) gives when the element is very small

$$D_t\sigma = \xi + c^{-1} . \Phi\nabla \quad \text{(cxxiv)}$$

(where ∇ operates on Φ *in situ*) for ultimately $M = cdv$.

(49.) Again we may write the couple equation (cxii) in the form

$$\int V\tau\ddot{\tau}dm = \int V\tau\xi dm + \int \eta dm + \int V\tau\Phi d\nu \quad \text{(cxxv)}$$

where the origin of vectors τ is at the centre of mass and where η is the voluminal distribution of impressed couple. By the principle of linear dimensions employed in the last section we must have separately

$$\int \eta dm + \int V\tau\Phi_0 d\nu = 0 \quad \text{(cxxvi)}$$

or ultimately if we take the element to be a small parallelepiped whose sides are parallel to α, β, γ,

$$\eta c S\alpha\beta\gamma + V\alpha\Phi_0 V\beta\gamma + V\beta\Phi_0 V\gamma\alpha + V\gamma\Phi_0 V\alpha\beta = 0 \quad \text{(cxxvii)}$$

or simply

$$\eta c = 2\epsilon \quad \text{(cxxviii)}$$

if ϵ is the spin-vector of Φ_0.

Thus if there is no impressed couple η the function Φ (or Φ_0) must be self-conjugate.

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* For example take a small tetrahedron whose directed faces are α, β, γ, and δ. Then $\Phi_0(\alpha + \beta + \gamma) = \Phi_0\alpha + \Phi_0\beta + \Phi_0\gamma$ because $\alpha + \beta + \gamma = -\delta$.

(50.) Neglecting small terms of the second order in (cxxiv) and elsewhere the motion of the substance is completely given by

$$\frac{\partial\sigma}{\partial t} - S\sigma\nabla . \sigma = \xi + c^{-1} . \Phi\nabla ; \quad \frac{\partial c}{\partial t} = S\nabla(c\sigma) \qquad \text{(cxxix)}$$

when we employ Euler's method (compare (cxiv), (cxv), (cxxiv)); or by

$$\ddot{\rho} = \xi + C^{-1}\left(\frac{\partial}{\partial u}\Phi . V\rho_2\rho_3 + \frac{\partial}{\partial v}\Phi . V\rho_3\rho_1 + \frac{\partial}{\partial w}\Phi . V\rho_1\rho_2\right) \qquad \text{(cxxx)}$$

when we employ Lagrange's (compare (cxvii), (cxx)), the function Φ being linear, vector and self-conjugate, and this function, not the vectors $V\rho_2\rho_3$, &c., being differentiated with respect to u, v, and w.

(51.) We shall now apply Lord Kelvin's great conception of the *flow* along a finite curve drawn in the medium and moving with it so that it always threads the same elements. The flow is the integral of the component velocities of the various points of the curve along the corresponding tangents and is given by

$$F = -\int S\sigma d\rho = -\int S\dot{\rho}d\rho. \qquad \text{(cxxxi)}$$

It is convenient to suppose ρ and σ or its equal $\dot{\rho}$ expressed in terms of the time and the necessary parameters as in Lagrange's method. The time rate of change is thus

$$\dot{F} = -\frac{d}{dt}\int S\dot{\rho}d\rho = -\int S\ddot{\rho}d\rho - \int S\dot{\rho}d\dot{\rho}. \qquad \text{(cxxxii)}$$

The second integral on the right is simply half the difference of the squares of the velocities of the extremities of the curve. The first integral depends generally on the nature of the curve connecting these extremities. It is however quite independent of the curve if (compare section (24.))

$$V\nabla\ddot{\rho} = 0 \quad \text{or} \quad V\nabla D_t\sigma = 0 \qquad \text{(cxxxiii)}$$

for then the expression under the sign of integration is integrable without a factor.* By (cxxiv) we have in this case

$$V\nabla\xi + V\nabla . c^{-1} . \Phi\nabla = 0 \qquad \text{(cxxxiv)}$$

and when this is satisfied we may speak of the rate of change of flow from one point to another without mentioning a connecting curve.

* The vectors $\ddot{\rho}$ and $D_t\sigma$ are identical in as much as they represent the same acceleration.

(52.) For a perfect fluid $\Phi d\nu = -p d\nu$ and $\Phi\nabla = -p\nabla = -\nabla p$ so (cxxxiv) reduces to

$$V\nabla\xi + V.\nabla c^{-1}.\nabla p = 0 \qquad \text{(cxxxv)}$$

and is satisfied if the density (c) is a function of the pressure (p) and if the forces (ξ) have a force function (P).

Under these conditions we find without trouble

$$\dot{F} = \left[\tfrac{1}{2}T\sigma^2 - P - \int \frac{dp}{c}\right] \qquad \text{(cxxxvi)}$$

where the square brackets indicate that the difference is to be taken of the values of the enclosed expression at the extremities of the curve.

In general when we integrate round a closed curve the flow or circulation changes at the rate (lxxx)

$$-\frac{d}{dt}\int S\dot{\rho}d\rho = -\int S\ddot{\rho}d\rho = -\int Sd\nu V\nabla\ddot{\rho}. \qquad \text{(cxxxvii)}$$

This vanishes under the supposed conditions (cxxxiii) so whenever the density of a perfect fluid is a function of the pressure conservative forces are powerless to alter the circulation in any circuit moving with the fluid.

(53.) It appears from (cxxxvii) that the component of $V\nabla\ddot{\rho}$ or $V\nabla D_t\sigma$ normal to any small unit circuit measures the rate of change of circulation in that circuit; and $V\nabla D_t\sigma$ determines the aspect and the rate of change of circulation of the unit circuit in which this rate of change is a maximum.

On the other hand $V\nabla\sigma$ determines the aspect and the circulation of the unit circuit* in which the circulation is a maximum, and $D_tV\nabla\sigma$ measures the rate of change (following the motion of the fluid) from one principal unit circuit to another. A principal unit circuit obviously does not remain fixed in the fluid.

The difference between these vectors is easily seen to be

$$V\nabla D_t\sigma - D_tV\nabla\sigma = -V\nabla_{,}S\sigma_{,}\nabla.\sigma = -V\nabla_{,}V.\sigma_{,}V\nabla\sigma \qquad \text{(cxxxviii)}$$

for ∇ and $\frac{\partial}{\partial t}$ are commutative in order of operation so that as a first step the difference is†

$$-V\nabla(S\sigma\nabla.\sigma) + S\sigma\nabla.V\nabla\sigma.$$

It vanishes as it ought if $V\nabla\sigma = 0$. In Lagrange's method the equivalent equation is

$$V\nabla\ddot{\rho} - \frac{d}{dt}V\nabla\dot{\rho} = -V\dot{\nabla}\dot{\rho}. \qquad \text{(cxxxix)}$$

* This has been called the principal circuit (23.).

† It is useful to observe that a term such as $V\nabla\nabla_{,}S\sigma\sigma_{,}$ vanishes for it should remain unchanged when the suffixes are transposed but it apparently changes sign.

(54.) As an additional example on the application of the operator ∇, we shall consider the nature of the stress in a viscous fluid. We assume as usual that the stress consists of a hydrostatic pressure p and a part linear in the rate of distortion or in the constituents of the strain function $\phi_0 = \frac{1}{2}(\phi + \phi')$ of section (27.), and that the principal planes of the stress-function (Φ) and the strain function ϕ_0 coincide. These considerations lead to the equation

$$\Phi\alpha = -p\alpha + 2n\phi_0\alpha + n'm''\alpha \qquad \text{(cxl)}$$

where α is an arbitrary vector, where n and n' are scalars independent of the rate of distortion and where $m''(= -\mathrm{S}\nabla\sigma)$ is the first invariant of ϕ_0. For this is the most general linear function involving p in the manner specified and linear in the constituents of ϕ_0 and having the same principal planes.

(55.) Defining p more particularly by the condition that the hydrostatic pressure is equal to the mean of the magnitudes of the principal stresses, we have, for i, j, and k along the principal axes,

$$-\Sigma \mathrm{S}i\Phi i = M'' = -3p + (2n + 3n')m''; \qquad \text{(cxli)}$$

and the condition requires

$$2n + 3n' = 0. \qquad \text{(cxlii)}$$

Therefore when we replace n' in terms of n and ϕ_0 in terms of ∇ and σ (section (27.))

$$\Phi\alpha = -p\alpha - n(\mathrm{S}\alpha\nabla\,.\,\sigma + \nabla\,.\,\mathrm{S}\alpha\sigma) + \tfrac{2}{3}n\alpha\mathrm{S}\nabla\sigma. \qquad \text{(cxliii)}$$

If n is constant, the equation of motion (cxxiv) becomes

$$\mathrm{D}_t\sigma = \xi - c^{-1}\nabla p - c^{-1}n(\nabla^2\sigma + \tfrac{1}{3}\nabla\mathrm{S}\nabla\sigma). \qquad \text{(cxliv)}$$

(56.) In like manner for an isotropic elastic solid if σ is the displacement, the stress is given by (cxl) when p is put equal to zero, and the equation analogous to (cxliv) is

$$\mathrm{D}_t\sigma = \xi - c^{-1}n\nabla^2\sigma - c^{-1}(n + n')\nabla\mathrm{S}\nabla\sigma. \qquad \text{(cxlv)}$$

(57.) The rate of change of the kinetic energy of the substance in any region fixed in space is evidently

$$\frac{\partial}{\partial t}\int \tfrac{1}{2}c\mathrm{T}\sigma^2 dv. \qquad \text{(cxlvi)}$$

This is due to the activity of the forces acting on the substance and to the transference of portions of the substance through the walls of the fixed enclosure.

Transforming and utilizing the equations of continuity and of motion (cxxix), so as to remove the differentials with respect to the time*

$$\frac{\partial}{\partial t}\int \tfrac{1}{2}c\mathrm{T}\sigma^2 dv = \int (\tfrac{1}{2}\mathrm{T}\sigma_0{}^2\mathrm{S}\nabla(c\sigma) - c\mathrm{S}\sigma_0\nabla\mathrm{S}\sigma_0\sigma - \mathrm{S}\sigma_0\Phi\nabla - c\mathrm{S}\sigma\xi)dv,$$

where σ_0 is free from the operation of ∇. Or again this is

$$\frac{\partial}{\partial t}\int \tfrac{1}{2}c\mathrm{T}\sigma^2 dv = \int (\tfrac{1}{2}\mathrm{S}\nabla(c\sigma\mathrm{T}\sigma^2) - \mathrm{S}\sigma\Phi(\nabla) + \mathrm{S}\sigma\Phi_0\nabla - c\mathrm{S}\sigma\xi)dv,$$

where (∇) operates both on σ and Φ and where Φ_0 is free from ∇.

Finally on integrating by parts

$$\frac{\partial}{\partial t}\int \tfrac{1}{2}c\mathrm{T}\sigma^2 dv = \int \tfrac{1}{2}c\mathrm{T}\sigma^2\mathrm{S}\sigma d\nu - \int \mathrm{S}\sigma\Phi d\nu + \int \mathrm{S}\sigma\Phi_0\nabla dv - \int c\mathrm{S}\sigma\xi dv. \quad \text{(cxlvii)}$$

The first integral on the right is the rate of increase of kinetic energy due to the influx of fresh matter; the second is the activity of the surface stress; the fourth that of the external forces; and the third with sign changed measures the rate at which energy is stored in the substance and dissipated (see section (59.)).

(58.) On the other hand for a definite portion of the substance the rate of change of kinetic energy is

$$\mathrm{D}_t\int \tfrac{1}{2}c\mathrm{T}\sigma^2 dv = \mathrm{D}_t\int \tfrac{1}{2}\mathrm{T}\sigma^2 dm = -\int \mathrm{S}\sigma\mathrm{D}_t\sigma dm = -\int \mathrm{S}\sigma(c\xi + \Phi\nabla)dv. \quad \text{(cxlviii)}$$

This reduces as in the last section the only difference being that there is no contribution due to influx across the boundary.

(59.) When Φ is given by the equation (cxliii),

$$-\mathrm{S}\sigma\Phi_0\nabla = +p\mathrm{S}\nabla\sigma + n(\mathrm{S}\nabla\nabla_{,}\mathrm{S}\sigma\sigma_{,} + \mathrm{S}\nabla\sigma_{,}\mathrm{S}\nabla_{,}\sigma) - \tfrac{2}{3}n(\mathrm{S}\nabla\sigma)^2 \quad \text{(cxlix)}$$

is the rate of storage and waste of energy per unit volume.

The term in p may be modified as follows. By the equation of continuity

$$\mathrm{S}\nabla\sigma = \mathrm{D}_t \log c = -\mathrm{D}_t \log b \quad \text{(cl)}$$

if b is the reciprocal of the density (c) or the *bulkiness* of the fluid. Hence as p is a function of c and therefore of b

$$p\mathrm{S}\nabla\sigma = -\mathrm{D}_t\int pb^{-1}db. \quad \text{(cli)}$$

Also we have† for the rate of change of the intrinsic energy of a given mass

$$\int p\mathrm{S}\nabla\sigma dv = -\int p\mathrm{D}_t c^{-1}dm = -\mathrm{D}_t\int dm\int pdb. \quad \text{(clii)}$$

* Namely from $\int\left(\tfrac{1}{2}\frac{\partial c}{\partial t}\mathrm{T}\sigma^2 - c\mathrm{S}\sigma\frac{\partial\sigma}{\partial t}\right)dv$.

† Compare Lamb's *Hydrodynamics*, Art. 287.

(60.) The part of (cxlix) quadratic in σ has been called by Lord Rayleigh the Dissipation Function. It measures the rate at which energy is wasted by the viscosity and it admits of many transformations which may serve as exercises.

It is essentially positive, for if we write the invariant m' of section (28.) in the form

$$m' = \tfrac{1}{2}(S\nabla\sigma)^2 - \tfrac{1}{2}S\nabla\sigma_{,}S\nabla_{,}\sigma \qquad \text{(cliii)}$$

and

$$S\nabla\nabla_{,}S\sigma\sigma_{,} = S\nabla\sigma_{,}S\sigma\nabla_{,} - SV\nabla\sigma V\nabla_{,}\sigma_{,} \qquad \text{(cliv)}$$

we have in the notation of the section cited

$$2F = n(S\nabla\nabla_{,}S\sigma\sigma_{,} + S\nabla\sigma_{,}S\nabla_{,}\sigma - \tfrac{2}{3}(S\nabla\sigma)^2) = \tfrac{4}{3}n(m''^2 - 3m' - 3\epsilon^2) \qquad \text{(clv)}$$

when we utilize (cliii) and (cliv) to eliminate $S\nabla\nabla_{,}S\sigma\sigma_{,}$ and $S\nabla\sigma_{,}S\nabla_{,}\sigma$.

But (p. 520, vol. i.) $m' + \epsilon^2$ is the sum of the products of the roots of the self-conjugate function $\phi_0 = \frac{1}{2}(\phi + \phi')$, so if these roots are e_1, e_2, and e_3,

$$2F = \tfrac{2}{3}n\,((e_2 - e_3)^2 + (e_3 - e_1)^2 + (e_1 - e_2)^2). \qquad \text{(clvi)}$$

If then the dissipation function vanishes every spherical element must remain spherical, for the condition is

$$e_1 = e_2 = e_3. \qquad \text{(clvii)}$$

Again as $\nabla\sigma^2 = 2\nabla S\sigma_{,}\sigma$ if $\sigma_{,}$ is free from $\nabla_{,}$ we have

$$\nabla^2\sigma^2 = 2\nabla^2 S\sigma_{,}\sigma + 2\nabla_{,}\nabla S\sigma_{,}\sigma = 2S\sigma_{,}\nabla^2\sigma + 2S\nabla_{,}\nabla S\sigma_{,}\sigma. \qquad \text{(clviii)}$$

Hence by (cliv), we obtain the relation

$$2F = n(\nabla^2\sigma^2 - 2S\sigma_{,}\nabla^2\sigma + (V\nabla\sigma)^2 - \tfrac{2}{3}(S\nabla\sigma)^2)$$

in which the operator is contiguous to the operand.

Integrating and supposing n constant we may transform as follows:*

$$2\int F\mathrm{d}v = n\int S\mathrm{d}\nu\nabla\,.\,\sigma^2 - 2n\int S\sigma_{,}\mathrm{d}\nu\nabla\sigma + n\int(\tfrac{4}{3}(S\nabla\sigma)^2 - (V\nabla\sigma)^2)\mathrm{d}v \qquad \text{(clix)}$$

because

$$\int S\sigma_{,}\nabla^2\sigma\mathrm{d}v = \int S\sigma_{,}\mathrm{d}\nu\nabla\sigma - \int S\sigma_{,}\nabla_{,}\nabla\sigma\mathrm{d}v$$

and

$$S\sigma_{,}\nabla_{,}\nabla\sigma = (S\nabla\sigma)^2 - (V\nabla\sigma)^2.$$

(61.) Before passing on to other matters, we shall consider the expression of stress in terms of strain.† By Hooke's law stress is a linear function of strain and

* Compare Lamb, *loc. cit.*

† Here, as elsewhere in this Appendix, my object is to provide suggestive illustrations of quaternion methods rather than short solutions of special problems.

therefore of the space variations of the displacement. Consequently the stress across any small plane area (ω) is a linear function of ω, of ∇, and of the displacement σ, the operand of ∇. Thus we may write

$$\Phi\omega = \theta(\omega, \nabla, \sigma) \quad \text{(clx)}$$

and we shall investigate in the first place the nature of this trilinear vector function θ. We have seen that $\Phi\omega$ is a self-conjugate function of ω. Therefore for any pair of vectors ω and ϖ,

$$S\varpi\theta(\omega, \nabla, \sigma) = S\omega\theta(\varpi, \nabla, \sigma). \quad \text{(clxi)}$$

Again we know when a potential function exists that the expression (comp. (cxlix))

$$S\Phi\nabla_{,}\sigma_{,} = S\sigma_{,}\theta(\nabla_{,}, \nabla, \sigma) \quad \text{(clxii)}$$

is symmetrical in the strain arising from the displacement σ and in that arising from the displacement $\sigma_{,}$, it being understood that ∇ operates on σ alone and $\nabla_{,}$ on $\sigma_{,}$. Therefore identically

$$S\sigma_{,}\theta(\nabla_{,}, \nabla, \sigma) = S\sigma\theta(\nabla, \nabla_{,}, \sigma_{,}). \quad \text{(clxiii)}$$

The two properties expressed by the equations (clxi) and (clxiii) furnish us with sufficient data to determine the nature of the function θ, or in other words to express stress in terms of strain.

(62.) On account of the arbitrariness of the vectors σ and $\sigma_{,}$ we may replace the equations just referred to by

$$S\alpha\theta(\beta, \gamma, \delta) = S\beta\theta(\alpha, \gamma, \delta) = S\delta\theta(\gamma, \beta, \alpha) \quad \text{(clxiv)}$$

where α, β, γ, and δ are four arbitrary vectors. Using as a matter of convenience the symbol $(\alpha, \beta, \gamma, \delta)$ defined by the equation

$$(\alpha, \beta, \gamma, \delta) = - S\alpha\theta(\beta, \gamma, \delta), \quad \text{(clxv)}$$

we see by (clxiv) that it is permissible to reverse the order of the vectors and to transpose the first and second vectors. Hence ringing the changes on these allowable alterations we have

$$\begin{aligned}(\alpha, \beta, \gamma, \delta) &= (\beta, \alpha, \gamma, \delta) = (\delta, \gamma, \alpha, \beta) = (\gamma, \delta, \alpha, \beta)\\ &= (\beta, \alpha, \delta, \gamma) = (\alpha, \beta, \delta, \gamma) = (\gamma, \delta, \beta, \alpha) = (\delta, \gamma, \beta, \alpha),\end{aligned} \quad \text{(clxvi)}$$

and the laws of the symbols $(\alpha, \beta, \gamma, \delta)$ may be summed up in the statement, the pair composed of the first and second vectors is interchangeable with the pair composed of the third and fourth and the members of each pair are likewise interchangeable.

(63.) Since $(\delta, \alpha, \beta, \gamma) = (\delta, \alpha, \gamma, \beta)$ we have generally as the vectors are arbitrary,

$$\theta(\alpha, \beta, \gamma) = \theta(\alpha, \gamma, \beta). \qquad \text{(clxvii)}$$

In particular

$$\Phi\omega = \theta(\omega, \nabla, \sigma) = \theta(\omega, \sigma, \nabla); \qquad \text{(clxviii)}$$

or more fully for any mutually rectangular system i, j, k

$$\Phi\omega = \Sigma\theta(\omega, i, i)\, Si\nabla Si\sigma + \Sigma\theta(\omega, j, k)\,(Sj\nabla Sk\sigma + Sk\nabla Sj\sigma) \qquad \text{(clxix)}$$

because $\theta(\omega, j, k) = \theta(\omega, k, j)$, or again in a usual notation for the strains,

$$\Phi\omega = \Sigma\theta(\omega, i, i)\, a + \Sigma\theta(\omega, j, k) \,.\, 2f. \qquad \text{(clxx)}$$

The constituents of the six vector functions $\theta(\omega, i, i)$, $\theta(\omega, j, k)$, &c. are the *elastic constants*. They are all of the type $(\alpha, \beta, \gamma, \delta)$ (comp. (clxv)) where α, β, γ, and δ stand for i, j, and k; and they fall into the following groups:—three of the type (i, i, i, i); six (i, i, i, j); three (i, i, j, j); three (i, j, i, j); three (j, k, i, i); and three (j, i, k, i); twenty-one in all bearing in mind the laws of the symbol $(\alpha, \beta, \gamma, \delta)$ (clxvi).

(64.) We saw at the beginning of the last section that the second and third vectors are interchangeable in $\theta(\alpha, \beta, \gamma)$. We shall now investigate the effect of interchanging the first and second vectors and we shall prove that

$$\theta(\alpha, \beta, \gamma) - \theta(\beta, \alpha, \gamma) = 2V\Theta V\alpha\beta \,.\, \gamma \qquad \text{(clxxi)}$$

where Θ is a linear and self-conjugate vector function of the ordinary kind. The left-hand member obviously vanishes if α and β are parallel. We are therefore entitled to assume

$$\theta(\alpha, \beta, \gamma) - \theta(\beta, \alpha, \gamma) = \chi(V\alpha\beta, \gamma) \qquad \text{(clxxii)}$$

where χ is a bi-linear function of $V\alpha\beta$ and of γ. Operating by $S\gamma$ and referring again to (clxvi) we find $S\gamma\chi(V\alpha\beta, \gamma) = 0$ for all vectors γ. The form of the right-hand member of (clxxi) is therefore justified and it only remains to prove that Θ is self-conjugate. To do so we operate by $S\delta$; and the law of interchanges again shows us that

$$S\delta\theta(\alpha, \beta, \gamma) - S\delta\theta(\beta, \alpha, \gamma) = S\alpha\theta(\delta, \gamma, \beta) - S\alpha\theta(\gamma, \delta, \beta)$$

when we find almost immediately

$$SV\gamma\delta\Theta V\alpha\beta = SV\alpha\beta\Theta V\gamma\delta, \qquad \text{(clxxiii)}$$

and Θ is self-conjugate as asserted.

(65.) We have now at our disposal two distinct geometrical methods of investigating the arrangement of the elastic properties of a body with respect to certain natural directions of reference. The first and the most obvious method consists in the study of the quartic surface

$$(\rho, \rho, \rho, \rho) = -\,S\rho\theta(\rho, \rho, \rho) = const. \qquad \text{(clxxiv)}$$

whose radii vectors are inversely proportional to the fourth roots of the elastic constants depending on a single direction—that of the corresponding radius vector.

When the body has a plane of symmetry normal to i, the elastic constants which involve i an odd number of times must vanish. Perhaps the most instructive way of seeing the truth of this is to equate the reflection, with respect to the plane of symmetry, of the stress across any small area to the stress due to the reflection of the strain across the reflection of the area. In this case the quartic surface has also a plane of symmetry. The converse is not generally true for the quartic depends on but fifteen constants, for example $2(jijk) + (jjik)$.

The surface must evidently be closed and finite; otherwise the potential energy might vanish for an actual strain. To discover the planes of symmetry, when they exist, we may calculate the positions of the summits of the surface,* or the points at which a concentric sphere can touch it. The vectors to these points have the directions of the solutions of

$$V\rho\theta(\rho, \rho, \rho) = 0 \qquad \text{(clxxv)}$$

for by the rule of interchanges

$$dS\rho\theta(\rho, \rho, \rho) = 4Sd\rho\theta(\rho, \rho, \rho).$$

The normal to a plane of symmetry obviously cuts the surface at a pair of summits. The radius of a touching sphere may be obtained by equating to zero the discriminant of the cone through its intersection with the surface, the centre being the vertex.† It is easy to see geometrically that three at least of the vector solutions of (clxxv) must be real.

(66.) When the potential energy involves the strains only in the combinations $a + b + c$ and the minors $bc - f^2$, &c., $gh - af$, &c. of the well-known determinant of a conic, that is when

$$2W = m(a + b + c)^2 + \Sigma m_1(bc - f^2) + \Sigma l_1(gh - af), \qquad \text{(clxxvi)}$$

the equation of the quartic reduces to

$$m\rho^4 = const. \qquad \text{(clxxvii)}$$

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* A more convenient process will be found in section (67.).

† When the surface has three planes of symmetry the equation has thirteen roots, one quadruple and three double.

The surface is spherical and fails to afford special directions of reference. In this case the second method to which we now proceed must be selected.

(67.) This method depends on the self-conjugate function Θ of section (64.). The coefficients of the quadric

$$S\rho\Theta\rho = const. \qquad \text{(clxxviii)}$$

are easily calculated in terms of the elastic constants by means of equations such as

$$\theta(j, k, \gamma) - \theta(k, j, \gamma) = 2V\Theta i\gamma \qquad \text{(clxxix)}$$

which is merely a modification of (clxxi). We find

$$2Si\Theta i = (jjkk) - (jkjk)\,; \quad 2Sj\Theta k = (ijik) - (iijk) \qquad \text{(clxxx)}$$

and the remaining coefficients may be written down from symmetry.

If the body has a plane of symmetry it must be a principal plane of this quadric, for if i is normal to a plane of symmetry $Sj\Theta i = Sk\Theta i = 0$ or $Vi\Theta i = 0$. The converse of course is not true. But (compare (65.)) when the quartic has a principal plane of the quadric for a plane of symmetry, we have from the equation of the quartic $2(jijk) + (jjik) = 0$ and from that of the quadric $(jijk) - (jjik) = 0$, &c. The elastic constants vanish separately and the plane is a plane of symmetry of the body.

Thus provided the quadric has determinate axes they form a natural system of lines of reference, and planes of symmetry may be at once detected by expressing the equation of the quartic in terms of these vectors. In the most general case having selected this system of axes we have only eighteen constants to deal with, the last group of (63.) being then merged in the preceding group. As an example for the case noticed in (66.)

$$(iiii) = m\,;\ (jjkk) = m + 2n_1\,;\ (jkkj) = -n_1\,;\ (iijk) = -2l_1\,;\ (jiik) = l_1 \qquad \text{(clxxxi)}$$

but when i, j, and k are along the axes of the quadric the constants l_1, l_2, and l_3 vanish.

(68.) It is only when the quadric is of revolution that the body can have two planes of symmetry not at right angles to one another; and moreover when the quadric is of revolution and when the quartic has a plane of symmetry through the axes of revolution it must be a plane of symmetry of the body, for every plane through the axis is a principal plane of the quadric. Taking the axis of revolution as axis of cylindrical coordinates z, p, u the equation of the quartic becomes

$$p^4 U_4 + zp^3 U_3 + z^2p^2 U_2 + z^3p U_1 + z^4 U_0 = \text{const.} \qquad \text{(clxxxii)}$$

where the suffixes denote the order in which cos u and sin u enter in the functions U.

If $u = 0$ is a plane of symmetry of the quartic the angle u must enter only in cosines and we may write

$$U_4 = a + a' \cos 2u + a'' \cos 4u\,; \quad U_3 = b \cos u + b' \cos 3u\,;$$
$$U_2 = c + c' \cos 2u\,; \quad U_1 = d \cos u. \quad \text{(clxxxiii)}$$

If $u = v$ is a second plane of symmetry, substitution of $v + w$ and of $v - w$ for u must lead to the same results. Hence

$$b \sin v = d \sin v = a' \sin 2v = c' \sin 2v = b' \sin 3v = a'' \sin 4v = 0. \quad \text{(clxxxiv)}$$

If the quartic is not a surface of revolution, the only admissible values of v are evidently $\frac{1}{2}\pi$, $\frac{1}{3}\pi$, and $\frac{1}{4}\pi$. Thus the planes of symmetry of the body must intersect at angles of 90°, 60°, or 45° if every plane through their intersection is not a plane of symmetry.

(69.) When the quadric is a sphere it fails of course to afford a natural system of lines of reference. This want may be supplied by the axes of the new quadric

$$\nabla^2 . (\rho\rho\rho\rho) = const. \quad \text{(clxxxv)}$$

for it is easy to see that a plane of symmetry of the quartic must be a principal plane of the quadric. In case this quadric is a sphere we can derive a third quadric by means of the operator ∇ to take its place. If for brevity $(\rho\rho\rho\rho) = f$, the equation of this quadric is

$$(S\nabla_1\nabla_2)^3 . f_1 f_2 = const., \quad \text{(clxxxvi)}$$

the suffixes being omitted after operation.

Even when this is a sphere, the quadric*

$$(S\nabla_1\nabla_3)^2\,(S\nabla_2\nabla_3)^2\,S\nabla_1\nabla_2 . f_1 f_2 f_3 = const. \quad \text{(clxxxvii)}$$

is available and must of necessity determine a natural system of axes if such exists. For when any one quadric becomes a sphere five conditions are established connecting the elastic constants. If the four quadrics are spheres but one constant remains in the equation of the quartic as in the case noticed in section (66.).

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* The equations of these third and fourth quadric may be obtained by operating by ∇^6 and ∇^{10} on f^2 and f^3 respectively and rejecting terms in $T\rho^2$. In Cartesians (clxxxvi) becomes

$$\Sigma(D_x^3 f)^2 + 3\Sigma(D_x^2 D_y f)^2 + 6(D_x D_y D_z f)^2 = const.$$

In Aronhold's notation if $f = a_x^4 = b_x^4$, the equation is

$$(a_1 b_1 + a_2 b_2 + a_3 b_3)^3 a_x b_x = const.$$

(70.) Although the subject is foreign to this Note on Hamilton's operator, it may be useful to offer here a few remarks on functions linear and distributive in several vectors as such functions have occurred in the treatment of stress. Though the process is general we take the case of a trilinear function and write in analogy with the notation for conjugates

$$S\alpha\theta(\beta, \gamma, \delta) = S\beta\theta'(\gamma, \delta, \alpha) = S\gamma\theta''(\delta, \alpha, \beta) = S\delta\theta'''(\alpha, \beta, \gamma). \qquad \text{(clxxxviii)}$$

If the function is self-conjugate in the first vector so that α and β may be interchanged in these equations, we must have in general

$$\theta(\alpha, \beta, \gamma) = \theta'(\beta, \gamma, \alpha); \quad \theta''(\alpha, \beta, \gamma) = \theta''(\alpha, \gamma, \beta);$$
$$\theta'''(\alpha, \beta, \gamma) = \theta'''(\beta, \alpha, \gamma). \qquad \text{(clxxxix)}$$

If it is self-conjugate in the second vector

$$\theta(\alpha, \beta, \gamma) = \theta''(\gamma, \beta, \alpha); \quad \theta'(\alpha, \beta, \gamma) = \theta'(\gamma, \beta, \alpha);$$
$$\theta'''(\alpha, \beta, \gamma) = \theta'''(\gamma, \beta, \alpha) \qquad \text{(cxc)}$$

and if it is self-conjugate in both of these

$$\theta(\alpha, \beta, \gamma) = \theta(\beta, \alpha, \gamma). \qquad \text{(cxci)}$$

If finally it is self-conjugate in all three they may be interchanged in all possible ways.

There is the closest analogy between these completely self-conjugate functions and Aronhold's notation $a_j a_s a_t a_u$ ($j = 1$, 2, or 3). We may imitate his notation by writing

$$\theta(\alpha, \beta, \gamma) = (\lambda)S(\lambda)\alpha S(\lambda)\beta S(\lambda)\gamma \qquad \text{(cxcii)}$$

where (λ) is a symbolic vector devoid of interpretation unless it occurs in a term involving three other vectors (λ). We may extend this notation to the case of non-conjugate functions by writing

$$\theta(\alpha, \beta, \gamma) = (\lambda)S(\mu)\alpha S(\nu)\gamma S(\varpi)\delta \qquad \text{(cxciii)}$$

where (λ), (μ), (ν), and (ϖ) are symbolic and uninterpretable unless they occur together in a term.

Reference to Aronhold's notation is sufficient to suggest a number of interpretations of quaternion forms. For example* if

$$\theta(\alpha, \beta) = 0 \qquad \text{(cxciv)}$$

where generally $\theta(\rho, \varpi) = \theta(\varpi, \rho)$, the vectors α and β are *corresponding* edges of the Hessian of a cubic cone $S\rho\theta(\rho, \rho) = 0$. The equation of the Hessian is $S\theta(\rho, \xi)\theta(\rho, \eta)\theta(\rho, \zeta) = 0$, ξ, η, and ζ being arbitrary constant vectors.

* This vector equation may be compared with the scalar $f(\alpha, \beta) = 0$, where generally $f(\rho, \varpi) = f(\varpi, \rho)$, which expresses that α and β are conjugate with respect to the cone $f(\rho, \rho) = 0$.

(71.) The operations performed in deducing the quadrics of section (69.) are related to the application of ∇ to the theory of Spherical Harmonics.* If $f_n(\nabla)$ is any integral and rational function of ∇ of degree n and with constant coefficients, $f_n\nabla \,.\, T\rho^{-1}$ is obviously a solid harmonic of order $-(n+1)$. In fact this function is of the degree $-(n+1)$ in $T\rho$ and it vanishes under the operator ∇^2.

It is always possible to determine a function $f_{n-2}\rho$ so that

$$f_n\rho + \rho^2 f_{n-2}\rho = S\alpha_1\rho S\alpha_2\rho \ . \ . \ . \ S\alpha_n\rho. \qquad \text{(cxcv)}$$

For draw n planes through distinct pairs of the $2n$ common edges of the cones $f_n\rho = 0$, $\rho^2 = 0$; and through $\frac{1}{2}(n-2)(n-2+3)$ of the remaining intersections of the planes with the cone $f_n\rho = 0$ draw a cone $f_{n-2}\rho = 0$. The complex cone $\rho^2 f_{n-2}\rho = 0$ passes through $2n + \frac{1}{2}(n-2)(n+1)$ or $\frac{1}{2}n(n+3) - 1$ of the intersections of $f_n\rho = 0$ with the n planes; it must consequently pass through all the remaining intersections as $\frac{1}{2}n(n+3) - 1$ is one less than the number of edges requisite to determine a cone of the n^{th} degree. The relation (cxcv) is therefore justified. Again the common edges of the cones $f_n\rho = 0$, $\rho^2 = 0$, group themselves into pairs $\alpha'_1 \pm \alpha''_1\sqrt{-1}$ and each group lies in a real plane. The reduction may therefore be uniquely effected in such a manner that the planes shall be all real. But in operating on $T\rho^{-1}$, any function $\nabla^2 f_{n-2}\nabla$ may be added to $f_n\nabla$ without altering the result. Thus we may always suppose†

$$f_n\nabla \,.\, T\rho^{-1} = S\alpha_1\nabla S\alpha_2\nabla \ . \ . \ . \ S\alpha_n\nabla \,.\, T\rho^{-1}. \qquad \text{(cxcvi)}$$

This enables us to expand any homogeneous function of ρ in a series of spherical harmonics. When we effect the operations indicated and multiply across by $T\rho^{2n+1}$, we have, ($F_{n-2}\rho$ being a determinate function of degree $n-2$),

$$T\rho^{2n+1} f_n\nabla \,.\, T\rho^{-1} = [n)S\alpha_1\rho S\alpha_2\rho \ . \ . \ . \ S\alpha_n\rho + T\rho^2 F_{n-2}\rho. \qquad \text{(cxcvii)}$$

where for the sake of brevity

$$[n) = (-)^n \,.\, 1 \,.\, 3 \,.\, 5 \ . \ . \ . \ (2n-1). \qquad \text{(cxcviii)}$$

Comparing (cxcv) and (cxcvii) we see that

$$f_n\rho = \frac{T\rho^{2n+1}}{[n)} \,.\, f_n\nabla \,.\, \frac{1}{T\rho} + T\rho^2 \,.\, g_{n-2}\rho \qquad \text{(cic)}$$

where $g_{n-2}\rho$ is a homogeneous function of ρ. Treating this new function $g_{n-2}\rho$ in the same manner we obtain the second harmonic in the series and the process may be repeated.

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* Much of the following is adapted from Clerk Maxwell's most interesting and instructive chapter on Spherical Harmonics, *Electricity and Magnetism.*

† The extremities of the vectors $U\alpha_1$, $U\alpha_2$, &c. are the *poles* of the spherical harmonic.

(72.) The potential due to any distribution of matter at any point (ρ) external to a sphere which encloses all the matter may be expressed by a relation of the form

$$P = f\nabla \,.\, \mathrm{T}(\rho - \alpha)^{-1} \qquad \text{(cc)}$$

$f\nabla$ being a function of ∇ expansible in ascending powers and the centre of the sphere being at the extremity of α. For if dm is the element of matter at the extremity of ϖ,

$$P = \int \frac{\mathrm{d}m}{\mathrm{T}(\rho - \varpi)} = \int \mathrm{d}m e^{\mathrm{S}(\varpi - \alpha)\nabla} \,.\, \frac{1}{\mathrm{T}(\rho - \alpha)} = f\nabla \,.\, \frac{1}{\mathrm{T}(\rho - \alpha)}.$$

If $Q_\alpha = \int \mathrm{d}m'\mathrm{T}(\rho - \alpha)^{-1}$ is the potential at the extremity of α of a second distribution of matter wholly exterior to the sphere enclosing the first, the mutual potential energy is

$$W = \int \mathrm{d}m'P = \int \mathrm{d}m' f(\nabla) \,.\, \mathrm{T}(\rho - \alpha)^{-1} = \int \mathrm{d}m' f(-\nabla_\alpha)\mathrm{T}(\rho - \alpha)^{-1} = f(-\nabla_\alpha) \,.\, Q_\alpha. \qquad \text{(cci)}$$

Or more conveniently if we take the origin at the centre of the sphere

$$W = f(-\nabla) \,.\, Q_0 \qquad \text{(ccii)}$$

provided we put $\rho = 0$ after the operations have been performed as indicated by the suffix.

If Q is due to a surface distribution of density s over the sphere

$$W = \int Ps\mathrm{d}S = f(-\nabla) \,.\, Q_0. \qquad \text{(cciii)}$$

When $Q = r^n Y_n$, Y_n being a spherical harmonic so that $4\pi s = (2n+1)a^{n-1}Y_n$ if a is the radius of the sphere, this equation becomes

$$4\pi f(-\nabla)r^n Y_n = (2n+1)a^{n+1}\int PY_n \mathrm{d}\Omega \qquad \text{(cciv)}$$

if dΩ is an element of solid angle. It is manifest that the terms in $f(-\nabla)$ of the n^{th} order in ∇ alone contribute to the left-hand member. For the operation of terms of higher order destroys $r^n Y_n$, and the results of operation of terms of lower order vanish when r is put equal zero. Hence in particular

$$4\pi f_n(-\nabla)r^n Y_n = (2n+1)\int Z_n Y_n \mathrm{d}\Omega\,; \quad \int Z_m Y_n \mathrm{d}\Omega = 0 \qquad \text{(ccv)}$$

if Z_n and Z_m are spherical harmonics and if

$$f_n(+\nabla) \,.\, r^{-1} = r^{-n-1}Z_n.$$

(73.) Up to the present we have scarcely considered the analytical structure of the operator ∇. In section (7.) we obtained an expression (xiii) depending on three arbitrary differentials and the corresponding differentiating symbols. In section (12.)

we employed the well-known Cartesian form (xviii) for purposes of illustration, and a third form (cxx) depending like (xviii) on the highly artificial method of determining a vector by means of those systems of surfaces occurred in section (47.) in connexion with Lagrange's method in fluid motion. Of all these forms (xiii) is the most accordant with the spirit of the *Elements* because there is perfect freedom in the choice of differentials most suitable for special purposes and because the conception of a vector as an entity is not obscured by any system of coordinates.

(74.) To leave as little obscurity as possible about the method of arbitrary differentials we shall consider the square of the operator (xiii) which we write for brevity in the form

$$\nabla = \delta d + \delta' d' + \delta'' d'' \qquad \text{(ccvi)}$$

where the vectors δ, δ', and δ'' are determined by the equations

$$\delta = -\frac{Vd'\rho d''\rho}{Sd\rho d'\rho d''\rho}, \quad \delta' = -\frac{Vd''\rho d\rho}{Sd\rho d'\rho d''\rho}, \quad \delta'' = -\frac{Vd\rho d'\rho}{Sd\rho d'\rho d''\rho}. \qquad \text{(ccvii)}$$

It must be observed however that any advantage that may arise from the use of this form is concealed when the operator is separated from the operand; and owing to the generality of the expression the result is apparently cumbrous. Squaring we find

$$\nabla^2 = \Sigma\delta^2 d^2 + \Sigma(\delta'\delta'' d'd'' + \delta''\delta' d''d') + \Sigma\nabla\delta\,.\,d. \qquad \text{(ccviii)}$$

In the third sum ∇ operates on the vectors δ alone and not on the operand of ∇^2.

Remembering that ∇^2 is a scalar operator this equation breaks up into two, a scalar and a vector,

$$\nabla^2 = \Sigma\delta^2 d^2 + \Sigma S\delta'\delta''(d'd'' + d''d') + \Sigma S\nabla\delta\,.\,d \qquad \text{(ccix)}$$

and

$$0 = \Sigma V\delta'\delta''(d'd'' - d''d') + \Sigma V\nabla\delta\,.\,d. \qquad \text{(ccx)}$$

It is only when the differentials are independent that the order in which the differentiations are performed is indifferent and in this case only is it generally lawful to suppress the terms involving $d'd'' - d''d'$ and similar expressions.

(75.) When independent differentials are employed, we may fall back on the equation (cxx) or

$$\nabla = -\frac{V\rho_2\rho_3}{S\rho_2\rho_3\rho_1}\cdot\frac{\partial}{\partial u} - \frac{V\rho_3\rho_1}{S\rho_3\rho_1\rho_2}\cdot\frac{\partial}{\partial v} - \frac{V\rho_1\rho_2}{S\rho_1\rho_2\rho_3}\cdot\frac{\partial}{\partial w}$$

which we shall write for brevity in the form

$$\nabla = \nu\frac{\partial}{\partial u} + \nu'\frac{\partial}{\partial v} + \nu''\frac{\partial}{\partial w} \qquad \text{(ccxi)}$$

where ν, ν', ν'' are normals to the surfaces determined by constant values of u, v and w

respectively. The following equations among others are satisfied by these normal vectors,

$$S\rho_1\nu + 1 = 0, \quad S\rho_2\nu = 0, \quad S\rho_3\nu = 0. \qquad \text{(ccxii)}$$

Also when we consider u, v and w to be functions of ρ we have

$$\nabla = \nabla u \frac{\partial}{\partial u} + \nabla v \frac{\partial}{\partial v} + \nabla w \frac{\partial}{\partial w}. \qquad \text{(ccxiii)}$$

So the vectors ν may be expressed by the equations

$$\nu = \nabla u, \quad \nu' = \nabla v, \quad \nu'' = \nabla w \qquad \text{(ccxiv)}$$

whence we find

$$V\nabla\nu = 0, \quad V\nabla\nu' = 0, \quad V\nabla\nu'' = 0. \qquad \text{(ccxv)}$$

(76.) These equations may be deduced from (ccx) as a particular case. In fact

$$\nabla^2 = \Sigma\nu^2 \frac{\partial^2}{\partial u^2} + 2\Sigma S\nu'\nu'' \frac{\partial^2}{\partial v\,\partial w} + \Sigma S\nabla\nu \frac{\partial}{\partial u} \qquad \text{(ccxvi)}$$

and

$$0 = V\nabla\nu \frac{\partial}{\partial u} + V\nabla\nu' \frac{\partial}{\partial v} + V\nabla\nu'' \frac{\partial}{\partial w}. \qquad \text{(ccxvii)}$$

The vector equation furnishes the three equations (ccxv) as appears by operating for example on u, v and w respectively.

(77.) But there is still another form for ∇, namely

$$\nabla q = -\frac{1}{S\rho_1\rho_2\rho_3}\left\{\frac{\partial}{\partial u}(V\rho_2\rho_3 . q) + \frac{\partial}{\partial v}(V\rho_3\rho_1 . q) + \frac{\partial}{\partial w}(V\rho_1\rho_2 . q)\right\} \qquad \text{(ccxviii)}$$

in which for greater clearness the operand is inserted. On expansion this obviously reduces to (cxx). Hence we have

$$\nabla^2 = +\frac{1}{S\rho_1\rho_2\rho_3}\left\{\Sigma\frac{\partial}{\partial u}\left(\frac{(V\rho_2\rho_3)^2}{S\rho_1\rho_2\rho_3}\frac{\partial}{\partial u}\right) + \Sigma\frac{\partial}{\partial u}\left(\frac{V\rho_2\rho_3 V\rho_3\rho_1}{S\rho_1\rho_2\rho_3}\frac{\partial}{\partial v}\right)\right\} \qquad \text{(ccxix)}$$

where the second sum includes six terms and to this the sign S may be prefixed. This may also be written in the more compact form

$$\nabla^2 = +S\nu\nu'\nu''\left\{\Sigma\frac{\partial}{\partial u}\left(\frac{\nu^2}{S\nu\nu'\nu''}\frac{\partial}{\partial u}\right) + \Sigma\frac{\partial}{\partial u}\left(\frac{S\nu\nu'}{S\nu\nu'\nu''}\frac{\partial}{\partial v}\right)\right\}. \qquad \text{(ccxx)}$$

(78.) The analytical expression for ∇^2 becomes immensely simplified in two important cases; (I.) whenever the parameters are Cartesian coordinates, rectangular or otherwise, for then the vectors ν, ν', ν'' are constant instead of being as in general variable with ρ; and (II.) whenever the three families of surfaces are mutually rectangular.

In the second case, which includes the most important application of the first, we may remove at once the superfluous symbols S and V from (cxx) or the first expression of (75.) we then have

$$\nabla = -\frac{1}{\rho_1}\frac{\partial}{\partial u} - \frac{1}{\rho_2}\frac{\partial}{\partial v} - \frac{1}{\rho_3}\frac{\partial}{\partial w}. \qquad \text{(ccxxi)}$$

Forming the square of this directly or replacing $S\rho_1\rho_2\rho_3$ by $-T\rho_1\rho_2\rho_3$ and $(V\rho_2\rho_3)^2$ by $-T\rho_2{}^2\rho_3{}^2$ in (ccxix) we obtain

$$\nabla^2 = -\frac{1}{T\rho_1\rho_2\rho_3}\Sigma\frac{\partial}{\partial u}\left(T\rho_1{}^{-1}\rho_2\rho_3\frac{\partial}{\partial u}\right) \qquad \text{(ccxxii)}$$

which is equivalent to the usual expression for ∇^2 in orthogonal curvilinear coordinates.

(79.) If the family of surfaces u *constant* is isothermal or equipotential, we must have

$$\nabla^2 f(u) = 0 \qquad \text{(ccxxiii)}$$

where $f(u)$ is the potential. Operating by ∇^2 as given by (ccxxii) on $f(u)$ we obtain

$$\frac{\partial}{\partial u}\left(T\rho_1{}^{-1}\rho_2\rho_3\frac{\partial f}{\partial u}\right) = 0 \quad \text{or} \quad T\rho_1{}^{-1}\rho_2\rho_3\frac{\partial f}{\partial u} = F(v, w). \qquad \text{(ccxxiv)}$$

If the parameter u is the potential, so that $\nabla^2 u = 0$, the product $T\rho_1\rho_2{}^{-1}\rho_3{}^{-1}$ is a function of v and w. If moreover the three families are equipotential (ccxxii) reduces to

$$\nabla^2 = \frac{1}{\rho_1{}^2}\frac{\partial^2}{\partial u^2} + \frac{1}{\rho_2{}^2}\frac{\partial^2}{\partial v^2} + \frac{1}{\rho_3{}^2}\frac{\partial^2}{\partial w^2} \qquad \text{(ccxxv)}$$

when u, v, and w are the corresponding potentials.

(80.) More generally we shall find the condition that the family of surfaces

$$f(\rho, u) = 0 \qquad \text{(ccxxvi)}$$

should be isothermal. If we suppose the parameter (u) of the family to be found by solution as a function of ρ we may treat (ccxxvi) as an identity and may equate to zero the results of operating by ∇ and ∇^2 on $f(\rho, u)$ when we operate both on u and on ρ. Hence

$$\nabla f + \nabla u \,.\, \frac{\partial f}{\partial u} = 0 \qquad \text{(ccxxvii)}$$

where ρ alone is operated on in ∇f. Again

$$\nabla^2 f + \nabla u \,.\, \frac{\partial \nabla f}{\partial u} + \nabla\frac{\partial f}{\partial u} \,.\, \nabla u + \nabla^2 u \,.\, \frac{\partial f}{\partial u} + (\nabla u)^2 \,.\, \frac{\partial^2 f}{\partial u^2} = 0. \qquad \text{(ccxxviii)}$$

But evidently ∇ as operating on ρ is commutative with $\frac{\partial}{\partial u}$, so we have the simpler expression

$$\nabla^2 f + 2S\nabla u \,.\, \frac{\partial \nabla f}{\partial u} + \nabla^2 u \frac{\partial f}{\partial u} + (\nabla u)^2 \frac{\partial^2 f}{\partial u^2} = 0. \qquad \text{(ccxxix)}$$

Now if P is a function of u which satisfies $\nabla^2 P = 0$, as a particular case of this equation

$$\frac{\partial}{\partial u} \log \frac{\partial P}{\partial u} + \frac{\nabla^2 u}{(\nabla u)^2} = 0 \qquad \text{(ccxxx)}$$

Eliminating ∇u and $\nabla^2 u$ between (ccxxvii), (ccxxix) and (ccxxx) we obtain without difficulty

$$\frac{\partial}{\partial u} \log \frac{\partial P}{\partial u} = \frac{\partial}{\partial u} \log \frac{\partial f}{du} - \frac{\partial}{\partial u} \log (\nabla f)^2 + \frac{\nabla^2 f}{(\nabla f)^2} \cdot \frac{\partial f}{\partial u}. \qquad \text{(ccxxxi)}$$

When the operations* indicated have been performed on the right-hand member of this equation, it must be possible to reduce it by means of the equation (ccxxvi) of the family of surfaces to a function of u alone if the family is isothermal. This condition being satisfied, two integrations afford P, the temperature (or the potential) appropriate to the surfaces. The condition may be obtained explicitly, for if $F(\rho, u)$ can thus be reduced to a function of u,

$$\nabla \,.\, F + \nabla u \frac{\partial F}{\partial u} \parallel \nabla u \parallel \nabla f \text{ or } V\nabla f \nabla F = 0. \qquad \text{(ccxxxii)}$$

Hence the condition may be written as a partial differential equation in the form

$$V\nabla f \,.\, \nabla \left\{ \frac{\partial}{\partial u} \log \left(\frac{\partial f}{\partial u} \frac{1}{(\nabla f)^2} \right) + \frac{\partial f}{\partial u} \frac{\nabla^2 f}{(\nabla f)^2} \right\} = 0. \qquad \text{(ccxxxiii)}$$

(81.) As an example take the system of confocals

$$f(\rho, u) = S\rho (\Phi + u)^{-1} \rho - 1 = 0. \qquad \text{(ccxxxiv)}$$

For this

$$\nabla f = -2(\Phi + u)^{-1}\rho; \quad \nabla^2 f = 2((a^2 + u)^{-1} + (b^2 + u)^{-1} + (c^2 + u)^{-1}); \quad \frac{\partial f}{\partial u} = -\tfrac{1}{4}(\nabla f)^2. \qquad \text{(ccxxxv)}$$

The differential equation for P is simply

$$\frac{\partial}{\partial u} \log \frac{\partial P}{\partial u} = -\tfrac{1}{4}\nabla^2 f = -\tfrac{1}{2}((a^2 + u)^{-1} + (b^2 + u)^{-1} + (c^2 + u)^{-1});$$

* The fact that these operations are partial must be borne in mind. This may be illustrated for the cases $f = \rho^2 + u^2$, $f = \rho^2 u^2 + 1$.

the condition (ccxxxiii) is obviously satisfied and

$$P = P_0 \int \frac{du}{((a^2+u)(b^2+u)(c^2+u))^{\frac{1}{2}}}, \qquad \text{(ccxxxvi)}$$

a second arbitrary constant being understood to accompany the sign of integration.

(82.) It is also easy to find in terms of ∇ the condition that the family of surfaces

$$F\rho = u \qquad \text{(ccxxxvii)}$$

should compose one of three mutually orthogonal systems.*

The vectors ρ_1, ρ_2, ρ_3 being the deriveds of ρ with respect to three parameters u, v, and w, the corresponding surfaces will be mutually orthogonal if the equations

$$S\rho_2\rho_3 = S\rho_3\rho_1 = S\rho_1\rho_2 = 0 \qquad \text{(ccxxxviii)}$$

are true for all values of u, v, and w. Under these conditions we may differentiate and equate the results to zero. Thus we obtain

$$S(\rho_{12}\rho_3 + \rho_2\rho_{31}) = S(\rho_{23}\rho_1 + \rho_3\rho_{12}) = S(\rho_{31}\rho_2 + \rho_1\rho_{23}) = 0,$$

or what is equivalent

$$S\rho_{23}\rho_1 = S\rho_{31}\rho_2 = S\rho_{12}\rho_3 = 0 \qquad \text{(ccxxxix)}$$

or again by the conditions of perpendicularity,

$$S\rho_{23}\rho_2\rho_3 = S\rho_{31}\rho_3\rho_1 = S\rho_{12}\rho_1\rho_2 = 0. \qquad \text{(ccxl)}$$

These equations show that the surfaces intersect along their lines of curvature for they are of the form

$$S\nu d\nu d\rho = 0 \qquad \text{(ccxli)}$$

which is the well-known equation of the lines of curvature. They may be replaced by vector equations, one of which is

$$\rho_{23} = y\rho_2 + z\rho_3 \qquad \text{(ccxlii)}$$

where y and z are certain scalars. Differentiating with respect to u, we may write the result in the form

$$\frac{\partial^2\rho_1}{\partial v\partial w} = y\frac{\partial\rho_1}{\partial v} + z\frac{\partial\rho_1}{\partial w} + \frac{\partial y}{\partial u}\rho_2 + \frac{\partial z}{\partial u}\rho_3 \qquad \text{(ccxliii)}$$

and this implies

$$S\rho_1\left(\frac{\partial^2\rho_1}{\partial v\partial w} - y\frac{\partial\rho_1}{\partial v} - z\frac{\partial\rho_1}{\partial w}\right) = 0. \qquad \text{(ccxliv)}$$

Now

$$\frac{\partial}{\partial v} = -S\rho'_2\nabla, \quad \frac{\partial}{\partial w} = -S\rho'_3\nabla, \quad \frac{\partial^2}{\partial v\partial w} = S\rho'_2\nabla S\rho'_3\nabla - S\rho'_{23}\nabla \qquad \text{(ccxlv)}$$

* Compare Salmon's *Geometry of Three Dimensions*, fourth edition, pages 436–450.

in which the accents signify that the marked vectors are free from the operation of ∇. Hence we see immediately by (ccxlii) that generally and symbolically

$$\frac{\partial^2}{\partial v \partial w} - y\frac{\partial}{\partial v} - z\frac{\partial}{\partial w} = S\rho'_2\nabla S\rho'_3\nabla, \tag{ccxlvi}$$

while the condition (ccxliv) may be replaced by

$$S\rho'_2\nabla S\rho'_3\nabla S\rho'_1\rho_1 = 0 \tag{ccxlvii}$$

in which ∇ operates solely upon the unaccented vector ρ_1.

It only remains to replace ρ_1, ρ_2, and ρ_3 in terms of ∇ and $F\rho$ in order to obtain the differential equation which the equation of the family of surfaces $F\rho = u$ must satisfy.

In the first place (ccxxi)

$$\nabla F = \nabla u = -\rho_1^{-1} \quad \text{or} \quad \rho_1 = -\nabla F^{-1}; \tag{ccxlviii}$$

and again if we put $\nu = \nabla F$ and write

$$d\nu = \phi d\rho = \quad Sd\rho\nabla.\nabla F \tag{ccxlix}$$

in the equation of the lines of curvature (ccxli), we find

$$\rho_n \parallel (\phi + t_n)^{-1}\nu \parallel (\psi + t_n\chi + t_n^2)\nu \tag{ccl}$$

where the suffix $n = 2$ or 3 and where t_2 and t_3 are the roots of the quadratic

$$S\nu(\phi + t_n)^{-1}\nu = 0 \quad \text{or} \quad S\nu(\psi + t_n\chi + t_n^2)\nu = 0. \tag{ccli}$$

We may also write

$$\rho_n \parallel \nu^{-1}V\nu(\psi + t_n\chi)\nu \parallel \lambda + t_n\mu, \tag{cclii}$$

the vectors λ and μ being introduced for the sake of brevity and being known in terms of ∇ and F by the results of sections (27.) and (28.). Substituting in (ccxlvii) we obtain

$$S(\lambda_0 + t_2\mu_0)\nabla S(\lambda_0 + t_3\mu_0)\nabla S(\nabla F)^{-1}(\nabla F)_0 = 0; \tag{ccliii}$$

and finally by the aid of the quadratic (ccli) we arrive at the equivalent of Cayley's differential equation of the third order in the form

$$\{(S\lambda_0\nabla)^2 - (S\nu^{-1}\chi\nu)_0 S\lambda_0\nabla S\mu_0\nabla + (S\nu^{-1}\psi\nu)_0(S\mu_0\nabla)^2\} S(\nabla F)^{-1}(\nabla F)_0 = 0. \tag{ccliv}$$

In this the suffixes are intended to indicate that the quantities distinguished by them are exempt from the operation of ∇.

INDEX TO VOLUMES I. AND II.

[*The Roman Numerals* i, ii, *refer to the First and Second Volumes respectively; the Arabic figures to the pages.*]

Ablatitious force, 322 ii.
Abstraction of symbol from subject of operation, T, 168 i; K, U, R, 142 i; K, S, V, 204 i; ϕ, 494 i; ∇, 434 ii.
Academy, Royal Irish, first communication on quaternions, note, 160 i, 316 i, 380 i.
construction of ellipsoid, note, 230 i.
spherical quadrilaterals, note, 392 i.
lines of curvature, note, 239 ii.
hodograph, notes, 300 ii, 310 ii, 318 ii, 319 ii.
rotation of a solid, note, 290 ii.
development of disturbing force on planet, note, 321 ii.
Third Supplement on Systems of Rays, notes, 330 ii, 341 ii.
Conical Refraction, note, 341 ii.
Operator ∇, note, 548 i.
Acceleration, vector of, 100 i.
with respect to moving axes, 289 ii, 403 ii.
Action of a dynamical system, 296 ii.
Actual vector, 3 i, 110 i.
Addition, extended meaning of, 5 i.
of vector and point, 5 i.
of vectors, 5 i; is commutative and associative, 7 i.
of quaternions, 116 i; is commutative and associative, 176 i, 204 i, 207 i.
of vector-arcs, is not commutative, 156 i; is associative, 304 i, 408 i.
of vector-angles, 406 i.
of amplitudes, 264 i.
spherical, 406 i.
Algebra, imaginary symbol of, 133 i, 224 i, 253 i, 258 i, 289 i, 316 i.
paradox in, 149 i.
principles adopted from, 108 i.
extended use of signs in, 5 i, 6 i, 108 i, 123 i, 256 i.
Algebraic form, equations of, 77 i.
Alternation and inversion, equidifference of points, 4 i.
equality of geometric quotients, 118 i.
of vector-arcs, 144 i.
Am, symbol for amplitude, 262 i.
Am_n or $\angle_n$, 263 i; Am_0, principal amplitude, 263 i.
Amplitude of quaternion, 262 i; note, 120 i.
of a twist, 390 ii.
Amplitudes, addition of, 264 i.
Analytical expressions for ∇, 469 ii.
Angle of quaternion, 120 i, 111 i.
differential of, 458 i, 277 ii.
Euclidean, 120 i.
representative, 151 i.
vector-, 151 i, 406 i.
for curves, auxiliary, H, 89 ii, P, 89 ii, C, 91 ii, D, 98 ii, J, 116 ii, j, 118 ii.
of contact, 403 ii, of torsion, 403 ii.
Angular acceleration, note, 289 ii.
momentum, 289 ii.
velocity, 288 ii, of emanent, 84 ii, 119 ii.
Anharmonic coordinates, 23 i, 378 ii; in space, 55 i, 388 ii.
construction of cubic curve, 38 i.
equations of curves, 32 i; of surfaces, 87 i.
function or quotient, 15 i.

Anharmonic coordinates of four points on a line, 15 i; in space, 294 i; on a circle, 297 i.
properties of ruled surface, 408 ii.
of triangle, 21 i.
quaternion, 294 i, 350 i.
Anti-parallel sections of a cone, 183 i.
Apollonius of Perga, locus, 130 i, 165 i, 191 i, 182 ii.
cyclic cone, 181 i.
Applicable surfaces, 264 ii, 402 ii.
Arc, vector-, 143 i.
representative, 143 i.
cyclic, 185 i, 308 i.
of curve, element of, 32 ii, 87 ii; of surface, 263 ii, 411 ii.
on surface, variation of, notes, 48 ii, 271 ii.
affections of curve involving the third power of, 88 ii; fourth power of, 112 ii; fifth power of, 156 ii.
of cusp-edge of polar developable, 120 ii; of rectifying developable, 123 ii.
of evolute, 169 ii.
Arch, oblique, 283 ii.
Arcual sum, 156 i, 303 i, 369 i.
Area, sign of, 18 i.
directed, 482 i.
of parallelogram, 246 i.
of spherical triangle, 364 i.
of spherical polygon or curve, 368 i, 370 i.
of spherical cap, 482 i.
Areal velocity, 299 ii.
Arithmetic, illustration of differential from, 434 i.
Aronhold's notation, 467 ii.
Arrangement of axes of a function and its conjugate, 512 i, 384 ii.
of rays of complex and congruency, 419 ii.
Aspect of plane, 112 i; note, 207 i.
Associative law for addition of vectors, 7 i; of quaternions, 207 i; of vector-arcs, 304 i.
for multiplication of *i*, *j*, *k*, 159 i; of quaternions, 308 i, 245 i; of vectors, 337 i.
for multiplication, enunciations of, 301 i.
Assumption as to equality of a right quaternion with its index-vector, 334 i; note, 175 i.
Asymptote of hyperbola, 34 i; 9 ii; of twisted cubic, 131 ii, 162 ii.
Asymptotic cones, 189 ii, 186 ii.
Attracting bodies, system of, 293 ii.
Atwood's machine, 100 i.
Auxiliary functions, linear vector, 486 i, 495 i; quaternion, 560 i.
spherical curves, 28 ii, 51 ii, 110 ii.
angles for curves, *H*, 89 ii; *P*, 89 ii; *C*, 91 ii; *D*, 98 ii; *J*, 116 ii; *j*, 118 ii.
vector function for confocals, 199 ii.
Ax., symbol for axis of quaternion, 120 i.
examples on, 121 i.
equals IUV, 203 i.
replaced by UV, 334 i.
Axes or directions of linear vector function, 508 i.
arrangement of, and of its conjugate, 512 i, 384 ii.
coincidence of two, 503 i, 379 ii.
of three, 505 i, 379 ii.
determination of, note, 512 i.
imaginary, 514 i.
indeterminate, 501 i, 525 i, 369 ii, 379 ii, 385 ii.
rectangularity of two, note, 513 i.
of three, note, 513 i, 514 i, 517 i.
of system of functions, 384 ii.
of ellipsoid, principal, 238 i; of quadric, 536 i, 187 ii, 272 ii.
of section of quadric, 238 i, 525 i, 231 ii, 253 ii.
of sphero-conic, 192 ii.
of confocals, 199 ii; of touched confocal, 228 ii.
of inertia, 291 ii.
of crystal, optic, 348 ii.
moving, 403 ii, 404 ii, 410 ii; note, 289 ii.
Axis of quaternion, 119 i, 112 i, 203 i; differential of, 458 i.
of parabola, 34 i.
of screw rotation, 84 ii, 95 ii. *See* Screw.
of system of applied forces, central, 285 ii.
of instantaneous rotation, 288 ii.
polar, 57 ii.

Ball, Sir R. S., Theory of Screws, Note VIII., 390 ii; notes, 285 ii, 287 ii; 374 ii.
Barycentres, 85 i.
Barycentric Calculus, referred to in notes, 22 i, 50 i, 61 i, 62 i, 85 i, 162 ii.
Bertrand, geodetics on cylinder, note, 111 ii.
Biaxal crystal, 323 ii.
Biconcyclic surfaces, 527 i, 187 ii, 272 ii.

Bicouple, 289 i.
Bifocal form of linear vector function, 545 i, 195 ii.
Binet's theorem on principal axes, note, 292 ii.
Binomial form of quaternion, 254 i.
of linear vector function, 498 i.
Binormal, 28 ii, 82 ii.
Biquadratic equation of linear quaternion function, 560 i.
of united points, 387 ii.
Biquaternion, 133 i, 225 i, 289 i, 316 i.
Biscalar, 225 i, 289 i.
Bisecting sides of spherical triangle, triangle, 358 i.
Bisectors of a triangle, 18 i.
Bivector, 225 i; note, 171 ii.
Bonnet on geodetic torsion, 415 ii.
Booth, tangential coordinates, notes, 40 i, 255 ii.

Calculus, Barycentric, notes, 22 i, 50 i, 61 i, 62 i, 85 i, 162 ii.
of finite differences, 83 i.
of functions, 202 i, 205 i. *See* Abstraction.
of partial differentials, 48 ii.
of variations, notes, 48 ii, 271 ii.
Cambrure, note, 81 ii.
Cap, area of spherical, 482 i.
Carnot on transversals, notes, 65 i, 377 i.
Cartesian coordinates and quaternions, note, 248 i, 351 ii.
expressions for functions of a quaternion, 242 i.
Cassinian oval, 281 i, 285 i.
Cauchy on the wave-surface, note, 324 ii.
Cayley on the theorem of the polar plane, note, 357 ii.
on quaternion determinants, 361 ii.
on orthogonal surfaces, 474 ii.
Central sections of quadric, 238 i, 525 i, 231 ii, 253 ii.
surfaces, 186 ii. *See* Quadric.
axis of system of applied forces, 285 ii.
of finite displacement, 398 ii.
orbit, 298 ii.
Centre of involution, 16 i, 409 ii.
of homology, 60 i.
of conic inscribed to triangle, 36 i.
of ruled hyperboloid, 92 i; vector to, 96 i.
of osculating circle, vector to, 50 ii, 57 ii.
of spherical curvature, 74 ii, 134 ii, 155 ii, 168 ii.
of quadric, 280 ii.
of applied forces, 286 ii.
Centres of curvature of a surface, 246 ii.
quadric, 260 ii.
surface of, 254 ii.
reciprocal of surface of, for a quadric, 255 ii.
Centro-focal ellipses, 203 ii.
Characteristic function, 297 ii; for central orbit, 314 ii.
of operation. *See* Symbol.
Chasles, referred to in notes, 16 i, 31 i, 72 i, 89 i, 183 i, 300 i, 308 i, 340 i, 227 ii, 236 ii.
Chiastic homography of screws, 394 ii.
Circle, inscribed or exscribed to triangle, 33 i; to spherical triangle, 401 i.
quaternion equation of, 133 i.
expressed by square of right radial, 134 i.
examples on, 174 i.
inverse of line, 296 i, 349 i.
through three points, vector equation of, 355 i.
exponential equation of, 417 i, 462 i, 5 ii.
touching three small circles on a sphere, 427 i.
osculating a curve, 50 ii, 86 ii, 174 ii.
vector equation of, 58 ii.
and intersecting, 60 ii.
deviation of, from curve, 97 ii, 133 ii, 138 ii, 404 ii.
locus of, 140 ii.
geodetic, or Didonia, 271 ii.
of excentricity, 306 ii.
Circular sections of cyclic cone, 184 i.
of ellipsoid, 232 i, 239 i.
of index surface, 234 ii.
of Fresnel's wave surface, 332 ii.
group of four points, 297 i.
successions, 297 i, 305 i, 311 i.
logarithmic spiral, 419 i.
points at infinity, note, 516 i.
hodograph, 300 ii.
ridges on Fresnel's wave-surface, 337 ii, 344 ii.
Circumscribed developable of curve, 116 ii.
to confocals, 222 ii.
to surfaces, 232 ii.
Cis (symbol), 260 i.
Class of a curve, 42 i, 93 i.
surface, 88 i.
congruency, 431 ii.
Classification of points of construction, 55 i, 75 i.
of quadrics by roots of function, 187 ii.
by centres, 280 ii.

Clifford, note, 289 ii.
Co-axial quaternions, 250 i.
linear functions, 364 ii.
Coefficient of non-sphericity, 80 ii, 120 ii, 135 ii, 138 ii.
Coefficients of vectors, 9 i.
differential, 99 i, 444 i.
Coincidence of axes of linear function, 378 ii.
of cusp-edges of envelope, 145 ii.
Collinear, condition that three points should be, 14 i, 49 i, 52 i, 57 i.
three right quaternions, 247 i.
quaternions, 210 i.
termino-, vectors, 14 i, 343 i.
Comet, distance from earth of, 320 ii.
Commutative law for addition of vectors, 6 i.
of quaternions, 176 i, 207 i.
not valid for addition of vector arcs, 156 i.
multiplication of quaternions, 147 i, 156 i.
of linear functions, 364 ii.
Complanar points, 14 i, 45 i, 56 i.
vectors, 14 i, 340 i; proportion of, 250 i, 256 i.
termino-, 45 i, 344 i.
quaternions, 116 i, 211 i, 250 i.
Complanarity, sign of, 117 i.
condition of, of points, 14 i, 45 i, 52 i.
of vectors, 14 i, 338 i.
of quaternions, 148 i.
Complex mean of n vectors, 85 i.
of lines, note, 211 ii; Note XII., 417 ii.
Composition of two quadrantal rotations, 149 i.
of rotations, 415 i.
of wrenches and twists, 390 ii.
Concircularity, condition of, of four points, 297 i, 355 i, 177 ii.
Concurrence of three lines, 18 i.
of four planes, 57 i, 342 i.
of three circles, 311 i.
Concyclic quadrics, 527 i, 187 ii, 272 ii.
Condition of collinearity of three points, 14 i, 49 i, 52 i, 57 i.
right quaternions, 247 i.
commutation of quaternions, 148 i.
linear vector functions, 364 ii.
complanarity of points, 14 i, 45 i, 52 i.
of vectors, 14 i, 338 i, 345 i.
of quaternions, 148 i.
of versors, 148 i.
Condition of concircularity, 297 i, 177 ii.
concurrence of lines, 18 i.
of planes, 57 i, 342 i.
of contact of line and sphere, 224 i, 427 i, 428 i.
and quadric, 209 ii.
homoconicism, 375 ii.
homosphericity, 352 i, 354 i.
indeterminateness of axes, 379 ii.
integrability of $S\nu d\rho$, 275 ii.
parallelism, 10 i, 194 i, 325 i.
perpendicularity, 180 i, 325 i, 345 i.
planarity of curve, 25 ii.
rectangularity of axes of function, note, 513 i.
sphericity of curve, 80 ii, 120 ii, 135 ii, 138 ii, 406 ii.
termino-collinearity of vectors, 14 i, 343 i.
complanarity, 45 i, 344 i.
Cone, asymptotic, 186 ii, 189 ii.
of axes of system $\phi + t\theta$, 384 ii.
of complex, 430 ii.
confocal, 213 ii.
cubic, vector expression for, 95 i; monomial equation of, 281 ii, 384 ii.
on curve, 27 ii, 34 ii.
cyclic or quadric, 95 i, 181 i, 309 i, 189 ii.
expressed by S, 181 i.
vector expression for, 95 i, 101 i.
differential equation of, 45 ii.
focal, 202 ii, 216 ii.
geodetics on, 31 ii.
motion of body about fixed point represented by rolling, 290 ii, 291 ii.
of normals, 243 ii.
osculating, related to curves, 91 ii, 99 ii, 125 ii, 129 ii, 163 ii.
of parallels to tangents, 6 ii, 27 ii, 29 ii.
Pascal's theorem deduced from equation of, 375 ii.
of revolution, 183 i; note, 184 ii.
expressed by $\angle$, 121 i, by S, 180 i.
equation of one sheet of, 121 i.
tangent to sphere, 225 i; to quadric, 217 ii.
tangent, to sphere, 225 i; to quadric, 209 ii.
to confocals, 213 ii; note, 224 ii.
right, 217 ii.
at wave-cusp, 336 ii, 341 ii.

Confocal quadrics, 196 ii.
tangent cones, 213 ii; note, 224 ii.
right, 217 ii.
equation of, 207 ii.
a line touches two, 214 ii.
developable circumscribing, 222 ii.
corresponding points on, note, 225 ii.
vector equation of, note, 225 ii.
axes of inertia related to, note, 293 ii.
wave-surface and, 346 ii.
as isothermal surfaces, 473 ii.
Congruence for quinary symbols, formula of, 51 i.
Congruency of lines, 417 ii.
Conic, anharmonic equation of, 32 i.
asymptotes of, 34 i, 9 ii.
circumscribed to triangle, 36 i.
conjugate diameters of, 94 i.
curvature of, 52 ii, 54 ii.
evolute of, 53 ii, note, 168 ii.
excentric anomaly of, 417 i, 5 ii.
exponential equation of, 417 i, 462 i, 5 ii, 25 ii.
exscribed to triangle, 33 i, 36 i.
focal, 202 ii, 219 ii.
index-, 231 ii.
intersection of cone and plane, 181 i.
cylinder and plane, 196 i, 199 i.
invariants, 380 ii.
quaternion equation of, 9 ii.
spherical, intersection of cone and sphere, 182 i.
of ellipsoid and sphere, 239 i.
associative principle illustrated by, 302 i, 308 i.
curvature of, 63 ii.
axes and foci of, 192 ii.
on wave surface, 345 ii.
vector equation of general, note, 10 ii.
Conical points on wave-surface, 332 ii, 335 ii, 341 ii.
refractions, 341 ii.
rotation, 154 i, 172 i, 359 i, 398 i, 429 i, 397 ii.
linear function of, 367 ii.
surfaces, family of, 45 ii.
Conjugate diameters of ellipse, 94 i; ellipsoid, 94 i.
of quadric, mutually, 211 ii, note, 250 ii, 374 ii.
directions to two quadrics, mutually, note, 250 ii, 364 ii.
harmonic, 16 i.
Conjugate of linear vector function, 485 i.
quaternion function, 555 i.
transformation function, 388 ii.
point of cubic, 41 i.
points to quadric 229 i, 428 i, 209 ii.
quadrics, note 188 ii.
quaternions, 115 i, 123 i.
screws of potential, 395 ii.
tangents, 231 ii.
of vector, 346 i.
Conjugation, characteristic of, K, 124 i.
equation of (linear vector function), 485 i.
(linear quaternion function), 555 i.
(pole and polar), 229 i, 428 i, 209 ii.
Connected region, multiply, 449 ii.
Constants determining a linear vector function, 486 i, 530 i, 374 ii.
quaternion function, 556 i.
dynamical, of a body, 396 ii.
elastic, 463 ii.
or invariants of linear vector function, 491 i, 376 ii.
vector-, of ellipsoid, 201 i, 236 i.
Constituents of a quaternion, 242 i.
Construction for centre of curvature of hyperbola, 54 ii; of sphero-conic, 65 ii.
of spherical curvature, 77 ii.
for cubic curve, 38 i.
for ellipsoid by diacentric spheres, 234 i, 241 i, 184 ii.
by variable ellipses 201 i, 204 ii.
modular and umbilicar, 279 ii.
of fourth proportional to three diplanar vectors, 362 i.
points of, for plane net, 17 i, 22 i; for net in space, 61 i. *See* Points.
for potential in orbit, 307 ii
of rays of congruency, 419 ii.
of series of spherical parallelograms, 390 i.
for wave-surface, 327 ii, 253 ii.
Contact of line with sphere, 225 i.
quadric 209 ii.
See Tangent, Osculating.
Continued proportion of complanar vectors, 256 i, 251 i.
Continuity, equation of, 455 ii.
Convention respecting sign of area, 18 i.
of volume, 48 i.

Convention respecting sense of rotation, 119 i, notes, 111 i, 152 i, 369 i.
position of operator and operand, 110 i, 147 i.
expansion of quaternion determinants, 361 ii.
Convergency of series, 269 i, 424 i.
Coordinates, anharmonic, 23 i, 55 i.
of a plane, 60 i.
Cartesian, 248 i.
of a couple, 254 i.
curvilinear, 471 ii.
elliptic, note, 225 ii.
geodetic, 264 ii.
idea of, foreign to quaternions, note, 112 i, 404 i, 351 ii.
of a line, note, 211 ii; 426 ii.
screw, 392 ii.
tangential, 40 i, 255 ii.
Co-reciprocal screws, 392 ii.
Co-residuals on cubic, 385 ii.
Corresponding points on confocals, note, 225 ii.
Cosine of quaternion, 275 i, 424 i.
Couple (quaternion in given plane), 254 i.
or moment, 284 ii.
Criterion as to nature of conic inscribed to triangle, 34 i.
Crystal, propagation of light in biaxal, 323 ii.
incidence of light on, 352 ii.
Cube-root of quaternion, 256 i; principal, 257 i.
Cube-roots of unity, nine, 291 i.
Cubic, anharmonic construction of plane, 38 i.
equation of, 41 i.
cone, 95 i, 281 ii, 384 ii.
surface, general equation of, 281 ii.
related to curve, 164 ii.
symbolic, of linear function, 494 i.
depressed, 501 i, 505 i, 379 ii.
twisted, having contact of third order with a curve, 92 ii, 104 ii; of fourth order, 125 ii, 128 ii; of fifth order, 129 ii, 167 ii, 404 ii.
vector equation of, 131 ii, note, 562 i.
Curl of vector, 444 ii.
Curvature of curve, absolute, 50 ii.
second, or torsion, 88 ii, 81 ii, 92 ii, 108 ii, 109 ii.
spherical, 74 ii.
vector of, 52 ii, 56 ii.
of hodograph, 302 ii.
of surfaces, 246 ii.
geodetic, 267 ii.
Curvature of surfaces, lines of, 230 ii, 235 ii, 236 ii.
through umbilic, 242 ii.
measure of, 261 ii, 412 ii, 415 ii.
of section, 267 ii.
total, 264 ii, 416 ii.
Curvatures, constant ratio of, 111 ii, 405 ii.
difference of, of surface, 249 ii.
Curve, affections of, depending on third power of arc, 88 ii.
on fourth power, 112 ii.
on fifth power, 156 ii.
auxiliary angles for a. *See* Angle.
cubic. *See* Cubic.
deviation of. *See* Deviation.
element of arc of, 32 ii.
emanating line and planes, 83 ii, 85 ii, 109 ii, 114 ii, 408 ii.
osculating planes and absolute normals to, 24 ii.
reciprocal of, 41 ii, 111 ii, 193 ii.
spherical, 62 ii. *See* Spherical.
tangents and normal planes to, 4 ii.
unicursal, note, 10 ii.
vector equation of, 94 i.
Curvilinear coordinates, 471 ii.
Cusp or conical point on wave-surface, 332 ii, 335 ii, 341 ii.
-cones of wave-surface, 343 ii.
-edge of developable, 93 i, 100 i, 36 ii.
(polar), 120 ii.
(rectifying), 122 ii.
of envelope of sphere, 144 ii.
-rays, 332 ii.
Cyclic arcs, 185 i, 308 i.
cones, 95 i, 181 i, 309 i, 189 ii.
form of linear vector function, 520 i, 528 i, 535 i, 538 i.
applied to quadric, 535 i, 187 ii.
to wave-surface, 332 ii.
to strain, 369 ii, 373 ii.
planes, normals, of cone, 183 i, 549 i.
of ellipsoid, 232 i, 235 i.
of quadric, 18 ii.
quadrilateral, 296 i, 347 i.
Cyclical law of *i*, *j*, *k*, 158 i.
permutation under S of vectors, 350 i; of quaternions, 248 i.
of linear vector functions, 363 ii.

Cylinder of revolution expressed by TV, 195 i.
by V, 199 i.
centre indeterminate, 280 ii.
geodetic on, 30 ii, 405 ii.
osculating curve, right, 95 ii.
tangent, to sphere, 201 i; spheroid, 201 i; ellipsoid, 202 i.
right, 201 i, 202 i, 236 i, 241 i, 220 ii.
Cylindrical surfaces, equation of, 44 ii, 45 ii.
Cylindroid, 391 ii, 419 ii, 422 ii.

D'Alembert's principle, 288 ii.
Darboux, referred to in notes, 401 ii, 415 ii.
Decomposition of quaternion $q = Tq \,.\, Uq$, 169 i; $q = Sq + Vq$, 193 i; in terms of four quaternions, 242 i, 564 i.
of vector along and perpendicular to a given direction, 193 i.
along three given directions, 338 i.
at right angles to three planes, 339 i.
into tensor and versor, 164 i.
of strain, 366 ii, 370 ii, 375 ii.
Definite integral, analogue of, 368 i.
over sphere, 482 i, 483 i.
total curvature, 265 ii, 416 ii.
principal function, 294 ii, 314 ii.
characteristic function, 296 ii, 314 ii, 317 ii.
time in orbit, 310 ii, 317 ii.
Definition of addition, 5 i, 109 i, 116 i.
of differential, 97 i, 430 i, 438 i.
of multiplication, 108 i, 10 i.
of power of a vector, 396 i.
of a quaternion, 421 i.
of reciprocal of vector, 293 i.
of subtraction of vectors, 5 i; of quaternions, 109 i, 116 i.
of a sum of vectors, 7 i.
of a vector, 3 i.
Deformation of sphere, 232 i.
theorem of geometrical, 525 i.
of surfaces, 264 ii, 402 ii, 416 ii.
strain, 365 ii.
Degenerate quaternions, 120 i, 178 i, 333 i.
Degree of plane curve, 32 i.
of twisted curve, 93 i, 10 ii.
of surface, 87 i.
Delaunay on isoprometrical problem, note, 271 ii.
De Moivre's theorem, 264 i.
De Morgan, *Double Algebra*, note, 278 i.
on geodetics on surfaces of revolution, 49 ii.
on evolutes, 169 ii.
Denominator, reduction of two quaternions to a common, 116 i.
Depressed equation of linear vector functions, 501 i, 505 i, 379 ii.
Derivative of a vector, 99 i.
partial, 294 ii.
Derived from scalar, vector (ν), 548 i, 13 ii, 432 ii.
Descartes on powers of lines, notes, 394 i, 404 i.
Desk, illustration of quaternion by, 113 i.
Determinants, quaternion, 361 ii, 382 ii, 393 ii.
Developable surface, 100 i, 36 ii, 409 ii.
circumscribed, 116 ii.
to surface (conjugate tangents), 232 ii.
circumscribing confocals, 222 ii.
cusp-edge of, 93 i, 100 i, 36 ii.
differential equation of, 45 ii.
as envelope, 42 ii.
generated by emanants, 85 ii, 409 ii.
geodetic on, 37 ii, 272 ii.
normal surface, 256 ii.
polar, 58 ii.
reciprocal of, 41 ii.
rectifying, 81 ii, 122 ii.
Development of a vector, 102 i.
of functions of quaternions, 465 i, 476 i.
of vector to point on curve, 112 ii, 156 ii, 404 ii.
of disturbing force, 320 ii.
Deviation of curve from osculating sphere, 79 ii, 132 ii, 138 ii.
circle, 87 ii, 97 ii, 133 ii, 138 ii; helix, 95 ii.
parabola, 96 ii.
of helix from twisted cubic, 161 ii.
in general, 404 ii.
Diacentric sphere, 234 i, 241 i; note, 204 ii.
Diameters, conjugate, of ellipse, 95 i; ellipsoid, 95 i; quadric, 211 ii; note, 250 ii.
Didonia, note, 271 ii.
Difference of two points, 3 i; of two vectors, 5 i.
of two quaternions, 176 i.
of curvatures of a surface, 249 ii.
Differenced equation of quadric, 21 ii.
Differences and differentials, 102 i, 431 i, 434 i, 469 i, 174 ii, 179 ii.

Differences and differentials, successive, 479 i.
finite, equation in, 84 i.
Differential coefficient, 99 i, 444 i.
definition of, 97 i, 431 i, 440 i.
s and differences, 102 i, 431 i, 434 i, 469 i, 174 ii, 179 ii.
distributive property of, 441 i.
elimination of, 448 i.
equation of families of surfaces, 44 ii, 47 ii.
of geodetics, 29 ii, 34 ii, 226 ii, 414 ii.
of lines of curvature, 230 ii, 412 ii.
for principal function, 296 ii, 314 ii.
for characteristic function, 297 ii, 315 ii.
exact, 275 ii, 554 i.
with multiplier, 276 ii.
finite, 99 i, 103 i, 432 i, note, 179 ii.
fluxion compared with, 431 i.
illustrations of, 432 i, 435 i, 469 i.
Lacroix on, 473 i.
Lagrange on, 441 i.
Maxwell on, 102 i.
Newton on, 431 i.
partial, 101 i, 446 i, 479 i, 48 ii.
of quaternion function, 438 i, 440 i, 445 i; of q^2, 438 i; of q^{-1}, 439 i, 447 i; of q^m, 451 i; of $q^{\frac{1}{2}}$, 452 i, 560 i; of ϵ^q, 453 i; of Kq, Sq, Vq, 455 i; of Tq and Uq, 456 i; of Ax q, 458 i; of $\angle q$, 459 i.
of function of function, 449 i.
of implicit quaternion function, 484 i.
quotient, 443 i, 98 i.
of a scalar function of vectors, 459 i.
s simultaneous, 431 i.
s successive, 100 i, 465 i, 479 i.
of a vector, 96 i, 462 i.
Differentiation formula of, 98 i, 438 i.
examples of quaternion, 451 i.
symbol $\text{ɑ} = (\text{S} \,.\, d\rho)^{-1}d$, note, 548 i.
with respect to moving axes, note, 289 ii, 403 ii.
See also Differential.
Dilatation, 372 ii.
Dimensions, principle of, applied to linear function of a vector, 488 i.
Diplanar quaternions, 116 i.
Direct and inverse similitude, 115 i, 365 ii.
circular succession, 297 i.
Directed area, 482 i.
Direction, relative, 110 i, 138 i.
unchanged by strain, 368 ii.
Directions of linear vector function, 508 i. *See* Linear vector function *or* Axes.
Displacement, axis of, 95 ii.
finite, 397 ii.
Dissipation function, 461 ii.
Distance of comet from earth, 320 ii.
Distribution, parameter of, 408 ii.
Distributive property of multiplication of vectors by co-efficients, 9 i.
of quaternions, 212 i, 219 i.
of sign I, 206 i; K, 176 i, 207 i; S, 185 i; V, 204 i.
of differential, 441 i.
Disturbing force on planet, 320 ii.
Division of vector by parallel vector, 10 i.
of quaternions, defined, 109 i, 116 i.
homographic, 16 i.
Double algebra, note, 278 i.
Duality, 388 ii, 427 ii. *See* Reciprocity.
Dupin, indicatrix, 231 ii.
conjugate tangents, 232 ii.
lines of curvature on confocals, 235 ii.
focal conics, note, 236 ii.
Dynamical constants of a body, 396 ii.
Dynamics, general equation of, 287 ii.
general method in, note, 295 ii.

Edge of regression, 93 i, 100 i, 36 ii.
Effective vector, 3 i.
Eight square roots of linear vector function, note, 225 ii, 367 ii.
umbilicar generators, 222 ii, 235 ii.
Elastic constants, 463 ii.
force, Fresnel's, 324 ii.
Electro-magnetic force, 445 ii.
wave surface, note, 326 ii.
Elimination of arbitrary functions, 48 ii.
of differential, 448 i.
of a vector 342 i, 355 i.
Ellipse, centro-focal, 203 ii.
curvature of, 52 ii.
evolute of, 53 ii, note, 168 ii.
focal, 202 ii, 219 ii.
section of cylinder, 196 i, 199 i, 418 i.
vector expression for, 95 i, 417 i.

Ellipsoid, axes of, 238 i.
of section, 238 i.
circular sections of, 232 i, 235 i.
construction by diacentric sphere, 234 i, notes, 184 ii, 204 ii, 351 ii.
cyclic normals, 235 i.
cylinder (right) tangent to, 202 i.
equation of, focal, 201 i, 230 i.
cyclic, 527 i.
$T(\iota\rho + \rho\kappa) = \kappa^2 - \iota^2$, note, 241 i, 314 i, 325 i, 328 i, 537 i, 15 ii, 185 ii, 274 ii, 350 ii.
homologous deformation of mean sphere, 232 i, 314 i.
strain-, 366 ii.
vector-constants of, 201 i, 236 i, 537 i.
vector expression for, 95 i.
Ellipsoidal linear function, 366 ii.
Elliptic coordinates, note, 225 ii.
functions, 385 ii.
logarithmic spiral, 419 i.
Elongation in strain, 372 ii.
Emanant lines and planes, 83 ii, 85 ii, 109 ii, 114 ii, 408 ii.
Energy equation, dynamics, 288 ii, 294 ii.
for light, 354 ii.
hydrodynamics, 460 ii.
Enunciation of the associative principle, 301 i.
Taylor's theorem, 22 ii
Envelope of confocals, 222 ii.
involute as limit of, 171 ii.
lines of curvature on quadric, 235 ii.
planes, developable as, 42 ii.
osculating sphere, 141 ii.
sphere with varying radius, 143 ii, 171 ii.
wave-surface as, note, 326 ii.
Equality of points, 3 i, 13 i.
quaternions, 109 i, 115 i, 243 i, 408 i.
vectors, 3 i.
vector-arcs, 144 i.
versors, 409 i.
Equation of algebraic form in quaternions, 277 i; n^4 roots of, 292 i.
anharmonic of curve, 32 i; local and tangential, 39 i; of surface, 87 i.
of confocals, 207 ii.
of conjugation (poles and polars), 229 i, 186 ii.
(linear functions), 485 i, 555 i.
Equation of continuity, 455 ii.
depressed, of linear function, 501 i, 505 i, 379 ii.
differential. *See* Differential.
of dynamics, 287 ii, 289 ii, 295 ii, 396 ii.
of equilibrium, 284 ii.
exponential, for spherical triangle, 404 i, 409 i.,
in finite differences, 84 i.
functional, of families of surfaces, 45 ii.
of loci involving signs Ax. and ∠, 121 i; K, 127 i; S, 180 i, 190 i; V, 195 i, 199 i; T, 165 i, 167 i, 190 i; U, 142 i.
powers of a vector, 417 i.
of second degree, homogeneous and vector, 508 i.
of six segments, 21 i, 18 i.
symbolic, for K, S, T, U, V, 142 i, 205 i.
for linear functions, 494 i, 560 i.
vector for curve, 94 i; surface, 94 i.
Equi-difference of points, 4 i.
Equilibrium, equation of, 284 ii.
Euclidean angle, 120 i, 262 i.
Euler, identity, law of the norms, 244 i.
theorem on curvature, 268 ii.
equations of motion, 289 ii.
fluid motion, 457 ii.
Evolute, 167 ii.
of ellipse, 53 ii, note, 168 ii.
of spherical curve, 169 ii.
Evolutionary quaternion, 295 i.
Exact differential, 275 ii.
Examples, geometrical, on signs ∠, Ax, 121; K, 127 i; R, 296 i; S, 180 i, 190 i; T, 165 i, 167 i, 190 i; U, 142 i; V, 195 i, 199 i.
depending on powers of vectors, 417 i.
of quaternion differentiation, 451 i.
Excentric anomaly, 5 ii.
Excess, spherical, 364 i.
spheroidal (total curvature), 266 ii.
Expansion. *See* Development.
Exponential of quaternion, 421 i.
form for sine and cosine, 266 i, 274 i, 424 i.
equation for spherical triangle, 404 i, 409 i.
of quadric, 206 ii.
transformation of Taylor's series, 468 i, 473 i, 551 i.
Exponents, scalar, 264 i.
quaternion, 274 i, 421 i.
Exscribed or circumscribed conic, 36 i.

Extensions of algebraic notation, 5 i, 6 i, 108 i, 123 i, 256 i.
Extreme points of congruency, 420 ii.

Factor or operator, 108 i, 135 i.
integrating, 276 ii.
Factorials, notation of, 476 i.
Families of surfaces, 45 ii.
isothermal, 472 ii.
orthogonal, 474 ii.
Finite differences, equation in, 84 i.
differentials, 99 i, 103 i, 432 i, note, 179 ii.
displacements, 397 ii.
Five quaternions, identical relation connecting, 563 i.
vectors, 47 i; terminating on a sphere, 354 i.
system of screws, 392 ii, 394 ii.
Flexion, note, 81 ii.
Flow, Lord Kelvin on, 457 ii.
Fluid motion, 454 ii.
Fluxions, note, 431 i.
Focal conics, 202 ii, 219 ii.
ellipses, centro-, 203 ii.
equation of quadrics, 535 i.
generations of quadrics, 278 ii.
lines of cones, 545 i, 549 i, 213 ii, 243 ii.
notation for, relations, 310 i.
properties of sphero-conics, 310 i, 393 i, 72 ii.
surface of congruency, 431 ii.
Foci of involution, 16 i.
modular, 278 ii.
principal, 420 ii.
by projection, 423 ii.
umbilicar, 278 ii.
virtual, 419 ii, 420 ii.
Force, Fresnel's elastic, 324 ii.
conservative, 444 ii.
disturbing, on planet, 320 ii.
function, 293 ii.
Formula (A), 160 i, 243 i, 344 i, 182 ii, 351 ii.
of addition, 5 i.
of association, 7 i, 245 i, 302 i.
of collinearity of three right quaternions, 247 i.
of commutation, 7 i.
of complanarity of three vectors, 338 i, 341 i, 247 i.
of congruence for quinary symbols, 51 i.
of differentials, 98 i, 438 i.
Formula of inversion of linear function, 486 i.
of parallelism of two vectors, 325 i.
of perpendicularity of two vectors, 325 i.
of relation between + and −, 5 i.
Formulæ of spherical trigonometry, fundamental, 400 i.
Four constituents of quaternion, 114 i, 242 i.
identical rows, determinant with, 363 ii.
points, group of, linear, 15 i; circular, 297 i.
complanarity of, 14 i, 45 i, 344 i.
concircularity of, 355 i.
system of screws, 392 ii, 394 ii.
vectors, linear relations between 44 i, 338 i.
proportion of, 250 i.
Fourth proportional to three vectors, complanar, 250 i, 293 i; diplanar, 356 i: rectangular, 377 i.
unit in space, 394 i, 380 i.
power of arc, affections of curve depending on, 112 ii.
Fraction or quotient, geometric, 107 i.
Fresnel, wave-surface, 323 ii, note, 253 ii.
Function, anharmonic quaternion, 294 i.
s, calculus of, 205 i, 202 i. *See* Abstraction of symbol.
characteristic, 297 ii.
dissipation, 461 ii.
elimination of arbitrary, 48 ii.
elliptic, 385 ii.
force, 293 ii, 394 ii.
implicit, differential of, 484 i.
linear. *See* Linear.
principal, in dynamics, 294 ii, 314 ii.
strain, 365 ii.
transcendental, of quaternions, 421 i, 453 i.
trigonometrical, of quaternions, 424 i.
of vectors, a quaternion, 332 i, 394 i.
Functional notation, 205 i, 202 i.
equations of families of surfaces, 45 ii.
Fundamental formulæ of trigonometry, plane 214 i; spherical, 400 i.

Gauche curve. *See* Cubic, etc.
hexagon inscribed to sphere, 305 i, 354 i.
polygons inscribed to sphere, 347 i.
quadrilateral, 82 i.
on quadric, 88 i, 95 i.
Gauss on measure of curvature, 253 ii, 261 ii, 264 ii, 412 ii, 416 ii.

Generation of ellipsoid, 241 i, notes, 184 ii, 204 ii, 351 ii.
of quadric, modular and umbilicar, 278 ii.
of ruled hyperboloid, 90 i.
of wave surface, 327 ii, 253 ii.
Generatrix of ruled hyperboloid, 89 i, 213 ii.
umbilicar, 221 ii, note, 225 ii.
Geocentric distance of comet, 320 ii.
Geodetic lines, 29 ii, 225 ii, 264 ii, 415 ii.
circles, 271 ii.
coordinates, 264 ii.
curvature, 267 ii.
on cylinder, 111 ii, 405 ii.
on developable, 37 ii, 272 ii.
Didonia, 271 ii.
on quadric, 225 ii; P · D = *const.* 226 ii.
on surface of revolution, 48 ii.
torsion, 415 ii.
triangle, curvature of, 266 ii.
Geometric quotient, 107 i.
inversion aud alternation of, 118 i.
Geometrical examples on V, 195 i, 199 i; S, 180 i; Ax. and ∠, 121 i; K, 127 i; R, 296 i; U, 142 i; S and T, 190 i.
deformation, theorem of, 525 i.
Geometrical illustration of differential, 436 i.
of ratio of vanishing quantities, 470 i.
nets, plane, 20 i, 29 i; in space, 61 i.
"Géométrie de position" of Carnot, note, 377 i.
"Géométrie supérieure," of Chasles, notes, 16 i; 72 i, 89 i, 300 i.
Geometry of Three Dimensions. *See* Salmon.
Grammarithm, note, 335 i.
Graves, C., note, 308 i.
Graves, J. T., note, 276 i.
Gravitation, Newton's law of, 99 i, 302 ii.
Group of four points on a line, 15 i; in space, 294 i; on a circle, 297 i.
of six points, evolutionary of, 295 i.
Guide-points, 239 i.
Guiding curve, 35 ii.

Half-line or ray, equation of, 121 i, 142 i.
Handle, versor compared to a, 336 i, note, 345 i.
Harmonic mean of two vectors, 231 i, 298 i; of *n*, 300 i.
polar of point to triangle, 21 i.
Harmonic property of quadrilateral, 20 i.
of pole and polar, 229 i.
spherical, 468 ii.
Helicoid or screw-surface, 419 i, 12 ii, 28 ii, 83 ii, 281 ii.
Helix, 419 i, 5 ii, 25 ii, 28 ii, 51 ii, 77 ii, 92 ii, 95 ii, 112 ii, 157 ii.
as curve of constant curvatures, 112 ii.
osculating curve, 95 ii.
related to complex, 428 ii.
Heptagon, inscribed to sphere, 354 i.
Herschel, Sir John F. W., *Treatise on Light*, note, 327 ii.
Hexagon, spherical, 303 i.
inscribed to sphere, 305 i, 354 i.
evolutionary of, 295 i.
"Higher Plane Curves," Salmon's, notes, 37 i, 40 i, 41 i, 42 i.
Historical notes on quaternions, 206 i, 258 i, 262 i, 278 i, 351 ii.
Hodograph, 300 ii, 99 i, 29 ii, note, 112 ii.
curvature of, 302 ii.
of evolute, 173 ii.
Hodographic isochronism, 310 ii.
geometrical proof, 319 ii.
relation to Lambert's theorem, 314 ii.
Homographic division, 16 i.
nets (plane), 31 i; (in space), 79 i.
property of ruled hyperboloid, 89 i.
surfaces, 408 ii.
screw-systems, 396 ii.
Homologies of ellipsoid and sphere, 315 i, 232 i.
Homology, centre of, 60 i; plane of, 60 i.
Homospheric property of cyclic sections, 18 ii.
Homosphericity, equation of, 354 i.
Huyghens, note, 353 ii.
Hydrodynamics, 454 ii.
Hydrostatics, 483 i.
Hyperbola, 33 i, 9 ii, 54 ii, 60 ii.
curvature of, 54 ii.
focal, 201 ii, 219 ii.
Hyperbolic paraboloid, 93 i, 96 i.
orbit, note, 303 ii.
Hyperboloid, ruled, anharmonic equation of, 88 i.
generators of, 89 i, 213 ii.
vector equation of, 95 i.
Hypotenuse, proof of theorem of square on, 212 i.

I, symbol for index of right quotient, 187 i.
is distribution, 206 i.
Ax. = IUV, 203 i; IV = V, 335 i.
i, j, k, laws of, 157 i, 344 i.
early use of, notes, 160 i, 316 i, 345 i, 351 ii.
formula (A), 160 i, 243 i, 344 i, 182 ii, 351 ii.
links between quaternions and coordinates, 351 ii.
quaternion in terms of, 242 i.
vector in terms of, 344 i.
Ideal or imaginary intersections, 223 i.
Identical rows, determinant with, 362 ii, 383 ii, 393 ii.
Identification of a right quaternion with its index, 331 i, note, 193 i.
Identity connecting three vectors, 337 i, 344 i, 375 i, 426 i.
four vectors, 338 i, 339 i, 376 i.
five quaternions, 563 i.
six spherical arcs, 377 i.
seven screws, 393 ii.
on square root of quaternion, 315 i.
Illustration of differential and difference, 434 i, 436 i.
of ratio of vanishing quantities, 470 i.
of a quaternion by a desk, 113 i.
Imaginary* of algebra, 224 i, 289 i.
geometrically real, notes, 133 i, 253 i.
as mark of geometrical impossibility, note, 404 i.
axes and roots of linear vector function, 514 i.
i, j, k formerly called, note, 316 i.
part, or vector part, note, 316 i.
quaternion. *See* Biquaternion.
Imponential, 274 i.
Independent differentials, 553 i.
variable, change of, 24 ii, 33 ii.
Indeterminateness of axis of linear vector function, 501 i, 525 i, 364 ii, 369 ii, 379 ii, 385 ii.
of interpretation of $\sqrt{-1}$, 133 i.
of versor of null quaternion, note, 120 i, 139 i.
of construction for fourth proportional, 379 i.
Index of right quotient, 122 i.
symbol of, 187 i.
equals right quotient, 331 i, note, 193 i.
curve (indicatrix), 231 ii.
surface, 233 ii, 237 ii.
surface or surface of wave-slowness, 324 ii.
Index of vector, 323 ii.
connexion with ray, 328 ii.
Indicatrix, 231 ii.
Indices of right quotients, quotient of, 175 i.
sum of, 206 i.
product of, 329 i.
Inertia, axis of, 287 ii, 291 ii.
linear function of, 289 ii.
principal screws of, 395 ii.
Infinitesimals, 170 ii, note 230 ii.
osculating circles treated by, 174 ii.
Infinity, line at, 27 i.
circular points at, note, 516 i.
circle at, 222 ii.
Inflexion points of, note, 25 ii.
Inflexional tangents to cubic, 37 i.
Inscription of polygons in sphere, 354 i.
Integrability, condition of, 277 ii.
Integral as limit of a sum, 482 i, 170 ii.
definite, analogue of, 368 i.
over sphere, 482 i, 483 i.
total curvature, 265 ii, 416 ii.
principal function, 294 ii, 314 ii.
characteristic function, 296 ii, 314 ii, 317 ii.
time in orbit, 310 ii, 317 ii.
of differential equation of family of surfaces, 45 ii, 48 ii.
of geodetics, 35 ii, 37 ii, 48 ii, 226 ii.
intermediate, of general equations of dynamics, 295 ii, 297 ii.
s, volume and surface, 448 ii.
s, line and surface, 449 ii.
Integration of differential equations, examples on, 30 ii, 52 ii, 277 ii.
See Integral.
Intensity of a wrench, 390 ii.
Intermediate integrals of dynamics, 295 ii, 297 ii.
Interpretation of a product of vectors as a quaternion, 321 i, 337 i, 394 i.
of $\sqrt{-1}$, 133 i, 253 i.
Intersection, ideal or imaginary, 223 i, note, 87 i.
real, 220 i.
of line and plane, 47 i.
of two planes, 338 i.
of confocals, 198 ii, 235 ii.

* The words *ideal* and *symbolical* are occasionally used by Hamilton as synonymous with *imaginary*.

Intersection of normals, 230 ii, 239 ii.
of osculating circle and curve, 60 ii; and spheroconic, 67 ii.
surfaces, 103 ii, 160 ii.
Invariants, 491 i, 506 i, 376 ii, 364 ii, 386 ii.
screw, 393 ii.
Inverse or reciprocal of a vector, 293 i, 322 i.
symbol for, R, 141 i, 293 i, 328 i.
called vector of proximity, 20 ii, 461 i.
of a quaternion, 122 i; differential of, 439 i.
of line, 296 i; of circle, 296 i, 349 i; of sphere, 353 i.
similitude, 115 i, 129 i, 365 ii.
Inversion and alternation, equidifference of points, 4 i.
equality of geometric quotients, 118 i.
of vector-axes, 144 i.
geometrical, 293 i. *See* Inverse.
of linear vector function, 485 i, 280 ii.
of linear quaternion function, 557 i.
Inversor, 135 i.
semi-, 135 i.
Involute, 167 ii.
as limit of envelope, 171 ii.
Involution, 16 i.
double, 72 i.
in space, 295 i, 300 i.
Irish Academy, Royal. *See* Academy.
Isochronism, hodographic, 310 ii. *See* Hodographic.
Isothermal surfaces, 472 ii.

Jacobi, generators of confocal, note, 213 ii.
Jamin, experiments on light, 358 ii.
Joachimstal, first integral for geodetics, note, 226 ii.

K introduced, 124 i.
examples on, 127 i.
differential of Kq, 455 i.
Kelvin, Lord, on flow, 457 ii.
Kinematical treatment of curves, 83 ii, 92 ii, 114 ii, 118 ii, 403 ii.
of surfaces, 410 ii.
Kummer surface of complex, 430 ii.

Lacroix, on Taylor's theorem, note 473 i.
Lagrange, definition of a derived function, note, 441 i.
ratio of vanishing quantities, note, 472 i.
virtual velocities, 288 ii.
motion of fluid, 457 ii.
Lambert's theorem, 314 ii, 316 ii.
Lancret, angle H for curves, note, 90 ii.
on evolutes, note, 172 ii.
Laplace, rule of heliocentric distances, note, 320 ii.
Latent roots of linear function, 517 i, 508 i, 500 i, 562 i, 369 ii, 376 ii.
Law of the Norms, 173 i, 244 i.
Lectures on Quaternions referred to in notes, 160 i, 206 i, 219 i, 304 i, 311 i, 315 i, 345 i, 354 i, 380 i, 479 i, 525 i, 38 ii, 48 ii, 129 ii, 203 ii, 204 ii, 219 ii, 226 ii, 264 ii, 269 ii, 270 ii, 271 ii, 279 ii, 283 ii, 321 ii, 351 ii.
Left-handed, 111 i. *See* Rotation.
Lemniscata, 286 i.
Length of line. *See* Tensor.
relative of two lines, 111 i.
"Letters on Quaternions," note, 311 i.
Light, Fresnel's theory of, 323 ii.
Limit, of sum, integral as, 482 i, 170 ii.
Limiting ratios, 469 i.
Line, expressed by ∠ and Ax., 121 i; K, 127 i; U, 142 i; V. 195 i.
anharmonic equation of, 26 i; coordinates of, 27 i.
s, complex of, 417 ii, 427 ii.
concurrent 18 i.
congruency of, 417 ii.
contact of, with sphere, 225 i.
with confocals, 214 ii.
of curvature, 230 ii, 235 ii, 236 ii.
emanant, 83 ii, 408 ii.
focal, 545 i, 549 i, 213 ii, 243 ii.
geodetic, 29 ii, 225 ii, 264 ii.
group of points on, 15 i, 72 i.
half, 121 i, 142 i.
at infinity, 27 i.
integral, 449 ii.
intersection of, and plane, 47 i; and sphere, 220 i; 223 i.
of intersection of two planes, 338 i.
inverse of, 296 i.
number added to line, 335 i.
parallel, 10 i, 194 i, 325 i.
perpendicular to line, 179 i, 194 i, 427 i.
to plane, 180 i, 342 i, 353 i.
to two lines, 83 ii.
to itself, note, 236 ii.
point added to, 5 i.

Line, projection of, 179 i.
rational, 29 i, 54 i, 79 i.
reflexion of, 129 i, 358 i.
regulus of, 408 ii, 417 ii.
relative length and direction of two, 111 i.
shortest, 83 ii.
singular on envelope of sphere, 144 ii.
six coordinates of, note, 211 ii; 426 ii.
of striction, 408 ii.
symbol of, ternary, 56 i; anharmonic, 57 i; quinary, 63 i.
tangent, 99 i, 4 ii, 7 ii. *See* Tangent.
vector expression for, 15 i, 94 i; equation of, 195 i.
Linear complex, 427 ii.
Linear equation between two vectors, 12 i.
three vectors, 13 i.
four vectors, 44 i, 338 i.
five quaternions, 563 i.
seven screws, 393 ii.
function of several vectors, 467 ii.
transformation, 387 ii.
Linear quaternion function, 485 i, 555 i.
number of constants in, 556 i.
standard quadrinominal form of, 565 i.
symbolic biquadratic of, 560 i.
Linear vector function, 484 i.
auxiliary (ψ), 485 i; (χ), 495 i.
axes or directions of, 508 i.
arrangement of, and of its conjugate, 512 i.
coincidence of two, 503 i, 379 ii.
of three, 505 i, 379 ii.
determination of, note, 512 i.
imaginary, 514 i.
indeterminate, 501 i, 525 i, 369 ii, 379 ii, 385 ii.
rectangularity of two, note, 513 i.
of three, note, 513 i, 514 i, 517 i.
of system of, 384 ii.
bifocal, 545 i, 195 ii.
binomial, 498 i.
conjugate, 485 i.
constants in, number of, 486 i, 530 i.
cyclic ($\phi\rho = g\rho + V\lambda\rho\mu$), 520 i, 528 i, 535 i, 549 i, 187 ii, 193 ii, 233 ii, 240 ii, 332 ii, 369 ii, 373 ii.
with depressed equation, 501 i, 505 i, 379 ii.
derived, 551 i.
Linear vector function—*continued.*
dimensions of, 488 i.
focal ($\phi\rho = a\alpha V\alpha\rho + b\beta S\beta\rho$) 531 i, 533 i, 535 i, 538 i, 373 ii.
invariants of, 491 i, 376 ii, 364 ii, 386 ii.
inversion of, 485 i.
monomial, 501 i, 505 i.
non-conjugate, part of, 492 i.
powers of, 491 i.
principal planes of, 512 i, 384 ii.
reduction of two, 363 ii.
roots of, 517 i, 508 i.
equal, 500 i.
imaginary, 515 i.
self-conjugate, 513 i, 519 i, 525 i, 272 ii.
specification of, 374 ii.
square-root of, 225 ii, 367 ii.
standard trinomial form of, 486 i.
strain represented by, 365 ii.
symbolic cubic of, 494 i.
of self-conjugate part, 520 i.
unifocal. *See* Focal.
Liouville on confocals, 228 ii.
on geodetic curvature, 267 ii.
Liouville's Monge referred in notes, 92 ii, 111 ii, 145 ii, 153 ii, 235 ii, 239 ii, 261 ii, 264 ii, 266 ii, 267 ii, 271 ii.
Lloyd, experiments on conical refraction, note, 341 ii.
Local equations, 39 i.
Loci, equations of, involving, ∠, Ax., 121 i; K, 127 i; R, 296 i; S, 180 i, 190 i; T, 165 i, 167 i; U, 142 i; V, 195 i, 199 i.
powers of a vector, 417 i.
Locus, Apollonian, 130 i, 165 i, 191 i, 182 ii.
of centres of curvature of curve, 77 ii, 106 ii, 168 ii.
of spherical curvature, 120 ii.
of osculating circle, 140 ii.
of vertices of right cones tangent to confocals, 223 ii.
Logarithm of quaternion, 268 i, 275 i, 421 i, 83 ii.
Logarithmic spiral, 418 i, 54 ii.

Mac Cullagh, modular generation of quadrics, note, 279 ii.
motion about a fixed point, 291 ii.
theorem of the polar plane, 352 ii.
referred to in notes, 323 ii, 324 ii.

Magnetic analogies for ∇, 445 ii.
Maxwell, on differentials, note, 102 i.
on sense of rotation, note, 111 i.
electro-magnetic equations, note, 326 ii.
Mean point, projection of, 19 i, 81 i.
of gauche quadrilateral, 82 i.
of partial systems, 83 i.
proportional between two vectors, 251 i.
of two vectors, harmonic, 298 i; of n, 300 i.
of n vectors, simple, 81 i; complex, 85 i.
Measure of curvature, 261 ii, 412 ii, 415 ii.
Mécanique Célèste, note 320 ii.
Meusnier's theorem, note, 257 ii, 413 ii.
Mixed transformations of functions, 545 i, 278 ii.
Möbius referred to in notes, 22 i, 31 i, 61 i, 62 i, 65 i, 66 i, 77 i, 79 i, 162 ii, 236 ii.
Modular generation of quadric, 278 ii.
Moment, central, 285 ii.
of inertia, 292 ii.
total, 286 ii.
Monge, families of surfaces, 48 ii.
envelope of sphere, 144 ii, 153 ii.
evolutes, 167 ii.
lines of curvature, 235 ii.
referred to in notes, 53 ii, 90 ii. *See* Liouville's Monge.
Monomial form of linear vector function, 501 i, 505 i.
equations of circle and sphere, 355 i.
of cubic cone, 281 ii, 384 ii, 467 ii.
Motion on surface of revolution, 49 ii.
about fixed point, 290 ii.
screw-, 83 ii, 84 ii, 285 ii. *See* Screw.
fluid, 454 ii.
Mourey, note, 278 ii.
Moving axes, note, 289 ii, 403 ii, 404 ii, 410 ii.
Multiplicand, 147 i, 110 i, 159 i.
Multiplication of i, j, k, 159 i.
of quaternions, definition of, 116 i.
is not commutative, 147 i.
is doubly distributive, 219 i, 212 i.
is associative, 245 i, 308 i.
reduced to multiplication of versors, 172 i.
by coefficients, 119 i.
of vectors is associative, 337 i.
by coefficients, 9 i.
-arcs by scalars is not distributive, 156 i.
of versors illustrated by vector arcs, 147 i.
Multiplier, 147 i, 110 i, 159 i.
or integrating factor, 276 ii.

N, symbol for norm, 130 i.
Nq, differential of, 455 i.
Negative unity, square of right radial, 132 i.
square root of, has geometrically real value, notes, 133 i, 253 i.
as an uninterpreted symbol, 224 i, 289 i.
square of vector, 203 i, 345 i.
Net, plane geometrical, 20 i, 29 i; in space, 61 i.
Nets are homographic figures, 31 i, 79 i.
Newton on fluxions, note, 97 i; 431 i, 471 i.
Non-commutative multiplication, 147 i.
addition, 156 i.
Non-conjugate part of linear vector function, 492 i.
Non-distributive multiplication, 156 i.
Non-scalar, 110 i.
Non-sphericity, coefficient of, 80 ii, 120 ii, 135 ii, 138 ii.
Norm, 130 i, note, 128 i.
of sum, 189 i, 219 i, 476 i.
differential of, 455 i.
s, law of the, 173 i, 244 i.
Normal, absolute, 24 ii, 38 ii.
bi-, 27 ii.
to confocals, 199 ii.
cyclic of cone, 183 i; of ellipsoid, 232 i, 235 i.
developable, surface, 256 ii, 259 ii.
emanants, 109 ii, 117 ii.
s, intersection of, 230 ii, 239 ii.
planes to curves, 4 ii, 8 ii.
to quadric, 16 ii, 199 ii, 239 ii.
to surfaces, 11 ii.
at umbilic, 241 ii.
Notation, extended meaning of algebraic, 5 i, 6 i, 108 i, 123 i, 256 i.
of factorials, 476 i.
for focal relations, 310 i.
functional, 205 i, 202 i.
simplification of, 334 i.
See Symbol.
Null quaternion, 125 i, 139 i.
vector, 3 i; vector-arc, 146 i.
Number added to line, 335 i.
of constants in linear vector function, 486 i, 530 i.
in linear quaternion function, 556 i.
in equation of twisted cubic, 131 ii.

Number of points of construction, 22 i, 73 i.
signless, 111 i, 170 i.
Numbers, *norm* borrowed from theory of, note, 130 i.

Oblique cone, 183 i, note, 181 i.
quotient, 321 i, 337 i.
Ohm on logarithms, note, 276 i.
Olivier, note, 91 ii.
Opening, spherical, 366 i.
Operation, characteristic of. *See* Symbol.
Operations, Calculus of, 202 i, 205 i.
Operator or factor, 108 i, 135 i.
∇, 432 ii; notes, 548 i, 554 i, 251 ii.
disguised as $-D_\alpha$, 294 ii.
Opposite quaternions, 126 i,
vectors, 3 i.
Optic axes of crystal, 348 ii.
Orbit, central, 298 i.
Order of curve (plane), 32 i; (twisted), 93 i; 10 ii.
of surface, 87 i.
of congruency, 431 ii.
of complex, 430 ii.
of factors, indifferent when one is scalar, 119 i.
generally vital, 147 i, 153 i, 158 i.
Origin of vectors, 12 i.
Orthogonal axes of self-conjugate function, 513 i, 519 i, 524 i, 272 ii.
vectors for curve, 82 ii.
section of confocals, 198 ii.
surfaces, 474 ii.
Osculating circles, 58 ii, 86 ii, 174 ii.
intersections with curve, 60 ii.
deviation from curve, 97 ii, 133 ii, 138 ii, 404 ii.
cone (right), 99 ii; to cone of chords, 102 ii; (oblique), 99 ii.
cylinder, 126 ii, 128 ii.
helix, 95 ii, 404 ii.
parabola, 96 ii.
plane, 24 ii.
quadric, to surface, note, 249 ii.
screw-surface, 85 ii.
sphere, 50 ii, 74 ii, 178 ii; to surface, 269 ii.
surfaces, 103 ii.
twisted cubic, 129 ii, 156 ii, 404 ii.
Oval, 279 i.

P, symbol for ponential, 268 i.
Pairs of vectors, note, 393 ii.
Parabola inscribed to triangle, 34 i.
osculating curve, 96 ii.
Parabolic time in orbit, 311 ii.
Paraboloid, centre at infinity, 280 ii.
ruled, anharmonic equation of, 93 i.
vector expression for, 96 i.
normal to, 11 ii.
Paradox, apparent, 149 i.
Parallelepiped, volume of, 247 i, 338 i.
Parallelism, condition of, 325 i.
Parallelogram, area of, 246 i.
spherical, 388 i.
Parameters in vector equations, 94 i.
of confocal system, 196 ii, note, 224 ii.
of distribution, 408 ii.
Part, right or vector, of a quaternion, 193 i, 204 i.
Partial differentials, 101 i, 446 i, 479 i, 48 ii.
equations for dynamical functions, 296 ii, 297 ii.
Pencil of lines, 23 i, 307 i; planes, 57 i.
of a congruency, 424 ii.
Pentagon inscribed to a sphere, 351 i.
Permanent screws, 396 ii.
Permutation, cyclical, of *i*, *j*, *k*, 158 i.
of quaternions under S, 248 i, 350 i.
of linear vector functions, 363 ii.
Perpendicular from point on line, 179 i, 194 i, 427 i.
on plane, 180 i, 342 i, 353 i.
to two lines, 83 ii.
to itself, line, note, 236 ii.
of spherical triangle, 217 i.
of tetrahedron, note, 568 i.
Perpendicularity, condition of, 325 i, 345 i.
Perturbing force, 320 ii.
Pitch of a screw, 285 ii, 390 ii.
Plane, expressed by $\angle$. 121 i; Ax., 121 i; K, 127 i; T, 165 i, 167 i; U, 143 i; S, 180 i.
anharmonic equation of, 56 i.
concurrence of four, 57 i, 342 i.
cyclic, of cone, 183 i; of ellipsoid, 232 i.
equation of, scalar, 180 i; vector, 24 i, 94 i.
geometrical nets, 20 i, 29 i.
intersection of and line, 47 i.
s, intersection of two, 338 i.
inverse of, 353 i.
normal, 4 ii, 8 ii.

Planes, pencil of, 57 i.
polar, to sphere, 228 i; to quadric, 210 ii.
polar, Mac Cullagh's theorem of, 352 ii.
rational, 54 i.
of quaternion, 111 i.
quinary symbol of, 53 i.
strain, 370 ii.
of symmetry, elastic, 464 ii, 466 ii.
s, system of, related to linear vector function, 512 i.
quaternion function, 568 i.
through three points, 344 i.
Planet, distance of, from earth, 320 ii.
Plücker's six coordinates of a line, note, 211 ii, 426 ii.
Poinsot, representation of motion about fixed point, 290 ii.
Point, addition of, to vector, 5 i.
s, circular, note, 516 i.
s, condition of three on a line, 14 i, 49 i, 52 i, 57 i.
four in a plane, 14 i, 45 i.
five on a sphere, 354 i.
s of construction in plane, 17 i, 22 i.
in space, (first) 61 i; (second) 62 i: table of types, 75 i; diagram of, 78 i.
s, corresponding, on confocals, note, 225 ii.
s, difference of two, 3 i.
s, equality of, 3 i, 13 i.
s, equidifference of, 4 i.
s, extreme, 420 ii.
s, group of, on line, 15 i; in space, 294 i; on circle, 297 i.
guide, 239 i.
mean, of triangle, 19 i; of gauche quadrilateral, 82 i; in general, 81 i.
s rational, in a plane, 29 i; in space, 54 i, 79 i; types of, 55 i, 75 i.
symbol of, ternary, 25 i; quinary, 51 i; quaternary, 55 i.
of transformation, united, 387 ii.
Polar axis, 57 ii.
developable, 58 ii; cusp-edge of, 120 ii.
plane to quadric, 210 ii.
to sphere, 228 i.
Mac Cullagh's theorem of, 352 ii.
reciprocals, 547 i, 20 ii, 41 ii.
Pole and polar of a sphere, 228 i.
of plane curve, 35 i.
Poles of a plane with respect to confocals, note, 224 ii.
s of a spherical harmonic, 468 ii.
Polygon, area for spherical, 368 i.
exponential equation for spherical, 404 i.
geodetic, 266 ii, 416 ii.
inscribed to sphere, 347 i, note, 129 ii.
spherical sum of angles of, 415 i.
Polynomial equations, 277 i.
Poncelet, plane of homology, note, 60 i.
Ponential of a quaternion, 268 i.
Position, vector of, 100 i.
function of, 432 ii.
Positive or signless number, 170 i, note, 111 i.
Potential of attracting system, 293 ii, 305 ii.
function, 469 ii.
Power of a vector, a quaternion, 396 i, 399 i.
development of a, 476 i.
differential of a, 451 i.
equation of loci involving a, 417 i.
transformations of a, 420 i.
of a quaternion, 264 i, 274 i, 421 i.
of a linear vector function, 491 i.
Pressure, hydrostatic, 483 i, 440 ii, 459 ii.
Principal amplitude, am_0, 263 i.
axes of a body, 292 ii.
of ellipsoid, 238 i.
of quadric, 536 i, 187 ii, 272 ii.
of a section of, 238 i, 525 i, 231 ii, 253 ii.
foci of congruency, 420 ii.
function in dynamics, 294 ii.
for central orbit, 314 ii.
root, 259 i.
screws of inertia, 395 ii.
Principia, Newton's, 97 i, 431 i, 471 i.
Principles adopted from algebra, 108 i.
Prism showing properties of curve, 100 ii.
Product of quaternions defined, 90 1i.
conjugate of, 173 i.
differential of, 451 i.
reciprocal of, 173 i.
scalar of, 187 i, 245 i.
tensor of, 171 i.
vector of, 245 i.
versor of, 171 i.
of two quaternions, 109 i, 116 i, 171 i.
of two vectors, interpreted, 321 i, 329 i.
rectangular vectors, 333 i.

Product of vectors, 337 i, 346 i, 394 i.
of indices, 329 i.
of linear vector functions, 363 ii.
of sides of a triangle, 348 i.
of an inscribed polygon, 347 i.
Projection of closed figure, 8 i.
of line, 179 i.
of mean point, 19 i, 81 i.
stereographic, 311 i.
foci by, 423 ii.
Property, associative, of addition, 7 i, 207 i, 304 i.
of multiplication, 159 i, 245 i, 308 i, 337 i.
commutative, of addition, 7 i, 176 i, 207 i.
distributive, 9 i, 212 i, 219 i.
harmonic, of quadrilateral, 20 i.
homographic, of ruled surface, 408 ii.
Proportion of vectors, 118 i, 175 i, 250 i.
continued, 251 i, 256 i.
mean, 251 i.
Proportional to three vectors, fourth (complanar), 250 i; (diplanar), 356 i, 362 i; rectangular, 377 i.
Provector, 3 i, 146 i.
Proximity, vector of, 461 i, 20 ii.
Pure strain, 366 ii.
Pyramid, volume of, 247 i, 338 i; sign of, 48 i.

Quadrantal rotations, 149 i, 157 i.
triangle, 377 i.
Quadratic equation in quaternions, 281 i.
vector equation, 508 i.
Quadric, anharmonic equation of, 88 i.
axes of, 536 i, 187 ii, 272 ii.
of section of, 238 i, 525 i, 231 ii, 253 ii.
biconcyclic, 527 i, 187 ii, 272 ii.
bifocal equation of, 545 i, 195 ii.
centre of, 280 ii.
s, confocal, 196 ii.
s, classification of, 187 ii, 280 ii.
conjugate radii, 211 ii, note, 250 ii, 374 ii.
curvature of, 249 ii.
cyclic equation of, 535 i, 187 ii.
elongation, 372 ii.
exponential equation of, 206 ii.
focal equation of, 535 i.
generation of, modular and umbilicar, 278 ii.
generators of, 89 i, 213 ii.
geodetics on, 225 ii.
Quadratic, normals to, 16 ii, 199 ii, 239 ii.
osculating surface, note, 249 ii.
s, reciprocal, 389 ii.
species of, 187 ii, 280 ii.
sub-contrary circular sections are on a sphere, 18 ii.
tetrahedron self-conjugate to, 389 ii.
zero pitch, 392 ii.
Quadrilateral, cyclic, 296 i, 347 i.
gauche, 82 i, 88 i, 95 i.
harmonic property of, 20 i.
of a plane net, 31 i.
product of sides of, 347 i.
spherical, area of, 368 i.
Quadrinomial form for quaternion, 242 i.
for linear quaternion function, 565 i.
Qualitative element of a quaternion, note, 167 i.
Quantitative element of a quaternion, notes, 138 i, 167 i.
Quantities, ratio of vanishing, 470 i.
Quartic, Steiner's, 392 ii.
Quaternary symbols, 55 i.
Quaternion addition, 116 i, 176 i, 207 i.
amplitude of, 262 i.
angle of, 119 i.
anharmonic, 294 i, 296 i, 350 i.
axis of, 119 i, 203 i.
binomial, 254 i.
s, collinear, 116 i, 210 i.
s, complanar, 116 i, 148 i, 211 i, 250 i.
conjugate of, 123 i; of product, 173 i.
s, conjugate, 115 i, 123 i.
convergence of, series, 269 i, 424 i.
cosine of, 275 i, 424 i.
cube-root of, 256 i.
cyclical permutation under S, 248 i.
decomposition of, $TqUq$, 169 i, $Sq + Vq$, 193 i.
determinants, 361 ii, 382 ii, 393 ii.
development, 473 i, 465 i, 320 ii.
differentials, 438 i.
s, diplanar, 116 i.
division, 109 i, 116 i.
elements of, 112 i, 113 i, note, 167 i.
equality of, 109 i, 115 i, 243 i, 408 i.
equations, 243 i; algebraic, 292 i; (complanar), 277 i.
evolutionary, 295 i.
exponent, 274 i, 421 i.
exponential, 421 i.

Quaternion, as a factor or operator, 135 i.
as a fourth proportional, 357 i, 362 i.
functions of vectors a, 332 i, 394 i.
s, historical notes on, 206 i, 258 i, 262 i, 278 i, 351 ii.
identities, 426 i, 563 i.
imaginary. *See* Biquaternion.
imponential of, 274 i.
index of right, 122 i.
integration, 482 i, 170 ii.
invariants, 491 i, 506 i, 382 ii.
inverse of, 122 i.
inversion, 557 i.
Lectures on. *See* Lectures.
Letters on, note, 311 i.
linear function, 485 i, 555 i.
logarithm of, 275 i, 421 i, 83 ii.
moment, 286 ii.
multiplication, 116 i, 172 i, 219 i, 245 i, 301 i, 308 i.
null, 125 i, 139 i.
as number added to line, 335 i.
oblique quotient, 321 i, 337 i.
opposite of, 126 i.
plane of, 111 i.
ponential of, 268 i.
as power of a vector, 396 i, 399 i.
power of, 268 i, 274 i, 421 i.
s product of two, 109 i, 116 i, 171 i.
product of vectors a, 321 i, 337 i.
quadrinomial form of, 242 i.
as quotient of two vectors, 110 i.
s, quotient of two, 109 i, 116 i, 171 i.
radial, 131 i.
reciprocal of, 122 i.
s reciprocal of product of two, 173 i.
right part, 192 i.
quotient, 121 i.
root of, 259 i.
scalar, 120 i.
scalar of, 177 i.
as scalar plus vector, 11 i, 335 i.
sine of, 275 i, 424 i.
square of, 132 i, 141 i, 170 i, 187 i.
square-root of, 188 i, 315 i, 367 i; differential of, 452 i, 560 i.
as square-root of a positive plus square-root of a negative, 203 i.

Quaternion, subtraction of, 116 i.
s, sum of, 176 i, 207 i.
Taylor's series adapted to, 473 i, 7 ii, 22 ii.
tensor of, 167 i.
transcendental functions of a, 421 i, 453 i.
and trigonometry (plane), 178 i, 197 i, 214 i, 208 i.
(spherical), 216 i, 209 i.
vector of a, 192 i.
vector as a, 335 i.
versor of, 137 i.
versor as a, 143 i.
Quinary symbols, 50 i; types, 55 i.
Quotients, differential, 443 i.
geometric, 107 i.
inversion and alternation of, 118 i.
two with common denominator, 109 i, 116 i.
oblique, 321 i, 337 i.
of quaternions, 109 i, 116 i, 171 i; scalar of, 187 i.
radial, 131 i.
right, 121 i.
index of, 122 i, 331 i.
quotient of, 175 i.
sum of, 206 i.
of vectors, 107 i.

R, symbol for reciprocal, 141 i, 293 i, 328 i.
Radial quotient, 131 i.
right, 132 i.
Radical plane, 328 i.
Radius of absolute curvature of curve, 51 ii.
second curvature, 88 ii, 108 ii.
spherical curvature, 79 ii.
Ratio of vanishing quantities, 470 i.
Rational points, lines and planes, 29 i, 54 i, 79 i.
Ray or half-line, equation of, 121 i, 142 i.
s, systems of, 417 ii.
Third supplement on Systems of, notes, 330 ii, 341 ii.
-velocity, 323 ii; lines of single, 332 ii.
Rayleigh, Lord, on the dissipation function, 461 ii.
Reality of roots of self-conjugate function, 513 i, 519 i, 525 i, 272 ii.
of principal screws of inertia, 395 ii.
Reals, 11 i, 258 i.
Reciprocal of quaternion, 122 i; development of, 475 i.
versor of, 138 i.

Reciprocal of vector, 293 i.
curves, 41 ii; on sphere, 111, ii.
of sphero-conic, 193 ii.
surfaces, 20 ii, 275 ii, 389 ii.
of surface of centres, 255 ii.
of system of confocals, 198 ii.
of wave-surface, 326 ii.
screws, 390 ii.
Reciprocity of forms, 547 i, 41 ii, 328 ii, 388 ii, 427 ii.
Rectangle, spherical, note, 388 i.
Rectangular system for confocals, 199 ii.
curves, 82 ii, 403 ii.
surfaces, 410 ii.
self-conjugate function, 513 i, 519 i, 525 i, 272 ii.
transformations of linear vector function 528 i.
vectors, fourth proportional to three, 377 i.
versors, multiplication of, 149 i, 157 i; vectors, 333 i.
Rectifying vector, 89 ii.
developable, cusp-edge, 122 ii.
Reduced wrench, 395 ii.
Reduction of two geometric quotients to a common denominator, 116 i.
two linear vector functions, 363 ii.
quaternion to a power of a vector, 399 i.
multiplication, 171 i.
Reflexion of a line, 129 i, 358 i.
successive, 361 i.
at surface of crystal, 352 ii.
strain accompanied by, 365 ii.
Refraction, conical, 341 ii.
crystalline, 352 ii.
Regression, edge of, 93 i, 100 i, 36 ii.
of polar developable, 120 ii.
of rectifying developable, 122 ii.
of envelope of sphere, 144 ii.
Relation connecting three vectors, 337 i, 344 i, 375 i, 426 i.
four vectors, 44 i, 338 i, 376 i.
five vectors, 47 i.
five quaternions, 563 i.
six spherical arcs, 377 i.
seven screws, 393 ii.
Relative length and direction, 111 i, 138 i.
Remainder of a series, 474 i.
Representative angle, 151 i; arc, 143 i; point, 143 i.
Resolution of vector along and at right angles to line, 193 i.
along three lines, 338 i.
normal to three planes, 339 i.
of quaternion $TqUq$, 169 i; $Sq + Vq$, 193 i.
Resultant of forces, condition for unique, 284 ii.
wrenches, 390 ii.
Revector, 3 i.
Reversor, 139 i.
Revolution, cone of, 183 i; tangent to confocals, 217 ii.
cylinder of, 195 i, 199 i.
spheroid of, 201 i.
surfaces of, 47 ii.
Ribaucour, 402 ii.
Ridges on wave-surface, 337 ii, 344 ii.
Right-hand rotation, 119 i.
part of quaternion, 193 i.
quotient or quaternion, 121 i.
index of, 122 i.
identification of, with index, 331 i, note, 193 i.
quotients, quotient of, 175 i.
index of sum of, 206 i.
radial, 132 i.
versor as unit vector, 335 i.
Roberts, Michael, note, 235 ii.
Root, cube, of quaternion, 256 i; of unity (nine), 291 i.
of equation of algebraic form, 277 i.
of linear vector function or latent, 517 i, 500 i, 508 i, 562 i, 369 ii, 376 ii.
of negative unity, imaginary symbol, 224 i, 289 i.
geometrically real values of, notes, 133 i, 253 i.
principal, 259 i.
square, of quaternion, 188 i, 252 i, 315 i, 367 i, 452 i, 560 i.
of linear vector function, note, 225 ii, 367 ii.
of unity n^{th}, geometrically real, 259 i; imaginary, 290 i.
of zero, 316 i, 291 i.
Rotations, composition of, 415 i.
convention of sense of, 111 i, 119 i, notes, 49 i, 369 i.
conical, 154 i, 172 i, 359 i, 398 i, 429 i, 397 ii.
of emanant, 85 ii.
finite, 397 ii.
instantaneous, 288 ii.
linear function for, 367 ii.

90

Rotation, molecular, 446 ii.
quadrantal, 149 i, 157 i.
of radius of spherical curvature, 114 ii.
of solid, 361 i, 398 ii.
Royal Irish Academy, *see* Academy.
Ruled hyperboloid, 88 i, 95 i.
paraboloid, 96 i, 93 i.
surface, 408 ii.
helicoid osculating, 83 ii.

S, symbol for scalar, 177 i; $= \frac{1}{2}(1+K)$ 177 i, $= 1 - V$, 193 i; S^{-1}, 202 i.
cyclical permutation under, 248 i, 350 i.
distributive, 185 i.
examples on, 180 i, 190 i.
Sadleir, Rev. W. D., note, 375 i.
Saint Venant, de, notes, 27 ii, 81 ii, 91 ii, 94 ii, 111 ii, 124 ii.
Salmon, Geometry of Three Dimensions, notes, 27 ii, 42 ii, 92 ii, 104 ii, 129 ii, 213 ii, 223 ii, 228 ii, 235 ii, 236 ii, 259 ii, 260 ii, 261 ii.
Higher Plane Curves, notes, 37 i, 41 i, 42 i.
on centres of curvature of quadric, note, 260 ii.
Scalar, differential of, 455 i.
exponents, 264 i.
integration, 482 i.
origin of term, 11 i.
plus vector equals quaternion, 11 i, 335 i.
of a product, 245 i.
of a quaternion, 177 i, 186 i.
of a sum, 185 i.
symbol of, 177 i. *See* S.
unit in space, 394 i, 380 i.
Screw, axis, 83 ii, 285 ii, 390 ii, 430 ii.
coordinates of a, 392 ii.
s, chiastic, 39 ii.
cylindroid of, 391 ii, 419 ii, 422 ii.
homography, 396 ii.
invariants, 393 ii.
motion, 83 ii, 84 ii, 285 ii.
pitch of, 285 ii, 390 ii.
surface, 419 i, 12 ii, 28 ii, 83 ii. 281 ii.
osculating, 85 ii.
system, 391 ii, 393 ii.
translation, 85 ii.
Second curvature, 88 ii, 81 ii, 92 ii, 108 ii, 109 ii.
Section of cone, 181 i.
cyclic of cone, 183 i.
of ellipsoid, 232 i.
of cylinder, 196 i, 199 i.
of ellipsoid, 238 i.
harmonic, 16 i.
homographic, 16 i.
normal, curvature of, 247 ii.
component of curvature of, 267 ii, 413 ii.
of quadric, axes of, 523 i, 253 ii.
sub-contrary of cone, 183 i.
of quadric, 18 ii.
of wave-surface, 332 ii.
Segments, equation of six, 18 i, 21 i.
Self-conjugate function, 492 i.
reality of roots of, 513 i, 519 i, 525 i, 272 ii.
of linear transformation, 389 ii.
tetrahedron, 389 ii.
Semi-inversor, 135 i.
Sense of rotation, 111 i, 119 i, notes, 49 i, 369 i.
Series, convergance of, 269 i, 424 i.
exponential, 274 i, 421 i.
pouential, 268 i.
remainder of, 474 i.
of spherical parallelograms, 388 .
Taylor's, 102 i, 473 i, 7 ii, 22 ii.
Serret, referred to in notes, 92 ii, 108 ii.
Sexiant of screws, 393 ii.
Shortest distance between two lines, 83 ii.
Sign of area of plane triangle, 18 i.
of spherical area, 370 i.
of volume of a pyramid, 48 i, 342 i.
Signless number, tensor a, 170 i, 111 i.
Similitude, direct and inverse, 115 i, 365 ii.
Simplification of notation, 334 i.
Simultaneous differentials, 431 i.
Sine, exponential form for, 266 i, 274 i.
of a quaternion, 424 i.
Six planes, arrangements of, to illustrate the associative principle, 302 i, 304 i, 305 i.
points, evolutionary of, 295 i.
co-reciprocal screws, 392 ii.
Skew, centre of arch, 283 ii.
surface of emanants, 85 ii.
Solution of exponential equation, 409 i.
Space, scalar unit in, 380 i.
symmetry of, 394 i.

Space, transformation, linear, 387 ii.
Species of quadrics, 187 ii, 280 ii.
Sphere equation of, in terms of K, 130 i; S, 180 i; V, 199 i; N, T, 165 i, 167 i; $S^2 - V^2$, 200 i.
Apollonian locus, 130 i, 165 i, 191 i.
monomial, 355 i.
square of vector, 327 i.
various, 180 ii.
diacentric, 234 i, 241 i.
envelope of, 143 ii, 171 ii.
geodetic on, 30 ii.
homologies of ellipsoid and, 315 i, 232 i.
intersection of right line and, 220 i, 223 i.
of cone and, 181 i.
of ellipsoid and, 240 i.
inverse of, 353 i.
osculating curve, 50 ii, 74 ii, 178 ii; surface, 269 ii.
poles and polars of, 228 i.
tangent cone to, 225 i.
cylinder to, 201 i.
Spherical addition, 406 i.
area, 364 i, 368 i, 482 i.
cap, 482 i.
conic, intersection of cone and sphere, 182 i.
of ellipsoid and, 239 i.
associative principle illustrated by, 302 i, 308 i.
curvature of, 63 ii.
axis and foci of, 192 ii.
on wave-surface, 345 ii.
cubic, 281 ii.
curvature, 74 ii.
curve of absolute normals, 28 ii.
osculating circle of, 62 ii.
defect, 266 ii.
excess, 364 i.
total curvature, 266 ii.
harmonics, 468 ii.
hexagon, 303 i.
inscription of polygons in, 347 i, note, 129 ii.
opening, 366 i.
parallelogram, 388 i.
polygon, 414 i.
sum, 156 i, 406 i, 415 i.
tetragonometry, 417 i.
triangle, area of, 364 i.
exponential equation for, 404 i.
Spherical trigonometry, 209 i, 216 i, 325 i, 358 i, 400 i
fundamental formula, 400 i.
Sphericity, coefficient of non-, 80 ii, 120 ii, 135 ii, 138 ii.
Spheroid of revolution, 201 i.
Spin-vector, 492 i, 373 ii, 382 ii.
Spiral, 418 i.
Square of quaternion, 132 i, 141 i, 170 i, 187 i.
of right radial, 132 i.
of vector, 327 i, 345 i.
root of − 1, geometrically real values, notes, 133 i, 253 i.
as uninterpreted symbol, 224 i, 289 i.
of linear vector function, note, 225 ii, 367 ii.
of quaternion, 188 i, 252 i, 315 i, 367 i.
differential of, 452 i, 560 i.
of zero, 316 i, 291 i.
theorem of, on hypotenuse, 212 i.
Standard form, quaternion, quadrinomial, 242 i.
binomial, 254 i.
vectors, 344 i.
linear function, vector, 486 i.
quaternion, 565 i.
Statics, 283 ii.
Steiner's quartic, 392 ii.
Stereographic projection 311 i.
Stokes's theorem, 44 ii, 449 ii, 416 ii.
Strain, 365 ii.
Stress, 456 ii.
in terms of strain, 461 ii.
Striction, line of, 408 ii.
Sub-contrary sections of cone, 183 i.
of quadric, circular, are homospherical, 18 ii.
Subtraction of vectors, 5 i.
of quaternions, 116 i.
of amplitudes, 264 i.
Succession, direct or indirect, 297 i.
Successive differentiation, 100 i, 465 i, 479 i.
Sum, arcual, 156 i, 369 i.
of quaternions, 176 i.
scalar of, 185 i.
spherical, 156 i, 406 i, 415 i.
tensor and norm of a, 189 i, 219 i, 476 i.
of vectors, 7 i.
Summand, 5 i.
Supplementary triangle, 217 i.
s, formed by axis of ϕ and ϕ', note, 512 i.

Surface, anharmonic equation of, 87 i.
properties of ruled, 408 ii.
of centres, 254 ii; vector equation of, 260 ii.
s, confocal, 196 ii.
curvature of, 246 ii.
deformation of, 264 ii, 402 ii.
developable, 100 i, 36 ii, 232 ii, 409 ii.
emanant, 85 ii, 408 ii.
s, families of, 45 ii.
focal, of congruency, 431 ii.
geodetics on, 29 ii, 225 ii, 264 ii, 415 ii.
integral, line and, 449 ii.
volume and, 448 ii.
kinematical treatment of, 410 ii.
Kummer, of complex, 430 ii.
normals to, 11 ii.
order of, 87 i.
osculating, related to curve, 103 ii.
quadric osculating, note, 249 ii.
of revolution, 47 ii.
ruled, 408 ii.
screw, 419 i, 12 ii, 28 ii, 83 ii, 281 ii.
of second order, 179 ii. *See* Quadric.
vector equation of, 94 i.
wave-, 326 ii.
Symbol,* Am. 262 i; Am_0., Am_n., 263 i.
$\angle$ 120 i, $\angle_n$ 263 i.
Ax. 120 i; replaced by UV, 334 i.
cis, 260 i.
∇, note, 548 i. *See* Operator.
I, 187 i; suppressed, 334 i.
i, *j*, *k*, note, 160 i.
K, 124 i.
l, 276 i.
N, 130 i, note, 128 i.
P, 268 i.
R, 141 i.
S, 177 i, 166 i, note, 127 i; $S^{-1}0$, 202 i.
T, 163 i, note, 131 i.
U, 136 i; U0, note 140 i.
V, 193 i, note, 124 i; $V^{-1}0$, 202 i.
of complanarity (|||), 117 i.
of focal relation (. .), 393 i.
of intersection (OA·BC), 17 i.
Symbol of similarity, direct ($\propto$), 115 i.
inverse ($\propto'$), 115 i.
for spherical addition [(+)], 406 i.
$\sqrt{-1}$, indeterminate, 133 i; uninterpreted, 289 i.
s, equations in, 202 i, 205 i.
s of algebra, extended use of, 5 i, 6 i, 108 i, 123 i, 256 i.
+ and −, formula of relation, 5 i.
s, notes on, 262 i, 334 i, 351 ii.
of point ternary, 25 i, 56 i; quaternary, 55 i; quinary, 50 i.
Symbolic cubic of linear and vector function, 494 i.
depressed, 501 i, 505 i, 379 ii.
biquadratic of linear quaternion function, 560 i.
expression for Taylor's Series, 468 i, 473 i, 551 i.
Symbolical or imaginary roots of quaternion equation, 288 i.
See also Imaginary.
Symmetry of space, 394 i.
elastic, 464 ii, 466 ii.
Synchronism, hodographic, 310 ii.
Syntypical points, 55 i, 75 i.
System of linear vector functions, 384 ii.
of lines and planes related to linear vector function, 512 i, 568 i.
of rays, 417 ii.
of screws, 391 ii, 393 ii, 429 ii.
of six planes (association principal), 302 i, 304 i, 305 i.
of three right versors, 157 i.

T, symbol for tensor, 163 i.
examples on, 165 i, 167 i, 190 i.
T*q*, differential of, 456 i.
Table of types of points of construction, 75 i.
Tait, Prof. P. G., on electro-magnetic wave-surface, note, 326 ii.
on wave-surface, note, 350 ii.
on strain, 365 ii.
Tangent cone to sphere, 225 i.
to quadric, 209 ii.
to system of confocals, 213 ii, note, 224 ii.
right, 217 ii.
to wave-surface at cusp, 335 ii, 342 ii.

* References are given to an early or the earliest usage of the following symbols in this work. Fuller particulars will be found under the letters appropriate to each symbol.

Tangent to curve, 97 i, 4 ii, 7 ii.
cylinder, right, 201 i, 202 i, 236 i, 241 i, 220 ii.
developable, 36 ii, 91 ii, 98 ii.
circumscribed to surface, 222 ii, 232 ii.
to geodetic, 227 ii.
to locus of centres of curvature of curve, 77 ii, 106 ii, 168 ii.
plane, 11 ii.
Tangential co-ordinates, 40 i, 255 ii.
equations, 39 i.
Taylor's series, 102 i, 469 i, 473 i, 7 ii, 22 ii.
symbolic form of, 468 i, 473 i, 551 i.
Tension, act of, 164 i.
total,* 287 ii.
Tensor of quaternion, 167 i.
a signless number, 170 i, note, 111 i.
of a scalar, 168 i.
of a sum, 189 i, 219 i.
development of, 476 i.
of a vector, 163 i.
Term, 3 i.
Termino-collinear vector, 14 i, 343 i.
-complanar vectors, 45 i, 344 i.
Ternary product of vectors, 337 i.
symbols, 25 i, 56 i.
types, 76 i.
Tetragonometry, spherical, 417 i.
Tetrahedron, pyramid or gauche quadrilateral, 82 i.
self-conjugate, 389 ii.
Total curvature, 264 ii, 416 ii.
differentiation, 479 i.
moment, 286 ii.
tension, 287 ii.
Torsion, 88 ii, 178 ii, note, 81 ii.
expressions for, 108 ii.
geodetic, 415 ii.
Transcendental equations of surfaces, 206 ii, 281 ii.
functions of a quaternion, 421 i, 453 i.
Transformation, S and T, 190 i, U, 141 i.
exponential, of Taylor's series, 468 i, 473 i, 551 i.
of equation of wave-surface, 326 ii, 331 ii, 332 ii, 342 ii, 344 ii, 346 ii, 352 ii.
linear, 387 ii.
Translation in finite displacement, 397 ii.
screw, 85 ii, 390 ii.
Transport, 4 i.
Transvector, 3 i, 147 i.
Transversal, triangle cut by, 21 i.
of spherical triangle, 362 i.
Triangle, geodetic, 266 ii.
plane, conics and, 32 i.
harmonic relations, 21 i.
product of vector sides of, 348 i.
spherical, area of, 364 i.
exponential equation for, 404 i.
sum of angles, 406 i.
Trigonometry, plane, fundamental formula, 214 i.
examples, 178 i, 197 i, 265 i, 272 i.
S and V, 208 i.
spherical, fundamental formula, 216 i, 400 i.
examples, 357 i, 325 i, 400 i.
S and V, 209 i.
solution of exponential equation of, 409 i.
Trigonometrical functions of quaternion, 424 i.
Trinomial form for vectors, 242 i, 344 i.
for linear vector functions, 486 i.
Tube of flow, 441 ii.
Twist on a screw, 390 ii.
Twisted cubic, contact with curve of third order, 92 ii, 104 ii.
fourth, 125 ii, 128 ii.
osculating curve, 129 ii, 156 ii, 404 ii.
vector equation of, 129 ii, 131 ii.
Two-system of screws, 391 ii, 393 ii.
Types of points of construction, 55 i, 75 i.

U, symbol for unit-vector, 136 i.
versor, 137 i.
examples on, 142 i.
Uq, differential of, 456 i.
Ultimate ratio, 469 i.
Umbilic, 218 ii, 249 ii.
lines of curvature through, 242 ii.
normals at, 241 ii.
Umbilicar generation of quadric, 278 ii.
generatrix, 221 ii, 235 ii.
as line of curvature, 242 ii.
as envelope, 235 ii.
locus of, 222 ii.
Unicursal curves, note, 10 ii.

* Clausius has called this the *virial.*

Unifocal or focal form of function, 531 i, 533 i, 535 i. 538 i, 373 ii.
equation of wave-surface, 342 ii.
Uninterpreted symbol, $\sqrt{-1}$, 224 i, 289 i.
Unique resultant, condition for, 284 ii.
Unit in space, scalar or fourth, 380 i, 394 i.
sphere, 120 i.
-vector, 120 i.
conception of right versor as, 335 i.
-vectors of curve, 82 ii, 403 ii.
of surface, 410 ii.
United points of transformation, 387 ii.
Unity, cube root of, principal, 258 i.
nine, 291 i.
geometrical real n^{th} roots of, 259 i.
imaginary n^{th} roots of, 290 i.
negative, inversor, 135 i.
square of right radial, 132 i.
square root of, geometrically real, notes, 133 i, 253 i.
uninterpreted symbol, 224 i, 289 i.
views of other writers, 258 i, 278 i.
particular case of radial quotient, 132 i.
of linear function, 376 ii.

V, symbol for vector or right part, 193 i; = IV, 335 i.
$= 1 - S = \frac{1}{2}(1 - K)$, 197 i.
distributive character of, 204 i.
examples on, 195 i, 197 i, 199 i, 208 i.
Vq, differential of, 455 i.
Vanishing quantities, ratio of, 470 i.
Variable, change of independent, 24 ii, 33 ii.
Variations, calculus of, note, 48 ii, 271 ii.
Vection, 5 i.
Vector of acceleration, 100 i, 289 ii.
addition, commutative, 6 i; associative, 7 i.
to point, 5 i.
angle, 3 i.
representative of versor, 151 i.
associative property, 304 i.
sum of, 156 i, 406 i.
arc, 3 i.
determination of, 144 i.
addition of, 156 i.
is associative, 304 i.
representative of versor, 143 i.
coefficients of, 9 i.
Vectors, complanar, 340 i.
continued proportion of, 251 i.
s, complanarity of three, 338 i.
conjugate of, 346 i.
of conjugate, 197 i.
constants, 201 i, 236 i, 491 i.
curl of, 442 ii.
of curvature, 50 ii, 267 ii.
second, 88 ii.
definition of, 3 i.
difference of points, 3 i.
differential of, 96 i.
division of, by number, 11 i.
by parallel vector, 10 i.
by vector, 107 i.
elimination of, 342 i, 355 i.
equation of curve, 94 i; of surface, 94 i.
of confocals, note, 225 ii.
of congruency, 417 ii; of complex, 417 ii.
of cylindroid, 391 ii.
of second degree, 508 i.
of surface of centres, 260 ii.
of twisted cubic, 129 ii, 131 ii, note, 10 ii.
of unicursal curves, note, 10 ii.
as factor, 335 i.
s, fourth proportional to three complanar, 250 i; diplanar, 357 i; rectangular, 377 i.
function. *See* Linear.
s harmonic mean of two, 298 i; of n, 300 i.
identities. *See* Identity.
or imaginary part, note, 316 i.
imaginary or bi-vector, 224 i, note, 171 ii.
integral, 483 i. *See* Integral.
invariant. *See* Invariant.
s, linear relations in four, 44 i, 338 i; five, 47 i.
s, mean between two, 251 i.
multiplication, 323 i.
is associative, 337 i.
null, 3 i.
origin and term of, 3 i, 12 i.
pairs of, note, 393 ii.
parallel, 10 i, 325 i; perpendicular, 325 i, 345 i.
plus scalar is quaternion, 11 i, 335 i.
power of, 396 i, 399 i, 420 i, 476 i.
of product, 245 i.
product of two, a quaternion, 321 i, 333 i; of n, 337 i, 346 i, 394 i.

Vectors, proportion of, 118 i, 175 i, 250 i.
of proximity, 20 ii, 461 i.
quotient, 107 i.
reciprocal of, 293 i.
resolution of, 194 i, 338 i, 339 i.
is right part of quaternion, 335 i.
scalar of, 346 i.
special case of quaternion, 335 i.
spin-, 492 i, 373 ii, 382 ii.
square of, 327 i, 345 i.
subtraction of, 5 i.
sum of, defined, 7 i.
tensor of, 163 i.
s, termino-collinear, 14 i, 343 i; complanar, 45 i, 344 i.
three conditions determine, 341 i.
trinomial form for, 344 i.
umbilical, 218 ii.
units of curve, 82 ii, 403 ii.
versor of, note 137 i.
Velocity, vector of, 99 i.
angular, 84 ii, 119 ii, 288 ii.
areal, 299 ii.
single ray-, 332 ii.
single wave-, 335 ii.
virtual, 287 ii.
twist, 390 ii.
Version, 164 i.
Versor, of conjugate, 138 i.
conjugate of, 139 i.
depends on relative direction, 138 i.
as factor, 135 i.
of null quaternion, 139 i.
power of unit vector, 261 i.
of product, 171 i.
of quaternion, 137 i, note, 124 i.
of reciprocal, 138 i.
reciprocal of, 138 i.
right, as unit-vector, 335 i.
Versor of scalar, 139 i.
of sum, 476 i.
development of, 476 i.
of vector, note 137 i.
Versors, condition of complanarity of, 148 i.
equation between, 409 i.
Vibration, lines of, on wave-surface, 330 ii.
orthogonals, to, 345 ii.
principle of equivalent, 354 ii.
Virial or total tension, 287 ii.
Virtual focus, 419 ii.
velocities, 287 ii.
Viscous fluid, 459 ii.
Volume, sign of a, 48 i, 342 i.
of parallelepiped, 247 i, 338 i.
of sphere, spheric shell, 483 i.
and surface integrals, 448 ii.
Vortex motion, 453 ii.

Warren, note, 278 i.
Wave-cusps, 332 ii.
tangent cones at, 335 ii, 342 ii.
ridges, 337 ii, 344 ii.
-slowness, surface of, or index surface, 325 ii.
-surface, equation of, 326 ii, 331 ii.
cyclic form of, 332 ii.
unifocal form of, 342 ii.
bi-focal form of, 344 ii.
confocal form of, 346 ii.
Cartesian, 352 ii.
electro-magnetic, note, 326 ii.
vibrations, 347 ii.
Wrench, 390 ii.

Zero, square root of, 316 i, 291 i.
versor of, 139 i.
null quaternion is, 125 i.
pitch quadric, 392 ii.
Zone, area of, 482 i.

END OF VOLUME II.

Hamilton, (Sir) William Rowan
Elements of quaternions
2d ed.