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ELEMENTS

OF

SOLID GEOMETRY

BY

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The Sphere

INDEX OF DEFINITIONS

SYMBOLS AND ABBREVIATIONS.

ax.	axiom.	$\odot$	circle.
cor.	corollary.	$+$	plus.
def.	definition.	$-$	minus.
iden.	identity.	$\times$	multiplied by.
prop.	proposition.	$\div$, /, :	divided by.
post.	postulate.	$=$	is equal to or equivalent to.
cons.	construction.	$\sim$	is similar to.
hyp.	hypothesis.	$\cong$	is congruent to.
rect.	rectangle.	$>$	is greater than.
rt.	right.	$<$	is less than.
st.	straight.	$\perp$	is perpendicular to, or a perpendicular.
$\measuredangle$	angle.	$\parallel$	is parallel to, or a parallel.
$\triangle$	triangle.		
$\square$	parallelogram.		

Q. E. D. (quod erat demonstrandum), which was to be proved.

Q. E. F. (quod erat faciendum), which was to be done.

NOTE. The foregoing are used also in the plural, as $=$ means "are equal to," as well as "is equal to."

REFERENCES TO PLANE GEOMETRY

63. The sum of all the angles about a point is equal to two straight angles.

65. If two straight lines intersect, the vertical angles are equal.

85. In congruent figures homologous parts are equal.

86. Any side of a triangle is less than the sum of the other two, and greater than their difference.

91. Two triangles are congruent if they have two sides and the included angle of the one equal, respectively, to two sides and the included angle of the other.

92. Two right triangles are congruent if the legs of the one are equal, respectively, to the legs of the other.

94. Two triangles are congruent if they have two angles and the included side of the one equal, respectively, to two angles and the included side of the other.

95. Two right triangles are congruent if a leg and an adjacent acute angle of the one are equal, respectively, to a leg and an adjacent acute angle of the other.

108. The perpendicular bisector of a line is the locus of points equidistant from the extremities of the line.

109. Two points each equidistant from the extremities of a line determine the perpendicular bisector of the line.

115. Only one perpendicular can be drawn from a given external point to a given straight line.

116. The perpendicular is the shortest line that can be drawn from a given point to a given line.

117. Two oblique lines from the same point in the perpendicular to a given line, cutting off equal segments from the foot of the perpendicular, are equal; and of two lines cutting off unequal segments from the foot of the perpendicular, the one cutting off the greater segment is the greater line.

121. Two right triangles are congruent if they have the hypotenuse and an acute angle of the one equal, respectively, to the hypotenuse and an acute angle of the other.

122. Two right triangles are congruent if the hypotenuse and a leg of the one are equal, respectively, to the hypotenuse and a leg of the other.

123. Two triangles are congruent if they have the three sides of the one equal, respectively, to the three sides of the other.

132. If two triangles have two sides of the one equal, respectively, to two sides of the other, but the third side of the first greater than the third side of the second, the angle opposite the third side of the first is greater than the angle opposite the third side of the second.

137. Two straight lines in the same plane perpendicular to the same straight line are parallel.

139. If a straight line is perpendicular to one of two parallels, it is perpendicular to the other.

151. Two angles having their right sides respectively parallel and also their left sides parallel are equal, whereas if the right side of each is parallel to the left side of the other they are supplementary.

155. The sum of three angles of a triangle is equal to a straight angle.

166. A diagonal divides a parallelogram into congruent triangles.

167. The opposite sides and the opposite angles of a parallelogram are equal.

169. Parallels included between parallels are equal.

173. If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.

176. Two parallelograms are congruent if two sides and the included angle of one are equal, respectively, to two sides and the included angle of the other.

213. Radii of the same circle or of equal circles are equal.

233. In the same circle, or in equal circles, equal arcs are subtended by equal chords and intercepted by equal central angles.

234. In the same circle, or in equal circles, equal chords subtend equal central angles and equal arcs.

240. Ten propositions, in all, may be obtained by selecting any two of the following conditions for the hypothesis and any one of the remaining three for the conclusion;

A straight line that
1. passes through the center of the circle.
2. bisects the chord.
3. is perpendicular to the chord.
4. bisects the minor arc.
5. bisects the major arc.

246. Through three points not in the same straight line one circle, and only one, can be drawn.

275. If two variables are constantly equal and each approaches a limit, their limits are equal.

277. The limit of the product of a constant and a variable is the product of the constant by the limit of the variable.

283. In the same circle, or in equal circles, central angles have the same ratio as their intercepted arcs.

287. A central angle is measured by its intercepted arc.

330. In any proportion the terms are in proportion by alternation.

331. In any proportion the terms are in proportion by inversion.

338. In any proportion like powers of the terms are in proportion.

340. A line parallel to one side of a triangle divides the other sides proportionally.

343. If a straight line parallel to the side BC of a triangle ABC cuts AB at D and AC at E, then $DE:BC = AD:AB$.

363. Similar polygons are those whose homologous angles are equal and whose homologous sides are proportional.

369. Two triangles are similar if their sides are respectively proportional.

370. Two triangles are similar if the sides of the one are, respectively, parallel or perpendicular to the sides of the other.

374. The homologous altitudes of two similar triangles have the same ratio as any two homologous sides.

376. Two similar polygons may be divided into the same number of similar triangles, similarly placed.

391. The square of the hypotenuse of a right triangle is equal to the sum of the squares of the two legs.

392. The square of either leg of a right triangle is equal to the difference of the square of the hypotenuse and the square of the other leg.

420. The area of a parallelogram is equal to the product of its base and altitude.

421. Parallelograms having equal bases and equal altitudes are equivalent.

425. The area of a triangle is equal to half the product of its base and altitude.

428. Triangles of equal altitudes are to each other as their bases.

430. Triangles of equal bases and altitudes are equivalent.

434. The areas of two triangles, having an angle in the one equal to an angle in the other, are to each other as the products of the sides including the equal angles.

436. The areas of two similar polygons are to each other as the squares of any two homologous sides.

486. If the number of sides of an inscribed polygon be indefinitely increased, the apothem of the polygon will approach the radius of the circle as a limit.

488. The circle is the limit of the perimeters of regular inscribed and circumscribed polygons and the area of the circle is the limit of the areas of these polygons when the number of their sides is indefinitely increased.

SOLID GEOMETRY.

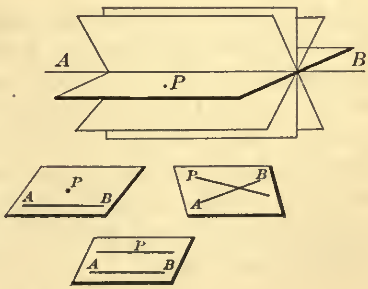
Book VI.

Solid geometry, or geometry of three dimensions, treats of figures whose elements are not all in the same plane. (§ 35.)

522. A **plane** is a surface such that the straight line joining any two of its points, lies wholly in the surface. (§ 16.) While regarded as indefinite in extent, it is usually represented in diagrams, by parallelograms that lie in the plane.

A plane is said to be determined by points or lines, when these points or lines fix its position in space.

523. A plane is determined by any three given points which are not in the same straight line.



Two points determine a line (§ 9), but do not determine a plane, because a plane may be rotated about any given line, assuming, in turn, an indefinite number of positions, but a third point, without this line, fixes the plane.

524. Because two points determine a line and three points a plane, so a plane is determined by the equivalent of three points; namely, a straight line and a point without this line; two intersecting straight lines; two parallel straight lines.

525. The point in which a straight line meets a plane is called the foot of the line.

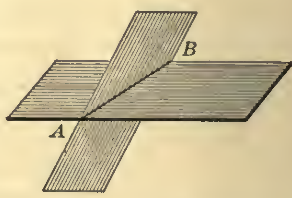
526. A straight line is perpendicular to a plane when it is perpendicular to every line in the plane drawn through its foot.

527. A straight line is parallel to a plane, or a plane is parallel to a straight line, when the two will not meet if produced indefinitely.

528. An **oblique line** is one which is neither perpendicular nor parallel to a plane.

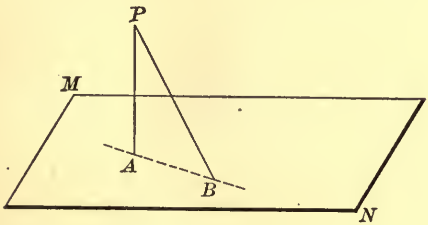
529. Two planes are parallel if they will not meet if produced indefinitely.

530. If two planes are not parallel, they must intersect in a line common to the two planes. For two planes cannot have, in common, a straight line and any point without that line. If they could, the two planes would coincide. (§ 523.) That is, the common points, or intersection, cannot be other than a straight line.



Proposition I. Theorem.

531. *The perpendicular is the shortest line that can be drawn from a point to a plane.*



Given the point P, the line PA ⊥ the plane MN, and any other line PB from P to MN.

To prove $PA < PB$.

Proof. Through A, the foot of PA, draw the line AB.

Then PA is $\perp$ AB. § 526

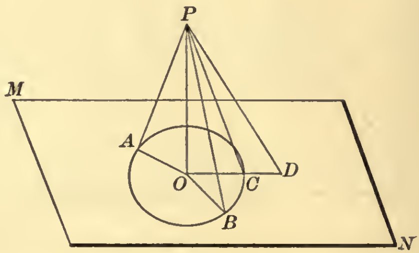
$\therefore PA < PB$. § 116

(*The ⊥ is the shortest line that can be drawn from a given point to a given line.*) Q. E. D.

532. The **distance** from a point to a plane is the length of the perpendicular from the point to the plane.

Proposition II. Theorem.

533. *Oblique lines drawn from a point to a plane, meeting the plane at equal distances from the foot of the perpendicular, are equal; and of two oblique lines meeting the plane at unequal distances from the foot of the perpendicular, the more remote is the greater.*



Given $PO \perp$ plane MN, and $AO = BO$, and $OD > OB$.

To prove I. $PA = PB$.

II. $PD > PB$.

Proof. I. The rt. $\triangle$s POA and POB are congruent, having PO common and $OA = OB$. Hyp. § 92

$\therefore PA = PB$. § 85

II. On OD take $OC = OB$ and draw PC.

Then $PD > PC$. § 117

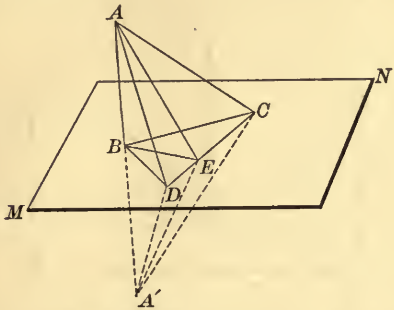
But $PC = PB$. Proof I

$\therefore PD > PB$. Q. E. D.

534. Cor. *Conversely: Equal oblique lines from a point to a plane meet the plane at equal distances from the foot of the perpendicular; and of two unequal lines the greater meets the plane at the greater distance from the foot of the perpendicular.*

PROPOSITION III. THEOREM.

535. *A straight line perpendicular to each of two straight lines at their point of intersection is perpendicular to the plane of those lines.*



Given $AB \perp BC$ and BD at B.

To prove $AB \perp MN$, *the plane of these lines.*

Proof. Draw BE, any other line, in the plane MN.

Draw DC meeting BE at E, and prolong AB to A' so that $BA' = AB$.

Join A and A' with D, E, and C.

BD and BC are each $\perp AA'$ at its mid-point. Hyp. and Cons.

Hence in $\triangle$s ADC and $A'DC$,

	$AD = DA'$,	§ 108
also,	$AC = A'C$,	§ 108
and	$DC = DC$.	Iden.
	$\therefore \triangle ADC \cong \triangle A'DC$.	§ 123
	$\therefore \measuredangle ADE = \measuredangle A'DE$.	§ 85
Hence	$\triangle ADE \cong \triangle A'DE$.	§ 91

$\therefore AE = EA'$, and $BE \perp AA'$ at B. §§ 109, 85

$\therefore AB \perp$ any line in MN passing through its foot.

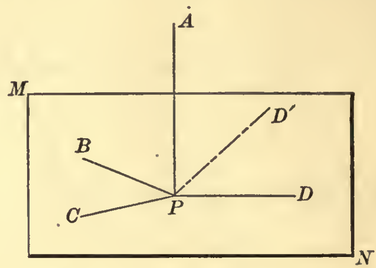
$\therefore$ it is $\perp$ the plane MN. § 526

Q. E. D.

Th2

Proposition IV. Theorem.

536. *All the perpendiculars to a straight line at the same point lie in a plane perpendicular to that line* (Converse of § 535).



Given PB, PC, PD, each ⊥ line AP at P.

To prove *PB, PC, PD, lie in a common plane ⊥ AP.*

Proof. Let *MN* be the plane of *PB* and *PC*. § 524

Then *AP* ⊥ plane *MN*. § 535

Let the plane of *AP* and *PD* intersect the plane *MN* in the line *PD′*. § 530

Then *AP* ⊥ *PD′*. § 526

But *AP* ⊥ *PD*. Hyp.

Since in the plane *APD* but one ⊥ can be drawn to *AP* at *P*, § 115

PD′ must coincide with *PD*.

∴ *PB*, *PC*, and *PD* lie in a common plane ⊥ *AP*. Q. E. D.

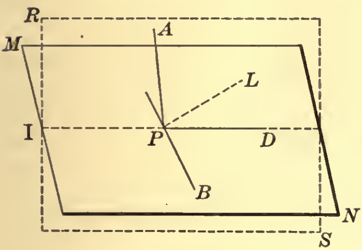
537. Cor. *Through a given point but one plane can be passed perpendicular to a given line, and the plane which is the perpendicular bisector of a straight line is the locus of points equidistant from the extremities of the line.*

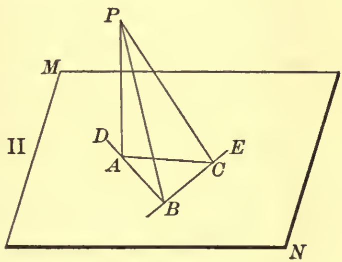
Proposition V. Problem.

538. *Through a given point, to draw a line perpendicular to a given plane.*

I. When the given point is within the plane.

II. When the given point is without the plane.





I. Given point P within the plane MN.

Required through P to draw a $\perp$ MN.

Construction. From P in plane MN draw any line PB.

In same plane draw $PD \perp PB$. § 113

Through P pass a plane intersecting MN in the line PB, and in this plane draw $PL \perp PB$.

In RS, the plane of PD and PL, draw $AP \perp PD$. § 113

Then $PA \perp MN$.

Proof. Because PB is $\perp$ PD and PL, it is $\perp$ their plane. § 535

$\therefore PB \perp AP$. § 526

And because AP is $\perp$ PB and PD, it is $\perp$ MN, their plane. § 535

Ex. 556. Find the locus of points in space equidistant from all points of a circle.

II. Given point P without the plane MN.

Required through P to draw a $\perp$ to MN.

Construction. Pass through P a plane intersecting MN in some line BC, and in this plane draw $BP \perp BC$. § 114

In the plane MN draw $BD \perp BC$. § 113

And in the plane of BD, BP, draw $PA \perp BD$. § 114

Then is $PA \perp MN$.

Proof. The $\triangle$s PAB, ABC, and PBC are rt. $\triangle$s. Cons.

$$\therefore \overline{PA}^2 = \overline{PB}^2 - \overline{AB}^2. \quad \text{§ 392}$$

$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2. \quad \text{§ 391}$$

$$\overline{PB}^2 + \overline{BC}^2 = \overline{PC}^2. \quad \text{§ 391}$$

Adding, $$\overline{PA}^2 + \overline{AC}^2 = \overline{PC}^2.$$

$$\therefore \triangle PAC \text{ is a rt. } \triangle. \quad \text{§ 391}$$

$$\therefore PA \perp AC.$$

$$\therefore PA \perp MN. \quad \text{Q. E. F.}$$

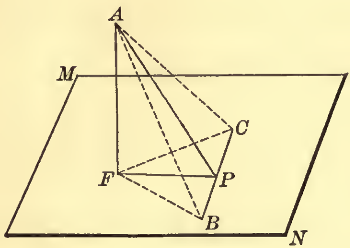
539. Cor. *Through a given point but one perpendicular to a plane can be drawn.*

Ex. 557. Find the locus of a point in space equidistant from three given points not in the same straight line.

Ex. 558. Find a point equidistant from four given points not all in the same plane.

PROPOSITION VI. THEOREM.

540. *If from the foot of a perpendicular to a plane a line be drawn at right angles to any line of the plane, and the point of intersection be joined with any point of the perpendicular, the last line will be perpendicular to the line of the plane.*



Given AF ⊥ plane MN, FP ⊥ any line CB in plane MN, and PA drawn from P to any point of AF.

To prove $AP \perp BC$.

Proof. From P on CB take $PC = PB$.

Join A and F with C and B.

Then $FC = FB$. § 108

$\therefore AB = AC$. § 533

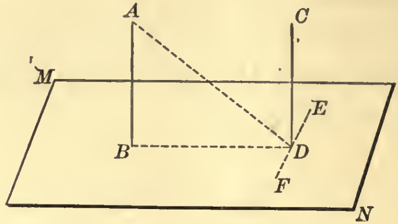
Hence $AP \perp BC$. § 109

Q. E. D.

Th 7

Proposition VII. Theorem.

541. *Two perpendiculars to the same plane are parallel.*



Given AB and CD ⊥ the plane MN.

To prove $AB \parallel CD$.

Proof. Join D to A and B. Draw $EF \perp BD$. § 113

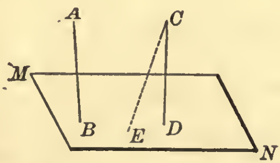
$\therefore AD \perp EF$. § 540

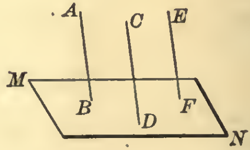
Because BD, AD, and CD are each $\perp$ EF, they lie in same plane. § 536

AB also lies in this plane. Because the three points A, B, and D determine a plane. § 523

$\therefore$ AB and CD lying in the same plane and $\perp$ the same line, BD, are $\parallel$. § 137

Q. E. D.





542. Cor. 1. *If one of two parallels is perpendicular to a plane, the other is perpendicular to the same plane.*

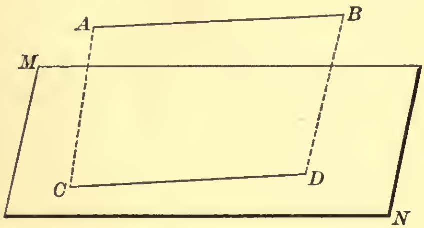
If CD is not $\perp$ MN, draw $CE \perp MN$. Then $CE \parallel AB$. § 541.
$\therefore$ CE and CD must coincide. $\therefore$ $CD \perp MN$.

543. Cor. 2. *If two lines are ∥ a third line they are ∥ each other.*

If MN is drawn $\perp$ AB, it must be $\perp$ CD and also $\perp$ EF. $\therefore$ the lines are ∥.

Proposition VIII. Theorem.

544. *A straight line parallel to a line in a plane is parallel to the plane.*



Given AB ∥ CD in plane MN.

To prove AB ∥ *plane* MN.

Proof. AB and CD being ∥ lie in the same plane AD. § 524

$\therefore$ if AB meets MN, it must meet it in line CD. § 530

But AB and CD are ∥ and cannot meet.

$\therefore$ AB ∥ plane MN. Q.E.D.

545. Cor. 1. *If a line is parallel to a plane, the intersection of the plane with any plane through the line is parallel to the line.*

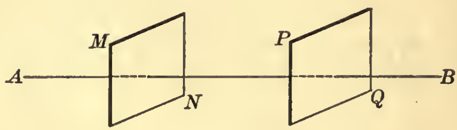
546. Cor. 2. *If two intersecting lines are each parallel to a given plane, the plane of these lines is parallel to the given plane.*

TWO PLANES.

Th 6

Proposition IX. Theorem.

547. *Two planes perpendicular to same straight line are parallel.*



Given planes MN and PQ ⊥ AB.

To prove *the plane MN ∥ the plane PQ.*

Proof. If MN and PQ are not ∥, they will meet if sufficiently produced.

Suppose them to meet. We would then have two planes through the same point ⊥ to the same line, which is impossible. § 537

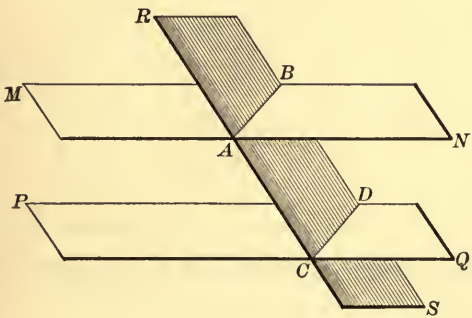
Therefore, plane MN ∥ plane PQ. Q. E. D.

548. Cor. *If a straight line and a plane are perpendicular to the same straight line, they are parallel.*

Th 15

PROPOSITION X. THEOREM.

549. *The intersections of two parallel planes by a third plane are parallel lines.*



Given two planes, MN and PQ, intersected by the plane RS in AB and CD.

To prove $AB \parallel CD$.

Proof. AB and CD lie in the same plane RS. Hyp.
Because they also lie in the planes MN and PQ they cannot meet. § 529

$$\therefore AB \parallel CD.$$ Q. E. D.

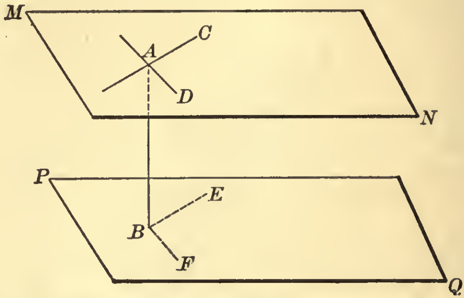
550. COR. 1. *Parallel lines included between parallel planes are equal.*

551. COR. 2. *Parallel planes are everywhere equally distant.*

Th 16

Proposition XI. Theorem.

552. *A straight line perpendicular to one of two parallel planes is perpendicular to the other, also.*



Given MN and PQ, ∥ planes, and AB ⊥ plane PQ.

To prove $AB \perp$ *plane* MN.

Proof. Draw BE and BF, any two lines in PQ intersecting at B.

Suppose planes passed through AB–BE and AB–BF, intersecting MN in AC and AD, respectively.

Then $AC \parallel BE$ and $AD \parallel BF$. § 549

But $AB \perp BE$ and to BF. § 526

$\therefore AB \perp AC$ and to AD. § 139

$\therefore AB \perp$ *plane* MN. § 535

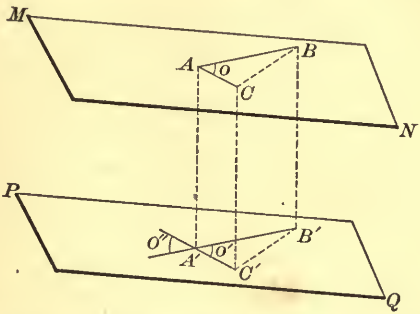
Q. E. D.

553. Cor. 1. *Reciprocally, a plane perpendicular to one of two parallel lines is perpendicular to the other, also.*

554. Cor. 2. *Through a given point one plane, and only one, can be drawn parallel to a given plane.*

Proposition XII. Theorem.

555. *Two angles not in the same plane, having their sides respectively parallel, right side to right side and left side to left side, are equal and their planes are parallel.*



Given $\measuredangle$s o and o' lying in planes MN and PQ, respectively, and having sides AB $\parallel$ A'B', AC $\parallel$ A'C'.

To prove $\measuredangle o = \measuredangle o'$ *and* $MN \parallel PQ$.

Proof. Take $A'B' = AB$ and $A'C' = AC$. Draw AA', BB', CC'.

Because AB and AC are respectively $\parallel$ and $=$ $A'B'$ and $A'C'$,

$AA'B'B$ and $AA'C'C$ are $\square$s. § 173

$\therefore AA' =$ and $\parallel BB'$ and $AA' =$ and $\parallel CC'$. § 167

Hence $BB' =$ and $\parallel CC'$. § 543

$\therefore CB'$ is a $\square$, and $BC = B'C'$. § 167

$\therefore \triangle ABC \cong \triangle A'B'C'$. § 123

$\therefore \measuredangle o = \measuredangle o'$. § 85

But $\measuredangle o' = \measuredangle o''$. § 65

$\therefore \measuredangle o = \measuredangle o''$.

Also PQ is $\parallel AB$ and AC. § 545

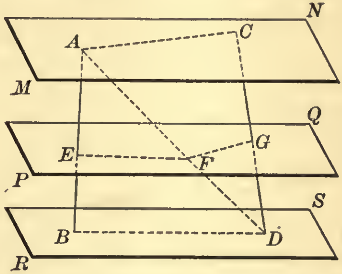
$\therefore MN \parallel PQ$. § 546

Q. E. D.

556. Cor. *Two angles not in the same plane, having their sides respectively parallel, right side to left side and left side to right side, are supplementary and their planes are parallel.*

Proposition XIII. Theorem.

557. *If two straight lines are intersected by parallel planes, the corresponding segments are proportional.*



Given AB and CD, intersected by planes MN, PQ, and RS, in A, E, B, and C, G, D, respectively.

To prove $$\frac{AE}{EB}=\frac{CG}{GD}.$$

Proof. Draw AD intersecting PQ in F. Draw EF, BD, FG, AC.

Then $EF \parallel BD$ and $FG \parallel AC$. § 549

$$\therefore \frac{AE}{EB}=\frac{AF}{FD} \text{ and } \frac{CG}{GD}=\frac{AF}{FD} \qquad \text{§ 340}$$

$$\therefore \frac{AE}{EB}=\frac{CG}{GD}.$$

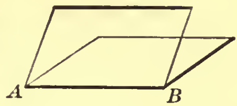
Q. E. D.

DIHEDRAL ANGLES.

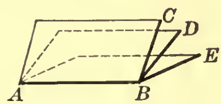
558. When two planes intersect, their divergence from the line of intersection is called a **dihedral angle**.

559. The two planes are its **faces** and the line of intersection is its **edge**.

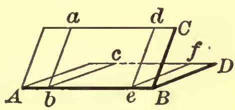
560. When a dihedral angle stands alone it may be designated by the two letters on its edge, as the dihedral angle AB. When two or more dihedral angles have a common edge, each angle is designated by four letters, as angles $C–AB–D$, or $D–AB–E$.



561. The **plane angle** of a dihedral angle is the angle formed by two lines, one in each face, drawn perpendicular to the edge at the same point. Thus $\measuredangle\ abc$ and $\measuredangle\ def$ are plane angles of the dihedral $\measuredangle\ C–AB–D$.



A plane perpendicular to the edge of a dihedral angle intersects the faces in lines which form the plane angle of the dihedral.

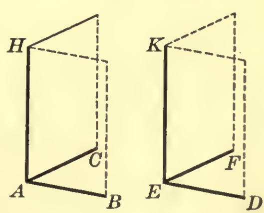


562. Any two plane angles of a dihedral angle or of equal dihedral angles are equal. $\measuredangle\ abc = \measuredangle\ def$, their sides being ∥. § 555

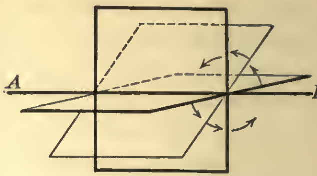
Since the points b and e are taken anywhere on the edge, *the plane angle of a dihedral angle is of the same magnitude at every point of the edge.*

Conversely, two dihedral angles are equal if their plane angles are equal.

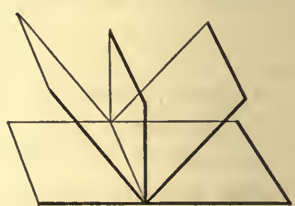
Given the plane $\measuredangle$ BAC = $\measuredangle$ DEF. The edge AH ⊥ the plane BAC and EK ⊥ plane DEF. Then by superposition it may be shown that the dihedral $\measuredangle$ B–HA–C = dihedral $\measuredangle$ D–KE–F. § 524



563. A dihedral angle may be conceived as generated by the revolution of a plane about a line taken as an edge. As the magnitude of the plane angle depends upon the amount of the revolution of one of its sides, from an initial line, independent of the length of its sides (§ 57), so the magnitude of a dihedral angle depends upon the amount of the revolution of a plane from an initial plane, and is independent of the extent of the planes.



564. By passing a plane perpendicular to the edge of the dihedral angle it is evident that *the plane angle is the measure of the dihedral angle.* A dihedral angle is acute, right, obtuse, straight, or reflex, according as its plane angle is acute, right, obtuse, straight, or reflex.



565. Dihedral angles are adjacent, vertical, complementary, or supplementary just as their plane angles hold these relations.

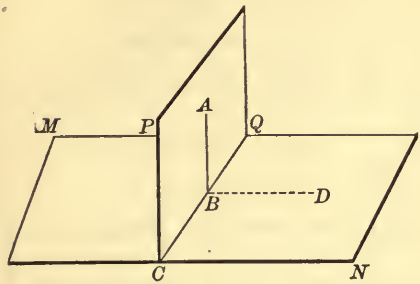
566. Many of the theorems of lines and angles have analogous theorems with planes and dihedral angles. By forming right sections of the dihedral angles the proofs of the following theorems may be obtained from the corresponding theorems of plane geometry:

1. The supplements of equal dihedral angles are equal.
2. The complements of equal dihedral angles are equal.
3. Vertical dihedral angles are equal.
4. If two parallel planes are cut by a third plane, the alternate interior dihedral angles are equal; the interior dihedral angles on the same side of the transverse plane are supplementary; the corresponding dihedral angles are equal; and conversely.
5. Two dihedral angles whose faces are respectively parallel are either equal or supplementary.
6. Two dihedral angles whose edges are parallel and whose faces are respectively perpendicular are either equal or supplementary.

Th 18

PROPOSITION XIV. THEOREM.

567. *If a straight line is perpendicular to a plane, every plane containing that line is perpendicular to the plane.*



Given AB ⊥ plane MN, and PQ any plane passing through AB intersecting MN in CQ.

To prove plane $PQ \perp$ *plane* MN.

Proof. In MN draw $BD \perp CQ$. § 113

But $AB \perp CQ$. § 526

$\therefore \measuredangle\, ABD$ is the plane $\measuredangle$ of the dihedral $\measuredangle\, P\text{–}QC\text{–}N$. § 561

And $\measuredangle\, ABD$ is a rt. $\measuredangle$. § 526

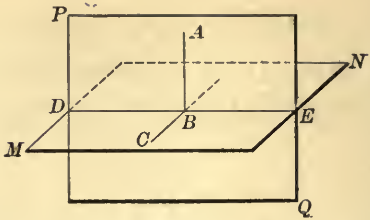
$\therefore$ plane $PQ \perp$ *plane* MN. § 564

Q. E. D.

568. COR. *A plane perpendicular to the edge of a dihedral angle is perpendicular to its faces.*

PROPOSITION XV. THEOREM.

569. *If two planes are perpendicular to each other, any straight line in one plane drawn perpendicular to their intersection is perpendicular to the other.*



Given plane PQ ⊥ plane MN, and AB in PQ ⊥ DE, their intersection.

To prove $AB \perp$ *plane* MN.

Proof. In plane MN draw $BC \perp DE$. § 113

Then $\measuredangle\ ABC$ is the plane $\measuredangle$ of the dihedral $\measuredangle\ P\text{–}DE\text{–}M$. § 561

But dihedral $\measuredangle\ P\text{–}DE\text{–}M$ is a rt. dihedral $\measuredangle$. Hyp.

$\therefore \measuredangle\ ABC$, its measure, is a rt. $\measuredangle$. § 564

Also $AB \perp DE$. Hyp.

$\therefore AB$, being $\perp BC$ and DE at their intersection, is $\perp$ their plane. § 535

$\therefore AB \perp$ plane MN. Q. E. D.

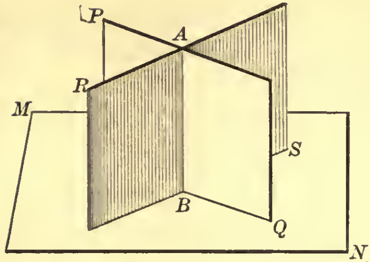
570. Cor. 1. *If two planes are perpendicular to each other, and a perpendicular to one of them is drawn from any point in their intersection, it will lie in the other plane.*

571. Cor. 2. *If two planes are perpendicular to each other, and a perpendicular to one of them is drawn from any point in the other plane, it will lie in that plane.*

Th 19

Proposition XVI. Theorem.

572. *If two intersecting planes are each perpendicular to a third plane, the line of intersection is perpendicular to that plane.*



Given planes PQ and RS, each $\perp$ plane MN, and intersecting in AB.

To prove $AB \perp$ *plane* MN.

Proof. At B in plane MN erect a $\perp$ to MN.

This $\perp$ lies in each of the planes RS and PQ. § 570

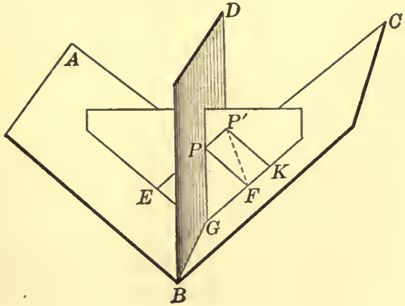
$\therefore$ it coincides with AB, their intersection. § 530

$\therefore AB \perp$ plane MN. Q. E. D.

573. Cor. *Conversely, if a plane is perpendicular to each of two intersecting planes, it is perpendicular to their line of intersection.*

Proposition XVII. Theorem.

574. *The plane bisecting a dihedral angle is the locus of points equidistant from its faces.*



I. Given the dihedral $\measuredangle$ A–BG–C, the plane BD bisecting this $\measuredangle$, any point P in the plane BD, PE, and PF, $\perp$s drawn from P to the faces AB and BC.

To prove P is equidistant from the faces AB and BC, or

$$PE = PF.$$

Proof. 1. Through PE and PF pass a plane intersecting AB in EG, BC in GK, and BD in PG.

Because $PE \perp AB$ and $PF \perp BC$, Cons.

the plane $PEF \perp BG$, their intersection. § 573

$\therefore$ $BG \perp EG$, PG, and FG. § 535

$\therefore$ $\measuredangle$s EGP and PGF are the plane $\measuredangle$s of the dihedral $\measuredangle$s $A–BG–D$ and $D–BG–C$. § 561

But the $\measuredangle A–BG–D = \measuredangle D–BG–C$. Hyp.

$\therefore \measuredangle EGP = \measuredangle PGF$. § 562

$\therefore$ rt. $\triangle PEG \cong$ rt. $\triangle PFG$. § 121

$\therefore PE = PF$. § 85

II. Given P′ any point without the bisecting plane BD.

To prove P′ unequally distant from the faces AB and AC.

Proof. Draw $P'K \perp BC$, $P'E \perp AB$, and from P draw $PF \perp BC$. Pass a plane through $P'E$ and $P'K$ intersecting AB in EG, BD in PG, and BC in GK.

Then $PP' + PF > P'F$. (§ 86.) Also $P'F > P'K$. § 531

$\therefore PP' + PF > P'K$ and $PF = PE$. Proof I

$\therefore PP' + PE > P'K$ or $P'E > P'K$.

$\therefore$ any point without the bisecting line is unequally distant from the faces of the dihedral angle. Q. E. D.

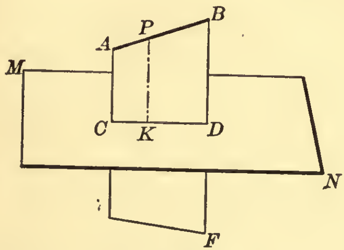
575. Def. The **projection of a point on a plane** is the foot of the perpendicular from the point to the plane.

The **projection of a line on a plane** is the locus of the projection of its points on the plane.

This may be illustrated by a shadow cast by a ruler on a wall, when the rays of light casting the shadow fall perpendicularly upon the wall.

Proposition XVIII. Problem.

576. *Through a straight line not perpendicular to a given plane to pass a plane perpendicular to that plane.*



Given the line AB not ⊥ plane MN.

Required to pass a plane through $AB \perp MN$.

Construction. From P, any point in AB, draw $PK \perp$ plane MN. § 538

Through PK and AB pass a plane AF.

AF is the plane required.

Proof. The plane AF through AB intersects MN. Cons.

$\therefore$ plane $AF \perp$ plane MN. § 567

Q. E. F.

577. Cor. 1. *Through a straight line not perpendicular to a given plane only one plane can be passed perpendicular to that plane.*

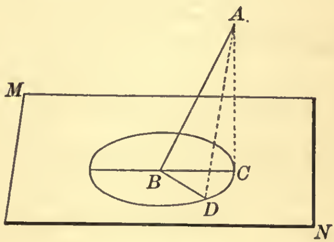
578. Cor. 2. *The projection of a straight line, not perpendicular to a plane, upon that plane, is a straight line.*

579. Cor. 3. *The projection of a straight line, perpendicular to a plane, is a point.*

580. Def. The plane containing all the perpendiculars drawn from a straight line to a plane is called the **projecting plane.**

Proposition XIX. Theorem.

581. *The acute angle which a line makes with its projection on a plane is the least angle which it makes with any line of that plane.*



Given BC the projection of the line AB on the plane MN, and BD any other line, drawn in plane MN, through B.

To prove $\measuredangle ABC < \measuredangle ABD$.

Proof. Take $BD = BC$, draw AD and AC.

In the $\triangle$s ABC and ABD,

$AB = AB$. Iden.

$BD = BC$. Cons.

But $AC < AD$. § 531

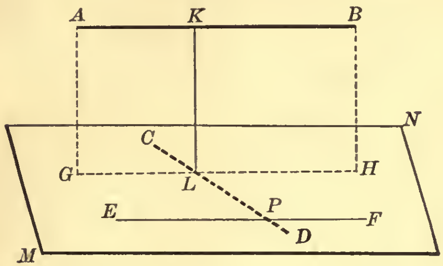
$\therefore \measuredangle ABC < \measuredangle ABD$. § 132

Q. E. D.

582. Def. **The inclination of a line to a plane** is the acute angle which the line makes with its projection on that plane.

PROPOSITION XX. PROBLEM.

583. *To draw a common perpendicular to two lines not in the same plane.*



Given two lines AB and CD not in the same plane.

Required to draw a common $\perp$ *to* AB *and* CD.

Construction. Through any point P of CD draw $EF \parallel AB$.

Through EF and CD pass a plane MN.

Then MN is $\parallel AB$. § 544

Through AB pass a plane $AH \perp MN$ (§ 576), intersecting MN in GH. Then $GH \parallel EF$ (§ 545), and must intersect CD in some point L.

At L in the plane AH draw $LK \perp GH$. § 113

Then LK is the $\perp$ required.

Proof. Since $GH \parallel AB$, and $LK \perp GH$ (cons.), $LK \perp AB$. § 139

But $LK \perp MN$. § 569

$\therefore LK \perp CD$. § 526

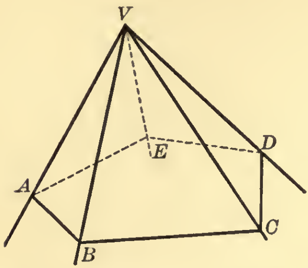
Q. E. F.

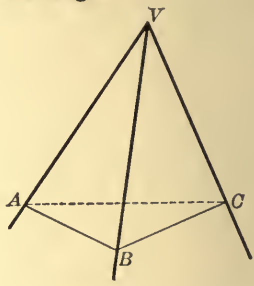
584. COR. 1. *Only one common perpendicular can be drawn to two lines not in the same plane.*

585. COR. 2. *The common perpendicular is the shortest path between two lines not in the same plane.*

POLYHEDRAL ANGLES.

586. When three or more planes meet at a common point the angle thus formed is called a **polyhedral angle.**





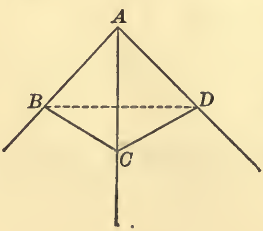
The common point is the **vertex** of the angle, the intersections of the planes are the **edges**, the portions of the planes between the edges are the **faces**, and the plane angles formed by the edges at the vertex are the **face angles.**

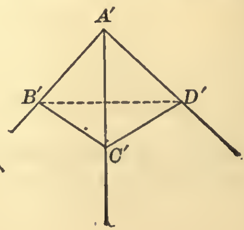
Thus in the diagram, V is the vertex, VA, VB, VC, etc., are the edges, VAB, VBC, etc., are the faces, and the angles AVB, BVC, etc., are face angles.

587. A polyhedral angle is designated by the letter at the vertex, as the polyhedral angle V; or by the letter at the vertex and a letter at a point on each edge, as the polyhedral angle $V\text{–}ABC$.

588. A polyhedral angle whose base is a convex polygon is called a **convex polyhedral angle.**

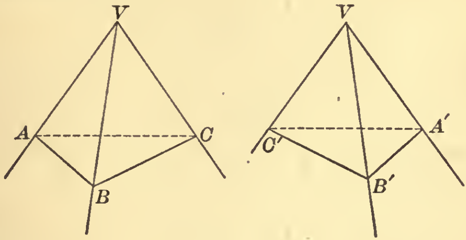
589. A polyhedral angle of three faces is called a **trihedral angle.**





590. Two polyhedral angles are congruent when their face angles and their dihedral angles are equal, each to each, and are arranged in the same order, for they may be made to coincide.

591. Two polyhedral angles are **symmetrical** when their face angles and their dihedral angles are equal each to each but arranged in a reverse order.

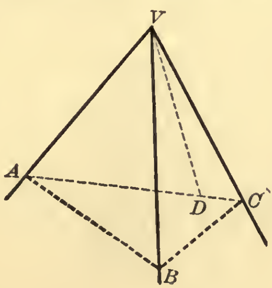


Two symmetrical polyhedral angles cannot, generally, be made to coincide; hence to show their equivalence, an indirect method is necessary.

592. Two polyhedral angles are **vertical** if the edges of one are the prolongations through the vertex of the edges of the other.

PROPOSITION XXI. THEOREM.

593. *The sum of any two face angles of a trihedral angle is greater than the third face angle.*



Given the trihedral $\measuredangle$ V, with $\measuredangle$ AVC > $\measuredangle$ AVB or $\measuredangle$ BVC.

To prove $\measuredangle AVB + \measuredangle BVC > \measuredangle AVC$.

Proof. In $\measuredangle AVC$ draw VD, making $\measuredangle AVD = \measuredangle AVB$.

Through D, any point in VD, draw AC.

Take $VB = VD$, and draw AB and BC.

Then $\triangle AVB \cong \triangle AVD$. § 91

$\therefore AB = AD$. § 85

In $\triangle ABC$, $AB + BC > AC$. § 86

But $\underline{AB = AD.}$

By subtraction, $BC > (AC - AD)$ or $BC > DC$.

In $\triangle$s BVC and DVC, VC is common,

$VB = VD$, Cons.

and $BC > DC$. Just proved

$\therefore \measuredangle BVC > \measuredangle DVC$. § 132

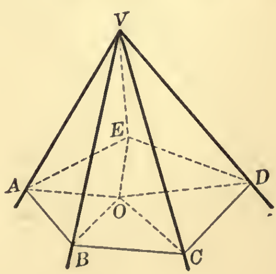
But $\measuredangle AVB = \measuredangle AVD$. Cons.

Adding, $\measuredangle AVB + \measuredangle BVC > \measuredangle AVD + \measuredangle DVC$.

$\therefore \measuredangle AVB + \measuredangle BVC > \measuredangle AVC$. Q. E. D.

Proposition XXII. Theorem.

594. *The sum of the face angles of any convex polyhedral angle is less than two straight angles.*



Given the polyhedral ∡ V.

To prove the sum of the ∡s at V less than 2 st. *∡s.*

Proof. Pass a plane cutting the polyhedral ∡ in the polygon *ABCDE.*

From *O*, any point in this polygon, draw *OA, OB, OC, OD, OE.*

The number of ∡s having a common vertex *O* is the same as the number of ∡s having a common vertex *V.*

∴ the sum of all the ∡s in the △s having a vertex at $O =$ the sum of all the ∡s of the △s having a vertex at *V.* § 155

But $\quad\quad ∡\, VBA + ∡\, VBC > ∡\, ABC,$

and $\quad\quad ∡\, VCB + ∡\, VCD > ∡\, BCD,$ etc. § 593

∴ the sum of the base ∡s of the △s having a common vertex at *V* is > the sum of the base ∡s of the △s having a common vertex at *O.*

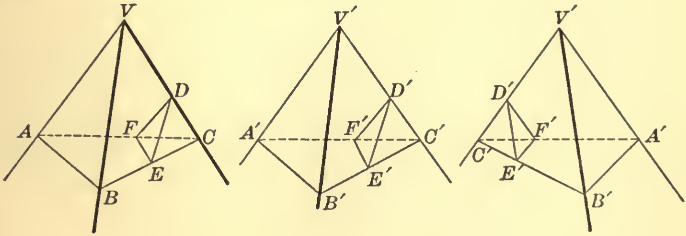
∴ the sum of the ∡s at the vertex at $V <$ the sum of the ∡s at the vertex *O.*

But the sum of the angles at the vertex $O = 2$ st. ∡s. § 63

∴ the sum of the ∡s at $V < 2$ st. ∡s. Q. E. D.

PROPOSITION XXIII. THEOREM.

595. *If two trihedral angles have the three face angles of one equal to the three face angles of the other, they are either congruent or symmetrical.*



Given the trihedral $\measuredangle$s V and V′, having the face $\measuredangle$s AVB, AVC, and BVC = the face $\measuredangle$s A′V′B′, A′V′C′, and B′V′C′, respectively.

To prove the corresponding dihedral $\measuredangle$s of V and V' are =, and that the trihedral $\measuredangle$s V and V' are congruent or symmetrical.

Proof. On the edges of V and V' take VA, VB, VC, $V'A'$, $V'B'$, $V'C'$ all equal, and draw AB, BC, AC, $A'B'$, $B'C'$, $A'C'$.

Then $\triangle$s VAB, VBC, $VAC \cong \triangle$s $V'A'B'$, $V'B'C'$, $V'A'C'$. § 91

$\therefore \triangle ABC \cong \triangle A'B'C'$. § 123

In the edges VC, $V'C'$, take $VD = V'D'$ and in the faces VBC, VAC, and $V'B'C'$, $V'A'C'$, draw DE, DF, and $D'E'$, $D'F'$ $\perp$ the edges VC and $V'C'$ respectively, meeting BC, AC, $B'C'$, and $A'C'$ in E, F, E', and F'. Draw EF, $E'F'$.

Then $\triangle CED \cong \triangle C'E'D'$. § 95

$\therefore CE = C'E'$,

and $DE = D'E'$.

Likewise $CF = C'F'$,

and $DF = D'F'$.

$\therefore \triangle CFE \cong \triangle C'F'E'$. § 91

$\therefore FE = F'E'$. § 85

$\therefore \triangle FED \cong \triangle F'E'D'$. § 123

$\therefore \measuredangle FED = \measuredangle F'E'D'$. § 85

$\therefore$ dihedral $\measuredangle$ VC = dihedral $\measuredangle$ $V'C'$.

In like manner it can be proved that dihedral $\measuredangle$ VB = dihedral $\measuredangle$ $V'B'$ and dihedral $\measuredangle$ VA = dihedral $\measuredangle$ $V'A'$. § 562

$\therefore$ the trihedrals V and V' are congruent or symmetrical according as the equal face angles are arranged in the same or reverse order. Q. E. D.

596. Cor. *If two trihedral angles have three face angles of the one equal to the three face angles of the other, then the dihedral angles of the one are respectively equal to the dihedral angles of the other.*

Book VII.

POLYHEDRONS.

597. A **polyhedron** is a solid bounded by planes. These planes are the **faces**; the intersections of the faces, the **edges**, and the intersections of the edges, the **vertices** of the polyhedron.

A line joining any two vertices, not in the same face, is a diagonal.

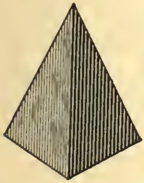
A polyhedron cannot have less than four faces.

598. A **section** of a polyhedron is the figure formed by its intersection with a plane.

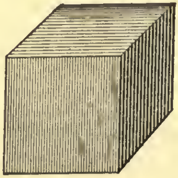
599. A polyhedron is **convex** when every section of it, formed by the intersection of a plane with its faces, is a **convex polygon**.

All polyhedrons considered in this book are convex.

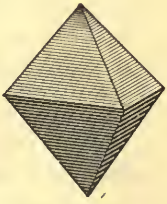
600. A polyhedron of four faces is called a **tetrahedron**; of six faces, a **hexahedron**; of eight faces, an **octahedron**; of twelve faces, a **dodecahedron**; of twenty faces, an **icosahedron**.



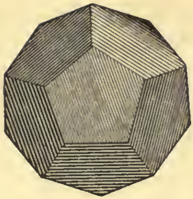
Tetrahedron.



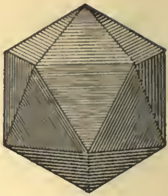
Hexahedron.



Octahedron.



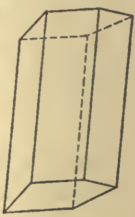
Dodecahedron.



Icosahedron.

PRISMS AND PARALLELOPIPEDS.

601. A **prism** is a polyhedron, of which two faces are congruent polygons in parallel planes, and the other faces are parallelograms. The congruent polygons are the **bases** and the parallelograms are the **lateral faces** of the prism. The intersections of the lateral faces are the **lateral edges.** The sum of the areas of the lateral faces is the **lateral area.** The perpendicular distance between its bases is the **altitude.** The lateral edges of a prism are all equal, because parallel lines included between parallel planes are equal.



Prism.

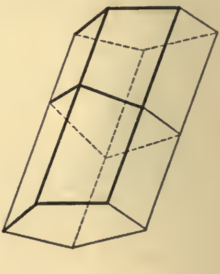
602. A prism whose lateral edges are perpendicular to its bases is a **right prism.**

Hence, the lateral edge of a right prism is equal to its altitude.

603. A right prism whose bases are regular polygons is a **regular prism.**

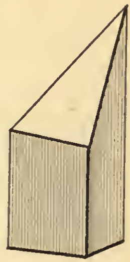
604. A prism whose lateral edges are not perpendicular to its bases is an **oblique prism.**

605. A prism is triangular, quadrangular, etc., according as the bases are triangles, quadrilaterals, etc.



606. The section of a prism made by a plane perpendicular to its lateral edges is a **right section.**

607. A **truncated prism** is that part of a prism included between a base and a section made by a plane oblique to the base.



608. A **parallelopiped** is a prism whose bases are parallelograms. Hence all the faces of a parallelopiped are parallelograms.

609. A **right parallelopiped** is a parallelopiped whose lateral edges are perpendicular to its bases.

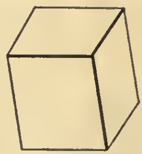
610. A **rectangular parallelopiped** is a right parallelopiped whose bases are rectangles.

Hence all its faces are rectangles.

611. A **cube** is a parallelopiped whose faces are squares.

Hence all of its edges are equal.

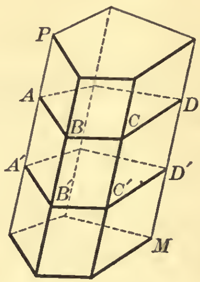
612. The **volume** of any solid is the ratio of the solid to another solid taken arbitrarily as the unit of volume. Because of its convenience a cube whose edge is a linear unit is adopted as the unit of volume.



613. **Equal** solids are solids whose volumes are equal.

Proposition I. Theorem.

614. *Sections of a prism made by parallel planes cutting all the lateral edges are congruent polygons.*



Given the prism PM cut by ∥ planes making the sections AD and A′D′.

To prove section $AD \cong$ *section* $A'D'$.

Proof. AB, BC, CD, etc. ∥ $A'B'$, $B'C'$, $C'D'$, etc., respectively. § 549

∴ AB, BC, CD, etc. $=$ $A'B'$, $B'C'$, $C'D'$, etc., respectively. § 169

Also ∡s ABC, BCD, etc. $=$ ∡s $A'B'C'$, $B'C'D'$, etc., respectively. § 555

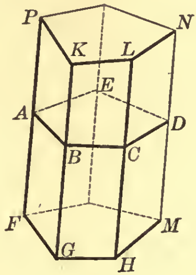
∴ polygon $AD \cong$ polygon $A'D'$.

(for the polygons have their sides and ∡s equal each to each, and are therefore congruent). Q. E. D.

615. Cor. *Any section of a prism made by a plane parallel to the base is equal to the base; also all right sections of a prism are equal.*

Proposition II. Theorem.

616. *The lateral area of a prism is equal to the product of the perimeter of a right section by a lateral edge.*



Given the prism PM. Denote its lateral area by s, a lateral edge by e, and the perimeter of a right section by p.

To prove $s = p \times e$.

Proof. $PF = KG = LH$, etc. $= e$. § 601

Also $AB \perp KG$, $BC \perp LH$, etc. § 606

$\therefore$ area $\square\, FK = AB \times PF, = AB \times e$,

area $\square\, GL = BC \times e$,

and area $\square\, HN = CD \times e$, etc. § 420

But the sum of these $\square$s equals the lateral area s.

$$\therefore s = (AB + BC + CD + \text{etc.}) \times e.$$

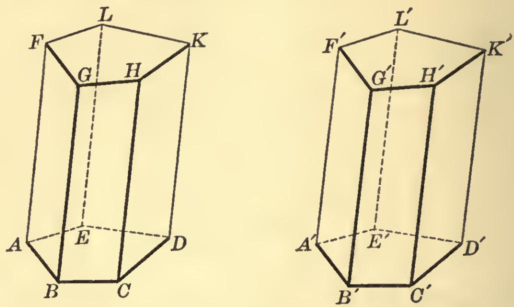
And the sum of $AB + BC + CD +$ etc. $= p$.

$$\therefore s = p \times e. \qquad \text{Q. E. D.}$$

617. Cor. *The lateral area of a right prism equals the product of the perimeter of the base by the altitude.*

PROPOSITION III. THEOREM.

618. *Two prisms are congruent when the three faces, including a trihedral angle of the one, are congruent, respectively, to the three faces, including a trihedral angle of the other, and are similarly placed.*



Given the prisms, AK and A′K′, having the faces AD, AL, AG = respectively to faces A′D′, A′L′, A′G′, and similarly placed.

To prove $AK \cong A'K'$.

Proof. The face $\measuredangle$s BAE, EAF, BAF are equal respectively to the face $\measuredangle$s $B'A'E', E'A'F', B'A'F'$. Hyp.

$\therefore$ trihedral $\measuredangle A =$ trihedral $\measuredangle A'$. § 595

Apply the prism $F'D'$ to the prism FD so that trihedral $\measuredangle A'$ will fall on its congruent trihedral $\measuredangle A$, and the faces of trihedral $\measuredangle A'$ shall coincide with the congruent faces of trihedral $\measuredangle A$, and the points C, D shall fall on the points C', D'.

Since the points L', F', G' coincide with L, F, G, the planes $F'K'$ and FK coincide (§ 523), because the lateral edges of the prism are ∥.

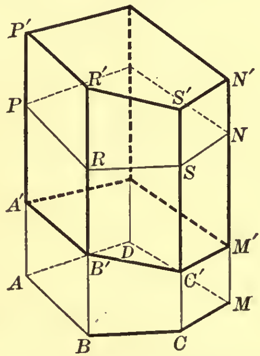
$\therefore$ the edges $C'H', D'K'$ coincide with CH, DK, and the points H', K' coincide with H, K. $\therefore AK \cong A'K'$. Q. E. D.

619. Cor. 1. *Two truncated prisms are congruent when the three faces including a trihedral angle of the one are congruent to the three faces including a trihedral angle of the other.*

620. Cor. 2. *Two prisms with equivalent bases and equal altitudes are equal.*

Proposition IV. Theorem.

621. *An oblique prism is equivalent to a right prism whose base is a right section of the oblique prism and whose altitude is a lateral edge of the oblique prism.*



Given the oblique prism PM with the right section A′M′ and lateral edge PA; also the right prism P′M′ with base A′M′ and lateral edges each equal to PA.

To prove	$PM = P'M'$.	
Proof.	$P'A' = PA$.	Hyp.
Subtracting PA' from each of these equals,		
	$P'P = A'A$.	
Likewise	$R'R = B'B$.	
Also	$PR = AB$ and $P'R' = A'B'$.	§ 167
And	$\angle$s of $PR' = \angle$s of AB'.	§ 151
	$\therefore$ face $PR' \cong$ face AB'.	§ 176
Similarly,	face $RS' \cong$ face BC'.	

And base $PN = AM$. § 601

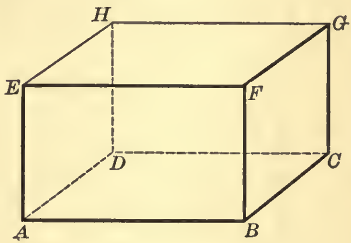
$\therefore$ truncated prism $PN' =$ truncated prism AM', § 619

Hence $P'M - PN' = P'M - AM'$,

or $PM = P'M'$. Q. E. D.

PROPOSITION V. THEOREM.

622. *The opposite lateral faces of a parallelopiped are parallel and congruent.*



Given parallelopiped AG.

To prove

$\square$ AF $\parallel$ and $\cong$ $\square$ DG, and $\square$ AH $\parallel$ and $\cong$ $\square$ BG.

Proof. Because AC is a $\square$, § 608

$\therefore AB \parallel$ and $= DC$. § 167

But AH is a $\square$, § 608

$\therefore AE \parallel$ and $= DH$. § 167

$\therefore \angle EAB = \angle HDC$,

and plane $AF \parallel$ plane DG. § 555

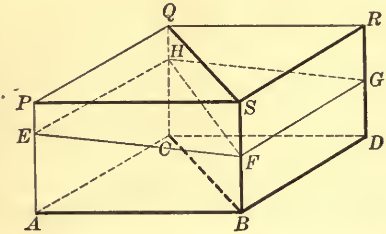
$\therefore \square AF \cong \square DG$. § 176

Similarly, $\square AH \parallel$ and $\cong \square BG$. Q. E. D.

623. COR. *Any two opposite faces of a parallelopiped may be taken as bases.*

Proposition VI. Theorem.

624. *The plane passed through two diagonally opposite edges of a parallelopiped divides it into two equivalent triangular prisms.*



Given the plane QSBC passing through the diagonally opposite edges QC and SB of the parallelopiped PD.

To prove PD is divided by the plane into two equivalent triangular prisms, ACB–S and BCD–R.

Proof. Let $EFGH$ be a right section of PD intersected by the plane $QSBC$ in the diagonal HF.

Then $EF \parallel HG$ and $EH \parallel FG$. § 549

$\therefore EFGH$ is a $\square$. § 164

$\therefore \triangle EFH \cong \triangle HFG$. § 166

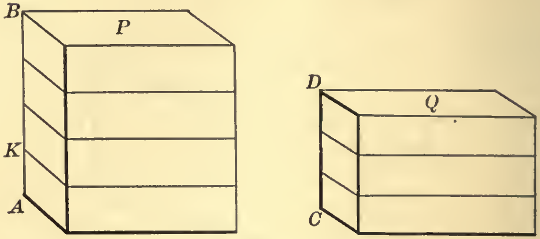
The triangular prism $ACB–S$ = a right prism whose base is EHF and altitude AP, and the prism $BCD–R$ = a right prism whose base is FHG and altitude AP.

But these right prisms are equal, having equal bases and equal altitudes. § 621

$\therefore ACB–P = BCD–R$. Q. E. D.

PROPOSITION VII. THEOREM.

625. *Two rectangular parallelopipeds having congruent bases are to each other as their altitudes.*



Given two rectangular parallelopipeds P and Q having congruent bases and the altitudes AB and CD.

To prove $$\frac{P}{Q}=\frac{AB}{CD}.$$

CASE I. When AB and CD are commensurable.

Proof. Let AK be the common measure of AB and CD. Suppose it is contained m times in AB and n times in CD.

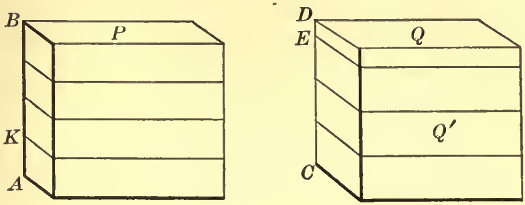
Then $$\frac{AB}{CD}=\frac{m}{n}.$$

At the points of division of AB and CD pass planes ∥ to the bases.

These planes divide P into m and Q into n equal parallelopipeds. § 620

$$\therefore \frac{P}{Q}=\frac{m}{n} \text{ or } \frac{P}{Q}=\frac{AB}{CD}.$$

Case II. When AB and CD are incommensurable.



Proof. Divide AB into any number of equal parts and apply one of these parts, as AK, to CD as a unit of measure. Since AB and CD are incommensurable, there will be a remainder ED less than one of the parts.

Through E pass a plane ∥ the bases of Q and let Q' be the parallelopiped between this plane and the lower base of Q.

Then $$\frac{Q'}{P}=\frac{CE}{AB}.$$ Case I

If the unit of measure for AB be continually decreased, the remainder ED, which is always less than the unit of measure, may be made smaller than any assignable quantity, but not equal to zero, since AB and CD are incommensurable.

$\therefore$ CE will approach CD as a limit.

$\therefore$ $\frac{CE}{AB}$ will approach $\frac{CD}{AB}$ as a limit.

$\therefore$ $\frac{Q'}{P}$ will approach $\frac{Q}{P}$ as a limit.

$$\therefore \frac{Q}{P}=\frac{CD}{AB}.$$ § 275

$$\therefore \frac{P}{Q}=\frac{AB}{CD}.$$ § 331

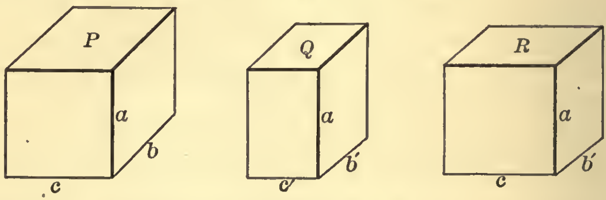
Q. E. D

626. Def. The three edges of a rectangular parallelopiped meeting at a common vertex are called its **dimensions.**

627. Cor. *Two rectangular parallelopipeds which have two dimensions in common are to each other as their third dimension.*

Proposition VIII. Theorem.

628. *Two rectangular parallelopipeds having equal altitudes are to each other as their bases.*



Given two rectangular parallelopipeds P and Q, having a common altitude a and the dimensions of their bases b, c, and b', c', respectively.

To prove $$\frac{P}{Q}=\frac{b\times c}{b'\times c'}.$$

Proof. Construct a third rectangular parallelopiped R, having the same altitude a and the dimensions of the base c and b'.

Then Q and R have two dimensions, a and b' in common.

$$\therefore \frac{R}{Q}=\frac{c}{c'}. \qquad \text{§ 627}$$

Likewise P and R have two dimensions a and c in common.

$$\therefore \frac{P}{R}=\frac{b}{b'}. \qquad \text{§ 627}$$

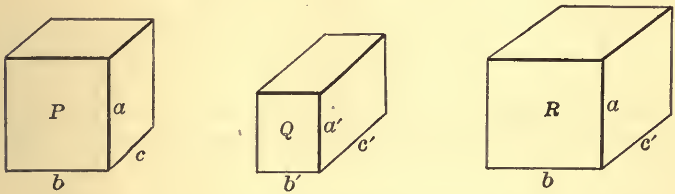
Multiplying the equations,

$$\frac{P}{Q}=\frac{b\times c}{b'\times c'}. \qquad \text{Q.E.D.}$$

629. Cor. *Two rectangular parallelopipeds having one dimension in common are to each other as the products of the other two dimensions.*

Proposition IX. Theorem.

630. *Two rectangular parallelopipeds are to each other as the products of their three dimensions.*



Given the two rectangular parallelopipeds P and Q having the dimensions a, b, c and a′, b′, c′, respectively.

To prove $$\frac{P}{Q}=\frac{a\times b\times c}{a'\times b'\times c'}.$$

Proof. Construct a third rectangular parallelopiped R having the dimensions a, b, and c'.

Then $$\frac{R}{Q}=\frac{a\times b}{a'\times b'}$$ § 629

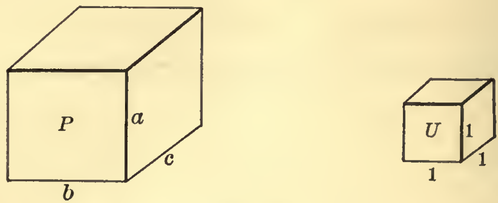
and $$\frac{P}{R}=\frac{c}{c'}.$$ § 627

Multiplying the equations,

$$\frac{P}{Q}=\frac{a\times b\times c}{a'\times b'\times c'}.$$ Q. E. D.

Proposition X. Theorem.

631. *The volume of a rectangular parallelopiped is equal to the product of its three dimensions.*



Given any rectangular parallelopiped P, with dimensions a, b, c, and the cube U, the unit of volume, whose edge is the linear unit.

To prove $P = a \times b \times c.$

Proof. $\frac{P}{U} = \frac{a \times b \times c}{1 \times 1 \times 1}.$ § 630

Because U is the unit of volume, $\frac{P}{U}$ is the numerical measure of the volume P. § 612

$\therefore$ the volume of $P = a \times b \times c.$ Q. E. D.

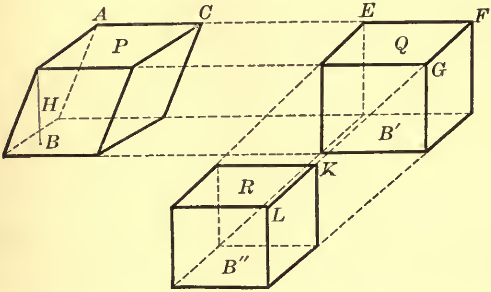
632. Cor. 1. *The volume of a cube is equal to the cube of its edge.*

633. Cor. 2. *The volume of a rectangular parallelopiped is equal to the product of its base by its altitude.*

Note. By the statement of Proposition X is meant that the number of unit cubes in the volume of any rectangular parallelopiped is equal to the product of the numerical measures of its length, breadth, and thickness.

Proposition XI. Theorem.

634. *The volume of any parallelopiped is equal to the product of its base by its altitude.*



Given the oblique parallelopiped P, with base B and altitude H.

To prove $\quad$ *volume* $P = B \times H$.

Proof. Produce the edge AC and the edges ∥ it. On AC produced take $EF = AC$ and through E and F pass planes ⊥ the produced edges, forming the oblique parallelopiped Q whose base B' is a rectangle.

Produce the edge FG and the edges ∥ it. Take $KL = FG$ and through L and K pass planes ⊥ these edges, forming the rectangular parallelopiped R.

Then $\quad P = Q$ and $Q = R$. $\quad$ § 621

Also $\quad B = B'$ and $B' = B'$. $\quad$ § 421

Because the planes of the upper and lower bases are ∥ the three parallelopipeds have a common altitude H.

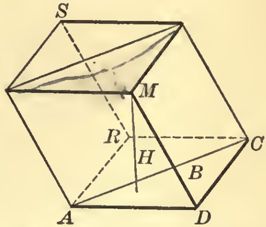
But $\quad$ volume $R = B'' \times H$. $\quad$ § 633

$\therefore$ volume $Q = B' \times H$,

and $\quad$ volume $P = B \times H$. $\quad$ Q. E. D.

PROPOSITION XII. THEOREM.

635. *The volume of a triangular prism is equal to the product of its base and altitude.*



Given the triangular prism ADC–M having its base B and altitude H.

To prove volume $ADC\text{–}M = B \times H$.

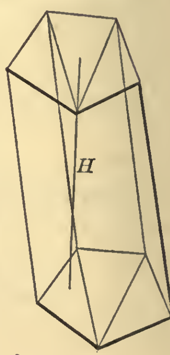
Proof. Upon DA, DC, DM as edges construct the parallelopiped $ADCR\text{–}M$.

Then volume $ADC\text{–}M = \frac{1}{2}$ volume of DS. § 624

But volume $DS = ADCR \times H = 2\,B \times H$. § 634

$\therefore$ volume $ADC\text{–}M = \frac{1}{2} \times 2\,B \times H = B \times H$. Q.E.D.

636. COR. 1. *The volume of any prism is equal to the product of its base by its altitude.*

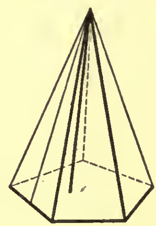


Any prism may be divided by diagonal planes into triangular prisms. These prisms have a common altitude. The volume of each is equal to the base multiplied by the common altitude. Hence the sum of the volumes of the triangular prisms is equal to the sum of the bases multiplied by the common altitude H. The sum of the triangular prisms is equal to the given prism and the sum of the bases is equal to B. $\therefore V = B \times H$.

637. Cor. 2. *Prisms that have equivalent bases and equal altitudes are equal; prisms are to each other as the product of their bases by their altitudes; prisms having equivalent bases are to each other as their altitudes; prisms having equal altitudes are to each other as their bases.*

THE PYRAMID.

638. A **pyramid** is a polyhedron, one of whose faces, called the **base**, is a polygon, and the **lateral faces** are triangles having a common **vertex.**



The **lateral edges** are the intersections of the lateral faces.

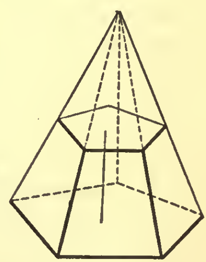
The **altitude** of a pyramid is the perpendicular from the vertex to the plane of the base.

The **lateral area** is the sum of the areas of the lateral faces.

639. When the base is a regular polygon, the center of which coincides with the foot of the altitude, the pyramid is **regular** and its altitude is its **axis.**

640. A pyramid is triangular, quadrangular, pentagonal, etc., as its base is a triangle, quadrilateral, pentagon, etc.

641. A triangular pyramid is also called a **tetrahedron** because it has four faces.



642. The **slant height** of a regular pyramid is the altitude of any one of its lateral faces.

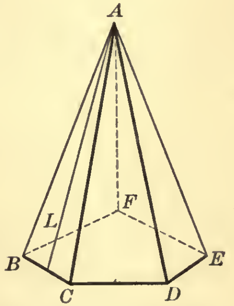
643. A **truncated pyramid** is the part of a pyramid contained between the base and a section made by a plane cutting all its lateral edges. If the plane of the section is parallel to the base, the part between this section and the base is called a **frustum** of a pyramid.

644. The **altitude** of a frustum of a pyramid is the perpendicular between the planes of its bases. The lateral faces of a frustum of a regular pyramid are congruent isosceles trapezoids and its **slant height** is the altitude of any lateral face.

645. The lateral edges of a regular pyramid are equal, and the lateral faces of a regular pyramid are equal.

Proposition XIII. Theorem.

646. *The lateral area of a regular pyramid is equal to half the product of the slant height by the perimeter of the base.*



Given A–BCDEF, a regular pyramid, L its slant height, P the perimeter of its base, and S its lateral area.

To prove $S = \frac{1}{2} L \times P.$

Proof. The lateral faces ABC, ACD, etc., are equal isosceles $\triangle$s. § 645

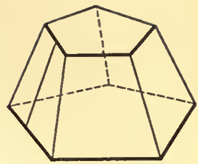
The common slant height L is the altitude of each $\triangle$.

$\therefore$ the area of each lateral face is $= \frac{1}{2} L \times$ its base. § 425

$\therefore$ the sum of all the lateral faces $= \frac{1}{2} L \times$ sum of bases.

But the sum of the bases $= P$.

$\therefore S = \frac{1}{2} L \times P.$ Q. E. D.



647. Cor. *The lateral area of the frustum of a regular pyramid is equal to half the product of the slant height multiplied by the sum of the perimeters of its bases. If P is the perimeter of its lower base and p is the perimeter of its upper base, then* $S = \frac{1}{2}(P + p)L$.

Ex. 559. Find the locus of a point in space equidistant from two given intersecting lines.

Ex. 560. The planes bisecting the dihedral angles of a trihedral angle intersect in the same straight line.

Ex. 561. The lateral faces of a right prism are rectangles.

Ex. 562. The diagonals of a parallelopiped bisect one another.

Ex. 563. The diagonals of a rectangular parallelopiped are equal.

Ex. 564. The square of a diagonal of a rectangular parallelopiped is equal to the sum of the squares of its three dimensions.

Ex. 565. The sum of the squares of the diagonals of a parallelopiped is equal to the sum of the squares of its twelve edges.

Ex. 566. Two rectangular parallelopipeds with equal altitudes have the dimensions of their bases 5 and 12, and 9 and 20, respectively. What is the ratio of their volumes?

Ex. 567. Find the volume and the area of the surface of a cube whose edge is 12 ft.

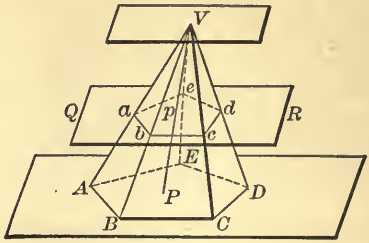
Ex. 568. Find the edge of a cube equal in volume to a rectangular parallelopiped whose dimensions are 6 ft., 8 ft., and 36 ft., respectively.

Proposition XIV. Theorem.

648. *If a pyramid is cut by a plane parallel to the base,*

I. *The edges and altitude are divided proportionally.*

II. *The section is a polygon similar to the base.*



Given the pyramid V–ABCDE cut by the plane QR ∥ the base and intersecting the lateral edges in a, b, c, d, e and the altitude in p.

To prove I. $\frac{Va}{VA}=\frac{Vb}{VB}=\frac{Vc}{VC}=\cdots=\frac{Vp}{VP}.$

II. The section $abcde \sim$ base $ABCDE$.

Proof. I. Through vertex V pass a plane ∥ QR.

Then the edges VA, VB, VC, etc., and the altitude VP, being intersected by parallel planes, are cut proportionally. § 557

$$\therefore \frac{Va}{VA}=\frac{Vb}{VB}=\frac{Vc}{VC}=\cdots=\frac{Vp}{VP}.$$

II. Since $ab \parallel AB$, $bc \parallel BC$, $cd \parallel CD$, etc. (§ 549), and $\measuredangle\, abc = \measuredangle\, ABC$, $\measuredangle\, bcd = \measuredangle\, BCD$, etc. (§ 555), then the two polygons $abcde$ and $ABCDE$ are mutually equiangular.

Since $\frac{ab}{AB}=\frac{Vb}{VB}$, $\frac{bc}{BC}=\frac{Vb}{VB}$, $\therefore\ \frac{ab}{AB}=\frac{bc}{BC}.$ § 343

Similarly, $\frac{bc}{BC}=\frac{cd}{CD}$, etc.

Hence the homologous sides of the polygons are proportional.

$\therefore$ section $abcde \sim$ base $ABCDE$. § 360

Q. E. D.

Ex. 569. How many bricks, each 8 in. long, 4 in. wide, and 2 in. thick, are equal in volume to a wall 16 ft. long, 4 ft. wide, and 12 ft. high?

649. Cor. 1. *Parallel sections of a pyramid are to each other as the squares of their distances from the vertex.*

For $$\frac{abcde}{ABCDE}=\frac{\overline{ab}^2}{\overline{AB}^2}. \qquad \S\ 436$$

But $$\frac{ab}{AB}=\frac{Vp}{VP} \text{ or } \frac{\overline{ab}^2}{\overline{AB}^2}=\frac{\overline{Vp}^2}{\overline{VP}^2}. \qquad \S\ 338$$

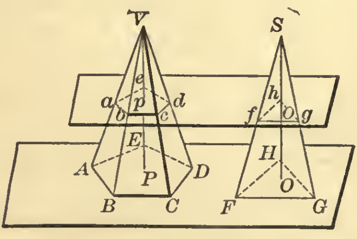
$$\therefore \frac{abcde}{ABCDE}=\frac{\overline{Vp}^2}{\overline{VP}^2}.$$

650. Cor. 2. *If two pyramids having equal altitudes are cut by planes parallel to their bases, and at equal distances from their vertices, the sections have the same ratio as their bases.*

For $$\frac{abcde}{ABCDE}=\frac{\overline{Vp}^2}{\overline{VP}^2}, \text{ also } \frac{fgh}{FGH}=\frac{\overline{so}^2}{\overline{SO}^2}. \qquad \S\ 649$$

But $$VP=So \text{ and } VP=SO. \qquad \text{Hyp.}$$

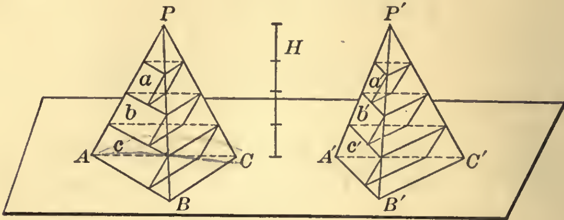
$$\therefore \frac{abcde}{ABCDE}=\frac{fgh}{FGH} \text{ or } \frac{abcde}{fgh}=\frac{ABCDE}{FGH}. \qquad \S\ 330$$



651. Cor. 3. *If two pyramids have equal altitudes and equivalent bases, sections made by planes parallel to the bases, and at equal distances from the vertices, are equivalent.*

PROPOSITION XV. THEOREM.

652. *Two triangular pyramids having equal altitudes and equivalent bases are equivalent.*



Given two triangular pyramids P–ABC and P′–A′B′C′, with equivalent bases, ABC and A′B′C′, and the same altitude H.

To prove $P\text{–}ABC = P'\text{–}A'B'C'$.

Proof. So place the pyramids that their bases shall be in the same plane.

Divide the altitude H into n equal parts, and through the points of division pass planes ∥ to the bases.

The corresponding sections of the pyramids will be equivalent. § 651

Upon these sections as upper bases inscribe a series of prisms, a, b, c, etc., a', b', c', etc., in each pyramid.

$\therefore a = a'$, $b = b'$, and each prism in $P =$ corresponding prism in P' and $a + b + c$, etc., $= a' + b' + c'$, etc.

Let the number of equal parts into which H is divided be increased indefinitely, then the sum $a + b + c +$ etc., will approach the pyramid P as a limit, and the sum $a' + b' + c' +$ etc., will approach P' as a limit.

But $a + b + c +$ etc., $= a' + b' + c' +$ etc., always.

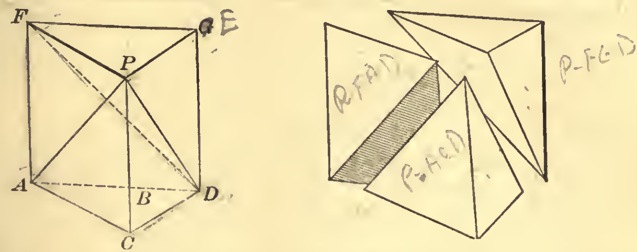
$\therefore P = P'$. § 275

Q. E. D.

653. COR. *Any two pyramids having equal altitudes and equal bases are equivalent.*

Proposition XVI. Theorem.

654. *The volume of a triangular pyramid is equal to one third the product of its base by its altitude.*



Given V the volume, B the base, and H the altitude of the triangular pyramid P–ACD.

To prove $V = \frac{1}{3} B \times H.$

Proof. On the base B construct the prism F–ACD with its lateral edges ∥ and $= PC$.

Pass a plane through P and FD.

Then the prism F–ACD is composed of three triangular pyramids P–ACD, P–FAD, and P–FGD.

But pyramid P–ACD = pyramid D–PFG, § 653

and D–PFG = P–FGD. Iden.

Since $\triangle AFD \cong \triangle DFG$, § 166

$\therefore P$–FGD = P–FAD. § 653

$\therefore P$–ACD = P–FGD = P–FAD.

$\therefore$ the prism is composed of three equivalent pyramids.

But the volume of the prism $= B \times H$. § 636

$\therefore$ the volume of a triangular pyramid $= \frac{1}{3} B \times H$. Q.E.D.

655. Cor. 1. *The volume of any pyramid is equal to one third the product of its base by its altitude.*

For, by drawing diagonals from one vertex of the base of any pyramid the base can be divided into triangles. By passing planes through the vertex of the pyramid and these diagonals the pyramid can be divided into triangular pyramids.

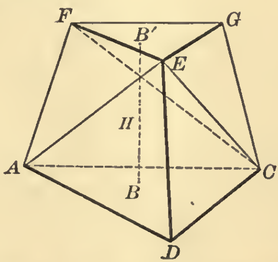
656. Cor. 2. *The volumes of two pyramids are to each other as the product of their bases and altitudes; pyramids having equivalent bases and equal altitudes are equivalent.*

657. Cor. 3. *Pyramids having equivalent bases are to each other as their altitudes; pyramids having equal altitudes are to each other as their bases.*

658. Note. The volume of any polyhedron may be determined by dividing the polyhedron into pyramids, finding the volume of each pyramid and taking their sum.

Proposition XVII. Theorem.

659. *A frustum of a triangular pyramid is equivalent to the sum of three pyramids whose common altitude is the altitude of the frustum, and whose bases are the upper base, the lower base of the frustum, and a mean proportional between them.*



Given the frustum of any triangular pyramid, F–ADC, with lower base B, upper base B′, and altitude H.

To prove, F–ADC = three pyramids whose altitude is H and whose bases are B, B′, and a mean proportional between B and B′.

Proof. Divide the frustum, by passing planes through E and FC, E and AC, into three triangular pyramids: $E\text{–}ADC$, $C\text{–}EFG$, and $E\text{–}AFC$.

Denoting these pyramids by P, Q, and R, respectively.

(*P and Q have a common altitude, H.*)

Then $P = \frac{1}{3}H \times B$, and $Q = \frac{1}{3}H \times B'$. § 654

It remains to prove that $E\text{–}AFC$ or $R =$ a pyramid whose altitude is H and whose base is a mean proportional between B and B' or $\sqrt{B \times B'}$.

Because P and R have a common vertex C, and their bases in the same plane $AFED$,

$$\therefore \frac{P}{R} = \frac{\triangle AED}{\triangle AEF}. \qquad \text{§ 657}$$

But $\triangle$s AED and AEF have a common altitude.

$$\therefore \frac{\triangle AED}{\triangle AEF} = \frac{AD}{EF}. \qquad \text{§ 428}$$

$$\therefore \frac{P}{R} = \frac{AD}{EF}.$$

Likewise pyramids Q and R may be considered as having a common vertex E, with their bases in the same plane $AFGC$.

$$\therefore \frac{R}{Q} = \frac{\triangle AFC}{\triangle FGC},$$

and

$$\frac{\triangle AFC}{\triangle FGC} = \frac{AC}{FG}.$$

$$\therefore \frac{R}{Q} = \frac{AC}{FG}.$$

Because $\triangle ADC \sim \triangle FGE$,

$$\frac{AD}{EF} = \frac{AC}{GF} \qquad \S\ 363$$

$$\therefore \frac{P}{R} = \frac{R}{Q} \text{ or } P \times Q = R^2.$$

$$\therefore R^2 = (\tfrac{1}{3}\, H \times B') \times (\tfrac{1}{3}\, H \times B') = (\tfrac{1}{3}\, H)^2 \times B \times B'.$$

$\therefore R = \frac{1}{3}\, H\sqrt{B \times B'}$, or $E\text{–}AFC =$ pyramid with altitude H and base $\sqrt{B \times B'}$.

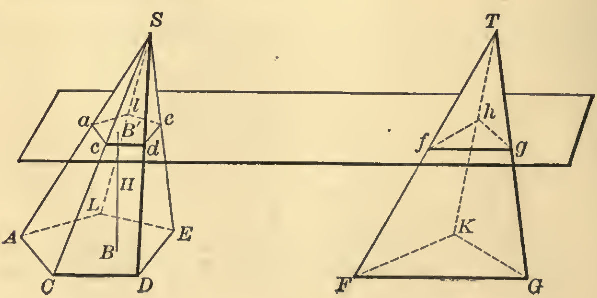
$\therefore$ frustum $F\text{–}ABC = \frac{1}{3}\, H\,(B + B' + \sqrt{B \times B'})$. Q.E.D.

660. Formula for volume of frustum of a triangular pyramid is

$$V = \tfrac{1}{3}\, H\,(B + B' + \sqrt{B \times B'}).$$

PROPOSITION XVIII. THEOREM.

661. *A frustum of any pyramid is equivalent to the sum of three pyramids whose common altitude is the altitude of the frustum, and whose bases are the upper base, the lower base of the frustum, and a mean proportional between them.*



Given the frustum of any pyramid Ad, with its upper base B′, its lower base B, its altitude H, and its volume V.

To prove $V = \frac{1}{3} H(B + B' + \sqrt{B \times B'})$.

Proof. Produce the lateral edges of the frustum Ad to meet in S.

Construct a triangular pyramid $T\text{–}FKG$ with altitude $= H$ and base $= ABCDE$ and lying in the same plane.

Produce the plane of $abcde$ to cut the pyramid $T\text{–}FKG$ in fhg.

$$\therefore fhg = abcde. \qquad \S\ 651$$

But $$S\text{–}ACDE = T\text{–}FKG \ (1), \qquad \S\ 653$$

and $$S\text{–}abcde = T\text{–}fhg \ (2).$$

Subtracting (2) from (1), frustum $aD =$ frustum fG.

But $$\text{volume } fG = \frac{1}{3} H(B + B' + \sqrt{B \times B'}). \qquad \S\ 660$$

$$\therefore V = \frac{1}{3} H(B + B' + \sqrt{B \times B'}). \qquad \text{Q. E. D.}$$

Ex. 570. The diagonal of a cube is to its edge as $\sqrt{3}$ is to 1.

Ex. 571. Find the diagonal of the rectangular parallelopiped whose edges are 8 ft., 9 ft., and 12 ft., respectively.

Ex. 572. Find the area of the entire surface of a triangular pyramid each of whose edges is 10 ft.

Ex. 573. Find the volume of a regular triangular pyramid each of whose edges is 8 ft.

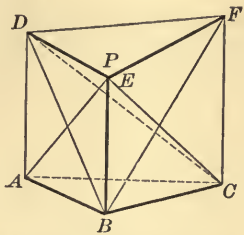
Ex. 574. The volume of a truncated parallelopiped is equal to the area of a right section multiplied by one-fourth the sum of the lateral edges.

Ex. 575. Find the volume of a right prism, if its altitude is 15 ft., and the sides of its base are, respectively, 10, 17, and 21 ft.

Ex. 576. A regular pyramid, whose base is an equilateral triangle, each side of which is 12 ft., has an altitude of 20 ft. Find its volume.

Proposition XIX. Theorem.

662. *A truncated triangular prism is equivalent to the sum of three pyramids whose common base is the base of the prism, and whose vertices are the vertices of the inclined section.*



Given the truncated triangular prism P, with base ABC and inclined section DEF.

To prove $P =$ *pyramid* $F\text{–}ABC +$ *pyramid* $D\text{–}ABC +$ *pyramid* $E\text{–}ABC$.

Proof. Let the planes determined by DE and C, and by AC and E divide the truncated prism into three pyramids, $E\text{–}FDC$, $E\text{–}DAC$, and $E\text{–}ABC$.

1. $E\text{–}FDC = B\text{–}ACF$, for their bases DFC and AFC are equal (§ 430) and their altitudes are equal because their vertices E and B lie in the edge of the prism which is ∥ to the face in which the bases lie.

But $B\text{–}ACF$ is identical with $F\text{–}ABC$.

$$\therefore E\text{–}FDC = F\text{–}ABC.$$

2. $E\text{–}DAC = B\text{–}DAC$, having same base and equal altitudes because their vertices E and B lie in the edge of the prism ∥ the face opposite.

But $B\text{–}DAC = D\text{–}ABC$, Iden. $\therefore E\text{–}DAC = D\text{–}ABC$.

3. $E\text{–}ABC$ has the required base and vertex.

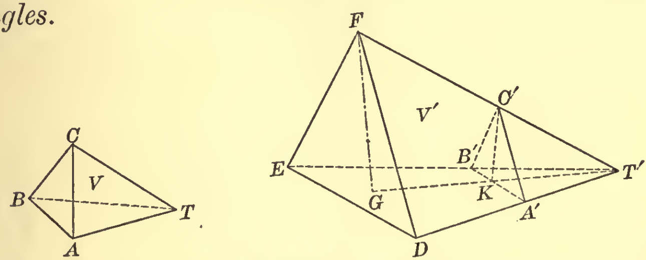
$$\therefore P = F\text{–}ABC + D\text{–}ABC + E\text{–}ABC. \qquad \text{Q. E. D.}$$

663. Cor. 1. *The volume of a right truncated triangular prism is equal to the product of its base by one third the sum of its lateral edges.*

664. Cor. 2. *The volume of any truncated triangular prism is equal to the product of its right section by one third the sum of its lateral edges.*

Proposition XX. Theorem

665. *Tetrahedrons having a trihedral angle of one equal to a trihedral angle of the other are to each other as the products of the edges about the equal trihedral angles.*



Given two tetrahedrons T–ABC and T′–DEF, with equal trihedral ∡s at T and T′ and volumes V and V′, respectively.

To prove $$\frac{V}{V'}=\frac{TA\times TB\times TC}{T'D\times T'C\times T'D}.$$

Proof. Apply the tetrahedron $T–ABC$ to $T'–DEF$ so that the equal trihedral ∡s T and T' coincide. From C' and F drop ⊥s upon the plane $T'ED$.

The three points T', K, and G lie in a straight line, § 578

(*the projection of a straight line upon a plane is a straight line*).

Then with $T'DE$ and $T'A'B'$ as the bases and FG and $C'K$ the altitudes of the tetrahedrons,

$$\frac{V}{V'}=\frac{T'A'B'\times C'K}{T'DE\times FG}=\frac{T'A'B'}{T'DE}\times\frac{C'K}{FG}, \qquad \S\ 656$$

But $$\frac{T'A'B'}{T'DE} = \frac{T'A' \times T'B'}{T'D \times T'E}. \qquad \S\ 434$$

In the right similar △s $T'C'K'$ and $T'FG$,

$$\frac{C'K'}{FG} = \frac{T'C'}{T'F}. \qquad \S\ 374$$

$$\therefore \frac{V}{V'} = \frac{T'A' \times T'B'}{T'D \times T'E} \times \frac{T'C'}{T'F} = \frac{TA \times TB \times TC}{T'D \times T'E \times T'F}. \qquad \text{Q.E.D.}$$

666. Polyhedrons are **similar** if they have the same number of faces, similar each to each, and similarly placed, and have their corresponding polyhedral ∡s equal.

Ex. 577. The altitude of the frustrum of a given pyramid is 18 ft. The lower base is a triangle whose sides are, respectively, 8 ft., 26 ft., and 30 ft. The shortest side of the upper base is 4 ft. Find the volume of the frustrum.

Ex. 578. The diagonal of one of the faces of a cube is d. Find the volume of the cube.

Ex. 579. The diagonal of a cube is D. Find the volume of the cube.

Ex. 580. Find the lateral area of a regular pyramid if its base is a square 16 ft. to the side, and the slant height 28 ft.

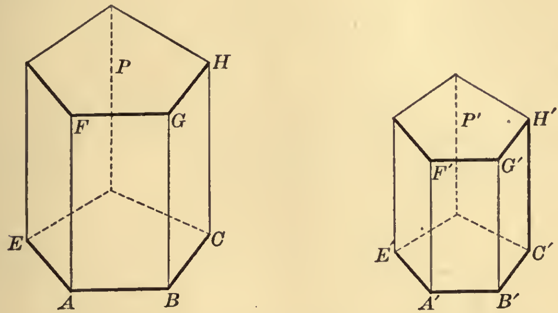
Ex. 581. The specific gravity of mercury being 14, and water weighing $62\frac{1}{2}$ lb. per cubic foot, what is the edge of a cubical box that would hold 40 lb. of mercury?

Ex. 582. Find the area of the surface of a regular icosahedron, whose edge is 2 in.

Ex. 583. In a regular pyramid with square base the lateral edge is 41 ft. and the slant height is 40 ft. Find the volume of the pyramid.

Proposition XXI. Theorem.

667. *The homologous edges of similar polyhedrons are proportional.*



Given the similar polyhedrons P and P′, with edges AB and CH homologous to A′B′ and C′H′.

To prove $$\frac{AB}{A'B'}=\frac{CH}{C'H'}.$$

Proof. Because the face $ABGF \sim A'B'G'F'$; § 666

$$\therefore \frac{AB}{A'B'}=\frac{BG}{B'G'},$$ § 363

and because face $BCHG \sim B'C'H'G'$, § 666

$$\therefore \frac{BG}{B'G'}=\frac{CH}{C'H'}.$$ § 363

$$\therefore \frac{AB}{A'B'}=\frac{CH}{C'H'}.$$ Q. E. D.

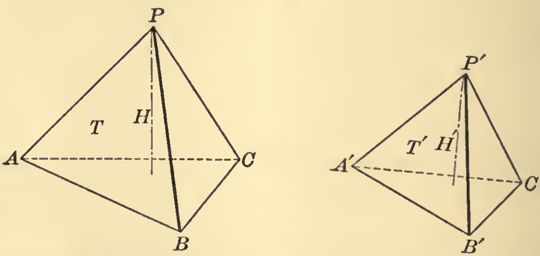
Ex. 584. The base of a regular pyramid is an equilateral triangle whose side is 6 ft. The altitude of the pyramid is 20 ft. Find the volume.

668. Cor. 1. *Any two homologous lines of similar polyhedrons are proportional to any other two homologous lines of the polyhedrons.*

669. Cor. 2. *Any two homologous faces of similar polyhedrons are proportional to the squares of any two homologous lines of the polyhedrons, and the total surface of two similar polyhedrons are proportional to the squares of any two homologous lines of the polyhedrons.*

Proposition XXII. Theorem.

670. *Similar tetrahedrons are to each other as the cubes of their homologous edges.*



Given two similar tetrahedrons T and T′, whose volumes are V and V′, and whose altitudes are H and H′, respectively.

To prove $$\frac{V}{V'}=\frac{\overline{AB}^3}{\overline{A'B'}^3}.$$

Proof. $$\frac{V}{V'}=\frac{\text{base } ABC\times H}{\text{base } A'B'C'\times H'}.\qquad \S\ 656$$

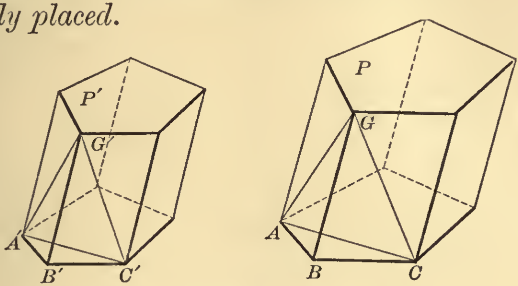
But $$\frac{\text{base } ABC}{\text{base } A'B'C'}=\frac{\overline{AB}^2}{\overline{A'B'}^2}.\qquad \S\ 669$$

And $$\frac{H}{H'}=\frac{AB}{A'B'}.\qquad \S\ 668$$

$$\therefore \frac{V}{V'}=\frac{\overline{AB}^2\times AB}{\overline{A'B'}^2\times A'B'}=\frac{\overline{AB}^3}{\overline{A'B'}^3}.$$ Q. E. D.

Proposition XXIII. Theorem.

671. *Similar polyhedrons may be divided into the same number of tetrahedrons similar each to each and similarly placed.*



Given the two similar polyhedrons, P and P′.

To prove P and P′ may be divided into the same number of tetrahedrons similar each to each, and similarly placed.

Proof. Take any homologous trihedral ∡s in P and P', as B and B', and through points G, A, C pass a plane; also through points G', A', C' pass a plane. Then in the two tetrahedrons thus cut off, $G\text{–}ABC$ and $G'\text{–}A'B'C'$,

$$\frac{AG}{A'G'}=\frac{BG}{B'G'}=\frac{CG}{C'G'}=\frac{AC}{A'C'}. \qquad \S\ 363$$

$\therefore$ the faces BAG, BAC, and BGC are similar to $B'A'G'$, $B'A'C'$, and $B'G'C'$ respectively. § 376

$\therefore$ the homologous faces of these tetrahedrons are similar. § 369

But the homologous trihedral ∡s of these tetrahedrons are equal. § 595

$\therefore$ tetrahedron $G\text{–}ABC \sim$ tetrahedron $G'\text{–}A'B'C'$. § 666

After removing tetrahedron $G\text{–}ABC$ from P and $G'\text{–}A'B'C'$ from P' the polyhedrons which remain will be similar, for their faces are similar and the polyhedral ∡s are equal. By this process P and P' may be divided into the same number of tetrahedrons, similar each to each and similarly placed. Q.E.D.

672. Cor. *The volumes of any two similar polyhedrons are to each other as the cubes of any two homologous lines of the polyhedrons.*

673. A **regular polyhedron** has all its faces congruent regular polygons and all its polyhedral angles congruent.

Proposition XXIV. Theorem.

674. *Only five regular polyhedrons are possible.*

Given, congruent regular polygons of any number of sides.

To prove, only five regular polyhedrons, with congruent regular polygons for faces, can be constructed.

Proof. A polyhedron ∡ must have at least three faces and the sum of the face ∡s must be $< 360°$.

1. With equilateral △s where each ∡ is 60°;

$60° \times 3 = 180°$; $60° \times 4 = 240°$; $60° \times 5 = 300°$; but $60° \times 6 = 360°$.

∴ only three regular polyhedrons can be formed with equilateral △s for faces.

2. With squares where each ∡ is 90°,

$$90° \times 3 = 270°;\ \text{but } 90° \times 4 = 360°.$$

∴ only one regular polyhedron can be formed with squares for faces.

3. With regular pentagons where each ∡ is 108°.

$$108 \times 3 = 324°;\ \text{but } 108° \times 4 = 432°$$

∴ only one regular polyhedron can be formed with regular pentagons for faces.

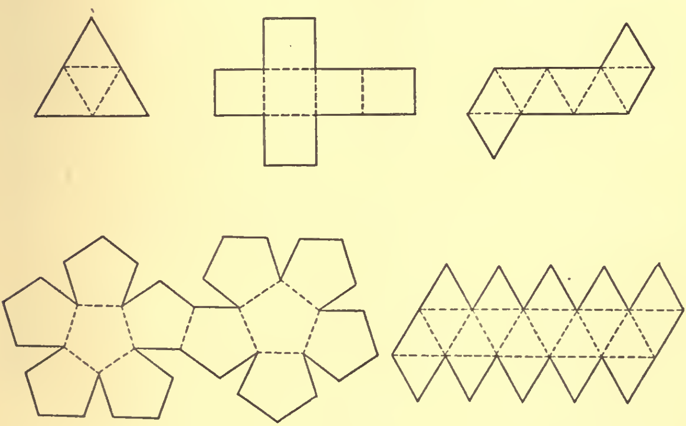
4. But with regular hexagons where each ∡ is 120°, because $120° \times 3 = 360°$, no regular polyhedron can be formed. Hence

no regular polyhedron can be formed with polygons having more than five sides.

∴ only five regular polyhedrons are possible. Q. E. D.

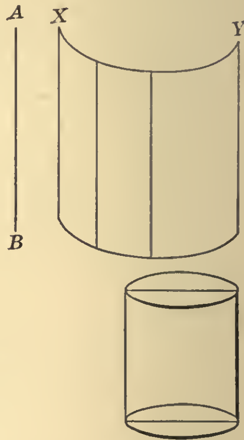
675. Note. The five regular polyhedrons are the tetrahedron, the octahedron, and the icosahedron from equilateral triangles; the hexahedron or cube from squares; and the dodecahedron from pentagons.

676. These regular polyhedrons may be formed by cutting out cardboard as indicated in the following diagrams. Cut entirely through on the full lines, half through on the broken lines, and bring the edges together. The edges may be held in place by pasting narrow strips of paper over them.



CYLINDERS.

677. A **cylindrical surface** is a curved surface generated by a moving straight line that constantly touches a given curve and is always parallel to a fixed straight line.



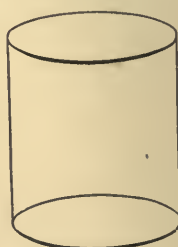
The moving line is called the **generatrix** and the given curve the **directrix.** The generatrix in any position is called an **element** of the cylindrical surface.

678. A **cylinder** is a solid bounded by a cylindrical surface, called the **lateral surface** and two parallel planes which are the bases of the cylinder.

679. The **altitude** of a cylinder is the perpendicular distance between the bases.

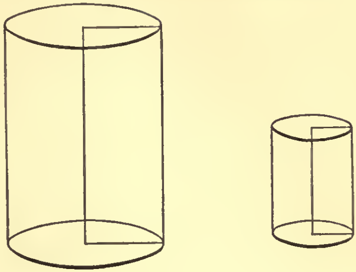
680. Because parallel lines included between parallel planes are all equal, the elements of a cylindrical surface are all equal.

681. A **circular cylinder** is a cylinder whose bases are circles. The term "cylinder," as hereafter used, will mean circular cylinder, as the circle is the only curve discussed in elementary plane geometry.



682. A **right cylinder** is a cylinder whose elements are perpendicular to its bases. An oblique cylinder is one whose elements are oblique to its bases.

A right cylinder is called a **cylinder of revolution** because it may be generated by the revolution of a rectangle about one of its sides as an **axis**. The radius of the base is the **radius** of the cylinder.



683. **Similar cylinders** of revolution are cylinders generated by similar rectangles revolving about corresponding sides as axes.

684. A plane is **tangent** to a cylinder when it passes through one element of the cylinder but does not cut it.

685. A prism is **inscribed** in a cylinder when its base is a polygon inscribed in the base of the cylinder and its lateral edges are elements of the cylinder.

686. A prism is **circumscribed** about a cylinder when its bases are polygons circumscribed about the bases of the cylinder and its lateral faces are tangent to the cylinder.

687. A **right section** of a cylinder is the figure formed by the intersection of a plane with the cylinder perpendicular to its elements. The right section of a cylinder is a circle.

688. Because the circle is the limit of the perimeters of regular inscribed and circumscribed polygons and the area of a circle is the limit of the areas of these polygons when the number of their sides is indefinitely increased, § 488, hence:

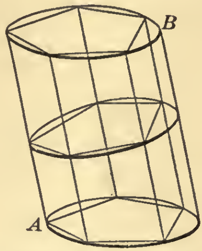
1. The circle of a right section of a cylinder is the limit of the perimeter of a right section of an inscribed or circumscribed prism.

2. The lateral area of a cylinder is the limit of the lateral area of the inscribed or circumscribed prism.

3. The volume of a cylinder is the limit of the volume of an inscribed or circumscribed prism.

Proposition XXV. Theorem.

689. *The lateral area of a cylinder is equal to the product of the circle of a right section by an element.*



Given the cylinder AB, with lateral area S, circle of a right section C, and element H.

To prove $S = C \times H$.

Proof. Inscribe in the cylinder a regular prism, with lateral area S', and perimeter of a right section P. Its lateral edge is E.

Then $S' = P \times H$. § 616

Let the number of lateral faces of the prism be indefinitely increased.

Then S' approaches S as a limit. § 688, 2

P approaches C as a limit § 688, 1

and $P \times H$ approaches $C \times H$ as a limit. § 277

But $S' = P \times H$ always. § 616

$\therefore\ S = C \times H$. § 275

Q. E. D.

690. Cor. 1. *The lateral area of a cylinder of revolution is equal to the circle of the base multiplied by the altitude.*

691. Cor. 2. *If S is the lateral area, T the total area, and H an element, then*

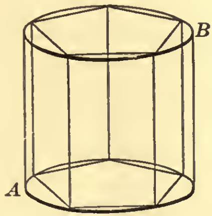
$$S = 2\,\pi R \times H.$$

$$T = 2\,\pi R \times H + 2\,\pi R^2 = 2\,\pi R(H + R).$$

692. Cor. 3. *Lateral areas or total areas of two similar cylinders are to each other as the squares of their like dimensions.*

Proposition XXVI. Theorem.

693. *The volume of a cylinder is equal to the product of its base and altitude.*



Given the cylinder AB, with volume V, base B, and altitude H.

To prove $V = B \times H$.

Proof. Inscribe in the cylinder a regular prism with volume V' and base B'. Its altitude is equal to H.

Then $V' = B' \times H$. § 636

Let the number of the lateral faces of the prism be indefinitely increased.

Then V' approaches V as a limit, § 688, 3

B' approaches B as a limit, § 688, 1

and $B' \times H$ approaches $B \times H$ as a limit. § 277

But $V' = B' \times H$ always. § 636

$\therefore V = B \times H$. § 275

Q. E. D.

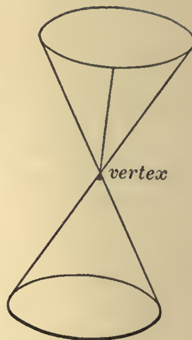
694. Cor. 1. *If the radius of the cylinder is R, then*

$$V = \pi R^2 H.$$

695. Cor. 2. *Volumes of two similar cylinders are to each other as the cubes of their like dimensions.*

CONES.

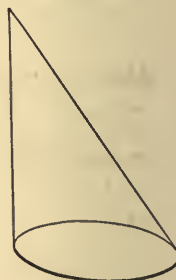
696. A **conical surface** is a curved surface generated by a moving straight line that constantly touches a given curve and passes through a fixed point. The moving line is called the **generatrix**, the fixed point, the **vertex**, and the given curve, the **directrix**.



The generatrix in any position is called an **element** of the conical surface.

697. The conical surface may consist of two parts, one above and the other below the vertex, called the upper and lower **nappes**, respectively.

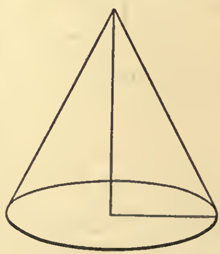
698. A **cone** is a solid bounded by a conical surface, called the **lateral surface** and a plane which is the **base** of the cone.



699. The **altitude** of a cone is the perpendicular from the vertex to the plane of the base.

700. A **circular cone** is a cone whose base is circular. The term "cone" as hereafter used, will mean a circular cone of one nappe.

701. The **axis** of a cone is the straight line from the vertex to the center of the base.

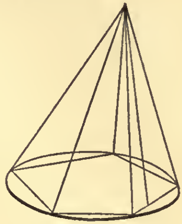


702. A **right cone** is a cone in which the axis is perpendicular to the base. A right cone is also called a **cone of revolution** because it may be generated by revolving a right triangle about one of its legs as an axis.

703. The **slant height** is an element of the conical surface.

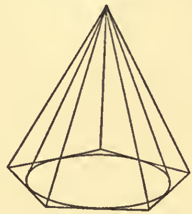
704. **Similar cones** of revolution are cones generated by similar right triangles revolving about corresponding legs.

705. A plane is **tangent** to a cone when it touches it in one element of the cone but does not cut it.



706. A pyramid is **inscribed** in a cone when its base is a polygon inscribed in the base of the cone and its lateral edges are elements of the cone.

707. A pyramid is **circumscribed** about a cone when its base is a polygon circumscribed about the base of the cone, and its lateral faces are tangent to the cone.



708. When the number of sides of regular inscribed or circumscribed polygons is indefinitely increased, §§ 488, 688,

1. The circle of a right section of a cone is the limit of the perimeter of a right section of an inscribed or circumscribed pyramid.

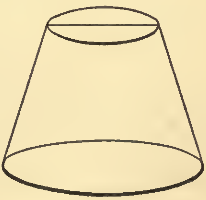
2. The lateral area of a cone is the limit of the lateral area of an inscribed or circumscribed pyramid.

3. The volume of a cone is the limit of the volume of an inscribed or circumscribed pyramid.

709. The axis of a right cone is its altitude.

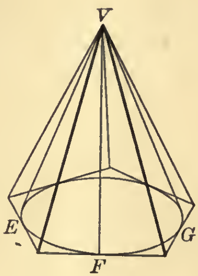
710. The elements of a right cone are all equal.

711. The **frustum** of a cone is the part of a cone contained between its base and a plane parallel to its base. The base of the cone is the lower base of the frustum and the section made by the plane parallel to the base is the upper base of the frustum.



Proposition XXVII. Theorem.

712. *The lateral area of a cone of revolution is equal to half the product of the slant height by the circle of the base.*



Given V–EFG a cone of revolution, L its slant height, C the circle of its base, and S its lateral area.

To prove $S = \frac{1}{2} L \times C.$

Proof. Circumscribe about the cone a regular pyramid, denoting its lateral area by S' and the perimeter of its base by P.

Then $S' = \frac{1}{2} L \times P.$ § 646

Let the number of lateral faces of the pyramid be indefinitely increased.

Then S' will approach S as a limit, § 708, 2

P' will approach P as a limit, § 708, 1

and $\frac{1}{2} L \times P$ will approach $\frac{1}{2} L \times C$ as a limit. § 277

But $S' = \frac{1}{2} L \times P$ always.

$\therefore S = \frac{1}{2} L \times C.$ Q.E.D.

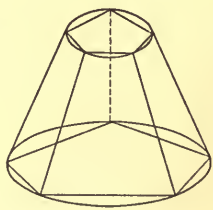
713. Cor. 1. *If R is the radius of a cone and T is its total surface, because the circle of the base is $2\pi R$, and the area is πR^2,*

$$S = \tfrac{1}{2} \times 2\pi R \times L = \pi RL. \quad T = \pi RL + \pi R^2 = \pi R(L + R).$$

714. Cor. 2. *The lateral areas or the total areas of two similar cones are to each other as the squares of their like dimensions.*

Proposition XXVIII.

715. *The lateral area of the frustum of a right cone is equal to half the product of the slant height of the frustum by the sum of the circles of the bases.*



Given L' the slant height, c the upper base of the frustum, C the lower base, and S' the lateral area.

To prove $S' = \frac{1}{2}(c + C)L'$.

Proof. The lateral area of the frustum of the inscribed pyramid is

$$S = \tfrac{1}{2}(p + P)L. \qquad \S\ 647$$

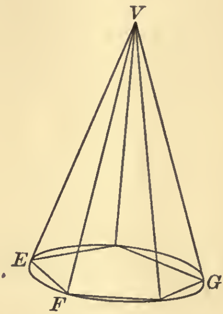
When the number of lateral faces of the inscribed frustum of the pyramid is indefinitely increased, P approaches C, p approaches c, L approaches L', and S approaches S' as a limit.

$$\therefore S' = \tfrac{1}{2}(c + C)L'. \qquad \text{Q.E.D.}$$

716. Cor. *The lateral area of the frustum of a cone of revolution is equal to the product of the circle of a section midway between the bases, by the slant height.*

Proposition XXIX. Theorem.

717. *The volume of a cone is equal to one third the product of its base by its altitude.*



Given V the volume, B the base, and H the altitude of the cone V–EFG.

To prove $V = \frac{1}{3} B \times H.$

Proof. Inscribe in the cone a regular pyramid, denoting its volume by V', its base by B', and H will be its altitude.

Then $V' = \frac{1}{3} B' \times H.$ § 655

Let the number of lateral faces of the pyramid be indefinitely increased.

Then V' approaches V as a limit. § 708, 3

Also B' approaches B as a limit. § 708, 1

But $V' = \frac{1}{3} B' \times H$ always. § 655

$\therefore V = \frac{1}{3} B \times H.$ § 275

Q.E.D.

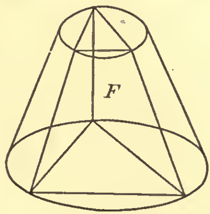
718. Cor. 1. *If* $\pi R^2 =$ *area of the base,*

$$V = \tfrac{1}{3} \pi R^2 H.$$

719. Cor. 2. *The volumes of similar cones are to each other as the cubes of their like dimensions.*

Proposition XXX. Theorem.

720. *A frustum of a cone is equivalent to the sum of three cones of the altitude of the frustum, and whose bases are the upper base, the lower base of the frustum, and a mean proportional between them.*



Given V the volume, b the upper base, B the lower base, and H the altitude of the frustum F.

To prove $V = \frac{1}{3} H(b + B + \sqrt{b \times B})$.

Proof. Inscribe in F the frustum of a regular pyramid, denoting its volume by V', its bases by b' and B'. H will be its altitude.

Then $V' = \frac{1}{3} H(b' + B' + \sqrt{b' \times B'})$. § 661

Let the number of lateral faces of the inscribed frustum be indefinitely increased.

Then V' will approach V as a limit. § 708, 3

Also B' will approach B as a limit. § 708, 1

And b' will approach b as a limit. § 708, 1

But $V' = \frac{1}{3} H(b' + B' + \sqrt{b' \times B'})$ always. § 661

$\therefore V = \frac{1}{3} H(b + B + \sqrt{b \times B})$. § 275

Q. E. D.

721. If r and R denote the radii of the upper and lower bases respectively, then πr^2 and πR^2 denote their areas, and

$$V = \frac{1}{3} \pi H(r^2 + R^2 + rR).$$

Ex. 585. Find the volume of a cube whose entire surface is 54 sq. ft.

Ex. 586. The radius of the lower base of a frustum of a cone is 21, the radius of the upper base 10, and the slant height 61. Find its volume.

Ex. 587. A chimney 60 ft. high is in the shape of the frustrum of a cone. Its lower diameter is 28 ft. and the upper diameter 20 ft. The conical flue has the lower diameter 12 ft. and the upper diameter 6 ft. Find the volume of the chimney.

Ex. 588. The volumes of two similar prisms are to each other as 5 to 6. What is the ratio of their surfaces?

Ex. 589. Find the volume of a frustum of a regular quadrangular pyramid, the sides of whose bases are 10 and 6 and whose altitude is 12.

Ex. 590. The radius of the lower base of the frustum of a cone is 34, the radius of the upper base 20, and the altitude 48. Find its lateral area.

Ex. 591. The altitude of the Great Pyramid is 488 ft. and its base is 764 ft. square. What is its volume?

Ex. 592. The altitude of a pyramid is 9 ft. and its base is a rhombus whose diagonals are, respectively, 10 and 12 ft. What is its volume?

Ex. 593. The lateral edge of a pyramid is 10 ft. and its inclination to the base is 30°. The base is an equilateral triangle whose side is 12 ft. Find the volume of the pyramid.

Ex. 594. Find the volume of a truncated right triangular prism, the sides of whose base are, respectively, 13, 14, and 15 ft., and whose lateral edges are 6, 8, and 10 ft., respectively.

Ex. 595. Find the volume of the cube, in which the diagonal of each face is 16 in.

Ex. 596. The base of a right pyramid is a regular hexagon of side 18 in. and the lateral faces are inclined to the base at an angle of 60°. Find the volume.

Book VIII.

THE SPHERE.

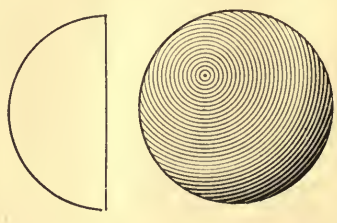
722. A **sphere** is a solid bounded by a surface, all points of which are equidistant from a point within, called the **center.**

723. The **radius** of a sphere is a straight line from the center to any point on the surface.

724. The **diameter** of a sphere is a straight line through the center terminated at both ends by the surface.

725. It follows from the definition of the sphere that *all radii of the same sphere or of equal spheres are equal,* that *all diameters of the same sphere or of equal spheres are equal,* and that *spheres are equal if their radii or their diameters are equal.*

726. A sphere may be generated by the revolution of a semicircle about its diameter as an axis.



727. A line or a plane which has one, and only one, point in common with the surface of a sphere is *tangent to the sphere.* The sphere is then, also, *tangent to the line or the plane.*

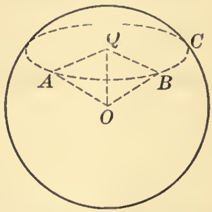
728. Two spheres whose surfaces have one, and only one, point in common are *tangent to each other.*

729. A polyhedron is *inscribed in a sphere* when all its vertices are in the surface of the sphere. The sphere is then *circumscribed about the polyhedron.*

730. A polyhedron is *circumscribed about a sphere* when all its faces are tangent to the sphere. The sphere is then *inscribed in the polyhedron.*

PROPOSITION I. THEOREM.

731. *The intersection of a plane and the surface of a sphere is a circle.*



Given ABC, a section made by a plane cutting the sphere whose center is O.

To prove ABC *is a* ⊙.

Proof. Let OQ be $\perp$ plane ABC.

From A and B, any two points in the boundary of the section ABC, draw AO and BO, and draw also AQ and BQ. Then in rt. $\triangle$s OAQ and BOQ,

$OQ = OQ$. Iden.

$AO = BO$. § 725

$\therefore AQ = BQ$. §§ 122, 85

But A and B are any two points on the boundary of the section ABC.

$\therefore ABC$ is a ⊙. § 208

Q. E. D.

732. COR. 1. *The line from the center of a sphere to the center of a circle of the sphere is perpendicular to the plane of the circle.*

733. DEF. A **great circle** of a sphere is a section of the sphere made by a plane that passes through the center of the sphere.

734. Def. A **small circle** of a sphere is a section of the sphere made by a plane that does not pass through the center of the sphere.

735. Def. The diameter of a sphere perpendicular to the plane of a circle of the sphere is called the **axis** of the circle, and the extremities of the diameter are called the **poles** of the circle.

736. Cor. 2. *A great circle has the same center and the same radius as the sphere, hence all* ***great circles of the same sphere or of all equal spheres are equal.***

737. Cor. 3. *A great circle bisects the sphere and the surface of the sphere,*

For, if the two parts into which it divides the sphere be so placed that their plane surfaces coincide, then the curved surfaces must coincide, otherwise there would be points in the surface of the sphere at different distances from the center.

738. Def. The **distance** between any two points on the surface of a sphere is the arc of a great circle, not greater than a semicircle, that joins the points.

Ex. 597. Of two given cubes the diagonal of the first is three times that of the second. What is the ratio of their volumes ?

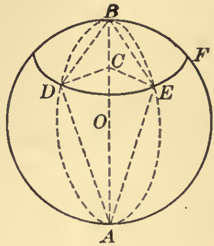
Ex. 598. The edges of a given rectangular parallelopiped are, respectively, 9 ft., 24 ft., and 32 ft. What is the volume of a similar parallelopiped, whose diagonal is 82 ft. ?

Ex. 599. A pyramid has an altitude of 26 ft. At what distance from the base must it be cut by a plane parallel to the base that the frustum may be half the pyramid ?

Ex. 600. If the area of the entire surface of a tetrahedron is 200 sq. ft. and the altitude 60 ft., what is the altitude of a similar tetrahedron whose entire surface is 800 sq. ft. ?

PROPOSITION II. THEOREM.

739. *All points on a circle of a sphere are equidistant from either of its poles.*



Given two points, D and E on the ⊙ DEF, and A and B, the poles of the ⊙ DEF.

To prove the great circle arcs AD and AE are equal, and the great circle arcs BD and BE are equal.

Proof. The straight lines AD and AE are equal. § 533

$\therefore$ arc $AD =$ arc AE. § 234

Similarly, arc $BD =$ arc BE. Q. E. D.

740. DEF. The distance on the surface of a sphere from the nearer pole of a circle to any point of the circle is called the **polar distance of the circle.**

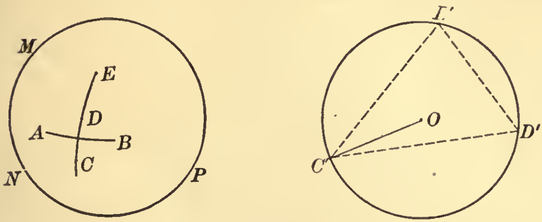
741. COR. *The polar distance of a great circle of a sphere is the quadrant of a great circle.*

Ex. 601. The base of a regular pyramid is an equilateral triangle whose side is 18 ft. The slant height of the pyramid is 30 ft. Find the volume.

Ex. 602. Of two similar pyramids the entire surface of the first is four times that of the second. What is the ratio of the volume of the first to that of the second?

Proposition III. Problem.

742. *To construct the radius of a material sphere.*



Let MNP represent a material sphere.

Required to construct its radius.

Construction. Take any two points, A and B, on the surface of the sphere, as poles, and with the same radius construct two arcs intersecting each other at C; then with the same poles and with other equal radii, construct two arcs intersecting at D; finally, with still the same poles and with other equal radii, construct two arcs intersecting at E.

The three points, C, D, and E, thus determined, determine the plane that is the perpendicular bisector of the straight line joining A and B. §§ 523, 538

∴ the plane CDE passes through the center of the sphere. §§ 722, 538

∴ C, D, E lie on the great circle of the sphere. § 733

Construct a plane triangle $C'D'E'$, whose sides are equal, respectively to CD, DE, and CE. § 303

Circumscribe the circle O about the plane triangle $C'D'E'$. § 306

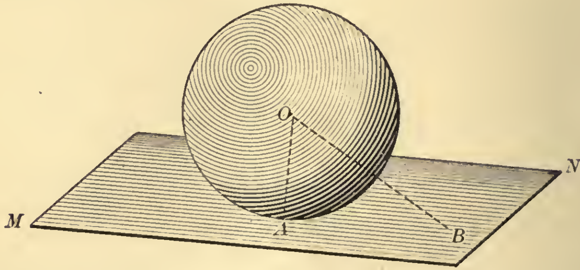
Then ⊙ O = great ⊙ of the sphere. § 246

∴ OC, the radius of ⊙ O = radius of great ⊙ (§ 246) = radius of the sphere. § 213

Q. E. F.

PROPOSITION IV. THEOREM.

743. *A plane perpendicular to a radius at its outer extremity is tangent to the sphere.*



Given O, the center of a sphere, MN, a plane $\perp$ radius OA at A.

To prove *the plane MN is tangent to the sphere.*

Proof. Let B be any other point except A in the plane MN.

Then $OB > OE$. § 531

$\therefore$ the point B is without the sphere. § 722

But B is any point in the plane MN other than A.

$\therefore$ the plane MN is tangent to the sphere.

Q. E. D.

744. COR. 1. *A plane tangent to a sphere is perpendicular to the radius drawn to the point of tangency.*

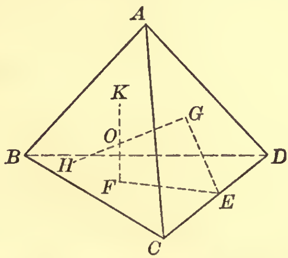
745. COR. 2. *A line in a tangent plane drawn through the point of tangency is tangent to the sphere at that point.*

746. COR. 3. *A line tangent to a circle of a sphere lies in the plane that is tangent to the sphere at the point of contact.*

Ex. 603. If the volumes of two similar prisms are to each other as 8 to 27, what is the ratio of their altitudes?

Proposition V. Theorem.

747. *Through any four points, not all in the same plane, one, and only one, sphere may be passed.*



Given A, B, C, and D, four points, not all in the same plane.

To prove one sphere, and only one, may be passed through A, B, C, and D.

Proof. Let F and G be the centers of ⊙s circumscribing △s BCD and ACD, respectively. Let FK be ⊥ the plane BCD and GH ⊥ the plane ACD.

Then every point in FK is equidistant from B, C, and D, and every point in GH is equidistant from A, C, and D. § 533

Join F and G to E, the mid-point of CD.

Then FE and GE are each ⊥ CD. § 240

∴ the plane GEF ⊥ CD. § 535

∴ the plane GEF ⊥ planes BCD and ACD. § 567

Then, since GH is ⊥ plane ACD by construction, GH lies in the plane GEF. § 569

Similarly, FK lies in the plane GEF. Therefore ⊥s GH and and FK lie in the same plane and being ⊥ to non-parallel planes, they meet in some point, as O.

∴ O lies in ⊥ GH and FK equidistant from B, C, and D, and from A, C, and D. § 533

∴ O is equidistant from A, B, C, and D.

Hence the sphere whose center is O and radius OA will pass through A, B, C, and D.

Again, since the center of any sphere through A, B, C, and D must lie in GH and FK (§ 534), their intersection O is the center of the only sphere that will pass through the four points A, B, C, and D. Q.E.D.

748. Cor. *Four points not all in the same plane determine a sphere.*

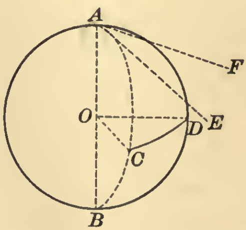
SPHERICAL ANGLES.

749. Def. The angle formed by two intersecting curves is the angle formed by the tangents to the curves at the point of intersection.

750. Def. The angle formed by two intersecting great circles of a sphere is called a **spherical angle.**

Proposition VI. Theorem.

751. *A spherical angle is measured by the arc of a great circle described from the vertex as a pole and included between its sides, or its sides produced.*



Given the great ⊙s BCA and BDA, intersecting at A, and CD, the arc of a great ⊙ described with A as a pole.

To prove the spherical angle CAD is measured by the arc CD.

Proof. Draw radii OC and OD and tangents AE and AF.

Arcs AC and AD are quadrants. § 216

OC and OD each $\perp OA$, and $\measuredangle COD = \measuredangle EAF$. § 555

$\therefore$ the spherical $\measuredangle CAD$ is measured by the arc CD. Q.E.D.

752. COR. *A spherical angle has the same measure as the dihedral angle formed by the planes of the two circles.*

SPHERICAL POLYGONS.

753. DEF. A **spherical polygon** is a portion of the surface of a sphere bounded by three or more great circles.

The bounding arcs are called the **sides**, their points of intersection, the **vertices**, and the spherical angles formed by the sides, the **angles** of the spherical polygon.

754. DEF. The **diagonal** of a spherical polygon is the arc of a great circle drawn between two non-consecutive vertices.

755. The planes of the sides of a spherical polygon form a polyhedral angle at the center of the sphere. A spherical polygon is **convex** if the corresponding polyhedral angle is convex. Unless stated otherwise a spherical polygon is assumed to be convex. From any property of polyhedral angles may be inferred an analogous property of spherical polygons, and conversely.

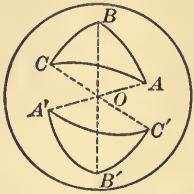
756. The measures of the sides of a spherical polygon are usually expressed in degrees.

757. DEF. Two spherical polygons are **vertical** when their corresponding polyhedral angles are vertical.

758. Two spherical polygons are **symmetrical** when the parts of one are, respectively, equal to the parts of the other and arranged in reverse order.

PROPOSITION VII. THEOREM.

759. *Two vertical spherical triangles are symmetrical.*



Given two vertical spherical triangles, ABC and A′B′C′.

To prove spherical $\triangle$*s* ABC *and* $A'B'C'$ *are symmetrical.*

Proof. Let O be the center of the sphere.

Plane $\triangle$s AOC and $A'OC'$ have two sides and the included $\measuredangle$s respectively equal, and are, therefore, congruent. §§ 65, 91

$\therefore$ chord $AC =$ chord $A'C'$. § 85

$\therefore$ arc $AC =$ arc $A'C'$. § 234

Similarly, arc $BC =$ arc $B'C'$,

arc $AB =$ arc $A'B'$, etc.

$\therefore$ spherical $\triangle$s ABC and $A'B'C'$ are symmetrical.

Q. E. D.

Ex. 604. Find the volume of a right circular cylinder the diameter of whose base is 12 ft. and whose altitude is 20 ft.

Ex. 605. If the diameter of a right circular cylinder is 8 ft., and its total surface is 128 sq. ft., what is its altitude?

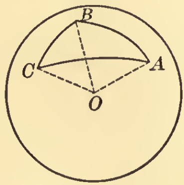
Ex. 606. Reckoning $7\frac{1}{2}$ gal. to the cubic foot, how many gallons will a cylindrical standpipe hold if its diameter is 20 ft. and the altitude 60 ft.?

Ex. 607. Given the lateral surface of a right circular cylinder, S, and altitude H, to find the volume.

760. Two symmetrical spherical polygons may be placed in position such that each is the vertical of the other.

Proposition VIII. Theorem.

761. *The sum of two sides of a spherical triangle is greater than the third side.*



Given the spherical △ ABC.

To prove $AB + BC > AC$.

Proof. Draw radii OA, OB, and OC.

Then in the trihedral ∡ $O\text{–}ABC$, ∡ AOB + ∡ BOC

> ∡ AOC. § 593

But the central angle is measured by the intercepted arc.

∴ arc AB + arc BC > arc AC.

Q. E. D.

Ex. 608. The slant height of a regular pyramid is divided by a plane parallel to the base in the ratio 1 : 4, the longer segment being next the base. What is the ratio of the section to the base?

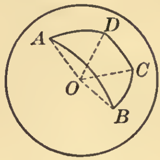
Ex. 609. Given V, the volume, and the altitude equal to the radius, of a right circular cylinder, to find the entire surface.

Ex. 610. A pyramid is divided into two parts by a plane parallel to the base and bisecting the altitude. What is the ratio of the two parts?

Ex. 611. Two similar right circular cones have their volumes in the ratio 8 : 27. What is the ratio of their lateral surfaces?

Proposition IX. Theorem.

762. *The sum of the sides of a spherical polygon is less than* 360°.



Given the spherical polygon ABCD.

To prove $AB + BC + CD + DA < 360°$.

Proof. Arcs AB, BC, CD, DA are the measures of the central angles AOB, BOC, COD, and DOA.

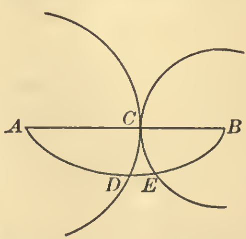
But the sum of these central angles $< 360°$. § 594

$\therefore AB + BC + CD + DA < 360°$. Q.E.D.

763. Cor. *The sum of the sides of a spherical polygon is less than a great circle.*

Proposition X. Theorem.

764. *The shortest line that can be drawn on the surface of a sphere between two points on the surface is the arc of a great circle not greater than a semicircle.*



Given two points, A and B, on the surface of a sphere, and AB, the arc of a great circle not greater than a semicircle.

To prove AB is the shortest line on the surface of the sphere between A and B.

Proof. Take any point C on the arc AB, and with A and B as poles with radii equal, respectively, to AC and BC, describe $\odot$s.

These $\odot$s cannot meet in any other point, for should they meet in any point, as K, the spherical $\triangle ABK$ would have the sum of two sides, AK and $BK = AC$, which is impossible. § 761

Therefore, any other line, as $ADEB$, between A and B must intersect the $\odot$s in two points, as D and E.

But $ADEB$ cannot be shorter than AB, for by revolving AD about A and BE about B until D and E coincide with C there would be a line between A and B shorter than $ADEB$ by the part EF.

Hence the shortest line between A and B must pass through C.

But by hypothesis C is any point on the great $\odot$ arc AB.

$\therefore$ AB is the shortest line that can be drawn on the surface between A and B. Q.E.D.

Ex. 612. Given the lateral surface, S, and altitude, equal to radius of the base, of a right circular cone, to find the volume.

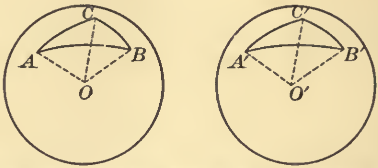
Ex. 613. Two similar right circular cones have their altitudes in the ratio 6 : 7. What is the ratio of their volumes?

Ex. 614. The altitude of the frustum of a pyramid is 36 ft. The lower base is a triangle whose sides are, respectively, 8, 26, and 30 ft. The longest side of the upper base is 15 ft. Find the volume of the frustum.

Ex. 615. The altitude of the frustum of a cone is 24 ft. The diameters of the bases are, respectively, 32 ft. and 18 ft. How far from the lower base must a plane parallel to the base be passed to divide the frustum into two equivalent frustums?

Proposition XI. Theorem.

765. *Two mutually equilateral triangles on the same sphere or equal spheres are mutually equiangular, and are congruent or symmetrical.*



Given the spherical $\triangle$s ABC and A'B'C' on equal spheres,

$$AB = A'B',\ BC = B'C',\ AC = A'C'.$$

To prove $\triangle$s ABC and $A'B'C'$ are either congruent or symmetrical.

Proof. The face angles of the corresponding polyhedral angles at the center of the spheres are, respectively, equal. § 233

$\therefore$ the corresponding dihedral angles are equal. § 596

$\therefore$ the angles of the spherical $\triangle$s are respectively equal. § 752

Therefore the $\triangle$s ABC and $A'B'C'$ are congruent or symmetrical according as the equal sides are arranged in the same or reverse order. Q. E. D.

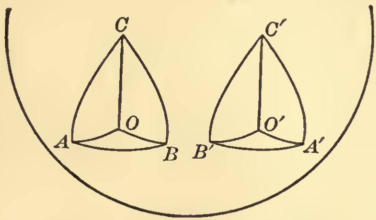
766. Cor. *Two symmetrical isosceles triangles are congruent.*

Ex. 616. The altitude of the frustum of a cone is 24 ft. The diameters of the bases are, respectively, 31 ft. and 18 ft. How far from the lower base must a plane be passed in order to divide the frustum into two similar frustums ?

Ex. 617. The edges of a rectangular parallelopiped are, respectively, 12 ft., 10 ft., and 21 feet. What is the area of the surface of a similar parallelopiped whose diagonal is 87 ft. ?

Proposition XII. Theorem.

767. *Two symmetrical spherical triangles are equal in area.*



Given two symmetrical $\triangle$s, ABC and A'B'C'.

To prove $\triangle$s ABC and $A'B'C'$ are equal in area.

Proof. Let $\odot$s be circumscribed about the plane $\triangle$s ABC and $A'B'C'$. § 306

Let O and O', respectively, be the poles of these $\odot$s

Then the chord $AB =$ the chord $A'B'$,

the chord $BC =$ the chord $B'C'$,

and the chord $AC =$ the chord $A'C'$. § 233

$\therefore \odot ABC = \odot A'B'C'$.

$\therefore$ radius of $\odot ABC =$ radius of $\odot A'B'C'$. § 213

$\therefore$ arcs AO, BO, CO, $A'O'$, $B'O'$, and $C'O'$ are all equal. § 234

$\therefore$ the spherical $\triangle$s AOB and $A'O'B'$ are congruent, therefore equal in area. § 766

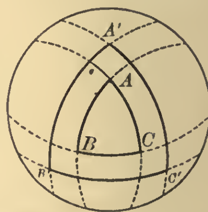
Similarly, spherical $\triangle$s BOC and AOC are respectively equivalent in area to spherical $\triangle$s $B'O'C'$ and $A'O'C$.

Whence, by addition, spherical $\triangle$s ABC and $A'B'C'$ are equal in area. Q.E.D.

Ex. 618. A regular cone 18 in. in height and 24 in. in diameter at the base is cut by a plane parallel to the base and 10 in. from it. Find the volume of the frustum so formed.

768. Def. If from the vertices of any spherical triangle as poles arcs of great circles are drawn, another triangle is formed which is called the **polar triangle** of the first triangle.

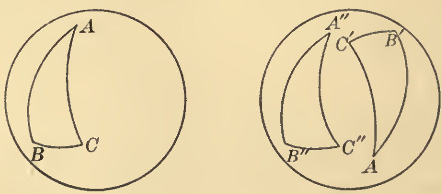
Thus, if A is the pole of the great circle arc $B'C'$, B the pole of the great circle arc $A'C'$, and C the pole of the great circle arc $A'B'$, the triangle $A'B'C'$ is the polar triangle of the triangle ABC.



769. The great circles of which $A'B'$, $A'C'$, and $B'C'$ are arcs form by their intersections eight spherical triangles. Of these eight triangles that one is the polar of ABC in which the vertex A', homologous to A, lies on the same side of the arc BC as the vertex A, etc.

Proposition XIII. Theorem.

770. *Two spherical triangles on the same or equal spheres are equal in area if they have two sides and the included angle, or two angles and the included side, of the one equal, respectively, to the corresponding parts of the other.*



Proof. If the equal parts of spherical $\triangle$s ABC and $A'B'C'$ are arranged in the same order, they are superposable, as in cases of plane triangles. §§ 91, 94

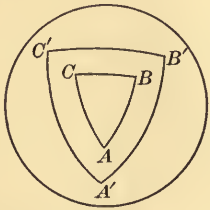
If the equal parts are arranged in reverse order, the $\triangle ABC$ and the symmetrical $\triangle$ of $A'B'C'$ will be superposable.

But $\triangle A'B'C'$ and its symmetrical $\triangle$ are equal in area. § 767

$\therefore \triangle ABC$ and $\triangle A'B'C'$ are equal in area. Q. E. D.

Proposition XIV. Theorem.

771. *If one spherical triangle is the polar of another, then the second spherical triangle is the polar of the first.*



Given $\triangle$ **A'B'C', the polar of ABC.**

To prove $\triangle ABC$ *is the polar of* $\triangle A'B'C'$.

Proof. Since A is the pole of $B'C'$, C is the pole of $A'B'$. § 768

$\therefore B'$ is a quadrant's distance from A and C. § 741

$\therefore B'$ is the pole of AC.

Similarly, A' and C' are the poles of BC and AB, respectively.

$\therefore \triangle ABC$ is the polar $\triangle$ of $\triangle A'B'C'$. Q. E. D.

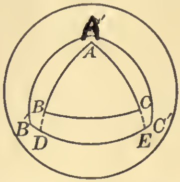
Ex. 619. The radius of a sphere is given, R, and the radius of a small circle of the sphere, r. Find the distance of the plane of the circle from the center of the sphere.

Ex. 620. The base of a pyramid is a triangle whose sides are, respectively, 17 ft., 25 ft., and 28 ft., and the altitude is 18 ft. Find the volume.

Ex. 621. The base of a regular pyramid is a square whose side is 10 ft. and the lateral edge of the pyramid is 24 ft. Find the volume.

Proposition XV. Theorem.

772. *In two polar triangles each angle of the one is the supplement of the side of the other of which it is the pole.*



Given the polar spherical triangles ABC and A′B′C′, A being the pole of B′C′, etc.

To prove $\angle A + \text{arc } B'C' = 180°.$

Proof. Produce the sides of $\angle A$ to meet $B'C'$ at D and E.

$\angle A$ is measured by arc DE. § 751

Since B' is the pole of arc AE and C' of arc AD, arcs $B'E$ and $C'D$ are both quadrants. § 741

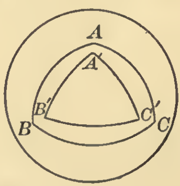
But $DE + B'C' = B'E + C'D = 180°,$

i.e. $DE + B'C' = 180°.$

$\therefore \angle A + \text{arc } B'C' = 180°.$ Q. E. D.

Proposition XVI. Theorem.

773. *The sum of the angles of a spherical triangle is less than* 540° *and greater than* 180°.



Given a spherical △ ABC.

To prove $\measuredangle A + \measuredangle B + \measuredangle C < 540°$ and $> 180°$.

Proof. Construct spherical $\triangle A'B'C'$, the polar $\triangle$ of $\triangle ABC$, and denote the sides $B'C'$, $A'C'$, $A'B'$, expressed in degrees by a, b, and c, respectively.

Then $\measuredangle A = 180° - a,$

$\measuredangle B = 180° - b,$

and $\measuredangle C = 180° - c.$ § 772

$$\therefore \measuredangle A + \measuredangle B + \measuredangle C = 540° - (a + b + c).$$

$$\therefore \measuredangle A + \measuredangle B + \measuredangle C < 540°.$$

But $a + b + c < 360°.$ § 762

$$\therefore \measuredangle A + \measuredangle B + \measuredangle C > 180°.$$ Q. E. D.

774. Cor. *A spherical triangle may have one, two, or three right angles ; also one, two, or three obtuse angles.*

775. Def. A **birectangular** spherical triangle is one that contains two right angles.

776. Def. A **trirectangular** spherical triangle is one that contains three right angles.

777. Def. The **spherical excess** of a spherical triangle is the difference between the sum of its angles and 180°.

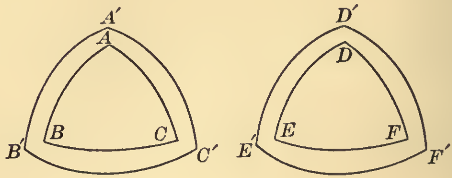
778. Cor. 1. *In a birectangular spherical triangle the sides opposite the right angles are quadrants, and the side opposite the third angle measures that angle.*

779. Cor. 2. *Each side of a trirectangular triangle is a quadrant.*

780. Cor. 3. *If three planes be passed through the center of a sphere each perpendicular to the other two, they divide the surface of the sphere into eight congruent trirectangular spherical triangles.*

Proposition XVII. Theorem.

781. *Two mutually equiangular spherical triangles on the same sphere or equal spheres are mutually equilateral, and are either congruent or symmetrical.*



Given the spherical △s ABC and DEF mutually equiangular, on equal spheres.

To prove △s ABC and DEF are mutually equilateral, and are either congruent or symmetrical.

Proof. Let $A'B'C'$ and $D'E'F'$ be the polar △s of ABC and DEF, respectively.

Then because △ ABC and △ DEF are mutually equiangular, △ $A'B'C'$ and △ $D'E'F'$ are mutually equilateral. § 772

∴ △s $A'B'C'$ and $D'E'F'$ are mutually equiangular. § 765

But ABC is the polar triangle of $A'B'C'$ and DEF is the polar triangle of $D'E'F'$. § 771

∴ △s ABC and DEF are mutually equilateral. § 772

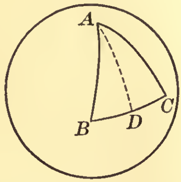
∴ △s ABC and DEF are congruent or symmetrical. § 765

Q.E.D.

782. Cor. *If two spherical triangles on the same or equal spheres are mutually equiangular, they are congruent if their equal parts are arranged in the same order, or symmetrical if their equal parts are arranged in reverse order.*

Proposition XVIII. Theorem.

783. *In an isosceles spherical triangle the angles opposite the equal sides are equal.*



Given the spherical triangle ABC, in which AB = AC.

To prove $\angle B = \angle C$.

Proof. Let AD, the arc of a great ⊙, be drawn from C to D, the mid-point of the arc BC.

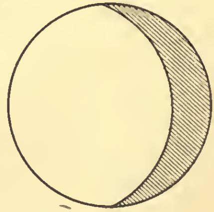
Then $\triangle$s ABD and ACD are mutually equilateral.

$\therefore$ $\triangle$s ABD and ACD are mutually equiangular § 765

$$\therefore \angle B = \angle C.$$ Q. E. D.

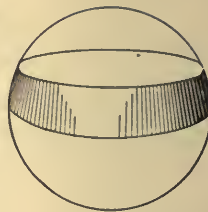
MEASUREMENT OF SPHERICAL SURFACES.

784. Def. A **lune** is a portion of the surface of a sphere bounded by two great semicircles.



785. The **angle** of a lune is the spherical angle between the semicircles that bound it.

It is evident that lunes on the same sphere are congruent if their angles are equal.

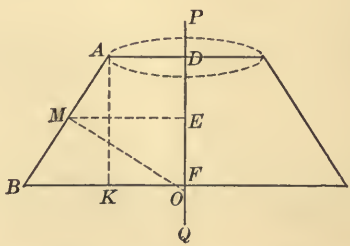


786. Def. A **zone** is a portion of the surface of a sphere included between two parallel planes.

787. Def. The common sections of the sphere and the planes are the bases of the zone, and the perpendicular distance between the planes is the **altitude** of the zone.

Proposition XIX. Theorem.

788. *The area generated by the revolution of a straight line about an axis in its plane is equal to the product of the projection of the line on the axis by the circle whose radius is the perpendicular from the mid-point of the line terminated by the axis.*



Given AB, a line revolving about the line PQ in its plane, M, the mid-point of AB, DF, the projection of AB on PQ, and MO $\perp$ AB.

To prove the area of surface generated by $AB = DF \times 2\pi MO$.

Proof. Draw $ME \perp$ and $AK \parallel PQ$.

The area generated by AB is the lateral area of the frustum of a cone of revolution whose slant height is AB and altitude DF.

$\therefore$ area $AB = AB \times 2\pi ME$. § 716

$\triangle ABK \sim \triangle EOM.$ § 370

$\therefore AB : MO = AK : ME.$

Hence $AB \times ME = MO \times AK = MO \times DF.$

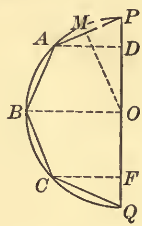
$\therefore$ area $AB = DF \times 2\pi MO.$

Hence, if AB meets PQ or is $\parallel PQ$, the result is the same. §§ 712, 689

Q. E. D.

PROPOSITION XX. THEOREM.

789. *The area of the surface of a sphere is equal to the product of its diameter by a great circle.*



Let the sphere be generated by the revolution of a semicircle about the diameter PQ.

Let O be the center of the sphere, and let its radius OM be denoted by R and the surface by S.

To prove $S = PQ \times 2\pi R.$

Proof. Let PA, AB, BC, CQ, be equal chords of the $\odot$.

Draw BO, then $BO \perp PQ$. § 240

Draw $AD, CF \perp PQ$.

Draw $OM \perp PA$.

OM bisects PA. § 240

Then the area generated by $PA = PD \times 2\pi OM.$

Similarly, the area generated by $AB = DO \times 2\pi OM$, etc. § 788

But the sum of the projections of PA, AB, etc., on $PQ = PQ$, the diameter.

$\therefore$ the surface generated by the polygon $PABCQ = PQ \times 2\pi OM$.

Now, let the number of sides of the polygon be indefinitely increased, then the perimeter will approach the semicircle on PQ as a limit and OM will approach R as a limit. § 488, 486

$\therefore$ the surface generated by the revolution of the polygo[n] will approach the surface of the sphere as a limit.

$$\therefore S = PQ \times 2\pi R. \qquad \text{Q. E. D.}$$

790. Cor. 1. *Since* $PQ = 2R$, $S = 4\pi R^2$.

791. Cor. 2. *The area of the surface of a sphere is equal to the area of four great circles.*

792. Cor. 3. *The areas of the surfaces of two spheres are to each other as the squares of their radii or the squares of their diameters.*

793. Cor. 4. *The area of a zone is equal to the product of its altitude by a great circle;* $Z = 2\pi RH$.

794. Cor. 5. *The areas of zones on the same sphere or equal spheres are to each other as their altitudes.*

Ex. 622. Prove that the surface of a sphere is equal to the lateral surface of the circumscribed cylinder.

Ex. 623. The lateral edge of a regular pyramid is 73 ft. and its altitude 55 ft., the base of the pyramid being a square. Find its volume.

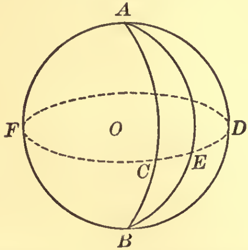
Ex. 624. Find the volume of a truncated right triangular prism, the sides of the base being, respectively, 33 in., 34 in., and 65 in., and the lateral edges, respectively, 18 in., 21 in., and 27 in.

Ex. 625. Find the capacity in bushels of a bin 12 ft. long, 10 ft. wide, 8 ft. high, a bushel being 2150.42 cu. in.

Ex. 626. A zone whose altitude is 16 in. is one-third the surface of the sphere. What is the radius of the sphere?

Proposition XXI. Theorem.

795. *The area of a lune is to the area of the surface of a sphere as the number of degrees in its angle is to* 360°.



Given ACBE, a lune on the surface of the sphere whose center is O.

Let A denote the angle of the lune, L the area of the lune, and S the surface of the sphere.

To prove $L : S :: A : 360°$.

Proof. Let $CEDF$ be the great ⊙ whose pole is A.

CE measures $\measuredangle A$. § 751

$\therefore CE : \odot\ CEDF = A : 360°$. § 287

If CE and ⊙ $CEDF$ are commensurable, let their common measure be contained in CE m times and in ⊙ $CEDF$ n times.

Then arc $CE : \odot\ CEDF = m : n$.

$\therefore \measuredangle A : 360° = m : n$. § 287

By passing great ⊙s through the points of division of the arc CE and the ⊙ $CEDF$, the arcs will divide the surface of the sphere into n equal lunes of which the lune $ACBE$ will contain m.

$\therefore L : S = m : n$.

$\therefore L : S = A : 360°$.

If CE and ⊙ $CDEF$ are incommensurable, by the method of limits as used in § 283 the same conclusion is reached.

Q. E. D.

796. Cor. 1. *The areas of two lunes on the same or equal spheres are to each other as their angles.*

797. Cor. 2. *If the right angle is the unit of angle and the trirectangular triangle the unit of surface, the area of the surface of the sphere being eight trirectangular triangles* (§ 780), *then*

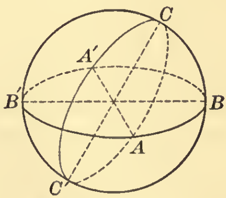
$$L : 8 = A : 4,$$

or $$L = 2\,A.$$

That is, ***the measure of the area of a lune is twice its angle.***

Proposition XXII. Theorem.

798. *If the unit angle is the right angle and the unit surface the area of the trirectangular triangle, the area of a spherical triangle is equal to its spherical excess.*



Given the spherical △ ABC.

To prove area $\triangle ABC = \angle A + \angle B + \angle C - 2$, *the right* △ *being the unit* ∡, *and the trirectangular* △ *being the unit surface.*

Proof. Complete the ⊙ of which BC is an arc, and let AB and AC intersect it again at B' and A'.

Then, since △s ABC and $AB'C$ together form the lune whose angle is B,

$$\text{area } \triangle ABC + \text{area } \triangle AB'C = 2\angle B. \qquad \S\ 796$$

Similarly, $\text{area } \triangle ABC + \text{area } \triangle ABC' = 2\angle C.$

Also $\triangle$s ABC and $AB'C'$ together form the lune whose angle is A.

$$\therefore \text{area } \triangle ABC + \text{area } \triangle AB'C' = 2 \measuredangle A.$$

$$\therefore 2 \triangle ABC + (\triangle ABC + \triangle AB'C + \triangle ABC' + \triangle AB'C') = 2(\measuredangle A + \measuredangle B + \measuredangle C).$$

But $\triangle ABC + \triangle AB'C + \triangle ABC' + \triangle AB'C'$ make up the surface of a hemisphere.

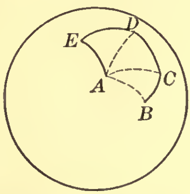
$\therefore \triangle ABC + \triangle AB'C + \triangle ABC' + \triangle AB'C' = 4$ trirectangular $\triangle$s. § 779

$$\therefore 2 \triangle ABC + 4 = 2(\measuredangle A + \measuredangle B + \measuredangle C).$$

Hence area $\triangle ABC = \measuredangle A + \measuredangle B + \measuredangle C - 2$. Q.E.D.

Proposition XXIII. Theorem.

799. *If the right angle is the angular unit and the trirectangular triangle the unit of surface, the area of a spherical polygon is equal to the sum of its angles diminished by the number of its sides less two.*



Given ABCDE, a spherical polygon of n sides.

To prove area $ABCDE = \measuredangle A + \measuredangle B + \measuredangle C + \measuredangle D + \measuredangle E - 2(n-2)$.

Proof. Draw all possible diagonals from the vertex A. These will divide the spherical polygon into $(n-2)$ spherical $\triangle$s.

The area of each spherical $\triangle$ = the sum of its $\measuredangle$s less two. § 798

$$\therefore \text{ area } ABCDE = \measuredangle A + \measuredangle B + \measuredangle C + \measuredangle D + \measuredangle E - 2(n-2).$$

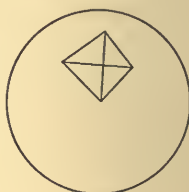
Q. E. D.

800. Def. The **spherical excess** of a spherical polygon is the spherical excess of the triangles into which its diagonals divide it.

801. Cor. *The area of a spherical polygon is to the area of the sphere as the spherical excess of the polygon expressed in degrees is to* 720.

SPHERICAL VOLUMES.

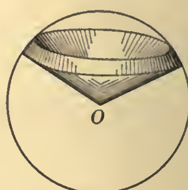
802. Def. A **spherical pyramid** is a portion of a sphere bounded by a spherical polygon and the faces of the corresponding polyhedral angle.



803. Def. The spherical polygon is called the **base** of the pyramid.

804. Def. A **spherical sector** is the volume generated by the revolution of a circular sector about the diameter of the circle of which the sector is a part.

805. Def. The **base** of a spherical sector is the zone generated by the arc of the circular sector.



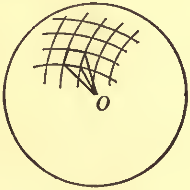
806. Def. A **spherical segment** is a portion of a sphere bounded by two parallel planes.

807. Def. The **bases** of a spherical segment are the sections of the sphere made by the parallel planes.

808. Def. The **altitude** of a spherical segment is the perpendicular distance between its bases. If one of the planes is tangent to the sphere, the segment is called a segment of one base.

Proposition XXIV. Theorem.

809. *The volume of a sphere is equal to the product of the area of its surface by its radius.*



Given V, the volume, S, the area of the surface, and R, the radius of a sphere.

To prove $V = \frac{1}{3} RS.$

Proof. Suppose the surface of the sphere to be divided into any number of congruent spherical polygons.

Then let pyramids be formed by joining the vertices of the polygons successively and drawing radii of the sphere to the vertices.

It is obvious that these pyramids will be congruent and will have equal altitudes.

The volume of each pyramid is equal to its base by one third its altitude. § 655

Therefore, the sum of the volumes of the pyramids is equal to the sum of the bases multiplied by one third the common altitude.

Then let the number of spherical polygons be indefinitely increased. Then the sum of the bases of the pyramids will approach the surface of the sphere as a limit.

$$\therefore V = \frac{1}{3} RS.$$ Q. E. D.

Formula: $V = \frac{4}{3} \pi R^3.$

810. Cor. 1. *The volume of a spherical pyramid is equal to the product of its base by one third its altitude.*

811. Cor. 2. *The volumes of two spheres are to each other as the cubes of their radii or as the cubes of their diameters.*

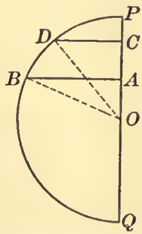
812. *The volume of a spherical sector is equal to one third the area of the zone which forms its base multiplied by the radius of the sphere.*

For, if Z denote the area of the zone, H its altitude, R the radius of the sphere, and V the volume of the sector, then

$$Z = 2\pi RH (\S\ 793), \text{ and } V = 2\pi RH \times \tfrac{1}{3}R = \tfrac{2}{3}\pi R^2 H.$$

Proposition XXV. Problem.

813. *To find the volume of a spherical segment.*



Given a spherical segment generated by the revolution of arc ABCD about PQ as an axis, O being the center of the sphere.

Draw OB and OD.

Denote the radius OD by R, AB by r, CD by r', and the volume of the segment by V.

The volume generated by $ABCD$ is equal to the spherical sector DBO + the cone generated by ODC − the cone generated by BAO.

Hence

$$V = \tfrac{2}{3}\pi R^2 H + \tfrac{1}{3}\pi \overline{CD}^2 \times CO - \tfrac{1}{3}\pi \overline{AB}^2 \times AO \qquad \text{§§ 812, 717}$$

$$= \tfrac{1}{3}\pi[2 R^2 H + (R^2 - \overline{CO}^2)CO - (R^2 - AO^2)AO]$$

$$= \tfrac{1}{3}\pi[2 R^2 H + R^2(CO - AO) - (\overline{CO}^3 - \overline{AO}^3)].$$

Factoring $\overline{CO}^3 - \overline{AO}^3$, and substituting H for its equal, $CO - AO$,

$$V = \tfrac{1}{3}\pi[2 R^2 H + R^2 H - H(\overline{CO}^2 + CO \times AO - \overline{AO}^2)]$$

$$= \tfrac{1}{3}\pi H[3 R^2 - (\overline{CO}^2 + CO \times AO + \overline{AO}^2)]. \qquad (1)$$

But $\qquad H^2 = (CO - AO)^2 = \overline{CO}^2 - 2\, CO \times AO + \overline{AO}^2,$

and $\overline{CO}^2 + CO \times AO + \overline{AO}^2$

$$= \tfrac{3}{2}\overline{CO}^2 + \tfrac{3}{2}\overline{AO}^2 - \left(\frac{\overline{CO}^2}{2} - \frac{2\, OC \times AO}{2} + \frac{\overline{AO}^2}{2}\right)$$

$$= \tfrac{3}{2}(\overline{CO}^2 + \overline{AO}^2) - \frac{H^2}{2}$$

$$= \tfrac{3}{2}(R^2 - r^2 + R^2 - r'^2) - \frac{H^2}{2}$$

$$= 3\, R^2 - \frac{3(r^2 + r'^2)}{2} - \frac{H^2}{2}.$$

Substituting this value in (1),

$$V = \frac{\pi H}{3}\left[\tfrac{3}{2}(r^2 + r'^2) + \frac{H^2}{2}\right]$$

$$= \frac{\pi H}{2}(r^2 + r'^2) + \frac{\pi H^3}{6}. \qquad \text{Q. E. F}$$

814. Cor. *In a spherical segment of one base,* $r' = 0$.

$$\therefore V = \frac{\pi r^2 H}{2} + \frac{\pi H^3}{6}.$$

Ex. 627. The section of a tunnel being a semicircle whose diameter is 50 ft., how many cubic feet of earth is removed in excavating the tunnel 500 ft.?

Ex. 628. Find the volume of a sphere whose radius is 14 ft.

Ex. 629. Find the surface of a sphere whose radius is 22 ft.

Ex. 630. Find the radius of the sphere whose surface contains the same number of units of surface that its volume contains of units of volume.

Ex. 631. What part of the surface of the sphere is the lune whose angle is 80°?

Ex. 632. How many spheres 2 ft. in diameter are required to equal in volume one sphere 6 ft. in diameter?

Ex. 633. If a sphere of iron 1 ft. in diameter weigh 230 lb., what would be the weight of a spherical iron shell whose inner diameter is 3 ft. and outer diameter 4 ft.?

Ex. 634. If the area of a lune whose angle is 30° is 200 sq. ft., what is the volume of the sphere?

Ex. 635. If the area of a lune whose angle is 60° is 360 sq. ft., what is the surface of the sphere?

Ex. 636. On a sphere whose surface is 600 sq. ft. the area of a lune is 200 sq. ft. What is the angle of the lune?

Ex. 637. What is the area of a spherical triangle whose angles are, respectively, 80°, 100°, and 130°, if the surface of the sphere is 100 sq. ft.?

Ex. 638. What is the area of a spherical triangle on a sphere whose radius is 30 yd., if the angles of the triangle are, respectively, 120°, 130°, and 140°?

Ex. 639. If the volume of a sphere is given 972 cu. ft., what is the area of a spherical triangle whose angles are, respectively, 140°, 150°, and 160°?

Ex. 640. What is the area of the zone whose altitude is 12 ft. on a sphere whose radius is 26 ft.?

Ex. 641. What part of the surface of the sphere is a zone whose altitude is one-third the diameter?

Ex. 642. If the radius of the earth be 4000 mi. and the altitude of the torrid zone 3200 mi., what part of the surface of the earth is the torrid zone?

Ex. 643. The sides of a spherical triangle are 80°, 110°, and 120°. What is the area of its polar triangle, the surface of the sphere being 144 sq. ft.?

Ex. 644. Find the area of the zone of a sphere of radius R, illuminated by a light at a distance d from the surface of the sphere.

Ex. 645. At what distance from the surface of a sphere whose radius is 20 ft. must a light be placed so as to illuminate one-eighth its surface?

Ex. 646. What is the area of a spherical pentagon whose angles are, respectively, 80°, 100°, 130°, 150°, and 160°, on a sphere whose radius is 18 ft.?

Ex. 647. If a light 20 ft. from the surface of a sphere illuminates one-eighth its surface, what is the volume of the sphere?

Ex. 648. A ball 3 in. in diameter is dropped into a conical glass 8 in. high and 5 in. in diameter at the top. What part of the volume of the glass does the ball occupy?

Ex. 649. What is the area of the surface of a spherical polygon of four sides, the angles being, respectively, 125°, 135°, 145°, and 155°, the diameter of the sphere being 80 ft.?

Ex. 650. What is the area of the section 8 in. from the center of a sphere whose radius is 17 in.?

Ex. 651. Find the volume of a spherical segment of one base which is 21 in. from the center of the sphere whose radius is 29 in.

Ex. 652. The altitude of a zone is 6 ft. and its area 30 sq. ft. Find the area of a lune whose angle is 60° on the same sphere.

Ex. 653. If the angle of a lune is 72° and equal to a zone with altitude 4 ft. on the same sphere, what is the diameter of the sphere?

Ex. 654. If a sphere is divided into two segments, the altitude of one being 2 ft. and the other 4 ft., what is the ratio of their volumes?

Ex. 655. The angles of a spherical triangle are, respectively, 130°, 135°, 140°; its area is equivalent to that of a lune on the same sphere whose angle is how many degrees?

Ex. 656. Considering the moon as a sphere of diameter of 2160 mi., and whose surface as 240,000 mi. from the earth, what part of the surface of the moon can be seen?

Ex. 657. In a sphere whose radius is R, what is the altitude of the zone whose area is equal to the area of a great circle of the sphere?

Ex. 658. If a zone of one base is the mean proportional between the remainder of the surface of the sphere and the entire surface of the sphere, what is the distance of the base of the zone from the center of the sphere?

Ex. 659. A spherical triangle is one-tenth the area of the surface of the sphere. Two of its angles are right angles. How many degrees in the third angle?

Ex. 660. What fraction of the surface of the sphere is the spherical triangle whose angles are 100°, 105°, and 115°, respectively?

Ex. 661. What is the ratio of the area of a lune whose angle is 100° to that of an equiangular triangle on the same sphere, each of whose angles is 100°?

Ex. 662. The angles of a spherical quadrilateral are, respectively, 120°, 130°, 140°, 150°. Find the angle of an equivalent lune on the same sphere.

Ex. 663. The volume of a sphere is to the volume of the circumscribed cube as π is to 6.

Ex. 664. The diameters of two spheres are to each other as 7 to 8 What is the ratio of their volumes?

Ex. 665. Prove that the volume of a sphere is two-thirds the volume of the circumscribing cylinder.

Ex. 666. Prove that the surface of a sphere is two-thirds the entire surface of the circumscribing cylinder.

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