ELEMENTS

OF

TRIGONOMETRY,

AND

TRIGONOMETRICAL ANALYSIS,

PRELIMINARY TO THE DIFFERENTIAL CALCULUS:

FIT FOR THOSE WHO HAVE STUDIED THE PRINCIPLES OF ARITHMETIC AND ALGEBRA, AND SIX BOOKS OF EUCLID.

BY

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Tant que l'algèbre et la géométrie ont été séparées, leur progrès ont été lents et leurs usages bornés ; mais lorsque ces deux sciences se sont réunies, elles se sont prêtées des forces mutuelles, et ont marché ensemble d'un pas rapide vers la perfection.—LAGRANGE.

LONDON:

PRINTED FOR TAYLOR AND WALTON,

BOOKSELLERS AND PUBLISHERS TO THE UNIVERSITY OF LONDON,

30 UPPER GOWER STREET.

M.DCCC.XXXVII.

LONDON:
PRINTED BY JAMES MOYES, CASTLE STREET,
LEICESTER SQUARE.

PREFACE.

The three great branches of elementary mathematics, meaning all that should precede the study of the Differential Calculus, are Arithmetic, Algebra, and Geometry. Each pair of these gives rise to a new inquiry, namely, the connexion which exists between the two. Thus, it is practicable to consider separately, 1. The application of arithmetic to algebra, and of algebra to arithmetic. 2. That of arithmetic to geometry, and of geometry to arithmetic. 3. That of algebra to geometry, and of geometry to algebra.

This will, at first sight, appear something like an undue quantity of distinction: nevertheless, with the exception only of the two-fold comparison of arithmetic with geometry, there is in it no degree of separation which cannot be fully justified, either as matter of necessity or convenience.

The application of arithmetic to algebra is made in the very formation of the latter science, though based upon considerations neither known nor admitted in the former. But, nevertheless, those considerations are derived from arithmetic in this sense,—that the want of them is suggested by the imperfections of the latter, and the method of arriving at them by the manner in which those imperfections appear.

The ideas to which arithmetic cannot fail to bring us, are more than its language has power to express; and the passage from that science to algebra consists in the methodical arrangement of the ideas to be expressed, and the invention of a language proper for the purpose. The application of algebra to arithmetic is a department of a more special character. It is of little consequence whether it be made a separate study, or not; the indispensable branches of it appear in their proper places, and nothing more is necessary than to point out the connexion. The higher parts of what is called the theory of numbers have offered, as yet, singularly little aid in the application of mathematics to the sciences of matter, and may, therefore, be omitted entirely by the student whose wishes on this subject are bounded by the possession of an instrument for physical inquiry. Neither will the more exclusively mathematical student find any thing in that theory for which he should make special preparation in his elementary reading: I have, therefore, omitted it altogether in my *Algebra*.

The application of arithmetic to geometry must be made as soon as the study of proportion begins. But, viewed by the side of arithmetic, geometry becomes the science of continuous magnitude in general: that is to say, the considerations on which it is necessary to dwell are such as apply equally to all magnitudes, as well as to spaces or lengths. In the accompanying* treatise, *On the Connexion of Number and Magnitude*, I have endeavoured, at least,

* I have placed this treatise at the end of the *Trigonometry;* but it should be understood as intended to be read first, or, at least, that the two should be read together.

to place the real difficulties of the subject before the higher class of students; guaranteeing nothing more than this, that a larger proportion of readers will understand the tract in question than would, by themselves, be able to master the Fifth Book of Euclid. The extension of the arithmetical notion of ratio, (shewn to be necessary, as well as furnished, by the consideration of magnitude in general, but principally of space magnitudes), constitutes the primary portion of the application of geometry to arithmetic.

The application of algebra to geometry is divisible into two distinct subjects. 1. The science usually called by that name, but which may be styled the theory of curves and surfaces. I may say of this part of the study, that though, on various accounts, it is *very desirable* that it should be made a separate branch, still it is not *indispensable*. A student of more than average intelligence might pick up, as he went along, enough of this part of mathematics to enable him to pursue his career: no new principles are insisted on, and the independent value of the subject mostly lies in the very extensive field of practice which it opens in the elements of geometry and the operations of algebra. 2. Trigonometry: the subject of the present treatise; on which I proceed to speak more at length.

The notion which is now studied under the name of Trigonometry, is that of magnitude in a state of alternating increase and decrease, or *periodic magnitude*. The term itself merely implies the measurement of triangles, as geometry does that of the earth; and it is still convenient to refer the measurement of triangles (and other figures) to trigonometry, but only as a minute and isolated application.

Taking the primary idea of quantity alternately increasing and decreasing, it is obviously of fundamental importance to detect a proper method of measurement. The circle presents itself for the purpose in the following way. Conceiving a periodic change of magnitude to run through its whole cycle in a given time, let a point revolve uniformly round a circle in the same time, starting from the end of a fixed diameter. The height of the point above the diameter is a periodic magnitude, which goes through all its changes in the same time as the given magnitude: and it is, in fact, one of the great objects of trigonometry to express periodic variation whose law is known in any way, by means of the simple species of variation just described.

Upon further examining the question of periodic variation, we discover in geometry, and in geometry only, a species of magnitude which is of necessity periodic, and is utterly exclusive of indefinite increase: namely, *direction*. In speaking of the direction of a line as a magnitude, we mean to imply that all direction is relative, inasmuch as we only judge the direction of one straight line by comparing it with another. No straight line can increase its difference of direction from that of another indefinitely: after a certain quantity of change, coincidence is reproduced. The connexion of direction with length is found to lead to an extension of the algebra of positive and negative quantities, which gives the same power of interpreting $a+b\sqrt{-1}$ relatively to a, as exists already in the case of $+a$ or $-a$ relatively to a. This is an application of geometry to algebra; and, though there does exist a point of view in which geometry may appear not absolutely indispensable, and which I have described in Chapter IV., yet it is pretty clear that abstraction must advance considerably before it will be safe to abandon reference to the science of space in presenting algebra complete to the beginner. The progress of algebra, in this respect, is very curious. In its infancy, geometrical interpretation was rendered necessary by want of power in its symbolic language; was abandoned as the latter grew to maturity; and is finally had recourse to again, because symbols are now sufficient to express relations of magnitude which do not yet exist except in regard to space.

This modern application of geometry to algebra is traced in Professor Peacock's Report on Analysis to the British Association, printed in the second volume of their Transactions (A.D. 1834). The account there given of this particular point, as well as the rest of the article, should be read by the student of the higher mathematics with great attention, as being equal, in the elementary point of view, and superior in the historical, to any thing which has yet appeared on the subject. The Report cited has rendered any reference to authorities unnecessary.

I have not inserted any thing on the solution of spherical triangles; the subject being one which, though of primary use in Astronomy, is but little connected with the fundamental part of Trigonometry. I should have added a chapter upon the subject, had I not already published a treatise in the Library of Useful Knowledge, to which I am thereby enabled to refer the reader.

A. DE MORGAN.

University College, London,
March 1, 1837.

ADDENDUM

TO THE

TREATISE ON TRIGONOMETRY.

In page 42, line 18, there is an obvious mistake in the reasoning contained in the words, " But $\sec\theta$ is greater than 1; therefore $\tan\theta$ is greater than θ." To set this right, let the student omit that sentence and the preceding, and supply their places by the following proof, that $\tan\theta$ is always greater than θ, *whenever the latter is less than a right angle.*

1. Let $\theta = n\phi$, where n is a whole number, whence ϕ, 2ϕ, 3ϕ .... $n\phi$, are severally less than a right angle. Then

$$\tan 2\phi = \frac{2\tan\phi}{1-\tan^2\phi} \qquad \text{or } \tan 2\phi \text{ is greater than } 2\tan\phi$$

$$\tan 3\phi = \frac{\tan 2\phi + \tan\phi}{1-\tan 2\phi . \tan\phi} > \tan 2\phi + \tan\phi > 3\tan\phi$$

and so on: observing that all the denominators must be positive (since the fractions themselves and their numerators are positive) and less than unity. Proceeding in this way, we shew that $\tan n\phi$ is greater than $n\tan\phi$: whence

$$\tan\theta > \frac{\theta}{\phi}\tan\phi, \text{ or } \frac{\tan\theta}{\theta} > \frac{\tan\phi}{\phi}.$$

2. If then we can shew that, by taking n sufficiently great, or ϕ sufficiently small, $\tan\phi \div \phi$ is greater than unity, it follows that $\tan\theta \div \theta$ is also greater than unity. Let a polygon of k sides be circumscribed about a circle whose radius is r; then each side of the polygon is $2r\tan\frac{\pi}{k}$, and the whole periphery is $2kr\tan\frac{\pi}{k}$, which is greater than the circumference of the circle or $2\pi r$. Hence,

$$\tan\frac{\pi}{k} \text{ is greater than } \frac{\pi}{k} \text{ (k being a whole number).}$$

3. Now, $\frac{n\pi}{k}$ may be made as near as we please to θ, and either greater or less, by properly assuming n and k (whole numbers). But, by combining the results of (1) and (2),

$$\tan\frac{n\pi}{k} \text{ is greater than } \frac{n\pi}{k}$$

or $\qquad \tan(\theta \pm \alpha)$ .... $\theta \pm \alpha$

where α may be as small as we please. Hence it follows that $\tan\theta$ is greater than θ.

ELEMENTS

OF

TRIGONOMETRY.

CHAPTER I.

DEFINITIONS AND FUNDAMENTAL FORMULÆ OF TRIGONOMETRY IN THE CASE OF ONE ANGLE.

(1.) TRIGONOMETRY originally meant simply the measurement of triangles. It now means measurement generally by means of the properties of triangles, in all cases in which the connexion between sides and angles is concerned, not sides only, nor angles only: together with all consequences of such measurements as are useful in the higher parts of mathematics. If algebraical symbols and operations be adopted, as is now universally the case, it is a branch of the application of algebra to geometry.

(2.) If a line U, taken at pleasure, be called the linear unit, or 1 of length, then 2 U is called 2, $\frac{1}{2}$U is called $\frac{1}{2}$, and so on. And any line incommensurable with U is denoted by a general symbol such as a, where, if the line specified were commensurable with U, a would be a number or fraction, and aU would represent the line. Let aU still represent the line, where, in the theory, a is a symbol for the ratio of the line to U; and, in practice, a line as near as we please to the incommensurable line is taken, namely, $\frac{m}{n}$U, and $\frac{m}{n}$ is substituted in results instead of a. I here suppose the student to have read the preliminary treatise: if not, he must be content with the application of the following proposition.

Two lines being given, A and B, either two whole numbers m and n can be found, so that $B = \frac{m}{n}A$; or so that $\frac{m}{n}A$ shall be as near to B as we please.

The same considerations apply to any other magnitudes; to angles for instance.

(3.) Before proceeding to ascertain how a line may depend upon an angle, or an angle upon a line, it may be useful to shew that we are, by means of geometry, able to determine lines from lines, without consideration of angles, and angles from angles, without consideration of lines. The complete determination of some angles of a figure by means of the rest, whenever that is possible, is contained in Prop. 32 of the first book of Euclid. If $a_1, a_2 \ldots\ldots a_n$ be the n angles of a rectilinear figure which has no re-entering angles (pointing inwards), and if π be the angle made by a line and its continuation, or *twice* a right angle, then

$$a_1 + a_2 + a_3 + \quad \cdots\cdots\cdots \quad + a_n = (n-2)\,\pi$$

In a triangle $$a_1 + a_2 + a_3 = \pi$$

Hence all the angles of a triangle are known when two are known.

(4.) The determination of lines by means of lines depends mostly upon the two following propositions:

I. 47. If A, B, and C, be the sides of a triangle, and if A and B contain a right angle, then the squares on A and B make together an area equal to the square on C.

Let A, B, and C, contain respectively a, b, and c, of any linear unit. Then (Arithmetic, § 234.), if a be a whole number or fraction, the square on the unit (which call T) is contained aa times in the square on A, or the square on A is aaT. Similarly, the square on B is bbT, that on C is ccT; whence

$$aa\mathrm{T} + bb\mathrm{T} = cc\mathrm{T}$$

or T taken $aa + bb$ times is T taken cc times. Therefore,

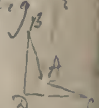
$$aa + bb = cc$$

[When* a, b, and c, are either of them incommensurable, this equation no longer exists arithmetically. We shall give the strict developement of this proposition in such a case. Firstly, the ratio of the square on A to the square on U (the linear unit) is that compounded of A : U and A : U. Let A be to U as U to X; then (VI. prop. 16.), the rectangle whose sides are A and X is equal in area

* The parts in brackets are for the student who has read the Introductory Treatise in such a manner as to believe he understands it.

The references are to the *articles* of my treatise on *Arithmetic*, the *pages* of my treatise on *Algebra*, and to the books of Euclid.

The prop^n is true when there are reentering ∠s. For take a fig & say with 4 sides—then Sum of int. ∠s = (n−2) π; [illegible]



to the square on U. Therefore, the squares on A and U are two rectangles with a common altitude A, and bases A and X; consequently (VI. 1),

Square on A : Square on U :: A : X

But A : X is compounded of A : U and U : X, that is, of A : U and A : U. Now, let the ratio of A : U be represented by $a : 1$, and the compound ratio by $aa : 1$. Then we have, proceeding in a similar way with the other sides,

Sq. on A : sq. on U :: $aa : 1$
Sq. on B : sq. on U :: $bb : 1$

Sq. on A + sq. on B : sq. on U :: $aa + bb : 1$
But, Sq. on C : sq. on U :: $cc : 1$

Therefore, Sq. on A + sq. on B :: sq. on C :: $aa + bb : cc$
But, Sq. on A + sq. on B = sq. on C, therefore, $aa + bb = cc$

or let the ratio of A : U have any symbol $a : 1$, &c., and let $aa : 1$ represent the ratio compounded of $a : 1$ and $a : 1$, and let $m + n : 1$ mean the ratio of the sum of two magnitudes to U which are severally to U as $m : 1$ and $n : 1$, agreeably to the definitions in the preliminary Treatise, and we have what should in strictness be written

$$aa + bb : 1 :: cc : 1$$

When A, B, and C, are commensurable with U, then a, b, c, mean whole numbers or fractions, and the preceding is reduced to $aa + bb = cc$ as before. If a, b, and c, be whole numbers or fractions, such that aU, bU, cU, are very near to A, B, C, then we have $aa + bb = cc$ very nearly; but if a, b, and c, be really symbols of incommensurable ratios, or rather $a : 1$, $b : 1$, $c : 1$, then the preceding proposition can only be interpreted with reference to magnitude; namely, as $aa\mathrm{U} + bb\mathrm{U} = cc\mathrm{U}$ where aaU : U is the ratio compounded of aU : U and aU : U, or A : U and A : U.]

(5.) VI. 4. If A, B, C, and P, Q, R, be the sides of equiangular triangles; namely, the angles opposite to A and P equal, &c. then,

$$A : B :: P : Q, \quad B : C :: Q : R, \quad C : A :: R : P$$

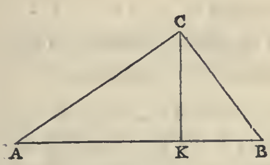
If the ratios of the sides of any triangle be known, the angles are *geometrically* known; that is, we can construct the angles, but cannot yet find their ratios to one another.

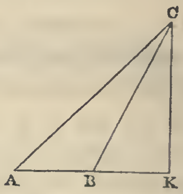
If the sides of the first triangle be aU, bU, cU, and those of the second pV, qV, rV, we have (a, b, and c, being numbers or fractions)

$$a : b :: p : q \qquad aq = pb$$
$$b : c :: q : r \qquad br = qc$$
$$c : a :: r : p \qquad cp = ra$$

[If these represent incommensurable ratios, let the student treat these equations in the same manner as $aa + bb = cc$ preceding].

(6.) I now proceed to a case in which lines are numerically determined by lines. Let there be a triangle ABC, in which, U being





an arbitrary linear unit, we have

$$\left.\begin{array}{l} AB = cU \\ BC = aU \\ CA = bU \end{array}\right\}$$ required CK, the perpendicular on cU; and AK and KB, the segments * of AB.

Let $\quad AK = xU, \quad KB = yU, \quad KC = pU$

Then, in the first case, $\quad x + y = c$

...... second .. $\quad x - y = c$

In both $\quad x^2 + p^2 = b^2 \qquad y^2 + p^2 = a^2$

First Case.	Second Case.
$x^2 - y^2 = b^2 - a^2$	$x^2 - y^2 = b^2 - a^2$
$c(x - y) = b^2 - a^2$	$c(x + y) = b^2 - a^2$
$c(x + y) = c^2$	$c(x - y) = c^2$
$2cx = b^2 + c^2 - a^2$	$2cx = b^2 + c^2 - a^2$

* If K be a point in AB, or AB *produced*, AK and KB are called *segments* of AB, *in all cases*. If K lie between A and B, AK + KB = AB; if K lie beyond B, AK − KB = AB; if K lie beyond A, BK − KA = AB.

First Case.

$$x = \frac{b^2+c^2-a^2}{2c}$$

$$2cy = c^2+a^2-b^2$$

$$y = \frac{c^2+a^2-b^2}{2c}$$

Second Case.

$$x = \frac{b^2+c^2-a^2}{2c}$$

$$2cy = b^2-a^2-c^2$$

$$y = \frac{b^2-a^2-c^2}{2c}$$

But both cases might have been contained in one, by adopting the conventions of algebra, instead of keeping to arithmetic; for, if we suppose the second case to be made from the first by moving B to the left, we see that BK will be measured in a direction contrary to that which it had at first. If, then, the formulæ of the first case had been used to determine BK in the second case, we should have been warned to measure BK in a direction contrary to that assumed, by its negative sign. For instance, let $c=2$, $a=3$, $b=4$; then we have, taking the formula of the first case,

$$y = \frac{4+9-16}{4} = -\frac{3}{4}$$

That is, BK is $\frac{3}{4}$U in magnitude, not on the same side of K as in the case from whence the formula was arithmetically derived, but on the contrary side. (*Algebra*, chapters I. and II.)

To find p, we observe that

$$p^2 = a^2 - y^2 = \frac{4a^2c^2-(c^2+a^2-b^2)^2}{4c^2}$$

$$= \frac{\{2ac-(c^2+a^2-b^2)\}\ \{2ac+c^2+a^2-b^2\}}{4c^2}$$

$$= \frac{\{b^2-(a-c)^2\}\ \{(a+c)^2-b^2\}}{4c^2} = \frac{(a+b+c)(a+c-b)(c+b-a)(b+a-c)}{4c^2}$$

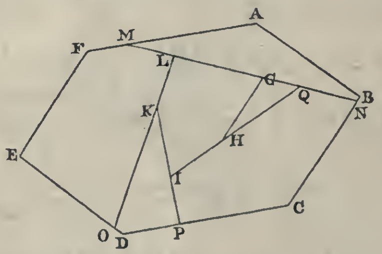
Let $\left.\begin{array}{l} a+b+c=2s \\ b+c-a=2(s-a) \\ c+a-b=2(s-b) \\ a+b-c=2(s-c) \end{array}\right\}$ then $\left\{\begin{array}{l} c^2p^2=4s(s-a)(s-b)(s-c) \\ \\ cp=2\sqrt{s(s-a)(s-b)(s-c)} \end{array}\right.$

As this result will appear in the sequel, we have chosen it as our instance; and by proceeding with the two propositions in question, it is thus possible to determine lines in terms of lines, without the necessity of employing angles as means of expression. I now pass on to the consideration of an angle connected with a line.

(7.) From the Sixth Book of Euclid it appears, that though given lines will determine angles (as, for instance, the three sides of a triangle being given, the angles can be constructed, both internal and external), yet that it is not necessary to give the lengths; for any other lengths which have the same relative magnitudes will give the same angles. Indeed, the Sixth Book of Euclid is, in great part, an inductive proof of the following proposition. If the absolute lengths of all the lines in a figure be altered in the same ratio, the angles of the figure are not altered: hence, angles depend upon the ratios of lines. The line with which an angle is most connected is the arc of a circle, and it will be necessary to know something more of this figure than can be directly found from the elements. We shall indicate the principal steps necessary, which may be readily filled up by a student who understands the Sixth Book.

(8.) Definition. A bounded figure is called *convex*, when no straight line whatsoever can meet its boundary in more than two points, unless it be itself part of the boundary.

(9.) Theorem. If one convex rectilinear figure be entirely contained within another, the boundary of the contained figure must be less than that of the containing.



Let ABCDEF and GHIKL be the containing and contained figures, then, 1. MN is less than MABN, or MNCDEFM is less than ABCDEFA. 2. Similarly, LODCNL is less than MNCDEFM; 3. KPCNLK is less than LODCNL; 4. KIQLK is less than KPCNLK; 5. KIHGLK is less than KIQLK. Whence the proposition.

(10.) Postulate. Let it be granted that this theorem is also true of convex curvilinear or mixtilinear figures.

(11.) Theorem. If the homologous sides of two similar rectilinear figures may be made as nearly equal as we please, the figures themselves may be made as nearly equal as we please, both in length of boundary and area. We leave the demonstration to the student, as it is a very simple consequence from the Sixth Book.

(12.) Theorem. For every rectilinear figure which can be described *in* any one circle, a similar figure can be described *about* any other circle. We shall merely indicate the construction, and leave the student to finish the demonstration. Let C be a circle, and P any inscribed polygon; let C′ be another circle; in it inscribe P′ similar to P. Bisect every side of P′, and draw radii through the points of bisection. From the extremity of each such radius draw a tangent; then will the polygon P″, formed by all the tangents, be similar to P′ and to P.

Corollary 1. The circles C and C′ are contained, both as to length of boundary and area, between P′ and P″

Corollary 2. By making the radii sufficiently near to equality, P′ and P″ may be made (11.), both in boundary length, and area, as near as we please to each other: and if we are at liberty to give P′ as many sides as we please, and as small (10), the same follows of the circles C and C′ themselves.

Corollary 3. And all these results are equally true of sectors of circles which contain the same angle.

(13.) Theorem. If in a circle a polygon be described, of which no one side exceeds Z in length; then, if Z may be made as small as we please, the polygon and circle may be made as nearly equal as we please, both in boundary length, and area.

For, under these circumstances, as may be easily shewn, the inscribed and similarly superscribed polygons may be made as nearly equal as we please in both respects; and (10.) the circle lies between them in both.

Corollary. The same is true of any sector of a circle.

(14) Theorem. The circumferences of circles (or of similar sectors) are as their radii, and their areas as the squares on the radii.

Let the radii be R and R′, and describe in the circles two similar polygons, having all their sides severally less than Z. Let P and P′ be the whole boundary lengths of the polygons, and let C and C′ be the circumferences of the circles. But the boundary lengths are proportional to the lengths of two homologous sides, which are to each other as the radii, and (12.) by making Z sufficiently small, these same lengths may be $C-K$ and $C'-K'$, where K and K′ are as small as we please. Thence, we have (however small K and K′ may be),

$$C-K : C'-K' :: R : R'$$

If possible, let C be to C′ not as R to R′, but, firstly, in a greater ratio. Then

$$C \text{ is to } C' \text{ more than } C-K \text{ is to } C'-K';$$

or mC exceeds nC' where $mC-mK$ does not exceed $nC'-nK'$.

Let $mC=nC'+V$; then $nC'+V-mK$ does not exceed $nC'-nK'$, and, still more, does not exceed nC'

Consequently, mK is at least equal to V; but V is determined without reference to K from C and C′, and K may be as small as we please. Therefore, it may be so taken that mK shall be less than V. But if the supposition under trial be true, mK must at least be equal to V; therefore, that supposition is not true, or C is not to C′ more than R to R′. If possible, let it be less; then we have, by similar reasoning,

mC is less than nC' where $mC-mK$ is not less than $nC'-nK'$.

Let $mC=nC'-V$; then $nC'-V-mK$ is not less than $nC'-nK'$, still more then $nC'-V$ is not less than $nC'-nK'$;

or, nK' is at least equal to V. This, as before, shews that the supposition cannot be true. Whence the first part of the theorem follows; and the same may be similarly proved of the sectors.

Let A and A′ be the areas of the circles, and Q and Q′ the squares on the radii; whence it follows, that $A-L$ and $A'-L'$ being the areas of the similar polygons, (12.) L and L′ may be made as small as we please. But (VI. prop. 19.) we must have,

$$A-L : A'-L' :: Q : Q'$$

and, precisely as in the former case, we find that A is to A′ neither more nor less than Q is to Q′. Hence, A is to A′ as Q to Q′.

Now, let the student shew, that the area of a regular circumscribed polygon of n sides is $\frac{1}{2}nrs$T (see next article), sU being one side: and thence, that cU and aT being the circumference and area, we must have $a=\frac{1}{2}cr$.

(15.) Algebraically, let rU and r'U be the two radii, cU and c'U the circumferences; then we have,

$$c : c' :: r : r' \quad \text{or} \quad \frac{c}{r} = \frac{c'}{r'}$$

If T be the square on U, the areas aT and a'T' are as rrT and $r'r'$T, or,

$$a : a' :: rr : r'r' \qquad \frac{a}{r^2} = \frac{a'}{r'^2}$$

(16.) The ratios represented by $\frac{c}{r}$ and $\frac{a}{r^2}$ are incommensurable; but we have very nearly

$$\frac{c}{r} = 2 \times \frac{355}{113} \qquad \frac{a}{r^2} = \frac{355}{113}$$

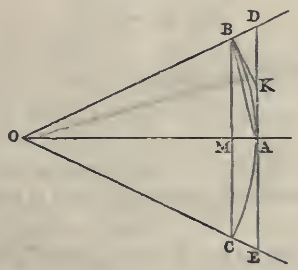
Let the ratio, which is incommensurable, but very near that of 355 : 113, be represented by $\pi : 1$; or, in the more common way of speaking, which uses incommensurables as commensurables, let $\frac{c}{r}$, which is the same for all circles, be called 2π. Then we have

$$c = 2\pi r \qquad a = \pi r^2$$

$$\pi = \text{nearly } \frac{355}{113} = \text{nearly } 3{\cdot}14159$$

To prove the preceding, we have not yet the means. Let us agree, however, to denote $\frac{c}{r}$, whatever it may be, by 2π.

[But for those students who dislike to leave any thing behind which is afterwards to be proved, we shall establish the following proposition, which assigns the approximate value of π.



Let BC, DE, be the sides of regular polygons of n sides, inscribed and circumscribed about the circle where radius is OA.

Draw a tangent at B, and complete the figure as shewn. Then will BA, AC, be sides of the polygon of $2n$ sides inscribed, and BK, KA, are halves of sides of the circumscribed polygon of $2n$ sides. Let I_p and E_p mean the areas of the interior and exterior polygons of p sides. Thence, we have,

$$
\begin{aligned}
I_n : I_{2n} &:: \text{triangle OBC} : 2 \times \text{triangle OBA} \\
&:: \ldots\ldots \text{OBM} : \ldots\ldots \text{OBA} \\
&:: \text{OM} : \text{OA} \\
I_{2n} : E_n &:: \text{triangle OBA} : \text{triangle ODA} \\
&:: \text{OB} : \text{OD} \\
&:: \text{OM} : \text{OA}
\end{aligned}
$$

Therefore, $I_n : I_{2n} :: I_{2n} : E_n$

If, then, T be the square on the linear unit U, and if $I_n = i_n T$, &c. we have

$$i_{2n} = \sqrt{i_n e_n}$$

Again (VI. 3.),

$$
\begin{aligned}
\text{AK} : \text{KD} &:: \text{AO} : \text{OD} :: \text{MO} : \text{OB} \\
\text{AK} : \text{AD} &:: \text{MO} : \text{MO}+\text{OB} \\
&:: \text{MO} : \text{MO}+\text{OA} \\
&:: I_n : I_n + I_{2n}
\end{aligned}
$$

And

$$
\begin{aligned}
E_{2n} : E_n &:: 2 \times \text{triangle AOK} : \text{triangle AOD} \\
&:: 2\text{AK} : \text{AD} \\
&:: 2 I_n : I_n + I_{2n}
\end{aligned}
$$

whence

$$e_{2n} = \frac{2\, i_n\, e_n}{i_n + i_{2n}}$$

which formulæ, namely,

$$i_{2n} = \sqrt{i_n e_n} \qquad e_{2n} = \frac{2\, i_{2n}^2}{i_n + i_{2n}}$$

enable us to pass in numbers approximately from the areas of any inscribed or circumscribed polygon, to those of double the number of sides.

Let the linear unit be the radius itself, and T, therefore, the square on the radius. We have, then, on inspection (or IV. 6, 7.)

$$I_4 = 2T \qquad E_4 = 4T$$

$$i_4 = 2 \qquad e_4 = 4$$

$$i_8 = \sqrt{8} = 2\sqrt{2} \qquad e_8 = \frac{16}{2+\sqrt{8}} = 8(\sqrt{2}-1)$$

$$i_{16} = 4\sqrt{2-\sqrt{2}} \qquad e_{16} = \frac{32(2-\sqrt{2})}{2\sqrt{2}+4\sqrt{2-\sqrt{2}}} \text{ \&c.}$$

These expressions shew no very easy law; but if we begin with approximate values of the second pair, namely,

$$i_4 = 2 \qquad i_8 = 2{\cdot}8284271$$

$$e_4 = 4 \qquad e_8 = 3{\cdot}3137085$$

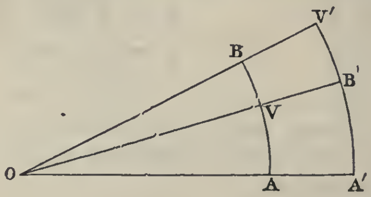
and proceed with these approximations, it will be found, by a calculation which may appear laborious (but, as we have said, this digression is for students whose industry exceeds their disposition to believe without proof), that we have

Number of sides in the polygon.	Approximate fraction of T contained in the area of the inscribed polygon.	Ditto, Ditto, of the circumscribed polygon.
4	2·0000000	4·0000000
8	2·8284271	3·3137085
16	3·0614674	3·1825979
32	3·1214451	3·1517249
64	3·1365485	3·1441184
128	3·1403311	3·1422236
256	3·1412772	3·1417504
512	3·1415138	3·1416321
1024	3·1415729	3·1416025
2048	3·1415877	3·1415951
4096	3·1415914	3·1415933
8192	3·1415923	3·1415928
16384	3·1415925	3·1415927
32768	3·1415926	3·1415926

Now, the area of the circle itself always lies between those of the inscribed and circumscribed polygon; and the polygons of 32768 sides, inscribed and circumscribed, do not differ, (slight errors of approximation excepted, which may affect the last place), by the ten-millionth part of a square unit. We may say, then, that the area of the circle is approximately 3·14159 T, or $\frac{355}{115}$T very nearly; that is, $\pi = 3·14159$ very nearly.]

(17.) *Lemma.* If the same magnitude be m units of one kind, and m' units of another kind, or if $m\text{U} = m'\text{U}'$, then $m : m' :: \text{U}' : \text{U}$, and if $\text{U}' = k\text{U}$, then $m = m'k$.

(18.) Let there be two different angles, B O A, B′O A′, and describe different circles AB, A′B′. Let the given quantities be



the radius OA, and the arc AB; the radius OA′ and the arc A′B′. Required the ratio of the angles.

(VI. 33.) Angle BOA : Angle B′OA′ :: BA : VA
(14.) OA : OA′ :: VA : B′A′

Therefore, BA : B′A′, the ratio of the arcs, is that compounded of the ratios of the angles and the radii. Let U be the linear unit, Θ the angular unit; and let

$$\text{BA} = s\text{U}, \quad \text{B}'\text{A}' = s'\text{U}, \quad \text{OA} = r\text{U}, \quad \text{OA}' = r'\text{U},$$
$$\text{Angle BOA} = \theta\,\Theta, \quad \text{Angle B}'\text{OA}' = \theta'\Theta$$

Then $$r\theta : r'\theta' :: s : s'$$

or $$\frac{\theta'}{\theta} = \frac{r}{r'} \cdot \frac{s'}{s}$$

(19.) We have already seen that there has happened a case (*Algebra*, p. 226.) in which one system of suppositions is most convenient in analysis, while another is so in practice; namely, in the choice of a base for logarithms, where 2·7182818 ...... is the base of analysis, and 10 that used in applying logarithms to computations. Just so, in the present instance, there is an angular unit which it is convenient to adopt in investigations, while another unit is universally supposed in practical applications. And the neglect of distinction between these two units is the stumbling-block of the beginner, though the necessity for the distinction is too great to allow us for a moment to think of abandoning one or the other unit.

(20.) 1. The *analytical* or *theoretical* unit (there is no distinct term for it in general use) is the angle which has *an arc equal to the radius.* If, in a solid circular plate, we stretch a thread equal to the radius from point to point of the edge, the thread is then the side of a regular hexagon (IV. 15.) and subtends two-thirds of a right angle. If, then, we bend the thread over the edge, it will (no stretching being supposed) subtend at the centre *somewhat less than two-thirds of a right angle*, which is the first rough notion of the analytical unit.

(21.) In the process of (18.), let $s=r$, and let Θ be the analytical unit; then $\theta\,\Theta$ is also the same, or $\theta=1$, and we have

$$\theta' = \frac{s'}{r'}$$

or to determine the number of analytical units in any angle, divide the number of linear units in the arc by that in the radius.

Hence we can easily ascertain how many analytical units there are in one, two, &c. right angles. The radius being $r\,\mathrm{U}$, the whole circumference is $2\pi r\,\mathrm{U}$, and its fourth (or the arc of a right angle) is $\frac{\pi}{2}r\,\mathrm{U}$, the number of units in which is $\frac{\pi}{2}r$. This divided by the number of units in the radius, or r, gives $\frac{\pi}{2}$.

The right angle in analytical units is	$\frac{\pi}{2}$,	approximately,	1·5707963
Two right angles .... are	π	....	3·1415926
Three right angles	$\frac{3}{2}\pi$		
Four right angles	2π		

(22.) 2. The practical method of measuring an angle is well known to be as follows. Let the 90th part of a right angle be called a *degree;* the *sixtieth* part of a degree, a *minute;* the *sixtieth* part of a minute, a second. Let these be denoted by 1°, 1′, 1″, which are not symbols of number, but of magnitude. *They are angles.* We shall always denote the theoretical unit by Θ.

$$1^\circ = 60.1' = 3600.1''$$

A right angle $= 90.1^\circ = 5400.1' = 324000.1'' = 1{\cdot}5707963\ \Theta$ nearly.

Hence $1^\circ = \cdot 01745329 \quad \Theta$ nearly $\qquad \Theta = 206264\cdot 8 \,.\, 1''$ nearly

$1' = \cdot 000290888 \quad \Theta \; \ldots\ldots \qquad \Theta = 3437\cdot 747 \,.\, 1' \; \ldots\ldots$

$1'' = \cdot 0000048481 \quad \Theta \; \ldots\ldots \qquad \Theta = 57\cdot 29578 \,.\, 1^\circ \; \ldots\ldots$

(23.) It is usual to divide the circumference of a circle into 360 equal parts, and to call each part a degree; the sixtieth part of a degree a minute, &c. To avoid confusion, I shall call these *linear* degrees, minutes, &c. Thus, every circle has its own linear degree; and the greater the circle, the greater the linear degree. On a circle of the earth, the linear degree is 69 miles and a half (a little less) in length. The student will remember that a *linear degree* and *a degree* are two things as different as a line and an angle.

(24.) It is very common among writers on this subject to confound π and 180° or 180.1°, $\frac{\pi}{2}$ and 90° or 90.1°. Thus we see sometimes such equations as

$$\pi = 180^\circ \qquad\qquad \frac{\pi}{2} = 90^\circ$$

which are equations of as little title to existence as

$$1760 \quad = \quad 1 \qquad \text{or} \quad 20 \qquad\qquad = 4$$

instead of 1760 yards = 1 mile, or 20 shillings = 4 crowns.

The angle, which in theoretical units is π, in degrees is 180. But it does not, therefore, follow that $\pi = 180$, any more than that $12 = 4$, because that length which in feet is 12, in yards is 4.

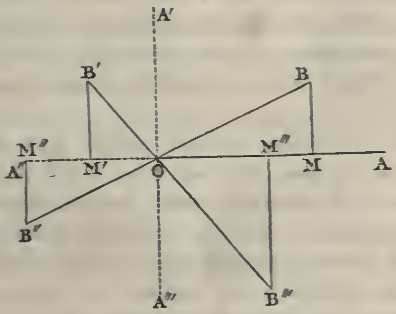
(25.) We now proceed to a more general use of the word *measure*, which frequently occurs in practice. One quantity is said to *measure* another, even when the two are of different kinds, if any change whatever made in the one is accompanied by a proportional change in the other, so that if A of the one give B of the second, mA of the one always gives mB of the second, whether m be whole or fractional, or the representative of an incommensurable numerical symbol.

Thus, angles are measured by arcs of given circles; for on the same circle any alteration of either arc or subtended angle produces a proportionate alteration in the other. Between given parallels, the areas of rectangles are measured by their bases. But a square is not measured by its side; for if the side be doubled, for instance, the square is not doubled, but quadrupled.

(26.) One magnitude or ratio is determined by another, when, the first being given, the second is given; or at least when, the first being given, the second cannot be any thing we please, but must have one or other of a certain finite number of values. Thus, one angle of a triangle being given (VI. 6.), the ratio of the containing sides determines the other two angles, or rather, determines one of them; for, one angle being given, the sum of the other two is given: the ratio of the containing sides, with the relation just mentioned, determines both the remaining angles.

(27.) If one angle be given (VI. 7.), the ratio of two sides, which do not contain that angle, absolutely determines the other angles, if the given angle be a right angle or more; but if the given angle be less than a right angle, the ratio of the two sides (which do not contain it) determines only two values of each angle, one of which it must be. We shall afterwards have to return to this point.

(28.) An angle, in Euclid, is only one of the angles or openings made by two straight lines, namely, that opening which is less than the opening made by a line and its continuation, or less than two right angles. But any two lines which terminate in their point of meeting, make two angles, one less and one greater than two right angles, the sum of both being four right angles.



(29.) The foundations of trigonometrical notation are as follows: Let a straight line OB, setting out from the position OA, revolve round the point O. Let AA″ and A′A‴ be at right angles, and let lines measured from O towards A, or from O towards A′, be positive, while lines measured from O towards A″, or from O towards A‴, are negative. Let positive angles be described when the revolution makes

OB proceed from OA to OA′, and let negative angles be described when the revolution makes OB proceed from OA to OA‴. And let ratios be positive when both their terms have the same sign, and negative when both their terms have different signs.

Let the Euclidean angle AOB be $\theta\Theta$, where Θ is the analytical unit, so that θ is the algebraical symbol for the angle expressed in analytical units; then the whole revolution, or 4 right angles, being $2\pi\Theta$, the angle of two revolutions, or eight right angles, being $4\pi\Theta$, and so on, the line OB is said to make, with OA, an angle

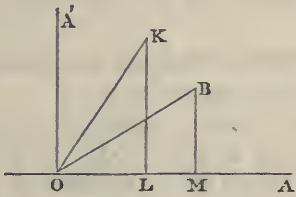
$$\theta\Theta, \quad \text{or} \quad (2\pi+\theta)\Theta, \quad \text{or} \quad (4\pi+\theta)\Theta, \quad \&c.$$

according as we consider OB to be in its first, second, third, &c. revolution. Or the method by which we take into account the necessity of considering the following proposition, "If a straight line revolve round a point continually, it will, in every succeeding revolution, pass again through all the positions which it had in the first," is by making the following assertion: The angles θ, $2\pi+\theta$, $4\pi+\theta$, &c., and, generally, $2n\pi+\theta$, n being a whole number, are the angles made by OB and OA in the first, second, third, &c., and, generally, in the $(n+1)$th revolution. The angles which it will be necessary to consider as denoting distinct positions, are those which are less than four right angles or $2\pi\Theta$, if we only consider positive angles; or those which are less than two right angles in magnitude, if we consider positive and negative angles; that is, which lie between $+\pi\Theta$ and $-\pi\Theta$. We shall, for the present, confine ourselves to these angles.

(30.) We now proceed to define what we shall call the *primary trigonometrical functions* of an angle, the names of which are the *sine*, *cosine*, *tangent*, *cotangent*, *secant*, *cosecant*, *versed sine*, and *coversed sine* of the angle. And, firstly, whereas in old books on trigonometry these functions are *lines*, in every modern system they are, or should be, defined to be *numbers*, or, in the widest sense, *ratios*. Secondly, the term *sine* is probably from an Arabic word, but the meaning is not well known; *tangent* and *secant* are of obvious Latin derivation; but the definition we shall give has nothing to do with their etymology.

Consider cosine as the abbreviation of "sine of the complement."
...... cotangent .... "tangent of the complement."
...... cosecant .... "secant of the complement."

where by the complement of an angle is meant the algebraical excess of a right angle over the angle in question. Thus, $\theta\,\Theta$ being an angle, $\frac{\pi}{2}\Theta-\theta\Theta$ or $\left(\frac{\pi}{2}-\theta\right)\Theta$ is the complement, and $\frac{\pi}{2}-\theta$ its algebraical representation. If the angle be greater than a right angle, the complement is negative. Similarly, if $A\,1^\circ$ be the angle, A being a number or fraction, $(90-A)\,1^\circ$ is the complement.



First, consider an angle AOB less than a right angle. From any point B draw a perpendicular BM; then the ratio of BM to BO is the *sine* of the angle AOB, which we may denote by $\frac{BM}{BO}$, meaning that if BM and BO be expressed in linear units, then the number of units in BM, divided by that in BO, gives the *number* which is called the sine of AOB.

The cosine, or sine of the complement, is thus deduced: Let the triangles OLK and OMB be equal in all respects, namely, KL to OM, &c.; then AOB and AOK make up a right angle, or AOK is the complement of AOB. Its sine is, therefore, $\frac{KL}{KO}$, or $\frac{OM}{OB}$, which is the *cosine* of AOB.

By the tangent of AOB is meant $\frac{BM}{MO}$

The cotangent of AOB is, therefore, $\frac{KL}{LO}$, or $\frac{MO}{MB}$

By the secant of AOB is meant $\frac{BO}{OM}$

The cosecant of AOB is, therefore, $\frac{KO}{OL}$, or $\frac{BO}{BM}$

By the versed sine of AOB is meant $\frac{BO-OM}{BO}$, or 1 — cosine of AOB

The coversed sine of AOB is, then, $\frac{KO-OL}{KO}$, or 1 — sine of AOB

(31.) When the angle is between one and two right angles, as A O B′, then let the same definitions be adopted, as follows (p. 15.):

	sine.	cosine.	tangent.	cotangent.	secant.	cosecant.
	$\frac{B'M'}{B'O}$	$\frac{OM'}{OB'}$	$\frac{B'M'}{OM'}$	$\frac{OM'}{B'M'}$	$\frac{B'O}{OM'}$	$\frac{B'O}{B'M}$
which is	pos.	neg.	neg.	neg.	neg.	pos.

Remember that O B and O B′, &c. have no sign. Lines only are considered as having signs + and —, which must be in one of two opposite directions.

When the angle is between two and three right angles, as A O B″ (positively measured), then we have for the

sine.	cosine.	tangent.	cotangent.	secant.	cosecant.
$\frac{B''M''}{B''O}$	$\frac{OM''}{OB''}$	$\frac{B''M''}{OM''}$	$\frac{OM''}{B''M''}$	$\frac{B''O}{OM''}$	$\frac{B''O}{B''M}$
neg.	neg.	pos.	pos.	neg.	neg.

When the angle is between three and four right angles, as A O B‴ (positively measured), then we have for the

sine.	cosine.	tangent.	cotangent.	secant.	cosecant.
$\frac{B'''M'''}{B'''O}$	$\frac{OM'''}{OB'''}$	$\frac{B'''M'''}{OM'''}$	$\frac{OM'''}{B'''M'''}$	$\frac{B'''O}{OM'''}$	$\frac{B'''O}{B'''M}$
neg.	pos.	neg.	neg.	pos.	neg.

The student should verify each of these assertions, which we shall proceed to systematise.

(32.) When an angle is less than a right angle, say it is in the first right angle; when between one and two right angles, say it is in the second right angle, &c. Now, remember the following table:

sine and cosecant	+ + — —
cosine and secant	+ — — +
tangent and cotangent	+ — + —

which will be found, on examination, to contain the results of the preceding articles thus. I wish to know whether the *sine*, in the *third* right angle, be positive or negative; repeat

sine and cosecant, plus, plus, *minus*, minus,

the *third* of which is *negative*, the answer required. Examine the preceding results, and see that this table contains them all.

(33.) Now, from the definitions, make it appear that the second column of assertions below is a translation of the first.

The side of a right angled triangle cannot exceed the hypothenuse.	No sine or cosine can exceed unity.
The hypothenuse of a right angled triangle cannot be less than a side.	No secant or cosecant can be less than unity.
Two lines, of any ratio whatsoever, may be the sides of a right angled triangle.	A tangent or cotangent may have any value whatsoever.

All this is relative to numerical magnitudes, independently of sign.

Corollary. Versed sines and coversed sines are always positive, and never exceed 2.

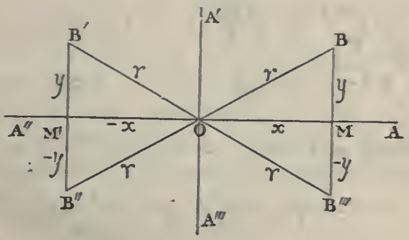
(34.) Now make the following abbreviations.

Let $\theta\,\Theta$ or $A.1^\circ$ be the angle.

Let sine of $\theta\,\Theta$	be written	$\sin\theta$	and read *sine* θ
cosine	....	$\cos\theta$	...... *cosine* θ
tangent	....	$\tan\theta$	...... *tangent* θ
cotangent	....	$\cot\theta$	...... *cotangent* θ
secant	....	$\sec\theta$	...... *secant* θ
cosecant	....	$\text{cosec}\,\theta$	...... *cosecant* θ
versed sine	....	$\text{vers}\,\theta$	...... *versed sine* θ
coversed sine	....	$\text{covers}\,\theta$	...... *coversed sine* θ

Similarly, let sine of $A.1^\circ$ be written $\sin A$, &c. Observe, that if $\theta\,\Theta$ and $A.1^\circ$ be the same angle, expressed in the two different units, then $\sin\theta = \sin A$, $\cos\theta = \cos A$, &c.; for it is obvious, that the angle itself has the same sine, &c. in whatever units it may be expressed.

(35.) *Fundamental properties implied in the definitions.*



Let U be the linear unit, and let O M and O M′ contain x linear units, the latter being marked $-x$, on account of the contrary direction of O M′. And, first, let A O B (in first right angle) $= \theta\Theta$;

$$\text{Then} \qquad \sin\theta \times \operatorname{cosec}\theta = \frac{y}{r} \times \frac{r}{y} = 1$$

$$\cos\theta \times \sec\theta = \frac{x}{r} \times \frac{r}{x} = 1$$

$$\tan\theta \times \cot\theta = \frac{y}{x} \times \frac{x}{y} = 1$$

Let A O B′ (in second right angle) $= \theta\Theta$; then, in the same way,

$$\sin\theta \times \operatorname{cosec}\theta = \frac{y}{r} + \frac{r}{y} = 1 \qquad \cos\theta \times \sec\theta = \frac{-x}{r} + \frac{r}{-x} = 1 \text{ \&c.}$$

so that these formulæ are universal, and we have the following consequences of definition.

The sine and cosecant are reciprocals.
The cosine and secant are reciprocals.
The tangent and cotangent are reciprocals.

We have also

$$\tan\theta = \frac{y}{x} = \frac{y \div r}{x \div r} = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{x}{y} = \frac{x \div r}{y \div r} = \frac{\cos\theta}{\sin\theta}$$

(36.) *Consequences of Euclid I. 47.* We have the equations (4.)

$$x^2+y^2=r^2,\ x^2+(-y)^2=r^2,\ (-x)^2+y^2=r^2,\ (-x)^2+(-y)^2=r^2$$

The first of which (and the rest may be treated in the same way) may be written in three different forms, namely,

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1 \quad \text{or} \quad \cos^2\theta + \sin^2\theta = 1$$

$$1 + \left(\frac{y}{x}\right)^2 = \left(\frac{r}{x}\right)^2 \quad \text{or} \quad 1 + \tan^2\theta = \sec^2\theta$$

$$1 + \left(\frac{x}{y}\right)^2 = \left(\frac{r}{y}\right)^2 \quad \text{or} \quad 1 + \cot^2\theta = \operatorname{cosec}^2\theta$$

$$\text{Hence} \quad \sin^2\theta = \frac{y^2}{x^2+y^2} = \frac{(y \div x)^2}{1+(y \div x)^2} = \frac{\tan^2\theta}{1+\tan^2\theta}$$

$$\cos^2\theta = \frac{x^2}{x^2+y^2} = \frac{1}{1+(y \div x)^2} = \frac{1}{1+\tan^2\theta}$$

or $\sin\theta = \dfrac{\tan\theta}{\sqrt{1+\tan^2\theta}}$ $\cos\theta = \dfrac{1}{\sqrt{1+\tan^2\theta}}$

Deduce also $\sin\theta = \dfrac{1}{\sqrt{1+\cot^2\theta}}$ $\cos\theta = \dfrac{\cot\theta}{\sqrt{1+\cot^2\theta}}$

To these we may add the equations of definition

$$\text{vers}\,\theta = 1 - \cos\theta, \qquad \text{covers}\,\theta = 1 - \sin\theta,$$

(37.) The student should now, as an exercise, express each of the primary functions in terms of every other, as in the following table, where the meaning of t in each column stands at the head, and the value of the expressions in each horizontal line at the beginning.

	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\text{cosec}\,\theta$
$\sin\theta =$	t	$\sqrt{1-t^2}$	$\frac{t}{\sqrt{1+t^2}}$	$\frac{1}{\sqrt{1+t^2}}$	$\frac{\sqrt{t^2-1}}{t}$	$\frac{1}{t}$
$\cos\theta =$	$\sqrt{1-t^2}$	t	$\frac{1}{\sqrt{1+t^2}}$	$\frac{t}{\sqrt{1+t^2}}$	$\frac{1}{t}$	$\frac{\sqrt{t^2-1}}{t}$
$\tan\theta =$	$\frac{t}{\sqrt{1-t^2}}$	$\frac{\sqrt{1-t^2}}{t}$	t	$\frac{1}{t}$	$\sqrt{t^2-1}$	$\frac{1}{\sqrt{t^2-1}}$
$\cot\theta =$	$\frac{\sqrt{1-t^2}}{t}$	$\frac{t}{\sqrt{1-t^2}}$	$\frac{1}{t}$	t	$\frac{1}{\sqrt{t^2-1}}$	$\sqrt{t^2-1}$
$\sec\theta =$	$\frac{1}{\sqrt{1-t^2}}$	$\frac{1}{t}$	$\sqrt{1+t^2}$	$\frac{\sqrt{1+t^2}}{t}$	t	$\frac{t}{\sqrt{t^2-1}}$
$\text{cosec}\,\theta =$	$\frac{1}{t}$	$\frac{1}{\sqrt{1-t^2}}$	$\frac{\sqrt{1+t^2}}{t}$	$\sqrt{1+t^2}$	$\frac{t}{\sqrt{t^2-1}}$	t

Thus, if we would know from the preceding the value of the cosecant in terms of the secant only, we find

$$\text{cosec}\,\theta = \frac{\sec\theta}{\sqrt{\sec^2\theta - 1}}$$

(38.) The next question is to determine what we may call the *transition* values of the primary functions, namely, those which they should have when the angle is *nothing*, or *one*, *two*, or *three* right angles exactly. For, in these cases, a look at the figure of (28.) will shew that the right angled triangle, which is the basis of the definitions, disappears altogether, so that, by any definition yet existing, there are no sines, cosines, &c. But, agreeably to the conventions of algebra, we shall use the following extensions as abbreviations (*Algebra*, p. 156.)

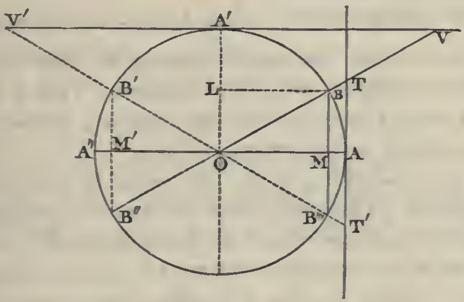
If, when x approaches without limit to a, X diminish without limit, let it be said that when $x = a$, $X = 0$: if, in such case, X approach without limit to A, let it be said that when $x = a$, $X = A$: and if X increase without limit, let it be said that when $x = a$ $X = \propto$, or is infinite.

(39.) According to these definitions, and observing what species of variation of magnitude each of the functions undergoes, we have the following table:

	0	1 right angle, or $\frac{\pi}{2}\Theta$ or 90.1°	2 right angles, or $\pi\Theta$ or 180.1°	3 right angles, or $\frac{3\pi}{2}\Theta$ or 270.1°
sine	0	1	0	-1
cosine	1	0	-1	0
tangent	0	$\propto$	0	$\propto$
cotangent	$\propto$	0	$\propto$	0
secant	1	$\propto$	-1	$\propto$
cosecant	$\propto$	1	$\propto$	-1
versed sine	0	1	2	1
coversed sine	1	0	1	2

The first three lines are the most important. The student may shew that as the angle approaches either of the four intermediate values, its sine, &c. approach the value in the table.

(40.) But the following method will be more clear. We shall determine *linear measures* of the sine, cosine, &c. in the same manner as the arc of a circle is the measure of an angle. Let there be any linear unit U, and with centre O, and radius $OA = U$ describe a circle, and let the angle OAB, or $\theta\Theta$, be that in question. The rest of the figure will need no description. Now, because $\sin\theta = \frac{BM}{BO}$ when BM and BO are expressed in units (let them be xU and U) we have $\sin\theta = x$, or $\sin\theta.U = BM$. Consequently, BM and the sine of θ change in the same proportions; and BM may stand for the sine of θ (as it might represent a sum of money, an area, or any other magnitude) being a linear measure of it, at the rate, to use a common phrase, of AO to a *unit*. Or let OA be one *measure*, then the fraction of a measure in BM is the sine of the angle AOB, and the fraction of a measure contained in the arc AB, is the fraction of an analytical unit contained in the angle AOB. In precisely a similar manner, OM is a linear measure of the cosine. But when we come to consider the tangent, we see that BM is not a measure of it: for, if the angle increase, the change of the tangent is the



combined effect upon the ratio of a simultaneous increase of BM, and decrease of MO. But both are represented in the increase of the line AT; for, by similar triangles, BM : MO :: AT : AO, of which AO remains the same, and AT changes in the same ratio as the tangent. Hence, AT is a linear measure of the tangent of AOB. Similarly, since OM : MB :: A′V : A′O, or AO, A′V is a linear measure of the cotangent; and, since BO : OM :: TO : OA, then TO is a linear measure of the secant. Similarly, OB : BM :: VO : OA′, or VO is a linear measure of the cosecant. Again AM, the linear measure of $1-\cos\theta$ and A′L, that of $1-\sin\theta$, are those of vers θ and covers θ. Consequently, understanding by the linear unit U the radius OA, and also that the question "How many times?" also implies "What fraction of a time?" and "What times and fraction of a time?" we have the following list of synonymous questions, in which the blanks may be filled up out of the same horizontal line.

What is the ( ) of the angle AOB?	How many linear units are contained in ( )?
sine	BM
cosine	OM
tangent	AT

What is the ( ) of the angle AOB?	How many linear units are contained in ( )?
cotangent	A′V
secant	OT
cosecant	OV
versed sine	AM
coversed sine	A′L
number of analytical units	the arc AB.

The student may now readily construct the table in (39.) by observing the final states of the linear measures. Thus, as the angle diminishes without limit, the line A′V increases without limit, or (38.) the cotangent of 0 is infinite; and so on.

(41.) (Now, according to the old system of trigonometry, not yet exploded for the beginner, though every person must practically get rid of it before he can advance far in analysis, the line BM is the sine, and not of the angle, but of the arc AB. In this system, there is an infinite number of sines to the same angle, corresponding to all the arcs which that angle can subtend; and, consequently, it is always with reference to the radius supposed that all formulæ must be constructed. For example, it is not true that

$$\sin^2\theta + \cos^2\theta = 1 \quad \text{unless when the radius is 1}$$

but $\sin^2\theta + \cos^2\theta = r^2$ where r is the number of linear units in the radius. This embarrassing consideration is always avoided in practice by making the radius the linear unit, and then substituting for the lines called sines, &c. their numerical proportions to the radius: which amounts in fact to the more modern method).

(42.) We are now to consider, in connection with θ, the angles which exceed or fall short of any whole number of right angles by θ, or which are contained in the following series.

$$\theta - \frac{3\pi}{2} \qquad \theta - \pi \qquad \theta - \frac{\pi}{2} \qquad \theta \qquad \theta + \frac{\pi}{2} \qquad \theta + \pi \text{ \&c.}$$

$$-\frac{3\pi}{2} - \theta \qquad -\pi - \theta \qquad -\frac{\pi}{2} - \theta \qquad -\theta \qquad -\theta + \frac{\pi}{2} \qquad -\theta + \pi \text{ \&c.}$$

all contained in the form $\pm m\frac{\pi}{2} \pm \theta$, where m is a whole number.

But, in the first place, we must observe that any addition or subtraction of four right angles, or multiples of four right angles, produces no change in the position of OB (29.) being, in fact, equivalent to supposing complete revolutions, one or more, to have taken place, leaving O B the same in position as before. The sines, cosines, &c. depending entirely upon the position of O B, and in no way upon the number of revolutions supposed in attaining that position, we must have (2π being the numerical symbol of four right angles)

$$\sin\theta = \sin(2\pi+\theta) = \sin(4\pi+\theta) = \sin(6\pi+\theta) \text{ \&c.}$$
$$= \sin(\theta-2\pi) = \sin(\theta-4\pi) = \sin(\theta-6\pi) \text{ \&c.}$$

and generally, F representing the operation by which we pass from an angle to any primary function (*Algebra*, p. 203.) we must have

$$F(\theta) = F(\theta+2m\pi)$$

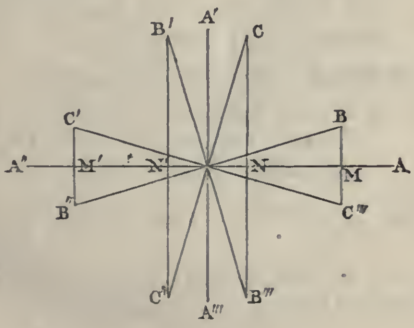
where m is any whole number, positive or negative.

Hence, we may reduce the list in the last page: for, we find that $F\left(\theta-\frac{\pi}{2}\right)$ is the same as $F\left(2\pi+\theta-\frac{\pi}{2}\right)$ or $F\left(\frac{3\pi}{2}+\theta\right)$.

We shall limit ourselves first, to the consideration of the following,

$$\theta \qquad \frac{\pi}{2}\pm\theta \qquad \pi\pm\theta \qquad \frac{3\pi}{2}\pm\theta \qquad 2\pi-\theta$$

omitting $2\pi+\theta$, because its primary functions are those of θ. And, first, let θ be less than a right angle, so that $m\frac{\pi}{2}+\theta$ must fall in the $(m+1)$th right angle, and $m\frac{\pi}{2}-\theta$ in the mth. We have called $\frac{\pi}{2}-\theta$ the complement of θ; it is also usual to call $\pi-\theta$ the supplement of θ; being, when θ is less than two right angles, the *adjacent* angle of Euclid. Now, draw the following figure, making θ a small angle for convenience.



O is at the centre (not marked), let the angles AOB, AOC‴, A′OB′, A′OC, A″OB″, A″OC′, A‴OB‴, A‴OC″ be all equal to each other, and to $\theta\Theta$. Let the triangles MOB, MOC‴, NOC, NOB‴, N′OB′, N′OC″, M′OC′, M′OB″ be all made equal to each other in every respect; namely, ON to MB, OM to NC, &c. &c. Then we have (all angles being measured positively)

$$\angle AOB = \theta\Theta \qquad \angle AOB' = \left(\frac{\pi}{2}+\theta\right)\Theta$$

$$\angle AOB'' = (\pi+\theta)\Theta \qquad \angle AOB''' = \left(3\frac{\pi}{2}+\theta\right)\Theta$$

$$\angle AOC = \left(\frac{\pi}{2}-\theta\right)\Theta \qquad \angle AOC' = (\pi-\theta)\Theta$$

$$\angle AOC'' = \left(3\frac{\pi}{2}-\theta\right)\Theta \qquad \angle AOC''' = (2\pi-\theta)\Theta$$

From hence we can immediately find any primary function in terms of a primary function of $\theta\Theta$, as follows. Suppose it required to find $\cot\left(3\frac{\pi}{2}-\theta\right)$: we have immediately

$$\cot(AOC'') = \frac{N'O}{C''N'} = \frac{BM\,(\text{with contrary sign})}{OM\,(\text{with contrary sign})} = +\frac{BM}{OM}$$

or $$\cot\left(3\frac{\pi}{2}-\theta\right) = \tan\theta$$

(43.) Now, in this investigation, there are 42 cases, but they all fall under the following rules for expressing a function of $m\frac{\pi}{2}\pm\theta$ by means of a function of θ. Let $F\left(m\frac{\pi}{2}+\theta\right)$ be required.

1. *If* m *be odd,* change F into its co-function; namely, sine into cosine, cosine into sine;* tangent into cotangent, cotangent into tangent, &c.: *if* m *be even,* let F remain.

2. Look at the *scale of signs* (32.) of F, namely,

for	sine and cosecant	+ + − −
	cosine and secant	+ − − +
	tangent and cotangent	+ − + −

and, observing in which right angle $m\frac{\pi}{2}\pm\theta$ falls, prefix the

* According to our definitions (30.) the co-cosine means the cosine of the complement, or the sine.

sign which answers to the number of that right angle in the scale.

For instance, for $\cot\left(3\frac{\pi}{2}-\theta\right)$, the number of right angles is *odd*, or we write *tan* for *cot*: and $3\frac{\pi}{2}-\theta$ is in the third right angle; $\left(cot\ \overset{+}{1}\ \overset{-}{2}\ \overset{+}{3}\right)$ the proper sign is $+$

$$\cot\left(3\frac{\pi}{2}-\theta\right) = \tan\theta$$

Let the student go through the cases *from the figure*, and satisfy himself that they agree with this rule.

(44.) The following are some results. The first set is in the definitions.

$$\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta \qquad \sin\left(\frac{\pi}{2}+\theta\right) = \cos\theta \qquad \sin(\pi-\theta) = \sin\theta$$

$$\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta \qquad \cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta \qquad \cos(\pi-\theta) = -\cos\theta$$

$$\tan\left(\frac{\pi}{2}-\theta\right) = \cot\theta \qquad \tan\left(\frac{\pi}{2}+\theta\right) = -\cot\theta \qquad \tan(\pi-\theta) = -\tan\theta$$

$$\sin(\pi+\theta) = -\sin\theta \qquad \sin\left(\frac{3\pi}{2}-\theta\right) = -\cos\theta \qquad \sin\left(\frac{3\pi}{2}+\theta\right) = -\cos\theta$$

$$\cos(\pi+\theta) = -\cos\theta \qquad \cos\left(\frac{3\pi}{2}-\theta\right) = -\sin\theta \qquad \cos\left(\frac{3\pi}{2}+\theta\right) = \sin\theta$$

$$\tan(\pi+\theta) = \tan\theta \qquad \tan\left(\frac{3\pi}{2}-\theta\right) = \cot\theta \qquad \tan\left(\frac{3\pi}{2}+\theta\right) = -\cot\theta$$

$$\sin(2\pi-\theta) = -\sin\theta \qquad \sin(-\theta) = -\sin\theta$$

$$\cos(2\pi-\theta) = \cos\theta \qquad \cos(-\theta) = \cos\theta$$

$$\tan(2\pi-\theta) = -\tan\theta \qquad \tan(-\theta) = -\tan\theta$$

The last set is deduced from that immediately preceding, by subtracting four right angles (42.). They may be deduced immediately, by observing that $MOC''' = -\theta\Theta$ (29.).

To obtain versed and coversed sines, remember that

$$\text{vers}\,\theta = 1-\cos\theta, \qquad \text{covers}\,\theta = 1-\sin\theta$$

Thus, $\text{covers}\left(3\frac{\pi}{2}-\theta\right) = 1-\sin\left(\frac{3\pi}{2}-\theta\right) = 1+\cos\theta$

(45.) When $\theta\Theta$ is greater than a right angle, the results are the same as if it were less than a right angle. An easier demonstration will afterwards apply; in the meantime, suppose $\theta = \pi + \theta'$, and we want, for instance, $\cot\left(3\frac{\pi}{2} - \theta\right)$. We have then,

$$3\frac{\pi}{2} - \theta = \frac{\pi}{2} - \theta' \qquad \cot\left(\frac{3\pi}{2} - \theta\right) = \cot\left(\frac{\pi}{2} - \theta'\right) = \tan\theta'$$

But $$\tan\theta' = \tan(\pi + \theta') = \tan\theta;$$

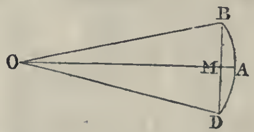
therefore, $$\cot\left(3\frac{\pi}{2} - \theta\right) = \tan\theta$$

the same as before. Let the student work a number of instances of this kind.

By means of the last set of the preceding formulæ, we can easily ascertain $F\left(\theta - m\frac{\pi}{2}\right)$. Suppose $\tan(\theta - \pi)$ is required; then we have

$$\tan(\theta - \pi) = -\tan(\pi - \theta) = -(-\tan\theta) = \tan\theta$$

(46.) We shall now proceed to some theorems connected with the limits of the ratios of trigonometrical functions, (*Alg.* p. 162.).



The ratio of an angle (in analytical units) to its sine, approximates without limit to unity, when the angle is diminished without limit. Let AOB, AOD, be equal angles; then $BM = MD$, arc $AB =$ arc AD; and, AOB being $\theta\Theta$; or, $\frac{AB}{AO}$ being θ, we have

$$\theta : \sin\theta :: \frac{AB}{AO} : \frac{BM}{AO} :: AB : BM$$
$$:: 2AB : 2BM :: \text{arc } BD : \text{chord } BD$$

Let the chord BD be the side of an inscribed polygon of n sides; then the greater n is taken, the less does the whole boundary of the polygon (which is $n \times$ chord BD in length) differ from the circumference of the circle (which is $n \times$ arc BD). Let $n \times$ arc BD $=$ $n \times$ chord BD $+ Z$; then can the length Z be made as small as we please, by taking n sufficiently great.

But chord BD $= \frac{\sin\theta}{\theta}$ arc BD, and substitution gives

$$n \times \text{arc BD} - n \times \frac{\sin\theta}{\theta} \times \text{arc BD} = Z,$$

or $$\left(1 - \frac{\sin\theta}{\theta}\right) \times n \times \text{arc BD} = Z$$

that is, $$\left(1 - \frac{\sin\theta}{\theta}\right) \times \text{circumference} = Z$$

But as n increases without limit, the angle BOD, and, therefore, BOM, diminishes without limit; and since, in such a case, Z also diminishes without limit, the fraction $1 - \frac{\sin\theta}{\theta}$ diminishes without limit, the circumference being always the same. Hence, $\frac{\sin\theta}{\theta}$ approaches without limit to unity. Let the student try to demonstrate this in the manner of (4.), supposing θ and $\sin\theta$ to be incommensurable.

The angle of 5 degrees, which is not for any practical purpose a small angle, and which, in analytical units, is ·0872665, has, for its sine, ·0871557; which gives,

$$\frac{\sin(\cdot 0872665)}{\cdot 0872665} = \frac{\cdot 0871557}{\cdot 0872665} = \frac{872}{873} \text{ very nearly.}$$

or, when AOB is the eighteenth part of a right angle, if the arc BA were divided into 800 equal parts, BM would be more than 799 of these parts.

(47.) As the angle diminishes without limit, the cosine approaches without limit to unity; and $1 - \cos\theta$ diminishes without limit, as also does $\sin\theta$. It will be necessary to examine the ratio of $1 - \cos\theta$ to $\sin\theta$ under this change.

$$\frac{1 - \cos\theta}{\sin\theta} = \frac{(1 - \cos\theta)(1 + \cos\theta)}{\sin\theta\ (1 + \cos\theta)} = \frac{\sin^2\theta}{\sin\theta(1 + \cos\theta)} = \frac{\sin\theta}{1 + \cos\theta}$$

of which the numerator diminishes without limit, while the denominator increases with the limit $1 + 1$ or 2. The fraction, therefore, diminishes without limit, or $\frac{1 - \cos\theta}{\sin\theta}$ diminishes without limit.

When θ is 5 degrees, its sine and cosine are ·0871557 and ·9961947 very nearly. Whence,

$$\frac{1 - \cos\theta}{\sin\theta} = \frac{\cdot 0038053}{\cdot 0871557} < \frac{4}{87}$$

or, for this angle, $1 - \cos\theta$ is less than the twentieth part of $\sin\theta$.

If we take 5 minutes instead of 5 degrees (5 minutes in analytical units being ·0014544, its sine the same to seven places of decima's, and its cosine ·9999989), we find, that if θ were divided into 14000 equal parts, the sine would not be less by so much as one of those parts, and that $1-\cos\theta$ is not the 1300th part of $\sin\theta$.

(48.) Again, since $\frac{\tan\theta}{\sin\theta}=\frac{1}{\cos\theta}$, the limit is unity when θ diminishes without limit; and since $\frac{\tan\theta}{\theta}=\frac{\tan\theta}{\sin\theta}\cdot\frac{\sin\theta}{\theta}$, the limit of this, in the same circumstance, is 1×1, or 1. And, because

$$\frac{1-\cos\theta}{\theta}=\frac{1-\cos\theta}{\sin\theta}\cdot\frac{\sin\theta}{\theta}$$

therefore also $\frac{1-\cos\theta}{\theta}$ diminishes without limit at the same time as θ.

(49.) We shall now propose, as a problem, the solution of the equation

$$\sin x = \sin y$$

or rather, to find certain solutions; for we have no means as yet of ascertaining that any given number of solutions is the total number. Looking among the results of (44.), we find the following solutions, premising, first, that $x=y$ is one solution: an angle has but one sine.

$$x=y\pm 2\pi \quad x=y\pm 4\pi \quad x=y\pm 6\pi, \text{ \&c.}$$
$$x=\pi-y \quad x=(\pi-y)\pm 2\pi=3\pi-y \text{ or } -\pi-y$$
$$x=(\pi-y)\pm 4\pi=5\pi-y \text{ or } -3\pi-y, \text{ \&c.}$$

We now propose $\tan mx=\cot ny$. Since $\cot ny=\tan\left(\frac{\pi}{2}-ny\right)$ we have, firstly, $mx=\frac{\pi}{2}-ny$, or $x=\frac{1}{m}\left(\frac{\pi}{2}-ny\right)$. We have, also,

$$mx=\frac{\pi}{2}-ny\pm 2\pi \qquad mx=\frac{\pi}{2}-ny\pm 4\pi, \text{ \&c.}$$

And since $\tan x=\tan(x\pm\pi)$, the following are also solutions:

$$mx=\frac{\pi}{2}-ny\pm\pi \text{ or } \frac{\pi}{2}-ny\pm 3\pi, \text{ \&c.}$$

(50.) The following propositions will be readily proved, especially from the figure in (40.). In the same right angle there are no two sines, or cosines, or tangents, &c., which are equal to each other.

And of angles which do not exceed four right angles, there are two to every sine (or cosecant), and x being one, $\pi - x$ is the other; two to every cosine (or secant), and x being one, $2\pi - x$ is the other; two to every tangent (or cotangent), and x being the lesser, $\pi + x$ is the greater.

Now, x being less than two right angles, so is $\pi - x$; but $\pi + x$ and $2\pi - x$ are greater than two right angles. Consequently, where there is question of the angles of a triangle, the cosine of an angle (or secant), or the tangent (or cotangent), being given, the angle is absolutely determined; for there is but one angle which is contained within the limits of the angles of a triangle (0 and π), to which such cosine, &c. can belong. But, when the sine of an angle is given, or found, as that by which the angle of a triangle is to be determined, there may be two angles within the limits of the problem; for if x be one answer, $\pi - x$ is another.

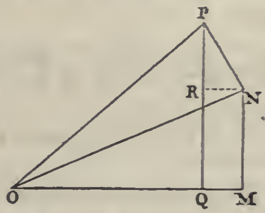
CHAPTER II.

FORMULÆ CONNECTED WITH TWO OR MORE ANGLES.

(51.) The doctrine of ratios, as in Euclid, presents the notions which answer to multiplication, division, raising of powers, extraction of roots; every fundamental operation, in fact, *except only addition and subtraction.* The truth is, that it considers two ratios just as two lines are treated in the first book; that is, as subject to the relations of greater, equal, and less, but without any classification or comparison of the various modes of greater and less. From the definitions, it appears that the ratio of X + Y to Z is more than the ratio of X to Z; and here the Fifth Book stops. But ratio is a magnitude; we apply the words greater, &c. to two ratios. It is true that the definition of ratio looks more like that of a criterion than of a magnitude; but, as we have seen, the word angle is in the same predicament: the definition of opening, or inclination, is only a rough primary conception, while the useful definition of an angle is the criterion which determines the greater or less, or the equality, of two angles. The rough conception of ratio is *relative magnitude;* that notion, by which a spectator who knew nothing about numbers, would decide whether the picture of a known object was in or out of proportion; that notion, by aid of which savages, who have as little idea of numbers as it is possible for a human being to have, comprehend a map as soon as it is shewn to them, and point out the various sites which they know, as soon as they know whereabouts in the map they are for the time, and the direction of north or south upon it. With this notion comes the following: That the relative magnitudes of X and Y to Z, make up the relative magnitude of X + Y to Z: whence, we subsequently come to the general definition of addition of ratios; namely, that to add the ratios of A to B and C to D, reduce both to other ratios having the same consequent, say X to Z, and Y to Z; then the sum of the preceding ratios is that of X + Y to Z.

We have introduced these considerations again, in order to point out the difference between geometry and algebra, which the following question, being the fundamental proposition of the present chapter, will exemplify. Given the primary functions of two angles, of which the sum is less than a right angle, required the primary functions of their sum. The geometrical solution is as follows:

The first angle being less than a right angle, take any straight



line O M, and erect M N, so that the ratio of N M to M O shall be the tangent of the first angle; then will N O M be the angle in question. Then draw N P perpendicular to O N of such length that the ratio of P N to N O shall be the given tangent of the second angle; whence P O N is the second angle, and M O P is the sum of the angles. Draw P Q perpendicular to O M, then is the ratio of P Q to P O the sine of the sum required, &c. This geometrical construction is a complete solution within the meaning of the terms *geometrical solution*, with regard to which it is matter of definition that lengths are determined, found, or given, when the extreme points are given. But it is not an algebraical solution, of which it is a condition that no magnitude is given, determined, or found, unless its ratio to some given magnitude of the same kind be given, &c. The geometrical solution is the more easy, because it assumes the harder point, and requires only determination of position; the algebraical solution, which requires ratios, carries the geometrical solution further, and demands consequences with which the geometrical solution, by an express definition of exclusion, has nothing to do. And the student will do well to remember this when he comes to read controversy about the relative value of algebraical and geometrical solutions.

The algebraical solution is as follows, without symbolic language. Draw N R parallel to O M. Then, since P Q is made up of P R and N M, the ratio P Q : P O is the sum of P R : P O and N M : PO. But P R : P O is compounded of P R : P N and P N : P O, or by similar triangles, of OM : ON and PN : PO, and being compounded of given ratios, may be expressed by whatever symbol we adopt to signify composition. Similarly, NM : PO is compounded of NM : NO and NO : PO, two given ratios. Under the common meaning of terms in algebra, we may, if all the pairs be commensurable, and if OP stand for the number of linear units in OP, &c. we proceed thus: let NOM $= \theta\,\Theta$, MOP $= \varphi\,\Theta$, then MOP $= (\varphi + \theta)\,\Theta$, and we have

$$\sin(\varphi+\theta) = \frac{PQ}{PO} = \frac{PR}{PO} + \frac{NM}{PO} = \frac{PR}{PN}\cdot\frac{PN}{PO} + \frac{NM}{NO}\cdot\frac{NO}{PO}$$

$$= \frac{MO}{NO}\cdot\frac{PN}{PO} + \frac{NM}{NO}\cdot\frac{NO}{PO} = \cos\theta\sin\varphi + \sin\theta\cos\varphi$$

If the ratios be incommensurable, we must either, 1. Imagine commensurables very nearly equal to MO, &c. to be substituted, and the real meaning of the equation will then be (meaning by (a) a very near approximation to a), $(\cos\theta)\,(\sin\varphi) + (\sin\theta)(\cos\varphi)$ is very near to $\sin(\varphi+\theta)$; or, 2. adopt the more general ideas of ratio in the preliminary treatise, and interpret the symbols of operation accordingly. Leaving the student to take which course he can, we now proceed, having obtained in every sense of the terms

$$\sin(\varphi+\theta) = \sin\varphi\cos\theta + \cos\varphi\sin\theta$$

similarly $\cos(\varphi+\theta) = \cos\varphi\cos\theta - \sin\varphi\sin\theta$ as follows:

$$\cos(\varphi+\theta) = \frac{OQ}{OP} = \frac{OM-NR}{OP} = \frac{OM}{ON}\cdot\frac{ON}{OP} - \frac{NR}{NP}\cdot\frac{NP}{NO}\ \&c.$$

Now, construct a figure in which $\varphi\,\Theta$ and $\theta\,\Theta$ are each less than a right angle, but their sum greater; shew that the process for the sine remains exactly the same, and that in that for the cosine of the sum, which is negative, OM — RN also becomes negative; whence we still have $\cos(\varphi+\theta)$ *with its sign* $=$ (OM — RN) $\div$ OP, and the two formulæ are precisely as before. Shew also that by aid of

$$\cos^2\varphi + \sin^2\varphi = 1 \qquad \cos^2\theta + \sin^2\theta = 1$$

the sum of the squares of the preceding developements is $= 1$.

(52.) Since these formulæ are universally true, independently of all values of the angles, *within a right angle* (as far as we know yet) they will remain true if instead of φ, we write $\varphi - \theta$. Do this, which gives

$$\sin\varphi = \sin(\varphi-\theta)\cos\theta + \cos(\varphi-\theta)\cdot\sin\theta$$
$$\cos\varphi = \cos(\varphi-\theta)\cos\theta - \sin(\varphi-\theta)\sin\theta$$

Multiply the first by $\cos\theta$, and subtract the second multiplied by $\sin\theta$, remembering that $\cos^2\theta + \sin^2\theta = 1$, which gives, transposing the sides,

$$\sin(\varphi-\theta) = \sin\varphi\cos\theta - \cos\varphi\sin\theta$$
$$\cos(\varphi-\theta) = \cos\varphi\cos\theta + \sin\varphi\sin\theta$$

Which proves that the two first formulæ are true when one of the angles is negative. In $\sin(\varphi+\theta)$ write $-\theta$ for θ, and we have

$$\sin(\varphi-\theta) = \sin\varphi\cos(-\theta) + \cos\varphi\sin(-\theta)$$

or (44.) $\qquad = \sin\varphi\cos\theta - \cos\varphi\sin\theta$, as just proved.

Endeavour to deduce these propositions from an adaptation of the construction in (51.), and verify the value of the sum of the squares as before.

(53.) We now shew that these formulæ remain true whatever may be the magnitude of the angles. We shall take a case, and recommend the student to acquire dexterity in the management of the formulæ, by trying various others. Let us suppose our first angle to be in the third right angle, and our second in the fourth, so that the sum must be in the sixth at least, or in the seventh. Let $\pi+\theta$, and $\frac{3\pi}{2}+\varphi$ be the analytical units in the angles, and θ and φ must therefore be severally less than a right angle. Then the sum is $\frac{5\pi}{2}+\theta+\varphi$, or $2\pi+\frac{\pi}{2}+\theta+\varphi$, and, therefore, its sine is (42.)

$$\sin\left(\frac{\pi}{2}+\overline{\theta+\varphi}\right) \text{ or } \cos(\theta+\varphi) \text{ or } \cos\theta\cdot\cos\varphi - \sin\theta\cdot\sin\varphi \quad .. \text{ (A)}$$

But
$$\sin(\pi+\theta) = -\sin\theta \quad \text{or} \quad \sin\theta = -\sin(\pi+\theta)$$
$$\cos(\pi+\theta) = -\cos\theta \quad .. \quad \cos\theta = -\cos(\pi+\theta)$$
$$\sin\left(\frac{3\pi}{2}+\phi\right) = -\cos\phi \quad .. \quad \cos\phi = -\sin\left(\frac{3\pi}{2}+\phi\right)$$
$$\cos\left(\frac{3\pi}{2}+\phi\right) = \sin\phi \quad .. \quad \sin\phi = \cos\left(\frac{3\pi}{2}+\phi\right)$$

and, by substitution in (A), we have

$$\sin\left(\pi+\theta+3\frac{\pi}{2}+\phi\right) = -\cos(\pi+\theta)\times -\sin\left(\frac{3\pi}{2}+\phi\right) - \left(-\sin\overline{\pi+\theta}\right)\times\cos\left(\frac{3\pi}{2}+\phi\right)$$

$$\sin\left(\pi+\theta+3\frac{\pi}{2}+\phi\right)=\cos(\pi+\theta)\sin\left(\frac{3\pi}{2}+\phi\right)+\sin(\pi+\theta)\cos\left(\frac{3\pi}{2}+\phi\right)$$

let $\pi+\theta=\theta'$, $\frac{3\pi}{2}+\phi=\phi'$, and we have

$\sin(\theta'+\phi')=\cos\theta'\sin\phi'+\sin\theta'\cos\phi'$, the same as before.

We have then as general formulæ, true for all angles, positive and negative,

$$\sin(\phi+\theta)=\sin\phi\cos\theta+\cos\phi\sin\theta \qquad \sin(\phi-\theta)=\sin\phi\cos\theta-\cos\phi\sin\theta$$
$$\cos(\phi+\theta)=\cos\phi\cos\theta-\sin\phi\sin\theta \qquad \cos(\phi-\theta)=\cos\phi\cos\theta+\sin\phi\sin\theta$$

(54.) We can now, with very slight labour, acquire a large number of very useful formulæ so quickly, that previous description will be unnecessary.

$$\sin(\phi+\theta)+\sin(\phi-\theta) = 2\sin\phi\cos\theta$$
$$\sin(\phi+\theta)-\sin(\phi-\theta) = 2\cos\phi\sin\theta$$
$$\cos(\phi+\theta)+\cos(\phi-\theta) = 2\cos\phi\cos\theta$$
$$\cos(\phi+\theta)-\cos(\phi-\theta) = -2\sin\phi\sin\theta$$

Since these are always true, we may for ϕ and θ write $\frac{1}{2}(\phi+\theta)$ and $\frac{1}{2}(\phi-\theta)$. Do this, which gives

$$\sin\phi+\sin\theta = 2\sin\tfrac{1}{2}(\phi+\theta)\cos\tfrac{1}{2}(\phi-\theta)$$
$$\sin\phi-\sin\theta = 2\cos\tfrac{1}{2}(\phi+\theta)\sin\tfrac{1}{2}(\phi-\theta)$$
$$\cos\phi+\cos\theta = 2\cos\tfrac{1}{2}(\phi+\theta)\cos\tfrac{1}{2}(\phi-\theta)$$
$$\cos\phi-\cos\theta = -2\sin\tfrac{1}{2}(\phi+\theta)\sin\tfrac{1}{2}(\phi-\theta)$$

$$\frac{\sin\phi-\sin\theta}{\sin\phi+\sin\theta}=\frac{\tan\frac{1}{2}(\phi-\theta)}{\tan\frac{1}{2}(\phi+\theta)} \qquad \frac{\sin\phi+\sin\theta}{\cos\phi+\cos\theta}=\tan\tfrac{1}{2}(\phi+\theta)$$

(55.) Now, from $2\theta=\theta+\theta$, and from $\sin(\theta+\theta)$ &c. deduce

$$\sin 2\theta = 2\sin\theta\cos\theta \qquad \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$\cos 2\theta = \cos^2\theta-\sin^2\theta \qquad \cos\theta = \cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}$$

$$= 1-2\sin^2\theta \qquad = 1-2\sin^2\frac{\theta}{2}$$

$$= 2\cos^2\theta-1 \qquad = 2\cos^2\frac{\theta}{2}-1$$

$$1+\cos 2\theta = 2\cos^2\theta \qquad 1+\cos\theta = 2\cos^2\frac{\theta}{2}$$

$$1-\cos 2\theta = 2\sin^2\theta \qquad 1-\cos\theta = 2\sin^2\frac{\theta}{2}$$

$$\tan^2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta} \qquad \tan^2\frac{\theta}{2} = \frac{1-\cos\theta}{1+\cos\theta}$$

$$\cot^2\theta = \frac{1+\cos 2\theta}{1-\cos 2\theta} \qquad \tan^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right) = \frac{1-\sin\theta}{1+\sin\theta}$$

(56.) $\tan(\phi+\theta) = \dfrac{\sin(\phi+\theta)}{\cos(\phi+\theta)} = \dfrac{\sin\phi.\cos\theta + \cos\phi\sin\theta}{\cos\phi\ \cos\theta - \sin\phi\sin\theta}$

divide both terms of the last fraction by $\cos\phi.\cos\theta$

$$\tan(\phi+\theta) = \frac{\tan\phi+\tan\theta}{1-\tan\phi\tan\theta};$$

similarly, $$\tan(\phi-\theta) = \frac{\tan\phi-\tan\theta}{1+\tan\phi\tan\theta};$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} \qquad \tan\theta = \frac{2\tan\frac{1}{2}\theta}{1-\tan^2\frac{1}{2}\theta}$$

$$\tan\theta+\tan\phi = \frac{\sin(\theta+\phi)}{\cos\theta\cos\phi} \qquad \tan\theta-\tan\phi = \frac{\sin(\theta-\phi)}{\cos\theta.\cos\phi}$$

(57.) The formulæ in (54.) reduce multiplication to addition in a way which may remind us of logarithms, and we shall see more of the same sort of analogy before we have finished. They give,

$$\sin\phi\cos\theta = \tfrac{1}{2}\sin(\phi+\theta) + \tfrac{1}{2}\sin(\phi-\theta)$$
$$\cos\phi\sin\theta = \tfrac{1}{2}\sin(\phi+\theta) - \tfrac{1}{2}\sin(\phi-\theta)$$
$$\cos\phi\cos\theta = \tfrac{1}{2}\cos(\phi+\theta) + \tfrac{1}{2}\cos(\phi-\theta)$$
$$\sin\phi\sin\theta = \tfrac{1}{2}\cos(\phi-\theta) - \tfrac{1}{2}\cos(\phi+\theta)$$

Let it be required to reduce the product $\cos m\cos n\cos p$;

$\cos m\ \cos n\ \cos p = \frac{1}{2}\cos(m+n)\,.\cos p + \frac{1}{2}\cos(m-n)\cos p =$ $\frac{1}{4}\cos(m+n+p) + \frac{1}{4}\cos(m+n-p) + \frac{1}{4}\cos(p+m-n) +$ $\frac{1}{4}\cos(p+n-m)$, by applying the same formulæ twice.

(58.) We may apply this method to ascertain the nth power of a sine or cosine in terms of the sines and cosines of the multiples of the angle, as follows; by (55.)

$\cos^2\theta = \frac{1}{2}+\frac{1}{2}\cos 2\theta \qquad \cos^3\theta = \frac{1}{2}\cos\theta + \frac{1}{2}\cos 2\theta\cos\theta$
$= \frac{1}{2}\cos\theta + \frac{1}{2}(\frac{1}{2}\cos 3\theta + \frac{1}{2}\cos\theta) \qquad (2\theta+\theta = 3\theta,\ 2\theta-\theta=\theta)$
$= \frac{3}{4}\cos\theta + \frac{1}{4}\cos 3\theta$ or $4\cos^3\theta = 3\cos\theta + \cos 3\theta$

Multiply by $2\cos\theta$ (we thus avoid fractions)

$$8\cos^4\theta = 6\cos^2\theta + 2\cos 3\theta\cos\theta$$
$$= 3+3\cos 2\theta + \cos 4\theta + \cos 2\theta = 3+4\cos 2\theta + \cos 4\theta$$

Proceeding in this way we get the following set of equations:

$$\cos\theta = \cos\theta$$
$$2\cos^2\theta = \cos 2\theta + 1$$
$$4\cos^3\theta = \cos 3\theta + 3\cos\theta$$
$$8\cos^4\theta = \cos 4\theta + 4\cos 2\theta + 3$$
$$16\cos^5\theta = \cos 5\theta + 5\cos 3\theta + 10\cos\theta$$
$$32\cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10$$
$$64\cos^7\theta = \cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta$$
$$128\cos^8\theta = \cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35$$

&c. &c. &c. &c. &c.

Again, $2\sin^2\theta = 1 - \cos 2\theta$, $4\sin^3\theta = 2\sin\theta - 2\cos 2\theta . \sin\theta$

(54.) $(2\cos 2\theta . \sin\theta = \sin 3\theta - \sin\theta)$ $\quad = 2\sin\theta - \sin 3\theta + \sin\theta$

$= 3\sin\theta - \sin 3\theta$

$$8\sin^4\theta = 6\sin^2\theta - 2\sin 3\theta . \sin\theta$$
$$= 3 - 3\cos 2\theta - (\cos 2\theta - \cos 4\theta) = 3 - 4\cos 2\theta + \cos 4\theta$$

Proceeding in this way, we get equations which may be thus most systematically arranged:

$$\sin\theta = \sin\theta$$
$$-2\sin^2\theta = \cos 2\theta - 1$$
$$-4\sin^3\theta = \sin 3\theta - 3\sin\theta$$
$$8\sin^4\theta = \cos 4\theta - 4\cos 2\theta + 3$$
$$16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$$
$$-32\sin^6\theta = \cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10$$
$$-64\sin^7\theta = \sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta$$
$$128\sin^8\theta = \cos 8\theta - 8\cos 6\theta + 28\cos 4\theta - 56\cos 2\theta + 35$$

&c. &c. &c. &c. &c.

Between these two sets there are strong resemblances and strong differences. It appears that the cosine is a much more simple function, in its relations with other cosines, than is the sine in relation to other sines. The alternation of positive and negative signs, *in pairs*, here occurs for the first time. We shall now shew how to form the inverse expressions, namely, $\cos n\theta$, &c. in terms of powers of $\cos\theta$, &c.

(59.) By (55.) we have

$$\cos 2\theta = \cos^2\theta - \sin^2\theta \qquad \sin 2\theta = 2\sin\theta . \cos\theta$$

Let the sine and cosine of θ be denoted by s and c.

Then $\cos 2\theta = c^2 - s^2 \qquad \sin 2\theta = 2cs$

$$\cos 3\theta = \cos(2\theta + \theta) = \cos 2\theta . c - \sin 2\theta . s = (c^2 - s^2)c - 2cs.s = c^3 - 3cs^2$$

$$\sin 3\theta = \sin(2\theta + \theta) = \sin 2\theta . c + \cos 2\theta . s = 2cs.c + (c^2 - s^2)s = 3c^2 s - s^3$$

$$\cos 4\theta = \cos 3\theta . c - \sin 3\theta . s = c^4 - 6c^2 s^2 + s^4$$

$$\sin 4\theta = \sin 3\theta . c + \cos 3\theta . s = 4c^3 s - 4cs^3$$

and thus we get the following equations:

$$\begin{array}{ll} \cos\theta = c & \sin\theta = s \\ \cos 2\theta = c^2 - s^2 & \sin 2\theta = 2cs \\ \cos 3\theta = c^3 - 3cs^2 & \sin 3\theta = 3c^2 s - s^3 \\ \cos 4\theta = c^4 - 6c^2 s^2 + s^4 & \sin 4\theta = 4c^3 s - 4cs^3 \\ \cos 5\theta = c^5 - 10c^3 s^2 + 5cs^4 & \sin 5\theta = 5c^4 s - 10c^2 s^3 + s^5 \end{array}$$

and so on; the law of which will be hereafter investigated.

(60.) It is easily proved that

$$(\cos\theta + \sin\theta)^2 = 1 + \sin 2\theta \qquad (\cos\theta - \sin\theta)^2 = 1 - \sin 2\theta$$

$$\cos\theta = \pm \tfrac{1}{2}\sqrt{1 + \sin 2\theta} \pm \tfrac{1}{2}\sqrt{1 - \sin 2\theta}$$

$$\sin\theta = \pm \tfrac{1}{2}\sqrt{1 + \sin 2\theta} \mp \tfrac{1}{2}\sqrt{1 - \sin 2\theta}$$

in which the ambiguity of signs will be afterwards discussed.

Also $\cos\theta = \sqrt{\tfrac{1}{2}(1 + \cos 2\theta)} \qquad \sin\theta = \sqrt{\tfrac{1}{2}(1 - \cos 2\theta)}$

(61.) Multiply together the fifth and sixth in (54.), and obtain

$$\sin(\varphi + \theta)\sin(\varphi - \theta) = \sin^2\varphi - \sin^2\theta$$

(62.) We shall now proceed to some cases, in which the sines, &c. may be exhibited numerically. But, first, by means of this theorem, namely, that $(m^2 + n^2)U$, $2mnU$, and $(m^2 - n^2)U$, are the sides of a right angled triangle, U being any linear unit, we can at pleasure find the means of verifying the preceding formulæ in the most exact manner.

For, if $\sin\theta = \dfrac{2mn}{m^2 + n^2}$, then $\cos\theta = \dfrac{m^2 - n^2}{m^2 + n^2} \quad \tan\theta = \dfrac{2mn}{m^2 - n^2}$

Let $m = 2$, $n = 1$; then $\sin\theta = \frac{4}{5}$, $\cos\theta = \frac{3}{5}$, $\tan\theta = \frac{4}{3}$, $\sin 2\theta = \frac{24}{25}$, $\cos 2\theta = -\frac{7}{25}$, $\sin 3\theta = \frac{44}{125}$, $\cos 3\theta = -\frac{117}{125}$, &c. In such a case as this we do not know the *angle* in question; but we will shew that this rude method, though the labour would be very considerable, is, in theory, an unfailing means of finding the angle to a given sine or cosine, as nearly as we please. Suppose an angle to have a sine and cosine, both positive; that is, to be less than a right angle, or in the first right angle. By finding $\sin\theta$, $\sin 2\theta$, &c., and $\cos\theta$, $\cos 2\theta$, &c., we are able to find $\tan\theta$, $\tan 2\theta$, &c. Now, since the angle in question is less than a right angle, there will be multiples of it in every right angle (one at least); that is, no right angle can be left out, or there must be values of $n\theta$ lying between m and $m+1$ right angles. Consequently, since the tangent in the several right angles is alternately positive and negative, we shall always be warned of the value of the multiple angles passing a whole number of right angles, by a change of sign. The first negative sign will indicate that the multiple has become greater than a right angle; the next change, namely, from negative to positive, that the multiple now exceeds two right angles; and, generally, the mth change of sign shews that the multiple in which it appears lies between m and $m+1$ right angles. If, then, we wish to know the angle which belongs to the given tangent within, say one vth part of a right angle, we proceed step by step, and find within what right angles $v\theta$ lies, by noting the number of changes of sign in the series $\tan\theta$, $\tan 2\theta$, $\tan 3\theta$ ...... $\tan v\theta$. Let it be between m and $m+1$ right angles; then θ lies between $\frac{m}{v}$ and $\frac{m+1}{v}$ of a right angle, or is known within one vth part of a right angle.

The preceding process would be too long and laborious for practical purposes; but it shews us, theoretically, that *the determination of the angle which has a given primary function, to any degree of nearness, is within the means of common algebra.*

(63.) Coming now to the determination of some primary functions, we shall express the angles both in analytical and practical units. By (34.), all that we have proved of the primary functions of angles represented in the former way, is true of the latter; except only the theorems in (46, &c.), where the angle enters directly with its primary functions. For instance, though $10''$ is a very small angle, it is obviously neither proved in (64.), nor true, that $\sin 10'' = 10$ nearly. If we now represent n degrees by $n°$, n minutes by n', &c., we have the following equations:

$\pi\Theta = 180°$, $\frac{\pi}{2}\Theta = 90°$, $\frac{\pi}{4}\Theta = 45°$, $\frac{\pi}{6}\Theta = 30°$, $\frac{\pi}{3}\Theta = 60°$,

$\frac{\pi}{12}\Theta = 15°$, $\frac{3\pi}{4}\Theta = 135°$, $\frac{2\pi}{3}\Theta = 120°$, &c. &c.

1. $\frac{\pi}{4}\Theta$ or 45°. The sine and cosine are evidently equal. In (30.) if $OM = MB = U$, then $OB = \sqrt{2}U$, and

$$\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2} \qquad \tan\frac{\pi}{4} = \cot\frac{\pi}{4} = 1$$

2. $\frac{\pi}{6}\Theta$ or 30°. A right-angled triangle, having an angle of 30°, is the half of an equilateral triangle. The side opposite to 30° is half the hypothenuse; hence

$$\sin\frac{\pi}{6} = \frac{1}{2} \quad \cos\frac{\pi}{6} = \sqrt{1-\frac{1}{4}} = \frac{1}{2}\sqrt{3} \quad \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} \quad \cot\frac{\pi}{6} = \sqrt{3}$$

3. $\frac{\pi}{3}\Theta$ or 60°, is the complement of $\frac{\pi}{6}\Theta$ or 30°. Therefore,

$$\cos\frac{\pi}{3} = \frac{1}{2} \quad \sin\frac{\pi}{3} = \frac{1}{2}\sqrt{3} \quad \tan\frac{\pi}{3} = \sqrt{3} \quad \cot\frac{\pi}{3} = \frac{1}{\sqrt{3}}$$

4. $\frac{\pi}{12}\Theta$ or 15°, is the half of $\frac{\pi}{6}\Theta$ or 30°. Hence, by the formula in (60.)

$$\cos\frac{\pi}{12} \text{ is either } \frac{1}{2}\sqrt{1+\frac{1}{2}} + \frac{1}{2}\sqrt{1-\frac{1}{2}}$$

or the same, made negative; but it must be positive, whence we have

$$\cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}} \quad \text{similarly, } \sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

$$= \frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right) \qquad = \frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$$

5. $\frac{\pi}{24}\Theta$ or 7°.30′ is the half of $\frac{\pi}{12}\Theta$ or 15°. And by (60.)

$$\cos\frac{\pi}{24} = \frac{1}{2}\sqrt{\left(1+\frac{1}{4}\sqrt{6}-\frac{1}{4}\sqrt{2}\right)} + \frac{1}{2}\sqrt{\left(1-\frac{1}{4}\sqrt{6}+\frac{1}{4}\sqrt{2}\right)}$$

$$\sin\frac{\pi}{24} = \frac{1}{2}\sqrt{\left(1+\frac{1}{4}\sqrt{6}-\frac{1}{4}\sqrt{2}\right)} - \frac{1}{2}\sqrt{\left(1-\frac{1}{4}\sqrt{6}+\frac{1}{4}\sqrt{2}\right)}$$

In this way we may successively find the sine and cosine of 3° 45′, 1° 52′ 30″, 56′ 15″, 28′ 7″·5, 14′ 3″·75, 7′ 1″·875, 3′ 30″·9375, 1′ 45″·46875, 52″·734375, the latter angle being ·8789063 of a minute. But since the sines of small angles are nearly in the ratio of the angles themselves, and since the ratio of magnitudes are the same in whatever units they may be expressed, we have very nearly,

As $\cdot 8789063$ of $1'$: $1'$ :: sine of the first : sine of $1'$

the fourth term of which can be found from the preceding three. This method of finding the sine of one minute supposes that we do not know how to express the angle of $1'$ in analytical units; if, however, we assume the results of (22.), we see that the sine and angle of $1'$ are nearly equal, when the angle is expressed in terms of Θ, and, therefore, that sine $1' = \cdot 000290888$, nearly. But *nearly* is a vague term; we must endeavour to find *how* nearly the ~~sines of the~~ preceding are equal. If we look at the figure in (40.), and the postulate in (10.), we see that $AT + TB$ is greater than the arc AB, or that

$$\frac{AT+TB}{OA} \text{ or } \frac{AT+OT-OA}{OA} \text{ is greater than } \frac{\text{Arc } AB}{OA}$$

that is $\qquad \tan\theta + \sec\theta - 1 \qquad$ is greater than θ

But $\sec\theta$ is greater than 1; therefore, $\tan\theta$ is greater than θ. Consequently, we find that $\theta - \sin\theta$ is less than $\tan\theta - \sin\theta$, or than $\sin\theta\,(1-\cos\theta) \div \cos\theta$, or than $2\sin\theta\,\sin^2\frac{\theta}{2} \div \cos\theta$; or

$$\theta - \sin\theta \quad \text{is less than} \quad \frac{2\sin\theta\left(\sin\frac{\theta}{2}\right)^2}{\cos\theta}$$

If, then, we increase the numerator of the preceding, and diminish the denominator, we in both ways increase the fraction; consequently,

$$\frac{2\theta\left(\frac{\theta}{2}\right)^2}{\cos^2\theta} \quad \text{is greater than} \quad \frac{2\sin\theta\left(\sin\frac{\theta}{2}\right)^2}{\cos\theta}$$

or $\theta - \sin\theta$ is less than $\frac{1}{2}\,\frac{\theta^3}{1-\sin^2\theta}$; diminish the denominator still further by substituting θ for $\sin\theta$, and we have finally

$$\theta - \sin\theta \quad \text{is less than} \quad \frac{1}{2}\,\frac{\theta^3}{1-\theta^2}$$

Or $\cdot 000290888 - \sin\cdot 000290888$ is less than a fraction very near to

$$\frac{1}{2}\,\frac{(\cdot 00029)^3}{1-(\cdot 00029)^2} \quad \text{or} \quad \cdot 00000000001 \quad \text{very nearly.}$$

Hence, to ten places of decimals, the angle and sine of one minute are the same things: that is, we may assume $\sin 1' = \cdot 000290888$. Now, $\cos\theta = 1 - 2\sin^2\frac{\theta}{2} = 1 - 2\left(\frac{\theta}{2}\right)^2$ very nearly, or $1 - \frac{1}{2}\theta^2$. We have, therefore, $\cos 1' = 1 - \frac{1}{2}(\cdot 0002909)^2$ very nearly $=$ $\cdot 999999958$ very nearly. Knowing thus the sine and cosine of one minute, we might calculate those of 2′, 3′ .... 1°, 1° 1′, ... up to 89° 59′, 90°, and by dividing sines by cosines, we might find the tangents. After which, by taking reciprocals, we might find the cotangents, &c. To put this method in practice would increase all necessary difficulties some hundreds of times; but here, as in (62.), we are not pointing out how a table of sines, &c. should be formed, but merely shewing how it may be done, that the student may not go to the tables, as to results of the possibility of arriving at which he has no comprehension whatever. We can, however, make him see that even the labour of this process may be materially lessened.

1. No cosines need be calculated, nor cotangents, nor cosecants; for in sin 1′ we have cosine 89° 59′; in sin 2′ we have cos 89° 58′ &c.; so that a complete table of sines for all minutes of the right angle is also a table of cosines. For a similar reason, a table of tangents is also one of cotangents, &c.

2. Half of the secants and cosecants may be formed by simple addition, when the rest are known, by the formulæ

$$\operatorname{cosec}\theta = \frac{1}{2}\left(\tan\frac{1}{2}\theta + \cot\frac{1}{2}\theta\right) \quad \sec\theta = \frac{1}{2}\left(\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)+\cot\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right)$$

which we leave to the student to prove.

(64.) Suppose, however, that our table is calculated for every minute of the right angle, it remains to see how the truth of the calculated results may be verified. It must be observed, that in all mathematical tables, the danger of an error of printing is greater than that of an error of calculation. An error of the former kind is one to which all places of figures are equally subject; one of the latter is only to be feared in the last figures. The method of verifying a doubtful figure is to calculate the function in question by means of any other functions, using one of the formulæ already obtained. Thus, if there be a doubt about sin 16°, as given in the tables, we may remember that it ought to be the same as 2 sin 8° cos 8°, and we may, therefore, double the product of sin 8° and cos 8°, and compare it with what is given for sin 16°. But, as multiplication and division are tedious operations, compared with addition and subtraction, we should, for our present purpose, prefer such formulæ as contain only the latter pair of operations. And such formulæ have received the name of *formulæ of verification*, meaning, that they are peculiarly applicable for that purpose. We shall now deduce a few of the kind. Let A be an angle measured in degrees, &c.

(54.) (63.) $\sin(30^\circ + A) + \sin(30^\circ - A) = 2\sin 30^\circ . \cos A = \cos A$

Again, the sine of $\frac{\pi}{10}\theta$, or 18°, may be thus expressed. If $5\theta = \frac{\pi}{2}$, we have

$\cos 3\theta = \sin 2\theta$, or (59.) $\cos^3\theta - 3\cos\theta . \sin^2\theta = 2\cos\theta . \sin\theta$ divide by $\cos\theta$, and substitute $1 - \sin^2\theta$ for $\cos^2\theta$,

$1 - 4\sin^2\theta = 2\sin\theta$; or, taking the positive value of $\sin\theta$

$$\sin\frac{\pi}{10} = \frac{\sqrt{5} - 1}{4} \text{ from which we find}$$

$$\cos\frac{\pi}{5} = 1 - 2\sin^2\frac{\pi}{10} = \frac{\sqrt{5} + 1}{4}$$

Hence we find (54.)

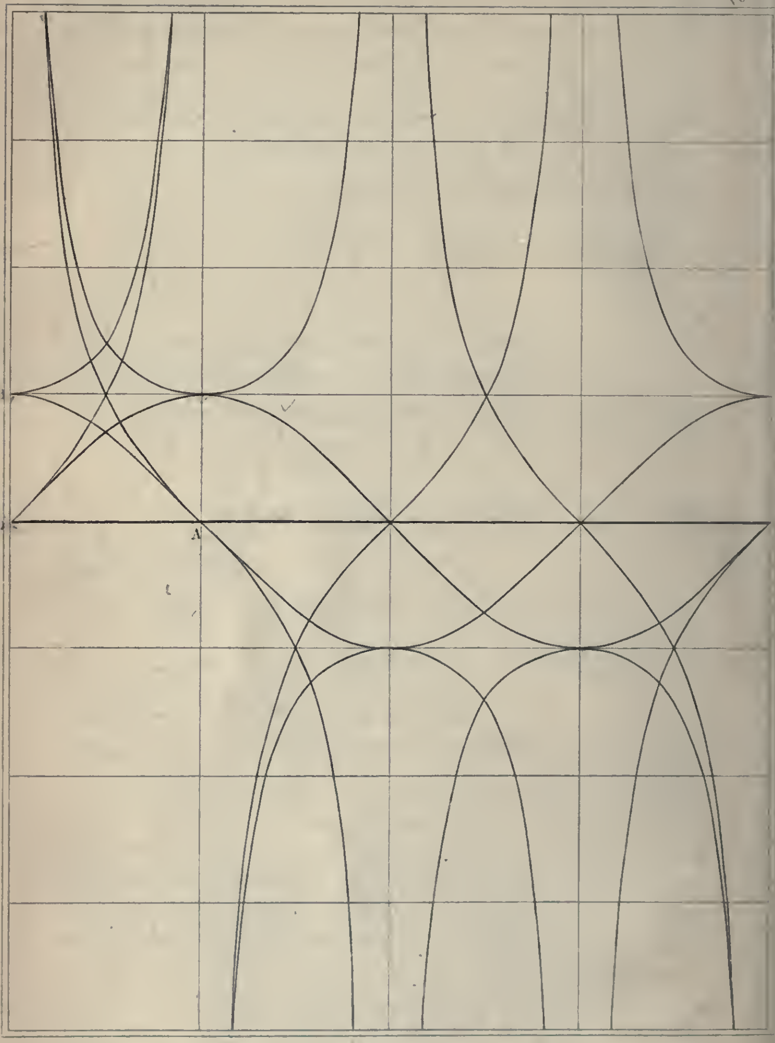
$$\sin(18^\circ + A) + \sin(18^\circ - A) = \frac{\sqrt{5} - 1}{2}\cos A$$

$$\cos(36^\circ + A) + \cos(36^\circ - A) = \frac{\sqrt{5} + 1}{2}\cos A$$

Subtract the first from the second, which gives

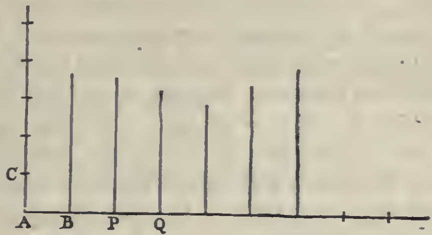
$$\cos(36^\circ + A) + \cos(36^\circ - A) = \cos A + \sin(18^\circ + A) + \sin(18 - A)$$

(65.) Before beginning to use the tables, the student should have a good notion of the changes which take place in the magnitudes of the several functions through the first right angle. And he should also take some method of readily remembering the changes of relative magnitude which take place through the four right angles. The best method of doing it is by remembering the general character of the forms of certain curves, which we shall presently proceed to explain, first premising a more simple illustration of the method. Suppose we wished to take a view of the prices of corn (the average per quarter) in different succeeding years.



London. Taylor & Walton, Upper Gower Street.

Take two scales perpendicular to each other, as in the figure: let A B represent a year of time, A C a pound sterling of money. If we



begin, say at the year 1790, let B be marked 1790, P 1791, Q 1792, &c. and at each point erect a perpendicular the length of which, A C being twenty shillings, shall represent the price of the quarter of corn for the year. We have thus a better idea of the magnitude of the changes than we could get by looking at a table of prices.

Now, take a similar scale for angles and their primary functions. Let angles be measured on O A, at the rate of a right angle, or 90° to O A; let numbers be measured on the perpendiculars to O A at the rate of O B to a unit. The curves are so drawn that if any angle be laid down on O A (that is, if the proper line be measured from O, which is to O A as the angle in degrees, &c. is to 90°), then the six curves will cut off from the perpendicular six lines which represent (if O B represent 1) the sine, cosine, &c. of the angle. It is now for the student to find out which is the curve of sines, which that of cosines, &c., to examine them attentively, until he perceives the truth of all the theorems in (32.) (39.) &c. and to remember the forms of the curves in such manner that the mere words sine, cosine, &c. shall call up the ideas of the variations of magnitude which are peculiar to the function in question.

For instance, it is long before it is as familiar to a beginner as the word cosine, that the cosine of 0 is 1; the notion always being, that the cosine of *nothing* is *nothing*. A recollection of the manner in which the curve of cosines begins with the angle, would completely remove the liability to this mistake.

It would be one of the most improving exercises which the student could impose upon himself, to draw a considerable number of such curves, provided he can obtain paper ruled * horizontally and vertically at small intervals. It will be quite sufficient to divide a right angle into six equal parts, or to take six intervals of length for the right angle. Four intervals of perpendicular length should represent the unit of the functions in question, by which means that same unit will very nearly represent the *analytical unit* Θ on the line on which angles are measured. Suppose, for instance, the paper is ruled at intervals of a tenth of an inch. Take three inches for a right angle, and two inches for a unit of sine, &c. Suppose the curve required to be that which cuts off $\sin 2\theta - \sin\theta$: then, taking two places of decimals (which will here be sufficient) and remembering that 20 perpendicular subdivisions on the paper count as 1, or each subdivision as ·05, divide the two places by 5, and the result is the number of subdivisions. Also, five subdivisions on the line OA represent 15 degrees, or each subdivision represents 3 degrees. The rest of the process is as follows:

Angle A	sin A	sin 2 A	sin 2 A — sin A	Subdivisions for the angle.	Subdivisions for sin 2 A — sin A.
0	·00	·00	·00	0	0
15°	·26	·50	·24	5	$4\frac{4}{5}$
30°	·50	·87	·37	10	$7\frac{2}{5}$
45°	·71	1·00	·29	15	$5\frac{4}{5}$
60°	·87	·87	·00	20	0
75°	·97	·50	—·47	25	$-9\frac{2}{5}$
90°	1·00	·00	—1·00	30	—20
105°	·97	—·50	—1·47	35	$-29\frac{2}{5}$
120°	·87	—·87	—1·74	40	$-34\frac{4}{5}$
135°	·71	—1·00	—1·71	45	$-34\frac{1}{5}$
150°	·50	—·87	—1·37	50	$-27\frac{2}{5}$
165°	·26	—·50	—·76	55	$-15\frac{1}{5}$
180°	·00	·00	·00	60	—0

* The common ruling machine used by stationers will rule paper very well to tenths of inches, with each inch-line broader than the rest. Some years ago, I caused a quantity of paper to be so ruled, which is still on sale with the publisher of this work. In looking at ruled paper, the eye is too accurate a judge: when parallel lines are ruled close together, a very trifling defect is perfectly visible.

The student should now lay down and continue this curve, and others of the same kind.

(66.) Let h be a small angle, of about the value of $1'$. It has been shewn that for more than eight places of decimals, $\cos h = 1$, $\sin h = h$, and hence $\tan h = h$. We have then, with quite sufficient exactness for the tables which it is necessary to use (which never exceed seven places of decimals).

$$\sin(\theta + h) = \sin\theta \cos h + \cos\theta \sin h = \sin\theta + \cos\theta . h$$

$$\cos(\theta + h) = \cos\theta \cos h - \sin\theta \sin h = \cos\theta - \sin\theta . h$$

(56.) $$\tan(\theta + h) - \tan\theta = \frac{\sin h}{\cos(\theta + h)\cos\theta} = \frac{h}{\cos^2\theta} \text{ very nearly}$$

or $$\tan(\theta + h) = \tan\theta + \frac{1}{\cos^2\theta} . h$$

We shall leave the following to the student.

$$\cot(\theta + h) = \cot\theta - \frac{1}{\sin^2\theta} . h$$

$$\sec(\theta + h) = \sec\theta + \frac{\sin\theta}{\cos^2\theta} . h$$

$$\text{cosec}(\theta + h) = \text{cosec}\,\theta - \frac{\cos\theta}{\sin^2\theta} . h$$

$$\text{vers}(\theta + h) = \text{vers}\,\theta + \sin\theta . h$$

$$\text{covers}(\theta + h) = \text{covers}\,\theta - \cos\theta . h$$

That is, $F\theta$ representing any primary function of θ, we have

for sin. tan. sec. or vers. $F(\theta + h) = F\theta + Mh$

for cos. cot. cosec. or covers. $F(\theta + h) = F\theta - Mh$

where M is not a function of h, but of θ only. (Remember that functions beginning with *co*. are all decreasing when the angle increases, in the first right angle). Let $h' = \cdot 000290888$ the minute expressed in analytical units; and let h be any angle less than h': then we have

$$\left.\begin{array}{l} F(\theta + h') - F\theta = Mh' \\ F(\theta + h) - F\theta = Mh \end{array}\right\} \text{ for functions which increase with the angle}$$

whence $$F(\theta + h) = F\theta + \frac{h}{h'}\left\{F(\theta + h') - F\theta\right\}$$

But $F(\theta + h') - F\theta$ is the increment of the function, when the angle receives an increment of one minute in value; it is, therefore, immediately found from the tables, in which the values of $F(\theta+h')$ and $F\theta$ are those which follow each other, if the tables be to every minute, as is usual. Let this increment be denoted by Dif. for *difference*, as in the tables, where the subtraction is made in a separate column. And let h contain s seconds and decimals of seconds. Then will $\frac{h}{h'} = \frac{s}{60}$ and we have (if A be the angle θ Θ in degrees and minutes.

$$F(A+s) = FA + \frac{s}{60} \times \text{Dif.}$$

and in the same way, if $F\theta$ be a function which decreases when θ increases, we have

$$F(\theta+h) \text{ or } F(A+s) = FA - \frac{s}{60} \times \text{Dif.}$$

where Dif. now stands for $FA - F(A+1')$ and is taken from the tables.

The preceding is, in a more exact form, the representation of a notion which may be more easily given. If we ask what is that function of h which increases uniformly when h increases uniformly (the more easy phrase is, which grows at the same rate as long as h grows at the same rate) the answer is, that the function can only be $P+Mh$ where P and M are independent of h. In this function, if for h we write successively

$$h, \quad h+t, \quad h+2t, \quad h+3t, \quad \text{\&c.}$$

the values of the function are

$$(P+Mh), \ (P+Mh)+Mt, \ (P+Mh)+2Mt, \ \text{\&c.}$$

and, similarly, the function $P-Mh$ is of the only form which decreases uniformly when h increases uniformly. If, then, there be a function of h which does not increase uniformly when h undergoes considerable changes of value, but which increases very nearly uniformly when h is small, and undergoes small changes; that is, is very nearly equal to some form of $P+Mh$; the consequence is, that we may for very small values of h, and very small changes, treat the function as if it were one which increased uniformly.

To illustrate this, I take out of the table the cosine and sine of 6°, 6° 1′, 6° 2′, and 6° 3′, as follows :

Angle	Sine	Dif.	Ccsine	Dif.
6° 0′	·1045285		·9945219	
		·0002893		·0000305
6° 1′	·1048178		·9944914	
		·0002892		·0000305
6° 2′	·1051070		·9944609	
		·0002893		·0000306
6° 3′	·1053963		·9944303	

Hence, at and near 6° 1′, the sine is a function which increases uniformly at the rate of ·0002893 per minute of angle, and the cosine diminishes at the rate of ·0000305 per minute. This uniformity of increase or decrease, which obtains when the angle changes through successive minutes, will, *à fortiori*, still remain when the angle changes from second to second in the interval between two minutes. That is, we must, to find the *sine* of 60° 1′ 37″·5, *add* to sin 6° 1′ such a part of ·0002893 as 37½ is of 60, and to find cos 6° 1′ 37″·5 we must subtract from cos 6° 1′ such part of ·0000305 as 37½ is of 60. The process may be performed either by common multiplication or division, or in the manner of the rule called *practice* in commercial arithmetic, as follows:

```
      2893
        37
    ------
     20251
     8679
    ------
    107041
      1447
    ------
6,0)10848,8
    ------
      1808
```

sin 6° 1′ = ·1048178
proportional part for 37″·5 = + ·0001808
sin 6° 1′ 37″·5 = ·1049986

for 60		2893
for 30	½	1447
for 6	⅕	289
for 1	⅙	48
for ·5	½	24
for 37·5		1808

```
       305
        37
    ------
      2135
      915
    ------
     11285
       153
    ------
6,0)1143,8
    ------
       191
```

cos 6° 1′ = ·9944914
proportional part for 37·5 = − ·0000191
cos 6° 1′ 37″·5 = ·9944723

for 60		305
for 30	½	153
for 6	⅕	31
for 1	⅙	5
for ·5	½	3
for 37·5		192

Nothing will secure accuracy to a single unit in the last place of tables, which are, therefore, always carried a unit further than would otherwise be requisite.

The inverse process to the preceding follows immediately from it. Given FA, F(A + s), and F(A + 1′), required s. Let P be the given value of F(A + s); then we have

$$P = FA + \frac{s}{60}\,\text{Dif.} \qquad \text{Dif.} = F(A+1') - FA$$

or

$$s = \frac{(P - FA)\,60}{\text{Dif.}}$$

But, if FA be a decreasing function, we have

$$P = FA - \frac{s}{60}\,\text{Dif.} \qquad s = \frac{(FA - P)\,60}{\text{Dif.}}$$

Thus, suppose it is required to find the angle which has ·1052111 for its sine, and also that which has ·9945000 for its cosine.

P	·1052111	P	·9945000
sin 6° 2′, or FA	·1051070	cos 6°, or FA	·9945219
P — FA	1041	FA — P	219
	60		60
	2893)62460(21·6		305)13140(43·1
	5786 = s		1220
	4600		940
	2893		915
	17070		250
Angle required	6° 2′ 21″·6	Angle required	6° 2′ 43″·1

(67.) We must proceed exactly in the same manner with the logarithms of the primary functions; and the law of the increase or decrease may be exhibited as follows. If we have (making $\mu = {\cdot}43429\ ..$)

$$F(A+s) = FA + \frac{s}{60}\cdot\text{Dif. this gives } \log F(A+s) = \log\left(FA + \frac{s}{60}\cdot\text{Dif.}\right)$$

$$= \log FA + \log\left(1 + \frac{s}{60}\,\frac{\text{Dif.}}{FA}\right) = \log FA + \frac{s}{60}\,\frac{\mu.\text{Dif.}}{FA} \text{ nearly.}$$

(See *Algebra*, pp. 226 and 237.) But by making $s = 60''$, or $1'$, we find

$$\log F(A+1') = \log FA + \frac{\mu.\text{Dif.}}{FA} \quad \text{or} \quad \frac{\mu.\text{Dif.}}{FA} = \log F(A+1') - \log FA$$

therefore μ.Dif. $\div$ FA is what is found in the column of differences of the logarithms, and may be taken from the tables. We have then

$$\log F(A+s) = \log FA + \frac{s}{60} \times \text{Dif. of log}$$

or we may remember that the reasoning in page 48 applies to any functions which increase continuously, and, therefore, to the logarithms of the primary functions, as well as to the functions themselves.

(68.) It depends, then, upon the amount of the difference between $F(A+1')$ and $F(A)$ whether we can pretend very nearly to find the angle which belongs to any intermediate between them. Look, for instance, at the beginning of the table of logarithmic cosines, which by the arrangement of the tables is the end of the table of logarithmic sines. We have for instance,

$$\begin{array}{l} \text{log. cos } 0^\circ\, 6' = \cdot 9999993 \\ \text{log. cos } 0^\circ\, 7' = \cdot 9999991 \end{array} \quad \text{Dif.} = \cdot 0000002$$

We cannot, out of this 2 of difference, make different cosines for every second between 6′ and 7′. The log. cosine here is increasing so slowly, that many successive increments of 1″ will not make it shew a difference of a unit in seven places of decimals. We come to 2° 41′ before an angle increased by 1″ has its log. cosine increased by ·0000001. And if log. cosines are to shew tenths of seconds, that is, if the log. cosine is to increase so rapidly that 0″·1 added to the angle shall make a difference in seven places of decimals, the angle must be upwards of 25°. But when we come to tangents of angles very near 90°, we find that the preceding method fails, because the increases of the log. tangent for successive increases of the angle are far from uniform. Consequently, *when the angle to be found is small, avoid expressing it by means of its cosine, if possible; when it is nearly a right angle, avoid its sine and tangent, if possible.* In the case of a tangent which is very great, denoting an angle near 90°, proceed as follows. Let $\tan A = a$, a being a considerable number, so that the angle is nearly a right angle. Remember that (56.)

$$\tan(A - 45^\circ) = \frac{\tan A - 1}{\tan A + 1} = \frac{a-1}{a+1}$$

Find, not A, but A — 45°, from this formula, and the difficulty will disappear; for near 45° the increase of the tangent is very nearly uniform, and also that of its logarithm. For instance, I wish to know, with great exactness, the angle whose tangent is 3000. On looking at the tables I see that it is between 89° 58′ and 89° 59′, but on examining the increase of the tangent, I see as follows:

tan 89° 57′	1145·9	Dif. = 573·0	not nearly
tan 89° 58′	1718·9	Dif. = 1718·8	equal
tan 89° 59′	3437·7		

But, the angle wanted diminished by 45° has for its tangent,

$$\frac{2999}{3001} = {\cdot}9993336$$

tan 44° 58′ ·9988371

4965
60

Dif. = 5813)297900(51·3
29065

7250
5813 $\qquad A - 45° = 44° \, 58' \, 51''{\cdot}3$

14370 $\qquad A = 89° \, 58' \, 51''{\cdot}3$

(69.) Suppose $\cos A = a$, a being very near unity, or the angle very small. We have then

$$\sin^2 \frac{1}{2} A = \frac{1}{2}(1-a) \qquad \sin \frac{1}{2} A = \sqrt{\frac{1-a}{2}}$$

which may easily be calculated by logarithms; and from $\frac{1}{2}$A, A can be found. Similarly, if we have $\sin A = a$, where a is very near unity, we have

$$1 - a = 1 - \cos(90° - A) = 2 \sin^2\left(45° - \frac{1}{2}A\right)$$

or

$$\sin\left(45° - \frac{1}{2}A\right) = \sqrt{\frac{1-a}{2}}$$

and from $45° - \frac{1}{2}$A, A can be found.

(70.) We have seen that the cosine approaches very near to unity when the angle is small; so near that $1-\cos\theta$ is a small quantity by the side of θ itself, when θ is small (48.). But our future purposes will require a theorem which we shall introduce here, to give a further notion of the rate at which $\cos\theta$ approaches unity. If we take an angle θ, less than $\frac{\pi}{2}$, and form the series $\theta, \frac{\theta}{2}, \frac{\theta}{3}$, &c. we have a series the terms of which diminish without limit. If we then take

$$\cos\theta, \qquad \cos\frac{\theta}{2}, \qquad \cos\frac{\theta}{3}, \qquad \cos\frac{\theta}{4}, \quad \&c.$$

we have a series of terms approximating without limit to unity. Let us now take

$$\cos\theta, \qquad \left(\cos\frac{\theta}{2}\right)^2 \quad \left(\cos\frac{\theta}{3}\right)^3 \quad \left(\cos\frac{\theta}{4}\right)^4 \quad \&c.$$

We know that the powers of a fraction less than unity decrease, and without limit as the exponents increase (*Algebra*, p. 159), that is, if the fraction operated upon remain the same. But here we have in passing from

$$\cos\theta \qquad \text{to} \qquad \cos\frac{\theta}{n}, \quad \text{increase}$$

$$\text{from} \quad \cos\frac{\theta}{n} \qquad \text{to} \qquad \left(\cos\frac{\theta}{n}\right)^n \text{decrease}$$

the question is, as n grows greater and greater, which will predominate, the increase or the decrease. Will $\left(\cos\frac{\theta}{n}\right)^n$, by the decrease which takes place in raising the power, tend to a limit less than unity, or may it be brought as near to unity as we please.

To try this, write

$$\left(\cos\frac{\theta}{n}\right)^n \text{ in the form } \left(1 - 2\sin^2\frac{\theta}{2n}\right)^n$$

and remember that if $2\left(\sin\frac{\theta}{2n}\right)^2 = 2\mu\left(\frac{\theta}{2n}\right)^2$ then μ continually approaches to unity as n is increased. Substitute, which gives (*Algebra*, p. 209, and p. 218, for a similar process)

$$\left(\cos\frac{\theta}{n}\right)^n = \left(1 - \frac{\mu\theta^2}{2n^2}\right)^n = 1 - n\frac{\mu\theta^2}{2n^2} + n\frac{n-1}{2}\frac{\mu^2\theta^2}{4n^4} - \&c.$$

$$= 1 - \frac{\mu\theta^2}{2n} + \frac{1-\frac{1}{n}}{2}\frac{\mu^2\theta^2}{4n^2} - \&c.$$

every term of which, except the first, diminishes without limit when n increases without limit; for θ remains the same, μ approaches to unity, and n has increase without limit, causing the same in all the denominators. Hence

$$\left(\cos\frac{\theta}{n}\right)^n \text{ has the limit 1, the same as that of } \cos\frac{\theta}{n}$$

(71.) We shall now examine the limit of $\left(\cos\frac{\theta}{n} + k\sin\frac{\theta}{n}\right)^n$ under like circumstances. It is evident that

$$\left(\cos\frac{\theta}{n} + k\sin\frac{\theta}{n}\right)^n = \left(\cos\frac{\theta}{n}\right)^n \left(1 + k\tan\frac{\theta}{n}\right)^n$$

for $$(p+q)^n = p^n\left(1+\frac{p}{q}\right)^n \text{ and } \frac{\sin}{\cos} = \tan$$

in which product the first factor, as just shewn, has unity for its limit, and we must examine that of the second factor. If we make $\tan\frac{\theta}{n} = \mu\frac{\theta}{n}$ then (48.), as n increases without limit, the limit of μ is unity. Substitute and develope by the binomial theorem, which gives (as in *Algebra*, p. 218),

$$\left(1 + k\mu\frac{\theta}{n}\right)^n = 1 + nk\mu\frac{\theta}{n} + n\frac{n-1}{2}k^2\mu^2\frac{\theta^2}{n^2} + n\frac{n-1}{2}\,\frac{n-2}{3}k^3\mu^3\frac{\theta^3}{n^3}\ \ldots.$$

$$= 1 + \mu.k\theta + \frac{1-\frac{1}{n}}{2}\mu^2.k^2\theta^2 + \frac{1-\frac{1}{n}}{2}\,\frac{1-\frac{2}{n}}{3}\mu^3.k^3\theta^3 + \ldots.$$

Take the limit of both sides (limit of $\mu = 1$)

$$\text{Limit of } \left(1 + k\tan\frac{\theta}{n}\right)^n = 1 + k\theta + \frac{k^2\theta^2}{2} + \frac{k^3\theta^3}{2.3} + \ldots. = \varepsilon^{k\theta}$$

or $$\text{Limit of } \left(\cos\frac{\theta}{n} + k\sin\frac{\theta}{n}\right)^n = \varepsilon^{k\theta}$$

That is the equation $$\left(\cos\frac{\theta}{n} + k\sin\frac{\theta}{n}\right)^n = \varepsilon^{k\theta}$$

1. is never absolutely true; 2. is very nearly true, if n be great; 3. can be brought as near to truth as we please, by making n sufficiently great. Extract the *arithmetical* nth root (*Algebra*, p. 110) of both sides, and we have, as nearly as we please

$$\left.\begin{array}{l}\cos\frac{\theta}{n} + k\sin\frac{\theta}{n} = \varepsilon^{k\frac{\theta}{n}} \\ \text{if } n \text{ may be as great as we please.}\end{array}\right\} \quad \text{or} \quad \left\{\begin{array}{l}\cos\omega + k\sin\omega = \varepsilon^{k\omega} \\ \text{if } \omega \text{ may be as small as we please.}\end{array}\right.$$

Observe that this is independent of the value of k. In the last form the result is easy to establish, for when ω is small $\cos\omega + k\sin\omega$ is nearly $1 + k\omega$, which (*Algebra*, p. 187) is by much the greater part of the developement of $\varepsilon^{k\omega}$. This process, therefore, is one more experience of the confidence to be placed in the developements of $(1+x)^n$ and ε^x (*Algebra*, c. xi, xii.); and also suggests the propriety of examining further the form $\cos\theta + k\sin\theta$. Multiply two such forms together, and we have

$$(\cos\theta + k\sin\theta)(\cos\theta' + k\sin\theta') = \cos\theta\cos\theta' + k^2\sin\theta\sin\theta' + k\sin(\theta+\theta')$$
$$= \cos\theta\cos\theta' - \sin\theta\sin\theta' + (1+k^2)\sin\theta\sin\theta' + k\sin(\theta+\theta')$$
$$= \cos(\theta+\theta') + k\sin(\theta+\theta') + (1+k^2)\sin\theta\sin\theta'$$

or, if this function of θ, $\cos\theta + k\sin\theta$ be denoted by $f\theta$, we have

$$f\theta \times f\theta' = f(\theta+\theta') + (1+k^2)\sin\theta . \sin\theta' \quad \ldots . \quad (1)$$

Multiply again by $f\theta''$ or $\cos\theta'' + k\sin\theta''$, which gives

$$f\theta \times f\theta' \times f\theta'' = f(\theta+\theta'+\theta'') + (1+k^2)\{\sin\theta''\sin(\theta+\theta') + \sin\theta.\sin\theta' f\theta''\}$$

since by (1) $f(\theta+\theta')f\theta'' = f(\theta+\theta'+\theta'') + (1+k^2)\sin(\theta+\theta')\sin\theta''$

By proceeding in this way we see that if we multiply together $f\theta$, $f\theta'$, $f\theta''$, ... we have an equation of the following form:

$$f\theta . f\theta' . f\theta'' \ldots . = f(\theta+\theta'+\theta''+\ldots .) + (1+k^2)\,\mathrm{V}$$

where V is a function of the angles, and of k. This factor $1+k^2$, might be made to simplify the expression materially, if there were such a value of k as that $1+k^2$ should be $= 0$, but there is evidently no such algebraical value, positive or negative, for k^2 is always positive, and $1+k^2$ greater than 1. We shall hereafter see the consequences of this hint, but we shall leave this formula for the present. As far as we have yet proceeded, every thing seems to render it most likely that, if any function of sines and cosines be identical with a function of common algebra, it is of the form $\cos\theta + k\sin\theta$, which, though not found to be such, is very nearly represented by $\varepsilon^{k\theta}$, when θ is small. To try this supposition, let us (as an experiment) make $\qquad \cos\theta + k\sin\theta = \varepsilon^{k\theta}$ for *all* values of θ,

merely to see whether the consequences coincide with those already obtained or not. Then, if this equation be universally true, we have, writing $-\theta$ for θ (44.),

$$\cos\theta - k\sin\theta = \varepsilon^{-k\theta} = \frac{1}{\varepsilon^{k\theta}}. \quad \text{Let } \varepsilon^{k\theta} = x$$

Then $\qquad 2\cos\theta = x + \frac{1}{x} \qquad 2k\sin\theta = x - \frac{1}{x}$

Again, $\qquad \cos n\theta + k\sin n\theta = \varepsilon^{kn\theta} = \left(\varepsilon^{k\theta}\right)^n = x^n$

$$\cos n\theta - k\sin n\theta = \varepsilon^{-kn\theta} = \left(\varepsilon^{-k\theta}\right)^n = \frac{1}{x^n}$$

Whence $2\cos n\theta = x^n + \frac{1}{x^n}$ $2k\sin n\theta = x^n - \frac{1}{x^n}$

(72.) Let us try some properties of the sine and cosine with these supposed values.

$\cos 2\theta = 2\cos^2\theta - 1$ or $2\cos 2\theta = (2\cos\theta)^2 - 2$.

If our preceding equations be correct, we should have,

$x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2$; but this is true, therefore this case does not contradict our assumptions.

Again $\sin 2\theta = 2\sin\theta\cos\theta$ or $2k\sin 2\theta = (2k\sin\theta)(2\cos\theta)$

But $x^2 - \frac{1}{x^2} = \left(x - \frac{1}{x}\right)\left(x + \frac{1}{x}\right)$ this case, therefore, is no contradiction.

Again, $\cos 2\theta = 1 - 2\sin^2\theta$ $2k^2\cos 2\theta = 2k^2 - (2k\sin\theta)^2$

But $2k^2\cos 2\theta = k^2x^2 + \frac{k^2}{x^2}$

$$2k^2 - (2k\sin\theta)^2 = 2k^2 - \left(x - \frac{1}{x}\right)^2 = 2k^2 + 2 - x^2 - \frac{1}{x^2}$$

to equate these two is therefore to make $k^2 + 1 = 0$, which cannot be.

We may next prove that all the equations contained in

$$2\cos n\theta = x^n + \frac{1}{x^n} \qquad 2k\sin n\theta = x^n - \frac{1}{x^n}$$

for all whole values of n from 0 upwards, *must* be true *if* the two first are true; namely,

$n = 0$ $2 = x^0 + \frac{1}{x^0}$ $2k\sin 0 = x^0 - \frac{1}{x^0}$ which are true

$n = 1$ $2\cos\theta = x + \frac{1}{x}$ $2k\sin\theta = x - \frac{1}{x}$ ...... (A)

For it is readily shewn that the truth of any two of these equations involves the truth of the next, as follows. Let a and $a+1$, any two successive values of n, give true results, that is, assume

$$2\cos a\theta = x^a + \frac{1}{x^a} \qquad 2k\sin a\theta = x^a - \frac{1}{x^a}$$

$$2\cos(a+1)\theta = x^{a+1} + \frac{1}{x^{a+1}} \qquad 2k\sin(a+1)\theta = x^{a+1} - \frac{1}{x^{a+1}}$$

Now, $\frac{(a+2)\theta + a\theta}{2} = (a+1)\theta$ $\frac{(a+2)\theta - a\theta}{2} = \theta$

(54.) $\cos(a+2)\theta + \cos a\theta = 2\cos(a+1)\theta.\cos\theta$

$\sin(a+2)\theta + \sin a\theta = 2\sin(a+1)\theta\ \cos\theta$

Therefore $2\cos(a+2)\theta = 2\cos(a+1)\theta.2\cos\theta - 2\cos a\theta$

$$= \left(x^{a+1} + \frac{1}{x^{a+1}}\right)\left(x+\frac{1}{x}\right) - \left(x^a + \frac{1}{x^a}\right)$$

$$= x^{a+2} + \frac{1}{x^{a+2}}$$

And $2k\sin(a+2)\theta = 2k.\sin(a+1)\theta.2\cos\theta - 2k\sin a\theta$

$$= \left(x^{a+1} - \frac{1}{x^{a+1}}\right)\left(x+\frac{1}{x}\right) - \left(x^a - \frac{1}{x^a}\right)$$

$$= x^{a+2} - \frac{1}{x^{a+2}}$$

From these it appears that the third follows from the two first; the fourth, from the second and third; the fifth, from the third and fourth, with the second, &c.

We now try the equation

$$(\cos\theta + \sin\theta)^2 = 1 + \sin 2\theta$$

or $$(k.2\cos\theta + 2k\sin\theta)^2 = 4k^2 + 2k.2k\sin 2\theta$$

or $$\left(kx + \frac{k}{x} + x - \frac{1}{x}\right)^2 = 4k^2 + 2k\left(x^2 - \frac{1}{x^2}\right)$$

But $$\left((k+1)x + \frac{k-1}{x}\right)^2 = (k+1)^2x^2 + 2(k^2-1) + \frac{(k-1)^2}{x^2}$$

$$= 2k\left(x^2 - \frac{1}{x^2}\right) + (k^2+1)\left(x^2 + \frac{1}{x^2}\right) + 2k^2 - 2$$

which becomes identical with $4k^2 + 2k\left(x^2 - \frac{1}{x^2}\right)$ only on the impossible supposition of $1 + k^2 = 0$.

We shall try one more case.

Since $x^n = \cos n\theta + k\sin n\theta$, and $x = \cos\theta + k\sin\theta$, we have,

$$\cos n\theta + k\sin n\theta = (\cos\theta + k\sin\theta)^n = (c + k\,s)^n$$

or $$\cos 2\theta + k\sin 2\theta = c^2 + 2k\,c\,s + k^2s^2 = (c^2 + 2k\,c\,s - s^2)$$

$$\cos 3\theta + k\sin 3\theta = c^3 + 3k\,c^2s + 3k^2c\,s^2 + k^3s^3$$

$$= (c^3 + 3k\,c^2s - 3c\,s^2 - k\,s^3)$$

$$\cos 4\theta + k\sin 4\theta = c^4 + 4kc^3s + 6k^2c^2s^2 + 4k^3cs^3 + k^4s^4$$

$$= (c^4 + 4kc^3s - 6c^2s^2 - 4kcs^3 + s^4).$$

The values in parentheses are those already found in (59.), simply multiplying the sines by k, and forming $\cos n\theta + k\sin n\theta$, case after case, altering only the order of the terms, so as to put them under those which they resemble in the results of our present method. We see, then, that the results of our present method coincide with those of (59.), by writing -1 for k^2, $-k$ for k^3, 1 for k^4, &c., which are all algebraical consequences of the single assumption, $k^2 = -1$, which gives $k^3 = -k$, $k^4 = -k^2 = 1$, $k^5 = k$, &c. And from all that has preceded, we deduce the following remark, which we have as much reason to suppose generally true, as instances can give.

(73.) The functions $x + \frac{1}{x}$ and $x - \frac{1}{x}$ have properties corresponding in all respects to the trigonometrical functions $2\cos\theta$ and $2k\sin\theta$; and such that, under the following limitations, the properties of the first may be deduced from those of the second. If an equation which is true of the first functions, undergo substitution of the second for the first, then, if the result do not contain k at all, it is absolutely true of the second; but if it contain powers of k it is never true of the second. Nevertheless, it becomes true of the second, if for the set of even powers of k, namely, k^2, k^4, k^6, &c. we substitute -1, $+1$, -1, &c. and for the set of odd powers of k, namely, k, k^3, k^5, k^7, &c. we substitute k, $-k$, $+k$, $-k$, &c.; in which case, the part independent of k on one side is equal to the part independent of k on the other, and the coefficient of k on one side equal to the coefficient of k on the other. And the representative of x is $\cos\theta + k\sin\theta$, and also $\varepsilon^{k\theta}$.

Nevertheless, $2\cos\theta = x + \frac{1}{x}$ is an impossible equation, except only when $x = 1, \cos\theta = 1$. For, whereas $2\cos\theta$ is never greater than 2, $x + \frac{1}{x}$ is never less than 2. For, if $x + \frac{1}{x}$ were less than 2, $x^2 + 1$ would be less than $2x$, or $x^2 - 2x + 1$, a *square*, would be negative. And, in fact, if we solve

$$x + \frac{1}{x} = 2\cos\theta \text{ we find } x = \cos\theta \pm \sqrt{-1}\sin\theta$$

$$x - \frac{1}{x} = 2k\sin\theta \text{ gives } x = k\sin\theta \pm \sqrt{k^2\sin^2\theta + 1}$$

which agrees in form with the preceding only when $k^2 = -1$.

We have thus laid the foundation of the application of a more abstruse analysis to the primary functions of an angle. We shall first consider the application of our formula to the *solution of triangles,* as it is called, that is, the determination of the remaining parts of a triangle, when enough are given to distinguish it from all others.

CHAPTER III.

ON THE SOLUTION OF TRIANGLES.

(74.) Let aU, bU, cU, be the sides of a triangle, U being any given linear unit, and let A, B, C, be the opposite angles, expressed in degrees, minutes, and seconds. When one of the angles is a right angle, it is evident that the tables of sines, cosines, &c. are nothing but registers of the proportions of the sides of such a triangle. Knowing, therefore, any one side, and an angle, we look to the table for the proportion of the other side to it, preferring, of course, the logarithm of the proportion for convenience of calculation.

(75.) The best tables in common use are those of Hutton, which may be procured of any bookseller. The arrangement by which the sine of 18°, for instance, is prevented from being again printed as the cosine of 72°, will be better understood by consulting the table (and remarking the description of the functions at the top and bottom of the page, and the reckoning in minutes downwards on the left hand, and upwards on the right) than by any explanation.

But the following point requires some notice. In every mathematical table which contains both positive and negative quantities, there is such a liability to error in taking out the signs, that it is most useful, and almost necessary, to form the table in such a way that all shall have the same sign. Suppose, for example, that the following table was in frequent use.

$$+6,\ +4,\ -3,\ +2,\ -10,\ -1,\ +8,\ -11.$$

Now 12 being greater than any one of these, add 12 to each, which converts the table into

$$+18,\ +16,\ +9,\ +14,\ +2,\ +11,\ +20,\ +1.$$

Every result in this table is to be added; but 12 is to be subtracted whenever the table is used. There is always both an addition and a subtraction; the sign of the table will not be liable to be read wrong, and the correction of the table is uniform—always a subtraction.

The trigonometrical tables consist of—sines and cosines with logarithms always negative—tangents and cotangents with the same sometimes positive and sometimes negative—and secants and cosecants with logarithms always positive. The plan which is followed is to add 10 to every logarithm in the table, without exception: so that,

True log. FA = *Tabular* log. FA — 10

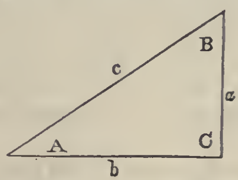
Tabular log. FA = True log. FA + 10

For instance, $\sin 30^\circ = \frac{1}{2}$ log. $\sin 30^\circ = -\cdot 3010300$.

In the tables we have, $10 - \cdot 3010300$ or $9\cdot 6989700$.

(76.) The formulæ for the solution of right-angled triangles, are as follows:

Let C be right angle, cU the hypothenuse; then we have,



$$\frac{a}{c} = \sin A = \cos B \qquad a = c \sin A = c \cos B$$

$$\frac{a}{b} = \tan A = \cot B \qquad a = b \tan A = b \cot B$$

$$c^2 = a^2 + b^2 \qquad b = \sqrt{(\overline{c-a} \,.\, \overline{c+a})}$$

But the following formulæ should be remembered in words.

side = hypothenuse into sine of *opposite* angle

side = hypothenuse into cosine of *adjacent* angle

hypothenuse = *side* by sine of *opposite* angle

hypothenuse = *side* by cosine of *adjacent* angle

side = other side into tangent of *opposite* angle

side = other side by tangent of *adjacent* angle

(77.) The following are the cases which may occur, and the logarithmic equations for the solution (L sin, &c. mean *tabular* log. sin, &c.; *side*, or *angle*, in Italics, means given side).

1. Given the hypothenuse and a side; required the rest.

log. remaining side = $\frac{1}{2}$ (log. $\overline{hyp. + side}$ + log. $\overline{hyp. - side}$)

L. sin. angle opp. *side* = 10 + log. *side* — log. *hyp.*

angle opp. other side = 90° — angle opp. *side*

2. Given the hypothenuse and an angle; required the rest.

log. side opp. *angle* = log. *hyp.* + L. sin. *angle* — 10
log. side adj. *angle* = log. *hyp.* + L cos. *angle* — 10
other angle = 90° — *angle*

3. Given a side and an angle; required the rest.

Other angle = 90° — *angle*
log. hyp. = 10 + log. *side* — L sin ∠ opp. *side*
log. other side = log. *side* + L tan ∠ adj. *side* — 10

4. Given the two sides; required the rest.

L tan. AN ANGLE = 10 + log. ITS opp. *side* — log. other *side*
Other angle = 90° — the ANGLE found
log. hyp. = 10 + log. *A SIDE* — L. sin. *ITS* opp. angle.

(78.) There are two cases in which the sine and tangent of an angle are severally to be found; from two equations of these forms

$$\sin A = \frac{a}{c} \qquad \tan A = \frac{a}{b}$$

If in the first case $\frac{a}{c}$ be very near unity (68.), use the equation

$$2\cos^2\left(\frac{90^\circ - A}{2}\right) = \frac{c-a}{c} \quad \text{or} \quad \sin\left(45^\circ - \frac{A}{2}\right) = \sqrt{\frac{c-a}{2c}}$$

If, in the second case, b be very small in comparison of a, use

$$\frac{\tan A - 1}{\tan A + 1} = \frac{a-b}{a+b} \quad \text{or} \quad \tan(A - 45^\circ) = \frac{a-b}{a+b}$$

When a side (b) is given and a very small adjacent angle (A), the hypothenuse may be determined by its excess above the given side (which is small) as follows:

$$c - b = \frac{b}{\cos A} - b = b\frac{(1-\cos A)}{\cos A} = 2b\sin^2\frac{A}{2} \text{ very nearly.}$$

(79.) Given the hypothenuse and the sum of the two sides; required the rest.

$$(a+b)^2 - c^2 = 2ab = 2c^2\sin A . \cos A = c^2 \sin 2A$$

$$\text{L.} \sin 2A = 10 + \log(a+b+c) + \log(a+b-c) - 2\log c$$

Given the excess of the hypothenuse over a side, and the difference of the angles; required the parts of the triangle.

Let $\quad c - b = h, \qquad A - B = M, \qquad A + B = 90^\circ,$

$$A = \frac{90^\circ + M}{2}, \quad B = \frac{90 - M}{2}, \quad b = \frac{h.\cos A}{2\sin^2\frac{A}{2}} \text{ \&c.}$$

(80.) From the following table, in which all the parts of a right-angled triangle are given, any *data* may be chosen, and the preceding formulæ verified.

$$c = 128{\cdot}4327 \quad \log c = 2{\cdot}1086756$$
$$b = 66{\cdot}1364 \quad \log b = 1{\cdot}8204405$$
$$a = 110{\cdot}0951 \quad \log a = 2{\cdot}0417681$$
$$A = 59^\circ\ 0'\ 21''{\cdot}25 \qquad B = 30^\circ\ 59'\ 38''{\cdot}75$$
$$L.\sin A = L\cos B = 9{\cdot}9330925$$
$$L\cos A = L\sin B = 9{\cdot}7117649$$
$$L\tan A = L\cot B = 10{\cdot}2213276$$

(81.) We shall now proceed to the cases of oblique-angled triangles. The three angles are connected by any of the following relations.

$$A + B + C = 180^\circ \qquad A + B = 180^\circ - C \qquad \frac{A+B}{2} = 90^\circ - \frac{C}{2} \text{ \&c.}$$

$$\sin(A + B) = \sin C, \quad \cos(A + B) = -\cos C, \quad \tan(A + B) = -\tan C \text{ \&c.}$$

$$\sin\tfrac{1}{2}(A + B) = \cos\tfrac{1}{2}C, \quad \cos\tfrac{1}{2}(A + B) = \sin\tfrac{1}{2}C, \quad \tan\tfrac{1}{2}(A + B) = \cot\tfrac{1}{2}C \text{ \&c.}$$

Again, $\sin^2(A + B) = \sin^2 A.\cos^2 B + \cos^2 A.\sin^2 B + 2\sin A\cos A\sin B\cos B$

for $\cos^2 A$ and $\cos^2 B$ write $1 - \sin^2 A$ and $1 - \sin^2 B$, which gives

$$\sin^2(A + B) = \sin^2 A + \sin^2 B + 2\sin A.\sin B(\cos A\cos B - \sin A.\sin B)$$

But, $\quad \sin(A + B) = \sin C \qquad \cos(A + B) = -\cos C$

or $\quad \sin^2 C = \sin^2 A + \sin^2 B - 2\sin A\sin B\cos C$

Similarly, $\quad \sin^2 B = \sin^2 C + \sin^2 A - 2\sin C\sin A\cos B$

$$\sin^2 A = \sin^2 B + \sin^2 C - 2\sin B\sin C\cos A$$

Again, $\quad \tan(A + B) = -\tan C = \dfrac{\tan A + \tan B}{1 - \tan A.\tan B}$

or $\quad \tan A + \tan B + \tan C = \tan A.\tan B.\tan C$

(82.) Let pU, qU, and rU, be the three perpendiculars let fall from the vertices of the triangle upon the sides aU, bU, and cU. And let A′, B′, and C′ be the exterior angles of the triangle adjacent to A, B, and C. Then, if A be an obtuse angle, there are right-angled triangles, having for hypothenuses bU and cU, and for sides opposite to A′, rU and qU. If A be a right angle we have $b = r$, and $c = q$. If A be an acute angle, we have right-angled triangles having for hypothenuses bU and cU, and for sides opposite to A, rU and qU. And the same of B and C. Again, $A + A' = 180°$, $B + B' = 180°$, $C + C' = 180°$, and we have

$$p = b \sin C \quad \text{or} \quad b \sin C' = b \sin C, \text{ in both cases}$$
$$p = c \sin B \quad \text{or} \quad c \sin B' = c \sin B, \text{ in both cases}$$

That is, $$b \sin C = c \sin B \quad \text{or} \quad \frac{b}{c} = \frac{\sin B}{\sin C}$$

Similarly, $$\frac{c}{a} = \frac{\sin C}{\sin A} \quad \text{and} \quad \frac{a}{b} = \frac{\sin A}{\sin B}$$

That is, any two sides are proportional to the sines of the opposite angles. This is the formula upon which all others relative to triangles will be made to depend.

(83.) Divide both sides of the value of $\sin^2 C$ in (81.) by $\sin^2 C$; substitute the ratios of the sides for those of the sines of angles, and we have

$$1 = \frac{a^2}{c^2} + \frac{b^2}{c^2} - 2\frac{a}{c} \cdot \frac{b}{c} \cdot \cos C$$

or $$c^2 = a^2 + b^2 - 2ab \cos C, \qquad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Similarly, $$b^2 = c^2 + a^2 - 2ca \cos B, \qquad \cos B = \frac{c^2 + a^2 - b^2}{2ca}$$

$$a^2 = b^2 + c^2 - 2bc \cos A, \qquad \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

These formulæ may be readily deduced from the triangle itself, it being obvious that $p = b \sin C$, and also that

$$c^2 = (b \sin C)^2 + (a - b \cos C)^2$$

(84.) We now proceed to put the preceding expressions in a form convenient for logarithmic computations:

$$\cos C = \frac{(a+b)^2 - c^2 - 2ab}{2ab} = \frac{(a+b)^2 - c^2}{2ab} - 1$$

Also $$\cos C = \frac{(a-b)^2 - c^2 + 2ab}{2ab} = \frac{(a-b)^2 - c^2}{2ab} + 1$$

or $$1 + \cos C = \frac{(a+b+c)(a+b-c)}{2ab} \qquad 1 - \cos C = \frac{(c+a-b)(c+b-a)}{2ab}$$

Let $a+b+c = 2s$ then $a+b-c = 2(s-c)$

$b+c-a = 2(s-a)$, $c+a-b = 2(s-b)$

Substitute these, and also the values of $1+\cos C$, &c. which gives

$$\cos^2\frac{C}{2} = \frac{s(s-c)}{ab} \qquad \sin^2\frac{C}{2} = \frac{(s-a)(s-b)}{ab}$$

Similarly $$\cos^2\frac{B}{2} = \frac{s(s-b)}{ac} \qquad \sin^2\frac{B}{2} = \frac{(s-a)(s-c)}{ac}$$

$$\cos^2\frac{A}{2} = \frac{s(s-a)}{bc} \qquad \sin^2\frac{A}{2} = \frac{(s-b)(s-c)}{bc}$$

$$\tan^2\frac{A}{2} = \frac{(s-b)(s-c)}{s(s-a)} \quad \tan^2\frac{B}{2} = \frac{(s-a)(s-c)}{s(s-b)} \quad \tan^2\frac{C}{2} = \frac{(s-a)(s-b)}{s(s-c)}$$

$$\sin A = \frac{2V}{bc} \qquad \sin B = \frac{2V}{ac} \qquad \sin C = \frac{2V}{ab}$$

where $$V = \sqrt{\{s(s-a)(s-b)(s-c)\}}$$

This is derived from the preceding, by aid of $\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \frac{2V}{abc}$$

$$\tfrac{1}{2}bc\sin A = \tfrac{1}{2}ca\sin B = \tfrac{1}{2}ab\sin C = V$$

(85.) To reduce $c^2 = a^2+b^2-2ab\cos C$ to a form adapted for logarithmic computation, proceed as follows:

1. $c^2 = (a+b)^2-2ab(1+\cos C)$

$$= (a+b)^2-4ab\cos^2\frac{C}{2} = (a+b)^2\left\{1-\frac{4ab}{(a+b)^2}\cos^2\frac{C}{2}\right\}$$

Now, $(a+b)^2-4ab$ being $(a-b)^2$ is positive; therefore $(a+b)^2$ is greater than $4ab$ and

$$\frac{4ab}{(a+b)^2} \text{ and still more } \frac{4ab}{(a+b)^2}\cos^2\frac{C}{2} \text{ is less than } 1.$$

Compute the positive square root of the last expression, and find in the tables the angle of which it is the sine; or find K from

$$\sin K = \frac{2\sqrt{ab}.\cos\frac{1}{2}C}{a+b}$$

Then $c^2 = (a+b)^2\{1-\sin^2 K\}$ or $c = (a+b)\cos K$

2. $$c^2 = (a-b)^2+2ab(1-\cos C)$$

$$= (a-b)^2\left\{1+\frac{4ab}{(a-b)^2}\sin^2\tfrac{1}{2}C\right\}$$

Compute K′ from $\tan K' = \frac{2\sqrt{ab}\sin\frac{1}{2}C}{a-b}$

Then $c^2 = (a-b)^2\left\{1+\tan^2 K'\right\}$ $c = \frac{a-b}{\cos K'}$

(86.) Lastly, $\frac{a}{b} = \frac{\sin A}{\sin B}$ gives (54.) and (81.)

$$\frac{a-b}{a+b} = \frac{\sin A - \sin B}{\sin A + \sin B} = \frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)} = \frac{\tan\frac{1}{2}(A-B)}{\cot\frac{1}{2}C}$$

or $$\tan\tfrac{1}{2}(A-B) = \frac{a-b}{a+b}\cot\tfrac{1}{2}C$$

(87.) The preceding formulæ are sufficient for our general purpose. We shall now proceed to different cases.

The student may verify the methods as they are produced upon the sides, &c. of the following triangle:

$a = 15{\cdot}236$ $\log a = 1{\cdot}1828710$ $s-a = 3{\cdot}098$ $\log(s-a) = 0{\cdot}4910814$
$b = 12{\cdot}414$ $\log b = 1{\cdot}0939117$ $s-b = 5{\cdot}920$ $\log(s-b) = 0{\cdot}7723217$
$c = 9{\cdot}018$ $\log c = 0{\cdot}9551102$ $s-c = 9{\cdot}316$ $\log(s-c) = 0{\cdot}9692295$

$s = 18{\cdot}334$ $\log s = 1{\cdot}2632572$ $V = 55{\cdot}96866$ $\log V = 1{\cdot}7479449$

$a+b = 27{\cdot}650$ $\log(a+b) = 1{\cdot}4416951$ $a-b = 2{\cdot}822$ $\log(a-b) = 0{\cdot}4505570$
$b+c = 21{\cdot}432$ $\log(b+c) = 1{\cdot}3310627$ $b-c = 3{\cdot}396$ $\log(b-c) = 0{\cdot}5309677$
$a+c = 24{\cdot}254$ $\log(a+c) = 1{\cdot}3847834$ $a-c = 6{\cdot}218$ $\log(a-c) = 0{\cdot}7936507$

	° ′ ″	L. sin.	L. cos.	L. tan.
A	= 89 9 23·54	9·9999530	8·1679268	11·8320262
B	= 54 33 25·12	9·9109937	9·7633479	10·1476458
C	= 36 17 11·48	9·7721922	9·9063714	9·8658208
½A	= 44 34 41·77	9·8462647	9·8526583	9·9936064
½B	= 27 16 42·56	9·6611649	9·9487989	9·7123661
½C	= 18 8 35 74	9 4933102	9·9778520	9·5154583
½(A—B)	= 17 17 59·21	9·4722989	9·9798951	9·4924039
½(B—C)	= 9 8 6·82	9·2007551	9·9944564	9·2062988
½(A—C)	= 26 26 6·03	9·6485379	9·9520365	9·6965016
K_a	= 44 41 30·6	9·8471366	9·8518083	....
K_b	= 59 12 50·4	9·9340361	9·7091283	....
K_c	= 70 57 53·6	9·9755783	9·5134140	....
K'_a	= 77 7 15·5	....	9·3408970	10·6408380
K'_b	= 59 56 28·9	....	9·6997390	10·2375348
K'_c	= 71 45 50·9	....	9·4954464	10·4821746

where, by K_a is meant the angle K of (85.), as obtained when a is to be found, or from $a^2 = b^2 + c^2 - 2bc\cos A$. In the following table, pU, qU, and rU, are, as before, the perpendiculars on a, b, and c, and a_bU means the segment of aU adjacent to bU, &c.

$p = 7{\cdot}347$	$\log p = 0{\cdot}8661039$		
$q = 9{\cdot}017$	$\log q = 0{\cdot}9550632$		
$r = 12{\cdot}413$	$\log r = 1{\cdot}0938647$		
$a_b = 10{\cdot}006$	$\log a_b = 1{\cdot}0002831$	$a_c = 5{\cdot}230$	$\log a_c = 0{\cdot}7184581$
$b_a = 12{\cdot}281$	$\log b_a = 1{\cdot}0892424$	$b_c = {\cdot}133$	$\log b_c = 9{\cdot}1230370$
$c_a = 8{\cdot}835$	$\log c_a = 0{\cdot}9462189$	$c_b = {\cdot}183$	$\log c_b = 9{\cdot}2618385$

In all the following article, a is the greatest side, b the mean, and c the least. Consequently, A is the greatest angle, &c. $\{\quad\}$ means that the quantity enclosed is the one found by the process; all the others in it being given or previously found.

(88.) First Case. Given the three sides, required the angles.

First method. (6.) $a_b^2 - a_c^2 = b^2 - c^2$ or $(a_b - a_c)a = (b+c)(b-c)$

$$\log\{a_b - a_c\} = \log(b+c) + \log(b-c) - \log a$$

Hence, $a_b - a_c$ is found: let it be h.

$$\{a_b\} = \tfrac{1}{2}(a+h) \qquad \{a_c\} = \tfrac{1}{2}(a-h)$$

$$\log.\cos\{C\} = \log a_b - \log b \qquad \log.\cos(B) = \log a_c - \log c$$

$$\{A\} = 180° - (B + C)$$

This method is the shortest when *all* the angles are wanted, but should not be used (68.) when one of the angles is very small; or one of the sides very small in proportion to the rest. When one angle only is wanted, use the

Second method. To find A, use (84.) one of these,

$$\text{L.}\sin\{\tfrac{1}{2}A\} = 10 + \tfrac{1}{2}(\log\overline{s-b} + \log\overline{s-c} - \log b - \log c)$$

$$\text{L.}\cos\{\tfrac{1}{2}A\} = 10 + \tfrac{1}{2}(\log s + \log\overline{s-a} - \log b - \log c)$$

$$\text{L.}\tan\{\tfrac{1}{2}A\} = 10 + \tfrac{1}{2}(\log\overline{s-b} + \log\overline{s-c} - \log s - \log(s-a))$$

$$\{A\} = 2 \times \tfrac{1}{2}A$$

(89.) Second Case. Given two sides (a and b, a the greater), and the included angle C; required the rest.

First method. When both the other angles and the remaining side are required, use

$$\text{L.tan}\left\{\tfrac{1}{2}\overline{A-B}\right\} = \log(a-b) + \text{L.cot}\,\tfrac{1}{2}C - \log(a+b)^*$$
$$\left\{\tfrac{1}{2}\overline{A+B}\right\} = 90° - \tfrac{1}{2}C$$
$$\{A\} = \tfrac{1}{2}(A+B) + \tfrac{1}{2}(A-B) \qquad \{B\} = \tfrac{1}{2}(A+B) - \tfrac{1}{2}(A-B)$$
$$\log\{c\} = \text{L.}\sin C + \log a - \text{L}\sin A \quad \dagger$$

Second method. When the side only is required (85.), use either of the following:

$$\left.\begin{aligned}\text{L.sin}\{K\} &= \tfrac{1}{2}(\log a + \log b) + \log 2 + \text{L}\cos\tfrac{1}{2}C - \log(a+b)\\ \log\{c\} &= \log(a+b) + \text{L.}\cos K - 10\end{aligned}\right\}$$
$$\left.\begin{aligned}\text{L.tan}\{K'\} &= \tfrac{1}{2}(\log a + \log b) + \log 2 + \text{L.sin}\,\tfrac{1}{2}C - \log(a-b)\\ \log\{c\} &= \log(a-b) + 10 - \text{L.}\cos K'\end{aligned}\right\}$$

Third method. When the given angle is very nearly 180°, let it be $180° - C_{\prime}$, where $C_{\prime}$ is small; we have then

$$c^2 = a^2 + b^2 - 2ab\cos(180° - C_{\prime}) = a^2 + b^2 + 2ab\cos C_{\prime}$$
$$= (a+b)^2 - 4ab\sin^2\tfrac{1}{2}C_{\prime} = (a+b)^2\left\{1 - \frac{4ab}{(a+b)^2}\sin^2\tfrac{1}{2}C_{\prime}\right\}$$

By the binomial theorem $\sqrt{1-x} = 1 - \tfrac{1}{2}x$ nearly, x being small

$$c = (a+b)\left(1 - \frac{2ab}{(a+b)^2}.\sin^2\tfrac{1}{2}C_{\prime}\right) = a + b - \frac{2ab}{a+b}\sin^2\tfrac{1}{2}C_{\prime}$$

very nearly. But $\sin C_{\prime} = 2\sin\tfrac{1}{2}C_{\prime}.\cos\tfrac{1}{2}C_{\prime}$, or $\cos\tfrac{1}{2}C_{\prime}$ being very nearly 1, we have

$$\sin\tfrac{1}{2}C_{\prime} = \tfrac{1}{2}\sin C_{\prime}, \quad \sin^2\tfrac{1}{2}C_{\prime} = \tfrac{1}{4}\sin^2 C_{\prime}, \text{ very nearly:}$$
$$c = a + b - \tfrac{1}{2}\frac{ab\sin^2 C_{\prime}}{a+b} \quad \text{very nearly}$$

$$\log\{k\} = 2\,\text{L}\sin C_{\prime} + \log a + \log b - \log(a+b) - 20$$
$$c = a + b - \tfrac{1}{2}k$$

Fourth method. When b is very small compared with a, and the small angle B is wanted, we have

$$\frac{a}{b} = \frac{\sin A}{\sin B} = \frac{\sin(B+C)}{\sin B} = \cos C + \cot B.\sin C$$

* 10 is not added here to give the tabular logarithm, because L. cot is already too great by 10.

† The excess of the one tabular log. compensates that of the other.

$$\cot B = \frac{a - b\cos C}{b\sin C} \qquad \tan B = \frac{b\sin C}{a - b\cos C} = \frac{b}{a}\sin C$$

very nearly; hence b is readily found, very nearly.

(90.) THIRD CASE. Given two sides and an angle *not included*, required the rest.

It can be shewn immediately that this is a problem of the second degree, admitting sometimes of two solutions. Let a and b be the given sides, B the given angle; we have then

$$b^2 = a^2 + c^2 - 2ac\cos B$$

or

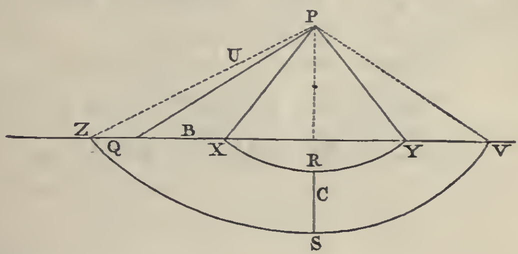
$$\{c\} = a\cos B \pm \sqrt{b^2 - a^2\sin^2 B}$$

$$\sin\{C\} = \frac{c\sin B}{b} \qquad \sin\{A\} = \frac{a\sin B}{b}$$

1. There is no such triangle at all when b is less than $a\sin B$.

2. Only positive values of c must be taken, for a negative value of C would give $\sin C$ negative, or C greater than two right angles, which is impossible in a triangle. The roots of the equation are both positive (*Algebra*, p. 139.) when $a^2 - b^2$ is positive, or a greater than b; in this case there are two triangles satisfying the conditions in question. When $a = b$ one triangle disappears, for then one value of c is 0; when a is less than b there is only one solution.

The geometrical construction will also shew this. Lay down the angle B equal to the given angle, and take $PQ = aU$. With



centre P and radius $PY = bU$ describe a circle, then, if bU be less than PR or $a\sin B.U$, there is no such triangle; if $bU = a\sin B.U$, there is a right-angled triangle QPR, under the given conditions; if, PY being greater than PR, it be still less than PQ, there are two triangles, QPX, and QPY, satisfying the conditions. But if PY be greater than PQ there is only one such triangle, QPV, for the triangle QPZ obviously has not the given angle, but its supplement.

For logarithmic solution, proceed as follows :

$$\mathrm{L.}\sin\{A\} = \log a + \mathrm{L.}\sin B - \log b$$

Here (50.) are two angles which satisfy the conditions; one less than a right angle, which call $A_{,}$, the other $A_{,,}$ or $180° - A_{,}$ greater. Or it may happen that $a\dfrac{\sin B}{b}$ may be greater than unity, in which case $\mathrm{L.}\sin A$ will be found greater than 10, or $\log.\sin A$ positive, and there is no such angle. In the case where this does not happen, we proceed as follows :

Either $\{C\} = 180° - A_{,} - B$ or $180 - A_{,,} - B$

Or $C_{,} = 180 - A_{,} - B$ $C_{,,} = A_{,} - B$

if $A_{,}$ be greater than B; for otherwise, $C_{,,}$ is negative, and is here inadmissible. Consequently, the two values of c being $c_{,}$ and $c_{,,}$, we have,

$$c_{,} = \frac{b\sin C_{,}}{\sin B} = \frac{b\sin(A_{,}+B)}{\sin B} \qquad c_{,,} = \frac{b\sin(A_{,}-B)}{\sin B}$$

$$\log\{c_{,}\} = \log b + \mathrm{L}\sin C_{,} - \mathrm{L}\sin B$$

$$\log\{c_{,,}\} = \log b + \mathrm{L}\sin C_{,,} - \mathrm{L}\sin B \text{ (if } C_{,,} \text{ be positive)}$$

(91.) Fourth Case. Given a side and two angles; (a U the side), required the rest.

$$\{\text{Third angle}\} = 180° - (\text{sum of given angles})$$

$$\log b = \log a + \mathrm{L}\sin B - \mathrm{L}\sin A$$

$$\log c = \log a + \mathrm{L}\sin C - \mathrm{L}\sin A$$

CHAPTER IV.

ON THE EXTENSION OF THE MEANINGS OF SYMBOLS, AND ON THE SQUARE ROOTS OF NEGATIVE QUANTITIES.

(92.) WE have as yet explicitly used only two different kinds of symbols: those of *quantity*, specific and general (arithmetical and algebraical), and those of *operation;* specific, as $+$ $-$, &c. and general, f, ϕ, &c. (*Algebra*, p. 203). But we are not therefore bound never to use any other symbols; the only laws by which our right to such aids is limited, are the following.

1. Neither the symbols themselves, nor any expressions in which they are used, must have different meanings of any such kind, such that the consequences of one meaning may be confounded with, and used for, the consequences of another.

2. The consequences of all assumptions must follow logically from the assumptions themselves.

It therefore becomes of interest to consider in what other possible ways we might use symbols. And first it must strike us that all yet employed may be styled under one name, more general than either operation or quantity. They are, in fact, symbols of *discrimination* or *distinction*. Thus, in a, b, and c, in which the symbolic difference is only difference of shape, that circumstance is made the distinction between difference of numerical magnitude. In $+$ and $-$ the same distinction, namely, of form, is made that of the direction given, as to which of two fundamental operations is to be performed. In ab and $\frac{a}{b}$ we see that difference of position, with the employment of a new, and not altogether necessary, symbol, is the distinction which implies difference of operations.

(93.) Let us now look at the extension of arithmetic into algebra, not confining ourselves to the notion of *operation* or *quantity*, but generalising our idea of symbols to that of mere discrimination; the object being to consider, whether, in this point of view, extension is *possible*, preliminary to the further question of whether it is *advisable*.

Looking narrowly at the steps by which we ascended (*Algebra*, p. 57. and 110.), we see that, so far as the discriminative quality of the symbols is concerned, we found the following consequence of our fundamental and *arithmetical* definitions of + — &c. namely, that *old rules were sufficient to express new distinctions* consistently with the old distinctions; in such sort that, whenever the new distinctions disappeared, the result was a legitimate consequence of the older notions, such as would have been obtained if the more circuitous method had been adopted, to avoid which the new distinctions were introduced. For example, when we came to the expressions $a+(+b)$ and $a+(-b)$, which have no meaning under the original meaning of + and —, we had extended our ideas to the following question. The distinction between a magnitude of any one kind and its diametrically opposite, being denoted by $*b$ and ¶b, if there occur a case of a problem in which $*b$ requires to be added to a, what will the similar problem require, in which ¶b is used where $*b$ was used in the other. And we found, 1. That where $*b$ requires addition, ¶b requires subtraction. 2. That $+a$ and $-a$ would themselves consistently express $*a$ and ¶a, provided that the rules appertaining to the original meaning of + and —, *and no others*, should be also applied to them in their new and distinctive capacity. And we also found that the disappearance of + and —, in their new character, was accompanied by the disappearance of all traces of the new distinctions; or that all theorems which in any case took the old forms only, were true under the old meanings. The old algebra (general arithmetic) was, in every sense, part of the new one.

(94.) Now, this we shall lay down as the restriction under which any *further* extension is to be made; namely, that all theorems at present existing, are to be theorems which are true, whenever they are the consequences of any further extension; and true in the sense in which they exist at present. We might propose infinite numbers of changes of meaning, under which *some* theorems would remain true, but in which others would not be so. For instance, if × placed between two quantities, were made to signify that their *sines* (not the numbers themselves) should be multiplied together, then (61.)

$$a\times a-b\times b = (a+b)\times(a-b)$$

would be true; but $a\times 2b = b\times 2a$ would not be true.

(95.) There might arise cases in which the answers of problems could not be expressed without more power of symbols than is possessed at present. For instance, will any one undertake to say, that of all possible problems, there is no one of which the answer is as follows: Divide 6754321 by 12, in the common way, with this exception, that whenever the remainder is an even number, it is to stand, but whenever it is an odd number, it is to be increased by 1, if the preceding remainder were even, and diminished by 1 if odd. The answer to this question would be,

$$12\overline{)6754321}$$
$$570367\tfrac{8}{12}$$

but no symbols, at present possessed, would describe the operation. Again, if we ask what is that expression which, when x is positive, is x^2, but when x is negative, is x^3? This cannot be expressed by present symbols; and the same may be said of many imaginable results. So much for the possibility of further extension, or a new symbol of distinction; the next question must be, is it wanted, and can it be made?

(96.) When the earlier algebraists first began to occupy themselves with questions expressed in general terms, the difficulties of subtraction soon became obvious, inasmuch as the greater would sometimes demand to be subtracted from the less. The science has been brought to its present state through three distinct steps. The first was tacitly to contend for the principle that human faculties, at the outset of any science, are judges both of the extent to which its results can be carried, and of the form in which they are to be expressed. *Ignorance*, the necessary predecessor of knowledge, was called *nature;* and all conceptions which were declared unintelligible by the former, were supposed to have been made impossible by the latter. The first who used algebraical symbols in a general sense, Vieta, concluded that subtraction was a defect, and that expressions containing it should be in every possible manner avoided. *Vitium negationis*, was his phrase. Nothing could make a more easy pillow for the mind, than the rejection of all which could give any trouble; but if Euclid had altogether dispensed with the *vitium parallelorum*, his geometry would have been confined to twenty-six propositions of the first book.

The next and second step, though not without considerable fault, yet avoided the error of supposing that the learner was a competent critic. It consisted in treating the results of algebra as necessarily true, and as representing some relation or other, however inconsistent they might be with the suppositions from which they were deduced. So soon as it was shewn that a particular result had no existence as a quantity, it was permitted, by definition, to have an existence of another kind, into which no particular inquiry was made, because the rules under which it was found that the new symbols would give true results, did not differ from those previously applied to the old ones. A symbol, the result of operations upon symbols, either meant quantity, or nothing at all; but in the latter case it was conceived to be a certain new kind of quantity, and admitted as a subject of operations, though not one of distinct conception. Thus, $1-2$, and $a-(a+b)$, appeared under the name of negative quantities, or quantities less than nothing. These phrases, incongruous as they always were, maintained their ground, because they always produced true results, whenever they produced any result at all which was intelligible: that is, the quantity less than nothing, in defiance of the common notion that all conceivable quantities are greater than nothing, and the square root of the negative quantity, an absurdity constructed upon an absurdity, always led to truths when they led back to arithmetic at all, or when the inconsistent suppositions destroyed each other. This ought to have been the most startling part of the whole process. That contradictions might occur, was no wonder; but that contradictions should uniformly, and without exception, lead to truth in algebra, and in no other species of mental occupation whatsoever, was a circumstance worthy the name of a mystery.

Nothing could prevail against the practical result, that theorems so produced were true; and at last, when the interpretation of the abstract negative quantity shewed that a part, at least, of the difficulty admitted of rational solution, the remaining part, namely, that of the square root of a negative quantity, was received, and its results admitted, with increased confidence.

(97.) The complete explanation of the embarrassing circumstances is comparatively modern; the latter arise from a very simple logical misconception, the assumption of the truth of a converse, namely, that if B follow from A, B follows from nothing else but A; or if A always yield B, B when it appears, must have been produced from A. We can imagine a, b, c, &c. $+$, $-$, $\times$, &c., defined in many different ways, so that certain of the theorems of algebra should severally be true under more than one set of meanings; and we have shewn an instance in page 72. We can further imagine it possible that one set of meanings should be so connected with another, that all theorems which are true in the first, should also be true in the second, and that, beside this, there should be other classes of theorems which are true in the second, but not true in the first. This is as possible as that one figure should be entirely contained in another, without filling it. We might thus conceive a succession of extensions of definitions, giving a series of sciences, each containing the whole of its predecessors, and more. Whence it is clear, that if the methods of operation, under any science, wandered beyond its limits, without that corresponding extension of definitions being made which converted the logic of one science into that of the other, the consequence would be, the appearance of some of the symbolic means of expressing truths in the wider science, without the key to their interpretation. This happened when we first came to the negative symbol in algebra (*Algebra*, p. 12), and we were under the necessity of adopting more extensive definitions. The proof of undue extension in the operations again occurred (p. 110.), where $\sqrt{-1}$ first appeared; but it was not then necessary to follow up the extensions necessary for its elucidation.

This matter is one of difficulty to a beginner, unused to the idea of a finished language having the meanings of all its terms extended so that the old meanings are only part of the new ones. But, in reality, he has gone through the process, by insensible steps, in his childhood.* Let him compare the first impression he was made to receive by the words "I see," with the sense he puts upon the phrase when, if he understand the preceding, he says he *sees* my meaning. That which I here call the extended use of the term, he will call the allegorical or metaphorical mode, that is, if we translate these Greek terms, the *other-speaking*, or *transferred* mode of expression. But in what way is the speech changed? By using the word to "see," not only as denoting perception by the eyes, but perception by the understanding in any way whatever. If we *heard* any one speak, we might still *see* his meaning.

(98.) Let us consider the most general meaning of any fundamental equation of algebra: for instance, $a + b = b + a$. We restrict

* The similarity of the views here given, with some in the review of Mr. Peacock's "Algebra," in the ninth volume of the "Journal of Education," makes it necessary for the author to state that he was also the writer of those articles.

ourselves to this only, that b and a, and $+$, mean the same thing in both places, that $=$ denotes, *gives the same result as*, or *is the same in effect as*, and that the order of the expression is from left to right, that is, on the first side we first examine a, and b on the second side. Then $a+b$ merely implying that something, b, is done in some manner, $+$, after something else, a, has been done; the above shews no more than that it is indifferent whether a is done first or b, or that the result is the same in both cases. Consistently with $a+b=b+a$, many meanings might be shewn to be impossible, and many possible. Neither need a and b stand in any way for quantities: for instance, the preceding would be true if a stood for a line not considered as a length, b for another line, $+$ for the formation of a rectangle out of a and b, and $=$ for equality of space enclosed. The preceding relation may therefore be a truth under an infinite number of different meanings. Let us now take another form, $ab=ba$, which admits of exactly as many meanings as the first, and denotes merely indifference of order. Suppose we pick out two such meanings at pleasure, and assign them, only requiring that a and b shall mean the same in both relations. We then have definitions for a, b, $+$ and the meaning of juxtaposition, and have explained $a+b=b+a$, and $ab=ba$, from which it follows that if $+$, as defined, will intelligibly apply to ab, we have $ab+b=b+ab$, &c. But if we would have our new algebra identical with the old one in forms, we must choose such meanings for the symbols in the two relations as will also make

$$a(a+b)=aa+ab \text{ represent a new truth.}$$

This is a restriction upon all the possible allowances of meaning which might be made: for it does not follow that every meaning which makes $a+b=b+a$, and $ab=ba$, true, also makes the last true. And other relations might be introduced which would still more restrict the meanings, and so on, until every fundamental relation necessary to algebra had been considered. If we could really collect all the possible meanings of each separate relation, and find the method of ascertaining which must be struck out for each and every new combination which the mechanism of algebra introduces, we should, if we could classify the remainder uniting those which are particular cases of a general meaning under their general head, be left with an *algebra* in the widest possible sense of the word.

We do not want to change operations; but we want to find all the definitions under which those operations will demonstratively enable us to pass from one truth to another.

(99.) Two explanations may be given of the manner in which $\sqrt{-1}$ may rationally be used; the first purely symbolical, that is, employing $\sqrt{-1}$ as a symbol, the meaning of which is given for convenience only; the second derived from geometry, and an extension of the method by which lines measured in opposite directions are represented by letters with different signs. The first is wholly algebraical; the second (for a reason which will afterwards appear) is an application of geometry to algebra.

Let k be a symbol which does not stand for quantity, but for a distinction, in whatever way it may be required to distinguish; that is, ka and a both stand for the same magnitude, but the first has the mark of either being used in a certain way, or appropriated for certain purposes, or liable to be rejected under certain circumstances, or whatever other distinction k may indicate. When two separated terms are multiplied together, as ka and kb, let the product be written k^2ab, in which k^2 merely implies the presence, in a product, of two terms which had the mark of distinction. Similarly, k^3ac is the distinction of a product made up of three terms which had the distinction, and so on. Let $k(a+b)$ be the distinction between $a+b$ and $ka+kb$, and so on. We are at liberty to assign to k any *discriminative power* we please. Let it be as follows: k is to be a distinction which ceases altogether in terms marked with k^4, k^8, k^{12}, &c. or k^{4n}, and with k^{-4}, k^{-8}, k^{-12}, &c. or k^{-4n}, and which is preserved in k^5, k^9, k^{13}, &c. k^{-3}, k^{-7}, k^{-11}, &c. or in k^{4n+1} where n is any whole number positive or negative. And let k^2, k^6, k^{10}, &c. k^{-2}, k^{-6}, k^{-10}, &c. be distinctions amounting to a change of sign in the terms denoted by them; so that if for k^2a we write $-a$, the object of the distinction is fulfilled, and the term need no longer be distinguished. Let k^3, k^7, k^{11}, &c. or k^{-1}, k^{-5}, k^{-9}, &c. imply both a change of sign and also the continuance of the distinction denoted by k; so that k^3a means $-ka$. We have then defined every thing except k itself, by the following identities;

$$k^{4n}a \text{ means } a, \quad k^{4n+1}a \text{ means } ka,$$
$$k^{4n+2}a \text{ means } -a, \quad k^{4n+3}a \text{ means } -ka$$

And thus every algebraical expression, when its distinctions are all marked, is reducible to the form $P+kQ$ where P and Q are algebraical. Thus,

$$a_0+a_1k+a_2k^2+\ \ldots \text{ means } a_0-a_2+a_4+\ \cdots\ +k(a_1-a_3+a_5-\&c.)$$

Now k itself is to have meaning as follows: in the equation

$$P+kQ = P'+kQ'$$

let the presence of k indicate that equality will still remain as well between the parts independent of k, as between those affected by k. If we had merely $P+Q=P'+Q'$ it would not at all follow that $P=Q$ and $P'=Q'$: but when we mean to make this additional supposition, let us signify the same by the presence of k.

Now, it will be obvious upon looking from (70.) to (73.) that we have here only made a notation to *express* distinctions which have been actually *arrived* at by process of reasoning. We found a method of embodying all the results previously obtained, of this kind: $2\cos\theta$ cannot be $x+\frac{1}{x}$; but if we work in any manner with $x+\frac{1}{x}$ and $x-\frac{1}{x}$, and produce an equation, then that same equation will correspond to one or two true equations, if we work in precisely the same manner with $2\cos\theta$ and $2k\sin\theta$, and then let k have its discriminative powers. And we shall then find that the result is the same as if

$2\cos n\theta$ had taken the place of $x^n+\frac{1}{x^n}$ and $2k\sin n\theta$ of $x^n-\frac{1}{x^n}$

When none but even numbers of ks occur, it is obvious that the result can be only one equation of the form $P=P'$; but when odd numbers of ks also occur, the result will be of the form $P+kQ=P'+kQ'$, giving two equations, $P=P'$, and $Q=Q'$.

(100.) Next, observe that the algebraical symbol $\sqrt{-1}$, which is certainly no quantity, positive or negative, and therefore not to be reasoned upon as a quantity, yet has this property, that if those rules be applied which would have been applied had it been a quantity, the results will be expressive of the distinction denoted by k. For in that case we have

$$\sqrt{-1} \text{ is } \sqrt{-1};\ (\sqrt{-1})^2 \text{ is } -1;\ (\sqrt{-1})^3 \text{ is } -\sqrt{-1};$$

$(\sqrt{-1})^4$ is 1; $(\sqrt{-1})^5$ is $\sqrt{-1}$; $(\sqrt{-1})^6$ is -1;
$(\sqrt{-1})^7$ is $-\sqrt{-1}$; $(\sqrt{-1})^8$ is 1.

Again,

$(\sqrt{-1})^{-1}$ is $-\sqrt{-1}$; $(\sqrt{-1})^{-2}$ is -1; $(\sqrt{-1})^{-3}$ is $\sqrt{-1}$, &c. from which the coincidence is apparent. Therefore, by making $\sqrt{-1}$ the symbol of the distinction meant by k, all the remaining distinctions will be drawn by applying the common rules of algebra to $\sqrt{-1}$ as if it were a quantity.

When in (71.) we began to compare $\cos\theta + k\sin\theta$ with $\varepsilon^{k\theta}$, we were also, in fact, comparing it with $1 + k\theta + k^2\frac{\theta^2}{2} + \ldots$, or (if we apply the meaning of k) with

$$1 - \frac{\theta^2}{2} + \frac{\theta^4}{2.3.4.} + \ldots + k\left(\theta - \frac{\theta^3}{2.3} + \frac{\theta^5}{2.3\,4.5} + \ldots\right)$$

Now, if the method of proceeding be valid, which extends the distinctive property of k to the developement of $\varepsilon^{k\theta}$, then

$$\cos\theta + k\sin\theta = 1 - \frac{\theta^2}{2} + \ldots + k\left(\theta - \frac{\theta^3}{2.3} + \ldots\right)$$

or $$\cos\theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{2.3.4} - \frac{\theta^6}{2.3.4.5.6} + \ldots$$

$$\sin\theta = \theta - \frac{\theta^3}{2.3} + \frac{\theta^5}{2.3.4.5} - \frac{\theta^7}{2.3.4.5.6.7} + \ldots$$

(101.) We shall now go through a strict deduction of these equations, which will shew that what we have done would, had we seen how, itself have been one. It is evident that

$$\cos\theta + k\sin\theta = \cos\theta + k\sin\theta \left\{ \begin{array}{c} \text{which amounts to} \\ \cos\theta = \cos\theta,\ \sin\theta = \sin\theta \end{array} \right.$$

Square both sides, which gives

$$(\cos\theta + k\sin\theta)^2 = \cos^2\theta + k.2\sin\theta.\cos\theta + k^2\sin^2\theta$$
$$= \cos^2\theta - \sin^2\theta + k.2\sin\theta.\cos\theta \quad = \quad \cos 2\theta + k\sin 2\theta$$

Generally, if $\quad \cos n\theta + k.\sin n\theta = (\cos\theta + k\sin\theta)^n$

$\times (\cos\theta + k\sin\theta)$ and $\left.\begin{array}{r}\cos n\theta.\cos\theta \\ + k^2\sin n\theta.\sin\theta\end{array}\right\} + k\left\{\begin{array}{l}\cos n\theta.\sin\theta \\ + \sin n\theta.\cos\theta\end{array}\right. = (\cos\theta + k\sin\theta)^{n+1}$

or $$\cos(n+1)\theta + k.\sin(n+1)\theta = (\cos\theta + k\sin\theta)^{n+1}$$

So that this relation, if true for one whole value of n, is true for the next. But it is true for $n = 1$ and $n = 2$, therefore it is true for all. This is called *De Moivre's Theorem.*

Develope the second side, substitute the meanings of k^2, k^3, &c., and form the two resulting equations, which will be found to be

$$\cos n\theta = c^n - n\frac{n-1}{2}c^{n-2}s^2 + n.\frac{n-1}{2.}\ \frac{n-2}{3}\ \frac{n-3}{4}c^{n-4}s^4 - \ \ldots.$$

$$\sin n\theta = nc^{n-1}s - n\frac{n-1}{2}\ \frac{n-2}{3}c^{n-3}s^3 + n\frac{n-1}{2}\ \frac{n-2}{3}\ \frac{n-3}{4}\ \frac{n-4}{5}c^{n-5}s^5 - \ \ldots$$

where c means $\cos\theta$, and s, $\sin\theta$. Divide both sides by c^n, and put t ($\tan\theta$) for $\frac{c}{s}$; at the same time divide and multiply n, $n\frac{n-1}{2}$, &c. by a power of n of the same number of factors, which gives

$$\frac{\cos n\theta}{c^n} = 1 - \frac{1-\frac{1}{n}}{2}(nt)^2 + \frac{1-\frac{1}{n}}{2}\ \frac{1-\frac{2}{n}}{3}\ \frac{1-\frac{3}{n}}{4}\ (nt)^4 - \ \ldots.$$

$$\frac{\sin n\theta}{c^n} = nt - \frac{1-\frac{1}{n}}{2}\ \frac{1-\frac{2}{n}}{3}(nt)^3 + \frac{1-\frac{1}{n}}{2}\ \frac{1-\frac{2}{n}}{3}\ \frac{1-\frac{3}{n}}{4}\ \frac{1-\frac{4}{n}}{5}(nt)^5 - \ \ldots.$$

which is true for all whole values of n, and all values of θ. Now, consider all those whole values of n, and values of θ, which make $n\theta = x$, a given angle: whence (*Algebra*, p. 157.) the limits of the two sides of each of the preceding, made by increasing n without limit, will be equal. We proceed to find these limits. We have

$c^n = \left(\cos\frac{x}{n}\right)^n$ the limit of which (70.) is 1; $\frac{1}{n}$, $\frac{2}{n}$, $\frac{3}{n}$, &c. the limits of which are severally 0; nt or $n\tan\frac{x}{n}$ or $x\left(\tan\frac{x}{n} \div \frac{x}{n}\right)$ the limit of which (48.) is $x \times 1$, or x; and $n\theta$, which is x throughout.

Take the limits of both sides, which give

$$\left.\begin{aligned}\cos x &= 1 - \frac{x^2}{2} + \frac{x^4}{2.3.4} - \text{ \&c.}\\ \sin x &= x - \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} - \text{ \&c.}\end{aligned}\right\}\text{ the theorem in question.}$$

Hence we have

$$\cos\theta + k\sin\theta = 1 - \frac{\theta^2}{2} + \cdot\cdot + k\left(\theta - \frac{\theta^3}{2.3} + \ldots.\right)$$

$$= 1 - k\theta + k^2\frac{\theta^2}{2} + k^3\frac{\theta^3}{2.3} + \ldots. = \varepsilon^{k\theta}$$

if we abbreviate the preceding series into the formula of which it would be the algebraical developement, if k were a quantity. We shall now adopt the symbol $\sqrt{-1}$ for k, which gives (for all values of θ)

$$\cos\theta + \sin\theta.\sqrt{-1} = \varepsilon^{\theta\sqrt{-1}}$$

For θ write $-\theta$ (44.) $\cos\theta - \sin\theta.\sqrt{-1} = \varepsilon^{-\theta\sqrt{-1}}$

$$\cos\theta = \frac{\varepsilon^{\theta\sqrt{-1}} + \varepsilon^{-\theta\sqrt{-1}}}{2} \qquad \sin\theta = \frac{\varepsilon^{\theta\sqrt{-1}} - \varepsilon^{-\theta\sqrt{-1}}}{2\sqrt{-1}}$$

that is, if we develope the preceding exponential expressions, paying attention to the difference of developement denoted by $\sqrt{-1}$, $(\sqrt{-1})^2$ &c. we shall arrive at series which we have shewn to be the developements of $\sin\theta$ and $\cos\theta$.

(102.) Some marks of distinction are definition, and all the rest are consequences. Thus $\varepsilon^{k\theta}$ has k a symbol of distinction; but it does not mean $k.\varepsilon^{\theta}$, but $\cos\theta + k\sin\theta$. And there are some results which when enunciated as if $\sqrt{-1}$ were a quantity, are yet more inconceivable than less than nothing, in an arithmetical sense (*Algebra*, p. 62.) For instance, calling $k\theta$ the logarithm of $\varepsilon^{k\theta}$, we have, making $\theta = 2n\pi$, n being a whole number ($\cos\theta = 1$, $\sin\theta = 0$)

$$\varepsilon^{2n\pi\sqrt{-1}} = 1 \qquad \log 1 = 2n\pi\sqrt{-1}$$

thus 1 has, under our present extension, an infinite number of logarithms, corresponding to all whole values of n, positive and negative, namely,

$$\ldots\ldots -2\pi\sqrt{-1}, \quad -\pi\sqrt{-1}, \quad 0, \quad \pi\sqrt{-1}, \quad 2\pi\sqrt{-1}, \ldots\ldots$$

among which the arithmetical logarithm is found, namely, 0. Similarly we may deduce

$$\varepsilon^{(2n+1)\pi\sqrt{-1}} = -1 \qquad \log(-1) = (2n+1)\pi\sqrt{-1}$$

Let x be the arithmetical logarithm of y, then we have

$$y = \varepsilon^{x} = \varepsilon^{x} \times 1 = \varepsilon^{x} \times \varepsilon^{2n\pi\sqrt{-1}} = \varepsilon^{x+2n\pi\sqrt{-1}}$$

or all the values of $x + 2n\pi\sqrt{-1}$ are also logarithms of y. The theorem $\log a + \log b = \log ab$ now exists in this form: if any logarithm of a be added to any logarithm of b, the sum is *one of the* logarithms of ab. Thus, if $\log. a$ stand for the arithmetical logarithm of a, and λa for its general logarithm, we have

$$\lambda a + \lambda b = \log a + 2\pi n\sqrt{-1} + \log b + 2\pi n'\sqrt{-1}$$
$$= \log(ab) + 2\pi(n + n')\sqrt{-1} = \lambda ab$$

(103.) We can now give the forms of all the different roots of a number, which was done to the fourth degree in p.113 of the *Algebra*. If we take the equation

$$1 = \varepsilon^{2n\pi\sqrt{-1}} = \cos 2n\pi + \sin 2n\pi \sqrt{-1}$$

we have $$(1)^{\frac{1}{m}} = \varepsilon^{2\frac{n}{m}\pi\sqrt{-1}} = \cos\frac{2n\pi}{m} + \sin\frac{2n\pi}{m}\sqrt{-1}$$

from which it might appear that there are as many mth roots of unity as there are whole values, positive and negative, of n. But it will be found that these values recur as we give to n different values in succession; as follows:

First root $n = 0$ First value of $(1)^{\frac{1}{m}}$ is $\cos 0 + \sin 0 . \sqrt{-1}$ or 1

Second root $n = 1$ Second . . . . $\cos\frac{2\pi}{m} + \sin\frac{2\pi}{m}\sqrt{-1}$

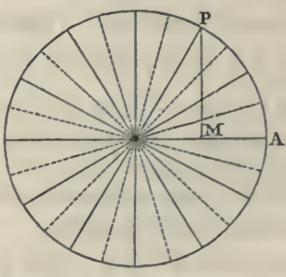
Third root $n = 2$ Third . . . . $\cos\frac{4\pi}{m} + \sin\frac{4\pi}{m}\sqrt{-1}$

. . . .

. . . .

mth root $n = m - 1$ mth . . . . $\cos\frac{2(m-1)\pi}{m} + \sin\frac{2(m-1)\pi}{m}\sqrt{-1}$

The $(m+1)$th value is $\cos\frac{2m\pi}{m} + \sin\frac{2m\pi}{m} . \sqrt{-1}$ or $\cos 0 + \sin 0 \sqrt{-1}$, the same as the first. The $(m+2)$th value is $\cos\frac{2(m+1)\pi}{m} + \sin\frac{2(m+1)\pi}{m}\sqrt{-1}$ or $\cos\left(2\pi + \frac{2\pi}{m}\right) + \sin\left(2\pi + \frac{2\pi}{m}\right)\sqrt{-1}$, the same as the second, and so on. To find the twelfth roots, for



instance, draw a circle, with the centre O. Divide its circumference into 12 equal parts, beginning at A, any given point. Then every point of subdivision shews a root as follows. If the radius be the linear unit, and if PM and MO represent the fractions of a linear unit which are in those lines; then,

$$\text{one value of } (1)^{\frac{1}{12}} \text{ is } OM + PM . \sqrt{-1}$$

(104.) It may be very soon shewn, that if $c + s\sqrt{-1}$ be a root of unity, $c - s\sqrt{-1}$ is also a root. Firstly, because if $\varepsilon^{x\sqrt{-1}}$ be a root, or if

$$\varepsilon^{nx\sqrt{-1}} = 1 \text{ we have } \left(\varepsilon^{nx\sqrt{-1}}\right)^{-1} = 1 = \left(\varepsilon^{-x\sqrt{-1}}\right)$$

or, if $\cos x + \sin x . \sqrt{-1}$ be a root, $\cos x - \sin x\sqrt{-1}$ is another. Secondly, in the preceding list of roots we see from

$$\frac{2(m-1)\pi}{m} \text{ or } 2\pi - \frac{2\pi}{m} \text{ that the last root is } \cos\left(-\frac{2\pi}{m}\right) + \sin\left(-\frac{2\pi}{m}\right)\sqrt{-1}$$

$$\text{or } \cos\frac{2\pi}{m} - \sin\frac{2\pi}{m}\sqrt{-1} \text{ the second being } \cos\frac{2\pi}{m} + \sin\frac{2\pi}{m}\sqrt{-1}$$

The same will also appear from consideration of the figure.

(105.) By proceeding in the same manner with $\varepsilon^{(2n+1)\pi\sqrt{-1}}$ or -1, we find

$$(-1)^{\frac{1}{m}} = \cos\frac{(2n+1)\pi}{m} + \sin\frac{(2n+1)\pi}{m}\sqrt{-1}$$

and, as before, it may be proved, that there are m roots, and no more. But the roots of -1 may be more easily set forth to the eye, by means of the roots of $+1$, as follows. Every root of -1 is twice as high a root of $+1$; for if $x^m = -1$, then $x^{2m} = 1$. Consequently, if we take all the twenty-fourth roots of unity, or divide the circle in last article into 24 parts, all those 24th roots of $+1$ which are not also 12th roots, are 12th roots of -1.

(106.) Let all roots of unity of the same order be called corresponding roots: thus there are m corresponding roots of $+1$ of the mth order. Then, all powers of a root are corresponding roots. For if $\mu^m = 1$, then μ^{2m} or $(\mu^2)^m = 1$, &c. And this holds equally of all negative whole powers. But it does not therefore follow that, among the powers of any one root, will be found all the other corresponding roots.

Let us denote $\varepsilon^{\pi x\sqrt{-1}}$ by $[x]$ for the present. Then we have

$$[nx] = [x]^n,\quad [x+2p] = [x],\quad [2p] = 1$$

where n and p are whole numbers, positive or negative. The mth roots of unity are

$$1 = \left[0\right],\ \left[\frac{2}{m}\right],\ \left[\frac{4}{m}\right],\ \left[\frac{6}{m}\right],\ \text{up to}\ \left[\frac{2(m-1)}{m}\right].$$

Let us consider the case of $m = 8$. Then the powers of $[0]$ are severally $= [0]$, and never produce more than one root. The powers of $[2 \div 8]$ are as follows:

$$\left[\frac{2}{8}\right],\ \left[\frac{4}{8}\right],\ \left[\frac{6}{8}\right],\ \left[\frac{8}{8}\right] = -1,\ \left[\frac{10}{8}\right],\ \left[\frac{12}{8}\right],\ \left[\frac{14}{8}\right],\ \left[\frac{16}{8}\right] = 1,\ \&c$$

or all the roots are produced in order. But the powers of $\left[\frac{4}{8}\right]$ are

$$\left[\frac{4}{8}\right],\ \left[\frac{8}{8}\right] = -1,\ \left[\frac{12}{8}\right],\ \left[\frac{16}{8}\right] = 1,\ \left[\frac{20}{8}\right] = \left[\frac{4}{8}\right]\ \&c.$$

and there is a continual recurrence of half the roots only. The powers of $\left[\frac{6}{8}\right]$ are

$$\left[\frac{6}{8}\right],\ \left[\frac{12}{8}\right],\ \left[\frac{18}{8}\right] = \left[\frac{2}{8}\right],\ \left[\frac{24}{8}\right] = -1,$$

$$\left[\frac{30}{8}\right] = \left[\frac{14}{8}\right],\ \left[\frac{36}{8}\right] = \left[\frac{4}{8}\right],\ \left[\frac{42}{8}\right] = \left[\frac{10}{8}\right],$$

$\left[\frac{48}{8}\right] = 1$, followed by recurrence.

Here again are all the roots. Those roots, whose powers give all the roots of their kind, are called *primitive*. It is enough for our present purpose to know that there is one primitive root of any order.

(107.) The roots of $+\sqrt{-1}$ and $-\sqrt{-1}$ may be now obtained. For we have

$$\varepsilon^{\frac{\pi}{2}\sqrt{-1}} = \sqrt{-1} = \varepsilon^{(2n+\frac{1}{2})\pi\sqrt{-1}},\quad \varepsilon^{\frac{3\pi}{2}\sqrt{-1}} = -\sqrt{-1} = \varepsilon^{(2n+\frac{3}{2})\pi\sqrt{-1}}$$

$$(\sqrt{-1})^{\frac{1}{m}} = \varepsilon^{\frac{4n+1}{2m}\pi\sqrt{-1}}\qquad (-\sqrt{-1})^{\frac{1}{m}} = \varepsilon^{\frac{4n+3}{2m}\pi\sqrt{-1}}$$

which may, as before, be shewn to have m values only.

To take an instance for verification: the cube roots of $+\sqrt{-1}$ are

$$\cos\frac{1}{6}\pi + \sin\frac{1}{6}\pi\sqrt{-1} \quad \cos\frac{5}{6}\pi + \sin\frac{5}{6}\pi\sqrt{-1} \quad \cos\frac{9}{6}\pi + \sin\frac{9}{6}\pi\sqrt{-1}$$

or $$\frac{1}{2}\sqrt{3} + \frac{1}{2}\sqrt{-1} \quad -\frac{1}{2}\sqrt{3} + \frac{1}{2}\sqrt{-1} \quad -\sqrt{-1}$$

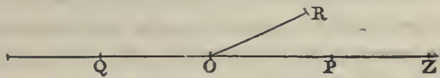
$$\left(\frac{1}{2}\sqrt{3} + \frac{1}{2}\sqrt{-1}\right)^3 = \frac{1}{8}\left(3\sqrt{3}+9\sqrt{-1}-3\sqrt{3}-\sqrt{-1}\right) = \sqrt{-1}$$

$$(-\sqrt{-1})^3 = \sqrt{-1}$$

(108.) Having shewn that the rules which *would* apply to $\sqrt{-1}$ *if* it could be considered as a quantity, will of themselves make all the necessary distinctions between the formulæ of common algebra, and the shape in which they become formulæ of trigonometry, we shall now proceed to the second method of explanation, or rather to the method of application, which shews the geometrical meaning of the symbols in question. As a starting point, we return again to the method of explanation of the negative sign. In looking at $a+(-b)$ we found that the sign $+$ no longer preserved the meaning of arithmetical addition, while the quantity operated on, $-b$, was no longer simply a number, but a number with a sign of direction. We might apply the preceding method of explanation to the passage from arithmetic to algebra. In this case ka would signify a distinction of this kind; terms having k^2, k^4, &c. are all to be of one kind, unmarked, while terms having k, k^3, k^5, &c. are all to be of another kind, marked with the distinction k. If we adopted this signification we should soon find that all theorems which are true when the distinction k means nothing, and may be entirely abolished, are also true when the distinction k means simply *change of sign*, *if it be arithmetically allowable*. And it would also be found that were it allowable to consider $0-1$ as a quantity, the rules which would apply to this latter symbol are precisely those by which the necessary distinctions would be drawn in the course of the process, without any particular attention. We might thus dispense with subtraction at the outset, and establish all theorems in which arithmetically additive terms only occur. And subtraction might be introduced in time by means of a distinctive symbol. We cannot make this an illustration for a beginner, because to place it on the same footing as the subject of (73.), we must require him to imagine himself divested of some of his most simple notions, and thereby reduced to learn things which now are axioms, by a long process of reasoning. He must conceive himself unable to form a distinct notion of subtraction, other than the *inverse of addition;* that is, though the notion of taking away that which he himself has just added may be simple, yet the idea of instituting a subtraction independently of any previously expressed addition, must be one to be learnt with some difficulty. If this were the case, we could make the use of a distinctive symbol facilitate the acquirement of the general operation of subtraction; as it is, the last-mentioned process is one of which we have a clear idea, independently of addition.

(109.) We found that the interpretation of a negative quantity was a magnitude taken in precisely the opposite sense and meaning to that which we imagined, when we applied arithmetical process to the determination of that magnitude. And the affairs of life contain so many such interpretations that the extension looks natural; so natural indeed, that many have drawn a great distinction between the negative quantity, and the square root of the negative quantity. Of this the application of the term impossible to the latter symbol only, is a sufficient instance: both are impossible, according to arithmetical notions, but the latter only has received the name. There is hardly a phenomenon in nature, or a relation of life, which does not admit of the modes of neutrality, excess on one side, or on the other. Where-ever there are two opposites, with a state between them which does not belong to either, we have the means of illustrating the terms, *positive, nothing,* and *negative.* Without specifying what we are speaking of, it would at once be granted that where the establishment of one state of things would cause increase, that of the opposite state would cause diminution, and that of the neutral state neither increase nor diminution. If we consider time as divided into *before, after,* and *at;* pecuniary relations divided into those of debtor, creditor, and neither debtor nor creditor; a balloon with weights, as either rising, sinking, and neither rising nor sinking; electricity as producing attraction, repulsion, or neither attraction nor repulsion, &c. &c. we have the same common modes of existence: *and in all the cases we have mentioned, no others whatsoever.* It is in considering *space* only that we have these modes, *and others,* as follows:

(110.) As long as we consider ourselves at liberty to change the position of a point in a given straight line, and in that straight line only, we have the three modes already considered, and no others. The point P may lie on one side or the other of a standard point O, given to measure from, or it may coincide with the point O. Let our



notion of space be contained in length, and we pass with perfect continuity from P to Q, by continually diminishing OP, until P coincides with O, and then continuing the motion of P on the left. We have here perfect analogy with all the other kinds of quantity which we can conceive. Let O represent the commencement of the Christian era; take an inch to a year, and we have the means of making a table of all conceivable modes of time, and by laying down points corresponding to different events, we might reduce chronology to a science of feet and inches. *But* if we allow all space to enter the question, we may bring OP into the position OQ without diminishing its length, by turning it round through two right angles. And if we now consider the first passage from P to Q, relatively to the new notion we have introduced, we see direct *discontinuity*. The line OP always continues making an angle *nothing** with its first position, until P coincides with O, when OP is not a line, and the idea of opening, actual or possible, ceases altogether: immediately afterwards, OP, now become OQ, makes two right angles, or half a revolution, with its first position. But it might have made this at two steps of a right angle each, or at four of half a right angle each, &c. &c. Consequently, in geometry, we can pass from positive to negative by an infinite number of gradations, but which, as yet, we have no means of noting, though the conception is clearly attained.

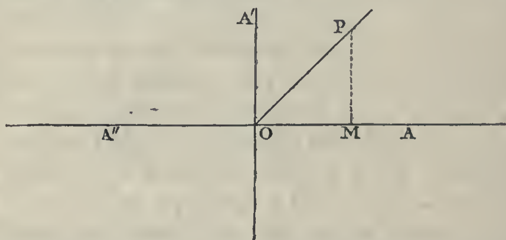
Now let us try to extend our notion of chronology in the same manner. Let O denote the commencement of the Christian era; let years A.C be measured to the left in inches, and A.D to the right.

* The student must carefully distinguish between *no opening where opening might have been*, and *no opening because opening is inconceivable*. And this distinction must be made throughout the mathematics. Our language wants a phrase to distinguish between the nonexistence which arises from incoherence of ideas, and that which is not, but might have been, the case.

We can easily say what point of time P represents, or where P should be in order that it might represent any given event. But what is the answer to the question, What point of time does R denote? This question is wholly unanswerable. All the time we can form a notion of, is already expressed on one side or other of O; there remains no idea of time answering to the point R. We have no idea of time except *quantity* of duration: we have two ideas of a straight line, *quantity of length*, and *direction.*

(111.) We see then, that in preparing an algebra for geometry, we are making one which will be more than we can apply to any thing else. But we shall carry on our geometrical algebra, and then shew that, by a process of pure reasoning, which contains no assailable point,* we can use this method in all other sciences.

Let all lines be considered as having *direction* as well as *magnitude*, and both essential to their definition; so that two lines shall not be called equal, unless they be in fact equal and parallel. Lines of equal length in different directions are to have distinct symbols, and to be different things. Let there be only one direction of revolution (for the present), namely, from OA through OA′, a direction, OA,



having been chosen which is to be permanently that of positive quantity, or that in which a line is to be expressed by its simple symbol aU, where U is the linear unit, and a the number of linear units, or (if the student think he comprehends the phrase) the symbol for the ratio of the line to the unit. Consequently, a line measured in the direction OA″ has such a symbol as $-aU$, supposed to be already understood. Call OA .... the *line of arithmetic*, which is part

* When I say this, I mean that all the objections which have been made relatively to negative and impossible quantities in the usual sense, right or wrong, have nothing to do with the reasoning which will be employed.

of .... A″O A .... that of the algebra of positive and negative quantities. *In this line the meaning of* $+$ *and* $-$ *is fixed by all that has preceded in ordinary algebra;* but as relates to lines not coinciding with O A or O A″, nothing must be conceived to have been yet laid down. The meaning of $+$ and $-$ remains to be determined, subject only to the *law of extension,* that *all* the limited meaning is to be contained in the extended meaning. The question to be determined is, what is the proper representation of the line O P, rU in *length,* and θ in direction, meaning that it makes an angle θ with O A.

Firstly, suppose θ to be commensurable with two right angles, or that $\theta\Theta = \frac{m}{n}\pi\Theta$ (m and n whole numbers) whence $2n\theta = m.2\pi$.

Let $k^\theta r$ be the distinctive symbol which marks that rU makes an angle θ with the arithmetical line; where k not being a symbol of quantity, neither is θ that of an exponent, in the algebraical sense. Then it is evident that, by turning O P until it has revolved through $2n$ times the angle θ, we bring O P into the position of a line making m sets of four right angles with O A, that is, we bring it into the direction O A for the mth time. And because coincidence is restored after every revolution, we must have

$k^{\theta+2m\pi}r$U signifying the same length and direction as $k^\theta r$U

$k^{2m\pi}$U .... U

or $k^{\theta+2m\pi}r$U *equals* $k^\theta r$U $\qquad k^{2\pi m}\text{U} = \text{U}$

for we have said, let lines be *equal* when they are the same in length and direction.

If then we want such a symbol as is capable of expressing the necessary distinctions, and which shall be pointed out by the rules of ordinary algebra, we must so assign k^θ as that $k^\theta.k^\theta.k^\theta....(2n) = k^{2\pi}\,k^{2\pi}....(m) = 1$, which can be done by making k^θ a $2n$th root of 1, in the manner already laid down, where $\sqrt{-1}$ is treated* as a quantity. If we ask which root of 1 is to be adopted, the answer

* Let the student carefully note the difference between *treating* $\sqrt{-1}$ as a quantity, when it has been proved that certain purposes of distinction are answered by so doing, and *considering* it as a quantity. In *operations,* this difference amounts to nothing, which is precisely the reason for which we adopt the method; but in *theory,* the first method gives *good reasoning;* the second gives only a part of it, *logical deduction.*

evidently is, that root which, when raised to the $2n$th power, gives the unit at the end of its mth revolution, or exhibits it in the form $\cos 2m\pi + \sin 2m\pi \,.\, \sqrt{-1}$. That is, we must let

$$k^{\theta} \text{ be signified by } \cos\frac{2m\pi}{2n} + \sin\frac{2m\pi}{2n}.\sqrt{-1}$$

(Be careful to remember that $\sin x \sqrt{-1}$ means, *not* $\sin(x\sqrt{-1})$ *but* $(\sin x) \,.\, \sqrt{-1})$. By De Moivre's Theorem,

$$\left(\cos\frac{2m\pi}{2n} + \sin\frac{2m\pi}{2n}\right)^{2n} = \cos 2m\pi + \sin 2m\pi \sqrt{-1}$$

a form of unity.

But $\quad \dfrac{2m\pi}{2n} = \dfrac{m}{n}\pi = \theta$; hence k^{θ} is $\cos\theta + \sin\theta\sqrt{-1}$

Secondly, let θ and π be incommensurable; the same notation must still be preserved, extending to the symbol of quantity θ all those considerations which have been heretofore introduced, in respect to the connexion of commensurables and incommensurables. The general result then is,

$r(\cos\theta + \sin\theta\sqrt{-1})$ U signifies a line rU inclined at an angle θ to the arithmetical line.

(112.) The meaning of the sign $+$ before a term distinguished by $\sqrt{-1}$, is to be determined from the preceding. If OM $= x$U ($=$ is here correctly applied), and if MP equals in length yU (the limited definition of $=$), we find that the line is

$$x + y\sqrt{-1};$$

consequently, $x + y\sqrt{-1}$, or OM + MP$\sqrt{-1}$, is the symbol for OP; whence we see that our extension of meaning is as follows: whereas, in the arithmetical line, OM + MP (MP being carried forward on that line) would have been OP on that line, then OM + MP$\sqrt{-1}$, when MP is distinguished as being at right angles to OM, *still means* OP. That $\sqrt{-1}$ is the distinction of *perpendicularity*, as -1 is of *contrariety of direction*, appears as follows: the unit perpendicular to OA is U $\left(\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\sqrt{-1}\right)$ or U$\sqrt{-1}$; but more evidently from the consideration that if kU denote perpendicularity, $k(k\text{U})$ must denote motion through a right angle more, or k^2U must denote $-1 \,.\, $U.

(113.) Our extension is now complete: it is yet for us to see what extension the preceding makes in the remaining notions of algebra.

1. What is $k^{\theta}r + k^{\theta'}r'$. Let x and x' be values of OM, and $y\sqrt{-1}$, and $y'\sqrt{-1}$ those of MP for the lines just stated. Then we have

$$k^{\theta}r + k^{\theta'}r' = x + x' + (y + y')\sqrt{-1}$$

Now, the general form $x + y\sqrt{-1}$ may be converted as follows:

$$x + y\sqrt{-1} = \sqrt{x^2 + y^2}\,(\cos\theta + \sin\theta\sqrt{-1})$$

where $\tan\theta = \dfrac{y}{x}$. This follows from the treatment of $\sqrt{-1}$ as a quantity; for if we assume

$$r\cos\theta = x \quad r\sin\theta = y;\ \text{then}\ r = \sqrt{x^2 + y^2} \quad \tan\theta = \frac{y}{x}.$$

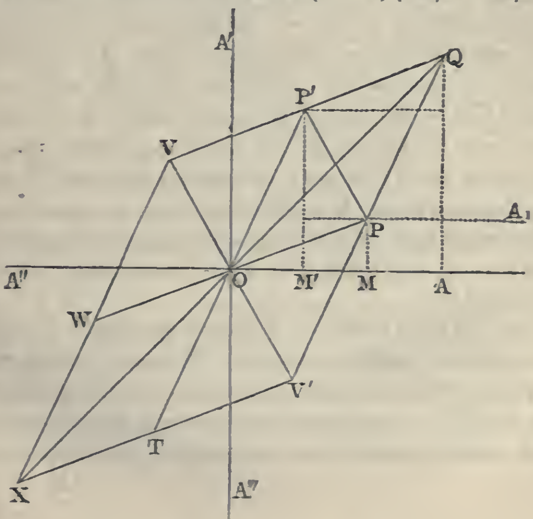
$$x + y\sqrt{-1} = r\cos\theta + r\sin\theta\sqrt{-1} = \sqrt{x^2+y^2}(\cos\theta + \sin\theta\sqrt{-1})$$

Applying this to $x + x' + (y + y')\sqrt{-1}$, and making also $x' = r'\cos\theta'$ $y' = r'\sin\theta'$, we find

$$\sqrt{(x+x')^2 + (y+y')^2}\,(\cos\phi + \sin\phi\sqrt{-1}),\ \tan\phi = \frac{y+y'}{x+x'},$$

$$\sqrt{x^2 + y^2 + x'^2 + y'^2 + 2\,(xx' + yy')}\,(\cos\phi + \sin\phi\sqrt{-1}),\ \text{or}$$

$$k^{\theta}r + k^{\theta'}r' = \sqrt{r^2 + r'^2 + 2rr'\cos(\theta - \theta')}\,(\cos\phi + \sin\phi\sqrt{-1})$$



$= \mathrm{OQ}\,(\cos \mathrm{QOA} + \sin \mathrm{QOA}\sqrt{-1})$ since $\tan \mathrm{QOA} = \dfrac{\mathrm{QA}}{\mathrm{AO}} = \dfrac{y+y'}{x+x'}$: or the sum of OP and OP′ is OQ, the diagonal of the completed parallelogram, which ends at O. The difference of the two lines will be found in the same way to be

$$x-x'+(y-y')\sqrt{-1} = \sqrt{r^2+r'^2-2rr'\cos(\theta-\theta')}\,(\cos\phi'+\sin\phi'\sqrt{-1})$$

$$\left(\text{where } \tan\phi' = \frac{y-y'}{x-x'} = \tan \mathrm{P'PA_1}\right)$$

$$= \mathrm{PP'}(\cos \mathrm{P'PA_1} + \sin \mathrm{P'PA_1}\sqrt{-1})$$

or the difference $k^\theta r - k^{\theta'} r'$ is PP′, the other *diagonal.* Let $\overline{\mathrm{OP}}$ signify the line OP in *length and direction.* The way of settling the distinction between $\overline{\mathrm{OP}} - \overline{\mathrm{OP'}}$ and $\overline{\mathrm{OP'}} - \overline{\mathrm{OP}}$, is as follows. The opposite of any line is found by merely changing the sign of r, if we allow θ to remain the same. For the opposite of $r\cos\theta + r\sin\theta\sqrt{-1}$ is $r\cos(\theta+\pi) + r\sin(\theta+\pi)\sqrt{-1} = (-r)(\cos\theta + \sin\theta\sqrt{-1})$: that is, two opposite lines may be expressed, either by changing the sign of the symbol of length, or adding two right angles to that of direction. We see then that the opposite of $r(\cos\theta + \sin\theta\sqrt{-1})$ may either be defined as r at the angle $\theta+\pi$, or as $-r$ at the angle θ. Now, to determine the proper interpretation of $\overline{\mathrm{OP}} - \overline{\mathrm{OP'}}$, since we are to have all algebraical formulæ remain true, let us write it thus: $\overline{\mathrm{OP}} + (-\overline{\mathrm{OP'}})$, and add $\overline{\mathrm{OP}}$ and $-\overline{\mathrm{OP'}}$, or $\overline{\mathrm{OP}}$ and $\overline{\mathrm{OT}}$; the result is $\overline{\mathrm{OV'}}$. Similarly $\overline{\mathrm{OP'}} - \overline{\mathrm{OP}} = \overline{\mathrm{OP'}} + (-\overline{\mathrm{OP}}) = \overline{\mathrm{OP'}} + \overline{\mathrm{OW}} = \overline{\mathrm{OV}}$. In truth, when we came to the first square root, namely, that which gave OQ, we should have ascertained that the sum was $\overline{\mathrm{OQ}}$ and not $\overline{\mathrm{OX}}$. This must be, for the extended definitions are entirely to contain the limited ones. Let OP and OP′ revolve towards the arithmetical line, and it is finally OQ, which becomes their arithmetical sum, and not OX.

(114.) The terms *greater* and *less* cannot have meaning as applied to lines defined in length and direction. We may have lines greater in length, or greater in direction, than others; but $\overline{\mathrm{OP}}$ is not greater or less than $\overline{\mathrm{OP'}}$. Watching this more narrowly, we see that we have defined *equal* in a sense which only applies to lines in the same direction; this *limitation*, for such it is, requires a corresponding limitation of the relative terms, greater and less.

(115.) As to multiplication, we see that

$$r(\cos\theta + \sin\theta\sqrt{-1}) \times r'(\cos\theta' + \sin\theta\sqrt{-1}) =$$
$$rr'(\cos(\theta+\theta') + \sin(\theta+\theta')\sqrt{-1})$$

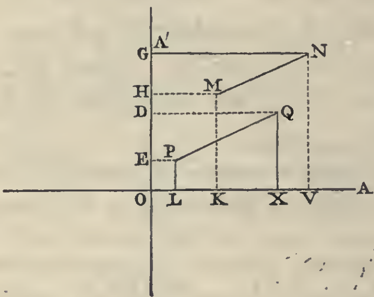
or, in multiplication of r and r', the result belongs to a line rr'U, with a direction, the sum of the directions of r and r'. Similarly, $\frac{r}{r'}$, belongs to a line $\frac{r}{r'}$U, at an angle $\theta - \theta'$, and so on. We see then that the angles of direction have the properties of logarithms relatively to the lines, which is also plainly shewn by (or, if not, is a confirmation of)

$$r(\cos\theta + \sin\theta\sqrt{-1}) = r\varepsilon^{\theta\sqrt{-1}}$$

Taking this convenient abbreviation, we have a set of equations, as follows:

$$r\varepsilon^{\theta\sqrt{-1}} \times r'\varepsilon^{\theta'\sqrt{-1}} = rr'\varepsilon^{(\theta+\theta')\sqrt{-1}},$$
$$a\varepsilon^{\alpha\sqrt{-1}} \times b\varepsilon^{\beta\sqrt{-1}} \div c\varepsilon^{\gamma\sqrt{-1}} = \frac{ab}{c}\varepsilon^{(\alpha+\beta-\gamma)\sqrt{-1}},$$
$$(r\varepsilon^{\theta\sqrt{-1}})^m = r^m\varepsilon^{m\theta\sqrt{-1}} \quad (r\varepsilon^{\theta\sqrt{-1}})^{\frac{m}{n}} = r^{\frac{m}{n}}\varepsilon^{\frac{m}{n}\theta\sqrt{-1}}$$

(116.) If we look at our first method of introducing $\sqrt{-1}$, (99.) we see that the one just explained agrees with it, except in one important particular. That first method did not bind us to any meaning of the sign + and — before k, but left us, as we then might suppose, to imagine that $P + kQ$ must be the same as $P + Q$, with a distinctive mark on Q, reserving it for future operations. But I did not notice at the time, wishing to avoid any generality which was not required by the subject, that the definition we assigned to k entirely destroyed all specific meaning for the sign + before k, though it continued that meaning in relations among the terms independent of k, or among those affected with it. For if we agree to mean by $P + kQ = P' + kQ'$ an equivalent to laying down both that $P = P'$ and $Q = Q'$, the preceding relation remains equally true, whatever may be the meaning of + before k; since identical operations performed upon equal quantities must give the same result. In applying the result to geometry, we find ourselves led to a more special meaning of k, for kQ means the line Q perpendicular to the arithmetical line. And with it we find a special meaning of + between a term without k and one with k, namely, the hypothenuse of a right-angled triangle. But still the meaning of k remains in force, for, in the geometrical application, $P = P'$ and $Q = Q'$. By expressly defining two lines as equal which have both equal lengths and directions, we require also equal projections on both OA and OA′, as is readily proved from the accompanying figure. For, if MN and PQ have



the same length and direction, then $KV = LX$, and $GH = DE$. But we have

$$\overline{MN} = KV + GH\sqrt{-1}$$
$$\overline{PQ} = LX + DE\sqrt{-1}$$

(117.) We have thus converted every theorem of algebra into one of geometry, not belonging to a class very useful at present, but, geometrically considered, of great complexity. And we have thus the satisfaction of putting upon every theorem of algebra a meaning which is as intelligible as an arithmetical operation, when the latter is as complicated; and this on the supposition that every letter is a (till now) impossible binomial of the form $x + y\sqrt{-1}$. Thus, in $a+b+c = c+b+a = b+a+c$, &c. we see the following theorem. If there be any number of straight lines meeting in a point, and if the conterminous diagonal of the parallelogram formed by any two be made one side of a new parallelogram, and a third line another; and if the conterminous diagonal of the last be similarly combined with a fourth line, and so on till all the lines are exhausted; then shall the last found diagonal be the same in magnitude and position, in what order soever the several lines were introduced. But theorems which are algebraically very simple lead to problems of great complexity. To shew this we shall enunciate the geometrical theorem which answers to

$$(a+b)^2 = a^2 + 2ab + b^2$$

as an exercise for the student in the meaning of the extension.

If there be two finite straight lines making angles with a third finite line on the same side, and all meeting in one point, and if we take the third proportional to the third line and each of the two first, and incline such third proportionals to the third line at angles twice as great as the first and second line; and if we also take the double of a fourth proportional to the third line and the two first and incline it to the third at an angle as great as the angles of the first and second together; and if we then take the conterminous diagonal of the parallelogram whose sides are the two third proportionals, and also the diagonal of the parallelogram which has the last-mentioned diagonal, and the double of the fourth proportional for its sides: then shall this last-mentioned diagonal be in length a third proportional to the third line and the diagonal of the parallelogram on the first two, and shall make, with the third line, an angle double of the angle made by the diagonal of the first two with the third.

The student should *construct* some problems of this kind, both as an exercise in the meaning of the extensions, and the use of the instruments.

(118.) I now pass to a question of much greater importance, namely, the use of the preceding extensions in reasoning. As a certain species of fallacy is called reasoning in a circle, I think the method I am going to describe might be called reasoning in a triangle; for, when there is an obstacle in one side, we pass from one end of it to the other over the other two, which we shall shew are free. To return to our illustration; suppose a question involving *times* before and after a certain epoch to be given, and suppose that ordinary algebraical reasoning, though it produce an answer positive or negative, does it by means of square roots of negative quantities fallaciously entering in the process. I say fallaciously (110.), because we cannot extend to our notions of duration those relations which answer to various directions of lines, except only, that if time before may be represented by a line north, time after may be represented by a line south. But a line east has no correlative in our notions of time. As long as we keep the notion of time, we must, so to speak, keep in the algebraical line (111.) of positive and negative quantity. But let us proceed as follows. Whatever ratios exist between the given times are made to exist between the geometrical lines which represent them in .... A″OA .... ; and if there be an answer to the question *in time*, there is an answer in the line A″OA to a given geometrical problem, corresponding to the given question. And, conversely, if there be in the line A″OA an answer to the geometrical question, namely, a line whose length has all the required ratios to the given lines, there must be a time whose ratio has all the required ratios to the given times; for ratios are the same things whether they be of portions of duration or of length. We cannot reason on the problem throughout when the concrete magnitudes are times: for, by our supposition, modes of duration of which we cannot conceive the existence are introduced. But we can reason on the geometrical problem, because geometry can put an intelligible construction upon correlative modes which exist among lines in different directions. But the geometrical result when obtained, gives an answer to the problem upon the relations of time; not depending upon the methods which gave the geometrical answer in any way, but upon a circumstance altogether different; namely, that relations among lines, positive or negative, however obtained, have their correlative relations among times. That is to say, we may depend upon the results of *general* algebra, even when the concrete magnitudes under discussion are such as do not admit of the geometrical extension. But, if the geometrical problem give an answer which amounts to supposing a line OP not in the line of algebraical positives or negatives, then we know that the corresponding problems relative to such concretes as do not admit of the extension, are strictly impossible, and (as yet, at least) inconceivable in any sense whatsoever. The same reasoning applies if the given problem be upon abstract numbers, with this additional limitation, that the question is impossible unless the answer of the corresponding geometrical problem lie upon the *arithmetical* line OA ...

We shall not in future consider it necessary to make any distinction between $\sqrt{-1}$ and other symbols. Those equations in which it occurs are rationally true in the extended sense of the symbols, and those in which it does not occur are true in the ordinary algebraical sense; for, as seen, the extended and ordinary meanings coincide when the symbol $\sqrt{-1}$ is neither expressed nor implied.

CHAPTER V.

ON THE DISTINCTION OF *PERIODIC* QUANTITIES, AND PRELIMINARY CONSIDERATIONS ON THE INVERSION OF PERIODIC FUNCTIONS.

(119.) WE have, in the notion of continual revolution, an idea of quantity, the whole effect of which is represented by the angle which the revolving line makes with the line of commencement; and in which, so far as position is concerned, or any ratios which depend upon position, it is indifferent whether the place which the line occupies be in the first, second, or any other revolution. If we were to say, let the angle $2\pi + \theta$ be the same angle as θ, we should then call an angle a *periodic* quantity; but this notion would not be very accurate. Yet $\sin\theta$, $\cos\theta$, &c. are really *periodic* functions of θ, for as θ increases they do not increase or decrease without end, but circulate in value through various changes, in such manner that what value soever any one has when $\theta = a$, it has the same for $a + 2\pi$, $a + 4\pi$, &c. It is usual to distinguish the primary functions of angles from magnitudes which depend on lines, or other continually increasing quantities, by the name of *periodic* quantities.

(120.) The question of finding the double, treble, &c. of a quantity of revolution is not affected by the multiplicity of values which answer to that quantity, considered as the determiner of angular position. Thus $m\theta$, $m(2\pi+\theta)$, $m(4\pi+\theta)$, &c. are $m\theta$, $2m\pi+m\theta$, $4m\pi+m\theta$, which all give the revolving line the same position as $m\theta$, when m is a whole number. But when m is a fraction, the case is altered. Suppose we ask, how many different quantities of revolution are there, each of which repeated 10 times, will leave the revolving line at an angle θ with the primary line OA. This position may be obtained by the revolutions θ, $\theta + 2\pi$, $\theta + 4\pi$, &c. consequently, the tenth part, or the solution necessary, is contained in the formula

$$\frac{2m\pi + \theta}{10} \quad \text{or} \quad \frac{2m}{10}.\pi + \frac{\theta}{10},$$ the values of which are

$$\frac{\theta}{10}, \quad \frac{\pi}{5}+\frac{\theta}{10}, \quad \frac{2\pi}{5}+\frac{\theta}{10} \ldots\ldots \text{ up to } \frac{9\pi}{5}+\frac{\theta}{10},$$

after which there is only recurrence; for $\frac{10\pi}{5}+\frac{\theta}{10}$ is $2\pi+\frac{\theta}{10}$. If then, we were to set out with a problem involving the given angle θ, and the answer were,

$$\text{the value required is } \cos\frac{\theta}{10}$$

there would, in fact, be ten answers; for θ might at the outset have been called $2\pi+\theta$, or $4\pi+\theta$, &c.; and though these angles have equal cosines, their tenth parts have not.

Thus, in the formula $\sin\theta = 2\sin\frac{1}{2}\theta.\cos\frac{1}{2}\theta$, we see that

$$\sin\theta \text{ is either } 2\sin\tfrac{1}{2}\theta.\cos\tfrac{1}{2}\theta \text{ or } 2\sin(\pi+\tfrac{1}{2}\theta)\cos(\pi+\tfrac{1}{2}\theta)$$

the latter arising from writing $2\pi+\theta$ for θ. And in

$$\cos\tfrac{1}{2}\theta+\sqrt{-1}\sin\tfrac{1}{2}\theta = \pm\sqrt{\cos\theta+\sqrt{-1}\sin\theta}$$

the second side has two values; and so has the first side, its second value appearing by writing $2\pi+\theta$ for θ.

(121.) The actually periodic character of the series in (101.) is a point of interest, which it may be advisable to verify. An angle is expressed in analytical units by an absolute number, and every number belongs to some quantity of revolution, 1 belonging to that which makes the arc and radius equal. And the remark in (63.) must be particularly attended to. No result yet obtained, which involves angles themselves, is true of degrees, minutes, and seconds, but only of analytical units. If we take a high number for θ, the formation of a few terms will make the series appear very divergent; but we see that convergence must come, for, in the sine, we have

$$(n+1)\text{th term} = n\text{th term} \times \frac{\theta^2}{2n(2n+1)}$$

And in the cosine

$$(n+1)\text{th term} = n\text{th term} \times \frac{\theta^2}{(2n-1)2n}$$

So that, however great θ may be, terms must arrive at which $2n(2n+1)$ and $(2n-1)2n$ bear as great a ratio as we please to θ^2, or the $(n+1)$th terms may be made parts as small as we please of the nth. And the terms being alternatively positive and negative, the magnitude of the latter kind will compensate that of the former, leaving a difference always less than unity, as should be the case from (36.). To shew this, we shall take some terms of the sine and cosine of the angle 10, something greater than 3π. We begin with 10, and the process is continual multiplication by 10, and division by successive numbers. Let (n) mean 10^n divided by $1.2.3 \ldots. n$, then we have

$$\sin 10 = (1) - (3) + (5) - (7) + \ldots.$$
$$\cos 10 = 1 - (2) + (4) - (6) + \ldots.$$

To pass from (n) to $(n+1)$ we must multiply by 10 (move the decimal point one place to the right) and divide by $n+1$. To get a sine and cosine from this (at the beginning) very diverging series, to a single place of decimals, will require about 29 places of decimals to be considered in the first term. Let us begin with

$$(1) = 10{\cdot}0000 \ldots. \text{ 29 ciphers}$$
$$(2) = 50{\cdot}000 \ldots.$$
$$(3) = 166{\cdot}66 \ldots. \quad \&c.$$

Proceeding in this way, and keeping two decimals from each term, we find as follows:

(1) = 10·00	(11) = 2505·21	(21) = 19·57
(2) = 50·00	(12) = 2087·68	(22) = 8·90
(3) = 166·67	(13) = 1605·90	(23) = 3·87
(4) = 416·67	(14) = 1147·07	(24) = 1·61
(5) = 833.33	(15) = 764·72	(25) = ·64
(6) = 1388·89	(16) = 477·95	(26) = ·25
(7) = 1984·13	(17) = 281·15	(27) = ·09
(8) = 2480·16	(18) = 156·19	(28) = ·03
(9) = 2755·73	(19) = 82·21	(29) = ·01
(10) = 2755·73	(20) = 41·10	

Hence, $(1) + (5) + (9) + \ldots.$ up to $(29) = 5506{\cdot}33 = A$
$(3) + (7) + (11) + \ldots.$ up to $(27) = 5506{\cdot}90 = B$
$\sin 10 = A - B$ nearly $= -{\cdot}57$

$1 + (4) + (8) + \ldots.$ up to $(28) = 5506{\cdot}20 = C$
$(2) + (6) + (10) + \ldots.$ up to $(26) = 5507{\cdot}03 = D$
$\cos 10 = C - D$ nearly $= -{\cdot}83$

$10 = 3\pi + {\cdot}58$ nearly, or $\cos 10 = -\cos{\cdot}58 \quad \sin 10 = -\sin{\cdot}58$
${\cdot}58\,\theta = {\cdot}58 \times 57{\cdot}31^\circ = 33{\cdot}23$ nearly *in degrees.*

And from the tables $\sin 55°\frac{1}{4} = \cdot 54$, $\cos 55°\frac{1}{4} = \cdot 84$, whence the first decimal is right, as we proposed. We have thus shewn how to verify the actual coincidence of the series with the sine and cosine of the tables.

(122.) When an operation is performed successively upon a quantity and its results, if we denote the operation by f, the results of the repeated operations may be denoted by ff, fff, &c., which may be abbreviated into f^2, f^3, &c. So that suppose x the quantity operated upon, and $2x+1$ the operation to be performed (that is, let f represent a direction to double, and then add one), we have

$$fx = 2x+1, \quad f^2x = 2(2x+1)+1 = 4x+3,$$

$$f^3x = 2\{2(2x+1)+1\}+1 = 8x+7, \text{ \&c.}$$

The convenience of this notation consists in the analogy which exists between the indices of f and algebraical exponents. Thus we have

$$f^2f^2x = f^4x = 2\left[2\{2(2x+1)+1\}+1\right]+1;$$

the only difference being, that in f^2f^2 we consider the preceding as having all between $\{\}$ first finished, which total operation is then repeated; and in f^4x we consider all the operations as separate. But both results, when developed, are, in fact, $ffffx$. To preserve the same equation, we must let f^0x signify x, in order that

$$ff^0x \text{ may be } f^{1+0}x, \text{ or } f^1x, \text{ or } fx;$$

and the meaning of $f^{-1}x$ may thus be given. Let it be such that the equation $f^{m+n}x = f^m(f^nx)$ still exists, and we must then have

$$ff^{-1}x = f^{1-1}x = f^0x = x;$$

or $f^{-1}x$ means the inverse of x, the function whose effect is reversed or undone by f; so that if f^{-1} be first performed upon x, and then f, the latter restores x again. Thus, if fx be x^2, $f^{-1}x$ is $\sqrt{x}$, because $ff^{-1}x$ or $(\sqrt{x})^2$ is x.

Since $f^{m+n}x$ is either f^mf^nx, or f^nf^mx, to preserve this equation, we should stipulate that $f^{-1}fx$ should be the same as $ff^{-1}x$, or that we should only allow functions satisfying this condition to be called $f^{-1}x$. Thus in x^2 and $\sqrt{x}$, we see that we have

$$(\sqrt{x})^2 = x, \text{ and, changing order of operation, } \sqrt{x^2} = x.$$

But if we take $-\sqrt{x}$ for $f^{-1}x$, fx being x^2, we find

$$(-\sqrt{x})^2 = x, \quad -\sqrt{x^2} = -x, \quad f^{-1}fx, \text{ not being } = x.$$

To elucidate this point, we must observe that, in algebra, direct operations, which produce but *one value* out of *one value*, are generally accompanied by inverses which produce more values than one; addition and multiplication only excepted. What do we mean by a square root? The expression, *which squared*, gives the original. This, with regard to x^2, answers both to x and $-x$. But if we had been considering x^2 as an operation to be repeated, giving $(x^2)^2$, &c. and had from thence, in the preceding manner, deduced an idea of an inverse operation, we should have found that if

$$fx = x^2, \text{ and if } ff^{-1}x \text{ and } f^{-1}fx, \text{ are both to be } = x,$$

we can only admit $f^{-1}x = \sqrt{x}$, and not $f^{-1}x = -\sqrt{x}$. But because it has been customary in algebra to admit all functions to the name of *inverses* of f, which satisfy $ff^{-1}x = x$, without inquiring whether they satisfy $f^{-1}fx = x$, we must here adopt the name, but with a distinction of notation. Let those forms be called *convertible* inverse functions of f, and be denoted by f^{-1}, which satisfy both $ff^{-1}x = x$ and $f^{-1}fx = x$. And let those forms be called *inconvertible* inverse functions of f, and be denoted by $f_{-1}x$, which satisfy only $ff_{-1}x = x$, and *not* $f_{-1}fx = x$. Thus, when $fx = x^2$, $f^{-1}x = \sqrt{x}$, $f_{-1}x = -\sqrt{x}$. But it is right to warn the student that, in other works, f^{-1} and f_{-1} are not distinguished.

(123.) Let us now turn to $\cos x$. What is its inverse function? Let $\cos x$ be denoted by fx; then

$$\varepsilon^{x\sqrt{-1}} = \cos x + \sqrt{1-\cos^2 x}\ \sqrt{-1} = fx + \sqrt{1-(fx)^2}\ \sqrt{-1},$$

(as in my *Algebra*, p. 123, I always mean by $\sqrt{x}$ the positive square root, and by $x^{\frac{1}{2}}$ the general form, either positive or negative). This is true only for angles less than π (of all in the first revolution); for if x be greater than π, we must have

$$\varepsilon^{x\sqrt{-1}} = \cos x + \sin x\sqrt{-1},$$

and the second term (44.) is negative;

whence $$\varepsilon^{x\sqrt{-1}} = fx + \sqrt{1-(fx)^2}\,\sqrt{-1},$$
(x between 0 and π, 2π and 3π, &c.)

$$\varepsilon^{x\sqrt{-1}} = fx - \sqrt{1-(fx)^2}\,\sqrt{-1},$$
(x between π and 2π, 3π and 4π, &c.)

both included in $$\varepsilon^{x\sqrt{-1}} = fx + \left(1-(fx)^2\right)^{\frac{1}{2}}\sqrt{-1},$$
(for all values of x.)

Now, if for x we write $x+2n\pi$, fx remains as before, and so does $\varepsilon^{x\sqrt{-1}}$: we have then every possible form of the preceding equation in

$$\varepsilon^{x\sqrt{-1}+2n\pi\sqrt{-1}} = f(x+2n\pi) + \left\{1-\left(f\overline{x+2n\pi}\right)^2\right\}^{\frac{1}{2}}\sqrt{-1}; \ldots (A)$$

in which there is identity of value in both sides of the equation for all values of n. But before we proceed a step further, we have an important remark to make, the neglect of which for a long time embarassed this subject.

(124.) *Identity of form* means positive and absolute identity in every respect. We see it in

$$x = x, \quad x+a = x+a, \quad \cos x = \cos x.$$

Identity of value includes all other cases in which the sign = applies. We see it in

$$x = x+a-a, \ (b-a)^2 = (a-b)^2, \ \cos(x+2\pi) = \cos x.$$

All operations, general or limited, performed upon identities of form, produce the same results. We see this in

$$\sqrt{x} = \sqrt{x} \text{ as well as } x^{\frac{1}{2}} = x^{\frac{1}{2}}.$$

This proposition is only worth stating as a contradictory of the next.

Limited operations performed upon identities of value with differences of form, do not necessarily produce the same results. Thus,

$$(b-a)^2 = (a-b)^2 \text{ does not give } \sqrt{(b-a)^2} = \sqrt{(a-b)^2},$$
$$\text{or } b-a = a-b,$$

but it does give $$\left\{(b-a)^2\right\}^{\frac{1}{2}} = \left\{(a-b)^2\right\}^{\frac{1}{2}},$$

that is, *every value of the first side is* a value *of the second side, but not* *at our pleasure.* It is $\sqrt{x}$ on the first side, which is equal to the $-\sqrt{x}$ of the second, and *vice versá:* if we attempt to displace this arrangement, and make $\sqrt{x}$ of the first side equal to $\sqrt{x}$ of the second, we have no longer an *identity*, but an *equation of condition* (*Algebra*, p. ix.), for then $b-a = a-b$, or $a = b$.

(125.) Conversely, when we find that an operation performed upon both sides of an identity of value gives different results, it will be most judicious, before proceeding further, to examine first the operation in question, to see whether it be not a limited case of a more general operation. And we see that an algebraical equation, of which the sides admit of different values, such as arises from performing a general operation upon a general form, may be considered as an ambiguity of this form,

$$A_1, \text{ or } A_2, \text{ or } A_3 \ldots\ldots = B_1, \text{ or } B_2, \text{ or } B_3 \ldots\ldots$$

in which, though we know that A_1 has its equal on the other side, we have no right to say that it is B_1, for it may be any other. And the same of A_2, &c. And all we know as yet, and indeed all we shall find, will shew us that, in operations conducted in all their generality, there will be complete identity of this kind: every value of the first side will have its value on the other. We shall not have A_1 and A_2 both equal to B_1, and B_2 left without a value.

As long as our inversions only involved the production of two values, as in the square root, or of three, as in the cube root, there was little need to embarrass the subject by general ideas upon inversion: each case was the best index to its own peculiarities. But now, when the simplest cases we have to consider give infinite numbers of inversions (seeing, for example, that though an angle has but one cosine, a cosine has an infinite number of angles), we must watch our processes very narrowly.. By trusting to limited notions of inversion, many errors have been introduced, some of which we shall point out in the sequel.

Returning to equation A, we must first find the arithmetical logarithm of the second side. We shall need some considerations, which the student will read with the more attention, when he knows that he is thus in reality entering upon a most essential part of his future studies, namely, the Differential and Integral Calculus.

CHAPTER VI.

NOTIONS OF THE DIFFERENTIAL AND INTEGRAL CALCULUS, SUCH AS WILL BE REQUISITE IN THE SUCCEEDING CHAPTERS.

(126.) Let any function of x, fx, have a certain value given to x, namely, a. Then let x have the value $a+h$ given to it, and let the difference of the resulting values of the function be taken, namely, $f(a+h)-fa$. Instances:

$$\varepsilon^{a+h}-\varepsilon^{a},\ \sin(a+h)-\sin a,\ (a+h)^2-a^2.$$

This difference, if the function be continuous (*Algebra*, p. 102), is one which diminishes without limit at the same time as h. When the fraction $\frac{f(a+h)-fa}{h}$ (*Algebra*, p. 156) has a finite limit, let it be called the *derived* function of fx, for the value $x=a$. Or, in general, let the limit of $\frac{f(x+h)-fx}{h}$, made by diminishing h without limit, be styled the *derived function* of fx, if it be a rational and continuous function of x. Let it be denoted by D, that is,

$$\text{Limit (if there be one) of } \frac{f(x+h)-fx}{h}=\mathrm{D}fx.$$

For example (*Algebra*, p. 225),

$$\frac{\varepsilon^{x+h}-\varepsilon^{x}}{h}=\varepsilon^{x}\frac{\varepsilon^{h}-1}{h}=\varepsilon^{x}\left(1+\frac{h}{2}+\frac{h^2}{2.3}+\ldots.\right)$$

Therefore (*Algebra*, p. 157.) $\mathrm{D}\varepsilon^{x}=\varepsilon^{x}$

$$\frac{\sin(x+h)-\sin x}{h}=\sin x\,\frac{\cos h-1}{h}+\cos x\,\frac{\sin h}{h}\text{ and by (46,47.)}$$

$$\mathrm{D}\sin x=\sin x\times 0+\cos x\times 1=\cos x$$

$$\frac{\cos(x+h)-\cos x}{h}=\cos x\,\frac{\cos h-1}{h}-\sin x\,\frac{\sin h}{h}\quad (46,\ 47.)$$

$$\mathrm{D}\cos x=\cos x\times 0-\sin x\times 1=-\sin x$$

(127.) Now precisely the same results are obtained if we use

$$\frac{f(x+h)-f(x+k)}{h-k}$$

and diminish h and k without limit. For instance,

$$\frac{\sin(x+h)-\sin(x+k)}{h-k} = \sin x \frac{\cos h - \cos k}{h-k} + \cos x \frac{\sin h - \sin k}{h-k}$$

$$= \sin x \times 2\left(-\frac{\sin\frac{1}{2}(h+k)\sin\frac{1}{2}(h-k)}{h-k}\right) + \cos x \times 2\frac{\sin\frac{1}{2}(h-k)\cos\frac{1}{2}(h+k)}{h-k}$$

$$- \sin x \sin\tfrac{1}{2}(h+k)\frac{\sin\frac{1}{2}(h-k)}{\frac{1}{2}(h-k)} + \cos x \,.\, \cos\tfrac{1}{2}(h+k)\frac{\sin\frac{1}{2}(h-k)}{\frac{1}{2}(h-k)}$$

the limit of which, diminishing h and k without limit, is as already obtained,

$= \sin x \times 0 \times 1 + \cos x \times 1 \times 1$, or $\cos x$, that is, $\mathrm{D}\sin x$.

Let the student try the other instances in a similar manner.

(128.) Theorem. Required the relation which must exist between two functions which have the same derived function.

Let the derived function in question be P, and let Q and R be two functions, of both of which the derived function is P. Let $\mathrm{R} = \mathrm{Q} + \mathrm{T}$, T being the difference of Q and R; and when x is made $x + h$, let R′, Q′, and T′ be the values of the three, whence, since $\mathrm{R} = \mathrm{Q} + \mathrm{T}$ is supposed to be an identical equation, we must have it true for all forms of x, and therefore we must have $\mathrm{R}' = \mathrm{Q}' + \mathrm{T}'$; whence

$$\frac{\mathrm{R}'-\mathrm{R}}{h} = \frac{\mathrm{Q}'-\mathrm{Q}}{h} + \frac{\mathrm{T}'-\mathrm{T}}{h}.$$

Take the limits of both sides (*Algebra*, p. 156), diminishing h without limit, and we have

$$\mathrm{DR} = \mathrm{DQ} + \mathrm{DT}, \quad \text{or} \quad \mathrm{DT} = 0,$$

because $\mathrm{DR} = \mathrm{DQ} = \mathrm{P}$. Consequently T must be a function, whose derived function is nothing for all values of x. Now let $\mathrm{T}(x)$ denote this function, where we make T the symbol of the functional operation, and express the quantity x, because we wish to express different values of it. Then let us suppose

$$T(x+h)-T(x) = V_1$$
$$T(x+2h)-T(x+h) = V_2$$
$$T(x+3h)-T(x+2h) = V_3$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$T(x+\overline{n-1}\,h)-T(x+\overline{n-2}\,h) = V_{n-1}$$
$$T(x+nh)-T(x-\overline{n-1}\,h) = V_n$$

And by addition, $T(x+nh)-Tx = V_1+V_2+\ldots\ldots+V_n$.

Now, let us diminish h without limit, increasing n at the same time in such manner that nh shall be always equal to a quantity *fixed in value, but what we please* (keep this in mind), and called y. (See the process in (101.)). By the nature of this derived function, $\frac{V_1}{h}, \frac{V_2}{h}$ .... must all diminish without limit with h, for that is implied in the limit of these fractions (which are values of the derived function) being always $= 0$. Consequently, each of V_1, V_2, &c. can be made as small a part as we please of h, and V_1+V_2+ .. .. $+V_n$ as small a part as we please of nh, which is always y, or the sum preceding can be made as small a part as we please of a finite quantity, *i e.* diminishes without limit. Hence, $T(x+y)-T(y)$ a quantity independent of h, diminishes without limit with h, which is absurd; for, not containing h at all, or any quantity whose value depends on h, it is the same whatever h may be. Whence does this absurd conclusion arise? Not from the reasoning, but from the initial supposition, namely, $T(x+h)-T(x)$, &c. are quantities depending on h. If they depend on h at all, all the preceding reasoning follows, and the inadmissible conclusion. The truth must be, then, that $T(x+h)-T(x)$ is not a function of h at all. Neither then, does $\{T(x+h)-T(x)\}+T(x)$ contain h, for we have not added a function of h. That is, $T(x+h)$ is not a function of h; that is, $T(x)$ is not a function of x. For, if it were, h would be found in $T(x+h)$. Therefore $T(x)$, supposed to be a function of x, and, therefore so expressed, turns out to be a quantity independent of x. Such a quantity is called a *constant* with respect to x. Hence the conclusion is, that if Q and R have the same derived function P, then Q can only differ from R by a constant quantity.

To verify the positive part of the conclusion, suppose $fx = Fx + C$, C not changing when x changes. Then $f(x+h) = \text{F}(x+h) + \text{C}$, and

$$\frac{f(x+h)-fx}{h} = \frac{\text{F}(x+h)-\text{F}(x)}{h}$$

consequently the limits (h diminishing) are the same, or $\text{D}fx = \text{DF}x$.

(129.) THEOREM. If ϕx be the derived function of fx, then the derived function of $f_{-1}x$, any inverse of fx satisfying $ff_{-1}x = x$, is $\dfrac{1}{\phi(f_{-1}x)}$

For $$\frac{f_{-1}(x+h)-f_{-1}x}{h} = \frac{f_{-1}(x+h)-f_{-1}x}{x+h-x} = \frac{f_{-1}(x+h)-f_{-1}x}{ff_{-1}(x+h)-ff_{-1}x}$$

$$= 1 \text{ divided by } \frac{ff_{-1}(x+h)-ff_{-1}x}{f_{-1}(x+h)-f_{-1}x}$$

Let $f_{-1}x = y$, $f_{-1}(x+h) = y+k$; then k and h diminish without limit together. Substitution gives

$$\frac{f_{-1}(x+h)-f_{-1}x}{h} = 1 \text{ divided by } \frac{f(y+k)-fy}{k}$$

Let h diminish without limit, in which case k does the same, and we have

$$\text{D}f_{-1}x = 1 \text{ divided by } \text{D}fy = \frac{1}{\phi y} = \frac{1}{\phi(f_{-1}x)}$$

For instance, let $fx = x^2$, $f_{-1}x = -\sqrt{x}$

Then $$\text{D}fx = 2x = \phi x \qquad \frac{1}{\phi x} = \frac{1}{2x}$$

$$\text{D}(-\sqrt{x}) = \frac{1}{\phi(-\sqrt{x})} = \frac{1}{-2\sqrt{x}} = -\frac{1}{2\sqrt{x}}$$

Let $fx = \sin x$, then we make $f_{-1}x =$ an angle whose sine is x.

$$\text{D}fx = \cos x = \phi x \qquad \frac{1}{\phi x} = \frac{1}{\cos x}$$

$$\text{D}f_{-1}x = \text{D}\left(\begin{matrix}\text{angle whose}\\ \text{sine} = x\end{matrix}\right) = \frac{1}{\phi(\text{ang. wh. sine is } x)} = \frac{1}{\cos(\text{angle, \&c.})}$$

$$= \frac{1}{\cos(\text{angle whose } cosine \text{ is } (1-x^2)^{\frac{1}{2}})} = \frac{1}{(1-x^2)^{\frac{1}{2}}}$$

The ambiguity of the sign is a subject to be hereafter considered; when actual application takes place, it depends upon the following proposition.

(130.) Dfx is positive for all values of x, at which, when x increases or diminishes, fx does the same; and negative when fx diminishes by increase of x, or increases by diminution of x. We must suppose in this proposition, that the increments or decrements may be as small as we please; and the general algebraical meaning of increase and decrease is contemplated. Its converse is also true.

Firstly, let $f(x+h)$ be greater than fx, for any value of h, however small; that is (*Algebra*, p. 63.) let $f(x+h)-fx$ be positive. Then, h being positive, we have $(f\overline{x+h}-fx) \div h$ is always positive, and its limit Dfx must be positive. Conversely, let Dfx be positive, then there are values of h which are so small, that $f(x+h)-fx$ must be positive. For if we make

$$\frac{f(x+h)-fx}{h} = Dfx + H,$$

then H must be an expression which diminishes without limit with h. The latter can then be taken so small, that $Dfx + H$ shall have the sign of Dfx, that is, shall be positive; that is, h being positive, $h(Dfx+H)$ shall be positive, or $f(x+h)$ greater than fx.

The other cases of the direct and converse proposition can easily be established in the same way.

(131.) Let x increase from $-a$ to $+a$, where a may be as great as we please, so that we thus suppose a variation of x between any limits of magnitude which it may be necessary to consider. We abbreviate all this into: Let x increase from $-\infty$ to $+\infty$: then the preceding shews, that whenever (x increasing) Dfx changes from $+$ to $-$, the function ceases increasing, and begins decreasing, and *vice versâ*.

Apply this theorem to $D\sin x = \cos x$, and shew that the sign of the cosine corresponds to the current method of variation of the sine in the manner described in the theorem.

The following theorems may be established by the direct process in (126.). Symbols not containing x, are supposed independent of x.

If $\varphi x = afx,\ D\varphi x = aDfx$

If $\varphi x = af_1x + bf_2x - cf_3x,\ D\varphi x = aDf_1x + bDf_2x - cDf_3x$

$$Dx^m = \text{limit of} \frac{(x+h)^m - x^m}{h} = \text{limit}\left(mx^{m-1} + m\frac{m-1}{2}x^{m-2}h + \cdots\right)$$

(*Algebra*, p. 217), or $Dx^m = mx^{m-1}$.

$$Dx^2 = 2x,\ Dx^3 = 3x^2,\ Dx^{\frac{1}{2}} = \tfrac{1}{2}x^{-\frac{1}{2}},\ Dx^{-1} = -x^{-2}, \&c.$$

$Dx =$ limit of $\frac{x+h-x}{h}$, or limit of 1. But 1 being invariable, as h diminishes without limit, it remains the same, or $Dx = 1$. Similarly, $Dax = a$, $D(ax+b) = a$.

$$D(a_0 + a_1x + a_2x^2 + \ldots) = a_1 + 2a_2x + 3a_3x^2 + \ldots$$

This being universally true, we may write a_2 for $2a_2$, a_3 for $3a_3$, &c. which gives

$$D\left(a_0 + a_1x + a_2\frac{x^2}{2} + a_3\frac{x^3}{3} + \ldots\right) = a_1 + a_2x + a_3x^2 + \&c.$$

(132.) We have hardly thought it necessary to state that if two functions be always the same in value, their derived functions must be the same in value. But (124.) we must not take a limited form of one, and equate its derived function to that of a limited form of the other, without first ascertaining that our limited forms are really equals.

The derived function of a function of a function of x is thus found.

Let us suppose $\phi(fx)$. Let $f(x+h) = fx + H$; then we have

$$D\phi fx = \text{limit of } \frac{\phi f(x+h) - \phi fx}{h} = \text{limit of } \frac{\phi(fx+H) - \phi fx}{H} \cdot \frac{H}{h}$$

$$= \text{limit of } \frac{\phi(fx+H) - \phi fx}{H} \times \text{limit of } \frac{f(x+h) - fx}{h}$$

Now h and H diminish without limit together, and the first limit would be $D\phi x$, but that fx is in the place of x. It is, therefore, what $D\phi x$ becomes when fx is written for x. The second limit is Dfx; whence

$$D\phi fx = D\phi x\left(\begin{matrix}\text{with } x \text{ afterwards} \\ \text{changed into } fx\end{matrix}\right) \times Dfx$$

Thus $\quad D\cos mx = -\sin x\,(x \text{ changed into } mx) \times Dmx$

(131.) $\quad = -\sin mx \times m$

$D\sin(x^5) = \cos x\,(\text{change } x \text{ into } x^5) \times Dx^5$

$= \cos x^5 \times 5x^4$: and so on.

(133.) $D\log x$ is limit of $\frac{1}{h}\left(\log\overline{x+h} - \log x\right)$; or of $\frac{1}{h}\log\left(1+\frac{h}{x}\right)$;

or of $\frac{1}{h}\left(\frac{h}{x} - \frac{1}{2}\frac{h^2}{x^2} + \ldots\right)$ (*Algebra*, p. 226); or of $\frac{1}{x} - \frac{1}{2}\frac{h}{x^2} + \ldots$ that is,

$$\mathrm{D}\log x = \frac{1}{x} \qquad \mathrm{D}\log fx = \frac{1}{fx} \times \mathrm{D}fx$$

We shall now be able to proceed with the expression of the inverse trigonometrical functions, and with the extension of what has been already done with the direct functions.

CHAPTER VII.

CONTINUATION OF THE CONNEXION OF DIRECT AND INVERSE TRIGONOMETRICAL FUNCTIONS.

(134.) By $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, we mean in analogy with (122.) the angles which have x for their sine, or cosine, or tangent, when we have both $\sin(\sin^{-1}x) = x$, and $\sin^{-1}(\sin x) = x$. That is, of all the possible angles contained in such a formula as $\theta \pm 2m\pi$, and having the same sine, there is one specific angle which is denoted by $\sin^{-1}x$, answering to the convertible inverse $f^{-1}x$. But all other angles contained in the same formula will be denoted by $\sin_{-1}x$, answering to the inconvertible inverses denoted generally by $f_{-1}x$.

Whatever a sine, or a tangent, may be in value, there is an angle lying between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$, to which that sine or tangent belongs. Not so with the cosines, all of which are positive between those limits. But all cosines are found to belong to angles between 0 and π. Let the fundamental angles to which any given primary function is attached, be chosen between those limits. For instance, by $\sin^{-1}(-\frac{1}{2})$, is meant $-\frac{\pi}{6}$; but we have

$$\sin_{-1}(-\tfrac{1}{2}) = -\frac{\pi}{6} \pm 2m\pi, \text{ or } \pi - \left(-\frac{\pi}{6} \pm 2m\pi\right)$$

that is $$\frac{\pi}{6} \pm (2m+1)\pi$$

We find, in fact, the following general equations:

$$\sin_{-1}x = \sin^{-1}x \pm 2m\pi \quad \text{or} \quad \pm(2m+1)\pi - \sin^{-1}x$$
$$\cos_{-1}x = \cos^{-1}x \pm 2m\pi \quad \text{or} \quad \pm 2m\pi - \cos^{-1}x$$
$$\tan_{-1}x = \tan^{-1}x \pm 2m\pi \quad \text{or} \quad \tan^{-1}x \pm (2m+1)\pi$$

These are, in fact, but a combination of the propositions that there are in the first revolution two distinct angles, having the same sine, cosine, or tangent, and that any number of revolutions added or subtracted, does not alter the sine, cosine, or tangent.

(135.) We have shewn (129.) that

$$D\sin^{-1}x = (1 - x^2)^{-\frac{1}{2}},$$

but which sign we should take has not been determined. If we consider that while x *increases* from -1 to $+1$, the angle *increases* from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$, we see (130.) that the positive sign must be chosen, or that we have

$$D\sin^{-1}x = \text{the positive value of } (1 - x^2)^{-\frac{1}{2}}.$$

Expand this by the binomial theorem, which gives

$$D\sin^{-1}x = 1 + \frac{1}{2}x^2 + \frac{1.3}{2.4}x^4 + \frac{1.3.5}{2.4.6}x^6 + \ldots\ldots$$

a series which (in common with all others of the kind derived from the binomial theorem, *Algebra*, p. 210) is convergent when $x^2 < 1$, or when x lies between $+1$ and -1, that is, in every case we propose to apply it to. And by (131.) we have

$$D\sin^{-1}x = D\left(x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \ldots\ldots\right)$$

But (128.) two expressions which have equal derived functions, can only differ by a quantity independent of x, whence we have

$$\sin^{-1}x = C + x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \ldots\ldots$$

where C is as yet undetermined. We find it thus: since it is independent of x, whatever value it has for one value of x, it has the same for all. Let then $x = 0$ (read carefully *Algebra*, p. 189.) The only angle between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$, which has 0 for sine, is 0, and the preceding then becomes $0 = C + 0$, or $C = 0$. That is,

$$\sin^{-1}x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \ldots.$$

Now, the only angle between 0 and π which has x for cosine, is $\frac{\pi}{2} - \sin^{-1}x$, so that we have

$$\cos^{-1}x = \frac{\pi}{2} - x - \frac{1}{2}\frac{x^3}{3} - \frac{1.3}{2.4}\frac{x^5}{5} - \ldots.$$

which is the same result as we should obtain by going through the same process from the beginning, as we now proceed to do.

(136.) Returning to (129.), let $fx = \cos x$, then Dfx or $\phi x = -\sin x$; therefore,

$$D\cos^{-1}x = -\frac{1}{\sin(\cos^{-1}x)} = -\frac{1}{(1-x^2)^{\frac{1}{2}}}$$

where the sign is undetermined; but as x diminishes from $+1$ to -1, while $\cos^{-1}x$ increases from 0 to π (the limits (134.) between which it is contained), we must take a negative sign for $D\cos^{-1}x$, or a positive sign for $(1-x^2)^{\frac{1}{2}}$. We have, therefore, by the same steps as before,

$$D\cos^{-1}x = -\left(1 + \frac{1}{2}x^2 + \frac{1.3}{2.4}.x^4 + ...\right)$$

or $$\cos^{-1}x = C - 1 - \frac{1}{2}x^2 - \frac{1.3}{2.4}x^4 - \&c.$$

and C may be determined, as before, by making $x = 0$, which gives $\cos^{-1}0 = C$. But between the limits of definition, $\frac{\pi}{2}$ is the only angle of which the cosine is 0, and we have, therefore,

$$\cos^{-1}x = \frac{\pi}{2} - x - \frac{1}{2}x^2 - \frac{1.3}{2.4}x^4 - .... \text{ as before.}$$

(137.) But we must now avoid all appearance of an unusual degree of caution without any particular reason shewn, by further examination of the proposition in (128.), namely, that if $D\phi x = Dfx$, then $\phi x - fx$ must be a constant independent of x. That demonstration, conducted by reducing $\phi x - fx = Tx$, a real function of x, to an absurdity, supposed throughout that DTx presented, between Tx and $T(x+y)$, no peculiarities which would remove it out of ordinary rules, x and y being specific quantities: and also that $D(\phi x - fx)$, or $D\phi x - Dfx$, fulfilled the same condition. But there may happen cases in which $\phi(x+h) - \phi x$, or $f(x+h) - fx$, do not diminish without limit when h diminishes without limit, or at least do not exhibit a form which will entitle us to draw that conclusion without further examination. If, for instance, $\phi x = \frac{1}{x}$, and $x = 0$, $x + h = h$, we can apply no conclusion of ordinary algebra to $\frac{1}{h} - \frac{1}{0}$, when h diminishes without limit. We might investigate a large number of cases, but we shall confine ourselves to one, namely, that in which either of the derived functions becomes infinite. And first, having shewn that so long as there are derived functions $D\phi x$ and Dfx, within the sense of the definition, we must arrive at $T(x+y) = Tx$, or Tx is independent of x; we may assume $\phi(x) = fx +$ (the same constant from $x = a$ to $x = b$) provided that between these limits $D\phi x$ and Dfx are finite; if, therefore, a change take place in the value of the constant, it must be at the point which our reasoning does not include. For example, let Dfx be finite and intelligible from $x = a$ to $x = b$, and also from $x = b + k$ to $x = c$: but between $x = b$ and $x = b + k$ let there be a value which makes $Dfx = \infty$. We know, then, that if $D\phi x = Dfx$,

from $x = a$ to $x = b$, $\varphi x = fx + \left\{ \begin{array}{l} \text{(C, to be determined, but} \\ \text{of one value between} \\ \text{those values of } x). \end{array} \right.$

from $x = b + k$ to $x = c$, $\varphi x = fx +$ (C_1 ditto ditto).

Are we at liberty to say that C_1 must be C? Certainly not: for if we attempt to reason by the process in (128.), from $x = a$ to $x = c$, we include the case about which we can draw no conclusion, where $Dfx = \infty$, and where it is by no means evident that $V_1 + V_2 + \ldots + V_n$ (in that article) will diminish without limit. We cannot shew, nor is it always true, that when Dfx is infinite, a change takes place in the constant; but we have shewn the converse, namely, that if a change do take place in the constant, it must be when Dfx undergoes some remarkable change of algebraical condition, either passing through infinity, becoming impossible, or the like. Let us suppose ourselves at a point at which the change of value takes place, and let it be when $x = t$. Then, when x is $t - h$, the function in question is represented by $f(t-h) + C$, but when $x = t + h$, by $f(t+h) + C_1$.

Hence, calling ϕx the general form of the function,

$$\varphi(t+h) - \varphi(t-h) = f(t+h) - f(t-h) + C_1 - C.$$

Now, in cases where $D\phi x$ and Dfx are finite for $x = t$, $\dfrac{\phi(t+h) - \phi(t-h)}{2h}$ and $\dfrac{f(t+h) - f(t-h)}{2h}$, will (127.), by diminishing h without limit, give the derived functions of ϕx and fx; and from the preceding it appears, that the difference of those derived functions will be the limit of $C' - C) \div h$, or infinite. But the difference of two finite quantities cannot be infinite; whence we may conclude, that $D\phi x$ and Dfx are not both finite, and as they are always equal, both will increase without limit together.

(138.) The preceding results (135.) may be immediately made to confirm this conjectural reasoning, for it is hardly more: we see that from $x = -\frac{\pi}{2}$ to $x = +\frac{\pi}{2}$, we have

$$\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \ldots.$$

which, when $x = 1$, gives a slowly converging series for $\frac{\pi}{2}$. But x never increases beyond 1, therefore this series can never represent $2\pi + \sin^{-1} x$, which is the first value of $\sin_{-1} x$ greater than $\frac{\pi}{2}$, belonging to the supposition on which this series was deduced (135.), namely, that x and $\sin^{-1} x$ increase together. In fact, if we suppose x to vary from -1 to $+1$, $\sin^{-1} x$ *then increasing*, we shall find that

$$\begin{array}{rll} \sin^{-1} x & \text{varies from} & -\frac{\pi}{2} \text{ to } +\frac{\pi}{2} \\ 2\pi + \sin^{-1} x & \text{varies from} & \frac{3\pi}{2} \text{ to } \frac{5\pi}{2} \\ 2m\pi + \sin^{-1} x & \text{varies from} & (4m-1)\frac{\pi}{2} \text{ to } (4m+1)\frac{\pi}{2} \end{array}$$

so that the general value of $\sin_{-1} x$ (when x and $\sin^{-1} x$ increase together) is as follows. That value which lies between $4m-1$ and $4m+1$ right angles, is

$$2m\pi + x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \ldots.$$

for all values of m, positive and negative.

Similarly, the value of $\cos_{-1} x$, which lies between $4m$ and $4m+2$ right angles (which includes the cases where $\cos^{-1} x$ decreases as x increases, as in (136.)) is

$$\left(4m+1\right)\frac{\pi}{2} - x - \frac{1}{2}\frac{x^3}{3} - \frac{1.3}{2.4}\frac{x^5}{5} - \ldots.$$

And at the points where the constant changes x is $+1$, or -1, and $D\sin_{-1} x$, and $D\cos_{-1} x$ (129.) become infinite. In a note at the end of the work we shall collect the various cases, including the change in the form of the series itself, arising from the change of sign in the differential coefficients.

(139.) We have now to consider $\tan x$ and $\tan_{-1} x$. The developement of the former in a series of powers of x is of little consequence as a result; but the methods by which it will here be obtained are important in other and more useful deductions.

THEOREM. A function of x, which is unaltered by changing the sign of x, or which satisfies the equation $\phi x = \phi(-x)$, cannot be expanded in a series of even and odd powers of x, but can only be expanded in even powers: and a function which is changed in sign only, and not in numerical value, by changing the sign of x, or which satisfies the equation $\phi(-x) = -\phi(x)$, can only be expanded in odd powers of x.

Let $\varphi x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots\cdot$

Then $\varphi(-x) = a_0 - a_1 x + a_2 x^2 - a_3 x^3 + a_4 x^4 - \cdots\cdot$

If $\phi(x) = \phi(-x)$, addition and division by 2 gives

$$\varphi x = a_0 + a_2 x^2 + a_4 x^4 + \cdots\cdot$$

If $\phi(-x) = -\phi x$, subtraction and division by 2 gives

$$\varphi x = a_1 x + a_3 x^3 + a_5 x^5 + \cdots\cdot$$

In the first case, then, $a_1 x + a_3 x^3 + \cdots = 0$ or (*Algebra*, p. 188)

$$a_1 = 0 \quad a_3 = 0 \text{ \&c.}$$

In the second case, $a_0 + a_2 x^2 + \cdots = 0$ or (*Algebra*, p. 188)

$$a_0 = 0 \quad a_2 = 0 \text{ \&c.}$$

(140.) The derived function of $\tan x$ is $\dfrac{1}{\cos^2 x}$, or $1 + \tan^2 x$; for (56.)

$$\frac{\tan(x+h) - \tan x}{h} = \frac{\sin(x+h-x)}{h.\cos(x+h)\cos x} = \frac{1}{\cos(x+h)\cos x} \cdot \frac{\sin h}{h}$$

the limit of which gives $\mathrm{D}\tan x = \dfrac{1}{\cos^2 x} = 1 + \tan^2 x$.

Again, $\tan x$, when the sign of x is changed, becomes $-\tan x$, or $\tan(-x) = -\tan x$; whence (139.) the series for $\tan x$ must consist of odd powers only. We have, therefore, to investigate what series of odd powers, P, will give $\mathrm{DP} = 1 + \mathrm{P}^2$.

Let $\mathrm{P} = a_1 x + a_3 x^3 + a_5 x^5 + a_7 x^7 + a_9 x^9 + \cdots\cdot$

Then, $\mathrm{DP} = a_1 + 3a_3 x^2 + 5a_5 x^4 + 7a_7 x^6 + 9a_9 x^8 + \cdots\cdot$

$$\begin{aligned}P^2 = a_1^2x^2+2a_1a_3x^4+2a_1a_5x^6+2a_1a_7x^8+ \dots\\ +a_3^2x^6+2a_3a_5x^8+\dots\\ +\dots\end{aligned}$$

Make DP equivalent with $1+P^2$, which gives

$$a_1 = 1,\ 3a_3 = a_1^2 \text{ or } a_3 = \frac{1}{3},\ 5a_5 = 2a_1a_3 \text{ or } a_5 = \frac{2}{15},$$

$$7a_7 = 2a_1a_5 + a_3^2 = \frac{4}{15} + \frac{1}{9} = \frac{17}{3.15} \text{ or } a_7 = \frac{17}{3.7.15},$$

$$9a_9 = 2a_1a_7 + 2a_3a_5 = \frac{34}{3.7.15} + \frac{4}{3.15} = \frac{62}{3.7.15},\ a_9 = \frac{62}{3.7.9.15}$$

whence, $\quad x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \dots$

is the only series of odd powers which has a property such as the tangent must have; namely, to be a function of x, whose derived function is 1 + the square of the function. But $\tan x$, if developable at all in powers of x, must be developable in *odd* powers only (139.); consequently, the preceding series must be the tangent of x, if there be any series whatever of whole powers of x which is $= \tan x$. But, by common division, we may see that $\sin x$, or $x - \frac{x^3}{2.3} + \dots$ divided by $\cos x$, or $1 - \frac{x^2}{2} + \dots$ does give an equivalent series of powers of x; consequently, we must have

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \dots$$

The actual division of the sine by the cosine will verify this, as follows:

$$1 - \frac{x^2}{2} + \frac{x^4}{2.3.4} - \dots\Big)\, x - \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} - \dots \Big(x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \dots$$

$$\frac{x - \frac{x^3}{2} + \frac{x^5}{2.3.4} - \dots}{\frac{1}{3}x^3 - \frac{4}{2.3.4.5}x^5 + \dots}$$

$$\frac{\frac{1}{3}x^3 - \frac{1}{2.3}x^5 + \dots}{\frac{4}{2.3.5}x^5 + \dots}$$

The preceding series must cease to be convergent when $x = \frac{\pi}{2}$, if not before; and must, therefore, be regarded (*Algebra*, chap. ix.) as an *algebraical* equivalent of the tangent.

(141.) We shall now proceed to the determination of $\tan_{-1} x$. Let us first take that value which lies between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$, (134.) which we have agreed to call $\tan^{-1} x$.

From (101.) $$\tan x = \frac{1}{\sqrt{-1}} \frac{\varepsilon^{x\sqrt{-1}} - \varepsilon^{-x\sqrt{-1}}}{\varepsilon^{x\sqrt{-1}} + \varepsilon^{-x\sqrt{-1}}} = \frac{1}{\sqrt{-1}} \frac{\varepsilon^{2x\sqrt{-1}} - 1}{\varepsilon^{2x\sqrt{-1}} + 1}$$

the latter fraction being made by multiplying numerator and denominator of the preceding by $\varepsilon^{x\sqrt{-1}}$. From this we find

$$\varepsilon^{2x\sqrt{-1}} = \frac{1 + \sqrt{-1}\,.\tan x}{1 - \sqrt{-1}\,\tan x}$$ and (*Algebra*, p. 226, formula 3.)

$$\log \varepsilon^{2x\sqrt{-1}} = 2\left(\sqrt{-1}\,.\tan x + \tfrac{1}{3}(\sqrt{-1}\tan x)^3 + \tfrac{1}{5}(\sqrt{-1}\tan x)^5 + \ldots\ldots\right)$$
$$= 2\sqrt{-1}\,(\tan x - \tfrac{1}{3}\tan^3 x + \tfrac{1}{5}\tan^5 x - \&c.)$$

Before proceeding further, annex $2\pi m\sqrt{-1}$ to the second side (102.), because (124.) we have no right to conclude that any one logarithm of the first side is equal to any one logarithm of the other. And, to make the question clear of all difficulties, except one, take a specific value for the tangent of x, say $\cdot 1$. We want to determine the angle or angles which have $\cdot 1$ for their tangent. Calculate

$$\cdot 1 - \frac{\cdot 001}{3} + \frac{\cdot 00001}{5} - \&c. \text{ and call it } v.$$

Then, taking the general logarithms of both sides, we have

$$2x\sqrt{-1} + 2\pi n\sqrt{-1} = 2v\sqrt{-1} + 2\pi m\sqrt{-1}$$

or $$x + n\pi = v + m\pi, \text{ and } x = v + (m - n)\pi,$$

which is the same as $$x = v + p\pi,$$

where p is any whole number, positive or negative. By proceeding similarly with the general series, we obtain

$$x = (\tan x - \tfrac{1}{3}\tan^3 x + \tfrac{1}{5}\tan^5 x - \&c.) + p\pi,$$

a result of a true form; for (44.) whatever angle v has a given tangent, $v + p\pi$ has the same. The only question is, with a given tangent, to which of the angles does the series itself belong? To investigate this, we shall produce the series itself by aid of the last chapter. Firstly, we must investigate $\mathrm{D}\tan_{-1}x$ (129.)

Here $\quad fx = \tan x, \ \mathrm{D}fx = \varphi x = 1 + \tan^2 x,$

$$\mathrm{D}f_{-1}x = \frac{1}{\varphi(f_{-1}x)} = \frac{1}{1+(\tan.\tan_{-1}x)^2} = \frac{1}{1+x^2}$$

or $\quad \mathrm{D}\tan_{-1}x = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots\ldots$

(*Algebra*, p. 16), from which, by reasoning similar to that in (135.)

$$\tan_{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots. + \mathrm{C}.$$

Now, if we look at (128.), we see that the only supposition on which the reasoning there given can fail, is where x itself is infinite; for $\frac{1}{1+x^2}$ never becomes infinite for possible values of x. Consequently, between $\tan_{-1}x = -\frac{\pi}{2}$, and $\tan_{-1}x = +\frac{\pi}{2}$ there is no change of value of the constant. Let C_1 be the value of the constant which belongs to the interval between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$ (in which $\tan_{-1}x$ is $\tan_{-1}x$); then

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\ldots + \mathrm{C}_1;$$

but when $x = 0$, $\tan^{-1}x = 0$, and therefore, as in (135.), $\mathrm{C}_1 = 0$, or we have

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots$$

for x write $\tan x$, and (134.) we have

$$x = \tan x - \tfrac{1}{3}\tan^3 x + \tfrac{1}{5}\tan^5 x - \&c.$$

that is, the value of the series is the angle which has the tangent specified in it, and which lies between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$.

This series is convergent only when $\tan x$ is less than 1. But we may immediately produce a series which shall always be convergent, as follows. From (135.)

$$\sin^{-1}x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \ldots\ldots$$

for x write $\sin x$, which gives

$$x = \sin x + \frac{1}{2}\frac{1}{3}\sin^3 x + \frac{1.3}{2.4}\frac{1}{5}\sin^5 x + \dots\dots$$

for $\sin x$ write $\tan x \div \sqrt{(1+\tan^2 x)}$, and then write $\tan^{-1}x$ for x, which gives

$$\tan^{-1}x = \frac{1}{\sqrt{1+x^2}}\left\{1 + \frac{1}{2}\frac{1}{3}\frac{1}{1+x^2} + \frac{1.3}{2.4}.\frac{1}{5}\frac{1}{(1+x^2)^2} + \dots\right\}$$

which will be found (*Algebra*, p. 182) to be always convergent.

(142.) As an exercise, deduce from $x = \tan x - \frac{1}{3}\tan^3 x + \&c.$ by taking derived functions of both sides, the following series:

$$\left.\begin{array}{l}\cos^2 x = 1 - \tan^2 x + \tan^4 x - \dots\dots\dots \\ \sin^2 x = \tan^2 x - \tan^4 x + \tan^6 x - \dots\dots\end{array}\right\}\text{and verify them otherwise.}$$

(143.) A series of periodic quantities, such as

$$\sin\theta + \sin 2\theta + \sin 3\theta + \sin 4\theta + \dots\dots \textit{ ad inf.}$$

can neither be called convergent nor divergent. If for θ we take any value, these sines will be a succession of positive and negative quantities, in parcels, no one of which exceeds unity. If, for example, we had $\theta = \frac{\pi}{4}$, the series would be

$$\left(\tfrac{1}{2}\sqrt{2} + 1 + \tfrac{1}{2}\sqrt{2} + 0 - \tfrac{1}{2}\sqrt{2} - 1 - \tfrac{1}{2}\sqrt{2} - 0\right)$$

+ a second parcel of the same form;

+ a third; + ... *ad inf.*

to which we may give the form

$$(1 + \sqrt{2})\,\{1 - 1 + 1 - 1 + \dots\dots\dots \textit{ ad inf.}\}$$

The meaning of this series has been already discussed in *Algebra*, p. 197. The question is, are we now to say that the preceding series is *one-half* of $1 + \sqrt{2}$. In the chapter cited, we found the series $1 - 1 + 1 - 1 + \dots$ to be a (then) unintelligible limiting form of $1 - x + x^2 - x^3 + \dots$ which was always arithmetically equal to $\frac{1}{1+x}$, as long as x was less than 1; whence we assumed $\frac{1}{2} = 1 - 1 + 1 - \dots$, and found that the one side was an algebraical equivalent of the other. Let us then examine the series

$$x\sin\theta + x^2\sin 2\theta + x^3\sin 3\theta + \dots\dots\ (x < 1),$$

which we may now call convergent, as its terms are severally less than those of the series $x + x^2 + x^3 + \ldots.$, which *is* convergent. We may sum this series in various ways, of which we shall choose two.

1. $\sin(n+1)\theta + \sin(n-1)\theta = 2\sin n\theta . \cos\theta$ gives

$$n = 1,\ x\sin 2\theta + 0 \qquad = 2x\sin\theta . \cos\theta$$
$$n = 2,\ x^2\sin 3\theta + x^2\sin\theta = 2x^2\sin 2\theta . \cos\theta$$
$$n = 3,\ x^3\sin 4\theta + x^3\sin 2\theta = 2x^3\sin 3\theta . \cos\theta \ \text{\&c.}$$

If, then, we call the sum of the whole series S, and sum these equations *ad infinitum*, the sum of the first column, $x \sin 2\theta + x^2 \sin 3\theta + \ldots.$ is $(S - x\sin\theta) \div x$; that of the second column is xS; that of the third column is $2S\cos\theta$. Consequently,

$$\frac{S - x\sin\theta}{x} + xS = 2S\cos\theta, \quad \text{or} \quad S = \frac{x\sin\theta}{x^2 - 2x\cos\theta + 1}.$$

2. Let the sines be replaced by their exponential values, which gives

$$(\varepsilon^{\theta\sqrt{-1}} = y) \quad \frac{1}{2\sqrt{-1}}\left\{x\left(y - \frac{1}{y}\right) + x^2\left(y^2 - \frac{1}{y^2}\right) + \ldots.\right\}$$

$$= \frac{1}{2\sqrt{-1}}\left(xy + x^2y^2 + \ldots.\right) - \frac{1}{2\sqrt{-1}}\left(\frac{x}{y} + \frac{x^2}{y^2} + \ldots.\right)$$

$$= \frac{1}{2\sqrt{-1}}\,\frac{xy}{1 - xy} - \frac{1}{2\sqrt{-1}}\,\frac{xy^{-1}}{1 - xy^{-1}} = \frac{1}{2\sqrt{-1}}\,\frac{xy - xy^{-1}}{1 - (y + y^{-1})x + x^2}$$

$$= \frac{x}{1 - (y + y^{-1})x + x^2}\,\frac{y - y^{-1}}{2\sqrt{-1}} = \frac{x\sin\theta}{1 - 2\cos\theta . x + x^2}$$

the same as before. Now, if in this result we make $x = 1$, we find it become

$$\frac{\sin\theta}{2(1 - \cos\theta)} \quad \text{or} \quad \frac{2\sin\frac{1}{2}\theta . \cos\frac{1}{2}\theta}{4\sin^2\frac{\theta}{2}} \quad \text{or} \quad \tfrac{1}{2}\cot\frac{\theta}{2}$$

that is, $\quad \frac{1}{2}\cot\frac{\theta}{2} = \sin\theta + \sin 2\theta + \sin 3\theta + \ldots.$

on which we must remark, that arithmetical equality has ceased, and that the sign $=$ can only mean (*Algebra*, chap. ix.) that one side may be substituted for the other without producing discordances in any arithmetical consequence of the substitution. For instance, we shall consider the following result, which those who have any idea of arithmetical identity between the two will be surprised at. The smaller we make θ, the greater is the value of the series. On which remark, that this is not a series of terms with fixed signs, but one in which the signs are positive and negative in parcels, the number of terms in each parcel depending upon the value of θ. First, let θ be diminished until the first three angles lie in the first two right angles the signs then are,

$$+ + + - - \ldots\ldots$$

let further diminution take place until six angles are less than π; the signs are then,

$$+ + + + + + - - \ldots\ldots$$

and so on. Now, one very simple case of the series is that in which θ is a measure of π, say $n\theta = \pi$. We have then $\sin(n\theta + \theta) = -\sin\theta$, $\sin(n\theta + 2\theta) = -\sin 2\theta$, &c.; so that, if we put a for $\sin\theta + \sin 2\theta + \ldots\ldots + \sin n\theta$, we have $a - a + a - a + \ldots\ldots$ or $\frac{1}{2}a$ for the series. It remains then, only to find an expression for a; now we have

$$a = \frac{1}{2\sqrt{-1}}\left(y - y^{-1} + y^2 - y^{-2} + \cdots + y^n - y^{-n}\right)$$

$$= \frac{1}{2\sqrt{-1}}\left\{\frac{y - y^{n+1}}{1 - y} - \frac{y^{-1} - y^{-n-1}}{1 - y^{-1}}\right\}$$

$$= \frac{1}{2\sqrt{-1}} \frac{y - y^{-1} + y^n - y^{-n} - (y^{n+1} - y^{-n-1})}{2 - (y + y^{-1})}$$

$$= \frac{\sin\theta + \sin n\theta - \sin(n+1)\theta}{2(1 - \cos\theta)}$$

which, if $n\theta = \pi$, $\sin n\theta = 0$, $\sin(n+1)\theta = -\sin\theta$, gives $a = \dfrac{2\sin\theta}{2(1 - \cos\theta)}$; and $\dfrac{a}{2}$ is the sum of the series already found.

(144.) If $\theta = \dfrac{m}{n}\pi$, m and n being whole numbers, and if $F\theta$ be a primary function of θ, or any function whatsoever of primary functions which is really periodic, the series $F\theta + F2\theta + \ldots\ldots$ can always be reduced to the form $a - a + a - a + \ldots\ldots$; and the whole question of assigning an algebraical equivalent to the series, in *finite* terms, is, therefore, made to depend upon the considerations developed in the ninth chapter of my *Algebra*. And, when θ is not commensurable with π, we know that θ may be found to lie between $\dfrac{m}{n}\pi$ and $\dfrac{m+1}{n}\pi$ where n is greater than any number named: so that the question in any other case is resolved into the general question discussed in the treatise on Number and Magnitude.

(145.) When both the series and its algebraical invelopement* are periodic in value, there is generally no difficulty with regard to the constants which may enter. But it may happen that one side increases without limit while the other side is periodic, in which case a discontinuous constant must enter.

For instance, let the series be

$$\sin\theta + \frac{1}{2}\sin 2\theta + \frac{1}{3}\sin 3\theta + \ldots.$$

which differs from that in (143.) by having terms which decrease without limit; but which are positive and negative in parcels determined by the value of θ. This series is,

$$\frac{1}{2\sqrt{-1}}\left\{y - y^{-1} + \frac{1}{2}\left(y^2 - y^{-2}\right) + \frac{1}{3}\left(y^3 - y^{-3}\right) + \ldots.\right\}$$

Now, (*Algebra*, p. 226) $y + \frac{1}{2}y^2 + \frac{1}{3}y^3 + \ldots = -\log(1-y)$

$$y^{-1} + \frac{1}{2}y^{-2} + \frac{1}{3}y^{-3} + \ldots = -\log\left(1 - \frac{1}{y}\right)$$

whence the series $= \dfrac{1}{2\sqrt{-1}}\left\{\log\left(1 - \frac{1}{y}\right) - \log(1-y)\right\}$

$$= \frac{1}{2\sqrt{-1}}\log\frac{y-1}{y(1-y)} = \frac{1}{2\sqrt{-1}}\log\left(-\frac{1}{y}\right) = \frac{1}{2\sqrt{-1}}\log \varepsilon^{-(\theta+\pi)\sqrt{-1}}$$

Because $\frac{1}{y} = \varepsilon^{-\theta\sqrt{-1}}$ $\;-\frac{1}{y} = \varepsilon^{-(\theta+\pi)\sqrt{-1}}$; and, as in (102.), the general logarithm gives

$$\sin\theta + \frac{1}{2}\sin 2\theta + \ldots = \frac{2\pi n\sqrt{-1} - (\theta+\pi)\sqrt{-1}}{2\sqrt{-1}} = \pi n - \frac{\theta+\pi}{2}$$

It remains to determine the value of n, corresponding to the condition that θ lies between certain limits, which limits are also to be found. Let $\theta = \frac{\pi}{2}$; the series becomes $1 - \frac{1}{3} + \frac{1}{5} - \ldots.$ or (141.) $\tan^{-1} 1$, that is $\frac{\pi}{4}$, whence we find

* I use this word as opposed to *developement;* thus $\frac{1}{2}$ is the invelopement of $1 - 1 + 1 - 1 + \ldots\ldots$

$$\pi n - \frac{3\pi}{4} = \frac{\pi}{4}, \text{ or } n = 1 \text{ when } \theta = \frac{\pi}{2}$$

Similarly, $$\pi n - \frac{\pi}{4} = -\frac{\pi}{4} \text{ or } n = 0 \text{ when } \theta = -\frac{\pi}{2}$$

Now, the addition of 2π to the first side of the equation creates no difference; neither must it do so on the second side. To satisfy this condition, consistently with those just deduced, and also with another which appears immediately, namely, that a change of sign in θ changes the sign of the first side, and, therefore, of the second, let n_1 be the value of n between $\theta = 0$ and $\theta = \frac{\pi}{2}$, and n_2 between $\theta = 0$ and $\theta = -\frac{\pi}{2}$. We have then

$$\pi n_1 - \frac{\theta + \pi}{2} = -\left(\pi n_2 - \frac{-\theta + \pi}{2}\right) \text{ or } n_1 = -n_2 + 1$$

the only value of which is $n_2 = 0$, $n_1 = 1$, since the only change consistent with periodicity is the addition of π, whenever θ has received an increase of 2π: and since n is 1 when $\theta = \frac{\pi}{2}$, and 0 when $\theta = -\frac{\pi}{2}$, and the change takes place when θ changes from negative to positive, it must be as just stated. Such an expression as $a - \frac{\theta}{2}$ can only thus be made identical with a periodic series; for, when θ becomes $\theta + 2\pi$, then the preceding becomes $a - \pi - \frac{\theta}{2}$, in which, if the same succession of values is to recur while the angle changes from $\theta + 2\pi$ to $\theta + 4\pi$, a must be increased by π, so that $a - \pi$ may be then what a was. The result is that

from $\theta = -2\pi$ to $\theta = 0$ the series is $-\frac{\theta + \pi}{2}$ lying between $\frac{\pi}{2}$ and $-\frac{\pi}{2}$

$\theta = 0$ to $\theta = 2\pi$ ...... $\pi - \frac{\theta + \pi}{2}$ ...... $\frac{\pi}{2}$ .. $-\frac{\pi}{2}$

$\theta = 2\pi$ to $\theta = 4\pi$ .... $2\pi - \frac{\theta + \pi}{2}$ ...... $\frac{\pi}{2}$ .. $-\frac{\pi}{2}$

and so on.

(146.) Let us consider the series

$$a \sin\theta + \frac{a^2}{2} \sin 2\theta + \frac{a^3}{3} \sin 3\theta + \ldots .$$

which is reduced to the preceding by $a = 1$. Substitute for $\sin\theta$, as in (145.), and the preceding becomes

$$\frac{1}{2\sqrt{-1}}\left\{\log\left(1-\frac{a}{y}\right)-\log(1-ay)\right\} \text{ or } \frac{1}{2\sqrt{-1}}\log\frac{1-ay^{-1}}{(1-ay)}$$

But $\quad y = \cos\theta + \sqrt{-1}\sin\theta \qquad \frac{1}{y} = \cos\theta - \sqrt{-1}\sin\theta$

whence the series is $\quad \dfrac{1}{2\sqrt{-1}}\log\dfrac{1-a\cos\theta+a\sin\theta\sqrt{-1}}{1-a\cos\theta-a\sin\theta\sqrt{-1}}$

Let $1-a\cos\theta = r\cos\phi$, $a\sin\theta = r\sin\phi$, whence $\phi = \tan_{-1}\left(\dfrac{a\sin\theta}{1-a\cos\theta}\right)$. The value of the series becomes

$$\frac{1}{2\sqrt{-1}}\log\frac{\cos\phi+\sin\phi\sqrt{-1}}{\cos\phi-\sin\phi\sqrt{-1}} = \frac{1}{2\sqrt{-1}}\log\frac{\varepsilon^{\phi\sqrt{-1}}}{\varepsilon^{-\phi\sqrt{-1}}}$$

$$= \frac{1}{2\sqrt{-1}}\log\varepsilon^{2\phi\sqrt{-1}} = \frac{2\phi\sqrt{-1}+2n\pi\sqrt{-1}}{2\sqrt{-1}} = \phi + n\pi$$

$$a\sin\theta + \frac{a^2}{2}\sin 2\theta + \ldots\ldots = n\pi + \tan^{-1}\left(\frac{a\sin\theta}{1-a\cos\theta}\right)$$

where $\tan_{-1}$ is changed into $\tan^{-1}$, as the difference between the two is expressed in the remaining term. In the form here given, n will have only one value; for $\tan^{-1}x$ is not a continually increasing term, being limited to that angle which lies between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$. And, when a is very small, the preceding series is very small, which cannot be true of the second side, unless $n = 0$. Consequently, we have

$$\tan^{-1}\left(\frac{a\sin\theta}{1-a\cos\theta}\right) = a\sin\theta + \frac{a^2}{2}\sin 2\theta + \ldots\ldots$$

This brings us to consider the difference between $\tan^{-1}\tan\theta$ and θ, which are equals only when θ lies between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$: but, if θ be greater than $\frac{\pi}{2}$, it is not $\tan^{-1}\tan\theta$ which is $= \theta$, but one of the values of $\tan_{-1}\tan\theta$. Consequently, as θ increases in successive revolutions, the expression $\tan^{-1}\tan\theta$ is periodic, varying continually between $\frac{\pi}{2}$ and $-\frac{\pi}{2}$. And similar considerations may be applied to $\sin^{-1}\sin\theta$, which has the same limits, and to $\cos^{-1}\cos\theta$ which has the limits 0 and π. We shall immediately see the use of this when we proceed to the series of last article, considered as a particular case of the present one. Let $a = 1$, then we have

$$\frac{a\sin\theta}{1-a\cos\theta} = \frac{2\sin\frac{\theta}{2}.\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \cot\frac{\theta}{2} = \tan\left(\frac{\pi}{2}-\frac{\theta}{2}\right)$$

or $\tan^{-1}\tan\left(\frac{\pi}{2}-\frac{\theta}{2}\right) = \sin\theta + \frac{1}{2}\sin 2\theta + \frac{1}{3}\sin 3\theta + \ldots.$

if we write the first side $\frac{\pi}{2}-\frac{\theta}{2}$, we are in error, for the two are not the same, except when $\frac{\pi}{2}-\frac{\theta}{2}$ lies between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$, or when θ lies between 0 and 2π. Such a transition would require a continual correction, amounting, in fact, to the process of the last article: but $\tan^{-1}\tan x$ is, by definition, the angle which lies between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$, and has x for its tangent.

The student should now prove the following theorems, illustrative of the limitation of F^{-1}, as distinguished from F_{-1}.

$$\sin^{-1}\cos x = \frac{\pi}{2} - x \text{ from } x = 0 \text{ to } x = \pi$$

$$\cos^{-1}\sin x = \frac{\pi}{2} - x \text{ from } x = -\frac{\pi}{2} \text{ to } x = +\frac{\pi}{2}$$

$$\cot^{-1}\tan x = \frac{\pi}{2} - x \text{ from } x = 0 \text{ to } x = +\pi$$

(147.) We now proceed to some adaptations of the theorems in chap. ii. to the notation of inverse functions.

$$\sin(\sin^{-1}x+\sin^{-1}y) = \sin\sin^{-1}x.\cos\sin^{-1}y + \cos\sin^{-1}x.\sin\sin^{-1}y$$
$$= x\sqrt{1-y^2} + y\sqrt{1-x^2}$$

$$\cos(\sin^{-1}x+\sin^{-1}y) = \sqrt{1-x^2}.\sqrt{1-y^2} - xy$$

$$\sin(\sin^{-1}x+\cos^{-1}y) = xy + \sqrt{1-x^2}\ \sqrt{1-y^2}$$

$$\sin(2\sin^{-1}x) = 2x\sqrt{1-x^2} \quad \cos(2\sin^{-1}x) = 1-2x^2$$

$$\sin(2\cos^{-1}x) = 2x\sqrt{1-x^2} \quad \cos(2\cos^{-1}x) = 2x^2-1$$

Is it true that $\sin(2\sin^{-1}x) = \sin(2\cos^{-1}x)$? if not, why? Investigate the consequences of the ambiguity of sign implied in $\sqrt{\quad}$, and also ascertain within what limits the following is true, and when and how it must be corrected.

$$\sin^{-1}x + \sin^{-1}y = \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$$

Next, $\tan(\tan^{-1}x + \tan^{-1}y) = \dfrac{x+y}{1-xy}$, $\tan(2\tan^{-1}x) = \dfrac{2x}{1-x^2}$

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

This latter equation is true whenever the sum of $\tan^{-1}x$ and $\tan^{-1}y$ lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$: but if it do not, say, for instance, it lies between $\frac{\pi}{2}$ and π, then either π must be subtracted from the first side, or $\tan_{-1}$ must be used on the second; and so on. In the same manner deduce (with similar limitation),

$$\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)$$

(148.) Let $\dfrac{x+y}{1-xy} = 1$ or $y = \dfrac{1-x}{1+x}$ which gives

$$\tan^{-1}x + \tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{\pi}{4}$$

Let $x = \frac{1}{2}$, $y = \frac{1}{3}$, and then $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \frac{\pi}{4}$,

which gives an easy method of computing the value of π. For (141.)

$$\tan^{-1}\frac{1}{2} = \frac{1}{2} - \frac{1}{3}\,\frac{1}{2^3} + \frac{1}{5}\,\frac{1}{2^5} - \frac{1}{7}\,\frac{1}{2^7} + \ldots\ldots$$

$$\tan^{-1}\frac{1}{3} = \frac{1}{3} - \frac{1}{3}\,\frac{1}{3^3} + \frac{1}{5}\,\frac{1}{3^5} - \frac{1}{7}\,\frac{1}{3^7} + \ldots\ldots$$

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{3} - \frac{1}{3}\left(\frac{1}{2^3} + \frac{1}{3^3}\right) + \frac{1}{5}\left(\frac{1}{2^5} + \frac{1}{3^5}\right) - \&c.$$

Write p and q for $\frac{1}{2}$ and $\frac{1}{3}$ (divide by 4 and 9 at every step).

$p = \cdot 50000000000$	$q = \cdot 33333333333$
$p^3 = \cdot 12500000000$	$q^3 = \cdot 03703703704$
$p^5 = \cdot 03125000000$	$q^5 = \cdot 00411522634$
$p^7 = \cdot 00781250000$	$q^7 = \cdot 00045724737$
$p^9 = \cdot 00195312500$	$q^9 = \cdot 00005080526$
$p^{11} = \cdot 00048828125$	$q^{11} = \cdot 00000564503$
$p^{13} = \cdot 00012207031$	$q^{13} = \cdot 00000062723$
$p^{15} = \cdot 00003051758$	$q^{15} = \cdot 00000006969$
$p^{17} = \cdot 00000762940$	$q^{17} = \cdot 00000000774$
$p^{19} = \cdot 00000190735$	$q^{19} = \cdot 00000000086$
$p^{21} = \cdot 00000047684$	$q^{21} = \cdot 00000000009$
$p^{23} = \cdot 00000011921$	
$p^{25} = \cdot 00000002980$	
$p^{27} = \cdot 00000000745$	
$p^{29} = \cdot 00000000186$	
$p^{31} = \cdot 00000000047$	
$p^{33} = \cdot 00000000012$	
$p^{35} = \cdot 00000000003$	

Now let $(p^n + q^n) \div n$ be denoted by r_n.

$r_1 = \cdot 83333333333$	$r_3 = \cdot 05401234568$
$r_5 = \cdot 00707304527$	$r_7 = \cdot 00118139248$
$r_9 = \cdot 00022265892$	$r_{11} = \cdot 00004490239$
$r_{13} = \cdot 00000943827$	$r_{15} = \cdot 00000203915$
$r_{17} = \cdot 00000044924$	$r_{19} = \cdot 00000010043$
$r_{21} = \cdot 00000002271$	$r_{23} = \cdot 00000000518$
$r_{25} = \cdot 00000000119$	$r_{27} = \cdot 00000000028$
$r_{29} = \cdot 00000000006$	$r_{31} = \cdot 00000000002$
$\cdot 84063894899$	$\cdot 05524078561$
	$\cdot 84063894899$
	$\frac{1}{4}\pi = \cdot 78539816338$
	4
	$\pi = 3\cdot 14159265352$

which is correct, with the exception of the last place.

(149.) $\sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$, &c. Form all similar equations, and explain under what limitations they are true.

It must be observed that every such equation becomes true (134.) when $\sin_{-1}$, &c. are substituted for $\sin^{-1}$, &c. But it does not follow that the same value of m in the general equation

$$\sin_{-1}x = 2m\pi + \sin^{-1}x = (2m+1)\pi - \sin^{-1}x$$

must be applied on both sides of the equation.

(150.) The angle $\sin^{-1}x$ is made, *by convention*, a *periodic* angle. It changes through $0, \frac{\pi}{2}, 0, -\frac{\pi}{2}, 0$, &c. while x changes through $0, 1, 0, -1, 0$, &c.

If x be greater than 1, the angle $\sin^{-1}x$ becomes impossible. If in the expressions for $\sin x$ and $\cos x$, we write $x\sqrt{-1}$ for x, we have

$$\sin\left(x\sqrt{-1}\right) = \frac{\varepsilon^{-x} - \varepsilon^{x}}{2\sqrt{-1}} \text{ or } \frac{\varepsilon^{x} - \varepsilon^{-x}}{2} = \frac{\sin\left(x\sqrt{-1}\right)}{\sqrt{-1}}$$

$$\cos\left(x\sqrt{-1}\right) = \frac{\varepsilon^{-x} + \varepsilon^{x}}{2} \text{ or } \frac{\varepsilon^{x} + \varepsilon^{-x}}{2} = \cos\left(x\sqrt{-1}\right).$$

The right hand sides of the equations are no longer periodic; and in the same way as all functions of sines, cosines, &c. may be expressed by exponential functions of x and $\sqrt{-1}$, so all exponential functions may be expressed by forms of sines and cosines of $x\sqrt{-1}$.

(151.) We may thus always give to periodic series their correct periodic values, either in terms of the primary functions, sines, &c. which are periodic, or by the introduction of *periodic angles*, $\sin^{-1}\sin\theta$, $\cos^{-1}\cos\theta$, &c. Having illustrated this point, on which freedom from error mainly depends, I shall proceed in the next chapter to some applications of trigonometry, which will give a first view of the manner in which it is used in astronomy and other branches of physics.

CHAPTER VIII.

APPLICATIONS OF THE PRECEDING CHAPTERS.

(152.) The following question occurs several times in astronomy:— If $\tan\phi = m\tan\theta$, required the developement of ϕ in a series of terms which shall be functions of m and θ: the supposition being that m is nearly unity, so that one value of ϕ is nearly θ, or $\phi = \theta$ nearly is the solution to which better approximation is required. Let $\varepsilon^{\sqrt{-1}}$ be called η; then we have

$$\frac{1}{2\sqrt{-1}}\frac{\eta^{\varphi}-\eta^{-\varphi}}{\eta^{\varphi}+\eta^{-\varphi}} = \frac{m}{2\sqrt{-1}}\frac{\eta^{\theta}-\eta^{-\theta}}{\eta^{\theta}+\eta^{-\theta}} \quad\text{or}\quad \frac{\eta^{2\varphi}-1}{\eta^{2\varphi}+1} = m\frac{\eta^{2\theta}-1}{\eta^{2\theta}+1}$$

Now, $$\frac{1-\frac{x-1}{x+1}}{1+\frac{x-1}{x+1}} = \frac{1}{x} \qquad \frac{1-m\frac{x-1}{x+1}}{1+m\frac{x-1}{x+1}} = \frac{(1-m)x+1+m}{(1+m)x+1-m}$$

whence $$\eta^{-2\varphi} = \frac{(1-m)\eta^{2\theta}+(1+m)}{(1+m)\eta^{2\theta}+(1-m)} = \frac{c\eta^{2\theta}+1}{\eta^{2\theta}+c} \text{ where } c = \frac{1-m}{1+m}$$

or $$\log\eta^{-2\varphi} = \log(1+c\eta^{2\theta}) - \log(1+c\eta^{-2\theta}) - \log\eta^{2\theta}$$

$$= -\log\eta^{2\theta} + c(\eta^{2\theta}-\eta^{-2\theta}) - \frac{c^2}{2}\left(\eta^{4\theta}-\eta^{-4\theta}\right)$$

$$+\frac{c^3}{3}\left(\eta^{6\theta}-\eta^{-6\theta}\right) - \text{\&c.}$$

But $\eta^{-2\varphi} = \varepsilon^{-2\varphi\sqrt{-1}}$ $\quad \log\eta^{-2\varphi} = -2\varphi\sqrt{-1}+2\pi n\sqrt{-1}$, &c.

whence $$-2\varphi\sqrt{-1}+2\pi n\sqrt{-1} = -2\theta\sqrt{-1}-2\pi n'\sqrt{-1}$$

$$+c\left(\varepsilon^{2\theta\sqrt{-1}}-\varepsilon^{-2\theta\sqrt{-1}}\right) - \text{\&c.}$$

Divide both sides by $-2\sqrt{-1}$, and since $n+n'$ is simply any whole number positive or negative, which call r,

$$\varphi - r\pi = \theta - 2c\sin 2\theta + \frac{2c^2}{2}\sin 4\theta - \frac{2c^3}{3}\sin 6\theta + \text{\&c.}$$

Since ϕ is to be the value nearest to θ, we must have $r = 0$, or

$$\varphi = \theta - 2c\sin 2\theta + \frac{2c^2}{2}\sin 4\theta - \frac{2c^3}{3}\sin 6\theta + \text{\&c.}$$

(153.) Let it now be proposed to expand the first side of the following equation in a series of the form of the second.

$$\frac{1}{1+e\cos\theta} = a_0 + a_1\cos\theta + a_2\cos 2\theta + a_3\cos 3\theta + \ldots\ldots$$

Assume $\quad x^2+y^2=1 \quad 2xy=e$

$$1+e\cos\theta = x^2 + 2xy \times \tfrac{1}{2}(\varepsilon^{\theta\sqrt{-1}} + \varepsilon^{-\theta\sqrt{1}}) + y^2$$

$$= (x+y\varepsilon^{\theta\sqrt{-1}})\,(x+y\varepsilon^{-\theta\sqrt{-1}})$$

$$= x^2\left(1+\frac{y}{x}\varepsilon^{\theta\sqrt{-1}}\right)\quad\left(1+\frac{y}{x}\varepsilon^{-\theta\sqrt{-1}}\right)$$

Let $\quad \frac{y}{x}=z \quad \varepsilon^{\theta\sqrt{-1}}=\omega.$ Then,

$$\frac{x^2}{1+e\cos\theta} = \frac{1}{1+z\omega}\cdot\frac{1}{1+z\omega^{-1}}$$

$$= (1-z\omega+z^2\omega^2-\ldots)\,(1-z\omega^{-1}+z^2\omega^{-2}-\ldots)$$

$$= (1+z^2+z^4+\cdot\cdot)-(z+z^3+\cdot\cdot)(\omega+\omega^{-1})$$
$$+(z^2+z^4+\cdot\cdot)\,(\omega^2+\omega^{-2})-\ldots\ldots$$

$$= \frac{1}{1-z^2}\left\{1-2z\cos\theta+2z^2\cos 2\theta-2z^3\cos 3\theta+\ldots.\right\}$$

Now, $2x = \sqrt{1+e}+\sqrt{1-e} \quad 2y = \sqrt{1+e}-\sqrt{1-e}$

$$z = \frac{y}{x} = \frac{\sqrt{1+e}-\sqrt{1-e}}{\sqrt{1+e}+\sqrt{1-e}} = \frac{1-\sqrt{1-e^2}}{e} \quad (\textit{Algebra}, \text{ p. } 119.)$$

$$x^2(1-z^2) = x^2-y^2 = \sqrt{1-e^2}$$

$$\frac{1}{1+e\cos\theta} = \frac{1}{\sqrt{1-e^2}}\cdot\left\{1-2\left(\frac{1-\sqrt{1-e^2}}{e}\right)\cos\theta+2\left(\frac{1-\sqrt{1-e^2}}{e}\right)^2\cos 2\theta-\cdot\cdot\right\}$$

If we take the other sign for $\sqrt{1-e^2}$ throughout, we have a developement which is an algebraical equivalent of the first side, but which is divergent when the one just found is convergent. It appears also that this form of developement is not arithmetical, unless e be less than 1. We shall now develope the series in powers of e, so as to include every term up to the third power of e. Firstly we have (rejecting every term above the third power)

$$(1-e^2)^{-\frac{1}{2}} = 1+\tfrac{1}{2}e^2+\ldots. \quad \sqrt{1-e^2} = 1-\tfrac{1}{2}e^2-\tfrac{1}{8}e^4-\ldots$$

$\dfrac{1-\sqrt{1-e^2}}{e} = \tfrac{1}{2}e+\tfrac{1}{8}e^3+\ldots.$ whence the series is

$$(1+\tfrac{1}{2}e^2)\Big(1-2(\tfrac{1}{2}e+\tfrac{1}{8}e^3)\cos\theta+2(\tfrac{1}{2}e+\tfrac{1}{8}e^3)^2\cos 2\theta - 2(\tfrac{1}{2}e+\tfrac{1}{8}e^3)^3\cos 3\theta+\ldots.\Big)$$

Develope the second factor, rejecting all powers of e above the third, which gives

$$(1+\tfrac{1}{2}e^2)\left(1-\frac{4e+e^3}{4}\cos\theta+\frac{e^2}{2}\cos 2\theta-\frac{e^3}{4}\cos 3\theta\right)$$

Multiply, rejecting e^4, &c. which gives

$$1+\tfrac{1}{2}e^2-\frac{4e+3e^3}{4}\cos\theta+\frac{e^2}{2}\cos 2\theta-\frac{e^3}{4}\cos 3\theta$$

Multiply both sides by $1-e^2$ with similar rejection, which gives

$$\frac{1-e^2}{1+e\cos\theta} = 1-\frac{e^2}{2}-\frac{4e-e^3}{4}\cos\theta+\frac{e^2}{2}\cos 2\theta-\frac{e^3}{4}\cos 3\theta$$

(154.) We might also obtain the preceding result as follows (*Algebra*, p. 161).

$$\frac{1-e^2}{1+e\cos\theta} = (1-e^2)\,(1-e\cos\theta+e^2\cos^2\theta-e^3\cos^3\theta), \text{ rejecting } e^4, \text{ \&c.}$$

$$= 1-e^2-(e-e^3)\cos\theta+e^2\cos^2\theta-e^3\cos^3\theta$$

$$= 1-e^2-(e-e^3)\cos\theta+e^2(\tfrac{1}{2}+\tfrac{1}{2}\cos 2\theta)-e^3(\tfrac{3}{4}\cos\theta+\tfrac{1}{4}\cos 3\theta)$$

which, reduced, gives the same result as before.

(155.) THEOREM. While θ varies from 0 to 2π, or from α to $\alpha+2\pi$, every expression of the form $a\sin\theta+b\sin 2\theta+\ldots.$ or $a\tan\theta+b\tan 2\theta+\ldots.$ &c. has as many positive as negative values; that is, for every positive value which the series has for one value of θ, there is a negative value which it has for another, numerically equal to the positive value.

This theorem depends upon another, namely, that in the same revolution there is always a second angle θ' to any given angle θ, such that

$$\sin\theta' = -\sin\theta,\ \sin 2\theta' = -\sin 2\theta,\ \sin 3\theta' = -\sin 3\theta\ldots. \textit{ ad inf.}$$

$$\tan\theta' = -\tan\theta,\ \tan 2\theta' = -\tan 2\theta,\ \tan 3\theta' = -\tan 3\theta\ldots. \textit{ ad inf.}$$

The angle in question is $2\pi-\theta$, as appears from (44). It is, therefore, evident that $\mathrm{F}\theta$ being either series mentioned, $\mathrm{F}(2\pi-\theta)$

$= -F\theta$. The cosine has not a similar property, but is contained with the others in the following general theorem, of which the one just given is a more simple equivalent as regards the sine and tangent.

THEOREM. If θ be an aliquot part of 2π and $F\theta = \sin\theta$ or $\cos\theta$, &c.

$$F\theta + F(2\theta) + F(3\theta) + \cdots\cdots + F(2\pi \text{ or } n\theta) = 0$$

This is already proved for the sine and tangent, since the series may be made from the beginning and end, into one of terms such as $Fx + F(2\pi - x)$, which are severally $= 0$; and the middle term, if there be one, is $F(\pi) = 0$, and the last term $F(2\pi) = 0$. For the cosine, if there be a middle term, $F(\pi)$, that is, if n be even, all the series preceding $F\pi$ may be arranged in terms of the form $Fx + F(\pi - x)$, each of which $= 0$; and all from $F(\pi + \theta)$ to $F(2\pi - \theta)$, the last but one, may be arranged in terms of the form $F(\pi + x) + F(2\pi - x)$, which are severally $= 0$. And the remaining terms, $F\pi$ and $F2\pi$, give $F\pi + F2\pi = 0$. When there is not a middle term $F\pi$, or when n is odd, we must make use of a general demonstration derived from the method in (143), by which it it is proved that

$$\cos\theta + \cos 2\theta + \cos 3\theta + \ldots + \cos n\theta = \frac{\cos\theta + \cos n\theta - 1 - \cos(n+1)\theta}{2(1-\cos\theta)}$$

the second side of which becomes 0 when $n\theta = 2\pi$.

Similarly, in (143), we have the series

$$\sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin n\theta = \frac{\sin\theta + \sin n\theta - \sin(n+1)\theta}{2(1-\cos\theta)}$$

which becomes $= 0$ when $n\theta = 2\pi$. *And the same, in both cases, if* $n\theta = 2m\pi$.

Hence it appears, that if we have such an expression as $a\sin\theta + b\sin 2\theta + \ldots$, and if we substitute for θ successively $\frac{2\pi}{n}\ \frac{4\pi}{n} \ldots$ up to 2π, the sum of each of the sets of terms corresponding to $a\sin\theta$, to $b\sin 2\theta$, &c., is $= 0$, whence the following result. If we take angles uniformly distributed in value between 0 and 2π, the sum of the negative values of $a\sin\theta + b\sin 2\theta + \ldots$ will be equal to the sum of the positive values; and the same of $a\cos\theta + b\cos 2\theta + \ldots$; more briefly expressed thus: the *mean* *value* of either of the preceding expressions is *nothing*. If, then, we have an expression such as $a_0 + a_1 \sin\theta + a_2 \sin 2\theta + \ldots$ the sum of n values, the angles of which are uniformly distributed through four right angles, will be $na_0 + 0$; that is, a_0, or a_0 is the *mean value* of the series. Hence an infinite number of ways of determining a magnitude which oscillates on one side and the other of a given magnitude: such as is almost every magnitude which is considered in astronomy.

(156.) We shall now inquire into the method of forming the product of two series such as $a_0 + a_1 \cos\theta + a_2 \cos 2\theta + \ldots$ and $b_0 + b_1 \cos\theta + b_2 \cos 2\theta + \ldots$ Let $\Sigma\phi(n)$ be the abbreviation of the series whose terms are the values of ϕn for all whole numbers from 0 to ∞. Then the preceding series are denoted by $\Sigma a_n \cos n\theta$ and $\Sigma b_n \cos n\theta$; their sum is $\Sigma(a_n + b_n)\cos n\theta$, and its mean value $a_0 + b_0$; their difference $\Sigma(a_n - b_n)\cos n\theta$, and its mean value $a_0 - b_0$. Their product is $\Sigma a_n b_m \cos n\theta \cos m\theta$, for every possible simultaneous pair of values of m and n, both being positive whole numbers. It is evident, moreover, that $\Sigma(p_n + q_n) = \Sigma p_n + \Sigma q_n$. If, then, we give the general term

$$a_n b_m \cos n\theta \cos m\theta \text{ its value } \tfrac{1}{2} a_n b_m \cos(m+n)\theta + \tfrac{1}{2} a_n b_m \cos(m-n)\theta$$

we have

$$\Sigma a_n \cos n\theta \times \Sigma b_m \cos m\theta = \tfrac{1}{2}\Sigma a_n b_m \cos(m+n)\theta + \tfrac{1}{2}\Sigma a_n b_m \cos(m-n)\theta$$

To find the co-efficient of a given cosine, $\cos p\theta$, in the product, we must inquire in how many ways $m+n$ may be made to give p, and $m-n$ either p or $-p$, for $\cos(-p\theta) = \cos p\theta$. Let us, then, first ask for the *mean value* of the product, or the term independent of θ. The only way in which $m+n$ gives nothing (m and n always positive whole numbers, 0 included) is when $m=0$ $n=0$, which contributes $\frac{1}{2}a_0 b_0$ to the mean value in question. But $m-n = 0$ whenever $m=n$, which contributes $\frac{1}{2}a_0 b_0 + \frac{1}{2}a_1 b_1 + \frac{1}{2}a_2 b_2 + \ldots$ *ad inf.*; so that the mean value of the product is $a_0 b_0 + \frac{1}{2}a_1 b_1 + \frac{1}{2}a_2 b_2 + \ldots$ The student might perhaps think that the mean value ought to have been $a_0 b_0$, and may reason thus: If there be a magnitude which, one time with another, is a_0, and a second which is b_0, the product is as often above as below $a_0 b_0$, and the latter should therefore be the mean value. But this is not true, as the most simple instance will shew. The average of 7 and 3 is 5; that of 4 and 12 is 8; the *product of the averages* is 40; the *average of the products* is $34\frac{1}{2}$. In the series deduced in (153), the mean value of the square of $1 \div (1 + e\cos\theta)$ is, by the preceding rule, $a_0^2 + \frac{1}{2}a_1^2 + \frac{1}{2}a_2^2 + \ldots\ldots$

or $$\frac{1}{1-e^2} + \frac{1}{2}\frac{4z^2}{1-e^2} + \frac{1}{2}\frac{4z^4}{1-e^2} + \ldots\ldots$$

or $$\frac{1}{1-e^2} + \frac{2z^2}{1-e^2} \cdot \frac{1}{1-z^2} \quad \text{or} \quad (1-e^2)^{-\frac{3}{2}}$$

(157.) I now ask for the term of the product which contains $\cos 5\theta$. The number of ways in which $m+n$ can be made $= 5$ is finite, namely,

$$n=0\ m=5;\ n=1\ m=4;\ n=2\ m=3;$$
$$n=3\ m=2;\ n=4\ m=1;\ n=5\ m=0$$

which contributes

$$\tfrac{1}{2}(a_0b_5 + a_1b_4 + a_2b_3 + a_3b_2 + a_4b_1 + a_5b_0)$$

to the co-efficient. Now all the ways of making $m-n=5$ or -5, are infinite in number, giving

$$m=5,\ n=0,\ \text{or}\ m=0,\ n=5;\ m=6,\ n=1,$$
$$\text{or}\ m=1,\ n=6;\ m=7,\ n=2,\ \text{or}\ m=2,\ n=7,\ \&c.$$

which contribute

$$\tfrac{1}{2}(a_0b_5 + a_5b_0) + \tfrac{1}{2}(a_1b_6 + a_6b_1) + \tfrac{1}{2}(a_2b_7 + a_7b_2) + \ldots\ldots$$

The co-efficient is therefore

$$\text{the half of} \left\{ \begin{array}{l} a_0b_5 + a_1b_4 + a_2b_3 + a_3b_2 + a_4b_1 + a_5b_0 \\ + a_0b_5 + a_1b_6 + a_2b_7 + a_3b_8 + a_4b_9 + \ldots\ldots \\ + a_5b_0 + a_6b_1 + a_7b_2 + a_8b_3 + a_9b_4 + \ldots\ldots \end{array} \right\}$$

Similarly, the co-efficient of $\cos n\theta$ is

$$\text{the half of} \left\{ \begin{array}{l} a_0b_n + a_1b_{n-1} + \ldots\ldots + a_nb_0 \\ + a_0b_n + a_1b_{n+1} + \ldots\ldots + a_nb_0 + a_{n+1}b_1 + \ldots\ldots \end{array} \right.$$

The student should now actually multiply some terms of $\Sigma a_n \cos n\theta$ and $\Sigma b_m \cos m\theta$, and thus produce the result here condensed, in a more expanded form.

(158.) The product $\Sigma a_n \sin n\theta \times \Sigma b_m \sin m\theta$ must be developed in a series of cosines: shew how the several terms may be deduced, and compare them with those of the last. The product $\Sigma a_n \sin n\theta \times \Sigma b_m \cos m\theta$ must be expanded in a series both of sines and cosines: proceed in the same way.

CHAPTER IX.

MISCELLANEOUS ADDITIONS TO THE PRECEDING CHAPTERS.

In this chapter I propose to touch slightly on several points which it is desirable the student should consider, but not necessary, so far as Trigonometry is concerned, that he should enter to any great depth.

(159.) Problem. Required a solution of any equation of the second degree which has possible roots, by help of the trigonometrical tables.

Let the equation be reduced to the form $x^2 + 2ax + b = 0$, or $x^2 + 2ax - b = 0$, a being either positive or negative, and b being positive. Under one or other of these forms every equation can be reduced.

1. $x^2 + 2ax + b = 0$ gives $x = -a \pm \sqrt{a^2 - b}$

Assume $b = a^2 \sin^2\theta$, or find θ from $\sin\theta = \sqrt{b} \div a$; which can be done, for, the roots being possible, a is greater than $\sqrt{b}$.

$$x = -a \pm \sqrt{a^2 - a^2 \sin^2\theta} = -a(1 \mp \cos\theta)$$

and $-2a\sin^2\frac{\theta}{2}$ and $-2a\cos^2\frac{\theta}{2}$ are the roots required.

2. $x^2 + 2ax - b = 0$ gives $x = -a \pm \sqrt{a^2 + b}$

Assume $b = a^2\tan^2\theta$, or find θ from $\tan\theta = \sqrt{b} \div a$, which can be done, as the tangent of an angle may have any value.

$$x = -a(1 \mp \sec\theta) = -a\frac{\cos\theta \pm 1}{\cos\theta} = -\sqrt{b}\frac{\cos\theta \pm 1}{\sin\theta}$$

and $\sqrt{b}.\tan\frac{\theta}{2}$ and $-\sqrt{b}\cot\frac{\theta}{2}$ are the roots.

N.B. If a be negative, as in $x^2 - 2x - 2 = 0$, it will be more convenient to solve $x^2 + 2x - 2 = 0$, and to change the sign of the roots of the latter. (*Algebra*, Chapter V.)

Examples (with obvious roots for verification.)

$x^2 + 3x + 2 = 0$

$a = 1{\cdot}5 \qquad b = 2$

log b	2)·3010300
	·1505150
log a	·1760913
	$\bar{1}$·9744237
	10
L sin 70° 31′ 43″·5	9·9744237

log sin 35° 15′ 51″·8	9·7614394—10
	2
	19·5228788—20
log a	·1760913
	·3010300
log 1·000000	20·0000001—20

log cos 35° 15′ 51″·8	9·9119544—10
	2
	19·8239088—20
	·1760913
	·3010300
log 2·000000	20·3010301—20

Roots, — 1 and — 2.

$x^2 + x - 12 = 0$

$a = {\cdot}5 \qquad b = 12$

log b	2)1·0791812
	·5395906
log a	$\bar{1}$·6989700
	0·8406206
	10
L.tan 81° 47′ 12″·5	10·8406206

log tan 40° 53′ 36″·3	9·9375310—10
log $\sqrt{b}$	·5395906
log 3·000000	10·4771216—10

log cot 40° 53′ 36″·3	0·0624690
log $\sqrt{b}$	·5395906
log 4·000000	0·6020596

Roots, 3 and — 4.

(160.) On the preceding there is a remark to be made, in continuation of (120). Having found one root of the first equation to be $-2a\sin^2\frac{1}{2}\theta$, we perceive that at the point where θ was first introduced, in the expression $\sin\theta$, we might, whatever θ may be, have substituted $\pi - \theta$. This shews us that in the result, since $-2a\sin^2\frac{1}{2}\theta$ is a root, $-2a\sin^2\frac{1}{2}(\pi-\theta)$, or $-2a\cos^2\frac{1}{2}\theta$ is another root; that is, *the* other root, since there can be but two. We shall illustrate this subject by giving a method of solving an equation of the third degree. Let the equation be

$$x^3 + 3kx^2 + 3lx + m = 0$$

Assume $x = y - k$, which gives

$$y^3 - (3k^2 - 3l)y + 2k^3 - 3kl + m = 0$$

Let $k^2 - l = p$, $2k^3 - 3kl + m = q$, whence $q = 3py - y^3$

But (59.) writing $1 - s^2$ for c^2, we have $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$; if, then, we assume $3py = 3\text{A}\sin\theta$, and $y^3 = 4\text{A}\sin^3\theta$, we have $q = \text{A}\sin 3\theta$. But the first two equations give $\text{A} = 2\sqrt{p^3}$, whence the equation can be solved if

$$\sin 3\theta = \sqrt{\frac{q^2}{4p^3}} \quad \text{for then} \quad y = 2\sqrt{p}.\sin\theta$$

This requires that q^2 should not exceed $4p^3$, in which case the first equation is rational. But $3\theta \pm 2\pi$ has the same sine as 3θ, whence, by the same process, we discover that the three following expressions are values of y, which make $q = 3py - y^3$.

$$2\sqrt{p}.\sin\left(\theta - \frac{2\pi}{3}\right) \quad 2\sqrt{p}\sin\theta, \quad 2\sqrt{p}\sin\left(\theta + \frac{2\pi}{3}\right) \ldots. \text{(A)}$$

(161.) If for $\varepsilon^{3\theta\sqrt{-1}}$ we write z, we find

$$\sin 3\theta = \frac{1}{2\sqrt{-1}}\left\{z - \frac{1}{z}\right\} = \sqrt{\frac{q^2}{4p^3}}$$

$$z = \frac{\sqrt{-q^2} \pm \sqrt{4p^3 - q^2}}{2\sqrt{p^3}}$$

If for z we take either of these values, $\frac{1}{z}$ is the other with its sign changed. Let us assume, then,

$$2\sqrt{p^3}z = \sqrt{4p^3 - q^2} + \sqrt{-q^2} \quad \frac{2\sqrt{p^3}}{z} = \sqrt{4p^3 - q^2} - \sqrt{-q^2}$$

Extract the cube root of both sides, and we have

$$2^{\frac{1}{3}}p^{\frac{1}{2}}z^{\frac{1}{3}} \text{ or } 2^{\frac{1}{3}}p^{\frac{1}{2}}\varepsilon^{\theta\sqrt{-1}} = \sqrt[3]{\sqrt{4p^3 - q^2} + \sqrt{-q^2}}$$

$$2^{\frac{1}{3}}p^{\frac{1}{2}}z^{-\frac{1}{3}} \text{ or } 2^{\frac{1}{3}}p^{\frac{1}{2}}\varepsilon^{-\theta\sqrt{-1}} = \sqrt[3]{\sqrt{4p^3 - q^2} - \sqrt{-q^2}}$$

$$y = 2p^{\frac{1}{2}}\sin\theta = \frac{p^{\frac{1}{2}}}{\sqrt{-1}}\left(\varepsilon^{\theta\sqrt{-1}} - \varepsilon^{-\theta\sqrt{-1}}\right)$$

$$= \frac{1}{2^{\frac{1}{3}}\sqrt{-1}}\left(\sqrt[3]{\sqrt{4p^3 - q^2} + \sqrt{-q^2}} - \sqrt[3]{\sqrt{4p^3 - q^2} - \sqrt{-q^2}}\right)$$

which, since $\sqrt{-1} = \sqrt[3]{-\sqrt{-1}}$, and $-\sqrt[3]{a} = \sqrt[3]{-a}$, is

$$= \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} - p^3}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} - p^3}} \quad \ldots . \text{ (B)}$$

This is the formula known by the name of Cardan, for the solution of $y^3 - 3py + q = 0$. It may be verified as follows:—Assume $y = u + v$, which gives

$$u^3 + v^3 + 3uv(u+v) - 3p(u+v) + q = 0$$

Let $uv = p$, then $u^3 + v^3 + q = 0$, or $\left\{\begin{matrix} u^6 + qu^3 + p^3 = 0 \\ v^6 + qv^3 + p^3 = 0 \end{matrix}\right\}$

whence u^3 and v^3 are the two roots of the last equation, treated as of the second degree. The expression (B) for y is then easily deduced.

The utility of this result (which is of great historical importance) is not considerable, and I shall give the following only as exercises for the student.

1. The expressions (A) and (B) cannot both have possible forms.

2. If $1, k$, and k^2 (106) be the cube roots of unity, and if A and B be the arithmetical cube roots of u^3 and v^3; then $u + v$, considered alone, may have the following forms

$$A + B, \quad A + Bk, \quad A + Bk^2, \quad B + Ak, \quad B + Ak^2,$$
$$Bk + Ak^2, \quad Ak + Bk^2,$$

none of which (from $uv = p$) are admissible, except

$$A + B, \quad Ak + Bk^2, \quad Ak^2 + Bk,$$

which are the three roots of the equation, or values of y.

3. The expressions (A) are possible when *all* the roots are possible; and (B) when one only is possible.

(162.) The equation $x^{2n} - 2\cos\theta \cdot x^n + 1$, is solved by every value of x which solves $x^2 - 2\cos\left(\frac{\theta + 2m\pi}{n}\right) . x + 1 = 0$: for (72)

$$2\cos\varphi = x + \frac{1}{x} \quad \text{gives} \quad 2\cos n\varphi = x^n + \frac{1}{x^n}.$$

Shew from this that the roots of the given equation are found by giving successive whole values to m in the formula

$$\cos\left(\frac{\theta + 2m\pi}{n}\right) + \sin\left(\frac{\theta + 2m\pi}{n}\right) . \sqrt{-1}$$

and shew, as in (103), that there cannot be more than n roots.

(163.) Any quantity of the form $a+b\sqrt{-1}$ can be reduced to the form $r(\cos\theta+\sin\theta\sqrt{-1})$. The conditions evidently are, that $\tan\theta=b\div a$, and $r=\sqrt{a^2+b^2}$. Hence it is easy to shew, that all functions of $a+b\sqrt{-1}$, &c. may be reduced to the same form: thus (115) we have

$$(a+b\sqrt{-1})(a'+b'\sqrt{-1})=rr'\left\{\cos(\theta+\theta')+\sin(\theta+\theta')\sqrt{-1}\right\}$$
$$(a+b\sqrt{-1})^n\qquad=r^n\left\{\cos n\theta+\sin n\theta\sqrt{-1}\right\}$$

We may express $a+b\sqrt{-1}$ in the form $r\varepsilon^{\theta\sqrt{-1}}$, which is the most compendious method of expressing any algebraical quantity whatsoever, and allows of r being supposed positive. A negative quantity is expressed by making θ any odd multiple of π (102); a positive quantity by making θ any even multiple of π: a quantity of the form $+b\sqrt{-1}$ or $-b\sqrt{-1}$, by making θ of the form $(4m+1)\frac{\pi}{2}$ or $(4m+3)\frac{\pi}{2}$, and one of the form $a+b\sqrt{-1}$ by a value of θ, which is no whole multiple whatsoever of a right angle.

(164.) Among possible expressions, those of the form $A\cos(a\theta+\alpha)$ or $A\sin(a\theta+\alpha)$, are such that the sum or difference of any of them is always reducible to the same form, whatever the values of A and α may be, provided only that a always remains the same. The two forms are not essentially different, for

$$A\sin(a\theta+\alpha)\quad\text{is}\quad A\cos\left(a\theta+\overline{a-\frac{\pi}{2}}\right)$$

To prove the theorem, observe that

$$A\cos(a\theta+\alpha)+A'\cos(a\theta+\alpha')=(A\cos\alpha+A'\cos\alpha')\cos a\theta$$
$$-(A\sin\alpha+A'\sin\alpha')\sin a\theta$$

Assume

$$A\cos\alpha+A'\cos\alpha'=L\cos\lambda\quad A\sin\alpha+A'\sin\alpha'=L\sin\lambda$$

or
$$L=\sqrt{A^2+A'^2+2AA'\cos(\alpha-\alpha')},$$
$$\tan\lambda=\frac{A\sin\alpha+A'\sin\alpha'}{A\cos\alpha+A'\cos\alpha'}$$

whence $A\cos(a\theta+\alpha)+A'\cos(a\theta+\alpha')=L\cos(a\theta+\lambda)$

(165.) I shall conclude this chapter with some examples of a process known by the name of successive substitution. Let us suppose any function of x, ϕx, and that, beginning with a value of x, say a, we form $\phi a = b$, $\phi b = c$, $\phi c = e$, &c. Or, agreeably to the notation of (122), suppose that we form ϕa, $\phi^2 a$, $\phi^3 a$, &c. Then the limit towards which we approach, if we approach any limit at all, must be a solution of $\phi x = x$.

For instance, take the function $1 + \frac{1}{2}x$: begin with $x = 0$, and successive substitution then gives

$$1,\quad 1+\tfrac{1}{2},\quad 1+\tfrac{1}{2}(1\tfrac{1}{2}) \text{ or } 1\tfrac{3}{4},\quad 1+\tfrac{1}{2}(1\tfrac{3}{4}) \text{ or } 1\tfrac{7}{8},\quad \&c.\ \&c.$$

the limit is 2, which is the solution of $1 + \frac{1}{2}x = x$.

To prove this generally, suppose it found that we can bring the results of successive substitutions as near as we please by carrying the process far enough. Let k and $k+z$ be the results of two successive substitutions: then, by hypothesis, $k + z = \phi k$. The smaller z is, the more nearly does k solve the equation $\phi x = x$. Let l be the limit, and let $k = l + \delta$, where δ may be as small as we please. Then $\phi l + \delta + z = \phi(l + \delta)$, which being true for values of δ and z as small as we please, gives (*Algebra*, page 157) $\phi l = l$.

(166.) As an example, let the function be $20 + \sin x$, where 20 means 20 degrees, and $\sin x$ is to be reckoned as a fraction of a degree. We begin with $x = 0$, which gives 20°; then $20° + \sin 20°$ is 20°·342, or 20° 20′$\frac{1}{2}$, and $20° + \sin(20°\ 20'\tfrac{1}{2})$ is 20°·347; which is a near approximation to the solution of $x = 20 + \sin x$ where 1 means one degree.

(167.) Successive substitutions, finite in number, will sometimes give theorems by which the logarithmic tables may be examined in many parts at once. For instance,

$$\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2} \quad \text{or} \quad 2 \sin \frac{x}{2} = \sin x \,.\, \sec \frac{x}{2}$$

$$4 \sin \frac{x}{4} = 2 \sin \frac{x}{2} \sec \frac{x}{4} = \sin x \sec \frac{x}{2} \sec \frac{x}{4}$$

$$8 \sin \frac{x}{8} = 2 \sin \frac{x}{2} \sec \frac{x}{4} \sec \frac{x}{8} = \sin x \sec \frac{x}{2} \sec \frac{x}{4} \sec \frac{x}{8}$$

and so on; whence it follows that

$$2^n \sin \frac{x}{2^n} = \sin x \,.\, \sec \frac{x}{2} \sec \frac{x}{4} \ldots\ldots \sec \frac{x}{2^n}$$

or $$\cos \frac{x}{2} \,.\, \cos \frac{x}{4} \,.\, \cos \frac{x}{8} \ldots\ldots \cos \frac{x}{2^n} = \sin x \div 2^n \sin \frac{x}{2^n}$$

If n increase without limit, the limit of the last dividor is x; whence

$$\cos\frac{x}{2}\ \cos\frac{x}{4}\ \cos\frac{x}{8}\ \cos\frac{x}{16}\ \ldots\ \textit{ad inf.}\text{ has the limit }\frac{\sin x}{x}$$

This affords an easy method of verifying the value of π, as follows: assume $x=\frac{\pi}{2}$, we have then

$$\cos\frac{\pi}{4}\,.\cos\frac{\pi}{8}\,.\cos\frac{\pi}{16}\ \ldots\ldots\ldots = \frac{2}{\pi}$$

log cos	45°	0′	0″	9·8494850 — 10
....	22	30	0	9·9656153 — 10
....	11	15	0	9·9915739 — 10
....	5	37	30	9 9979037 — 10
....	2	48	45	9·9994766 — 10
....	1	24	22·5	9·9998692 — 10
....	0	42	11·3	9 9999673 — 10
....	0	21	5·7	9·9999918 — 10
....	0	10	32·9	9·9999980 — 10
....	0	5	16·5	9·9999995 — 10
....	0	2	38·3	9·9999999 — 10
				109·8038802 — 110
				·3010300
log 3·141593				·4971498

I should, however, recommend the student not to proceed further in the subject of developement of trigonometrical forms, until he is able to apply the Differential Calculus, without which the theory of series will always be very incomplete.

THE END.

LONDON:
PRINTED BY J. MOYES, CASTLE STREET, LEICESTER SQUARE.

THE CONNEXION OF NUMBER AND MAGNITUDE:

AN ATTEMPT TO EXPLAIN

THE FIFTH BOOK OF EUCLID.

BY

AUGUSTUS DE MORGAN,

OF TRINITY COLLEGE, CAMBRIDGE.

La seule manière de bien traiter les élémens d'une science exacte et rigoureuse, c'est d'y mettre toute la rigueur et l'exactitude possible. — D'ALEMBERT.

LONDON:

PRINTED FOR TAYLOR AND WALTON,

BOOKSELLERS AND PUBLISHERS TO THE UNIVERSITY OF LONDON,

30 UPPER GOWER STREET.

M DCCC XXXVI.

LONDON:
J. MOYES, CASTLE STREET, LEICESTER SQUARE.

PREFACE.

This Treatise is intended ultimately to form part of one on Trigonometry. The place in which most students consider number and magnitude together for the first time, is in the elements of the latter science, unless they have understood the Fifth Book of Euclid better than is usually the case. Previously, therefore, to commencing Trigonometry, I consider it advisable to enter upon the consideration of proportion in its strict form; that is, upon the Fifth Book of Euclid. There is no other method with which I am acquainted which gives any thing like demonstration of the general properties of ratios, though there is a *doux oreiller pour reposer une tête bien faite*, which many of the continental mathematicians have agreed shall be called demonstration, and which is beginning to make its way in this country.

Hitherto, however, it has been customary for mathematical students among us to read the Fifth Book of Euclid; frequently without understanding it. The form in which it appears in Simson's edition is certainly unnecessarily long, and the tedious repetition of "AB is the same multiple of CD which EF is of GH," in all the length of words, renders the reasoning not easy to follow. The use of general symbols of concrete magnitude, instead of the straight line of Euclid, and of a general algebraical symbol for whole number, seems to me to remove a great part of the difficulty. Throughout this work it must be understood, that a capital letter denotes a magnitude; not a numerical representation, but the magnitude itself: while a small letter denotes a number, and mostly a whole number. And by the term *arithmetical proportion*, when it occurs, is signified, not the common and now useless meaning of the words, but the proportion of two magnitudes which are arithmetically related, or which are commensurable.

The subject is one of some real difficulty, arising from the limited character of the symbols of arithmetic, considered as representatives of ratios, and the consequent introduction of incommensurable ratios, that is, of ratios which have no arithmetical representation. The whole number of students is divided into two classes: those who do not feel satisfied without rigorous definition and deduction; and those who would rather miss both ~~that~~ than take a long road, while a shorter one can be cut at no greater expense, than that of declaring that there *shall be* propositions which arithmetical demonstration declares there *are not*. This work is intended for the former class.

AUGUSTUS DE MORGAN.

London, May 1, 1836.

CONNEXION

OF

NUMBER AND MAGNITUDE.

When a student has acquired a moderate knowledge of the operations and principles of algebra, with as many theorems of geometry as are contained in the first four books of *Euclid's Elements*, it becomes most desirable that he should gain some more exact knowledge of the connexion between the ideas which are the foundation of one and the other science, than would present itself either to an inattentive reader, or to one whose whole attention is engrossed by the difficulty of comprehending terms which cannot yet have become familiar to him. Before proceeding, therefore, to explain Trigonometry (the measurement of triangles), which, in the widest sense, includes all the applications of algebra to geometry, it will be right to inquire on what sort of demonstration we are to pass from an arithmetical to a geometrical proposition, or *vice versâ*.

Geometry cannot proceed very far without arithmetic, and the connexion was first made by Euclid in his Fifth Book, which is so difficult a speculation, that it is either omitted, or not understood by those who read it for the first time. And yet this same book, and the logic of Aristotle, are the two most unobjectionable and unassailable treatises which ever were written.

The reason of the difficulty which is found in the Fifth Book is twofold. Firstly;—It is all reasoning, unhelped by the senses: most of the propositions have no portion of that intrinsic evidence which is seen in " two sides of a triangle are greater than the third;" but, at the same time, the propositions of arithmetic which correspond to those of the Fifth Book are very evident, and the student is therefore led to escape from the notion of *magnitude*, and fly to that of *number*. Secondly;—The non-existence of any very easy notation and system of arithmetic in the time of Euclid, made geometrical considerations relatively so much more simple, that the form of his book is (to us) unnecessarily remote from all likeness to a treatise connected with numbers. The difference between our day and his lies in this: that in the former the exactness of geometry was gained with some degree of prolixity and (to a beginner) obscurity; in the latter, the facility of arithmetic is preferred, and perfect demonstration is more or less sacrificed to it. I shall now endeavour to present the Fifth Book of Euclid in a form which will be more easy than the original, to those who have some acquaintance with algebra.

By *number* is here meant what is called *abstract* number, which merely conveys the notion of times or repetitions, considered independently of the things counted or repeated. By *magnitude*, or *quantity*, is meant a thing presented to us, not as to its form, if it have form, or as to colour, weight, or any other circumstance, but simply as that which is made up of parts, not differing from the whole in any thing but in being less; so that, if we consider separately a part and the whole, we have only two inferences:

The part is less than the whole.

The whole is greater than the part.

Every thing we can see or feel presents to us the notion of magnitude or quantity. And here we must observe, that we have to pick our words from among those in common use, which never have very precise meanings. For instance, we have *magnitude*, the nearest English word to which is *greatness;* and *quantity*, for which the word, if it existed, should be *so-much-ness*. These words are of the same meaning, and the more indefinite we now leave them (except only in assigning that they are to be considered as applied to any thing which can be made *more* or *less*), the better for our purpose; since it is the object of this treatise to deduce from that indefinite notion a method of making mathematical comparisons of quantities, by aid of the notion of number.

Upon two magnitudes, our senses will enable us to draw one or other of the following conclusions:

1. The first is sensibly *greater* than the second.
2. The first is sensibly *less* than the second.

3. The first is sensibly *equal* to the second; meaning that the difference, if any, is so small that our senses cannot perceive it. This is what is meant by equality of magnitudes in common life. The English foot and the Florence foot are equal for common purposes: they differ by about the twentieth part of an inch, which in a foot is called nothing.

Perfect equality is a mathematical conception, which never can be absolutely verified in practice; for so long as the senses cannot perceive a certain quantity, be it ever so small, so long it must always be possible that two quantities, which appear equal, may differ by as much as the imperceptible quantity. But we are not reasoning upon what we can carry into effect, but upon the conceptions of our own minds, which are the exact limits we are led to imagine by the rough processes of our bands. The following, then, is the postulate upon which we construct our results:

Any one magnitude being given, let it be granted that any number of others may be found, each of which is (positively and mathematically) equal to the first.

Let A represent a magnitude—not as in algebra, the number of units which it contains, but the magnitude itself—so that if it be, for instance, weight of which we are speaking, A is not a number of pounds, but the weight itself. Let B represent another magnitude of the same kind; we can then make a third magnitude, either by putting the two magnitudes together, or by taking away from the greater a magnitude equal to the less. Let these be represented by $A + B$ and $A - B$, A being supposed the greater. We can also construct other magnitudes, by taking a number of magnitudes each equal to A, and putting any number of them together. Thus we have

$A + A$	which abbreviate into	$2A$
$A + A + A$	....	$3A$
$A + A + A + A$	....	$4A$

and so on. We have thus a set of magnitudes, depending upon A, and all known when A is known; namely,

$$A \quad 2A \quad 3A \quad 4A \quad 5A \quad \&c.$$

which we can carry as far as we please. These (except the first) are distinguished from all other magnitudes by the name of *multiples* of A; and it is evident that they increase continually. Let the preceding be called the *scale of multiples* of A. It is clear that the multiples of multiples are multiples; thus, 7 times 3 A is 21 A, m times nA is (mn)A, where mn is the arithmetical product of the whole numbers m and n. The following propositions may then be proved.

Prop. I. If A be made up of B and C, then any multiple of A is made up of the same multiples of B and C; for 2 A must be made up of

B C B C

of which B and B make 2 B, C and C make 2 C; so that 2 A is made up of 2 B and 2 C. Similarly, 3 A is made up of

B C B C B C

or of 3 B and 3 C.

Corollary. Hence it follows, that if A be less than B by C, any multiple of A is less than the same multiple of B by the same multiple of C. For, since A is less than B by C, A and C together make up B; therefore, 2 A and 2 C make up 2 B, or 2 A is less than 2 B by 2 C. The algebraical representations of these theorems are as follows:

If $\quad A = B + C \qquad mA = mB + mC$

If $\quad A = B - C \qquad mA = mB - mC$

m being any of the numbers 2, 3, 4, &c.......

Prop. II. However small A may be, or however great B may be, the multiples in the scale

A, 2 A, 3 A, 4 A, 5 A, &c.

will come in time to exceed B, by continuing the scale sufficiently far: B and A being magnitudes of the same kind. This is a proposition which must be considered as self-evident: it must be remembered that B remains the same, while we pass from one multiple of A to the next. Put feet together and we shall come in time to exceed any number of miles, say a thousand. But the best illustration of the reason why we formally put forward so self-evident a proposition, will be to remark, that it is not *every* way of adding magnitude to magnitude without end, which will enable us to surpass any given magnitude. To a magnitude add its half; to that sum add half of the half; to which add the half of the last: and so on. No continuation of this process, were it performed a hundred million of times, could ever double the first magnitude.

Prop. III. If A be greater than B, any multiple of A is greater than the same multiple of B. This follows from Prop. I. And if A be less than B, any multiple of A is less than the same multiple of B. This follows from the corollary, Prop. I. And if A be equal to B, any multiple of A is equal to the same multiple of B. This is self-evident.

Prop. IV. If any multiple of A be greater than (equal to, or less than) the same multiple of B, then A is greater than (equal to, or less than) B. For example, let 4 A be greater than 4 B; then A must be greater than B; for, if not, 4 A would be equal to, or less than, 4 B (Prop. III.).

Prop. V. If from a magnitude the greater part be taken away; and if from the remainder the greater part of itself be taken away, and so on: the given magnitude may thus be made as small as we please, meaning as small as, or smaller than, any second magnitude we choose to name.

Let A and Z be the two magnitudes, and let A diminished by more than its half be B, then 2 B is less than A. Let B diminished by more than half be C; then 2 C is less than B, 4 C is less than 2 B, and still more less than A. Let C diminished by more than its half be D, then 2 D is less than C, 8 D is less than 4 C, and still more than A. This process must end by bringing one of the quantities A, B, C, D, &c. below Z in magnitude. For, if not, let A, B, C, &c. always remain greater than Z. Then, since 2 B, 4 C, 8 D, 16 E, &c. are all less than A (just proved) still more must 2 Z, 4 Z, 8 Z, 16 Z, &c. be less than A. But this cannot be; therefore, one of the set A, B, C, &c. must be less than Z.

[The *reductio ad absurdum*, as this sort of argument is usually called, is a difficult form of a simple inference. Suppose it proved that whenever P is Q, then X is Y. It follows that whenever X is *not* Y, P is not Q. It is usually held enough to say, for if P *were* Q X *would be* Y. But the form in which Euclid argues, supposes an opponent; and the whole argument then stands as follows. "When X is Y, you grant that P is Q; but you grant that P is not Q. I say that X is not Y. If you deny this you must affirm that X *is* Y, of which you admit it to be a consequence that P *is* Q. But you grant that P is not Q; therefore, you say at one time that P is Q and that P is not Q. Consequently, one or other of your assertions is wrong, either ' P is not Q ' or ' X is Y.' If the first be right, the second is wrong: that is, ' X is not Y ' is right."

The preceding argument runs as follows;—when A, B, C, &c. are all greater than Z, then 2 Z, 4 Z, &c. are all less than A: but 2 Z, 4 Z, &c. are not all less than A; therefore, A, B, C, &c. are not all greater than Z].

Corollary. The preceding proposition is equally true when, instead of taking more than the half at each step, we take the half itself in some or all of the steps.

PROP. VI. If there be two magnitudes of the same kind, A and B, and if the scales of multiples be formed

$$A, \quad 2A, \quad 3A, \quad \&c. \qquad B, \quad 2B, \quad 3B, \quad \&c.$$

then one of these two things must be true; EITHER, there are multiples in the first scale which are equal to multiples in the second scale; OR, there are multiples in the first scale which are as nearly equal as we please to multiples (not the same perhaps) in the second set: that is, we can find one of the first set, say mA, which shall either be equal to another in the second set, say nB, or shall exceed or fall short of it by a quantity less than a given quantity Z, which we may name as small as we please.

Let us take a multiple out of each set, any we please, say pA and qB. If pA and qB be equal, the first part of the alternative exists; if not, one must exceed the other. Let pA exceed qB, say by E; then we have

$$pA = qB + E \quad \dots\dots\dots\dots \quad (1)$$

Now E is either less than B, or equal to B, or greater than B. If the first, let it remain for the present; if the second, we have $pA = (q + 1)B$, or the first alternative exists: if the third, then B can be so multiplied as to exceed E. Let $(t + 1)B$ be the *first* multiple of B which exceeds E; that is, let the next below, or tB, be less than E, say by G, then we have

$$E = tB + G \qquad\qquad pA = qB + tB + G$$

or

$$pA = (q + t)B + G$$

Now G must be less than B; for E or $tB + G$ is less than $(t + 1)B$, or $tB + B$. We have then made this first step (observe that $q + t$ is only *some* multiple of B; call it rB). Either the first alternative exists, or we can find pA and rB, so that

$$pA = rB + G \text{ where } G \text{ is less than } B \cdots\cdots\cdots (2)$$

Now G can be so multiplied as to exceed B; let vG and $(v+1)G$ be the multiples of G, between which B lies, so that

$$vG \text{ is less than } B, \text{ say } vG = B - K$$
$$(v+1)G \text{ is greater than } B, \text{ say } vG + G = B + L$$

and it follows that $K + L = G$, for since $(v+1)G$ and vG differ by G, if a magnitude lie between them, their difference must be made up of the excess of that magnitude over the lesser, together with its defect from the greater. Consequently, either K and L are both halves of G, or one of them falls short of the half. Suppose K is less than the half of G: then take both sides of (2), v times, and we have

$$vpA = vrB + vG$$

or $$vpA = vrB + B - K$$

or $$vpA = (vr + 1)B - K \text{ (K less than half G)}$$

But if L be less than the half of G, take both sides of (2) $v+1$ times which gives

$$(v+1)pA = (v+1)rB + (v+1)G$$

or $$(v+1)pA = \overline{v+1}\cdot rB + B + L$$

or $$\overline{v+1}\; pA = (\overline{v+1}\, r + 1)B + L \text{ (L less than half G)}$$

If K and L are both halves of G, we may take either. And if (a case not yet included) a multiple of G, vG, be exactly equal to B, we have then

$$vpA = vrB + vG = (vr+1)B$$

which gives the first alternative. Consequently, we either prove the first alternative, or we reduce the equation

$$pA = rB + G \quad \text{(G less than B)}$$

to an equation of the form

$$p'A = r'B \pm G' \left\{ G' \text{ not greater than the half of } G. \right.$$

We may now proceed as before; but, to exemplify all the cases that may arise, let us take

$$p'A = r'B - G'$$

If $v'G'$ be exactly B, we prove the first alternative, as before; but if B lie between $v'G'$ and $(v'+1)G'$, let us suppose

$$\left.\begin{array}{r} v'G' = B - K' \\ (v'+1)G' = B + L' \end{array}\right\} \text{ and } K' + L' = G' \text{ as before,}$$

in which one of the two, K' or L', will not be greater than the half of G', so that we obtain by the same process, an equation of the form

$$p''A = q''A \pm G'' \left.\right\} \begin{array}{l} G'' \text{ not greater than} \\ \text{the half of } G'. \end{array}$$

By proceeding in this way, we prove either, 1. The first alternative of the proposition; or, 2. the possibility of forming a continued set of equations

$$pA = qB \pm G, \quad p'A = q'B \pm G', \quad p''A = q''B \pm G'', \text{ \&c.}$$

where, in the scale of quantities G, G', G'', &c., no one exceeds the half of the preceding. Consequently, we may (unless interrupted by the first alternative) carry on this process until one of the quantities G, G', G'' &c. is smaller than Z (Prop. V.) that is, we have either the first or second alternative of the problem. And exactly the same demonstration may be applied to the case, where at the outset $pA = qB - E$.

This proposition proves nothing of a single magnitude, but it establishes two apparently very distinct relations between magnitudes considered in pairs. There may be cases in which the first alternative is established at last: and there may be cases in which it is never established. We shall first take the case in which the first alternative is established.

Suppose it ascertained by the preceding process that

$$8A = 5B$$

Here is an arithmetical equation between the magnitudes: and therefore any processes of *concrete* arithmetic will apply. Take the 40th part ($8 \times 5 = 40$) of both sides,

which gives $\qquad \dfrac{8A}{40} = \dfrac{5B}{40} \quad \text{or} \quad \dfrac{A}{5} = \dfrac{B}{8}$

consequently the fifth part of A is the same as the eighth part of B, or that which is contained 5 times in A is also that which is contained

8 times in B. Let this fifth of A or eighth of B be called M; then $A = 5M$, $B = 8M$, and A and B are both multiples of M. Consequently, when the first alternative of Prop. VI. exists, both A and B are multiples of some third magnitude M. The converse is readily proved, namely, that when A and B are both multiples of any third magnitude, the first alternative of Prop. VI. is true. For if $A = xM$, $B = yM$, we have $yA = yxM$, $xB = xyM$, or $yA = xB$. The term *measure* is used conversely to multiple, thus: if A be a multiple of M, M is said to be a measure of B. Hence in the case we are now considering, A and B have a *common measure*, and are said to be *commensurable*. We have therefore shewn that all commensurable magnitudes, and commensurable magnitudes only, satisfy this first alternative.

There remains, then, only the second case to consider, which it is now evident contains those magnitudes (if any such there be) which have no common measure whatsoever. The question therefore is, Are there such things as incommensurable magnitudes? On this point the second alternative shews that our senses cannot judge, for let Z be the least magnitude of the kind in question, which they are capable of perceiving (of course with the best telescopes, or other means of magnifying small quantities which can be obtained) then we know that pA may be made to differ from qB by less than Z, that is, we may say that all magnitudes are *sensibly* commensurable. But it evidently does not follow that all magnitudes are mathematically commensurable; and it has been shewn, by process of demonstration, that there are * incommensurable quantities in such abundance, that take almost any process of geometry we please, the odds are immense against any two results being commensurable.

The suspicion that all magnitudes must be commensurable led to the attempt, which lasted for centuries, to find the exact ratio of the circumference of a circle to its diameter. And even now, though the adventure is never tried by those who have knowledge enough to read demonstration of its impossibility, no small number of persons

* Legendre, and others before him, have shewn that the diameter and circumference of a circle are incommensurable; and the student will find in my Algebra, p. 98, or in the *Lib. Useful Know.*, treatise on the *Study of Mathematics*, p. 81, proof that the side and diagonal of a square are incommensurables. Also in Legendre's Geometry, or Sir D. Brewster's Translation.

exercise themselves by endeavouring to make an elementary acquaintance with geometry (and sometimes none at all) overcome this difficulty. It is our business here to shew how strict deductions may be made upon quantities which are incommensurable, with the same facility, and in the same manner, as upon commensurables. If we call any length (say that known by the name of a foot), the unit of its kind, and denote it in calculation by 1, we must call twice such a magnitude 2, and so on; half such a magnitude $\frac{1}{2}$, and so on. We may then apply arithmetic, every possible subject of which is contained in the following infinitely extended table.

0						1						2	&c.
0			$\frac{1}{2}$			1			$\frac{3}{2}$			2	&c.
0		$\frac{1}{3}$		$\frac{2}{3}$		1		$\frac{4}{3}$		$\frac{5}{3}$		2	&c.
0	$\frac{1}{4}$		$\frac{2}{4}$		$\frac{3}{4}$	1	$\frac{5}{4}$		$\frac{6}{4}$		$\frac{7}{4}$	2	&c.
		&c.			&c.			&c.			&c.		

And every length which is commensurable with the foot is included, in many different forms, in this table. Let F represent the foot, L any other length, let M be their common measure: let

$$\mathrm{F}=f\mathrm{M},\ \ \mathrm{L}=l\mathrm{M},\ \text{ then }\ l\mathrm{F}=f\mathrm{M},\ \text{ or}$$

$$\mathrm{L}=\frac{l}{f}\mathrm{F}=\frac{l}{f}\ \text{ when F is called 1.}$$

But it is plain that we cannot, by any arithmetic of length founded upon the foot as a unit, draw conclusions as to lengths which are incommensurable with the foot, though we can perhaps do so for any practical purpose. Let L be a length which is such, and let Z be a length so small as to be immaterial for the purpose in question. Then, we can determine l and f, so that

$$f\mathrm{L}=l\mathrm{F}\pm\mathrm{G} \qquad (\text{G less than Z})$$

or

$$\mathrm{L}=\frac{l}{f}\mathrm{F}\pm\frac{\mathrm{G}}{f}:$$

so that, by assuming $\mathrm{L}=\frac{l}{f}\mathrm{F}$, we commit an error, in excess or defect, less than G, and therefore immaterial. With such a process many minds would rest contented; but there is a consideration which will stand in the way of perfect satisfaction, or at least ought to do so. Granting that in the preceding case the error at the outset is immaterial, let us suppose the student disposed to substitute for all incommensurables, magnitudes very near to them which are commensurables, and thus to continue his career till he comes to the highest branches of applied mathematics. Let us suppose a set of processes, beginning in arithmetic, continued through algebra, the differential calculus, &c., up to a point in optics or astronomy, in a series of results, embracing, we may suppose, ten thousand inferences. If he set out with an erroneous method, what security has he that the error will not be multiplied ten thousand fold at the end, and thus become of perceptible magnitude. If somebody acquainted with the subject have told him that it will not so happen, he might as well skip the intermediate sciences and receive the result he wants to obtain on the authority of that person, as study them in a manner, the correctness or incorrectness of which depends on that person's authority. If he answer that the result, namely, such multiplication of errors, appears extremely improbable, it may be replied, *firstly*, that that is more than he can undertake to decide; *secondly*, that by pursuing his mathematical studies on such a presumption, he makes all the pure sciences present *probable* results only, not demonstrated results; more probable, perhaps, than many parts of history, but resting on an impression which must, in his mind, be the result of testimony.

It appears, however, that we may expect series of collateral results, the one for commensurables, the other for incommensurables, and presenting great resemblances to each other; for we may, by any alteration, however minute, convert the latter kind of magnitude into the former. But this we may prevent, by extending our notions of arithmetical operations, or rather by applying to *magnitude* processes which are usually applied to *number* only, as follows:

If we examine the processes of arithmetic, we find, 1st, Addition and substraction, to which abstract number is not necessary, since the concrete magnitudes themselves can be added or subtracted. 2d, Multiplication, the raising of powers and the extraction of roots, in all of which abstract number is essentially supposed to be the subject of operation. 3d, Division, in which it is not necessary to suppose abstract number in finding the whole part of the quotient, but in which we cannot, without reference to numbers, compare the remainder and divisor, in order to form the finishing fraction of the quotient.

4th, The process of finding the greatest common measure of two quantities, in which the remainder is not compared with the divisor, except in a manner which is as applicable to the case of concrete magnitudes as of abstract numbers. To shew this, we shall demonstrate the method of finding the greatest common measure of two *magnitudes*.

Let A and B be two magnitudes, which have a common measure M; let $A=aM$, $B=bM$. Then, it is clear that

$$xA+yB \text{ or } (xa+yb)M,\ xA-yB \text{ or } (xa-yb)M$$

have the same measure, unless it should happen that in the latter case $xa=yb$, in which case $xA=yB$. Let A be the greater of the two, and let A contain B more than β and less than $\beta+1$ times, so that $A=\beta B+B'$, when B' is less than B. Then B' being $A-\beta B$, is measured by M. Let B contain B' more than β' and less than $\beta'+1$ times; or let $B=\beta' B'+B''$ where B'' is less than B'. Let B' contain B'' more than β'' times, &c., or let $B'=\beta'' B''+B'''$, and so on. And B'' or $B-\beta' B'$ is measured by M, &c. We have then the following conditions:

A is a multiple of M
B .... M

$A=\beta B+B'$	$B'<B$,	but is a multiple of M
$B=\beta' B'+B''$	$B''<B'$	....
$B'=\beta'' B''+B'''$	$B'''<B''$	....

Now, since B B' B'' are decreasing quantities, and all multiples of M, they are all to be found in the series,

M, 2 M, 3 M, 4 M, &c.

in which continual decrease must bring us at last to nothing, or we must end with an equation of the form.*

$$B^{(n)}=\beta^{(n+1)}B^{(n+1)}+0$$

that is, one remainder is a multiple of the next. To take a case, let the fifth equation finish the process; so that, in addition to the preceding, we have

$$B''=\beta''' B'''+B^{iv}$$
$$B'''=\beta^{iv} B^{iv}$$

* When a letter denotes an indefinite number of accents, it is distinguished from an exponent by being placed in brackets, and higher numbers of accents than three are usually denoted by *Roman* numerals.

In the fourth, substitute B‴ from the fifth, giving

$$B'' = (\beta''' \beta^{iv} + 1)\, B^{iv}$$

In the third, substitute B″ and B‴ as found, giving

$$B' = (\beta'' \beta''' \beta^{iv} + \beta'' + \beta^{iv})\, B^{iv}$$

In the second, substitute B′ and B″ as found, giving

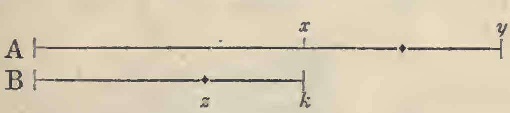
$$B = (\beta' \beta'' \beta''' \beta^{iv} + \beta' \beta'' + \beta' \beta^{iv} + \beta''' \beta^{iv} + 1)\, B^{iv}$$

In the first, substitute B and B′ as found, giving

$$= (\beta \beta' \beta'' \beta''' \beta^{iv} + \beta \beta' \beta'' + \beta \beta' \beta^{iv} + \beta \beta''' \beta^{iv} + \beta'' \beta''' \beta^{iv} + \beta + \beta'' + \beta^{iv})\, B^{iv}$$

Consequently, B^{iv} is a common measure of A and B; but, since M at the outset is any common measure we please, let it be the greatest common measure. Then B^{iv} must be M, for it is in the series M, 2 M, &c.; and were it any other than M, there would be B^{iv} a common measure, greater than the greatest. Hence this process determines the greatest common measure, and also the number of times which each of the two, A and B, contains the greatest common measure.

It is here most essential to observe, that this whole process is independent of any arithmetic, except pure addition and subtraction, which can be performed on the magnitudes themselves, without any numerical relation whatsoever; the only thing required being the axiom in page 3. We shall actually exemplify this on two right lines.



$$A = B + (xy) \quad B = (xy) + (zk) \quad (xy) = 2(zk)$$

Therefore $\quad B = 3(zk) \qquad A = 5(zk)$

In this case, by actual measurement (supposed geometrically exact) B and A are found to be respectively 3 and 5 times zk.

When the preceding process has an end, we therefore detect the greatest common measure; and we have shewn, that where there is a common measure, the process must give it, with the converse. Consequently, in the case where there is no common measure, this process must go on for ever, and we have an interminable series of equations, $A = \beta B + B'$, $B = \beta' B' + B''$, $B' = \beta'' B'' + B'''$, &c., the conditions of which are, that B, B′, B″, B‴, &c., are a continually decreasing series, though it does not follow that each one was less than the half of the preceding. We shall now examine the effect of successive substitutions from the beginning, first making the following remark: If there be any two incommensurable quantities A and B, of which A is the greater, then there follows an interminable set of whole numbers, β, β', β'', ... which are not subject to any particular law, but can be found when A and B are given; and an interminable set of quantities, A, B, B′, B″, ... connected with the former by this law, that A contains B between β and $\beta+1$ times; B contains B′ between β' and $\beta'+1$ times, and so on.

We have $\mathrm{B}' = \mathrm{A} - \beta\mathrm{B}$

$$\mathrm{B}'' = \mathrm{B} - \beta'\mathrm{B}' = \mathrm{B} - \beta'(\mathrm{A} - \beta\mathrm{B})$$

or

$$\mathrm{B}'' = (\beta\beta' + 1)\mathrm{B} - \beta'\mathrm{A}$$

$$\mathrm{B}''' = \mathrm{B}' - \beta''\mathrm{B}'' = \mathrm{A} - \beta\mathrm{B} - (\beta\beta' + 1)\beta''\mathrm{B} + \beta'\beta''\mathrm{A}$$

$$= (\beta'\beta'' + 1)\mathrm{A} - (\beta\beta'\beta'' + \beta + \beta'')\mathrm{B}$$

and thus we go on representing the remainders alternately, in the form $p\mathrm{A} - q\mathrm{B}$ and $q\mathrm{B} - p\mathrm{A}$. We may easily find the law of the coefficients, as follows:

Suppose we come to

$$\mathrm{B}^{(n)} = q\mathrm{B} - p\mathrm{A}$$

$$\mathrm{B}^{(n+1)} = p'\mathrm{A} - q'\mathrm{B}$$

Then we have

$$\mathrm{B}^{(n)} = \beta^{(n+1)}\,\mathrm{B}^{(n+1)} + \mathrm{B}^{(n+2)}$$

or

$$\mathrm{B}^{(n+2)} = \mathrm{B}^{(n)} - \beta^{(n+1)}\,\mathrm{B}^{(n+1)}$$

$$= q\mathrm{B} - p\mathrm{A} - \beta^{(n+1)}(p'\mathrm{A} - q'\mathrm{B})$$

$$= (\beta^{(n+1)}q' + q)\mathrm{B} - (\beta^{(n+1)}p' + p)\mathrm{A}$$

or if, continuing the preceding notation, we suppose

$$\mathrm{B}^{(n+2)} = q''\mathrm{B} - p''\mathrm{A}$$

we have

$$p'' = \beta^{(n+1)}p' + p \qquad q'' = \beta^{(n+1)}q' + q$$

so that, if we write the values of B′, B″, ... with the following notation, putting opposite to each the $\beta^{(n)}$ which occurs *for the first time* in the $\mathrm{B}^{(n)}$ of the equation; namely,

$$\mathrm{B}' = p_1\mathrm{A} - q_1\mathrm{B} \quad \ldots\ldots\ldots\ldots\ldots \quad \beta$$

$$\mathrm{B}'' = q_2\mathrm{B} - p_2\mathrm{A} \quad \ldots\ldots\ldots\ldots\ldots \quad \beta'$$

$$\begin{array}{lll} B''' = p_3 A - q_3 B & \ldots\ldots\ldots\ldots\ldots & \beta'' \\ B^{iv} = q_4 B - p_4 A & \ldots\ldots\ldots\ldots\ldots & \beta''' \\ \&c. \quad \&c. & & \&c. \end{array}$$

we have the following uniform method of forming p_n and q_n for different values of n in succession.

$$\begin{array}{ll} p_1 = 1 & q_1 = \beta \\ p_2 = \beta' & q_2 = \beta'\beta + 1 \\ p_3 = \beta'' p_2 + p_1 & q_3 = \beta'' q_2 + q_1 \\ p_4 = \beta''' p_3 + p_2 & q_4 = \beta''' q_3 + q_2 \\ \&c. \quad \&c. & \&c. \quad \&c. \end{array}$$

in which it is plain, from the method of formation, that p_1 p_2 &c. q_1 q_2 &c. are increasing whole numbers, so that we may continue, supposing B′ B″ .... never fail, till p_n and q_n are greater than any number named. And since B′ B″ .... are all less than B, and therefore less than A, we have the following succession of results, *ad infinitum.*

p_1A	is greater than	q_1B	but less than	$(q_1 + 1)$B
p_2A	is less than	q_2B	but greater than	$(q_2 - 1)$B
p_3A	is greater than	q_3B	but less than	$(q_3 + 1)$B
&c.		&c.		&c.

Hence, it appears that

A is greater than $\frac{q_1}{p_1}$ B

less than $\frac{q_2}{p_2}$ B

greater than $\frac{q_3}{p_3}$ B

less than $\frac{q_4}{p_4}$ B &c. *ad inf.*

Now, from this table of relations, we can determine whether any given multiple of A, xA, is greater or less than any given multiple of B, yB. To do this we must inquire between what two consecutive multiples of B does xA lie.

We now proceed as follows:

1. We must shew that any fraction, such as

$$\frac{a+m}{b+n} \text{ lies between } \frac{a}{b} \text{ and } \frac{m}{n}$$

unless where the two latter are equal, in which case the first also is the same. The preceding must be true if

$$a+m \text{ lies between } \frac{a}{b}(b+n) \text{ and } \frac{m}{n}(b+n)$$

$$\text{or} \quad a+\frac{an}{b} \text{ and } \frac{bm}{n}+m$$

$$\text{or} \quad a+m+\frac{an}{b}-m \text{ and } a+m+\frac{bm}{n}-a$$

$$\text{or} \quad a+m+n\frac{a}{b}-n\frac{m}{n} \text{ and } a+m+b\frac{m}{n}-b\frac{a}{b}$$

which is evidently true: for if $\frac{a}{b}$ be greater than $\frac{m}{n}$, the first is greater than $a+m$, and the second less; if $\frac{a}{b}$ be less than $\frac{m}{n}$, *vice versá.*

2. We now see that

$$\frac{q_3}{p_3} \text{ or } \frac{\beta'' q_2+q_1}{\beta'' p_2+p_1} \text{ lies between } \frac{\beta'' q_2}{\beta'' p_2} \text{ and } \frac{q_1}{p_1}$$

$$\text{or} \quad \frac{q_2}{p_2} \text{ and } \frac{q_1}{p_1}$$

$$\frac{q_4}{p_4} \text{ or } \frac{\beta''' q_3+q_2}{\beta''' p_3+p_2} \ \ldots\ldots\ldots\ldots \ \frac{q_3}{p_3} \quad \frac{q_2}{p_2}$$

and so on. Consequently, to arrange all the fractions thus considered, in order of magnitude, we must write them thus,

$$\frac{q_1}{p_1} \quad \frac{q_3}{p_3} \quad \frac{q_5}{p_5} \ \ldots\ldots \qquad \ldots\ldots \ \frac{q_6}{p_6} \quad \frac{q_4}{p_4} \quad \frac{q_2}{p_2}$$

3. We can thus bring two fractions as near together as we please: to prove this, take three consecutive fractions

$$\frac{q_m}{p_m} \quad \frac{q_{m+1}}{p_{m+1}} \quad \left(\frac{q_{m+2}}{p_{m+2}} \text{ or } \frac{\beta^{(m+1)} q_{m+1}+q_m}{\beta^{(m+1)} p_{m+1}+p_m}\right)$$

which reduced to a common denominator, the first and second, and the second and third, give

$$\frac{q_m\, p_{m+1}}{p_m\, p_{m+1}} \qquad \frac{q_{m+1}\, p_m}{p_{m+1}\, p_m}$$

$$\text{and} \quad \frac{\beta^{(m+1)} q_{m+1}\, p_{m+1}+p_m\, q_{m+1}}{p_{m+1}\, p_{m+2}} \quad \text{and} \quad \frac{\beta^{(m+1)} q_{m+1}\, p_{m+1}+p_{m+1}\, q_m}{p_{m+1}\, p_{m+2}}$$

in which it is clear that the difference of the numerators is the same in each couple, but that if the first numerator be the greater of the first couple, the second numerator is that of the second; a result we might have foreseen, having proved that

$$\frac{q_{m+2}}{p_{m+2}} \text{ lies between } \frac{q_{m+1}}{p_{m+1}} \text{ and } \frac{q_m}{p_m}.$$

Hence it follows that the numerator of the difference of any two successive fractions of the set

$$\frac{q_1}{p_1} \quad \frac{q_2}{p_2} \quad \frac{q_3}{p_3} \quad \ldots\ldots\ldots\ldots$$

is the same as that of the difference immediately preceding, that is,

the difference of $\frac{q_n}{p_n}$ and $\frac{q_{n-1}}{p_{n-1}}$ has the same numerator as

the difference of $\frac{q_{n-1}}{p_{n-1}}$ and $\frac{q_{n-2}}{p_{n-2}}$ ...... which has the same

numerator as the difference of $\frac{q_2}{p_2}$ and $\frac{q_1}{p_1}$; but

$$\frac{q_1}{p_1} - \frac{q_2}{p_2} = \frac{1}{\beta} - \frac{\beta'}{\beta'\beta+1} = \frac{1}{\beta(\beta\beta'+1)}$$

therefore this numerator of the differences is always 1, or

$$\frac{q_n}{p_n} \text{ and } \frac{q_{n+1}}{p_{n+1}} \text{ differ by } \frac{1}{p_n\, p_{n+1}}$$

Hence the difference may be made as small as we please, or smaller than any fraction $\frac{1}{m}$ named by us, since p_n itself can be made greater than m, much more $p_n\, p_{n+1}$.

4. These fractions cannot for ever lie alternately on one side and the other of any *given* fraction $\frac{v}{x}$.

For if this were possible, then, since A lies between

$$\frac{q_n}{p_n}\text{B} \quad \text{and} \quad \frac{q_{n+1}}{p_{n+1}}\text{B}$$

and since by the supposition $\frac{v}{x}$B does the same, and since the couple just mentioned can be made to differ by as small a fraction of B as we please, then we should have

$$\text{A} = \frac{v}{x}\text{B} \pm \text{K}$$

where K may be made as small as we please. Now this is saying that $A = \frac{v}{x}B$; for A must either be

$$\frac{v}{x}B \text{ or } \frac{v}{x}B \pm \text{ some definite magnitude;}$$

but the latter it is not; for the supposition we are trying leads to

$$A = \frac{v}{x}B + \text{ a magnitude as small as we please.}$$

Consequently, our supposition that the series of fractions lie alternately on one side and the other of a definite fraction $\frac{v}{x}$, leads to the conclusion that A and B are commensurable, or the process of finding B′ B″ .... finishes, as we have shewn. But it does not finish, by hypothesis; therefore the series of fractions cannot lie alternately on one side and the other of $\frac{v}{x}$.

We can now shew between what multiples of B xA must lie. It is clear that

$$xA \text{ lies between } \frac{xq_n}{p_n}B \text{ and } \frac{xq_{n+1}}{p_{n+1}}B:$$

now it is not possible that any whole number v should always lie between $\frac{xq_n}{p_n}$ and $\frac{xq_{n+1}}{p_{n+1}}$; for if so, then would

$$\frac{v}{x} \text{ always lie between } \frac{q_n}{p_n} \text{ and } \frac{q_{n+1}}{p_{n+1}}$$

which has been proved to be impossible. Consequently,

$$\frac{xq_n}{p_n}B \text{ and } \frac{xq_{n+1}}{p_{n+1}}B \quad \text{(which approach each other without limit)}$$

must come at last always to lie between two multiples of B; and still more must xA, which lies between *them*. Hence, by proceeding far enough, we can always find between what multiples of B lies xA; and thence whether xA is greater or less than yB.

We have thus divided all pairs of magnitudes into two classes, 1. *Commensurables*, in which we can always say that $A = \frac{q}{p}B$, q and p being whole numbers, and can always tell exactly by what fraction of A or B, xA exceeds or falls short of yB. For we have

$$xA - yB = \left(x - y\frac{p}{q}\right)A = \left(x\frac{q}{p} - y\right)B \quad \text{if} \quad xA > yB$$

$$(yB - xA) = \left(\frac{p}{q}y - x\right)A = \left(y - x\frac{q}{p}\right)B \quad \text{if} \quad xA < yB$$

2. *Incommensurables*, in which we can never say $A = \frac{q}{p}B$, but in which we can assign a series of fractions alternately increasing and decreasing, but making less and less change at every step,

$$\frac{q_1}{p_1} \qquad \frac{q_2}{p_2} \qquad \frac{q_3}{p_3} \quad \ldots\ldots$$

and such that A is greater than $\frac{q_1}{p_1}B$, less than $\frac{q_2}{p_2}B$, &c. *ad infinitum*: so that we can always assign

$$A = \frac{q_n}{p_n}B + K$$

where K is less than any magnitude we name; and such that we can always tell by them whether xA exceeds or falls short of yB, but not exactly *how much*.

Let us suppose, as an example, that we have two magnitudes A and B, which tried by the process in page 13, give

$A = B + B'$, $B = B' + B''$, $B' = B'' + B'''$, &c. *ad inf.*

or suppose $\beta = 1$, $\beta' = 1$, $\beta'' = 1$, &c. *ad inf.*

Hence the several values of p and q are $p_1 = 1$, $p_2 = 1$, $p_3 = 2$, &c. as in this table,

	1	2	3	4	5	6	7	8	9	10	11	12	&c.
p	1	1	2	3	5	8	13	21	34	55	89	144	
q	1	2	3	5	8	13	21	34	55	89	144	233	&c.

or $$A > B < 2B > \frac{3}{2}B < \frac{5}{3}B > \frac{8}{5}B < \frac{13}{8}B \quad \text{\&c.}$$

Hence	A	lies between	B	and 2B
	2A	....	3B	.. 4B
	3A	....	4B	.. 5B
	4A	....	6B	.. 7B
	5A	....	8B	.. 9B &c.

If we wish to know between what multiples of B 100A lies, we find

$$A > \frac{144}{89}B < \frac{233}{144}B;\ 100A > 161\frac{71}{89}B < 161\frac{116}{144}B$$

or 100 A lies between 161 B and 162 B.

We can thus form what we may call a *relative* multiple scale made by writing down the multiples of A, and inserting the multiples of B in their proper places; or *vice versâ*. In the instance just given the commencement of this scale is

B, A, 2 B, 3 B, 2 A, 4 B, 3 A, 5 B, 6 B, 4 A, 7 B, 8 B, 5 A, 9 B, &c

which we may continue as far as we please by simple arithmetic. If the magnitudes in question be lines, we may represent this multiple scale as follows:

Measuring from O, the crosses mark off multiples of A, and the bars multiples of B. Thus

$$O\,1_1 = B \qquad O\,1_2 = 2B \qquad O\,1_3 = 3B \text{ \&c.}$$
$$O \times_1 = A \qquad O \times_2 = 2A \qquad O \times_3 = 3A \text{ \&c.}$$

We shall now proceed to some considerations connected with a multiple scale, for the purpose of accustoming the mind of the student to its consideration. We may imagine a scale like the preceding to be equivalent to an infinite number of assertions or negations, each one connected with the interval of magnitude lying between two multiples of B. Thus, the preceding scale contains the following list *ad infinitum.*

1.	Between	0	and	B	lies no multiple of A	
2.	Between	B	and	2 B	lies A	
3.	Between	2 B	and	3 B	lies no multiple of A	
4.	Between	3 B	and	4 B	lies 2 A	
&c.		&c.		&c.	&c.	

Now, on this we remark, 1st, That the negatives of the above series, though they appear at first to prove nothing, yet in reality have each an infinite number of negative consequences. From the third assertion of the preceding list, namely, neither A, nor 2 A, nor 3 A, &c. lies between 2 B and 3 B, we immediately deduce all the following: A does not lie between 2 B and 3 B, nor between $\frac{2}{2}$ B and

$\frac{3}{2}$B, nor between $\frac{2}{3}$B and $\frac{3}{3}$B, nor between $\frac{2}{4}$B and $\frac{3}{4}$B, &c. &c. 2d, Observe that every affirmative assertion in the above includes a certain number of the affirmative ones which *precede*, and an infinite number of parts of the negative ones preceding and following. For instance, we find that 100 A lies between 161 B and 162 B, or A lies between $\frac{161}{100}$B and $\frac{162}{100}$B, that is between B and 2 B. Again, 2 A lies between $\frac{322}{100}$B and $\frac{324}{100}$B, or between 3 B and 4 B. Similarly, 3 A lies between $\frac{483}{100}$B and $\frac{486}{100}$B, or between 4 B and 5 B; and it might thus seem at first as if every affirmation made all the affirmations preceding its necessary consequences. But if we try 90 A by the preceding, we shall find that it lies between

$$\frac{14490}{100}\text{B and }\frac{14580}{100}\text{B or between } 144\,\text{B and } 146\,\text{B}$$

so that we can only affirm 90 A to lie either between 144 B and 145 B, or between 145 B and 146 B, but we do not (from this) know which. But we can say that 90 A does not lie between 146 B and 147 B, or between 147 B and 148 B, &c. The points at which any affirmation does not determine those preceding may be thus found. Let kA lie between lB and $(l+1)$B; or

$$\text{A lies between } \frac{l}{k}\text{B and } \frac{l+1}{k}\text{B}$$

$$m\text{A} \;\ldots\ldots\ldots\ldots\; \frac{ml}{k}\text{B and } \frac{m(l+1)}{k}\text{B}$$

If $\frac{ml}{k}$ and $\frac{m(l+1)}{k}$ lie between t and $t+1$, then mA lies between tB and $(t+1)$B: but if, in going from the first to the second, we pass through a whole number, or if ml, divided by k, gives a quotient t and remainder r, and $m(l+1)$, divided by k, gives a quotient $t+1$ and remainder r', then we have

$$\frac{ml}{k} = t + \frac{r}{k} \qquad \frac{m(l+1)}{k} = t+1+\frac{r'}{k}$$

$$\text{or} \quad m = k - r + r' \qquad \text{or} \quad r + m = k + r'$$

and in all cases where $r+m$ is greater than k, this condition can be fulfilled. The process may be shortened, by using instead of l the remainder arising from dividing l by k. Suppose, for instance, it is required to determine what preceding affirmatives are ascertained by the proposition 10A lies between 33B and 34B. We have then $l=33$, $k=10$, remainder of $l \div k=3$.

$m=2$	$t=6$	$r=6$	2A lies between 6B and 7B
$m=3$	$t=9$	$r=9$	————
$m=4$	$t=13$	$r=2$	4A .... 13B and 14B
$m=5$	$t=16$	$r=5$	5A .... 16B and 17B
$m=6$	$t=19$	$r=8$	————
$m=7$	$t=23$	$r=1$	7A .... 23B and 24B
$m=8$	$t=26$	$r=4$	————
$m=9$	$t=29$	$r=7$	————

By proceeding thus, it will appear that there is no perceptible law regulating the places of A, 2A, ... among B, 2B, &c., derivable from the sole condition of kA lying between lB and $(l+1)$B. Nevertheless, it is easy to prove, that if all the rest of the relative scale be given from and after any given point, that the whole of the preceding part can then be determined. For, suppose kB to be the commencement of the part of the scale given, and let the place of mA be asked for, which precedes hA, the first multiple of A appearing in the scale. Multiply m by g, so that mg shall be greater than h. Then mgA appears in the portion of the scale given, say between wB and $(w+1)$B. Therefore

$$m\text{A lies between } \frac{w}{g}\text{B and } \frac{w+1}{g}\text{B}$$

and if $\frac{w}{g}$ and $\frac{w+1}{g}$ lie between t and $t+1$, the question is settled; but this must always be the case, if we include the case where $\frac{w}{g}$ or $\frac{w+1}{g}$ is itself a whole number.

From all that precedes, we draw the following conclusions:

1. Having given A and B, two incommensurable magnitudes of the same species (both lengths, both weights, &c.), we can assign, by a process resembling that of finding the greatest common measure in arithmetic, the relative scale of multiples of A and B, which points out between what two multiples of B any given multiple of A lies, or *vice versâ*.

2. Any part of the beginning of this scale being deficient, we can construct it by means of the rest.

3. We can find a magnitude which shall be commensurable with A, differing from B by less than any magnitude we name; and can assign the fraction which it is of A.

Given the two magnitudes, their relative multiple scale is given; but when the scale is given, the two magnitudes are not given. For it is easily proved that there is an infinite number of couples of magnitudes which have the same scale with any given one. Let the scale of A and B be given; then will the scale of $\frac{p}{q}$A and $\frac{p}{q}$B be the same, where p and q are any whole numbers whatsoever.

For if $\quad kA$ lie between lB and $(l+1)B$

then $\quad k\frac{p}{q}A$ lies between $l\frac{p}{q}B$ and $(l+1)\frac{p}{q}B$

or making $\quad \frac{p}{q}A = A' \qquad \frac{p}{q}B = B'$

kA' lies between lB' and $(l+1)B'$

whence the scale of A and B is the same as that of A′ and B′ for any value of k.

What is it, then, which is *given when the scale is given?* Not the magnitudes themselves; for if the scale belong to A and B, it also belongs to every one of the infinite cases of $\frac{p}{q}$A and $\frac{p}{q}$B. The scale, therefore, only defines such a relation between the magnitudes as belongs to 2 A and 2 B, 3 A and 3 B, &c., as well as to A and B. It is usual to call this relation the *proportion* between the two quantities in common life, and in mathematics their *ratio;* in Euclid the term is λόγος.

Two magnitudes, A and B, are said to have the same ratio as two other magnitudes, P and Q, when the relative scales of the two are the same; that is, when the multiples of Q are distributed as to magnitude among those of P, in the same way precisely as those of B are distributed among those of A. And P and Q may be two magnitudes of one kind, two areas, for instance, while A and B may be of another, two lines, for instance.

It is easy to shew that this accordance of scales is equivalent to the common idea of proportion, such as it would become if we took all means of comparison away, except that of multiples. Let us imagine A and B to be two lines in a picture, and P and Q the two corresponding lines in what is meant for an exact copy on a larger scale. Set an artist to determine whether P and Q are in the proper proportion to each other, without any assistance except the means of repeating A, B, P, Q, as many times as he pleases. He will reason as follows: "If Q be ever so little out of proportion to P, though it may not be visible to the eye, yet every multiplication of the two will increase the error, so that at last it will become perceptible. If there be a line 100A laid down in the first picture, and if it be found to lie between 51B and 52B, then should 100P lie between 51Q and 52Q. But if Q be a little wrong, then 100P may not lie between 51Q and 52Q."

It only remains to see whether this definition of proportion will include the case of commensurable quantities. These satisfy such an equation as $k\mathrm{A} = l\mathrm{B}$, k and l being two whole numbers, and it is easy to shew that the whole relative scale is divided into an infinite succession of similar portions. Firstly, this one equation determines the whole scale; for we have

$$\mathrm{A} = \frac{l}{k}\mathrm{B} \qquad\qquad m\mathrm{A} = \frac{ml}{k}\mathrm{B}$$

or if $\frac{ml}{k}$ lie between t and $t+1$, $m\mathrm{A}$ lies between $t\mathrm{B}$ and $(t+1)\mathrm{B}$

$$\text{if} \quad \frac{ml}{k} = t \qquad\qquad m\mathrm{A} = t\mathrm{B}$$

Let us suppose, for instance, $\mathrm{A} = \frac{7}{4}\mathrm{B}$. Then we have

A	lies between	B	and	2B
2A	....	3B	and	4B
3A	....	5B	and	6B
4A	is equal to	7B :	or the scale is	

$$0 \;\; \mathrm{B} \;\; \mathrm{A} \;\; 2\mathrm{B} \;\; 3\mathrm{B} \;\; 2\mathrm{A} \;\; 4\mathrm{B} \;\; 5\mathrm{B} \;\; 3\mathrm{A} \;\; 6\mathrm{B} \;\; \genfrac{}{}{0pt}{}{7\mathrm{B}}{4\mathrm{A}}$$

From this point the scale begins again in the same order. Thus, the second portion is

$$\genfrac{}{}{0pt}{}{7\mathrm{B}}{4\mathrm{A}} \;\; 8\mathrm{B} \;\; 5\mathrm{A} \;\; 9\mathrm{B} \;\; 10\mathrm{B} \;\; 6\mathrm{A} \;\; 11\mathrm{B} \;\; 12\mathrm{B} \;\; 7\mathrm{A} \;\; 13\mathrm{B} \;\; \genfrac{}{}{0pt}{}{14\mathrm{B}}{8\mathrm{A}}$$

and so on *ad infinitum*. The arithmetical definition of A having the same ratio to B which P has to Q, is simply that of A being the same fraction of B which P is of Q: or if

$$A = \frac{l}{k} B \qquad\qquad P = \frac{l}{k} Q$$

Now, since the scale depends entirely on $\frac{l}{k}$, it is the same for both; conversely, if the scale of A and B be the same as that of P and Q, then if $kA = lB$, kP must $= lQ$. Hence the two definitions are synonymous: if one applies, the other does also.

When the multiple scale of A and B is the same as that of P and Q, we have recognised the proportionality of A and B to P and Q. But these scales may differ. The question now is, may they differ in all possible ways, or how far will their manner of differing in one part of the scale affect their manner of differing in others? Am I, to take an instance, at liberty to say, that there may be four magnitudes such that 20 A exceeds 18 B, while 20 P falls short of 18 Q; but that, *for the same magnitudes*, 13 A falls short of 17 B, while 13 P exceeds 17 Q? Such questions as this we proceed to try.

When only two things are possible, which cannot co-exist, each is the complete and only contradiction of the other: the assertion of one is a denial of the other, and *vice versâ*. But when three different things are possible, one only of which can be true, the assertion of one contradicts both of the other two; the denial of one does not establish either of the other two.

The want of a common term, which may simply mean *not less*, that is, either *equal* or *greater*, without specifying which, and so on, causes some confusion in mathematical language. To remind the student that *not less* does not mean greater, but either equal or greater, we shall put such words in italics. Thus, *not less* and *less*, *not greater* and *greater*, are complete contradictions: the denial of one is the assertion of the other.

If A and B be two magnitudes of one kind, and P and Q two others, of the same or another kind, such that

mA is *less* than nB, $\qquad$ mP is *not less* than nQ

then it is impossible that there should be any multiples such that

$m'A$ is *greater* than $n'B$, $\quad$ $m'P$ is *not greater* than $n'Q$

For we find, from the first of each pair, A is less than $\frac{n}{m}$ B, A is greater than $\frac{n'}{m'}$ B

still more is $\frac{n}{m}$ B greater than $\frac{n'}{m'}$ B or $\frac{n}{m}$ greater than $\frac{n'}{m'}$

But P is *not less* than $\frac{n}{m}$ Q, P is *not greater* than $\frac{n'}{m'}$ Q

Now, all the four combinations of this latter assertion contradict

$\frac{n}{m}$ is greater than $\frac{n'}{m'}$; as follows:

$P = \frac{n}{m}Q$, $P = \frac{n'}{m'}Q$, gives $\frac{n}{m} = \frac{n'}{m'}$

P greater than $\frac{n}{m}$ Q, $P = \frac{n'}{m'}Q$, gives $\frac{n}{m}$ less than $\frac{n'}{m'}$,

$P = \frac{n}{m}Q$, P less than $\frac{n'}{m'}$ Q, gives $\frac{n}{m}$ less than $\frac{n'}{m'}$

P greater than $\frac{n}{m}$ Q, P less than $\frac{n'}{m'}$ Q, gives $\frac{n}{m}$ less than $\frac{n'}{m'}$,

Hence the two suppositions above cannot be true together: the happening of any one case of either proves every case of the other to be impossible.

If we range all the possible assertions which can be made, we have as follows:

A_3	mA is greater than nB	P_3	mP is greater than nQ
A_2	mA is equal to nB	P_2	mP is equal to nQ
A_1	mA is less than nB	P_1	mP is less than nQ
a_3	$m'A$ is greater than $n'B$	p_3	$m'P$ is greater than $n'Q$
a_2	$m'A$ is equal to $n'B$	p_2	$m'P$ is equal to $n'Q$
a_1	$m'A$ is less than $n'B$	p_1	$m'P$ is less than $n'Q$

Four of these must be true, one out of each triad; and there are 81 ways of taking one of each, so as to put four together. But we shall take the sets A and *a* together, and find what inference we can draw by taking one out of each.

$A_3\ a_3$ proves nothing as to $\frac{n}{m}$ and $\frac{n'}{m'}$. It merely says that $\frac{n}{m}$ B and $\frac{n'}{m'}$ B are both exceeded by A, which may be whether $\frac{n}{m}$ is greater than, equal to, or less than $\frac{n'}{m'}$. The same for $P_3\ p_3$

$A_3\ a_2$ proves $\frac{n'}{m'}$ greater than $\frac{n}{m}$; as does $P_3\ p_2$

$A_3\ a_1$ proves $\frac{n'}{m'}$ greater than $\frac{n}{m}$; as does $P_3\ p_1$

$A_2\ a_3$ proves $\frac{n'}{m'}$ less than $\frac{n}{m}$; as does $P_2\ p_3$

$A_2\ a_2$ proves $\frac{n'}{m'}$ equal to $\frac{n}{m}$; as does $P_2\ p_2$

$A_2\ a_1$ proves $\frac{n'}{m'}$ greater than $\frac{n}{m}$; as does $P_2\ p_1$

$A_1\ a_3$ proves $\frac{n'}{m'}$ less than $\frac{n}{m}$; as does $P_1\ p_3$

$A_1\ a_2$ proves $\frac{n'}{m'}$ less than $\frac{n}{m}$; as does $P_1\ p_2$

$A_1\ a_1$ proves nothing; neither does $P_1\ p_1$

Now, if we put these pairs together, or make pairs of assertions, in the manner already done, we have 81 distinct sets of four assertions, divisible into those which *may* be true together, and those which *cannot* be true together. An inconsequential supposition, such as A_3a_3, may co-exist with any of the rest from the other set P*p*; but those which give $\frac{n'}{m'}$ necessarily greater, equal to, or less than $\frac{n}{m}$ in the set A*a*, can only co-exist either with the similar ones from the set P*p*, or with those which are inconsequential. Thus we have

$A_3\,a_3$	may be true with any marked	P*p*				
$A_3\,a_2$	requires either	P_3p_3,	P_3p_2,	P_3p_1,	P_2p_1,	or P_1p_1
$A_3\,a_1$	......	P_3p_3,	P_3p_2,	P_3p_1,	P_2p_1,	.. P_1p_1
$A_2\,a_3$	......	P_3p_3,	P_2p_3,	P_1p_3,	P_1p_2,	.. P_1p_1
$A_2\,a_2$	......	P_3p_3,	P_2p_2,			.. P_1p_1
$A_2\,a_1$	......	P_3p_3,	P_3p_2,	P_3p_1.	P_2p_1,	.. P_1p_1
$A_1\,a_3$	......	P_3p_3,	P_2p_3,	P_1p_3,	P_1p_2,	.. P_1p_1
A_1a_2	......	P_3p_3,	P_2p_3,	P_1p_3,	P_1p_2,	.. P_1p_1
$A_1\,a_1$	may be true with any marked	P*p*				

The remaining thirty cannot be true; but it is unnecessary to specify them, as a simple induction from the preceding will shew how to classify those which may and cannot be true. Attach an idea of magnitude to the phrases *greater, equal,* and *less;* say that "*A is greater than B,*" is higher than "*A is equal to B,*" and this again higher than "*A is less than B.*" We have marked the highest phrases by the highest numbers. Say that in $A_3 a_2$, $A_3 a_1$, &c. (calling A and a the *antecedent* clauses of any four marked A, a, P, p), the antecedents are *descending;* in $A_3 a_3$, $A_2 a_2$, and $A_1 a_1$, *stationary;* and in $A_1 a_2$, $A_1 a_3$, &c. *ascending.* Then all the propositions which imply the co-existence of any two antecedents, and any two consequents of the form $A a \; P p$, may be divided into those which *may* be true, and those which cannot be true, by the two following rules:

Ascending antecedents cannot have descending consequents.

Descending antecedents cannot have ascending consequents.

Precisely the same rules will apply if we take two propositions A P for antecedents, and two others $a p$ for consequents; as we may either deduce in the same manner, or by simple inversion. For if $A a \; P p$, with any numerals subscribed, do not contradict either of the preceding rules, neither will the corresponding case of $A P \; a p$ do so, and the contrary. Instances, $A_1 a_2 \; P_3 p_3$ and $A_1 P_3 \; a_2 p_3$; $A_2 a_1 \; P_3 p_2$ and $A_2 P_3 \; a_1 p_2$, &c.

Let us then take a case of A, B, P, Q, in which we find one ascending assertion relative to mA, nB, mP, nQ, for some particular values of m and n; for instance

$$A_1 P_3 \begin{cases} 3A & \text{is less than} \quad 4B \\ 3P & \text{is greater than} \quad 4Q \end{cases}$$

which, as we have seen, is never contradicted in form by any assertion that can be true of any other multiples. These four quantities are not proportionals: for 3 A being less than 4 B, and 3 P greater than 4 Q, P cannot lie in the scale of P and Q in the same place as A in the scale of A and B. But to what more common notion can we assimilate this sort of relation between A, B, P, and Q, namely, that all true assertions of the form (A P) are either ascending or stationary, and never descending? Have we any thing corresponding to this in the arithmetic of commensurable quantities? Let us suppose A and B commensurable, and also P and Q: say that

$$A = \frac{t}{v} B \qquad\qquad P = \frac{t'}{v'} Q$$

Then $3\frac{t}{v}B$ is less than $4B$; $3\frac{t'}{v'}Q$ is greater than $4Q$;

$\frac{t}{v}$ is less than $\frac{4}{3}$; $\frac{t'}{v'}$ is greater than $\frac{4}{3}$; $\frac{t'}{v'}$ is greater than $\frac{t}{v}$

or A is a less fraction of B than P is of Q; which in arithmetic is also said thus, A bears a less proportion to B than P does to Q, or P bears to Q a greater proportion than A bears to B. Hence we get the following definitions, in which we insert the previous definition of proportion, and the accordance of the whole will be seen.

When all true assertions on (mA, nB) (mP, nQ) are either ascending or stationary, and never descending, A is said to have to B a *less* ratio than P to Q; when always stationary, the *same* ratio; when always descending or stationary, and never ascending, a *greater* ratio.

This amounts in fact to the definition given by Euclid, the opening part of whose Fifth Book we shall now make some extracts from, with a few remarks.

DEFINITION III. Ratio is a certain mutual habitude (σχέσις, method of holding or having, mode or kind of existence) of two magnitudes of the same kind, depending upon their quantuplicity (πηλικότης, for which there is no English word; it means relative greatness, and is the substantive which refers to the number of times or parts of times one is in the other).

In this definition, Euclid gives that sort of inexact notion of ratio which defines it in commensurable quantities, and gives some light as to its general meaning. It stands here like the definition of a straight line, "that which lies evenly between its extreme points" prior to the common notion, "two straight lines cannot enclose space," which is the actual subsequent test of straightness. In most of the editions of Euclid we see "Ratio is a mutual habitude of two magnitudes with respect to *quantity*," which makes the definition unmeaning. For quantity and magnitude in our language are very nearly, if not quite, synonymous; or if any distinction can be drawn, it is this: magnitude is the quantity of space in any part of space. But as Euclid is here speaking of magnitude generally (not of space magnitudes only) the words magnitude and quantity are the same.*

* Euclid again uses the word πηλικότης (Book VI. def. 5) in a manner which settles its meaning conclusively. The more advanced reader may consult Wallis, *Opera Mathematica*, v. II. p. 665.

Definition IV. Magnitudes are said to have a ratio to each other which can, being multiplied, exceed "one the other." This means that quantities have a ratio when, any multiples of both being taken, the relation of greater or less exists. It is usually rendered "Two magnitudes are said to have a ratio when the lesser can be multiplied so as to exceed the greater." But the above is literally translated, and the sense here given to ratio makes the next definition consistent. It is a way of expressing that the two magnitudes must be of the same kind, which requires that the notion of greater and less should be applicable to them. That this notion should be applicable to the quantities themselves as well as their multiples, being the necessary and sufficient condition of the possibility of the comparison implied in the next definition, is here assumed* as the distinction of quantities which have a ratio.

Definition V. Magnitudes are said to be in the same ratio the first to the second, and the third to the fourth: when the same multiples of the first and third being taken, and also of the second and fourth, with any multiplication, the first and third (multiples) are greater than the second and fourth together, or equal to them together, or less than them together.

This amounts to our definition of proportion, namely, that the relative multiple scale of A and B is the same as that of P and Q. For, take the same multiples of A and P, namely, mA and mP, and the same multiples of B and Q, namely, nB and nQ. Then, if the relative multiple scales be the same, let mA lie between vB and $(v+1)B$, it follows that mP lies between vQ and $(v+1)Q$. If, then, n be less than v, nB is less than vB, and nQ less than vQ. And mA being greater than vB must be greater than nB, while, for the same reason, mP is simultaneously greater than nQ. In the same way the other parts of the definition V. may be shewn to be included in that of identity of multiple scales. Now, reverse the supposition

* The common version is several times referred to afterwards, and the definition 4 expressly alluded to, in the editions of Euclid. But it must be remembered that the Greek of Euclid contains no references to preceding propositions, these having been supplied by commentators. The reader may, if he can, make Λόγον ἔχειν πρὸς ἄλληλα μεγέθη λέγεται, ἃ δύναται πολλαπλασιαζόμενα ἀλλήλων ὑπερέχειν mean, "Magnitudes are said to have a ratio, when the less can be multiplied so as to exceed the greater."

and assume Euclid's definition. If, then, mA lie between vB and $(v+1)B$, it follows that mA is greater than vB, whence, by the assumption mP is greater than vQ. Similarly, because mA is less than $(v+1)B$, mP is (by that definition) less than $(v+1)Q$. Therefore, mP lies between vQ and $(v+1)Q$, or in this instance, or for any one value of m, the scales are accordant, and the same may be proved in any other case. It follows, then, that the two definitions are mutually inclusive of each other.

The manner in which Euclid arrived at this definition has been matter of inquiry. But any one who will examine the first nine propositions of the *tenth** book, will see that he had precisely the same means of arriving at it as we have used. But, besides this, he might have come by the definition from a common notion of practical mensuration, as follows. Suppose two rods given, one of which is the English yard, the other the French metre, but neither of them subdivided. The only indication which looking at them will offer, is that the metre exceeds the yard apparently by about ten per cent. To get a more exact notion, the obvious plan will be to measure some great distance with both. Suppose 100 yards to be taken off with the yard measure, it will be found that that 100 yards contains about 91 metres and a half, the half being taken by estimation, and we will suppose the eye could not thus err by a quarter of a metre. Then the yard must be ·915 nearly of a metre, and the error upon one yard cannot exceed the hundredth part of the quarter of a metre, or ·0025 of the metre. But the mathematician, to make this process perfectly correct, will suppose distance *ad infinitum*, measured from a point both in yards and metres, or in fact will form what we call the relative multiple scale. He then looks along this scale for a point at which a multiple of a yard, and a multiple of a metre end together. If this happen, and it thus appear that m yards is exactly equal to n metres, the question is settled, for a yard must be $\frac{n}{m}$ of a metre. But it will immediately suggest itself to a mind which is accustomed not to receive assumptions without inquiry, that it may be no two

* There are two English editions of the *whole* of Euclid, and there may be more: that of John Dee (now old and very scarce) and that of J. Williamson, London, 1788, in two thin quarto volumes. The dissertations in the latter are a strange mixture of good and bad, but the text is very literally Euclid, in general.

points ever coincide on the multiple scale. But in this case it is very soon proved that mA may be made as nearly equal to nB as we please, by properly finding m and n; so that a fraction $\frac{n}{m}$ may be found such that A shall be as nearly $\frac{n}{m}$B as we please. Even admitting that this would do to assign A in terms of B, it leaves us no method of establishing any definite connexion between A considered as a part of B, and P considered as a part of Q.

The word part usually means *arithmetical* part, namely, the result of division into equal parts. Thus $\frac{3}{7}$ is a part of 1 made by dividing 1 into 7 *equal* parts, and taking 3 of them. The phrase of Euclid in the books on number (VII. to X. both inclusive) is that $\frac{1}{7}$ is *part* of 1, $\frac{3}{7}$ is *parts* of 1. And it is easily shewn that, in this use of the word, every quantity is either *part* or *parts* of every other quantity which is commensurable with it. And of two incommensurable quantities, neither is part or parts of the other. But in the original sense of the word part, any less is always part of the greater. This notion of incommensurability, the non-existence of the equation $m\text{A} = n\text{B}$, for any values of m or n, obliges us to have recourse to a *negative* definition of proportionality, a term which we proceed to explain. Examine the definition of a square, namely, " a plane foursided figure, with four equal sides and one right angle." It is clear that the examination of a finite number of questions will settle whether or no a figure is a square. Has it four sides? are they in the same plane? are the sides equal? is one angle a right angle? *Proof of the affirmative* of these four propositions proves the figure to be a square. Now, examine the number of ways in which a figure can be shewn to be not a square. All propositions are either affirmative or negative; A is B or A is not B. The affirmative can be proved or the negative disproved, with one result only, for both give A is B. But the affirmative can be disproved, or the negative proved, with an infinite number of results; it is done by proving that A is C, or D, or E, &c. &c. *ad infinitum*. Thus there may be an infinite number of ways of shewing that a figure is not a square, but there is only one way of shewing that it is a square. This we call a *positive* definition.

Now examine the definition of parallel lines, " those which are in the same plane, but being produced ever so far do not meet." We are not considering where the lines meet, if they do meet, or distinguishing between lines which meet in one point and in another, but simply dividing all possible pairs of lines into two classes, *parallels* and *intersectors*. Now here it is impossible to prove * the affirmative of the proposition, "A and B are parallels," by means of the definition only, without proving an infinite number of cases. To see this more clearly, remember that every proposition relative to the intersection or non-intersection of straight lines, is an assertion which either includes or excludes every possible couple of points which can be taken, one on each straight line. "Lines intersect" means there is a couple of such points which coincide. "Lines are parallel" means that there is no such couple whatsoever, of all the infinite number which can be taken.

The first proposition in which Euclid *proves* the existence of parallels (the 27th) does not shew that the lines *are* parallels, but that the proposition, "the lines are intersectors," is inconsistent with preceding results. The proposition, "A and B are parallels," though it appears affirmative, yet is in Euclid a negative, for his express definition of parallels does not define what they are, but what they are not, "not intersectors." This we call a *negative* definition.

Now, to examine further Euclid's definition of equal ratios, we must consider his definition of greater and less ratios. They amount to the following. A is said to have to B a greater ratio than P has to Q where there is, among all possible whole numbers m and n, *any one pair* which give mA greater than nB, but mP equal to or less than nQ; or which give mA equal to nB, but mP less than nQ: which give in fact, *in any one case*, what we have called a *descending* assertion. And A is said to have to B a less ratio than P has to Q, when *any one pair* of whole numbers m and n gives mA less than mB, but mP equal to or greater than nQ, or mA equal to nB, but mP greater than nQ: which give in fact, in any one case, what we have called an *ascending* assertion. Here, to a mind the least inquisitive, appears at once a decided objection. Our notions of the terms

* The celebrated axiom of Euclid evades this, and in point of fact amounts to another and a positive definition of parallels, the assumption being that the old definition agrees with it. Or rather we should say, that the first twenty-five propositions of the first book establish a part of the connexion of the definitions, and the axiom assumes the rest.

greater and *less* will never allow us to suppose that any thing, quantity, ratio, or any thing else, can be both greater and less than another quantity, or ratio; and yet, on looking at the definition of Euclid, we see that for any thing which appears to the contrary, one pair of values of m and n may shew that A has a greater ratio to B than P to Q, while another pair may shew that it has a less. The objection is perfectly valid; the only fault to be found is, that it should not have arisen before, when the definitions of the first book were proposed. How is it then known that there can be such a thing as a foursided figure with equal sides and one right angle, or as lines which never meet? The confusion arises from placing the definitions in the form of assertions, before the possibility of the assertions which they imply are proved. The defect may be remedied (we take the square as an instance) in two ways.

1. Write all definitions in the following manner. To define a square, for example, "If it be possible to construct a plane figure having four equal sides and one right angle, let that figure be called a square."

2. Omit the definition of a square, head the 46th proposition of the first book as follows.

"Theorem. On a given straight line, a four-sided figure can be constructed which shall have all its sides equal to the given straight line, and all its angles right angles." Having demonstrated this, add the following definition: Let the figure so constructed be called a square.

We have shewn that all sets of four magnitudes, A and B of one kind, P and Q both of the same kind with the first, or both of one other kind, can be divided into three classes.

1. Those in which simultaneous assertions on mA and nB, and on mP and nQ, are *all* (for all values of m and n) either ascending or stationary.

2. Those in which they are all stationary.

3. Those in which they are all either descending or stationary.

For we have shewn that the only remaining possible case *à priori*, namely, that in which there are both ascending and descending assertions for different values of m and n, is a contradiction amounting in fact to supposing one fraction to be both greater and less than another. And it has been shewn that all the three cases are possible, for commensurable quantities at least. We are now, therefore, in a condition to say, let A and B in the first case be said to have a less ratio to B than P has to Q; in the second, the same ratio; in the third, a ~~less~~ greater ratio. The only question now is, are these definitions properly negative or positive. It will immediately appear that, out of the three, the first and third can be directly and affirmatively shewn to be true of particular magnitudes, and that the second cannot. By which is meant, that the comparison of individual multiples may, by a single instance, establish the first or third, but that no comparison of individual multiples, however extensive, can establish the second. For the second consists in stationary assertions *ad infinitum*, and the first and third are proved by a single ascending or descending assertion.

As an instance, suppose

A = 951 feet	B = 497 feet	P = 1300 lbs.	Q = 679 lbs.
1902	994	2600	1358
2853	1491	3900	2037
3804	1988	5200	2716
4755	2485	6500	3395

In these first five multiples, there are none but stationary assertions, of twenty five which might be made. Thus

$$\left.\begin{matrix}4755 > 994\\ 6500 > 1358\end{matrix}\right\} \quad \left.\begin{matrix}2853 > 2485\\ 3900 > 3395\end{matrix}\right\} \quad \left.\begin{matrix}951 < 994\\ 1300 < 1358\end{matrix}\right\} \text{ \&c.}$$

but neither of the three definitions is thereby shewn to belong to these four magnitudes. Now, take the first and third 498 times, and the second and fourth 952 times, and we have, going on with the series of multiples,

473598	473144	647400	646408
474549	473641	648700	647087

and here the process may close, for we have 473598 less than 473641, while 647400 is greater than 647087. Consequently, we *have* proved, by comparison, that 951 feet has to 497 feet a less ratio than 1300 lbs. to 679 lbs.

But the case in which neither greater nor less ratio exists can never be established by actual comparison of multiples, except only in the case where the pairs of magnitudes are commensurable. For, remark that the mere circumstance of the relative multiple scale of A and B agreeing with that of P and Q up to any point, is neither proof nor presumption that the two magnitudes given are actually proportional, though, as we shall see, it is certain evidence that they are nearly proportional, if the multiple scales agree for a great number of multiples. Proportion is not established until the similarity of the multiple scales is shewn to continue for ever. Now, though it would not be remarked at first, this insertion of an infinite number of conditions to be fulfilled, is tantamount to a negative definition, if we wish to make the definition specifically speak of one absolute criterion of disproportion or proportion. Disproportion is where there is an ascending or descending assertion somewhere in the comparison of the multiple scales. Proportion is where there is no descending or ascending assertion.

In the case of commensurable quantities the definition is positive, because there is then a single stationary assertion, which, being proved, all the rest are shewn to follow. If A and B be commensurable, let $m\text{A} = n\text{B}$; then if $m\text{P} = n\text{Q}$, there is proportion; if not, there is disproportion. See page 24 for the proof as to the rest of the multiple scales.

We have said, that, when the multiple scales agree for a long period, there is proportion nearly; and it is proved thus: Suppose that the scale of A and B agrees with that of P and Q, up to 10,000 P and 10,000 Q, but that we have disagreement as follows: 9326 A lies between 10,000 B and 10,001 B, whereas 9326 P lies between 10,001 Q and 10,002 Q. Or the scales run thus:

10,000 B	9326 A	10,001 B		10,002 B
10,000 Q		10,001 Q	9326 B	10,002 Q

How much must we alter A to produce absolute proportion? Not more than would be necessary to make 9326 A greater than 10,002 B, or less than would still keep it less than 10,001 B. That is, we must so alter A as to add somewhere between 0 and 2 B to 9326 A, or somewhere between

$$0 \quad \text{and} \quad \frac{2}{9326}\text{B} \quad \text{to} \quad \text{A}$$

Consequently, the addition of a small part of B to A would make an accurate proportion.

We might now proceed to the propositions of the Fifth Book of

Euclid; but there are three difficulties in the way of the student's perfect satisfaction with the definition. 1st, He may have a mysterious idea of incommensurables. 2d, He may not be satisfied of the necessity of departing from arithmetic. 3d, He may find it difficult to imagine how the existence of proportionals can ever be established, with, apparently, an infinite number of conditions of definition to satisfy. We suppose that the gravity of tone which elementary writers adopt, is inconsistent with the statement of a beginner's difficulties, in the words in which he would express them. We shall remove all necessity for preserving such dignity in a case where it may be inconvenient, by a simple supposition. Let *A* be a beginner in the stricter parts of mathematics; that is, a person apt to mix previously acquired notions with the meaning he attaches to definitions which are intended to exclude all but the ideas literally conveyed in the words which are used; much better pleased with the apparent simplicity of an incorrect definition, gained either by omitting what should not be omitted, or by supposing what cannot be supposed, than with the comparatively cumbrous forms which provide for all cases, and distinguish differences which really exist; and, finally, when a doubt exists, rather predisposed against, than in favour of, the necessity of demonstration. Let *B* be another person, who has subjected his mind to that sort of discipline which has a tendency to remove the propensities abovementioned. We can imagine them talking together in this manner:

A.—I have been trying to understand the meaning of incommensurable quantities, and cannot at all make out how it can be that one given line may be no fraction whatsoever of another given line, though both remain fixed, and certain lines ever so little greater or less than the first are fractions of the second.

B.—A little consideration will teach you, that neither in arithmetic nor geometry are we at all concerned with how things *can be*, but only with whether they *are* or not. Do you admit it to be demonstrated that the side and diagonal of a square, for instance, are incommensurable? (*Algebra*, page 98).

A.—I cannot deny the demonstration, but the result is incomprehensible. Does it really prove, that if I were to cut the diagonal of a square into ten equal parts, each of these again into ten equal parts, and so on for ever, I should never, by any number of subdivisions, succeed in placing a point of subdivision exactly upon the point which cuts off a length equal to the side.

B.—I take it for granted you have sufficiently comprehended the definitions of geometry, to be aware that a thin rod of black lead, or a canal of ink, are not geometrical lines; and that the excavations which you perforate by the compasses are not points.

A.—Certainly; I now have no difficulty in imagining mere length intersected by partition marks, which are not themselves lengths.

B.—Then, in the case you proposed, you need not go so far for a difficulty; for your method of subdivision will never succeed in cutting off so simple a fraction as the third part of the diagonal.

A.—Why not?

B.—You see that 9, 99, 999, &c., are all divisible by 3, so that 10, 100, 1000, &c., cannot in any case be divisible by 3, but must leave a remainder. Your method of subdivision can never put together any thing but tenths, hundredths, &c. If possible, suppose one-third to be made up of tenths, a in number, added to hundredths, b in number, added to thousandths, c in number. Then we must have

$$\frac{1}{3} = \frac{a}{10} + \frac{b}{100} + \frac{c}{1000}$$

Clear the second side of fractions, and we have

$$\frac{1000}{3} = a \times 100 + b \times 10 + c$$

or $\frac{1000}{3}$ is a whole number, which is not true. And the same reasoning might be applied to any other case.

A.—This is conclusive enough; but it seems to follow that the third part of a line is incommensurable with the whole.

B.—So it is, as far as the one method of subdividing which you propose is concerned. Let tenths, hundredths, &c., be the *measurers*, and one-third and unity are *incommensurable*. But the word with which we set out implies all the possible subdivisions of halves, thirds, fourths, fifths, &c. &c., to be tried, and all to fail.

A.—But here is an infinite number of ways of subdividing. Can it be possible that no one of them will give a side of a square, when the diagonal is a unit?

B.—In the first place, it would be a sufficient answer to this sort of difficulty to say, that, for any thing you know to the contrary, the number of ways in which you may fail is as infinite as the number of ways in which you may try to succeed. In the second place, there is also an infinite number of ways of subdividing, which will not give one-third. Let your first subdivision be into any number of equal parts, except only 3, 6, 9, 12, &c.; and your second subdivision the same, or any other, with the same exceptions, &c. The same reasoning will prove that you can never get one-third.

A.—But look at the matter in this way. Suppose the halves, the thirds, the fourths, the fifths, &c. &c. of a diagonal laid down upon it *ad infinitum*, so that there is no method of subdividing into aliquot parts, how many soever, but what is done and finished. Would not the whole line be then absolutely filled with subdivision points, and would not one of them cut off a line equal to the side of the square.

B.—You have now changed your use of the word *infinite*, and applied it in the sense of *infinity attained*, not *infinity unattainable*. As long as you used the word to signify succession, which might be carried as far as you pleased, and of which you were not obliged to make an end, the word was rational enough, though likely to be misunderstood; but as it is, you may as well suppose you have got beyond infinite space, at the rate of four miles an hour, and are looking back upon the infinite time which it took you to do it, as imagine that you *have* subdivided a line *ad infinitum*. But if the idea of *infinity attained* be a definite conception of your mind, you meet the difficulty of incommensurable quantities in another form. The definition of the term incommensurable was shaped in accordance with the exact notion, that, subdivide a line as far as you may, you must stop at some finite subdivision; and incommensurable parts of a whole are those which you never exactly separate arithmetically, stop at what finite subdivision you please. But, if you will contend for infinite subdivision attained, and imagine the line thus filled up by points, then it will be necessary to divide all parts of a whole into two classes, those which are cut off by finite subdivision, and those which are not attainable, except by infinite subdivision; the former answering to commensurable, the latter to incommensurable, parts. The difficulty remains then just as before; in other words, why should the side of a *square* be not attainable from its diagonal except by infinite subdivision, when the sides of a rectangle, which are as 3 to 4 (instead of 3 to 3), are attainable by a finite number of subdivisions?

In the next place, you have spoken of a line filled up by points, the infinitude of the number of points being the compensation for each of the points having no length whatsoever; at least, it is not easy to see what else you can mean.

A.—Certainly that is what I mean; and the common expressions of algebra are in accordance with what I say. For, if I cut a line into n equal parts, it is plain that the sum of the n parts makes up the whole, be the number n great or small. But by making n sufficiently great, each of the parts may be made as small as I please; and, therefore, allowing it to be rational to say that P takes place when n is infinite, in all cases in which we may come as near to P as we please, by making n sufficiently great (which is the expressed meaning of *infinite* in algebra), it follows that we may say, that the line is made up of the infinite number of points into which it is cut when divided into an infinite number of equal parts.

B.—I see every thing but the last consequence.

A.—Why, surely, the smaller a line grows, the more nearly does it approximate to a point.

B.—How is that proved?

A.—Suppose two points to approach each other, they continually inclose a length which is less and less, and finally vanishes altogether when the two points come to coincide in one point. So that the smaller the straight line is, the more near is it to its final state—a point.

B.—You have not kept strictly to your own idea (which is a correct one) of the way in which the words *nothing* and *infinite* may be legitimately used. You have supposed a line to be entirely made up of points, each of which has no length whatsoever, because you may compose a line of a very large number of very small lines, each of which, you say, is nearly a point. Let us now consider whether your final supposition is one to which we can approach as near as we please by diminution of a length. Any line, however small, can be divided into other lines by an infinite number of different points; for any line, however small, admits of its halves, its thirds, &c. &c. So that there is a theorem which is not lessened in the numbers it speaks of, or altered in force or meaning, in any the smallest degree, by diminishing the line supposed in it; namely, any line whatsoever admits of as many different points as we please being laid down in it. Now, of your *final length*, or limit of length—the point—this is not true: consequently, you throw away a result at the end, which you cannot throw away as nearly as you please during the process by which you attain that end; nor will the denial of it, near the end, be less in the consequence or amount of the error, than if the rejection were made further from the end. Therefore, in asserting that a diminishing straight line approximates to a point, you have abandoned the condition under which you are allowed to speak of *nothing* or *infinite*.

Again, the nth part of a line taken twice is certainly greater than the simple nth part, however great n may be. Now, what do you suppose two points to be, which are laid side by side without any interval of length between them?

A.—They are, of course, one and the same point.

B.—But in your infinite subdivision, two nth parts must be greater than one nth part, or two of your points must be greater than one; but these two points are the same point, which is therefore twice as great as itself. Such are the consequences to which the supposition of a line made up of points will lead.

A.—I have frequently heard of lines being divided into an infinite number of equal parts.

B.—But you never heard those equal parts called *points*. I can soon shew you that, in the mode of allowing infinity to be spoken of, this fundamental condition is preserved, namely, that no theorem, limitation, number, nor other idea whatsoever, which forms a part of any question, is allowed to be rejected or modified when n is infinite, unless it can be shewn that such rejection or modification may be made with little error when n is great, with less error when n is greater, and so on; finally, with as small an error as we please, by making n sufficiently great. Now, remark the following truths, and the form of speech which accompanies them, when n is supposed infinite.

General Theorem.	Terminal Theorem.
The greater the number of equal parts into which a line is divided, the less *line* is each of the parts: so that an aliquot part of any line, however great, may be made less than any given line, however small.	If a straight line be divided into an infinite number of equal parts, each part is an infinitely small *line*.

General Theorem.	Terminal Theorem.
Any line, however small, may be cut by as many points as we please.	An infinitely small line may be cut by as many points as we please.
No straight line, however small, ceases to be a length terminated by points.	An infinitely small straight line is a length terminated by points.

Now, taking your notion of infinite subdivision attained, it may be shewn that incommensurable parts necessarily follow. For, however far you carry the subdivision, you do not, by means of the subdivision points, lessen the number of points which may be laid down. For each interval defined by the subdivisions contains an infinite number of points. Consequently, if you will suppose the infinite subdivision attained, you cannot do it without supposing an infinite number of points left in the intervals, or an infinite number of incommensurable quantities. This I intend only to shew that the proof of the existence of incommensurable quantities is, upon your own supposition, somewhat better than that of their non-existence. But it would be better to use nothing and infinity as convenient phrases of abbreviation, not as containing definite conceptions which may be employed in demonstration.

A.—I do not see how your objection applies against *nothing;* if we cannot attain infinity by continual augmentation, we can certainly attain nothing by continual diminution.

B.—So it may seem at first, and in truth you are right as to one sort of diminution, that which is implied in the word *subtraction.* From the place in which there is something take away all there is, and you get *nothing* by a legitimate process. But subtraction is the only process which leaves nothing; division, for example, never leaves it. Halve a quantity, take the half of the half, and so on, *ad infinitum:* you will never reduce the result to nothing.

A.—But however clearly you may shew that incommensurable quantities actually exist, as a necessary consequence of our definitions of length, number, &c., I should feel better satisfied if you could give something like an account of the way in which they arise.

B.—If you will consider the way in which number and length are conceived, perhaps the difficulty may be somewhat lessened. Let a point set out from another point, and move uniformly along a straight line until the two are a foot distant from each other. It is clear that every possible length between 0 and one foot will have been in existence at some part or other of the motion. Now, suppose a number of points as great as you please, to set off from the first point together; but, instead of moving in the straight line let them move off in curves, the first coming to the straight line at $\frac{1}{2}$ and 1 of a foot; the second at $\frac{1}{3}$ $\frac{2}{3}$ and 1; the third at $\frac{1}{4}$ $\frac{2}{4}$ $\frac{3}{4}$ and 1 of a foot; and so on, as in this diagram.

Can you feel sure that these contacts of curves with the line, separated as they must always be from each other by finite intervals, will ever fill up the whole line described by a continuous motion. If not, this figure will always supply presumption in favour of incommensurable parts, which will of course be increased to certainty by the actual proof of their existence. And this should be sufficient to overturn a doubt which after all is derived from confounding the mathematical point with the excavation made by the points of a pair of compasses. The *practical* commensurability of all parts with the whole is a consequence of there being magnitudes of all sorts below the limits of perception of the senses (see page 3).

A.—Granting, then, that there are such things as incommensurable quantities, it is admitted, that though A and B are incommensurable, yet A and B + K may be made commensurable, though it be insisted on that K shall be less than any given quantity, say less than the hundred thousand million millionth of the smallest quantity which the senses could perceive, if they were a hundred thousand million of million of times keener than they are at present. Would it not be sufficient, when incommensurable quantities, A and B, occur, to suppose so slight an alteration made in B as is implied in the above, and reason upon A and B + K so obtained, instead of upon A and B. Surely such a change could never produce any error which would be of any consequence?

B.—Of consequence to what?

A.—To any purpose of life for which mathematics can be made useful.

B.—I am still at a loss.

A.—What process in astronomy, optics, mechanics, engineering, manufactures, or any other part either of physics or the arts of life, would be vitiated by such an alteration, or its consequences, to any extent which could be perceived, were the error multiplied a million fold?

B.—None whatever, that I know of.

A.—What, then, would be the harm of introducing a supposition which would save much trouble, and do no mischief?

B.—I am not aware that I admitted such a supposition would do no mischief, when I said that it would not sensibly vitiate the application of mathematics to what are commonly called the arts of life. I see that your idea of mathematics is very much like that which a shoemaker has of his tools. If they make shoes which keep the weather out, and bring customers, he need not wish them to do more, or inquire further into any use, actual or possible, which they may or might have. The end he proposes to himself is answered, when he has sewed the upper leather firmly to the sole. But whether his art serves any higher purpose—whether the possibility of obtaining conveniencies, and avoiding hardships (which it creates in one respect), excites industry and ingenuity, creates property to equalise the fluctuations of harvests and commerce, and prevent the community from undergoing periodical pests and famines—makes men so dependent on each other that internal war is next to impossible, and external war a grave and serious consideration, &c. &c., are not matters for the thoughts of a working shoemaker; nor will similar considerations ever enter the mind of a working mathematician. You have spoken of the purposes of life; I do not know what the purposes of your life may be, but if among them you count such a discipline of the mind as may always render your perception of the force of an argument properly dependent upon the probability of the premises, and the method by which the inferences are drawn, it will be one of your first wishes to propose to yourself, as a standard and a model, some branch of study in which the first are self-evident, or as evident as any thing can be, and the second indisputable and undisputed. For though you may find no other science which will compete with this in accuracy, yet you will be more likely to infer correctly, when you have seen what you know to be correct inference, than you would have been if you had never, in any case, distinguished between demonstration of certainties and presumptions from probabilities. And still more, will you be qualified to refute, and refuse admission to, that which takes the form of accuracy without the reality. If the mathematical sciences be good as a weapon, they are a hundred fold better as a shield. I have seen many who were visibly little the better for their mathematical studies in what they advanced; but very few indeed who were not made sensibly more cautious in what they received.

A.—But is not my notion adopted in practice by a great part of the mathematical world, particularly on the continent.

B.—It is certainly true, and it is particularly the case with the French, who, though they have done more than any other nation, since the time of Newton, to advance the mathematical sciences, have been by no means anxious to consider them as resting on other evidence than that—not of the senses—but of the limits of the senses. One of their most celebrated elementary writers considers none but arithmetical proportion, and begins his work by shewing either that two straight lines have a common measure, as in page 12, or that the remainder " échappe aux sens par sa petitesse." All his propositions, therefore, *in geometry*, are either true, or so nearly true, that the difference is imperceptible. The phrase we have quoted is an honest and a valuable admission ; it shews you, that in the opinion of one of the most useful and extensive elementary writers that ever lived, *arithmetical* proportion makes geometry a science of approximate, not absolute, truth.

A.—I see as much ; but cannot the slight shifting of one of the quantities which I proposed be somehow or other corrected, so as to make a strict and useful theory of the proportions of incommensurable quantities?

B.—Yes, and in a very simple way; by adopting the definition of Euclid. This may surprise you, but I will soon shew that the most natural correction of your notion leads direct to the definition of Euclid. Let it be granted that A and B being commensurable, and $m\mathrm{A}=n\mathrm{B}$, proportion between A, B, P, and Q means that $m\mathrm{P}=n\mathrm{Q}$. Now you want, when A and B are *incommensurable*, to be allowed to substitute $\mathrm{B}+\mathrm{K}$ instead of B, where K is excessively small. I suppose you would be perfectly content if it could not be made visible by any microscope. Now I am of a somewhat more abstract turn, and should not like my geometry to be put in peril by the abolition of the excise on glass; which it might be by the allowance of experiments for the improvement of that article, which are now effectually prevented. I cannot admit $B + K$, where the magnitude I want to reason upon is B. But as the definition of proportion of incommensurables is not yet settled, let us examine this case: A and B being incommensurable, let P and Q be quantities of such a kind that A and $B + K$ are commensurable, and also P and $Q + Z$, and that the four just named are arithmetically proportionals. Let it be possible, these conditions subsisting, to make K and Z as small as we please: not as small as this, that, or the other small quantity, but smaller than any whatsoever which ~~may be~~ named, being still *some* quantities. You wish to substitute $B + K$ and $Q + Z$ for B and Q: I prefer to use the conditions laid down to ascertain how B and Q themselves stand related to A and P. Let us suppose we name two small magnitudes, K' and Z', of the same kind as A and P, or B and Q, or K and Z, which we are at liberty to make as small as we please. We can then find K and Z less than K' and Z', and such that A, $B + K$, P, $Q + Z$ are proportional. Suppose A and $B + K$ commensurable, and let

$$mA = n(B + K) \quad \text{whence} \quad mP = n(Q + Z)$$

whence it is easily proved, as in page 24, that the relative scale of multiples of A and $B + K$ is the same as that of P and $Q + Z$. I say it follows, that the relative scale of A and B is the same as that of P and Q; for, if not, the two latter scales must differ somewhere. Let it be that vA is greater than wB, but vP less than wQ. Then, since vA is greater than wB, let K be taken so small (it may be as small as we please) that vA shall also exceed $w(B + K)$, whence, by the proportion assumed in the hypothesis, vP exceeds $w(Q + Z)$, while, by the hypothesis we are trying, vP is less than wQ. This is a contradiction, for vP cannot exceed $wQ + wZ$ and fall short of wQ at the same time. In the same way, any other case may be treated; and it follows that our suppositions, if K may be as small as we please, amount to an hypothesis from which Euclid's definition follows. If, in the above, we suppose vA less than wB, while vP is greater than wQ, we see that vA also falls short of $v(B + K)$, whence, by the proportion, vP falls short of w(Q + Z), which cannot be if it exceed wQ, and Z may be as small as we please.

A.—But is not this deduction, namely, Euclid's definition, more cumbrous than the form from which it has just been deduced?

B.—How so?

A.—Does it not involve an infinite number of considerations, extending the whole length of the multiple scales?

B.—And does not your definition do the same thing, unless you stop somewhere with the values of K and Z? Is it not necessary, if we would not be merely microscopically correct, but absolutely correct, to suppose that K and Z may be diminished and diminished *ad infinitum?* And what difference is there, as to the number of considerations in question, between two magnitudes which are to diminish without limit, and a set of increasing multiples of two given magnitudes?

A.—But Euclid's definition seems to wander such a way from the quantities in question, while the other remains close to them, and we never seem to quit them, except for something very near to them. The actual application of the definition I prefer will require nothing but the division of all magnitudes into aliquot parts.

B.—Your objection amounts to this; that you feel the fractions of a quantity to be more closely connected in your mind with the quantity itself than its multiples. This may be the case; and, if so, it is some reason for preferring the form to which you seem most inclined. But there may be a stronger reason for preferring the other; and, undoubtedly, as long as difficulties exist, every system of science must be a balance of inconveniences. But Euclid is, of all men who ever wrote, the one who has a reason for the course which he takes, where there are two or more. I suppose you cannot but admit that it is better to found a definition *in geometry* upon the result of something which can actually be done, *by the means of geometry*, than upon something which can only be conceived or imagined to be done, with what certainty soever; for instance, you would not wish to be obliged to use other means than the straight line and circle, or to suppose an object gained without using any means at all?

A.—Certainly not. That there shall be no assumption of mechanical power beyond that of drawing a straight line or circle, is the foundation of pure geometry.

B.—Then the question is settled in favour of Euclid's definition;

for, without either assuming more mechanical means, or making a gratuitous assumption, no angle, nor arc, nor sector of a circle, can be divided into 3, or 6, or 9, &c. parts, unless it be a right angle, or a given half, fourth, eighth, &c. of a right angle. There are some other exceptions; but, generally, to cut any angle into three equal parts is a geometrical impossibility, and certain algebraical considerations furnish the highest presumption that it will always remain so.

A.—But this difficulty is still left: how are we ever to shew that there are such things as proportional quantities?

B.—We can do this so easily, that the greatest stumbling-block of the process lies in its being so easy and perceptible, that a beginner does not very well see where lies the knowledge he has gained, unless he has paid profitable attention to the definition of proportion. From the first book of Euclid it is evident that a rectangle is doubled by doubling the base, trebled by trebling it, and so on; and also, that of two rectangles between the same parallels, the greater base belongs to the greater, and the lesser base to the lesser. Now, let B and B′ represent two bases, and R and R′ the rectangles upon them, the altitudes, or distances of the parallels, being the same. If then we take the first base m times, giving mB, the rectangle upon that base is mR: if we take the second base n times, giving nB′, the rectangle upon that base is nR′, the parallels always remaining the same. Hence it follows, that mB and nB′ are bases to the rectangles mR and nR′ between the same parallels; accordingly, therefore, as mB is greater than, equal to, or less than nB′, so is mR greater than, equal to, or less than nR′: and this being true for all values of m and n, it follows that B has to B′ the ratio of R to R′, or the bases of rectangles between the same parallels, and the rectangles themselves are proportionals.

A.—Am I to understand then that there are difficulties in the way of considering magnitude in general, which are not found in arithmetic, or the science of abstract number?

B.—Quite the reverse: the difficulty arises from the deficiences of arithmetic itself, and from their being ratios which the ratio of number to number cannot represent.

A.—But how is that? arithmetic always seemed clear of such difficulties as we have been considering.

B.—And so would this subject, if the disposition to be satisfied with what is in the book, which is part and parcel of almost every beginner had been permitted to rest quietly upon a theory of commensurable ratios. But did you never, in arithmetic, hear of the creation of a nonexistent number or fraction, in spite of there being no such thing, by agreeing that there should be such a thing, and drawing a picture to represent it?

A.—I do not understand the jest; but I suppose you allude to algebra, and to quantities less than nothing?

B.—Not at all; I am speaking of pure arithmetic. To me, -2 is a much easier symbol, or picture, than $\sqrt{2}$; and even the difficulties of $\sqrt{-2}$ lie as much in the $\sqrt{\ }$ as in the $-$.

A.—But I do not understand what you mean by saying that $\sqrt{2}$ does not exist; it is the square root of 2, and multiplied by itself it gives 2. You may find it as nearly as you please.

B.—If it be the object of arithmetic, commonly so called, it is either a whole number or a fraction. Which of these is it?

A.—It is a fraction; 1·4142136, very nearly.

B.—I did not ask you what it is *very nearly*, but what it is?

A.—It cannot be given exactly, but we all know there is such a thing as the square root of 2.

B.—If the objects of arithmetic were numbers, fractions, and things, and the latter term had a definition, I might admit what you say. And in *concrete* arithmetic, where 1 is a *thing*, a foot, a pound, or an acre, I admit that there is such a thing as $\sqrt{2}$. But that thing is not attainable arithmetically by taking any aliquot part of the thing 1, and repeating it any number of times. In abstract arithmetic the square root of 2 is an impossibility; and having no existence, I do not see how one fraction can be said to be nearer to it than another, except in this sense, that $2+z$ may be made to have a square root where z may be less than any fraction we name. The independent existence of $\sqrt{2}$ is an algebraical consideration of some difficulty; that is, belongs to the science which has relations of symbols, under prescribed definitions, for its object, without reference to their numerical interpretation. The difficulties of $\sqrt{2}$ are precisely those of incommensurable magnitudes; in fact $\sqrt{2}$ is the diagonal of a square whose side is 1. But it is to algebra that difficulties of this kind should be referred. The student, if he use $\sqrt{2}$ in pure arithmetic, must expressly understand it as a fraction whose square is nearly 2, and must consider this part of arithmetic (without algebra as a science of approximation, unless geometry, or some other science of concrete quantity, be supposed to lend its aid.

A.—But I cannot divest myself of the idea that $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{6}$ are really fractions, and that the product of the two first gives the last. I suppose, in some sense or other, you admit this proposition?

B.—Certainly. If $\sqrt{2+x}$ and $\sqrt{3+y}$ and $\sqrt{6+z}$ be made to exist, by giving proper values to x, y, and z, which may all be as small as I please, and if, moreover, x, y, and z be so related that $z = 3x + 2y + xy$, which condition does not interfere with the last; I can then admit that

$$\sqrt{2+x} \times \sqrt{3+y} = \sqrt{6+z}$$

But I do not allow myself to suppose that (understanding by multiplication the taking of one number or fraction as many times or parts of times as there are units or fractions of a unit in another), there can be such a truth as that

$\sqrt{2}$ (neither number nor fraction) *multiplied* by $\sqrt{3}$ (do. do.) $=$ $\sqrt{6}$ (do. do.)

But this is beyond our subject, except so far as it shews that the difficulty lies more in arithmetical than in geometrical considerations.

A.—Might we not then dispense with arithmetic altogether, and make a definition corresponding to proportion for geometry?

B.—Yes; but the difficulty would appear in another shape, of the very same substance. Let four lines be called proportional when, being straightened without alteration of length, if necessary, the rectangle made by the first and fourth is equal to that made by the second and third. Let areas be proportional when, being converted into rectangles with a common altitude, their bases are proportional. Let angles be proportional, when they are angles at the centre of proportional arcs of the same circle. But here would immediately arise this difficulty,—to make a straight line equal to a given arc of a circle; which is out of the power of the geometry of straight lines and circles.

A.—Is not the *reductio ad absurdum* (which is very much used in the establishment of the theory of proportion) rather a suspected method. I have heard it called *indirect* demonstration; and it is frequently stated as a defective method, not to be used if it can possibly be avoided.

B.—The complaints against this method of demonstration have become much more frequent, if not entirely made their appearance, since the time when logic was a necessary part of a liberal education, as it once was, and as I hope it will be again. I have sometimes wondered whether this argument would have been considered objectionable if it had been reduced to the form "A is B, B is C, therefore A is C;" as follows: "Every contradiction of P is a contradiction of the proposition that the whole is greater than its part; but every contradiction of this proposition is false: therefore every contradiction of P is false; or P is true." The *reductio ad absurdum* is as conclusive, and may be made as intelligible, as any other argument. And if any argument be good in proportion to the effect upon the mind, where is the affirmative proposition, in geometry or not, which the mind seizes as readily as it recoils from an absolute contradiction in terms? Where is the likeness or resemblance between things which are alike, that is so forcible as the unlikeness or want of resemblance of two ideas which palpably contradict, such as black is white?

A.—Is there then no advantage in the direct over the indirect demonstrations?

B.—D'Alembert has said that the former are to be preferred "parce qu'elles éclairent en même-temps qu'elles convainquent," which is a good description of the difference. But even this must be taken with some allowance, for there are many indirect demonstrations which are highly instructive.

Recapitulation. By the *ratio* of A to B, we mean (without any further specification at present) a relation between the magnitudes of A and B, determined by the manner in which the multiples of A are distributed, if each be written between the nearest multiples of B in magnitude. That is, if B, 2 B, 3 B, &c., be formed, and A, 2 A, 3 A, &c., and if A lie between B and 2 B, 2 A between 3 B and 4 B, and so on, the relative scale

B, A, 2B, 3B, 2A, 4B, &c.

is to be the sole determining element of the ratio, so that there is to be nothing but the order of this scale on which the ratio depends. And if P and Q be two other magnitudes with the same order in their scale, P compared with A, and Q with B, then A and B are to be said to have the same ratio as P and Q. But if any multiple of A precede among the multiples of B the place which the corresponding multiple of P occupies among the multiples of Q, then A is to be said to have to B a less ratio than P has to Q. But if a multiple of A come later in the series of multiples of B than its corresponding multiple of P in the series of multiples of Q, then A is said to have to B a greater ratio than P has to Q. It is plain that the ratio of A to B must be greater than, equal to, or less than that of P to Q, and also, that in saying A is to B as P to Q, we also say that B is to A as Q to P.

[We must remind the student that we have now nothing to do with the reasons of this definition, or the accordance of its parts with each other, or with any notion of ratio more than is contained in it. We are merely now concerned to know what follows from this definition. The numbering of the following propositions is that in Euclid.]

When A has to B the same ratio as P to Q, the four are said to be proportionals, and are written thus:

$$A : B :: P : Q$$

which is read A is to B as P is to Q.

IV. If $A : B :: P : Q$
Then $mA : mB :: nP : nQ$ } m and n being any whole numbers.

This we know when we see that any quantities being arranged in order of magnitude, so will be their multiples. If the scales be

B	A	2B	3B	......
Q	P	2Q	3Q	......

the following scales

mB	mA	$2mB$	$3mB$	......
nQ	nP	$2nQ$	$3nQ$	......

will also be arranged according to magnitude. Whence the proposition.

VII. If A, B, C, be three *homogeneous* magnitudes (all lines, or all weights, &c.) and if $A = B$; then

$$A : C :: B : C$$

and

$$C : A :: C : B;$$

for the scales must evidently be identical.

VIII. A+M has a greater ratio to B than A has to B, and B has a less ratio to A+M than B has to A. Let M be multiplied so many times that it exceeds B; say $m\text{M}=\text{B}+\text{K}$: then

$$m(\text{A}+\text{M}) = m\text{A}+\text{B}+\text{K}$$

Let mA lie between vB and $(v+1)$ B; then $m(\text{A}+\text{M})$ lies between $v\text{B}+\text{B}+\text{K}$ and $(v+1)\text{B}+\text{B}+\text{K}$, and certainly beyond $(v+1)\text{B}$. Consequently; in the scales of A+M and B, and A and B, a multiple of A + M is found to be in a higher place among the multiples of B than the same multiple of A among the multiples of B. Whence, by definition, A+M has to B a greater ratio than A to B. The second part of the proposition is but another way of stating the first, as appears from definition. Thus we may also say that A has to B a less ratio than A + M has to B.

IX. If $\text{A} : \text{C} :: \text{B} : \text{C}$ then $\text{A}=\text{B}$
or if $\text{C} : \text{A} :: \text{C} : \text{B}$ then $\text{A}=\text{B}$

For (VIII.), if A be greater than B, A has to C a greater ratio than B to C, which is not true. If A be less than C, A has to C a less ratio than B to C, which is not true: therefore A = B. The same reasoning proves the second case.

X. If A have to C a greater ratio than B has to C, then A is greater than B. For if A were equal to B, then these ratios would be the same: if A were less than B (VIII.), then would A have to C a less ratio than B has to C. Therefore, A is greater than B. Similarly, if A have to C a less ratio than B has to C, A is less than B. And if C have a greater ratio to A than to B, A is less than B; if C have a less ratio to A than to B, A is greater than B.

XI. If the ratios of C to D and of E to F, be severally the same as that of A to B, then C has to D the same ratio as E to F. This answers to a case of the general axiom, that two things which are perfectly like to a third in any respect, are perfectly like each other in that respect. The multiples of C are distributed among those of D in the same manner as those of A among those of B, as are those of E among those of F. Therefore, the multiples of C are distributed among those of D as are those of E among those of F. Whence the proposition.

XII. If A be to B as C to D, and as E to F, then as A is to B so is A+C+E to B+E+F.

For mA lying between nB and $(n+1)B$, then mC lies between nD and $(n+1)D$, and mE between nF and $(n+1)F$, and, consequently, $mA+mC+mE$, or $m(A+C+E)$ between $n(B+D+F)$ and $(n+1)(B+D+F)$. Whence the proposition.

XIII. If A have to B the same ratio as C has to D, but C to D a greater ratio than E to F, then A has to B a greater ratio than E to F.

This is one of a class of propositions which come under this general theorem: for any ratio, an equal ratio may be substituted, and all consequences of the first ratio are consequences of the second. This, which seems very evident, may appear so upon mistaken evidence. Ratios, as far as we have yet gone, are not quantities, but expressions of that relation between quantities upon which the order of magnitude of their multiples depends. For quantity, we may substitute other quantity equal to the first in magnitude wherever the relation is one which depends only on quantity; we may not substitute a triangle of the same area instead of a square, except there be question of nothing but superficial magnitude, or area. Ratio, again, is to us at present the order of the multiples, so that if A and B have their multiples arranged among each other in a given order, if P and Q have the same, we may say that whatever is true of the order of multiples of A and B, is also true of the order of P and Q; whatever connexion the order of multiples of A and B establishes between A and B and other magnitudes, the same connexion exists between P and Q and those other magnitudes, because the accident of A and B, which is the sole connexion between them and the consequence inferred, is also an accident of P and Q. The necessity for going over such considerations, arises from its never being allowed to be taken for granted that a mathematician has studied logic. Hence Euclid* is frequently obliged to reiterate the same assertions in different forms. To take the proof of the present proposition; to say that C has to D a greater ratio than E to F, is to say that mC can be found greater than

* Euclid was a contemporary of Aristotle, as is generally supposed, and may, therefore, never have seen the science of the latter. It is free to us to suppose that if he had, he would have distinguished between a purely logical and a geometrical consequence: that is, would not have reiterated the same proposition in different forms; or, if you please, different cases of the same verbal truth as if they were distinct truths: and we will suppose so accordingly.

nD, while mE is equal to or less than nF. But to say that A has to B the ratio of C to D, is to say that whenever mC is greater than nD, mA is greater than nB. Therefore, to say that the ratios of A and B and C and D are the same, but the latter greater than that of E to F, is to say that mA may be greater than nB, while mE is equal to or less than nF; or that A has to B a greater ratio than E has to F.

Now, let the student compare this with the following proposition. A and B are greens of exactly the same shade: but B is a darker green than C, therefore, A is a darker green than C. Would it be unnecessary to prove this? then it is equally unnecessary to prove the preceding. But we will prove this in the same manner as we prove the preceding. Let there be a test of greenness, which decides between two greens (there is a test of comparison of ratios in Euclid), and apply the test to B and C. The result is, of course, that B is the darker. But A being by hypothesis exactly the same as B, the testing operation would be self contradictory if it did not exhibit, when applied to A and C, the very same intermediate process by which we were able to compare B and C, with the same result. If the above be unnecessary, then the demonstration of Euclid's proposition is unnecessary.

The fact is, that there are in geometry two distinct sorts of demonstration, the first of which is only a portion of the second. The first is the verbal treatment of the terms of an hypothesis, and the developement of all assertions which are necessarily included in the terms of the proposition, without drawing upon any other axioms or theorems for evidence. It is the purely logical process, by which we make two assertions put together shew their joint meaning, and express what, without deduction, they only imply. Thus, from "Every A is B," and "no B is C," we make it evident that in these assertions is necessarily contained a third, that "no A is C." Thus it has been shewn that we cannot allow simultaneous existence to the two propositions, "A is to B as C is to D," and "C is to D more than E is to F," without almost expressing, and certainly implying, by the mere meaning of our terms, this third proposition, that "A is to B more than E is to F."

The second process is that in which the demonstration, besides the purely logical process of extracting implied meanings out of the expressions of the hypothesis, appeals to propositions which are not in the hypothesis, and which, for any thing the hypotheses tell us to the contrary, may or may not be true. Of course—not logic, but—reason requires that these propositions should have been previously proved, or assumed on their own evidence expressly. Let us take the following proposition, "The sum of the circles described upon the two sides of a right-angled triangle is equal to the circle described upon the hypothenuse." Now, take every notion implied in this hypothesis, "Let there be a right-angled triangle, and let circles be described on its three sides." The united faculties of man never proved that the sum of the circles on the sides was equal to the circle on the hypothenuse, without assuming with Euclid, to the effect that only one parallel can be drawn through a point to a given right line; with Archimedes, to the effect that the chord of a curve is shorter than its arc, &c. &c.; and various consequences. But are any of these propositions necessary to our complete definition of a right angle, a triangle, or a circle? If not, we have a broad and easily recognised distinction between the first and second method of demonstration; the first, an operation of logic, or deduction from the premises of the hypothesis; the second, introducing premises from without.

There are two classes of reasoners whose ideas we recommend the student closely to examine, before he finally decides: 1. Geometrical writers in general, who pay no attention to the methods which they are using, but let the first book and the fifth book of Euclid contain no difference by which it may be remarked that the processes contained in the two are different acts of mind. Did they ever think that geometry could be made the engine by which the student could examine certain operations of his own faculties, or did they only imagine that it was a method of making very sure that squares, circles, &c. had such and such properties? 2. The class of metaphysical writers, who express themselves to the effect that all mathematical propositions are contained in the definitions and axioms, in a sense in which other results of reasoning are not. Put them to the proof of this assertion as to geometry, and then as to arithmetic.

The whole of the process in the fifth book is purely logical, that is, the whole of the results *are* virtually contained in the definitions, in the manner and sense in which metaphysicians (certain of them) imagine all the results of mathematics to be contained in their definitions and hypotheses. No assumption is made to determine the truth of any consequence of this definition, which takes for granted more about number or magnitude than is necessary to understand the definition itself. The latter being once understood, its results are deduced by inspection—of itself only, without the necessity of looking at any thing else. Hence, a great distinction between the fifth and the preceding books presents itself. The first four are a series of propositions, resting on different fundamental assumptions; that is, about different kinds of magnitudes. The fifth is a definition and its developement; and if the analogy by which names have been given in the preceding books had been attended to, the propositions of that book would have been called *corollaries of the definition.*

XIV. If A be to B as C is to D, all four being of the same kind, then if A be greater than C, B is greater than D; if equal, equal, and if less, less.

A must either be $>$ $=$ or $<$ C. Let A be greater than C; then mA is greater than mC. Let mA lie between nB and $(n+1)$B; then will mC lie between nD and $(n+1)$D. But because A exceeds C, 2A exceeds 2C by twice as much, &c., and mA exceeds mC by m times as much; or mA may be made to exceed mC by a quantity greater than any one named, say greater than B and D together. Then the order of magnitude of the four multiples mC $(n+1)$D, nB, mA must be as written: for $(n+1)$D does not exceed mC by so much as D, and nB does not fall short of mA by as much as B, while mA exceeds mC by more than B and D put together. Therefore, nB is greater than $(n+1)$D, and still more than nD. That is, B is greater than D.

Let A be equal to C. If B exceed D at all, mB may be made to exceed mD by more than D, or mB may be made, from and after some value of m, greater than $(m+1)$D. That is, the order of magnitude may be made

$$mD \quad (m+1)D \qquad mB \quad (m+1)B$$

Having gone so far on the scales that this order becomes permanent, go on till a multiple of C (kC) falls between the two first. Then, by the definition, kA falls between the two last, which is absurd; for, because A$=$C, kA$=k$C; therefore, B does not exceed D. In the same way it may be shewn that B does not fall short of D. Therefore, B$=$D.

The remaining case (A less than C) may be proved like the first.

XV. A is to B as mA is to mB

The scale of multiples of A and B is nowhere altered in the order of magnitude by multiplying every term by m. If pA lie between qB and $(q+1)$B, (pm)A which is $p(m$A$)$ lies between $q(m$B$)$ and $(q+1)$ $(m$B$)$.

XVI. If A be to B as C is to D

and if all four be of the same kind,

Then A is to C as B is to D.

(IV.) mA is to mB as nC is to nD

(XIV.) If mA be greater than nC, mB is greater than nD, if equal, equal; if less, less. Therefore, A is to C as B to D.

XVII. If A + B be to B as C + D to D, then A is to B as C is to D. If mA lie between nB and $(n+1)$B, it follows that mA+mB, or m(A + B) lies between $(m+n)$B and $(m+n+1)$B. Then, by the proportion, m(C + D) lies between $(m+n)$D and $(m+n+1)$D, or mC + mD lies between mD + nD and mD + $(n+1)$D, or mC lies between nD and $(n+1)$D. Therefore, the scales of A and B, and of C and D, are the same; whence the proposition.

XVIII. If A be to B as C is to D, then A + B is to B as C + D is to D. A proof of exactly the same kind as the last should be given by the student.

XIX. If A : B :: C : D, C and D being less than A and B, then A : B :: A — C : B — D. For the hypothesis gives A to C as B to D, and A is C + (A — C), and B is D + (B — D), whence,

C + (A — C) is to C as D + (B — D) is to D

(XVII.) A — C is to C as B — D is to D

(XVI.) A — C is to B — D as C to D, or as A to B

XX. If A be to B as D to E

and B to C as E to F

Then A is $\left\{\begin{matrix}\text{greater than}\\ \text{equal to}\\ \text{less than}\end{matrix}\right\}$ C, when D is $\left\{\begin{matrix}\text{greater than}\\ \text{equal to}\\ \text{less than}\end{matrix}\right\}$ F

Let A be greater than C; then A is to B more than C is to B; but A is to B as D to E, and C to B as F to E; therefore, D is to E more than F is to E, or D is greater than F. In a similar way the other cases may be proved.

Hence it follows, that A is to C as D to F. For,

(VI.)	mA	is to	mB	as	mD	to	mE
	nB	is to	nC	as	nE	to	nF

therefore, mA is $> =$ or $< nC$ when mD is $> =$ or $< nF$

whence, A is to C as D to F.

XXI. If of the magnitudes

$$\begin{matrix} A & B & C \\ D & E & F \end{matrix} \quad \text{we have} \quad \begin{matrix} A : B :: E : F \\ B : C :: D : E \end{matrix}$$

Then $A > =$ or $< C$ when $D > =$ or $< F$

Let A be greater than C; then A is to B more than C is to B: as before E is to F more than E is to D, or D is greater than F. Similarly for the other cases.

XXII. If there be any number of magnitudes,

$$\begin{matrix} A & B & C & D & \ldots\ldots\ldots\ldots \\ P & Q & R & S & \ldots\ldots\ldots\ldots \end{matrix}$$

and if any two *adjoining* be proportional to the two under or above them, then any two whatsoever are proportional to the two under or above them. For, since (XX.)

$$\left.\begin{matrix} A : B :: P : Q \\ B : C :: Q : R \end{matrix}\right\} \begin{matrix} \text{Therefore, } A : C :: P : R \\ \text{But, } \quad C : D :: R : S \end{matrix}\Big\} \text{Therefore, } A : D :: P : S, \text{ \&c.}$$

XXIII. In the hypothesis of (XXI.), by proof as before in (XX.), A is to C as D to F.

XXIV. If A be to B as C to D, and E be to B as F to D, then A + E is to B as C + F to D

$$\begin{matrix} \text{If} & A \text{ be to } B \text{ as } C \text{ to } D \\ \text{and} & E \text{ be to } B \text{ as } F \text{ to } D \end{matrix}\Big\} \text{then } A + E \text{ is to } B \text{ as } C + F \text{ to } D$$

For, $A : B :: C : D$
and $B : E :: D : F$ } whence, $A : E :: C : F$

(XVIII.) $A + E : E :: C + F : F$

But, $E : B :: F : D$

Therefore, $A + E : B :: C + F : D$

XXV. If A : B :: C : D, all being of the same kind, the sum of the greatest and least is greater than that of the other two. First, which are the greatest and least? If A be the greatest, then C is greater than D; and because A : C :: B : D, B is greater than D;

therefore, D is the least. Now, prove that if B be the greatest, C is the least; and that, by inverting the proportion, if necessary, it may always be written with the greatest term first, and the least last.

When A is the greatest, since A — B : A :: C — D : D, A — B is greater than C — D; therefore, (A — B) + B + D is greater than (C — D) + B + D, or A + D is greater than C + B.

If there be a given ratio, that of A to B, and another magnitude P, there must be a fourth magnitude Q, of the same kind as P, such that A is to B as P to Q, or Q to P as B to A.

Firstly; Q may certainly be taken so small that (mB being greater than nA) mQ shall be less than nP. Find m and n to satisfy the first conditions, and let K satisfy the second. Then K is to P less than B is to A. Now (mB being less than nA), Q may be taken so that mQ shall be greater than nP. Find m and n to satisfy the first, and let L satisfy the second. Then, L is to P in a greater ratio than A to B. And it is immediately shewn that every magnitude less than K is to P less than B to A, and every magnitude greater than L is to P more than B to A. Whence, it is between K and L that the fourth proportional Q is found, if any where. There cannot be more than one such value of Q; for, if there be two different magnitudes V and W, since, then, by taking m sufficiently great, we may make mV and mW differ by more than P, it is impossible that both mV and mW can lie between the same consecutive multiples of P, as those of B which contain between them mA. And the above also evidently shews, that if we suppose a magnitude Q, changing its value from K to L, it cannot during its increase become of the same kind as L, namely, more to P than B is to A, and then again become of the same kind as K. For, whatever magnitude has this property of L, every greater one has the same. There is then only one point between K and L at which this change takes place, and we have, therefore, this alternative: EITHER G (between K and L) is less to P than B is to A, and *every* magnitude greater than G is more; OR, some magnitude G between K and L is the same to P as B to A, and is the intermediate limit lying above all those which are less to P, and below all those which are more. By disproving the first alternative, we prove our proposition. If possible, let G be less to P than A to B, G + V more, however small V may be.

Then may mG be made less than nP (mA being greater than nB), while m(G+V) is greater than $\overset{n}{m}$P. For the *ascending* assertion must be converted at least into a *stationary* one. Let m G fall short of $\overset{n}{m}$P by Z; then V may be taken so small that mV shall not be so great as Z, or mG+mV not so great as mG+Z, that is, not so great as $\overset{n}{m}$P. But the first clause of the alternative supposes that m(G+V) must be greater than $\overset{n}{m}$P, how small soever V may be; therefore this clause cannot be true, or the second must be true.

This fourth proportional to A, B, and P, then, must exist; but whether it can be expressed by the notation, or determined by the means of any science, is another question. It can be expressed in arithmetic when A and B are commensurable: it can be found in geometry (by the straight line and circle) when A and B are lines or rectilinear areas. But if they be angles, arcs of circles, solids, &c. it cannot be assigned by the straight line and circle, except in particular cases.

Let us suppose the ratio of A to B given, that is, not A and B themselves, but only the answer to this question for all values of m, "Between what consecutive multiples of B lies mA?" Suppose also the ratio of B to C given; how are we to find the ratio of A to C, or can it be found at all? that is, is it given or determined by the two preceding ratios. Take any magnitude P, and determine Q so that P is to Q as A to B, and then determine R so that Q is to R as B to C. Then the ratio of P to R (page 59) is that of A to C; not that P is A or R is C (for they may even be magnitudes of different kinds), but P is to R as A is to C.

The process by which the ratio of A to C is found by means of those of A to B and B to C, is called by Euclid *composition* of these ratios; or the ratio of A to C is compounded of the ratios of A to B and B to C. What, then, ought to be meant by the ratio compounded of the ratios of A to B and X to Y. Our guide in the assimilation of processes, and the extension of names, is always the following axiom.

Let names be so given, that the substitution of one magnitude for another equal magnitude shall not change the name of the process; and, generally, that the same operations (in name) performed upon equal magnitudes, shall produce the same result.

Let X be to Y as B to N, where N is a fourth proportional to be determined. Then the ratio of A to N is that compounded of A to B and B to N, and is what must be meant by that compounded of

A to B, and X to Y. It is proved in Prop. 20, that ratios compounded of equal ratios are equal ratios.

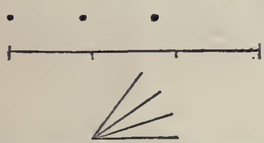
Again, to find the ratio compounded of the ratios of A to B, C to D, and E to F; let the process by which the ratio of A to D is derived from those of A to B, B to C, and C to D, still be called composition. Then take B to M as C to D, and M to N as E to F: the ratio of A to N is that compounded of the three ratios.

In the beginning of this work, we deduced the necessity for considering incommensurables in some such manner as that of Euclid, from the notion which, as applied to commensurables, admits of a definite representation, derived from the idea of proportion. But the method of the fifth book is different. It is there implied, that where-ever two magnitudes exist, their joint existence gives rise to a third magnitude, called their ratio, of which magnitude no conception is given except what is contained in certain directions how to apply the terms equal, greater, and less, to two of the kind. On this the natural question is, what sort of magnitude is this, and how do we know that there is any magnitude whatsoever which admits of this apparently arbitrary exposition of definitions? This question is very much to the point, and the want of an answer at the outset is a main cause of the difficulty of the Fifth Book. The answer implied in the work of Euclid is this: Let us first consider what will follow if there be such things as ratios, or magnitudes to which these definitions of equal, greater, and less apply; we shall then shew (in the Sixth Book) that there are different pairs of magnitudes, of which it may be said that they have ratios, and we shall never have occasion to inquire what ratio is.

We may take a case parallel to the preceding from the First Book. The notion of a straight line suggests nothing but length; that of two straight lines which meet, suggests a relation, which we may conceive stated in this way. If A, B, C, and D, be straight lines, of which A and B, and C and D, meet; let A and B be said to make the same angle as C and D, when, if A be applied to C, and B and D fall on the same side, B and D also coincide: but let A be said to make a greater angle with B than C with D, when, in a similar case, B falls outside of C and D, &c. To this it would be answered, that the preceding definitions are a circuitous way of saying that the angle made by two lines is their opening or inclination; an indefinite term, which, though it distinguishes angle from length, does not serve to compare one angle with another. And just in the same manner, if it were not that the definition is more complicated, and refers to an abstract, not a visible or tangible, conception, it would immediately be seen that *ratio* is *relative* magnitude,—a term which is sufficient to distinguish the thing in question from *absolute* magnitude, but which does not give any means of comparing one thing of the kind with another. The immediate deduction of this idea is as follows: If, whenever mA lies between nB and $(n+1)$B, it also happens that mP lies between nQ and $(n+1)$Q, it follows that A, lying between two certain fractions of B, $\frac{n}{m}$B, and $\frac{n+1}{m}$B, then P lies between the same two fractions of Q. Or, if $m\text{A}=n\text{B}$, that is, if $\text{A}=\frac{n}{m}\text{B}$, then P is the same fraction of Q. Or we may state it thus: if B be made unity, for the measurement of A, and Q for the measurement of P, then A and P are the same numbers or fractions of their respective units.

Euclid has commenced the subject with a rough definition, as we have seen, p. 29, and the translators have spoiled it, by not distinguishing between quantity, and relative quantity; that is, by so wording the definition as to say nothing more than that ratio is a relation of magnitudes with respect to magnitude.

We now come to consider the application of the preceding notions to arithmetic. Let us first separate all that part* of arithmetic which relates to abstract and definite numbers, from the rest, and let us call it *primary* arithmetic. A little observation will shew that *abstract* number as distinguished from concrete, is really the same thing as ratio of magnitude to magnitude. What is *three*, for example? It is an idea which we obtain equally from looking at



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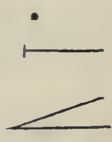
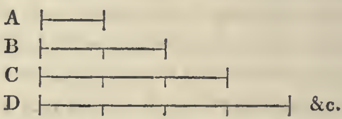


From putting such concretes together, we bring away a notion of there being the same relative magnitudes existing between the individuals

* The whole of the First Book of my *Treatise on Arithmetic*, with the exception of § 158, 165–169.

of each pair. In the first, it is *repetition*, in the second, it is length, in the third, it is opening, we are reminded of; but in all three, we say the first is *three times* the second. Now this word *times* is, in fact, a limitation, which will not do for our present purpose; it implies that we will have no other ratios except those of line to line in the series



made by repetitions only: but there may be ratios which are not those of line to line in any *repetition*, how far soever carried.

Here is a point at which we are compelled to pause, to adjust the well-known terms of number to the new idea we have put upon them. *Abstract numbers are certain ratios*; *abstract fractions are certain other ratios: but all possible ratios are not found among numbers and fractions; whence it arises, that primary arithmetic, though it may be, so far as it goes, a theory of ratios, is not a theory of all ratios, nor are its operations such as can be performed upon all ratios.*

That ratios are magnitudes, we must have supposed from the beginning, seeing that they bear the terms equal, greater, and less. But there was still this defect, that our test of A being to B more than C to D, was one which left us with no idea *how much* more A was to B than C to D; which amounts but to this, that we could not define the *ratio* of *ratios* without having first defined *ratio*. But, in like manner as arithmetic was made the guide to that notion which is properly* called the ratio of incommensurable quantities, so will the ratio of two ratios in arithmetic lead us, after a little consideration, to the meaning of the ratio of ratios of incommensurables.

When we say *two*, we refer to the repetitions of the smaller in a ratio of magnitudes, thus visibly related:

When we say twice two, there is a change of idiom in our language. It might be, instead of *twice two* is *four*, *two twos* are *four*; that is, where there exists that idea of relative magnitude which we signify by

* Consistently; so as to couple with operations upon problems of commensurables those operations which apply to the same problems upon incommensurables.

two, let the idea of *relation* be coupled with the idea of a *larger relation*, in exactly the same manner as our idea of *magnitude*, when we look at ——, is increased when we look at —— ——; and we shall then, by considering the result as one of relative magnitude, be led to the idea of the relation between —— —— —— —— and ——. This, of course, does not give a better comprehension of twice two is four; but what it explains is, that we are using the term *ratio* in a consistent sense, when we say that the ratio of 2 to 1, increased in the ratio of 2 to 1, is the same as the ratio of 4 to 1; and, generally, that the ratio of m to 1, increased in the ratio of n to 1, is the ratio of mn to 1. And the notion of relative magnitude contained in the words, ratio of m to p, must be the same as that contained in the words, ratio of mn to pn; and, conversely, the notion in the latter is that implied in the former. I doubt if any thing that deserves the name of proof can be given of this proposition, which seems to be worthy the name of an axiom. What idea we form of magnitude as portion of magnitude from A and B, the same do we form from 2 A and 2 B. Nor can I imagine these propositions extended to fractions in any more fundamental manner than by observing, that as $\frac{m}{n}$ taken $\frac{p}{q}$ times is $\frac{mp}{nq}$ times (times mean times, or parts of times, either separately or both together,) a unit, the ratio of $\frac{m}{n}$ to 1, altered in the ratio of $\frac{p}{q}$ to 1, is the ratio of $\frac{mp}{nq}$ to 1; or that the ratio of m to n, altered in the ratio of p to q, is the ratio of mp to nq. These are propositions in which the line between deduction and mere establishment of the synonymous character of terms is very indefinite. I recommend the student to examine his own idea of what he would have meant by "the proportion of 3 to 2 increased in the proportion of 5 to 4, is the proportion of 15 to 8." If he be a metaphysician, I refer him to his oracle, on condition only that the response shall not contradict the preceding proposition.

The multiplication of m and n is, then, the alteration of the ratio of m to 1 in the proportion of n to 1; and the ratio of magnitudes mA and nA is the same as the ratio of magnitudes mB and nB, and of m to n. Hence, to alter $m\text{A} : n\text{A}$ (which is $m : n$) in the ratio of pB to qB, which is $(p : q)$, is the formation of $mp : nq$, or mpA to nqA, or mpB to nqB. Now, this is precisely what Euclid has termed the *composition* of these ratios; for, let $mA : nA :: vB : pB$, then $vB : pB$ compounded with $pB : qB$, is $vB : qB$, or $v : q$. But

$$mA : nA :: m : n \qquad vB : pB :: v : p$$

Therefore $\quad m : n$ is $v : p$ or $\frac{m}{n} : 1$ is $\frac{v}{p} : 1$

$v = \frac{pm}{n}$ $\; v : q$ is $\frac{pm}{n} : q$ or $pm : nq$ or $pmA : nqA$ or $\quad pmB : nqB$.

Hence, *composition* is multiplication of terms, when the ratios are those of number to number. Let, then, composition of ratios stand for multiplication of terms, and be considered as the corresponding operation in the case of incommensurable magnitudes.

Prove from this, that if $U : A$ and $U : B$ be compounded, giving $U : C$, that when $A = aU$ and $B = bU$, we have $C = abU$, and that if $U : A$ and $B : U$ be thus compounded, giving $U : D$, we have $D = \frac{a}{b}U$, $D : U :: \frac{a}{b} : 1$; in which operations, corresponding to multiplication and division.

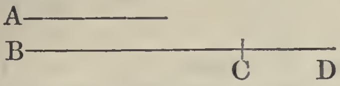
It may be a matter of some curiosity to know whether Euclid carried with him the notion of multiplication of numbers in the composition of ratios. In the Fifth Book, the notion of the numerical magnitude of a ratio is entirely suppressed, except only in the single word πηλικοτης (see page 29.) Composition* is defined to be the taking an antecedent of one ratio with the consequent of another; and it is not even specified that the intermediate terms are to be the same. But in the Sixth Book we find composition, or collocation of ratios, to mean the *multiplication of their quantuplicities* (see page 29).

* Σύνθεσις λόγου ἐστὶ λῆψις τοῦ ἡγουμένου μετὰ τοῦ ἑπομένου ὡς ἑνὸς πρὸς αὐτὸ τὸ ἑπόμενον.—V. Def. 15.

Λόγος ἐκ λόγων συγκεῖσθαι λέγεται ὅταν αἱ τῶν λόγων πηλικότητες ἐφ' ἑαυτὰς πολλαπλασιασθεῖσαι ποιῶσί τινά.—VI. Def. 5.

The second of these definitions has usually been omitted in modern editions. But it is worthy of remark, that, in the first, to compound is συντίθεσθαι; in the second, συγκεῖσθαι; and the second is the word afterwards used by Euclid, though in the sense of the first. The reason of the omission appears to have been a disposition on the part of commentators to consider Euclid as a perfect book, and every thing which did not accord with their notions of perfection, as the work of unskilful editors or interpolators.

The addition and subtraction of ratios can only be primarily conceived when the latter terms of the ratios are alike. Thus,



we must imagine the idea of relative magnitude given by BC compared with A, and by CD compared with A, to be put together, in order to make up the relative magnitude of BD to A. Addition and subtraction are, as to ratios, ideas not so simple as multiplication and division. Shew that the preceding is the only way in which $m:1$, increased in the ratio of $n:1$, will give $mn:1$, consistently with the notion of multiplication of whole numbers being successive additions.

When ratios have not the same consequents, they must be reduced to the same consequents. Thus, A : B and C : D are added by taking A : B :: P : Z and C : D :: Q : Z, and P + Q : Z is the sum of the ratios. This answers to addition of fractions.

Let P be the mean proportional between A and B, meaning that A : P as P : B. It may be proved, as in page 60, that there must be such a magnitude as this mean proportional, and we may also prove that we can find A : P as P : Q, and P : Q as Q : B, thus forming two mean proportionals. It is readily proved, that if $A = aU$ and $B = bU$, then $P = cU$ where $cc = ab$. If, then, ab be a number or fraction which has a square root, P can be found commensurable with A and B; but if ab have no square root, number or fraction, then P is incommensurable with A and B, but not, therefore, unassignable as a magnitude, though unassignable as a numerical fraction of A or B. Consequently, when we speak of $\sqrt{2}$, it must be with reference to magnitude, and we mean $\sqrt{2}\,M$, an accurate representative (if we choose to define it so) of the mean proportional between M and 2 M. Similarly, when there are two mean proportionals, we find P, if $A = aU$ and $B = bU$, to be cU where $ccc = ab$, and this is incommensurable unless ab be a cube number or fraction. But we may define $\sqrt[3]{2}\,M$ to be the first of two mean proportionals between M and 2 M; and so on.

Are we, then, to use long processes and comparatively obscure definitions, whenever the ratios of a problem are incommensurables? By no means; we proceed to shew that it may always be made possible to let the processes of arithmetic (or rather of algebra) be used as if the ratios in question were commensurable; and that we may thus deduce a result which may either be interpreted strictly at the end of the process, or made to give a result as near as we please to the truth in arithmetical terms. Let us suppose this PROBLEM: Two pounds are spent in buying yards of stuff, and as many yards are bought as shillings are given for a yard. Let x be the number of yards, then x yards at x shillings a yard, gives xx shillings; whence $xx = 40$, which is arithmetically impossible. Now, turn from numbers of pounds to quantities of silver, and let S be the silver in a shilling, X that in the price given; let L be a yard, and Y the length bought. Then it is required that 40S should be given, and that X should bear the same ratio to S as Y bears to L. Now, if X be given for L, what must be given for Y? Take P of such relative magnitude to X, as Y is to L; that is, let

$$L : Y :: X : P = 40S$$

But as $L : Y :: S : X$ Therefore $S : X :: X : 40S$

or X must be a mean proportional between S and 40S. Now, if we make our symbols general, and let x stand for any ratio, numerically possible or not, but proceed as we should do if it were arithmetical, we proceed as in the first case, and find $x = \sqrt{40}$, which, being interpreted as a magnitude, with reference to its ratio to S, means, when the symbols are general, $\sqrt{40}\,S$, the mean proportional between S and 40S. If we wish for an approximate numerical result, we must suppose $40 + a$ to be the sum, where $40 + a$ has a square root, and then we have $x = \sqrt{40 + a}$; and since a may be made as small as we please, we can make this problem as near the given one as we please.

The following table should be attentively considered. In the first column, an incommensurable ratio x, of X to U, is given, or a function of it and other ratios, under arithmetical symbols; in the second is the ratio which the function really gives, when the symbols on the first side are extended in meaning.

x or $x : 1$	the ratio of	X to	U
y .. $y : 1$	....	Y ..	U
z .. $z : 1$	....	Z ..	V

$\frac{1}{x}$	$\frac{1}{x} : 1 :: 1 : x$ or U : X as $\frac{1}{x} : 1$
x^2	compounded of $x : 1$ and $x : 1$ or X : U and X : U. Let U : X :: X : P then P : X and X : U compounded give P : U or $x^2 : 1$ is the ratio which a third proportional to U and X bears to U.
xyz	$(x : 1)\ (y : 1)\ (z : 1)$ ratio compd. of X : U, Y : U, and Z : V. Let X : U :: P : Y; the above is then compounded of P : U and Z : V. Let P : U :: Q : Z. The result is then Q : V or $xyz : 1$ is Q : V P a fourth prop. to U, X, and Y Q .... U, P, .. Z
$xy + yz$	$xy : 1$ is P : U when U : X :: Y : P $yz : 1$ is Q : V .... U : Y :: Z : Q Take Q : V :: M : U or V : Q :: U : M P + M : U is the ratio required.
$\frac{x}{z}$	$x\frac{1}{z} : 1$ compd. of X : U and V : Z Let X : U :: P : V and P : Z is the ratio required.

Now, we have assumed the operations of finding a fourth proportional, a mean proportional, two mean proportionals, &c. Whether these can be done, or whether any or all cannot be done, is a question for every particular application. In arithmetic, we will suppose the data arithmetical; a fourth proportional can always be found. In geometry, a fourth proportional can be found to lines or rectilinear areas; but not to angles, &c. And a mean proportional cannot generally be found in arithmetic, but can be found in geometry, between two straight lines, or two rectilinear areas. But two mean proportionals cannot be found in geometry or in arithmetic.

It must be remembered, that while we are here speaking of geometry or arithmetic, we are not speaking of every conception we can form of these sciences, but of the subjects as limited by the definitions of what it has been agreed shall be called arithmetic and geometry. Elementary arithmetic means the science of numbers and fractions: elementary geometry, the science of space, so far as the same has properties which can be deduced by allowing of *fixed* straight lines and circles. To say that an angle cannot be trisected *geometrically*, means, that it cannot be trisected by means of straight lines and circles as defined. But there is an abundance of curves, the stipulation to draw any one of which would secure the means of trisecting an angle. And, by simply granting that a circle should be allowed to roll along a straight line, and that the curve described by one of its points should be granted, we can either square the circle, or find the ratio of any two arcs. And, just in the same way, if we were to define a journey to be 100 miles or less, it would be perfectly true that we could not make a journey from London to York, but that we could from London to Brighton.

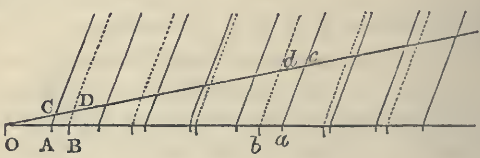
It is surely time that the verbal distinction between different parts of the same sciences should be done away with. Every conception which can be shewn to be not self contradictory, can be as easily realised by assumption as the drawing of a circle, which is itself a perfect geometrical idea, and can only be roughly represented by mechanical means. Whatever can be distinctly conceived, exists for all mental purposes; whatever can be approximately found, for all practical uses.

It may be worth while to make the student remark the close similarity which exists between the process in page 64, and that by which we enlarge our ideas in algebra, from the simple consideration of numerical magnitude to that of positive and negative quantities. In both, we set out with a notation insufficient to express all the results of problems; in both, this circumstance is marked by the appearance of unexplained results, the examination of which, on wider grounds, shews the necessity for attaching more extensive ideas to symbols; and in both, the partial view first taken is wholly included in the more general one: while in both, the processes conducted under the wider meanings are precisely the same in form and rules as those which are restricted to the original meanings of the symbols. The principal difference is, that in extending arithmetic to the general science of ratios, we are not engaged in interpreting difficulties arising from contradictions, but from results which are only approximately attainable. But in both the reason is, that we set out with our symbols so constructed, that we cannot undertake a problem without tacitly dictating conditions to the result. In beginning algebra, we make quantities indeterminate in magnitude, with symbols of operation so fixed in meaning, that they cannot be used without an assumption that we know which is the greater and which is the less of two unknown quantities. We have, therefore, to examine the different cases of problems which present different results according as one datum is greater or less than another; and thus we obtain those extensions of meaning which will make the problems and the symbols equally general. In beginning arithmetic, we invent no symbols of ratio, except those which represent the ratios of magnitudes formed by the repetitions of a given magnitude. These we find to be not sufficient to represent all ratios; though it is shewn that we can make them represent any ratio which magnitudes can have, as nearly as we please. The invention of new symbols of ratio must require the generalisation of operations; that is, we cannot speak of *multiplication* or *division* of ratios generally, while these words have a definition which applies only to ratios of repetitions, or commensurable ratios.

There is a difference between the *impossible* of primary arithmetic, and that of geometry. The first is unattainable by a restricted definition, the second by restricting the cases of general definitions which shall be allowed to be used. In arithmetic, we attempt a science of relative magnitudes, by running from the general notion of relative magnitude to the more precise and easy notion of the relative magnitudes of one certain set of magnitudes, A, an arbitrary, $A + A$, $A + A + A$, &c. We are very soon taught that our symbols will not express all ratios, that is, if we have a general notion of ratio to think about: whence our definitions are not sufficiently extensive. But in geometry, having assumed notions and definitions from which we cannot help conceiving an infinite number of different lines and curves, we immediately proceed to cut ourselves off from the use of all except the straight line and circle; that is, the straight line between or beyond two given points, and the circle which has a given centre and a given radial line. Until these demands or postulates are looked upon as *restrictions,* their sense is never understood. (See the Appendix.)

This difference is, however, not very essential; since it is much the same whether we define in too limited a manner, or whether we limit ourselves to the use of only a part of a general definition. We shall in the sequel discard the restrictive postulates, and suppose ourselves able to draw any line which we can shew to be made by the motion of a point.

The method by which Euclid first exhibits four proportional straight lines, though elegant and ingenious, has not the advantage of exhibiting the notion of ratio directly applied to two straight lines. The following theorem is directly proved from the first book, and may be made the guide. If a series of parallels cut off consecutive equal parts from any one line which they cut, they do the same from every other. This premised, suppose any two lines OA, OB, and take a succession of lines equal to OA and OB, drawing through every point a parallel to a given line. Draw any other line, OCD, intersecting all the parallels: from which the preliminary proposition shews, that whatever multiple O*a* is of OA, the same is O*c* of OC; and whatever O*b* is of OB, the same is O*d* of OD. And if O*a* be greater than, equal to, or less than O*b*, O*c* is greater than, equal to, or less than, O*d*. Hence the definition of equal ratios applies precisely to the lines OA, OB, OC, and OD, which are, therefore, proportionals. This gives the construction of Book VI. Prop. 12, or one analogous to it.



The method of finding a mean proportional between two straight lines is given in Prop. 13; but as we now wish to make the straight line the foundation of general conceptions of magnitude, we shall pass at once to those considerations which involve any number of mean proportionals. It adds considerably to the interest of this part of the subject, that we are thus brought to the notions on which the first theory of logarithms was founded.

Let there be any number of lines, V, V_1, V_2, V_3, ... in continued proportion; that is, let all the ratios of V to V_1, V_1 to V_2, V_2 to V_3, &c. be the same. And let V_1 be greater than V; in which case V_2 is greater than V_1, &c. If V_1 were equal to V, then would V_2 be equal to V_1, &c. And, first, we have the following

Theorem. By however little V_1 exceeds V, the series V, V_1, &c. is a series of magnitudes increasing without limit: so that, however great A may be, a point may be attained from and after which every term is greater than A: but in all cases whatsoever, V_1 may be taken so near to V, that the terms of the series V, V_1, &c. between which A lies, shall be as near to A in magnitude as we please.

Firstly, the series increases without limit. For, since $V : V_1 :: V_1 : V_2$, and V and V_2 are the greatest and least, we have

$$V + V_2 \text{ is greater than } V_1 + V_1$$

or
$$V_2 - V_1 \text{ is greater than } V_1 - V$$

Or, V_2 exceeds V_1 by more than V_1 exceeds V. Similarly, V_3 exceeds V_2 by more than V_2 exceeds V_1; and so on. But if to V were added continually the same quantity, the result would come in time to exceed any given magnitude; still more when a greater quantity is added at every step.

Secondly, since then we come at last to V_n less than A, while V_{n+1} exceeds A, it is plain that A will not differ from either by so much as they differ from each other. But because

$$V_n : V_{n+1} :: V : V_1$$

we have
$$V_{n+1} - V_n : V_n :: V_1 - V : V$$

If then $V_1 - V$ be so small that $m(V_1 - V)$ shall not exceed V, neither will $m(V_{n+1} - V_n)$ exceed V_n, and of course not A. Let m be any given number, however great, and let $V_1 - V$ be less than the mth part of V; then will $V_{n+1} - V_n$ be less than the mth part of A; or, by taking m sufficiently great, may be made as small as we please. Whence the second part of the theorem.

THEOREM. In the preceding series, the selection

$$V \quad V_n \quad V_{2n} \quad V_{3n} \quad \&c.$$

constitutes a similar series of continued proportionals. For, since any two consecutives in the upper line next given are proportiona to those under them in the lower,

$$V, \quad V_1, \quad V_2 \quad \cdots\cdots\cdots \quad V_n$$
$$V_n \quad V_{n+1} \quad V_{n+2} \quad \cdots\cdots\cdots \quad V_{2n}$$

we have (XXII.) $V : V_n :: V_n : V_{2n}$: and so on.

If between each of the terms of the series we insert the same number of mean proportionals, the series thus formed will have the same properties as the original. Let us say we insert two mean proportionals between each two terms. Then we have

$$\text{V K K}' \ \text{V}_1 \ \text{L L}' \ \text{V}_2 \ \text{M M}' \ \text{V}_3 \ldots\ldots$$

Now the only question about the continuance of the same ratio from term to term is in the ratios $V_1 : L$, $V_2 : M$, &c. But I say that since

$$\begin{array}{ll} V : K :: K : K' :: K' : V_1 & \quad V : V_1 \\ V_1 : L :: L : L' :: L' : V_2 & \text{and} \quad V_1 : V_2 \end{array}$$

that $V : K :: V_1 : L$. For if not, let these latter ratios differ; say V is to K more than V_1 is to L. Then is K to K′ more than L is to L′; and hence (presently will be shewn) the ratio compounded of V to K and K to K′, or V : K′, is greater than that compounded of $V_1 : L$ and $L : L'$ or $V_1 : L'$. Similarly, V to K′ and K′ to V_1 being more than $V_1 : L'$ and $L' : V_2$, we have $V : V_1$ is more than V_1 to V_2, which is not true. Therefore V is not to K more than V_1 to L; a similar process shews that it is not less: consequently,

$$V : K :: V_1 : L$$

or the continuance of the primary ratio is uninterrupted.

The theorem assumed in the above is thus proved. If A : B more than P : Q we have mA greater than nB, while mP is less than nQ; or any other descending assertion. And if B : C more than Q : R, we have xB greater than yC, while xQ is less than yR. Or we have

$$\begin{array}{l} mxA \text{ greater than } nxB,\ nxB \text{ greater than } nyC, \text{ or } mxA \text{ greater than } nyC \\ mxP \text{ less than } nxQ,\ nxQ \text{ less than } nyR, \text{ or } mxP \text{ less than } nyR \end{array}$$

that is, A is to C more than P is to R; which is what we assumed.

If then we insert a mean proportional between V and A, giving

$$V \quad M \quad A$$

if between each we insert a mean proportional, we have

$$V \quad M' \quad M \quad M'' \quad A$$

If we proceed in this way, we shall come at last to a series of the form

$$V \quad V_1 \quad V_2 \ldots\ldots\ldots\ldots \ V_{n-1}\ (V_n = A)$$

in which no two quantities differ by so much as a given quantity K. We can actually insert one mean proportional between any two quantities; it is done in geometry between two lines, and (page 60) two magnitudes of any sort may be made (one being given) proportional to two lines. Thus, let A, B, C, be continually proportional lines, or let B be a mean proportional between A and C. Then if A and C were taken proportional to (say angles) M and K, it follows that if A : B :: M : L, that M, L, and K are continued proportionals, by a proof of the sort given in the lemma of the last theorem. Granting, then, that every two magnitudes have one mean proportional, we may now shew that they have any number of intermediate proportionals; as follows:

We set out with 2 quantities, and the first insertion adds 1, the second 2, the third 2^2, the fourth 2^3 .... and the nth 2^{n-1}. Consequently, n complete insertions add

$$1+2+2^2+ \cdots\cdots +2^{n-1} \quad \text{or} \quad 2^n-1$$

to the first 2; giving 2^n+1 in all. Now, let us suppose that 2^n+1 divided by p leaves a quotient q, and a remainder r which is not greater than p. Consequently, we have for the whole number (V and A inclusive) after n insertions,

$$v = pq + r \quad \text{which is also} \quad p(q+1)-(p-r)$$

and $p-r$ is also not greater than p; and $V_m = A$ when $m = v$ and is greater or less than A, according as m is greater or less than v. If then out of the series (the proportion being continued up to $V_{p(q+1)}$) we select

$$\begin{array}{llll} V & V_q \quad V_{2q} & \cdots\cdots\cdots\cdots & (V_{pq} \quad \text{less than} \quad A) \\ V & V_{q+1}\, V_{2(q+1)} & \cdots\cdots\cdots & (V_{p(q+1)} \text{ greater than } A) \end{array}$$

We see V and V_{pq}, and V and $V_{p(q+1)}$ each with $p-1$ mean proportionals inserted between them, namely,

$$V_q \quad V_{2q} \quad \cdots\cdots \quad V_{(p-1)q} \quad \text{and} \quad V_{q+1} \quad V_{2(q+1)} \quad \cdots\cdots \quad V_{(p-1)(q+1)}$$

But from V_{pq} to $V_{p(q+1)}$ there are p passages from term to term of the complete series, consequently, since each passage may be made by an augmentation less than K, the difference between the two may be made less than p K, which call Z. Hence we have the following

Theorem. To find two magnitudes, one greater and the other less than A, but differing from it by less than a given quantity Z, between each of which and V, $p-1$ mean proportionals shall exist, obtained by continual insertion of one mean proportional, continue the insertion until no two successive terms shall differ by so much as the

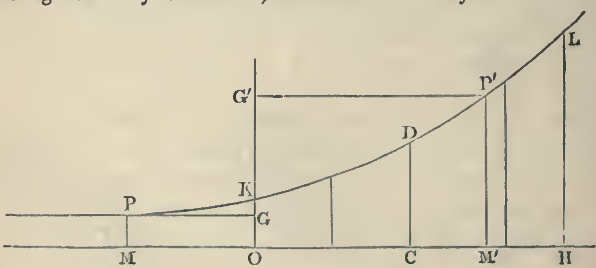
pth part of the quantity Z : then the quantities required and the mean proportionals shall be in the set so found.

Hence it can be shewn that there *are* $p-1$ magnitudes (whether attainable or not with any given means is not the question) which are mean proportionals between V and A. Let P_p and Q_p be magnitudes, one greater and one less than A, which have such mean proportionals, namely, let the following be continued proportionals,

$$V \quad P_1 \quad P_2 \cdots\cdots P_{p-1} \ (P_p \text{ greater than } A)$$
$$V \quad Q_1 \quad Q_2 \cdots\cdots Q_{p-1} \ (Q_p \text{ less than } A)$$

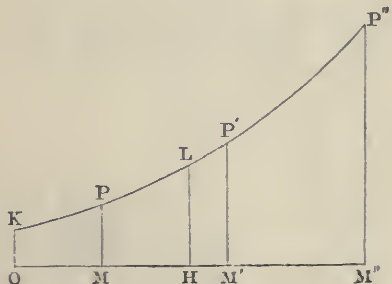
obtained by the preceding method, from which it is apparent that P_1 is greater than Q_1. Now, exactly as in page 60, if we assume X_1 to set out in value $=Q_1$, so that $V : X_p$ more than $V : A$ (X_p bring the pth of the set of continued proportionals $V, X_1, X_2, \ldots\ldots$) and to change through all possible intermediate magnitudes up to $X_1 = P_1$, or $V : X_p$ less than $V : A$, there is but this alternative; EITHER at some intermediate point $V : X_p$ as $V : A$, or $X_p = A$, OR, there is a point at which $V : X_p$ more than $V : A$, being always less when X_1 is greater by any magnitude however small. The latter may be disproved, or the former proved, as in the page cited.

To resume the original subject. It appears, then, 1st, that if between V and A we continually insert mean proportionals, in such manner that at every step one mean proportional is inserted between every two consecutive results of the preceding step. 2d, If the series be continued beyond A, preserving still the same ratio between the consecutive terms of the continuation which exists between consecutive terms lying between V and A; then will this process leave us at last with a series of consecutive proportionals, having consecutive terms so near together in magnitude, that every magnitude lying between V and any we please to name, shall have a term of the series differing from it by less than Z, however small Z may be.



Let us now make OK and HL perpendicular to any chosen line OM, and let V be the line OK, A the line HL. Bisect OM in C, and erect CD the mean proportional between OK and HL. Bisect OC and CH, and erect the mean proportionals between OK and CD, and between CD and HL. Continue this process, and we shall thus get an increasing number of points between K and L, which will soon give to the eye the idea of a curve line rising from K to L. When we have thus divided OH into 2^n parts, by n insertions, giving $2^n + 1$ lines, we may, by setting off portions equal to those intercepted in OH, continue that line on one side and the other, and thus continue the scale of proportionals and the series of points on one side and on the other of O and H. However far we may go we can never complete this curve; but if we admit that *a curve exists*, wherever a series of points can be laid down, as many as we please, and consecutively as near as we please, then we have a right to assume this curve as existing, and, *for purposes of reasoning*, as constructed. Call this the *exponential* curve, (*exponere*, to set forth), which *expounds* ratios, a phrase to which we shall presently give meaning. That the student may not suppose we are using an old word in a new sense, it is necessary to inform him that this curve, or rather the process which we have illustrated by it, is older than the algebraical symbol a^x, and that x gets the name of exponent from it. We shall presently see the analogy.

The exponential curve being given, every line OG has its place MP among the *ordinates* of the curve, and its *abscissa* OM, which expounds or sets forth that place. From the nature of the formation, it is evident that a given line has but one exponent, and that the order of magnitude of lines (to the right of O), is also that of their exponents.



And the main property of the curve is this: that a fourth proportional to any three lines (V being one), OK, MP, M′P′, may be found by adding the exponents OM and OM′ (making OM″), and finding the line M″P″ expounded by that sum. To prove this, make n sets of insertions in OH, and suppose MP to lie between V_m and V_{m+1}, while M′P′ lies between $V_{m'}$ and $V_{m'+1}$. Now, in the series of continued proportionals,

$$V \quad V_1 \quad V_2 \quad \ldots\ldots\ldots\ldots \quad (V_{2^n+1} = A)\, V_{2^n+2} \quad \ldots\ldots$$

I say that $V : V_m : V_{m'} : V_{m+m'}$

~~For~~ From

$$V \quad V_1 \quad V_2 \quad \ldots\ldots\ldots\ldots \quad V_{m-1} \quad V_m$$

$$V_{m'} \quad V_{m'+1} \quad V_{m'+2} \quad \ldots\ldots\ldots\ldots \quad V_{m'+m-1} \quad V_{m'+m}$$

we have $V : V_1 :: V_{m'} : V_{m'+1}$ &c. &c.

whence $V : V_m : V_{m'} : V_{m+m'}$

Similarly, $V : V_{m+1} : V_{m'+1} : V_{m+m'+2}$

Now, by a lemma we shall presently shew, since MP lies between V_m and V_{m+1}, and M′P′ lies between $V_{m'}$ and $V_{m'+1}$, the fourth proportional required lies between $V_{m+m'}$ and $V_{m+m'+2}$. Let K be the value of one of the last subdivisions of OH; then we have supposed OM to lie between mK and $(m+1)$K, and OM′ between m'K and $(m'+1)$K. The preceding makes it evident that the fourth proportional has an exponent between $(m+m')$K and $(m+m'+2)$K; while the sum of the exponents OM and OM′ also lies between $(m+m')$K and $(m+m'+2)$K. Since K can be made as small as we please, it must follow that the sum of the exponents is the exponent of the fourth proportional; for two different magnitudes cannot lie between two quantities which can be made as near as we please, as can $(m+m')$K and $(m+m')$K $+ 2$K. If the two approximating magnitudes approach to each other, keeping one of two *different* magnitudes between them, they must, at last, leave out the other.

The lemma alluded to is as follows: If

$$A : B :: C : D$$

and

$$A : B + B' :: C + C' : D + D'$$

Then if A, B + X, C + Y, D + Z, be also proportionals, where X and Y are less than B′ and C′, then Z must be less than D′; for A is to $B+X$ more than A is to $B+B'$; or (substituting equal ratios), $C+Y$ is to $D+Z$ more than $C+C'$ is to $D+D'$. Still more is $C+C'$ (remember that C' is greater than Y) to $D+Z$ more than $C+C'$ to $D+D'$; that is, $D+Z$ is less than $D+D'$, or Z less than D'.

The following property we leave to the student to deduce from the last. If there be any three lines, X_1 X_2 X_3, expounding Y_1 Y_2 Y_3, any lines whatsoever greater than V, then the exponent of the fourth proportional is $X_2+X_3-X_1$.

These are all properties of algebraical exponents, or of logarithms, (λογων αριθμοι, numbers expounding ratios). We shall now make it appear, that the line expounded by x is of the form a^x.

Let the numerical symbol of V or OK be v; let that of HL or A be a. Then, if arithmetical mean proportions be continually inserted, we have

$$v \qquad (a\,v)^{\frac{1}{2}} \qquad a$$

$$v \qquad a^{\frac{1}{4}}v^{\frac{3}{4}} \quad a^{\frac{2}{4}}v^{\frac{2}{4}} \quad a^{\frac{3}{4}}v^{\frac{1}{2}} \qquad a$$

$$v \qquad a^{\frac{1}{4}}v^{\frac{7}{8}} \quad a^{\frac{2}{8}}v^{\frac{6}{8}} \quad a^{\frac{3}{8}}v^{\frac{5}{8}} \quad \ldots\ldots\ldots \quad a^{\frac{7}{8}}v^{\frac{1}{8}} \qquad a$$

or generally, when 2^n-1 (say $p-1$) mean proportionals are inserted between v and a, the mth of these proportionals is

$$(2^n=p) \qquad a^{\frac{m}{p}}\,v^{1-\frac{m}{p}} \quad \text{which is} \quad v\,k^{\frac{m}{p}}$$

if we suppose $a=vk$. Now, let us suppose a number y thus expounded by x; and after n insertions, let this number x lie between $m\alpha$ and $(m+1)\alpha$, α being the pth part of OH, (let OH be c). We have then

$$x \quad \text{lies between} \quad m\frac{c}{p} \quad \text{and} \quad (m+1)\frac{c}{p}$$

$$\text{or} \quad \text{between} \quad c\frac{m}{p} \quad \text{and} \quad c\frac{m}{p}+\frac{c}{p}$$

or $$x = c\frac{m}{p}+\beta \qquad \left(\beta<\frac{c}{p}\right)$$

Therefore, $\dfrac{m}{p}=\dfrac{x}{c}-\dfrac{\beta}{c}$

Consequently the number expounded by $m\alpha$, or $m\dfrac{c}{p}$, or $x-\beta$ is

$$v\,k^{\frac{x-\beta}{c}}$$

and since β diminishes without limit as the insertions continue, the number expounded by x is $vk^{\frac{x}{c}}$. That is, if we adopt general numerical symbols, let $OM = x$, $MP = y$, and we have

$$y = v\,k^{\frac{x}{c}}$$

or, if we let OK represent the linear unit ($v = 1$), and let $OH = OK = 1$, $HL = 10$, or $k = 10$, we have

$$y = 10^{x}$$

or x is the common logarithm of y.

From the curve we see how it is that magnitudes less than V are expounded by negative quantities, with other well-known properties of logarithms.

We see then, that the assertion "the common logarithm of 2 is ·30103 very nearly," may be thus made; which is perhaps the most distinct view that can be given of a *numerical* logarithm. If we make 10 V the hundred thousandth magnitude in a series of proportionals,

$$V,\ V_1,\ V_2,\ \dots\dots\ (V_{100,000} = 10)\ V_{100,001},\ \&c.$$

then will the 30103rd of these proportionals, or V_{30103}, be very nearly equal to 2 V.

If we chose, we might, granting that the exponential curve can be constructed, make $V\,k\,X$ by definition the line M P; where X stands for O M, and k for the ratio of H L to O K. From this it would readily be deduced, that when k represents a commensurable ratio, and X is $\frac{m}{p}$ linear units where $\sqrt[p]{k^m}$ has an arithmetical existence, the results of this theory are the same as those of common algebra. And from hence it appears, that the science known by the name of the application of algebra to geometry (of which it is the foundation, that a linear unit being given, every expression of algebra may be considered as a length, or at least the symbol of the ratio of a length to that unit) does, in point of fact, make this additional assumption, while an application of geometry (with this assumption) to algebra, would take away all want of rigorous conception of the meaning of algebraical formulæ, so far as the meaning of the exponent is concerned.

The view above given is very nearly that by which logarithms were first calculated, but the method was not so general. The *natural* logarithms (see my *Algebra*, p. 226) arose thus. If we suppose a very large number of mean proportionals, then V and V_1 will be very nearly equal. Let $V_1 = V + X$, then if we assume O H, so that X shall expound $V + X$ when X is very small, or more correctly, if we suppose the limit of $(V + X)$ divided by the magnitude expounded by X, as X diminishes without limit, to be unity, we have the first, or Napier's system.

APPENDIX.

ON THE DEFINITIONS, POSTULATES, AND AXIOMS OF EUCLID.

I HERE propose to endeavour to make such a subdivision of the definitions, &c. at the beginning of the First Book, as may enable the student to review the reasoning of the whole.

I shall consider the 10th and 11th axioms as among the postulates, firstly, because some old manuscripts support this change; secondly, because the older translations (from the Arabic) support it also, and even place the 12th axiom in the same list; thirdly, because it is utterly impossible to place them in Euclid's list of *common notions.* For he uses no such word as *axiom* (Greek though it be), but calls "the whole is greater than its part," κοινη ἐννοια, that which is in the conceptions of every one. Now, what is the probability that he considered "all right angles are equal," as a truth familiar to the understanding of every beginner in geometry? His *postulates* (αιτηματα, demands) do, according to the etymology of the word, include those axioms, if not the 12th also.

I also place out of view the axioms which belong to all kinds of magnitude as much as to space, namely, from the 1st to the 9th inclusive. There remains then in the shape of limitation, or assumption, six postulates, namely, three which I will call *restrictive*, being those commonly called *postulates*,* and three *assumptions*, being the 10th, 11th, and 12th axioms, so called.

Some of the definitions contain assumptions of certain conceptions existing to which names are to be given; namely, those of a point, a

* I have seen the word postulate defined as a self-evident problem; and axiom as a self-evident theorem. This definition is derived from the character of the postulates and axioms as usually given; but from no other source.

line, the extremities of a line, a straight line, a surface, the extremities of a surface, a plane surface, a plane angle, a plane rectilineal angle. Others assume the possibility of certain relations existing, as will appear from the form in which they are put. I shall now give the definitions, classified with the corresponding postulates, in the manner which appears to me to be most systematic, and placing in [ ] such additions as seem requisite.

1. A *point;* an indefinable notion; but two persons, whatever their idea of it may be, can reason together in geometry who deny a point all parts or magnitude. *Let it be granted* that a point has no parts or magnitude, and that we are concerned with no other property of it, if there be any.

2. A *line;* also indefinable, but those whose ideas of it allow it length, and deny it breadth, can proceed. Let it be granted that all reasoning upon lines is to be founded only upon the assumption that they have length without breadth. [Thickness should have been added, but breadth may mean breadth in any direction.]

3. The extremities of a line are points. [If this define any term, it must be the term *extremities*, for the other two have been defined. To me it appears something like a theorem, as follows: That which ends a line cannot have length, for it would be a part of the line; it cannot have breadth or thickness, which a line has not; it has therefore the only qualities of a point on which we reason, or comes within the definition of a point.]

4. A *straight line;* an indefinable notion, except by the rough idea that it does not go on one side or the other of the two points, [which is no definition, because it assumes the thing in question.] Let it be granted, as a common notion, that two straight lines do not enclose a space, or have not two points in common, without having all intermediate points in common. Whatever the idea of a straight line may be, this is the only property which will be appealed to.

5. *Surface;* an indefinable notion; those whose ideas give it length and breadth, but deny it thickness, have the means of reasoning upon it in geometry.

6. The extremities of a surface are lines. (See remarks on 3.)

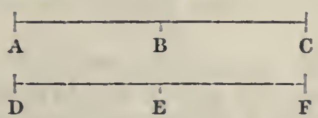
7. A *plane* surface; an obvious notion, roughly defined by lying evenly between bounding straight lines. [This notion, however obvious, does admit of a stricter definition. It is a surface of such kind that *any* two points in it being joined by a straight line, all

intermediate points of the straight line are on the surface. This property is tacitly appealed to throughout.]

8. A plane *angle;* the inclination, or bending towards each other of two lines in a plane. [This definition is superseded by the next; no angle except one made by straight lines is ever used.]

9. A rectilineal angle (plane), the inclination of two straight lines. [An obvious notion of *opening;* it is tacitly assumed that we know how to determine when two angles are equal, or when one of them exceeds the other, as in the fourth proposition.]

10. Right angles are those made by a straight line, called a perpendicular, which falls on another straight line, making equal angles on both sides. *Postulate;* let it be granted that all right angles are equal. [This is far from an obvious postulate; the reason for it seems to have been as follows: That two straight lines which coincide in two points coincide in every point *between them*, has been admitted; it is sufficiently obvious to sense that they coincide beyond or on each side of the two common points; that is, they coincide altogether, throughout all possible length. This seems an infinite assumption; and if it be assumed instead that all right angles are equal, it may be proved afterwards that no two straight lines have a common segment; that is, that two straight lines which coincide for any length, never afterwards separate. But it may be shewn, that the assumption of all right angles being equal, amounts to the same infinity of assumption; as follows: The right angle is by definition the half of the opening which two straight lines make, when one is the continuation of the other, as AB, BC. To assume that all right angles are equal, is to assume that the doubles



of right angles are equal; that is, that if we lay B on E, with ED coinciding with BA, then EF and BC will coincide. Now it is precisely the same thing to assume, that when AB is made to coincide with DE up to the point E, that the two coincide beyond it.

I should recommend the student to make to the assumption that two straight lines cannot coincide *in* two points without coinciding between them, the addition that they also must coincide beyond them.

It may then be directly proved that all doubles of right angles are equal, and thence that all right angles are equal.]

The definitions 11, 12, 13, 14, need no remark, being purely nominal.

14. The *circle*, a *plane* figure, having all points of its boundary (15. the *circumference*) equally distant from a given point (16. the centre) within it. [Here is tacitly a postulate, namely, that this point lies within the figure. It is also assumed in the first proposition, that if any point of a circle be within another, the two circles must intersect. There are several assumptions of this kind, which shew that Euclid did not affect that extreme *form* of accuracy which subsequent commentators have attributed to him. The assumption of a circle assumes the existence of an isosceles triangle.]

17. A *diameter* of a circle is a line passing through the centre, and terminated both ways by the circumference; it divides the circle into two equal parts, or (18. semicircles). [Here is a demonstrable theorem positively assumed. The application of one part of the circle to the other (as by revolution of one-half round the diameter) as in the fourth proposition, would prove it.]

From (19.) to (23.), the definitions are merely nominal.

24. If there be a triangle having three equal sides, let it be called *equilateral*. [In this form I give all definitions, the existence of the objects of which is to be established.]

25. An isosceles triangle is one having two sides equal.

26. A scalene triangle has the three sides unequal.] This definition is never used.]

(27.) and (28.) are nominal; (29.) tacitly refers to the thirty-second proposition; and from (30.) to (33.), should be written in the manner of (24.)

(35.) If there be two right lines, which being produced ever so far on the same side never meet, let them be called parallels. And let it be granted, that if two right lines falling upon a third make interior angles together less than two right angles, they are not parallels. [This bone of contention, when reduced to the form in which it is most palpable to the senses, is as follows: Let it be granted that two right lines which meet in a point, are not *both* parallel to any third line. This assumed, Euclid's axiom follows. For he is able to shew that the one parallel which he afterwards draws, through a point to a given line, has the property of making the two internal angles equal to right angles: *there is but one parallel;* consequently all lines which have not that property are not parallels.]

It remains to add what I have called the *restrictive* postulates. I cannot believe that Euclid, who appears to assume *very* obvious propositions, even when he might prove them, could have intended to require formally the admissions that a straight line may join two points, and may be continued, and that a circle may be drawn with a given centre and radius. If this had been the case, why not assume (Prop. IV.) that two straight lines may be drawn making equal angles with two other straight lines,—a conception more difficult than that a straight line may be drawn. I conceive, therefore, that the meaning of the three assertions commonly called postulates, is as follows: Let it be considered as intended, that no assumption of processes shall be made, except only the drawing of a straight line between two given points, the continuation of any terminated straight line to any indefinite (not given) distance, and the construction of a circle with a given centre and radius.

THE END.

LONDON:—PRINTED BY JAMES MOYES,
Castle Street, Leicester Square.

ERRATA IN "ELEMENTS OF TRIGONOMETRY."

Page 7, *Corollaries* 1 and 2, *for* the circles C and C′, *read* the circle C′; *for* P *read* P′.

—— 20, line 8, *for* $\frac{y}{r} + \frac{r}{y}$ *read* $\frac{y}{r} \times \frac{r}{y}$.

—— 20, line 8, *for* $\frac{-x}{r} + \frac{r}{-x}$ *read* $\frac{-x}{r} \times \frac{r}{-x}$.

—— 26, line 2, *for* A″OC″, *read* A‴OC″.

—— 26, line 14, *for* C″N, *read* C″N′.

—— 31, line 11, *for* O, *read* 0.

—— 42, line 12, *strike out* sines of the.

—— 81, line 6, *for* $\frac{\varepsilon^{\theta\sqrt{-1}} - \varepsilon^{-\theta\sqrt{-1}}}{2}$ *read* $\frac{\varepsilon^{\theta\sqrt{-1}} - \varepsilon^{-\theta\sqrt{-1}}}{2\sqrt{-1}}$.

—— 81, line 8 from the end, *for* $\varepsilon^{x} + \varepsilon^{2n\pi\sqrt{-1}}$ *read* $\varepsilon^{x} \times \varepsilon^{2n\pi\sqrt{-1}}$

—— 122, line 20, *for* $y-$ *read* y^{-n}.

ERRATA IN "CONNEXION OF NUMBER AND MAGNITUDE."

Preface, page iv. line 17, *for* that, *read* than.

Page 5, lines 7, 8, and 9 from the end, transpose P and X; and Q and Y.

—— 9, line 9, *for* B, *read* A.

—— 10, line 19, *for* $l\text{F} = f\text{M}$, *read* $l\text{F} = f\text{L}$.

—— 22, line 4 from the end, *for* processes embling, *read* process resembling.

—— 27, line 16, *for* $\frac{m'}{n'}$, *read* $\frac{n'}{m'}$.

—— 34, line 13, *for* are, *read* is.

—— 35, line 3, *for* less, *read* greater.

—— 46, line 13, *for* may be named, *read* is named.

—— 46, last line, *for* $v(\text{B}+\text{K})$, *read* $w(\text{B}+\text{K})$.

—— 47, line 1, *for* $v(\text{Q}+\text{Z})$, *read* $w(\text{Q}+\text{Z})$.

—— 47, line 2, *for* $v\text{Q}$ *read* $w\text{Q}$, and Z may be as small as we please.

—— 53, last line, *for* B + C + F, *read* B + D + F.

—— 58, line 5 from the end, *for* B, *read* D.

—— 61, lines 2, 4, 6, 7, *for* $m\text{P}$, *read* $n\text{P}$.

—— 66, line 14, *for* in which, *read* in which we see.

—— 74, line 10, *for* V : L′, *read* V_1 : L′.

—— 76, line 17, *for* X_n, *read* X_p.

—— 78, line 10, *for* for, *read* from.

—— 79, line 18, *for* $a^{\frac{1}{4}} v^{\frac{1}{6}}$, *read* $a^{\frac{1}{4}} v^{\frac{7}{8}}$.

—— 80, line 16, *for* 10, *read* 10V.

—— 80, line 20, *for* V *k* X, *read* $\text{V} k^{x}$.