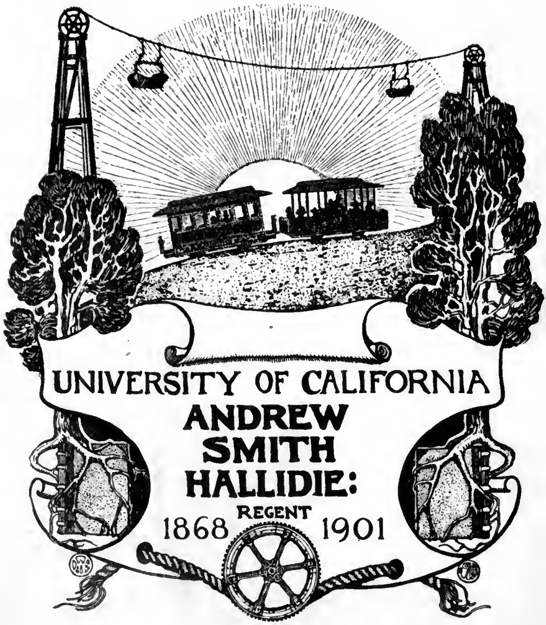
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ELEMENTS

OF

PLANE AND SPHERICAL

TRIGONOMETRY,

WITH THEIR APPLICATIONS TO

MENSURATION, SURVEYING, AND

NAVIGATION.

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PREFACE.

The following treatise constitutes the third volume of a course of Mathematics designed for colleges and high schools, and is prepared upon substantially the same model as the works on Algebra and Geometry. It does not profess to embody every thing which is known on the subject of Trigonometry, but it contains those principles which are most important on account of their applications, or their connection with other parts of a course of mathematical study. The aim has been to render every principle intelligible, not by the repetition of superfluous words, but by the use of precise and appropriate language. Whenever it could conveniently be done, the most important principles have been reduced to the form of theorems or rules, which are distinguished by the use of italic letters, and are designed to be committed to memory. The most important instruments used in Surveying are fully described, and are illustrated by drawings.

The computations are all made by the aid of natural numbers, or with logarithms to six places; and by means of the accompanying tables, such computations can be performed with great facility and precision. This volume, having been used by several successive classes, has been subjected to the severest scrutiny, and the present edition embodies all the alterations which have been suggested by experience in the recitation room.

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TRIGONOMETRY.

BOOK I.

THE NATURE AND PROPERTIES OF LOGARITHMS.

ARTICLE 1. Logarithms are numbers designed to diminish the labor of Multiplication and Division, by substituting in their stead Addition and Subtraction. All numbers are regarded as powers of some one number, which is called the *base* of the system; and the exponent of that power of the base which is equal to a given number, is called the logarithm of that number.

The base of the common system of logarithms (called, from their inventor, Briggs' logarithms) is the number 10. Hence all numbers are to be regarded as powers of 10. Thus, since

$10^0=1$,	0 is the logarithm of		1	in Briggs' system;	
$10^1=10$,	1	"	" 10	"	"
$10^2=100$,	2	"	" 100	"	"
$10^3=1000$,	3	"	" 1000	"	"
$10^4=10000$,	4	"	" 10,000	"	"
&c.,		&c.,		&c.;	

whence it appears that, in Briggs' system, the logarithm of every number between 1 and 10 is some number between 0 and 1, *i. e.*, is a proper fraction. The logarithm of every number between 10 and 100 is some number between 1 and 2, *i. e.*, is 1 plus a fraction. The logarithm of every number between 100 and 1000 is some number between 2 and 3, *i. e.*, is 2 plus a fraction, and so on.

(2.) The preceding principles may be extended to fractions by means of negative exponents. Thus, since

$10^{-1}=0.1$,	-1 is the logarithm of		0.1	in Briggs' system;	
$10^{-2}=0.01$,	-2	"	" 0.01	"	"
$10^{-3}=0.001$,	-3	"	" 0.001	"	"
$10^{-4}=0.0001$,	-4	"	" 0.0001	"	"
&c.,		&c.,		&c.	

Hence it appears that the logarithm of every number between 1 and 0.1 is some number between 0 and −1, or may be represented by −1 plus a fraction; the logarithm of every number between 0.1 and .01 is some number between −1 and −2, or may be represented by −2 plus a fraction; the logarithm of every number between .01 and .001 is some number between −2 and −3, or is equal to −3 plus a fraction, and so on.

The logarithms of most numbers, therefore, consist of an integer and a fraction. The integral part is called the *characteristic*, and may be known from the following

Rule.

The characteristic of the logarithm of any number greater than unity, is one less than the number of integral figures in the given number.

Thus the logarithm of 297 is 2 plus a fraction; that is, the characteristic of the logarithm of 297 is 2, which is one less than the number of integral figures. The characteristic of the logarithm of 5673.29 is 3; that of 73254.1 is 4, &c.

The characteristic of the logarithm of a decimal fraction is a negative number, and is equal to the number of places by which its first significant figure is removed from the place of units.

Thus the logarithm of .0046 is −3 plus a fraction; that is, the characteristic of the logarithm is −3, the first significant figure, 4, being removed three places from units.

(3.) Since powers of the same quantity are multiplied by adding their exponents (*Alg.*, Art. 50),

The logarithm of the product of two or more factors is equal to the sum of the logarithms of those factors.

Hence we see that if it is required to multiply two or more numbers by each other, we have only to add their logarithms: the sum will be the logarithm of their product. We then look in the table for the number answering to that logarithm, in order to obtain the required product.

Also, since powers of the same quantity are divided by subtracting their exponents (*Alg.*, Art. 66),

The logarithm of the quotient of one number divided by an-

other, is equal to the difference of the logarithms of those numbers.

Hence we see that if we wish to divide one number by another, we have only to subtract the logarithm of the divisor from that of the dividend; the difference will be the logarithm of their quotient.

(4.) Since, in Briggs' system, the logarithm of 10 is 1, if any number be multiplied or divided by 10, its logarithm will be increased or diminished by 1; and as this is an integer, it will only change the characteristic of the logarithm, without affecting the decimal part. Hence

The decimal part of the logarithm of any number is the same as that of the number multiplied or divided by 10, 100. 1000, &c.

Thus, the logarithm of		65430	is 4.815777;
"	"	6543	is 3.815777;
"	"	654.3	is 2.815777;
"	"	65.43	is 1.815777;
"	"	6.543	is 0.815777;
"	"	.6543	is $\bar{1}$.815777;
"	"	.06543	is $\bar{2}$.815777;
"	"	.006543	is $\bar{3}$.815777.

The minus sign is here placed *over* the characteristic, to show that *that* alone is negative, while the decimal part of the logarithm is positive.

TABLE OF LOGARITHMS.

(5.) A table of logarithms usually contains the logarithms of the entire series of natural numbers from 1 up to 10,000, and the larger tables extend to 100,000 or more. In the smaller tables the logarithms are usually given to five or six decimal places; the larger tables extend to seven, and sometimes eight or more places.

In the accompanying table, the logarithms of the first 100 numbers are given with their characteristics; but, for all other numbers, the decimal part only of the logarithm is given, while the characteristic is left to be supplied, according to the rule in Art. 2.

(6.) *To find the Logarithm of any Number between* 1 *and* 100.

Look on the first page of the accompanying table, along the column of numbers under N., for the given number, and against it, in the next column, will be found the logarithm with its characteristic. Thus,

opposite 13 is 1.113943, which is the logarithm of 13;
" 65 is 1.812913, " " 65.

To find the Logarithm of any Number consisting of three Figures.

Look on one of the pages of the table from 2 to 20, along the left-hand column, marked N., for the given number, and against it, in the column headed 0, will be found the decimal part of its logarithm. To this the characteristic must be prefixed, according to the rule in Art. 2. Thus

the logarithm of 347 will be found, from page 8, 2.540329;
" " 871 " " 18, 2.940018.

As the first two figures of the decimal are the same for several successive numbers in the table, they are not repeated for each logarithm separately, but are left to be supplied. Thus the decimal part of the logarithm of 339 is .530200. The first two figures of the decimal remain the same up to 347; they are therefore omitted in the table, and are to be supplied.

To find the Logarithm of any Number consisting of four Figures.

Find the three left-hand figures in the column marked N., as before, and the fourth figure at the head of one of the other columns. Opposite to the first three figures, and in the column under the fourth figure, will be found four figures of the logarithm, to which two figures from the column headed 0 are to be prefixed, as in the former case. The characteristic must be supplied according to Art. 2. Thus

the logarithm of 3456 is 3.538574;
" " 8765 is 3.942752.

In several of the columns headed 1, 2, 3, &c., small dots are found in the place of figures. This is to show that the two figures which are to be prefixed from the first column have changed, and they are to be taken from the horizontal line directly *below*. The place of the dots is to be supplied with ciphers. Thus

the logarithm of 2045 is 3.310693;
" " 9777 is 3.990206.

The two leading figures from the column 0 must also be taken from the horizontal line below, if any dots have been passed over on the same horizontal line. Thus

the logarithm of 1628 is 3.211654.

To find the Logarithm of any Number containing more than four Figures.

(7.) By inspecting the table, we shall find that, within certain limits, the logarithms are nearly proportional to their corresponding numbers. Thus

the logarithm of 7250 is 3.860338;
" " 7251 is 3.860398;
" " 7252 is 3.860458;
" " 7253 is 3.860518.

Here the difference between the successive logarithms, called *the tabular difference*, is constantly 60, corresponding to a difference of unity in the natural numbers. If, then, we suppose the logarithms to be proportional to their corresponding numbers (as they are nearly), a difference of 0.1 in the numbers should correspond to a difference of 6 in the logarithms; a difference of 0.2 in the numbers should correspond to a difference of 12 in the logarithms, &c. Hence

the logarithm of 7250.1 must be 3.860344;
" " 7250.2 " 3.860350;
" " 7250.3 " 3.860356.

In order to facilitate the computation, the tabular difference is inserted on page 16 in the column headed D., and the proportional part for the fifth figure of the natural number is given at the bottom of the page. Thus, when the tabular difference is 60, the corrections for .1, .2, .3, &c., are seen to be 6, 12, 18, &c.

If the given number was 72501, the characteristic of its logarithm would be 4, but the decimal part would be the same as for 7250.1.

If it were required to find the correction for a sixth figure in the natural number, it is readily obtained from the Proportional Parts in the table. The correction for a figure in the sixth place must be one tenth of the correction for the same figure if it stood in the fifth place. Thus, if the correction for .5 is 30, the correction for .05 is obviously 3.

As the differences change rapidly in the first part of the table, it was found inconvenient to give the proportional parts for each tabular difference; accordingly, for the first seven pages, they are only given for the *even* differences, but the proportional parts for the odd differences will be readily found by inspection.

Required the logarithm of 452789.

The logarithm of 452700 is 5.655810.

The tabular difference is 96.

Accordingly, the correction for the fifth figure, 8, is 77, and for the sixth figure, 9, is 8.6, or 9 nearly. Adding these corrections to the number before found, we obtain 5.655896.

The preceding logarithms do not pretend to be perfectly exact, but only the nearest numbers limited to six decimal places. Accordingly, when the fraction which is omitted exceeds half a unit in the sixth decimal place, the last figure must be increased by unity.

Required the logarithm of 8765432.

The logarithm of 8765000 is	6.942752
Correction for the fifth figure, 4,	20
" " sixth figure, 3,	1.5
" " seventh figure, 2,	0.1
Therefore the logarithm of 8765432 is	6.942774.

Required the logarithm of 234567.

The logarithm of 234500 is	5.370143
Correction for the fifth figure, 6,	111
" " sixth figure, 7,	13
Therefore the logarithm of 234567 is	5.370267.

To find the Logarithm of a Decimal Fraction.

(8.) According to Art. 4, the decimal part of the logarithm of any number is the same as that of the number multiplied or divided by 10, 100, 1000, &c. Hence, for a decimal fraction, we find the logarithm as if the figures were integers, and prefix the characteristic according to the rule of Art. 2.

EXAMPLES.

The logarithm of		345.6	is 2.538574;
"	"	87.65	is 1.942752;
"	"	2.345	is 0.370143;
"	"	.1234	is $\bar{1}.091315$;
"	"	.005678	is $\bar{3}.754195$.

To find the Logarithm of a Vulgar Fraction.

(9.) We may reduce the vulgar fraction to a decimal, and find its logarithm by the preceding article; or, since the value of a fraction is equal to the quotient of the numerator divided by the denominator, we may, according to Art. 3, *subtract the logarithm of the denominator from that of the numerator;* the difference will be the logarithm of the fraction.

Ex. 1. Find the logarithm of $\frac{3}{16}$, or 0.1875.

From the logarithm of 3,	0.477121,
Take the logarithm of 16,	1.204120.
Leaves the logarithm of $\frac{3}{16}$, or .1875,	$\bar{1}.273001$.

Ex. 2. The logarithm of $\frac{4}{55}$ is $\bar{2}.861697$.

Ex. 3. The logarithm of $\frac{123}{876}$ is $\bar{1}.147401$.

To find the Natural Number corresponding to any Logarithm.

(10.) Look in the table, in the column headed 0, for the first two figures of the logarithm, neglecting the characteristic; the other four figures are to be looked for in the same column, or in one of the nine following columns; and if they are exactly found, the first three figures of the corresponding number will be found opposite to them in the column headed N., and the fourth figure will be found at the top of the page. This number must be made to correspond with the characteristic of the given logarithm by pointing off decimals or annexing ciphers. Thus the natural number belonging to the log. 4.370143 is 23450;
" " " " 1.538574 is 34.56.

If the decimal part of the logarithm can not be exactly found in the table, look for the nearest less logarithm, and take out the four figures of the corresponding natural number as before; the additional figures may be obtained by means of the Proportional Parts at the bottom of the page.

Required the number belonging to the logarithm 4.368399.
On page 6, we find the next less logarithm .368287.

The four corresponding figures of the natural number are 2335. Their logarithm is less than the one proposed by 112. The tabular difference is 186; and, by referring to the bottom of page 6, we find that, with a difference of 186, the figure corresponding to the proportional part 112 is 6. Hence the five figures of the natural number are 23356; and, since the characteristic of the proposed logarithm is 4, these five figures are all integral.

Required the number belonging to logarithm 5.345678.
The next less logarithm in the table is 345570.
Their difference is 108.
The first four figures of the natural number are 2216.

With the tabular difference 196, the fifth figure, corresponding to 108, is seen to be 5, with a remainder of 10. To find the sixth figure corresponding to this remainder 10, we may multiply it by 10, making 100, and search for 100 in the same line of proportional parts. We see that a difference of 100 would give us 5 in the fifth place of the natural number. Therefore, a difference of 10 must give us 5 in the sixth place of the natural number. Hence the required number is 221655.

In the same manner we find
the number corresponding to log. 3.538672 is 3456.78;
" " " 1.994605 is 98.7654;
" " " $\bar{1}.647817$ is .444444.

Multiplication by Logarithms.

(11.) According to Art. 3, the logarithm of the product of two or more factors is equal to the sum of the logarithms of those factors. Hence, for multiplication by logarithms, we have the following

Rule.

Add the logarithms of the factors; the sum will be the logarithm of their product.

Ex. 1. Required the product of 57.98 by 18.

The logarithm of 57.98 is 1.763278
" " 18 is 1.255273

The logarithm of the product 1043.64 is 3.018551.

Ex. 2. Required the product of 397.65 by 43.78.

Ans., 17409.117.

Ex. 3. Required the continued product of 54.32, 6.543, and 12.345.

The word *sum*, in the preceding rule, is to be understood in its algebraic sense; therefore, if any of the characteristics of the logarithms are *negative*, we must take the difference between their sum and that of the positive characteristics, and prefix the sign of the greater. It should be remembered that the decimal part of the logarithm is invariably positive; hence that which is carried from the decimal part to the characteristic must be considered positive.

Ex. 4. Multiply 0.00563 by 17.

The logarithm of 0.00563 is $\bar{3}.750508$
" " 17 is 1.230449

Product, 0.09571, whose logarithm is $\bar{2}.980957$.

Ex. 5. Multiply 0.3854 by 0.0576. *Ans.*, 0.022199.

Ex. 6. Multiply 0.007853 by 0.00476.

Ans., 0.00003738.

Ex. 7. Find the continued product of 11.35, 0.072, and 0.017.

(12.) *Negative* quantities may be multiplied by means of logarithms in the same manner as positive, the proper sign being prefixed to the result according to the rules of Algebra. To distinguish the negative sign of a natural number from the negative characteristic of a logarithm, we append the letter *n* to the logarithm of a negative factor. Thus

the logarithm of −56 we write 1.748188 *n*.

Ex. 8. Multiply 53.46 by −29.47.

The logarithm of 53.46 is 1.728029
" " −29.47 is 1.469380 *n*.

Product, −1575.47, log. 3.197409 *n*.

Ex. 9. Find the continued product of 372.1, −.0054, and −175.6.

Ex. 10. Find the continued product of −0.137, −7.689, and −.0376.

DIVISION BY LOGARITHMS.

(13.) According to Art. 3, the logarithm of the quotient of one number divided by another is equal to the difference of the logarithms of those numbers. Hence, for division by logarithms, we have the following

RULE.

From the logarithm of the dividend, subtract the logarithm of the divisor; the difference will be the logarithm of the quotient.

Ex. 1. Required the quotient of 888.7 divided by 42.24.

The logarithm of 888.7 is 2.948755
" " 42.24 is 1.625724

The quotient is 21.039, whose log. is 1.323031.

Ex. 2. Required the quotient of 3807.6 divided by 13.7.
Ans., 277.927.

The word *difference*, in the preceding rule, is to be understood in its algebraic sense; therefore, if the characteristic of one of the logarithms is negative, or the lower one is greater than the upper, we must change the sign of the subtrahend, and proceed as in addition. If unity is carried from the decimal part, this must be considered as positive, and must be united with the characteristic before its sign is changed.

Ex. 3. Required the quotient of 56.4 divided by 0.00015.

The logarithm of 56.4 is 1.751279
" " 0.00015 is $\bar{4}$.176091

The quotient is 376000, whose log. is 5.575188.

This result may be verified in the same way as subtraction in common arithmetic. The remainder, added to the subtrahend, should be equal to the minuend. This precaution should always be observed when there is any doubt with regard to the sign of the result.

Ex. 4. Required the quotient of .8692 divided by 42.258.
Ans.

Ex. 5. Required the quotient of .74274 divided by .00928.

Ex. 6. Required the quotient of 24.934 divided by .078541.

Negative quantities may be divided by means of logarithms in the same manner as positive, the proper sign being prefixed to the result according to the rules of Algebra.

Ex. 7. Required the quotient of -79.54 divided by 0.08321.

Ex. 8. Required the quotient of -0.4753 divided by -36.74.

INVOLUTION BY LOGARITHMS.

(14.) It is proved in Algebra, Art. 340, that the logarithm of any power of a number is equal to the logarithm of that number multiplied by the exponent of the power. Hence, to involve a number by logarithms, we have the following

RULE.

Multiply the logarithm of the number by the exponent of the power required.

Ex. 1. Required the square of 428.

The logarithm of 428 is 2.631444
2
Square, 183184, log. 5.262888.

Ex. 2. Required the 20th power of 1.06.

The logarithm of 1.06 is 0.025306
20
20th power, 3.2071, log. 0.506120.

Ex. 3. Required the 5th power of 2.846.

It should be remembered, that what is carried from the decimal part of the logarithm is positive, whether the characteristic is positive or negative.

Ex. 4. Required the cube of .07654.

The logarithm of .07654 is $\bar{2}.883888$
3
Cube, .0004484, log. $\bar{4}.651664$.

Ex. 5. Required the fourth power of 0.09874.

Ex. 6. Required the seventh power of 0.8952.

EVOLUTION BY LOGARITHMS.

(15.) It is proved in Algebra, Art. 341, that the logarithm of any root of a number is equal to the logarithm of that number divided by the index of the root. Hence, to extract the root of a number by logarithms, we have the following

RULE.

Divide the logarithm of the number by the index of the root required.

Ex. 1. Required the cube root of 482.38.

The logarithm of 482.38 is 2.683389.

Dividing by 3, we have 0.894463, which corresponds to 7.842, which is therefore the root required.

Ex. 2. Required the 100th root of 365.

Ans., 1.0608.

When the characteristic of the logarithm is negative, and is not divisible by the given divisor, we may increase the characteristic by any number which will make it exactly divisible, provided we prefix an equal positive number to the decimal part of the logarithm.

Ex. 3. Required the seventh root of 0.005846.

The logarithm of 0.005846 is $\bar{3}.766859$, which may be written $\bar{7}+4.766859$.

Dividing by 7, we have $\bar{1}.680980$, which is the logarithm of .4797, which is, therefore, the root required.

This result may be verified by multiplying $\bar{1}.680980$ by 7; the result will be found to be $\bar{3}.766860$.

Ex. 4. Required the fifth root of 0.08452.

Ex. 5. Required the tenth root of 0.007815.

PROPORTION BY LOGARITHMS.

(16.) The fourth term of a proportion is found by multiplying together the second and third terms, and dividing by the first. Hence, to find the fourth term of a proportion by logarithms,

Add the logarithms of the second and third terms, and from their sum subtract the logarithm of the first term.

Ex. 1. Find a fourth proportional to 72.34, 2.519, and 357.48.

Ans., 12.448.

(17.) When one logarithm is to be subtracted from another, it may be more convenient to convert the subtraction into an addition, which may be done by first subtracting the given logarithm from 10, adding the difference to the other logarithm, and afterward rejecting the 10.

The difference between a given logarithm and 10 is called its *complement;* and this is easily taken from the table by beginning at the left hand, subtracting each figure from 9, except the last significant figure on the right, which must be subtracted from 10.

To subtract one logarithm from another is the same as to add its complement, and then reject 10 from the result. For $a-b$ is equivalent to $10-b+a-10$.

To work a proportion, then, by logarithms, we must

Add the complement of the logarithm of the first term to the logarithms of the second and third terms.

The characteristic must afterward be diminished by 10.

Ex. 2. Find a fourth proportional to 6853, 489, and 38750.

The complement of the logarithm of	6853 is	6.164119
The logarithm of	489 is	2.689309
" "	38750 is	4.588272

The fourth term is 2765, whose logarithm is 3.441700.

One advantage of using the complement of the first term in working a proportion by logarithms is, that it enables us to exhibit the operation in a more compact form.

Ex. 3. Find a fourth proportional to 73.84, 658.3, and 4872.

Ans.

Ex. 4. Find a fourth proportional to 5.745, 781.2, and 54.27.

BOOK II.

PLANE TRIGONOMETRY.

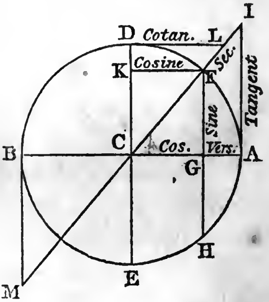
(18.) TRIGONOMETRY *is the science which teaches how to determine the several parts of a triangle from having certain parts given.*

Plane Trigonometry treats of plane triangles; Spherical Trigonometry treats of spherical triangles.

(19.) The circumference of every circle is supposed to be divided into 360 equal parts, called *degrees;* each degree into 60 *minutes*, and each minute into 60 *seconds.* Degrees, minutes, and seconds are designated by the characters °, ′, ″. Thus 23° 14′ 35″ is read 23 degrees, 14 minutes, and 35 seconds.

Since an angle at the center of a circle is measured by the arc intercepted by its sides, a right angle is measured by 90°, two right angles by 180°, and four right angles are measured by 360°.

(20.) The *complement* of an arc is what remains after subtracting the arc from 90°. Thus the arc DF is the complement of AF. The complement of 25° 15′ is 64° 45′.



In general, if we represent any arc by A, its complement is 90°−A. Hence, if an arc is greater than 90°, its complement must be negative. Thus, the complement of 100° 15′ is −10° 15′. Since the two acute angles of a right-angled triangle are together equal to a right angle, each of them must be the complement of the other.

(21.) The *supplement* of an arc is what remains after subtracting the arc from 180°. Thus the arc BDF is the supplement of the arc AF. The supplement of 25° 15′ is 154° 45′.

In general, if we represent any arc by A, its supplement is

180°−A. Hence, if an arc is greater than 180°, its supplement must be negative. Thus the supplement of 200° is −20°. Since in every triangle the sum of the three angles is 180°, either angle is the supplement of the sum of the other two.

(22.) *The sine of an arc is the perpendicular let fall from one extremity of the arc on the radius passing through the other extremity.* Thus FG is the sine of the arc AF, or of the angle ACF.

Every sine is half the chord of double the arc. Thus the sine FG is the half of FH, which is the chord of the arc FAH, double of FA. The chord which subtends the sixth part of the circumference, or the chord of 60°, is equal to the radius (*Geom.*, Prop. IV., B. VI.); hence the sine of 30° is equal to half of the radius.

(23.) The *versed sine* of an arc is that part of the diameter intercepted between the sine and the arc. Thus GA is the versed sine of the arc AF.

(24). *The tangent of an arc is the line which touches it at one extremity, and is terminated by a line drawn from the center through the other extremity.* Thus AI is the tangent of the arc AF, or of the angle ACF.

(25.) *The secant of an arc is the line drawn from the center of the circle through one extremity of the arc, and is limited by the tangent drawn through the other extremity.*

Thus CI is the secant of the arc AF, or of the angle ACF.

(26.) The *cosine* of an arc is the sine of the complement of that arc. Thus the arc DF, being the complement of AF, FK is the sine of the arc DF, or the cosine of the arc AF.

The *cotangent* of an arc is the tangent of the complement of that arc. Thus DL is the tangent of the arc DF, or the cotangent of the arc AF.

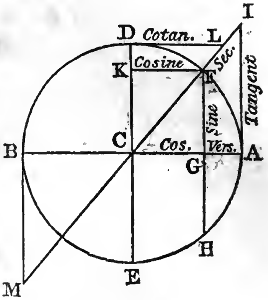
The *cosecant* of an arc is the secant of the complement of that arc. Thus CL is the secant of the arc DF, or the cosecant of the arc AF.

In general, if we represent any angle by A,

$$\text{cos. } A = \text{sine } (90° - A).$$
$$\text{cot. } A = \text{tang. } (90° - A).$$
$$\text{cosec. } A = \text{sec. } (90° - A).$$

Since, in a right-angled triangle, either of the acute angles is the complement of the other, the sine, tangent, and secant of one of these angles is the cosine, cotangent, and cosecant of the other.

(27.) The sine, tangent, and secant of an arc are equal to the sine, tangent, and secant of its supplement. Thus FG is the sine of the arc AF, or of its supplement, BDF. Also, AI, the tangent of the arc AF, is equal to BM, the tangent of the arc BDF. And CI, the secant of the arc AF, is equal to CM, the secant of the arc BDF.



The versed sine of an acute angle, ACF, is equal to the radius *minus* the cosine CG. The versed sine of an obtuse angle, BCF, is equal to radius *plus* the cosine CG; that is, to BG.

(28.) The relations of the sine, cosine, &c., to each other, may be derived from the proportions of the sides of similar triangles. Thus the triangles CGF, CAI, CDL, being similar, we have,

1. CG : GF : : CA : AI; that is, representing the arc by A, and the radius of the circle by R, cos. A : sin. A : : R : tang. A.

$$\text{Whence tang. A} = \frac{\text{R sin. A}}{\text{cos. A}}.$$

2. CG : : CF : CA : CI; that is, cos. A : R : : R : sec. A.

$$\text{Whence sec. A} = \frac{\text{R}^2}{\text{cos. A}}.$$

3. GF : CG : : CD : DL; that is, sin. A : cos. A : : R : cot. A.

$$\text{Whence cot. A} = \frac{\text{R cos. A}}{\text{sin. A}}.$$

4 GF : CF : : CD : CL; that is, sin. A : R : : R : cosec. A.

$$\text{Whence cosec. A} = \frac{\text{R}^2}{\text{sin. A}}.$$

5. AI : AC : : CD : DL; that is, tang. A : R : : R : cot. A.

$$\text{Whence tang. A.} = \frac{\text{R}^2}{\text{cot. A}}.$$

The preceding values of tangent and cotangent, secant and cosecant will be frequently referred to hereafter, and should be carefully committed to memory.

Also, in the right-angled triangle CGF, we find $CG^2+GF^2=CF^2$; that is, $\sin.^2A+\cos.^2A=R^2$; or,

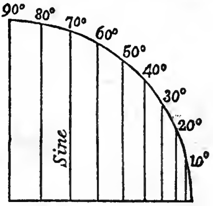
The square of the sine of an arc, together with the square of its cosine, is equal to the square of the radius.

$$\text{Hence } \sin. A=\pm\sqrt{R^2-\cos.^2A}.$$
$$\text{And } \cos. A=\pm\sqrt{R^2-\sin.^2A}.$$

(29.) A table of *natural sines, tangents*, &c., is a table giving the lengths of those lines for different angles in a circle whose radius is unity.

Thus, if we describe a circle with a radius of one inch, and divide the circumference into equal parts of ten degrees, we shall find

the sine	of	10°	equals	0.174	inch;
"	"	20°	"	0.342	"
"	"	30°	"	0.500	"
"	"	40°	"	0.643	"
"	"	50°	"	0.766	"
"	"	60°	"	0.866	"
"	"	70°	"	0.940	"
"	"	80°	"	0.985	"
"	"	90°	"	1.000	"



If we draw the tangents of the same arcs, we shall find

the tangent	of	10°	equals	0.176	inch;
"	"	20°	"	0.364	"
"	"	30°	"	0.577	"
"	"	40°	"	0.839	"
"	"	45°	"	1.000	"
"	"	50°	"	1.192	"
"	"	60°	"	1.732	"
"	"	70°	"	2.747	"
"	"	80°	"	5.671	"
"	"	90°	"	infinite.	

Also, if we draw the secants of the same arcs, we shall find that

the secant	of	10°	equals	1.015	inch;
"	"	20°	"	1.064	"
"	"	30°	"	1.155	"
"	"	40°	"	1.305	"

the secant of		50°	equals	1.556	inch;
"	"	60°	"	2.000	"
"	"	70°	"	2.924	"
"	"	80°	"	5.759	"
"	"	90°	"	infinite.	

In the accompanying table, pages 116–133, the sines, cosines, tangents, and cotangents are given for every minute of the quadrant to six places of figures.

(30.) *To find from the table the natural sine, cosine, &c., of an arc or angle.*

If a sine is required, look for the degrees at the *top* of the page, and for the minutes on the *left;* then, directly under the given number of degrees at the top of the page, and opposite to the minutes on the left, will be found the sine required. Since the radius of the circle is supposed to be unity, the sine of every arc below 90° is less than unity. The sines are expressed in decimal parts of radius; and, although the decimal point is not written in the table, it must always be prefixed. As the first two figures remain the same for a great many numbers in the table, they are only inserted for every ten minutes, and the vacant places must be supplied from the two leading figures next preceding. Thus, on

page 120, the sine of 25° 11′ is 0.425516;
page 126, " " 51° 34′ is 0.783332, &c.

The tangents are found in a similar manner. Thus

the tangent of 31° 44′ is 0.618417;
" " 65° 27′ is 2.18923.

The same number in the table is both the sine of an arc and the cosine of its complement. The degrees for the cosines must be sought at the bottom of the page, and the minutes on the right. Thus,

on page 130, the cosine of 16° 42′ is 0.957822;
on page 118, " " 73° 17′ is 0.287639, &c.

The cotangents are found in the same manner. Thus

the cotangent of 19° 16′ is 2.86089;
" " 54° 53′ is 0.703246.

It is not necessary to extend the tables beyond a quadrant, because the sine of an angle is equal to that of its supplement (Art. 27). Thus

the sine of 143° 24′ is 0.596225;
" cosine of 151° 23′ is 0.877844;
" tangent of 132° 36′ is 1.08749;
" cotangent of 116° 7′ is 0.490256, &c.

(31.) If a sine is required for an arc consisting of degrees, minutes, and *seconds*, we must make an allowance for the seconds in the same manner as was directed in the case of logarithms, Art. 7; for, within certain limits, the differences of the sines are proportional to the differences of the corresponding arcs. Thus

the sine of 34° 25′ is .565207;
" " 34° 26′ is .565447.

The difference of the sines corresponding to one minute of arc, or 60 seconds, is .000240. The proportional part for 1 ′ is found by dividing the tabular difference by 60, and the quotient, .000004, is placed at the bottom of page 122, in the column headed 34°. The correction for any number of seconds will be found by multiplying the proportional part for 1″ by the number of seconds.

Required the natural sine of 34° 25′ 37″.

The proportional part for 1″, being multiplied by 37, becomes 148, which is the correction for 37″. Adding this to the sine of 34° 25′, we find

the sine of 34° 25′ 37″ is .565355.

Since the proportional part for 1″ is given to *hundredths* of a unit in the sixth place of figures, after we have multiplied by the given number of seconds, we must reject the last two figures of the product.

In the same manner we find

the cosine of 56° 34′ 28″ is .550853.

It will be observed, that since the cosines decrease while the arcs increase, the correction for the 28″ is to be subtracted from the cosine of 56° 34′.

In the same manner we find

the natural sine of 27° 17′ 12″ is 0.458443;
" " cosine of 45° 23′ 23″ is 0.702281;
" " tangent of 63° 32′ 34″ is 2.00945;
" " cotangent of 81° 48′ 56″ is 0.143825.

(32.) *To find the number of degrees, minutes, and seconds belonging to a given sine or tangent.*

If the given sine or tangent is found exactly in the table, the corresponding degrees will be found at the top of the page, and the minutes on the left hand. But when the given number is not found exactly in the table, look for the sine or tangent which is next less than the proposed one, and take out the corresponding degrees and minutes. Find, also, the difference between this tabular number and the number proposed, and divide it by the proportional part for 1″ found at the bottom of the page; the quotient will be the required number of seconds.

Required the arc whose sine is .750000.

The next less sine in the table is .749919, the arc corresponding to which is 48° 35′. The difference between this sine and that proposed is 81, which, divided by 3.21, gives 25. Hence the required arc is 48° 35′ 25″.

In the same manner we find

the arc whose tangent is 2.00000 is 63° 26′ 6″.

If a cosine or cotangent is required, we must look for the number in the table which is next greater than the one proposed, and then proceed as for a sine or tangent. Thus

the arc whose cosine is .40000 is 66° 25′ 18″;
" " " cotangent is 1.99468 is 26° 37′ 34″.

(33.) On pages 134–5 will be found a table of natural secants for every ten minutes of the quadrant, carried to seven places of figures. The degrees are arranged in order in the first vertical column on the left, and the minutes at the top of the page. Thus

the secant of 21° 20′ is 1.073561;
" " 81° 50′ is 7.039622.

If a secant is required for a number of minutes not given in the table, the correction for the odd minutes may be found by means of the last vertical column on the right, which shows the proportional part for one minute.

Let it be required to find the secant of 30° 33′.

The secant of 30° 30′ is 1.160592.

The correction for 1′ is 198.9, which, multiplied by 3, becomes 597. Adding this to the number before found, we obtain 1.161189.

For a cosecant, the degrees must be sought in the right-hand vertical column, and the minutes at the bottom of the page. Thus

the cosecant of 47° 40′ is 1.352742;
" " 38° 33′ is 1.604626.

(34.) When the natural sines, tangents, &c., are used in proportions, it is necessary to perform the tedious operations of multiplication and division. It is, therefore, generally preferable to employ the *logarithms* of the sines; and, for convenience, these numbers are arranged in a separate table, called *logarithmic sines*, &c. Thus

the natural sine of 14° 30′ is 0.250380.
Its logarithm, found from page 6, is $\bar{1}.398600$.

The characteristic of the logarithm is *negative*, as must be the case with all the sines, since they are less than unity. To avoid the introduction of negative numbers in the table, we increase the characteristic by 10, making 9.398600, and this is the number found on page 38 for the logarithmic sine of 14° 30′. The radius of the table of logarithmic sines is therefore, properly, 10,000,000,000, whose logarithm is 10.

(35.) The accompanying table contains the logarithmic sines and tangents for every ten seconds of the quadrant. The degrees and seconds are placed at the top of the page, and the minutes in the left vertical column. After the first two degrees, the three leading figures in the table of sines are only given in the column headed 0″, and are to be prefixed to the numbers in the other columns, as in the table of logarithms of numbers. Also, where the leading figures change, this change is indicated by dots, as in the former table. The correction for any number of seconds less than 10 is given at the bottom of the page.

(36.) *To find the logarithmic sine or tangent of a given arc.*

Look for the degrees at the top of the page, the minutes on the left hand, and the next less number of seconds at the top; then, under the seconds, and opposite to the minutes, will be found four figures, to which the three leading figures are to be prefixed from the column headed 0″; to this add the proportional part for the odd seconds at the bottom of the page.

Required the logarithmic sine of 24° 27′ 34″.

The logarithmic sine of 24° 27′ 30″ is	9.617033
Proportional part for 4″ is	18
Logarithmic sine of 24° 27′ 34″ is	9.617051.

Required the logarithmic tangent of 73° 35′ 43″.

The logarithmic tangent 73° 35′ 40″ is	10.531031
Proportional part for 3″ is	23
Logarithmic tangent of 73° 35′ 43″ is	10.531054.

When a cosine is required, the degrees and seconds must be sought at the bottom of the page, and the minutes on the right, and the correction for the odd seconds must be subtracted from the number in the table.

Required the logarithmic cosine of 59° 33′ 47″.

The logarithmic cosine of 59° 33′ 40″ is	9.704682
Proportional part for 7″ is	25
Logarithmic cosine of 59° 33′ 47″ is	9.704657.

So, also, the logarithmic cotangent of 37° 27′ 14″ is found to be 10.115744.

It will be observed that for the cosines and cotangents, the seconds are numbered from 10″ to 60″, so that if it is required to find the cosine of 25° 25′ 0″ we must look for 25° 24′ 60″; and so, also, for the cotangents.

(37.) The proportional parts given at the bottom of each page correspond to the degrees at the top of the page, increased by 30′, and are not strictly applicable to any other number of minutes; nevertheless, the differences of the sines change so slowly, except near the commencement of the quadrant, that the error resulting from using these numbers for every part of the page will seldom exceed a unit in the sixth decimal place. For the first two degrees, the differences change so rapidly that the proportional part for 1″ is given for each minute in the right-hand column of the page. The correction for any number of seconds less than ten will be found by multiplying the proportional part for 1″ by the given number of seconds.

Required the logarithmic sine of 1° 17′ 33″.

The logarithmic sine of 1° 17′ 30″ is 8.352991.

The correction for 3″ is found by multiplying 93.4 by 3, which gives 280. Adding this to the above tabular number, we obtain for

the sine of 1° 17′ 33″, 8.353271.

A similar method may be employed for several of the first degrees of the quadrant, if the proportional parts at the bottom of the page are not thought sufficiently precise. This correction may, however, be obtained pretty nearly by inspection, from comparing the proportional parts for two successive degrees. Thus, on page 26, the correction for 1″, corresponding to the sine of 2° 30′, is 48; the correction for 1″, corresponding to the sine of 3° 30′, is 34. Hence the correction for 1″, corresponding to the sine of 3° 0′, must be about 41; and, in the same manner, we may proceed for any other part of the table.

(38.) Near the close of the quadrant, the tangents vary so rapidly that the same arrangement of the table is adopted as for the commencement of the quadrant. For the last, as well as the first two degrees of the quadrant, the proportional part to 1″ is given for each minute separately. These proportional parts are computed for the minutes placed opposite to them, increased by 30″, and are not strictly applicable to any other number of seconds; nevertheless, the differences for the most part change so slowly, that the error resulting from using these numbers for every part of the same horizontal line is quite small. When great accuracy is required, the table on page 114 may be employed for arcs near the limits of the quadrant. This table furnishes the differences between the logarithmic sines and the logarithms of the arcs expressed in seconds. Thus

the logarithmic sine of 0° 5′, from page 22, is	7.162696
the logarithm of 300″ (=5′) is	2.477121
the difference is	4.685575.

This is the number found on page 114, under the heading *log. sine A—log. A″*, opposite to 5 min.; and, in a similar manner, the other numbers in the same column are obtained. These numbers vary quite slowly for two degrees; and hence, to find the logarithmic sine of an arc less than two degrees, we have but to add the logarithm of the arc expressed in seconds to the appropriate number found in this table.

Required the logarithmic sine of 0° 7′ 22″.

Tabular number from page 114,	4.685575
The logarithm of 442″ is	2.645422
Logarithmic sine of 0° 7′ 22″ is	7.330997.

The logarithmic tangent of an arc less than two degrees is found in a similar manner.

Required the logarithmic tangent of 0° 27′ 36″.

Tabular number from page 114,	4.685584
The logarithm of 1656″ is	3.219060
Logarithmic tangent of 0° 27′ 36″ is	7.904644.

The column headed *log. cot. A+log. A″*, is found by adding the logarithmic cotangent to the logarithm of the arc expressed in seconds. Hence, to find the logarithmic cotangent of an arc less than two degrees, we must subtract from the tabular number the logarithm of the arc in seconds.

Required the logarithmic cotangent of 0° 27′ 36″.

Tabular number from page 114,	15.314416
The logarithm of 1656″ is	3.219060
Logarithmic cotangent of 0° 27′ 36″ is	12.095356.

The same method will, of course, furnish cosines and cotangents of arcs near 90°.

(39.) The secants and cosecants are omitted in this table, since they are easily derived from the cosines and sines. We have found, Art. 28, secant $=\frac{R^2}{\text{cosine}}$; or, taking the logarithms,

log. secant $=2.\ \log.\ R-\log.\ \text{cosine}$
$=20-\log.\ \text{cosine}.$

Also, cosecant $=\frac{R^2}{\text{sine}}$,

or log. cosecant $=20-\log.\ \text{sine}$. That is,

The logarithmic secant is found by subtracting the logarithmic cosine from 20; *and the logarithmic cosecant is found by subtracting the logarithmic sine from* 20.

Thus we have found the logarithmic sine of 24° 27′ 34″ to be 9.617051.

Hence the logarithmic cosecant of 24° 27′ 34″ is 10.382949.

The logarithmic cosine of 54° 12′ 40″ is 9.767008.

Hence the logarithmic secant of 54° 12′ 40″ is 10.232992.

(40.) *To find the arc corresponding to a given logarithmic sine or tangent.*

If the given number is found exactly in the table, the corresponding degrees and seconds will be found at the top of the page, and the minutes on the left. But when the given number is not found exactly in the table, look for the sine or tangent which is next less than the proposed one, and take out the corresponding degrees, minutes, and seconds. Find, also, the difference between this tabular number and the number proposed, and corresponding to this difference, at the bottom of the page, will be found a certain number of seconds which is to be added to the arc before found.

Required the arc corresponding to the logarithmic sine 9.750000.

The next less sine in the table is 9.749987.

The arc corresponding to which is 34° 13′ 0″.

The difference between its sine and the one proposed is 13, corresponding to which, at the bottom of the page, we find 4″ nearly. Hence the required arc is 34° 13′ 4″.

In the same manner, we find the arc corresponding to logarithmic tangent 10.250000 to be 60° 38′ 57″.

When the arc falls within the first two degrees of the quadrant, the odd seconds may be found by dividing the difference between the tabular number and the one proposed, by the proportional part for 1″. We thus find the arc corresponding to logarithmic sine 8.400000 to be 1° 26′ 22″ nearly.

We may employ the same method for the last two degrees of the quadrant when a tangent is given; but near the limits of the quadrant it is better to employ the auxiliary table on page 114. The tabular number on page 114 is equal to log. sin. A−log. A″. Hence log. sin. A−tabular number=log. A″; that is, if we subtract the corresponding tabular number on page 114, from the given logarithmic sine, the remainder will be the logarithm of the arc expressed in seconds.

Required the arc corresponding to logarithmic sine 7.000000.

We see, from page 22, that the arc must be nearly 3′; the corresponding tabular number on page 114 is 4.685575

The difference is 2.314425,

which is the logarithm of 206.″265.

Hence the required arc is 3′ 26.″265.

Required the arc corresponding to log. sine 8.000000.

We see from page 22, that the arc is about 34′. The corresponding tabular number from page 114 is 4.685568, which, subtracted from 8.000000, leaves 3.314432, which is the logarithm of 2062.″68. Hence the required arc is

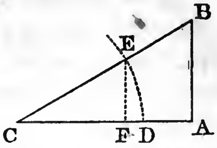
34′ 22.″68.

In the same manner, we find the arc corresponding to logarithmic tangent 8.184608 to be 0° 52′ 35″.

SOLUTIONS OF RIGHT-ANGLED TRIANGLES.

THEOREM I.

(41.) *In any right-angled triangle, radius is to the hypothenuse as the sine of either acute angle is to the opposite side, or the cosine of either acute angle to the adjacent side.*



Let the triangle CAB be right angled at A, then will

R : CB : : sin. C : BA : : cos. C : CA.

From the point C as a center, with a radius equal to the radius of the tables, describe the arc DE, and on AC let fall the perpendicular EF. Then EF will be the sine, and CF the cosine of the angle C. Because the triangles CAB, CFE are similar, we have

CE : CB : : EF : BA,

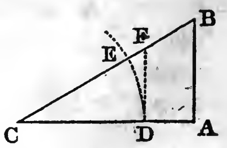
or R : CB : : sin. C : BA.

Also, CE : CB : : CF : CA,

or R : CB : : cos. C : CA.

THEOREM II.

(42.) *In any right-angled triangle, radius is to either side as the tangent of the adjacent acute angle is to the opposite side, or the secant of the same angle to the hypothenuse.*



Let the triangle CAB be right angled at A, then will

R : CA : : tang. C : AB : : sec. C : CB.

From the point C as a center, with a radius equal to the radius of the tables, describe the arc DE, and from the point D draw DF perpendicular to CA. Then DF will be the tangent, and CF the secant of the angle C. Because the triangles CAB, CDF are similar, we have CD : CA : : DF : AB,

or R : CA : : tang. C : AB.

Also, CD : CA : : CF : CB,

or R : CA : : sec. C : CB.

(43.) In every plane triangle there are *six* parts: three sides and three angles. Of these, any three being given, provided one of them is a side, the others may be determined. In a right-angled triangle, one of the six parts, viz., the right angle, is always given; and if one of the acute angles is given, the other is, of course, known. Hence the number of parts to be considered in a right-angled triangle is reduced to *four*, any two of which being given, the others may be found.

It is desirable to have appropriate names by which to designate each of the parts of a triangle. One of the sides adjacent to the right angle being called the base, the other side adjacent to the right angle may be called the perpendicular. The three sides will then be called the hypothenuse, base, and perpendicular. The base and perpendicular are sometimes called the legs of the triangle. Of the two acute angles, that which is adjacent to the base may be called the angle at the base, and the other the angle at the perpendicular.

We may, therefore, have four cases, according as there are given,

1. The hypothenuse and the angles;
2. The hypothenuse and a leg;
3. One leg and the angles; or,
4. The two legs.

All of these cases may be solved by the two preceding theorems.

CASE I.

(44.) *Given the hypothenuse and the angles, to find the base and perpendicular.*

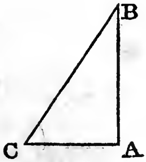
This case is solved by Theorem I.

Radius : *hypothenuse* : : *sine of the angle at the base* : *perpendicular;*

: : *cosine of the angle at the base* : *base.*

Ex. 1. Given the hypothenuse 275, and the angle at the base 57° 23′, to find the base and perpendicular.

The natural sine of 57° 23′ is .842296;
" cosine " .539016.



Hence 1 : 275 : : .842296 : 231.631=AB.
1 : 275 : : .539016 : 148.229=AC.

The computation is here made by natural numbers. If we work the proportion by logarithms, we shall have

Radius,	10.000000
Is to the hypothenuse 275	2.439333
As the sine of C 57° 23′	9.925465
To the perpendicular 231.63	2.364798.
Also, Radius,	10.000000
Is to the hypothenuse 275	2.439333
As the cosine of C 57° 23′	9.731602
To the base 148.23	2.170935.

Ex. 2. Given the hypothenuse 67.43, and the angle at the perpendicular 38° 43′, to find the base and perpendicular.

Ans. The base is 42.175, and perpendicular 52.612.

The student should work this and the following examples both by natural numbers and by logarithms, until he has made himself perfectly familiar with both methods. He may then employ either method, as may appear to him most expeditious.

CASE II.

(45.) *Given the hypothenuse and one leg, to find the angles and the other leg.*

This case is solved by Theorem I.

Hypothenuse : *radius* : : *base* : *cosine of the angle at the base.*

Radius : *hypothenuse* : : *sine of the angle at the base* : *perpendicular.*

When the perpendicular is given, perpendicular must be substituted for base in this proportion.

Ex. 1. Given the hypothenuse 54.32, and the base 32.11, to find the angles and the perpendicular.

By natural numbers, we have

54.32 : 1 : : 32.11 : .591127, which is the cosine of 53° 45′ 47″, the angle at the base.

Also, 1 : 54.32 : : .806617 : 43.813=the perpendicular.

The computation may be performed more expeditiously by logarithms, as in the former case.

Ex. 2. Given the hypothenuse 332.49, and the perpendicular 98.399, to find the angles and the base.

Ans. The angles are 17° 12′ 51″ and 72° 47′ 9″; the base, 317.6.

CASE III.

(46.) *Given one leg and the angles, to find the other leg and hypothenuse.*

This case is solved by Theorem II.

Radius : *base* : : *tangent of the angle at the base* : *the perpendicular.*

: : *secant of the angle at the base* : *hypothenuse.*

When the perpendicular is given, perpendicular must be substituted for base in this proportion.

Ex. 1. Given the base 222, and the angle at the base 25° 15′, to find the perpendicular and hypothenuse.

By natural numbers, we have

1 : 222 : : .471631 : 104.70, perpendicular;
: : 1.105638 : 245.45, hypothenuse.

The computation should also be performed by logarithms, as in Case I.

Ex. 2. Given the perpendicular 125, and the angle at the perpendicular 51° 19′, to find the hypothenuse and base.

Ans. Hypothenuse, 199.99; base, 156.12.

CASE IV.

(47.) *Given the two legs, to find the angles and hypothenuse.*

This case is solved by Theorem II.

Base : *radius* : : *perpendicular* : *tangent of the angle at the base.*
Radius : *base* : : *secant of the angle at the base* : *hypothenuse.*

Ex. 1. Given the base 123, and perpendicular 765, to find the angles and hypothenuse.

By natural numbers, we have

123 : 1 : : 765 : 6.219512, which is the tangent of 80° 51′ 57″, the angle at the base.

1 : 123 : : 6.300479 : 774.96, hypothenuse.

The computation may also be made by logarithms, as in Case I.

Ex. 2. Given the base 53, and perpendicular 67, to find the angles and hypothenuse.

Ans. The angles are 51° 39′ 16″ and 38° 20′ 44″; hypothenuse, 85.428.

Examples for Practice.

1. Given the base 777, and perpendicular 345, to find the hypothenuse and angles.

This example, it will be seen, falls under Case IV.

2. Given the hypothenuse 324, and the angle at the base 48° 17′, to find the base and perpendicular.

3. Given the perpendicular 543, and the angle at the base 72° 45′, to find the hypothenuse and base.

4. Given the hypothenuse 666, and base 432, to find the angles and perpendicular.

5. Given the base 634, and the angle at the base 53° 27′, to find the hypothenuse and perpendicular.

6. Given the hypothenuse 1234, and perpendicular 555, to find the base and angles.

(48.) When two sides of a right-angled triangle are given, the third may be found by means of the property that the square of the hypothenuse is equal to the sum of the squares of the other two sides.

Hence, representing the hypothenuse, base, and perpendicular by the initial letters of these words, we have

$$h=\sqrt{b^2+p^2};\ b=\sqrt{h^2-p^2};\ p=\sqrt{h^2-b^2}.$$

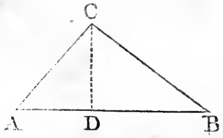
Ex. 1. If the base is 2720, and the perpendicular 3104, what is the hypothenuse? *Ans.*, 4127.1.

Ex. 2. If the hypothenuse is 514, and the perpendicular 432, what is the base?

SOLUTIONS OF OBLIQUE-ANGLED TRIANGLES.

THEOREM I.

(49.) *In any plane triangle, the sines of the angles are proportional to the opposite sides.*



Let ABC be any triangle, and from one of its angles, as C, let CD be drawn perpendicular to AB. Then, because the triangle ACD is right angled at D, we have

$$R : \sin. A :: AC : CD; \text{ whence } R \times CD = \sin. A \times AC.$$

For the same reason,

$$R : \sin. B :: BC : CD; \text{ whence } R \times CD = \sin. B \times BC.$$

Therefore, $\sin. A \times AC = \sin. B \times BC,$

or $\sin. A : \sin. B :: BC : AC.$

THEOREM II.

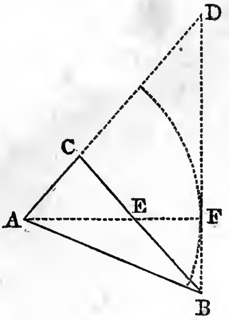
(50.) *In any plane triangle, the sum of any two sides is to their difference, as the tangent of half the sum of the opposite angles is to the tangent of half their difference.*

Let ABC be any triangle; then will

$$CB+CA : CB-CA :: \text{tang.} \frac{A+B}{2} : \text{tang.} \frac{A-B}{2}.$$

Produce AC to D, making CD equal to CB, and join DB. Take CE equal to CA, draw AE, and produce it to F. Then AD is the *sum* of CB and CA, and BE is their *difference*.

The sum of the two angles CAE, CEA, is equal to the sum of CAB, CBA, each being the supplement of ACB (*Geom.*, Prop. 27, B. I.). But, since CA is equal to CE, the angle CAE is equal to the angle CEA; therefore, CAE is the *half sum* of the angles CAB, CBA. Also, if from the greater of the two angles CAB, CBA, there be taken their half sum, the remainder, FAB, will be their *half difference* (*Algebra*, p. 68).



Since CD is equal to CB, the angle ADF is equal to the angle EBF; also, the angle CAE is equal to AEC, which is equal to the vertical angle BEF. Therefore, the two triangles DAF, BEF, are mutually equiangular; hence the two angles at F are equal, and AF is perpendicular to DB. If, then, AF be made radius, DF will be the tangent of DAF, and BF will be the tangent of BAF. But, by similar triangles, we have

$$AD : BE :: DF : BF;\text{ that is,}$$

$$CB+CA : CB-CA :: \text{tang.}\frac{A+B}{2} : \text{tang.}\frac{A-B}{2}.$$

THEOREM III.

(51.) *If from any angle of a triangle a perpendicular be drawn to the opposite side or base, the whole base will be to the sum of the other two sides, as the difference of those two sides is to the difference of the segments of the base.*

For demonstration, see Geometry, Prop. 31, Cor., B. IV.

(52). In every plane triangle, three parts must be given to enable us to determine the others; and of the given parts, one, at least, must be a side. For if the angles only are given, these might belong to an infinite number of different triangles. In solving oblique-angled triangles, four different cases may therefore be presented. There may be given,

1. Two angles and a side;
2. Two sides and an angle opposite one of them;
3. Two sides and the included angle; or,
4. The three sides.

We shall represent the three angles of the proposed triangle by A, B, C, and the sides opposite them, respectively, by *a*, *b*, *c*.

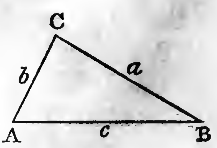
CASE I.

(53.) *Given two angles and a side, to find the third angle and the other two sides.*

To find the third angle, add the given angles together, and subtract their sum from 180°.

The required sides may be found by Theorem I. The proportion will be,

The sine of the angle opposite the given side : the given side :: the sine of the angle opposite the required side : the required side.



Ex. 1. In the triangle ABC, there are given the angle A, 57° 15′, the angle B, 35° 30′, and the side *c*, 364, to find the other parts.

The sum of the given angles, subtracted

from 180°, leaves 87° 15′ for the angle C. Then, to find the side a, we say, sin. C : c : : sin. A : a.

By natural numbers,

$$.998848 : 364 :: .841039 : 306.49 = a.$$

This proportion is most easily worked by logarithms, thus:

As the sine of the angle C, 87° 15′, comp.,	0.000500
Is to the side c, 364,	2.561101
So is the sine of the angle A, 57° 15′,	9.924816
To the side a, 306.49,	2.486417.

To find the side b:

sin. C : c : : sin. B : b.

By natural numbers,

$$.998848 : 364 :: .580703 : 211.62 = b.$$

The work by logarithms is as follows:

sin. C, 87° 15′, comp.,	0.000500
: c, 364,	2.561101
: : sin. B, 35° 30′,	9.763954
: b, 211.62,	2.325555.

Ex. 2. In the triangle ABC, there are given the angle A, 49° 25′, the angle C, 63° 48′, and the side c, 275, to find the other parts. *Ans.*, B=66° 47′; a=232.766; b=281.67.

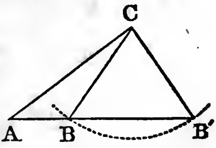
CASE II.

(54.) *Given two sides and an angle opposite one of them, to find the third side and the remaining angles.*

One of the required angles is found by Theorem I. The proportion is,

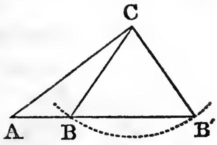
The side opposite the given angle : the sine of that angle : : the other given side : the sine of the opposite angle.

The third angle is found by subtracting the sum of the other two from 180°; and the third side is found as in Case I.



If the side BC, opposite the given angle A, is shorter than the other given side AC, the solution will be *ambiguous;* that is, two different triangles, ABC, AB′C, may be formed, each of which will satisfy the conditions of the problem.

The numerical result is also ambiguous, for the fourth term of the first proportion is a sine of an angle. But this may be the sine either of the *acute* angle AB′C, or of its supplement, the obtuse angle ABC (Art. 27). In practice, however, there will generally be some circumstance to determine whether the required angle is acute or obtuse. If the given angle is obtuse, there can be no ambiguity in the solution, for then the remaining angles must of course be acute.



Ex. 1. In a triangle, ABC, there are given AC, 458, BC, 307, and the angle A, 28° 45′, to find the other parts.

To find the angle B:

BC : sin. A : : AC : sin. B.

By natural numbers,

307 : .480989 : : 458 : .717566, sin. B, the arc corresponding to which is 45° 51′ 14″, or 134° 8′ 46″.

This proportion is most easily worked by logarithms, thus:

BC, 307, comp.,	7.512862
: sin. A, 28° 45′,	9.682135
: : AC, 458,	2.660865
: sin. B, 45° 51′ 14″, or 134° 8′ 46″,	9.855862.

The angle ABC is 134° 8′ 46″, and the angle AB′C, 45° 51′ 14″. Hence the angle ACB is 17° 6′ 14″, and the angle ACB′, 105° 23′ 46″.

To find the side AB:

sin. A : CB : : sin. ACB : AB.

By logarithms,

sin. A, 28° 45′, comp.,	0.317865
: CB, 307,	2.487138
: : sin. ACB, 17° 6′ 14″,	9.468502
: AB, 187.72,	2.273505.

To find the side AB′:

sin. A : CB′ : : sin. ACB′ : AB′.

By logarithms,

sin. A, 28° 45′, comp.,	0.317865
: CB′, 307,	2.487138
: : sin. ACB′, 105° 23′ 46″,	9.984128
: AB′, 615.36,	2.789131.

Ex. 2. In a triangle, ABC, there are given AB, 532, BC, 358, and the angle C, 107° 40′, to find the other parts.

Ans. A=39° 52′ 52″; B=32° 27′ 8″; AC=299.6.

In this example there is no ambiguity, because the given angle is obtuse.

CASE III.

(55.) *Given two sides and the included angle, to find the third side and the remaining angles.*

The *sum* of the required angles is found by subtracting the given angle from 180°. The *difference* of the required angles is then found by Theorem II. Half the difference added to half the sum gives the greater angle, and, subtracted, gives the less angle. The third side is then found by Theorem I.

Ex. 1. In the triangle ABC, the angle A is given 53° 8′; the side c, 420, and the side b, 535, to find the remaining parts.

The sum of the angles B+C=180°−53° 8′=126° 52′. Half their sum is 63° 26′.

Then, by Theorem II.,

535+420 : 535−420 : : tang. 63° 26′ : tang. 13° 32′ 25″,

which is half the difference of the two required angles.

Hence the angle B is 76° 58′ 25″, and the angle C, 49° 53′ 35″.

To find the side a:

sin. C : c : : sin. A : a=439.32.

Ex. 2. Given the side c, 176, a, 133, and the included angle B, 73°, to find the remaining parts.

Ans., b=187.022, the angle C, 64° 9′ 3″, and A, 42° 50′ 57″.

CASE IV.

(56.) *Given the three sides, to find the angles.*

Let fall a perpendicular upon the longest side from the opposite angle, dividing the given triangle into two right-angled triangles. The two segments of the base may be found by Theorem III. There will then be given the hypothenuse and one side of a right-angled triangle to find the angles.

Ex. 1. In the triangle ABC, the side a is 261, the side b, 345, and c, 395. What are the angles?

Let fall the perpendicular CD upon AB.

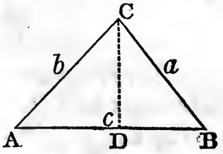
Then, by Theorem III.,

$$AB : AC+CB :: AC-CB : AD-DB;$$

or

$$395 : 606 :: 84 : 128.87.$$

Half the difference of the segments added to half their sum gives the greater segment, and subtracted gives the less segment.



Therefore, AD is 261.935, and BD, 133.065.

Then, in each of the right-angled triangles, ACD, BCD, we have given the hypothenuse and base, to find the angles by Case II. of right-angled triangles. Hence

$$AC : R :: AD : \cos. A=40° 36' 13'';$$
$$BC : R :: BD : \cos. B=59° 20' 52''.$$

Therefore the angle C=80° 2′ 55″.

Ex. 2. If the three sides of a triangle are 150, 140, and 130, what are the angles?

Ans., 67° 22′ 48″, 59° 29′ 23″, and 53° 7′ 49″

Examples for Practice.

1. Given two sides of a triangle, 478 and 567, and the included angle, 47° 30′, to find the remaining parts.

2. Given the angle A, 56° 34′, the opposite side, *a*, 735, and the side *b*, 576, to find the remaining parts.

3. Given the angle A, 65° 40′, the angle B, 74° 20′, and the side *a*, 275, to find the remaining parts.

4. Given the three sides, 742, 657, and 379, to find the angles.

5. Given the angle A, 116° 32′, the opposite side, *a*, 492, and the side *c*, 295, to find the remaining parts.

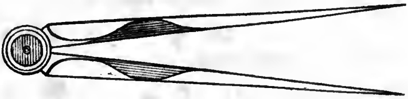
6. Given the angle C, 56° 18′, the opposite side, *c*, 184, and the side *b*, 219, to find the remaining parts.

This problem admits of two answers.

INSTRUMENTS USED IN DRAWING.

(57.) The following are some of the most important instruments used in drawing.

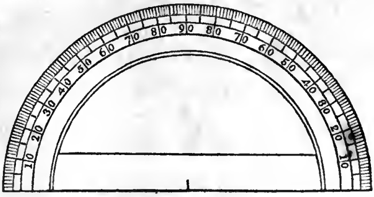
I. The *dividers* consist of two legs, revolving upon a pivot at one extremity. The joints should be composed of two different metals, of unequal hardness: one part, for example, of steel, and the other of brass or silver, in order that they may move upon each other with greater freedom. The points should be of tempered steel, and when the dividers are closed, they should meet with great exactness. The dividers are often furnished with various appendages, which are exceedingly convenient in drawing. Sometimes one of the legs is furnished with an adjusting screw, by which a slow motion may be given to one of the points, in which case they are called *hair compasses*. It is also useful to have a movable leg, which may be removed at pleasure, and other parts fitted to its place; as, for example, a long beam for drawing large circles, a pencil point for drawing circles with a pencil, an ink point for drawing black circles, &c.



(58.) II. The *parallel rule* consists of two flat rules, made of wood or ivory, and connected together by two cross-bars of

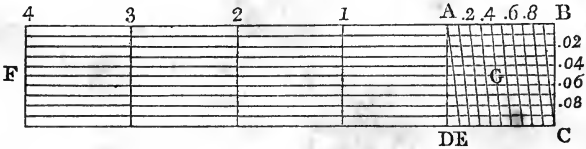
equal length, and parallel to each other. This instrument is useful for drawing a line parallel to a given line, through a given point. For this purpose, place the edge of one of the flat rules against the given line, and move the other rule until its edge coincides with the given point. A line drawn along its edge will be parallel to the given line.

(59.) III. The *protractor* is used to lay down or to measure angles. It consists of a semicircle, usually of brass, and is divided into degrees, and sometimes smaller portions, the center of the circle being indicated by a small notch.



To lay down an angle with the protractor, draw a base line, and apply to it the edge of the protractor, so that its center shall fall at the angular point. Count the degrees contained in the proposed angle on the limb of the circle, and mark the extremity of the arc with a fine dot. Remove the instrument, and through the dot draw a line from the angular point; it will give the angle required. In a similar manner, the inclination of any two lines may be measured with the protractor.

(60.) IV. The *plain scale* is a ruler, frequently two feet in length, containing a line of *equal parts*, *chords*, *sines*, *tangents*, &c. For a scale of equal parts, a line is divided into inches and tenths of an inch, or half inches and twentieths. When smaller fractions are required, they are obtained by means of the *diagonal scale*, which is constructed in the following manner. Describe a square inch, ABCD, and divide

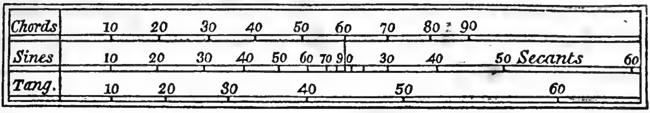


each of its sides into ten equal parts. Draw diagonal lines from the first point of division on the upper line, to the second on the lower; from the second on the upper line, to the third on the lower, and so on. Draw, also, other lines parallel to AB, through the points of division of BC. Then, in the triangle ADE, the base, DE, is one tenth of an inch; and, since the line AD is divided into ten equal parts, and through the points of division lines are drawn parallel to the base, forming nine smaller triangles, the base of the least is one tenth of DE, that is, .01 of an inch; the base of the second is .02 of an inch; the third, .03, and so on. Thus the diagonal scale furnishes us *hundredths* of an inch. To take off from the scale a line of given length, as, for example, 4.45 inches, place one foot of the dividers at F, on the sixth horizontal line, and extend the other foot to G, the fifth diagonal line.

A half inch or less is frequently subdivided in the same manner.

(61.) A *line of chords*, commonly marked CHO., is found on most plane scales, and is useful in setting off angles. To form this line, describe a circle with any convenient radius, and divide the circumference into degrees. Let the length of the chords for every degree of the quadrant be determined and laid off on a scale: this is called a line of chords.

Since the chord of 60° is equal to radius, in order to lay

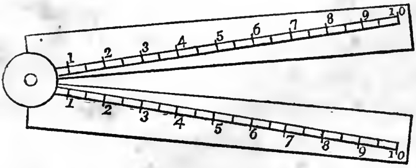


down an angle, we take from the scale the chord of 60°, and with this radius describe an arc of a circle. Then take from the scale the chord of the given angle, and set it off upon the former arc. Through these two points of division draw lines to the center of the circle, and they will contain the required angle.

The line of sines, commonly marked SIN., exhibits the lengths of the sines to every degree of the quadrant, to the same radius as the line of chords. The line of tangents and the line of secants are constructed in the same manner. Since the sine of 90° is equal to radius, and the secant of 0° is the same, the graduation on the line of secants begins where the line of sines ends.

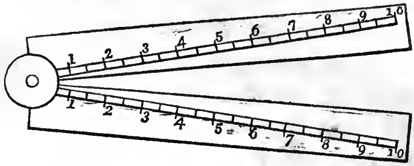
On the back side of the plane scale are often found lines representing the logarithms of numbers, sines, tangents, &c. This is called Gunter's Scale.

(62.) V. The *Sector* is a very convenient instrument in drawing. It consists of two equal arms, movable about a pivot as a center, having several scales drawn on the faces, some single, others double.



The single scales are like those upon a common Gunter's scale. The double scales are those which proceed from the center, each being laid twice on the same face of the instrument, viz., once on each leg. The double scales are a scale of lines, marked Lin. or L.; the scale of chords, sines, &c. On each arm of the sector there is a diagonal line, which diverges from the central point like the radius of a circle, and these diagonal lines are divided into equal parts.

The advantage of the sector is to enable us to draw a line upon paper to any scale; as, for example, a scale of 6 feet to the inch. For this purpose, take an inch with the dividers from the scale of inches; then, placing one foot of the dividers at 6 on one arm of the sector, open the sector until the other foot reaches to the same number on the other arm. Now, regarding the lines on the sector as the sides of a triangle, of which the line measured from 6 on one arm to 6 on the other arm is the base, it is



plain that if any other line be measured across the angle of the sector, the bases of the triangles thus formed will be proportional to their sides. Therefore, a line of 7 feet will be represented by the distance from 7 to 7, and so on for other lines.

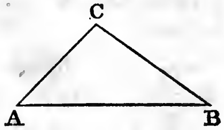
The sector also contains a line of chords, arranged like the line of equal parts already mentioned. Two lines of chords are drawn, one on each arm of the sector, diverging from the central point. This double line of chords is more convenient than the single one upon the plane scale, because it furnishes chords to *any radius.* If it be required to lay down any angle, as, for example, an angle of 25°, describe a circle with any convenient radius. Open the sector so that the distance from 60 to 60, on the line of chords, shall be equal to this radius. Then, preserving the same opening of the sector, place one foot of the dividers upon the division 25 on one scale, and extend the other foot to the same number upon the other scale: this distance will be the chord of 25 degrees, which must be set off upon the circle first described.

The lines of sines, tangents, &c., are arranged in the same manner.

(63.) By means of the instruments now enumerated, all the cases in Plane Trigonometry may be solved mechanically. The sides and angles which are *given* are laid down according to the preceding directions, and the *required* parts are then measured from the same scale. The student will do well to exercise himself upon the following problems:

I. *Given the angles and one side of a triangle*, to find, by construction, the other two sides.

Draw an indefinite straight line, and from the scale of equal parts lay off a portion, AB, equal to the given side. From each extremity lay off an angle equal to one of the adjacent angles, by means of a protractor or a scale of chords. Extend the two lines till they intersect, and measure their lengths upon the same scale of equal parts which was used in laying off the base.



Ex. 1. Given the angle A, 45° 30′, the angle B, 35° 20′, and the side AB, 432 rods, to construct the triangle, and find the lengths of the sides AC and BC.

The triangle ABC may be constructed of any dimensions whatever; all which is essential is that its angles be made equal to the given angles. We may construct the triangle upon a scale of 100 rods to an inch, in which case the side AB will be represented by 4.32 inches; or we may construct it upon a scale of 200 rods to an inch; that is, 100 rods to a half inch, which is very conveniently done from a scale on which a half inch is divided like that described in Art. 60; or we may use any other scale at pleasure. It should, however, be remembered, that the required sides must be measured upon the *same* scale as the given sides.

Ex. 2. Given the angle A, 48°, the angle C, 113°, and the side AC, 795, to construct the triangle.

II. *Given two sides and an angle opposite one of them*, to find the other parts.

Draw the side which is adjacent to the given angle. From one end of it lay off the given angle, and extend a line indefinitely for the required side. From the other extremity of the first side, with the remaining given side for radius, describe an arc cutting the indefinite line. The point of intersection will determine the third angle of the triangle.

Ex. 1. Given the angle A, 74° 45′, the side AC, 432, and the side BC, 475, to construct the triangle, and find the other parts.

Ex. 2. Given the angle A, 105°, the side BC, 498, and the side AC, 375, to construct the triangle.

III. *Given two sides and the included angle*, to find the other parts.

Draw one of the given sides. From one end of it lay off the given angle, and draw the other given side, making the required angle with the first side. Then connect the extremities of the two sides, and there will be formed the triangle required.

Ex. 1. Given the angle A, 37° 25′, the side AC, 675, and the side AB, 417, to construct the triangle, and find the other parts.

Ex. 2. Given the angle A, 75°, the side AC, 543, and the side AB, 721, to construct the triangle.

IV. *Given the three sides*, to find the angles.

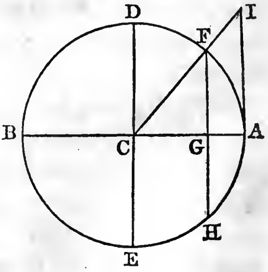
Draw one of the sides as a base; and from one extremity of the base, with a radius equal to the second side, describe an arc of a circle. From the other end of the base, with a radius equal to the third side, describe a second arc intersecting the former; the point of intersection will be the third angle of the triangle.

Ex. 1. Given AB, 678, AC, 598, and BC, 435, to find the angles.

Ex. 2. Given the three sides 476, 287, and 354, to find the angles.

Values of the Sines, Cosines, &c., of certain Angles.

(64.) We propose now to examine the changes which the sines, cosines, &c., undergo in the different quadrants of a circle. Draw two diameters, AB, DE, perpendicular to each other, and suppose one of them to occupy a horizontal position, the other a vertical. The angle ACD is called the *first* quadrant, the angle DCB the *second* quadrant, the angle BCE the *third* quadrant, and the angle ECA the *fourth* quadrant; that is, the first quadrant is above the horizontal diameter, and on the right of the vertical diameter; the second quadrant is above the horizontal diameter, and on the left of the vertical, and so on.



Suppose one extremity of the arc remains fixed in A, while the other extremity, marked F, runs round the entire circumference in the direction ADBE.

When the point F is at A, or when the arc AF is zero, the sine is zero. As the point F advances toward D, the sine increases; and when the arc AF becomes 45°, the triangle CFG being isosceles, we have FG : CF : : 1 : $\sqrt{2}$ (*Geom.*, Prop. 11, Cor. 3, B. IV.); or sin. 45° : R : : 1 : $\sqrt{2}$.

Hence, $$\sin. 45° = \frac{R}{\sqrt{2}} = \tfrac{1}{2}R\sqrt{2}.$$

The sine of 30° is equal to half radius (Art. 22). Also, since sin. A$=\sqrt{R^2-\cos.^2 A}$, the sine of 60°, which is equal to the cosine of 30°, $=\sqrt{R^2-\frac{1}{4}R^2}=\sqrt{\frac{3}{4}R^2}=\frac{1}{2}R\sqrt{3}$.

The arc AF continuing to increase, the sine also increases till F arrives at D, at which point the sine is equal to the radius; that is, the sine of 90°=R.

As the point F advances from D toward B, the sines diminish, and become zero at B; that is, the sine of 180°=0.

In the third quadrant, the sine increases again, becomes equal to radius at E, and is reduced to zero at A.

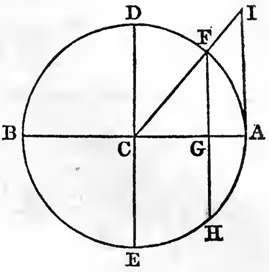
(65.) When the point F is at A, the cosine is equal to radius. As the point F advances toward D, the cosine decreases, and the cosine of 45°=sine 45°$=\frac{1}{2}R\sqrt{2}$. The arc continuing to increase, the cosine diminishes till F arrives at D, at which point the cosine becomes equal to zero. The cosine in the second quadrant increases, and becomes equal to radius at B; in the third quadrant it decreases, and becomes zero at E; in the fourth quadrant it increases again, and becomes equal to radius at A.

(66.) The tangent begins with zero at A, increases with the arc, and at 45° becomes equal to radius. As the point F approaches D, the tangent increases very rapidly; and when the difference between the arc and 90° is less than any assignable quantity, the tangent is greater than any assignable quantity. Hence the tangent of 90° is said to be infinite.

In the second quadrant the tangent is at first infinitely great, and rapidly diminishes till at B it is reduced to zero. In the third quadrant it increases again, becomes infinite at E, and is reduced to zero at A.

The cotangent is equal to zero at D and E, and is infinite at A and B.

(67.) The secant begins with radius at A, increases through the first quadrant, and becomes infinite at D; diminishes in the second quadrant, till at B it is equal to the radius; increases again in the third quadrant, and becomes infinite at E; decreases in the fourth quadrant, and becomes equal to the radius at A.



The cosecant is equal to radius at D and E, and is infinite at A and B.

(68.) Let us now consider the algebraic signs by which these lines are to be distinguished. In the first and second quadrants, the sines fall *above* the diameter AB, while in the third and fourth quadrants they fall *below*. This opposition of directions ought to be distinguished by the algebraic signs; and if one of these directions is regarded as positive, the other ought to be considered as negative. It is generally agreed to consider those sines which fall *above* the horizontal diameter as positive; consequently, those which fall *below* must be regarded as negative. That is, the sines are positive in the first and second quadrants, and negative in the third and fourth.

In the first quadrant the cosine falls on the *right* of DE, but in the second quadrant it falls on the *left*. These two lines should obviously have opposite signs, and it is generally agreed to consider those which fall to the right of the vertical diameter as positive; consequently, those which fall to the left must be considered negative. That is, the cosines are positive in the first and fourth quadrants, and negative in the second and third.

(69.) The signs of the tangents are derived from those of the sines and cosines. For tang. $=\frac{\text{R. sin.}}{\text{cos.}}$ (Art. 28). Hence, when the sine and cosine have like algebraic signs, the tangent will be positive; when unlike, negative. That is, the tangent is positive in the first and third quadrants, and negative in the second and fourth.

Also, cotangent $=\frac{\text{R}^2}{\text{tang.}}$ (Art. 28); hence the tangent and cotangent have always the same sign.

We have seen that sec. $=\frac{R^2}{\cos.}$; hence the secant must have the same sign as the cosine.

Also, cosec. $=\frac{R^2}{\sin.}$; hence the cosecant must have the same sign as the sine.

(70.) The preceding results are exhibited in the following tables, which should be made perfectly familiar:

	First quad.	*Second* quad.	*Third* quad.	*Fourth* quad.
Sine and cosecant,	+	+	−	−
Cosine and secant,	+	−	−	+
Tangent and cotangent,	+	−	+	−

	0°	90°	180°	270°	360°
Sine,	0	+R	0	−R	0
Cosine,	+R	0	−R	0	+R
Tangent,	0	∞	0	∞	0
Cotangent,	∞	0	∞	0	∞
Secant,	+R	∞	−R	∞	+R
Cosecant,	∞	+R	∞	−R	∞

(71.) In Astronomy we frequently have occasion to consider arcs greater than 360°. But if an entire circumference, or any number of circumferences, be added to any arc, it will terminate in the same point as before. Hence, if C represent an entire circumference, or 360°, and A any arc whatever, we shall have

$$\sin. A=\sin. (C+A)=\sin. (2C+A)=\sin. (3C+A)=, \&c.$$

The same is true of the cosine, tangent, &c.

We generally consider those arcs as positive which are estimated from A in the direction ADBE. If, then, an arc were estimated in the direction AEBD, it should be considered as negative; that is, if the arc AF be considered positive, AH must be considered negative. But the latter belongs to the fourth quadrant; hence its sine is negative. Therefore, $\sin. (-A)=-\sin. A$.

The cosine CG is the same for both the arcs AF and AH.

Hence, $\cos. (-A)=\cos. A.$

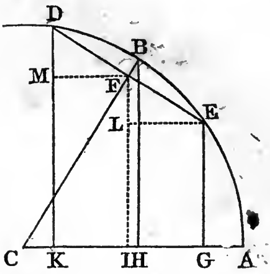
Also, $\text{tang.} (-A)=-\text{tang.} A.$

And $\cot. (-A)=-\cot. A.$

TRIGONOMETRICAL FORMULÆ.

(72.) *Expressions for the sine and cosine of the sum and difference of two arcs.*

Let AB and BD represent any two given arcs; take BE equal to BD: it is required to find an expression for the sine of AD, the sum, and of AE, the difference of these arcs.



Put AB$=a$, and BD$=b$; then AD$=a+b$, and AE$=a-b$. Draw the chord DE, and the radius CB, which may be represented by R. Since DB is by construction equal to BE, DF is equal to FE, and therefore DE is perpendicular to CB. Let fall the perpendiculars EG, BH, FI, and DK upon AC, and draw EL, FM parallel to AC.

Because the triangles BCH, FCI are similar, we have

$$\text{CB} : \text{CF} :: \text{BH} : \text{FI}; \text{ or } \text{R} : \cos. b :: \sin. a : \text{FI}.$$

Therefore, $$\text{FI}=\frac{\sin. a \cos. b}{\text{R}}.$$

Also, $\text{CB} : \text{CF} :: \text{CH} : \text{CI}$; or $\text{R} : \cos. b :: \cos. a : \text{CI}$.

Therefore, $$\text{CI}=\frac{\cos. a \cos. b}{\text{R}}.$$

The triangles DFM, CBH, having their sides perpendicular each to each, are similar, and give the proportions

$$\text{CB} : \text{DF} :: \text{CH} : \text{DM}; \text{ or } \text{R} : \sin. b :: \cos. a : \text{DM}.$$

Hence $$\text{DM}=\frac{\cos. a \sin. b}{\text{R}}.$$

Also, $\text{CB} : \text{DF} :: \text{BH} : \text{FM}$; or $\text{R} : \sin. b :: \sin. a : \text{FM}$.

Hence $$\text{FM}=\frac{\sin. a \sin. b}{\text{R}}.$$

But $\text{FI}+\text{DM}=\text{DK}=\sin. (a+b)$;

and $\text{CI}-\text{FM}=\text{CK}=\cos. (a+b)$.

Also, $\text{FI}-\text{FL}=\text{EG}=\sin. (a-b)$;

and $\text{CI}+\text{EL}=\text{CG}=\cos. (a-b)$.

Hence, $$\sin. (a+b)=\frac{\sin. a \cos. b+\sin. b \cos. a}{\text{R}} \quad (1)$$

$$\cos. (a+b)=\frac{\cos. a \cos. b-\sin. a \sin. b}{\text{R}} \quad (2)$$

$$\sin.(a-b)=\frac{\sin. a \cos. b-\sin. b \cos. a}{R} \quad (3)$$

$$\cos.(a-b)=\frac{\cos. a \cos. b+\sin. a \sin. b}{R} \quad (4)$$

(73.) *Expressions for the sine and cosine of a double arc.*

If, in the formulas of the preceding article, we make $b=a$, the first and second will become

$$\sin. 2a=\frac{2 \sin. a \cos. a}{R},$$

$$\cos. 2a=\frac{\cos.^2 a-\sin.^2 a}{R}.$$

Making radius equal to unity, and substituting the values of sin. a, cos. a. &c., from Art. 28, we obtain

$$\sin. 2a=\frac{2 \text{ tang. } a}{1+\text{tang.}^2 a},$$

$$\cos. 2a=\frac{1-\text{tang.}^2 a}{1+\text{tang.}^2 a}.$$

(74.) *Expressions for the sine and cosine of half a given arc.*

If we put $\frac{1}{2}a$ for a in the preceding equations, we obtain

$$\sin. a=\frac{2 \sin. \frac{1}{2}a \cos. \frac{1}{2}a}{R},$$

$$\cos. a=\frac{\cos.^2 \frac{1}{2}a-\sin.^2 \frac{1}{2}a}{R}.$$

We may also find the sine and cosine of $\frac{1}{2}a$ in terms of a.

Since the sum of the squares of the sine and cosine is equal to the square of radius, we have

$$\cos.^2 \tfrac{1}{2}a+\sin.^2 \tfrac{1}{2}a=R^2.$$

And, from the preceding equation,

$$\cos.^2 \tfrac{1}{2}a-\sin.^2 \tfrac{1}{2}a=R \cos. a.$$

If we subtract one of these from the other, we have

$$2 \sin.^2 \tfrac{1}{2}a=R^2-R \cos. a.$$

And, adding the same equations,

$$2 \cos.^2 \tfrac{1}{2}a=R^2+R \cos. a.$$

Hence,

$$\sin. \tfrac{1}{2}a=\sqrt{\tfrac{1}{2}R^2-\tfrac{1}{2}R \cos. a};$$

$$\cos. \tfrac{1}{2}a=\sqrt{\tfrac{1}{2}R^2+\tfrac{1}{2}R \cos. a}.$$

(75.) *Expressions for the products of sines and cosines.*

By adding and subtracting the formulas of Art. 72, we obtain

$$\sin.\ (a+b)+\sin.\ (a-b)=\frac{2}{R}\sin.\ a\cos.\ b.$$

$$\sin.\ (a+b)-\sin.\ (a-b)=\frac{2}{R}\sin.\ b\cos.\ a;$$

$$\cos.\ (a+b)+\cos.\ (a-b)=\frac{2}{R}\cos.\ a\cos.\ b;$$

$$\cos.\ (a-b)-\cos.\ (a+b)=\frac{2}{R}\sin.\ a\sin.\ b.$$

If, in these formulas, we make $a+b=A$, and $a-b=B$; that is, $a=\frac{1}{2}(A+B)$, and $b=\frac{1}{2}(A-B)$, we shall have

$$\sin.\ A+\sin.\ B=\frac{2}{R}\sin.\ \tfrac{1}{2}(A+B)\cos.\ \tfrac{1}{2}(A-B) \quad (1)$$

$$\sin.\ A-\sin.\ B=\frac{2}{R}\sin.\ \tfrac{1}{2}(A-B)\cos.\ \tfrac{1}{2}(A+B) \quad (2)$$

$$\cos.\ A+\cos.\ B=\frac{2}{R}\cos.\ \tfrac{1}{2}(A+B)\cos.\ \tfrac{1}{2}(A-B) \quad (3)$$

$$\cos.\ B-\cos.\ A=\frac{2}{R}\sin.\ \tfrac{1}{2}(A+B)\sin.\ \tfrac{1}{2}(A-B) \quad (4)$$

(76.) Dividing formula (1) by (2), and considering that $\frac{\sin.\ a}{\cos.\ a}=\frac{\text{tang.}\ a}{R}$ (Art. 28), we have

$$\frac{\sin.\ A+\sin.\ B}{\sin.\ A-\sin.\ B}=\frac{\sin.\ \frac{1}{2}(A+B)\cos.\ \frac{1}{2}(A-B)}{\sin.\ \frac{1}{2}(A-B)\cos.\ \frac{1}{2}(A+B)}=\frac{\text{tang.}\ \frac{1}{2}(A+B)}{\text{tang.}\ \frac{1}{2}(A-B)};$$

that is,

The sum of the sines of two arcs is to their difference, as the tangent of half the sum of those arcs is to the tangent of half their difference.

Dividing formula (3) by (4), and considering that $\frac{\cos.}{\sin.}=\frac{\cot.}{R}=\frac{R}{\text{tang.}}$ (Art. 28), we have

$$\frac{\cos.\ A+\cos.\ B}{\cos.\ B-\cos.\ A}=\frac{\cos.\ \frac{1}{2}(A+B)\cos.\ \frac{1}{2}(A-B)}{\sin.\ \frac{1}{2}(A+B)\sin.\ \frac{1}{2}(A-B)}=\frac{\cot.\ \frac{1}{2}(A+B)}{\text{tang.}\ \frac{1}{2}(A-B)};$$

that is,

The sum of the cosines of two arcs is to their difference, as the cotangent of half the sum of those arcs is to the tangent of half their difference.

From the first formula of Art. 74, by substituting A+B for a, we have

$$\sin.(A+B)=\frac{2\sin.\frac{1}{2}(A+B)\times\cos.\frac{1}{2}(A+B)}{R}.$$

Dividing formula (1), Art. 75, by this, we obtain

$$\frac{\sin.A+\sin.B}{\sin.(A+B)}=\frac{\sin.\frac{1}{2}(A+B)\cos.\frac{1}{2}(A-B)}{\sin.\frac{1}{2}(A+B)\cos.\frac{1}{2}(A+B)}=\frac{\cos.\frac{1}{2}(A-B)}{\cos.\frac{1}{2}(A+B)};$$

that is,

The sum of the sines of two arcs is to the sine of their sum, as the cosine of half the difference of those arcs is to the cosine of half their sum.

If we divide equation (1), Art. 72, by equation (3), we shall have

$$\frac{\sin.(a+b)}{\sin.(a-b)}=\frac{\sin.a\cos.b+\sin.b\cos.a}{\sin.a\cos.b-\sin.b\cos.a}.$$

By dividing both numerator and denominator of the second member by cos. a cos. b, and substituting $\frac{\text{tang.}}{R}$ for $\frac{\sin.}{\cos.}$, we obtain $$\frac{\sin.(a+b)}{\sin.(a-b)}=\frac{\text{tang.}a+\text{tang.}b}{\text{tang.}a-\text{tang.}b};$$ that is,

The sine of the sum of two arcs is to the sine of their difference, as the sum of the tangents of those arcs is to the difference of the tangents.

From equation (3), Art. 72, by dividing each member by cos. a cos. b, we obtain

$$\frac{\sin.(a-b)}{\cos.a\cos.b}=\frac{\sin.a\cos.b-\sin.b\cos.a}{R\cos.a\cos.b}=\frac{\text{tang.}a-\text{tang.}b}{R^2};$$

that is,

The sine of the difference of two arcs is to the product of their cosines, as the difference of their tangents is to the square of radius.

(77.) *Expressions for the tangents of arcs.*

If we take the expression $\text{tang.}(a+b)=\frac{R\sin.(a+b)}{\cos.(a+b)}$ (Art. 28), and substitute for sin. $(a+b)$ and cos. $(a+b)$ their values given in Art. 72, we shall find

$$\text{tang.}(a+b)=\frac{R(\sin.a\cos.b+\sin.b\cos.a)}{\cos.a\cos.b-\sin.a\sin.b}.$$

But $\sin. a = \frac{\cos. a \tan g. a}{R}$, and $\sin. b = \frac{\cos. b \tan g. b}{R}$ (Art. 28).

If we substitute these values in the preceding equation, and divide all the terms by $\cos. a \cos. b$, we shall have

$$\text{tang.}(a+b) = \frac{R^2(\text{tang.}\, a + \text{tang.}\, b)}{R^2 - \text{tang.}\, a\ \text{tang.}\, b}.$$

In like manner we shall find

$$\text{tang.}(a-b) = \frac{R^2(\text{tang.}\, a - \text{tang.}\, b)}{R^2 + \text{tang.}\, a\ \text{tang.}\, b}.$$

Suppose $b=a$, then

$$\text{tang.}\, 2a = \frac{2R^2\ \text{tang.}\, a}{R^2 - \text{tang.}^2 a}.$$

Suppose $b=2a$, then

$$\text{tang.}\, 3a = \frac{R^2(\text{tang.}\, a + \text{tang.}\, 2a)}{R^2 - \text{tang.}\, a\ \text{tang.}\, 2a}.$$

In the same manner we find

$$\cot.(a+b) = \frac{\cot. a \cot. b - R^2}{\cot. b + \cot. a},$$

$$\cot.(a-b) = \frac{\cot. a \cot. b + R^2}{\cot. b - \cot. a}.$$

(78.) When the three sides of a triangle are given, the angles may be found by the formula

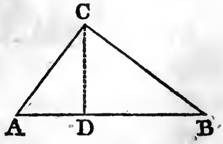
$$\sin. \tfrac{1}{2}A = R\sqrt{\frac{(S-b)(S-c)}{bc}},$$

where S represents half the sum of the sides a, b, and c.

Demonstration.

Let ABC be any triangle; then (*Geom.*, Prop. 12, B. IV.),

$$BC^2 = AB^2 + AC^2 - 2AB \times AD.$$



Hence, $AD = \frac{AB^2 + AC^2 - BC^2}{2AB}$.

But in the right-angled triangle ACD, we have

$$R : AC :: \cos. A : AD.$$

Hence, $\cos. A = \frac{R \times AD}{AC}$;

or, by substituting the value of AD,

$$\cos. A = R \times \frac{AB^2 + AC^2 - BC^2}{2AB \times AC}.$$

Let a, b, c represent the sides opposite the angles A, B, C; then,
$$\cos. A = R \times \frac{b^2+c^2-a^2}{2bc}.$$

By Art. 74, we have $2 \sin.^2 \frac{1}{2}A = R^2 - R \cos. A$.

Substituting for cos. A its value given above, we obtain
$$2 \sin.^2 \tfrac{1}{2}A = R^2 - R^2 \times \frac{b^2+c^2-a^2}{2bc} = R^2 \times \frac{2bc+a^2-b^2-c^2}{2bc},$$
$$= \frac{R^2 \times (a+b-c)(a+c-b)}{2bc}.$$

Put $S=\frac{1}{2}(a+b+c)$, and we obtain, after reduction,
$$\sin. \tfrac{1}{2}A = R\sqrt{\frac{(S-b)(S-c)}{bc}}.$$

In the same manner we find
$$\sin. \tfrac{1}{2}B = R\sqrt{\frac{(S-a)(S-c)}{ac}}.$$
$$\sin. \tfrac{1}{2}C = R\sqrt{\frac{(S-a)(S-b)}{ab}}.$$

Ex. 1. What are the angles of a plane triangle whose sides are 432, 543, and 654?

Here S=814.5; $S-b$=382.5; $S-c$=271.5.

log. 382.5		2.582631
log. 271.5		2.433770
log. b, 432	comp.	7.364516
log. c, 543	comp.	7.265200
	2)	19.646117
sin. $\frac{1}{2}$A, 41° 42′ 36½″.		9.823058.
Angle A=83° 25′ 13″.		

In a similar manner we find the angle B=41° 0′ 39″, and the angle C=55° 34′ 8″.

Ex. 3. What are the angles of a plane triangle whose sides are 245, 219, and 91?

(79.) *On the computation of a table of sines, cosines, &c.*

In computing a table of sines and cosines, we begin with finding the sine and cosine of *one minute*, and thence deduce the sines and cosines of larger arcs. The sine of so small an angle as one minute is nearly equal to the corresponding arc. The radius being taken as unity, the semicircumference is known to be 3.14159. This being divided successively by 180 and 60, gives .0002908882 for the arc of one minute, which may be regarded as the sine of one minute.

$$\text{The cosine of } 1' = \sqrt{1-\sin.^2} = 0.9999999577.$$

The sines of very small angles are nearly proportional to the angles themselves. We might then obtain several other sines by direct proportion. This method will give the sines correct to five decimal places, as far as two degrees. By the following method they may be obtained with greater accuracy for the entire quadrant.

By Art. 75, we have, by transposition,

$$\sin.\,(a+b)=2\sin.\,a\cos.\,b-\sin.\,(a-b),$$
$$\cos.\,(a+b)=2\cos.\,a\cos.\,b-\cos.\,(a-b).$$

If we make $a=b$, $2b$, $3b$, &c., successively, we shall have

$$\sin.\,2b=2\sin.\,b\cos.\,b;$$
$$\sin.\,3b=2\sin.\,2b\cos.\,b-\sin.\,b,$$
$$\sin.\,4b=2\sin.\,3b\cos.\,b-\sin.\,2b,$$
&c., &c.

$$\cos.\,2b=2\cos.\,b\cos.\,b-1;$$
$$\cos.\,3b=2\cos.\,2b\cos.\,b-\cos.\,b;$$
$$\cos.\,4b=2\cos.\,3b\cos.\,b-\cos.\,2b,$$
&c., &c.

Whence, making $b=1'$, we have

$$\sin.\,2'=2\sin.\,1'\cos.\,1' \qquad =.000582;$$
$$\sin.\,3'=2\sin.\,2'\cos.\,1'-\sin.\,1'=.000873;$$
$$\sin.\,4'=2\sin.\,3'\cos.\,1'-\sin.\,2'=.001164,$$
&c., &c.

$$\cos.\,2'=2\cos.\,1'\cos.\,1'- \quad 1=0.999999;$$
$$\cos.\,3'=2\cos.\,2'\cos.\,1'-\cos.\,1'=0.999999;$$
$$\cos.\,4'=2\cos.\,3'\cos.\,1'-\cos.\,2'=0.999999,$$
&c., &c.

The tangents, cotangents, secants, and cosecants are easily derived from the sines and cosines. Thus,

$$\text{tang. } 1'=\frac{\sin.\,1'}{\cos.\,1'}; \qquad \cot.\,1'=\frac{\cos.\,1'}{\sin.\,1'};$$

$$\sec.\,1'=\frac{1}{\cos.\,1'}; \qquad \text{cosec. } 1'=\frac{1}{\sin.\,1'};$$

&c., &c.

BOOK III.

MENSURATION OF SURFACES.

(80.) The *area* of a figure is the space contained within the line or lines by which it is bounded. This area is determined by finding how many times the figure contains some other surface, which is assumed as the *unit of measure.* This unit is commonly a *square;* such as a square inch, a square foot, a square rod, &c.

The *superficial* unit has generally the same name as the *linear* unit, which forms the side of the square. Thus,

the side of a square inch is a linear inch;
" " of a square foot is a linear foot;
" " of a square yard is a linear yard, &c.

There are some superficial units which have no corresponding linear units of the same name, as, for example, an acre.

The following table contains the square measures in common use:

Table of Square Measures.

Sq. Inches.	*Sq. Feet.*	*Sq. Yards.*	*Sq. Rods.*	*S. Ch's.*	*Acres.*	*M.*
144=	1					
1296=	9 =	1				
39204=	$272\frac{1}{4}$=	$30\frac{1}{4}$=	1			
627264=	4356 =	484 =	16=	1		
6272640=	43560 =	4840 =	160=	10=	1	
4014489600=	27878400 =	3097600 =	102400=	6400=	640=	1

Problem I.

(81.) *To find the area of a parallelogram.*

Rule I.

Multiply the base by the altitude.

For the demonstration of this rule, see Geometry, Prop. 5, B. IV.

Ex. 1. What is the area of a parallelogram whose base is 17.5 rods, and the altitude 13 rods?

Ans., 227.5 square rods.

Ex. 2. What is the area of a square whose side is 315.7 feet? *Ans.*, 99666.49 square feet.

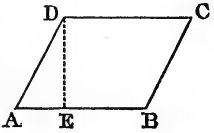
Ex. 3. What is the area of a rectangular board whose length is 15.25 feet, and breadth 15 inches?

Ans., 19.0625 square feet.

Ex. 4. How many square yards are there in the four sides of a room which is 18 feet long, 15 feet broad, and 9 feet high?

Ans., 66 square yards.

(82.) If the sides and angles of a parallelogram are given, the perpendicular height may be found by Trigonometry. For DE is one side of a right-angled triangle, of which AD is the hypothenuse. Hence,



$$R : AD :: \sin. A : DE;$$

from which $$DE = \frac{AD \times \sin. A}{R}.$$

Therefore, *the area* $= AB \times DE = \dfrac{AB \times AD \times \sin. A}{R}$.

Hence we derive

RULE II.

Multiply together two adjacent sides, and the sine of the included angle.

Ex. 1. What is the area of a parallelogram having an angle of 58°, and the including sides 36 and 25.5 feet?

Ans. The area$=36 \times 25.5 \times .84805$ (natural sine of 58°)$=$ 778.508 square feet.

The computation will generally be most conveniently performed by logarithms.

Ex. 2. What is the area of a rhombus, each of whose sides is 21 feet 3 inches, and each of the acute angles 53° 20′?

Ans., 362.209 feet.

Ex. 3. How many acres are contained in a parallelogram one of whose angles is 30°, and the including sides are 25.35 and 10.4 chains? *Ans.*, 13 acres, 29.12 rods

PROBLEM II.

(83.) *To find the area of a triangle.*

RULE I.

Multiply the base by half the altitude.

For demonstration, see Geometry, Prop. 6, B. IV.

Ex. 1. How many square yards are contained in a triangle whose base is 49 feet, and altitude $25\frac{1}{4}$ feet?

Ans., 68.736.

Ex. 2. What is the area of a triangle whose base is 45 feet, and altitude 27.5 feet? *Ans.*, 618.75 square feet.

(84.) When two sides and the included angle are given, we may use

RULE II.

Multiply half the product of two sides by the sine of the included angle.

The reason of this rule is obvious, from Art. 82, since a triangle is half of a parallelogram, having the same base and altitude.

Ex. 1. What is the area of a triangle of which two sides are 45 and 32 feet, and the included angle 46° 30′?

Ans. The area$=45\times16\times.725374$ (natural sine of 46° 30′)$=$ 522.269 feet.

Ex. 2. What is the area of a triangle of which two sides are 127 and 96 feet, and the included angle 67° 15′?

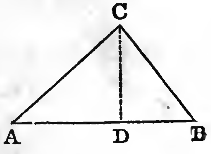
Ans.

(85.) When the three sides are known, we may use

RULE III.

From half the sum of the three sides subtract each side severally; multiply together the half sum and the three remainders, and extract the square root of the product.

Demonstration.



Let a, b, c denote the sides of the triangle ABC; then, by Geometry, Prop. 12, B. IV., we have $BC^2=AB^2+AC^2-2AB\times AD$, or $a^2=b^2+c^2-2c\times AD$; whence,

$$AD=\frac{b^2+c^2-a^2}{2c}.$$

But $$CD^2=AC^2-AD^2;$$

hence $$CD^2=b^2-\frac{(b^2+c^2-a^2)^2}{4c^2}=\frac{4b^2c^2-(b^2+c^2-a^2)^2}{4c^2},$$

or $$CD=\frac{\sqrt{4b^2c^2-(b^2+c^2-a^2)^2}}{2c}.$$

But *the area*$=\frac{AB\times CD}{2}=\frac{1}{4}\sqrt{4b^2c^2-(b^2+c^2-a^2)^2}.$

The quantity under the radical sign being the difference of two squares, may be resolved into the factors $2bc+(b^2+c^2-a^2)$ and $2bc-(b^2+c^2-a^2)$; and these, in the same manner, may be resolved into $(b+c+a)\times(b+c-a)$,
and $(a+b-c)\times(a-b+c)$.

Hence, if we put S equal to $\frac{a+b+c}{2}$, we shall have

the *area*$=\sqrt{S(S-a)(S-b)(S-c)}$.

Ex. 1. What is the area of a triangle whose sides are 125, 173, and 216 feet?

Here $S=257$, $S-b=84$,
$S-a=132$, $S-c=41$.

Hence the *area*$=\sqrt{257\times132\times84\times41}=10809$ square feet.

Ex. 2. How many acres are contained in a triangle whose sides are 49, 50.25, and 25.69 chains?

Ans., 61 acres, 1 rood, 39.68 perches.

Ex. 3. What is the area of a triangle whose sides are 234, 289, and 345 feet?

Ans.

(86.) In an equilateral triangle, one of whose sides is a, the expression for the area becomes

$$\sqrt{\tfrac{3}{2}a\times\tfrac{1}{2}a\times\tfrac{1}{2}a\times\tfrac{1}{2}a}$$
$$=\frac{a^2\sqrt{3}}{4};$$

that is, the area of an equilateral triangle is equal to one fourth the square of one of its sides multiplied by the square root of 3.

Ex. What is the area of a triangle whose sides are each 37 feet? *Ans.*, 592.79 feet.

PROBLEM III.

(87.) *To find the area of a trapezoid.*

RULE.

Multiply half the sum of the parallel sides into their perpendicular distance.

For demonstration, see Geometry, Prop. 7, B. IV.

Ex. 1. What is the area of a trapezoid whose parallel sides are 156 and 124, and the perpendicular distance between them 57 feet?

Ans., 7980 feet.

Ex. 2. How many square yards in a trapezoid whose parallel sides are 678 and 987 feet, and altitude 524 feet?

Ans.

PROBLEM IV.

(88.) *To find the area of an irregular polygon.*

RULE.

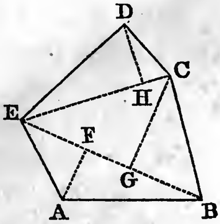
Draw diagonals dividing the polygon into triangles, and find the sum of the areas of these triangles.

Ex. 1. What is the area of a quadrilateral, one of whose diagonals is 126 feet, and the two perpendiculars let fall upon it from the opposite angles are 74 and 28 feet?

Ans., 6426 feet.

Ex. 2. In the polygon ABCDE, there are given EC=205, EB=242, AF=65, CG=114, and DH=110, to find the area.

Ans.



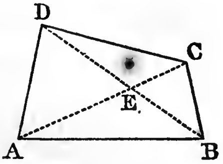
(89.) If the diagonals of a quadrilateral are given, the area may be found by the following

RULE.

Multiply half the product of the diagonals by the sine of the angle at their intersection.

Demonstration.

The sines of the four angles at E are all equal to each other, since the adjacent angles AED, DEC are the supplements of each other (Art. 27). But, according to the Rule, Art. 84, the area of



the triangle ABE=$\frac{1}{2}$AE×BE×sine E;
" " AED=$\frac{1}{2}$AE×DE×sine E;
" " BEC=$\frac{1}{2}$BE×EC×sine E;
" " DEC=$\frac{1}{2}$DE×EC×sine E.

Therefore,

the area of ABCD=$\frac{1}{2}$(AE+EC)×(BE+ED)×sine E
=$\frac{1}{2}$AC×BD×sine E.

Ex. 1. If the diagonals of a quadrilateral are 34 and 56 rods, and if they intersect at an angle of 67°, what is the area?

Ans., 876.32.

Ex. 2. If the diagonals of a quadrilateral are 75 and 49, and the angle of intersection is 42°, what is the area?

Ans.

PROBLEM V.

(90.) *To find the area of a regular polygon.*

RULE I.

Multiply half the perimeter by the perpendicular let fall from the center on one of the sides.

For demonstration, see Geometry, Prop. 7, B. VI.

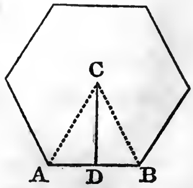
Ex. 1. What is the area of a regular pentagon whose side is 25, and the perpendicular from the center 17.205 feet?

Ans., 1075.31 feet.

Ex. 2. What is the area of a regular octagon whose side is 53, and the perpendicular 63.977?

Ans.

(91.) When the perpendicular is not given, it may be computed from the perimeter and number of sides. If we divide 360 degrees by the number of sides of the polygon, the quotient will be the angle ACB at the center, subtended by one of the sides. The perpendicular CD bisects the side AB, and the angle ACB. Then, in the triangle ACD, we have (Art. 42),



R : AD : cot. ACD : CD; that is,

Radius is to half of one of the sides of the polygon, as the cotangent of the opposite angle is to the perpendicular from the center.

Ex. 3. Find the area of a regular hexagon whose side is 32 inches.

The angle ACD is $\frac{1}{12}$ of $360° = 30°$. Then

$$R : 16 :: \cot. 30° : 27.7128 = CD, \text{ the perpendicular;}$$
$$\text{and the area} = 27.7128 \times 16 \times 6 = 2660.4288.$$

Ex. 4. Find the area of a regular decagon whose side is 46 feet. *Ans.*, 16280.946.

(92.) In this manner was computed the following table of the areas of regular polygons, in which the side of each polygon is supposed to be a unit.

TABLE OF REGULAR POLYGONS.

Names.	*Sides.*	*Areas.*
Triangle,	3	0.4330127.
Square,	4	1.0000000.
Pentagon,	5	1.7204774.
Hexagon,	6	2.5980762.
Heptagon,	7	3.6339124.
Octagon,	8	4.8284271.
Nonagon,	9	6.1818242.
Decagon,	10	7.6942088.
Undecagon,	11	9.3656399.
Dodecagon,	12	11.1961524.

By the aid of this table may be computed the area of any other regular polygon having not more than twelve sides. For, since the areas of similar polygons are as the *squares* of their homologous sides, we derive

RULE II.

Multiply the square of one of the sides of the polygon by the area of a similar polygon whose side is unity.

Ex. 5. What is the area of a regular nonagon whose side is 63? *Ans.*, 24535.66.

Ex. 6. What is the area of a regular dodecagon whose side is 54 feet? *Ans.*, 32647.98 feet.

PROBLEM VI.

(93.) *To find the circumference of a circle from its diameter.*

RULE.

Multiply the diameter by 3.14159.

For the demonstration of this rule, see Geometry, Prop. 13, Cor. 2, B. VI.

When the diameter of the circle is small, and no great accuracy is required, it may be sufficient to employ the value of π to only 4 or 5 decimal places. But if the diameter is large, and accuracy is required, it will be necessary to employ a corresponding number of decimal places of π. The value of π to ten decimal places is 3.14159,26536,

and *its logarithm is* 0.497150.

Ex. 1. What is the circumference of a circle whose diameter is 125 feet?

Ans., 392.7 feet.

Ex. 2. If the diameter of the earth is 7912 miles, what is its circumference?

Ans., 24856.28 miles.

Ex. 3. If the diameter of the earth's orbit is 189,761,000 miles, what is its circumference?

Ans., 596,151,764 miles.

To obtain this answer, the value of π must be taken to at least eight decimal places.

PROBLEM VII.

(94.) *To find the diameter of a circle from its circumference.*

RULE I.

Divide the circumference by 3.14159.

This rule is an obvious consequence from the preceding. To divide by a number is the same as to multiply by its *reciprocal;* and, since multiplication is more easily performed than division, it is generally most convenient to multiply by the reciprocal of π, which is 0.3183099. Hence we have

RULE II.

Multiply the circumference by 0.31831.

Ex. 1. What is the diameter of a circle whose circumference is 875 feet?

Ans., 278.52 feet.

Ex. 2. If the circumference of the moon is 6786 miles, what is its diameter?

Ans., 2160 miles.

Ex. 3. If the circumference of the moon's orbit is 1,492,987 miles, what is its diameter?

Ans., 475,233 miles.

PROBLEM VIII.

(95.) *To find the length of an arc of a circle.*

RULE I.

As 360 *is to the number of degrees in the arc, so is the circumference of the circle to the length of the arc.*

This rule follows from Prop. 14, B. III., in Geometry, where it is proved that angles at the center of a circle have the same ratio with the intercepted arcs.

Ex. 1. What is the length of an arc of 22°, in a circle whose diameter is 125 feet?

The circumference of the circle is found to be 392.7 feet.

Then 360 : 22 : : 392.7 : 23.998 feet.

Ex. 2. If the circumference of the earth is 24,856.28 miles, what is the length of one degree?

Ans., 69.045 miles.

RULE II.

(96.) *Multiply the diameter of the circle by the number of degrees in the arc, and this product by* 0.0087266.

Since the circumference of a circle whose diameter is unity is 3.14159, if we divide this number by 360, we shall obtain the length of an arc of *one* degree, viz., 0.0087266. If we multiply this decimal by the number of degrees in any arc, we shall obtain the length of that arc in a circle whose diameter is unity; and this product, multiplied by the diameter of any other circle, will give the length of an arc of the given number of degrees in that circle.

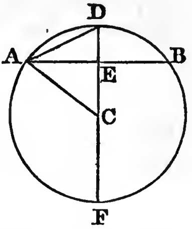
Ex. 3. What is the length of an arc of 25°, in a circle whose radius is 44 rods?

Ans., 19.198 rods.

Ex. 4. What is the length of an arc of 11° 15′, in a circle whose diameter is 1234 feet?

Ans., 121.147 feet.

(97.) If the number of degrees in an arc is not given, it may be computed from the radius of the circle, and either the chord or height of the arc. Thus, let AB be the chord, and DE the height of the arc ADB, and C the center of the circle. Then, in the right-angled triangle ACE,



$$AC : R :: \begin{cases} AE : \text{sin. } ACE, \\ CE : \text{cos. } ACE, \end{cases}$$

either of which proportions will give the number of degrees in half the arc.

If only the chord and height of the arc are given, the diameter of the circle may be found. For, by Geometry, Prop. 22, Cor., B. IV.,

$$DE : AE :: AE : EF.$$

Ex. 5. What is the length of an arc whose chord is 6 feet, in a circle whose radius is 9 feet?

Ans., 6.116 feet.

PROBLEM IX.

(98.) *To find the area of a circle.*

RULE I.

Multiply the circumference by half the radius.

For demonstration, see Geometry, Prop. 12, B. VI.

RULE II.

Multiply the square of the radius by 3.14159.

See Geometry, Prop 13, Cor. 3, B. VI.

Ex. 1. What is the area of a circle whose diameter is 18 feet?

Ans., 254.469 feet.

Ex. 2. What is the area of a circle whose circumference is 74 feet?

Ans., 435.766 feet.

Ex. 3. What is the area of a circle whose radius is 125 yards?

Ans., 49087.38 yards.

PROBLEM X.

(99.) *To find the area of a sector of a circle.*

RULE I.

Multiply the arc of the sector by half its radius.

See Geometry, Prop. 12, Cor., B. VI.

RULE II.

As 360 *is to the number of degrees in the arc, so is the area of the circle to the area of the sector.*

This follows from Geometry, Prop. 14, Cor. 2, B. III.

Ex. 1. What is the area of a sector whose arc is 22°, in a circle whose diameter is 125 feet?

The length of the arc is found to be 23.998.

Hence the area of the sector is 749.937.

Ex. 2. What is the area of a sector whose arc is 25°, in a circle whose radius is 44 rods?

Ans., 422.367 rods.

Ex. 3. What is the area of a sector less than a semicircle, whose chord is 6 feet, in a circle whose radius is 9 feet?

Ans., 27.522 feet.

PROBLEM XI.

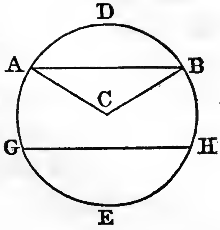
(100.) *To find the area of a segment of a circle.*

RULE.

Find the area of the sector which has the same arc, and also the area of the triangle formed by the chord of the segment and the radii of the sector.

Then take the sum of these areas if the segment is greater than a semicircle, but take their difference if it is less.

It is obvious that the segment AEB is equal to the sum of the sector ACBE and the triangle ACB, and that the segment ADB is equal to the difference between the sector ACBD and the triangle ACB.



Ex. 1. What is the area of a segment whose arc contains 280°, in a circle whose diameter is 50?

The whole circle	=	1963.495
The sector	=	1527.163
The triangle	=	307.752
The segment	=	1834.915, *Ans.*

Ex. 2. What is the area of a segment whose chord is 20 feet, and height 2 feet?

Ans., 26.8788 feet.

Ex. 3. What is the area of a segment whose arc is 25°, in a circle whose radius is 44 rods?

Ans.

(101.) The area of the zone ABHG, included between two parallel chords, is equal to the difference between the segments GDH and ADB.

Ex. 4. What is the area of a zone, one side of which is 96, and the other side 60, and the distance between them 26?

Ans., 2136.7527.

The radius of the circle in this example will be found to be 50.

PROBLEM XII.

(102.) *To find the area of a ring included between the circumferences of two concentric circles.*

RULE.

Take the difference between the areas of the two circles; or, *Subtract the square of the less radius from the square of the greater, and multiply their difference by* 3.14159.

For, according to Geometry, Prop. 13, Cor. 3, B. VI.,

the area of the greater circle is equal to πR^2,

and the area of the smaller, πr^2.

Their difference, or the area of the ring, is $\pi (R^2 - r^2)$.

Ex. 1. The diameters of two concentric circles are 60 and 50. What is the area of the ring included between their circumferences?

Ans., 863.938.

Ex. 2. The diameters of two concentric circles are 320 and 280. What is the area of the ring included between their circumferences?

Ans., 18849.55.

PROBLEM XIII.

(103.) *To find the area of an ellipse.*

RULE.

Multiply the product of the semi-axes by 3.14159.

For demonstration, see Geometry, Ellipse, Prop. 21.

Ex. 1. What is the area of an ellipse whose major axis is 70 feet, and minor axis 60 feet?

Ans., 3298.67 feet.

Ex. 2. What is the area of an ellipse whose axes are 340 and 310?

Ans., 82780.896

PROBLEM XIV.

(104.) *To find the area of a parabola.*

RULE.

Multiply the base by two thirds of the height.

For demonstration, see Geometry, Parabola, Prop. 12.

Ex. 1. What is the area of a parabola whose base is 18 feet, and height 5 feet?

Ans., 60 feet.

Ex. 2. What is the area of a parabola whose base is 525 feet, and height 350 feet?

Ans., 122500 feet.

MENSURATION OF SOLIDS.

(105.) The common measuring unit of solids is a *cube*, whose faces are squares of the same name; as, a cubic inch, a cubic foot, &c. This measuring unit is not, however, of necessity a cube whose faces are squares of the same name. Thus a bushel may have the form of a cube, but its faces can only be expressed by means of some unit of a different denomination. .The following is

The Table of Solid Measure.

1728	cubic inches	=	1 cubic foot.
27	cubic feet	=	1 cubic yard.
$4492\frac{1}{8}$	cubic feet	=	1 cubic rod.
231	cubic inches	=	1 gallon (liquid measure).
268.8	cubic inches	=	1 gallon (dry measure).
2150.4	cubic inches	=	1 bushel.

PROBLEM I.

(106.) *To find the surface of a right prism.*

RULE.

Multiply the perimeter of the base by the altitude for the convex surface. To this add the areas of the two ends when the entire surface is required.

See Geometry, Prop. 1, B. VIII.

Ex. 1. What is the entire surface of a parallelopiped whose altitude is 20 feet, breadth 4 feet, and depth 2 feet?

Ans., 256 square feet.

Ex. 2. What is the entire surface of a pentagonal prism whose altitude is 25 feet 6 inches, and each side of its base 3 feet 9 inches?

Ans., 526.513 square feet.

Ex. 3. What is the entire surface of an octagonal prism whose altitude is 12 feet 9 inches, and each side of its base 2 feet 5 inches?

Ans., 302.898 square feet.

PROBLEM II.

(107.) *To find the solidity of a prism.*

RULE.

Multiply the area of the base by the altitude.

See Geometry, Prop. 11, B. VIII.

Ex. 1. What is the solidity of a parallelopiped whose altitude is 30 feet, breadth 6 feet, and depth 4 feet?

Ans., 720 cubic feet.

Ex. 2. What is the solidity of a square prism whose altitude is 8 feet 10 inches, and each side of its base 2 feet 3 inches?

Ans., $44\frac{23}{32}$ cubic feet.

Ex. 3. What is the solidity of a pentagonal prism whose altitude is 20 feet 6 inches, and its side 2 feet 7 inches?

Ans., 235.376 cubic feet.

PROBLEM III.

(108.) *To find the surface of a regular pyramid.*

RULE.

Multiply the perimeter of the base by half the slant height for the convex surface. To this add the area of the base when the entire surface is required.

See Geometry, Prop. 14, B. VIII.

Ex. 1. What is the entire surface of a triangular pyramid whose slant height is 25 feet, and each side of its base 5 feet?

Ans., 198.325 square feet.

Ex. 2. What is the entire surface of a square pyramid whose slant height is 30 feet, and each side of the base 4 feet?

Ans., 256 square feet.

Ex. 3. What is the entire surface of a pentagonal pyramid whose slant height is 20 feet, and each side of the base 3 feet?

Ans., 165.484 square feet.

PROBLEM IV.

(109.) *To find the solidity of a pyramid.*

RULE.

Multiply the area of the base by one third of the altitude.

See Geometry, Prop. 17, B. VIII.

Ex. 1. What is the solidity of a triangular pyramid whose altitude is 25 feet, and each side of its base 6 feet?

Ans., 129.904 cubic feet.

Ex. 2. What is the solidity of a square pyramid whose slant height is 22 feet, and each side of its base 10 feet?

Ans., 714.143 cubic feet.

Ex. 3. What is the solidity of a pentagonal pyramid whose altitude is 20 feet, and each side of its base 3 feet?

Ans., 103.228 cubic feet.

PROBLEM V.

(110.) *To find the surface of a frustum of a regular pyramid.*

RULE.

Multiply half the slant height by the sum of the perimeters of the two bases for the convex surface. To this add the areas of the two bases when the entire surface is required.

See Geometry, Prop. 14, Cor. 1, B. VIII.

Ex. 1. What is the entire surface of a frustum of a square pyramid whose slant height is 15 feet, each side of the greater base being 4 feet 6 inches, and each side of the less base 2 feet 10 inches?

Ans., 248.278 square feet.

Ex. 2. What is the entire surface of a frustum of an octagonal pyramid whose slant height is 14 feet, and the sides of the ends 3 feet 9 inches, and 2 feet 3 inches?

Ans., 428.344 square feet.

PROBLEM VI.

(111.) *To find the solidity of a frustum of a pyramid.*

RULE.

Add together the areas of the two bases, and a mean proportional between them, and multiply the sum by one third of the altitude.

See Geometry, Prop. 18, B. VIII.

When the pyramid is regular, it is generally most convenient to find the area of its base by Rule II., Art. 92. If we put a to represent one side of the lower base, and b one side of the upper base, and the tabular number from Art. 92 by

T, the area of the lower base will be a^2T; that of the upper base will be b^2T; and the mean proportional will be abT. Hence, if we represent the height of the frustum by h, its solidity will be

$$(a^2+b^2+ab)\frac{hT}{3}.$$

Ex. 1. What is the solidity of a frustum of an hexagonal pyramid whose altitude is 15 feet, each side of the greater end being 3 feet, and that of the less end 2 feet?

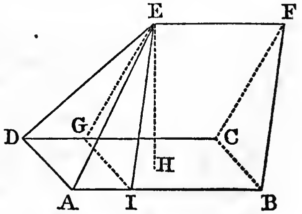
Ans., 246.817 cubic feet.

Ex. 2. What is the solidity of a frustum of an octagonal pyramid whose altitude is 9 feet, each side of the greater end being 30 inches, and that of the less end 20 inches?

Ans., 191.125 cubic feet.

Definition.

(112.) A *wedge* is a solid bounded by five planes, viz., a rectangular base, ABCD, two trapezoids, ABFE, DCFE, meeting in an edge, and two triangular ends, ADE, BCF. The *altitude* of the wedge is the perpendicular drawn from any point in the edge to the plane of the base, as EH.



PROBLEM VII.

(113.) *To find the solidity of a wedge.*

RULE.

Add the length of the edge to twice the length of the base, and multiply the sum by one sixth of the product of the height of the wedge and the breadth of the base.

Demonstration.

Put L=AB, the length of the base;
" l=EF, the length of the edge;
" b=BC, the breadth of the base;
" h=EH, the altitude of the wedge.

Now, if the length of the base is equal to that of the edge, it is evident that the wedge is half of a prism of the same base and height. If the length of the base is greater than that of the edge, let a plane, EGI, be drawn parallel to BCF. The wedge will be divided into two parts, viz., the pyramid E—AIGD, and the triangular prism BCF—G.

The solidity of the former is equal to $\frac{1}{3}bh(\text{L}-l)$, and that of the latter is $\frac{1}{2}bhl$. Their sum is

$$\tfrac{1}{2}bhl+\tfrac{1}{3}bh(\text{L}-l)=\tfrac{1}{6}bh3l+\tfrac{1}{6}bh2\text{L}-\tfrac{1}{6}bh2l=\tfrac{1}{6}bh(2\text{L}+l).$$

If the length of the base is less than that of the edge, the wedge will be equal to the difference between the prism and pyramid, and we shall have

$$\tfrac{1}{2}bhl-\tfrac{1}{3}bh(l-\text{L}),$$

which is equal to

$$\tfrac{1}{2}bhl+\tfrac{1}{3}bh(\text{L}-l),$$

the same result as before.

Ex. 1. What is the solidity of a wedge whose base is 30 inches long and 5 inches broad, its altitude 12 inches, and the length of the edge 2 feet?

Ans., 840 cubic inches.

Ex. 2. What is the solidity of a wedge whose base is 40 inches long and 7 inches broad, its altitude 18 inches, and the length of the edge 30 inches?

Ans., 2310 cubic inches.

Definition.

(114.) A *rectangular prismoid* is a solid bounded by six planes, of which the two bases are rectangles having their corresponding sides parallel, and the four upright sides of the solid are trapezoids.

PROBLEM VIII.

To find the solidity of a rectangular prismoid.

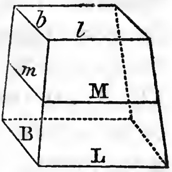
RULE.

Add together the areas of the two bases, and four times the area of a parallel section equally distant from the bases, and multiply the sum by one sixth of the altitude.

Demonstration.

Put L and B=length and breadth of one base;

Put l and b = length and breadth of the other base;
" M " m = length and breadth of middle sec.;
" h = the altitude of the prismoid.



It is evident that if a plane be made to pass through the opposite edges of the upper and lower bases, the prismoid will be divided into two wedges, whose bases are the bases of the prismoid, and whose edges are L and l. The solidity of these wedges, and, consequently, that of the prismoid, is

$$\tfrac{1}{6}\mathrm{B}h(2\mathrm{L}+l)+\tfrac{1}{6}bh(2l+\mathrm{L})=\tfrac{1}{6}h(2\mathrm{BL}+\mathrm{B}l+2bl+b\mathrm{L}).$$

But, since M is equally distant from L and l, we have

$$2\mathrm{M}=\mathrm{L}+l, \text{ and } 2m=\mathrm{B}+b;$$

hence $4\mathrm{M}m=(\mathrm{L}+l)(\mathrm{B}+b)=\mathrm{BL}+\mathrm{B}l+b\mathrm{L}+bl.$

Substituting $4\mathrm{M}m$ for its value in the preceding expression, we obtain for the solidity of the prismoid

$$\tfrac{1}{6}h(\mathrm{BL}+bl+4\mathrm{M}m).$$

Ex. 1. What are the contents of a log of wood, in the form of a rectangular prismoid, the length and breadth of one end being 16 inches and 12 inches, and of the other 7 inches and 4 inches, the length of the log being 24 feet?

Ans., $16\frac{1}{3}$ cubic feet.

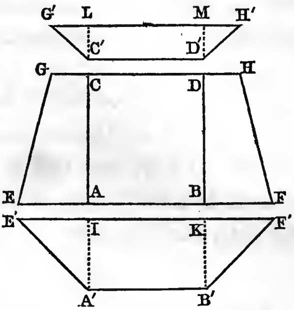
Ex. 2. What is the solidity of a log of hewn timber, whose ends are 18 inches by 15, and 14 inches by $11\frac{1}{2}$, its length being 18 feet? *Ans.*, $26\frac{3}{4}\frac{1}{8}$ cubic feet.

PROBLEM IX.

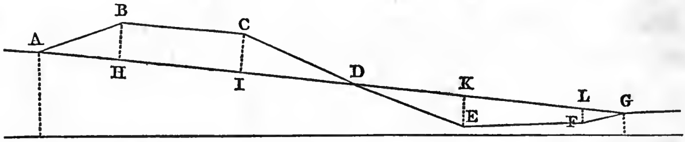
To compute the excavation or embankment for a rail-way.

(115.) By the preceding rule may be computed the amount of excavation or embankment required in constructing a railroad or canal. If we divide the line of the road into portions so small that each may be regarded as a straight line, and suppose an equal number of transverse sections to be made, the excavation or embankment between two sections may be regarded as a prismoid, and its contents found by the preceding rule.

Let ABCD represent the lower surface of the supposed excavation, which we will assume to be parallel to the horizon; and let EFGH represent the upper surface of the excavation projected on a horizontal plane. Also, let E′A′B′F′, G′C′D′H′ represent the vertical sections at the extremities. If we suppose vertical planes to pass through the lines AC, BD, the middle part of the excavation, or that contained between these vertical planes, will be a rectangular prismoid, of which A′B′KI will be one base, and C′D′ML the other base. Its solidity will therefore be given by Art. 114. The parts upon each side of the middle prismoid are also halves of rectangular prismoids; or, if the two parts are equal, they may be regarded as constituting a second prismoid, one of whose bases is the sum of the triangles A′E′I, B′F′K; and the other base is the sum of the triangles C′G′L, D′H′M. Therefore the volume of the entire solid is equal to the product of one sixth of its length, by the sum of the areas of the sections at the two extremities, and four times the area of a parallel and equidistant section.



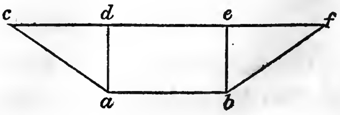
Ex. 1. Let ABCDEFG represent the profile of a tract of



land selected for the line of a rail-way; and suppose it is required, by cutting and embankment, to reduce it from its present hilly surface to one uniform slope from the point A to the point G.

The distance AH is 561 feet; the distance DK is 820 feet;
" " HI is 858 feet; " " KL is 825 feet;
" " ID is 825 feet; " " LG is 330 feet.
The perpendicular BH is 18 feet; the perpendicular KE is 19 feet;
" " CI is 20 feet; " " LF is 8 feet.

The annexed figure represents a cross section, showing the form of the excavation.



The base of the cutting is to

be 50 feet wide, the slope $1\frac{1}{2}$ horizontal to 1 perpendicular; that is, where the depth *ad* is 10 feet, the width of the slope *cd* at the surface will be 15 feet.

Calculation of the portion ABH.

Since BH is 18 feet, the length of *cd* in the cross section will be 27 feet, and *cf*, the breadth at the top of the section, will be 104 feet. We accordingly find, by Art. 87, the area of the trapezoid forming the cross section at BH equal to

$$\frac{104+50}{2}\times 18=1386 \text{ feet.}$$

For the middle section, the height is 9 feet, *cd* is 13.5 feet, and *cf* is 77 feet. The area of the cross section is therefore equal to

$$\frac{77+50}{2}\times 9=571.5.$$

The solid ABH will therefore be equal to

$$(1386+4\times 571.5)\,\frac{561}{6}=343332 \text{ cubic feet, or}$$

12716 cubic yards.

Calculation of the portion BCIH.

Since CI is 20 feet, the length of *cd* is 30 feet, and *cf* is 110 feet. The area of the section at CI is therefore equal to

$$\frac{110+50}{2}\times 20=1600.$$

For the middle section, the height is 19 feet, *cd* is 28.5 feet, and *cf* is 107 feet. The area of the cross section is therefore equal to

$$\frac{107+50}{2}\times 19=1491.5.$$

The solid BCIH will therefore be equal to

$$(1386+1600+4\times 1491.5)\,\frac{858}{6}=1280136 \text{ cubic feet, or}$$

47412.4 cubic yards.

Calculation of the portion CID.

The height of the middle section is 10 feet; therefore *cf* is 80 feet, and the area of the cross section is

$$\frac{80+50}{2}\times 10=650.$$

The solid CID will therefore be equal to

$$(1600+4\times 650)\,\frac{825}{6}=577500 \text{ cubic feet, or}$$

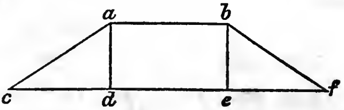
21388.9 cubic yards.

The entire amount of excavation therefore is,

ABH=	12716.0	cubic yards.
BCIH=	47412.4	"
CDI=	21388.9	"
Total excavation,	81517.3	"

The following is a cross section, showing the form of the embankment.

The top of the embankment is to be 50 feet wide, the slope 2 to 1; that is, where the height *ad* is 10 feet, the base *cd* is to be 20 feet.



Calculation of the portion DKE.

Since EK is 19 feet, the length of *cd* is 38 feet, and *cf* is 126 feet. The area of the cross section at EK is therefore equal to

$$\frac{126+50}{2}\times 19=1672.$$

For the middle section, the height is 9.5 feet; *cd* is therefore 19 feet, and *cf* is 88 feet. The area of the cross section is therefore

$$\frac{88+50}{2}\times 9.5=655.5$$

The solid DKE is therefore equal to

$$(1672+4\times 655.5)\,\frac{820}{6}=586846.7 \text{ cubic feet, or}$$

21735.1 cubic yards.

Calculation of the portion KEFL.

Since LF is 8 feet, *cd* is 16 feet, and *cf* is 82 feet. The area of the section at LF is therefore equal to

$$\frac{82+50}{2}\times 8=528.$$

The height of the middle section is 13.5 feet; therefore *cd* is 27 feet, and *cf* is 104 feet. The area of the cross section is therefore equal to

$$\frac{104+50}{2}\times 13.5=1039.5.$$

The solid KEFL will therefore be equal to

$$(1672+528+4\times 1039.5)\frac{825}{6}=874225 \text{ cubic feet, or}$$

32378.7 cubic yards.

Calculation of the portion LFG.

The height of the middle section is 4 feet; therefore *cf* is 66 feet, and the area of the cross section is equal to

$$\frac{66+50}{2}\times 4=232.$$

The solid LFG will therefore be equal to

$$(528+4\times 232)\frac{330}{6}=80080 \text{ cubic feet, or}$$

2965.9 cubic yards.

The entire amount of embankment therefore is

DKE=	21735.1	cubic yards.
KEFL=	32378.7	"
LFG=	2965.9	"
Total embankment,	57079.7	"

Ex. 2. Compute the amount of excavation of the hill ABCD from the following data:

The distance AH is 325 feet; the perpendicular BH is 12 feet;
" " HI is 672 feet; " " CI is 13 feet.
" " ID is 534 feet.

The base of the cutting to be 50 feet wide, and the slope $1\frac{1}{2}$ horizontal to 1 perpendicular. *Ans.*, 33969 cubic yards.

PROBLEM X.

(116.) *To find the surface of a regular polyedron.*

RULE.

Multiply the area of one of the faces by the number of *faces;* or, *Multiply the square of one of the edges by the surface of a similar solid whose edge is unity.*

Since all the faces of a regular polyedron are equal, it is evident that the area of one of them, multiplied by their number, will give the entire surface. Also, regular solids of the same name are similar, and similar polygons are as the squares of their homologous sides (*Geom.*, Prop. 26, B. IV.). The following table shows the surface and solidity of regular polyedrons whose edge is unity. The surface is obtained by multiplying the area of one of the faces, as given in Art. 92, by the number of faces. Thus the area of an equilateral triangle, whose side is 1, is 0.4330127. Hence the surface of a regular tetraedron

$$=.4330127\times 4=1.7320508,$$

and so on for the other solids.

A Table of the regular Polyedrons whose Edges are unity.

Names.	*No. of Faces.*	*Surface.*	*Solidity.*
Tetraedron,	4	1.7320508	0.1178513.
Hexaedron,	6	6.0000000	1.0000000.
Octaedron,	8	3.4641016	0.4714045.
Dodecaedron,	12	20.6457288	7.6631189.
Icosaedron,	20	8.6602540	2.1816950.

Ex. 1. What is the surface of a regular octaedron whose edges are each 8 feet?

Ans., 221.7025 feet.

Ex. 2. What is the surface of a regular dodecaedron whose edge is 12 feet?

Ans., 2972.985 feet.

PROBLEM XI.

(117.) *To find the solidity of a regular polyedron.*

RULE.

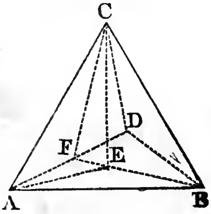
Multiply the surface by one third of the perpendicular let fall from the center on one of the faces; or, *Multiply the cube of one of the edges by the solidity of a similar polyedron whose edge is unity.*

Since the faces of a regular polyedron are similar and equal, and the solid angles are all equal to each other, it is evident that the faces are all equally distant from a point in the solid called the center. If planes be made to pass through the center and the several edges of the solid, they will divide it into as many equal pyramids as it has faces. The base of each pyramid will be one of the faces of the polyedron; and since their altitude is the perpendicular from the center upon one of the faces, the solidity of the polyedron must be equal to the areas of all the faces, multiplied by one third of this perpendicular.

Also, similar pyramids are to each other as the cubes of their homologous edges (*Geom.*, Prop. 17, Cor. 3, B. VIII.). And since two regular polyedrons of the same name may be divided into the same number of similar pyramids, they must be to each other as the cubes of their edges.

(118.) The solidity of a tetraedron whose edge is unity, may be computed in the following manner:

Let C—ABD be a tetraedron. From one angle, C, let fall a perpendicular, CE, on the opposite face; draw EF perpendicular to AD; and join CF, AE. Then AEF is a right-angled triangle, in which EF, being the sine of 30°, is one half of AE or BE; and therefore FE is one third of BF or CF. Hence the cosine of the angle CFE is equal to $\frac{1}{3}$; that is, the angle of inclination of the faces of the polyedron is 70° 31′ 44″. Also, in the triangle CAF, CF is the sine of 60°, which is 0.866025. Hence, in the right-angled triangle CEF, knowing one side and the angles, we can compute CE, which is found to be 0.8164966. Whence, knowing the base ABD (Art. 92), we obtain the solidity of the tetraedron = 0.1178513.



In a somewhat similar manner may the solidities of the other regular polyedrons, given in Art. 116, be obtained.

Ex. 1. What is the solidity of a regular tetraedron whose edges are each 24 inches?

Ans., 0.9428 feet.

Ex. 2. What is the solidity of a regular icosaedron whose edges are each 20 feet?

Ans., 17453.56 feet.

THE THREE ROUND BODIES.

PROBLEM I.

(119.) *To find the surface of a cylinder.*

RULE.

Multiply the circumference of the base by the altitude for the convex surface. To this add the areas of the two ends when the entire surface is required.

See Geometry, Prop. 1, B. X.

Ex. 1. What is the convex surface of a cylinder whose altitude is 23 feet, and the diameter of its base 3 feet?

Ans., 216.77 square feet.

Ex. 2. What is the entire surface of a cylinder whose altitude is 18 feet, and the diameter of its base 5 feet?

Ans.

PROBLEM II.

(120.) *To find the solidity of a cylinder.*

RULE.

Multiply the area of the base by the altitude.

See Geometry, Prop. 2, B. X.

Ex. 1. What is the solidity of a cylinder whose altitude is 18 feet 4 inches, and the diameter of its base 2 feet 10 inches?

Ans., 115.5917 cubic feet.

Ex. 2. What is the solidity of a cylinder whose altitude is 12 feet 11 inches, and the circumference of its base 5 feet 3 inches?

Ans., 28.3308 cubic feet.

PROBLEM III.

(121.) *To find the surface of a cone.*

RULE.

Multiply the circumference of the base by half the side for the convex surface; to which add the area of the base when the entire surface is required.

See Geometry, Prop. 3, B. X.

Ex. 1. What is the entire surface of a cone whose side is 10 feet, and the diameter of its base 2 feet 3 inches?

Ans., 39.319 square feet.

Ex. 2. What is the entire surface of a cone whose side is 15 feet, and the circumference of its base 8 feet?

Ans., 65.093 square feet.

PROBLEM IV.

(122.) *To find the solidity of a cone.*

RULE.

Multiply the area of the base by one third of the altitude.

See Geometry, Prop. 5, B. X.

Ex. 1. What is the solidity of a cone whose altitude is 12 feet, and the diameter of its base $2\frac{1}{2}$ feet?

Ans., 19.635 cubic feet.

Ex. 2. What is the solidity of a cone whose altitude is 25 feet, and the circumference of its base 6 feet 9 inches?

Ans.

PROBLEM V.

(123.) *To find the surface of a frustum of a cone.*

RULE.

Multiply half the side by the sum of the circumferences of the two bases for the convex surface; to this add the areas of the two bases when the entire surface is required.

See Geometry, Prop. 4, B. X.

Ex. 1. What is the entire surface of a frustum of a cone, the diameters of whose bases are 9 feet and 5 feet, and whose side is 16 feet 9 inches?

Ans., 451.6036 square feet.

Ex. 2. What is the convex surface of a frustum of a cone whose side is 10 feet, and the circumferences of its bases 6 feet and 4 feet?

Ans., 50 square feet.

PROBLEM VI.

(124.) *To find the solidity of a frustum of a cone.*

RULE.

Add together the areas of the two bases, and a mean proportional between them, and multiply the sum by one third of the altitude.

See Geometry, Prop. 6, B. X.

If we put R and r for the radii of the two bases, then πR^2 will represent the area of one base, πr^2 the area of the other, and πRr the mean proportional between them. Hence, if we represent the height of the frustum by h, its solidity will be

$$\frac{\pi h}{3}(R^2+r^2+Rr).$$

Ex. 1. What is the solidity of a frustum of a cone whose altitude is 20 feet, the diameter of the greater end 5 feet, and that of the less end 2 feet 6 inches?

Ans., 229.074 cubic feet.

Ex. 2. The length of a mast is 60 feet, its diameter at the greater end is 20 inches, and at the less end 12 inches: what is its solidity? *Ans.*, 85.521 cubic feet.

PROBLEM VII.

(125.) *To find the surface of a sphere.*

RULE.

Multiply the diameter by the circumference of a great circle; or, *Multiply the square of the diameter by* 3.14159.

See Geometry, Prop. 7, B. X.

Ex. 1. Required the surface of the earth, its diameter being 7912 miles. *Ans.*, 196,662,896 square miles.

Ex. 2. Required the surface of the moon, its circumference being 6786 miles. *Ans.*

PROBLEM VIII.

(126.) *To find the solidity of a sphere.*

RULE.

Multiply the surface by one third of the radius; or, *Multiply the cube of the diameter by* $\frac{1}{6}\pi$; *that is, by* 0.5236.

See Geometry, Prop. 8, B. X.

Where great accuracy is required, the value of $\frac{1}{6}\pi$ must be taken to more than four decimal places. Its value, correct to ten decimal places, is .52359,87756.

Ex. 1. What is the solidity of the earth, if it be a sphere 7912 miles in diameter?

Ans., 259,332,805,350 cubic miles.

Ex. 2. If the diameter of the moon be 2160 miles, what is its solidity? *Ans.*

PROBLEM IX.

(127.) *To find the surface of a spherical zone.*

RULE.

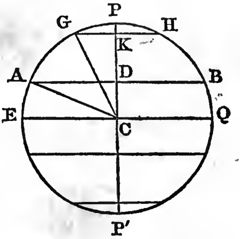
Multiply the altitude of the zone by the circumference of a great circle of the sphere.

See Geometry, Prop. 7, Cor. 1, B. X.

Ex. 1. If the diameter of the earth be 7912 miles, what is the surface of the torrid zone, extending 23° 27′ 36″ on each side of the equator?

Ans., 78,293,218 square miles.

Let PEP′Q represent a meridian of the earth; EQ the equator; P, P′ the poles; AB one of the tropics, and GH one of the polar circles. Then PK will represent the height of one of the frigid zones, KD the height of one of the temperate zones, and CD half the height of the torrid zone.



Each of the angles ACE, CAD, and GCK is equal to 23° 27′ 36″.

In the right-angled triangle ACD,

$$R : AC :: \sin. CAD : CD.$$

Also, in the right-angled triangle CGK,

$$R : CG :: \cos. GCK : CK,$$

Then $$PK = PC - KC.$$

Where great accuracy is required, the sine and cosine of 23° 27′ 36″ must be taken to more than six decimal places. The following values are correct to ten decimal places:

Natural sine of 23° 27′ 36″=.39810,87431.

" cosine of 23° 27′ 36″=.91733,82302.

Ex. 2. If the polar circle extends 23° 27′ 36″ from the pole, find the convex surface of either frigid zone.

Ans., 8,128,252 square miles.

Ex. 3. On the same suppositions, find the surface of each of the temperate zones.

Ans., 51,056,587 square miles.

Problem X.

(128.) *To find the solidity of a spherical segment with one base.*

Rule.

Multiply half the height of the segment by the area of the base, and the cube of the height by .5236, *and add the two products.*

See Geometry, Prop. 9, B. X.

Ex. 1. What is the solidity of either frigid zone, supposing the earth to be 7912 miles in diameter, the polar circles extending 23° 27′ 36″ from the poles?

Ans., 1,292,390,176 cubic miles.

(129.) The solidity of a spherical segment of two bases is the difference between two spherical segments, each having a single base.

Ex. 2. On the same supposition as in Ex. 1, find the solidity of either temperate zone.

Ans., 55,032,766,543 cubic miles.

Ex. 3. Find the solidity of the torrid zone.

Ans., 146,682,491,911 cubic miles.

Problem XI.

(130.) *To find the area of a spherical triangle.*

Rule.

Compute the surface of the quadrantal triangle, or one eighth of the surface of the sphere. From the sum of the three angles subtract two right angles; divide the remainder by 90, *and multiply the quotient by the quadrantal triangle.*

See Geometry, Prop. 20, B. IX.

Ex. 1. What is the area of a triangle on a sphere whose diameter is 10 feet, if the angles are 55°, 60°, and 85°?

Ans., 8.7266 square feet.

Ex. 2. If the angles of a spherical triangle measured on the surface of the earth are 78° 4′ 10″, 59° 50′ 54″, and 42° 5′ 37″, what is the area of the triangle, supposing the earth a sphere, of which the diameter is 7912 miles?

Ans., 3110.794 square miles.

If the excess of the angles above two right angles is expressed in seconds, we must divide it by 90 degrees also expressed in seconds; that is, by 324,000.

PROBLEM XII.

(131.) *To find the area of a spherical polygon.*

RULE.

Compute the surface of the quadrantal triangle. From the sum of all the angles subtract the product of two right angles by the number of sides less two; divide the remainder by 90, *and multiply the quotient by the quadrantal triangle.*

See Geometry, Prop. 21, B. IX.

Ex. 1. What is the area of a spherical polygon of 5 sides on a sphere whose diameter is 10 feet, supposing the sum of the angles to be 640 degrees?

Ans., 43.633 square feet.

Ex. 2. The angles of a spherical polygon, measured on the surface of the earth, are
62° 33′ 13″;
135° 8′ 26″;
149° 16′ 9″;
111° 45′ 8″;
105° 59′ 7″;
155° 19′ 12″.

Required the area of the polygon.

Ans., 5690.477 square miles.

BOOK IV.

SURVEYING.

(132.) The term *Surveying* includes the measurement of heights and distances, the determination of the area of portions of the earth's surface, and their delineation upon paper.

Since the earth is spherical, its surface is not a plane surface, and if large portions of the earth are to be measured, the curvature must be taken into account; but in ordinary surveying, the portions of the earth are supposed to be so small that the curvature may be neglected. The parts surveyed are therefore regarded as plane figures.

(133.) If a plummet be freely suspended by a line, and allowed to come to a state of rest, this line is called a *vertical line.*

Every plane passing through a vertical line is a *vertical plane.*

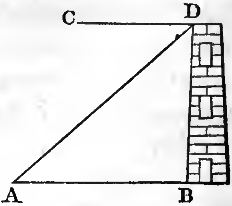
A line perpendicular to a vertical line is a *horizontal line.*

A plane perpendicular to a vertical line is a *horizontal plane.*

A *vertical angle* is one the plane of whose sides is vertical.

A *horizontal angle* is one the plane of whose sides is horizontal.

An *angle of elevation* is a vertical angle having one side horizontal and the other an ascending line, as the angle BAD.



An *angle of depression* is a vertical angle having one side horizontal and the other a descending line, as the angle CDA.

(134.) When distances are to be found by trigonometrical computation, it is necessary to measure at least one line upon the ground, and also as many angles as may be necessary to render three parts of every triangle known.

In the measurement of lines, the unit commonly employed by surveyors is a chain four rods or sixty-six feet in length, called *Gunter's Chain*, from the name of the inventor. This chain is divided into 100 links. Sometimes a half chain is used, containing 50 links.

Hence, 1 chain = 100 links = 66 feet;
1 rod = 25 links = $16\frac{1}{2}$ feet;
1 link = 7.92 inches = $\frac{2}{3}$ of a foot nearly.

(135.) *To measure a horizontal line.*

To mark the termination of the chain in measuring, ten iron pins should be provided, about a foot in length.

Let the person who is to go foremost in carrying the chain, and who is called the leader, take one end of the chain and the ten pins; and let another person take the other end of the chain, and hold it at the beginning of the line to be measured. When the leader has advanced until the chain is stretched tight, he must set down one pin at the end of the chain, the other person taking care that the chain is in the direction of the line to be measured. Then measure a second chain in the same manner, and so on until all the marking pins are exhausted. A record should then be made that ten chains have been measured, after which the marking pins should be returned to the leader, and the measurement continued as before until the whole line has been passed over.

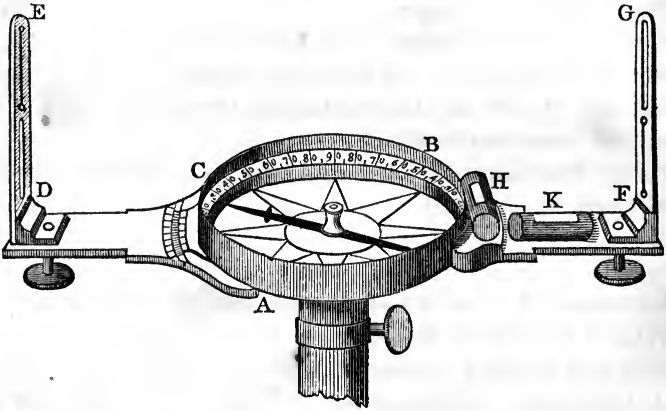
It is generally agreed to refer all surfaces to a horizontal plane. Hence, when an inclined surface, like the side of a hill, is to be measured, the chain should be maintained in a horizontal position. For this purpose, in ascending a hill, the hind end of the chain should be raised from the ground until it is on a level with the fore end, and should be held vertically over the termination of the preceding chain. In descending a hill, the fore end of the chain should be raised in the same manner.

INSTRUMENTS FOR MEASURING ANGLES.

In measuring angles, some instrument is used which contains a portion of a graduated circle divided into degrees and minutes. These instruments may be adapted to measuring either horizontal or vertical angles. The instrument most frequently employed for measuring horizontal angles is called

THE SURVEYOR'S COMPASS.

(136.) The principal parts of this instrument are a compass-box, a magnetic needle, two sights, and a stand for its support. The compass-box, ABC, is circular, generally about six inches in diameter, and at its center is a small pin on which the magnetic needle is balanced. The circumference of the box is divided into degrees, and sometimes to half degrees; and the degrees are numbered from the extremities of a diameter both ways to 90°. The sights, DE, FG, are placed at right angles



to the plane of the graduated circle, and in each of these there is a large and small aperture for convenience of observation The instrument, when used, is mounted on a tripod, or a single staff pointed with iron at the bottom, so that it may be firmly placed in the ground.

Sometimes two spirit levels, H and K, are attached, to indicate when the plane of the graduated circle is brought into a horizontal position.

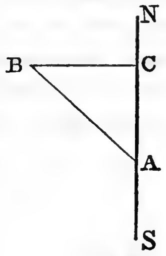
(137.) When the magnetic needle is supported so as to turn freely, and is allowed to come to a state of rest, the direction it assumes is called the *magnetic meridian*, one end of the needle indicating the north point and the other the south.

A horizontal line perpendicular to a meridian is an east and west line

All the meridians passing through a survey of moderate extent, are considered as straight lines parallel to each other.

The *bearing* or *course* of a line is the angle which it makes with a meridian passing through one end; and it is reckoned from the north or south point of the horizon, toward the east or west.

Thus, if NS represent a meridian, and the angle NAB is 40°, then the bearing of AB from the point A is 40° to the west of north, and is written N. 40° W., and read north forty degrees west.



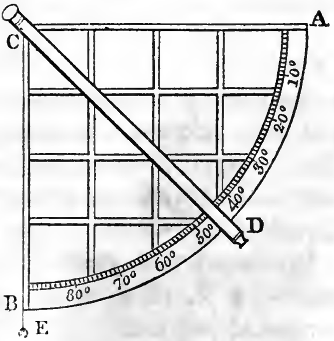
The *reverse bearing* of a line is the bearing taken from the other end of the line.

The forward bearing and reverse bearing of a line are equal angles, but lie between directly opposite points. Thus, if the bearing of AB from A is N. 40° W., the bearing of the same line from B is S. 40° E.

(138.) For measuring vertical angles, the instrument commonly used is

A QUADRANT.

It consists of a quarter of a circle, usually made of brass, and its limb, AB, is divided into degrees and minutes, numbered from A up to 90°. It is furnished either with a pair of plain sights or with a telescope, CD, which is to be directed toward the object observed. A plumb line, CE, is suspended from the center of the quadrant, and indicates when the radius CB is brought into a vertical position.

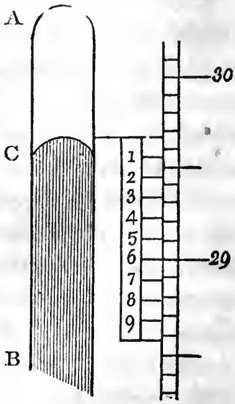


To measure the angle of elevation, for example, of the top of a tower, point the telescope, CD, toward the tower, keeping the radius, CB, in a vertical position by means of the plumb line, CE. Move the telescope until the given object is seen in the middle of the field of view. The center of the field is indicated by two wires placed in the focus of the object-glass of the telescope, one wire being vertical and the other horizontal. When the horizontal wire is made to coincide with the summit of the tower, the angle of elevation is shown upon the arc AB by means of an index which moves with the telescope.

As the arc is not commonly divided into parts smaller than half degrees, when great accuracy is required, some contrivance is needed for obtaining smaller fractions of a degree. This is usually effected by a vernier.

(139.) A *Vernier* is a scale of small extent, graduated in such a manner that, being moved by the side of a fixed scale, we are enabled to measure minute portions of this scale. The length of this movable scale is equal to a certain number of parts of that to be subdivided, but it is divided into parts one more or one less than those of the primary scale taken for the length of the vernier. Thus, if we wish to measure hundredths of an inch, as in the case of a barometer, we first divide an inch into ten equal parts. We then construct a vernier equal in length to 11 of these divisions, but divide it into 10 equal parts, by which means each division on the vernier is $\frac{1}{10}$th longer than a division of the primary scale.

Thus, let AB be the upper end of a barometer tube, the mercury standing at the point C; the scale is divided into inches and tenths of an inch, and the middle piece, numbered from 1 to 9, is the vernier that slides up and down, having 10 of its divisions equal to 11 divisions of the scale, that is, to $\frac{11}{10}$ths of an inch. Therefore, each division of the vernier is $\frac{11}{100}$ths of an inch; or one division of the vernier exceeds one division of the scale by $\frac{1}{100}$th of an inch. Now, as the sixth division of the vernier (in the figure) coincides with a division of the scale, the fifth division of the vernier will stand $\frac{1}{100}$th of an inch above the nearest division of the scale; the fourth division $\frac{2}{100}$ths of an inch, and the top of the vernier will be $\frac{6}{100}$ths of an inch above the next lower division of the scale; *i. e.*, the top of the vernier coincides with 29.66 inches upon the scale. In practice, therefore, we ob-



serve what division of the vernier coincides with a division of the scale; this will show the hundredths of an inch to be added to the tenths next below the vernier at the top.

A similar contrivance is applied to graduated circles, to obtain the value of an arc with greater accuracy. If a circle is graduated to half degrees, or 30′, and we wish to measure single minutes by the vernier, we take an arc equal to 31 divisions upon the limb, and divide it into 30 equal parts. Then each division of the vernier will be equal to $\frac{31}{60}$ths of a degree, while each division of the scale is $\frac{30}{60}$ths of a degree. That is, each space on the vernier exceeds one on the limb by 1′.

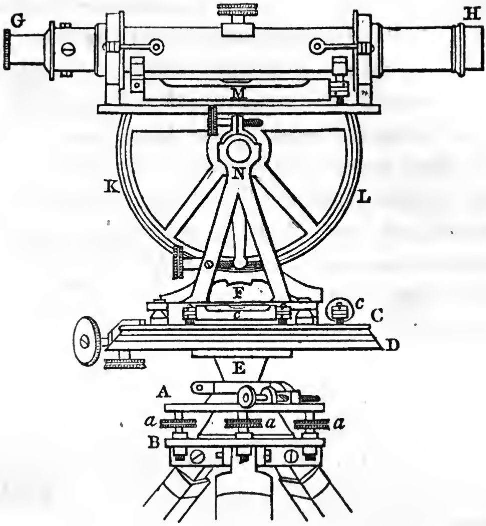
In order, therefore, to read an angle for any position of the vernier, we pass along the vernier until a line is found coinciding with a line of the limb. The number of this line from the zero point indicates the minutes which are to be added to the degrees and half degrees taken from the graduated circle. Sometimes a vernier is attached to the common surveyor's compass.

(140.) An instrument in common use for measuring both horizontal and vertical angles is

THE THEODOLITE.

The theodolite has two circular brass plates, C and D (see fig. next page), the former of which is called the *vernier plate*, and the latter the *graduated limb*. Both have a horizontal motion about the vertical axis, E. This axis consists of two parts, one external, and the other internal; the former secured to the graduated limb, D, and the latter to the vernier plate, C, so that the vernier plate turns freely upon the lower. The edge of the lower plate is divided into degrees and half degrees, and this is subdivided by a vernier attached to the upper plate into single minutes. The degrees are numbered from 0 to 360.

The parallel plates, A and B, are held together by a ball which rests in a socket. Four screws, three of which, *a*, *a*, *a*, are shown in the figure, turn in sockets fixed to the lower plate, while their heads press against the under side of the upper plate, by which means the instrument is leveled for observation. The whole rests upon a tripod, which is firmly attached to the body of the instrument.



To the vernier plate, two spirit-levels, *c*, *c*, are attached at right angles to each other, to determine when the graduated limb is horizontal. A compass, also, is placed at F. Two frames, one of which is seen at N, support the pivots of the horizontal axis of the vertical semicircle KL, on which the telescope, GH, is placed. One side of the vertical arc is divided into degrees and half degrees, and it is divided into single minutes by the aid of its vernier. The graduation commences at the middle of the arc, and reads both ways to 90°. Under and parallel to the telescope is a spirit-level, M, to show when the telescope is brought to a horizontal position. To enable us to direct the telescope upon an object with precision, two lines called wires are fixed at right angles to each other in the focus of the telescope.

To measure a Horizontal Angle with the Theodolite.

(141.) Place the instrument exactly over the station from which the angle is to be measured; then level the instrument by means of the screws, *a*, *a*, bringing the telescope over each pair alternately until the two spirit-levels on the vernier plate retain their position, while the instrument is turned entirely round upon its axis. Direct the telescope to one of the objects to be observed, moving it until the cross-wires and object coincide. Now read off the degrees upon the graduated limb, and the minutes indicated by the vernier. Next, release the upper plate (leaving the graduated limb undisturbed), and move it round until the telescope is directed to the second object, and make the cross-wires bisect this object, as was done by the first. Again, read off the vernier; the difference between this and the former reading will be the angle required.

The magnetic bearing of an object is determined by simply reading the angle pointed out by the compass-needle when the object is bisected.

To measure an Angle of Elevation with the Theodolite.

(142.) Direct the telescope toward the given object so that it may be bisected by the horizontal wire, and then read off the arc upon the vertical semicircle. After observing the object with the telescope in its natural position, it is well to revolve the telescope in its supports until the level comes uppermost, and repeat the observation. The mean of the two measures may be taken as the angle of elevation.

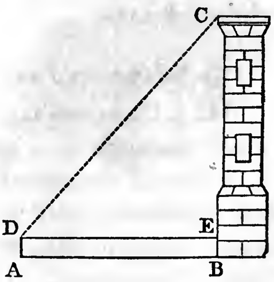
By the aid of the instruments now described, we may determine the distance of an inaccessible object, and its height above the surface of the earth.

HEIGHTS AND DISTANCES.

PROBLEM I.

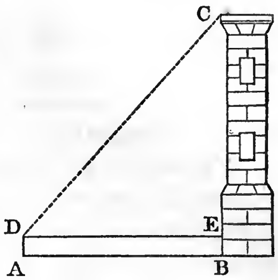
(143.) *To determine the height of a vertical object situated on a horizontal plane.*

Measure from the object to any convenient distance in a straight line, and then take the angle of elevation subtended by the object.



If we measure the distance DE, and the angle of elevation CDE, there will be given, in the right-angled triangle CDE, the base and the angles, to find the perpendicular CE (Art. 46). To this we must add the height of the instrument, to obtain the entire height of the object above the plane AB.

Ex. 1. Having measured AB equal to 100 feet from the bottom of a tower on a horizontal plane, I found the angle of elevation, CDE, of the top to be 47° 30′, the center of the quadrant being five feet above the ground. What is the height of the tower?



R : tang. CDE : : DE : CE=109.13. To which add five feet, and we obtain the height of the tower, 114.13 feet.

Ex. 2. From the edge of a ditch 18 feet wide, surrounding a fort, the angle of elevation of the wall was found to be 62° 40′. Required the height of the wall, and the length of a ladder necessary to reach from my station to the top of it.

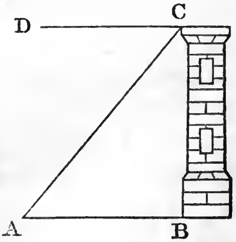
Ans. The height is 34.82 feet. Length of ladder, 39.20 feet.

PROBLEM II.

(144.) *To find the distance of a vertical object whose height is known.*

Measure the angle of elevation, and we shall have given the angles and perpendicular of a right-angled triangle to find the base (Art. 46).

Ex. 1. The angle of elevation of the top of a tower whose height was known to be 143 feet, was found to be 35°. What was its distance?



Here we have given the angles of the triangle ABC, and the side CB, to find AB.

Ans., 204.22 feet.

If the observer were stationed at the top of the tower BC, he might find the length of the base AB by measuring the angle of depression DCA, which is equal to BAC.

Ex. 2. From the top of a ship's mast, which was 80 feet above the water, the angle of depression of another ship's hull was found to be 20°. What was its distance?

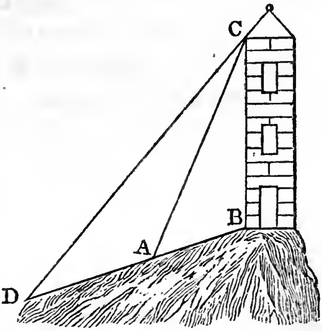
Ans., 219.80 feet.

PROBLEM III.

(145.) *To find the height of a vertical object standing on an inclined plane.*

Measure the distance from the object to any convenient station, and observe the angles which the base-line makes with lines drawn from its two ends to the top of the object.

If we measure the base-line AB, and the two angles ABC, BAC, then, in the triangle ABC, we shall have given one side and the angles to find BC.



Ex. 1. Wanting to know the height of a tower standing on an inclined plane, BD, I measured from the bottom of the tower a distance, AB, equal to 165 feet; also the angle ABC, equal to 107° 18′, and the angle BAC, equal to 33° 35′. Required the height of the object.

sin. ACB : AB : : sin. BAC : BC=144.66 feet.

The height, BC, may also be found by measuring the distances BA, AD, and taking the angles BAC, BDC. The difference between the angles BAC and BDC will be the angle ACD. There will then be given, in the triangle DAC, one side and all the angles to find AC; after which we shall have, in the triangle ABC, two sides and the included angle to find BC.

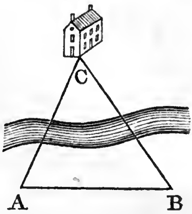
Ex. 2. A tower standing on the top of a declivity, I measured 75 feet from its base, and then took the angle BAC, 47° 50′; going on in the same direction 40 feet further, I took the angle BDC, 38° 30′. What was the height of the tower?

Ans., 117.21 feet.

PROBLEM IV.

(146.) *To find the distance of an inaccessible object.*

Measure a horizontal base-line, and also the angles between this line and lines drawn from each station to the object. Let C be the object inaccessible from A and B. Then, if the distance between the stations A and B be measured, as also the angles at A and B, there will be given, in the triangle ABC, the side AB and the angles, to find AC and BC, the distances of the object from the two stations.



Ex. 1. Being on the side of a river, and wanting to know the distance to a house which stood on the other side, I measured 400 yards in a right line by the side of the river, and found that the two angles at the ends of this line, formed by the other end and the house, were 73° 15′ and 68° 2′. What was the distance between each station and the house?

The angle C is found to be 38° 43′. Then

$$\text{sin. C} : \text{AB} :: \begin{cases} \text{sin. A} : \text{BC}=612.38; \\ \text{sin. B} : \text{AC}=593.09. \end{cases}$$

Ex. 2. Two ships of war, wishing to ascertain their distance from a fort, sail from each other a distance of half a mile, when they find that the angles formed between a line from one to the other, and from each to the fort, are 85° 15′ and 83° 45′. What are the respective distances from the fort?

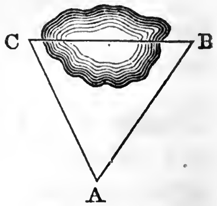
Ans., 4584.52 and 4596.10 yards.

PROBLEM V.

(147.) *To find the distance between two objects separated by an impassable barrier.*

Measure the distance from any convenient station to each of the objects, and the angle included between those lines.

If we wish to know the distance between the places C and B, both of which are accessible, but separated from each other by water, we may measure the lines AC and AB, and also the angle A. We shall then have given two sides of a triangle and the included angle to find the third side.



Ex. 1. The passage between the two objects C and B being obstructed, I measured from A to C 735 rods, and from A to B 840 rods; also, the angle A, equal to 55° 40′. What is the distance of the places C and B?

Ans., 741.21 rods.

Ex. 2. In order to find the distance between two objects, C and B, which could not be directly measured, I measured from C to A 652 yards, and from B to A 756 yards; also, the angle A equal to 142° 25′. What is the distance between the objects C and B?

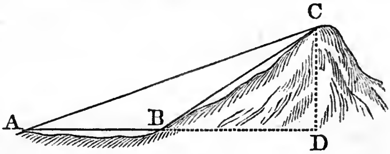
Ans.

PROBLEM VI.

(148.) *To find the height of an inaccessible object above a horizontal plane.*

First Method.—Take two stations in a vertical plane passing through the top of the object; measure the distance between the stations and the angle of elevation at each.

If we measure the base AB, and the angles DAC, DBC, then, since CBA is the supplement of DBC, we shall have, in the triangle ABC, one side and all the angles to find BC. Then, in the right-angled triangle DBC, we shall have the hypothenuse and the angles to find DC.



Ex. 1. What is the perpendicular height of a hill whose angle of elevation, taken at the bottom of it, was 46°; and 100 yards farther off, on a level with the bottom of it, the angle was 31°?

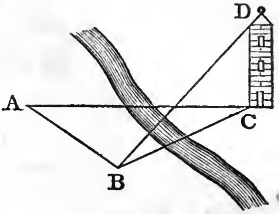
Ans., 143.14 yards.

Ex. 2. The angle of elevation of a spire I found to be 58°, and going 100 yards directly from it, found the angle to be only 32°. What is the height of the spire, supposing the instrument to have been five feet above the ground at each observation?

Ans., 104.18 yards.

(149.) *Second Method.*—Measure any convenient base-line, also the angles between this base and lines drawn from each of its extremities to the foot of the object, and the angle of elevation at one of the stations.

Let DC be the given object. If we measure the horizontal base-line AB, and the angles CAB, CBA, we can compute the distance BC. Also, if we observe the angle of elevation CBD, we shall have given, in the right-angled triangle BCD, the base and angles to find the perpendicular.



Ex. 1. Being on one side of a river, and wanting to know the height of a spire on the other side, I measured 500 yards, AB, along the side of the river, and found the angle ABC=74° 14′, and BAC=49° 23′; also, the angle of elevation CBD=11° 15′. Required the height of the spire. *Ans.*, 271.97 feet.

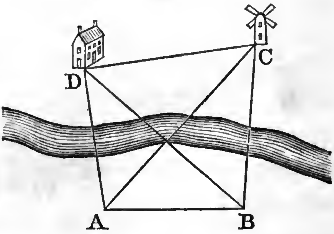
Ex. 2. To find the height of an inaccessible castle, I measured a line of 73 yards, and at each end of it took the angle of position of the object and the other end, and found the one to be 90°, and the other 61° 45′; also, the elevation of the castle from the latter station, 10° 35′. Required the height of the castle. *Ans.*, 86.45 feet.

PROBLEM VII.

(150.) *To find the distance between two inaccessible objects.*

Measure any convenient base-line, and the angles between this base and lines drawn from each of its extremities to each of the objects.

Let C and D be the two inaccessible objects. If we measure a base-line, AB, and the angles DAB, DBA, CAB, CBA, then, in the triangle DAB, we shall have given the side AB and all the angles to find BD; also, in the triangle ABC, we shall have one side and all the angles to find BC; and then, in the triangle BCD, we shall have two sides, BD, BC, with the included angle, to find DC.



Ex. 1. Wanting to know the distance between a house and a mill, which were separated from me by a river, I measured a base-line, AB, 300 yards, and found the angle CAB=58° 20′, CAD=37°, ABD=53° 30′, DBC=45° 15′. What is the distance of the house from the mill? *Ans.*, 479.80 yards.

Ex. 2. Wanting to know the distance between two inaccessible objects, C and D, I measured a base-line, AB, 28.76 rods, and found the angle CAB=33°, CAD=66°, DBA=59° 45′, and DBC=76°. What is the distance from C to D?

Ans., 97.696 rods.

THE DETERMINATION OF AREAS.

(151.) The *area* or *content* of a tract of land is the *horizontal* surface included within its boundaries.

When the surface of the ground is broken and uneven, it is very difficult to ascertain exactly its actual surface. Hence it has been agreed to refer every surface to a horizontal plane; and for this reason, in measuring the boundary lines, it is necessary to reduce them all to horizontal lines.

The measuring unit of surfaces chiefly employed by surveyors is the *acre*, or ten square chains.

One quarter of an acre is called a rood.

Since a chain is four rods in length, a square chain contains sixteen square rods; and an acre, or ten square chains, contains 160 square rods. Square rods are called *perches*. The area of a field is usually expressed in acres, roods, and perches, designated by the letters A., R., P.

When the lengths of the bounding lines of a field are given in chains and links, the area is obtained in square chains and square links. Now, since a link is $\frac{1}{100}$ of a chain, a square link will be $\frac{1}{100}\times\frac{1}{100}$ of a square chain; that is, $\frac{1}{10000}$ of a chain. Hence we have the following

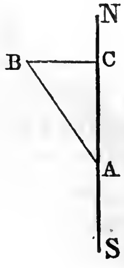
TABLE.

1 square chain=10,000 square links.
1 acre=10 square chains=100,000 square links.
1 acre=4 roods=160 perches.

If, then, the linear dimensions are links, the area will be expressed in square links, and may be reduced to square chains by cutting off *four* places of decimals; if *five* places be cut off, the remaining figures will be *acres*. If the decimal part of an acre be multiplied by 4, it will give the roods, and the resulting decimal, multiplied by 40, will give the perches.

(152.) The *difference of latitude*, or the *northing* or *southing* of a line, is the distance that one end is further north or south than the other end.

Thus, if NS be a meridian passing through the end A of the line AB, and BC be perpendicular to NS, then is AC the difference of latitude, or northing of AB.



The *departure*, or the *easting* or *westing* of a line, is the distance that one end is further east or west than the other end.

Thus BC is the departure or westing of the line AB.

It is evident that the distance, difference of latitude, and departure form a right-angled triangle, of which the distance is the hypothenuse.

The meridian distance of a point is the perpendicular let fall from the given point on some assumed meridian, and is east or west according as this point lies on the east or west side of the meridian.

The meridian distance of a line is the distance of the middle point of that line from some assumed meridian.

(153.) When a piece of ground is to be surveyed, we begin at one corner of the field, and go entirely around the field, measuring the length of each of the sides with a chain, and their bearings with a compass.

Plotting a Survey.

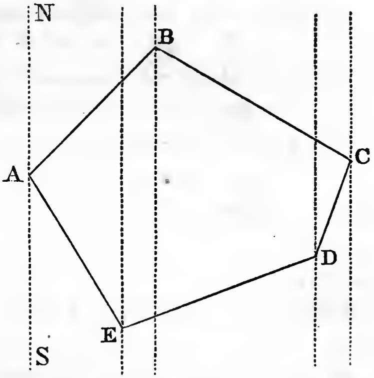
When a field has been surveyed, it is easy to draw a plan of it on paper. For this purpose, draw a line to represent the meridian passing through the first station; then lay off an angle equal to the angle which the first side of the field makes with the meridian, and take the length of the side from a scale of equal parts. Through the extremity of this side draw a second meridian parallel to the first, and proceed in the same manner with the remaining sides. This method will be easily understood from an example.

EXAMPLE 1.

Draw a plan of a field from the following courses and distances, as given in the field-book.

Stations.	Bearings.	Distances.
1	N. 45° E.	9.30 chains.
2	S. 60° E.	11.85 "
3	S. 20° W.	5.30 "
4	S. 70° W.	10.90 "
5	N. 31° W.	9.40 "

Draw NS to represent a meridian line; in NS take any convenient point, as A, for the first station, and lay off an angle, NAB, equal to 45°, the bearing from A to B, which will give the direction from A to B. Then, from the scale of equal parts, make AB equal to 9.30, the length of the first side; this will give the station B. Through B draw a second meridian parallel to NS; lay off an angle of 60°, and make the line BC equal to 11.85. Proceed in the same manner with the other sides. If the survey is correct, and the plotting accurately performed, the end of the last side, EA, will fall on A, the place of beginning. This plot is made on a scale of 10 chains to an inch.



(154.) To avoid the inconvenience of drawing a meridian through each angle of the field, the sides may be laid down from the angles which they make with each other, instead of the angles which they make with the meridian. Reverse one of the bearings, if necessary, so that both bearings may run from the same angular point; then the angle which any two contiguous sides make with each other may be determined from the following

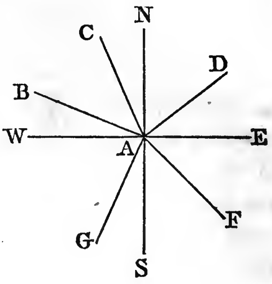
RULES.

1. If both courses are north or south, and both east or west, *subtract the less from the greater.*

2. If both are north or south, but one east and the other west, *add them together.*

3. If one is north and the other south, but both east or west, *subtract their sum from* 180°.

4. If one is north and the other south, one east and the other west, *subtract their difference from* 180°.



Thus the angle CAB is equal to NAB−NAC.

The angle CAD is equal to NAC +NAD.

The angle DAF is equal to 180° −(NAD+SAF).

The angle CAF is equal to 180° −(SAF−NAC).

In the preceding example we accordingly find the angle

ABC=105°. DEA=101°.
BCD=100°. EAB=104°.
CDE=130°.

With these angles the field may be plotted without drawing parallels.

EXAMPLE 2.

The following field notes are given to protract the survey:

Stations.	Bearings.	Distances.
1	N. 50° 30′ E.	16.50 chains.
2	S. 68° 15′ E.	14.20 "
3	S. 9° 45′ E.	8.45 "
4	S. 21° 0′ W.	6.84 "
5	S. 73° 30′ W.	12.31 "
6	N. 78° 15′ W.	9.76 "
7	N. 15° 30′ W.	11.55 "

THE TRAVERSE TABLE.

(155.) The accompanying traverse table shows the difference of latitude and the departure to four decimal places, for distances from 1 to 10, and for bearings from 0° to 90°, at intervals of 15′. If the bearing is less than 45°, the angle will be found on the left margin of one of the pages of the table, and the distance at the top or bottom of the page; the difference of latitude will be found in the column headed *Lat.* at the top of the page, and the departure in the column headed *Dep.* If the bearing is more than 45°, the angle will be found on the right margin, and the difference of latitude will be found in the column marked *Lat.* at the bottom of the page, and the departure in the other column. The latitudes and departures for different distances with the same bearing are proportional to the distances. Therefore the distances may be reckoned as tens, hundreds, or thousands, if the place of the decimal point in each departure and difference of latitude be changed accordingly.

Ex. 1. To find the latitude and departure for the course 45° and the distance 93.

Under distance 9 on page 141, and opposite 45°, will be found latitude 6.3640 and departure 6.3640. Hence, for distance 90, the latitude is 63.640, and adding the latitude for the distance 3, viz., 2.121, we find the latitude for distance 93 to be 65.761.

Ex. 2. To find the latitude and departure for the course 60° and the distance 11.85.

The latitude for 10 is	5.0000.	Departure for 10 is	8.6603.
" " " 1 is	.5000.	" " 1 is	.8660.
" " " .8 is	.4000.	" " .8 is	.6928.
" " " .05 is	.0250.	" " .05 is	.0433.
Latitude for 11.85 is	5.9250.	Depart. for 11.85 is	10.2624.

Ex. 3. To find the latitude and departure for the course 20° and the distance 5.30.

Ans. Latitude 4.98, and departure 1.81.

The traverse table may be used not only for obtaining departure and difference of latitude, but for finding by inspection the sides and angles of any right-angled triangle; for the latitude and departure form the two legs of a right-angled triangle, of which the distance is the hypothenuse, and the course is one of the acute angles.

In this manner we find the latitude and departure for each side of the field given in Example 1, page 105, to be as in the following table:

	Courses.	Dis-tances.	Latitude.		Departure.		Cor. Lat.	Cor. Dep.	Balanced.			
			N.	S.	E.	W.			N.	S.	E.	W.
1	N. 45° E.	9.30	6.58		6.58			+.01	6.58		6.59	
2	S. 60° E.	11.85		5.92	10.26		+.01	+.01		5.93	10.27	
3	S. 20° W.	5.30		4.98		1.81		—.01		4.98		1.80
4	S. 70° W.	10.90		3.73		10.24		—.01		3.73		10.23
5	N. 31° W.	9.40	8.06			4.84		—.01	8.06			4.83
	Perimeter	46.75	14.64	14.63	16.84	16.89			14.64	14.64	16.86	16.86

(156.) When a field has been correctly surveyed, and the latitudes and departures accurately calculated, the sum of the northings should be equal to the sum of the southings, and the sum of the eastings equal to the sum of the westings. If the northings do not agree with the southings, and the eastings with the westings, there must be an error either in the survey or in the calculation. In the preceding example, the northings exceed the southings by one link, and the westings exceed the eastings by five links. Small errors of this kind are unavoidable; but when the error does not exceed one link to a distance of three or four chains, it is customary to distribute the error among the sides by the following proportion:

As the perimeter of the field,
Is to the length of one of the sides,
So is the error in latitude or departure,
To the correction corresponding to that side.

This correction, when applied to a column in which the sum of the numbers is too small, is to be *added;* but if the sum of the numbers is too great, it is to be *subtracted.*

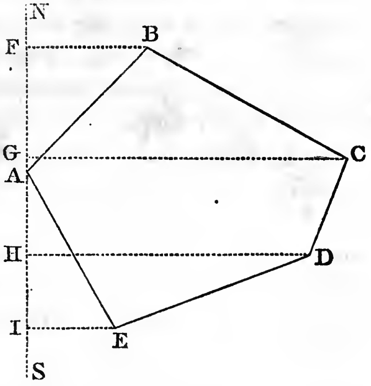
We thus obtain the corrections in columns 8 and 9 of the preceding table; and applying these corrections, we obtain the *balanced* latitudes and departures, in which the sums of the northings and southings are equal, and also those of the eastings and westings.

As the computations are generally carried to but two decimal places, the corrections of the latitudes and departures are only required to the nearest link, and these corrections may often be found by mere inspection without stating a formal proportion. Thus, in the preceding example, since the departures require a correction of five links, and the field has five sides which are not very unequal, it is obvious that we must make a correction of one link on each side.

It is the opinion of some surveyors that when the error in latitude or departure exceeds one link for every five chains of the perimeter, the field should be resurveyed; but most surveyors do not attain to this degree of accuracy. The error, however, should never exceed one link to a distance of two or three chains.

(157.) *To find the area of the field.*

Let ABCDE be the field to be measured. Through A, the most western station, draw the meridian NS, and upon it let fall the perpendiculars BF, CG, DH, EI.



Then the area of the required field is equal to FBCDEI−(ABF+AEI).

But FBCDEI is equal to the sum of the three trapezoids FBCG, GCDH, HDEI.

Also, if the sum of the parallel sides FB, GC be multiplied by FG, it will give twice the area of FBCG (Art. 87). The sum of the sides GC, DH, multiplied by GH, gives twice the area of GCDH; and the sum of HD, IE, multiplied by HI, gives twice the area of HDEI.

Now BF is the departure of the first side, GC is the sum of the departures of the first and second sides, HD is the algebraic sum of the three preceding departures, IE is the algebraic sum of the four preceding departures. Then the sum of the parallel sides of the trapezoids is obtained by adding together the preceding meridian distances two by two; and if these sums are multiplied by FG, GH, &c., which are the corresponding latitudes, it will give the double areas of the trapezoids.

(158.) It is most convenient to reduce all these operations to a tabular form, according to the following

RULE.

Having arranged the balanced latitudes and departures in *their appropriate columns, draw a meridian through the most eastern or western station of the survey, and, calling this the first station, form a column of double meridian distances.*

The double meridian distance of the first side is equal to its departure; and the double meridian distance of any side is equal to the double meridian distance of the preceding side, plus its departure, plus the departure of the side itself.

Multiply each double meridian distance by its corresponding northing or southing, and place the product in the column of north or south areas. The difference between the sum of the north areas and the sum of the south areas will be double the area of the field.

It must be borne in mind that by the term *plus* in this rule is to be understood the algebraic sum. Hence, when the double meridian distance and the departure are both east or both west, they must be added together; but if one be east and the other west, the one must be subtracted from the other.

The double meridian distance of the last side should always be equal to the departure for that side. This coincidence affords a check against any mistake in forming the column of double meridian distances.

The preceding example will then be completed as follows:

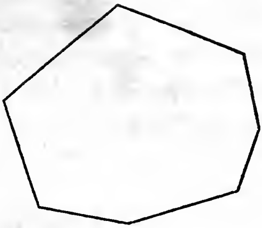
	N.	S.	E.	W.	D.M.D.	N. Areas.	S. Areas.
1	6.58		6.59		6.59	43.3622	
2		5.93	10.27		23.45		139.0585
3		4.98		1.80	31.92		158.9616
4		3.73		10.23	19.89		74.1897
5	8.06			4.83	4.83	38.9298	
						82.2920	372.2098

Twice the figure FBCDEI is 372.2098 square chains.
Twice the figure FBAEI is 82.2920 "
The difference is . . . . . 289.9178 "

Therefore the area of the field is 144.9589 square chains, or 14.49589 acres, which is equal to 14 acres, 1 rood, 39 perches.

Ex. 2. It is required to find the contents of a tract of land of which the following are the field notes:

Stations.	Bearings.	Distances.
1	N. 50° 30′ E.	16.50 chains.
2	S. 68° 15′ E.	14.20 "
3	S. 9° 45′ E.	8.45 "
4	S. 21° 0′ W.	6.84 "
5	S. 73° 30′ W.	12.31 "
6	N. 78° 15′ W.	9.76 "
7	N. 17° 0′ W.	11.64 "



Calculation.

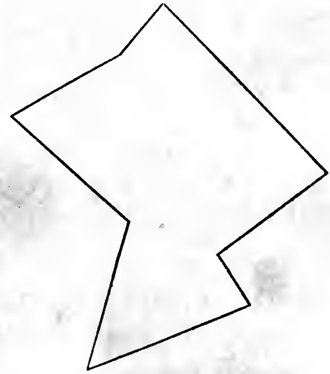
	Courses.	Dist.	Dif. Lat. N.	Dif. Lat. S.	Departure. E.	Departure. W.	Cor.	Balanced. N.	Balanced. S.	Balanced. E.	Balanced. W.	D.M.D.	N. Areas.	S. Areas.
1	N. 50° 30′ E.	16.50	10.50		12.73		.03	10.47		12.70		12.70	132.9690	
2	S. 68° 15′ E.	14.20		5.26	13.19		.03		5.29	13.16		38.56		203.9824
3	S. 9° 45′ E.	8.45		8.33	1.43		.01		8.34	1.42		53.14		443.1876
4	S. 21° 0′ W.	6.84		6.39		2.45	.01		6.40		2.46	52.10		333.4400
5	S. 73° 30′ W.	12.31		3.50		11.80	.02		3.52		11.82	37.82		133.1264
6	N. 78° 15′ W.	9.76	1.99			9.56	.02	1.97			9.58	16.42	32.3474	
7	N. 17° 0′ W.	11.64	11.13			3.40	.02	11.11			3.42	3.42	37.9962	
		79.70	23.62	23.48	27.35	27.21		23.55	23.55	27.28	27.28		203.3126	1113.7364
			Error .14		Error .14									203.3126

2)910.4238

Ans., 45 A., 2 R., 3 P. . . 455.2119.

Ex. 3. Required the area of a tract of land of which the following are the field notes:

Stations.	Bearings.	Distances.
1	N. 58° 45′ E.	19.84 chains.
2	N. 39° 30′ E.	10.45 "
3	S. 45° 15′ E.	37.26 "
4	S. 52° 30′ W.	21.53 "
5	S. 34° 0′ E.	9.12 "
6	S. 66° 15′ W.	27.69 "
7	N. 12° 45′ E.	24.31 "
8	N. 48° 15′ W.	24.60 "



Ans., 130 A., 2 R., 23 P.

Ex. 4. Required the area of a piece of land from the following field notes:

Stations.	Bearings.	Distances.
1	N. 5° 15′ E.	15.17 chains.
2	N. 45° 45′ E.	16.83 "
3	N. 32° 0′ W.	14.26 "
4	N. 88° 30′ E.	19.54 "
5	S. 28° 15′ E.	17.92 "
6	S. 40° 45′ W.	9.71 "
7	S. 31° 30′ E.	22.65 "
8	S. 14° 0′ W.	18.39 "
9	S. 82° 45′ W.	24.80 "
10	N. 23° 15′ W.	26.31 "

Ans., 173 A., 0 R., 23 P.

Ex. 5. Required the area of a field from the following notes:

Stations.	Bearings.	Distances.
1	N. 32° 15′ E.	28.74 chains.
2	N. 17° 45′ E.	21.59 "
3	S. 81° 30′ E.	13.38 "
4	S. 9° 45′ W.	11.92 "
5	S. 43° 0′ E.	19.65 "
6	N. 25° 30′ E.	17.26 "
7	S. 78° 15′ E.	18.87 "
8	S. 5° 45′ W.	31.41 "
9	S. 37° 30′ W.	26.13 "
10	N. 69° 0′ W.	23.86 "
11	S. 74° 15′ W.	20.91 "
12	N. 27° 30′ W.	23.20 "

Ans., 304 A., 2 R., 9 P.

Ex. 6. Required the area of a field from the following notes:

Stations.	Bearings.	Distances.
1	N. 36° 15′ E.	24.73 chains.
2	N. 7° 45′ E.	11.58 "
3	N. 79° 30′ E.	15.39 "
4	S. 86° 45′ E.	20.56 "
5	S. 12° 15′ W.	18.14 "
6	S. 25° 0′ E.	21.92 "
7	S. 58° 30′ W.	29.27 "
8	N. 34° 0′ W.	19.81 "
9	N. 81° 15′ W.	21.24 "

Ans.. 179 A., 1 R., 6 P.

(159.) The field notes from which the area is to be computed may be imperfect. There may be obstacles which prevent the measuring of one side, or the notes may be defaced so as to render some of the numbers illegible. If the bearings and lengths of all the sides of a field *except one* are given, the remaining side may easily be found by calculation. For the difference between the sum of the northings and the sum of the southings of the given sides will be the northing or southing of the remaining side; and the difference between the sum of the eastings and the sum of the westings of the given sides will be the easting or westing of the remaining side. Having, then, the difference of latitude and departure of the required side, its length and direction are easily found by Trigonometry (Art. 47).

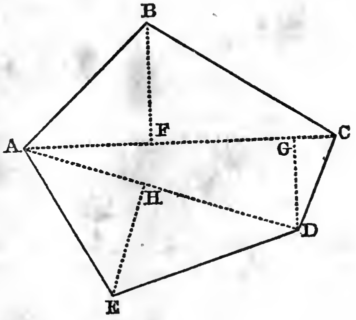
Ex. Given the bearings and lengths of the sides of a tract of land as follows:

Stations.	Bearings.	Distances.
1	N. 18° 15′ E.	8.93 chains.
2	N. 79° 45′ E.	15.64 "
3	S. 25° 0′ E.	14.27 "
4	*Unknown.*	*Unknown.*
5	N. 87° 30′ W.	18.52 chains.
6	N. 41° 15′ W.	12.18 "

Required the bearing and distance of the fourth side.

Ans., S. 15° 33′ E., distance 8.62 chains.

(160.) There is another method of finding the area of a field which may be practiced when great accuracy is not required. It consists in first drawing a plan of the field, as in Art. 153, then dividing the field into triangles by diagonal lines, and measuring the bases and perpendiculars of the triangles upon the same scale of equal parts by which the plot was drawn. Thus, if we take Ex. 1, and draw the diagonals AC, AD, the field will be divided into three triangles, whose area is easily found when we know



the diagonals AC, AD, and the perpendiculars BF, DG, EH. The diagonal AC is found by measurement upon the scale of equal parts to be 16.87; the diagonal AD is 15.67; the perpendicular BF is 6.30; DG is 4.92; and EH is 6.42. Hence

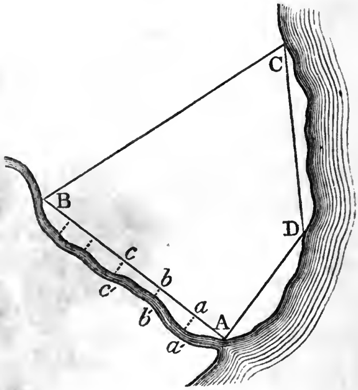
the triangle	$ABC=16.87\times3.15=$	53.14
" "	$ADC=16.87\times2.46=$	41.50
" "	$ADE=15.67\times3.21=$	50.30
the figure ABCDE	$=$	144.94 sq. chains.

This method of finding the area of a field is very expeditious, and when the plot is carefully drawn, may afford results sufficiently precise for many purposes.

(161.) *To survey an irregular boundary by means of offsets.*

When the boundaries of a field are very irregular, like a river or lake shore, it is generally best to run a straight line, coming as near as is convenient to the true boundary, and measure the perpendicular distances of the prominent points of the boundary from this line.

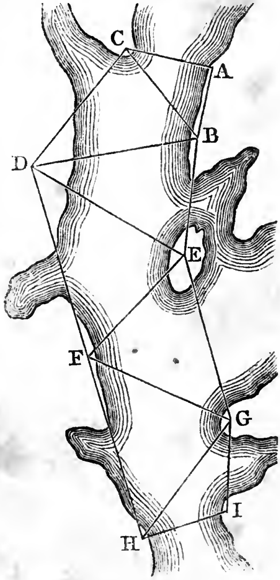
Let ABCD be a piece of land to be surveyed; the land being bounded on the east by a lake, and on the west by a creek. We select stations A, B, C, D, so as to form a polygon which shall embrace most of the proposed field, and find its area. We then measure perpendiculars aa', bb', cc', &c., as also the distances Aa, ab, bc, &c. Then, considering the spaces Aaa', $abb'a'$, &c., as triangles or trapezoids, their area may be computed; and, adding these areas to the figure ABCD, we shall obtain the area of the proposed field nearly.



(162.) *To determine the bearing and distance from one point to another by means of a series of triangles.*

When it is required to find the distance between two points remote from each other, we form a series of triangles such that the first and second triangles may have one side in common; the second and third, also, one side in common; the third and fourth, &c. We then measure one side of the first triangle for a *base line*, and all the angles in each of the triangles. These data are sufficient to determine the length of the sides of each triangle; for in the first triangle we have one side and the angles to find the other sides. When these are found, we shall have one side and all the angles of a second triangle to find the other sides. In the same manner we may calculate the dimensions of the third triangle, the fourth, and so on. We shall illustrate this method by an example taken from the Coast Survey of the United States.

The object here is to make a survey of Chesapeake Bay and its vicinity; to determine with the utmost precision the position of the most prominent points of the country, to which subordinate points may be referred, and thus a perfect map of the country be obtained. Accordingly, a level spot of ground was selected on the eastern side of the bay, on Kent Island, where a base line, AB, of more than five miles in length, was measured with every precaution. A station, C, was also selected upon the other side of the bay, near Annapolis, so situated that it was visible from A and B. The three angles of the triangle ABC were then measured with a large theodolite, after which the length of BC may be computed.

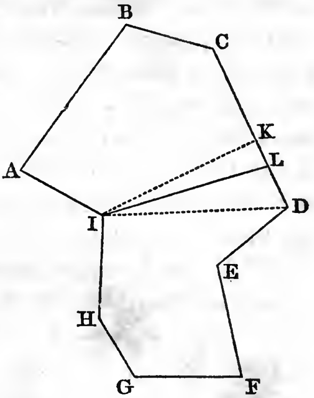


A fourth station, D, is now taken on the western shore of the bay, visible from C and B, and all the angles of the triangle BCD are measured, when the line BD can be computed. A fifth station, E, is now taken on an island near the eastern shore, visible both from B and D, and all the angles of the triangle BDE are measured, when DE can be computed. Also, all the angles of the triangle DEF are measured, and EF is computed. Then all the angles of the triangle EFG are measured, and FG is computed. So, also, all the angles of the triangle FGH are measured, and GH is computed; and thus a chain of triangles may be extended along the entire coast of the United States. To test the accuracy of the work, it is common to measure a side in one of the triangles remote from the first base, and compare its measured length with that deduced by computation from the entire series of triangles. This line is called a *base of verification.* Such a base has been measured on Long Island; and, indeed, several bases have been measured on different points of the coast. These are all connected by a triangulation, and thus the length of a side in any triangle may be deduced from more than one base line, and the agreement of these results is a test of the accuracy of the entire work. Thus the length of one of the sides of a triangle which was twelve miles, as deduced from the Kent Island base, differed only twenty inches from that derived from the Long Island base, distant two hundred miles.

The superiority of this method of surveying arises from the circumstance that it is necessary to measure but a small number of base lines along a coast of a thousand or more miles in extent; and for these the most favorable ground may be selected any where in the vicinity of the system of triangles. All the other quantities measured are angles; and the precision of these measurements is not at all impaired by the inequalities of the surface of the ground. Indeed, mountainous countries afford peculiar facilities for a trigonometrical survey, since they present heights of ground visible to a great distance, and thus permit the formation of triangles of very large dimensions.

(163.) *To divide an irregular piece of land into any two given parts.*

We first run a line, by estimation, as near as may be to the required division line, and compute the area thus cut off. If this is found too large or too small, we add or subtract a triangle, or some other figure, as the case may require. Suppose it is required to divide the field ABCDEFGHI into two equal parts, by a line IL, running from the corner I to the opposite side CD. We first draw a line from I to D, and compute the area of the part DEFGHI; and, knowing the area of the entire field, we learn the area which must be contained in the triangle DIL, in order that IL may divide the field into two equal parts. Having the bearings and distances of the sides DE, EF, &c., we can compute the bearing and distance of DI. Thus the angle IDK is known; and, having the hypothenuse ID, we can compute the length of the perpendicular IK let fall on CD. Now the base of a triangle must be equal to its area divided by half the altitude. Hence, if we divide the area of the triangle DIL by half of IK, it will give DL.



In a similar manner we might proceed if it was required to divide a tract of land into any two given parts.

Variation of the Needle.

(164.) The line indicated by a magnetic needle, when freely supported and allowed to come to a state of rest, is called the *magnetic meridian*. This does not generally coincide with the *astronomical meridian*, which is a true north and south line.

The angle which the magnetic meridian makes with the true meridian is called the *variation of the needle*, and is said to be east or west, according as the north end of the needle points east or west of the north pole of the earth.

The variation of the needle is different in different parts of the earth. In some parts of the United States it is 10° west, and in others 10° east, while at other places the variation has every intermediate value. Even at the same place, the variation does not remain constant for any length of time. Hence it is necessary frequently to determine the amount of the variation, which is easily done when we know the position of the true meridian. The latter can only be determined by astronomical observations. The best method is by observations of the pole star. If this star were exactly at the pole, it would always be on the meridian; but, being at a distance of about a degree and a half from the pole, it revolves about the pole in a small circle in a little less than 24 hours. In about six hours from its passing the meridian above the pole, it attains its greatest distance west of the meridian; in about six hours more it is on the meridian beneath the pole; and in about six hours more it attains its greatest distance east of the meridian. If the star can be observed at the instant when it is on the meridian, either above or below the pole, a true north and south line may be obtained.

(165.) The following table shows the time of the pole star's passing the meridian above the pole for every fifth day of the year:

	1st Day.	6th Day.	11th Day.	16th Day.	21st Day.	26th Day.
	h. m.	h. m.	h. m.	h. m.	h. m.	h. m.
January . . .	6 20 P.M.	6 0 P.M.	5 41 P.M.	5 21 P.M.	5 1 P.M.	4 42 P.M.
February. . .	4 18 "	3 58 "	3 39 "	3 19 "	3 0 "	2 40 "
March	2 28 "	2 8 "	1 49 "	1 29 "	1 9 "	0 50 "
April	0 26 "	0 7 "	11 47 A.M.	11 27 A.M.	11 8 A.M.	10 48 A.M.
May	10 28 A.M.	10 9 A.M.	9 49 "	9 29 "	9 9 "	8 50 "
June	8 26 "	8 7 "	7 47 "	7 27 "	7 8 "	6 48 "
July	6 28 "	6 9 "	5 49 "	5 29 "	5 10 "	4 50 "
August	4 27 "	4 7 "	3 47 "	3 27 "	3 8 "	2 48 "
September . .	2 25 "	2 5 "	1 45 "	1 26 "	1 6 "	0 46 "
October . . .	0 26 "	0 7 "	11 43 P.M.	11 24 P.M.	11 4 P.M.	10 44 P.M.
November . .	10 21 P.M.	10 1 P.M.	9 41 "	9 22 "	9 2 "	8 42 "
December . .	8 23 "	8 3 "	7 43 "	7 24 "	7 4 "	6 44 "

If the pole star passes the meridian in the daytime, it can not be observed without a good telescope; but $11^{h.}\ 58^{m.}$ after the dates in the above table, the star will be on the meridian *below* the pole, and during the whole year, except in summer, the pole star may be seen with the naked eye on the meridian either above or below the pole. These observations are best made with a theodolite, but they may be made with a common compass. At $5^{h.}\ 59^{m.}$ after the dates in the above table, the star will have attained its greatest distance west of the meridian; and $5^{h.}\ 59^{m.}$ before these dates, it will be at its greatest distance east of the meridian. In summer, therefore, we may observe the greatest eastern elongation of the pole star, at which time the star is $1^\circ\ 55'$ east of the true meridian for all places in the neighborhood of New York. Making this allowance, a true meridian is easily obtained; after which, the variation of the needle is determined by placing a compass upon this line, turning the sights in the same direction, and noting the angle shown by the needle.

The following table shows the angle which the plane of the meridian makes with a vertical plane passing through the pole star, when at its greatest eastern or western elongation, for any latitude from 30° to 44°.

Lat. 30°	Lat. 32°	Lat. 34°	Lat. 36°	Lat. 38°	Lat. 40°	Lat. 42°	Lat. 44°
1° 41′	1° 43′	1° 45′	1° 48′	1° 51′	1° 54′	1° 58′	2° 2′

(166.) The variation of the needle, in 1840, for several parts of the United States, was as follows:

Burlington, Vt. . .	9° 27′ W.	Buffalo, N. Y. . .	1° 37′ W.
Boston, Mass. . .	9° 12′ W.	Cleveland, Ohio .	0° 19′ E.
Albany, N. Y. . .	6° 58′ W.	Detroit, Mich. . .	1° 56′ E.
New Haven, Ct. .	6° 13′ W.	Charleston, S. C. .	2° 44′ E.
New York City .	5° 34′ W.	Cincinnati, Ohio .	4° 46′ E.
Philadelphia . . .	4° 8′ W.	Mobile, Ala. . . .	7° 5′ E.
Washington City .	1° 20′ W.	St. Louis, Mo. . .	8° 37′ E.

Since 1840, the variation in New England has increased about five minutes annually; in New York and Pennsylvania it has increased from three to four minutes annually. In the Western States it decreases at about the same rate, and in the Southern States it decreases about two minutes annually.

LEVELING.

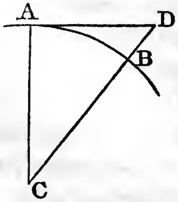
(167.) *Leveling* is the art of determining the difference of level between two or more places.

The surface of an expanse of tranquil water, or any surface parallel to it, is called a *level surface.* Points situated in a level surface are said to be on the *same level*, and a line traced on such a surface is called *a line of true level.*

On account of the globular figure of the earth, a level surface is not a plane surface. It is nearly spherical; and in the common operations of leveling it is regarded as perfectly so. Hence every point of a level surface is regarded as at the same distance from the center of the earth; and the *difference of level* of two places is the difference between their distances from the center.

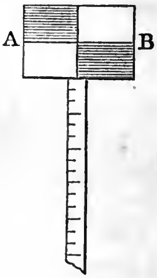
A line of *apparent level* is a straight line tangent to the surface of the earth.

Thus, if AB represent the surface of the ocean, the two places A and B are said to be on the same level; but if AD be drawn tangent to the arc AB at A, then AD is a line of apparent level.



This is the line which is indicated by a leveling instrument placed at A. The theodolite may be employed for tracing horizontal lines; but if nothing further were required, there would be no occasion for graduated circles, and several parts of the theodolite might be dispensed with. A leveling instrument, therefore, usually consists of a large spirit level attached to a telescope, mounted upon a stand in a manner similar to the theodolite.

(168.) The surveyor should also be provided with a pair of *leveling staves.* A leveling staff consists of a rectangular bar of wood six feet in length, divided to inches and sometimes tenths of an inch, and having a groove running its entire length. A smaller staff of the same length, called a *slide*, also divided into inches, is inserted in this groove, and moves freely along it.

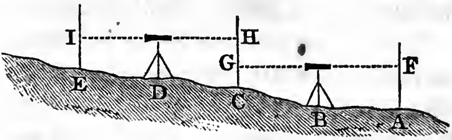


At the upper end of the slide is a rectangular board called a *vane*, AB, about six inches wide. The vane is divided into four equal parts by two lines, one horizontal and the other vertical. Two opposite parts of the vane are painted white, and the other two black, in order that they may be distinguished at a great distance.

To find the difference of level between any two points.

(169.) Set up the leveling staves perpendicular to the horizon, and at equal distances from the leveling instrument. Having adjusted the level by means of the proper screws, turn the telescope to one of the staves, and direct an assistant to slide up the vane until the line AB coincides with the center of the telescope, and note the height of this line from the ground. Turn the telescope to the other staff, and repeat the same operation. Level in the same manner from the second station to the third, from the third to the fourth, &c. Then the difference between the sum of the heights at the back stations and at the forward stations will be equal to the difference of level between the first station and the last.

If we wish to level from A to E, we set up the staves at a convenient distance, AC, and midway between them place the level B. Observe where the line of level, FG, cuts the rods, and note the heights AF, CG. Their difference is the difference of level between the first and second stations. Take up the level and place it at D, midway between the rods C and E, and observe where the line of level, HI, cuts the rods, and note the heights CH, EI.



Then FA−CG=the ascent from A to C,
and CH−EI =the ascent from C to E.

Therefore (FA+CH)−(CG+EI)=the entire ascent from A to E; and in the same manner we may find the difference of level for any distance; that is, *the difference between the sum of the heights at the back stations and at the forward stations is equal to the difference of level between the first station and the last.*

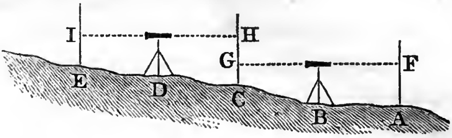
(170.) The following is a copy of the field notes for running a level from A to E:

	Back sights. Feet.	Inches.	*Fore sights.* Feet.	Inches.
	0	4	3	2
	5	10	5	7
	4	2	4	3
	5	6	1	2
	4	11	3	2
	4	7	1	3
	6	1	2	0
Sum	31	5	20	7

The back sights being greater in amount than the forward sights, it is evident that E is higher than A by 10 feet 10 inches.

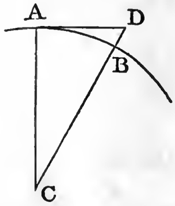
The heights indicated by the leveling staves are sometimes read off by the assistant, but it is better for the observer to read off the quantities himself through the telescope of his leveling instrument. This may easily be done provided the graduation of the staff is perfectly distinct; and in that case it is only necessary to rely upon the assistant to hold the staff perpendicularly. To enable him to this, a small plummet is suspended in a groove cut in the side of the staff.

(171.) It must be observed that the lines GF, HI are lines of *apparent* level, and not of *true* level; nevertheless, we shall obtain the true difference of level between A and E by this method if the leveling instrument is placed midway between the leveling staves, because the points G and F will in that case be at equal distances from the earth's center. If the level is not placed midway between the staves, then we must apply a correction for the difference between the true and apparent level.



(172.) *To find the difference between the true and apparent level.*

Let C be the center of the earth, AB a portion of its surface, and AD a tangent to the earth's surface at A; then BD is the difference between the true and apparent level for the distance AD.



Now, by Geom., Prop. 11, B. IV.,

$$CD^2 = AC^2 + AD^2.$$

Hence $$CD = \sqrt{AC^2 + AD^2},$$

and $$BD = \sqrt{AC^2 + AD^2} - BC.$$

If we put $R = BC$, the radius of the earth,

$d = AD$,

and $h = BD$, the difference between the true and apparent level, we shall have

$$h = \sqrt{R^2 + d^2} - R;$$

that is, to find the difference between the true and apparent level for any distance, *add the square of the distance to the square of the earth's radius, extract the square root of the sum, and subtract the radius of the earth.*

If BD represent a mountain, or other elevated object, then AD will represent the distance at which it can be seen in consequence of the curvature of the earth.

Ex. 1. If the diameter of the earth be 7912 miles, and if Mount Ætna can be seen at sea 126 miles, what is its height?

Ans., 2 miles.

Ex. 2. If a straight line from the summit of Chimborazo touch the surface of the ocean at the distance of 179 miles, what is the height of the mountain? *Ans.*, 4.05 miles.

From the preceding formula we obtain

$$R^2+d^2=(R+h)^2$$
$$=R^2+2Rh+h^2,$$

that is, $$d^2=2Rh+h^2.$$

But in the common operations of leveling, h is very small in comparison with the radius of the earth, and h^2 is very small in comparison with $2Rh$. If we neglect the term h^2, we have

$$d^2=2Rh;$$

whence $$h=\frac{d^2}{2R};$$

that is, *the difference between the true and apparent level is nearly equal to the square of the distance divided by the diameter of the earth.*

Ex. 1. What is the difference between the true and apparent level for one mile, supposing the diameter of the earth to be 7912 miles? *Ans.*, 8.008 inches, or 8 inches nearly.

Ex. 2. What is the difference between the true and apparent level for half a mile? *Ans.*, 2 inches.

In the equation $h=\frac{d^2}{2R}$, since $2R$ is a constant quantity, h varies as d^2; that is, *the difference between the true and apparent level varies as the square of the distance.*

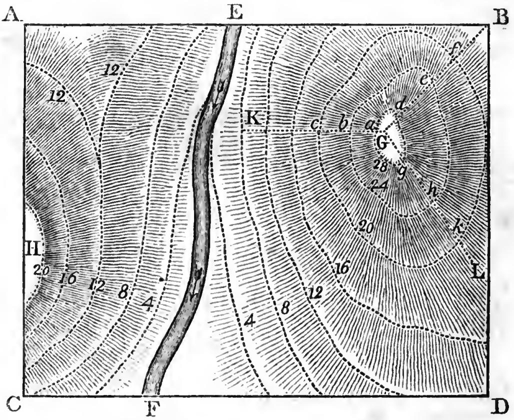
Hence, the difference for 1 mile being 8 inches,

							Ft.	In
the difference for	2	miles is	$8\times2^2=$	32	inches	=	2	8.
" "	3	"	$8\times3^2=$	72	"	=	6	0.
" "	4	"	$8\times4^2=$	128	"	=	10	8.
" "	5	"	$8\times5^2=$	200	"	=	16	8.
" "	6	"	$8\times6^2=$	288	"	=	24	0, &c.

Topographical Maps.

(173.) It is sometimes required to determine and represent upon a map the undulations and inequalities in the surface of a tract of land. Such a map should give a complete view of the ground, so as to afford the means for an appropriate location of buildings or extensive works. For this purpose, we suppose the surface of the ground to be intersected by a number of horizontal planes, at equal distances from each other. The lines in which these planes meet the surface of the ground, being transferred to paper, will indicate the variations in the inclination of the ground; for it is obvious that the curves will be nearer together or further apart, according as the ascent is steep or gentle.

Thus, let ABCD be a tract of broken ground, divided by a stream, EF, the ascent being rapid on each bank, the ground swelling to a hill at G, and also at H. It is required to represent these inequalities upon paper, so as to give an exact idea of the face of the ground. The lowest point of the ground is at F. Suppose the tract to be intersected by a horizontal plane four feet above F, and let this plane intersect the surface of the ground in the undulating lines marked 4, one on each side of the stream. Suppose a second horizontal plane to be drawn eight feet above F, and let it intersect the surface of the ground in the lines marked 8. Let other horizontal planes be drawn at a distance of 12, 16, 20, 24, &c., feet above the point F. The projection of these lines of level upon paper shows at a glance the outline of the tract. We perceive that on the right bank of the stream the ground rises more rapidly on the upper than on the lower portion of the map, as is shown by the lines of level being nearer to one another. On the right bank of the stream the ascent is uninterrupted until we reach G, which is the summit of the hill. Beyond G the ground descends again toward B. On the left



bank of the stream the ground rises to H; but toward A the level line of 12 feet divides into two branches, and between them the ground is nearly level.

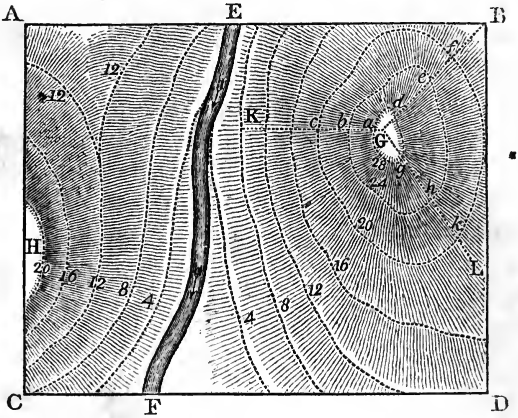
(174.) The surveys requisite for the construction of such a map may be made with a theodolite or common level.

The object is to trace a series of level lines upon the surface of the ground. For this purpose we may select any point on the surface of a hill, place the level there, and run a level line around the hill, measuring the distances, and also the angles, at every change of direction. We may then select a second point at any convenient distance above or below the former, and trace a second level line around the hill, and so on for as many curves as may be thought necessary. Such a method, however, would not always be most convenient in practice.

(175.) The following method may sometimes be preferable: Set up the level on the summit of the hill at G, and fix the vane on the leveling staff at an elevation of four feet in addition to the height of the telescope above the ground. Then direct an assistant to carry the leveling staff, holding it in a vertical position, toward K, till he arrives at a point, as *a*, where the vane appears to coincide with the cross wires of the telescope. This will determine one point of the curve line four feet below G. The assistant may then proceed to the line GB, and afterward to GL, moving backward or forward in each of those directions till he finds points, as *d* and *g*, at which the vane coincides with the cross wires of the telescope. The horizontal distance between G and *a*, G and *d*, G and *g*, must then be measured.

If the leveling staff is sufficiently long, the vane may be fixed on it at the height of eight feet, in addition to the height of the telescope at G; and the assistant, placing himself in the directions GK, GB, GL, must move till the vane appears to coincide with the cross wires as before. The horizontal distances *ab*, *de*, *gh*, must then be measured, and stakes driven into the ground at *b*, *e*, and *h*.

The level must now be removed to *b*; and the vane being fixed on the staff at a height equal to four feet, together with the height of the instrument from the ground at *b*, the assistant must proceed in the direction *b*K, and stop at *c* when



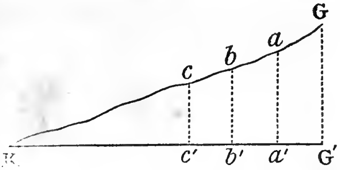
the vane coincides with the cross wires; then the horizontal distance of *c* from *b* must be measured. In a like manner, the operations may be continued from *b* or *c* as far as necessary toward K; then, commencing at *e*, and afterward at *h*, they may be continued in the same way toward B and L respectively. The angles which the directions GK, GB, GL make with the magnetic meridian being found with the compass, these directions may be represented on paper. Then the measured distances G*a*, *ab*, &c.; G*d*, *de*, &c.; G*g*, *gh*, &c., being set off on those lines of direction, curves drawn through *a*, *d*, *g*; *b*, *e*, *h*; *c*, *f*, *k*, &c., will show the contour of the hill.

The map is shaded so as to indicate the hills and slopes by drawing fine lines, as in the figure, perpendicular to the horizontal curves.

(176.) Another method, which may often be more convenient than either of the preceding, is as follows: From the summit of the hill measure any line, as GK, and at convenient points of this line let stakes be driven, and their distances from G be carefully measured. Then determine the difference of level of all these points; and if the assumed points do not fall upon the horizontal curves which are required to be delineated, we may, by supposing the slope to be uniform from one stake to another, compute by a proportion the points where the horizontal curves for intervals of four feet intersect the line GK. The same may be done for the lines GB and GL, and for other lines, if they should be thought necessary.

(177.) If the surface of the ground is gently undulating, it may be more convenient to run across the tract a number of lines parallel to one another. Drive stakes at each extremity of these lines, and also at all the points along them where there is any material change in the inclination of the ground, and find the difference of level between all these stakes, and their distances from each other. Then, if we wish to draw upon a map the level lines at intervals of 4, 6, or 10 feet, we may compute in the manner already explained the points where the horizontal curves intersect each of the parallel lines. The curve lines are then to be drawn through these points, according to the judgment of the surveyor.

(178.) If it is required to draw a *profile* of the ground, for example, from G to K, draw a straight line, G′K, to represent a horizontal line to which the heights are referred, and set off G′a′, G′b′, G′c′, &c., equal to the distances of the stations from the beginning of the line. At the points G′, a′, b′, &c., erect perpendiculars, G′G, a′a, &c., and make them equal to the heights of the respective stations. Through the tops of these perpendiculars draw the curved line GK, and it will be the profile of the hill in the direction of the line GK.

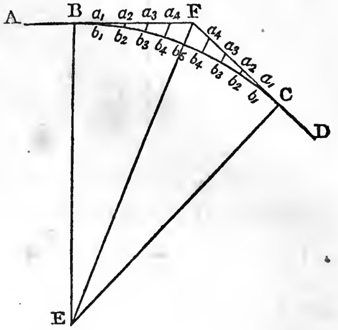


On setting out Rail-way Curves.

(179.) It is of course desirable that the line of a rail-way should be perfectly straight and horizontal. This, however, is seldom possible for any great distance; and when it becomes necessary to change the direction of the line, it should be done gradually by a curve. The curve almost universally employed for this purpose is the arc of a circle, and such an arc may be traced upon the ground by either of the following methods.

First Method.—When the center of the circle can be seen from every part of the curve.

Let AB, CD be two straight portions of a road which it is desired to connect by an arc of a circle. Set up a theodolite at B and another at C, and from each point range a line at right angles to the lines AB and CD respectively; and at the intersection of these lines, E, which will be the center of the circle, erect a signal which can be seen from any point between B and C. Produce the lines AB and CD until they meet in F, and on these lines drive stakes at equal distances, a_1, a_2, a_3, commencing from the points B and C. If r represents the radius of the circle, and d the distance between the points a_1, a_2, a_3, &c., then (Art. 172),



$$\sqrt{r^2+d^2}-r$$

will be the distance which must be set off from the first point a_1, in the direction a_1E, to obtain a point of the circular arc. In like manner,

$$\sqrt{r^2+(2d)^2}-r$$

will be the distance to be set off from the point a_2, in the direction a_2E; and, generally,

$$\sqrt{r^2+(nd)^2}-r$$

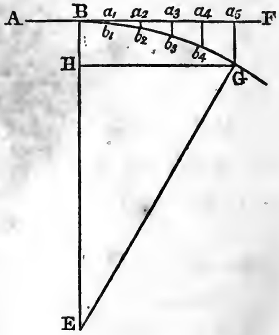
will be the distance to be set off at the nth points from B and C. For example, let r be one mile, or 5280 feet, and d equal to 100 feet; then,

$$\sqrt{5280^2+100^2}-5280=.94 \text{ feet},$$

will be the distance a_1b_1. In a similar manner, we find at

a_2, or 200 feet from B,	the offset	will be		3.79	feet.
a_3, or 300	"	"	"	8.52	"
a_4, or 400	"	"	"	15.13	"
a_5, or 500	"	"	"	23.62	"

(180.) *Second Method.*—When the center of the circle can not be seen from every part of the curve, the offsets may be set off perpendicularly to the tangent BF, in which case they must be computed from the formula



$$r-\sqrt{r^2-d^2}.$$

For, in the annexed figure,

$$\text{EH}=\sqrt{\text{GE}^2-\text{GH}^2};$$

that is, $\text{EH}=\sqrt{r^2-d^2}$.

And $$a_5\text{G}=\text{BH}=\text{BE}-\text{HE}=r-\sqrt{r^2-d^2}.$$

If $r=5280$ feet, we shall find the offsets at intervals of 100 feet to be

$a_1b_1=$ .95 feet.
$a_2b_2=$ 3.79 "
$a_3b_3=$ 8.53 "
$a_4b_4=$15.17 "
$a_5b_5=$23.73 "

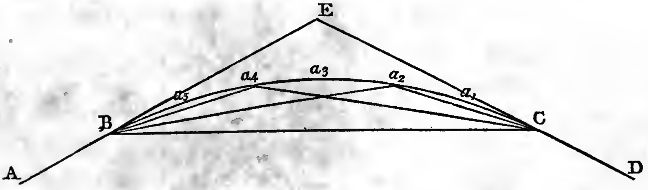
For small distances, the offsets will be given with sufficient accuracy by the formula

$$\frac{d^2}{2r},$$

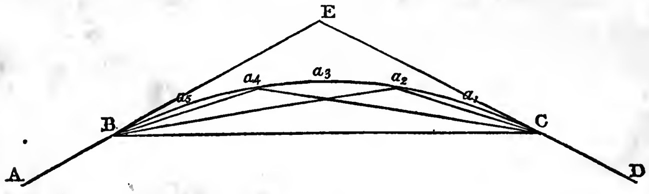
see Art. 172.

It is very common for surveyors, after they have found the first point, b_1, of the curve, to join the points B, b_1, and produce the line Bb_1 to the distance d, and from the end of this line set off an offset to determine the point b_2; then, producing the line b_1b_2, set off a third offset to determine the point b_3, and so on. The objection to this method is, that any error committed in setting out one of the points of the curve will occasion an error in every succeeding one. Whenever this method, therefore, is employed, it should be checked by determining the position of every fourth or fifth point by independent computation and measurement.

(181.) *Third Method.*—Where the radius of the curve is small, place a theodolite at B, and point its telescope toward



C. Place another theodolite at C, and point its telescope toward E, the point of intersection of the lines AB, CD produced. Then, if the former be moved through any number of degrees toward a_1, and the latter the same number of degrees toward a_1, the point a_1 will be a point of the curve, for the angle Ba_1C will be equal to BCD (*Geom.*, Prop. 16, B. III.). In the same manner, a_2, a_3, &c., any number of points of the curve, may be determined. It will be most convenient to move the theodolites each time through an even number of degrees, for example, an arc of two degrees, and a stake must be driven at each of the points of intersection a_1, a_2, a_3, &c. The accuracy of this method is independent of any undulations in the surface of the ground, so that in a hilly country this method may be preferable to any other.



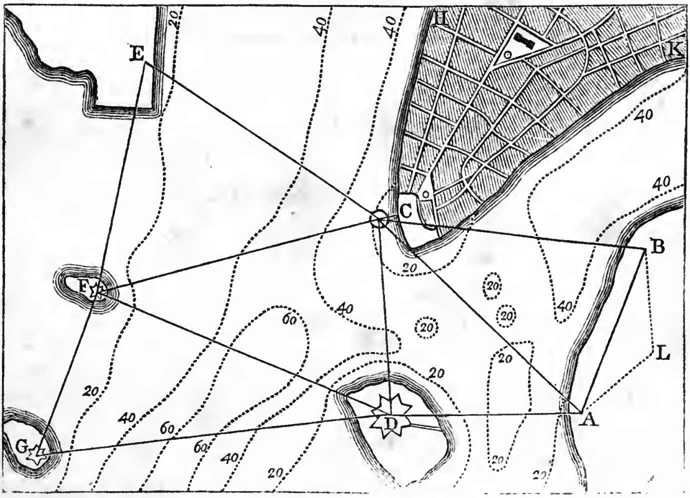
When the position of one end of the curve is not absolutely determined, the engineer may proceed more rapidly. Suppose it is required to trace an arc of a circle having a curvature of two degrees for a hundred feet.

Place a theodolite at C, the point where the curve commences, and lay off from the line CE, toward B, an angle of two degrees, and in the direction of the axis of the instrument set off a distance of 100 feet, which will give the first point a_1 of the curve. Next lay off from CE an angle of four degrees, and from a_1 set off a distance of 100 feet, and the point where this line cuts the axis of the instrument produced will be the second point a_2. In the same manner, lay off from CE an angle of six degrees, and from a_2 set off a distance of 100 feet, and the point where it cuts the axis of the instrument produced will be the third point a_3. All the points a_1, a_2, a_3, etc., thus determined lie in the circumference of a circle (*Geom.*, Prop. 15, B. III.). Circles thus drawn are generally made with a curvature of one or two degrees, or some convenient fraction of a degree, for every hundred feet. This method is very extensively practiced in the United States.

Surveying Harbors.

(182.) In surveying a harbor, it is necessary to determine the position of the most conspicuous objects, to trace the outline of the shore, and discover the depth of water in the neighborhood of the channel. A smooth, level piece of ground is chosen, on which a base line of considerable length is measured, and station staves are fixed at its extremities. We also erect station staves on all the prominent points to be surveyed, forming a series of triangles covering the entire surface of the harbor. The angles of these triangles are now measured with a theodolite, and their sides computed. After the principal points have been determined, subordinate points may be ascertained by the compass or plane table.

Let the following figure be a map of a harbor to be survey-



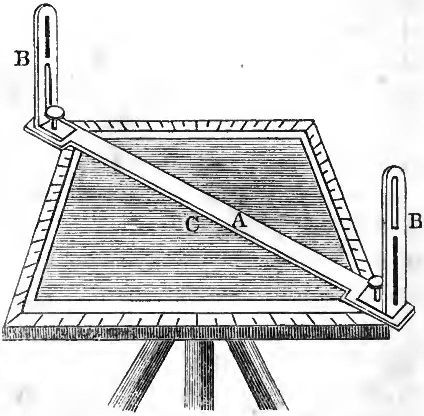
ed. We select the most favorable position for a base line, which is found to be on the right of the harbor, from A to B. We also erect station flags at the points C, D, E, F, and G. Having carefully measured the base line AB, we measure the three angles of the triangle ABC, which enables us to compute the remaining sides. We then measure the three angles of the triangle ACD, and by means of the side AC, just computed, we are enabled to compute AD and CD. We then measure the three angles of the triangle CDF, and by means of the side CD, just found, we are enabled to compute CF and DF. Proceeding in the same manner with the triangles CEF, DFG, we are enabled, after measuring the angles, to compute the sides.

(183.) Having determined the main points of the harbor, we may proceed to a more detailed survey by means of the chain and compass. If it is required to trace the shore, HCK, we commence at H, and observe the bearings with the compass, and measure the distances with the chain. Where the shore is undulating, it is most convenient to run a straight line for a considerable distance, and at frequent intervals measure offsets to the shore.

When a great many objects are to be represented upon a map, the most convenient instrument is

The Plane Table.

(184.) The plane table is a board about sixteen inches square, designed to receive a sheet of drawing paper, and has two plates of brass upon opposite sides, confined by screws, for stretching and retaining the paper upon the board. The margin of the board is divided to 360 degrees from a center C, in the middle of the board, and these are subdivided as minutely as the size of the table will admit. On one side of the board there is usually a diagonal scale of equal parts. A compass box is sometimes attached, which renders the plane table capable of answering the purpose of a surveyor's compass.



The ruler, A, is made of brass, as long as the diagonal of the table, and about two inches broad. A perpendicular sight-vane, B, B, is fixed to each extremity of the ruler, and the eye looking through one of them, the vertical thread in the other is made to bisect any required distant object.

To the under side of the table, a center is attached with a ball and socket, or parallel plate screws, like those of the theodolite, by which it can be placed upon a staff-head; and the table may be made horizontal by means of a detached spirit level.

(185.) To prepare the table for use, it must be covered with drawing paper. Then set up the instrument at one of the stations, for example, B (see fig. on p. 131), and fix a needle in the table at the point on the paper representing that station, and place the edge of the ruler against the needle. Then direct the sights to the station A, and by the side of the ruler draw a line upon the paper to represent the direction of AB. Then, with a pair of dividers, take from the scale a certain number of equal parts to represent the base, and lay off this distance on the base line. Having drawn the base line, move the ruler around the needle, direct the sights to any object, as L, and keeping it there, draw a line along the edge of the ruler. Then direct the sights in the same manner to any other objects which are required to be sketched, drawing lines in their respective directions, taking care that the table remains steady during the operation.

Now remove the instrument to the other extremity of the base A, and place the point of the paper corresponding to that extremity directly over it. Place the edge of the ruler on the base line, and turn the table about till the sights are directed to the station B. Then placing the edge of the ruler against the needle, direct the sights in succession to all the objects observed from the other station, drawing lines from the point A in their several directions. The intersections of these lines with those drawn from the point B will determine the positions of the several objects on the map.

In this manner the plane table may be employed for filling in the details of a map; setting it up at the most remarkable spots, and sketching by the eye what is not necessary should be more particularly determined, the paper will gradually become a representation of the country to be surveyed.

To determine the Depth of Water.

(186.) Let signals be established on the principal shoals and along the edges of the channel, by erecting poles or anchoring buoys, and let their bearings be observed from two stations of the survey. Then in each triangle there will be known one side and the angles, from which the other sides may be computed, and their positions thus become known. Then ascertain the precise depth of water at each of the buoys, and proceed in this manner to determine as many points as may be thought necessary.

If an observer is stationed with a theodolite at each extremity of the base line, we may dispense with the erection of permanent marks upon the water. One observer in a boat may make a sounding for the depth of water, giving a signal at the same instant to two observers at the extremities of the base line. The direction of the boat being observed at that instant from two stations, the precise place of the boat can be computed. In this way soundings may be made with great expedition.

There is also another method, still more expeditious, which may afford results sufficiently precise in some cases. Let a boat be rowed uniformly across the harbor from one station to another, for example, from D to G (see fig. on p. 131), and let a series of soundings be made as rapidly as possible, and the instant of each sounding be recorded. Then, knowing the entire length of the line DG, and the time of rowing over it, we may find by proportion the approximate position of the boat at each sounding.

If the soundings are made in tide waters, the times of high water should be observed, and the time of each sounding be recorded, so that the depth of water at high or low tide may be computed. In the maps of the United States Coast Survey, the soundings are all reduced to low-water mark, and the number of feet which the tide rises or falls is noted upon the map.

(187.) The results of the soundings may be delineated upon a map in the same manner as the observations of level on page 124. We draw lines joining all those points where the depth of water is the same, for example, 20 feet. Such a line is seen to be an undulating line running in the direction from E to G. We draw another line connecting all those points where the depth of water is 40 feet. This line runs somewhat to the east of the former line, but nearly parallel with it. We draw other lines for depths of 60 feet, &c. The lines being thus drawn, a mere glance at the map will show nearly the depth of water at any point of the harbor.

BOOK V.

NAVIGATION.

(188.) NAVIGATION is the art of conducting a ship at sea from one port to another.

There are two methods of determining the situation of a vessel at sea. The one consists in finding by astronomical observations her latitude and longitude; the other consists in measuring the ship's course, and her progress every day from the time of her leaving port, from which her place may be computed by trigonometry. The latter method is the one to be now considered.

(189.) The figure of the earth is nearly that of a sphere, and in navigation it is considered perfectly spherical. The *earth's axis* is the diameter around which it revolves once a day. The extremities of this axis are the terrestrial *poles;* one is called the north pole, and the other the south pole.

The *equator* is a great circle perpendicular to the earth's axis.

Meridians are great circles passing through the poles of the earth. Every place on the earth's surface has its own meridian.

(190.) The *longitude* of any place is the arc of the equator intercepted between the meridian of that place and some assumed meridian to which all others are referred. In most countries of Europe, that has been taken as the standard meridian which passes through their principal observatory. The English reckon longitude from the Observatory of Greenwich; and in the United States, we have usually adhered to the English custom, though we believe the time has come when longitude should be reckoned from the Observatory of Washington.

Longitude is usually reckoned east and west of the first meridian, from 0° to 180°.

The difference of longitude of two places is the arc of the equator included between their meridians. It is equal to the difference of their longitudes if they are on the same side of the first meridian, and to the sum of their longitudes if on opposite sides.

(191.) The *latitude* of a place is the arc of the meridian passing through the place, which is comprehended between that place and the equator.

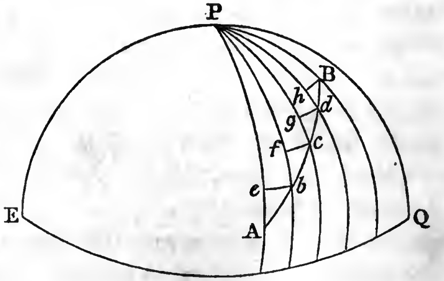
Latitude is reckoned north and south of the equator, from 0° to 90°.

Parallels of latitude are the circumferences of small circles parallel to the equator.

The *difference of latitude* of two places is the arc of a meridian included between the parallels of latitude passing through those places. It is equal to the difference of their latitudes if they are on the same side of the equator, and to the sum of their latitudes if on opposite sides.

The *distance* is the length of the line which a vessel describes in a given time.

The *departure* of two places is the distance of either place from the meridian of the other. If the two places are on the same parallel, the departure is the distance between the places. Otherwise, we divide the distance AB into portions A*b*, *bc*, *cd*, &c., so small that the curvature of the earth may be neglected. Through these points we draw the meridians P*b*, P*c*, &c., and the parallels *be*, *cf*, &c. Then the departure for A*b* is *eb*, for *bc* it is *fc*; and the whole departure from A to B is $eb+fc+gd+h\mathrm{B}$; that is, the sum of the departures corresponding to the small portions into which the distance is divided.



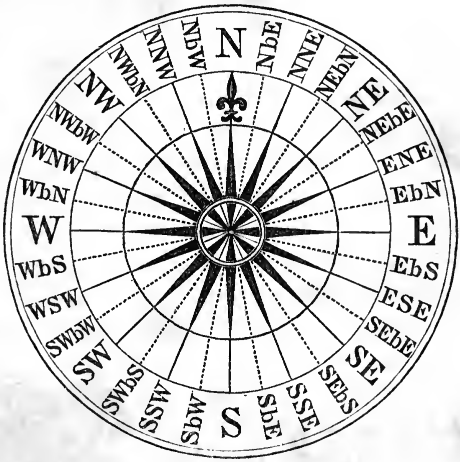
Distance, departure, and difference of latitude are measured in nautical miles, one of which is the 60th part of a degree at the equator. A nautical mile is nearly one sixth greater than an English statute mile.

The *course* of a ship is the angle which the ship's path makes with the meridian. A ship is said to continue on the same course when she cuts every meridian which she crosses at the same angle. The path thus described is not a straight line, but a curve called a *rhumb-line.*

The course of a ship is given by the mariner's compass.

(192.) The *mariner's compass* consists of a circular piece of paper, called a card, attached to a magnetic needle, which is balanced on a pin so as to move freely in any direction. Directly over the needle, a line is drawn on the card, one end of which is marked N, and the other S. The circumference is divided into thirty-two equal parts called rhumbs or *points*, each point being subdivided into four equal parts called *quarter points.*

The points of the compass are designated as follows, beginning at north and going east: north, north by east, north-northeast, northeast by north, northeast, and so on, as shown in the annexed figure.

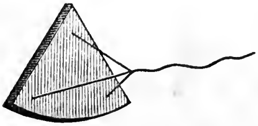


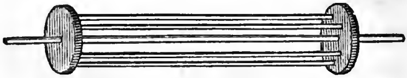
The interval between two adjacent points is 11° 15′, which is the eighth part of a quadrant. On the inside of the compass-box a black line is drawn perpendicular to the horizon, and the compass should be so placed that a line drawn from this mark through the center of the card may be parallel to the keel of the ship. The part of the card which coincides with this mark will then show the point of the compass to which the keel is directed. The compass is suspended in its box in such a manner as to maintain a horizontal position, notwithstanding the motion of the ship.

The following table shows the number of degrees and minutes corresponding to each point of the compass:

North.		Pts.		Pts.	South.	
N. by E.	N. by W.	1	11° 15′	1	S. by E.	S. by W.
N.N.E.	N.N.W.	2	22° 30′	2	S.S.E.	S.S.W.
N.E. by N.	N.W. by N.	3	33° 45′	3	S.E. by S.	S.W. by S.
N.E.	N.W.	4	45° 0′	4	S.E.	S.W.
N.E. by E.	N.W. by W.	5	56° 15′	5	S.E. by E.	S.W. by W.
E.N.E.	W.N.W.	6	67° 30′	6	E.S.E.	W.S.W.
E. by N.	W. by N.	7	78° 45′	7	E. by S.	W. by S.
East.	West.	8	90° 0′	8	East.	West.

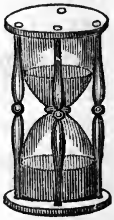
(193.) The ship's rate of sailing is measured by a *log-line.* The log-line is a cord about 300 yards long, which is wound





round a reel, one end being attached to a piece of thin board called a log. This board is in the form of a sector of a circle, the arc of which is loaded with lead sufficient to give the board a vertical position when thrown upon the water. This is designed to prevent the log from being drawn along after the vessel while the line is running off the reel.

The time is measured by a sand-glass, through which the sand passes in *half a minute*, or the 120th part of an hour.

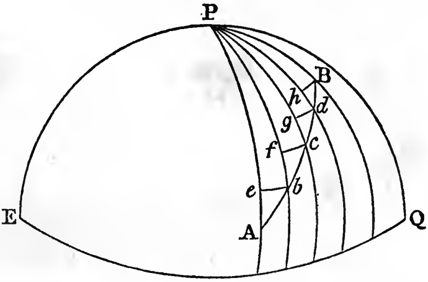


The log-line is divided into equal parts called *knots*, each of which is 50 feet, or the 120th part of a nautical mile. Now, since a knot has the same ratio to a nautical mile that half a minute has to an hour, it follows, that if the motion of a ship is uniform, she sails as many miles in an hour as she does knots in half a minute. If, then, seven knots are observed to run off in half a minute, the ship is sailing at the rate of seven miles an hour.

PLANE SAILING.

(194.) *Plane sailing* is the method of calculating a ship's place at sea by means of the properties of a plane triangle. The particulars which are given or required are four, viz., the distance, course, difference of latitude, and departure. Of these, any two being given, the others may be found.

Let the figure EPQ represent a portion of the earth's surface, P the pole, and EQ the equator. Let AB be a rhumb-line, or the track described by a ship in sailing from A to B on a uniform course. Let the whole distance be divided into portions A*b*, *bc*, &c., so small

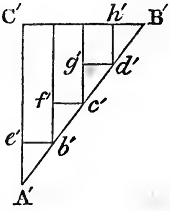


that the curvature of the earth may be neglected. Through the points of division draw the meridians P*b*, P*c*, &c., and the parallels *eb*, *fc*, &c. Then, since the course is every where the same, each of the angles *e*A*b*, *fbc*, &c., is equal to the course. The distances A*e*, *bf*, &c., are the differences of latitude of A and *b*, *b* and *c*, &c. Also, *eb*, *fc*, &c., are the departures for the same distances. Hence the difference of latitude from A to B is equal to

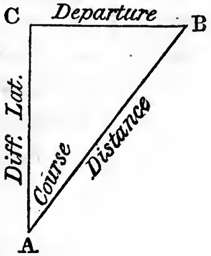
$$Ae+bf+cg+dh,$$

and the departure is equal to

$$eb+fc+gd+hB.$$



Construct the triangle A′B′C′ so that A′*b*′*e*′ shall be equal to A*be*, *b*′*c*′*f*′ shall be equal to *bcf*, *c*′*d*′*g*′ equal to *cdg*, and *d*′B′*h*′ equal to *d*B*h*. Then A′B′ represents the distance sailed, B′A′C′ the course, A′C′ the difference of latitude, and B′C′ the departure; that is, the distance, difference of latitude, and departure are correctly represented by the hypothenuse and sides of a right-angled triangle, of which the angle opposite to the departure is the course. Of these four quantities, any two being given, the others may be found.



Plane sailing does not assume the earth's surface to be a plane, and does not involve any error even in great distances.

EXAMPLES.

1. A ship sails from Vera Cruz N.E. by N. 74 miles. Required her departure and difference of latitude.

According to the principles of right-angled triangles, Art. 44,

Radius : *distance* : : *sin. course* : *departure.*

: : *cos. course* : *diff. latitude.*

The course is three points, or 33° 45′; hence we obtain

Departure =41.11 miles.

Diff. latitude=61.53 miles.

2. A ship sails from Sandy Hook, latitude 40° 28′ N., upon a course E.S.E., till she makes a departure of 500 miles. What distance has she sailed, and at what latitude has she arrived?

By Trigonometry, Art. 44,

Sin. course : *departure* : : *radius* : *distance,*

: : *cos. course* : *diff. latitude.*

Ans. Distance =541.20 miles.

Diff. latitude=207.11 miles, or 3° 27′.

Hence the latitude at which she has arrived is 37° 1′ N.

3. The bearing of Sandy Hook from Bermuda is N. 42° 56′ W., and the difference of latitude 486 miles. Required the distance and departure.

By Trigonometry, Art. 46,

Radius : *diff. latitude* : : *tang. course* : *departure,*

: : *sec. course* : *distance.*

Ans. Distance =663.8 miles.

Departure=452.1 miles.

4. A ship sails from Bermuda, latitude 32° 22′ N., a distance of 666 miles, upon a course between north and east, until she finds her departure 444 miles. What course has she sailed, and what is her latitude?

By Trigonometry, Art. 44,

Distance : *radius* : : *departure* : *sin. course,*

Radius : *distance* : : *cos. course* : *diff. latitude.*

Ans. Latitude= 40° 38′ N.

Course =N. 41° 49′ E.

5. The distance from Vera Cruz, latitude 19° 12′ N., to Pensacola, latitude 30° 19′ N., is 820 miles. Required the bearing and departure.

By Trigonometry, Art. 45,

Distance : *radius* : : *diff. latitude* : *cos. course*,
Radius : *distance* : : *sin. course* : *departure*.

Ans. Bearing =N. 35° 34′ E.
Departure=476.95 miles.

6. A ship sails from Sandy Hook upon a course between south and east to the parallel of 35°, when her departure was 300 miles. Required her course and distance.

By Trigonometry, Art. 47,

Diff. latitude : *radius* : : *departure* : *tang. course*,
Radius : *diff. latitude* : : *sec. course* : *distance*.

Ans. Course S. 42° 27′ E.
Distance 444.5 miles.

TRAVERSE SAILING.

(195.) A *traverse* is the irregular path of a ship when sailing on different courses.

The object of *traverse sailing* is to reduce a traverse to a single course, when the distances sailed are so small that the curvature of the earth may be neglected. When a ship sails on different courses, the difference of latitude is equal to the difference between the sum of the northings and the sum of the southings; and, neglecting the earth's curvature, the departure is equal to the difference between the sum of the eastings and the sum of the westings. If, then, the difference of latitude and the departure for each course be taken from the traverse table, and arranged in appropriate columns, the difference of latitude for the whole time may be obtained exactly, and the departure nearly, by addition and subtraction; and the corresponding distance and course may be determined as in plane sailing.

EXAMPLES.

1. A ship sails on the following successive tracks:

1.	N.E.	23	miles.
2.	E.S.E.	45	"
3.	E. by N.	34	"
4.	North	29	"
5.	N. by W.	31	"
6.	N.N.E.	17	"

Find the course and distance for the whole traverse.

We form a table as below, entering the courses from the table of rhumbs, page 138, and then enter the latitudes and departures taken from the traverse table.

Traverse Table.

No.	Course.	Distance.	N.	S.	E.	W.
1	N. 45° E.	23	16.26		16.26	
2	S. 67° 30′ E.	45		17.22	41.57	
3	N. 78° 45′ E.	34	6.63		33.35	
4	North.	29	29.00			
5	N. 11° 15′ W.	31	30.40			6.05
6	N. 22° 30′ E.	17	15.71		6.51	
Sum of columns			98.00	17.22	97.69	6.05
			17.22		6.05	
Diff. latitude			=80.78 N.		Dep.=91.64 E.	

Hence the course is found by plane sailing N. 48° 36′ E., and the distance =122.2 miles.

The proportions are

Diff. latitude : *radius* : : *departure* : *tang. course*,

Radius : *diff. latitude* : : *sec. course* : *distance*.

2. A ship leaving Sandy Hook makes the following courses and distances:

1. S.E. 25 miles.
2. E.S.E. 32 "
3. East 17 "
4. E. by S. 51 "
5. South 45 "
6. S. by E. 63 "

Required her latitude, the distance made, and the direct course.

Ans. Latitude=38° 1′ N.
Distance=193.7 miles.
Course =S. 40° 47′ E.

3. A ship from Pensacola, latitude 30° 19′, sails on the following successive courses:

1. South 48 miles.
2. S.S.W. 23 "
3. S.W. 32 "

4.	S.W. by S.	76	miles.
5.	West	17	"
6.	W.S.W.	54	"

Required her latitude, direct course, and distance.

Ans. Latitude= 27° 23′ N.
Course =S. 38° 39′ W.
Distance =225.0 miles.

4. A ship from Bermuda, latitude 32° 22′, sails on the following successive courses:

1.	N.E.	66	miles.
2.	N.N.E.	14	"
3.	N.E. by E.	45	"
4.	East	21	"
5.	E. by N.	32	"

Required her latitude, direct course, and distance.

Ans. Latitude= 33° 53′ N.
Course =N. 57° 22′ E.
Distance=168.4 miles.

(196.) When the water through which a ship is moving has a progressive motion, the ship's progress is affected in the same manner as if she had sailed in still water, with an additional course and distance equal to the direction and motion of the current.

Ex. 5. If a ship sail 125 miles N.N.E. in a current which sets W. by N. 32 miles in the same time, required her true course and distance.

Form a traverse table containing the course sailed by the ship and the progress of the current, and find the difference of latitude and departure. The resulting course and distance is found as in the preceding examples.

Traverse Table.

Courses.	Distance.	N.	E.	W.
N. 22° 30′ E	125	115.49	47.84	
N. 78° 45′ W.	32	6.24		31.39
Diff. latitude		=121.73	47.84	31.39
			31.39	
Departure			=16.45 E.	

Hence the course is found by plane sailing N. 7° 42′ E., and the distance=122.8 miles.

Ex. 6. A ship sails S. by E. for two hours at the rate of 9 miles an hour; then S. by W. for five hours at the rate of 8 miles an hour; and during the whole time a current sets W. by N. at the rate of two and a half miles an hour. Required the direct course and distance.

Ans. The course is S. 21° 51′ W.
Distance 57.6 miles.

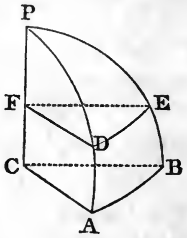
PARALLEL SAILING.

(197.) *Parallel sailing* is when a ship sails exactly east or west, and therefore remains constantly on the same parallel of latitude. In this case the departure is equal to the distance sailed, and the difference of longitude may be found by the following

THEOREM.

The cosine of the latitude of the parallel is to radius, as the distance run is to the difference of longitude.

Let P be the pole of the earth, C the center, AB a portion of the equator, and DE any parallel of latitude; then will CA be the radius of the equator, and FD the radius of the parallel. Let DE be the distance sailed by the ship on the parallel of latitude, then the difference of longitude will be measured by AB, the arc intercepted on the equator by the meridians passing through D and E.



Since AB and DE correspond to the equal angles ACB, DFE, they are similar arcs, and are to each other as their radii. Hence

FD : CA : : *arc* DE : *arc* AB.

But FD is the sine of PD, or the cosine of AD, that is, the cosine of the latitude, and CA is the radius of the sphere; hence

Cosine of latitude : R : : *distance* : *diff. longitude.*

Cor. Like portions of different parallels of latitude are to each other as the cosines of the latitudes.

The length of a degree of longitude in different parallels may be computed by this theorem. A degree of longitude at the equator being 60 nautical miles, a degree in latitude 40° may be found by the proportion

R : cosine 40° : : 60 : 45.96, the required length.

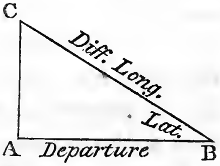
The following table is computed in the same manner.

(198.) *Table showing the length of a degree of longitude for each degree of latitude.*

Lat.	Miles.	Lat.	Miles.	Lat.	Miles.	Lat.	Miles.	Lat.	Miles.	Lat.	Miles.
1	59.99	16	57.68	31	51.43	46	41.68	61	29.09	76	14.52
2	59.96	17	57.38	32	50.88	47	40.92	62	28.17	77	13.50
3	59.92	18	57.06	33	50.32	48	40.15	63	27.24	78	12.47
4	59.85	19	56.73	34	49.74	49	39.36	64	26.30	79	11.45
5	59.77	20	56.38	35	49.15	50	38.57	65	25.36	80	10.42
6	59.67	21	56.01	36	48.54	51	37.76	66	24.40	81	9.39
7	59.55	22	55.63	37	47.92	52	36.94	67	23.44	82	8.35
8	59.42	23	55.23	38	47.28	53	36.11	68	22.48	83	7.31
9	59.26	24	54.81	39	46.63	54	35.27	69	21.50	84	6.27
10	59.09	25	54.38	40	45.96	55	34.41	70	20.52	85	5.23
11	58.90	26	53.93	41	45.28	56	33.55	71	19.53	86	4.19
12	58.69	27	53.46	42	44.59	57	32.68	72	18.54	87	3.14
13	58.46	28	52.98	43	43.88	58	31.80	73	17.54	88	2.09
14	58.22	29	52.48	44	43.16	59	30.90	74	16.54	89	1.05
15	57.96	30	51.96	45	42.43	60	30.00	75	15.53	90	0.00

Let ABC represent a right-angled triangle; then, by Trigonometry, Art. 41,

cos. B : R : : AB : BC.



But, by the preceding Theorem, we have

cos. lat. : R : : *depart.* : *diff. long.*,

from which we see that if one leg of a right-angled triangle represent the distance run on any parallel, and the adjacent acute angle be made equal to the degrees of latitude of that parallel, then the hypothenuse will represent the difference of longitude.

EXAMPLES.

1. A ship sails from Sandy Hook, latitude 40° 28′ N., longitude 74° 1′ W., 618 miles due east. Required her present longitude.

Cos. 40° 28′ : R : : 618 : 812′.3=13° 32′, the difference of longitude.

This, subtracted from 74° 1′, leaves 60° 29′ W., the longitude required.

2. A ship in latitude 40° sails due east through nine degrees of longitude. Required the distance run.

Ans. 413.66 miles.

3. A ship having sailed on a parallel of latitude 261 miles, finds her difference of longitude 6° 15′. What is her latitude?

Ans. Latitude 45° 54′.

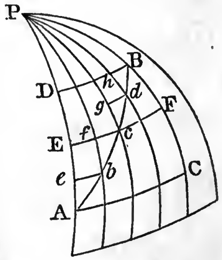
4. Two ships in latitude 52° N., distant from each other 95 miles, sail directly south until their distance is 150 miles. What latitude do they arrive at?

Ans. Latitude 13° 34′.

MIDDLE LATITUDE SAILING.

(199.) By the method just explained may be found the longitude which a ship makes while sailing on a parallel of latitude. When the course is oblique, the departure may be found by plane sailing, but a difficulty is found in converting this departure into difference of longitude.

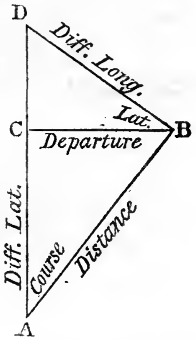
If a ship sail from A to B, the departure is equal to $eb+fc+gd+h\text{B}$, which is less than AC, but greater than DB. Navigators have assumed that the departure was equal to the distance between the meridians PA, PB, measured on a parallel EF, equidistant from A and B, called the *middle latitude.*



The middle latitude is equal to *half the sum* of the two extreme latitudes, if both are north or both south; but to *half their difference*, if one is north and the other south.

The principle assumed in middle latitude sailing is not perfectly correct. For long distances the error is considerable, but the method is rendered perfectly accurate by applying to the middle latitude a correction which is given in the accompanying tables, page 149.

(200.) It has been shown that when a ship sails upon an oblique course, the distance, departure, and difference of latitude may be represented by the sides of a right-angled triangle. The difference of longitude is derived from the departure, in the same manner as in parallel sailing, the ship being supposed to sail on the middle latitude parallel. Hence, if we combine the triangle ABC for plane sailing with the triangle BCD for parallel sailing, we shall obtain a triangle ABD, by which all the cases of middle latitude sailing may be solved.



In the triangle BCD,

$$\text{Cos. CBD} : \text{BC} :: \text{R} : \text{BD};$$

that is, *cosine of middle latitude is to the departure, as radius is to the difference of longitude.*

In the triangle ABD, since the angle D is the complement of CBD, which represents the middle latitude, we have

$$\text{Sin. D} : \text{AB} :: \text{sin. A} : \text{BD};$$

that is, *cosine of middle latitude is to the distance, as the sine of the course is to the difference of longitude.*

In the triangle ABC, we have the proportion

$$\text{AC} : \text{BC} :: \text{R} : \text{tang. A}.$$

But we have before had the proportion

$$\text{Cos. CBD} : \text{BC} :: \text{R} : \text{BD}.$$

The means being the same in these two proportions, we have

$$\text{Cos. CBD} : \text{AC} :: \text{tang. A} : \text{BD};$$

that is, *cosine of middle latitude is to the difference of latitude, as the tangent of the course is to the difference of longitude.*

The middle latitude should always be corrected according to the table on page 149. The given middle latitude is to be looked for either in the first or last vertical column, opposite to which, and under the given difference of latitude, is inserted the proper correction in minutes, which must be *added* to the middle latitude to obtain the latitude in which the meridian distance is exactly equal to the departure. Thus, if the middle latitude is 41°, and the difference of latitude 14°, the correction will be found to be 25′, which, added to the middle latitude, gives the corrected middle latitude 41° 25′.

EXAMPLES.

1. Find the bearing and distance of Liverpool, latitude 53° 22′ N., longitude 2° 52′ W., from New York, latitude 40° 42′ N., longitude 74° 1′ W.

Here are given two latitudes and longitudes to find the course and distance.

The difference of latitude is	12° 40′= 760′.
The difference of longitude is	71° 9′=4269′.
The middle latitude is	47° 2′.
To which add the correction from p. 149	22′.
The corrected middle latitude is	47° 24′.

Then, according to the third of the preceding theorems,

Diff. lat. : *cos. mid. lat.* : : *diff. long.* : *tang. course*=N. 75° 16′ E.

To find the distance by plane sailing,

Cos. course : *diff. latitude* : : R : *distance*=2988.4 miles.

2. A ship sailed from Bermuda, latitude 32° 22′ N., longitude 64° 38′ W., a distance of 500 miles, upon a course W.N.W. Required her latitude and longitude at that time.

By plane sailing,

R : *distance* : : *cos. course* : *diff. latitude*=191.3.

Therefore the required latitude is	35° 33′;
the middle latitude	33° 58′;
and the corrected middle latitude	33° 59′.

Then we have

Cos. mid. lat. : *distance* : : *sin. course* : *diff. long.*=557′.1.

Therefore the longitude required is 73° 55′.

3. A ship sails southeasterly from Sandy Hook, latitude 40° 28′ N., longitude 74° 1′ W., a distance of 395 miles, when her latitude is 34° 40′ N. Required her course and longitude.

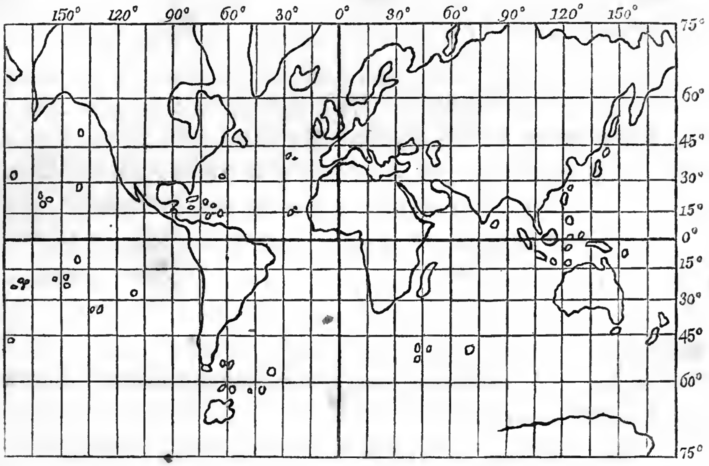
Ans. Course S. 28° 14′ E.
Longitude 70° 5′ W.

4. A ship sails from Brest, latitude 48° 23′ N., longitude 4° 29′ W., upon a course W.S.W., till her departure is 556 miles. Required the distance sailed and the place of the ship.

Ans. Distance 601.8 miles.
Latitude 44° 33′ N.
Longitude 17° 57′ W.

MERCATOR'S SAILING.

(201.) Mercator's sailing is a method of computing difference of longitude on the principles of Mercator's chart. On this chart, the meridians, instead of converging toward the poles as they do on the globe, are drawn *parallel* to each other, by which means the distance of the meridians is every where



made too great except at the equator. To compensate for this, in order that the outline of countries may not be too much distorted, the degrees of latitude are proportionally enlarged, so that the distance between the parallels of latitude increases from the equator to the poles. In latitude 60° the distance of the meridians is twice as great, compared with a degree at the equator, as it is upon a globe, and a degree of latitude is here represented twice as great as near the equator. The diameter of an island in latitude 60° is represented twice as great as if it was on the equator, and its area four times too great. In latitude 70° 32′ the distance of the meridians is three times too great, in latitude 75° 31′ four times too great, and so on, by which means the relative dimensions of countries in high latitudes is exceedingly distorted. On this account it is not common to extend the chart beyond latitude 75°.

(202.) The distance of any parallel upon Mercator's chart from the equator has been computed, and is exhibited in the accompanying tables, pages 142–8, which is called a Table of *Meridional Parts.* This table may be computed in the following manner:

According to Art. 197, cosine of latitude is to radius, as the departure is to the difference of longitude; that is, as a part of a parallel of latitude is to a like part of the equator, or any meridian.

But by Art. 28, cosine : R : : R : secant; hence

$1'$ *of a parallel* : $1'$ *of a meridian* : : R : *sec. latitude.*

But on Mercator's chart the distance between the meridians is the same in all latitudes; that is, a minute on a parallel of latitude is equal to a minute at the equator, or a geographical mile. Hence *the length of one minute, on any part of a meridian, is equal to the secant of the latitude.* Thus,

The *first*	minute of the	meridian	=	the secant of	1′;
second	"	"	=	"	2′;
third	"	"	=	"	3′,
&c.,		&c.			

The table of meridional parts is formed by adding together the minutes thus found. Thus,

Mer. parts of $1'$=sec. $1'$;
Mer. parts of $2'$=sec. $1'$+sec. $2'$;
Mer. parts of $3'$=sec. $1'$+sec. $2'$+sec. $3'$;
Mer. parts of $4'$=sec. $1'$+sec. $2'$+sec. $3'$+sec. $4'$,
&c., &c., &c.

Since the secants of small arcs are nearly equal to radius or unity, if the meridional parts are only given to one tenth of a mile, we shall have

The meridional	parts	of	1′=1.0	mile;	
"	"	"	2′=2.0	"	
"	"	"	3′=3.0	"	
"	"	"	4′=4.0	"	&c.,

as shown in the table on page 142.

At 2° 33′ the sum of the small fractions omitted becomes greater than half of one tenth, and the meridional parts of 2° 33′ is 153.1; that is, the meridional parts exceed by one tenth of a mile the minutes of latitude. At 3° 40′ the excess is two tenths of a mile; at 4° 21′ the excess is three tenths;

and as the latitude increases, the meridional parts increase more rapidly, as is seen from the table.

An arc of Mercator's meridian contained between two parallels of latitude is called *meridional difference of latitude.* It is found by subtracting the meridional parts of the less latitude from the meridional parts of the greater, if both are north or south, or by adding them together if one is north and the other south. Thus,

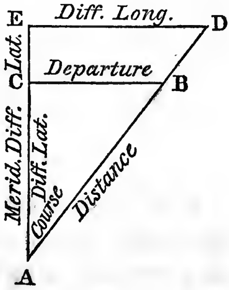
The lat. of New York is	40° 42′; meridional parts	=2677.8,
" New Orleans	29° 57′; " "	1884.9.
The true diff. of lat. is	10° 45′; mer. diff. lat. is	792.9.

If one latitude and the meridional difference of latitude be given, the true difference of latitude may be found by reversing this process. Thus,

The meridional parts for New Orleans . . .	=1884.9.
Meridional difference of latitude between New York and New Orleans	= 792.9.
Therefore the meridional parts for New York	=2677.8,

and the corresponding latitude from the table is 40° 42′.

(203.) If we take the figure ABC for plane sailing, as on page 139, and produce AC to E, making AE equal to the meridional difference of latitude, then will DE represent the difference of longitude corresponding to the departure BC. For we have seen (Art. 202) that the departure is to the difference of longitude as radius is to the secant of latitude, which is also the ratio of the true difference of latitude to the meridional difference of latitude.



Now, from the similarity of the triangles ABC, ADE, we have

$$AC : AE :: BC : DE;$$

that is, *the true difference of latitude is to the meridional difference of latitude, as the departure is to the difference of longitude.*

Also, in the triangle ADE, we have

$$R : \tan. A :: AE : DE;$$

that is, *radius is to the tangent of the course, as the meridional difference of latitude is to the difference of longitude.*

EXAMPLES.

1. Find the bearing and distance from Sandy Hook, latitude 40° 28′ N., longitude 74° 1′ W., to Havre, latitude 49° 29′ N., longitude 0° 6′ E.

The true difference of latitude is 9° 1′= 541′;
meridional difference of latitude =767.1;
difference of longitude is 74° 7′=4447.

Hence, to find the course by the preceding proportion,

Mer. diff. lat. : *diff. long.* : : R : *tan. course*=N. 80° 13′ E.

To find the distance by plane sailing,

Cos. course : *true diff. lat.* : : R : *distance*=3183.8 miles.

2. Find the bearing and distance from Nantucket Shoals, in latitude 41° 4′ N., longitude 69° 55′ W., to Cape Clear, in latitude 51° 26′ N., longitude 9° 29′ W.

Ans. Course N. 76° E.
Distance 2572.9 miles.

3. A ship sails from Sandy Hook a distance of 600 miles upon a course S. by E. Required the place of the ship.

The difference of latitude may be found by plane sailing, the difference of longitude by Mercator's sailing.

Ans. Latitude 30° 39′.5 N.
Longitude 71° 36′.7 W.

4. A ship sails from St. Augustine, latitude 29° 52′ N., longitude 81° 25′ W., upon a course N.E. by E., until her latitude is found to be 34° 40′ N. What is then her longitude, and what distance has she run?

Ans. Longitude=72° 55′ W.
Distance =518.4 miles.

5. A ship sails from Bermuda upon a course N.W. by W. until her longitude is found to be 69° 30′ W. What is then her latitude, and what distance has she run?

Ans. Latitude 35° 4′ N.
Distance 291.6 miles.

6. A ship sailing from Madeira, latitude 32° 38′ N., longitude 16° 55′ W., steers westerly until her latitude is 40° 2′ N.,

and her departure 2425 miles. Required her course, distance, and longitude.

Ans. Course N. 79° 37′ W.
Distance 2465.3 miles.
Longitude 67° 9′.3 W.

7. Find the bearing and distance from Sandy Hook, latitude 40° 28′ N., longitude 74° 1′ W., to the Cape of Good Hope, latitude 34° 22′ S., longitude 18° 30′ E.

Ans. Course
Distance

CHARTS.

(204.) The charts commonly used in navigation are *plane charts*, or *Mercator's chart*. In the construction of the former, the portion of the earth's surface which is represented is supposed to be a *plane*. The meridians are drawn parallel to each other, and the lines of latitude at equal distances. The distance between the parallels should be to the distance between the meridians, as radius to the cosine of the middle latitude of the chart. A chart of moderate extent constructed in this manner will be tolerably correct. The distance of the meridians in the *middle* of the chart will be exact, but on each side it will be either too great or too small.

When large portions of the earth's surface are to be represented, the error of the plane chart becomes excessive. To obviate this inconvenience Mercator's chart has been constructed. Upon this chart the meridians are represented by parallel lines, and the distance between the parallels of latitude is proportioned to the meridional difference of latitude, as represented on page 149.

We have seen that the meridional difference of latitude is to the difference of longitude as radius is to the tangent of the course. Hence, while the course remains unchanged, the ratio of the meridional difference of latitude to the difference of longitude is *constant;* and, therefore, every rhumb line will be represented on Mercator's chart by a *straight* line. This property renders Mercator's chart peculiarly convenient to navigators.

The preceding sketch affords a very incomplete view of the present state of the science of navigation. The most accurate method of ascertaining the situation of a vessel at sea is by means of astronomical observations. For these, however, the student must be referred to some treatise on Astronomy.

BOOK VI.

SPHERICAL TRIGONOMETRY.

(205.) Spherical trigonometry teaches how to determine the several parts of a spherical triangle from having certain parts given.

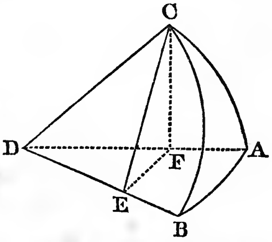
A spherical triangle is a portion of the surface of a sphere, bounded by three arcs of great circles, each of which is less than a semicircumference.

RIGHT-ANGLED SPHERICAL TRIANGLES.

Theorem I.

(206.) *In any right-angled spherical triangle, the sine of the hypothenuse is to radius, as the sine of either side is to the sine of the opposite angle.*

Let ABC be a spherical triangle, right-angled at A; then will the sine of the hypothenuse BC be to radius, as the sine of the side AC is to the sine of the angle ABC.



Let D be the center of the sphere; join AD, BD, CD, and draw CE perpendicular to DB, which will, therefore, be the sine of the hypothenuse BC. From the point E draw the straight line EF, in the plane ABD, perpendicular to BD, and join CF. Then, because DB is perpendicular to the two lines CE, EF, it is perpendicular to the plane CEF; and, consequently, the plane CEF is perpendicular to the plane ABD (*Geom.*, Prop. 6, B. VII.). But the plane CAD is also perpendicular to the plane ABD; therefore their line of common section, CF, is perpendicular to the plane ABD; hence CFD, CFE are right angles, and CF is the sine of the arc AC.

Now, in the right-angled plane triangle CFE,

$$CE : \text{radius} :: CF : \text{sine } CEF.$$

But since CE and FE are both at right angles to DB, the angle CEF is equal to the inclination of the planes CBD, ABD; that is, to the spherical angle ABC. Therefore,

sine BC : R : : sine AC : sine ABC.

(207.) *Cor.* 1. In any right-angled spherical triangle, *the sines of the sides are as the sines of the opposite angles.*

For, by the preceding theorem,

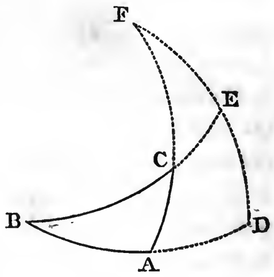
sine BC : R : : sine AC : sine ABC,

and sine BC : R : : sine AB : sine ACB;

therefore, sine AC : sine AB : : sine ABC : sine ACB.

Cor. 2. In any right-angled spherical triangle, *the cosine of either of the sides is to radius, as the cosine of the hypothenuse is to the cosine of the other side.*

Let ABC be a spherical triangle, right-angled at A. Describe the circle DE, of which B is the pole, and let it meet the three sides of the triangle ABC produced in D, E, and F. Then, because BD and BE are quadrants, the arc DF is perpendicular to BD. And since BAC is a right angle, the arc AF is perpendicular to BD. Hence the point F, where the arcs FD, FA intersect each other, is the pole of the arc BD (*Geom.*, Prop. 5, Cor. 2, B. IX.), and the arcs FA, FD are quadrants.



Now, in the triangle CEF, right-angled at the point E, according to the preceding theorem, we have

sine CF : R : : sine CE : sine CFE.

But CF is the complement of AC, CE is the complement of BC, and the angle CFE is measured by the arc AD, which is the complement of AB. Therefore, in the triangle ABC, we have

cos. AC : R : : cos. BC : cos. AB.

Cor. 3. In any right-angled spherical triangle, *the cosine of either of the sides is to radius, as the cosine of the angle opposite to that side is to the sine of the other angle.*

For, in the triangle CEF, we have

sine CF : R : : sine EF : sine ECF.

But sine CF is equal to cos. CA. EF is the complement of

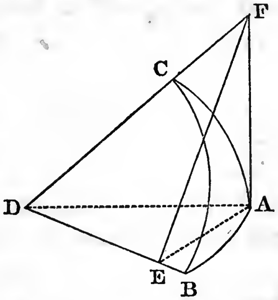
ED, which measures the angle ABC, that is, sine EF is equal to cos. ABC, and sine ECF is the same as sine ACB; therefore,

cos. AC : R : : cos. ABC : sine ACB

THEOREM II.

(208.) *In any right-angled spherical triangle, the sine of either of the sides about the right angle is to the cotangent of the adjacent angle, as the tangent of the remaining side is to radius.*

Let ABC be a spherical triangle, right-angled at A; then will the sine of the side AB be to the cotangent of the angle ABC, as the tangent of the side AC is to radius.



Let D be the center of the sphere; join AD, BD, CD; draw AE perpendicular to BD, which will, therefore, be the sine of the arc AB. Also, from the point E in the plane BDC, draw the straight line EF perpendicular to BD, meeting DC produced in F, and join AF. Then will AF be perpendicular to the plane ABD, because, as was shown in the preceding theorem, it is the common section of the two planes ADF, AEF, each perpendicular to the plane ADB. Therefore FAD, FAE are right angles, and AF is the tangent of the arc AC.

Now, in the triangle AEF, right-angled at A, we have

AE : radius : : AF : tang. AEF.

But AE is the sine of the arc AB, AF is the tangent of the arc AC, and the angle AEF is equal to the inclination of the planes CBD, ABD, or to the spherical angle ABC; hence

sine AB : R : : tang. AC : tang. ABC.

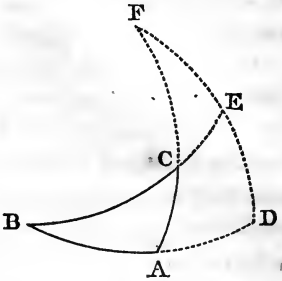
And because, Art. 28,

R : cot. ABC : : tang. ABC : R;

therefore, sine AB : cot. ABC : : tang. AC : R.

(209.) *Cor.* 1. In any right-angled spherical triangle, *the cosine of the hypothenuse is to the cotangent of either of the oblique angles, as the cotangent of the other oblique angle is to radius.*

Let ABC be a spherical triangle, right-angled at A. Describe the circle DEF, of which B is the pole, and construct the complemental triangle CEF, as in Cor. 2, Theorem I.



Then, in the triangle CEF, according to the preceding theorem, we have

sine CE : cot. ECF : : tan. EF : R.

But CE is the complement of BC, EF is the complement of ED, the measure of the angle ABC; and the angle ECF is equal to ACB, being its vertical angle; hence

cos. BC : cot. ACB : : cot. ABC : R.

Cor. 2. In any right-angled spherical triangle, *the cosine of either of the oblique angles is to the tangent of the adjacent side, as the cotangent of the hypothenuse is to radius.*

For, in the complemental triangle CEF, according to the preceding theorem, we have

sine EF : cot. CFE : : tan. CE : R;

hence, in the triangle ABC,

cos. ABC : tan. AB : : cot. BC : R.

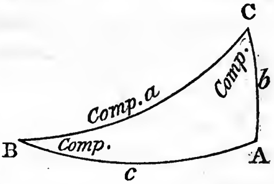
Napier's Rule of the Circular Parts.

(210.) The two preceding theorems, with their corollaries, are sufficient for the solution of all cases of right-angled spherical triangles, and a rule was invented by Napier by means of which these principles are easily retained in mind.

If, in a right-angled spherical triangle, we set aside the right angle, and consider only the five remaining parts of the triangle, viz., the three sides and the two oblique angles, then the two sides which contain the right angle, and the complements of the other three, viz., of the two angles and the hypothenuse, are called the *circular parts.*

Thus, in the triangle ABC, right-angled at A, the circular parts are AB, AC, with the complements of B, BC, and C.

When, of the five circular parts, any one is taken for the *middle part*, then, of the remaining four, the two which are immediately adjacent to it on the right and left are called the *adjacent parts;* and the other two, each of which is separated from the middle by an adjacent part, are called *opposite parts.*



In every question proposed for solution, three of the circular parts are concerned, two of which are given, and one required; and of these three, the middle part must be such that the other two may be equidistant from it; that is, may be either *both adjacent* or *both opposite parts.* The value of the part required may then be found by the following

RULE OF NAPIER.

(211.) *The product of the radius and the sine of the middle part, is equal to the product of the tangents of the adjacent parts, or to the product of the cosines of the opposite parts.*

It will assist the learner in remembering this rule to remark, that the first syllable of each of the words *tangent* and *adjacent* contains the same vowel *a*, and the first syllable of the words *cosine* and *opposite* contains the same vowel *o*.

It is obvious that the cosine of the complement of an angle is the sine of that angle, and the tangent of a complement is a cotangent, and vice versa.

In the triangle ABC, if we take the side b as the middle part, then the side c and the complement of the angle C are the adjacent parts, and the complements of the angle B and of the hypothenuse a are the opposite parts. Then, according to Napier's rule, $\text{R} \sin. b = \tan. c \cot. \text{C}$,
which corresponds with Theorem II.

Also, by Napier's rule,

$$\text{R} \sin. b = \sin. a \sin. \text{B},$$

which corresponds with Theorem I.

Making each of the circular parts in succession the middle part, we obtain the ten following equations:

$$\text{R} \sin. b = \sin. a \sin. \text{B} = \tan. c \cot. \text{C}.$$
$$\text{R} \sin. c = \sin. a \sin. \text{C} = \tan. b \cot. \text{B}.$$
$$\text{R} \cos. \text{B} = \cos. b \sin. \text{C} = \cot. a \tan. c.$$
$$\text{R} \cos. a = \cos. b \cos. c = \cot. \text{B} \cot. \text{C}.$$
$$\text{R} \cos. \text{C} = \cos. c \sin. \text{B} = \cot. a \tan. b.$$

(212.) In order to determine whether the quantity sought is less or greater than 90°, the algebraic sign of each term should be preserved whenever one of them is negative. If the quantity sought is determined by means of its cosine, tangent, or cotangent, the algebraic sign of the result will show whether this quantity is less or greater than 90°; for the cosines, tangents, and cotangents are positive in the first quadrant, and negative in the second. But since the sines are positive in both the first and second quadrants, when a quantity is determined by means of its sine, this rule will leave it ambiguous whether the quantity is less or greater than 90°. The ambiguity may, however, generally be removed by the following rule.

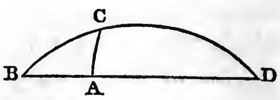
In every right-angled spherical triangle, an oblique angle and its opposite side are always of the same species; that is, both are greater, or both less than 90°.

This follows from the equation

$$R \sin. b = \tan. c \cot. C;$$

where, since sin. b is always positive, tan. c must always have the same sign as cot. C; that is, the side c and the opposite angle C both belong to the same quadrant.

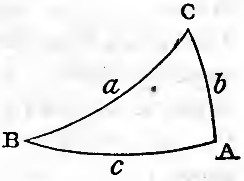
(213.) When the given parts are a side and its opposite angle, the problem admits of two solutions; for two right-angled spherical triangles may always be found, having a side and its opposite angle the same in both, but of which the remaining sides and the remaining angle of the one are the supplements of the remaining sides and the remaining angle of the other. Thus, let BCD, BAD be the halves of two great circles, and let the arc CA be drawn perpendicular to BD; then ABC, ADC are two right-angled triangles, having the side AC common, and the opposite angle B equal to the angle D; but the side DC is the supplement of BC, AD is the supplement of AB, and the angle ACD is the supplement of ACB.



EXAMPLES.

1. In the right-angled spherical triangle ABC, there are given $a=63° 56'$ and $b=40°$. Required the other side c, and the angles B and C.

To find the side c.



Here the circular parts concerned are the two legs and the complement of the hypothenuse; and it is evident that if the complement of a be made the *middle part*, b and c will be *opposite parts;* hence, by Napier's rule,

$$\text{R cos. } a = \text{cos. } b \text{ cos. } c;$$

or, reducing this equation to a proportion,

$$\text{cos. } b : \text{R} :: \text{cos. } a : \text{cos. } c = 54° \ 59' \ 49''.$$

To find the angle B.

Here b is the *middle part*, and the complements of B and a are *opposite parts;* hence

$$\text{R sin. } b = \text{cos. (comp. } a) \times \text{cos. (comp. B)} = \text{sin. } a \text{ sin. B},$$

or $$\text{sin. } a : \text{R} :: \text{sin. } b : \text{sin. B} = 45° \ 41' \ 25''.$$

B is known to be an acute angle, because its opposite side is less than 90°.

To find the angle C.

Here the complement of C is the *middle part;* also b and the complement of a are *adjacent parts;* hence

$$\text{R cos. C} = \text{cot. } a \text{ tan. } b,$$

or $$\text{R} : \text{tan. } b :: \text{cot. } a : \text{cos. C} = 65° \ 45' \ 57''.$$

Ex. 2. In a right-angled triangle ABC, there are given the hypothenuse $a = 91° \ 42'$, and the angle $\text{B} = 95° \ 6'$. Required the remaining parts.

To find the angle C.

Make the complement of the hypothenuse the middle part; then $$\text{R cos. } a = \text{cot. B cot. C}.$$

Whence $$\text{C} = 71° \ 36' \ 47''.$$

To find the side c.

Make the complement of the angle B the middle part; and we have $$\text{R cos. B} = \text{cot. } a \text{ tan. } c.$$

Whence $$c = 71° \ 32' \ 14''.$$

To find the side b.

Make the side b the middle part; then

$$R \sin. b = \sin. a \sin. B.$$

Whence $b = 95° 22' 30''$.

b is known to be greater than a quadrant, because its opposite angle is obtuse.

Ex. 3. In the right-angled triangle ABC, the side b is 26° 4′, and its opposite angle B 36°. Required the remaining parts.

$$Ans. \begin{cases} a = 48° 22' 52'', \text{ or } 131° 37' 8''. \\ c = 42° 19' 17'', \text{ or } 137° 40' 43''. \\ C = 64° 14' 26'', \text{ or } 115° 45' 34''. \end{cases}$$

This example, it will be seen, admits of two solutions, conformably to Art. 213.

Ex. 4. In the right-angled spherical triangle ABC, there are given the side c, 54° 30′, and its adjacent angle B, 44° 50′. Required the remaining parts.

$$Ans. \begin{cases} C = 65° 49' 53''. \\ a = 63° 10' 4''. \\ b = 38° 59' 11''. \end{cases}$$

Why is not the result ambiguous in this case?

Ex. 5. In the right-angled spherical triangle ABC, the side b is 55° 28′, and the side c 63° 15′. Required the remaining parts.

$$Ans. \begin{cases} a = 75° 13' 2''. \\ B = 58° 25' 47''. \\ C = 67° 27' 1''. \end{cases}$$

Ex. 6. In the right-angled spherical triangle ABC, there are given the angle B=69° 20′, and the angle C=58° 16′. Required the remaining parts.

$$Ans. \begin{cases} a = 76° 30' 37''. \\ b = 65° 28' 58''. \\ c = 55° 47' 46''. \end{cases}$$

(214.) A triangle, in which one of the sides is equal to a quadrant, may be solved upon the same principles as right-angled triangles, for its polar triangle will contain a right angle. See *Geom.*, Prop. 9, B. IX.

Ex. 7. In the spherical triangle ABC, the side BC=90°, the angle C=42° 10′, and the angle A=115° 20′. Required the remaining parts.

Taking the supplements of the given parts, we shall have in the polar triangle the hypothenuse $a'=180°-115° 20'=64° 40'$, and one of the sides, $c'=180°-42° 10'=137° 50'$, from which, by Napier's rule, we find

$$B'=115° 23' 20''.$$
$$C'=132° 2' 13''.$$
$$b'=125° 15' 36''.$$

Hence, taking the supplements of these arcs, we find the parts of the required triangle are

$$AC=64° 36' 40''.$$
$$AB=47° 57' 47''.$$
$$B=54° 44' 24''.$$

Ex. 8. In the spherical triangle ABC, the side AC=90°, the angle C=69° 13′ 46″, and the angle A=72° 12′ 4″. Required the remaining parts.

$$Ans. \begin{cases} AB=70° \ 8' 39''. \\ BC=73° 17' 29''. \\ B=96° 13' 23''. \end{cases}$$

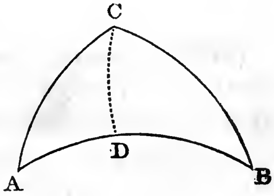
OBLIQUE-ANGLED SPHERICAL TRIANGLES.

THEOREM III.

(215.) *In any spherical triangle, the sines of the sides are proportional to the sines of the opposite angles.*

In the case of right-angled spherical triangles, this proposition has already been demonstrated. Let, then, ABC be an oblique-angled triangle; we are to prove that

sin. BC : sin. AC : : sin. A : sin. B.



Through the point C draw an arc of a great circle CD perpendicular to AB. Then, in the spherical triangle ACD, right-angled at D, we have, by Napier's rule,

R sin. CD=sin. AC sin. A.

Also, in the triangle BCD, we have

R sin. CD=sin. BC sin. B.

Hence sin. AC sin. A=sin. BC sin. B,

or sin. BC : sin. AC : : sin. A : sin. B.

(216.) *Cor.* 1. In any spherical triangle, *the cosines of the sides are proportional to the cosines of the segments of the base, made by a perpendicular from the opposite angle.*

For, by Theorem I., Cor. 2,

cos. CD : R : : cos. AC : cos. AD.
Also, cos. CD : R : : cos. BC : cos. BD.
Hence cos. AC : cos. BC : : cos. AD : cos. BD.

Cor. 2. *The cosines of the angles at the base are proportional to the sines of the segments of the vertical angle.*

For, by Theorem I., Cor. 3,

cos. CD : R : : cos. A : sin. ACD.
Also, cos. CD : R : : cos. B : sin. BCD.
Hence cos. A : cos. B : : sin. ACD : sin. BCD.

Cor. 3. *The sines of the segments of the base are reciprocally proportional to the tangents of the angles at the base.*

For, by Theorem II.,

sin. AD : R : : tan. CD : tan. A.
Also, sin. BD : R : : tan. CD : tan. B.
Hence sin. AD : sin. BD : : tan. B : tan. A.

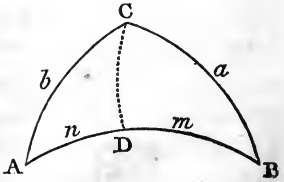
Cor. 4. *The cotangents of the two sides are proportional to the cosines of the segments of the vertical angle.*

For, by Theorem II., Cor. 2,

cos. ACD : cot. AC : : tan. CD : R.
Also, cos. BCD : cot. BC : : tan. CD : R.
Hence cos. ACD : cos. BCD : : cot. AC : cot. BC.

THEOREM IV.

(217.) *If from an angle of a spherical triangle a perpendicular be drawn to the base, then the tangent of half the sum of the segments of the base is to the tangent of half the sum of the sides, as the tangent of half the difference of the sides is to the tangent of half the difference of the segments of the base.*



Let ABC be any spherical triangle, and let CD be drawn from C perpendicular to the base AB; then tan. $\frac{1}{2}$(BD+AD) : tan. $\frac{1}{2}$(BC+AC) : : tan. $\frac{1}{2}$(BC−AC) : tan. $\frac{1}{2}$(BD−AD).

Let BC=a, AC=b, BD=m, and AD=n. Then, by Theorem III., Cor. 1,

$$\cos. a : \cos. b : : \cos. m : \cos. n.$$

Whence, Geom., Prop. 7, Cor., B. II.,

$\cos. b+\cos. a : \cos. b-\cos. a :: \cos. n+\cos. m : \cos. n-\cos. m.$

But by Trig., Art. 76,

$\cos. b+\cos. a : \cos. b-\cos. a :: \cot. \frac{1}{2}(a+b) : \tan. \frac{1}{2}(a-b).$

Also, by the same Art.,

$\cos. n+\cos. m : \cos. n-\cos. m :: \cot. \frac{1}{2}(m+n) : \tan. \frac{1}{2}(m-n).$

Therefore

$\cot. \frac{1}{2}(a+b) : \cot. \frac{1}{2}(m+n) :: \tan. \frac{1}{2}(a-b) : \tan. \frac{1}{2}(m-n).$

But, since tangents are reciprocally as their cotangents, Art. 28, we have

$\cot. \frac{1}{2}(a+b) : \cot. \frac{1}{2}(m+n) :: \tan. \frac{1}{2}(m+n) : \tan. \frac{1}{2}(a+b).$

Hence

$\tan. \frac{1}{2}(m+n) : \tan. \frac{1}{2}(a+b) :: \tan. \frac{1}{2}(a-b) : \tan. \frac{1}{2}(m-n).$

(218.) In the solution of oblique-angled spherical triangles, six cases may occur, viz.:

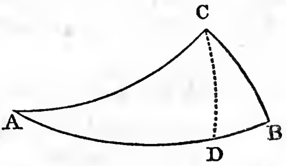
1. Given two sides and an angle opposite one of them.
2. Given two angles and a side opposite one of them.
3. Given two sides and the included angle.
4. Given two angles and the included side.
5. Given the three sides.
6. Given the three angles.

CASE I.

(219.) *Given two sides and an angle opposite one of them, to find the remaining parts.*

In the triangle ABC, let there be given the two sides AC and BC, and the angle A opposite one of them. The angle B may be found by Theorem III.

sin. BC : sin. AC : : sin. A : sin. B.



From the angle C let fall the perpendicular CD upon the side AB. The triangle ABC is divided into two right-angled triangles, in each of which there is given the hypothenuse and the angle at the base. The remaining parts may then be found by Napier's rule.

Ex. 1. In the oblique-angled spherical triangle ABC, the side AC=70° 10′ 30″, BC=80° 5′ 4″, and the angle A=33° 15′ 7″. Required the other parts.

sin. BC : sin. AC : : sin. A : sin. B=31° 34′ 38″.

Then, in the triangle ACD,

R cos. AC=cot. A cot. ACD.

Whence ACD=77° 27′ 47″.

Also, in the triangle BCD,

R cos. BC=cot. B cot. BCD.

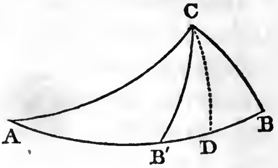
Whence BCD= 83° 57′ 29″.

Therefore ACB=161° 25′ 16″.

To find the side AB.

sin. A : sin. ACB : : sin. BC : sin. AB=145° 5′ 0″.

When we have given two sides and an opposite angle, there are, in general, two solutions, each of which will satisfy the conditions of the problem. If the side AC, the angle A, and the side opposite this angle are given, then, with the latter for radius, describe an arc cutting the arc AB in the points B and B′. The arcs CB, CB′ will be equal, and each of the triangles ACB, ACB′ will satisfy the conditions of the problem. There is the same ambiguity in the numerical computation. The angle B is found by means of its *sine*. But this may be the sine either of ABC, or of its *supplement* AB′C (Art. 27). In the preceding example, the first proportion leaves it ambiguous whether the angle B is 31° 34′ 38′, or its supplement 148° 25′ 22″. In order to avoid false solutions, we should remember that *the greater side of a spherical triangle must lie opposite the greater angle, and conversely* (*Geom.*, Prop. 17, B. IX.). Thus, since in the preceding example the side AC is less than BC, the angle B must be less than A, and, therefore, can not be obtuse.



If the quantity sought is determined by means of its cosine, tangent, or cotangent, the algebraic sign of the result will show whether this quantity is less or greater than 90°; for the cosines, tangents, and cotangents are positive in the first quadrant, and negative in the second. Hence the algebraic sign of each term of a proportion should be preserved whenever one of them is negative.

Ex. 2. In the spherical triangle ABC, the side $a=124° 53'$, $b=31° 19'$, and the angle $A=16° 26'$. Required the remaining parts.

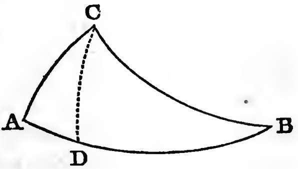
$$Ans. \begin{cases} B=\ 10° 19' 34''. \\ C=171° 48' 22''. \\ c=155° 35' 22''. \end{cases}$$

CASE II.

(220.) *Given two angles and a side opposite one of them, to find the remaining parts.*

In the triangle ABC let there be given two angles, as A and B, and the side AC opposite to one of them. The side BC may be found by Theorem III.

sin. B : sin. A : : sin. AC : sin. BC.



From the unknown angle C draw CD perpendicular to AB; then will the triangle ABC be divided into two right-angled triangles, in each of which there is given the hypothenuse and the angle at the base. Whence we may proceed by Napier's rule, as in Case I.

Ex. 1. In the oblique-angled spherical triangle ABC, there are given the angle $A=52° 20'$, $B=63° 40'$, and the side $b=83° 25'$. Required the remaining parts.

sin. B : sin. A : : sin. AC : sin. BC=61° 19′ 53″.

Then, in the triangle ACD,

cot. AC : R : : cos. A : tan. AD=79° 18′ 17″.

Also, in the triangle BCD,

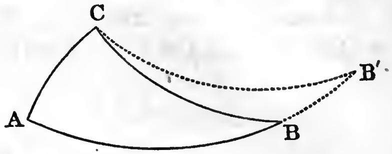
cot. BC : R : : cos. B : tan. BD=39° 3′ 8″.

Hence AB=118° 21′ 25″.

To find the angle ACB.

sin. BC : sin. AB : : sin. A : sin. ACB=127° 26′ 47″.

When we have given two angles and an opposite side, there are, in general, two solutions, each of which will satisfy the conditions of the problem. If the angle A, the side AC, and the angle opposite this side are given, then through the point C there may generally be drawn two arcs of great circles CB, CB′, making the same angle with AB, and each of the triangles ABC, AB′C will satisfy the conditions of the problem. There is the same ambiguity in the numerical computation, since the side BC is found by means of its *sine* (Art. 27). In the preceding example, however, there is no ambiguity, because the angle A is less than B, and, therefore, the side a must be less than b, that is, less than a quadrant.



Ex. 2. In the oblique-angled spherical triangle ABC, the angle A is 128° 45′, the angle C=30° 35′, and B=68° 50′. Required the remaining parts.

It will be observed that in this case the perpendicular BD, drawn from the angle B, falls without the triangle ABC, and therefore the side AC is the difference between the segments CD and AD.

Ans. { AB=37° 28′ 20″.
AC=40° 9′ 4″.
B =32° 37′ 58″.

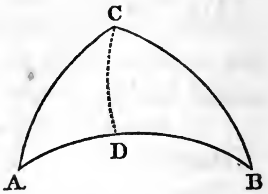
CASE III.

(221.) *Given two sides and the included angle, to find the remaining parts.*

In the triangle ABC let there be given two sides, as AB, AC, and the included angle A. Let fall the perpendicular CD on the side AB; then, by Napier's rule,

R cos. A=tan. AD cot. AC.

Having found the segment AD, the segment BD becomes known; then, by Theorem III., Cor. 3,



sin. BD : sin. AD : : tan. A : tan. B.

The remaining parts may now be found by Theorem III.

Ex. 1. In the spherical triangle ABC, the side AB=73° 20, AC=41° 45′, and the angle A=30° 30′. Required the remaining parts.

cot. AC : cos. A : : R : tan. AD=37° 33′ 41″.

Hence BD=35° 46′ 19″.

sin. BD : sin. AD : : tan. A. : tan. B=31° 33′ 43″.

Also, by Theorem III., Cor. 1,

cos. AD : cos. BD : : cos. AC : cos. BC=40° 12′ 59″.

Then, by Theorem III.,

sin. BC : sin. AB : : sin. A : sin. ACB=131° 8 46″.

Ex. 2. In the spherical triangle ABC, the side AB=78° 15′, AC=56° 20′, and the angle A=120°. Required the other parts.

Ans. { B = 48° 57′ 29″.
C = 62° 31′ 40″.
BC=107° 7′ 45″

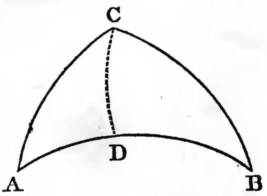
CASE IV.

(222.) *Given two angles and the included side, to find the remaining parts.*

In the triangle ABC let there be given two angles, as A and ACB, and the side AC included between them. From C let fall the perpendicular CD on the side AB. Then, by Napier's rule,

R cos. AC=cot. A cot. ACD.

Having found the angle ACD, the angle BCD becomes known; then, by Theorem III., Cor. 4,



cos. ACD : cos. BCD : : cot. AC : cot. BC.

The remaining parts may now be found by Theorem III.

Ex. 1. In the spherical triangle ABC, the angle A=32° 10′, the angle ACB=133° 20′, and the side AC=39° 15′. Required the other parts.

cot. A : cos. AC : : R : cot. ACD=64° 1′ 57″.

Hence BCD=69° 18′ 3″.

Then

cos. ACD : cos. BCD : : cot. AC : cot. BC=45° 20′ 43″.

Also, by Theorem III., Cor. 2,

sin. ACD : sin. BCD : : cos. A : cos. B=28° 15′ 47″.

Then, by Theorem III.,

sin. B : sin. ACB : : sin. AC : sin. AB=76° 23′ 5″.

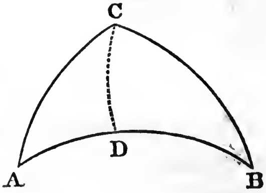
Ex. 2. In the spherical triangle ABC, the angle A=125° 20′, the angle C=48° 30′, and the side AC=83° 13′. Reauired the remaining parts.

Ans. { AB= 56° 39′ 9″. BC=114° 30′ 24″. B = 62° 54′ 38″.

Case V.

(223.) *Given the three sides of a spherical triangle, to find the angles.*

In the triangle ABC let there be given the three sides. From one of the angles, as C, draw CD perpendicular to AB. Then, by Theorem IV., tan. $\frac{1}{2}$AB : tan. $\frac{1}{2}$(AC+BC) : tan. $\frac{1}{2}$(AC−BC) : tan. $\frac{1}{2}$(AD−BD).



Hence AD and BD become known; then, by Napier's rule,

R cos. A=tan. AD cot. AC.

The other angles may now be easily found.

It is generally most convenient to let fall the perpendicular upon the longest side of the triangle.

Ex. 1. In the spherical triangle ABC, the side AB=112° 25′, AC=60° 20′, and BC=81° 10′. Required the angles.

tan. 56° 12$\frac{1}{2}$′ : tan. 70° 45′ : : tan. 10° 25′ : tan. 19° 24′ 26″.

Hence AD=36° 48′ 4″, and BD=75° 36′ 56″.

Then R : tan. AD : : cot. AC : cos. A=64° 46′ 36″.

Also, R : tan. BD : : cot. BC : cos. B=52° 42′ 12″.

Then sin. AC : sin. AB : : sin. B : sin. ACB=122° 11′ 6″.

Ex. 2. In the spherical triangle ABC, the side AB=40° 35′, AC=39° 10′, and BC=71° 15′. Required the angles.

Ans. { A=130° 35′ 55″. B= 30° 25′ 34″. C= 31° 26′ 32″.

Case VI.

(224.) *Given the three angles of a spherical triangle, to find the sides.*

If A, B, C are the angles of the given triangle, and *a*, *b*, *c* its sides, then 180°−A, 180°−B, and 180°−C are the sides of its polar triangle, whose angles may be found by Case V. Then the supplements of those angles will be the sides a, b, c of the proposed triangle.

Ex. 1. In the spherical triangle ABC, the angle A=125° 34′, B=98° 44′, and C=61° 53′. Required the sides.

The sides of the polar triangle are

54° 26′, 81° 16′, and 118° 7′.

From which, by Case V., the angles are found to be

134° 6′ 21″, 41° 28′ 17″, and 53° 34′ 47″.

Hence the sides of the proposed triangle are

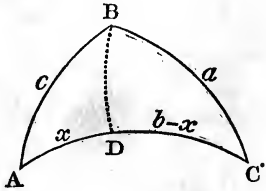
AB=45° 53′ 39″, BC=138° 31′ 43″, and AC=126° 25′ 13″.

Ex. 2. In the spherical triangle ABC, the angle A=109° 55′, B=116° 38′, and C=120° 43′. Required the sides.

Ans. $\begin{cases} a = 98° 21' 20''. \\ b = 109° 50' 10''. \\ c = 115° 13' 7''. \end{cases}$

TRIGONOMETRICAL FORMULÆ.

(225.) Let ABC be any spherical triangle, and from the angle B draw the arc BD perpendicular to the base AC. Represent the sides of the triangle by a, b, c, and the segment AD by x; then will CD be equal to $b-x$.



By Theorem II., Cor. 1,

$$\cos. c : \cos. a :: \cos. x : \cos. (b-x)$$

$$:: \cos. x : \frac{\cos. b \cos. x + \sin. b \sin. x}{R}$$

(Trig., Art. 72), formula (4).

Whence

$$R \cos. a \cos. x = \cos. b \cos. c \cos. x + \sin. b \cos. c \sin. x;$$

or, dividing each term by cos. x, and substituting the value of $\frac{\sin. x}{\cos. x}$ (Art. 28), we obtain

$$R^2 \cos. a = R \cos. b \cos. c + \sin. b \cos. c \tan. x.$$

But by Theorem II., Cor. 2, we have

$$\tan. x = \frac{R \cos. A}{\cot. c} = \frac{\cos. A \sin. c}{\cos. c} \text{ (Art. 28).}$$

Hence $R^2 \cos. a = R \cos. b \cos. c + \sin. b \sin. c \cos. A$, (1)
from which all the formulæ necessary for the solution of spherical triangles may be deduced.

In a similar manner we obtain

$$R^2 \cos. b = R \cos. a \cos. c + \sin. a \sin. c \cos. B, \qquad (2)$$
$$R^2 \cos. c = R \cos. a \cos. b + \sin. a \sin. b \cos. C. \qquad (3)$$

These equations express the following Theorem:

The square of radius multiplied by the cosine of either side of a spherical triangle, is equal to radius into the product of the cosines of the two other sides, plus the product of the sines of those sides into the cosine of their included angle.

(226.) From equation (1) we obtain, by transposition,

$$\cos. A = \frac{R^2 \cos. a - R \cos. b \cos. c}{\sin. b \sin. c},$$

a formula which furnishes an angle of a triangle when the three sides are known.

If we add R to each member of this equation, we shall have

$$R + \cos. A = \frac{R^2 \cos. a + R \sin. b \sin. c - R \cos. b \cos. c}{\sin. b \sin. c}.$$

But, by Art. 74, $R + \cos. A = \frac{2 \cos.^2 \frac{1}{2}A}{R}$.

And, by Art. 72, formula (2), by transposition,

$$R \sin. b \sin. c - R \cos. b \cos. c = -R^2 \cos. (b+c).$$

Hence, by substitution, we obtain

$$\frac{2 \cos.^2 \frac{1}{2}A}{R} = \frac{R^2(\cos. a - \cos. (b+c))}{\sin. b \sin. c}$$
$$= \frac{2R \sin. \frac{1}{2}(a+b+c) \sin. \frac{1}{2}(b+c-a)}{\sin. b \sin. c},$$

by Art. 75, formula (4).

If, then, we put $s = \frac{1}{2}(a+b+c)$, that is, half the sum of the sides, we shall find

$$\cos. \tfrac{1}{2}A = R\sqrt{\frac{\sin. s \sin. (s-a)}{\sin. b \sin. c}}. \qquad (4)$$

By subtracting cos. A from R instead of adding, we shall obtain, in a similar manner,

$$\sin. \tfrac{1}{2}A = R\sqrt{\frac{\sin. (s-b) \sin. (s-c)}{\sin. b \sin. c}}. \qquad (5)$$

Either formula (4) or (5) may be employed to compute the angles of a spherical triangle when the three sides are known, and this method may be preferred to that of Art. 223.

Ex. 1. In a spherical triangle there are given $a=63° 50'$, $b=80° 19'$, and $c=120° 47'$. Required the three angles.

Here half the sum of the sides is $132° 28'=s$.

Also, $s-a=68° 38'$.

Using formula (4), we have

log. sine s,	132° 28′	. .	9.867862
log. sine $(s-a)$,	68° 38′	. .	9.969075
$-$log. sine b,	80° 19′	comp.	0.006232
$-$log. sine c,	120° 47′	comp.	0.065952
		Sum	19.909121
log. cos. $\frac{1}{2}$A,	25° 45′ 19″		9.954560.

Hence the angle $A=51° 30' 38''$.

The remaining angles may be found by Theorem III., or by formulas similar to formula (4).

$$\cos. \tfrac{1}{2}B=R\sqrt{\frac{\sin. s \sin. (s-b)}{\sin. a \sin. c}}.$$

$$\cos. \tfrac{1}{2}C=R\sqrt{\frac{\sin. s \sin. (s-c)}{\sin. a \sin. b}}.$$

We thus find the angle $B= 59° 16' 46''$,

and $C=131° 28' 36''$.

Ex. 2. In a spherical triangle there are given $a=115° 20'$, $b=57° 30'$, and $c=82° 28'$. Required the three angles.

$$Ans. \begin{cases} A=126° 35' \ 2''. \\ B= 48° 31' 42''. \\ C= 61° 43' 58''. \end{cases}$$

(227.) By means of the polar triangle, we may convert the preceding formulæ for angles into formulæ for the sides of a triangle, since the angles of every triangle are the supplements of the sides of its polar triangle. Let, then, a', b', c', A′, B′, C′ represent the sides and angles of the polar triangle, and we shall have

$$A=180°-a',\ B=180°-b',\ C=180°-c,$$
$$a=180°-A',\ b=180°-B',\ c=180°-C'.$$

Therefore $\sin. \frac{1}{2}A=\sin. (90°-\frac{1}{2}a')=\cos. \frac{1}{2}a',$

$\cos. \frac{1}{2}A=\cos. (90°-\frac{1}{2}a')=\sin. \frac{1}{2}a',$

$$\sin. b = \sin. (180° - B') = \sin. B',$$
$$\sin. c = \sin. (180° - C') = \sin. C'.$$

Also, if we put S′=half the sum of the angles of the polar triangle, we shall have

$$a+b+c = 540° - (A'+B'+C'),$$

or $$s = 270° - S',$$

whence $$\sin. s = -\cos. S',$$

$$\sin. (s-a) = \sin. [90° - (S'-A')] = \cos. (S'-A'),$$
$$\sin. (s-b) = \cos. (S'-B'),$$
$$\sin. (s-c) = \cos. (S'-C').$$

By substituting these values in formula (5), Art. 226, and omitting all the accents, since the equations are applicable to any triangle, we obtain

$$\cos. \tfrac{1}{2}a = R\sqrt{\frac{\cos. (S-B) \cos. (S-C)}{\sin. B \sin. C}}; \qquad (6)$$

and formula (4) becomes

$$\sin. \tfrac{1}{2}a = R\sqrt{\frac{-\cos. S \cos. (S-A)}{\sin. B \sin. C}}, \qquad (7)$$

which formulæ enable us to compute the sides of a triangle when the three angles are known; and this method may be preferred to that of Art. 224.

In a similar manner, by means of the polar triangle, we derive from formula (1), Art. 225, the equation

$$R^2 \cos. A = \cos. a \sin. B \sin. C - R \cos. B \cos. C; \qquad (8)$$

that is, *the square of radius multiplied by the cosine of either angle of a spherical triangle, is equal to the product of the sines of the two other angles into the cosine of their included side, minus radius into the product of their cosines.*

Ex. 1. In a spherical triangle ABC, there are given A= 130° 30′, B=30° 50′, and C=32° 5′. Required the three sides.

Here half the sum of the angles is 96° 42′ 30″=S.

Also, S−A=−33° 47′ 30″,
S−B= 65° 52′ 30″,
S−C= 64° 37′ 30″.

Using formula (6), we have

log. cos. (S−B), 65° 52′ 30″	.	9.611435
log. cos. (S−C), 64° 37′ 30″	.	9.631992
−log. sin. B, 30° 50′ comp.		0.290270
−log. sin. C, 32° 5′ comp.		0.274781
	Sum	19.808478
log. cos. ½a, 36° 40′ 1″		9.904239.

Hence the side a=73° 20′ 2″.

The remaining sides may be found by Theorem III., or by formulas similar to formula (6).

$$\cos.\ \tfrac{1}{2}b = R\sqrt{\frac{\cos.\ (S-A)\cos.\ (S-C)}{\sin.\ A \sin.\ C}},$$

$$\cos.\ \tfrac{1}{2}c = R\sqrt{\frac{\cos.\ (S-A)\cos.\ (S-B)}{\sin.\ A \sin.\ B}}.$$

We thus find the side b=40° 13° 12″,
and c=42° 0′ 12″.

Ex. 2. In the spherical triangle ABC, the angle A=129° 30′, B=54° 35′, and C=63° 5′. Required the three sides.

Ans. a=120° 57′ 5″.
b= 64° 55′ 37″
c= 82° 19′ 0″.

(228.) Formula (1), Art. 225, will also furnish a new test for removing the ambiguity of the solution in Case I. of oblique-angled triangles. For we have

$$\cos.\ A = \frac{R^2 \cos.\ a - R \cos.\ b \cos.\ c}{\sin.\ b \sin.\ c}.$$

Now if cos. a is greater than cos. b, we shall have

$$R^2 \cos.\ a > R \cos.\ b \cos.\ c,$$

or the sign of the second member of the equation will be the same as that of cos. a, since the denominator is necessarily positive, and cos. c is less than radius. Hence cos. A and cos. a will have the same sign; or A and a will be of the same species when cos a>cos. b, or sin. a<sin. b; that is,

If the sine of the side opposite to the required angle is less than the sine of the other given side, there will be but one triangle.

But if cos. a is less than cos. b, then such a value may be given to c as to render

$$R^2 \cos. a < R \cos. b \cos. c,$$

or the sign of the second member of the equation will depend upon the value of cos. c; that is, c may be taken so as to render cos. A either positive or negative. Hence

If the sine of the side opposite to the required angle is greater than the sine of the other given side, there will be two triangles which fulfill the given conditions.

(229.) Formula (8), Art. 227, will furnish a test for removing the ambiguity in Case II. of oblique-angled triangles. For we have

$$\cos. a = \frac{R^2 \cos. A + R \cos. B \cos. C}{\sin. B \sin. C};$$

from which it follows, as in the preceding article, that if cos. A is greater than cos. B, A and a will be of the same species. But if cos. A is less than cos. B, then such values may be given to C as to render cos. a either positive or negative. Hence

If the sine of the angle opposite to the required side is less than the sine of the other given angle, there will be but one triangle;

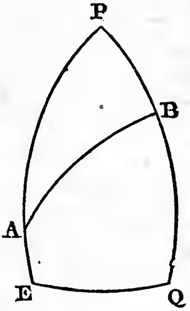
But, *if the sine of the angle opposite to the required side is greater than the sine of the other given angle, there will be two triangles which fulfill the given conditions.*

SAILING ON AN ARC OF A GREAT CIRCLE.

(230.) It is demonstrated in *Geom.*, Prop. 3, B. IX., that the shortest path from one point to another on the surface of a sphere is the arc of a great circle which joins the two given points. Hence, if it is desired to sail from one port to another by the shortest route, it is necessary to follow an arc of a great circle, and this arc generally does not coincide with a rhumb-line.

The bearing and distance from one place to another on the arc of a great circle may be computed from the latitudes and longitudes of the places by means of Spherical Trigonometry.

Thus, let P be the pole of the earth, EQ a part of the equator, and A and B the two given places comprehended between the meridians PE and PQ. Then PA is the complement of the latitude of A, PB is the complement of the latitude of B, and the angle P is measured by the arc EQ, which is the difference of longitude between the two places. Hence, in the triangle ABP, we have given two sides AP, BP, and the included angle P, from which we may compute the side AB, and the angles A and B, according to Case III. of oblique-angled triangles.



Ex. 1. Required the course and distance from Nantucket Shoals, in latitude 41° 4′ N., longitude 69° 55′ W., to Cape Clear, in latitude 51° 26′ N., longitude 9° 29′ W., on the arc of a great circle.

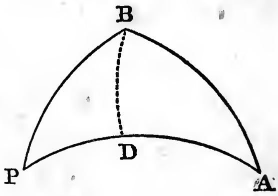
Here we have given

the angle P=69° 55′− 9° 29′=60° 26′;
the side PA=90° −41° 4′=48° 56′;
the side PB=90° −51° 26′=38° 34′.

Then cot. PB : cos. P : : R : tan. PD=21° 28′ 35″.
Whence AD=27° 27′ 25″.
Also sin. AD : sin. PD : : tan. P : tan. A =54° 27′ 21″,
and sin. A : sin. PB : : sin. P : sin. AB=41° 47′ 28″.

41° 47′ 28″ is equal to 2507.47 nautical miles.

Hence the course from Nantucket Shoals to Cape Clear is N. 54° 27′ E., and the distance is 2507.47 miles.



According to Mercator's sailing, the course on a rhumb-line, found on page 152, is N. 76° E., and the distance 2572.9 miles. Hence the distance on an arc of a great circle is 65.4 miles less than on a rhumb-line, and the former course is 21½ degrees more northerly than the latter.

While sailing on a rhumb-line the course of a ship remains always the same, but while sailing on an arc of a great circle the course is continually changing. The preceding course is that with which the ship starts from Nantucket, and a new computation of the course should be made every day or two; or it might be more convenient to compute beforehand the position of the points in which the great circle intersects the meridians for every five degrees of longitude, and the ship might be steered upon a direct course for these points successively.

Ex. 2. Required the course and distance from Nantucket Shoals to Gibraltar, in latitude 36° 6′ N., longitude 5° 20′ W., on the shortest route.

Ans. The course is N. 73° 29′ E.
Distance 2974.1 miles.

Ex. 3. Required the course and distance from Sandy Hook, in latitude 40° 28′ N., longitude 74° 1′ W., to Madeira, in latitude 32° 28′ N., longitude 16° 55′ W., on the shortest route.

Ans. The course is N. 80° 53′ E.
Distance 2744.1 miles.

Ex. 4. Required the course and distance from Sandy Hook to St. Jago, in latitude 14° 54′ N., longitude 23° 30′ W., on the shortest route.

Ans. The course is S. 74° 46′ E.
Distance 3037.6 miles.

Ex. 5. Required the course and distance from Sandy Hook to the Cape of Good Hope, in latitude 34° 22′ S., longitude 18° 30′ E., on the shortest route.

Ans. The course is S. 63° 48′ E.
Distance 6792 miles.

THE END.