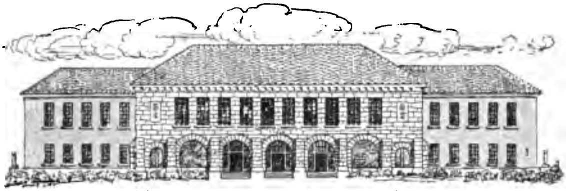

THE ELEMENTS

OF

SOLID GEOMETRY.

WITH NUMEROUS EXERCISES.

BY

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"Every study of a generalization or extension gives additional power over the particular form by which the generalization is suggested."

DE MORGAN, FORMAL LOGIC.

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PREFACE.

The key-note to the arrangement of the contents of the book, and to its *raison d'être*, is the quotation from De Morgan given on the title-page.

A number of supposed improvements are:

Improved Notation. The notation throughout the book is as brief as possible. The general features are the representation of lines, planes, and angles by single letters where possible, and planes and angles by two letters where one is objectionable.

In problems and construction diagrams, the known parts are generally indicated by the middle letters of the alphabet, and the construction points, etc., by the first letters of the alphabet, in the order in which they are found. A glance at the diagram shows the given, required, and intermediate parts, and moreover the *order* in which they are drawn.

Where possible, suggestive letters are used, as t for the trace of one plane upon another, h for height or altitude, r for radius, B for base, Sph. Z for surface of a spherical zone, σ for area of a spherical degree.

Corresponding lines in different figures are indicated by corresponding Greek and English letters. Similarly, subscripts indicate a *close relation* between the parts, equality, similarity of position or homology, etc.

Lines are indicated by the lower case letters, points and surfaces by capitals.

The new symbols representing angle, triangle, perpendicular, parallel, etc., have been judiciously used.

Improved Diagrams. The briefer notation results in increased clearness in both diagrams and text. The student sees much of the demonstration in the character of the typography, and the diagrams are not encumbered with a confusing array of letters.

The increased simplicity and clearness of the diagrams is shown by comparing those of §§ 135, 149, 153, etc., with the corresponding diagrams in other books. They are much easier for the student to grasp and remember.

Much of the difficulty experienced by students consists in not being able to "see the figure." This difficulty is due not to lack of mathematical power but to lack of imagination, made worse by the poor character of the diagrams. Pains has been taken to overcome this difficulty by careful attention to the perspective of the figures. See the figures of §§ 24, 43, 173, 261, 286, etc.

The accurate delineation of polar triangles is noteworthy and new. § 261.

Figures which are *dissimilar* or *irregular* are represented as markedly so, that the student may have no difficulty in recognizing from the diagram the dissimilarity or irregularity. See §§ 92, 167, 169, 170.

Clear Statements. In the text, demonstrations are presented in their constituent parts under distinctly designated headings : *Notation*, *To Prove*, *Construction*, *Analysis*, *Proof.*

No student need say he "can't see what the book is driving at."

Generalized Conceptions. In the chapter on cones, polyhedrons, etc., the frustum of a pyramid is considered as the primary solid, and the pyramid, prism, cone, and cylinder are considered as special cases of the frustum.

The student thus gets a broader view of the subject, and is impressed with the oneness of the different solids.

His attention is constantly directed to the limiting cases of surfaces and solids, and the manner in which they merge, one into the other, is constantly impressed upon his mind. See §§ 104–106, 188, 190, 298, 301½.

Condensation. Aside from enlarging the conception of the student and broadening his horizon generally, a large saving in space is made, twenty-one propositions by this arrangement covering about the same ground as thirty-nine or forty under the usual arrangement. Many propositions covering a page under the usual arrangement are here disposed of as a corollary of a few lines.

The student gets a clearer grasp of the subject, and is not deluded by a page of padded matter into thinking that he has acquired something more than a corollary. For instance, the student has pointed out to him once for all that the cone is the limiting case of the pyramid, and then the properties of the cone are given as corollaries to those of the pyramid, instead of repeating each time that the cone is a limiting case, etc., and making a whole demonstration out of what is properly disposed of in a few lines.

Chapter III, on the sphere, effects a similar condensation from thirty odd to twenty-one propositions.

This gain in space is effected, not by curtailing the demonstrations or omitting any necessary matter, but solely by the method of arrangement, making corollaries of what have usually been propositions which were padded out with repetitions of previous matter to give them the proper length.

The whilom popular idea that each proposition must occupy an entire page or pages is discarded. A *short* demonstration is made *short.*

The student is not deceived into thinking he has learned a page of geometrical truth, when in fact he has learned but a few lines.

In the present work the diagrams are generally so simply lettered, that one glance is sufficient to fix it in the student's mind, and he is under no necessity to turn to it again, this simplicity being secured by few letters, or where that is impossible, by suggestive or characteristic letters, as explained under the head of notation. Compare §§ 24, 34, 64, 67, 135, 153, 174, etc.

Special Characteristics. The problems in the construction of the regular polyhedrons (§ 209) are solved upon tentative rather than upon dogmatic lines, with marked gain for the student.

The student is not arbitrarily led *per saltum*, and the correctness of the step proved afterwards, but each solid is put together piece by piece, as one would do in the actual construction of a model.

In the propositions relating to spherical surfaces, increased clearness has been gained by the use of the *spherical degree* as the unit of spherical surface, and it is defined much more precisely and comprehensively than is usual.

Precision and comprehensiveness have been sought in all the definitions. See §§ 298, 300, 301, 301½, 315½.

The very general solid, the prismatoid has been introduced, and made the basis of demonstration for its limiting cases, the wedge, pyramid, cone, etc.

In the area of the spherical triangle, lune, etc., the expression of the angles in angular degrees or radians has been carefully distinguished.

The One object which has been kept in sight throughout the book has been the attainment by the student of a practical and comprehensive working knowledge of the principles of solid geometry considered as a unit: the attainment of something more than the accumulation of a quantity of isolated theorems of little use as a mental training and of no use for practical purposes.

One aid to this object is a large number of practical examples scattered through the book. Among the typographical aids to the student is the table of abbreviations and formulae at the end of each chapter.

Where the properties of parallel lines are referred to, the well-recognized and convenient formula that parallel lines meet at infinity has been used.

This need not mislead the student, and makes the treatment much more brief.

But, to insure that the language of the book shall not be misunderstood, it may be well to say here: Parallel lines are the limit to which intersecting lines tend as the intersection passes out to infinity. This is conveniently phrased in the formula referred to above.

Similar remarks apply to parallel planes, prisms considered as limiting cases of pyramids, etc.

CONTENTS.

CHAPTER III.

THE SPHERE.

CHAPTER IV.

CONIC SECTIONS.

SUGGESTIONS TO STUDENTS.

WHEN studying a proposition, *draw the figure for yourself on a separate piece of paper*, varying the shape and proportions, but keeping the lettering the same.

Write out the principal points, equations, etc., as you study them.

Consult other books and see if you can find a demonstration that suits you better than the one given here.

When through, write out the demonstration in your own language, using the one which seems simplest and best.

Use pencil and paper constantly. Mathematics must be written into the mind, not read into it. "No head for mathematics" nearly always means "will not use a pencil."

NOTATION.

The following notation is used generally throughout the book, though departed from in some instances where additional brevity could be secured.

Points are indicated by capital letters, A, B, C, M, N, etc., also by the two lines whose intersection determine the point, thus point *pq* means the point at the intersection of the lines *p* and *q*.

Lines are indicated by lower case letters, *a*, *b*, *c*, *m*, *p*, etc.

Planes and surfaces are indicated by capital letters, M, N, R, S, etc.

Angles are indicated by the two lines which form the angle, thus $\angle$ *mn* means the angle between the lines *mn*; or by Greek letters, e.g., α, β, γ, etc.

Plane figures are indicated by giving two or more of their sides, as many as are sufficient to completely determine them. Thus $\triangle$ *ab* means the $\triangle$ two of whose sides are *ab*. Figure *ab* means the figure which has *ab* for its sides. If necessary all the sides are named.

Volumes are denoted by heavy faced type, e.g., **V**, **P**, **Q**, etc.

The ordinary notation of geometry is used where that happens to be shorter and clearer.

The *given parts* are generally designated by the middle letters of the alphabet, the *unknown parts* by the last letters, and the *construction parts* by the first letters of the alphabet in the order in which they are found. Hence a glance at the diagram indicates the given, required, and intermediary parts, and moreover the *order* in which the intermediary parts are found.

Suggestive letters are used where practicable, thus *t* for the intersections (traces) of planes, *h* for height or altitude, *r* for radius, *B* for base.

Corresponding lines in two figures are indicated by corresponding English and Greek letters.

Subscripts indicate a close relation between the parts, equality, similarity of position or homology, etc.

CHAPTER I.

LINES AND PLANES IN SPACE.

Solid Geometry is that branch of geometry in which the forms (or figures) treated are not limited to a single plane.

1. **A plane** is a surface such that a straight line joining any two points in it lies wholly in the surface. A plane is indefinite in extent, so that however far the straight line is produced, all its points lie in the plane; but a limited portion of a plane is usually represented by a parallelogram.

2. A plane is said to be **determined** by certain lines or points, when it is the only plane which contains those lines or points.

3. Any number of planes may be passed through a straight line, for a plane passing through the line may be revolved about the line and made to occupy an infinite number of positions, each of which will be a different plane. Hence a single straight line does not determine a plane.

4. *A plane is determined by three points not in the same straight line.*

For if the plane be turned about the straight line containing two of the points until it contains the third point, the plane is evidently determined, since if it is then revolved either backward or forward, it will no longer contain the third point.

5. *A plane is determined by a straight line and a point without that line, by two intersecting straight lines, or by two parallel lines,* since each of these cases can be converted into that of § 4 by selecting three points, two in one of the given lines and the third in the other line.

6. A straight line is **perpendicular to a plane** when it is perpendicular to every straight line of the plane which passes through its *foot,* that is, the point where it meets the plane.

Conversely, the plane is perpendicular to the line.

7. A line is **oblique** to a plane if it is not perpendicular to all the straight lines drawn in the plane through its foot.

8. A line is **parallel to a plane** when it is the limiting case of an oblique line, that is when its point of intersection has passed out to infinity.

In this case it is said to meet the plane at infinity.

A plane cuts all lines in space which are not parallel to it.

9. The **distance** from a point to a plane is the perpendicular distance from the point to the plane.

10. Two planes are **parallel** when their line of intersection has passed out to infinity; and the planes are said to meet at infinity.

11. The **projection of a point** on a plane is the foot of the perpendicular let fall from the point to the plane.

12. The **projection of a line** on a plane is the locus of the projections of all its points.

13. The **angle which a line makes with a plane** is the angle which it makes with its projection on the plane.

Proposition I. Theorem.

14. *The intersection of two planes is a straight line.*

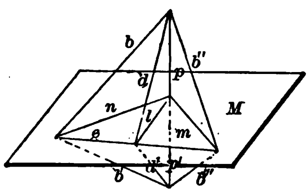
Proof. Join two points of the intersection by a straight line. By definition, § 1, this straight line lies wholly in each plane, and hence must be common to them, or, in other words, be their intersection. Q. E. D.

No point outside of this line can lie in the intersection, for only *one* plane can contain a straight line and a point without, § 5.

Proposition II. Theorem.

15. *If a straight line is perpendicular to each of two straight lines at their point of intersection it is perpendicular to the plane of those lines.*

Notation. Let m and n be two straight lines which determine the plane M, and p another straight line perpendicular to each of them at their intersection. Let l be *any* other straight line in the plane M passing through the intersection of m and n.



To prove $p \perp$ to l, and therefore to the plane M.

Construction. Lay off $m = n$, and prolong p making $p' = p$. Connect the extremities of m, n, p, p' by straight lines. From the intersection of l and one of these new lines, draw lines to the extremities of p and p'.

Analysis. 1°. b, b', b'', b''' are = since points in the $\perp$ bisectors m, n are equally distant from the ends of p, p', and $m = n$. Hence in the isosceles △'s $bb''e$, $b'b'''e$, the ∠ be = ∠ $b'e$, and the △ bed = △ $b'ed'$.

2°. d and d' are = because the △ bed = △ $b'ed'$ (having two sides and the included angles equal).

3°. p is ⊥ to l because the lines d, d' are equal (points equidistant from the ends of a straight line lie in the ⊥ bisector).

Hence p is perpendicular to M because it is perpendicular to l, any line in that plane. Q. E. D.

Is it necessary that $m = n$?

16. Cor. 1. *At a given point in a plane, only one perpendicular to the plane can be erected.*

Otherwise, if we pass a plane through the two perpendiculars, giving an intersection l with the plane M, we should have two ⊥'s in the same plane to the same straight line l at the same point, which is impossible.

17. Cor. 2. *From a point without a plane only one perpendicular can be drawn to the plane.*

For if p, b be two such ⊥'s, the △ pbn would contain two rt. ∠s, which is impossible.

18. Cor. 3. *Oblique lines drawn from a point to a plane and meeting the plane at equal distances from the foot of the perpendicular, are equal.* E.g., b and b'', § 15.

19. Cor. 4. *Of oblique lines drawn from a point to a plane the one which meets the plane further from the foot of the perpendicular is the longer.*

If n be extended, b, the hypothenuse of the rt. △ bn, must become longer than b''.

20. Cor. 5. *Equal oblique lines from a point to a plane meet the plane at equal distances from the foot of the perpendicular; and of two unequal oblique lines the greater* *meets the plane at a greater distance from the foot of the perpendicular.*

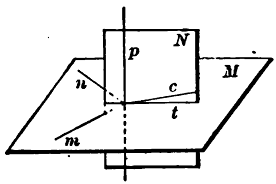
21. Cor. 6. *The perpendicular is the shortest line from a point to a plane.*

22. Cor. 7. *The locus of the point in a plane at a given distance from a fixed point, is a circle whose centre is the foot of the perpendicular through the point.*

23. Cor. 8. *The locus of a point in space equidistant from all points in the circumference of a circle is a straight line passing through the centre and perpendicular to the plane of the circle.*

Proposition III. Theorem.

24. *All the perpendiculars to a straight line at the same point lie in a plane perpendicular to the line.*



Notation. Let m, n and c be three perpendiculars to p at the same point, and M the plane of two of them, m and n.

To prove that M contains c.

Construction. Pass the plane N through p and c, cutting M in t.

Analysis. c and t are two perpendiculars to p in the same plane at the same point, which is impossible, ∴ c and t must coincide, and c lies in M. Q. E. D.

25. Cor. 1. *At a given point in a straight line, one and only one plane perpendicular to the line can be drawn.*

26. Cor. 2. *If a right angle be turned around one of its arms as an axis, the other arm will generate a plane perpendicular to the axis.*

27. Cor. 3. *The locus of a point in space equidistant from the extremities of a straight line is the plane perpendicular to this line at its middle point.*

For, all points in a ⊥ bisector of the line are equidistant from the ends of the line, and if this ⊥ bisector be revolved around the given line as an axis, it generates a plane, § 26, whose points must be equidistant from the ends of the given line.

28. Cor. 4. *Through a given point without a straight line one plane, and only one, can be drawn perpendicular to the line.*

Notation. In the figure let p be the given line, and the end of t the given point, and t a ⊥ to p through the given point, M the plane generated by revolving t around p as an axis, and N a plane through p and t.

To prove M ⊥ to p; and that there can be no other perpendicular plane.

Analysis. Cor. 2 proves the first part.

If there could be another perpendicular plane, its intersection with N would make a second perpendicular in the same plane to the line p from a point without, which is impossible, hence there cannot be a second perpendicular plane. Q. E. D.

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1. Show two nonparallel lines which will not meet.

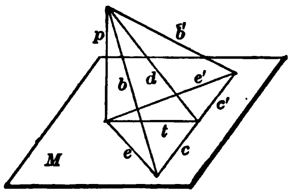
2. With a 12 foot pole how would you determine a point in the floor directly under a certain point in a ceiling which is 10 feet high?

3. How would you determine a perpendicular to a plane by the use of two carpenter's squares?

4. Find the locus of points equally distant from two given points.

PROPOSITION IV. THEOREM.

29. *If from the foot of a perpendicular to a plane a straight line is drawn at right angles to any line in the plane, and its intersection with that line is joined to any point of the perpendicular, this last line will be perpendicular to the line in the plane.*



Notation. Let p be $\perp$ to the plane M, t a $\perp$ from its foot to any line c, and d a line joining the intersection of c and t to any point in p.

To prove $d \perp c$.

Construction. Take $c = c'$ and draw b, b'.

Analysis. 1°. d is $\perp$ c because $b = b'$. 2°. $b = b'$ because $e = e'$.

Proof. $e = e'$ because $c = c'$ (if two sides and the included angle of one triangle are equal, etc.) $\therefore b = b'$, § 18.

But if $b = b'$ and $c = c'$, then d is $\perp$ to c. (If a straight line have two points each of which is equally distant from the extremities of a second line, the first line is a perpendicular bisector of the second.) Q. E. D.

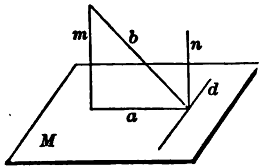
30. COR. *c is $\perp$ to the plane of the $\triangle$ pdt.* Why?

Find the locus of points

5. Equally distant from two given lines in a plane.
6. Equally distant from three given points.
7. Equally distant from three given planes.
8. Equally distant from three given lines in a plane.
9. Equally distant from three concurrent lines.
10. In a given plane equally distant from two exterior points.

PROPOSITION V. THEOREM.

31. *Two straight lines perpendicular to the same plane are parallel.*



Notation. Let m, n be $\perp$ to the plane M.

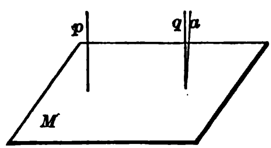
Construction. Connect the foot of the $\perp$s by the line a and draw the line b. Through the foot of n draw the line d in the plane M $\perp$ to a.

To prove $m \parallel$ to n.

Analysis. 1° m and n lie in the same plane.
2° they are both $\perp$ to the same line a.
3° $\therefore$ they are $\parallel$.

Proof. 1° n is $\perp$ to d, § 6: b is $\perp$ to d, § 29: a is $\perp$ to d by construction. $\therefore$ n, b and a lie in the same plane, § 24. $\therefore$ n and m lie in the same plane.
2° they are both $\perp$ to a, § 6.
3° $\therefore$ they are $\parallel$. Q. E. D.

32. COR. 1. *If one of two parallels is perpendicular to a plane, the other is also.*



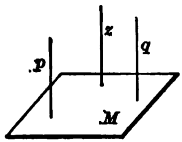
Notation. Let p be $\parallel$ to q and $\perp$ to M.

To prove $q \perp M$.

Construction. Through the foot of q, draw a, $\perp$ to M.

Analysis. a is $\parallel$ to p, § 31, but q is already $\parallel$ to p through the same point, and hence q and a must coincide. Hence q is $\perp$ to M. Q. E. D.

33. Cor. 2. *Two straight lines that are parallel to a third are parallel to each other.*



Notation. Let p and q be each ∥ to z.

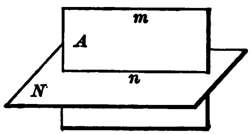
To prove p ∥ to q.

Construction. Draw the plane $M \perp$ to z.

Analysis. p and q are each $\perp$ to M, § 32: hence p ∥ to q, § 31. Q. E. D.

Proposition VI. Theorem.

34. *One of two parallel lines is parallel to every plane containing the other.*



Notation. Let m, n be two ∥ lines, and N a plane through n.

To prove m ∥ to N.

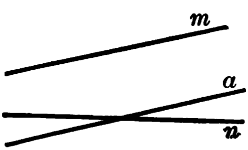
Construction. Draw the plane A through m and n.

Analysis. If m meets the plane N it must be somewhere in the line n, the intersection of the two planes, but m and n are ∥, hence m cannot meet N. Q. E. D.

Note. The limiting case of a line ∥ to a plane is a line in the plane.

35. Cor. 1. *A line parallel to the intersection of two planes is parallel to the planes.*

36. Cor. 2. *Through any given straight line, a plane can be passed parallel to any other straight line.*



Notation. Let n be the given line. Required to pass a plane through it ∥ to m.

Construction. Through a point of n, draw a, ∥ to m.

Analysis. The two lines *n*, *a* determine a plane which by § 34 is ∥ to *m*. Q. E. D.

37. Cor. 3. *Through a given point a plane can be passed parallel to any two lines.*

For through the point draw two lines ∥ to the given lines. These will determine a plane which by § 34 is ∥ to each of the given lines.

38. Cor. 4. *If a line is parallel to a plane, the intersection of this plane with any plane through the line is parallel to the line.*

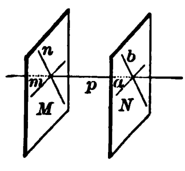
For, if the intersection is not ∥ to the line it must meet it (being in the same plane with it), and since the intersection cannot pass out of the given plane, it must meet it in that plane, but this is impossible since the line is ∥ to the plane, hence the intersection must be ∥ to the line. Q. E. D.

Are two lines which do not meet necessarily parallel?

39. Cor. 5. *If a straight line and a plane are parallel, a parallel to the line drawn through a point in the plane, lies in the plane.* § 38

Proposition VII. Theorem.

40. *Two planes perpendicular to the same straight line are parallel.*



Notation. Let *M* and *N* be ⊥ to the line *p*.

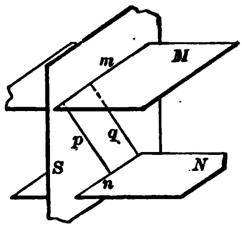
To prove *M* ∥ to *N*.

Analysis. If *M* and *N* could meet, we should have two planes perpendicular to the same line from the same point, which is impossible, § 28, hence *M* and *N* are parallel. Q. E. D

PROPOSITION VIII. THEOREM.

41. *The intersections of two parallel planes by a third plane are parallel lines.*

Notation. Let the ∥ planes M, N be cut by the plane S, giving the intersections m, n.



To prove m ∥ to n.

Analysis. m and n can not meet because the planes which contain them are ∥, and being in the same plane, S, they must be parallel. Q. E. D.

42. COR. *Parallel lines included between parallel planes are equal.*

Notation. Let p, q be two ∥ lines (whose plane is S.) included between the ∥ planes M, N.

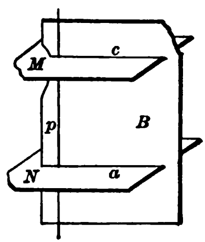
To prove $p = q$.

Analysis. p and q are ∥ lines included between the ∥ lines m, n, and hence are equal. Q. E. D.

PROPOSITION IX. THEOREM.

43. *A straight line perpendicular to one of two parallel planes is perpendicular to the other.*

Notation. Let M, N be ∥ planes and let p be ⊥ to M.



To prove p ⊥ to N.

Construction. Through p in the plane N draw the line a, and through ap pass the plane B, cutting M in c.

Analysis. a is ∥ to c, § 41 : p is ⊥

to c, § 6: $\therefore$ p is $\perp$ to a, and since a is any line in the plane N, p is $\perp$ to N. Q. E. D.

44. Cor. 1. *Two parallel planes are everywhere equally distant.*

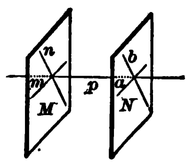
For perpendiculars between them must be equal by § 42.

45. Cor. 2. *Through a given point one plane can be passed parallel to a given plane, and only one.*

For if p is $\perp$ to the plane N, a plane M through any point of p $\perp$ to p, will be $\parallel$ to N, by § 40.

If another plane could be passed through the same point $\parallel$ to N, it would, by § 43, be $\perp$ to p, and we should have two planes through the same point $\perp$ to the same line which, by § 25, is impossible. Hence only one $\parallel$ plane can be passed through this point. Q. E. D.

46. Cor. 3. *If two intersecting lines are each parallel to a given plane, the plane of these lines is parallel to the given plane.*



Notation. Let m and n be the two lines (determining the plane M) $\parallel$ to the given plane N.

To prove $M \parallel$ to N.

Construction. Draw p through the intersection of m, n and $\perp$ to N. Where p pierces N, draw in N (§ 39) a and b $\parallel$ respectively to m and n.

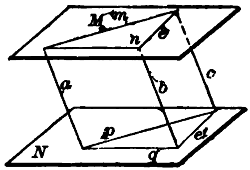
Analysis. p is $\perp$ to a and b, since it is $\perp$ to N, and is also $\perp$ to m and n, their parallels, consequently p is $\perp$ to M. Hence M and N are parallel, § 40. Q. E. D.

Find the locus of points

11. Equidistant from two given planes, and two given points.
12. Equidistant from two parallel planes.

Proposition X. Theorem.

47. *If two angles not in the same plane have their sides respectively parallel and lying in the same direction, they are equal and their planes are parallel.*



Notation. Let m, n (determining the plane M) and p, q (determining the plane N) be the sides of the two angles: $m \parallel$ to p and $n \parallel$ to q.

(1) **To prove** $\angle mn = \angle pq$.

Construction. Take $m = p$, $n = q$, and draw the lines a, b, c.

Analysis. The angles are = because the $\triangle mne = \triangle pqe'$.

Proof. Since m = and $\parallel$ to p, the figure a, c is a parallelogram and a = and $\parallel$ to c. Similarly, a = and $\parallel$ to b. Hence the figure b, c is a parallelogram and e = and $\parallel$ to e'. Hence, having the three sides =, each to each, $\triangle mne = \triangle pqe'$. Hence $\angle mn = \angle pq$. Q. E. D.

(2) **To prove** $M \parallel$ to N.

Proof. Since m, n are each $\parallel$ to N, § 34, their plane M is $\parallel$ to N, § 46. Q. E. D.

48. Cor. *If two angles have their sides parallel, they are equal or supplemental, and their planes are parallel.*

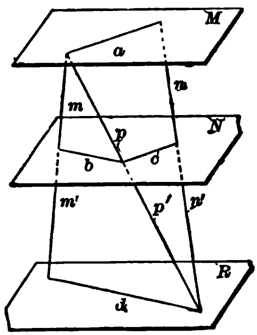
---

13. How would you test whether the side wall is a plane or not?

14. How would you test the perpendicularity of a rod projecting from a wall?

PROPOSITION XI. THEOREM.

49. *If two straight lines are cut by three parallel planes, the corresponding segments are proportional.*



Notation. Let m, n be cut by the three ∥ planes M, N, R, into the segments m, m' and n, n'.

To prove $\dfrac{m}{m'} = \dfrac{n}{n'}$.

Construction. Draw the line pp' which is cut into the segments, p, p'. Connect the points where these lines pierce the planes by the lines a, b, c, d.

Analysis. The segments m, m' and n, n' are proportional because they are the segments of the sides of triangles cut by a line ∥ to the base.

Proof. $b \parallel d$, by § 41, $\therefore \dfrac{m}{m'} = \dfrac{p}{p'}$.

Similarly $\dfrac{p}{p'} = \dfrac{n}{n'}$. Hence $\dfrac{m}{m'} = \dfrac{n}{n'}$.

Q. E. D.

50. COR. *Any number of straight lines cut by parallel planes are divided into proportional segments.*

---

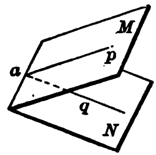
15. The heights of two rooms, one above the other, are 9 and 12 feet respectively, and the floor is one foot thick. A rod 32 feet long rests on the lower floor and projects into the room above, just reaching the ceiling. How much of the rod is in each room?

16. The guy ropes of a derrick are 75, 86, 93 and 97 feet respectively, and the derrick is 35 feet high. It is desired to extend horizontal spars from a point 10 feet from the top of the derrick to the guy ropes. At what points will they touch the guy ropes?

DIHEDRAL ANGLES.

51. The opening between two intersecting planes is called a **dihedral angle.**

The two planes are called the **faces,** and their line of intersection the **edge.**



52. A dihedral angle is designated by its edge, or its two faces and its edge, or its two faces. Thus the dihedral angle in the figure is designated by *a*, or *MaN*, or *MN*.

53. A dihedral angle is generated by the revolution of a plane about a line in the plane, or what is the same thing, about its intersection with another plane.

Consequently, the measure of a dihedral angle is the amount of this revolution from the position of coincidence.

If at a point in the intersection a line be drawn in each plane perpendicular to the intersection (p, q in the figure), these lines will coincide when the planes coincide, and will open or revolve about their intersection exactly the same amount that the planes revolve about their intersection, and evidently the angle between these lines will be the measure of the revolution of the planes.

This angle is called the **plane angle** of the dihedral angle, and will evidently be of the same magnitude at whatever point in the edge it is constructed, § 47.

54. *Two dihedral angles are equal if their plane angles are equal.*

55. The plane of the plane angle is evidently perpendicular to the edge, § 15.

56. If a plane is drawn perpendicular to the edge, its intersections with the faces form the plane angle of the dihedral angle.

57. Two dihedral angles are **equal** when they can be made to coincide, or when their plane angles are equal.

The **magnitude** of a dihedral angle depends upon the relative position of its faces, and not upon their extent.

58. Two dihedral angles are **adjacent** when they have a common edge and a common face between them, as in § 76.

59. When two intersecting planes make the **adjacent dihedral angles equal,** each of these dihedral angles is called a **right dihedral angle**: and one plane is said to be *perpendicular* to the other.

60. Vertical dihedral angles are those which have a common edge and the faces of one are prolongations of the faces of the other.

61. Dihedral angles are *acute, obtuse, complementary, supplementary,* under the same conditions that hold for plane angles.

62. The demonstrations of many properties of dihedral angles are the same as the demonstrations of analogous properties of plane angles.

For example:

Vertical dihedral angles are equal.

Dihedral angles whose faces are respectively parallel or perpendicular are either equal or supplementary.

Proposition XII. Theorem.

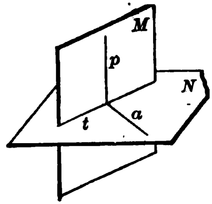
63. *The ratio of two dihedral angles is equal to the ratio of their plane angles.*

This follows from § 53, since the plane angles are the measures of the dihedral angles.

Proposition XIII. Theorem.

64. *If two planes are perpendicular to each other, a line in one of them perpendicular to the intersection is perpendicular to the other.*

Notation. Let M and N be $\perp$ to each other, and let p be drawn in M perpendicular to their intersection t.



To prove $p \perp$ to N.

Construction. In N draw $a \perp$ to t.

Analysis. 1°. The $\angle pa$ is a right angle, being the plane angle of the dihedral $\angle MN$. Hence

2°. p is $\perp$ to both t and a and $\therefore \perp$ to N.

Proof. 1°. The $\angle pa$ is the plane angle of the dihedral $\angle MN$ because p and a are both drawn $\perp$ to t at the same point, one in each plane, § 53.

2°. p is $\perp$ to t by construction, and $\perp$ to a because $\angle pa$ is the measure of the angle between the planes, which is by hypothesis a rt. $\angle$. Hence, § 15, p is $\perp$ to N. Q. E. D.

65. Cor. 1. *If two planes are perpendicular to each other, a perpendicular to one of them at any point of their intersection will lie in the other.*

By § 16, only one $\perp$ to the plane can be erected at the point, but p has already, § 64, been shown to be one $\perp$ to N, and p lies in M. Hence, if two planes, etc. Q. E. D.

66. Cor. 2. *If two planes are perpendicular to each other, a perpendicular to one of them from any point of the other will lie in the other.*

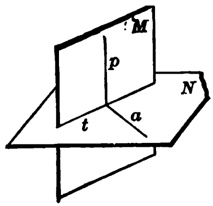
By § 17, only one $\perp$ can be drawn to a plane from a point without, but p has already, § 64, been shown to be one $\perp$ to N, and p lies in M. Hence, If two planes, etc.

Q. E. D.

PROPOSITION XIV. THOREM.

67. *If a straight line is perpendicular to a plane, every plane passed through the line is perpendicular to the first plane.*

Notation. Let p be $\perp$ to N, and M be any plane passed through p, intersecting N in t.



To prove $M \perp$ to N.

Construction. Through the foot of p draw a in the plane N, $\perp$ to t.

Analysis. Since p is $\perp$ to N, it is $\perp$ to t and to a, § 6, $\therefore \angle pa$ is the plane angle of the dihedral $\angle MN$. But $\angle pa$ is a rt. $\angle$ $\therefore M$ is $\perp$ to N. Q. E. D.

68. COR. *A plane perpendicular to the edge of a dihedral angle is perpendicular to its faces.*

For the faces are planes through the perpendicular line (edge).

69. *If three lines p, t, a are perpendicular to one another at the same point, each line is perpendicular to the plane of the other two, and the three planes determined by the lines are perpendicular to each other.*

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17. If a plane intersects two ∥ planes, are there any equal dihedral angles?

18. How would you determine if the floor is level, if you had a plumb line and a carpenter's square? If you had a carpenter's level?

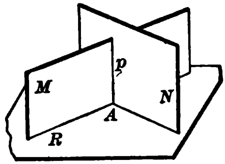
19. In example 18, what must be the relative positions of the square? Of the level?

20. If a line and a plane are $\perp$ to the same plane, they are ∥.

PROPOSITION XV. THEOREM.

70. *If two intersecting planes are each perpendicular to a third plane, their intersection is also perpendicular to that plane.*

Notation. Let the planes M and N intersecting in the line p be $\perp$ to the plane R.



To prove $p \perp$ to R.

Analysis. If a perpendicular to R be erected at the point A where the traces of M and N intersect, it will, by § 65, lie in both M and N, and hence must be the intersection p, hence p is $\perp$ to R. **Q. E. D.**

71. COR. 1. *A plane perpendicular to each of two intersecting planes is perpendicular to their intersection.*

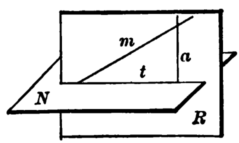
72. COR. 2. *If a plane R be perpendicular to two planes M and N which include a right dihedral angle, the intersection of any two of these planes is perpendicular to the third plane, and each of the three intersections is perpendicular to the other two.*

Compare § 69.

PROPOSITION XVI. THEOREM.

73. *Through a given line oblique to a plane, one, and only one plane, can be passed perpendicular to the given plane.*

Notation. Let m be oblique to the plane N.



To prove that one plane can be passed through $m \perp$ to N and only one.

Construction. From any point of m draw $a \perp$ to N, and through am pass the plane R.

Analysis. R is $\perp$ to N by § 67, since it passes through the perpendicular a. Q. E. D.

Also because any plane passed through $m \perp$ to N must contain the perpendicular a, § 66, $\therefore$ since a plane is determined by two intersecting straight lines, § 5, R is the only plane $\perp$ to N that can be passed through m. Q. E. D.

74. Cor. 1. *The projection of a straight line on a plane is a straight line.*

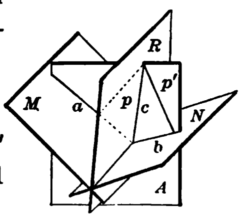
For the $\perp$s from all points of m to N lie in the plane R $\perp$ to N, § 66, and therefore all these perpendiculars meet N in the trace t, which is a straight line, § 14.

75. Cor. 2. *The projection of a line passes through the point where the line intersects the plane.*

Proposition XVII. Theorem.

76. *Every point in a plane which bisects a dihedral angle is equidistant from the faces of the angle.*

Notation. Let the plane R bisect the dihedral angle MN; and let p, p' be $\perp$s drawn from any point of R to M and N respectively.



To prove $p = p'$.

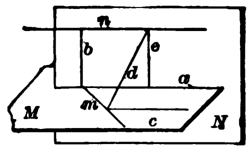
Construction. Through p and p' pass a plane A, intersecting M and N in the lines a and b, and R in c.

Analysis. 1°. The $\triangle pac = \triangle p'bc$, $\therefore$ 2° $p = p'$.

Proof. A is $\perp$ to M and N by § 67, and therefore is $\perp$ to their intersection by § 71. Hence, by § 56, $\angle ac = \angle bc$, since the dihedral angles are equal. Hence the rt. $\triangle$s pac and $p'bc$ are equal, having a hypothenuse and an $\angle =$ each to each. $\therefore p = p'$. Q. E. D.

Proposition XVIII. Theorem.

77. *Between two straight lines not in the same plane only one perpendicular can be drawn.*



Notation. Let m and n be the given lines.

(1) **To prove** $b \perp$ to m and to n.

Construction. Through m pass the plane $M \parallel$ to n, and through n pass the plane $N \perp$ to M, intersecting M in a. At the point am erect b in the plane $N \perp$ to a.

Analysis. 1°. b is $\perp$ to M and therefore to m.

2°. b is $\perp$ to a and therefore to n.

Proof. 1°. b is $\perp$ to M by § 64, and is therefore perpendicular to m, a line through its foot.

2°. a is $\parallel$ to n by § 38, and $\therefore b$, which is $\perp$ to a by construction, is also $\perp$ to its $\parallel$ n.

Hence b is $\perp$ to both m and n. Q. E. D.

(2) **To prove** that d, *any* other line, is not $\perp$ to both m and n.

Construction. Through the point md, draw $c \parallel$ to n (m and c determine the plane M); and through the point nd draw $e \perp$ to a.

Analysis. e is $\perp$ to M. If d is $\perp$ to m and n, it is also $\perp$ to c and therefore to M, thus giving two $\perp$s to M from the same point, which is impossible.

Proof. e is $\perp$ to M by § 64.

If d is $\perp$ to n it is $\perp$ to its $\parallel$ c. Being perpendicular to m and c at their intersection it is $\perp$ to M, § 15.

Hence we have two $\perp$s to M from the same point, which is impossible, $\therefore$ d is not $\perp$ to m and n. Q. E. D

78. Cor. *The common perpendicular is the shortest distance between two lines.*

For $b = e < d$. § 21.

Polyhedral Angles.

79. When three or more planes meet in a common point, they form what is called a **polyhedral angle** at that point.

80. The common point is called the **vertex** of the angle, the intersections of the planes are the **edges**, and the portions of the planes between the edges are the **faces**, and the plane angles formed by the edges are its **face angles.**

81. The edges of a polyhedral angle may be produced indefinitely, but are usually represented as cut off by a plane as in §§ 91, 92. The intersections of the faces with this plane form a polygon which is called the **base** of the polyhedral angle.

82. The **magnitude** of a polyhedral angle depends upon the *relative position* of its faces and not upon their extent.

83. In a polyhedral angle each pair of adjacent faces forms a dihedral angle, and each pair of adjacent edges forms a face angle.

84. Two polyhedral angles are **equal** when the face and dihedral angles of one are respectively equal to the face and dihedral angles of the other and are arranged in the *same order.* The two polyhedral angles can evidently be made to coincide.

85. A polyhedral angle is convex if its base is convex.

86. A polyhedral angle is called **trihedral, tetrahedral,** etc., according as it has *three* faces, *four* faces, etc.

87. A trihedral angle is called **isosceles** if it has two of its face angles equal: **equilateral** if it has all three of its face angles equal.

88. A trihedral angle is called **rectangular, bi-rectangular, tri-rectangular,** according as it has one, two or three right dihedral angles.

The corner of a cube is a tri-rectangular trihedral angle.

89. Two polyhedral angles are **symmetrical** when the face and dihedral angles of one are equal to those of the other, each to each, but *arranged in reverse order,* as shown in figures S and S_1 in § 93.

In general, two symmetrical polyhedral angles cannot be brought into coincidence.

The two hands are an illustration of symmetry. The right hand is symmetrical to the left hand, but cannot be made to coincide with it. So with the right and left shoe.

90. Opposite or **vertical** polyhedral angles are those in which the edges of one are prolongations of the edges of the other.

They are evidently symmetrical.

---

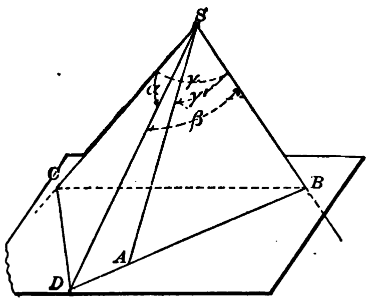
21. If two dihedrals of a trihedral are equal, the trihedral is isosceles.

22. If a line is drawn in each face of a dihedral from the same point in the edge, and oblique to the edge, is the angle between the lines greater or less than the plane angle? If the dihedral is a right angle how large or how small can it be?

Proposition XIX. Theorem.

91. *The sum of two face angles of a trihedral angle is greater than the third.*

Notation. Let α, β and γ be the three face angles of a trihedral angle, of which β is the greatest.



To prove $\alpha + \gamma > \beta$.

Construction. In the face β, draw the line SA making the angle $\gamma' = \gamma$ with the edge SB. Lay off $SC = SA$. Draw the line BA to D.

Analysis. $\alpha > (\beta - \gamma_1)$. $\therefore \alpha + \gamma > (\beta - \gamma_1 + \gamma = \beta)$.

Proof. The $\triangle SAB = \triangle SBC$ (having two sides and the included angle equal, etc.). $\therefore AB = BC$.

But $BC + CD > AB + AD$. Taking away the equals AB and BC we have $CD > AD$.

But $SC = SA$, $\therefore \angle ASD = \beta - \gamma_1 < \alpha$.

Adding γ to both sides of this inequality we have $\alpha + \gamma > (\beta - \gamma_1 + \gamma = \beta)$. Q. E. D.

---

23. Can three planes intersect without forming a trihedral angle?

24. If two intersecting planes contain two lines ∥ to each other, the intersection of the planes will be ∥ to the lines.

25. In how many points do three planes intersect?

26. How many planes can be determined by three parallel lines?

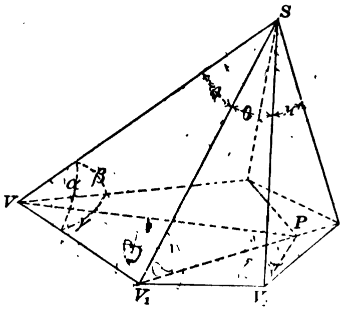
27. How many planes can be determined by four parallel lines?

28. How many planes can be determined by the sides of a *gauche* quadrilateral?

PROPOSITION XX. THEOREM.

92. *The sum of the face angles of any convex polyhedral angle is less than four right angles.*

Notation. Let S be any polyhedral angle formed by the face angles η, θ, κ, etc.



To prove
$\eta + \theta + \kappa + \ldots$ etc. < 4 rt. $\angle$s.

Construction. Pass a plane across the polyhedral angle giving a base, one point in the interior of which is P. From P draw lines to the vertices of the base, thus giving a set of triangles with their common vertex at P, and equal in number to the set of face triangles having their common vertex at S.

Notation. At any vertex V, V_1, etc., denote the angles of the face triangles by α, α_1, etc., β, β_1, etc., and of the base triangles by γ, γ_1, etc. Denote the sum of all the angles of the base triangles (having their common vertex at P) by ΣP, and the sum of all the angles of the face triangles (having their common vertex at S) by ΣS.

NOTE. — Σ stands for sum.

Analysis. $\Sigma S = \Sigma P$, since each set has the same number of triangles.

By § 91, $\alpha + \beta > \gamma$, $\alpha_1 + \beta_1 > \gamma_1$, etc.

$$\therefore (\alpha + \alpha_1 + \ldots + \beta + \beta_1 + \ldots) > (\gamma + \gamma_1 + \ldots).$$

Hence

$$\Sigma S - (\alpha + \alpha_1 + \ldots + \beta + \beta_1 + \ldots) < \Sigma P - (\gamma + \gamma_1 + \ldots).$$

But $\Sigma S - (\alpha + \alpha_1 + \ldots \beta + \beta_1 + \ldots) = \eta + \theta +$ etc.

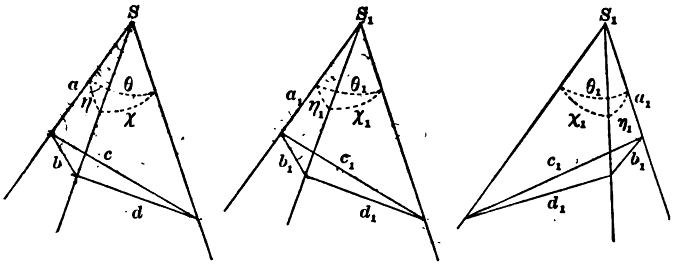
and $\Sigma P - (\gamma + \gamma_1 + ...) = 4$ rt. $\angle$s.

$\therefore \eta + \theta + \kappa +$ etc. < 4 rt. $\angle$s. Q. E. D.

The limits of the sum of the face angles of a convex polyhedral angle are 0° and 360°.

Proposition XXI. Theorem.

93. ***Two trihedral angles, which have three face angles of the one equal respectively to three face angles of the other, are either equal or symmetrical.***



Notation. In the trihedral $\angle$s S, S_1 the face angles η, θ, χ, and η_1, θ_1, χ_1, are respectively equal as shown in the figure.

To prove the dihedral angles between these faces respectively equal.

Construction. We will measure the dihedral $\angle$s $\eta\theta$ and $\eta_1\theta_1$, between the faces $\eta\theta$ and $\eta_1\theta_1$. On the edge of this dihedral $\angle$ lay off $a = a_1$. At this point erect the $\perp$s b, c, b_1, c_1, in the faces η and θ respectively. Connect the intersections of these lines with the other edges by the lines d, d_1.

Analysis. The $\triangle\ bcd = \triangle\ b_1c_1d_1$ $\therefore$ the $\angle\ bc = b_1c_1$, but this is the plane angle of the dihedral angle $\eta\theta$, and therefore the dihedral angles are equal.

Proof. $\triangle\ ab = \triangle\ a_1b_1$ (having two $\angle$s and a side $=$), $\therefore\ b = b_1$.

Similarly $c = c_1$, and $d = d_1$.

Hence $\triangle\ bcd = \triangle\ b_1c_1d_1$.

$\therefore \angle\ bc = \angle\ b_1c_1$, or dihedral $\angle\ \eta\theta =$ dihedral $\angle\ \eta_1\theta_1$.

Q. E. D.

In the same way it can be proved that the other dihedral $\angle$s are $=$.

Hence, §§ 84, 89, the trihedral $\angle$s S and S_1 are either equal or symmetrical. Q. E. D.

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29. Perpendiculars to the faces of a dihedral angle include an angle supplemental to the plane angle of the dihedral.

30. The sum of the dihedrals of a trihedral may be anything between two and six right angles.

31. An isosceles trihedral and its symmetrical trihedral are equal.

32. The dihedrals opposite the equal face angles of an isosceles trihedral are equal.

33. If the intersections of several planes are ‖, what about the normals to those planes through a given point.

34. The projections of ‖ lines upon any plane are ‖ lines.

35. The projection of an $\angle$ upon a plane is greatest, when? When least? When $=$ to the $\angle$?

36. Find the plane upon which the projections of two non ‖ and non-intersecting lines will be ‖.

37. Through the vertex of a trihedral angle draw a line equally inclined to the edges.

38. In any trihedral angle the planes through the edges, $\perp$ to the opposite faces, intersect in the same straight line. (See § 70).

39. If the projections of a line upon two non ‖ planes are straight lines, the given line is also a straight line.

CHAPTER II.

POLYHEDRONS, CYLINDERS, CONES.

Definitions.

94. A **polyhedron** is a solid bounded by planes. The portions of the bounding planes, limited by their mutual intersections, are the **faces** of the polyhedron; the intersections of the faces are the **edges**, which meet in points called the **vertices**.

95. A **diagonal** of a polyhedron is a straight line joining any two vertices not in the same plane.

96. A polyhedron of four faces is called a **tetrahedron**; one of six faces, a **hexahedron**; one of eight faces, an **octahedron**; one of twelve faces, a **dodecahedron**; one of twenty faces, an **icosahedron.**

97. A polyhedron is **convex** when every section formed by an intersecting plane is a convex polygon. Only convex polyhedrons will be treated of in this book.

98. The **volume** of a solid is the number which expresses its ratio to some other solid taken as a unit of volume. The **unit of volume** is a cube whose edge is a linear unit.

99. Two solids are **equivalent** when their volumes are equal.

100. Polyhedrons are **equal** when their homologous edges coincide.

101. A **pyramid** is a polyhedron all of whose faces except one are triangles meeting in a point called the **vertex.** The face not passing through the vertex is called the **base.** The faces which meet at the vertex are called **lateral faces** and their intersections are called **lateral edges.** The edges of the base are called **base edges.**

102. A **frustum** of a pyramid is the portion contained between the base and a plane parallel to it. Parallel planes evidently cut off frustums between them, one of the sections being considered as the base of a new pyramid.

103. The frustum is the general solid, and special cases of the frustum are, as defined below, the pyramid, prism, cone and cylinder.

104. A pyramid may be considered as a frustum in which the cutting plane parallel to the base passes through the vertex: a frustum whose upper base has become zero.

105. Another special case of a frustum is that in which the vertex is moved out to infinity, in which case the lateral edges become parallel lines, and we have what is called a **prism.**

106. Another special case of a frustum is that in which the number of lateral faces becomes infinite, in which case the pyramid becomes what is called a **cone,** and the prism a **cylinder,** and the frustum a frustum of a cone, and the lateral edges become **elements** of the cone and cylinder.

107. The following definitions apply to frustums, and, therefore, to the special cases of frustums: pyramids, prisms, cones and cylinders.

108. The **altitude** of a frustum (pyramid, prism, cone, cylinder) is the perpendicular distance between its bases.

109. A **regular** frustum (pyramid, prism, cone, cylinder) is one whose base is a regular polygon (circle for cone and cylinder) and whose edges (or elements) are equally inclined to the base. The cone is called a **cone of revolution,** because it can be considered as generated by revolving a right angled triangle about one of its legs as an axis. **Similar cones of revolution** are generated by similar triangles.

110. In the case of a regular pyramid (cone) the centre of the polygon (circle) coincides with the foot of the perpendicular let fall from the vertex. The perpendicular is called the **axis** of the pyramid (cone).

111. In the case of a **regular prism** (*i. e.*, frustum of an infinitely long pyramid) the edges as well as the axis are perpendicular to the base. The corresponding cylinder is called a **cylinder of revolution,** being considered as generated by a rectangle revolving about one of its sides as an axis. **Similar cylinders of revolution** are generated by similar rectangles.

If the base is distorted so as to be no longer regular, we still have what is called a **right** (though not a regular) **prism** or cylinder. Compare § 120.

112. The **slant height** of a regular frustum is the perpendicular distance between the corresponding base edges.

113. In the frustum of a cone, the slant height becomes the portion of an element included between the bases.

114. In the case of the prism and cylinder, the slant height becomes the altitude.

115. In the case of the pyramid the slant height becomes the distance from the vertex to one of the base edges, and in the cone it becomes one of the elements.

Special Definitions.

116. A frustum of a pyramid (pyramid, prism) is called **triangular, quadrangular, pentagonal,** etc., according as its base is a triangle, quadrangle, pentagon, etc.

117. A triangular pyramid is called a **tetrahedron**; and any one of its faces can be taken as its base.

118. A **circular** cone (cylinder) is one whose base is a circle. The line connecting the centre of the base with the vertex is called the *axis.*

119. A **right circular** cone (cylinder) is one whose axis is perpendicular to the base.

120. A **right** prism (cylinder) is one whose lateral edges (elements) are perpendicular to its bases.

121. An **oblique** prism (cylinder) is one whose lateral edges (elements) are not perpendicular to its bases.

122. A **right section** of a prism (cylinder) is a section made by a plane perpendicular to its lateral edges (elements).

123. A **truncated pyramid (prism)** is the part of a pyramid (prism) included between the base and a plane not parallel to the base.

124. A prism whose bases are parallelograms is called a **parallelopiped**: a **right parallelopiped** if it is a right prism, and a **rectangular parallelopiped** *or cuboid** if it is a right rectangular prism.

A parallelopiped is evidently a solid contained between three pairs of parallel planes.

125. A rectangular parallelopiped whose edges are equal is called a **cube.**

*Suggested by R. Baldwin Hayward, *Elements of Solid Geometry.*

126. A **tangent line** to a cone (cylinder) is a straight line (not an element) which touches but does not intersect the lateral surface.

127. A plane which contains an element of the cone (cylinder) and does not cut the surface is called a **tangent plane.**

128. The element contained by the tangent plane is called the **element of contact.**

129. A frustum of a pyramid (prism) is **inscribed** in a frustum of a cone (cylinder) when its lateral edges are elements of the frustum of the cone (cylinder), and its bases are inscribed in the bases of the frustum of the cone (cylinder).

130. If the faces are tangent externally and the bases are circumscribed instead of inscribed, the pyramid (prism) is said to be circumscribed about the cone (cylinder).

Secondary Definitions.

x **131.** A **conical surface** is the surface generated by a moving straight line called the **generatrix** passing through a fixed point called the **vertex** and constantly touching a fixed curve called the **directrix.**

The directrix may be an open or a closed curve.

The generatrix in any position is called an element of the surface.

Since the generatrix extends on both sides of the vertex, the whole surface consists of two portions lying on opposite sides of the vertex, which are called the upper and lower nappes or sheets.

132. If the vertex is carried out to infinity we get a **cylindrical** surface.

133. A **cone (cylinder)** is a solid bounded by a conical (cylindrical) surface which returns into itself, and a plane cutting all its elements. The conical (cylindrical) surface is called the **lateral surface,** and its area, the **lateral area.**

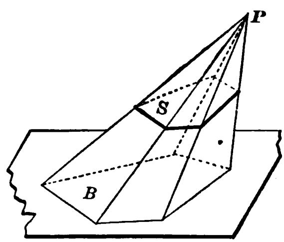
134. The **base** of a cone (cylinder) is its plane surface.

Proposition I. Theorem.

135. *If a pyramid is cut by a plane parallel to the base, the edges are divided proportionally, and the section is a polygon similar to the base.*

Notation. Let the pyramid P be cut by a plane ∥ to the base B, giving the section S.

(1) **To prove** the segments of the edges proportional.



Analysis. S and B have their edges respectively ∥. ∴ the faces are each divided into similar △s, and the edges are divided proportionally.

Proof. The edges of S and B are respectively ∥, because they are sections of ∥ planes by a third, § 41.

∴ the faces are similar △s (having their sides ∥).

∴ the edges are divided proportionally (sides of similar △s are proportional). Q. E. D.

(2) **To prove** S similar to B.

Analysis. They are ∥ polygons* (proved above) and similar, because the edges are proportional, being the bases of similar △s. Q. E. D.

* Parallel polygons are those having their sides respectively parallel. They are not necessarily similar.

136. Cor. 1. *Sections of a pyramid (cone) made by parallel planes are similar polygons.*

137. Cor. 2. *Sections of a circular cone by a plane parallel to the base are circles.*

138. Cor. 3. *The upper base of a frustum of a pyramid (cone) is similar to the lower base.*

139. Cor. 4. *Sections of a prism (cylinder) made by parallel planes are congruent polygons,* since the edges are mutually equal, being parallel lines between parallel lines.

140. Cor. 5. *Sections of a prism (cylinder) parallel to the base are congruent to the base.*

141. Cor. 6. *All right sections of a prism (cylinder) are equal.*

142. Cor. 7. *The upper and lower bases of a prism (cylinder) are congruent polygons.*

143. This furnishes ground for a new definition of a prism, viz., a polyhedron two of whose opposite faces, called *bases*, are parallel polygons, and whose other faces, called *lateral faces*, meet in parallel lines called *lateral edges.*

From this it easily follows that the lateral edges are equal, § 42; the lateral faces are parallelograms; and the bases are equal.

144. Cor. 8. *The areas of parallel sections of a pyramid (cone) are proportional to the squares of their distances from the vertex.*

This follows because similar polygons are proportional to the squares of homologous lines.

145. If the pyramid (cone) become a prism (cylinder) then the areas are equal. Compare Cor. 4.

146. Cor. 9. *If pyramids (cones) of equal altitudes are cut by planes parallel to the base and at equal distances from the vertex, the sections will be proportional to the bases.*

147. Cor. 10. *In pyramids (cones) of equal altitudes and equivalent bases, sections made by planes parallel to the bases and at equal distances from the vertices are equivalent.*

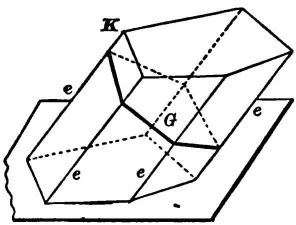
148. Cor. 11. *The lateral faces of a regular frustum are isosceles trapezoids.*

PROPOSITION II. THEOREM.

149. ***The lateral area of a prism is equal to the product of a lateral edge by the perimeter of a right section.***

Notation. Let K be a prism with the right section G (whose perimeter is p), lateral edge e, and lateral area S.

To prove $S = e \cdot p$.



Analysis. Each face is a parallelogram whose area is equal to its base e, multiplied by its altitude. The altitude is a side of the right section. The sum of all the altitudes = the perimeter of the right section $= p$. ∴ the sum of all the faces = the lateral area $= S = e \cdot p$. Q. E. D.

150. Cor. 1. *The lateral area of a cylinder is equal to the product of an element by the perimeter of a right section.*

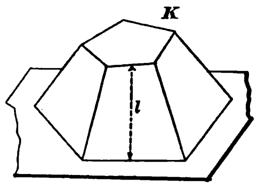
151. Cor. 2. *The lateral area of a right prism (cylinder) is equal to the altitude multiplied by the perimeter of the base.*

152. Cor. 3. *The lateral area of a cylinder of revolution is the product of the circumference of its base by its altitude.*

Proposition III. Theorem.

× **153.** *The lateral area of a regular frustum is equal to one half the product of the slant height by the sum of the perimeters of the bases.*

Notation. Let K be a frustum of a regular pyramid; its lateral surface S; slant height l; perimeter of bases p and p_1.



To prove $S = \frac{1}{2}\, l\, (p + p_1)$.

Analysis. The lateral faces are isosceles trapezoids, §§ 110, 18, 50, with the common altitude l. ∴ the sum of the faces $= S = \frac{1}{2}\, l \times$ sum of the perimeters of the bases $= \frac{1}{2}\, l\, (p + p_1)$. Q. E. D.

154. Cor. 1. *The lateral area of a frustum of a cone of revolution is equal to the product of the slant height by one half the sum of the circumferences of the bases.*

155. Cor. 2. *The lateral area of a frustum of a cone of revolution is equal to the slant height by the perimeter of a mid-section.*

156. Cor. 3. *The lateral area of a regular pyramid (cone of revolution) is equal to one half the perimeter of base multiplied by the slant height.*

157. Cor. 4. *The lateral area of a right prism (cylinder) is equal to the altitude multiplied by the perimeter of the base.*

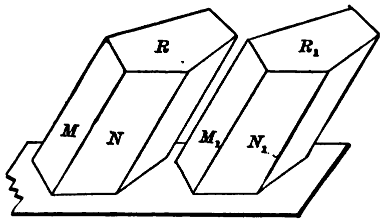
---

40. Sections of a prism made by planes parallel to the lateral edges are parallelograms.

Proposition IV. Theorem.

158. *Two prisms are equal when the three faces about a trihedral of one are equal respectively to the three faces about a trihedral of the other, and similarly arranged.*

Notation. Let the two prisms have the three faces M, N and R equal respectively to M_1, N_1 and R_1.



To prove the prisms equal.

Analysis. Since the face angles are equal, the trihedrals are equal, and can be made to coincide. Then the vertices of R coincide with the vertices of R_1; and likewise the vertices of the other = faces, respectively.

Since the vertices of R and R_1 coincide, the lateral edges extending from these vertices must coincide, since they are ∥ to the edge MN, and are limited by the plane determined by the three lower vertices of M, N.

But if the lateral edges of one prism coincide respectively with the lateral edges of the other, the prisms must be equal, § 100. Q. E. D.

159. Cor. 1. *Two right prisms which have equal bases and equal altitudes are equal.*

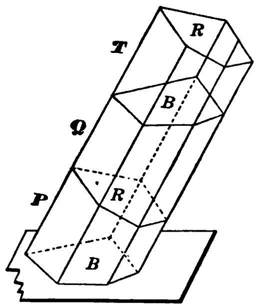
If the faces are not similarly placed, one of the prisms can be inverted and applied to the other.

160. Cor. 2. *The above proposition and demonstration applies equally well to truncated prisms.*

PROPOSITION V. THEOREM.

161. *An oblique prism (cylinder) is equivalent to a right prism (cylinder) whose base is a right section of the oblique prism (cylinder) and whose altitude is a lateral edge (element) of the oblique prism (cylinder).*

Notation. Let R be a right section of the oblique prism, and the base of a right prism whose lateral edges equal the edges of the oblique prism; $B =$ the base of the oblique prism. The combined figure is composed of three truncated prisms. $\mathbf{P} + \mathbf{Q}$ = the oblique prism. $\mathbf{Q} + \mathbf{T} =$ the right prism.



To prove $\mathbf{P} + \mathbf{Q} = \mathbf{Q} + \mathbf{T}$ or $\mathbf{P} = \mathbf{T}$.

Analysis. **P** and **T** have the same base B. If from the lateral edges of the prisms $\mathbf{P} + \mathbf{Q}$ and $\mathbf{Q} + \mathbf{T}$ we take the edges of **Q**, which are common to both, the remainders will be equal; that is, the edges of $\mathbf{P} =$ the edges of **T**. $\therefore$ the faces of $\mathbf{P} =$ the faces of **T**, being ∥ polygons with their homologous edges =.

Hence $\mathbf{P} = \mathbf{T}$, § 160.

$\therefore \mathbf{P} + \mathbf{Q} = \mathbf{Q} + \mathbf{T}$. Q. E. D.

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41. If the four diagonals of a quadrangular prism pass through a common point, the prism is a parallelopiped.

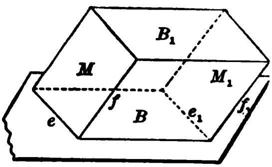
42. If two non-parallel diagonal planes of a prism are perpendicular to the base, the prism is a right prism. § 70.

43. In a cube the square of a diagonal is three times the square of an edge.

PROPOSITION VI. THEOREM.

162. *The opposite faces of a parallelopiped are equal and parallel.*

Notation. Let M, M_1 be two opposite faces of a parallelopiped, their edges being e, f and e_1, f_1 respectively. B and B_1 are the upper and lower bases of the parallelopiped.



To prove $M =$ and $\parallel$ to M_1.

Analysis. Since B is a parallelogram $e =$ and $\parallel$ to e_1. Similarly (or by § 143) $f =$ and $\parallel$ to f_1. $\therefore \angle ef = \angle e_1f_1$, § 47. Similarly with the other angles. Hence $M =$ and $\parallel$ to M_1. Q. E. D.

In the same way the other faces can be proved $\parallel$ and $=$.

Second Method. $f \parallel f_1$ by definition, § 143. $e \parallel e_1$ by definition of a parallelogram. Hence M and M_1 are $\parallel$ polygons. But the base edges not adjacent to M, M_1 are $\parallel$, § 143. $\therefore$ the definition of § 143 applies with M and M_1 as bases, and they must be $=$ and $\parallel$. Q. E. D.

163. **Scholium.** Any two opposite faces of a parallelopiped may be taken as bases.

---

44. The four diagonals of a parallelopiped mutually bisect each other.

45. The sum of the squares upon the four diagonals of a parallelopiped is equal to the sum of the squares upon its twelve edges.

46. The four diagonals of a rectangular parallelopiped are equal to each other, and the square is equal to the sum of the squares of the three edges which meet at any vertex.

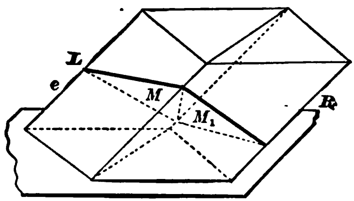
(The Pythagorean proposition is a special case of this one.)

47. Find the length of the diagonal of a rectangular parallelopiped whose edges are 1, 4 and 8.

PROPOSITION VII. THEOREM.

164. *The plane passed through two diagonally opposite edges of a parallelopiped divides it into two equivalent triangular prisms.*

Notation. Let the parallelopiped whose edge is e be cut by the plane into two triangular prisms R and L.



To prove $R \triangleq L$.

Construction. Take a right section of the parallelopiped giving a quadrilateral which is divided by the diagonal plane into two triangles M, M_1.

Analysis. $M = M_1$. $\therefore L = eM^* = eM_1 = R$.

Proof. The quadrilateral MM_1 is a parallelogram, its sides being ‖ since they are intersections of ‖ planes by a third, § 41.

$$\therefore \text{the } \triangle M = \triangle M_1.$$

By § 161, $L = eM$. Likewise $R = eM_1$.

Hence since $M = M_1$, $L = eM = eM_1 = R$. Q. E. D.

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48. The lateral surface of a pyramid is greater than the base.

49. The lines joining each vertex of a tetrahedron with the intersection of the medial lines of the opposite face, meet in a point, which divides each line in the ratio 1 : 4.

50. How often must a cylinder $5\frac{1}{4}$ feet long, whose diameter is 21 inches, revolve, to roll an acre?

51. A pyramid whose height is 12, and whose base is 8, is cut by a plane parallel to the base giving a section of 5. What is the distance of this section from the vertex?

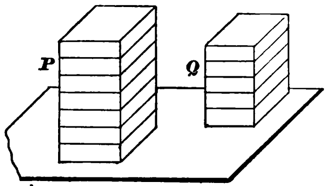
* eM, eM_1, etc., means the prism whose edge is e and whose base is M, M_1, etc.

Proposition VIII. Theorem.

165. *Two rectangular parallelopipeds having equal bases are to each other as their altitudes.*

Case I. *When the altitudes are commensurable.*

Notation. Let **P** and **Q** be two rectangular parallelopipeds having their altitudes m and n units respectively.



To prove $\frac{\mathbf{P}}{\mathbf{Q}} = \frac{m}{n}.$

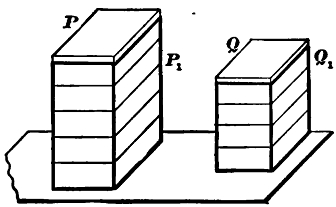
Construction. Divide **P** into m and **Q** into n rectangular parallelopipeds by planes ∥ to the base.

Analysis. These small prisms are all equal, § 159.
Since **P** contains m of them, and **Q** n of them,

then $$\frac{\mathbf{P}}{\mathbf{Q}} = \frac{m}{n}.$$ Q. E. D.

Case II. *When the altitudes are incommensurable.*

Notation. Let **P** and **Q** be two rectangular parallelopipeds having their altitudes as m and n respectively, where m or n, or both, are incommensurable* numbers. Let m_1 and n_1 be the units in m and n respectively. Let $\mathbf{P}_1$, $\mathbf{Q}_1$ be parallelopipeds having the same bases as **P** and **Q**, but altitudes of m_1 and n_1 units.



To prove $$\frac{\mathbf{P}}{\mathbf{Q}} = \frac{m}{n}.$$

* An incommensurable number is one whose value cannot be exactly expressed in figures, e.g., $\pi = 3.141592.....$, $\sqrt{2} = 1.4142136....$, etc.

Construction. Divide $\mathbf{P}_1$ and $\mathbf{Q}_1$ into m_1 and n_1 rectangular parallelopipeds respectively, by planes ∥ to the base.

Analysis. By Case I, $\frac{\mathbf{P}_1}{\mathbf{Q}_1} = \frac{m_1}{n_1}$.

If the unit of measure is indefinitely diminished, these ratios continue =, and approach the limiting ratios

$$\frac{\mathbf{P}}{\mathbf{Q}} \text{ and } \frac{m}{n}.$$

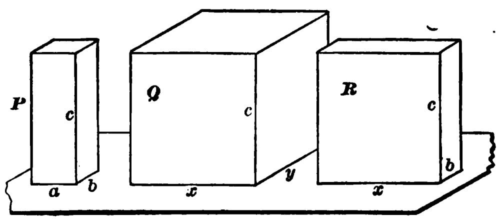
But if two variables are constantly equal, and each approaches a limit, the limits are equal.

Hence $$\frac{\mathbf{P}}{\mathbf{Q}} = \frac{m}{n}.$$ Q. E. D.

166. Scholium. This theorem may be expressed: *Two rectangular parallelopipeds which have two dimensions in common are to each other as their third dimension.*

Proposition IX. Theorem.

167. ***Two rectangular parallelopipeds having equal altitudes are to each other as their bases.***



Notation. Let **P** and **Q** be two rectangular parallelopipeds having the three dimensions a, b, c and x, y, c, respectively.

To prove $$\frac{P}{Q}=\frac{ab}{xy}.$$

Construction. Construct a third parallelopiped **R** with the dimensions c, x, b. (c, y, a would have done as well.)

Analysis. By § 166, $\frac{P}{R}=\frac{a}{x}$ and $\frac{R}{Q}=\frac{b}{y}$.

Multiplying these equations, we have

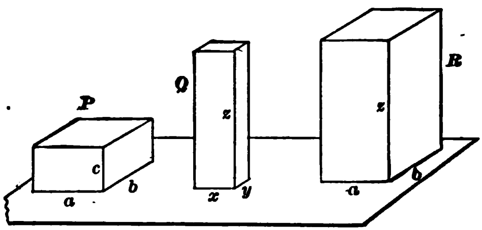
$$\frac{P}{Q}=\frac{ab}{xy}.$$

Q. E. D.

168. Scholium. *Two rectangular parallelopipeds having one dimension in common are to each other as the products of the other two dimensions.*

PROPOSITION X. THEOREM.

169. *Two rectangular parallelopipeds are to each other as the products of their three dimensions.*



Notation. Let **P** and **Q** be two rectangular parallelopipeds whose dimensions are a, b, c and x, y, z, respectively.

To prove $$\frac{P}{Q}=\frac{abc}{xyz}.$$

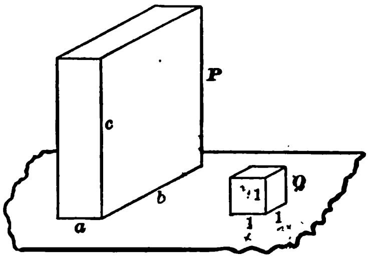
Construction. Make a third parallelopiped **R**, whose dimensions shall be a, b, z. (a, c, y or x, y, c or b, c, x, etc., would have done as well.)

Analysis. By §§ 166, 168, $\frac{\mathbf{P}}{\mathbf{R}} = \frac{c}{z}$ and $\frac{\mathbf{R}}{\mathbf{Q}} = \frac{ab}{xy}$.

Eliminating **R**, we get $\frac{\mathbf{P}}{\mathbf{Q}} = \frac{abc}{xyz}$. Q. E. D

PROPOSITION XI. THEOREM.

170. *The volume of a rectangular parallelopiped is equal to the product of its three dimensions.*



Notation. Let **P** be a rectangular parallelopiped whose dimensions are a, b, c, and **Q** the unit of volume.

To prove $\mathbf{P} = abc$.

Proof. By § 169, $\frac{\mathbf{P}}{\mathbf{Q}} = \frac{\mathbf{P}}{1} = \frac{abc}{1}$, $\therefore \mathbf{P} = abc$. Q. E. D.

171. COR. 1. *The volume of a rectangular parallelopiped is equal to the product of its base by its altitude.*

172. COR. 2. *The volume of a cube is the cube of its edge.*

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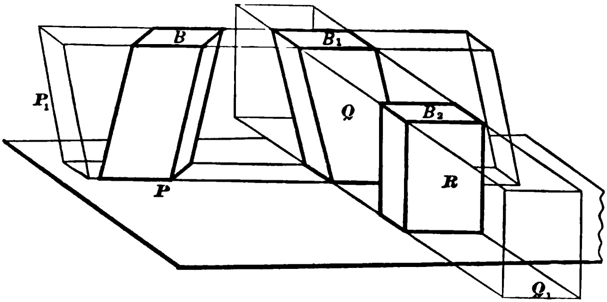
52. A pyramid 22 feet high has a base containing 324 square feet. How far from the vertex must a plane be passed parallel to the base so that the section may contain 81 square feet?

53. The base of a pyramid contains 196 square feet: a plane parallel to the base and 6 feet from the vertex cuts a section containing 121 square feet: find the height of the pyramid.

PROPOSITION XII. THEOREM.

173. *The volume of any parallelopiped is equal to the product of its base and altitude.*



Notation. Let **P** be an oblique parallelopiped with the base B and altitude h.

To prove $\mathbf{P} = Bh.$

Construction. Extend the edges of **P** as shown, making the prism of $\mathbf{P}_1$ of indefinite length. From this prism cut a right prism **Q** whose altitude = the lateral edge of **P**, with face B_1. Extend the edges of **Q** as shown, giving the indefinite prism $\mathbf{Q}_1$. From this cut a right prism **R** with its altitude = the lateral edge of **Q**, and face B_2.

Analysis. $\mathbf{P} = \mathbf{Q}$ by § 161. Likewise $\mathbf{Q} = \mathbf{R}$. $\therefore \mathbf{P} = \mathbf{R}$.
But R, since its faces are at rt. $\angle$s to each other, is a rectangular parallelopiped. $\therefore \mathbf{R} = B_2h,$ § 171.
But, being parallelograms between the same parallels, $B = B_1 = B_2$. $\therefore \mathbf{P} = \mathbf{R} = B_2h = Bh.$ Q. E. D.

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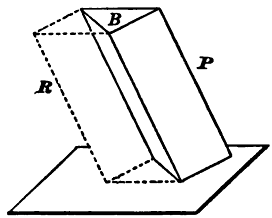
54. A pyramid 20 feet high has a square base 10 feet on a side. Find the area of a section parallel to the base and 5 feet from the vertex.

PROPOSITION XIII. THEOREM.

174. *The volume of a triangular prism is equal to the product of its base and altitude.*



Notation. Let **P** be a triangular prism with the base B and altitude h.

To prove $\mathbf{P} = Bh$.

Construction. Complete the parallelopiped **R**, having its edges = and ∥ to those of **P**, and its base $= 2B$.

Analysis. $\mathbf{P} = \frac{1}{2}\mathbf{R}$, § 164.

But $\mathbf{R} = 2B \cdot h$, § 173.

$\therefore \mathbf{P} = \frac{1}{2}\mathbf{R} = Bh$. Q. E. D.

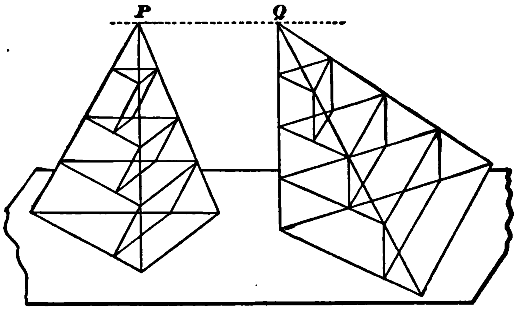
175. COR. 1. *The volume of any prism (cylinder) is equal to the product of its base and altitude.*

For, any prism may be considered as made up of triangular prisms, the sum of whose bases form the base of the given prism. Then the volume of the given prism equals the sum of the volumes of the triangular prisms, that is, the sum of their bases, or the base of the given prism, multiplied by the common altitude.

176. COR. 2. *The volumes of two prisms (cylinders) are to each other as the product of their bases and altitudes: prisms (cylinders) having equivalent bases are to each other as their altitudes: prisms (cylinders) having equal altitudes are to each other as their bases: prisms (cylinders) having equivalent bases and equal altitudes are equivalent.*

PROPOSITION XIV. THEOREM.

177. *Two triangular pyramids having equivalent bases and equal altitudes are equivalent.*



Notation. Let the two pyramids **P** and **Q** have equivalent bases and equal altitudes.

To prove **P** ⏢ **Q**.

Construction. Divide the common altitude into any number of equal parts, and through the points of division pass planes ∥ to the bases. On these sections as upper bases, construct prisms, as shown, with their edges ∥ to one edge of the pyramid.

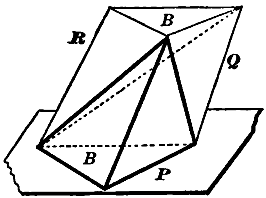
Analysis. The sum of the prisms in **P** = the sum of the prisms in **Q**. By increasing the number of divisions of the altitude indefinitely, the sums of the two sets of prisms approach the pyramids **P** and **Q**, respectively, as their limits. The sums remain =, ∴ **P** ⏢ **Q**.

Proof. Since the corresponding sections of *P* and *Q* are equal, § 147, each prism in **P** ⏢ its corresponding prism in **Q**. ∴ the sum of the prisms in **P** ⏢ sum of the prisms in **Q**. The limit of one sum is **P**, of the other **Q**. But if two variables are constantly equal, and each approaches a limit, the limits are equal. ∴ **P** ⏢ **Q**. Q. E. D.

Proposition XV.

178. *The volume of a triangular pyramid is equal to one third of a triangular prism of the same base and altitude.*

Notation. Let **P** be a triangular pyramid, of base B and altitude h.



To prove $\mathbf{P} = \frac{1}{3} Bh.$

Construction. Complete the triangular prism of the same base and altitude, composed of the three pyramids **P, Q** and **R**. **R** has for its base the upper base of the prism, B. **Q** may be considered as having for its base $\frac{1}{2}$ the rear face of the prism.

Analysis. $\mathbf{P} = \mathbf{R}$, since they have equal bases (B) and equal altitudes, § 177. **Q** and **R**, when compared, may be considered as having their bases in the rear face of the prism and their vertices at the front upper vertex of the prism. Their bases being in the same plane and their vertices together, their altitudes are =. Their bases are halves of the same parallelograms and ∴ =.

Hence $\mathbf{R} = \mathbf{Q}.$

$$\therefore \mathbf{P} = \mathbf{R} = \mathbf{Q} = \tfrac{1}{3}(\mathbf{P} + \mathbf{Q} + \mathbf{R}) = \tfrac{1}{3} Bh, \qquad \S\ 174.$$

Q. E. D.

179. Cor. 1. *The volume of a triangular pyramid is equal to one third the product of its base and altitude.*

180. Cor. 2. *The volume of any pyramid (cone) is equal to one third the product of its base and altitude.*

For it can be considered as the sum of triangular pyramids.

181. Cor. 3. *The volume of any pyramid (cone) is equal to one third the prism (cylinder) upon the same base and of the same altitude.*

182. Cor. 4. *The volumes of two pyramids (cones) are to each other as the product of their bases and altitudes: having equivalent bases they are to each other as their altitudes: having the same altitude they are to each other as their bases: having equivalent bases and altitudes, they are equivalent.*

183. Cor. 5. *If a triangle and a rectangle having the same base and equal altitudes be revolved about the common base as an axis, the volume generated by the triangle will be one third that generated by the rectangle.*

For the volumes generated are a double cone and a cylinder, and one is one third of the other, by Cor. 3.

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184. A **prismatoid** is a polyhedron whose bases are any two polygons in parallel planes, and whose lateral faces are triangles determined by so joining the vertices of the bases that each line in order forms a triangle with the preceding line and one side of one of the bases.

185. The **altitude** of a prismatoid is the perpendicular distance between the bases. A plane midway between the bases and parallel to them gives what is called a **midsection.** Its vertices halve the lateral edges of the prismatoid.

186. In general, the lateral faces are triangles in different planes, yet if two basal edges, which are respectively the sides of two adjoining faces, are parallel, then these two triangular faces fall into the same plane, and together form a trapezoid.

187. If the bases of the prismatoid become parallel polygons, we have what is called a **prismoid.**

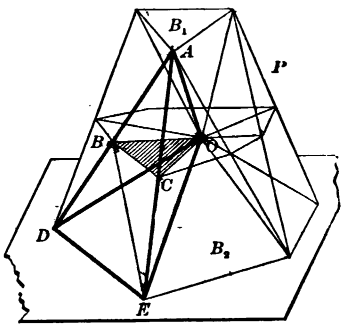
188. If the bases are parallel polygons and similar, the prismatoid becomes a frustum of a pyramid, with its limiting cases, pyramid, cone, etc. If both bases become straight lines, it is a tetrahedron.

189. If one base is a rectangle, and the other base a line parallel to a side of the rectangle, we have a **wedge.**

190. If one base becomes a point we have a **pyramid,** with its limiting cases, *cone*, *cylinder* and *prism.*

PROPOSITION XVI. THEOREM.

191. *The volume of a prismatoid is equal to the product of one-sixth the altitude into the sum of the two bases and four times the mid-section.*



Notation. Let **P** be a prismatoid with the bases B_1 and B_2, mid-section M, and the altitude h.

To prove

$\mathbf{P} = \frac{1}{6} h (B_1 + B_2 + 4M)$.

Construction. Take any point O in the mid-section and connect it with all the vertices of the prismatoid and of the mid-section. This divides the prismatoid into pyramids having, their vertices at O, and for bases, the upper base B_1, the lower base B_2, and the lateral faces, respectively. One of these lateral pyramids is $O-ADE$, composed of three pyramids $O-ABC$, $O-BCE$, $O-BDE$.

Let the part of the mid-section within these pyramids $=\triangle OBC=\beta_1$. It is shaded in the figure, to distinguish it.

Analysis. $O-ABC=O-BCE=\frac{1}{6}h\beta_1$ (having the same base β_1, and the same altitude $\frac{1}{2}h$). $DE=2BC$ since BC is half-way up the triangle. $\therefore \triangle BDE=2\triangle BCE$ since they have the same altitude. $\therefore O-BDE=2(O-BCE)$ $=2\cdot\frac{1}{6}h\beta_1$, their vertices being together and their bases in the same plane, $\therefore O-ADE=\frac{4}{6}h\beta_1$.

In like manner, the volume of every lateral pyramid $=\frac{4}{6}h$ times the area of its own piece of the mid-section, and their sum is $\frac{4}{6}hM$.

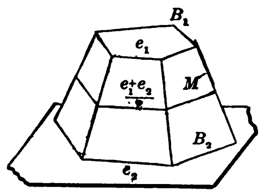
The pyramid having B_1 for its base $=\frac{1}{6}hB_1$, and the pyramid having B_2 for its base $=\frac{1}{6}hB_2$.

Adding all these pyramids, we get

$$\mathbf{P}=\tfrac{1}{6}h(B_1+B_2+4M). \qquad \text{Q. E. D.}$$

The prismatoidal formula and definitions are taken, by permission, from Dr. George Bruce Halsted's work on Mensuration.

192. Cor. *The volume of the frustum of a pyramid (cone) is equal to the sum of three pyramids (cones) whose common altitude is the altitude of the frustum and whose bases are respectively the upper base, the lower base, and a mean proportional between them.*



Notation. Let e_1 and e_2 be homologous edges of the upper and lower bases, B_1 and B_2, of a frustum of a pyramid of volume $\mathbf{V}$, and M its mid-section.

To prove $\mathbf{V}=\frac{1}{3}h(B_1+B_2+\sqrt{B_1B_2})$.

Analysis. The homologous edge of the mid-section will be $\frac{1}{2}(e_1+e_2)$, since it is the median of a trapezoid. Since similar areas are proportional to the squares of homologous lines,

$$e_1 : \frac{e_1 + e_2}{2} = \sqrt{B_1} : \sqrt{M},$$

and
$$e_2 : \frac{e_1 + e_2}{2} = \sqrt{B_2} : \sqrt{M}.$$

Adding the antecedents of these proportions, we have

$$e_1 + e_2 : \frac{e_1 + e_2}{2} = \sqrt{B_1} + \sqrt{B_2} : \sqrt{M},$$

or
$$2\sqrt{M} = \sqrt{B_1} + \sqrt{B_2},$$

whence
$$4M = B_1 + 2\sqrt{(B_1 B_2)} + B_2.$$

Inserting this in the prismatoidal formula, we have

$$\mathbf{V} = \tfrac{1}{6}\, h\, (B_1 + B_2 + 4M)$$
$$= \tfrac{1}{3}\, h\, [B_1 + B_2 + \sqrt{(B_1 B_2)}].$$ Q. E. D.

193. Scholium 1. If the upper base of the frustum becomes zero, the frustum becomes a pyramid (cone) and the formula becomes $\mathbf{V} = \frac{1}{3}\, h\, B_2$ as before.

194. Scholium 2. If the vertex is carried out to infinity, the frustum becomes a prism (cylinder) and the formula becomes, since the two bases become equal, $\mathbf{V} = \frac{1}{3}\, h\, (3B_1) = B_1\, h$ as before.

194½. Scholium 3. For a frustum of revolution, of height h, and bases whose radii are r' and r'', the formula becomes

$$\mathbf{V} = \tfrac{1}{3}\, \pi h\, (r'^2 + r''^2 + r'r'').$$

---

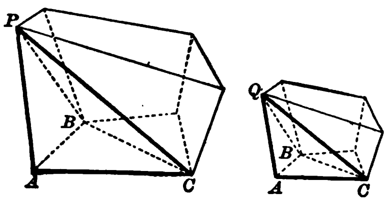
195. Similar polyhedrons are those having the same number of polyhedral angles, equal each to each, and the corresponding faces similar and similarly placed.

196. Homologous faces, edges, angles, lines, etc., in similar polyhedrons are faces, edges, angles, lines, etc., which are similarly placed.

Proposition XVII. Theorem.

197. *Two similar polyhedrons may be decomposed into the same number of tetrahedrons, similar each to each, and similarly placed.*

Notation. Let P and Q be two similar polyhedrons, the vertices P and Q being homologous.



To prove that they may be decomposed into the same number of tetrahedrons, similar each to each, and similarly placed.

Construction. Divide all the faces of each polyhedron except those adjacent to P and Q, into corresponding similar $\triangle$s, and draw straight lines from P and Q to their vertices.

Analysis. The polyhedrons are then decomposed into the same number of tetrahedrons similarly placed.

The tetrahedrons $P-ABC$ and $Q-ABC$ have the faces about the trihedral $\angle$s A similar, since they are the homologous parts of similar polygons. $\therefore$ the $\triangle\, PBC$ is similar to $\triangle\, QBC$, since their sides are the homologous diagonals of similar and proportional polygons. By § 93, the corresponding trihedral $\angle$s are $=$. $\therefore$ $P-ABC$ is similar to $Q-ABC$, § 195.

If these tetrahedrons be removed, the polyhedrons remaining will be similar, since similar parts have been removed from the faces, and equal parts from the polyhedral angles.

This process of removing similar tetrahedrons can be repeated until the two similar polyhedrons are reduced to similar tetrahedrons; that is, until they are decomposed into the same number of tetrahedrons, similar each to each, and similarly situated. Q. E. D.

198. Cor. 1. *Any two homologous lines in two similar polyhedrons are proportional to any two homologous edges.*

199. Cor. 2. *The homologous lines of two similar polyhedrons are proportional to each other.*

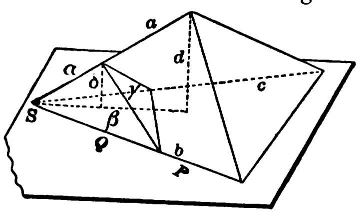
200. Cor. 3. *The homologous faces of two similar polygons are proportional to the squares of any two homologous lines.*

201. Cor. 4. *The entire surfaces, or proportional parts thereof, of two similar polyhedrons are proportional to the squares of any two homologous lines.*

Proposition XVIII. Theorem.

202. *Two tetrahedrons having a trihedral angle in each equal, are to each other as the products of the including edges.*

Notation. Let **P** and **Q** be two tetrahedrons having the common trihedral angle S. Let a, b, c and α, β, γ, be the corresponding edges about the angle S, and d and δ the altitudes of the tetrahedrons, as shown.



To prove $$\frac{\mathbf{P}}{\mathbf{Q}} = \frac{abc}{\alpha\beta\gamma}$$

Analysis. By § 179, $\frac{\mathbf{P}}{\mathbf{Q}} = \frac{\triangle bc \,.\, d}{\triangle \beta\gamma \,.\, \delta}$. But since the $\triangle$s bc and $\beta\gamma$ have an angle in common they are to each other as the products of the including sides: that is

$$\frac{\triangle bc}{\triangle\beta\gamma}=\frac{bc}{\beta\gamma}.$$

Since d and δ are the legs of similar rt. $\angle\triangle$s,

$$\frac{d}{\delta}=\frac{a}{\alpha}.$$

Substituting these expressions, we get

$$\frac{\text{P}}{\text{Q}}=\frac{abc}{\alpha\beta\gamma}.$$ Q. E. D.

203. Cor. 1. *Two similar tetrahedrons are to each other as the cubes of their homologous edges.*

If the tetrahedrons are similar, and therefore the faces similar, then

$$\frac{a}{\alpha}=\frac{b}{\beta}=\frac{c}{\gamma}\;\therefore\;\frac{abc}{\alpha\beta\gamma}=\frac{a^3}{\alpha^3}=\frac{b^3}{\beta^3}=\frac{c^3}{\gamma^3}.$$

Hence $$\frac{\text{P}}{\text{Q}}=\frac{a^3}{\alpha^3}=\frac{b^3}{\beta^3}=\frac{c^3}{\gamma^3}.$$ Q. E. D.

204. *Similar polyhedrons are to each other as the cubes of their homologous lines.*

For any two similar polyhedrons may be decomposed into the same number of tetrahedrons, similar each to each, § 197; and any two homologous tetrahedrons are to each other as the cubes of their homologous edges, or as the cubes of any two homologous edges or lines of the polyhedrons, § 199. . ∴ The polyhedrons, or sums of the similar tetrahedrons, are proportional to the cubes of the homologous lines.

205. Cor. 3. Similar pyramids (prisms, cones, cylinders) are to each other as the cubes of their altitudes, or of any other homologous lines.

206. Scholium. This is a partial application of the general rule, that similar surfaces *of any form* are to each other as the squares of homologous lines: and similar volumes as the cubes of homologous lines.

Regular Polyhedrons.

207. A **regular polyhedron** is a polyhedron all of whose faces are equal regular polygons, and all of whose polyhedral angles are equal.

Proposition XIX. Theorem.

208. *There can be only five regular convex polyhedrons.*

Analysis. At least three faces are necessary to form a polyhedral angle, and the sum of its face angles must be less than 360°, § 92.

1°. Because the angle of an equilateral triangle is 60°, a convex polyhedral angle may be formed with 3, 4, or 5 equilateral triangles, but not with 6, because the sum of 6 such angles is 360°. Therefore only three regular convex polyhedrons can be formed with equilateral triangles; the *tetrahedron, octahedron,* and *icosahedron.*

2°. Because the angle of a square is 90°, a convex polyhedral angle may be formed with 3 squares, but not with 4, because the sum of 4 such angles is 360°. Therefore only one regular convex polyhedron can be formed with squares; the *hexahedron* or *cube.*

3°. Because the angle of a regular pentagon is 108°, a convex polyhedral angle may be formed with 3 regular pentagons, but not with 4, because the sum of 4 such angles is greater than 360°. Therefore only one regular convex polyhedron can be formed with regular pentagons; the *dodecahedron.*

4°. Because the angles of a regular polygon of more than five sides are each equal to or greater than 120°, the sum of three such angles equals or exceeds 360°, and these polygons cannot be used to form regular polyhedrons.

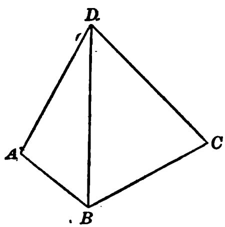
Hence only five regular polyhedrons are possible.

Proposition XX. Problem.

209. *To construct the regular polyhedrons, having given one of the edges.*

1. The regular tetrahedron.

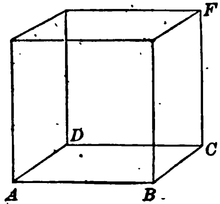
Upon the given edge AB construct the equilateral $\triangle ABC$. From the three vertices A, B and C, erect the three lines meeting in the point D, each line = the given edge. $D-ABC$ is a regular tetrahedron.



Proof. Each face is an equilateral triangle with its side $= AB$, the given edge, and the trihedral $\angle$s are = by § 93. $\therefore D-ABC$ is a regular tetrahedron.

2. The regular hexaedron, or cube.

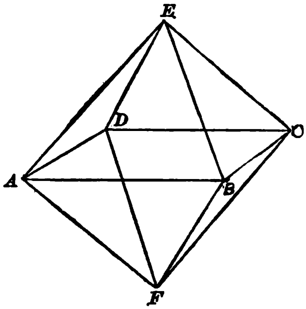
Upon the given edge AB, construct the square $ABCD$. Upon the sides of this square construct four equal squares $\perp$ to the plane $ABCD$, giving the figure $ABCD-F$, which is a regular hexaedron.



Proof. Since the faces are by construction equal squares, the eight trihedral angles are equal, § 93; and $ABCD-F$ is a regular hexaedron.

3. The regular octahedron.

Upon the given edge AB construct the square $ABCD$, and upon this square as a base construct the two regular pyramids with their vertices at E and F respectively, and their lateral edge $AE = BE = CE =$ etc., $= AB$. This gives us the octahedron EF, with



(a) All the lateral faces equal equilateral triangles.

The polyhedral angle E is evidently equal to the polyhedral angle F, since the two pyramids are equal.

Since the $\triangle$s ABC and AEC are = legged isosceles $\triangle$s upon the same base AC, they are equal and $\angle AEC =$ rt. $\angle$. Similarly, $\angle AFC =$ rt. $\angle$. Therefore $AFCE$ is a square $= ABCD$, and the pyramid $B - AECF =$ pyramid $D - AECF =$ pyramid $E - ABCD$. $\therefore$ polyhedral $\angle B =$ polyhedral $\angle D =$ polyhedral $\angle E$. Similarly,

(b) All the polyhedral angles can be proved equal.

Hence, we have a regular polyhedron.

4. The regular dodecahedron.

Upon the given edge AB construct a regular pentagon, and upon the edges of this construct five other regular pentagons, so inclined that their edges unite to form the convex surface shown in figure a. Into each of the open angular spaces thus formed around the upper edge fit a regular pentagon equal to the others, giving figure b, which is completed by fitting another regular pentagon into the top, giving a regular dodecahedron.

Proof. In figure a all the trihedral $\angle$s around the base are equal, § 93.

Since the faces *M* and *N* are regular pentagons the = edges e, e_1 must be inclined at the same angle as the = edges c, c_1, viz., 108°, the angle of a regular pentagon. Therefore a third regular pentagon can be fitted in at the vertex *C*. So with all the other vertices, *D*, etc. Let the student show that these faces will exactly join each other. The trihedral ∠s at these vertices are equal to thoso at *A*, § 93.

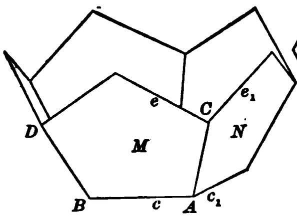


FIG. *a*.

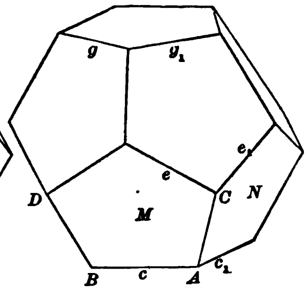


FIG. *b*.

In figure *b* the edges g, g_1 are at an angle of 108°, for the same reason as above. So with the other edges around the top. Hence the upper edges form a regular pentagon, equal to the other faces.

By § 93, all the trihedral ∠s at these vertices = the trihedral ∠ at *A*. ∴ the trihedral ∠s are equal, the faces are equal regular pentagons, and the solid is a regular dodecahedron.

> **5. The regular icosahedron.**

Upon the given edge *AB* construct a regular pentagon *ABCDE*. Upon this as a base construct the regular pyramid *F* — *ABCDE*, with its lateral edges = *AB*, giving figure *a*. Upon the base edges of this pyramid erect equilateral triangles *ABG*, *BCH*, etc., so inclined that the distances between their vertices *KG*, *GH*, *HI*, etc., shall equal AB, i. e., so that the inserted triangles AGK, BGH, etc., shall be equilateral triangles. This gives figure b, with the open base $GHIJK$.

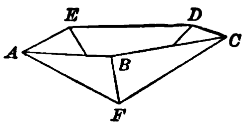


FIG. *a*.

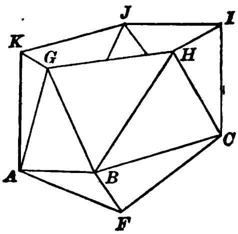


FIG. *b*.

Since the $=$ legged isosceles $\triangle$s AFC and ABC are upon the same base AC they are $=$, and $\angle AFC = \angle ABC = 108°$. $\therefore$ the pyramid $B - AFCHG$, having $=$ lateral edges, and standing upon an equilateral base, one of whose angles is $108°$, must be a regular pyramid $=$ pyramid $F - ABCDE$. $\therefore$ the polyhedral $\angle B =$ the polyhedral $\angle F$. Similarly all the other completed polyhedral $\angle$s A, C, etc., can be shown to be $=$.

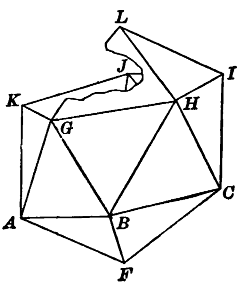


FIG. *c*.

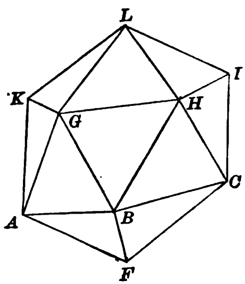


FIG. *d*.

Upon the edges GH, HI of the open base, erect the equilateral $\triangle$s GHL, HIL, giving us figure c. The pentagon $BCILG$ is regular, since it is equilateral, with two $\angle$s = 108° each. The = legged isosceles $\triangle$s GHI and GLI upon the same base are =, and $\angle GHI = \angle GLI =$ 108°. So with the $\angle$s between the other edges of the open base.

$\therefore$ the open base $GHIJK$ is a regular pentagon, and can be taken as the base of a regular pyramid = pyramid $F-ABCDE$, giving figure d.

The polyhedral $\angle$s G, H, etc., can be proved = as were the $\angle$s A, B, etc.

Therefore, the figure, being formed from equilateral $\triangle$s, and having its polyhedral $\angle$s =, is a regular polyhedron.

209.* If a plane be passed through each vertex of a regular polyhedron, perpendicular to the radius from the centre, these planes will be the faces of another regular polyhedron called the **sympolar** polyhedron.

The sympolar polyhedron evidently has as many faces as its primitive has vertices.

PROPOSITION XXI. THEOREM.

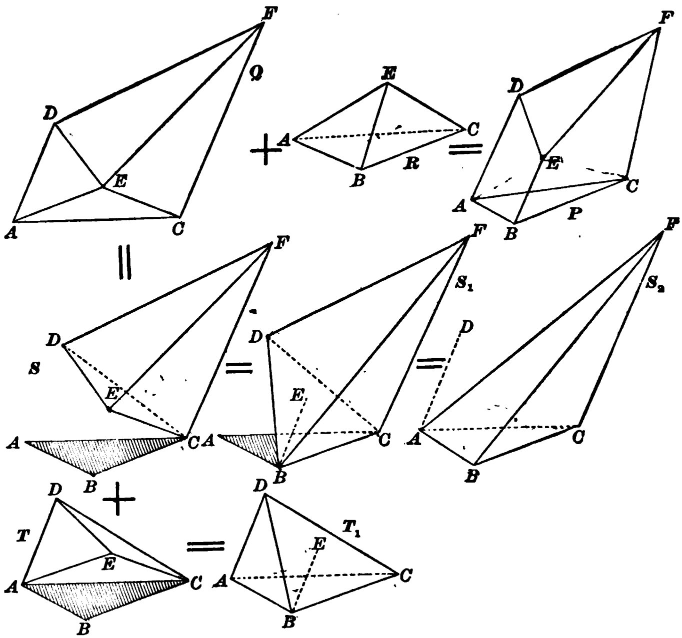
210. *A truncated triangular prism is equivalent to the sum of three pyramids whose common base is the base of the prism and whose vertices are the three vertices of the upper base.*

Notation. Let **P** be a truncated triangular prism and **R**, $\mathbf{S}_2$ and $\mathbf{T}_1$ three pyramids having for their bases the base of the frustum, and their vertices at the vertices of the upper base of **P**.

To prove $\mathbf{P} = \mathbf{R} + \mathbf{S}_2 + \mathbf{T}_1$.

Construction. Pass a plane so as to cut **P** into **Q** and **R**. **R** has the required base and vertex.

By another plane cut **Q** into **S** and **T**, which are respectively = to $\mathbf{S}_2$ and $\mathbf{T}_1$ which have the required bases. The required bases are represented by shaded areas.



Proof. $\mathbf{S} = \mathbf{S}_1$, because they have the same base DCF and the same altitude, their vertices being in the same line $EB \parallel$ to the base, § 34.

For a similar reason $\mathbf{S}_1 = \mathbf{S}_2$. Likewise $\mathbf{T} = \mathbf{T}_1$.

$$\therefore \mathbf{P} = \mathbf{Q} + \mathbf{R} = \mathbf{S} + \mathbf{T} + \mathbf{R} = \mathbf{S}_2 + \mathbf{T}_1 + \mathbf{R}.$$ Q. E. D.

211. Cor. 1. *The volume of a truncated right triangular prism is equal to the product of its base by one-third the sum of its lateral edges.*

For the lateral edges being ⊥ to the base, are the altitudes of the three pyramids whose sum is equivalent to the truncated prism. The sum of the volumes of the pyramids = the common base by $\frac{1}{3}$ the sum of the three altitudes.

212. Cor. 2. *The volume of any truncated triangular prism is equal to the product of its right section by one-third the sum of its lateral edges.*

For the right section divides the truncated prism into two truncated right prisms, whose volumes are by Cor. 1 ≐ the rt. section × $\frac{1}{3}$ the sum of the edges.

Proposition XXII. Theorem (Euler's).

212.* *In any pyramid the number of edges is two less than the sum of the faces and vertices.*

Notation. Let E denote the number of edges, V the number of vertices, and F the number of faces of the pyramid.

To prove $E = V + F - 2.$

Proof. Beginning with the base, we have (in the base)

Edges = vertices + faces − 1.

Annexing a lateral face, two edges are added to the left-hand member, and one vertex and one face to the right-hand member; so that the equation remains as before,

Edges = vertices + faces − 1.

Annexing another face, one unit is added to the *edges* (left-hand member) and one unit to the *faces* (right-hand member), so that the equation remains unchanged.

And so on, no change resulting in the equation by the annexing of additional lateral faces, until we come to the last one.

When this is annexed one unit is added to the faces only, and to preserve the equation an abstract unit must be subtracted, which puts the equation in the form

$$\text{Edges} = \text{vertices} + \text{faces} - 2,$$

or $$E = V + F - 2.$$ Q. E. D.

212.** Cor. *In any polyhedron* $E = V + F - 2$.

Assume the truth of the formula. Suppose an edge to disappear by the coalescence of two vertices which will cause the disappearance or loss of the joining edge and one of the vertices. In *addition* there will be lost, if the faces adjoining the vanishing edge are

Polygons	no edges,	no vertex,	no face.
Polygon and triangle .	one edge,	no vertex,	one face.
Triangles	two edges,	no vertex,	two faces.

In any case the equation $E = V + F - 2$ is not disturbed. By repeating this operation the polyhedron is finally reduced to a pyramid, for which the equation does hold, and therefore it holds for the given polyhedron. Q. E. D.

Proposition XXIII. Theorem.

212.*** *The sum of the face-angles of any polyhedron is equal to four right angles taken as many times as the polygon has vertices less two.*

Notation. Let E, F and V denote the number of edges, faces and vertices, respectively, and S the sum of the face $\angle$s of any polyhedron, expressed in rt. $\angle$s.

To prove $S = (V - 2)\ 4$ rt. $\angle$s.

Proof. Since each edge is common to two faces, $\therefore$ the whole number of *polygon* edges is $2E$, belonging to F polygons.

If we form an exterior $\angle$ at each vertex of the polygons, the sum of the exterior and interior angles at each vertex $= 2$ rt. $\angle$s. Since the whole number of polygon edges or vertices is $2E$, the sum of all the ext. $\angle$s + sum of all the int. $\angle$s = sum of all the ext. $\angle$s + S,

$$= 2E \cdot 2 \text{ rt. } \angle\text{s},$$
$$= E \cdot 4 \text{ rt. } \angle\text{s}.$$

But, sum of all the ext. $\angle$s $= F \cdot 4$ rt. $\angle$s.

$$\therefore F \cdot 4 \text{ rt. } \angle\text{s} + S = E \cdot 4 \text{ rt. } \angle\text{s}.$$
$$\therefore S = (E - F)\, 4 \text{ rt. } \angle\text{s},$$
$$= (V - 2)\, 4 \text{ rt. } \angle\text{s}.$$

Q. E. D.

212.**** The following table gives the number of edges, faces, etc., of the regular polyhedrons, and the corresponding sympolar:

Solid.	F	E	V	Faces of Sympolar.	Sympolar.
Tetrahedron	4	6	4	4	Tetrahedron
Cube	6	12	8	8	Octahedron
Octahedron	8	12	6	6	Cube
Dodecahedron	12	30	20	20	Icosahedron
Icosahedron	20	30	12	12	Dodecahedron

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55. The base of a regular pyramid is a hexagon of which the side is 6 feet. Find the height of the pyramid if the lateral area is 6 times the area of the base.

56. Find the lateral area of a right pyramid whose slant height is 6, and whose base is a regular octagon of which each side is 3.

57. Find the total surface of a regular pyramid when each side of its square base is 10 feet and the slant height is 20 feet.

58. Find the total surface of a regular pyramid when each side of its square base is 25 feet and the altitude is 80 feet.

NOTATION.

B = base.
r = radius.
h = altitude.
S = lateral or side surface.
T = total surface.
M = midsection.
$\mathbf{V}$ = volume.
p = perimeter of base.
e = edge.
l = slant height.

FORMULAE.

PRISMATOID.

$\mathbf{V} = \frac{1}{6} h (B_1 + B_2 + 4M)$ . . . . § 191.

FRUSTUM OF A PYRAMID (CONE).

$\mathbf{V} = \frac{1}{3} h [B_1 + B_2 + \sqrt{(B_1 B_2)}]$ . . . . § 192.

of Circular Cone, or Cone of Revolution.

$\mathbf{V} = \frac{1}{3} \pi h (r_1^2 + r_2^2 + r_1 r_2)$ . . . . § 194½.

of Cone of Revolution.

$S = \frac{1}{2} (p + p_1) l$ . . . . § 154.

PYRAMID (CONE).

$\mathbf{V} = \frac{1}{3} Bh$ . . . . § 180.

Circular Cone.

$\mathbf{V} = \frac{1}{3} \pi r^2 h$ . . . . § 180.

Cone of Revolution.

$S = \pi (r_1 + r_2) l = \frac{1}{2} (p + p_1) l$ . . . . . . § 155.

PRISM (CYLINDER).

$\mathbf{V} = Bh$ . . . . § 175.

Circular Cylinder.

$\mathbf{V} = \pi r^2 h$ . . . . § 175.

Cylinder of revolution.

$S = 2\pi r \cdot h$ . . . . § 152.

$\frac{\mathbf{V}}{\mathbf{V}_1} = \frac{e^3}{e_1^3}. \quad \frac{S}{S_1} = \frac{e^2}{e_1^2} = \frac{T}{T_1}$ . . . . § 206.

EXERCISES.

59. Find the total surface of a triangular pyramid when each side of the base is 9 feet and the slant height is 20 feet.

60. Find the volume of a regular pyramid when each side of its square base is 40 feet and the lateral edge is 154 feet.

61. Find the volume of a right quadrangular pyramid whose altitude is 9 and whose base is 6 units.

62. Find the lateral area of a right pentagonal pyramid whose slant height is 14 and each side of the base 6.

63. Find the volume of a regular pyramid whose base is an equilateral triangle inscribed in a circle of 80 feet radius, and whose slant height is 50 feet.

64. Find the volume of a regular square pyramid whose base edge is $2\frac{2}{3}$ and whose height is 7.2.

65. Find the volume of a regular triangular pyramid whose base edge is 3.2 feet and whose height is 12 feet.

66. Find the volume of a regular hexagonal pyramid whose base edge is 4.8 feet and whose height is 24 feet.

67. Find the total surface of a regular square pyramid whose base edge is 6.4 feet and the slant height 16 feet.

68. Find the tctal surface of a regular square pyramid whose base edge is 8 and whose slant height is 20.

69. Find the total surface of a regular triangular pyramid whose base edge is 4.8 and the slant height 7.5.

70. Find the total surface of a regular square pyramid whose base edge is 26 and whose height is 84.

71. Find the height of a square pyramid whose volume is 13.6 and whose base edge is 2.8.

72. Find the height of a triangular pyramid whose base edges are 3, 4 and 5, and volume 20.

73. Find the volume of a regular square pyramid whose base edge is 32 and whose lateral edge is 80.8.

74. Find the volume of a regular triangular pyramid whose slant height is 9.6 and whose base is inscribed in a circle of 8 units radius.

75. How many square inches of tin will be required to make a funnel if the diameters of the top and bottom are 22.4 and 11.2 inches respectively, and the height 19.2 inches?

76. Find the volume of the frustum of a regular square pyramid whose slant height is 16 and its base edges 32 and 12.8.

77. Find the height of the frustum of a regular hexagonal pyramid whose base edges are .8 and 1.6 respectively, and whose volume is 6.144.

78. The frustum of a right circular cone is 11.2 in height and its volume is 474. Find the radii of the bases if their sum is 7.2.

79. Find the height of a right circular cylinder of a radius r whose volume is equivalent to a rectangular parallelopiped of the dimensions a, b, c.

80. Find the height of right circular cone of radius r whose volume is equivalent to a right rectangular prism whose dimensions are a, b, c.

81. Find the height of a prism equivalent to a regular pyramid 120 feet high and of the same base.

82. Find the height of a right square prism whose base edge is 8, and whose volume is equivalent to right circular cylinder 4 units high and whose diameter is 28.

83. Find the height of a regular square prism with base edge a, which is equivalent to a cylinder of revolution whose altitude is h and diameter $2r$.

84. If one edge of a cube is a, what is the height of an equivalent right circular cylinder whose diameter is b?

85. The heights of two equivalent right circular cylinders are as $m : n$. The diameter of the first is d, what is the radius of the other?

86. What is the height of a cylinder 4.8 feet in diameter, which is equivalent to a right circular cone 5.6 feet in diameter and 6.4 feet high?

87. What is the height of a regular square pyramid whose base edge is 9.6, which is equivalent to a frustum of a regular four-sided pyramid whose base edges are 4 and 6.4 respectively, and whose height is 4.8?

88. Find the edge of a cube equivalent to a regular tetrahedron whose edge is 24.

89. Find the edge of a cube equivalent to a regular octahedron whose edge is 24.

90. The dimensions of a trunk are 4, 6, 8. If the trunk is to hold 5 times as much, what change must be made in one dimension? What similar change in all the dimensions?

91. The dimensions of a trunk are 1, 2, 3. What will be the dimensions of a similar trunk which holds 5 times as much?

92. How must the dimensions of a cylinder be increased in order to get a similar cylinder n times as large superficially?

93. How must the dimensions of a cylinder be increased to give a similar cylinder n times as large volumetrically?

94. Find the height of a right circular cylinder whose base is the mid-section of a frustrum of a cone of revolution, the frustum being 4 feet high, and the diameters of its bases being 1.6 and 2.4 feet respectively, the solids being equivalent.

95. A mid-section of a pyramid has how much of the pyramid above it?

96. How far from the vertex will a plane cut the lateral edge of a pyramid 6 feet high in order to cut off one-half the pyramid?

97. If you had a cone of maple sugar, what proportion of the altitude should be cut off in order to give away one-half the cone?

98. At what distance from the vertex must a pyramid (cone) be cut by two planes in order to divide the solid into three equal frustums?

99. In example 98, if the lateral edge is 6 feet where will it be cut by the two planes?

100. At what distance from the vextex must a pyramid (cone) be cut in order to divide the solid into n equal frustums.

101. At what distance from the vertex must a pyramid (cone) be cut so as to divide the solid into two frustums which are to each other as 3 : 4?

102. In example 101, suppose the solid be divided into 3 frustums which are to each other as a, b, c.

103. The volumes of two similar pyramids (cones) are to each other as 8 : 343. The height of the first one is 4. What is the height of the other?

104. The volumes of two similar pyramids (cones) are to each other as 8 : 343. A line in one (edge, diagonal, diameter, etc.) is 4. What is the corresponding line in the other?

105. The volumes of two similar pyramids (cones) are to each other as 8 : 343. A rod sticking through the first one has 4 feet within the solid. How much of a rod similarly placed in the other solid is within the solid?

106. In each of two cones of revolution the radius of the base is .3 times the slant height. The volume of one 4 : 5 that of the other. What is the ratio of the radii?

107. In example 106, what is the ratio of the altitudes?

108. The height of a cone of revolution is h and the radius of its base is r. What are the dimensions of similar cones 3, 4 and 5 times as large?

109. The bases of two similar pyramids (cones) are to each other as 2 : 3. What is the ratio of their volumes.

110. The volumes of two similar solids are in the ratio of $m : n$. What is the ratio of homologous sections.

111. Homologous faces of similar solids are in the ratio $m : n$. What is the ratio of their volumes? Of homologous lines?

112. The height of a frustum of a pyramid (cone) is $\frac{3}{4}$ the height of the entire pyramid (cone). What is the ratio between the volumes?

113. The frustum of a pyramid is 10 feet high, and two homologous edges of its bases are 5 and 6 feet respectively. What part of the pyramid is the frustum?

114. How far from the top must you cut a circular tent in order to cut the cloth in half?

115. If the slant height of a frustum of a cone is inclined to the base at 45°, what is the lateral surface, the radii of the bases being r and r'?

116. What is the volume of a wedge of which the base is 70×30, the length of the edge 110, and the altitude 24.8?

117. If in the frustum of a cone, the diameter of the upper base equals the slant height, find the lateral area, having given the altitude h and the perimeter p of a vertical section through the axis.

118. The frustum of a cone has its upper base circumscribing the base of a frustum of a regular square pyramid, and its lower base inscribed in the base of the pyramidal frustum. If the base edges of the pyramidal frustum are e and e' and its lateral edge ϵ, what is the slant height of the conical frustum?

119. Given l and l', the longest and shortest elements of a circular cone, and the altitude h, find the radius of the base.

120. Find the altitude of a frustum of a cone of revolution, having the lateral surface S and the bases B and B'.

121. Find the total surface of a regular hexagonal pyramid whose altitude is h and whose base edge is e.

122. In a regular square pyramid, given p, the perimeter of the base and Δ, the area of a section through two diagonally opposite edges, find the lateral surface.

123. How many square feet of canvas are required for a conical tent 20 feet high and 12 feet in diameter.

124. In the frustum of a right circular cone, on each base stands a cone with its vertex in the center of the other base; if the radii of the bases are r and r', what is the radius of the circle in which the two cones intersect?

125. If a piece of brass 8 by 6 by 12 inches is drawn into a wire $\frac{1}{30}$ of an inch in diameter, what will be the length of the wire?

126. If the area of a vertical section through the axis of a right cylinder is M, what is area of a parallel section which is distant from it half the radius of the cylinder.

127. What is the amount of metal in a pipe 2.55 feet long and whose internal and external diameters are 12.8 and 14.4 feet.

128. What is the area of the upper base of a conical frustum which divides the altitude of the entire cone in the ratio $m : n$, the radius of the base of the cone being r?

129. A cone and a cylinder have equal total surfaces, and their axis sections are regular polygons. What is the ratio of their volumes?

130. Find the ratio between the volumes of the cones inscribed and circumscribed to a regular tetrahedron whose edge is a.

131. Both faces of a prismatoid of altitude h are squares; the lateral faces are isosceles triangles; the sides of the upper base are parallel to the diagonals of the lower base, and half as long as these diagonals; b is a side of the lower base. What is the volume?

132. The upper base of a prismatoid of altitude $h = 6$ is a square of side $a = 7.07107$; the lower base is a square of side $b = 10$, with its diagonals parallel to the sides of the upper base; the lateral faces are isosceles triangles. Find the volume.

133. The base edges of a frustum of a square pyramid are 20 and 7.2, and the height is 192. What is the volume?

134. If the total surface of a right circular cylinder is T and the radius of the base is r, what is the height of the cylinder?

135. In example 134, what is the volume?

136. In example 134, what is the volume if the height is equal to the diameter of the base?

137. Having the total surface T of a right circular cone and the lateral surface, find the volume.

138. A stone is dropped into a circular tub 48 inches in diameter, causing the water therein to rise 24 inches. What is the size of the stone?

139. If the circumference of the base of a right circular cylinder is p and the height h, what is the volume?

140. Find the volume of a pyramid whose altitude is 18 feet and whose base is a rectangle 7 feet by 5.

141. Find the volume of the frustum of a square pyramid, the sides of whose bases are 8 and 10 feet, and whose altitude is 12 feet.

142. A regular hexagonal pyramid whose altitude is 16 feet, and the edge of whose base is 6 feet, is cut by a horizontal plane midway between the vertex and the base. What is the lateral area and volume of the frustum?

143. Find the volume of a pyramid whose height is 8 and whose base is a square, each side of which is 5.

144. Find the volume of the frustum of a regular triangular pyramid, the sides of whose bases are 6 and 8 and whose lateral edge is 7.

145. Find the convex surface and the volume of a frustum cut from a right circular cone whose slant height is 24, and the circumference of whose base is 8, the slant height of the frustum being 19.2.

146. What is the difference between the volumes of the frustum of a square pyramid whose base edges are 8 and 6 feet respectively, and the volume of a prism of the same altitude whose base is a mid-section of the frustum?

147. Find the volume V of the frustum of a cone of revolution, whose slant height is l, and height h, and convex surface S.

148. Find the number of edges in a pyramid whose base is a polygon of n sides.

149. Find the number of edges in a prism whose base is a polygon of n sides.

150. Find the lateral area of a regular pentagonal pyramid whose slant height is 18 and whose base edge is 6.

151. Find the ratio of two rectangular parallelopipeds whose dimensions are 5, 6, 7, and 6, 14, 17 respectively.

152. Find the surface of a rectangular parallelopiped whose base is 7 by 12 feet and whose volume is 504 cubic feet.

153. Find the volume of a rectangular parallelopiped whose surface is 480 and whose base is 3 by 6.

154. The volume of a triangular prism is equal to the product of the area of a lateral face by one-half the perpendicular distance of that face from the opposite edge.

155. The volume of a regular tetrahedron is equal to the cube of its edge multiplied by $\frac{1}{12}\sqrt{2}$.

156. The volume of a regular octahedron is equal to the cube of its edge multiplied by $\frac{1}{3}\sqrt{2}$.

157. To cut a tetrahedral angle by a plane, so that the section shall be a parallelogram.

158. Find the dimensions of a cube whose surface shall numerically equal its volume.

159. Find the volume of a rectangular prismoid whose bases are 9 and 5, and 5 and 3, respectively, and whose altitude is 5.

160. Find the volume of a rectangular pyramid whose base edges are 4 and 7 and whose altitude is 5.

161. Find the volume V of a regular four-sided pyramid whose total surface is T and whose base edge is e.

162. What is the height of a regular four-sided pyramid whose base edge is e and whose total surface is T?

163. Given the height h of a regular four-sided pyramid, and the surface T, find the base edge e.

164. Find the volume of a pyramid whose altitude is 22 and whose base is a regular hexagon, each side of which is 6.

165. How many edges has a regular tetrahedron?

166. How many edges has a regular dodecahedron?

167. How many edges has a regular icosahedron?

168. What is the sum of the face angles of a regular tetrahedron?

169. What is the sum of the face angles of a regular icosahedron?

CHAPTER III.

THE SPHERE.

213. A **sphere** is a solid bounded by a curved surface, every point of which is equally distant from a point within, called the *centre.*

A sphere may be considered as generated by the revolution of a semicircle about its diameter as an axis.

214. A **radius** of a sphere is a straight line drawn from its centre to its surface.

215. A **diameter** of a sphere is a straight line passing through the centre and limited by the surface.

Since all the radii of a sphere are equal, and a diameter is equal to two radii, all the diameters of a sphere are equal.

216. A line or plane is **tangent** to a sphere when it has but one point in common with the surface of the sphere.

217. Two spheres are **tangent** to each other when their surfaces have but one point in common.

218. The common point is called the **point of tangency.**

219. Two spheres are **concentric** when they have the same centre.

220. Two spheres are **equal** when they have equal radii.

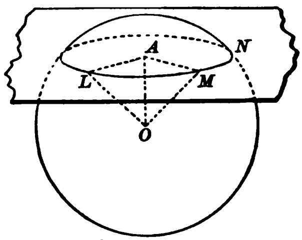
221. A polyhedron is said to be **inscribed in a sphere** when its vertices lie in the surface of the sphere. In this case the sphere is circumscribed about the polyhedron.

222. A polyhedron is said to be **circumscribed about a sphere** when its faces are tangent to the surface. In this case the sphere is inscribed in the polyhedron.

PROPOSITION I. THEOREM.

223. *Every section of a sphere by a plane is a circle.*

Notation. Let LMN be a plane section of the sphere whose centre is O.



To prove that $LMN = \odot$.

Construction. Draw $OA \perp$ to the plane of the section meeting it in A, and draw OL, OM to any two points in the section.

Proof. $OL = OM$, § 213. $\therefore AL = AM$, § 20. But AL, AM are any two lines. $\therefore LMN$ is a $\odot$ whose centre is A.

Q. E. D.

DEFINITIONS.

224. The circular section of a sphere by a plane is called a **circle of the sphere**; a **small circle** if the plane does not pass through the centre of the sphere; a **great circle** if the plane does pass through the centre, in which case its radius is the radius of the sphere.

225. The diameter of the sphere perpendicular to a circle of the sphere is called the **axis** of the circle, and its extremities are called the **poles** of the circle.

226. COR. 1. *The diameter of the sphere which is perpendicular to the plane of a circle passes through its centre,*

§ 223.

Therefore, *the axis of a circle passes through its centre and all parallel circles have the same axis and the same poles.*

227. Cor. 2. *All small circles at equal distances from the centre of the sphere are equal; and of two circles unequally distant from the centre, the nearer is the larger, and conversely.*

For with the same hypothenuse OL, AL is larger the shorter OA is; and vice versa.

228. Cor. 3. *All great circles of a sphere are equal,* § 224.

229. Cor. 4. *Every great circle bisects the sphere and its surface.*

For the two parts can be made to coincide, § 213.

230. Cor. 5. *Any two great circles bisect each other.*

For the intersection of their planes passes through the centre of the sphere, and is therefore a diameter of each circle.

231. Cor. 6. *An arc of a great circle may be drawn through any two points on the surface of a sphere.*

For the two given points and the centre of the sphere determine the plane of a great circle passing through the two points, § 4.

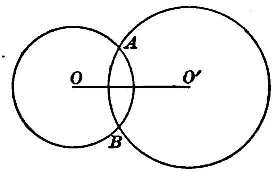
232. Cor. 7. *An arc of a small circle may be drawn through any three given points on the surface of a sphere,* § 4.

233. Definition. By the **distance** between two points on the surface of a sphere is meant the shorter of the two *arcs of a great circle* joining them. The length of the arc is generally expressed in angular measure, and the distance is called **angular distance.**

PROPOSITION II. THEOREM.

233.* *Every section of a sphere by a sphere is a circle.*

Notation. Let OA be the circle whose revolution generates one sphere, and $O'A$ the circle in the same plane whose revolution about the same axis generates the other sphere. The point A generates the line of intersection of the two surfaces.



To prove the line generated by $A =$ a circle.

Proof. The common chord AB is $\perp$ to and is bisected by OO'. When the circles are revolved the point A remains at a constant distance from the axis OO', and hence generates a circle, which must be the line of intersection.

Q. E. D.

233.** COR. 1. *All lines tangent to a sphere from a point without are equal and touch the sphere in a circle.*

The point without may be taken as the centre of another sphere whose radius = the length of the tangent.

233.*** COR. 2. *A sphere touches the circumscribed cone in a circle.*

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170. Through any four points not in the same plane, a single spherical surface can be passed.

171. A sphere may be inscribed in any tetrahedron.

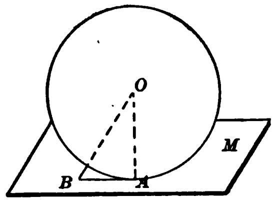
172. A sphere may be circumscribed about any tetrahedron.

173. The four perpendiculars to the faces of a tetrahedron through their centres meet at the same point.

174. The six planes which bisect at right angles the six edges of a tetrahedron all intersect in the same point.

PROPOSITION III. THEOREM.

234. *A plane perpendicular to a radius at its extremity is tangent to the sphere.*



Notation. Let O be the centre of the sphere and M a plane perpendicular to the radius OA at its extremity.

To prove M tangent to the sphere.

Construction. Draw any line OB, from O to the plane.

Proof. By § 21, $OB > OA$, $\therefore B$ is without the sphere. B is any point, except A, $\therefore M$ is tangent to the sphere, § 216.

Q. E. D.

235. COR. 1. *Every straight line perpendicularly to a radius at its extremity is tangent to the sphere,* § 216.

236. COR. 2. *Every line or plane tangent to a sphere is perpendicular to the radius through the point of contact.*

237. COR. 3. *A straight line tangent to any circle of a sphere lies in the tangent plane through the point of contact.*

238. COR. 4. *Any straight line drawn in a tangent plane through the point of contact is tangent to the sphere at that point.*

239. COR. 5. *Any two straight lines, tangent to a sphere at the same point, determine the tangent plane at that point.*

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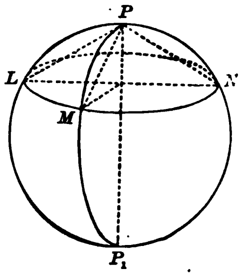
175. The six planes which bisect the six dihedral angles of a tetrahedron intersect in a point.

176. Lines from a point in the surface of a sphere to the end of a diameter are — what?

PROPOSITION IV. THEOREM.

240. *All points in the circumference of a circle of a sphere are equally distant from each of its poles.*



Notation. Let P and P_1 be the poles of the circle LMN.

To prove $PL = PM = PN$ and $P_1M = P_1L = P_1N$.

Proof. Since PP_1 is $\perp$ LMN, § 225, and passes through its centre, § 226, ∴ the chords PL, PM, PN are equal, § 20, ∵ the arcs PL, PM, PN are equal. Q. E. D.

Similarly, the arcs P_1L, P_1M, PN are equal. Q.E.D.

241. Definition. The **polar distance** of a small circle is the distance from the nearer pole to the circumference of the circle.

242. Scholium 1. The term **quadrant** in Spherical Geometry usually signifies a quadrant of a great circle.

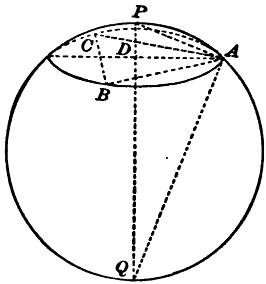
243. COR. 1. *The polar distance of a great circle is a quadrant.*

For it is the measure of a right angle whose vertex is at the centre of the sphere.

244. Scholium 2. The properties of the pole enable us to draw circles upon a sphere with the same facility as upon a plane surface. With one point of the compasses at the pole, the other point describes the circle required. In order to reach around the bulge of the sphere the compasses must be curved, like the callipers of the machinist.

PROBLEMS.

245. (*a*). *To find the radius of a given sphere.*



Notation. Let P be any point on the given sphere, and Q its antipodal point.

Construction. From any point P as a pole describe with the callipers a small circle ABC. With the three chords AB, BC and AC construct a plane △ ABC, and circumscribe a circle about it. This circle will = the small circle ABC. Hence, knowing its radius, we know AD. In the rt. ∠ △s ADP, PAQ, knowing AD and AP we can construct the △s, which will give us PQ the diameter of the sphere.

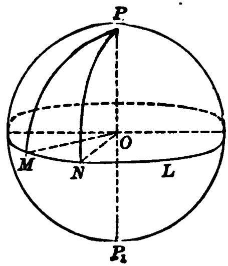
Q. E. F.

246. (*b*). *To draw the arc of a great circle through two given points.*

Construction. From the two points as poles, i. e. with the opening of the callipers = the *chord* of a quadrant, describe the arcs of two great circles. Their intersection will, § 243, be the pole of an arc through AB. Then with this pole as a centre, the arc passing through AB may be described.

PROPOSITION V. THEOREM.

248. *A point on the surface of a sphere, which is at the distance of a quadrant from each of two other points, not the extremities of a diameter, is a pole of the great circle passing through these points.*



Notation. Let P be a point at a quadrant's distance from M and N.

To prove P the pole of MNL.

Proof. $\angle POM = \angle PON =$ rt. $\angle$ (because they are measured by quadrants), $\therefore$ PO is $\perp$ to the plane MNO, § 15, $\therefore$ P is a pole of MNL, § 225.

Q. E. D.

Definitions.

249. The **angle of two curves** passing through the same point is the angle formed by the two straight lines tangent to the curves at that point.

250. When the two curves are arcs of *great circles* the angle is called a **spherical angle.** The tangents being $\perp$ to the common diameter of the great ⊙s, must be the sides of the plane angle of the dihedral angle between the planes of the ⊙s.

251. A **spherical polygon** is the portion of the surface of a sphere bounded by three or more arcs of *great circles.* The bounding arcs are the sides of the polygon; the spherical angles which they form are the **angles** of the polygon; their points of intersection are the **vertices** of the polygon.

252. A **diagonal** of a spherical polygon is an arc of a great circle joining any two vertices which are not consecutive.

253. The planes of the sides of a spherical polygon form a polyhedral angle whose vertex is the centre of the sphere, and whose face angles are measured by the sides of the polygon.

254. A spherical polygon is **convex** when its corresponding polyhedral angle is convex.

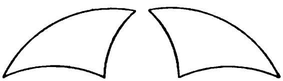
255. A **spherical pyramid** is the portion of the sphere bounded by a spherical polygon and the planes of its sides. The centre of the sphere is the **vertex** of the pyramid, and the spherical polygon is called the **base.**

256. The sides of a spherical polygon, being arcs, are usually expressed in angular measure (degrees).

257. Two spherical polygons are **equal** if they can be applied one to the other, so as to coincide.

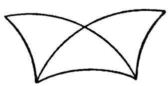
258. A **spherical triangle** is a polygon of three sides, and is **right** or **oblique, scalene, isosceles** or **equilateral,** under the same conditions that apply to a plane triangle.

259. Two spherical polygons are **symmetrical** when the sides and angles of one are equal respectively to the sides and angles of the other, but arranged *in the reverse order.*

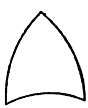


In general they cannot be made to coincide by superposition. They lean in different directions, as it were, as shown in the figure.

If, however, they are isosceles, that is lean neither to the right nor left, they may be superimposed.

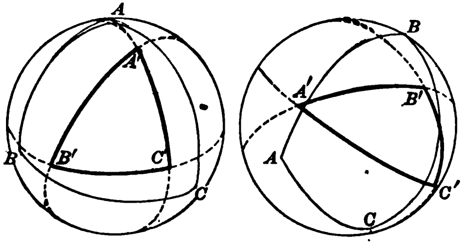


260. Symmetrical triangles may be considered as the bases of opposite or vertical spherical polyhedrons, § 90; their corresponding vertices will be at the opposite ends of the same diameter.

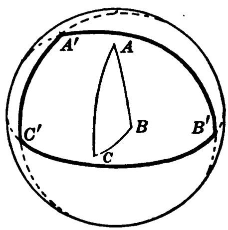


261. One spherical triangle is called the **polar triangle** of a second spherical triangle when the sides of the first triangle have their poles at the vertices of the second.

Thus A, B, and C are the poles of the arcs $B'C'$, $C'A'$ and $A'B'$, and $A'B'C'$ is the *polar* of ABC.



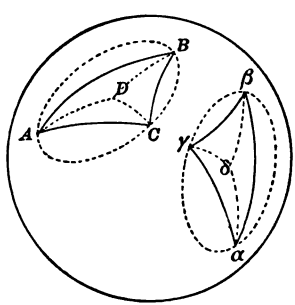
The arcs of the polar form by their intersection eight triangles, seven of which can be partly seen in the figure. The eighth is the symmetrical of $A'B'C'$. Of the eight triangles, that is the polar in which the vertex A', polar to A, lies on the same side of BC as the vertex A, and similarly for the other vertices. The vertex which is polar to A is the intersection of arcs drawn from B and C as poles.



Proposition VI. Theorem.

262. *Two symmetrical spherical triangles are equivalent.*

Notation. Let ABC and $\alpha\beta\gamma$ be two symmetrical triangles, and D and δ the poles of small circles passing through their vertices.



To prove

Sph. △ ABC = Sph. △ $\alpha\beta\gamma$.

Construction. Connect D and δ with the vertices of the triangles by arcs of great circles.

Proof. Since the sides of the

Sph. △s are respectively =, the chords of those arcs are respectively =, and the small circles, being circumscribed about equal plane △s, are =. Hence $AD = a\delta$, $BD = \beta\delta$, etc. ∴ Sph. △ ADB = Sph. △ $a\delta\beta$, Sph. △ ADC = Sph. △ $a\delta\gamma$, etc., § 257.

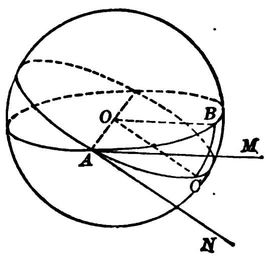
Hence Sph. △ $ABC \doteq$ Sph. △ $a\beta\gamma$, as the two are composed of equal parts. Q. E. D.

If the poles of the small circles fall without the given △s, they will be equivalent to the sum of two triangles minus a third.

Proposition VII. Theorem.

263. *A spherical angle is measured by the arc of a great circle described from its vertex as a pole and included between its sides (produced if necessary).*

Notation. Let AB, AC be two arcs of great circles intersecting at A: and AM, AN the tangents to these arcs at A: BC an arc of a great circle described from A as a pole.



To prove Sph. $\angle BAC$ = arc BC.

Construction. Draw the radii OB, OC.

Analysis. Sph. $\angle BAC = \angle MAN$, § 250, $= \angle BOC$, § 53, = arc BC. Q. E. D.

264. Cor. 1. *A spherical angle is equal to the dihedral angle between the planes of the two circles.*

265. Cor. 2. *If two arcs of great circles cut each other, their vertical angles are equal.*

266. Cor. 3. *The angles of a spherical polygon are equal to the dihedral angles between the planes of the sides of the polygon.*

267. Scholium. Since the sides and angles of a spherical polygon are measured by the face and dihedral angles of the polyhedral angle corresponding to the polygon, *we may from any property of polyhedral angles, infer an analogous property of spherical polygons.*

268. Cor. 4. *Each side of a spherical triangle is less than the sum of the other two,* § 91.

269. Cor. 5. *Any side of a polygon is less than the sum of the other sides.*

270. Cor. 6. *The sum of the sides of a spherical polygon is less than 360°,* § 92.

271. Cor. 7. *Two mutually equilateral triangles on equal spheres are mutually equiangular, and are equal or symmetrical,* § 93, *and equivalent,* § 262.

272. Cor. 8. *In an isosceles spherical triangle, the angles opposite the equal sides are equal.*

For, from the intersection of the = sides draw an arc to the middle point of the base. This divides the isosceles triangle into two mutually equilateral Sph. △s. Hence, by Cor. 7, they are mutually equiangular, and the Sph. ∠s at the base are equal, Q. E. D.

273. Cor. 9. *The arc of a great circle drawn from the vertex of an isosceles triangle to the middle of the base, bisects the vertical angle, is perpendicular to the base, and divides the triangle into two symmetrical triangles.*

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177. If from a point perpendiculars be dropped to other lines all of which pass through a common point, the feet of the perpendiculars will lie upon the surface of a sphere.

PROPOSITION VIII. THEOREM.

X **274.** *The shortest distance on the surface of a sphere between any two points on the surface, is the shorter arc of a great circle joining them.*

Proof. Let two points be joined by an arc of a great circle, and by some other line on the surface of the sphere. The second line may be considered as made up of a large number of very small arcs of great circles. These, with the arc joining the two points, form a polygon. By § 269, the arc of the great circle is less than the sum of the other sides of the polygon, that is, less than any other line joining the two points. Q. E. D.

PROPOSITION IX. THEOREM.

X **275.** *Two triangles on equal spheres, having two sides and the included angle of one equal respectively to two sides and the included angle of the other, are either equal or equivalent.*

Proof. By superposing one Sph. △ upon the other, or upon its symmetrical, the theorem can be proved exactly as in the case of plane △s. Q. E. D.

PROPOSITION. THEOREM.

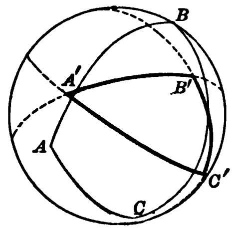
X **276.** *Two triangles on equal spheres, having a side and the two adjacent angles of the one equal respectively to the side and the two adjacent angles of the other, are either equal or symmetrical.*

Proof. One of the Sph. △s, or the Sph. △ symmetrical with it, can be superposed upon the other, as in the corresponding case of plane △s. Q. E. D.

Proposition X. Theorem.

277. *If one spherical triangle is the polar of another, then, reciprocally, the second triangle is the polar of the first.*

Notation. Let Sph. △ $A'B'C'$ be the polar of Sph. △ ABC.



To prove Sph. △ ABC the polar of Sph. △ $A'B'C'$.

Analysis. Since B is the pole of $A'C'$, the vertex A' lies at a quadrant's distance from B, § 243. And since C is the pole of $A'B'$, A' lies at a quadrant's distance from C.

∴ A' is the pole of BC, § 248.

Similarly B' is the pole of AC, and C' the pole of AB.

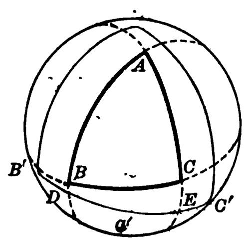
∴ Sph. △ ABC is the polar of Sph. △ $A'B'C'$, § 261.

Q. E. D.

Proposition XI. Theorem.

278. *In two polar triangles, each angle of one is the supplement of the side lying opposite it in the other.*

Notation. Let a, b, c and a', b', c' denote the sides, and A, B, C and A', B' C' the opposite angles in the two polar triangles.



To prove $A + a' = 180°$, $B + b' = 180°$, $A' + a = 180°$, etc.

Construction. Produce the arcs AB and AC until they meet $B'C'$ at D and E.

Analysis. By § 263,

$$\text{Sph. } \angle A = \text{arc } DE.$$

By § 277, $B'E = DC' = 90°.$

$\therefore B'D + DE + EC' + DE = B'C' + DE = a' + DE = 180°.$

But $DE = \text{Sph. } \angle A. \quad \therefore a' + A = 180°.$ Q. E. D.

Similarly, all the other relations may be proved.

279. Scholium. Polar triangles are sometimes called *supplementary* triangles.

PROPOSITION XII. THEOREM.

280. *The sum of the angles of a spherical triangle is greater than two, and less than six, right angles.*

Notation. Same as in § 278.

To prove $A + B + C > 180°$ and $< 540°.$

Proof. $A = 180° - a', B = 180° - b', C = 180° - c'.$

Adding, we get $A + B + C = 540° - (a' + b' + c').$

But, by § 270, $a' + b' + c' < 360°$ and $> 0°.$

$\therefore A + B + C > 180°$ and $< 540°.$ Q. E. D.

281. Cor. *A spherical triangle may have one, two or three obtuse or smaller angles.*

282. Definition. A **bi-rectangular** or **tri-rectangular** spherical triangle is one that has two and three right angles, respectively.

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178. The planes of intersection of three intersecting spheres intersect in one line.

179. If the sides of a spherical triangle are 80°, 105°, 110°, what are angles of its polar?

Proposition XIII. Theorem.

283. *Two mutually equiangular triangles upon equal spheres are mutually equilateral, and are either equal or ~~equivalent~~.*

Notation. Let A and B be two mutually equiangular Sph. $\triangle$s, and P and Q their polar triangles.

To prove Sides of $A =$ sides of B.

Proof. By hypothesis, angles of $A =$ angles of B.

$\therefore$ § 278, supplements of *sides* of $P =$ supplements of *sides* of Q.

$\therefore$ sides of $P =$ sides of Q.

$\therefore$ § 271, angles of $P =$ angles of Q.

$\therefore$ § 278, supplements of sides of $A =$ supplements of sides of B.

$\therefore$ sides of $A =$ sides of B. Q. E. D.

By §§ 275, 276 they are either equal or ~~equivalent~~.

Q. E. D.

284. Cor. 1. *If two angles of a spherical triangle are equal, the triangle is isosceles.*

285. Cor. 2. *If three mutually perpendicular planes meet at the centre of a sphere, they divide the sphere surface into eight equal tri-rectangular triangles.* § 283.

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180. If the angles of a spherical triangle are 72°, 89°, 122°, find the sides of its polar.

181. Show that the sum of the angles of a spherical pentagon is greater than six, and less than ten, right angles.

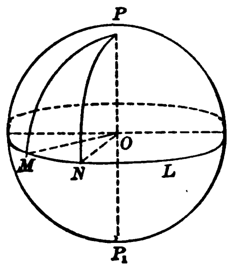
182. If the sides of a spherical triangle are respectively 62°, 87°, and 114°, how many degrees are there in each angle of its polar triangle.

183. Through a given point on a sphere, to draw a great circle tangent to a given small circle.

Proposition XIV. Theorem.

286. *In a bi-rectangular spherical triangle, the sides opposite the right angles are quadrants.*

Notation. Let MNP be a bi-rectangular Sph. $\triangle$, right angled at M and N.



To prove $MP = NP = 90°$.

Proof. The planes MOP and NOP are each $\perp$ to the plane MNO. $\therefore$ their intersection OP is $\perp$ to MNO. § 70.

$\therefore$ P is the pole of MN, § 225.

$\therefore$ $MP = NP = 90°$. Q. E. D.

287. Cor. 1. *If two sides of a triangle are quadrants, the opposite angles are right angles.*

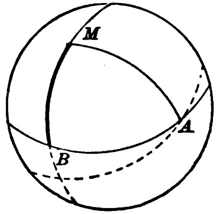
288. Cor. 2. *In a bi-rectangular triangle the third angle and its opposite side are equal,* § 263.

289. Cor. 3. *Each side of a tri-rectangular spherical triangle is a quadrant,* § 263.

Problem.

290. *Through a given point to drop an arc perpendicular to a given arc.*

Notation. Let M be the given point and AB the given arc.

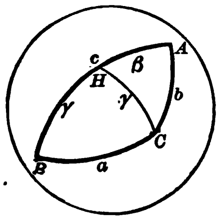


Construction. From M as a pole describe an arc of a great circle cutting the giving arc in A. From A as a pole describe the arc MB through M. Since $AM = AB = 90°$, Sph. $\triangle$ AMB

is a bi-rectangular Sph. $\triangle$, § 287, and the Sph. $\angle B$ is a rt. Sph. $\angle$. Q. E. F.

Proposition XV. Theorem.

291. *In a spherical triangle, the greater side is opposite the greater angle, and conversely.*



(1.) **Notation.** Let Sph. $\triangle ABC$ be a Sph. $\triangle$ having Sph. $\angle C >$ Sph. $\angle B$.

To prove $c > b$.

Construction. From C draw the arc CH making Sph. $\angle BCH =$ Sph. $\angle B$, and denote the segments of the side c by γ and β.

Proof. By § 284, $HC = \gamma$. But by § 268, $HC + \beta = \gamma + \beta > b$. $\therefore c > b$. Q. E. D.

(2.) *Hypothesis.* Suppose $c > b$.

To prove Sph. $\angle C >$ Sph. $\angle B$.

Proof. If Sph. $\angle C =$ Sph. $\angle B$. $\therefore c = b$, § 284, which is contrary to hypothesis.

If Sph. $\angle C <$ Sph. $\angle B$, $\therefore c < b$ (proved above) which is also contrary to hypothesis.

$\therefore$ Sph. $\angle C >$ Sph. $\angle B$. Q. E. D.

Spherical Surfaces. Definitions.

292. A **zone** is the portion of the surface of a sphere included between two parallel planes. The circumferences of the sections made by the planes, are called the **bases** of the zone, and the distance between the planes is its **altitude**.

293. When one of the planes is tangent to the sphere, we have a **zone of one base.** When a sphere is generated by the revolution of a semi-circle, any arc of the semi-circle generates a zone. A sphere is one limiting case of a zone, as a circle is the other.

294. A **lune** is the portion of the surface of a sphere bounded by two semi-circumferences of great circles. The limiting cases are a sphere and a semi-circle. A lune may be considered as a spherical triangle, one of whose sides is zero. The two opposite lunes formed by two intersecting great circles may be considered as two symmetrical triangles and therefore equal.

295. The **angle** of a lune is the angle between the arcs which form its boundaries.

296. A lune may be considered as generated by the revolution of a semi-circle about its diameter as an axis. Hence two lunes, generated by the same semi-circumference, are to each other as their angles.

297. Lunes having the same angle, but upon different spheres are called **similar lunes.**

298. A **spherical polygon** has already been defined, § 251.

The fundamental spherical surface is the spherical polygon, special cases of which are the triangle (polygon of three sides), lune (polygon of two sides), and sphere surface (lune of 360°).

Hence, an enlarged definition of **spherical polygon** would be, a portion of the surface of a sphere bounded by two or more arcs of great circles.

The limiting cases of a spherical polygon are the half sphere surface, a semi-circumference and a point.

Special cases are the tri-rectangular triangle and the hemisphere.

299. As a quadrant is divided into 90 parts called angular degrees, so for convenience a tri-rectangular triangle is divided into 90 semi-lunes called **spherical degrees.** There are 720 spherical degrees on the whole surface of the sphere.

A spherical degree is not angular measure but surface measure.* Its typical shape is a semi-lune of 1°. It is denoted by σ.

300. The **spherical excess of a spherical polygon** is the excess of the sum of its angles over what it would have been if the polygon had been plane, i. e., over a straight angle taken as many times as the polygon has sides less two. It is denoted by E.

301. The **spherical excess of a spherical triangle** is the excess of the sum of its angles over what the sum would have been if the triangle had been plane, i. e., the excess of the sum of its angles over a straight angle. It is denoted by ϵ. It varies from 0 to a round angle. 360°.

301½. The spherical excess of any spherical surface bounded by arcs of great circles, is the excess of the sum of its angles over what that sum would have been if the arcs had been straight lines.

This applies to lunes, triangles, polygons and sphere surfaces. The spherical excess of a lune is twice its angle; that is, we may say that as the correlative of the spherical triangle is the plane triangle with the sum of the angles equal to a straight angle, so the correlative of the lune is two coincident straight lines with the sum of the angles equal to zero. Since a sphere is a lune of 360°, the spherical excess of a sphere expressed in angular degrees is 720°.

* It must be impressed upon the mind of the student, that a spherical degree is *surface*, and not angular measure.

PROPOSITION XVI. THEOREM.

302. *The area of a lune in spherical degrees is equal to twice the angle of the lune expressed in angular degrees.*

Notation. Let $A°$ denote the angle of the lune L.

To prove Sph. $L = 2A°\sigma$.

Proof. Since two lunes generated by the same semicircumference are to each other as their angles, § 296, so two semi-lunes are to each other as their angles.

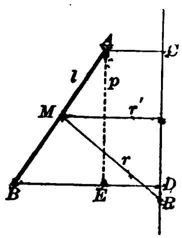
The spherical degree is a semi-lune of 1°.

$\therefore \frac{1}{2}$ Sph. $L : \sigma = A° : 1°$, or Sph. $L = 2A°\sigma$. Q. E. D.

PROPOSITION XVII. THEOREM.

303. *The area of the surface generated by a straight line revolving about an axis in its plane is equal to the product of the projection of the line on the axis by the circumference whose radius is a perpendicular erected at the middle point of the line and terminated by the axis.*

Notation. Let l be the length of the generating line AB, and $p = CD$ its projection on the axis, r' = the radius of the circumference described by its middle point, and $r = MR$, the length of the perpendicular.



To prove area $AB = p \cdot 2\pi r$.

Construction. Draw $AE \parallel$ to CD.

Proof. Area $AB = l \cdot 2\pi r'$, § 155.

In the similar triangles ABE, rr',

$$r' : p = r : l. \quad \therefore pr = r'l.$$

Substituting this in the above equation, we get

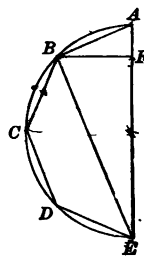
$$\text{area } AB = p \cdot 2\pi r. \qquad \text{Q. E. D.}$$

304. Scholium. The surface described by AB is a frustum of a cone of revolution. We get the limiting cases, a cylinder or a cone, according as AB is $\parallel$ to or touches the axis. In either case, this formula can be reduced to those of §§ 151, 156.

PROPOSITION XVIII. THEOREM.

305. *The surface of a sphere is equal to four great circles.*

Notation. Let $ABCDE$ be the semicircle which generates a sphere, and $ABCDE$ a regular semi-polygon inscribed therein.



To prove Sph. $S = 4\pi r^2$.

Proof. The semi-polygon generates a number of frustums of cones of revolution, the lateral surface of each of which = the projection of its slant height $\times 2\pi \cdot$ the apothem of the polygon, § 303. The sum of all the lateral areas = the apothem $\times 2\pi \cdot$ the sum of all the projections of the slant heights = the apothem $\times 2\pi \cdot AE$.

When the number of sides of the polygon is indefinitely increased, the apothem becomes the radius of the circle; the lateral area becomes Sph. S, and our formula becomes

$$\text{Sph. } S = r \cdot 2\pi \cdot 2r = 4\pi r^2. \qquad \text{Q. E. D.}$$

306. COR. 1. *In square units*, Sph. $L = \dfrac{\pi r^2 A^\circ}{90}$.

For $$\sigma = \frac{\text{Sph. } S}{720} = \frac{4\pi r^2}{720} = \frac{\pi r^2}{180}.$$

307. Cor. 2. *Similar lunes are to each other as the squares of the radii of the spheres upon which they are situated.* Compare § 206.

308. Cor. 3. *The surfaces of two spheres are to each other as the squares of their radii, or as the squares of their diameters.*

309. Cor. 4. *The area of a zone is equal to the product of its altitude by the circumference of a great circle* $= 2\pi r \cdot h$.

This follows by applying the reasoning of § 305 to the arc BC.

310. Cor. 5. *Zones on equal spheres are to each other as their altitudes.*

311. Cor. 6. *The area of a zone of one base is equal to the area of the circle whose radius is the chord of the generating arc.*

The zone generated by the arc $AB = AF \times 2\pi r = \pi \cdot AF \cdot AE = \pi \cdot AB^2$, since in the similar $\triangle$s ABE and ABF, $AF : AB = AB : AE$.

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184. Find the surface of a sphere if the diameter is 4.

185. Find the surface of a sphere if the diameter is 21.

186. Find the diameter of a sphere if the surface is 9856.

187. What is the numerical value of the radius of a sphere if its surface has the same numerical value as the circumference of a great circle?

188. Find the surface of a lune if its angle is $37\frac{1}{2}°$ and the total surface of the sphere is 6.

189. Two angles of a spherical triangle are 40° and 50°. What do we know about the third angle?

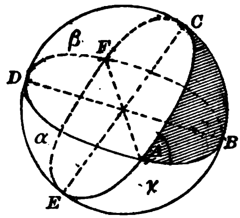
190. A lune of 40° on a sphere of 6 feet radius has how many square feet?

191. Lunes of 50° upon spheres of 2 and 3 feet radius bear what ratio to each other?

PROPOSITION XIX. THEOREM.

312. *The area of a spherical triangle, expressed in spherical degrees, is equal to its spherical excess expressed in angular degrees.*

Notation. Let A, B, C, denote the values of the angles of the Sph. $\triangle$ ABC expressed in degrees.



To prove Sph. $\triangle$ $ABC = \epsilon^\circ \cdot \sigma$.

Construction. Produce the sides of the Sph. $\triangle$ to complete circles. This gives us three lunes, $ABCF$, $ABCD$ and $ABCE$, the given Sph. $\triangle$ being common to all of them. Denote the $\triangle$s ADE, ACD and ABE by a, β, γ, respectively.

The Sph. $\triangle$s BCF and a being symmetrical, are $=$, § 262.

$\therefore$ lune $ABCF \simeq$ Sph. $\triangle + a$, Sph. $\triangle$ denoting the Sph. $\triangle$ ABC.

$$\text{Hence, lune } ABCF = 2\,A^\circ \cdot \sigma = \text{Sph. } \triangle + a.$$

$$\text{lune } ABCD = 2\,B^\circ \cdot \sigma = \text{Sph. } \triangle + \beta.$$

$$\text{lune } ABCE = 2\,C^\circ \cdot \sigma = \text{Sph. } \triangle + \gamma.$$

Adding these, we get

$$2\,(A + B + C)\,\sigma = 2 \text{ Sph. } \triangle + \text{Sph. } \triangle + a + \beta + \gamma.$$

$$= 2 \text{ Sph. } \triangle + 360\,\sigma. \qquad \text{§ 299.}$$

$$\therefore \text{Sph. } \triangle = (A + B + C - 180)\,\sigma = \epsilon^\circ \cdot \sigma.$$

Q. E. D.

313. COR. 1. *The area of a spherical triangle is to the surface of a sphere as its spherical excess in degrees is to 720°.*

314. Cor. 2. If Sph. $\triangle$ is required in the usual units of area, as square inches, etc., we get by substituting the value of σ,

$$\text{Sph.}\ \triangle = \epsilon^\circ \cdot \sigma = \epsilon^\circ \frac{4\pi r^2}{720} = \frac{\pi r^2 \epsilon^\circ}{180}.$$

315. Cor. 3. *The area of a spherical polygon, expressed in spherical degrees, is equal to its spherical excess expressed in angular degrees.*

If n denote the number of sides of the polygon, it can be divided into $(n-2)$ Sph. $\triangle$s. If ϵ, ϵ_1, ϵ_2 etc., be the spherical excess of the different Sph. $\triangle$s, we will have

$$\text{Sph. polyg.} = (\epsilon + \epsilon_1 + \epsilon_2 + \ldots)\,\sigma.$$

But $\epsilon + \epsilon_1 + \epsilon_2 + \ldots\ldots$ is evidently equal to the sum of the angles of the polygon less as many straight angles as the polygon has sides less two, i. e. $= E$.

$$\therefore \text{Sph. polyg.} = E^\circ \sigma. \qquad \text{Q. E. D.}$$

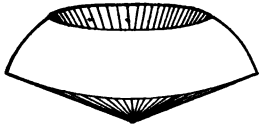
315½. *The area in spherical degrees of any spherical surface bounded by arcs of great circles is equal to its spherical excess expressed in angular degrees.*

Compare §§ 300, 301½, 302, 312, 315.

Spherical Volumes.

316. A **spherical pyramid** has already been defined in § 255. The limiting cases are the sphere, a radius, and a semicircle. Compare § 298.

317. A **spherical sector** is the portion of a sphere generated by the revolution of a circular sector about any diameter of its circle. The limiting cases are the sphere and a conical surface.



318. Its **base** is the zone generated by the arc of the spherical sector.

319. If the sector revolves about its bounding radius as an axis, it generates what may be called a **spherical cone.** A spherical cone may be considered as a spherical pyramid with a regular base of an infinite number of sides.

320. A **spherical segment** is the portion of a sphere contained between two parallel planes. Its **bases** are the sections made by the planes; and its **altitude** is the distance between the bases.

321. If one of the planes is tangent to the sphere the segment is called a **segment of one base.**

The sphere is a segment whose altitude is $2r$.

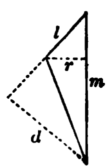
322. A **spherical ungula or wedge** is a portion of a sphere bounded by a lune and two great semicircles.

The **angle** of the wedge is the angle of the lune forming its base.

The sphere is a wedge of 360°.

Proposition XX. Theorem.

323. *The volume generated by a triangle revolving about one of its sides is equal to the area generated by its base multiplied by one-third its altitude.*



Notation. Let m be the side about which the triangle revolves, l its base, d its altitude, and r the radius described by its vertex, **V** the volume generated.

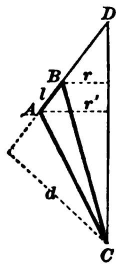
To prove $\mathbf{V} = \frac{1}{3}\, d \cdot \pi r l$. § 156.

Proof. By § 183, $\mathbf{V} = \frac{1}{3}\pi r^2 . m.$ In the similar rt. $\triangle$s lr and dm we have $rm = dl$.

$$\therefore \mathbf{V} = \frac{1}{3}\pi r \cdot dl = \frac{1}{3} d \cdot \pi rl.$$ Q. E. D.

324. Cor. 1. *The volume generated by any triangle revolving about a line passing through its vertex is equal to the area generated by the base multiplied by one-third the altitude.*

Notation. Let ABC be the given $\triangle$ and CD the axis about which it revolves, d the altitude and $l = AB$, its base, r = the radius of the circle described by B, and r' = the radius described by A.



To prove $\mathbf{V} = \frac{1}{3} d \cdot \pi (r + r') l.$ § 155.

Proof. Vol. ABC = vol. ADC − vol. BDC $= \frac{1}{3} d \cdot \text{area } AD - \frac{1}{3} d \cdot \text{area } BD = \frac{1}{3} d \cdot \text{area } AB$ $= \frac{1}{3} d \cdot \pi (r + r') l.$ § 155. Q. E. D.

325. Cor. 2. *The volume of a double cone is equal to the lateral surface of one cone multiplied by one-third the perpendicular let fall from the vertex of the second cone to the side of the first.*

Proposition XXI. Theorem.

326. *The volume of a sphere is equal to its surface multiplied by one-third its radius.*

Notation. Let Sph. $\mathbf{V}$ denote the volume of the sphere, Sph. S the area of its surface, and r its radius.

To prove Sph. $\mathbf{V}$ = Sph. $S \cdot \frac{1}{3} r.$

Proof. Conceive a polyhedron circumscribed about the sphere. If pyramids are formed having the faces of the polyhedron as bases and the centre of the sphere as a common vertex, these pyramids will have a common altitude equal to the radius of the sphere.

Then each pyramid = face $\times \frac{1}{3} r$.

$\therefore$ sum of pyramids = sum of faces $\times \frac{1}{3} r$. But at the limit, the sum of the pyramids = Sph. **V**, the sum of the faces = Sph. *S*.

$\therefore$ Sph. **V** = Sph. $S \cdot \frac{1}{3} r$. Q. E. D.

327. Scholium. The sphere might be considered as generated by the revolution of isosceles triangles about a common vertex, whence by § 323, Sph. **V** = $\frac{1}{3} r \cdot$ Sph. *S*.

328. Cor. 1. *The volume of a spherical pyramid (cone) is equal to the area of its base multiplied by one third the radius of the sphere.*

329. Cor. 2. *The volume of a spherical sector is equal to the area of the zone which forms its base multiplied by one third the radius of the sphere,* $= \frac{1}{3} r \cdot 2 \pi r h = \frac{2}{3} \pi r^2 h$.

For the spherical sector may be considered as the difference of two spherical cones, and its base as the difference of two one-based zones.

330. Cor. 3. *Spherical sectors on equal spheres are to each other as the zones which form their bases, or as the altitudes of those zones,* § 310.

331. Cor. 4.

The volume of a sphere $= \frac{4}{3} \pi r^3 = \frac{1}{6} \pi \cdot (2 r)^3$.

For Sph. $S = 4 \pi r^2$,

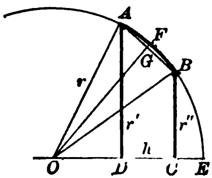
$\therefore$ Sph. **V** $= \frac{1}{3} r$ Sph. $S = \frac{4}{3} \pi r^3$.

332. Cor. 5. *The volumes of two spheres are to each other as the cubes of their radii.* Compare § 206.

333. Cor. 6. *The volume of a sphere is two thirds the circumscribing cylinder,* since the cylinder $= \pi r^2 \cdot 2 r = 2 \pi r^3$.

PROPOSITION XXII. THEOREM.

334. *The volume of a spherical segment is equal to the sum of two cylinders and a sphere, the altitudes of the cylinders being one half the altitude of the segment, and their bases the upper and lower bases of the segment, respectively, and the diameter of the sphere being the altitude of the segment.*



Notation. Let $ABCD$ be the plane semi-segment whose revolution about DC as an axis generates the spherical segment. Let $h = DC$, $h' = DE$, $h'' = CE$, $r' = AD$, $r'' = BC$, $r = OE$.

To prove

$$\text{segment } ABCD = \mathbf{V} = \pi r'^2 \frac{h}{2} + \pi r''^2 \frac{h}{2} + \tfrac{1}{6} \pi h^3.$$

Analysis. The segment $AFBCD =$ sector $OAFB$ — vol. $AOB +$ frustum $ABCD$.

Proof. By §§ 329, 324, 303,

$$\text{vol. } GAFB = \tfrac{2}{3} \pi r^2 h - \tfrac{2}{3} \pi\, OG^2 h,$$

$$= \tfrac{2}{3} \pi h \left[r^2 - OG^2 = AG^2 = \tfrac{1}{4} AB^2 = \frac{h^2 + (r' - r'')^2}{4} \right],$$

$$= \tfrac{1}{6} \pi h \left[(r' - r'')^2 + h^2 \right].$$

By § 194½, frustum $ABCD = \tfrac{1}{3} \pi h \,(r'^2 + r''^2 + r'r'')$.

$$\therefore \text{vol } AFBCD = \tfrac{1}{6} \pi h \left[(r' - r'')^2 + h^2 \right]$$

$$+ \tfrac{1}{3} \pi h \,(r'^2 + r''^2 + r'r''),$$

$$= \tfrac{1}{6} \pi h \,(r'^2 - 2 r'r'' + r''^2 + h^2 + 2 r'^2 + 2 r''^2 + 2 r'r''),$$

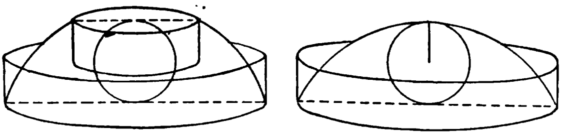
$$= \tfrac{1}{6} \pi h \,(3 r'^2 + 3 r''^2 + h^2),$$

$$= \pi r'^2 \frac{h}{2} + \pi r''^2 \frac{h}{2} + \tfrac{1}{6} \pi h^3.$$

Q. E. D.

335. Cor. 1. If the segment has one base only, $r''=0$, the second cylinder becomes zero, and therefore disappears from the equation, and the formula becomes

$$V=\pi r'^2\frac{h}{2}+\tfrac{1}{6}\pi h^3.$$



336. A second unit of angular measure is the *radian*, or the arc whose length is equal to the radius. Since a circumference $=360°$, 2π radians $=360°$, and

$$1 \text{ radian}=\frac{360°}{2\pi}=57°.3.$$

The radian is designated by ρ.

Evidently $\rho=\dfrac{180°}{\pi}$, and $1°=\dfrac{\pi}{180}\rho$.

Hence, *to convert angular degrees into radians, multiply by* $\dfrac{\pi}{180}$.

This rule can be used to convert our formulae for spherical area from spherical degrees to square radii.

Example: Sph. $L=\dfrac{\pi r^2}{90}A°$.

Multiplying by $\dfrac{\pi}{180}$ to convert the angular degrees into radians, and dividing by $\dfrac{\pi}{180}$ so as not to alter the value of the expression, we get

$$\text{Sph. } L=\frac{\pi r^2}{90}A°=\frac{\pi r^2}{90}\cdot\frac{A°\pi}{180}\cdot\frac{180}{\pi}$$

$$=\frac{\pi r^2}{90}\cdot A^\rho\cdot\frac{180}{\pi}=2r^2A^\rho.$$

Sph. S, Sph. $\triangle$, and Sph. P can be transformed in the same manner. This is left as an exercise for the student.

337. In § 299, we adopted as the unit of spherical surface measure, the spherical degree.

Another unit is the squared radius, or the surface equal to a square radius.

By § 301½, the spherical excess of a sphere surface expressed in radians is 4π. By § 305, its area equals $4\pi r^2$.

Hence, *The area of a sphere surface expressed in squared radii equals the spherical excess expressed in radians.*

This is a special case of the general

Theorem: *The area of any spherical surface bounded by arcs of great circles expressed in square radii equals the spherical excess expressed in radians.* Compare §§ 338, 339.

338. By § 296, since the sphere may be considered as a lune, Sph. $L : (\text{Sph. } S = 4\pi r^2) = A^\rho : 2\pi$,

whence Sph. $L = 2A^\rho r^2$. Compare § 302.

339. By using the formula of § 338 in § 312, we easily get Sph. $\triangle = \epsilon^\rho \cdot r^2$.

NOTATION.

ϵ = spherical excess of spherical triangle.
E = spherical excess of spherical polygon.
σ = area of as pherical degree = $\frac{1}{720}$ Sph. S = $\frac{1}{90}$ Sph. T.
Sph. S = area of a sphere = $720\,\sigma$.
Sph. T = area of a tri-rectangular triangle = $90\,\sigma$.
Sph. L = area of a lune.
Sph. $\triangle$ = spherical triangle.
r = radius.
h = altitude.
Sph. P = area of spherical polygon.
Sph. Z = zone.
A = spherical angle.
Sph. $\mathbf{V}$ = spherical volume.

FORMULAE.

Sph. $L = 2\,A^\circ . \sigma$ . . . . § 302.

$= \dfrac{\pi r^2 A^\circ}{90}$ . . . . § 306.

$= 2\,A^\rho \cdot r^2$ . . . . § 338.

Sph. $S = 4\pi r^2$ . . . . § 305.

Sph. $\triangle = \epsilon^\circ \cdot \sigma$ . . . . § 312.

$= \dfrac{\pi r^2 \cdot \epsilon^\circ}{180}$ . . . . § 314.

$= \epsilon^\rho \cdot r^2$ . . . . § 339.

Sph. $P = E^\circ \cdot \sigma$ . . . . § 315.

$= E^\rho \cdot r^2$ . . . . § 336.

Sph. $\mathbf{V} = \frac{1}{3}\,r \cdot$ Sph. S . . . . § 326.

$= \frac{4}{3}\,\pi r^3 = \frac{1}{6}\,\pi\,(2r = d)^3$ . . . . § 331.

Area generated by revolving straight line $= p \cdot 2\pi r$ . . § 303.

Volume generated by revolving $\triangle = \frac{1}{3}\,d \cdot \pi r l$ . . . . § 323.

$= \frac{1}{3}\,d \cdot \pi\,(r + r')\,l$. § 324.

Volume of a double cone = ... § 325.

Volume of a spherical pyramid $= \frac{1}{3}\,r \cdot$ Sph. Z . . . . § 328.

Volume of a spherical sector $= \frac{1}{3}\,r \cdot$ Sph. $Z = \frac{2}{3}\,\pi r^2 h$ . . § 329.

Volume of a spherical segment $= \pi r'^2 \frac{h}{2} + \pi r''^2 \frac{h}{2} + \frac{1}{6}\,\pi h^3$ § 334.

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192. The angles of a spherical triangle are 60°, 70° and 80°. The radius of the sphere is 56. What is the area of the triangle in square feet?

193. The sides of a spherical triangle are 42°, 82° and 116°. What is the area of its polar triangle?

194. In the last example what is the area of the polar in square radii? In square feet, if the radius is 6 feet?

195. A tin pan 1 foot high, with a spherical bottom, has its sides flared out 60°. The bottom measures 2 feet across. How much tin, neglecting the seams, does it take to make it?

196. What is the area of a zone of one base whose height is h, and the radius of whose base is r? What would be the area if the height were three times as great? If $\frac{1}{2}$ as great?

197. A globe 10 inches in diameter is how much larger than a marble 1 inch in diameter?

198. The slant height of a cone of revolution is equal to the diameter of its base: its total area is 27. What is the area of the inscribed sphere?

199. If an equilateral (slant height equal to diameter of the base) cylinder and cone be inscribed in a sphere, what is the relation subsisting between the total areas of the cylinder, the cone and the sphere?

200. In the last example what is the relation between the volumes?

201. In the last example, suppose the sphere inscribed in the cone.

202. A right circular cone whose vertical angle is 60°, is circumscribed about a sphere. Compare the lateral area of the cone and of the sphere.

203. In the last example, compare the volumes.

204. A sphere is cut by 3 planes 6 inches apart, the outside one being tangent to the sphere. The radius of the sphere is 13 inches. Compare the spherical surfaces included between the planes.

205. A spherical bowl is 14 inches across the top. A pencil 8 inches long just reaches from the edge to the center. What is the surface of the bowl?

206. Having the volume V of a sphere, find the volume of the inscribed cube.

207. Having the volume V of a sphere, find the volume of the circumscribed cube.

208. The two legs of a right triangle are a and b: find the area of the surface generated when the triangle revolves about its hypotenuse.

209. Find the volume generated in the last example.

210. The lateral area of a cone of revolution is double the area of its base: its altitude is 4: find the radius of the base.

211. Find the volume of a spherical pyramid the angles of whose base are 80°, 90° and 130°, the volume of the sphere being 120.

212. Find the area of a spherical pentagon whose angles are 138°, 112°, 131°, 168° and 153°, the surface of the sphere being 40.

213. A cylinder of height h is inscribed in a sphere. What is the volume of the sphere lying outside its convex surface?

214. A cone is circumscribed about a sphere, its altitude being $\frac{4}{3}$ times the radius of the sphere. If the surface of the sphere is 120, what is the total surface of the cone?

215. In the last example, if the volume of the sphere is 16 what is the volume of the cone?

216. A circular sector of 60° revolves about one side, the radius being 12, generating a spherical sector. What is the volume necessary to complete the hemisphere?

217. Find the surface of a sphere inscribed in a cube whose surface is 729.

218. The volume of a sphere is 226: find its diameter.

219. Find the surface of a sphere circumscribing a regular tetrahedron whose edge is 6.

220. What is the surface of a sphere inscribed in a cube whose surface is 48?

221. Find the volume of a spherical segment, the radii of whose bases are 3 and 5, and whose altitude is 4.

222. How much of the earth's surface would a man see if he were raised to the height of a diameter above it?

223. What is the volume of a wedge of 42° in a sphere of 5 feet radius?

224. What is the angle of a spherical wedge if its volume is 3, and the volume of the sphere is 15?

225. What is the volume of a spherical sector, if its basal zone is 5, and the radius of the sphere is 6?

226. The radii of the bases of a spherical segment are 6 and 8 feet, and its height is 3 feet: what is its volume?

227. Find the area of a spherical polygon whose angles are 100°, 120°, 140° and 160°, the radius of the sphere being $10\frac{1}{2}$.

228. A plane cuts a sphere of radius 5 feet, 3 feet from the centre. What is the area of the section?

229. In a sphere whose radius is r, find the height of a zone whose area is equal to that of a great circle.

230. Find the area of the zone illuminated by a candle h feet from the surface, the radius being r feet.

231. Find the volume V of a sphere in terms of the circumference c of a great circle.

232. Having the volume V of a sphere, what is the radius?

233. A plane bisects the radius of a sphere at right angles. What is the ratio of the two zones formed?

234. A cone has for its base the base of a hemisphere, and its vertex at the pole of the base. Compare the volumes of the hemisphere and cone.

235. The surface and volume of a sphere are expressed by the same number. What is its radius?

236. A zone of one base is a mean proportional between the remaining surface of the sphere and the total sphere surface. How far is the plane of its base from the centre?

237. Find the volume of a sphere circumscribed about a cube whose volume is 216.

238. Find the volume of a sphere circumscribed about a cube whose volume is V.

239. To what height must a man be raised above the earth in order to see one-fourth of its surface?

240. To what height must a man be raised above the earth in order to see $\frac{1}{n}$ of its surface?

241. A cone of revolution the radius of whose base is 12, is inscribed in a sphere of radius 20; what is the volume of the cone?

242. A cone of revolution the radius of whose base is a is inscribed in a sphere of radius r; what is the volume of the cone?

243. Find the spherical excess of a triangle from its area and the radius.

244. A section parallel to the base of a hemisphere bisects its altitude: what is the ratio of the two volumes?

245. A segment of one base, 4 feet high contains 150 cubic feet: what is the radius of the sphere?

246. Find the radius of a sphere equal to the sum of two spheres whose radii are 3 and 6.

247. Having the volume V and the height h, of a spherical segment of one base, find the radius of the sphere.

248. The radius of the base of a segment is 16 feet and its height 8 feet: what is its volume?

249. The edge of a spherical pyramid is 3.85 and the angles of its base are 80°, 100°, 120° and 150°: what is its volume?

250. If the volume of a sphere is V, and the area of a zone is S, what is its height?

251. If the two bases of a segment are πr_1^2 and πr_2^2, and the ratio of their distances from the centre of the sphere is $m:n$, what is the radius of the sphere?

252. The sphere surface has what ratio to the entire surface of the circumscribing cylinder.

253. If a man is h feet from the surface of a globe what proportion of the surface will he see, the radius of the sphere being r?

254. A light h feet from the surface of a globe of radius r, illuminates how many square feet of the globe?

255. How far from the centre must a plane be passed to divide a hemisphere into two equal zones?

256. What is the area of a lune whose angle is 42°?

257. If two angles of a spherical triangle are right angles what relation does its area bear to the third angle?

258. The volume generated by a triangle revolving about one of its sides is equal to its area multiplied by the path described by its centre of gravity. § 323.

259. Ditto, about any line through its vertex. § 324.

260. Ditto, about any exterior line. § 194½.

261. Ditto, any area. (Theorem of Pappus.)

262. A circle, radius r, revolves about an exterior line at a distance d from its centre: find the volume generated.

263. The surface generated by a line revolving about an axis is equal to its length multiplied by the path described by its centre of gravity.

264. Ditto, the surface by two (or more) contiguous lines.

265. Ditto, any curve. (Theorem of Pappus.)

266. Find the surface in example 262.

CONIC SECTIONS.

PREFACE.

The subject is treated purely from the standpoint of a plane section of a conical surface, and the elementary properties of the conic section determined *in situ*, upon the conic surface. The existence of a constant ratio between the distances of any point of the section from a fixed point and a fixed line is shown at the outset, and the three varieties of curves shown to be but special cases of the general section.

Particular attention is paid to the special and limiting cases, found by varying the slope of the cutting plane, and the angle of the cone.

The attempt has been, not to give the student many specialized properties of the conic sections, but to give a broad and comprehensive view of the correlations of the three varieties, and of the ways in which they merge into each other.

The essential unity of the different curves, their subjection to the same law of generation and their possession of the same general properties is constantly impressed upon the mind of the student.

CHAPTER IV.

CONIC SECTIONS.

1. A **conic section** is the curve made by the intersection of a plane with the conical surface of a circular cone.

A cone of revolution is generally selected, and will be the one used in what follows.

2. There are three cases :—

The cutting plane may intersect one nappe only, cutting all the elements, in which case we call the curve an **ellipse**: or it may cut all the elements but one, to which it is of course parallel, in which case we call the curve a **parabola.**

3. The cutting plane may cut both nappes, in which case we call the curve a **hyperbola.**

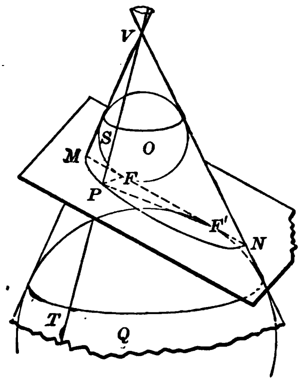
We will investigate the properties of these three curves in the order named.

PROPOSITION I. THEOREM.

4. *Every complete plane section of one nappe of a right circular cone is a curve, the sum of the distances of every point of which from two fixed points is constant.*

Notation. Let MPN be a plane section of one nappe of a right circular cone, passing through all the elements: O and Q the centres of two spheres inscribed in the cone, and tangent to the cutting plane at the points F and F'

respectively: P any point in the section, and S, T the points in which the element VP touches the spheres.



To prove $PF + PF' =$ constant.

Proof. $PS = PF$, being two tangents to a sphere from a common point. (S. G § 233.**)

Likewise $PF' = PT$.

$\therefore PF + PF'$

$= PS + PT =$ constant.

Q. E. D.

5. Scholium. The fixed points F, F' are called the **foci** of the curve, and the lines PF and PF' **focal radii**; the line MN passing through the foci is called the **focal axis** * of the curve, and the points MN are called **vertices.**

6. Cor. 1. *The sum of the focal radii of an ellipse is equal to the major axis.*

$$MF + MF' = 2\,MF + FF' = \text{constant}$$
$$= NF + NF' = 2\,NF' + FF'.$$
$$\therefore MF = NF'.$$
$$\therefore PF + PF' = MF + MF' = MF' + NF' = MN.$$

Q. E. D.

7. Cor. 2. *The foci are equidistant from the vertices.*

8. Scholium. The length of the major axis is denoted by $2\,a$. One half the distance between the foci, called the **linear eccentricity** of the curve, is denoted by c.

9. Cor. 3. *If the cutting plane is parallel to the base, the section is a circle.*

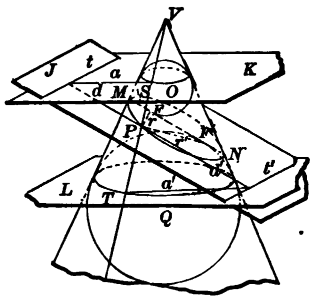
The circle is a special case of the ellipse.

* In the ellipse, also called major axis.

PROPOSITION II. THEOREM.

10. ***Every complete plane section of one nappe of a right circular cone is a curve, the ratio of the distances of whose points from a fixed point and a fixed line is constant.***

Notation. Same as in § 4. Also, let K and L be the planes determined by the circles of tangency of the cone and spheres, intersecting the cutting plane J in the two lines t, t'. Denote the distance of P from t by d, and from t' by d'. Through the lines ST and dd' pass a plane cutting K and L in the lines a, a', respectively.



To prove $\frac{r}{d} = \text{constant.}$

Proof. $a \parallel a'$ (S. G. § 41). $\therefore \triangle\, ad$ is similar to $\triangle\, a'd'$.

$$\therefore \frac{PS}{PT} = \frac{r}{r'} = \frac{d}{d'}.$$

By composition $$\frac{r}{r+r'} = \frac{d}{d+d'}.$$

$$\therefore \frac{r}{d} = \frac{r+r'}{d+d'} = m = \text{constant.} \qquad \S\ 4.$$

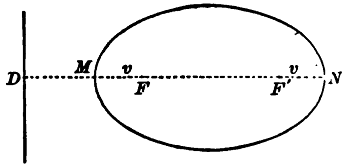
Q. E. D.

11. Scholium. A conic section which cuts all the elements of one nappe may be defined as a curve the ratio of the distances of whose points from a fixed point and a fixed line is constant. The fixed point is called the **focus**, and the fixed line is called the **directrix**.

12. COR. 1. *The directrix is perpendicular to the major axis.*

13. Cor. 2. *The constant ratio between the distances of a point on an ellipse from the focus and the directrix equals the linear eccentricity divided by the semi major axis.*

Notation. Let D be a point in the directrix and F, F' the foci of the conic whose vertices are M and N. Let $MF = NF' = v$, and $MD = \delta$.



To prove $m = \frac{r}{d} = \frac{c}{a}$.

Proof. For the vertex M

$$v = m\,\delta. \qquad \text{§ 10.}$$

For the vertex N

$$2c + v = m\,(2c + 2v + \delta).$$

Substituting the value of v in the first member, and the value of $2a = 2c + 2v$ in the second, we get

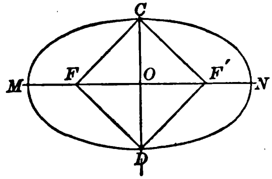
$$m = \frac{2c}{2a}.$$

Q. E. D.

14. Scholium. This ratio is called the **eccentricity** of the ellipse, and is denoted by e. Evidently $e < 1$.

15. Scholium. There will be two points, C and D, of the ellipse which are equidistant from the vertices. The line CD, connecting these two points, is called the **conjugate axis** (in the ellipse, minor axis), and is designated by $2b$.

In the rhombus $FCF'D$, the diagonals bisect each other at right angles. Hence the minor and major axis mutually bisect each other. The point of intersection O is called the **centre** of the ellipse.



16. Cor. 3. *An ellipse has two directrices.*

This follows since the above demonstrations apply equally well to the line t'.

17. Cor. 4. *The ellipse is symmetrical about either axis.*

This follows because § 4 applies to either focus, and §§ 10, 13 to either focus in connection with the corresponding directrix.

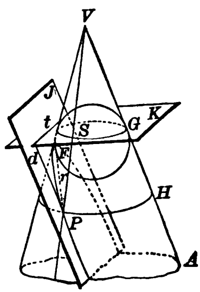
18. Cor. 5. *The eccentricity of a circle is zero.*

19. If the cutting plane be turned downward, the conic section will evidently become longer, until finally, when the cutting plane becomes parallel to one element of the cone, the ellipse has become infinitely long, and its centre is said to have passed out to infinity.

This special case of the ellipse is called a **parabola.**

Proposition III. Theorem.

20. *Every point of a parabola is equidistant from the focus and the directrix.*



Notation. Same as in § 10, except that J is ∥ to the element GH. Also, let PH be a section of the cone made by a plane ∥ to K.

To prove $r = d$.

Proof. $r = PS = GH = d$ (GH and d being ∥ lines between ∥ planes).

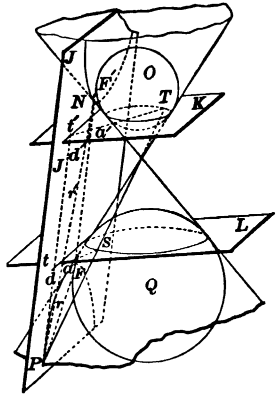
Q. E. D.

21. Cor. 1. *The eccentricity of the parabola equals unity.*

22. If the cutting plane be turned beyond the point of parallelism it must cut both nappes and we get a hyperbola.

PROPOSITION. THEOREM.

23. *Every plane section of both nappes of a right circular cone is a curve, the difference of the distances of whose points from two fixed points is constant.*



Notation. Same as in § 10, mutatis mutandis.

To prove $r' - r =$ constant.

Proof. $r' - r = PT - PS = ST =$ constant. Q. E. D.

24. Scholium. The remarks of § 5 apply to this section without change.

25. COR. 1. *The difference of the focal radii of a hyperbola is equal to the major axis.*

The proof is the same as for § 6, mutatis mutandis. This is left as an exercise for the student.

26. COR. 2. *The foci are equidistant from the vertices.*

PROPOSITION IV. THEOREM.

27. *Every plane section of both nappes of a right circular cone is a curve the ratio of the distances of whose points from a fixed point and a fixed line is constant.*

Notation. Same as in § 23.

To prove $\frac{r}{d} =$ constant.

Proof. Same as in § 10.

28. Scholium. The fixed point is called the **focus,** and the fixed line is called the **directrix.**

29. Cor. 1. *The constant ratio between the distances of the points of a hyperbola from the focus and the directrix equals the linear eccentricity divided by the* ***semi*** *major axis.*

See § 13. The proof is left as an exercise for the student.

30. Cor. 2. *The eccentricity of the hyperbola is greater than unity.*

31. Cor. 3. *A hyperbola has two directrices.*

32. From what has gone before, we see that the **general definition of a conic section is** — *a curve the distances of whose points from a fixed point and a fixed line bear a constant ratio: less than* 1 *for the ellipse: equal to* 1 *for the parabola: and greater than* 1 *for the hyperbola.*

Another definition might be: *A curve, the algebraic sum of the focal radii of any point of which is a constant, the focal radii having like signs when the foci are on the same side of the vertex, and unlike signs when on opposite sides of the vertex.*

If the algebraic sum is also the arithmetic sum, the resulting curve will be an ellipse; if the arithmetic difference, a hyperbola.

The middle of the transverse axis is a centre of symmetry, and is called the centre of the curve. Compare § 17.

Limiting and Special Cases.

33. Ellipse. If the cutting plane pass parallel to the base of the cone, the ellipse becomes a *circle,* as a special case.

If the plane pass through the vertex, we get a *point* as the limiting case of an ellipse, or of a circle.

The other limiting case of the ellipse is the *parabola*, as has already been mentioned.

34. Now suppose the angle of the cone to increase, the limit of which would be a circular disc.

The limit of the conic section would be an infinitely large curve, a finite portion of the arc of which would be a *straight line, or lines.*

35. If the cone lengthens to a cylinder the section still remains an ellipse, and we have the

Theorem. *Every complete section of a circular cylinder is an ellipse.*

36. Parabola. As the cutting plane approaches the element to which it is parallel, the section gets narrower and sharper, and finally, as it reaches the element, we have as a limiting case of the parabola *two coincident straight lines* perpendicular to the directrix. Before the lines coincide, we have for that portion of the parabola not in the vicinity of the vertex, *two parallel lines* as a limiting case, which are further apart the more obtuse the cone.

If the cone is flattened into a circular disc, the limiting case becomes a *straight line* parallel to the directrix, the directrix and focus being at an infinite distance.

37. If the cone lengthens out to a cylinder, the section, by a plane parallel to an element, becomes two parallel straight lines as a limiting case of a parabola.

38. Hyperbola. As the cutting plane approaches the vertex the two branches of the hyperbola become sharper, and at the vertex become *two intersecting straight lines* as a limiting case.

As the plane recedes from the vertex, the hyperbolas become less curved.

Now suppose the angle of the cone to increase, the limit of which would be two flat discs with a common centre at the vertex.

The limiting case would be *two coincident straight lines* (parallel to the directrix) if the plane passed through the vertex; and *two parallel straight lines* further and further apart as the plane passed from the vertex and the angle of the cone lessened.

39. *The parabola is the limiting case between the ellipse and the hyperbola. As the ellipse becomes longer and one focus passes out to infinity, the last ellipse is a parabola.*

As the hyperbola elongates and one focus passes out to infinity (as the cutting plane approaches parallelism to one of the elements of the cone), one branch of the hyperbola disappears with its focus, and the last of the series of hyperbolas is a parabola.

If the cutting plane has revolved about one of the vertices of the curve, this last hyperbola (or parabola) is exactly the same parabola which we arrived at as the last of the ellipses.

Hence, the parabola is the limiting case between the ellipse and the hyperbola. Whatever properties are possessed by the ellipse or hyperbola belong equally to the parabola, but the parabola possesses many properties which do not belong to the other curves, just as whatever is true of the ellipse is true of the circle (as §§ 40-43, 44), but not *vice versa*, as, a perpendicular to a tangent at the point of contact passes through the centre, equal chords are equally distant from the centre, etc.

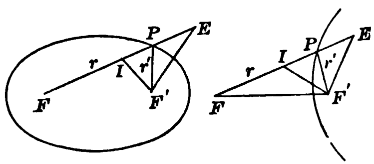
Many of the properties of the ellipse and hyperbola, though existent in the parabola, are rendered unavailable by reason of the introduction of infinite magnitudes; *e.g.*, §§ 4, 6, 7, 13, 16, 17, 25, 26, 40, 42.

Others, as §§ 10, 12, 23, 41, 43, 45, are available, notwithstanding these infinite magnitudes.

Except in §§ 20, 21, no special properties of the parabola are introduced here, though many exist.

40. *The algebraic sum of the distances of any point from the foci of a conic is greater or less than $2a$, according as the point is without or within* the curve.*

Notation. Let E and I denote respectively an exterior and an interior point of the conic, and P the point where the focal radius r through either of the points cuts the curve.



To prove $EF \pm EF' > 2a > IF \pm IF'$.

Construction. Draw the other focal radius r', and connect the points with the foci.

Proof ELLIPSE.

$$EF + EF' > r + r' > IF + IF'. \qquad \text{Q. E. D.}$$

HYPERBOLA. $r - r' = 2a$, by definition.

$$\therefore r + EP - (r' + EP) = 2a.$$

But $\qquad r + EP = EF$ and $r' + EP > EF'$.

$$\therefore EF - EF' > 2a.$$

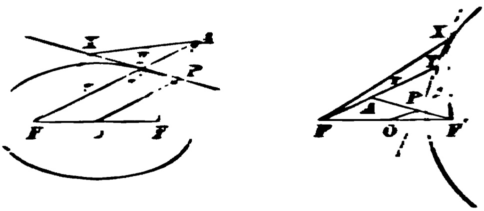
Similarly, $\qquad IF - IF' < 2a. \qquad$ Q. E. D.

41. *A line through a point on the ellipse (hyperbola) and bisecting the external (internal) angle between the focal radii is a tangent.*

Notation. Let TX be a line through the point T and

* Within means, on the same side of the curve as the centre.

bisecting the external (internal) angle between the focal radii and X any point except T.



To prove X on the convex side of the curve.

Construction. On FT lay off the distance $TA = r'$, so that $FA = r \pm r'$ respectively.

Proof. $\angle XAT = \angle XF'T.$

$\therefore XA = XF'.$

In the $\triangle XAF$, $XF \pm XA \gtrless FA.$

But $FA = r \pm r' = 2a,$

and $XA = XF'.$

$\therefore XF \pm XF' \gtrless 2a$

and X is on the convex side of the curve and XT is a tangent, § 40.

Q. E. D.

42. Cor. 1. $TX \perp$ to AF'.

43. Cor. 2. *The locus of the foot of the focal perpendicular upon the tangent is a circle upon the major axis as a diameter.*

For, O being the middle point of FF' and P of AF', $OP = \frac{1}{2} AF = \frac{1}{2}(2a) = a.$

Therefore P is on a circle of radius a and centre O.

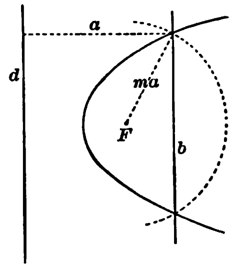
44. The circle on the focal axis as a diameter is called the **auxiliary circle.**

45. Cor. 3. *The perpendicular to the focal radius at its intersection with the auxiliary circle is tangent to the conic.*

CONSTRUCTION PROBLEMS.

46. *To construct a conic section having given the directrix, the focus and the eccentricity.*

Notation. Let d be the directrix, and F the focus, and m the eccentricity.

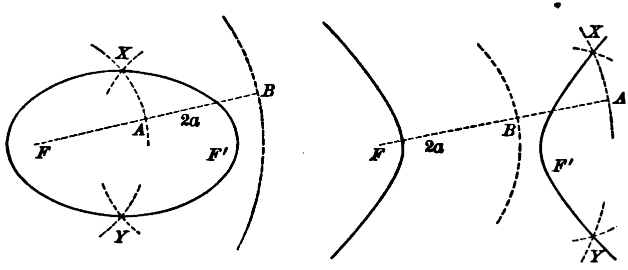


Construction. ∥ to d and at any assumed distance, a, draw the line b.

From F as a centre, and with a radius $= ma$, describe the circle cutting b in the points c, c'. c, c' will be points of the conic; an ellipse, a parabola, or a hyperbola, according as $m <=> 1$.

If $m > 1$, the arbitrary line b can be drawn on the other side of the directrix from the focus, and we will get two branches, or a hyperbola. Repetitions of the process will give different points of the locus through which a curve can be drawn.

47. *To construct a conic, having given the foci and the major axis.*



Notation. Let F and F' be the given foci, and $2a$ the length of the major axis.

Construction. About F as a centre describe a circle with radius $2a$.

On any circle described from F as a centre assume a point A and let the indefinite line FA cut the circle $2a$ in the point B.

From F' as a centre with radius AB describe a circle cutting the circle through A (centre F) in the points X, Y. X, Y will be points on the conic.

Proof. $FX \pm F'X = FA \pm AB = FB = 2a$. Q. E. D.

This gives the rule:—*From one focus with a radius equal to one segment of the major axis, describe a circle: from the other focus with the remaining segment describe another circle: where these circles intersect will be points on the conic: an ellipse if the segments are internal, a hyperbola if the segments are external.*

48. *To construct a conic, having given the focus and the auxiliary circle.*

Construction. Draw the line FA from the focus to the auxiliary circle, and at the point A erect the perpendicular AB. AB will be a tangent, and the different tangents will outline the curve: envelope it. § 45.

If the focus is outside the auxiliary circle, we get a hyperbola; if inside, an ellipse.

If the auxiliary circle is a straight line (radius $= \infty$) we get a parabola.

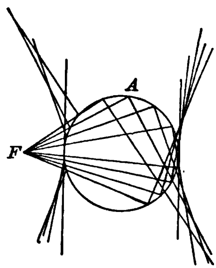


FIG. 1.

49. This gives a very convenient and accurate method of outlining a conic, the successive tangents as they are put in determining the curve to any degree of accuracy.

It also gives a very clear and easy method of following the curve in its successive changes.

50. Change of focus.

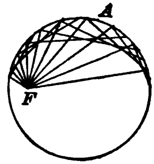


FIG. 2.

Starting with the focus on the left of the auxiliary circle of moderate size, we have a very obtuse hyperbola which grows sharper and narrower as the focus approaches the circle, and finally when the focus reaches the circle the hyperbola becomes two lines intersecting at an angle of zero. At the same time, we get the limiting case of an ellipse, viz., the diameter of the circle.

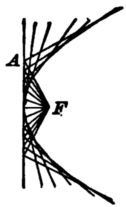


FIG. 3.

As the focus moves inside the circle the ellipse broadens out until the focus reaches the centre, when it becomes a circle, namely, the auxiliary circle.

As the focus moves on, the ellipse narrows again and we have the same succession of figures in a reverse order, the hyperbolas tending to become parallel straight lines perpendicular to the axis as the focus moves to infinity.

51. Change of auxiliary circle.

As the auxiliary circle increases in size, all these figures approach the form of a parabola, becoming a parabola when the circle becomes infinite, that is, when its circumference in the neighborhood of the focus is a straight line.

This shows again the parabola as the transition form between the ellipse and hyperbola.

As the auxiliary circle decreases, the hyperbolas tend to become parallel straight lines perpendicular to the axis, and the ellipses to vanish in a point.

If, as the circle diminishes, the focus approaches the circle, the limit of the hyperbolas will be two intersecting straight lines.

It will be noticed that diminishing the circle, or increasing the distance of the focus, has the same effect on the shape of the hyperbola, viz., a tendency to flatten it: and enlarging the circle or decreasing the distance of the focus has tendency to sharpen it.

SOLID GEOMETRY.

Examples to be Solved Mentally.

57, 61, 68, 79, 80–85, 90, 92, 93, 95–98, 100, 101, 108–112, 114, 128, 140, 141, 143, 151, 168, 197, 204, 205, 207, 220, 224, 233–235, 252.

Sections and Corresponding Examples.

Sections.	Examples.	Sections.	Examples.
118,	119.	194½,	94, 115.
119,	124.	201,	92, 114, 128.
120,	126.	204,	90, 91, 93, 95–113.
150,	134.	209,	88, 89, 130, 155, 156.
153,	50, 62, 67–70, 75, 150.	212*,	148, 149.
154,	120.	212**,	165–167.
156,	121–123.	212***,	168, 169.
170,	151–153, 158.	223,	228, 241, 242.
174,	116.	302,	256.
175,	79, 85, 125, 127, 135–139.	303,	208, 210.
180,	60, 61, 63–66, 71–74, 80–84, 86, 129, 140, 141, 143, 160–164.	305,	198, 199, 202, 214, 217, 219, 220, 252.
191,	116, 131–133, 154, 159.	309,	204, 205, 229, 233, 250, 255.
192,	76–78, 87, 117, 118, 142, 144–147.	311,	195, 196, 222, 230, 236, 239, 240, 253, 254.
		312,	192–194, 257.
		314,	243.

Sections.	Examples.	Sections.	Examples.
315,	212, 227.	331,	200, 201, 203, 206, 207, 215, 218, 231, 232, 234, 235, 237, 238, 246.
323,	209, 258.	332,	197.
324,	259–262.	334,	213, 221, 226, 244, 245, 247, 248, 251.
328,	211, 216, 223, 224, 249.		
329,	225.		

ANSWERS.

$\pi = 22/7$.

15. $13\frac{1}{11}$, $17\frac{5}{11}$.
16. $21\frac{3}{7}$, $24\frac{4}{7}$, $26\frac{4}{7}$, $27\frac{5}{7}$.
47. 9.
50. 1509 times.
51. $6\sqrt{(\frac{5}{2})}$.
52. 11.
53. $6 \cdot 14/11$.
54. 6.25.
55. $\sqrt{[3^3(6^2-1)]} = 30.74$.
56. 72.
57. 500.
58. $25^2\{1+\sqrt{(1+4^6/5^2)}\}$.
59. $3^3(10+\frac{3}{4}\sqrt{3})$.
60. $\frac{1}{8}4^3 10^2\sqrt{(7^2 11^2 - 2^8 \cdot 5^6)}$.
61. 18.
62. 210.
63. $2^5 \cdot 3 \cdot 5^3\sqrt{48}$.
64. 17.066.
65. 17.
66. 479.
67. 245.76.
68. 384.
69. 63.93.
70. 5096.
71. 5.2.
72. 10.
73. 26,475.99.
74. 241.826.
75. 1056.
76. 6815.
77. 1.58.
78. 4.8, 2.4.
79. $h = abc/\pi r^2$.
80. $3\,abc/\pi r^2$.
81. 40.
82. 38.5.
83. $\pi r^2 h/a^2$.
84. $4\,a^3/\pi b^2$.
85. $d\sqrt{m}/2\sqrt{n}$.
86. 2.9.

87. 4.3.
88. 11.76.
89. $18\frac{2}{3}$.
90. $\times 5$, $\times \sqrt[3]{5}$.
91. 1.7, 3.4, 5.1.
92. $\sqrt{n}$.
93. $\sqrt[3]{n}$.
94. 4.05.
95. $\frac{1}{8}$.
96. .79.
97. $1/\sqrt[3]{2} = .793$.
98. $1/\sqrt[3]{3}$, $\sqrt[3]{2}/\sqrt[3]{3}$.
99. $6\sqrt[3]{\frac{1}{3}}$, $6\sqrt[3]{\frac{2}{3}}$.
100. $1/\sqrt[3]{n}$, $\sqrt[3]{2}/\sqrt[3]{n}$, etc.
101. $\sqrt[3]{3}/\sqrt[3]{7}$.
102. $\sqrt[3]{a}/\sqrt[3]{(a+b+c)}$,
$\sqrt[3]{(a+b)}/\sqrt[3]{(a+b+c)}$.
103. 14.
104. 14.
105. 14.
106. $1/\sqrt[3]{\frac{1}{5}}$.
107. $1/\sqrt[3]{\frac{4}{5}}$.
108. $h\sqrt[3]{3}$, $h\sqrt[3]{4}$, $h\sqrt[3]{5}$, etc.
109. $2^{\frac{3}{2}}/3^{\frac{3}{2}}$.
110. $m^{\frac{2}{3}}/n^{\frac{2}{3}}$.
111. $m^{\frac{3}{2}}/n^{\frac{3}{2}}$, $m^{\frac{1}{2}}/n^{\frac{1}{2}}$.
112. 63/64.
113. 91/216.
114. $h/\sqrt{2}$.
115. $\pi\sqrt{2}(r'^2 - r^2)$.
116. 31,000.
117. $\pi[p^2 + 12h^2 + p\sqrt{(p^2 - 12h^2)}]/36$.
118. $\sqrt{[\epsilon - \frac{1}{4}e'^2 + (1 - \frac{1}{2}\sqrt{2})\,ee']}$.
119. $\frac{1}{2}\{\sqrt{(l^2 - h^2)} - \sqrt{(l'^2 - h^2)}\}$.
120. $\sqrt{\{S - (B - B')^2\}}/\sqrt{\{\pi(\sqrt{B} + \sqrt{B'})\}}$.
121. $3e\{\frac{1}{2}e\sqrt{3} + \sqrt{(h^2 + \frac{3}{4}a^2)}\}$.
122. $\frac{1}{2}\sqrt{(32\Delta^2 + \frac{1}{64}p^4)}$.
123. 393.73.
124. $rr'/(r + r')$.
125. 55,004 ft.
126. $\frac{1}{2}M\sqrt{3}$.
127. 87.39.
128. $\pi m^2 r^2/(m + n)^2$.
129. $\sqrt{2}/\sqrt{3}$.
130. $(\frac{1}{2}a\sqrt{\frac{1}{3}})^2 : (a/\sqrt{3})^2 = \frac{1}{4}$.
131. $\frac{5}{6}hb^2$.
132. $\frac{1}{3}h(b^2 + \sqrt{2}ab + a^2) = 500$.
133. 38,133.76.
134. $T/2\pi r - r$.
135. $\frac{1}{2}Tr - \pi r^2$.
136. $T\sqrt{T}/3\sqrt{(6\pi)}$.
137. $\sqrt{\{(T - S)(2S - T)T\}}/3\sqrt{\pi}$.
138. 43,429.
139. $p^2h/4\pi$.
140. 210.
141. 976.

142. 227.1, 436.46.

143. $66\frac{2}{3}$.

144. $\frac{1}{12}\sqrt{143\,(6^2+8^2+6{\cdot}8)}$.

145. 92.16, 40.3.

146. $\frac{1}{3}h$.

147. $V=\frac{1}{3}\pi h\,\{3\,S^2/4\,\pi^2 l^2+(l^2-h^2)/4\}$.

148. $2n$.

149. $3n$.

150. 270.

151. 5/34.

152. 396.6.

153. 444.

154. Use § 191.

158. Edge = 6.

159. $143\frac{1}{3}$.

160. $46\frac{2}{3}$.

161. $V=e\sqrt{\{T(T-2\,e^2)\}}/6$.

162. $h=\sqrt{\{T(T-2\,e^2)\}}/2\,e$.

163. $a=T\sqrt{2}/2\sqrt{(T+2\,h^2)}$.

164. $2^2 3^5 11$.

168. 8 rt. ∠s.

169. 40 rt. ∠s.

184. $50\frac{2}{7}$.

185. 1386.

186. 56.

187. $\frac{1}{2}$.

188. $\frac{5}{8}$.

190. $50\frac{2}{7}$.

191. 4 : 9.

192. $1642\frac{2}{3}$.

193. $120\,\sigma$.

194. $\frac{4}{6}\,\pi r^2$, $24\,\pi$.

195. $8\pi(\sqrt{3}-1)$.

196. $\pi(h^2+r^2)$, $3\pi(h^2+r^2)$, $\frac{1}{2}\pi(h^2+r^2)$.

197. 1000.

198. 12.

199. Cyl. = √(Sph. S. × Cone).

200. Cyl. = √(Sph. V. × Cone).

201. $4\sqrt{2}:3:4/3$.

202. 3:2.

203. 9 : 4.

204. Equal.

205. $\pi 8^2$.

206. $2\,V/\pi\sqrt{3}$.

207. $6\,V/\pi$.

208. $\pi ab\,(a+b)/(a^2+b^2)^{\frac{1}{2}}$.

209. $\pi a^2b^2/3\,(a^2+b^2)^{\frac{1}{2}}$.

210. $r=4\sqrt{3}/3$.

211. 20.

212. 9.

213. $\pi h^3/6$.

214. 240.

215. 32.

216. $\pi r^3/3$.

217. $\pi 121\frac{1}{2}$.

218. $\sqrt[3]{(113{\cdot}76/11)}$.

219. 6π.

220. 8π.

221. $236\,\pi/3$.

222. $\frac{1}{8}$.

223. $61\frac{1}{9}$.

224. 72°.

225. 10.

226. $485\frac{3}{7}$. 4

227. 308.

228. $\pi 16$.

229. $r/2$.

230. $2\pi r^2 h/(r+h)$.
231. $c^3/6\pi^2$.
232. $\sqrt[3]{(3V/4\pi)}$.
233. $3:1$.
234. $2:1$.
235. 3.
236. $r(\sqrt{5}-2)$.
237. $\pi 108\sqrt{3}$.
238. $\pi V\sqrt{3}/2$.
239. r.
240. $2r/(n-2)$.
241. $\pi 12^3$.
242. $\frac{1}{3}\pi a^2\{\sqrt{(r^2-a^2)}+r\}$.
243. $\triangle\ 180/\pi r^2$.
244. 11/5.
245. 4.3.
246. $3\sqrt[3]{9}$.
247. $r=V/\pi h^2+h/3$.
248. 3486.
249. 29.89.
250. $S/\sqrt[3]{(6\pi^2 V)}$.
251. $\sqrt{(m^2 r_1^2-n^2 r_2^2)}/\sqrt{(m^2-n^2)}$.
252. 2/3.
253. $h/2(r+h)$.
254. $2\pi r^2 h/(r+h)$.
255. $r/2$.
256. $7\pi r^2/15$.
257. $\triangle \propto$ 3d $\angle$.
262. $2\pi^2 r^2 d$.
266. $4\pi^2 rd$.