47

THE ELEMENTS

OF

SOLID GEOMETRY.

BY

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PREFACE.

I HAVE written this book, having in view the ultimate improvement of the course in mathematics offered at Bucknell University, and bearing in mind that the crowded curricula give to this course less time and more subjects than was the case twenty-five years ago.

In carrying forward a course of mathematical study, nothing can make amends for hasty or imperfect preparation. However, since Plane Geometry is an almost universal requirement for admission to college, it becomes possible, by means of the entrance examinations, to enter in the subject of Solid Geometry, only those students who are already well trained in Euclidian methods of demonstration and investigation.

Believing that for such students the course in Solid Geometry may be made quite brief with the ultimate advantage of having more time for advanced mathematics, I offer this short course. In it are a number of theorems for original demonstration and many illustrative examples. A section on Mensuration is introduced with the design of calling special attention, by means of illustrative examples, to all the important rules for finding volumes and surfaces of solids, demonstrated in the preceding sections. Also, methods for finding the volumes of the Regular Polyedron, the Wedge, and the Prismoid are deduced.

For the purpose of bringing the important theorems as near as possible to the definitions, postulates, etc., on which they rest, I have found it necessary to deviate somewhat from the usual sequence of propositions. Thus, I have grouped in the same section the prism and its limiting case, the cylinder, because they have so many properties in common. I have treated the pyramid and its limiting case, the cone, in like manner, etc.

Always, I have aimed to give the most direct proof possible, and to save the student, by means of corollaries, the labor of reproducing constructions unnecessarily.

An experience of twenty years in teaching mathematics leads me to think that the student who gets up the subject from this brief work, in the end will be at no disadvantage from not having used some one of our larger popular text-books.

Many of the diagrams used in illustration are, by permission, from Professor Wells' geometry. In thanking him for this act of courtesy, I desire also to acknowledge my indebtedness to him for valuable aid rendered me through the agency of his text-books, some of which I have had in class-room use from the date of their publication.

WILLIAM C. BARTOL.

Bucknell University,
Lewisburg, Pa., Aug., 1893.

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THE ELEMENTS OF SOLID GEOMETRY.

SECTION I.

LINES AND PLANES IN SPACE.

DEFINITIONS.

1. A *plane* is a surface such that, if a straight line be passed through any two of its points, the line will lie wholly in the surface.

When a line lies wholly in a plane, it may be said of the plane that it passes through the line.

2. The *intersection* of two surfaces is a line containing all the points which are common to the two surfaces.

3. The *intersection* of a line and a plane is the point where the line pierces the plane. This point is called the *foot* of the line.

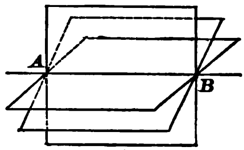
4. A line is *perpendicular* to a plane when it is perpendicular to every line of the plane, passing through its foot. And the plane is then perpendicular to the line.

5. Planes are *parallel* if they do not meet, however far extended.

6. A line and a plane are *parallel* if they do not meet, however far extended.

PROPOSITION I.

7. THEOREM. ***Through a straight line an indefinite number of planes may be passed.***



Let AB be a straight line lying in a plane (1). Now, we may conceive of the plane as rotating about the line as an axis, and thus occupying successively an indefinite number of positions. But the plane in any position is a plane through the line AB. Hence, through the line AB an indefinite number of planes may be passed. *Q. E. D.*

8. COROLLARY 1. *Through a straight line and a point without it, only one plane may be passed.*

For if the plane be rotated about the line until it includes the given point, any further rotation, in either direction, will cause the plane to no longer include the given point.

9. SCHOLIUM. Since only one plane may be passed through a straight line and a point without it, *a plane is determined by a straight line and a point without it.*

10. COROLLARY 2. *A plane is determined by three points not in the same straight line.*

For two of the points may be joined by a straight line, and the plane rotated about the line, as in COR. 1, until the third point is included.

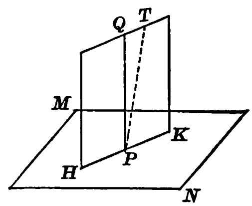
11. COROLLARY 3. *A plane is determined by two straight lines which intersect, or by two parallel lines.*

12. COROLLARY 4. *The intersection of two planes is a straight line.*

For if through two points of the intersection we pass a straight line, all the points of that line must be common to the two planes (1); and no point without this line can be common to the two planes (8). Hence, the straight line must be the line of intersection of the planes (2).

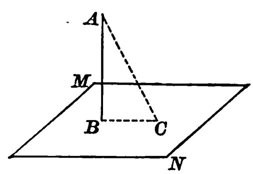
PROPOSITION II.

13. THEOREM. *Only one perpendicular can be drawn from a given point to a plane.*



Let QP be a perpendicular to the plane MN, drawn from P, a point in the plane; and let TP be any other line drawn from P; then TP cannot be perpendicular to MN.

For, pass through QP and PT a plane, intersecting MN in the line HK; then, since QP is perpendicular to HK (4), TP cannot be perpendicular to HK (350); neither can TP be perpendicular to MN (4).

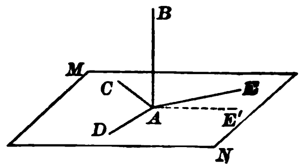


Again, let AB be a perpendicular to the plane MN, drawn from A, a point without the plane; and let AC be any other line drawn from A to the plane; then AC cannot be perpendicular to MN.

Draw the line BC. Now, since AB is perpendicular to the plane MN, it is also perpendicular to the line BC (4); and therefore AC cannot be perpendicular to BC (350); hence it cannot be perpendicular to MN (4). *Q. E. D.*

PROPOSITION III.

14. THEOREM. *All perpendiculars to a straight line at a given point lie in a plane which is perpendicular to the line at the given point.*



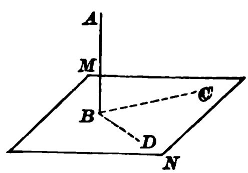
Let any line AE be drawn perpendicular to the line AB at a given point A; and let MN be a plane passing through A, and perpendicular to AB; then will AE lie in MN.

For, if AE does not lie in MN, pass through AE and AB a plane BAE; and let AE′ be the intersection of this plane with MN. Now, AB is perpendicular to AE′ (4), and AE was drawn perpendicular to AB; hence, at A there are two perpendiculars to the line AB, lying in the same plane BAE. But this is impossible. Hence the assumption that AE does not lie in MN is false; or, AE lies in MN. *Q. E. D.*

15. COROLLARY. *Through a given point in a straight line only one plane may be passed perpendicular to the line.*

PROPOSITION IV.

16. THEOREM. *If a line is perpendicular to each of two lines at their point of intersection, it is perpendicular to the plane of the two lines.*

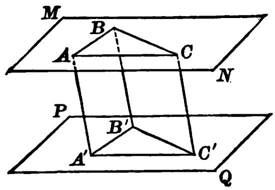


Let the line AB be perpendicular to the lines BC and BD at B, their intersection; and let MN be the plane of BC and BD; then will AB be perpendicular to MN.

By our hypothesis BC and BD are perpendiculars to AB, at a common point B; therefore they lie in a plane which is perpendicular to AB at B (14). But, there is but one plane in which both BC and BD can lie; *i.e.*, the plane of the lines (11), which is by hypothesis MN. Hence MN is perpendicular to AB; or, AB is perpendicular to MN. *Q. E. D.*

PROPOSITION V.

17. THEOREM. *Parallel lines included between two parallel planes are equal.*



Let the parallel lines AA′, BB′, and CC′ be included between the parallel planes MN and PQ; then will AA′, BB′, and CC′ be equal.

Draw the lines AC and A′C′. Now these lines are parallel. For, they lie in the same plane, *i.e.*, the plane of the parallel lines AA′ and CC′ (11); and they cannot meet, since they lie in the parallel planes MN and PQ (5); hence they are parallel.

AC′ is therefore a parallelogram (371); and AA′ is equal to CC′ (360). In like manner we may prove that AA′ equals BB′, and that BB′ equals CC′. *Q. E. D.*

18. COROLLARY 1. *The intersections of a plane with two parallel planes are parallel lines.*

We have shown that AC and A′C′ are parallel; *i.e.*, the intersections of the plane AC′ with the parallel planes MN and PQ are parallel lines.

19. COROLLARY 2. *Angles lying in different planes having their sides parallel and in the same direction each to each, are equal.*

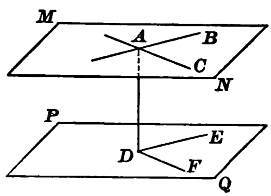
We have shown that AC′ is a parallelogram; therefore AC is equal and parallel to A′C′. In like manner it may be shown that AB is equal and parallel to A′B′, and that BC is equal and parallel to B′C′. It follows then that the triangle ABC is equal to the triangle A′B′C′ (357), in which triangles, any angle ABC is equal to the corresponding angle A′B′C′. And these angles have their sides parallel, each to each, and lying in different planes.

20. COROLLARY 3. *Lines parallel to the same line are parallel to each other.*

For, from the parallelograms AC′ and AB′, we derive that, the lines CC′ and BB′ are each parallel to the same line AA′; and they are parallel to each other, since BC′ is also a parallelogram.

PROPOSITION VI.

21. THEOREM. *If a straight line is perpendicular to one of two parallel planes, it is perpendicular to the other also.*



Let MN and PQ be parallel planes, and let AD be perpendicular to PQ; then will AD also be perpendicular to MN.

Through AD pass two planes whose intersections with MN and PQ are the lines DF, AC, and DE, AB. Now DF and AC are parallel (18); and since ADF is a right angle (4), DAC is also a right angle (368). In like manner we may prove that DAB is a right angle. Hence, since DA is perpendicular to both AC and AB at their point of intersection, AD is also perpendicular to MN (16). *Q. E. D.*

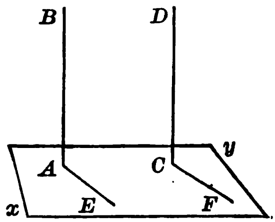
22. Corollary. *If two planes are perpendicular to the same straight line, they are parallel to each other.*

Let the two planes MN and PQ be perpendicular to the straight line AD at the points A and D; then will these planes be parallel to each other.

For, pass through A a plane which is parallel to PQ, and it will be perpendicular to AD (21). But only one plane can be passed through A which is perpendicular to AD (15). Therefore MN must coincide with this plane which has been passed through A, parallel to PQ; *i.e.*, MN is parallel to PQ.

Proposition VII.

23. Theorem. *If one of two parallel lines is perpendicular to a plane, the other is also perpendicular to the plane.*



Let the line BA be perpendicular to the plane *xy*; and let the line DC be parallel to BA; then will DC be perpendicular to *xy*.

In *xy*, draw through C, any line CF, and parallel to CF draw AE; then the angles BAE and DCF will be equal (19). But BAE is a right angle (4); therefore DCF is a right angle. Then, since DC is perpendicular to CF any line in the plane *xy*, DC must also be perpendicular to *xy* (4). *Q. E. D.*

24. Corollary 1. *If two lines are perpendicular to the same plane they are parallel to each other.*

Let the lines BA and DC be perpendicular to the plane *xy* at the points A and C. Now through A, draw a parallel to CD and it will be a perpendicular to xy, by our proposition; and it must therefore coincide with the perpendicular BA (13). Since then, BA coincides with a parallel to DC, BA must be parallel to DC.

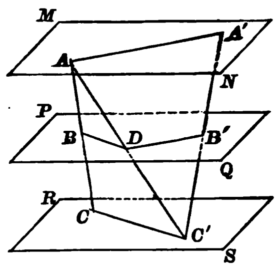
25. Corollary 2. *Parallel planes are everywhere equally distant.*

Consider the parallel planes MN and PQ, of Prop. V. Let AA′ and CC′ be any two perpendiculars to the plane PQ between the parallel planes; then will AA′ and CC′ be equal.

For, AA′ is parallel to CC′ (24),
Hence, AA′ is equal to CC′ (17).

Proposition VIII.

26. Theorem. *Straight lines cut by three parallel planes are divided proportionally.*



Let the lines AC and A′C′ be cut by the parallel planes MN, PQ, and RS, in the points A, B, C, and A′, B′, C′; then will

AB : BC :: A′B′ : B′C′.

Draw the line AC′ piercing PQ at D; then draw BD and DB′, also draw AA′ and CC′. Now BD is parallel to CC′, and DB′ is parallel to AA′ (18).

Hence, AB : BC :: AD : DC′,
and A′B′ : B′C′ :: AD : DC′ . . . (361).
Therefore, AB : BC :: A′B′ : B′C′. *Q. E. D.*

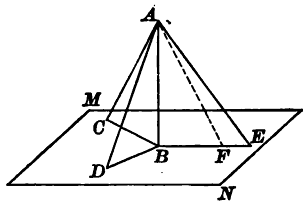
EXERCISES.

27. THEOREM. *If A be any point without a plane:*

(*a*). *The perpendicular from A to the plane is shorter than any oblique line.*

(*b*). *Oblique lines from A cutting off equal distances from the foot of the perpendicular are equal.*

(*c*). *Of two oblique lines cutting off unequal distances from the foot of the perpendicular, the more remote is the greater.*



Let the line AB be drawn from A, perpendicular to the plane MN, meeting it at any point B. Let the points C and D be taken in MN, equally distant from B; and let E be a point more remote.

Prove (*a*). AB is shorter than AD, any other line.
(*b*). AC is equal to AD.
(*c*). AE is greater than AC.

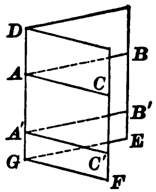
DEFINITIONS.

28. A *diedral* angle is the angle between two planes which intersect each other.

29. The line in which the planes intersect is called the *edge* of the angle; the planes themselves are called the *faces* of the angle.

30. Lines drawn in the faces of a diedral angle perpendicular to the edge and from the same point in it, form a plane angle which is taken as the measure of the diedral angle.

If the plane DE intersect the plane DF in the line DG, then F–GD–E, the angle between the planes, is a diedral angle. DF and DE are faces and DG is the edge of the diedral angle.



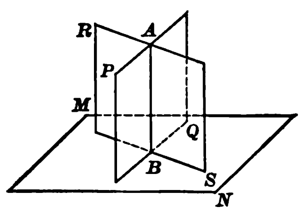
In the face DF draw the line AC perpendicular to the edge GD at the point A. And, in the face DE draw the line AB perpendicular to the edge GD at the point A. Then CAB, the plane angle between the lines CA and AB, is the measure of the diedral angle F–GD–E.

For, if we revolve DF about DG as an axis, the angle CAB will increase or diminish precisely as the angle F–GD–E increases or diminishes. And, since this is not the case with any other plane angle whose vertex is at A and whose sides are in DF and DE respectively, CAB is taken as the measure of F–GD–E.

Construct a second plane angle C′A′B′, as before; then CAB is equal to C′A′B′ . . . (19).

PROPOSITION IX.

31. THEOREM. *If a line is perpendicular to a plane, any plane passing through the line is perpendicular to the plane.*



Let the line BA be perpendicular to the plane MN; and let the plane PQ be passed through BA; then will PQ be perpendicular to MN.

In MN draw the line BS perpendicular to BQ, the intersection of PQ and MN. Now since AB and BS lying in their respective planes PQ and MN, are perpendicular to the intersection of these planes, at a common point, the angle ABS is the measure of the diedral angle between the planes PQ and MN . . . (30). But since ABS is a right angle (4), PQ is perpendicular to MN. *Q. E. D.*

32. COROLLARY 1. *A line drawn in one of two perpendicular planes, perpendicular to their intersection, is perpendicular to the other plane.*

Thus, as we see, the line BA drawn in the plane PQ perpendicular to the intersection BQ, is perpendicular to the plane MN.

33. COROLLARY 2. *If a line be drawn perpendicular to one of two perpendicular planes, at a point of their intersection, it will lie in the other plane.*

Thus, as we see, the line BA drawn perpendicular to the plane MN, at B in the intersection, lies in the plane PQ perpendicular to MN.

34. COROLLARY 3. *If two intersecting planes are perpendicular to a third plane, their line of intersection will also be perpendicular to the third plane.*

Thus, as we see, the line BA drawn perpendicular to MN, is the intersection of the planes PQ and RS passing through BA (12), which planes are perpendicular to MN (31).

DEFINITIONS.

35. A *polyedral* angle is an angle formed by three or more planes meeting at a common point.

36. The point in which the planes meet is called the *vertex* of the angle. The lines in which the planes meet are called the *edges* of the angle. Those portions of planes which lie between the edges are termed the *faces.*

37. The plane angles lying between the edges and having a common vertex, are called *face* angles.

38. The face angles and the diedral angles between the faces are called the *parts* of the polyedral angle.

39. Two polyedral angles in which all the parts of the one are equal to all the parts of the other, each to each, are said to be *mutually equal* in all their parts, or, with respect to all their parts.

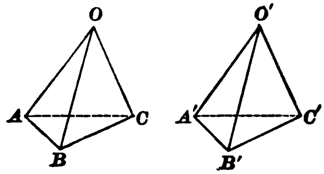


FIG. 1.

40. If the equal parts of the one are arranged in the *same order* as the equal parts of the other, throughout, the polyedral angles are ***superposable,*** and hence *equal.* See *Fig. 1.*

41. If the equal parts of the one are arranged in the *reverse order* of the equal parts of the other, the polyedral angles are evidently ***not superposable*** ; in this case they are *symmetrical.* See *Fig. 2.*

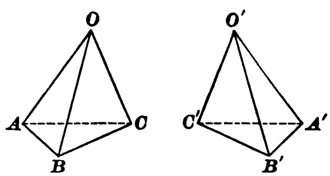


FIG. 2.

42. A good illustration of symmetrical forms is the case of an *object* and its *image* in a plane mirror.

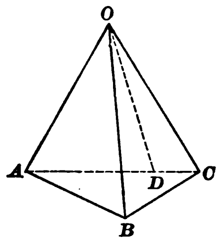
Place O′–A′B′C′ under O–ABC, so that A′B′C′ coincides with ABC and O′ falls below the common base. The solids will then be in the relative position of an object standing upon a horizontal mirror and the image of the object.

"A familiar example of the symmetry of non-superposable figures is afforded by a pair of outstretched hands. They can be so placed, palm opposite to palm, that each is the image of the other."

43. A *triedral angle* is a polyedral angle of three faces.

Proposition X.

44. Theorem. *Any face angle of a triedral angle is less than the sum of the other two.*



Of the three face angles forming the triedral angle O–ABC, let AOC be the greatest; then will AOC < AOB + BOC.

Draw OD making the angle AOD equal to the angle AOB. Take DO, AO, and BO equal to each other. Draw AD produced to meet OC in some point C. Draw AB and BC.

The triangles AOD and AOB are equal (355); and therefore the homologous sides AD and AB are equal.

Now,	AC < AB + BC . . . (353),
and,	AD = AB, just shown.
Hence,	AC − AD < BC, by subtraction,
or,	DC < BC.

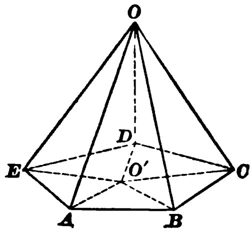
In the triangles

	DOC and BOC, since DC < BC
angle	DOC < BOC . . . (358).
But angle	DOA = AOB by construction.
Hence,	DOA + DOC < AOB + BOC;
or,	AOC < AOB + BOC.

Q. E. D.

Proposition XI.

45. Theorem. *The sum of the face angles of a convex polyedral angle is less than four right angles.*



Let O–ABCDE be a convex polyedral angle whose face angles are AOB, BOC, COD, DOE, and EOA; then will

$$AOB + BOC + COD + DOE + EOA < 4 \text{ rt. angles.}$$

Through the polyedral angle, pass a plane intersecting the faces in the lines AB, BC, CD, DE, and EA; and let O′ be a point within the polygon thus formed. Draw the lines O′A, O′B, O′C, O′D, and O′E.

Now, angle

$$OAE + OAB > O'AE + O'AB \qquad (44),$$

and angle

$$OBA + OBC > O'BA + O'BC, \text{ and so on,}$$

and, if we add all these inequalities, member to member, we shall obtain the inequality

$$OAE + OAB + OBA + \text{etc.} > O'AE + O'AB + O'BA + \text{etc.};$$

i.e., the sum of the angles at the bases of the triangles, whose common vertex is O is greater than the sum of the angles at the bases of the triangles whose common vertex is O′.

But the sum of all the angles of the triangles whose common vertex is O equals the sum of all the angles of the triangles whose common vertex is O′, since there is the same number of triangles in each set, and (354).

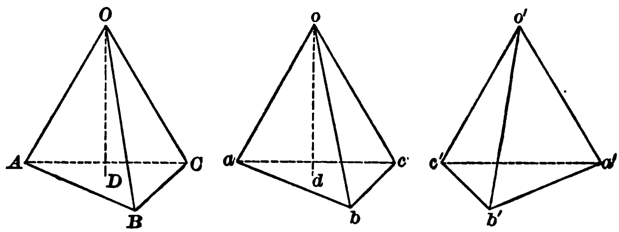
Hence, the sum of the angles whose common vertex is O is less than the sum of the angles whose vertex is O′. That is,

$$AOB + BOC + COD + DOE + EOA < 4 \text{ rt. angles.}$$

Q. E. D.

Proposition XII.

46. Theorem. *If two triedral angles have the three face angles of the one equal to the three face angles of the other, each to each, the two triedral angles are mutually equal in all their parts.*



In the triedral angles *o* and *o′*, let the face angles *aob, boc, coa,* be equal to the face angles *a′o′b′*, *b′o′c′*, *c′o′a′*, respectively. Then will the diedral angle *a-ob-c* be equal to the diedral angle *a′-o′b′-c′*, etc.

Take *ob* equal to *o′b′*. Draw *ba* and *bc*, each perpendicular to *ob*. In like manner draw *b′a′* and *b′c′*, each perpendicular to *o′b′*. The plane angles *abc* and *a′b′c′* are the measures of the diedral angles *a-ob-c* and *a′-o′b′-c′* (30).

The triangle *aob* equals the triangle *a′o′b′* (356); hence, *ab* equals *a′b′*, and *ao* equals *a′o′*. In like manner we can prove that *bc* equals *b′c′* and *co* equals *c′o′*. Hence, the triangle *aoc* equals the triangle *a′o′c′* (355), and hence *ac* equals *a′c′*.

Now, the triangles *abc* and *a′b′c′*, being mutually equilateral, are equal, and hence the angle *abc* equals the angle *a′b′c′*. But *abc* and *a′b′c′* are the measures of the diedral angles *a-ob-c* and *a′-o′b′-c′*; hence *a-ob-c* equals *a′-o′b′-c′*. In like manner the other two diedral angles of *o* may be shown to be equal to the corresponding diedral angles of *o′*, each to each. *Q. E. D.*

47. SCHOLIUM. The triedral angles *o* and *o′* have their equal parts arranged in the reverse order. The triedral angles O and *o* have their equal parts arranged in the same order. The above demonstration applies in either case. We observe that *o* and *o′* are symmetrical (41); and that O and *o* are equal (40).

EXERCISES.

48. THEOREM. *If two triedral angles have two face angles and the included diedral angle of the one, equal to two face angles and the included diedral angle of the other, each to each, the two triedral angles are mutually equal in all their parts.*

Prove the above theorem, using the diagram and method of (46).

49. THEOREM. *If two parallel lines intersect a plane, they make equal angles with it.*

50. THEOREM. *If a line is parallel to one plane and perpendicular to another, the two planes are perpendicular.*

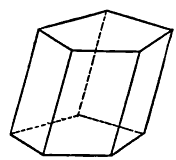
SECTION II.

SOLIDS: PRISMS AND CYLINDERS.

DEFINITIONS.

51. A *solid* is a portion of space bounded on all sides by plane or curved surfaces. A solid bounded by planes is called a *polyedron.* The bounding planes are called *faces.* The intersections of the faces are called *edges.* The points in which the edges meet are called *vertices.*

52. A *prism* is a polyedron whose edges are all parallel, except those formed by two parallel faces cutting all the other faces. The parallel faces are called *bases;* all others are called *lateral* faces.



53. The *lateral surface* is the sum of the lateral faces.

54. The *lateral edges* are the parallel edges.

55. Prisms are named from their bases; if the bases are pentagons, as in the figure of (52), the prism is called a *pentangular* prism. In like manner we have *quadrangular* prisms, *triangular* prisms, etc.

56. A *right* prism is one whose lateral edges are perpendicular to the bases; all other prisms are termed *oblique.*

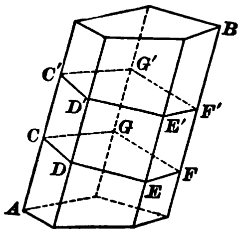
57. The *altitude* of a prism is the perpendicular distance between the bases.

58. Any section of a prism perpendicular to the lateral edges is called a *right* section.

From the definition of a prism we readily deduce the following:

59. *The lateral edges are equal to each other.*

60. *The lateral faces are parallelograms.*



61. *The bases are equal polygons.*

62. *Parallel sections cutting all the lateral edges are equal.*

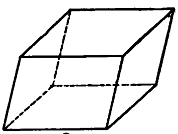
63. *Right sections are equal.*

64. *The lateral faces of a right prism are rectangles perpendicular to the bases.*

65. *Right prisms whose bases and altitude are equal, each to each, may be shown to be equal by superposition.*

The demonstration of the above is left to the student.

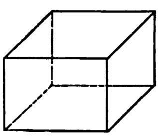
66. A *parallelopiped* is a prism whose bases are parallelograms. It is termed *right* or *oblique* according to the conditions of Art. 56.



From the definition the student may readily deduce the following: —

(*a*) *The opposite faces are parallel.*
(*b*) *Any right section is a parallelogram.*
(*c*) *All right sections are equal.*

67. A *rectangular* parallelopiped is a right prism whose bases are rectangles; all the faces are therefore rectangles (64).



68. A *cube* is a rectangular parallelopiped* all of whose faces are equal squares.

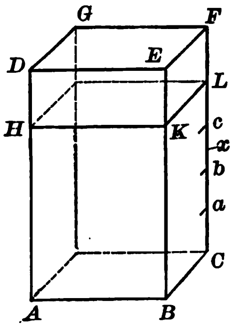
69. The *unit of cubic measure* is the cube whose edge is the linear unit.

70. The *volume* of a solid is the ratio of the solid to the unit of cubic measure.

* R. Baldwin Hayward, late President of the Association for the Improvement of Geometrical Teaching, calls the rectangular parallelopiped a cuboid. We shall adopt the term.

PROPOSITION XIII.

71. THEOREM. *Right prisms having a common base are to each other as their altitudes.*



Let the right prisms AF and AL have the common base AC, and the altitudes CF and CL respectively; then will

AF : AL :: CF : CL.

Let us suppose that CF and CL are commensurable; and that a common measure Ca is contained in CF five times, and in CL four times; then will CF : CL :: 5 : 4.

Apply Ca, the common measure, to CF so that C$a = ab = bc$, etc. Through the points of division a, b, c, etc., pass planes perpendicular to the edge FC. These planes divide AF into 5 right prisms, and AL into 4 (22 and 56). The small prisms thus formed are equal to each other; for they may be made to coincide by superposition (63 and 65).

Now, since AF contains 5 of these equal prisms, and AL contains 4, . . . AF : AL :: 5 : 4;
but, CF : CL :: 5 : 4;
hence, AF : AL :: CF : CL. *Q. E. D.*

Let us assume that CF and CL are incommensurable, still the proposition remains true.

For if AF is not to AL as CF is to CL, AF must be to AL as CF is to some altitude Cx, either greater or less than CL.

Assume AF : AL :: CF : Cx, in which Cx is less than CL.

Divide CF into equal parts, each of which is less than xL; at least one point of division c, will fall between x and L.

Construct a right prism ABC–c. Now, because CF and Cc are commensurable,

AF : Ac :: CF : Cc; and we have supposed
AF : AL :: CF : Cx; combine these proportions,
then, Ac : AL :: Cc : Cx.

Now, Ac is less than AL, being a part of it; hence our proportion determines Cc to be less than Cx. But this is absurd, since the whole is greater than any part.

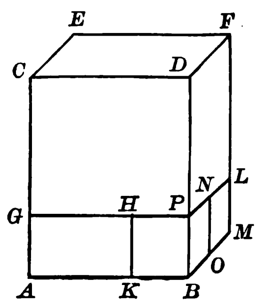
Hence, since our assumption that AF : AL :: CF : Cx, in which Cx is taken less than CL, leads to an absurdity it must be false.

Cx, then, cannot be taken less than CL; and in like manner it may be shown that Cx cannot be taken greater than CL. Hence, Cx, the fourth term of the proportion, must be taken equal to CL; or, AF : AL :: CF : CL. *Q. E. D.*

72. Corollary. *Right prisms whose bases are equal are to each other as their altitudes.*

Proposition XIV.

73. Theorem. *Cuboids are to each other as the products of their three dimensions.*



Let AF and KN be cuboids whose dimensions are respectively BA, BM, BD, and BK, BO, BP; then will

$$AF : KN :: (BA)(BM)(BD) : (BK)(BO)(BP).$$

Produce the faces of KN forming the cuboids AL and AN.

Regarding AM as a common base we may write,

$$AF : AL :: BD : BP \quad . \; . \; . \quad (71).$$

Regarding AP as a common base we may write,

$$AL : AN :: BM : BO.$$

Regarding BN as a common base we may write,

$$AN : KN :: BA : BK.$$

Multiplying the like terms of these three proportions we may write, after cancelling AL and AN,

$$AF : KN :: (BD)(BM)(BA) : (BP)(BO)(BK).$$

Q. E. D.

74. Corollary 1. *The volume of a cuboid is equal to the product of its three dimensions.*

For, let KN be taken as the unit of cubic measure, which we shall represent by U; BK, BO, and BP, each representing the linear unit; then,

$$AF : U :: (BA)(BM)(BD) : 1 \times 1 \times 1 \quad . \; . \; . \quad (73)$$

$$i.e., \; \frac{AF}{U} = \frac{(BA)(BM)(BD)}{1 \times 1 \times 1}.$$

But $\frac{AF}{U}$ is the volume of AF (70).

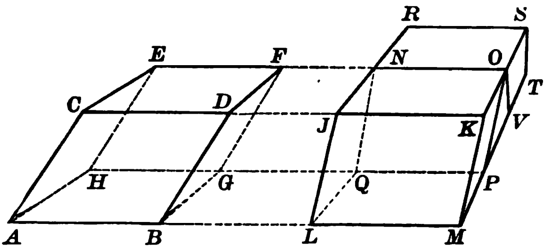
Hence, volume

$$AF = (BA)(BM)(BD).$$

75. Corollary 2. *The volume of a cuboid is equal to the product of its base and altitude.*

PROPOSITION XV.

76. THEOREM. *An oblique parallelopiped is equivalent to that right parallelopiped whose base and altitude are equal respectively to a right section and a lateral edge of the oblique parallelopiped.*



Let AF be an oblique parallelopiped. Produce its horizontal edges AB, CD, HG, and EF. On one of these lay off JK equal to CD; and pass through the points J and K the planes JQ and KP at right angles to the edge JK. Regarding JQ as a base, LO is a right parallelopiped whose base JQ equals a right section of the oblique parallelopiped AF; and the altitude JK of LO equals CD, a lateral edge of the oblique parallelopiped.

We are to prove AF equivalent to LO.

Designate the solid AHEC–LQNJ as the solid A′; and the solid BGFD–MPOK as the solid B′.

Since JQ equals KP (63), we may move solid A′ so that its face JQ will come into coincidence with KP a face of solid B′. Let this be done; then the edge CJ of solid A′ coincides with the edge DK of solid B′. For, these edges are perpendicular respectively to the coincident planes JQ and KP, at the point J in coincidence with K; and only one perpendicular to a plane can be drawn to the same point; and moreover, these edges CJ and DK are equal, since they are made up of equal parts CD and JK, and a common part DJ. In like manner we may prove that, by the proposed movement of the solid, EN comes into coincidence with FO, HQ with GP, and AL with BM. Hence, the solids A′ and B′ may be made to coincide throughout and are therefore equal.

Now, from these equal solids A′ and B′ take away the common part BGFD–LQNJ and there remains

AF equivalent to LO.

Q. E. D.

Since the above demonstration was made wholly independent of the form of the base or right section involved, it applies as well to any other prism as to the parallelopiped. Hence,

77. Corollary 1. *An oblique prism is equivalent to that right prism whose base and altitude are equal respectively to a right section and a lateral edge of the oblique prism.*

78. Corollary 2. *If a cuboid and a parallelopiped have equivalent bases and equal altitudes, they are equal to each other in volume.*

Produce the edges LQ, JN, MP, and KO; lay off OS equal to KO; and pass through O and S planes perpendicular to OS. Then, VR thus formed is a cuboid; and, since NV is a right section of LO, and OS equals KO, LO is equivalent to VR (76). But, AF is equivalent to LO, as we have shown (76); hence, VR is equivalent to AF.

79. Corollary 3. *The volume of any parallelopiped is equal to the product of its base and altitude.*

Regard CF, JO, and NS, as the bases of the parallelopipeds to which they belong. Represent the common altitude of the three solids by a . . . (25); then will the volume of AF be equal to (a) (CF).

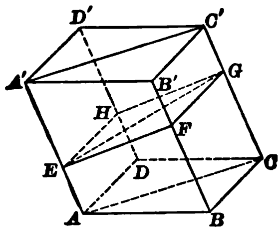
For CF, JO, and NS are equivalent (363); then, multiplying each of the above by a, we obtain (a) (CF), (a) (JO), and (a) (NS) equivalent to one another.

But (*a*) (NS) equals the volume of VR (75); and the volume of VR equals the volume of AF (78). Hence, the volume of AF equals (*a*) (CF). *Q. E. D.*

Note. We shall hereafter use the symbol ⩯ to represent equivalence.

Proposition XVI.

80. Theorem. *A plane passed through the diagonally opposite edges of a parallelopiped divides it into two equivalent triangular prisms.*



Let the plane AC′ pass through the diagonally opposite edges AA′ and CC′ of the parallelopiped BD′; then will the triangular prisms ABC–B′ and ADC–D′, thus formed, be equivalent.

Any right section EFGH is a parallelogram (66, *b*), and EG, drawn from vertex E to vertex G, is a diagonal. Hence the triangle EFG equals the triangle EHG. (359.)

The prisms ABC–B′ and ADC–D′ are equivalent, respectively, to those right prisms whose bases are the right sections EFG and EHG, and whose common altitude is the edge CC′ (77). But EFG and EHG are equal; therefore the right prisms of which they are bases, are equal in volume (65). Hence the prism ABC–B′ is equivalent to the prism ADC–D′. *Q. E. D.*

81. Scholium. ABC–B′ and ADC–D′ are *mutually equal* in all their parts; however, not being superposable, they are not equal solids but symmetrical (41 and 42).

82. Corollary. *The volume of any triangular prism is equal to the product of its base and altitude.*

Let a represent the altitude of ABC–B′; then will the volume of ABC–B′ be equal to (ABC)(a).

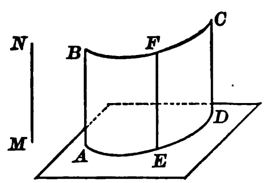
(1). For, vol. BD′ = (2)(ABC–B′) . . . (80),
(2). And, area DB = (2)(ABC) . . . (359),
(3). But, vol. BD′ = (DB)(a) . . . (79).

Now substitute in equation (3), the values of vol. BD′ and area DB from equations (1) and (2) respectively, and we find, (2)(ABC–B′) = (2)(ABC)(a).

Hence, ABC–B′ = (ABC)(a). *Q. E. D.*

DEFINITIONS.

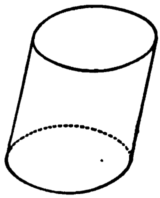
83. A *cylindrical surface* is a curved surface generated by a line moving parallel to a given fixed line and continually touching a guiding curve, the curve and the fixed line not lying in the same plane. The generating line in any position is termed an *element* of the surface.



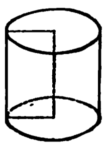
CD is a moving line always parallel to NM a given fixed line, and continually touching DEA, a guiding curve. The guiding curve and the given fixed line lie in different planes.

CD, FE, and BA are elements of the cylindrical surface DB.

84. A *cylinder* is a solid bounded by a cylindrical surface and two parallel planes called ***bases.*** The ***bases*** cut all the elements of the cylindrical surface. If the bases are circles as in the figure, the cylinder is said to be *circular;* and if the generating line is perpendicular to the planes of the circles, the cylinder is termed a *right* circular cylinder. The *axis* of the cylinder is the line joining the centres of the bases.

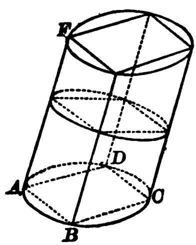


85. A right circular cylinder is called a *cylinder of revolution;* for if a rectangle be revolved about one of its sides as an axis, it will generate a right circular cylinder.



86. If two similar rectangles be revolved about homologous sides as axes, two ***similar*** cylinders of revolution will be generated.

87. A prism is *inscribed* in a cylinder when the bases of the prism are inscribed in the bases of the cylinder. The lateral edges of the prism then become elements of the surface of the cylinder. FC is a prism inscribed in the cylinder FC.



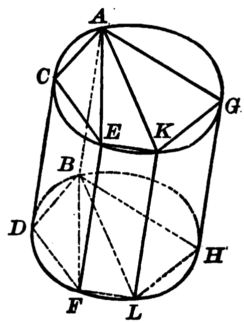
88. *A cylinder may be regarded as the limiting case of an inscribed prism, when the number of lateral faces of the prism is increased indefinitely.*

Thus if an inscribed prism of four lateral faces be changed to an inscribed prism of eight lateral faces, and then the prism of eight faces be again changed to one of sixteen faces, etc., the prism will approach the cylinder as a limit. By indefinitely increasing the number of faces of the inscribed prism, it may ultimately be brought into coincidence with the cylinder, in altitude, in surface, and in volume; that is, in its entirety.

89. For definitions of altitude, right section, etc., of cylinders, see corresponding definitions under prisms.

PROPOSITION XVII.

90. THEOREM. *The volume of any prism is equal to the product of its base and altitude.*



Let the base, altitude, and volume of any prism DG, be represented by B, a, and V, respectively; then will V = (B) (a).

Through the edge AB and the vertices E, K, etc., pass planes AF, AL, etc., dividing the prism DG into triangular prisms. All the prisms thus constructed have the common altitude a.

From (82) we may write the following equations:

vol. (DFB–C) = (DFB) (a).
vol. (FBL–A = (FBL) (a).
vol. (BLH–K) = (BLH) (a).

By axiom the sum of the first members of the three equations equals the sum of the second members; but the sum of the first members equals V; therefore,

$$V = [(DFB) + (FBL) + (BLH)]\ (a);$$

and since $DFB + FBL + BLH = DFLHB$, or B,

$$V = (B)\ (a).$$ *Q. E. D.*

91. Corollary 1. *The volume of any cylinder is equal to the product of its base and altitude.*

For the cylinder is the limiting case of the inscribed prism; and accordingly the theorem of (90) must be true alike for the prism and for the cylinder (88).

92. Corollary 2. *Prisms are to each other as the products of their bases and altitudes. When the altitudes are equal, the prisms are to each other as their bases. When the bases are equal, the prisms are to each other as their altitudes. This* Corollary *is also true of the cylinder.*

Use V, B, and a to represent the volume, base, and altitude, respectively, of a prism; and correspondingly for any other prism use V′, B′ and a'; then,

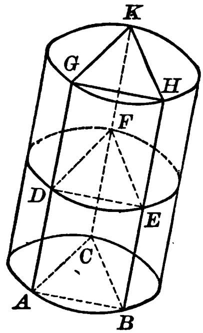
$\frac{V}{V'} = \frac{Ba}{B'a'}$; also when $a = a'$, then $\frac{V}{V'} = \frac{B}{B'}$; or if $B = B'$, then $\frac{V}{V'} = \frac{a}{a'}$.

93. Corollary 3. *Using V and a, as in* (92), *and R to denote the radius of the base, then for any circular cylinder*

$$V = (\pi R^2)\ (a).$$

Proposition XVIII.

94. Theorem. *The lateral surface of a prism is equal to the product of the perimeter of a right section by a lateral edge.*



Let DE + EF + FD be the perimeter of a right section of any prism ABC–K. Let l represent the length of any lateral edge (59). Let S represent the lateral surface; then will

$$S = l\,(DE + EF + FD).$$

Bearing in mind (4) and (372), we may write

$AH = l\,(DE)$, . . . for DE is the altitude of AH;
$BK = l\,(EF)$ . . . for EF is the altitude of BK;
$CG = l\,(FD)$ . . . for FD is the altitude of CG;
then, $AH + BK + CG = l\,(DE + EF + FD)$ by addition;
but, $AH + BK + CG = S$;
hence, $S = l\,(DE + EF + FD)$. *Q. E. D.*

95. Corollary 1. *The lateral surface of a cylinder is equal to the product of the perimeter of a right section by an element of the surface.*

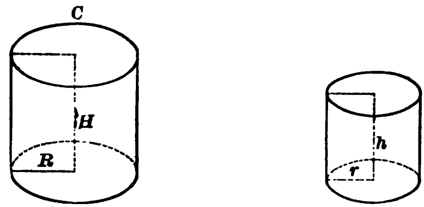
For the cylinder is the limiting case of the inscribed prism (88); and, accordingly, theorem (94) must be true alike for the prism and for the cylinder.

96. Corollary 2. *The lateral surface of a right prism or cylinder is equal to the product of the altitude and the perimeter of the base.*

97. Corollary 3. *Lateral surfaces of right prisms or cylinders, if the perimeters of the bases are equal, are to each other as the altitudes of the solids; if the altitudes are equal, the lateral surfaces are to each other as the perimeters of the bases.*

EXERCISES.

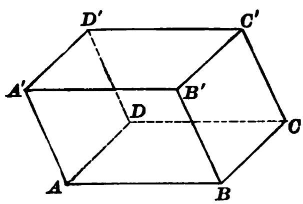
98. Theorem. *The volumes of two similar cylinders of revolution are to each other as the cubes of their altitudes, or as the cubes of the radii of their bases.*



99. Theorem. *The surfaces, lateral or total, of two similar cylinders of revolution are to each other as the squares of their altitudes or as the squares of the radii of their bases.*

Use diagram of (98) for (99).

100. Theorem. *The opposite faces of a parallelopiped are equal.*



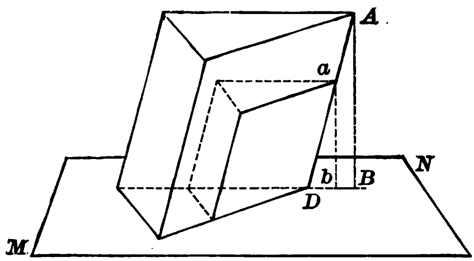
101. THEOREM. *The diagonals of any parallelopiped bisect each other.*

102. DEFINITION. Similar prisms are such as have the same number of faces, each face of the one being similar to a corresponding face of the other, and similarly placed with respect to adjoining faces.

103. THEOREM. *With respect to their triedral angles, two similar prisms are mutually equiangular.*

Use diagram of (104) for (103).

104. THEOREM. *Two similar prisms are to each other as the cubes of their corresponding edges.*



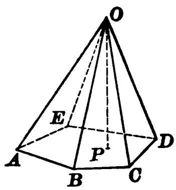
Place the prisms, as in the diagram, with their corresponding bases on the plane MN. Draw the perpendiculars AB and *ab*. Then AB, *ab*, AD, and DB lie in the same plane; DB being in the plane MN.

AB and *ab* are the altitudes of the respective prisms.

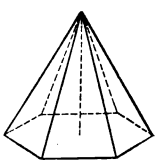
SECTION III.

PYRAMIDS AND CONES.

DEFINITIONS.

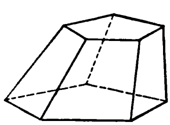


105. A *pyramid* is a solid enclosed by the faces of a polyedral angle and a plane cutting all these faces (36). The cutting plane bounded by its intersections with the faces is called the *base.* The line from the vertex (36) to the centre of the base is called the *axis.* The perpendicular distance from the vertex to the base is termed the *altitude.* The *lateral faces* are all the faces except the base. The *lateral surface* is the sum of the lateral faces. The *lateral edges* are the edges of the polyedral angle (36).



106. A *right* pyramid is one whose axis is perpendicular to the base; if the base is a regular polygon the right pyramid is termed a *regular* pyramid.

107. A *truncated* pyramid is a portion of a pyramid included between the base and any section cutting all the lateral edges. If the section is parallel to the base, the truncated pyramid is called a *frustum,* and the section is its upper base. The *altitude* of the frustum is the distance between its bases. The *lateral faces* are all the faces except the bases, and the *lateral surface* is the sum of the lateral faces.

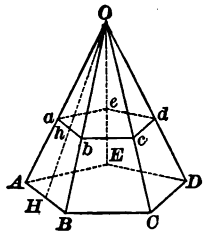


108. Pyramids and frustums, like prisms, are named from their bases (55).

From the above definitions we may readily deduce for the regular pyramid and for its frustum the following:

109. *The lateral edges are equal.*

110. *The altitudes of the lateral faces are equal.*



111. *The lateral faces are equal.*

Also for the frustum we may deduce:

112. *The lateral faces are trapezoids.*

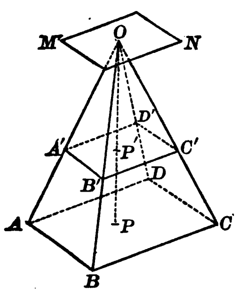
113. *The perimeter of a right section midway between the bases equals half the sum of the perimeters of the bases.*

The demonstration of the above is left to the student.

114. The common altitude of the lateral faces is called the *slant height* (110).

PROPOSITION XIX.

115. THEOREM. *Any section of a pyramid parallel to the base is a polygon similar to the base.*



Let A′B′C′D′ be a section of the pyramid O–ABCD, parallel to the base ABCD; then will A′B′C′D′ be similar to ABCD.

For, A′B′ is parallel to AB . . . (18),
and B′C′ is parallel to BC.

Hence, angle

A′B′C′ equals ABC . . . (19).

And in like manner we can prove that the section and the base have corresponding angles equal, each to each, throughout.

Now, triangle

OAB is similar to OA′B′ . . . (362);

and triangle

OBC is similar to OB′C′.

Hence, $$\frac{A'B'}{AB} = \left(\frac{OB'}{OB}\right) = \frac{B'C'}{BC} \ldots (370).$$

That is, A′B′ : AB :: B′C′ : BC.

And in like manner we may prove the sides of the section and of the base, proportional throughout.

The section and the base have their sides proportional, and are mutually equiangular, as we have shown; hence they are similar. *Q. E. D.*

116. Scholium: *The lateral edges and the altitude of a pyramid are divided proportionally by the section parallel to the base.*

The section A′C′ parallel to the base cuts the altitude OP in the point P′. A plane MN passed through the vertex O parallel to the base, cuts the altitude and all the lateral edges in the point O.

Then, AA′ : A′O :: BB′ : B′O :: PP′ : P′O, etc. (26).

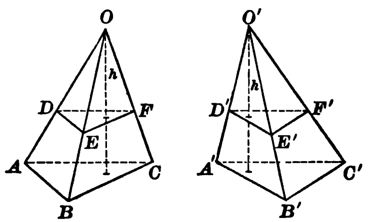
117. Corollary. *The base of a pyramid and the section parallel to the base are proportional to the squares of their distances from the vertex.*

For, $AA' : A'O :: PP' : P'O$ . . . (116);
hence, $AO : A'O :: PO : P'O$ (composition).
But $AO : A'O :: AB : A'B'$ (similar triangles);
hence, $AB : A'B' :: PO : P'O$
or, $\overline{AB}^2 : \overline{A'B'}^2 :: \overline{PO}^2 : \overline{P'O}^2$.
But, $AC : A'C' :: \overline{AB}^2 : \overline{A'B'}^2$ . . . (115);
hence, $AC : A'C' :: \overline{PO}^2 : \overline{P'O}^2$.

Q. E. D.

Proposition XX.

118. Theorem. *If two pyramids have equal altitudes and equivalent bases, sections made by planes parallel to their bases and at equal distances from their vertices are equivalent.*



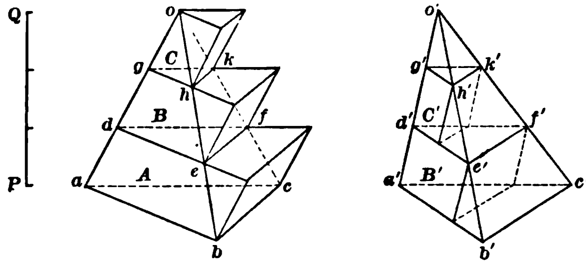
Let H be the common altitude of the pyramids O–ABC and O′–A′B′C′; and let h be the common distance from the vertices of the sections DEF and D′E′F′. Let the sections be parallel to the bases ABC and A′B′C′, and let the bases be equivalent. Then will the sections DEF and D′E′F′ be equivalent.

For, $ABC : DEF :: H^2 : h^2$ . . . (117);
and, $A'B'C' : D'E'F' :: H^2 : h^2$;
Hence, $ABC : DEF :: A'B'C' : D'E'F'$.
But, ABC ⇔ A′B′C′ (hypothesis);
hence DEF ⇔ D′E′F′.

Q. E. D.

PROPOSITION XXI.

119. THEOREM. *Triangular pyramids having equivalent bases and equal altitudes are equal in volume.*



Let *o* and *o'* represent two triangular pyramids having their bases *abc* and *a'b'c'* equivalent; and let any line QP be the common altitude of the pyramids. Then will *o* be equivalent to *o'*.

Place the pyramids upon the same horizontal plane. Divide the common altitude QP into equal parts; and, through the points of division pass planes parallel to the plane of their bases, forming sections *def* and *ghk* in pyramid *o*, and *d'e'f'* and *g'h'k'* in pyramid *o'*.

At pyramid *o*, construct the prisms *abc-d*, *def-g*, and *ghk-o*; and in pyramid *o'* construct the prisms *d'e'f'-a'* and *g'h'k'-d'*.

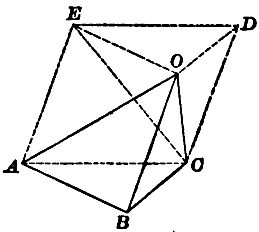
(*a*). It is evident that the difference between the sum of the prisms of *o* and the sum of the prisms in *o'*, is greater than the difference between pyramids *o* and *o'*.

Now *def-g* and *ghk-o* are equivalent respectively to *d'e'f'-a'* and *g'h'k'-d'*, the bases being equivalent and altitudes equal, (82). And hence the sum of all the prisms of *o* is greater than the sum of all the prisms of *o'*, by as much as the volume of the prism constructed on *abc* as a base.

By increasing the number of equal parts into which QP is divided, and then with these parts as altitudes constructing new series of prisms, we may make the volume of the prism on the base *abc* as small as we please; and hence make the sum of the prisms of *o* differ from the sum of the prisms of *o′*, by as small an amount as we please. Thus by dividing QP into a sufficient number of parts the difference between the sums of these series of prisms may be made indefinitely small; but, since this difference is greater than the difference between the pyramids *o* and *o′*, see paragraph (*a*), *o* must be regarded as equivalent to *o′*. *Q. E. D.*

Lemma XXII.

120. Theorem. *A triangular prism may be divided into three equivalent triangular pyramids.*



Let DEO–ACB be a triangular prism. Pass through it the planes AOC and EOC, forming three pyramids whose common vertex is O; then will these three pyramids ABC–O, AEC–O, and EDC–O be equivalent.

Now, AEC–O and EDC–O have their bases in the same plane, AD, and their vertices at the same point O; hence, they have a common altitude. Moreover, their bases are equal, being halves of the parallelogram AD (359).

Therefore AEC–O ⫏ EDC–O (119).

Pyramid EDC–O may be regarded as having EOD for its base and C for its vertex. It is therefore equivalent to ABC–O (119).

Hence, ABC–O, AEC–O, and EDC–O are equivalent.

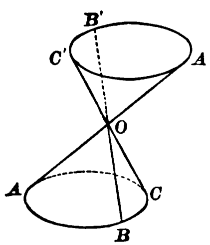
Q. E. D.

121. Corollary. *The volume of a triangular pyramid is equal to one-third the product of its base and altitude.*

For, the volume of the prism DEO–ACB equals the product of its base and altitude (82); and from (120), the pyramid ABC–O, constructed with the same base and altitude as the prism, has a volume one-third as great. Hence, the volume of a triangular pyramid is equal to one-third the product of its base and altitude.

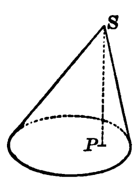
DEFINITIONS.

122. A *conical* surface is a curved surface generated by a moving line passing through a fixed point and continually touching a guiding curve, the curve and the fixed point not lying in the same plane. The generating line in any position is termed an *element* of the surface. The point in which the elements meet is called the *vertex*.



It follows from the definition that the conical surface will consist of two parts, each of which has its vertex at O. We shall, however, refer to only the one part, O–ABC, as the conical surface

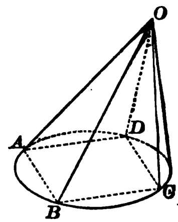
123. A *cone* is a solid bounded by a conical surface and a plane called the *base* passing through all the elements of the surface. The line from the vertex to the centre of the base is called the *axis*. If the base is a circle, the cone is said to be *circular;* and if the axis is perpendicular to the circle, the cone is termed a *right* circular cone.



124. A right circular cone is called a *cone of revolution;* for if a right triangle be revolved about one of its sides as an axis it will generate a right circular cone.

125. If two similar triangles are revolved about homologous sides as axes, two *similar* cones of revolution will be generated.

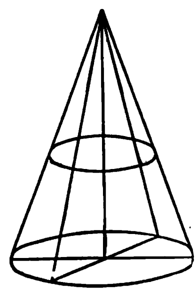
126. A pyramid is *inscribed* in a cone when the base of the pyramid is inscribed in the base of the cone and the two solids have a common vertex. The *lateral edges* of the pyramid are elements of the surface of the cone.



127. *A cone may be regarded as the limit of an inscribed pyramid when the number of its lateral faces is increased indefinitely.*

By indefinitely increasing the number of lateral faces of the pyramid O–ABCD inscribed in the cone O–ABCD, the pyramid is made to approach the cone as its limit. Ultimately, then, the pyramid in its entirety becomes the cone.

For definitions of altitude, right section, frustum, etc., of a cone, see corresponding definitions under pyramid.



128. From the above definitions we may readily deduce the following:

In the right circular cone:

129. *All elements of the surface are equal.*

130. *Sections embracing the axis are isosceles triangles.*

131. *Right sections are circles.*

132. *The circumference of the right section bisecting the altitude equals one-half the circumference of the base.*

In the frustum of the right cone:

133. *All elements of the surface are equal.*

134. *Sections embracing the axis are trapezoids.*

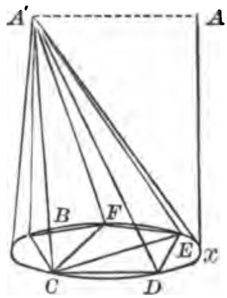
135. *The circumference of the right section bisecting the altitude equals one-half the sum of the circumferences of the bases.*

The demonstration of the above is left to the student.

136. *An element of the surface is taken as the slant height* (129 and 133).

Proposition XXIII.

137. Theorem. *The volume of any pyramid is equal to one-third the product of its base and altitude.*



Let V represent the volume of a pyramid, BCDEF the base, and Ax the altitude; then will V = $\frac{1}{3}$ (BCDEF) (Ax).

Divide the pyramid into triangular pyramids by the planes A′CE and A′CF. Represent the volumes of these triangular pyramids by V′, V″, etc.

Now, V′ = $\frac{1}{3}$ (BCF) (Ax) . . . (121);
and V″ = $\frac{1}{3}$ (FCE) (Ax), etc.
Hence, V′ + V″ + V‴ = $\frac{1}{3}$ (BCF + FCE + ECD) (Ax).
But, V′ + V″ + V‴ = V,
and, BCF + FCE + ECD = BCDEF.
Hence, V = $\frac{1}{3}$ (BCDEF) (Ax). *Q. E. D.*

138. Corollary. *The volume of any cone is equal to one-third the product of the base and altitude.*

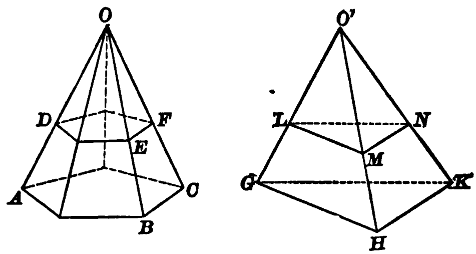
For, in any cone A′–BCDEF inscribe a pyramid A′–BCDEF.

By indefinitely increasing the number of its lateral faces the inscribed pyramid is made to approach the cone as a limit (127).

Hence, it is true of the cone as of the pyramid, that the volume is equal to one-third the product of the base and altitude.

FRUSTUMS OF PYRAMIDS AND CONES.

139. PROBLEM. *Given the bases and altitude of the frustum of a pyramid to find the altitude of the smaller pyramid removed to form the frustum.*



Let B represent the area of the lower base.

Let b represent the area of the upper base.

Let H represent the altitude of the frustum.

Let x represent the altitude of the smaller pyramid.

Then $H + x$ will represent the altitude of the entire pyramid.

We are to find the value of x in terms of B, b, and H.

Now, $B : b :: (H + x)^2 : x^2$. (117).

Hence, $\sqrt{B} : \sqrt{b} :: H + x : x$.

Hence, $x\sqrt{B} = H\sqrt{b} + x\sqrt{b}$;

or, $x\sqrt{B} - x\sqrt{b} = H\sqrt{b}$;

or, $$x = \frac{H\sqrt{b}}{\sqrt{B} - \sqrt{b}}$$ *Q. E. F.*

140. *From the conditions given in (139), to find the volume of the entire pyramid of which the frustum is a part.*

Let V represent the required volume, then we are to find the value of V in terms of B, b, and H.

$$V = \tfrac{1}{3}(B)(H + x) \ . \ . \ . \ (137).$$

Substitute the value of x found in . . . (139);

then, $$V = \tfrac{1}{3}(B)\left(H + \frac{H\sqrt{b}}{\sqrt{B} - \sqrt{b}}\right);$$

or, $$V = \tfrac{1}{3}B\left(\frac{H\sqrt{B} - H\sqrt{b} + H\sqrt{b}}{\sqrt{B} - \sqrt{b}}\right);$$

or, $$V = \tfrac{1}{3}B\frac{(H\sqrt{B})}{\sqrt{B} - \sqrt{b}}.$$ *Q. E. F.*

141. *From the conditions given in (139), to find the volume of the frustum.*

Let V′ represent the volume of the smaller pyramid.
Let F represent the volume of the frustum.
We are to find the value of F in terms of B, b, and H.

Now, $F = V - V'$.

But, $V' = \dfrac{\tfrac{1}{3}b(H\sqrt{b})}{\sqrt{B} - \sqrt{b}}$, . . . (137) and (139);

and, $V = \tfrac{1}{3}B\dfrac{(H\sqrt{B})}{\sqrt{B} - \sqrt{b}}$, . . . (140).

Hence, $$F = \frac{\tfrac{1}{3}B(H\sqrt{B})}{\sqrt{B} - \sqrt{b}} - \frac{\tfrac{1}{3}b(H\sqrt{b})}{\sqrt{B} - \sqrt{b}};$$

or, $$F = \frac{\tfrac{1}{3}H(B\sqrt{B} - b\sqrt{b})}{\sqrt{B} - \sqrt{b}};$$

or, $$F = \tfrac{1}{3}H(B + \sqrt{Bb} + b);$$

or, $$F = \tfrac{1}{3}HB + \tfrac{1}{3}Hb + \tfrac{1}{3}H\sqrt{Bb}.$$ *Q. E. F.*

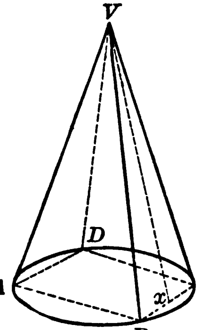
142. Scholium. The formulæ obtained in (139), (140), and (141), apply equally well to all frustums of pyramids or of cones.

These formulæ may be stated in general terms; (141) may be expressed as follows:

143. *The volume of any frustum is equal to the volume of three pyramids or cones having for their altitude the altitude of the frustum, and for their bases, the bases of the frustum, and a mean proportional between the bases.*

Proposition XXIV.

144. Theorem. *The lateral surface of a regular pyramid is equal to one-half the product of the slant height by the perimeter of the base.*



Let V–ABCD be a regular pyramid whose slant height is Vx. Let A represent the lateral surface, P the perimeter of the base, and S the slant height, then will A $= \frac{1}{2}$ (S) (P).

Now, S is the common altitude of all the lateral faces (114).

Hence, $A = \frac{1}{2} S (AB) + \frac{1}{2} S (BC) + \frac{1}{2} S (CD) + \frac{1}{2} S (DA)$ . . . (364);

or, $A = \frac{1}{2} S (AB + BC + CD + DA)$;

or, $A = \frac{1}{2} (S) (P)$. *Q. E. D.*

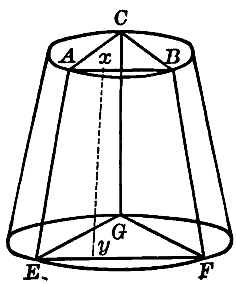
145. Corollary 1. *The lateral surface of a right cone is equal to one-half the product of the slant height by the circumference of the base.*

For the cone may be regarded as the limiting case of the inscribed pyramid (127). Accordingly (144) must be true, alike for the pyramid and for the cone.

146. Corollary 2. *When A represents the lateral surface, S the slant height, and R the radius of the base, cf a cone of revolution; then $A = \pi (R) (S)$.*

Proposition XXV.

147. Theorem. *The lateral surface of the frustum of a regular pyramid is equal to one-half the product of the slant height by the sum of the perimeters of the bases.*



Let EFG–ABC be the frustum of a regular pyramid whose slant height is xy. Let P represent the perimeter of the lower base, p the perimeter of the upper base, S the slant height, and A the lateral surface; then will

$$A = \tfrac{1}{2} S (p + P).$$

Now, S is the common altitude of all the lateral faces (114), and each face is a trapezoid (112).

Hence,

$$A = \tfrac{1}{2} S (AB + EF) + \tfrac{1}{2} S (BC+FG) + \text{etc.} \ldots (366);$$

or, $A = \tfrac{1}{2} S (AB + EF + BC + FG + CA + GE)$;

or, $A = \tfrac{1}{2} S [(AB + BC + CA) + (EF + FG + GE)]$;

or, $A = \tfrac{1}{2} S (p + P)$. *Q. E. D.*

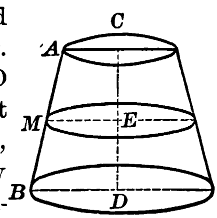
148. Corollary 1. *The lateral surface of the frustum of a right cone is equal to one-half the product of the slant height by the sum of the circumferences of the bases.*

For, the frustum of a cone is the limiting case of the inscribed frustum of a pyramid; and accordingly theorem (147) must be true alike for the frustum of the pyramid and for the frustum of the cone.

149. Corollary 2. *The lateral surface of the frustum of a right cone may be found by multiplying its slant height by the circumference of the right section equidistant from the bases.*

For, the circumference of the right section equidistant from the bases equals one-half the sum of the circumferences of the bases (135).

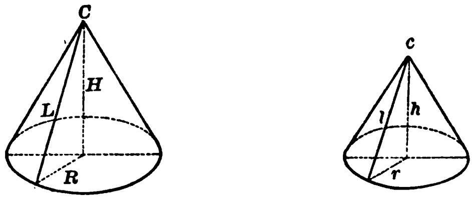
150. Scholium. Let the lines AB and CD be drawn in the same plane, and let M be the middle point of AB. Let AC, ME, and BD be perpendiculars drawn to CD. Now, if ABDC be revolved about CD as an axis, the frustum of a right circular cone will be generated, AC, BD, and ME generating respectively an upper base, a lower base, and a middle right section. If we represent the lateral surface of the frustum by A, then will



$$A = BA \,.\, 2\pi \,.\, ME \ldots (149).$$

EXERCISES.

151. THEOREM. *The volume of two similar cones of revolution are to each other as the cubes of their altitudes, or as the cubes of the radii of their bases.*

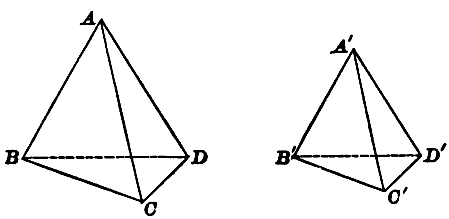


152. THEOREM. *The surfaces, lateral or total, of two similar cones of revolution are to each other as the squares of their altitudes, or as the squares of the radii of their bases.* Use diagram of (151).

153. DEFINITION. *Similar pyramids* are such as have the same number of faces, each face of the one being similar to a corresponding face of the other, and similarly placed with respect to adjoining faces.

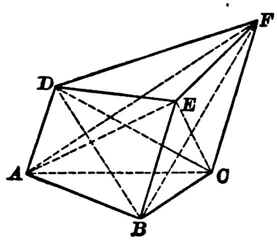
154. THEOREM. *With respect to their triedral angles, two similar pyramids are mutually equiangular.* Use diagram of (155). Also see (39), (40), and (43).

155. THEOREM. *Two similar pyramids are to each other as the cubes of their corresponding edges.*

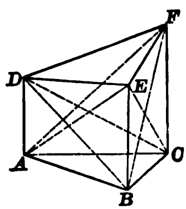


Place the equal triedral angles A and A′ in coincidence, and draw from the common vertex A, a perpendicular to the base BCD; thus representing the altitude of the pyramids.

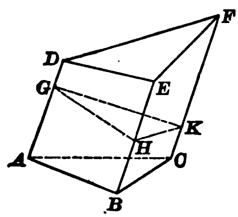
156. THEOREM. *In any truncated triangular prism ABC–F, the sum of the three pyramids ABC–F, ABC–E, and ABC–D is equivalent to the truncated prism.*



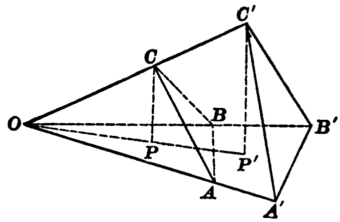
157. COROLLARY 1 to (156). *The volume of any truncated right triangular prism ABC–F is equal to the product of its base ABC by one-third the sum of its lateral edges AD, BE, and CF.*



158. COROLLARY 2 to (156). *The volume of any truncated triangular prism ABC–F is equal to the product of the right section GHK, by one-third the sum of the lateral edges AD, BE, and CF.*



159. THEOREM. *Two triangular pyramids having a triedral angle of the one equal to a triedral angle of the other, are to each other as the products of the edges including the equal triedral angles.*



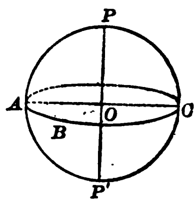
Place the equal triedral angles in coincidence at O. Draw CP and C'P' perpendicular to the face OA'B'.

SECTION IV.

THE SPHERE.

DEFINITIONS.

160. A *sphere* is a solid bounded by a curved surface, every point of which is equally distant from a fixed point within. The fixed point is the *centre*. Any line from the centre to the surface is a *radius*. Any line through the centre limited by the surface is a *diameter*.



161. A plane or a line is *tangent* to a sphere when it touches the surface of the sphere in only one point.

From the above definitions the student may readily deduce the following:

162. *Radii of the same or equal spheres are equal.*

163. *Diameters of the same or equal spheres are equal.*

164. *Two spheres of equal radii may be made to coincide by placing their centres in coincidence.*

165. *A sphere may be generated by the revolution of a semicircle about the diameter as an axis.*

166. *Sections of a sphere through the centre are equal circles of the same radius as the sphere. Such circles are called great circles.*

167. *Any great circle bisects the sphere.*

168. *Two great circles bisect each other.*

169. *A great circle is determined by two points in the surface of the sphere, unless the points are extremities of a diameter.*

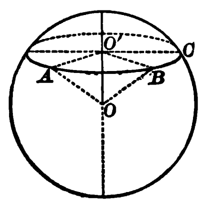
170. *The plane perpendicular to the radius of a sphere at its extremity is tangent to the sphere. And conversely the plane which is tangent to a sphere is perpendicular to the radius drawn to the point of contact.*

171. The *distance* between two points on the surface of a sphere is measured on the arc of a great circle joining the points.

172. A sphere is *inscribed* in a polyedron when the faces of the polyedron are tangent to the sphere. The polyedron is then *circumscribed* about the sphere.

Proposition XXVI.

173. Theorem. *Every section of a sphere made by a plane is a circle.*



The section through the centre is a circle (166). Let ABC be any section not through the centre O; then will ABC be a circle.

Draw a diameter perpendicular to the section ABC, piercing it at some point O′. From the centre O to any points B and A in the perimeter of the section ABC, draw OB and OA. Draw BO′ and AO′.

Now, in the triangles OBO′ and OAO′, the angle OO′A equals the angle OO′B (4); OB equals AO being radii, and OO′ is common; hence the triangles are equal, and therefore AO′ equals BO′. That is, any points B and A in the perimeter of the section ABC are equally distant from O′, a point within. By definition then, ABC is a circle having O′ for its centre. *Q. E. D.*

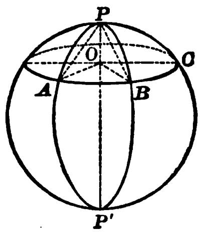
174. Scholium. A diameter of a sphere perpendicular to a circle is termed the *axis* of the circle. The extremities of the axis are the *poles* of the circle.

175. Corollary. *The axis of any circle of a sphere pierces the centre of the circle.*

For the diameter perpendicular to any circle ABC has been shown to pierce the centre of ABC.

Proposition XXVII.

176. Theorem. *Either pole of a circle of a sphere is equally distant from any two points in the circumference of the circle.*



Let A and B be any two points in the circumference of the circle ABC whose poles are P and P′; then will the arcs AP and BP be equal, also the arcs AP′ and BP′ will be equal.

Let O be the intersection of the axis PP′ with the circle ABC. Draw the straight lines AO, BO, AP, and BP.

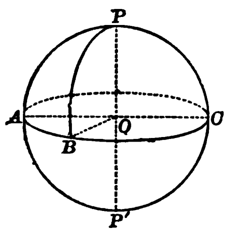
Now, in the triangles AOP and BOP, the angle AOP equals the angle BOP (4); the side AO equals the side BO (175); and OP is common to the two triangles; hence, the triangles are equal and homologous parts AP and BP are equal. Then the arc AP equals the arc BP (374). In like manner we may prove that the arc AP′ equals the arc BP′.

Q. E. D.

177. SCHOLIUM. The distance from the pole of a circle of a sphere to a point in its circumference is called the *polar distance* of the circle.

178. COROLLARY 1. *The polar distances of two equal circles of the same or equal spheres are equal.*

For, the diameters of equal circles of a sphere are chords of equal arcs of great circles, each of these equal arcs being twice the polar distance of one of these equal circles.



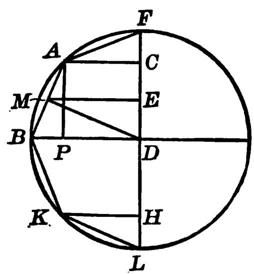
179. COROLLARY 2. *The polar distance of a great circle is a quadrant.*

For, AOP is a right angle (174); hence arc AP which is the measure of AOP is a quadrant. In like manner arc BP is found to be a quadrant.

180. COROLLARY 3. *That point on the surface of a sphere which is a quadrant's distance from two points in the circumference of a great circle is the pole of the great circle.*

PROPOSITION XXVIII.

181. THEOREM. *The surface of a sphere is equal to the area of four great circles.*



Let the sphere whose centre is D be generated by the revolution of the semicircle FBL about the diameter FL as an axis; and let R represent the radius and S the surface of the sphere; then will πR^2 represent the area of a great circle (373) and S will equal $4\pi R^2$.

In the semicircle inscribe FABKL one-half of the regular polygon. From M, the centre of the chord AB, and from the vertices A, B, and K, draw to FL the perpendiculars ME, AC, BD, and KH. Also draw AP, a perpendicular to BD. MD is the radius of the circle which may be inscribed in the regular polygon FABKL etc., (367); call this radius r.

In the revolution of the semicircle, the line AB will generate the surface of a frustum; and if A represent this surface, then will $A = BA \cdot 2\pi \cdot ME = 2\pi \cdot BA \cdot ME$ (150).

Now, by the similar triangles MDE and ABP,

$$MD : BA :: ME : PA \text{ or } CD;$$

hence, $BA \cdot ME = MD \cdot CD.$

But, $A = 2\pi BA \cdot ME;$

hence, $A = 2\pi \cdot MD \cdot CD = 2\pi \cdot r \cdot CD.$

182. This proves that the surface generated by a side of this regular polygon is equal to the circumference of the inscribed circle multiplied by the projection of the side upon the axis.

In the revolution, then, the sides will generate the following areas:

area $AF = 2\pi r \cdot FC$ . . . (182);

area $AB = 2\pi r \cdot CD;$

area $BK = 2\pi r \cdot DH;$

area $KL = 2\pi r \cdot HL.$

Call the sum of these areas S'; then,

$$S' = 2\pi r\,(FC + CD + DH + HL).$$

Now, $FC + CD + DH + HL =$ the diameter, or $2R$.

Hence, $S' = (2\pi r)(2R).$

If the number of sides of the inscribed semi-polygon be increased indefinitely, it will approach the semicircle as its limit and r will approach R as its limit; at the limit then,

$$S = (2\pi R)(2R);$$

or, $S = 4\pi R^2.$ *Q. E. D.*

183. Corollary 1. *The surfaces of two spheres are to each other as the squares of the radii of the spheres.*

For, if we represent the surfaces of two spheres by S and S′, and the radii by R and R′, then,

$$\frac{S}{S'} = \frac{4\pi R^2}{4\pi R'^2} = \frac{R^2}{R'^2}.$$

184. Scholium. The surface generated by an arc AB is termed the *zone* AB. CD, the projection of the arc upon the axis, is the *height* of the zone.

185. Corollary 2. *The area of a zone is equal to the circumference of a great circle multiplied by the height of the zone.*

For, at the limit, area AB $= 2\pi \cdot r \cdot$ CD becomes, —

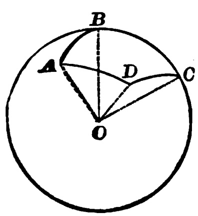
$$\text{zone AB} = 2\pi \cdot R \cdot CD.$$

186. Corollary 3. *Zones on the same or equal spheres are to each other as their height.*

For $$\frac{\text{zone AB}}{\text{zone BK}} = \frac{2\pi R \cdot CD}{2\pi R \cdot DH} = \frac{CD}{DH}.$$

DEFINITIONS.

187. A *spherical polygon* is that portion of the surface of a sphere intercepted by the faces of a polyedral angle having its vertex at the centre of the sphere. ABCD is a spherical polygon.



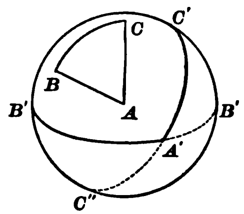
188. It follows from the definition that the sides of a spherical polygon are those *arcs of great circles* which measure the face angles of the polyedral angle at the centre; and that each angle of the polygon corresponds to a *diedral angle* between two faces of the polyedral angle. Thus, AB, a side of the polygon, is the arc of a great circle which measures

AOB, an angle at the centre. And BAD, an angle of the polygon, corresponds to the diedral angle B–AO–D; to which angle it is equal in value; *i.e.*, BAD and B–AO–D contain the same number of degrees.

189. A *spherical triangle* is a spherical polygon of three sides; it, like the plane triangle, may be right, oblique, isosceles, etc. Its three sides and three angles are called the *parts* of the triangle.

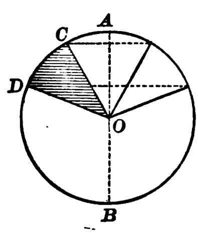
190. A *spherical angle* may be defined as the angle on the surface of the sphere between two intersecting arcs of great circles. The spherical angle between the arcs of two intersecting circles, and the corresponding diedral angle between the planes of these circles are equal in value (188).

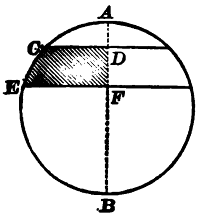
191. If, from the vertices of a spherical triangle as poles, arcs of great circles be described, a second spherical triangle will be formed which is called the *polar triangle* of the first. If A, B, and C are poles of B′C′, C′A′, and A′B′, respectively, then will A′B′C′ be the polar triangle of ABC.



192. A *spherical pyramid* is a solid enclosed by a spherical polygon and the faces of the polyedral angle which intercept the polygon. See O–ABCD (187). The spherical polygon is called the *base*.

193. *A spherical sector* is that portion of a sphere generated by revolving a sector of a great circle about any diameter of the sphere lying in the plane of the sector. The plane sector DOC revolved about AB as an axis generates a solid or spherical sector. The base of the sector is the zone generated by the arc DC.



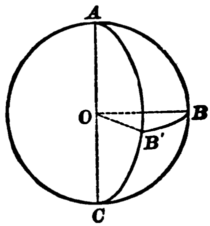


194. *A spherical segment of one base* is the portion of a sphere generated by revolving a semi-segment of a great circle about that diameter which bisects the segment. The plane semi-segment EAF revolved about AB as an axis generates a solid or spherical segment. The plane face is called the *base.*

195. A spherical segment of *two bases* is the portion of a sphere included between two parallel planes. It is generated by revolving CDFE about the diameter AB; CD and EF being perpendicular to AB. See diagram (194).

PROPOSITION XXIX.

196. THEOREM. *The measure of a spherical angle is the arc of a great circle described from its vertex as a pole, included between its sides produced if necessary.*



Let BB′ be the arc of a great circle described from the vertex of the spherical angle BAB′ as a pole, included between BA and B′A, the sides of the angle; then will BB′ be the measure of BAB′.

To O, the centre of the sphere, draw BO and B′O. Now, AB and AB′ are quadrants (179); hence, AOB and AOB′ are right angles (375). It follows, then, that BOB′, or BB′ which measures BOB′, is the measure of the diedral angle between the planes of the circles through AB and AB′ (30).

And since BB′ is the measure of the diedral angle between the planes through AB and AB′, it is the measure of the spherical angle between AB and AB′ (190).

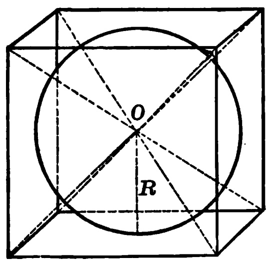
Q. E. D.

197. Corollary. *Tangents to the sides of a spherical angle, at its vertex, include a plane angle equal in value to the spherical angle.*

For such a plane angle measures the diedral angle between the planes through the sides of the spherical angle (376).

Proposition XXX.

198. Theorem. *The volume of a sphere is the product of its surface by one-third the radius.*



Let the volume of the sphere whose centre is O be represented by V, the radius by R, and the surface by S; then will $V = \frac{1}{3} R \cdot S$.

Let a cube be circumscribed about the sphere (172). To the vertices of the cube draw lines from the centre O. Conceive the cube to be formed of six pyramids whose bases are the faces of the cube and whose common vertex is the centre of the sphere.

Now, these pyramids have a common altitude R (172 and 170); and since the volume of each pyramid is $\frac{1}{3}$ R times its base (137), the volume of all these pyramids, *i.e.*, the volume of the cube, is $\frac{1}{3}$ R times the surface of the cube.

Let the polyedron circumscribed about the sphere have the number of its faces increased, all of them, however, remaining tangent; still the volume of the circumscribed solid is $\frac{1}{3}$ R times its surface.

When the number of faces of the circumscribed solid is made infinite, the solid will approach the sphere as a limit; and, at the limit, the volume of the sphere is $\frac{1}{3}$ R times the surface of the sphere; or $V = \frac{1}{3} R \cdot S$. *Q. E. D.*

199. COROLLARY 1. *The volume of a sphere equals $\frac{4}{3} \pi R^3$, or $\frac{1}{6} \pi D^3$, D being the diameter.*

For, $V = \frac{1}{3} R \cdot S$. and $S = 4 \pi R^2$; hence $V = \frac{4}{3} \pi R^3$ or $\frac{1}{6} \pi D^3$.

200. COROLLARY 2. *Spheres are to each other as the cubes of their radii or diameters.*

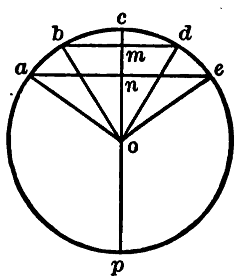
For, $$\frac{V}{V'} = \frac{\frac{4}{3} \pi R^3}{\frac{4}{3} \pi R'^3} = \frac{R^3}{R'^3}; \text{ and, } \frac{V}{V'} = \frac{\frac{1}{6} \pi D^3}{\frac{1}{6} \pi D'^3} = \frac{D^3}{D'^3}.$$

201. COROLLARY 3. *The volume of a spherical pyramid is equal to the product of its base by one-third the radius of the sphere.*

202. COROLLARY 4. *The volume of a spherical sector is equal to the product of the zone which forms its base by one-third the radius of the sphere.*

PROPOSITION XXXI.

203. PROBLEM. *To find the volume of a spherical segment.*



Let *a, b, c, p,* be a circle whose centre is *o*. From *a* and *b* draw perpendiculars *an* and *bm* to *cp*, a diameter. Revolve the semicircle *acp* about *cp* as an axis; then, the plane sectors *boc* and *aoc* will generate corresponding spherical sectors. Call these S and S′, respectively.

The triangles *bmo* and *ano* will generate right circular cones; call these C and C′ respectively.

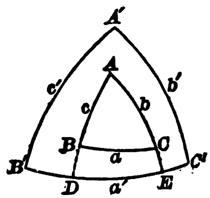
The semi-segments *bcm* and *acn* will generate corresponding spherical segments of one base, and *anmb* will generate a spherical segment of two bases; call these spherical segments O, O′, and T, respectively.

Now, $O = S - C$; and $O' = S' - C'$; and $T = O' - O$.

By means of these formulæ the volume of any spherical segment may be found. *Q. E. F.*

Proposition XXXII.

204. Theorem. *If one spherical triangle is the polar triangle of a second, then, reciprocally, the second is the polar triangle of the first.*



Let A′B′C′ be the polar triangle of ABC; then will ABC be the polar triangle of A′B′C′.

Since A is the pole of the arc B′C′, B′ is at a quadrant's distance from A. And since C is the pole of the arc B′A′, B′ is also a quadrant's distance from C. Now, since B′ is at a quadrant's distance from both A and C, B′ must be the pole of the arc AC (180).

In like manner we may prove that C′ is the pole of the arc BA, and A′ is the pole of the arc BC.

Hence, ABC is the polar triangle of A′B′C′ (191).

Q. E. D.

205. Corollary 1. *If A is any angle of a spherical triangle, and if B′C′ is the side opposite A, in the polar triangle, then will A be equivalent to 180° − B′C′.*

For, produce the arcs AB and AC until they meet B′C′ at some points D and E.

	(*a*)	Now, arc B′E = 90° (179).
	(*b*)	And arc C′D = 90°;
	(*c*)	hence, B′E + C′D = 180°, adding (*a*) and (*b*).
But,		B′E + C′D = B′E + (EC′ + ED);
or,		B′E + C′D = B′C′ + ED; substitute in (*c*),
then,		B′C′ + ED = 180°;
hence,		ED = 180° −B′C′.
But arc		ED is the measure of angle A (196);
hence,		A ⇔ 180° − B′C′. *Q. E. D.*

206 Scholium. If in two spherical triangles all the angles of the one are equal to all the angles of the other, each to each, the triangles are said to be *mutually equiangular;* if in the two triangles all the sides of the one are equal to all the sides of the other, each to each, the triangles are said to be *mutually equilateral.*

207. Corollary 2. *If two spherical triangles are mutually equiangular, their polar triangles are mutually equilateral.*

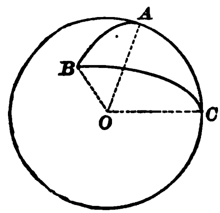
For, homologous sides in the polar triangles are supplements of equal angles in the primitive triangles (205).

208. Corollary 3. *If two spherical triangles are mutually equilateral, their polar triangles are mutually equiangular.*

For, homologous angles in the polar triangles are supplements of equal sides in the primitive triangles (205).

PROPOSITION XXXIII.

209. THEOREM. *Any side of a spherical triangle is less than the sum of the other two sides.*



In the spherical triangle ABC let the side BC be greater than either AC or AB; then will BC be less than AB + AC.

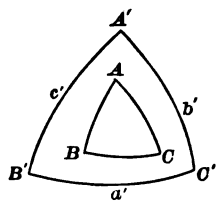
Complete the demonstration by (44).

210. THEOREM. *The sum of the sides of any spherical polygon is less than four right angles.*

Use the diagram of (209), and complete the demonstration by (45).

PROPOSITION XXXIV.

211. THEOREM. *The sum of the angles of a spherical triangle is greater than 180° and less than 540°.*



Let A, B, and C represent the angle of any spherical triangle ABC; then will $A + B + C > 180°$ and $< 540°$.

Construct A′B′C′, the polar triangle of ABC.

Now, $A = 180° - B'C'$ . . . (205)

and $B = 180° - A'C'$

and $C = 180° - A'B'$; adding

then, $A + B + C = 540° - (B'C' + A'C' + A'B')$.

But, $B'A' + A'C' + A'B' < 360°$ . . . (210).

Substituting this value of $B'A' + A'C' + A'B'$ in the preceding equation, we find $A + B + C > 180°$ and $< 540°$.

Q. E. D.

212. COROLLARY. *A spherical triangle may have one, two, or three right angles, or each of its angles may be obtuse.*

213. SCHOLIUM. A bi-rectangular triangle has two right angles; a tri-rectangular triangle has three right angles.

214. SCHOLIUM. The sum of all the angles of a spherical polygon minus two right angles gives a remainder which is called the *spherical excess* of the polygon.

PROPOSITION XXXV.

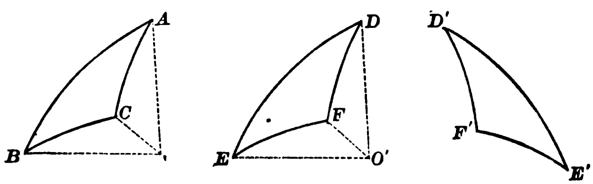
215. THEOREM. *Two spherical triangles on the same or equal spheres are mutually equal with respect to all their parts, under the following conditions:*

I. If they are mutually equilateral.

II. If they are mutually equiangular.

III. If they are mutually equal with respect to two sides and their included angle.

IV. If they are mutually equal with respect to two angles and their common side.

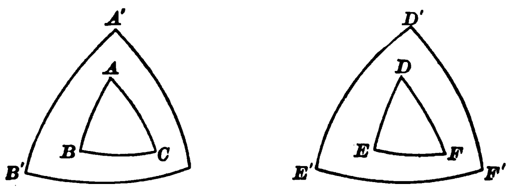


CASE I. Let the spherical triangles ABC and DEF, on equal spheres, be mutually equilartea ; then will they be mutually equal with respect to all their parts.

Let O and O′ be the triedral angles of the spherical triangles ABC and DEF (187 and 189).

Now, O and O′ have three face angles of the one equal to three face angles of the other, each to each (188). Hence O and O′ are mutually equal in all their parts (46).

But, if O and O′ are mutually equal with respect to all their parts, then ABC and DEF are mutually equal with respect to all their parts (188). *Q. E. D.*



CASE II. Let the spherical triangles ABC and DEF be mutually equiangular, then will they also be mutually equilateral. For, —

Let A′B′C′ and D′E′F′ be the polar triangles of ABC and DEF respectively.

Now, since ABC and DEF are mutually equiangular (Hyp.), A′B′C′ and D′E′F′ are mutually equilateral (207).

And, since A′B′C′ and D′E′F′ are mutually equilateral, they are also mutually equiangular (Case I.).

And because A′B′C′ and D′E′F′ are mutually equiangular, ABC and DEF are mutually equilateral (207).

Q. E. D.

CASE III. See diagram, Case I.

Let the angle B equal the angle E. Let the side AB equal the side DE, and let the side BC equal the side EF; then ABC and DEF will be mutually equal with respect to all their parts.

Now, the triedral angles O and O′ have two face angles and the included diedral angle of the one, equal to two face angles and the included diedral angle of the other, each to each (188). Hence O and O′ are mutually equal with respect to all their parts (48), then ABC and DEF are mutually equal with respect to all their parts (188). *Q. E. D.*

CASE IV. See diagram, Case II.

Let the spherical triangles ABC and DEF be mutually equal with respect to two angles and their common side; then ABC and DEF will be mutually equal with respect to all their parts.

Since ABC and DEF are mutually equal with respect to two angles and their common side, A′B′C′ and D′E′F′ are mutually equal with respect to two sides and their included angle; for the parts of a polar triangle are supplements of the opposite parts in the primitive triangle (205).

But, if A′B′C′ and D′E′F′ are mutually equal with respect to two sides and their included angle, they are mutually equal in all their parts (Case III).

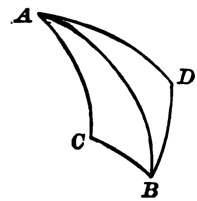
And, if A′B′C′ and D′E′F′ are mutually equal in all their parts, ABC and DEF are mutually equal in all their parts (207 and 208). *Q. E. D.*

216. SCHOLIUM 1. Our theorem is proved to be true, independent of the size or equality of the spheres involved if the sides of the triangles are expressed in degrees. It is also proved to be true, independent of the order in which the equal parts are arranged.

217. SCHOLIUM 2. Two spherical triangles on the same or equal spheres, mutually equal with respect to all their parts, cannot always be made to coincide by superposition.

(*a*). If the equal parts of the one triangle are arranged in the same order as the equal parts of the other, the triangles may be made to coincide, and hence are *equal.* ABC and DEF are equal. See diagram, Case I.

(*b*). If the equal parts of the one triangle are arranged in the reverse order of the equal parts of the other, coincidence is not always possible, and hence the triangles are not always equal. Such triangles are *symmetrical.* EFD and E′F′D′ are symmetrical. See diagram, Case I.



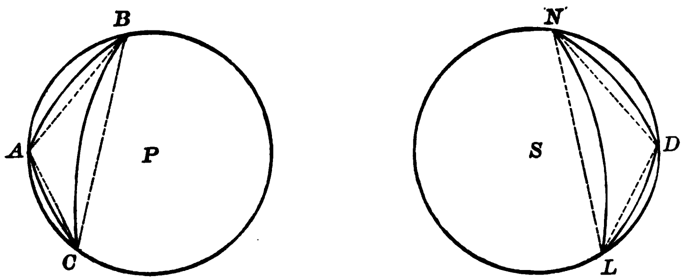
Let the triangles ABC and ABD be symmetrical. To make them coincide we must bring the equal sides AD and AC together; if, for this purpose, we turn ADB about AB as an axis, the convexity of the surfaces will prevent the coincidence of the triangles.

(*c*). If the triangles ABC and ABD are isosceles, they may be made to coincide by sliding the side AB of the triangle ABD over on the side AC of the triangle ACB; hence,

Symmetrical isosceles triangles on any sphere are superposable and therefore equal.

PROPOSITION XXXVI.

218. LEMMA. *Circles passed through the vertices of each of two symmetrical spherical triangles are equal.*



Let ABC and DNL be two symmetrical spherical triangles, and let a circle be passed through the vertices A, B, C; and similarly, let a circle be passed through D, N, L;

then will the circle determined by A, B, C be equal to the circle determined by D, N, L.

Draw chords forming the plane triangles ABC and DNL.

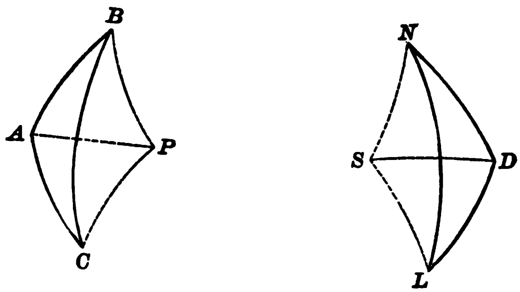
Now, since the arc AB equals the arc DN, the chord AB equals the chord DN. In like manner it follows that the chord AC equals the chord DL, and the chord BC equals the chord NL. The plane triangles ABC and DNL are therefore mutually equilateral, and hence equal.

But, if the plane triangles ABC and DNL are equal, the circles which may be circumscribed about them must be equal, as may be shown by superposition. *Q. E. D.*

219. Corollary. *The pole P of the circle ABC is the same distance from the vertices A, B, C, as the pole S of the circle DNL is from the vertices D, N, L (178).*

Proposition XXXVII.

220. Theorem. *Symmetrical spherical triangles are equal in area.*



Let ABC and DNL be two symmetrical spherical triangles in which AB equals DN, AC equals DL, and BC equals NL; then ABC and DNL will be equal in area.

Let P be the pole of the circle through A, B, and C; and let S be the pole of the circle through D, N, and L. Draw arcs of great circles, PA, PB, PC, and SD, SN, SL.

Now, PA $=$ SD, and PB $=$ SN (219); and since by hypothesis AB $=$ DN, it follows that the triangles APB and DSN are mutually equal with respect to all their parts (215, Case I.).

But, PA $=$ PB and SD $=$ SN (176). The triangles APB and DSN are thus found to be symmetrical isosceles triangles; they are therefore equal (217, *c*).

Similarly, we may prove that the triangle BPC equals the triangle NSL; and the triangle APC equals the triangle DSL.

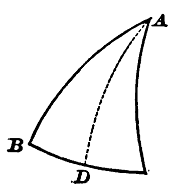
But, if APB $=$ DSN,
and APC $=$ DSL, then, adding,
APB $+$ APC $=$ DSN $+$ DSL.
But, BPC $=$ NSL, then, subtracting,
APB $+$ APC $-$ BPC $\Leftrightarrow$ DSF $+$ DSL $-$ NLS;
which from the diagram gives,

ABC $\Leftrightarrow$ DNL. *Q. E. D.*

221. Scholium. If the poles fall *within* the spherical triangles, the areas of the triangles may be shown to be equal by *adding* BPC $=$ NSL, instead of subtracting as above.

Proposition XXXVIII.

222. Theorem. *In an isosceles spherical triangle the angles opposite the equal sides are equal.*



Let ABC be an isosceles spherical triangle whose sides AB and AC are equal; then C and B the angles opposite the equal sides will be equal.

Through A the vertex and D the middle point in the base pass the arc of a great circle, forming triangles ADB and ADC.

Now, the triangles ADB and ADC being mutually equilateral are mutually equal with respect to all their parts (215, Case I.).

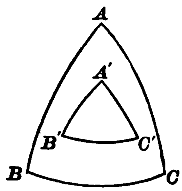
Hence B and C are equal, being homologous parts.

Q. E. D.

223. Corollary. *The arc from the vertex of an isosceles triangle to the middle of its base bisects both the triangle and its vertical angle, and it is also perpendicular to the base.*

Proposition XXXIX.

224. Theorem. *If two angles of a spherical triangle are equal, the sides opposite are equal.*



Let A′B′C′ be a spherical triangle whose angles B′ and C′ are equal, then will A′C′ and A′B′, the sides opposite, be equal.

Let ABC be the polar triangle of A′B′C′.

Now, $B' = 180° - AC$ . . . (205);

and $C' = 180° - AB$.

Hence, $180° - AC = 180° - AB$, since $B' = C'$;

or, $AC = AB$.

But if $AC = AB$,

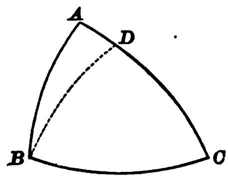
then, $B = C$ . . . (222);

and, in the polar triangle by the method used above,

$A'C' = A'B'$. *Q. E. D.*

PROPOSITION XL.

225. THEOREM. *In a spherical triangle of unequal parts the greater side lies opposite the greater angle; and conversely the greater angle lies opposite the greater side.*



(*a*). Let ABC be a spherical triangle in which the angle B is greater than the angle C, then will AC opposite B be greater than AB opposite C.

Construct the spherical angle DBC equal to C, completing the triangle BDC. Then, since the angle DBC equals C, the side DC equals the side DB (224).

Now, $BD + AD > AB$ . . . (209).

But, $BD = DC$, just shown.

Hence, $DC + AD > AB$ by substitution;

or, $AC > AB$. *Q. E. D.*

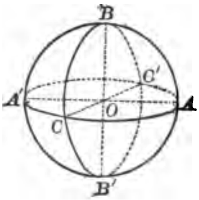
(*b*). Let the side AC be greater than the side AB, then will the angle B opposite AC, be greater than the angle C opposite AB.

If B were less than C, AC would be less than AB. See (*a*). If B were equal to C, AC would be equal to AB (224). But since AC is neither less than nor equal to AB, B cannot be less than or equal to C.

Hence B is greater than C. *Q. E. D.*

DEFINITIONS.

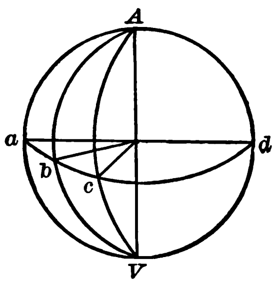
226. Let two great circles of a sphere, A′CAC′ and BA′B′A, bisect each other at right angles, they will have a common diameter AA′. Pass a third great circle CB′C′B through the sphere, whose poles are the extremities of this common diameter AA′. The sphere is now divided into eight equal parts by three great circles at right angles to each other. The spherical surface of one of these parts is a *tri-rectangular triangle* each of whose sides is a quadrant. The entire surface of a sphere then contains eight tri-rectangular triangles.



227. A *lune* is a spherical surface bounded by two semi-circumferences of great circles. A lune then has two equal angles and two equal sides. A′B′CB is a lune. See (226).

PROPOSITION XLI.

228. THEOREM. *A lune is to the entire surface of the sphere as the angle of the lune is to four right angles.*



Let A*a*V and A*c*V be two great circles through a common diameter AV, then the surface A*a*V*c*A will be a lune whose angle is *c*A*a*. Represent the surface A*a*V*c*A by L, the whole surface of the sphere by S, and the angle of the lune by A: then

$$L : S :: A : 4 \text{ rt. angles.}$$

Let *abcda* be the circumference of a great circle whose poles are A and V, and let *ac* be an arc of it intercepted by the sides of the lune; then *ac* is the measure of A (196).

Let *ab*, a common measure, be contained in *ac*, *m* times, and in the circumference *abcda*, *n* times; then,

$$\text{arc } ac : \text{circum. } abcda :: m : n;$$

or, $$A : 4 \text{ rt. angles} :: m : n \quad . \ . \ . \quad (196).$$

Through all the points of division in *abcda* pass great circles embracing AV. The whole spherical surface is thus divided into *n* smaller lunes which are superposable and therefore equal. The lune L contains *m* smaller lunes.

Hence, $L : S :: m : n.$
But, $A : 4 \text{ rt. angles} :: m : n.$
Hence, $L : S :: A : 4 \text{ rt. angles.}$

And the same may be shown when arc *ac* and circumference *abcda* are incommensurable. *Q. E. D.*

229. Corollary 1. *The area of a lune may be found by multiplying twice its angle expressed in right angles, by a tri-rectangular triangle.*

If a right angle be taken as the unit of angular measure, the value of the angle A is A right angles. And if a tri-rectangular triangle is taken as the unit of spherical surface, the value of the entire spherical surface is 8 T, in which T represents a tri-rectangular triangle (226). Whence the proportion,

$$L : S :: A : 4 \text{ rt. angles (228) becomes}$$
$$L : 8\,T :: A \text{ rt. angles} : 4 \text{ rt. angles};$$

or, $L : 8\,T :: A : 4;$

or, $4\,L = 8\,T \cdot A;$

or, $L = 2\,A \cdot T.$

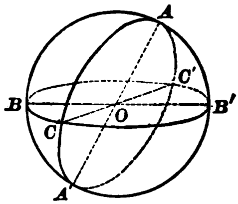
230. Corollary 2. *Lunes on the same sphere have the same ratio as their angles.*

If L and L′ are two lunes on the same sphere and A and A′ their angles, then,

$$\frac{L}{L'} = \frac{2T \cdot A}{2T \cdot A'} = \frac{A}{A'}.$$

Proposition XLII.

231. Theorem. *The area of a spherical triangle is equal to the product of the spherical excess by a tri-rectangular triangle.*



Let ABC be any spherical triangle, and let A + B + C express the sum of its angles in right angles; then A + B + C − 2 will express its spherical excess (214). Now, if T be the area of a tri-rectangular triangle, and S the area of the triangle ABC; then will S = (A + B + C − 2) T.

Complete the great circles of which the sides of the triangle are arcs. Now, since two great circles bisect each other these great circles will intersect in diameters AA′, BB′, and CC′.

And, since the vertical angles at the centre are equal, BC = B′C′, BA′ = AB′, and CA′ = AC′; whence the triangles AC′B′ and A′CB being mutually equilateral are mutually equal in all their parts (215), and must be equal or symmetrical (217) and equivalent (220); and, the lune which is made up of the triangles ABC and A′BC is equivalent to ABC + AC′B′.

(1). That is, lune $BACA' \simeq ABC + AB'C'$.

(2). But lune $ABCB' \simeq ABC + ACB'$.

(3). And, lune $ACBC' \simeq ACB + ABC'$.

Equations (1), (2), and (3) give by addition,

(4). $BACA' + ABCB' + ACBC' \simeq 2\,ABC + \overline{ACB + AB'C' + ACB' + ABC'}$

(5). But, $BACA' + ABCB' + ACBC' \simeq 2\,A \cdot T + 2\,B \cdot T + 2\,C \cdot T$ . . . (229);

(6). and $\overline{ACB + AB'C' + ACB' + ABC'} \simeq 4\,T$, the surface of the hemisphere BCB′A . . . (226).

(7). And $2\,ABC = 2\,S$, from our hypothesis.

Now, substituting in (4) values found in (5), (6), and (7); then,

(8). $2\,A \cdot T + 2\,B \cdot T + 2\,C \cdot T = 2\,S + 4\,T$;

(9). or, $2\,S = 2\,A \cdot T + 2\,B \cdot T + 2\,C \cdot T - 4\,T$;

or, $S = (A + B + C - 2)\,T$. *Q. E. D.*

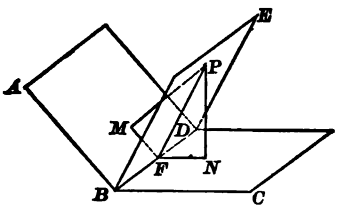
232. SCHOLIUM. By dividing a spherical polygon into a number of spherical triangles we may readily deduce from our theorem that,—

The area of a spherical polygon is equal to the product of its spherical excess by a tri-rectangular triangle.

233. GENERAL SCHOLIUM. In the foregoing demonstrations it has been shown that certain similar solids (see prisms, pyramids, cylinders, cones, and spheres) are to each other as the cubes of corresponding edges or dimensions, and that the surfaces of these similar solids are to each other as the squares of corresponding edges or dimensions. By the methods already used we may show that, *any two similar solids are to each other as the cubes of their corresponding dimensions;* and that, *the surfaces of any two similar solids are to each other as the squares of their corresponding dimensions.*

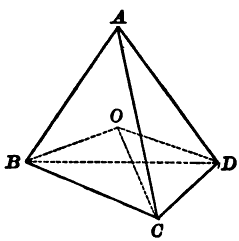
EXERCISES.

234. LEMMA. *Any point in the plane which bisects a diedral angle is equally distant from the faces of the angle.*



From any point P in the bisecting plane BE, draw perpendiculars PN and PM to the faces of the diedral angle. Through PM and PN pass a plane intersecting the faces of the angle and the bisecting plane in the lines FN, FM, and PF. Show that the plane angles PFN and PFM are the measures of corresponding diedral angles, and hence equal. From which find the triangles PNF and PMF to be equal, and hence PM equals to PN.

235. THEOREM. *A sphere may be inscribed in a triangular pyramid.*



Bisect the diedral angle between the base BCD and each face, by planes; then see (234).

236. THEOREM. *The surface of a sphere is to the surface of a circumscribed right cylinder as two is to three.*

237. THEOREM. *The volume of a sphere is to the volume of a circumscribed right cylinder as two is to three.*

238. THEOREM. *Tangents drawn to a sphere from a common external point are equal.*

SECTION V.

REGULAR POLYEDRONS.

DEFINITIONS.

239. A polyedron has already been defined as a solid bounded by plane surfaces, the bounding planes being called faces.

240. Polyedrons are named from the number of their faces; one of four faces is called a *tetraedron,* one of six faces a *hexaedron,* etc.

241. A *regular* polyedron is a polyedron of which the faces are equal regular polygons, and each polyedral angle is bounded by the same number of faces.

Proposition XLIII.

242. Theorem. *There cannot be more than five regular polyedrons.*

Each face is a regular polygon (241). And the value of one angle of a regular polygon is, —

in the case of the triangle 60°,
" " " " " square 90°,
" " " " " pentagon 108°,
" " " " " hexagon 120°.

But since the sum of the face angles at any vertex of a polyedral angle is less than 360° . . . (45); and since there are at least three face angles meeting at any vertex, the hexagon cannot be the face of any regular polyedron; for

120° × 3 is not less than 360°. In like manner the heptagon, octagon, etc., are excluded as faces.

There remains then to be considered only the *triangle*, the *square*, and the *pentagon*. With respect to these, —

The polyedral angle may be formed by the meeting at a point of, —

(*a*) 3 triangles, since 60° × 3 = 180°, or by
(*b*) 4 triangles, since 60° × 4 = 240°, or by
(*c*) 5 triangles, since 60° × 5 = 300°, or by
(*d*) 3 squares, since 90° × 3 = 270°, or by
(*e*) 3 pentagons, since 108° × 3 = 324°.

All other cases are excluded by the limiting value of a polyedral angle, *i.e.*, 360°.

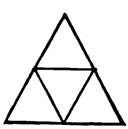
Thus then there cannot be more than, —

three regular polyedrons of *triangular* faces; see (*a*), (*b*), (*c*),
one regular polyedron of *square* faces; see (*d*),
one regular polyedron of *pentagonal* faces; see (*e*).
Five in all. *Q. E. D.*

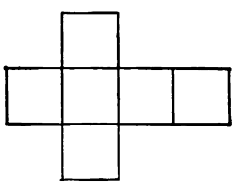
243. Scholium. The five regular polyedrons are named from the number of their faces, the *tetraedron*, the *hexaedron*, the *octaedron*, the *dodecaedron*, and the *icosaedron*.

CONSTRUCTION. — REGULAR POLYEDRONS.

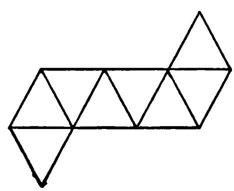
244. The five regular polyedrons (243) may be constructed out of cardboard in a very simple manner.



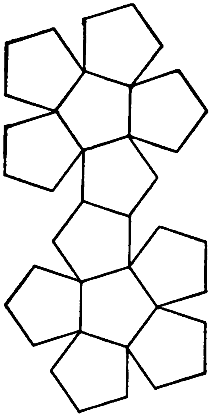
Tetraedron.



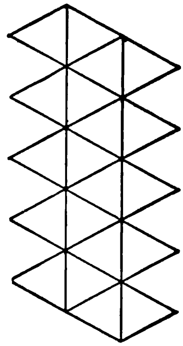
Hexaedron.



Octaedron.



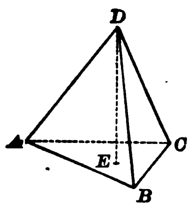
DODECAEDRON.



ICOSAEDRON.

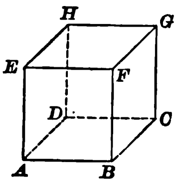
Draw on the cardboard as accurately as possible the foregoing diagrams and cut them out. On the interior lines cut the cardboard about half through its thickness. The parts will then readily bend about the half-cut lines into the required form, and can be retained in place by gluing over the edges a strip of paper or linen.

245. PROBLEM. *To construct a regular tetraedron.*

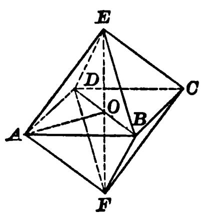


At E, the middle point of ABC, erect a perpendicular ED, and take the point D so that AD, which is equal to DB and to DC, shall be equal to AB.

246. PROBLEM. *To construct a regular hexaedron.*



247. PROBLEM. *To construct a regular octaedron.*



248. Fig. 1 and Fig. 2, following, represent the regular dodecaedron: completed in Fig. 2.

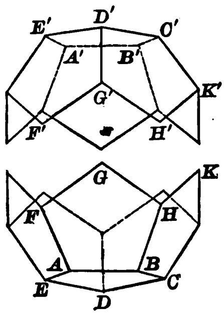


FIG. 1.

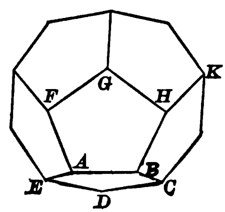


FIG. 2.

249. Fig. 1, Fig. 2, and Fig. 3, following, represent the regular icosaedron: completed in Fig. 3.

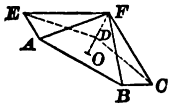


FIG. 1.

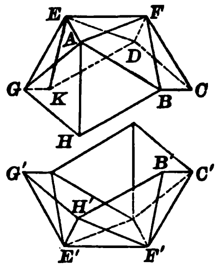


FIG. 2.

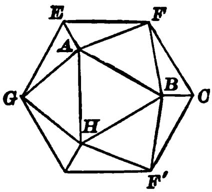


FIG 3.

SECTION VI.

MENSURATION OF SOLIDS.

250. In this section it is designed to call special attention, by means of illustrative examples, to all the important rules for finding volumes and surfaces of solids, demonstrated in the preceding sections. Also, methods will be deduced for finding the volumes of certain additional solids, such as the *Regular Polyedron*, the *Wedge*, and the *Prismoid.*

PRISMS AND CYLINDERS.

251. A piece of timber in the form of a right prism is 10 feet long, and contains 48 cubic feet. If a carpenter cuts off one linear foot, how many cubic feet will remain? See 71, Prop. XIII.

252. A block of marble in the form of a cuboid is 3 feet 6 inches long, 2 feet 3 inches wide, 6 inches thick, and weighs 800 lbs. What is the weight of a like piece of marble whose dimensions are 2 feet 6 inches, by 1 foot 3 inches, by 3 inches? See 73, Prop. XIV.

253. How many cubic feet in each piece of marble described in the foregoing Problem? See 74, Cor. 1.

254. Across a street 50 feet wide a ditch 2 feet broad and 3 feet deep is dug. The ditch is straight, but runs at an angle of 45 degrees with the direction of the street. How many cubic feet of earth were removed in making it? See 79, Cor. 3.

255. Each edge of the base of a triangular prism is 16 inches. Now, if the altitude is also 16 inches, what is the volume? See 82.

256. Each edge of the base of a hexagonal prism is 12 inches. Now, if the altitude of this prism is 12 inches, what is its volume? See 90.

257. A cylindrical column is 100 feet high and measures 31.416 feet in circumference. What is its volume? See 91 and 93.

258. From the column described in Problem 257, 10 linear feet were removed. How many cubic feet remained? See 92.

259. What is the total surface of the cylindrical column described in Problem 257? See 96.

260. A cylindrical column is 100 feet high and measures 62.832 feet in circumference. Compare its total surface with that of the column mentioned in Problem 259. See 97.

261. Compare the volumes of the two columns in question. See 92.

262. The corresponding dimensions of two similar cylinders of revolution are in the ratio of 2 to 3. If the volume of the larger cylinder is 24 cubic feet, what is the volume of the smaller? See 98.

263. Compare the surfaces, lateral and total, of the cylinders described in Problem 262. See 99.

PYRAMIDS AND CONES.

264. Each edge of the base of a right triangular pyramid is 24 feet. Now, if the altitude is also 24 feet, what is the volume? See 137.

265. The altitude of a right circular cone is 12 feet, and the circumference of its base is 24 feet. What is the volume? See 138.

266. The altitude of the frustum of a right circular cone is 64 inches, and the radii of the upper and lower bases are, respectively, 8 inches and 12 inches. What is the volume of the frustum? See 143.

267. The cone described in Problem 265 is cut by a plane, 8 feet above the base, and parallel to it. Find the volume of the frustum thus formed. See 143.

268. Find the total surface of the pyramid described in Problem 264. See 144.

269. Find the total surface of the cone described in Problem 265. See 146.

270. Find the total surface of the frustum described in Problem 266. See 149 and 150.

271. Two similar cones of revolution have their corresponding dimensions in the ratio of 3 to 4. If the volume of the smaller is 64 cubic feet, what is the volume of the larger? See 151.

272. If the lateral surface of the smaller cone is 48 square feet, in Problem 271, what is the lateral surface of the larger cone? See 152.

273. There is a truncated triangular prism the lateral edges of which are 18, 20, and 22 feet, respectively. A right cross section of this cone forms a triangle whose sides are 6, 8, and 10 inches, respectively. What is the volume of this prism? See 158.

THE SPHERE.

274. What is the surface of a sphere whose diameter is 200 feet? See 181.

275. The radii of two spheres are to each other as 10 to 40. What is the ratio of the surfaces? See 183.

276. What is the area of a zone on the sphere described in Problem 274, whose altitude is 10 feet? See 185.

277. What is the volume of the sphere described in Problem 274? See 198.

278. What is the volume of a sphere whose diameter is 100 feet? See 199.

279. What is the ratio of the volumes of the spheres described in Problem 275? See 200.

280. The values of the angles of a spherical triangle are $40° + 10'$, $80° + 20'$, and $140° + 30'$, respectively. What are the values of the sides of the polar triangles? See 205.

281. On a sphere whose diameter is 2000 feet, the angle of a lune is 30 degrees. What is the surface of the lune? See 228.

282. What is the area of a tri-rectangular triangle of the sphere in Problem 281? See 226.

283. What is the area of the triangle described in Problem 280, if located on the sphere of problem 281? See 231.

THE REGULAR POLYEDRON.

284. In the case of any one of the regular polyedrons, it follows from the definition (241), that if lines be drawn from the centre to each of the vertices, pyramids thus formed will have equal bases and equal altitudes.

Since the volume of a pyramid is equal to the product of the base by one-third the altitude, it follows that, —

285. *The volume of a regular polyedron is equal to one-third the product of its surface by the perpendicular distance from its centre to any face.*

It is plain too that, —

286. *Any two regular polyedrons of the same number of faces are similar, and hence are to each other as the cubes of their like dimensions.* Moreover, *the surfaces of these similar solids are to each other as the squares of their like dimensions.*

287. By rule 285 the volumes of all the regular polyedrons have been computed, the edge being taken in each case equal to the linear unit. The total surface for each of these solids has also been computed. Following are the results.

TABLE FOR SURFACES.

Name.	*No. of Faces.*	*Surface.*
Tetraedron	4	1.73205
Hexaedron	6	6.00000
Octaedron	8	3.46410
Dodecaedron	12	20.64573
Icosaedron	20	8.66025

TABLE FOR VOLUMES.

Name.	*No. of Faces.*	*Volumes.*
Tetraedron	4	0.11785
Hexaedron	6	1.00000
Octaedron	8	0.47140
Dodecaedron	12	7.66312
Icosaedron	20	2.18169

288. What is the volume and what is the surface of a regular tetraedron whose edge is 30? See 286.

289. Find the volume and the surface of a regular hexaedron whose edge is 30.

290. Find the volume and the surface of a regular octaedron whose edge is 30.

291. Find the volume and the surface of a regular dodecaedron whose edge is 30.

292. Find the volume and the surface of a regular icosaedron whose edge is 30.

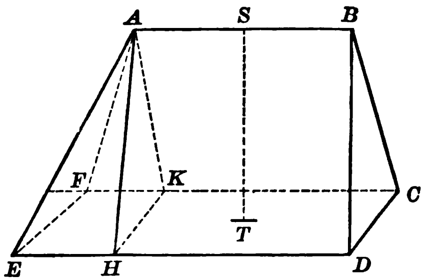
THE WEDGE.

293. A *wedge* is a solid bounded by five plane faces, viz., a rectangle called the *back*, two trapezoids called *faces*, and two triangles called *ends*. The intersection of the two faces forms the *edge*. See next diagram.

The *length* of the back is that dimension which is parallel to the edge. The length of the back may be equal to the length of the edge, or it may be shorter or longer.

PROPOSITION XLIV.

294. THEOREM. *The volume of a wedge is equal to the product obtained by multiplying twice the length of the back added to the length of the edge, by the breadth of the back, and that result by one-sixth of the altitude.*



Let ADF, *i.e.*, AB — CDEF be a wedge. Through A, one extremity of the edge, pass a plane AHK parallel to the end BCD. The wedge is thus found to be composed of a prism, ADK (52) and a pyramid A–EHKF (105).

Draw from S, any point in the edge, ST a perpendicular to the back.

Represent the length FC of the back by l, the breadth CD by b, the edge AB of the wedge equal to KC by e, the altitude ST by a; then will the volume of ADF equal

$$\tfrac{1}{6}\,(ab)\,(e + 2\,l).$$

Now, ADK is one-half of a parallelopiped whose base is KD, equal to eb, and whose altitude is ST equal to a (80).

(1) . . . Whence Vol. ADK $= \frac{1}{2}\,a\,(eb)$ . . . (79).

(2) . . . And Vol. A–FEHK $= \frac{1}{3}\,a\,(l-e)\,b$ . . . (137).

By adding, we obtain from (1) and (2) the volume of the wedge, or,

$$\text{Vol. ADF} = \tfrac{1}{2}\,aeb + \tfrac{1}{3}\,alb - \tfrac{1}{3}\,aeb;$$

or,

$$\text{Vol. ADF} = \tfrac{1}{6}\,aeb + \tfrac{1}{3}\,alb;$$

or,

$$\text{Vol. ADF} = \tfrac{1}{6}\,ab\,(e + 2\,l). \qquad \textit{Q. E. D.}$$

295. Corollary. *The same formula for the volume of ADF will be obtained when e is greater than l.*

296. Problem. Required the volume of a wedge; the edge is 12 feet, the altitude is 14 feet, the breadth of the base is 8 feet, and the length is 16 feet.

297. Problem. Required the volume of a wedge; the edge is 10 feet, the altitude is 8 feet, the breadth of the base is 3 feet, and the length 18 feet.

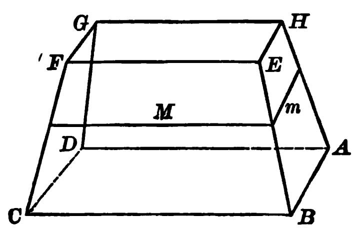
THE PRISMOID.

298. A *prismoid* is a solid bounded by six plane faces, viz., two rectangles, which are parallel, and four trapezoids.

The two rectangles are the *bases*, and the distance between them is the *altitude* of the prismoid. The prismoid is the frustum of a wedge.

Proposition XLV.

299. Theorem. *The volume of a prismoid is equal to the product obtained by multiplying the sum of the bases and four times a section midway between them, by one-sixth of the altitude.*



Let AF, *i.e.*, ABCD-EFGH be a prismoid. Let a represent its altitude, b and B the areas of its upper and lower bases, respectively, and Mm the area of the section midway between them; then will the volume of the prismoid be equal to $\frac{1}{6} a (B + b + 4 Mm)$.

By passing a plane through the lines BC and HG, the prismoid is found to be composed of two wedges, whose edges are BC and HG, and whose backs are HF and AC, respectively.

Now, we may write for the volumes of the wedges, using V′ and V″ to represent these volumes, and V to represent the volume of the prismoid,

$$V' = \tfrac{1}{6} a (2 \overline{BC} + \overline{HG}) \overline{AB}, \ldots (294), \text{ and}$$
$$V'' = \tfrac{1}{6} a (2 \overline{HG} + \overline{BC}) \overline{HE}.$$

We may change the form of the above equations to

$V' = \frac{1}{6} a (2 B + \overline{HG} \cdot \overline{AB}) \ldots$ and

$V'' = \frac{1}{6} a (2 b + \overline{BC} \cdot \overline{HE})$; by adding, we obtain

$V = \frac{1}{6} a [B + b + (B + b + \overline{HG} \cdot \overline{AB} + \overline{BC} \cdot \overline{HE})] \ldots$ (A).

Now, $2 m = \overline{AB} + \overline{HE} \ldots (377)$,

and $2 M = \overline{BC} + \overline{HG}$; by multiplying, we obtain

$$4 Mm = (\overline{AB} + \overline{HE}) (\overline{BC} + \overline{HG}),$$

or, $4 Mm = B + b + \overline{HG} \cdot \overline{AB} + \overline{BC} \cdot \overline{HE}$.

Now, substituting in equation (A), we obtain

$$V = \tfrac{1}{6} a (B + b + 4 Mm). \qquad \textit{Q. E. D.}$$

300. Scholium 1. This formula is of very extensive application. It may be used in computing volumes of earthwork in excavations and embankments for railroads, canals, etc.

301. Scholium 2. This formula may be applied, in general, to the computation of the volume of any one of the solids which we have considered in this book. Thus, in a pyramid, we may regard the upper base as zero. The area of the section midway between the bases is evidently one-fourth the area of the lower base. Applying the above formula, then,

$$V = \tfrac{1}{6}\,a\,(B + 0 + B), \text{ or,}$$

$V = \frac{1}{3}\,a \cdot B$, which is the formula for finding the volume of a pyramid.

302. What is the volume of a prismoid whose bases are 25 feet by 40, and 35 feet by 50, respectively, and whose altitude is 20 feet?

303. In making an excavation for a railroad, a prismoidal cut is made 300 yards long; the areas of the right cross sections at the ends are 1000 square feet and 1500 square feet, respectively. Find the volume of the prismoidal cut.

MISCELLANEOUS EXERCISES.

304. From a point 4 ft. above a plane a circle is described in the plane by a line 12 ft. long. What is the radius of the circle?

305. What is the length of a diagonal of a cuboid whose edges are 6 ft. 8ft. and 10 ft.?

306. There is a regular triangular pyramid whose altitude is 6 ft., and whose basal edges are each 8 ft. What is the lateral edge of the pyramid?

307. Find the entire surface of a regular triangular prism whose altitude is 12 ft., each side of the base being 12 ft.

308. Find the entire surface of a regular hexagonal prism whose altitude is 12 ft., each side of the base being 6 ft.

309. What is the entire surface of a regular triangular pyramid whose altitude is 12 ft., each side of the base being 10 ft.?

310. What is the entire surface of a right square pyramid whose altitude is 12 ft., each side of the base being 4 ft.?

311. Pass a plane through one of the diagonals of a parallelogram; draw perpendiculars from the extremities of the other diagonal to this plane. Prove that the perpendiculars are equal.

312. There are two similar octaedrons whose corresponding edges are as 4 : 5. What is the ratio of their surfaces, and of their volumes?

313. There is a pyramid 40 ft. high whose base contains 400 sq. ft. The top is cut off so as to form a frustum 10 ft. high. What is the area of the upper base of the frustum?

314. If the radius of the earth is 4000 miles, what is the area of a small circle parallel to the equator, through a point 60° from the equator?

315. What is the entire surface of a frustum of a cone of revolution, the radii of the bases being 6 ft. and 8 ft. and the altitude 10 ft?

316. How many bushels of wheat will the frustum of a hexagonal pyramid hold, each edge of the upper base being 4 ft., of the lower 10 ft., and the altitude 6 ft.?

317. Find the surface and volume of a sphere whose diameter is 10 ft.

318. What is the volume of the crust of the earth, assuming the earth to be a sphere with a radius of 4000 miles and the crust to be 50 miles thick?

319. Find the volume of the earth's atmosphere, assuming that it extends 50 miles above the surface.

320. How many miles from the centre of the earth is that point from which one third the earth's surface may be seen?

321. There is a sphere whose volume and surface are represented by the same number. Find the diameter of this sphere.

322. There is a cube whose volume and surface are represented by the same number. Find the edge of this cube.

323. There is a regular tetraedron whose volume and surface are represented by the same number. Find the edge of the tetraedron.

324. There is a right circular cone whose altitude and diameter of base are equal. The volume and surface are represented by the same number. What is the altitude of this cone?

325. A regular pyramid 40 ft. high is changed into a regular prism of equivalent base. What is the height of the prism?

326. How much of the earth's surface is visible from a point 4000 miles above it?

327. The diameter of a sphere is 40 ft. Find the curved surface of a segment whose height is 10 ft.

328. In a sphere whose radius is 4000 miles, find the height of a zone whose area is equal to that of a great circle.

329. A vessel is in the shape of the segment of a sphere; the distance across the top is 20 in., the greatest depth is 10 in. How many quarts of water will it hold?

330. A right triangle whose hypotenuse and sides are respectively 10, 8, and 6 ft., is revolved about the hypotenuse as an axis. Find the volume of the solid generated.

331. An equilateral triangle is circumscribed about a circle whose radius is 10 ft., and revolved about the altitude of the triangle as an axis. Find the volumes of the solids generated by the triangle and by the circle.

332. A conical glass having a diameter and altitude of 10 in. is filled with water. Into it is dropped a sphere of iron whose radius is 4 in. Find the number of cubic inches of water remaining in the glass.

333. If the sphere in problem 332 is drawn into a cylindrical wire one-eighth of an inch thick, what is the length of the wire?

334. If through the centre of a four-inch sphere a two-inch hole be bored from surface to surface, what are the contents of the remaining portion of the sphere?

335. Find a point in a given straight line equally distant from two points in space.

336. Draw a plane parallel to two given straight lines in space.

337. Find the surface and volume of a regular tetraedron whose edge is a.

338. Cut a cube by a plane so that its section shall be a regular hexagon.

339. Find the volume of the double cone generated by the revolution of an equilateral triangle about one of its sides.

340. Prove that the six planes which bisect the edges of a tetraedron at right angles will all pass through one point.

341. A cone is circumscribed about a sphere and its height is double the diameter of the sphere. Prove that the total surface and the volume of the cone are respectively double those of the sphere.

342. Show that a regular dodecaedron may be inscribed in a regular icosaedron.

343. Show that a regular icosaedron may be inscribed in a regular dodecaedron.

344. Find the radii of the sphere inscribed in a regular tetraedron whose edge is 12 ft.

345. To find a method of bisecting a given arc or given angle of a sphere.

346. There is a cube inscribed in a sphere; the surface of the cube is equal to the surface of the sphere. Find the diameter of the sphere. Are these conditions possible?

347. The angle which a line makes with a plane is the angle between the line and its projection upon the plane. Prove that if a line intersect two parallel planes it makes equal angles with them.

348. Prove that only one common perpendicular can be drawn to two lines not in the same plane.

THEOREMS OF PLANE GEOMETRY.

349. The demonstrations of the preceding pages refer to a few of the theorems of Plane Geometry. These theorems are collected here and numbered the same as in the text; they will be found convenient for reference.

350. At a point in a straight line only one perpendicular to that line can be drawn; and from a point without a straight line only one perpendicular to that line can be drawn.

351. Two parallel lines cannot meet.

352. Two lines perpendicular to the same straight line are parallel to each other.

353. Any side of a triangle is less than the sum of the other two sides.

354. The sum of the three angles of a triangle is equal to two right angles.

355. Two triangles are equal in all respects when two sides and the included angle of the one are equal respectively to two sides and the included angle of the other.

356. Two triangles are equal when a side and two adjacent angles of the one are respectively equal to a side and two adjacent angles of the other.

357. Two triangles are equal when the three sides of the one are equal respectively to the three sides of the other.

358. If two sides of a triangle be equal respectively to two sides of another, but the third side of the first triangle be greater than the third side of the second, then the angle opposite the third side of the first triangle is greater than the angle opposite the third side of the second.

359. The diagonal of a parallelogram divides the figure into two equal triangles.

360. In a parallelogram the opposite sides are equal and the opposite angles are equal.

361. If a line be drawn through two sides of a triangle parallel to the third side, it divides those sides proportionally.

362. Two triangles which have their sides respectively parallel are similar.

363. Parallelograms having equal bases and equal altitudes are equivalent.

364. The area of a triangle is equal to one-half the product of its base by its altitude.

365. Triangles having equal bases are to each other as their altitudes; triangles having equal altitudes are to each other as their bases; any two triangles are to each other as the product of their bases by their altitudes.

366. The area of a trapezoid is equal to one-half the sum of the parallel sides multiplied by the altitude.

367. I. A circle may be circumscribed about a regular polygon.

II. A circle may be inscribed in a regular polygon.

368. If a line is perpendicular to one of two parallel lines it is perpendicular to the other also.

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369. Parallel Lines are straight lines which lie in the same plane and have the same direction, or opposite directions.

370. Similar Polygons are polygons which have their homologous angles equal and their homologous sides proportional.

371. A Parallelogram is a quadrilateral which has its opposite sides parallel.

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372. The area of a parallelogram is equal to the product of its base and altitude.

373. The area of a circle equals the square of the radius multiplied by π, *i.e.* $A = \pi R^2$.

374. In the same or equal circles equal chords subtend equal arcs.

375. The angle at the centre of a circle is measured by the arc of the circumference intercepted by the sides of the angle.

376. A tangent to a circle is perpendicular to a radius drawn to the point of tangency.

377. The line which joins the middle points of the two sides, not parallel, of a trapezoid, equals half the sum of the parallel sides.