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Games with a Continuum of Players

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ABSTRACT

We prove random Nash equilibrium existence theorems as well as Bayesian Nash equilibrium existence results for games with a measure space of players.

1. INTRODUCTION

The main purpose of this paper is to prove the existence of a random (Nash) equilibrium for a game with a continuum of players. Moreover, we show how such a random equilibrium existence result can be used to obtain a Bayesian equilibrium existence theorem for a game with a continuum of players.

In a seminal paper Nash (1951) introduced the notion of a noncooperative equilibrium for a game with a finite number of players. In particular, according to Nash, a game consists of a finite number of players, each of whom is characterized by a strategy set and a payoff (utility) function, i.e., a real-valued function defined on the Cartesian product of the strategy sets of the players. A noncooperative equilibrium for such a game is a strategy vector having the property that no player can deviate from his/her optimal strategy and increase his/her payoff. Nash (1951) and subsequently Debreu (1952) proved the existence of such an equilibrium, using finite-dimensional fixed point theorems of the Brouwer-Kakutani type.

Three main extensions of the Nash-Debreu results have been obtained in the literature. The first one is due to Glicksberg (1952), who allowed the strategy set of each player to be a subset of an infinite-dimensional linear topological space. This equilibrium result necessitated an infinite-dimensional version of the Kakutani fixed point theorem [see also Fan (1952) or Browder (1968)]. The second extension is due to Schmeidler (1973). It allowed for the set of players to be an atomless measure space. This development was motivated by economic problems [see for instance Aumann (1964) among others]. In particular, economists are interested in perfectly competitive outcomes, that is, in situations where each player's effect is "negligible" (which means that he/she is assigned measure zero in the model). In Schmeidler's approach the strategy sets are finite-dimensional. Khan (1986) extended this to the infinite-dimensional case. This work, together with that of Khan-Papageorgiou (1988) and Yannelis (1987), allowed each player in addition to have a preference correspondence (instead of the original utility function) which need not be transitive or complete and therefore may not be representable by a utility function. The present paper is involved with the third of the extensions mentioned above. The value of such an extension is supported by empirical studies, which show that in many instances players do not behave in a transitive way [see also Shafer-Sonnenschein (1975) or Yannelis-Prabhakar (1983)]. We present results which do not only include the three main extensions of the work of Nash and Debreu, but which also allow for incomplete information. In particular, our first and main result is an equilibrium existence theorem which allows the preference correspondence of each agent to depend on the states of natures of the world, that is, it allows for random preference correspondences. This result is then used to obtain an equilibrium existence theorem for a Bayesian game with a continuum of players.

Our paper is organized as follows: Section 2 contains some preliminary notation and definitions. In Section 3 we state the main theorem, whose proof, which involves new ideas and techniques, can be found in Section 5. In Section 4 we discuss the assumptions of the main theorem. Section 6 contains an equilibrium existence result for a Bayesian game with a continuum of players. Some concluding remarks can be found in our closing Section 7.

2. PRELIMINARIES

We begin by introducing some notation:

2^A denotes the set of all nonempty subsets of the set A,

con A denotes the convex hull of the set A,

$/$ denotes the set theoretic subtraction,

int denotes interior,

dom denotes domain.

If X is a linear topological space, its dual is the space X' of all continuous linear functionals on X, and if $p \in X'$ and $x \in X$ the value of p at x is denoted by $\langle p,x \rangle$.

Let X, Y be topological spaces. The correspondence $\phi : X \to 2^Y$ is said to be <u>upper semicontinuous</u> (u.s.c.) if the set $\{x \in X : \phi(x) \subset V\}$ is open in X for every open subset V of Y. Let (T,τ) be a measurable space and $\phi : T \to 2^X$ be a correspondence. We say that the <u>graph</u> of ϕ, i.e., the set of all $(t,x) \in T \times X$ with $x \in \phi(t)$, is <u>measurable</u> if it belongs to the product σ-algebra $\tau \otimes \beta(X)$, where $\beta(X)$ denotes the Borel σ-algebra on X. Moreover, $\phi : T \to 2^X$ is said to be <u>lower measurable</u> if for every V open in X the set $\{t \in T : \phi(t) \cap V \neq \emptyset\}$ belongs to τ. If ϕ has closed values then lower measurability of ϕ implies that its graph is measurable, [Castaing-Valadier (1977, III)]; conversely, if ϕ has a measurable graph and τ is the σ-algebra of all universally measurable subsets of T (in particular, if τ is complete with respect to some σ-finite measure on T) then ϕ is lower measurable if it has a measurable graph, [Castaing-Valadier (1977, III)]. Let $(E,\|\cdot\|)$ be a separable Banach space and let E' be its topological dual. The weak topology $\sigma(E,E')$ on E will be referred to as the w-topology; thus, we speak of w-closed, w-compact, etc. There exists a countable subset $\{x_i'\}$ which is dense in E' for the topology $\sigma(E',E)$, [Castaing-Valadier (1977, III.32)]. Correspondingly, we define the <u>weak metric</u> d on E by

$$d(x,y) := \sum_{i=1}^{\infty} 2^{-i} \frac{|\langle x-y,x_i'\rangle|}{1 + |\langle x-y,x_i'\rangle|}$$

Note that the d-topology, induced by d on E, is weaker than the w-topology. Hence, it coincides with the w-topology of any w-compact subset of E. Note also that the Borel σ-algebras $\beta(E,d)$, $\beta(E,w)$ and $\beta(E,\|\cdot\|)$ coincide, since

$$\|x\| = \sup_{i\in N} \frac{|\langle x,x_i'\rangle|}{\|x_i'\|'},$$

where $\|\cdot\|'$ stands for the dual norm on E'. Let (T,τ,μ) be a finite measure space and let $X : T \to 2^E$ be a correspondence. The set of (equivalence classes of) all μ-integrable functions from (T,τ,μ) into $R[E]$ is denoted by $L_R^1[L_E^1]$, [cf. Diestel-Uhl (1977)]. We say that the correspondence X is μ-<u>integrably bounded</u> if there exists a μ-integrable function $g \in L_R^1$ such that for μ-a.e. $t \in T$

$$\|x\| \leq g(t) \text{ for all } x \in X(t).$$

Correspondingly, we define

$$L_X := \{x \in L^1_E : x(t) \in X(t) \ \mu\text{-a.e.}\}.$$

The usual L^1-norm on L^1_E is defined by

$$\|x\|_1 := \int_T \|x(t)\| \ \mu(dt).$$

The topological dual of $(L^1_E, \|\cdot\|_1)$ is homeomorphic to the space $L^\infty_{E'}[E]$ of all scalarly measurable essentially bounded (equivalence classes of) functions from (T,τ,μ) into $(E', \|\cdot\|')$, [see for instance Ionescu-Tulcea (1969)]. The topology $\sigma(L^1_E, L^\infty_{E'}[E])$ will be referred to as the weak topology on L^1_E, [Diestel-Uhl (1977)].

3. THE MAIN THEOREM

Below we introduce the notion of a random game with a continuum of players. This notion of a game extends the ones of Khan (1986), Khan-Papageorgiou (1988), Schmeidler (1973), Yannelis (1987) by allowing the preference correspondence of each player to depend on the (random) state of nature. The random game with a continuum of players is described as follows:

Let (T,τ,μ) be a complete finite measure space, where T is the set of players, τ the set of all possible coalitions, and μ is the set function assigning to each coalition its "weight" for the game. Let E be a separable Banach space, the set of possible decisions (or moves) for the players. The players are restricted in making their decisions as follows: Let $X : T \to 2^E$ be a given correspondence; then we require that μ-almost every player t selects his/her decision from the set $X(t)$, the <u>strategy set</u> of player t. (The exceptional set would represent a null coalition of players, whose erratic behavior does not carry any weight.) Let $(\Omega, \mathcal{A}, p)$ be a complete probability space, where Ω stands for <u>the possible states of nature</u>, $\mathcal{A}$ for set of all the outcomes, and p for the probability distribution of the outcomes. Let L_X denote the set $\{x \in L^1_E : x(t) \in X(t)\ \mu\text{-a.e.}\}$. Let $P : T \times \Omega \times L_X \to 2^E$ be a correspondence revealing the <u>individual preferences</u> of the players. We require that $P(t,\omega,x) \subset X(t)$ for all $t \in T$ and $\omega \in \Omega$. For μ-a.e. player $t \in T$ the set $P(t,\omega,x)$ consists of the decisions which he/she strictly prefers to his/her own decision $x(t)$, given that ω is the state of nature and given that the decisions of all participating players modulo null coalitions are represented by $x \in L_X$ (the implicit assumption that the players take their individual decisions in a measurable way is standard in this context). Thus, each player's preference pattern is influenced by the decisions of the other players (modulo null coalitions) and by the realized state of nature. However, each player must make his/her decision independently after having observed the realized state ω of nature. Thus, the resultant of the players' actions (even though they act independently and noncooperatively!) can be modelled as a <u>decision rule</u> $f : \Omega \to L_X$, which prescribes, for each possible state of nature, the decisions of all players, modulo null coalitions. For reasons of analytical tractability we require that the decision rule f be $(\mathcal{A}, \beta(L_X))$-measurable. Summing up, a <u>random game with a continuum of players</u> is formally a quadruple $\Gamma = [(T,\tau,\mu), (\Omega,\mathcal{A},p), X, P]$, where the measure space (T,τ,μ), $(\Omega,\mathcal{A},p)$ and the correspondences X, P are as described above. The game Γ is said to have a <u>random</u> (Cournot-Nash) <u>equilibrium</u> if there exists a decision rule $f^* : \Omega \to L_X$ such that for p-a.e. $\omega \in \Omega$

$$P(t,\omega,f^*(\omega)) = \emptyset \text{ for } \mu\text{-a.e. } t \in T.$$

Thus, with probability one there is at most a powerless null coalition of players having "something left to be desired" under the equilibrium rule f^*: no player t outside the null coalition disposes of a decision which he/she would strictly prefer to his/her actual decision $f^*(\omega)(t)$ prescribed by f^*. Clearly, the above definition is a generalization of the usual notion in economics and game theory, [see, e.g., Khan (1986), Khan-Papageorgiou (1988), Schmeidler (1973), Yannelis (1987)]. [We especially recommend the excellent survey of Khan (1985).]

We now state the conditions needed for the proof of our main theorem. A detailed discussion of these is given in Section 4.

(C0) the σ-algebra τ is countably generated,

(C1) X has nonempty convex w-compact values,

(C2) X is μ-integrably bounded and has a $\tau \otimes \beta(E)$-measurable graph,

(C3) dom (con P) is a $\tau \otimes \mathcal{A} \otimes \beta(L_X)$-measurable subset of $T \times \Omega \times L_X$,

(C4) there exists no decision rule $f : \Omega \to L_X$ such that for p-a.e. $\omega \in \Omega$, $f(\omega)(t) \in \text{con } P(t,\omega,f(\omega))$ for μ-a.e. $t \in T$.

We also require the existence of a correspondence $\alpha : \text{dom (con P)} \to 2^E$ with measurable graph and nonempty values, such that for every $t \in T$, $\omega \in \Omega$, (C5) $\alpha(t,\omega,x) \subset \text{con } P(t,\omega,x) \subset X(t)$ for every $x \in L_X$,

(C6) $\alpha(t,\omega,\cdot) : L_X \to 2^E$ is u.s.c. with w-closed convex values,

(C7) dom (con $P(t,\omega,\cdot)$) is w-open.

Our main result, whose proof is deferred to Section 5, can now be stated.

Main Theorem: Let $\Gamma = [(T,\tau,\mu), (\Omega,\mathcal{A},p), X, P]$ be a random game satisfying the conditions (C0)-(C7). Then Γ has a random equilibrium.

As a corollary of our Main Theorem, we can obtain an extension of the results of Khan (1986) and Schmeidler (1973) by allowing the utility function of each player to be random. In particular, let $G = [(T,\tau,\mu), (\Omega,\mathcal{A},p), X, u]$ be a random game with a continuum of players as defined above, with the exception that each player t in T is now equipped with a random utility function $u_t : \Omega \times X(t) \times L_X \to R$. The game G is said to have a random Cournot-Nash equilibrium if there exists an $(\mathcal{A},\beta(L_X))$-measurable function $f^* : \Omega \to L_X$ such that

$$u_t(\omega,f_t^*(\omega),f^*(\omega)) = \max_{y \in X(t)} u_t(\omega,y,f^*(\omega))$$

for μ-a.e. $t \in T$ and p-a.e. $\omega \in \Omega$.

Now we can state the following corollary of our main theorem:

Corollary 3.1: Let $G = [(T,\tau,\mu), (\Omega,\mathcal{A},p), X, u]$ be a random game with a continuum of players satisfying the conditions (C0)-(C2) and in addition:

(N1) The set $\{(t,\omega,y) \in T \times \Omega \times E : y \in X(t),\ u_t(\omega,y,x) \leq \delta\}$ belongs to $\tau \otimes \mathcal{A} \otimes \beta(E)$ for every $\delta \in R$, $x \in L_X$, (N2) $u_t(\omega,\cdot,x)$ is quasi-concave on $X(t)$ for every $t \in T$, $\omega \in \Omega$, $x \in L_X$,

(N3) $u_t(\omega,\cdot,\cdot)$ is continuous on $X(t) \times L_X$ for every $t \in T$, $\omega \in \Omega$.

Then G has a random Cournot-Nash equilibrium.

Proof: Define the correspondence $P : T \times \Omega \times L_X \to 2^E$ by

$$P(t,\omega,x) := \{y \in X(t) : u_t(\omega,y,x) > u_t(\omega,x(t),x)\}.$$

Then in view of Remarks 4.4, 4.5 in the next section it is easily seen (given the conditions (N1)-(N3)) that the correspondence P satisfies all the conditions of the Main Theorem. The resulting random equilibrium is easily seen to form a random equilibrium for G.

The above Corollary will be used in Section 6 to prove the existence of a symmetric Bayesian equilibrium. In the next section we will discuss the conditions used in the Main Theorem.

4. DISCUSSION OF THE ASSUMPTIONS

In this section we discuss a number of cases where the technical conditions of the Main Theorem are fulfilled. The conditions of the first two cases presented here are not attractive from a more practical standpoint, but in the third and fourth case the conditions are of a rather standard nature. The key instrument there is formed by recent Carathéodory selection results of Kim-Pikry-Yannelis (1987, 1988).

<u>Remark 4.1</u>: Condition (C4) is obviously weaker than

(C4') for p-a.e. $\omega \in \Omega$ there does not exist $x \in L_X$ such that $x(t) \in \text{con } P(t,\omega,x)$ for μ-a.e. $t \in T$.

Condition (C3) is weaker than

(C3') $\text{con } P : T \times \Omega \times L_X \to 2^E$ is a measurable correspondence.

<u>Remark 4.2</u>: Suppose that

(E1) $\text{con } P : T \times \Omega \times L_X \to 2^E$ has a $\tau \otimes \mathcal{A} \otimes \beta(L_X \times E)$-measurable graph,

(E2) $\text{con } P(t,\omega,\cdot) : L_X \to 2^E$ is u.s.c. and has nonempty w-closed values for every $t \in T$, $\omega \in \Omega$.

Then conditions (C5)-(C7) hold.

<u>Proof</u>: Define $\alpha := \text{con } P$; then (C5)-(C7) hold trivially.

<u>Remark 4.3</u>: Suppose that (E1) holds and that

(E3) $\text{con } P(t,\omega,\cdot) : L_X \to 2^E$ is l.s.c. and has w-closed values (possibly empty) for every $t \in T$, $\omega \in \Omega$,

(E4) the σ-algebra $\tau \otimes \mathcal{A}$ is complete with respect to $\mu \times p$.

Then conditions (C5)-(C7) hold.

<u>Proof</u>: By the definition of lower semicontinuity, (E3) implies that (C7) holds. Also, it follows from (E3)-(E4) that the Carathéodory selection result of Kim-Pikry-Yannelis (1987, Thm. 3.1) obtains. Hence, there exists a $(\tau \otimes \mathcal{A} \otimes \beta(L_X), \beta(E))$-measurable function $a : \text{dom (con } P) \to E$ such that for all $t \in T$ and $\omega \in \Omega$

$a(t,\omega,\cdot)$ is continuous on dom (con $P(t,\omega,\cdot)$) and

$a(t,\omega,x) \in \text{con } P(t,\omega,x)$ for all $x \in \text{dom con } P(t,\omega,\cdot)$.

Therefore, $a(t,\omega,x) := \{a(t,\omega,x)\}$ satisfies (C5), (C7).

In (E2) and (E3) the correspondence con P is required to have w-closed values. In view of condition (C4) this is hardly satisfactory, since $P(t,\omega,x)$ represents for player t the decisions which he/she would strictly prefer over $x(t)$ under the state of nature ω. Fortunately, in the following cases the values of con P need not be closed.

The following case is a variant of the one above. It follows by applying the variant in Kim-Pikry-Yannelis (1987, Thm. 3.2) of the Carathéodory selection result used above. The proof in this case remains exactly the same.

<u>Remark 4.4</u>: Suppose that (E1) and (E4) hold and

(E5) $\quad \text{con } P(t,\omega,\cdot) : L_X \to 2^E$ is l.s.c. for every $t \in T$, $\omega \in \Omega$,

and at least one of the following two conditions are satisfied:

(E6) $\quad$ E is finite-dimensional,

(E6') $\quad \text{int } (\text{con } P(t,\omega,x))$ is nonempty for every $(t,\omega,x) \in \text{dom } (\text{con } P)$.

Then conditions (C5)-(C7) hold.

<u>Remark 4.5</u>: Suppose that (E1), (E4) hold and that for all $t \in T$ and $\omega \in \Omega$

(E7) $\quad \text{con } P(t,\omega,\cdot)^{-1}(x) := \{y \in L_X : x \in \text{con } P(t,\omega,y)\}$ is weakly open for every $x \in E$,

(E8) $\quad \text{rel int con } P(t,\omega,x)$ is nonempty for every $x \in \text{dom } (\text{con } P(t,\omega,\cdot))$, where the relative interior is taken with respect to $X(t)$.

Then conditions (C5)-(C7) hold.

Proof: The proof is entirely similar to the proof of Remark 4.3; only this time we invoke to the Carathéodory selection result of Kim-Pikry-Yannelis (1988, Main Theorem).

5. PROOF OF THE MAIN THEOREM

We begin by proving some preparatory results that are needed for the proof of our main theorem.

The compactness part of the following result is commonly referred to as Diestel's theorem. This was given by Diestel (1977) for a "dominating" correspondence X(t) which did not vary with t. In its present form the theorem was first stated by Byrne (1978, Thm. 3), [see also Balder (1990) for an extension involving a.e. convergence of arithmetic averages]. Note also that the nonemptiness part of the result below is a direct consequence of the von Neumann-Aumann measurable selection theorem, [Castaing-Valadier (1977, III.22)].

Theorem 5.1: Suppose that conditions (C1)-(C2) hold. Then L_X is a nonempty convex and weakly compact subset of L^1_E.

Proposition 5.2: Suppose that conditions (C0)-(C2) hold. Then the weak topology of L_X coincides with the topology induced by the weak metric D of L^1_E.

Proof: By (C0) the Banach space $(L^1_E, \|\cdot\|_1)$ is separable. Thus, L^1_E has a well-defined weak metric D (see the definition given in Section 2). By (C1), (C2) L_X is a weakly compact subset of L^1_E (apply Theorem 5.1), so by what was said following the introduction of the weak metric, the weak topology and the D-topology coincide on L_X.

Let (Y,ρ) be a separable metric space, and let $\gamma : T \times \Omega \times Y \to 2^E$ be a given correspondence such that $\gamma(t,\omega,y) \subset X(t)$ for every $t \in T$, $\omega \in \Omega$. Correspondingly, we define $\psi : \Omega \times Y \to 2^{L_X}$ by

$$\psi(\omega,y) := \{x \in L_X : x(t) \in \gamma(t,\omega,y) \ \mu\text{-a.e.}\}.$$

The first part of the next result is well-known [see for instance Khan-Papageorgiou (1987) or Yannelis (1987, 1989)]. We give a new proof, which matches the proof of its second part.

<u>Proposition 5.3</u>: Suppose that (C0), (C1), and (C2) are valid. If

$\gamma(t,\omega,\cdot) : Y \to 2^E$ is w-u.s.c. with w-closed, convex values for every $t \in T$, $\omega \in \Omega$,

then

$\psi(\omega, \cdot) : Y \to 2^{L_X}$ is weakly u.s.c. for every $\omega \in \Omega$.

If in addition

γ has a $\tau \otimes \mathcal{A} \otimes \beta(Y \times E)$-measurable graph,

then there exists a correspondence $\psi' : \Omega \times Y \to 2^{L_X}$ such that

$\psi'(\omega,\cdot) = \psi(\omega,\cdot)$ for p-a.e. $\omega \in \Omega$, and

ψ' has a $\mathcal{A} \otimes \beta(L_X)$-measurable graph.

<u>Proof</u>: To prove the first statement, it is enough, in view of Theorem 5.1, to prove that for arbitrary fixed $\omega \in \Omega$ the correspondence $\psi(\omega,\cdot)$ has a closed graph in $Y \times L_X$ [Aubin-Cellina (1984,

Cor. 1.1.1)]. To this end, let $\{y_k\}$ converge in Y to $\bar{y}$, and let $\{x_k\}$ converge weakly to $\bar{x}$ in L_X, $x_k \in \psi(\omega, y_k)$. Define $m : T \times \Omega \times Y \times E \to \{0, +\infty\}$ by

$$m(t,\omega,y,x) := \begin{cases} 0 \text{ if } x \in \gamma(t,\omega,y) \\ +\infty \quad \text{if not} . \end{cases}$$

Since $\omega \in \Omega$ is fixed, we shall abbreviate as follows:

$$l(t,y,x) := m(t,\omega,y,x).$$

The desired conclusion $\bar{x} \in \psi(\omega,\bar{y})$ follows now immediately from applying a classical lower semicontinuity theorem to the outer integral functional $I_l : Y \times L_X \to \{0,+\infty\}$, given by

$$I_l(y,x) := \tilde{\int}_T l(t,y,x(t))\, \mu(dt),$$

Note that for every $t \in T$ the function $l(t,\cdot,\cdot)$ is l.s.c. on $Y \times (E,w)$, since the graph of $\gamma(t,\omega,\cdot)$ is a closed subset of $Y \times X(t)$. Note also that the function $l(t,y,\cdot)$ is convex. Thus, all the conditions of the lower semicontinuity result in Balder (1984, Thm. 3.1) hold, and we obtain

$$I_l(\bar{y},\bar{x}) \leq \liminf_k I_l(y_k,x_k) = 0.$$

(Note the y_k's act as constant functions on T, which converge to the constant function $\bar{y}$.) In view of the definition of outer integration it follows immediately that μ-a.e. $l(t,\bar{y},\bar{x}(t)) \leq 0$, i.e., $\bar{x}(t) \in \gamma(t,\omega,\bar{y})$.

For the proof of the second part we employ a very similar argument, which is based on the key observation that the Borel σ-algebras

$\beta(L^1_E, \|\cdot\|_1)$ and $\beta(L^1_E, D)$ coincide (as was noted following the definition of the weak metric). Note first that measurability of the graph of γ is actually equivalent to $\tau \otimes \mathcal{A} \otimes \beta(Y \times E)$-measurability of m, defined above. Also, as we already saw above, $l(t,\cdot,\cdot) := m(t,\omega,\cdot,\cdot)$ is l.s.c. on $Y \times (E,w)$--hence a fortiori on $Y \times (E,\|\cdot\|)$ for every $t \in T$, $\omega \in \Omega$. Thus, m is a so-called normal integrand on $T \times \Omega \times Y \times (E,\|\cdot\|)$. By the construction given in Balder (1984, Appendix) there exist a nondecreasing sequence $\{m_n\}$ of $\tau \otimes \mathcal{A} \otimes \beta(Y \times E)$-measurable functions $m_n : T \times \Omega \times Y \times E \to [0,+\infty)$ and a $\tau \otimes \mathcal{A}$-measurable subset N of $T \times \Omega$, $(\mu \times p)(N) = 0$, such that for every $(t,\omega) \in T \times \Omega$, $n \in \mathbb{N}$

$$|m_n(t,\omega,y,x) - m_n(t,\omega,y',x')| \leq n\rho(y,y') + n\|x - x'\|$$

for all $y \in Y$, $x \in E$ and for every $(t,\omega) \in (T \times \Omega) \setminus N$, $n \in \mathbb{N}$

$$\lim_n \uparrow m_n(t,\omega,y,x) = m(t,\omega,y,x) \text{ for all } y \in Y,\ x \in E.$$

From the second property it follows immediately, by the monotone convergence that for p-a.e. $\omega \in \Omega$ for all $y \in Y$ and $x \in L_X$

$$(5.1) \qquad \lim_n \uparrow I_{m_n}(\omega,y,x) = I_m(\omega,y,x),$$

where the integral functionals $I_m, I_{m_n} : \Omega \times Y \times L_X \to [0,+\infty]$ are defined by

$$I_m(\omega,y,x) := \int_T m(t,\omega,y,x(t))\, \mu(dt),$$

etc. (note that there is no longer need for outer integration). By the Lipschitz continuity property of the $\{m_n\}$ it follows elementarily that for every $n \in N$

$$I_{m_n}(\omega,\cdot,\cdot) \text{ is continuous on } Y \times (L^1_E, \|\cdot\|_1) \text{ for every } \omega \in \Omega.$$

Also, it is a basic fact, [e.g., see Neveu (1964, III)] that for every $n \in N$

$$I_{m_n}(\cdot,y,x) \text{ is } \mathcal{A}\text{-measurable for every } y \in Y,\ x \in L_X.$$

Therefore, it follows from Castaing-Valadier (1977, III.14), in view of the separability of $Y \times (L^1_E, \|\cdot\|_1)$, that for every $n \in N$

$$I_{m_n} \text{ is } \mathcal{A} \otimes \beta(Y \times L_X)\text{-measurable.}$$

Hence, by (5.1)

$$I_m \text{ is } \mathcal{A} \otimes \beta(Y \times L_X)\text{-measurable.}$$

Finally, it remains to define $\psi' : \Omega \times Y \to 2^{L_X}$ by specifying its graph to be the set

$$\{(\omega,y,x) \in \Omega \times Y \times L_X : \lim_n \uparrow I_{m_n}(\omega,y,x) \leq 0\};$$

then the proof is over, in view of (5.1) and the definition of m.

Remark 5.4: An alternative proof of the measurability part of Proposition 5.3 can be given using Castaing-Valadier (1977, III.15). In contrast to the proof given above, one then has to exploit the fact that γ has convex values.

We are now ready to complete the proof of our Main Theorem.

Proof of the Main Theorem: To begin with, note that by Theorem 5.1 and Proposition 5.2 the space (L_X, D) is metrizable, compact and hence separable for the weak topology. (This space will play the role of (Y,ρ) in the previous section.) We define the correspondence $F : T \times \Omega \times L_X \to 2^E$ by

$$F(t,\omega,x) := \begin{cases} \alpha(t,\omega,x) \text{ if } x \in \text{dom } (\text{con } P(t,\omega,\cdot)) \\ X(t) \quad \text{if not.} \end{cases}$$

From the definition of upper semicontinuity it follows from (C5), (C6) that for every $t \in T$, $\omega \in \Omega$, $F(t,\omega,\cdot) : L_X \to 2^E$ is upper semicontinuous; also, $F(t,\omega,\cdot)$ has w-compact convex values in $X(t)$ for every $t \in T$, $\omega \in \Omega$ by (C1), (C5) and the properties of α. Therefore, by Proposition 5.3 the convex-valued correspondence $B : \Omega \times L_X \to 2^{L_X}$, defined by

$$B(\omega,y) := \{x \in L_X : x(t) \in F(t,\omega,y) \text{ for } \mu\text{-a.e. } t \in T\},$$

is such that for every $\omega \in \Omega$

$$B(\omega,\cdot) : L_X \to 2^{L_X} \text{ is weakly u.s.c.}$$

Also, for every $\omega \in \Omega$, $y \in L_X$ the correspondence $F(\cdot,\omega,y)$ has a measurable graph and nonempty values, by (C3) and the definitions of α, F. Hence, the values of B are nonempty by the von Neumann-Aumann measurable selection theorem [Castaing-Valadier (1977, III.22)]. By Theorem 5.1, L_X is a nonempty convex weakly compact subset of L^1_E.

Hence, it follows from the Fan-Glicksberg fixed point theorem [Fan (1952), Glicksberg (1952)] that for every $\omega \in \Omega$

(5.2) there exists $x \in L_X$ such that $x \in B(\omega,x)$.

By (C3) and the definitions of α, X the correspondence F is seen to have a $\tau \otimes \mathcal{A} \otimes \beta(L_X \times E)$-measurable graph. Hence, by Proposition 4.3 there exists a correspondence $B' : \Omega \times L_X \to 2^{L_X}$ with $\mathcal{A} \otimes \beta(L_X \times L_X)$-measurable graph such that $B'(\omega,\cdot) = B(\omega,\cdot)$ p-a.e. Therefore, it is immediate that the graph

$$G := \{(\omega,y,x) \in \Omega \times L_X \times L_X : y = x, \ x \in B'(\omega,y)\}$$

of the correspondence $\Phi \cap B'$ is also $\mathcal{A} \otimes \beta(L_X \times L_X)$-measurable, where $\Phi : \Omega \times L_X \to 2^{L_X}$ is defined by $\Phi(\omega,y) := \{y\}$ (note that the graph of Φ is measurable by virtue of the fact that L_X is separable and metric). By (5.2) we have that for p-a.e. $\omega \in \Omega$ the section of G at ω is nonempty. Hence, by the von Neumann-Aumann measurable selection theorem [Castaing-Valadier (1977, III.22)] there exists a $(\mathcal{A} \otimes \beta(L_X), \beta(L_X \times L_X))$-measurable function $h : \Omega \to L_X \times L_X$ such that

$$(\omega,h_1(\omega),h_2(\omega)) \in G \text{ for p-a.e. } \omega \in \Omega,$$

where h_1, h_2 stand for the coordinate functions of h. In particular, this implies the following

$$h_2(\omega) \in B'(\omega,h_2(\omega)) \text{ for p-a.e. } \omega \in \Omega.$$

By the definition of B' and B, this implies that for p-a.e. $\omega \in \Omega$

$$h_2(\omega)(t) \in F(t,\omega,h_2(\omega)) \text{ for } \mu\text{-a.e. } t \in T.$$

If there were a set $C \in \mathcal{A}$, $p(C) > 0$, such that $\text{dom}\,(\text{con}\, P(t,\omega,h_2(\omega))) \neq \emptyset$ μ-a.e. for every $\omega \in C$, then by the definition of F for every $\omega \in C$

$$h_2(\omega)(t) \in \alpha(t,\omega,h_2(\omega)) \in \text{con}\, P(t,\omega,h_2(\omega)) \text{ for } \mu\text{-a.e. } t \in T.$$

This would contradict (C4). Thus we conclude that for p-a.e. $\omega \in \Omega$

$$P(t,\omega,h_2(\omega)) \subset \text{con}\, P(t,\omega,h_2(\omega)) = \emptyset \text{ for } \mu\text{-a.e. } t \in T,$$

which proves that h_2 is the desired rule to achieve a random equilibrium for the game Γ.

6. THE EXISTENCE OF A SYMMETRIC BAYESIAN EQUILIBRIUM

We now show how our main result can be used to prove the existence of an equilibrium for a Bayesian game with a continuum of players. For somewhat similar finite player Bayesian games, existence results have been obtained by Balder (1988), Harsanyi (1967), Milgrom-Weber (1985) and Yannelis-Rustichini (1988).

A symmetric Bayesian game with a continuum of players is a sextuple $B = [(T,\tau,\mu),(\Omega,\mathcal{A},p),X,u,\mathcal{J},q]$, where (T,τ,μ) is the measure space of players, $(\Omega,\mathcal{A},p)$ a probability space denoting all the possible states of nature, whose probability distribution p may be unknown, $X : T \to 2^E$ the decision correspondence from T into the separable Banach space E, $u_t : \Omega \times X(t) \times L_X \to R$ the random utility function of player t in T, $\mathcal{S}$ a sub-σ-algebra of $\mathcal{A}$ representing the information pattern available to the players (supposed to be the same for all players; hence the adjective symmetric for the game), and q_t a transition probability from $(\Omega, \mathcal{S})$ into $(\Omega, \mathcal{A})$ representing player t's a posteriori belief about p : the realized $\omega \in \Omega$, (by means of the possibly incompletely known distribution p) gives rise to player t observing an outcome S in $\mathcal{S}$; then $q_t(\omega;\cdot)$ represents player t's belief about p, based on the observed ω. A symmetric Bayesian equilibrium for B now consists of an $\mathcal{S}$-measurable function $f^* : \Omega \to L_X$ such that

$$v_t(\omega, f_t^*(\omega), f^*(\omega)) = \max_{y \in X(t)} v_t(\omega, y, f^*(\omega))$$

$$\text{for } \mu\text{-a.e. } t \in T \text{ and } p\text{-a.e. } \omega \in \Omega.$$

Here we define player t's a posteriori expected utility function by

$$v_t(\omega, y, x) := \int_\Omega u_t(\omega', y, x)\, q_t(\omega; d\omega')$$

The result below may be seen as an extension of Theorem 3.3 in Yannelis-Rustichini (1988) to Bayesian games with a continuum of players.

Corollary 6.1. Let $B = [(T,\tau,\mu),(\Omega,\mathcal{A},p),X,u,\mathcal{S},q]$ be a Bayesian game with a continuum of players satisfying the conditions (C1)-(C2), (N1)-(N3). Moreover, suppose that

$\mathcal{S}$ is countably generated,

and that for every $t \in T$, $\omega \in \Omega$

$$u_t(\omega,\cdot,\cdot) \text{ is } q_t(\omega;\cdot)\text{-integrably bounded.}$$

Then B has a symmetric Bayesian equilibrium.

Proof: By the dominated convergence theorem it follows immediately from (N3) that for every $t \in T$, $\omega \in \Omega$, $v_t(\omega,\cdot,\cdot)$ is continuous on $X(t) \times L_X$. Moreover, it is easily seen that v is measurable in the sense that the set $\{(t,\omega,y) \in T \times \Omega \times E : y \in X(t),\ u_t(\omega,y,x) \leq \delta\}$ belongs to $\tau \otimes \mathcal{S} \otimes \beta(E)$ for every $\delta \in R$, $x \in L_X$. It follows from (N2) that $v_t(\omega,\cdot,x)$ is quasi-concave on $X(t)$ for every $t \in T$, $\omega \in \Omega$, $x \in L_X$. We can now consider the Bayesian game B as a random game $[(T,\tau,\mu),(\Omega,\mathcal{A},p),X,v]$. This random game satisfies all the conditions of Corollary 3.1, so there exists an $\mathcal{S}$-measurable equilibrium decision rule $f^* : \Omega \to L_X$ for the random game. This rule is precisely the desired symmetric Bayesian equilibrium rule for B.

7. CONCLUDING REMARKS

Remark 7.1: The Main Theorem as well as Corollary 3.1 can be extended in a straightforward manner to random abstract economies, that is, abstract economies as defined in Khan-Papageorgiou (1988) or Yannelis (1987), with the only exception that preference correspondences as well as constraint correspondences are now allowed to depend on the states of nature of the world. Obviously, such results can be used to obtain equilibrium existence results for either random or Bayesian exchange economies.

Remark 7.2: The form of the Bayesian game with a continuum of players described in Section 6 can be generalized by replacing the random utility function $u_t : \Omega \times X(t) \times L_X \to R$ of each player by a random preference correspondence $P : T \times \Omega \times L_X \to 2^E$. In this new setting the a posteriori preference correspondence of player t is now defined by

$$\Pi(t,\omega,x) := \int_\Omega q_t(\omega;d\omega')\ P(t,\omega',x),$$

where the integral of the correspondence $P(t,\cdot,x)$ is defined in the usual way [see for instance Yannelis (1989)]. By setting in Theorem 6.1, $E = R^n$ and replacing the assumptions (N1)-(N3) by:

(N1') con $\Pi(\cdot,\cdot,\cdot)$ is lower measurable,

(N2') for each measurable function $x : \Omega \to L_X$,
$x_t(\omega) \notin \text{con}\ \Pi(t,\omega,x(\omega))$ for almost all t in T and for almost all ω in Ω,

(N3') for each fixed t in T and ω in Ω, $P(t,\omega,\cdot)$ is l.s.c., P is integrably bounded and it has a measurable graph,

then one can prove the existence of a symmetric Bayesian equilibrium for this more general form of a Bayesian game. In particular, by Theorem 3.3 in Yannelis (1989) for each fixed $t \in T$ and $\omega \in \Omega$, $\Pi(t,\omega,\cdot)$ is l.s.c. Hence, (in view of Remark 4.4) it follows from the Main Theorem that there exists an $\mathcal{F}$-measurable function $f^* : \Omega \to L_X$ such that for almost all t in T, $\Pi(t,\omega,f^*(\omega)) = \emptyset$ for almost all $\omega \in \Omega$.

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