AN ESSAY

ON

ALGEBRAIC DEVELOPMENT,

CONTAINING

THE PRINCIPAL EXPANSIONS

IN COMMON ALGEBRA, IN THE DIFFERENTIAL

AND INTEGRAL CALCULUS,

AND IN

THE CALCULUS OF FINITE DIFFERENCES;

THE GENERAL TERM

BEING IN EACH CASE IMMEDIATELY OBTAINED
BY MEANS OF

A NEW AND COMPREHENSIVE NOTATION.

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PREFACE.

THE following pages are intended to illustrate and apply a system of Algebraic Notation submitted to the Cambridge Philosophical Society in the year 1827, and published in the third Volume of their Transactions. In that paper the applications were necessarily few, and the whole was deficient in that development which was indispensable to render the introduction of the system into general use at all probable; but in the present Work it is applied to the demonstration of the most important series in pure Analysis. The methods by which these are demonstrated are partly original, and partly taken from one or other of the works of which a list follows this Preface, but they are in general so much modified that a distinct reference to the inventor of each demonstration appeared useless; so much, however, is due to the admirable works of Schweins, that it would be unjust not to make a distinct acknowledgment of the great use that has been made of his "Analysis." The demonstration of the legitimacy of the separation of the symbols of operation and quantity, with certain limitations, belongs to Servois, and will be found in the "Annales des Mathématiques;" and the proof, that the coefficients of the binomial, (the index being a positive integer,) are integers, is due to Mr. Miller, of St. John's College, and is the only independent proof with which I am acquainted.

The following apparent innovations in the ordinary notation are not original:

(1) $E_x\phi(x)$, for $\phi(x+Dx)$, is partly due to Arbogast, who uses $E\phi(x)$ for the same function.

(2) $d_x^n u$, for $\frac{d^n u}{dx^n}$, is due to Lacroix, although not used by him, being merely pointed out in a single line*; it was suggested to the writer of these pages by the analogous integral notation invented by Professor Airy.

(3) $(u)_{x=a}$, for the value assumed by u when x is put equal to a, belongs to Schweins.

In order that the work may be as independent as possible, the Reader is supposed to be acquainted only with the first rules of Algebra, and the fundamental theorems of Trigonometry; and, for the sake of facility of reference, the whole of the theorems have been arranged in an index at the end of the volume.

The additions contain a few Theorems of importance that did not suggest themselves till too late to be inserted in the text, together with a few simplifications of the demonstrations inserted in the body of the Work.

In conclusion, the Author has to acknowledge the great liberality of the Syndics of the University Press, in defraying a considerable part of the expense of publishing.

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* Calcul Diff. Tome II, page 527.

LIST OF WORKS

WHICH HAVE BEEN CONSULTED.

ANNALES des Mathématiques.

Arbogast, Calcul des Dérivations.

Herschel, Examples on Finite Differences.

Hindenburg, Sammlung combinatorisch-analytischer Abhandlungen.

Lacroix, Calcul Différentiel et Intégral.

Laplace, Théorie Analytique des Probabilités.

Schweins, Analysis.

——— Theorie der Differenzen und Differentiale.

Wronski, Introduction à la Philosophie des Mathématiques.

INDEX TO THE CHAPTERS.

INDEX TO THE SYMBOLS.

ERRATA.

The Reader is requested to correct the following Errata before he proceeds to the perusal of the Work.

PAGE	LINE	ERRATUM.	CORRECTION.
5	8	$a_5 + a_{2n-1}$	$a_5 + \ldots + a_{2n-1}$
6	*last*	S_n	$\overset{s}{S}_n$
19	1	$c\}_{n+1}\ldots\}_1$	c
	6	$\overset{m}{P}$	$\overset{n}{P}$
21	3	$\overset{n,m}{C}$	$\overset{m,n}{C}$
22	17	$\overset{n-m+1,n}{C}$	$\overset{n-m+1,m}{C}$
	last but one	$+a_1 a_3$	$+a_1 a_2 a_1 + a_1 a_3$
	last	$2a_2 a_1^2$	$3a_2 a_1^2$
24	19	$\overset{m}{A}$	$\overset{m-1}{A}$
47	*last*	$D_x(x)$	$D_x(u)$
48	3	$\phi(u)$	$\phi(x)$
	14	$a.D.$	$a.D_x.$
62	11	a_x	a^x
88	11	x^{2m+3}	x^{2m-3}
89	10	$\underline{\vert 2m8}$	$\underline{\vert 2m}$
98	6 *and last*	(201)	(200)
104 105	15 9	$\overset{\infty}{S}_m$	$\overset{n}{S}_m$
	last but one	S_n	S_m
126	8 *and* 9	x^n	x^{-n}
135	6	S_m	$\overset{\infty}{S}_m$
146	*last but one*	$\Delta_x^n(u_x)$	$\Delta_x^m(u_x)$

ON THE DEVELOPMENT

OF

ALGEBRAIC FUNCTIONS.

CHAPTER I.

ON SERIES IN GENERAL.

1. In the expansion of Algebraic Functions it has been usual to investigate the first three or four terms, and from these to deduce the remainder of the series by analogy. The unsatisfactory nature of this method in all cases, and the errors into which it may readily lead us in very many instances, must have been obvious to all who have made use of it. In some cases indeed, the connection between the consecutive terms at the commencement of the series is so obscure, that the most patient of analysts have given up the search, and have been compelled to state that "the law of the series is not obvious." In order to avoid this obscurity and embarrassment, we shall adopt a notation by means of which the general term will be obtained in every case, and which will enable us to perform any operation whatever on a series, with the same facility as on a single term.

2. The m^{th} term of a series being usually some function of m, we shall denote it by a_m; and, taking the letter S as an abridgment of the word *Sum*, the symbol $\overset{n}{\text{S}}_m a_m$ will be used to

denote *the sum of* n *terms of which the* m^{th} *is* a_m: that is,

$$\overset{n}{S}_m a_m = a_1 + a_2 + a_3 + \dots + a_n.$$

In this notation it will be seen that the index placed over the S denotes the *number* of terms, and that the index placed under the same letter, is that to which the successive values 1, 2, 3, ..., n, must be given, in the function a_m, in order to form the consecutive terms: that is,

$$\overset{n}{S}_m \phi(m) = \phi(1) + \phi(2) + \phi(3) + \dots + \phi(n).$$

Or, to give examples of a more simple nature,

$$\overset{n}{S}_m m^2 = 1^2 + 2^2 + 3^2 + \dots + n^2.$$

$$\overset{n}{S}_m m^r = 1^r + 2^r + 3^r + \dots + n^r.$$

$$\overset{n}{S}_m (r-2m+1)^s = (r-1)^s + (r-3)^s + (r-5)^s + \dots + (r-2n+1)^s.$$

3. The symbol $\overset{n;r}{S}_m a_m$ denotes that the r^{th} term must be omitted.

4. *Theorem.* If $a_m = b_m$, $\binom{m=1}{m=n}$*, then $\overset{n}{S}_m a_m = \overset{n}{S}_m b_m$.

For, since $a_1 = b_1$

$a_2 = b_2$

$a_3 = b_3$

&c. = &c.

$a_n = b_n$.

$$\therefore\ a_1 + a_2 + a_3 + \text{\&c.} + a_n = b_1 + b_2 + b_3 + \text{\&c.} + b_n,$$

$$\text{or } \overset{n}{S}_m a_m = \overset{n}{S}_m b_m.$$

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* By this notation is meant, that this equation is to hold for every integral value of m, from 1 to n.

5. *Theorem.* $\overset{n}{\mathrm{S}}_m (a_m + b_m) = \overset{n}{\mathrm{S}}_m a_m + \overset{n}{\mathrm{S}}_m b_m$.

$$\text{For, } \overset{n}{\mathrm{S}}_m (a_m + b_m) = a_1 + b_1 + a_2 + b_2 + \ldots + a_n + b_n$$

$$= a_1 + a_2 + \ldots + a_n + b_1 + b_2 + \ldots + b_n$$

$$= \overset{n}{\mathrm{S}}_m a_m + \overset{n}{\mathrm{S}}_m b_m.$$

6. *Theorem.* If b is independent of m, then $\overset{n}{\mathrm{S}}_m a_m b = b \,.\, \overset{n}{\mathrm{S}}_m a_m$.

$$\text{For, } \overset{n}{\mathrm{S}}_m a_m b = a_1 b + a_2 b + a_3 b + \ldots + a_n b$$

$$= b\,(a_1 + a_2 + a_3 + \ldots + a_n)$$

$$= b \,.\, \overset{n}{\mathrm{S}}_m a_m.$$

7. Cor. $\overset{n}{\mathrm{S}}_m b = nb$.

8. *Problem.* To invert the order of the terms of a given series.

$$\text{Now, } \overset{n}{\mathrm{S}}_m a_m = a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_n$$

$$= a_n + a_{n-1} + a_{n-2} + \ldots + a_1, \text{ by inverting the series;}$$

$$= \overset{n}{\mathrm{S}}_m a_{n-m+1}.$$

In order therefore to invert the series, we must substitute $n - m + 1$ for m in the expression for a_m.

9. *Theorem.* $\overset{n}{\mathrm{S}}_m a_m = \overset{r}{\mathrm{S}}_m a_m + \overset{n-r}{\mathrm{S}}_m a_{r+m}$.

$$\text{For, } \overset{n}{\mathrm{S}}_m a_m = \{a_1 + a_2 + a_3 + \ldots a_r\} + \{a_{r+1} + a_{r+2} + \ldots + a_n\}$$

$$= \overset{r}{\mathrm{S}}_m a_m + \overset{n-r}{\mathrm{S}}_m a_{r+m}.$$

By means of this theorem we can separate from the rest any number of terms, taken either at the beginning or end of a given series.

10. *Theorem.* $\overset{n}{\mathrm{S}}_m x^{m-1} = \dfrac{1-x^n}{1-x}$.

For, $x^{m-1} = x^{m-1} \cdot \dfrac{1-x}{1-x}$

$$= \frac{x^{m-1}}{1-x} - \frac{x^m}{1-x}.$$

$$\therefore \overset{n}{\mathrm{S}}_m x^{m-1} = \overset{n}{\mathrm{S}}_m \frac{x^{m-1}}{1-x} - \overset{n}{\mathrm{S}}_m \frac{x^m}{1-x}, \quad (4) \text{ and } (5);$$

$$= \frac{x^0}{1-x} + \overset{n-1}{\mathrm{S}}_m \frac{x^m}{1-x} - \overset{n-1}{\mathrm{S}}_m \frac{x^m}{1-x} - \frac{x^n}{1-x}, \quad (9);$$

$$= \frac{1-x^n}{1-x}.$$

11. Cor. 1. $\dfrac{a^n - x^n}{a-x} = \overset{n}{\mathrm{S}}_m a^{n-m} \cdot x^{m-1}$.

12. Cor. 2. $\dfrac{1}{1-x} = \overset{n}{\mathrm{S}}_m x^{m-1} + \dfrac{x^n}{1-x}$;

and $\dfrac{1}{1+x} = \overset{n}{\mathrm{S}}_m (-1)^{m-1} \cdot x^{m-1} + (-1)^n \cdot \dfrac{x^n}{1+x}$.

If x is < 1, the term $\dfrac{x^n}{1-x}$ will diminish as n increases, and therefore, by taking n sufficiently great, $\overset{n}{\mathrm{S}}_m x^{m-1}$ may be made to differ from $\dfrac{1}{1-x}$ by a quantity less than any assignable quantity, although that difference will never vanish for any finite value of n. In this case $\dfrac{1}{1-x}$ is said to equal an *infinite* series of which the m^{th} term is x^{m-1}; and this relation is denoted by the equation $\dfrac{1}{1-x} = \overset{\infty}{\mathrm{S}}_m x^{m-1}$.

13. If the law which determines the value of a_m in the series $\overset{n}{S}_m a_m$ is such, that $a_m = 0$ for every value of m greater than n, we may substitute $\overset{\infty}{S}_m a_m$ for $\overset{n}{S}_m a_m$; and this substitution will frequently facilitate our investigations.

14. *Theorem.* $\overset{2n}{S}_m a_m = \overset{n}{S}_m a_{2m-1} + \overset{n}{S}_m a_{2m}$,

and $\overset{2n-1}{S}_m a_m = \overset{n}{S}_m a_{2m-1} + \overset{n-1}{S}_m a_{2m}$.

For, $\overset{2n}{S}_m a_m = a_1 + a_2 + a_3 + a_4 + a_5 + \ldots + a_{2n}$

$$= (a_1 + a_3 + a_5 + a_{2n-1}) + (a_2 + a_4 + a_6 \ldots + a_{2n})$$

$$= \overset{n}{S}_m a_{2m-1} + \overset{n}{S}_m a_{2m}.$$

and $\overset{2n-1}{S}_m a_m = a_1 + a_2 + a_3 + \ldots + a_{2n-1}$

$$= (a_1 + a_3 + \ldots + a_{2n-1}) + (a_2 + a_4 + \ldots + a_{2n-2})$$

$$= \overset{n}{S}_m a_{2m-1} + \overset{n-1}{S}_m a_{2m}.$$

Cor. $\overset{\infty}{S}_m a_m = \overset{\infty}{S}_m a_{2m-1} + \overset{\infty}{S}_m a_{2m}$.

By means of these theorems we can separate the odd and even terms of a given series.

15. The symbol $\overset{r}{S}_m \overset{s}{S}_n a_{m,n}$ denotes the sum of r terms of which the m^{th} is $\overset{s}{S}_n a_{m,n}$: that is,

$$\overset{r}{S}_m \overset{s}{S}_n a_{m,n} = \overset{s}{S}_n a_{1,n} + \overset{s}{S}_n a_{2,n} + \overset{s}{S}_n a_{3,n} + \ldots + \overset{s}{S}_n a_{r,n}$$

$$= a_{1,1} + a_{1,2} + a_{1,3} + \ldots + a_{1,s}$$

$$+ a_{2,1} + a_{2,2} + a_{2,3} + \ldots + a_{2,s}$$

$$+ a_{3,1} + a_{3,2} + a_{3,3} + \ldots + a_{3,s}$$

$$+ \quad \&c. \quad + \quad \&c.$$

$$+ a_{r,1} + a_{r,2} + a_{r,3} + \ldots + a_{r,s}.$$

It is obvious that the same principle may be extended to any number of symbols of summation.

16. *Theorem.* $(\overset{r}{\mathrm{S}}_m a_m) \times (\overset{s}{\mathrm{S}}_n b_n) = \overset{r}{\mathrm{S}}_m a_m \,.\, \overset{s}{\mathrm{S}}_n b_n$ *.

$$\text{For, } \overset{r}{\mathrm{S}}_m a_m = a_1 + a_2 + a_3 + \ldots + a_r.$$

$$\therefore\ (\overset{r}{\mathrm{S}}_m a_m) \times (\overset{s}{\mathrm{S}}_n b_n) = a_1 . \overset{s}{\mathrm{S}}_n b_n + a_2 . \overset{s}{\mathrm{S}}_n b_n + a_3 . \overset{s}{\mathrm{S}}_n b_n + \ldots + a_r . \overset{s}{\mathrm{S}}_n b_n$$
$$= \overset{r}{\mathrm{S}}_m a_m . \overset{s}{\mathrm{S}}_n b_n.$$

17. *Theorem.* If r is independent of n, and s of m,

$$\text{then } \overset{r}{\mathrm{S}}_m \overset{s}{\mathrm{S}}_n a_{m,n} = \overset{s}{\mathrm{S}}_n \overset{r}{\mathrm{S}}_m a_{m,n}.$$

$$\text{For } \overset{s}{\mathrm{S}}_n a_{m,n} = a_{m,1} + a_{m,2} + a_{m,3} + \ldots + a_{m,s}.$$

$$\therefore\ \overset{r}{\mathrm{S}}_m \overset{s}{\mathrm{S}}_n a_{m,n} = \overset{r}{\mathrm{S}}_m a_{m,1} + \overset{r}{\mathrm{S}}_m a_{m,2} + \overset{r}{\mathrm{S}}_m a_{m,3} + \ldots + \overset{r}{\mathrm{S}}_m a_{m,s}, \quad (4) \text{ and } (5);$$
$$= \overset{s}{\mathrm{S}}_n \overset{r}{\mathrm{S}}_m a_{m,n}.$$

18. *Theorem.* $\overset{\infty}{\mathrm{S}}_m \overset{\infty}{\mathrm{S}}_n a_{m,n} = \overset{\infty}{\mathrm{S}}_m \overset{m}{\mathrm{S}}_n a_{m-n+1,n}.$

$$\text{For, } \overset{\infty}{\mathrm{S}}_m \overset{\infty}{\mathrm{S}}_n a_{m,n} = \overset{\infty}{\mathrm{S}}_n \overset{\infty}{\mathrm{S}}_m a_{m,n}, \quad (17);$$

$$\begin{array}{rl}
= \overset{\infty}{\mathrm{S}}_m a_{m,1} + \overset{\infty}{\mathrm{S}}_m a_{m,2} + \overset{\infty}{\mathrm{S}}_m a_{m,3} + \ldots + \overset{\infty}{\mathrm{S}}_m a_{m,n} & + \&c. \\
= a_{1,1} + a_{2,1} + a_{3,1} + a_{4,1} + \ldots + a_{m,1} & + \&c. \\
+ a_{1,2} + a_{2,2} + a_{3,2} + \ldots + a_{m-1,2} & + \&c. \\
+ a_{1,3} + a_{2,3} + \ldots + a_{m-2,3} & + \&c. \\
+ a_{1,4} + \ldots + a_{m-3,4} & + \&c. \\
+ \&c. + \ \&c. & + \&c. \\
+ a_{1,n} + \&c. + a_{m-n+1,n} & + \&c. \\
+ \&c. + \ \&c. & + \&c. \\
+ a_{1,m} & + \&c. \\
 & + \&c.
\end{array}$$

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* That is, the product of $\overset{r}{\mathrm{S}}_m a_m$ multiplied by $\overset{s}{\mathrm{S}}_n b_n$, is a series consisting of r terms of which the m^{th} is $a_m . \mathrm{S}_n b_n$.

$$=\overset{1}{S}_n a_{2-n,n}+\overset{2}{S}_n a_{3-n,n}+\overset{3}{S}_n a_{4-n,n}+\overset{4}{S}_n a_{5-n,n}+\&c.+\overset{m}{S}_n a_{m-n+1,n}+\&c.$$

by summing vertically;

$$=\overset{\infty}{S}_m\overset{m}{S}_n a_{m-n+1,n}.$$

19. Cor. 1. $\overset{\infty}{S}_m\overset{m}{S}_n a_{m,n}=\overset{\infty}{S}_m\overset{\infty}{S}_n a_{m+n-1,n}.$

20. Cor. 2. $\overset{\infty}{S}_m\overset{m-1}{S}_n a_{m,n}=\overset{0}{S}_n a_{1,n}+\overset{\infty}{S}_m\overset{m}{S}_n a_{m+1,n},$ (9);

$$=\overset{\infty}{S}_m\overset{m}{S}_n a_{m+1,n}$$

$$=\overset{\infty}{S}_m\overset{\infty}{S}_n a_{m+n,n} \quad (19);$$

21. *Theorem.* $\overset{\infty}{S}_m\overset{r}{S}_n a_{m,n}=\overset{r}{S}_m\overset{m}{S}_n a_{m-n+1,n}+\overset{\infty}{S}_m\overset{r}{S}_n a_{m+r-n+1,n}.$

For, $\overset{\infty}{S}_m\overset{r}{S}_n a_{m,n}=\overset{r}{S}_n a_{1,n}+\overset{r}{S}_n a_{2,n}+\overset{r}{S}_n a_{3,n}+\ldots+\overset{r}{S}_n a_{m,n}+\&c.$

$$\begin{aligned}
&=a_{1,1}+a_{1,2}+a_{1,3}+\ldots+a_{1,n}+\ldots+a_{1,r}\\
&+a_{2,1}+a_{2,2}+a_{2,3}+\ldots+a_{2,n}+\ldots+a_{2,r}\\
&+a_{3,1}+a_{3,2}+a_{3,3}+\ldots+a_{3,n}+\ldots+a_{3,r}\\
&+\ \&c. \qquad +\ \&c. \qquad +\ \&c.\\
&+a_{m,1}+a_{m,2}+a_{m,3}+\ldots+a_{m,n}+\ldots+a_{m,r}\\
&+\ \&c. \qquad +\ \&c. \qquad +\ \&c.
\end{aligned}$$

therefore, summing diagonally,

$$\overset{\infty}{S}_m\overset{r}{S}_n a_{m,n}=a_{1,1}+(a_{2,1}+a_{1,2})+(a_{3,1}+a_{2,2}+a_{1,3})+\ldots$$

$$+(a_{r,1}+a_{r-1,2}+a_{r-2,3}+\ldots+a_{1,r})+(a_{r+1,1}+a_{r,2}+a_{r-1,3}+\ldots+a_{2,r})$$

$$+(a_{r+2,1}+a_{r+1,2}+a_{r,3}+\ldots+a_{3,r})+\&c. \quad + \quad \&c.$$

$$=\left\{\overset{1}{S}_n a_{2-n,n}+\overset{2}{S}_n a_{3-n,n}+\overset{3}{S}_n a_{4-n,n}+\ldots+\overset{r}{S}_n a_{r-n+1,n}\right\}$$

$$+\left\{\overset{r}{S}_n a_{r-n+2,n}+\overset{r}{S}_n a_{r-n+3,n}+\&c.\right\}$$

$$=\overset{r}{S}_m\overset{m}{S}_n a_{m-n+1,n}+\overset{\infty}{S}_m\overset{r}{S}_n a_{m+r-n+1,n}.$$

22. *Theorem.* $\overset{m}{\mathrm{S}}_n \overset{n}{\mathrm{S}}_r a_{n,r} = \overset{m}{\mathrm{S}}_n \overset{m-n+1}{\mathrm{S}}_r a_{n+r-1,r}$

and $= \overset{m}{\mathrm{S}}_n \overset{m-n+1}{\mathrm{S}}_r a_{n+r-1,}$

For, $\overset{m}{\mathrm{S}}_n \overset{n}{\mathrm{S}}_r a_{n,r} = \overset{1}{\mathrm{S}}_r a_{1,r} + \overset{2}{\mathrm{S}}_r a_{2,r} + \overset{3}{\mathrm{S}}_r a_{3,r} + \ldots + \overset{m}{\mathrm{S}}_r a_{m,r}$

$$= a_{1,1}$$
$$+ a_{2,1} + a_{2,2}$$
$$+ a_{3,1} + a_{3,2} + a_{3,3}$$
$$+ a_{4,1} + a_{4,2} + a_{4,3} + a_{4,4}$$
$$+ a_{5,1} + a_{5,2} + a_{5,3} + a_{5,4} + a_{5,5}$$
$$+ \ \&c. \quad + \ \&c.$$
$$+ a_{m,1} + a_{m,2} + a_{m,3} + \ldots\ldots + a_{m,m};$$

therefore, summing diagonally,

$$\overset{m}{\mathrm{S}}_n \overset{n}{\mathrm{S}}_r a_{n,r} = a_{1,1} + a_{2,2} + a_{3,3} + \ldots + a_{m,m}$$
$$+ a_{2,1} + a_{3,2} + a_{4,3} + \ldots + a_{m,m-1}$$
$$+ a_{3,1} + a_{4,2} + a_{5,3} + \ldots + a_{m,m-2}$$
$$+ \ \&c. \quad + \ \&c.$$
$$+ a_{m-1,1} + a_{m,2}$$
$$+ a_{m,1}$$
$$= \overset{m}{\mathrm{S}}_r a_{r,r} + \overset{m-1}{\mathrm{S}}_r a_{r+1,r} + \overset{m-2}{\mathrm{S}}_r a_{r+2,r} + \ldots + \overset{m-n+1}{\mathrm{S}}_r a_{r+n-1,r} + \ldots + a_{m,1}$$
$$= \overset{m}{\mathrm{S}}_n \overset{m-n+1}{\mathrm{S}}_r a_{n+r-1,r};$$

and, summing vertically,

$$\overset{m}{\mathrm{S}}_n \overset{n}{\mathrm{S}}_r a_{n,r} = \overset{m}{\mathrm{S}}_r a_{r,1} + \overset{m-1}{\mathrm{S}}_r a_{1+r,2} + \overset{m-2}{\mathrm{S}}_r a_{2+r,3} + \ldots + \overset{m-n+1}{\mathrm{S}}_r a_{n+r-1,n} + \ldots + a_{m,m}$$
$$= \overset{m}{\mathrm{S}}_n \overset{m-n+1}{\mathrm{S}}_r a_{n+r-1,n}.$$

23. *Theorem.* $\overset{2m}{S}_n \overset{n}{S}_r a_{n,r} = \overset{m}{S}_n \overset{n}{S}_r (a_{2n-r,r} + a_{2n-r+1,r})$

$$+\overset{m}{S}_r a_{2m-r+1,r+1} + \overset{m-1}{S}_n \overset{m-n}{S}_r (a_{2m-r+1,r+n+1} + a_{2m-r+1,r+n+2}),$$

and $\overset{2m-1}{S}_n \overset{n}{S}_r a_{n,r} = \overset{m-1}{S}_n \overset{n}{S}_r (a_{2n-r,r} + a_{2n-r+1,r}) + \overset{m-1}{S}_r a_{2m-r-1,r+1}$

$$+\overset{m-2}{S}_n \overset{m-n-1}{S}_r (a_{2m-r-1,r+n+1} + a_{2m-r-1,r+n+2}) + \overset{2m-1}{S}_r a_{2m-1,r}.$$

For, $\overset{2m}{S}_n \overset{n}{S}_r a_{n,r} = \overset{1}{S}_r a_{1,r} + \overset{2}{S}_r a_{2,r} + \overset{3}{S}_r a_{3,r} + \ldots + \overset{2m}{S}_r a_{2m,r}$

$$\begin{aligned}
&= a_{1,1} \\
&+ a_{2,1} + a_{2,2} \\
&+ a_{3,1} + a_{3,2} + a_{3,3} \\
&+ a_{4,1} + a_{4,2} + a_{4,3} + a_{4,4} \\
&+ \ \&c. \qquad + \ \&c. \\
&+ a_{2m,1} + a_{2m,2} + \ldots + a_{2m,2m};
\end{aligned}$$

therefore, summing diagonally,

$$\overset{2m}{S}_n \overset{n}{S}_r a_{n,r} = \left\{ \begin{aligned}
&(a_{1,1}) + (a_{2,1}) + (a_{3,1} + a_{2,2}) \\
&+ (a_{4,1} + a_{3,2}) + (a_{5,1} + a_{4,2} + a_{3,3}) + (a_{6,1} + a_{5,2} + {}_{4,3}) \\
&+ (a_{7,1} + a_{6,2} + a_{5,3} + a_{4,4}) + (a_{8,1} + a_{7,2} + a_{6,3} + a_{5,4}) \\
&+ \ \&c. \qquad + \ \&c. \qquad + \&c. \\
&+ a_{2m-1,1} + a_{2m-2,2} + a_{2m-3,3} + a_{2m-4,4} + a_{2m-5,5} + \ldots + a_{m,m} \\
&+ a_{2m,1} + a_{2m-1,2} + a_{2m-2,3} + a_{2m-3,4} + a_{2m-4,5} + \ldots + a_{m+1,m} \\
&+ a_{2m,2} + a_{2m-1,3} + a_{2m-2,4} + a_{2m-3,5} + a_{2m-4,6} + \ldots + a_{m+1,m+1} \\
&+ a_{2m,3} + a_{2m-1,4} + a_{2m-2,5} + a_{2m-3,6} + a_{2m-4,7} + \ldots + a_{m+2,m+1} \\
&+ a_{2m,4} + a_{2m-1,5} + a_{2m-2,6} + a_{2m-3,7} + a_{2m-4,8} + \ldots + a_{m+2,m+2} \\
&+ a_{2m,5} + a_{2m-1,6} + a_{2m-2,7} + a_{2m-3,8} + a_{2m-4,9} + \ldots + a_{m+3,m+2} \\
&+ a_{2m,6} + a_{2m-1,7} + a_{2m-2,8} + a_{2m-3,9} + a_{2m-4,10} + \ldots + a_{m+3,m+3} \\
&+ \&c. \\
&+ (a_{2m,2m-3} + a_{2m-1,2m-2}) + (a_{2m,2m-2} + a_{2m-1,2m-1}) \\
&+ (a_{2m,2m-1}) + (a_{2m,2m}).
\end{aligned} \right.$$

$$=\overset{m}{S}_n \overset{n}{S}_r (a_{2n-r,r}+a_{2n-r+1,r})+\overset{m}{S}_r a_{2m-r+1,r+1}$$

$$+\overset{m-1}{S}_n \overset{m-n}{S}_r (a_{2m-r+1,r+n+1}+a_{2m-r+1,r+n+2});$$

and $\overset{2m-1}{S}_n \overset{n}{S}_r a_{n,r}=\overset{2m-2}{S}_n \overset{n}{S}_r a_{n,r}+\overset{2m-1}{S}_r a_{2m-1,r}$, (9);

$$=\overset{m-1}{S}_n \overset{n}{S}_r (a_{2n-r,r}+a_{2n-r+1,r})+\overset{m-1}{S}_r a_{2m-r-1,r+1}$$

$$+\overset{m-2}{S}_n \overset{m-n-1}{S}_r (a_{2m-r-1,r+n+1}+a_{2m-r-1,r+n+2})+\overset{2m-1}{S}_r a_{2m-1,r},$$

by the former case.

24. *Theorem.* If $a_{m+1}=a_m+b_m$, $\binom{m=1}{m=n-1}$, then $a_n=a_1+\overset{n-1}{S}_m b_m$.

For, $\overset{n-1}{S}_m a_{m+1}=\overset{n-1}{S}_m a_m+\overset{n-1}{S}_m b_m$, (4) & (5).

$\therefore \overset{n-2}{S}_m a_{m+1}+a_n=a_1+\overset{n-2}{S}_m a_{m+1}+\overset{n-1}{S}_m b_m$, (9).

$\therefore a_n=a_1+\overset{n-1}{S}_m b_m$, cancelling identical terms.

25. *Theorem.* If $a_{m+1}=c\,a_m+b_m$, $\binom{m=1}{m=n-1}$,

then $a_n=c^{n-1}.a_1+\overset{n-1}{S}_m c^{m-1}.b_{n-m}$.

For, $a_{n-m+1}=c\,a_{n-m}+b_{n-m}$;

$\therefore c^{m-1}.a_{n-m+1}=c^m.a_{n-m}+c^{m-1}.b_{n-m}$, multiplying by c^{m-1};

and $\overset{n-1}{S}_m c^{m-1}.a_{n-m+1}=\overset{n-1}{S}_m c^m a_{n-m}+\overset{n-1}{S}_m c^{m-1}.b_{n-m}$, (4) & (5);

$$a_n+\overset{n-2}{S}_m c^m a_{n-m}=\overset{n-2}{S}_m c^m a_{n-m}+c^{n-1}.a_1+\overset{n-1}{S}_m c^{m-1} b_{n-m}, \quad (9).$$

$$\therefore a_n=c^{n-1}.a_1+\overset{n-1}{S}_m c^{m-1}.b_{n-m}.$$

26. *Theorem.*

$$(\overset{\infty}{\mathrm{S}}_m a_{m-1} x^{m-1}) (\overset{\infty}{\mathrm{S}}_n b_{n-1} \cdot x^{n-1}) = \overset{\infty}{\mathrm{S}}_m x^{m-1} \cdot \overset{m}{\mathrm{S}}_n a_{m-n} \cdot b_{n-1}.$$

For, $(\overset{\infty}{\mathrm{S}}_m a_{m-1} x^{m-1}) (\overset{\infty}{\mathrm{S}}_n b_{n-1} x^{n-1})$

$$= \overset{\infty}{\mathrm{S}}_m a_{m-1} \cdot x^{m-1} \cdot \overset{\infty}{\mathrm{S}}_n b_{n-1} x^{n-1}, \quad (16);$$

$$= \overset{\infty}{\mathrm{S}}_m \overset{\infty}{\mathrm{S}}_n a_{m-1} \cdot b_{n-1} \cdot x^{m+n-2}, \quad (6);$$

$$= \overset{\infty}{\mathrm{S}}_m \overset{m}{\mathrm{S}}_n a_{m-n} \cdot b_{n-1} \cdot x^{m-1}, \quad (18);$$

$$= \overset{\infty}{\mathrm{S}}_m x^{m-1} \cdot \overset{m}{\mathrm{S}}_n a_{m-n} \cdot b_{n-1}, \quad (6).$$

27. Cor. $(\overset{\infty}{\mathrm{S}}_m a_{m-1} \cdot x^{m-1})^2 = \overset{\infty}{\mathrm{S}}_m x^{m-1} \cdot \overset{m}{\mathrm{S}}_n a_{m-n} \cdot a_{n-1}.$

CHAPTER II.

ON PRODUCTS AND FACTORIALS.

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28. THE *Product* of n factors, of which the m^{th} is a_m, will be denoted by $\overset{n}{\mathrm{P}}_m a_m$; that is,

$$\overset{n}{\mathrm{P}}_m a_m = a_1 \,.\, a_2 \,.\, a_3 \ldots a_n \,.$$

The symbol $\overset{n;r}{\mathrm{P}}_m a_m$ will denote that the r^{th} factor is to be omitted.

29. *Theorem.* If $a_m = b_m$, $\binom{m=1}{m=n}$, then $\overset{n}{\mathrm{P}}_m a_m = \overset{n}{\mathrm{P}}_m b_m$.

For, since
$$\begin{aligned} a_1 &= b_1 \\ a_2 &= b_2 \\ a_3 &= b_3 \\ \&\text{c.} &= \&\text{c.} \\ a_n &= b_n\,; \end{aligned}$$

therefore, multiplying,

$$a_1 \,.\, a_2 \,.\, a_3 \ldots a_n = b_1 \,.\, b_2 \,.\, b_3 \ldots b_n \,,$$

$$\text{or } \overset{n}{\mathrm{P}}_m a_m = \overset{n}{\mathrm{P}}_m b_m.$$

30. *Theorem.*

If b is independent of m, then $\overset{n}{\mathrm{P}}_m (a_m b) = b^n \,.\, \overset{n}{\mathrm{P}}_m a_m$.

$$\begin{aligned} \text{For, } \overset{n}{\mathrm{P}}_m (a_m b) &= a_1 b \,.\, a_2 b \,.\, a_3 b \ldots a_n b \,. \\ &= a_1 a_2 a_3 \ldots a_n \,.\, b^n \\ &= b^n \,.\, \overset{n}{\mathrm{P}}_m a_m. \end{aligned}$$

31. *Problem.* To invert the order of the factors of a given product.

Now, $\overset{n}{\mathrm{P}}_m a_m = a_1 . a_2 . a_3 \ldots a_n$

$= a_n . a_{n-1} . a_{n-2} \ldots a_1$, by inverting the order of the factors;

$= \overset{n}{\mathrm{P}}_m a_{n-m+1}$.

If, therefore, we substitute $n-m+1$ for m, in the expression for a_m, the order of the factors will be inverted. See Art. 8.

32. *Theorem.* $\overset{n}{\mathrm{P}}_m a_m = \overset{r}{\mathrm{P}}_m (a_m) . \overset{n-r}{\mathrm{P}}_m (a_{r+m})$.

For, $\overset{n}{\mathrm{P}}_m a_m = (a_1 . a_2 . a_3 \ldots a_r)(a_{r+1} . a_{r+2} . a_{r+3} \ldots a_n)$

$$= \overset{r}{\mathrm{P}}_m (a_m) . \overset{n-r}{\mathrm{P}}_m (a_{r+m}).$$

By means of this theorem we can separate from the rest any number of factors, taken either at the beginning or the end of a given product. See Art. 9.

33. Cor. $\overset{r}{\mathrm{P}}_m (a_m) . \overset{n}{\mathrm{P}}_m (a_{r+m}) = \overset{n+r}{\mathrm{P}}_m a_m$.

34. *Theorem.* $\overset{0}{\mathrm{P}}_m a_m = 1$.

For, $\overset{0}{\mathrm{P}}_m (a_m) . \overset{n}{\mathrm{P}}_m (a_{0+m}) = \overset{0+n}{\mathrm{P}}_m a_m$, (33);

or $\overset{0}{\mathrm{P}}_m (a_m) . \overset{n}{\mathrm{P}}_m a_m = \overset{n}{\mathrm{P}}_m a_m$;

$\therefore \overset{0}{\mathrm{P}}_m a_m = 1$, by division.

35. *Theorem.* $\overset{-n}{\mathrm{P}}_m a_m = \dfrac{1}{\overset{n}{\mathrm{P}}_m a_{m-n}} = \dfrac{1}{\overset{n}{\mathrm{P}}_m a_{-(m-1)}}$.

For, $\overset{n-n}{\mathrm{P}}_m a_{m-n} = \overset{n}{\mathrm{P}}_m (a_{m-n}) . \overset{-n}{\mathrm{P}}_m a_m$, (32);

and $= 1$, (34).

$\therefore \overset{-n}{\mathrm{P}}_m (a_m) . \overset{n}{\mathrm{P}}_m a_{m-n} = 1$;

$$\text{and } \overset{-n}{\mathrm{P}}_m a_m = \frac{1}{\overset{n}{\mathrm{P}}_m a_{m-n}}, \text{ by division;}$$

$$= \frac{1}{\overset{n}{\mathrm{P}}_m a_{-(m-1)}}. \quad (31).$$

36. *Theorem.* If $a_{m+1} = a_m \cdot b_m$, then $a_n = a_1 \cdot \overset{n-1}{\mathrm{P}}_m b_m$.

For, $b_m = \dfrac{a_{m+1}}{a_m}$;

$$\therefore \overset{n-1}{\mathrm{P}}_m b_m = \overset{n-1}{\mathrm{P}}_m \left(\frac{a_{m+1}}{a_m}\right), \quad (29);$$

$$= \frac{\overset{n-2}{\mathrm{P}}_m (a_{m+1}) \cdot a_n}{a_1 \cdot \overset{n-2}{\mathrm{P}}_m a_{m+1}}, \quad (32);$$

$$= \frac{a_n}{a_1}, \text{ cancelling the identical factors.}$$

$$\therefore a_n = a_1 \cdot \overset{n-1}{\mathrm{P}}_m b_m.$$

37. *Theorem.* If $a_{m+2} = a_m \cdot b_m$,

then $a_{2n} = a_2 \cdot \overset{n-1}{\mathrm{P}}_m b_{2m}$, and $a_{2n-1} = a_1 \cdot \overset{n-1}{\mathrm{P}}_m b_{2m-1}$.

For, $b_{2m} = \dfrac{a_{2m+2}}{a_{2m}}$.

$$\therefore \overset{n-1}{\mathrm{P}}_m b_{2m} = \overset{n-1}{\mathrm{P}}_m \left(\frac{a_{2m+2}}{a_{2m}}\right), \quad (29);$$

$$= \frac{\overset{n-2}{\mathrm{P}}_m (a_{2m+2}) \cdot a_{2n}}{a_2 \cdot \overset{n-2}{\mathrm{P}}_m a_{2m+2}}, \quad (32);$$

$$= \frac{a_{2n}}{a_2}, \text{ cancelling the identical factors.}$$

$$\therefore a_{2n} = a_2 \cdot \overset{n-1}{\mathrm{P}}_m b_{2m}.$$

$$\text{Also } b_{2m-1} = \frac{a_{2m+1}}{a_{2m-1}};$$

$$\therefore \overset{n-1}{\mathrm{P}}_m b_{2m-1} = \overset{n-1}{\mathrm{P}}_m \left(\frac{a_{2m+1}}{a_{2m-1}}\right), \quad (29);$$

$$= \frac{\overset{n-2}{\mathrm{P}}_m (a_{2m+1}) \cdot a_{2n-1}}{a_1 \cdot \overset{n-2}{\mathrm{P}}_m a_{2m+1}}, \quad (32);$$

$$= \frac{a_{2n-1}}{a_1}.$$

$$\therefore a_{2n-1} = a_1 \cdot \overset{n-1}{\mathrm{P}}_m (b_{2m-1}).$$

38. The symbol $\underline{|a}_{\,n,m}$ denotes the product of n factors forming an arithmetical progression, of which the first term is a, and the common difference m; if $m = -1$, the m may be omitted; and, if, in the same case, $n = a$, the n also may be omitted: thus

$$\underline{|a}_{\,n,m} = a(a+m)(a+2m)\ldots(a+\overline{n-1}\,.\,m),$$

$$\underline{|a}_{\,n} = a(a-1)(a-2)\ldots(a-n+1), \text{ and}$$

$$\underline{|a} = a(a-1)(a-2)\ldots 2\,.\,1.$$

39. *Theorem.* $\underline{|ab}_{\,n,m} = b^n \,.\, \underline{|a}_{\,n,\frac{m}{b}}$.

$$\text{For, } \underline{|ab}_{\,n,m} = ab\,.\,(ab+m)(ab+2m)\ldots(ab+\overline{n-1}\,.\,m)$$

$$= b^n\,.\,a\,.\left(a+\frac{m}{b}\right)\left(a+\frac{2m}{b}\right)\ldots\left(a+\frac{\overline{n-1}\,.\,m}{b}\right).$$

$$= b^n\,.\,\underline{|a}_{\,n,\frac{m}{b}}. \quad \text{See Art. 30.}$$

40. *Theorem.* $\underline{|a}_{\,n,\,m} = \underline{|a + \overline{n - 1} \,.\, m}_{\,n,\,-m}\,.$

For, $\underline{|a}_{\,n,\,m} = a \,.\, (a + m)(a + 2m) \ldots (a + \overline{n - 1} \,.\, m),$

$$= (a + \overline{n-1} \,.\, m)(a + \overline{n-2} \,.\, m)(a + \overline{n-3} \,.\, m) \ldots (a + m)\, a,$$

by inverting the order of the factors;

$$= \underline{|a + \overline{n - 1} \,.\, m}_{\,n,\,-m}. \quad \text{See Art. 31.}$$

41. *Theorem.* $\underline{|a}_{\,n,\,m} = \underline{|a}_{\,r,\,m} \,.\, \underline{|a + rm}_{\,n-r,\,m}\,.$

For, $\underline{|a}_{\,n,\,m} = \{a \,.\, (a + m)(a + 2m) \ldots (a + \overline{r - 1} \,.\, m)\} \times$

$$\{(a + rm)(a + \overline{r + 1} \,.\, m) \ldots (a + \overline{n - 1} \,.\, m)\}$$

$$= \underline{|a}_{\,r,\,m} \,.\, \underline{|a + rm}_{\,n-r,\,m}. \quad \text{See Art. 32.}$$

42. *Theorem.* $\underline{|a}_{\,0,\,m} = 1.$

For, $\underline{|a}_{\,0,\,m} \,.\, \underline{|a + 0 \,.\, m}_{\,n,\,m} = \underline{|a}_{\,0+n,\,m}, \quad (41);$

$$\text{or } \underline{|a}_{\,0,\,m} \,.\, \underline{|a}_{\,n,\,m} = \underline{|a}_{\,n,\,m}\,.$$

$$\therefore \underline{|a}_{\,0,\,m} = 1, \text{ by division. See Art. 34.}$$

43. *Theorem.* $\underline{|a}_{\,-n,\,m} = \dfrac{1}{\underline{|a - nm}_{\,n,\,m}} = \dfrac{1}{\underline{|a - m}_{\,n,\,-m}}\,.$

For, $\underline{|a - nm}_{\,n-n,\,m} = \underline{|a - nm}_{\,n,\,m} \,.\, \underline{|a}_{\,-n,\,m}, \quad (41),$

$$\text{and } = 1, \quad (42);$$

$$\therefore \underline{|a - nm}_{\,n,\,m} \,.\, \underline{|a}_{\,-n,\,m} = 1.$$

$$\therefore\ \underline{|a}_{\,-n,m} = \frac{1}{\underline{|a - nm}_{\,n,m}},\ \text{by division;}$$

$$= \frac{1}{\underline{|a - m}_{\,n,-m}},\quad (40).\quad \text{See Art. 35.}$$

44. *Theorem.* $\dfrac{1}{\underline{|-m}} = 0.$

For, $$\frac{1}{\underline{|-m}} = \frac{1}{\underline{|-m}_{\,-m,-1}}$$

$$= \underline{|-m+m}_{\,m,-1},\quad (43);$$

$$= \underline{|0}_{\,m}$$

$$= 0.$$

45. The theorems in Articles 34, 35, 42, and 43 are analogous to the equations $a^0 = 1$, and $a^{-n} = \dfrac{1}{a^n}$; which last equations indeed may be deduced from them as particular cases.

46. *Theorem.* $\dfrac{\underline{|n}_{\,m}}{\underline{|m}} = \dfrac{\underline{|n}_{\,n-m}}{\underline{|n-m}}.$

For, $$\underline{|n}_{\,m} \cdot \underline{|n-m} = \underline{|n}$$

$$= \underline{|n}_{\,n-m} \cdot \underline{|m}.$$

$$\therefore\ \frac{\underline{|n}_{\,m}}{\underline{|m}} = \frac{\underline{|n}_{\,n-m}}{\underline{|n-m}}.$$

47. *Theorem.* $\dfrac{\underline{|n}_{m}}{\underline{|m}} + \dfrac{\underline{|n}_{m+1}}{\underline{|m+1}} = \dfrac{\underline{|n+1}_{m+1}}{\underline{|m+1}}$.

For,
$$\frac{\underline{|n}_{m}}{\underline{|m}} + \frac{\underline{|n}_{m+1}}{\underline{|m+1}} = \frac{\underline{|n}_{m}}{\underline{|m}}\left(1 + \frac{n-m}{m+1}\right)$$
$$= \frac{\underline{|n}_{m} \,.\, (m+1+n-m)}{\underline{|m+1}}$$
$$= \frac{(n+1)\ \underline{|n}_{m}}{\underline{|m+1}}$$
$$= \frac{\underline{|n+1}_{m+1}}{\underline{|m+1}}.$$

48. *Problem.* To shew that $\dfrac{\underline{|n}_{m}}{\underline{|m}}$ is a whole number; n and m being integers.

Now,
$$\frac{\underline{|r+1}_{m+1}}{\underline{|m+1}} = \frac{\underline{|r}_{m+1}}{\underline{|m+1}} + \frac{\underline{|r}_{m}}{\underline{|m}}, \quad (47);$$
$$\therefore \frac{\underline{|n}_{m+1}}{\underline{|m+1}} = \frac{\underline{|1}_{m+1}}{\underline{|m+1}} + \overset{n-1}{S_r}\frac{\underline{|r}_{m}}{\underline{|m}}, \quad (24);$$
$$= \overset{n-1}{S_r}\frac{\underline{|r}_{m}}{\underline{|m}}.$$

If, therefore, $\dfrac{\underline{|n}_{m}}{\underline{|m}}$ were an integer, $\dfrac{\underline{|n}_{m+1}}{\underline{|m+1}}$ would be an integer; but $\dfrac{\underline{|n}_{1}}{\underline{|1}}$ *is* an integer, and therefore $\dfrac{\underline{|n}_{m}}{\underline{|m}}$ is an integer.

49. The symbol $\overset{n}{\underset{m}{\{}} a_m + b_m \underset{m+1}{\{} \ldots \underset{n+1}{\{} c \underset{n+1}{\}} \ldots \underset{1}{\}}$ denotes the result of the combination of the symbols

$$\underset{1}{\{} a_1 + b_1 \underset{2}{\{} a_2 + b_2 \underset{3}{\{} \ldots \underset{n}{\{} a_n + b_n \underset{n+1}{\{} c \underset{n+1}{\}} \ldots \underset{1}{\}} ;$$

the brackets being omitted after the expansion, if they are then without signification.

50. *Theorem.* $\overset{n}{\underset{m}{\{}} a_m + b_m \underset{m+1}{\{} \ldots \underset{n+1}{\{} c = \overset{n}{\mathrm{S}}_m a_m \,.\, \overset{m-1}{\mathrm{P}}_r b_r + c \,.\, \overset{m}{\mathrm{P}}_r b_r .$

For, $\overset{n}{\underset{m}{\{}} a_m + b_m \underset{m+1}{\{} \ldots \underset{n+1}{\{} c = \underset{1}{\{} a_1 + b_1 \underset{2}{\{} a_2 + b_2 \underset{3}{\{} \ldots \underset{n+1}{\{} c \underset{n+1}{\}} \ldots \underset{1}{\}}$

$$= \underset{1}{\{} a_1 + \underset{2}{\{} b_1 a_2 + \underset{3}{\{} b_1 b_2 a_3 + \underset{4}{\{} \ldots$$

$$+ \underset{n}{\{} b_1 b_2 \ldots b_{n-1} \,.\, a_n + \underset{n+1}{\{} b_1 b_2 \ldots b_n c \underset{n+1}{\}} \ldots \underset{1}{\}}$$

$$= a_1 + b_1 a_2 + b_1 b_2 a_3 + \ldots$$

$$+ b_1 b_2 \ldots b_{n-1} a_n + b_1 b_2 \ldots b_n c$$

$$= \overset{n}{\mathrm{S}}_m a_m \,.\, \overset{m-1}{\mathrm{P}}_r b_r + c \,.\, \overset{n}{\mathrm{P}}_r b_r .$$

51. *Theorem.* If $a_n = b_n + c_n \,.\, a_{n+\alpha}$,

then $a_n = \overset{m}{\mathrm{S}}_s b_{n+\overline{s-1}.\alpha} \,.\, \overset{s-1}{\mathrm{P}}_t c_{n+\overline{t-1}.\alpha} + a_{n+m\alpha} \,.\, \overset{m}{\mathrm{P}}_t c_{n+\overline{t-1}.\alpha} .$

For, $a_n = \underset{1}{\{} b_n + c_n \underset{2}{\{} b_{n+\alpha} + c_{n+\alpha} \underset{3}{\{} \ldots \underset{s}{\{} b_{n+\overline{s-1}.\alpha} + c_{n+\overline{s-1}.\alpha} \underset{s+1}{\{} \ldots \underset{m+1}{\{} a_{n+m\alpha} \underset{m+1}{\}} \ldots \underset{1}{\}}$,

by substitution;

$$= \overset{m}{\underset{s}{\{}} b_{n+\overline{s-1}.\alpha} + c_{n+\overline{s-1}.\alpha} \underset{s+1}{\{} \ldots \underset{m+1}{\{} a_{n+m\alpha}, \quad (49);$$

$$= \overset{m}{\mathrm{S}}_s b_{n+\overline{s-1}.\alpha} \,.\, \overset{s-1}{\mathrm{P}}_t c_{n+\overline{t-1}.\alpha} + a_{n+m\alpha} \,.\, \overset{m}{\mathrm{P}}_t c_{n+\overline{t-1}.\alpha}, \quad (50).$$

52. Cor. If $a_{m,n} = b_{m,n} + c_{m,n} \,.\, a_{m+\alpha,\, n+\beta}$, then

$$a_{m,n} = \overset{r}{\mathrm{S}}_s b_{m+\overline{s-1}.\alpha,\, n+\overline{s-1}.\beta} \,.\, \overset{s-1}{\mathrm{P}}_t (c_{m+\overline{t-1}.\alpha,\, n+\overline{t-1}.\beta})$$

$$+ a_{m+r\alpha,\, n+r\beta} \,.\, \overset{r}{\mathrm{P}}_t (c_{m+\overline{t-1}.\alpha,\, n+\overline{t-1}.\beta}).$$

CHAPTER III.

ON COMBINATIONS AND ARRANGEMENTS.

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53. The symbol $\overset{m,n}{C}_r a_r$ will be used to denote the sum of every possible *combination*, (without repetitions of any one letter in the same combination,) that can be formed by taking m at a time of n quantities of which the r^{th} is a_r; and the symbol $\overset{m,n;s}{C}_r a_r$ will denote the same thing, with the condition that a_s is to be every where omitted.

54. *Theorem.* $\overset{m+1,n+1}{C}_r a_r = \overset{m+1,n}{C}_r a_r + a_{n+1} . \overset{m,n}{C}_r a_r$.

For, $\overset{m+1,n+1}{C}_r a_r$ must consist of terms into which a_{n+1} does not enter as a factor, and of others into which it enters as a factor once only; and it is obvious that $\overset{m+1,n}{C}_r a_r$ will express the first set, and that $a_{n+1} . \overset{m,n}{C}_r a_r$ will express the second. Hence the truth of the theorem is manifest.

55. *Theorem.* If b is independent of r, then

$$\overset{m,n}{C}_r (a_r b) = b^m . \overset{m,n}{C}_r a_r.$$

For, $\overset{m,n}{C}_r (a_r b)$ denotes the sum of a certain series, each term of which is the product of m quantities, and into each of which quantities b enters as a multiplier; and $\overset{m,n}{C}_r a_r$ denotes the sum of a series, each term of which is the product of the same m quantities, each being deprived of its multiplier b.

56. *Theorem.* If a is independent of r,

$$\text{then } \overset{m,n}{C_r}(a) = \frac{\underline{|n}}{\underline{|m}} \cdot a^m.$$

For, the number of terms in $\overset{n,m}{C_r} a_r$ is $\dfrac{\underline{|n}}{\underline{|m}}$; and each term consists of m factors. Since, therefore, in $\overset{m,n}{C_r}(a)$ each of these factors is equal to a, the truth of the theorem is manifest.

57. *Theorem.* $$\frac{\overset{m,n}{C_r} a_r}{\overset{n}{P_r} a_r} = \overset{n-m,n}{C_r} a_r^{-1}.$$

For, the numerator of the first member of this equation consists of every possible combination of n quantities, taken m at a time; and, hence, that side of the equation consists of a series of fractions, the numerator of each being unity, and in which the denominators are formed by taking away, in every possible manner, m of n given quantities, and will, therefore, consist of every possible combination of these n quantities taken $n-m$ at a time.

58. *Theorem.* $\overset{0,n}{C_r} a_r = 1.$

$$\text{For, } \overset{0,n}{C_r} a_r = \overset{n-n,n}{C_r} a_r$$

$$= \frac{\overset{n,n}{C_r} a_r^{-1}}{\overset{n}{P_r} a_r^{-1}}, \ (57);$$

$$= 1.$$

59. The symbol $\overset{\overline{m,n-m}}{C_{r,s}}(a_r \cdot b_s)$ denotes that there are n quantities of which the r^{th} is a_r, and n others of which the s^{th} is b_s, and that every possible combination, (without repetitions of the same quantity in any one combination,) is to be formed of the first series, by taking them m at a time;

and that each combination thus formed is to be multiplied by $n-m$ quantities of the second series, so taken that in each of the combinations the whole of the natural numbers from 1 to n shall appear as indices: thus,

$$\overset{\overline{2,3}}{C}_{r,s}(a_r \, . \, b_s) = a_1 a_2 b_3 b_4 b_5 + a_1 a_3 b_2 b_4 b_5 + a_1 a_4 b_2 b_3 b_5 + a_1 a_5 b_2 b_3 b_4$$
$$+ a_2 a_3 b_1 b_4 b_5 + a_2 a_4 b_1 b_3 b_5 + a_2 a_5 b_1 b_3 b_4$$
$$+ a_3 a_4 b_1 b_2 b_5 + a_3 a_5 b_1 b_2 b_4 + a_4 a_5 b_1 b_2 b_3.$$

60. Cor. If $b_s = b$, then $\overset{\overline{m,\,n-m}}{C}_{r,s}(a_r \, . \, b_s) = b^{n-m} \, . \, \overset{m,\,n}{C}_r \, a_r$.

61. *Theorem.*

$$\overset{\overline{n-m+1,\,m}}{C}_{r,s}(a_r \, . \, b_s) = a_{n+1} \, . \, \overset{\overline{n-m,\,m}}{C}_{r,s}(a_r \, . \, b_s) + b_{n+1} \, . \, \overset{\overline{n-m+1,\,m-1}}{C}_{r,s}(a_r \, . \, b_s).$$

For, $\overset{\overline{n-m+1,\,m}}{C}_{r,s}(a_r \, . \, b_s)$ will consist of terms into which a_{n+1} enters as a factor, and b_{n+1} does not; and of others into which b_{n+1} enters and a_{n+1} does not. Also each of these terms must consist of n factors, exclusive of the factors a_{n+1} or b_{n+1}; and each of them must contain $n-m+1$ factors of the series $a_1, a_2, \ldots a_{n+1}$, and m of the series $b_1, b_2, \ldots b_{n+1}$. Also $\overset{\overline{n-m+1,\,n}}{C}_{r,s}(a_r \, . \, b_s)$ must contain every possible term that can be formed consistently with these conditions. Hence $a_{n+1} \, . \, \overset{\overline{n-m,\,m}}{C}_{r,s}(a_r \, . \, b_s)$ will contain all the terms of the first kind, and $b_{n+1} \, . \, \overset{\overline{n-m+1,\,m-1}}{C}_{r,s}(a_r \, . \, b_s)$ all those of the second kind.

62. The symbol $\overset{n}{A}_{+m} \, a_m$ denotes the sum of every possible *Arrangement* that can be formed of *any* number of quantities of which the m^{th} is a_m, these arrangements being subject to the condition that the sum of the indices subscript shall in every single arrangement amount to n; repetitions of the same letter being allowed in any arrangement: thus

$$\overset{4}{A}_{+m} \, a_m = a_4 + a_3 a_1 + a_2 a_2 + a_2 a_1 a_1 + a_1 a_1 a_1 a_1 + a_1 a_1 a_2 + a_1 a_3$$
$$= a_4 + 2 a_3 a_1 + a_2^2 + 2 a_2 \, . \, a_1^2 + a_1^4.$$

63. *Theorem.* $\overset{n}{\mathrm{A}}_{+r}\, a_r = \overset{n}{\mathrm{S}}_m\, a_m \,.\, \overset{n-m}{\mathrm{A}}_{+r}\, a_r$

$$= \overset{n}{\mathrm{S}}_m\, a_{n-m+1} \,.\, \overset{m-1}{\mathrm{A}}_{+r}\, a_r.$$

For, $\overset{n}{\mathrm{A}}_{+r}\, a_r$ is the sum of all the terms that can be formed of any number of quantities a_1, a_2, &c. such that the sum of the indices subscript shall be n; now $a_m \,.\, \overset{n-m}{\mathrm{A}}_{+r}\, a_r$ will include every term in which a_m is a factor, and $\overset{n}{\mathrm{S}}_m\, a_m \,.\, \overset{n-m}{\mathrm{A}}_{+r}\, a_r$ will include all the admissible values of a_m, and therefore every term of $\overset{n}{\mathrm{A}}_{+r}\, a_r$.

$$\therefore\ \overset{n}{\mathrm{A}}_{+r}\, a_r = \overset{n}{\mathrm{S}}_m\, a_m \,.\, \overset{n-m}{\mathrm{A}}_{+r}\, a_r$$

$$= \overset{n}{\mathrm{S}}_m\, a_{n-m+1} \,.\, \overset{m-1}{\mathrm{A}}_{+r}\, a_r, \quad (8).$$

64. Cor. $\overset{0}{\mathrm{A}}_{+r}\, a_r = 1.$

65. *Theorem.* If $a_n = \overset{n}{\mathrm{S}}_m\, a_{n-m} \,.\, b_m$, then $a_n = a_0 \,.\, \overset{n}{\mathrm{A}}_{+r}\, b_r$.

For, $a_1 = a_0 \,.\, b_1 = a_0 \,.\, \overset{1}{\mathrm{A}}_{+r}\, b_r.$

$$a_2 = a_1 b_1 + a_0 b_2$$

$$= a_0 (b_1^2 + b_2) = a_0 \,.\, \overset{2}{\mathrm{A}}_{+r}\, b_r.$$

$$a_3 = a_2 b_1 + a_1 b_2 + a_0 b_3$$

$$= a_0 (b_1^3 + b_1 b_2 + b_2 b_1 + b_3)$$

$$= a_0 \,.\, \overset{3}{\mathrm{A}}_{+r}\, b_r.$$

$$a_4 = a_3 b_1 + a_2 b_2 + a_1 b_3 + a_0 b_4$$

$$= a_0 (b_1^4 + b_1^2 b_2 + b_1 b_2 b_1 + b_1 b_3 + b_2 b_1^2 + b_2^2 + b_3 b_1 + b_4)$$

$$= a_0 \,.\, \overset{4}{\mathrm{A}}_{+r}\, b_r.$$

Suppose, therefore, $a_n = a_0 \,.\, \overset{n}{\mathrm{A}}_{+r}\, b_r$;

$$\text{then } a_{n+1} = \overset{n+1}{\mathrm{S}}_m\, a_{n-m+1}\, b_m$$

$$= \overset{n+1}{\mathrm{S}}_m\, a_0 \,.\, b_m \,.\, \overset{n-m+1}{\mathrm{A}}_{+r}\, b_r$$

$$= a_0 \,.\, \overset{n+1}{\mathrm{S}}_m\, b_m \,.\, \overset{n-m+1}{\mathrm{A}}_{+r}\, b_r, \quad (6);$$

$$= a_0 \,.\, \overset{n+1}{\mathrm{A}}_{+r}\, b_r, \quad (63).$$

If, therefore, the law were true for n and all inferior integers, it would be true for $n+1$; but it *is* true for 1, 2, 3 and 4, and therefore for n.

66. *Theorem.* If $a_n = c_n + \overset{n}{S}_m a_{n-m} \cdot b_m$,

$$\text{then } a_n = \overset{n}{S}_m c_{n-m+1} \cdot \overset{m-1}{A}_{+r} b_r + a_0 \cdot \overset{n}{A}_{+r} b_r.$$

For, proceeding as in the last Article, we shall find that

$$a_4 = \overset{4}{S}_m c_{5-m} \cdot \overset{m-1}{A}_{+r} b_r + a_0 \cdot \overset{4}{A}_{+r} b_r.$$

Suppose, therefore, $a_n = \overset{n}{S}_m c_{n-m+1} \cdot \overset{m-1}{A}_{+r} b_r + a_0 \cdot \overset{n}{A}_{+r} b_r$,

$$\text{then } a_{n+1} = c_{n+1} + \overset{n+1}{S}_m b_m \cdot a_{n-m+1}$$

$$= c_{n+1} + \overset{n+1}{S}_m b_m \{ \overset{n-m+1}{S}_s c_{n-m+1-s+1} \cdot \overset{s-1}{A}_{+r} b_r + a_0 \cdot \overset{n-m+1}{A}_{+r} b_r \}, \text{ by substitution;}$$

$$= c_{n+1} + \overset{n+1}{S}_m b_{n-m+2} \cdot \overset{m-1}{S}_s c_{m-s} \cdot \overset{s-1}{A}_{+r} b_r + a_0 \cdot \overset{n+1}{S}_m b_m \cdot \overset{n-m+1}{A}_{+r} b_r, \text{ (8) and (6);}$$

$$= c_{n+1} + b_{n+1} \cdot \overset{0}{S}_s c_{1-s} \cdot \overset{s-1}{A}_{+r} b_r + \overset{n}{S}_m b_{n-m+1} \cdot \overset{m}{S}_s c_{m-s+1} \cdot \overset{s-1}{A}_{+r} b_r + a_0 \cdot \overset{n+1}{A}_{+r} b_r, \text{ (9) and (63);}$$

$$= c_{n+1} + \overset{n}{S}_m b_{n-m+1} \cdot \overset{m}{S}_s c_{m-s+1} \cdot \overset{s-1}{A}_{+r} b_r + a_0 \cdot \overset{n+1}{A}_{+r} b_r$$

$$= c_{n+1} + \overset{n}{S}_m \overset{n-m+1}{S}_s b_{n-m-s+2} \cdot c_m \cdot \overset{s-1}{A}_{+r} b_r + a_0 \cdot \overset{n+1}{A}_{+r} b_r, \text{ (6) and (22);}$$

$$= c_{n+1} + \overset{n}{S}_m c_m \cdot \overset{n-m+1}{S}_s b_{n-m-s+2} \cdot \overset{s-1}{A}_{+r} b_r + a_0 \cdot \overset{n+1}{A}_{+r} b_r, \text{ (6);}$$

$$= c_{n+1} + \overset{n}{S}_m c_m \cdot \overset{n-m+1}{A}_{+r} b_r + a_0 \cdot \overset{n+1}{A}_{+r} b_r, \text{ (63);}$$

$$= c_{n+1} + \overset{n}{S}_m c_{n-m+1} \cdot \overset{m}{A}_{+r} b_r + a_0 \cdot \overset{n+1}{A}_{+r} b_r, \text{ (8);}$$

$$= \overset{n+1}{S}_m c_{n-m+2} \cdot \overset{m}{A}_{+r} b_r + a_0 \cdot \overset{n+1}{A}_{+r} b_r, \text{ (9).}$$

If, therefore, the law were true for n and all inferior integers, it would be true for $n+1$; but it *is* true for 1, 2, 3 and 4, and therefore for n.

CHAPTER IV.

ON BINOMIALS AND EXPONENTIALS.

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67. *Theorem.* $\overset{n}{\mathrm{P}}_r(a_r + b_r) = \overset{n+1}{\mathrm{S}}_m \overset{\overline{n-m+1,\, m-1}}{\mathrm{C}}_{r,s}(a_r \,.\, b_s).$

For, by actual multiplication,

$$\overset{3}{\mathrm{P}}_r(a_r + b_r) = a_1 a_2 a_3 + a_1 a_2 b_3 + a_1 a_3 b_2 + a_2 a_3 b_1$$

$$+ a_1 b_2 b_3 + a_2 b_1 b_3 + a_3 b_1 b_2 + b_1 b_2 b_3$$

$$= \overset{4}{\mathrm{S}}_m \overset{\overline{4-m,\, m-1}}{\mathrm{C}}_{r,s}(a_r \,.\, b_s).$$

Suppose, therefore, $\overset{n}{\mathrm{P}}_r(a_r + b_r) = \overset{n+1}{\mathrm{S}}_m \overset{\overline{n-m+1,\, m-1}}{\mathrm{C}}_{r,s}(a_r \,.\, b_s),$

$$\text{then } \overset{n+1}{\mathrm{P}}_r(a_r + b_r) = (a_{n+1} + b_{n+1}) \,.\, \overset{n+1}{\mathrm{S}}_m \overset{\overline{n-m+1,\, m-1}}{\mathrm{C}}_{r,s}(a_r \,.\, b_s)$$

$$= \overset{n+1}{\mathrm{S}}_m a_{n+1} \,.\, \overset{\overline{n-m+1,\, m-1}}{\mathrm{C}}_{r,s}(a_r \,.\, b_s) + \overset{n+1}{\mathrm{S}}_m b_{n+1} \,.\, \overset{\overline{n-m+1,\, m-1}}{\mathrm{C}}_{r,s}(a_r \,.\, b_s) \quad (6);$$

$$= a_{n+1} \,.\, \overset{\overline{n,\, 0}}{\mathrm{C}}_{r,s}(a_r \,.\, b_s) + \overset{n}{\mathrm{S}}_m a_{n+1} \,.\, \overset{\overline{n-m,\, m}}{\mathrm{C}}_{r,s}(a_r \,.\, b_s)$$

$$+ \overset{n}{\mathrm{S}}_m b_{n+1} \,.\, \overset{\overline{n-m+1,\, m-1}}{\mathrm{C}}_{r,s}(a_r \,.\, b_s) + b_{n+1} \,.\, \overset{\overline{0,\, n}}{\mathrm{C}}_{r,s}(a_r \,.\, b_s), \quad (9);$$

$$= \overset{\overline{n+1,\, 0}}{\mathrm{C}}_{r,s}(a_r \,.\, b_s) + \overset{n}{\mathrm{S}}_m \left\{ a_{n+1} \,.\, \overset{\overline{n-m,\, m}}{\mathrm{C}}_{r,s} \,.\, (a_r \,.\, b_s) + b_{n+1} \,.\, \overset{\overline{n-m+1,\, m-1}}{\mathrm{C}}_{r,s}(a_r \,.\, b_s) \right\} + \overset{\overline{0,\, n+1}}{\mathrm{C}}_{r,s}(a_r \,.\, b_s), \quad (5);$$

$$= \overset{\overline{n+1,\, 0}}{\mathrm{C}}_{r,s}(a_r \,.\, b_s) + \overset{n}{\mathrm{S}}_m \overset{\overline{n-m+1,\, m}}{\mathrm{C}}_{r,s}(a_r \,.\, b_s) + \overset{\overline{0,\, n+1}}{\mathrm{C}}_{r,s}(a_r \,.\, b_s), \quad (61);$$

$$= \overset{n+2}{\mathrm{S}}_m \overset{\overline{n-m+2,\, m-1}}{\mathrm{C}}_{r,s}(a_r \,.\, b_s), \quad (9).$$

If therefore, the law were true for n factors, it would be true for $n+1$; but it *is* true for 3, and therefore, it is true for n.

68. Cor. 1. Put $b_r = x$, then $\overset{n}{\mathrm{P}}_r(x + a_r) = \overset{n+1}{\mathrm{S}}_m \overset{\overline{n-m+1,\, m-1}}{\mathrm{C}}_{r,s}(a_r . x)$

$$= \overset{n+1}{\mathrm{S}}_m\, x^{m-1} . \overset{n-m+1,\, n}{\mathrm{C}}_r\, a_r, \quad (60).$$

69. *Theorem.* If a_r is the r^{th} root of the equation

$$0 = \overset{n+1}{\mathrm{S}}_m a_{m-1} . x^{m-1}, \text{ then shall } a_{m-1} = \overset{n-m+1,\, n}{\mathrm{C}}_r(-a_r).$$

For, $\overset{n+1}{\mathrm{S}}_m a_{m-1} . x^{m-1} = \overset{n}{\mathrm{P}}_r(x - a_r)$

$$= \overset{n+1}{\mathrm{S}}_m\, x^{m-1} . \overset{n-m+1,\, n}{\mathrm{C}}_r(-a_r), \qquad (68);$$

$$\therefore\ a_{m-1} = \overset{n-m+1,\, n}{\mathrm{C}}_r(-a_r).$$

70. *Problem.* Given $b_{r-1} = \overset{n}{\mathrm{S}}_m a_m^{r-1} . x_m$, $\binom{r=1}{r=n}$, to find x_t.

Multiply both sides of the equation by $\overset{n-r,\, n\, ;\, t}{\mathrm{C}}_s(-a_s)$

then $b_{r-1} . \overset{n-r,\, n\, ;\, t}{\mathrm{C}}_s(-a_s) = \overset{n}{\mathrm{S}}_m x_m . a_m^{r-1} . \overset{n-r,\, n\, ;\, t}{\mathrm{C}}_s(-a_s)$, (6).

$\therefore\ \overset{n}{\mathrm{S}}_r b_{r-1} . \overset{n-r,\, n\, ;\, t}{\mathrm{C}}_s(-a_s) = \overset{n}{\mathrm{S}}_m x_m . \overset{n}{\mathrm{S}}_r a_m^{r-1} . \overset{n-r,\, n\, ;\, t}{\mathrm{C}}_s(-a_s)$, (4) and (17);

$$= \overset{n}{\mathrm{S}}_m x_m . \overset{n\, ;\, t}{\mathrm{P}}_r(a_m - a_r), \qquad (68).$$

But $\overset{n\, ;\, t}{\mathrm{P}}_r(a_m - a_r) = 0$, for every value from $m = 1$ to $m = n$, except for $m = t$;

$$\text{therefore, } \overset{n}{\mathrm{S}}_r b_{r-1} . \overset{n-r,\, n\, ;\, t}{\mathrm{C}}_s(-a_s) = x_t . \overset{n\, ;\, t}{\mathrm{P}}_r(a_t - a_r),$$

$$\text{and } x_t = \frac{\overset{n}{\mathrm{S}}_r b_{r-1} . \overset{n-r,\, n\, ;\, t}{\mathrm{C}}_s(-a_s)}{\overset{n\, ;\, t}{\mathrm{P}}_r(a_t - a_r)}.$$

71. Cor. If $b_{r-1} = b^{r-1}$, then $x_t = \dfrac{\overset{n}{\text{S}}_r b^{r-1} . \overset{n-r,\, n\, ;\, t}{\text{C}}_s(-a_s)}{\overset{n;\, t}{\text{P}}_r(a_t - a_r)}$,

$$= \overset{n;\,t}{\text{P}}_r \left(\frac{b - a_r}{a_t - a_r}\right), \quad (68).$$

72. *Theorem.*

$$\frac{\overset{n+1}{\text{S}}_m a_{m-1} . x^{m-1}}{x - b} = \overset{n}{\text{S}}_m x^{m-1} . \overset{n-m+1}{\text{S}}_r a_{m+r-1} . b^{r-1} + \frac{\overset{n+1}{\text{S}}_m a_{m-1} . b^{m-1}}{x - b}.$$

For, $\dfrac{x^{m-1} - b^{m-1}}{x - b} = \overset{m-1}{\text{S}}_r x^{r-1} . b^{m-r-1}$, (11);

$$\therefore \frac{a_{m-1} . x^{m-1}}{x - b} - \frac{a_{m-1} . b^{m-1}}{x - b} = a_{m-1} . \overset{m-1}{\text{S}}_r x^{r-1} . b^{m-r-1};$$

and $\dfrac{\overset{n+1}{\text{S}}_m a_{m-1} x^{m-1}}{x - b} - \dfrac{\overset{n+1}{\text{S}}_m a_{m-1} b^{m-1}}{x - b} = \overset{n+1}{\text{S}}_m a_{m-1} . \overset{m-1}{\text{S}}_r x^{r-1} . b^{m-r-1}$, (4);

$$= a_0 . \overset{0}{\text{S}}_r x^{r-1} . b^{-r} + \overset{n}{\text{S}}_m a_m . \overset{m}{\text{S}}_r x^{r-1} . b^{m-r}, \quad (9);$$

$$= \overset{n}{\text{S}}_m a_m . \overset{m}{\text{S}}_r x^{r-1} . b^{m-r}$$

$$= \overset{n}{\text{S}}_m x^{m-1} . \overset{n-m+1}{\text{S}}_r a_{m+r-1} . b^{r-1}, \quad (6) \text{ and } (22).$$

$$\therefore \frac{\overset{n+1}{\text{S}}_m a_{m-1} . x^{m-1}}{x - b} = \overset{n}{\text{S}}_m x^{m-1} . \overset{n-m+1}{\text{S}}_r a_{m+r-1} . b^{r-1} + \frac{\overset{n+1}{\text{S}}_m a_{m-1} . b^{m-1}}{x - b}.$$

73. Cor. If b is a root of the equation $0 = \overset{n+1}{\text{S}}_m a_{m-1} . x^{m-1}$, the second side of the equation is divisible by $x - b$.

For, the remainder after the performance of this division is $\overset{n+1}{\text{S}}_m a_{m-1} . b^{m-1}$; which $= 0$, since b is a root of the equation.

74. *Theorem.* $\underline{|a+b}_{\,n,r}=\overset{n+1}{\mathrm{S}}_m \dfrac{\underline{|n}_{\,m-1}}{\underline{|m-1}}\,.\,\underline{|a}_{\,n-m+1,r}\,.\,\underline{|b}_{\,m-1,r}$; n being any positive integer.

For, $\underline{|a+b}_{\,1,r}=a+b=\underline{|a}_{\,1,r}+\underline{|b}_{\,1,r}$.

$$\underline{|a+b}_{\,2,r}=(a+b+r)\left(\underline{|a}_{\,1,r}+\underline{|b}_{\,1,r}\right)=\underline{|a}_{\,1,r}\,.\,(\overline{a+r}+b)+\underline{|b}_{\,1,r}(a+\overline{b+r})$$

$$=\underline{|a}_{\,2,r}+\underline{|a}_{\,1,r}\,.\,\underline{|b}_{\,1,r}+\underline{|b}_{\,1,r}\,.\,\underline{|a}_{\,1,r}+\underline{|b}_{\,2,r}=\underline{|a}_{\,2,r}+2\,.\,\underline{|a}_{\,1,r}\,.\,\underline{|b}_{\,1,r}+\underline{|b}_{\,2,r}\,.$$

$$\underline{|a+b}_{\,3,r}=(a+b+2r)\left(\underline{|a}_{\,2,r}+2\,.\,\underline{|a}_{\,1,r}\,.\,\underline{|b}_{\,1,r}+\underline{|b}_{\,2,r}\right)$$

$$=\underline{|a}_{\,2,r}(\overline{a+2r}+b)+2\,.\,\underline{|a}_{\,1,r}\,.\,\underline{|b}_{\,1,r}(\overline{a+r}+\overline{b+r})+\underline{|b}_{\,2,r}(a+\overline{b+2r})$$

$$=\underline{|a}_{\,3,r}+\underline{|a}_{\,2,r}\,.\,\underline{|b}_{\,1,r}+2\,.\,\underline{|a}_{\,2,r}\,.\,\underline{|b}_{\,1,r}+2\,.\,\underline{|a}_{\,1,r}\,.\,\underline{|b}_{\,2,r}+\underline{|a}_{\,1}\,.\,\underline{|b}_{\,2,r}+\underline{|b}_{\,3,r}\,.$$

$$=\underline{|a}_{\,3,r}+3\,.\,\underline{|a}_{\,2,r}\,.\,\underline{|b}_{\,1,r}+3\,.\,\underline{|a}_{\,1,r}\,.\,\underline{|b}_{\,2,r}+\underline{|b}_{\,3,r}\,.$$

$$=\overset{4}{\mathrm{S}}_m \dfrac{\underline{|3}_{\,m-1}}{\underline{|m-1}}\,.\,\underline{|a}_{\,4-m,r}\,.\,\underline{|b}_{\,m-1,r}\,.$$

Similarly, $\underline{|a+b}_{\,4,r}=\overset{5}{\mathrm{S}}_m \dfrac{\underline{|4}_{\,m-1}}{\underline{|m-1}}\,.\,\underline{|a}_{\,5-m,r}\,.\,\underline{|b}_{\,m-1,r}$.

Suppose, therefore, $\underline{|a+b}_{\,n,r}=\overset{n+1}{\mathrm{S}}_m \dfrac{\underline{|n}_{\,m-1}}{\underline{|m-1}}\,.\,\underline{|a}_{\,n-m+1,r}\,.\,\underline{|b}_{\,m-1,r}$;

then $\underline{|a+b}_{\,n+1,r}=(a+b+nr)\,.\,\underline{|a+b}_{\,n,r}$

$$=\overset{n+1}{\mathrm{S}}_m \dfrac{\underline{|n}_{\,m-1}}{\underline{|m-1}}\;\underline{|a}_{\,n-m+1,r}\,.\,\underline{|b}_{\,m-1,r}\,.\,(a+\overline{n-m+1}\,.\,r+b+\overline{m-1}\,.\,r), \quad (6);$$

$$=\overset{n+1}{\mathrm{S}}_m \frac{\underline{|n}_{m-1}}{\underline{|m-1}} \cdot \left(\underline{|a}_{n-m+2,r} \cdot \underline{|b}_{m-1,r} + \underline{|a}_{n-m+1,r} \cdot \underline{|b}_{m,r}\right)$$

$$=\underline{|a}_{n+1,r} \cdot \underline{|b}_{0,r} + \overset{n}{\mathrm{S}}_m \frac{\underline{|n}_{m}}{\underline{|m}} \cdot \underline{|a}_{n-m+1,r} \cdot \underline{|b}_{m,r} + \overset{n}{\mathrm{S}}_m \frac{\underline{|n}_{m-1}}{\underline{|m-1}} \cdot \underline{|a}_{n-m+1,r} \cdot \underline{|b}_{m,r}$$

$$+\underline{|a}_{0,r} \cdot \underline{|b}_{n+1,r}, \quad (9);$$

$$=\underline{|a}_{n+1,r} \cdot \underline{|b}_{0,r} + \overset{n}{\mathrm{S}}_m \left(\frac{\underline{|n}_{m}}{\underline{|m}} + \frac{\underline{|n}_{m-1}}{\underline{|m-1}}\right) \cdot \underline{|a}_{n-m+1,r} \cdot \underline{|b}_{m,r} + \underline{|a}_{0,r} \cdot \underline{|b}_{n+1,r}$$

$$=\underline{|a}_{n+1,r} \cdot \underline{|b}_{0,r} + \overset{n}{\mathrm{S}}_m \frac{\underline{|n+1}_{m}}{\underline{|m}} \cdot \underline{|a}_{n-m+1,r} \cdot \underline{|b}_{m,r} + \underline{|a}_{0,r} \cdot \underline{|b}_{n+1,r}, \quad (47);$$

$$=\overset{n+2}{\mathrm{S}}_m \frac{\underline{|n+1}_{m-1}}{\underline{|m-1}} \cdot \underline{|a}_{n-m+2,r} \cdot \underline{|b}_{m-1,r}, \quad (9).$$

If, therefore, the law were true for n, it would be true for $n+1$: but it is true for 4, and therefore, for n.

75. Cor. 1. $\underline{|a-b}_{n,r} = \overset{n+1}{\mathrm{S}}_m \frac{\underline{|n}_{m-1}}{\underline{|m-1}} \cdot \underline{|a}_{n-m+1,r} \cdot \underline{|-b}_{m-1,r}$.

But $\underline{|-b}_{m-1,r} = (-1)^{m-1} \cdot \underline{|b}_{m-1,-r}$, (39).

$$\therefore \underline{|a-b}_{n,r} = \overset{n+1}{\mathrm{S}}_m (-1)^{m-1} \cdot \frac{\underline{|n}_{m-1}}{\underline{|m-1}} \cdot \underline{|a}_{n-m+1,r} \cdot \underline{|b}_{m-1,-r}.$$

76. Cor. 2. Since $\underline{|n} = \underline{|n}_{m-1} \cdot \underline{|n-m+1}$, $\therefore \underline{|n}_{m-1} = \frac{\underline{|n}}{\underline{|n-m+1}}$,

$$\text{and } \frac{\underline{|a \pm b}_{n,r}}{\underline{|n}} = \overset{n+1}{\mathrm{S}}_m (\pm 1)^{m-1} \cdot \frac{\underline{|a}_{n-m+1,r} \cdot \underline{|b}_{m-1,\pm r}}{\underline{|n-m+1}\ \underline{|m-1}}.$$

77. *Theorem.*

$$\left(\overset{\infty}{\mathrm{S}}_m \frac{\underline{|a}_{\,m-1,r}\,.\,x^{m-1}}{\underline{|m-1}}\right)\left(\overset{\infty}{\mathrm{S}}_n \frac{\underline{|b}_{\,n-1,r}\,.\,x^{n-1}}{\underline{|n-1}}\right)=\overset{\infty}{\mathrm{S}}_m \frac{\underline{|a+b}_{\,m-1,r}\,.\,x^{m-1}}{\underline{|m-1}}.$$

For, $\left(\overset{\infty}{\mathrm{S}}_m \frac{\underline{|a}_{\,m-1,r}\,.\,x^{m-1}}{\underline{|m-1}}\right)\left(\overset{\infty}{\mathrm{S}}_n \frac{\underline{|b}_{\,n-1,r}\,.\,x^{n-1}}{\underline{|n-1}}\right)$

$$=\overset{\infty}{\mathrm{S}}_m x^{m-1}\,.\,\overset{m}{\mathrm{S}}_n \frac{\underline{|a}_{\,m-n,r}\,.\,\underline{|b}_{\,n-1,r}}{\underline{|m-n}\,.\,\underline{|n-1}}, \quad (26);$$

$$=\overset{\infty}{\mathrm{S}}_m x^{m-1}\,.\,\frac{\underline{|a+b}_{\,m-1,r}}{\underline{|m-1}}, \quad (76).$$

78. Cor. 1. $$\left(\overset{\infty}{\mathrm{S}}_m \frac{\underline{|a}_{\,m-1,r}\,.\,x^{m-1}}{\underline{|m-1}}\right)^n=\overset{\infty}{\mathrm{S}}_m \frac{\underline{|na}_{\,m-1,r}\,.\,x^{m-1}}{\underline{|m-1}};$$

n being any positive integer.

79. Cor. 2. $$\left(\overset{\infty}{\mathrm{S}}_m \frac{\underline{|a-b}_{\,m-1,r}\,.\,x^{m-1}}{\underline{|m-1}}\right)\left(\overset{\infty}{\mathrm{S}}_n \frac{\underline{|b}_{\,n-1,r}\,.\,x^{n-1}}{\underline{|n-1}}\right)=\overset{\infty}{\mathrm{S}}_m \frac{\underline{|a}_{\,m-1,r}\,.\,x^{m-1}}{\underline{|m-1}};$$

$$\therefore\ \left(\overset{\infty}{\mathrm{S}}_m \frac{\underline{|a}_{\,m-1,r}\,.\,x^{m-1}}{\underline{|m-1}}\right)\div\left(\overset{\infty}{\mathrm{S}}_n \frac{\underline{|b}_{\,n-1,r}\,.\,x^{n-1}}{\underline{|n-1}}\right)=\overset{\infty}{\mathrm{S}}_m \frac{\underline{|a-b}_{\,m-1,r}\,.\,x^{m-1}}{\underline{|m-1}}.$$

80. Cor. 3. $$\left(\overset{\infty}{\mathrm{S}}_m \underline{\left|\frac{a}{n}\right.}_{\,m-1,r}\,.\,\frac{x^{m-1}}{\underline{|m-1}}\right)^n=\overset{\infty}{\mathrm{S}}_m \underline{|a}_{\,m-1,r}\,.\,\frac{x^{m-1}}{\underline{|m-1}};$$

$$\therefore\ \left(\overset{\infty}{\mathrm{S}}_m \underline{|a}_{\,m-1,r}\,.\,\frac{x^{m-1}}{\underline{|m-1}}\right)^{\frac{1}{n}}=\overset{\infty}{\mathrm{S}}_m \underline{\left|\frac{a}{n}\right.}_{\,m-1,r}\,.\,\frac{x^{m-1}}{\underline{|m-1}}.$$

81. Cor. 4. $\left(\overset{\infty}{\mathrm{S}}_m \underline{|a}_{\,m-1,r} \cdot \dfrac{x^{m-1}}{\underline{|m-1}}\right)^{\frac{t}{n}} = \left\{\left(\overset{\infty}{\mathrm{S}}_m \underline{|a}_{\,m-1,r} \cdot \dfrac{x^{m-1}}{\underline{|m-1}}\right)^{\frac{1}{n}}\right\}^t$

$$= \left\{\overset{\infty}{\mathrm{S}}_m \underline{\left|\frac{a}{n}\right.}_{\,m-1,r} \cdot \frac{x^{m-1}}{\underline{|m-1}}\right\}^t = \overset{\infty}{\mathrm{S}}_m \underline{\left|\frac{t}{n} \cdot a\right.}_{\,m-1,r} \cdot \frac{x^{m-1}}{\underline{|m-1}}.$$

82. Cor. 5. Put $\overset{\infty}{\mathrm{S}}_m \underline{|a}_{\,m-1,r} \cdot \dfrac{x^{m-1}}{\underline{|m-1}} = f(a)$,

then $\{f(a)\}^{-n} = \dfrac{1}{\{f(a)\}^n} = \dfrac{f(0)}{f(na)} = f(0-na)$, (79);

$$= f(-na).$$

$$\therefore \left(\overset{\infty}{\mathrm{S}}_m \underline{|a}_{\,m-1,r} \cdot \frac{x^{m-1}}{\underline{|m-1}}\right)^{-n} = \overset{\infty}{\mathrm{S}}_m \underline{|-na}_{\,m-1,r} \cdot \frac{x^{m-1}}{\underline{|m-1}}.$$

83. Cor. 6. $\{f(a)\}^{-1} = f(-a)$;

$$\therefore \{f(a)\}^{-\frac{t}{n}} = \{f(-a)\}^{\frac{t}{n}} = f\left(-\frac{t}{n} \cdot a\right);$$

$$\therefore \left(\overset{\infty}{\mathrm{S}}_m \underline{|a}_{\,m-1,r} \cdot \frac{x^{m-1}}{\underline{|m-1}}\right)^{-\frac{t}{n}} = \overset{\infty}{\mathrm{S}}_m \underline{\left|-\frac{t}{n} \cdot a\right.}_{\,m-1,r} \cdot \frac{x^{m-1}}{\underline{|m-1}}.$$

84. *Theorem.* $\left(\overset{t}{\mathrm{S}}_m \underline{|a}_{\,m-1,r} \cdot \dfrac{x^{m-1}}{\underline{|m-1}}\right)\left(\overset{t}{\mathrm{S}}_n \underline{|b}_{\,n-1,r} \cdot \dfrac{x^{n-1}}{\underline{|n-1}}\right)$

$$= \overset{t}{\mathrm{S}}_m \underline{|a+b}_{\,m-1,r} \cdot \frac{x^{m-1}}{\underline{|m-1}} + \text{terms in } x^t.$$

For, $\left(\overset{t}{\mathrm{S}}_m \underline{|a}_{\,m-1,r} \cdot \dfrac{x^{m-1}}{\underline{|m-1}}\right)\left(\overset{t}{\mathrm{S}}_n \underline{|b}_{\,n-1,r} \cdot \dfrac{x^{n-1}}{\underline{|n-1}}\right)$

$$= \overset{t}{\mathrm{S}}_m \overset{t}{\mathrm{S}}_n \frac{\underline{|a}_{\,m-1,r} \cdot \underline{|b}_{\,n-1,r}}{\underline{|m-1} \cdot \underline{|n-1}} \cdot x^{m+n-2}, \quad \text{(16) and (6);}$$

$$= \overset{t}{\mathrm{S}}_m \overset{m}{\mathrm{S}}_n \frac{\underline{|a}_{m-n,r} \cdot \underline{|b}_{n-1,r}}{\underline{|m-n} \cdot \underline{|n-1}} \cdot x^{m-1} + \text{terms in } x^t, \quad (18);$$

$$= \overset{t}{\mathrm{S}}_m x^{m-1} \cdot \overset{m}{\mathrm{S}}_n \frac{\underline{|a}_{m-n,r} \cdot \underline{|b}_{n-1,r}}{\underline{|m-n} \cdot \underline{|n-1}} + \text{terms in } x^t, \quad (6);$$

$$= \overset{t}{\mathrm{S}}_m \frac{\underline{|a+b}_{m-1,r} \cdot x^{m-1}}{\underline{|m-1}} + \text{terms in } x^t, \quad (76).$$

85. Cor. 1. $\left(\overset{t}{\mathrm{S}}_m \frac{\underline{|a}_{m-1,r} \cdot x^{m-1}}{\underline{|m-1}}\right)^n = \overset{t}{\mathrm{S}}_m \frac{\underline{|na}_{m-1,r} \cdot x^{m-1}}{\underline{|m-1}} + \text{terms in } x^t.$

86. *Theorem.* $(a+b)^n = \overset{n+1}{\mathrm{S}}_m \frac{\underline{|n}_{m-1}}{\underline{|m-1}} \cdot a^{n-m+1} \cdot b^{m-1}$; n being any positive integer.

For, $\overset{n}{\mathrm{P}}_r (a+b_r) = \overset{n+1}{\mathrm{S}}_m a^{n-m+1} \cdot \overset{m-1,n}{\mathrm{C}}_r (b_r)$, (68) and (8).

Put $b_r = b$ then $(a+b)^n = \overset{n+1}{\mathrm{S}}_m a^{n-m+1} \cdot \overset{m-1,n}{\mathrm{C}}_r (b)$

$$= \overset{n+1}{\mathrm{S}}_m \frac{\underline{|n}_{m-1}}{\underline{|m-1}} \cdot a^{n-m+1} \cdot b^{m-1}, \quad (56).$$

This theorem may also be proved as follows:

$$\underline{|a+b}_{n,r} = \overset{n+1}{\mathrm{S}}_m \frac{\underline{|n}_{m-1}}{\underline{|m-1}} \cdot \underline{|a}_{n-m+1,r} \cdot \underline{|b}_{m-1,r}, \quad (74).$$

Put $r=0$, then $\underline{|a+b}_{n,r} = (a+b)^n$, $\underline{|a}_{n-m+1,r} = a^{n-m+1}$, and $\underline{|b}_{m-1,r} = b^{m-1}$.

$$\therefore \ (a+b)^n = \overset{n+1}{\mathrm{S}}_m \frac{\underline{|n}_{m-1}}{\underline{|m-1}} \cdot a^{n-m+1} \cdot b^{m-1}.$$

87. Cor. 1. If we invert the series we shall get

$$(a+b)^n = \overset{n+1}{S_m} \frac{\underline{|n}_{\,n-m+1}}{\underline{|n-m+1}} \cdot a^{m-1} \cdot b^{n-m+1}, \quad (8);$$

$$\text{but } \frac{\underline{|n}_{\,n-m+1}}{\underline{|n-m+1}} = \frac{\underline{|n}_{\,m-1}}{\underline{|m-1}}, \quad (46);$$

therefore the coefficients are the same when taken in an inverted order.

88. Cor. 2. Since $\underline{|n} = \underline{|n}_{\,m-1} \cdot \underline{|n-m+1}$,

$$\therefore \frac{(a+b)^n}{\underline{|n}} = \overset{n+1}{S_m} \frac{a^{n-m+1}\, b^{m-1}}{\underline{|n-m+1}\ \underline{|m-1}}.$$

89. *Theorem.*

$$(a+b)^n = \overset{\frac{1}{2}n}{S_m} \frac{\underline{|n}_{\,m-1}}{\underline{|m-1}} \cdot (ab)^{m-1} (a^{n-2m+2} + b^{n-2m+2}) + \frac{\underline{|n}_{\,\frac{1}{2}n}}{\underline{|\tfrac{1}{2}n}} \cdot (ab)^{\frac{1}{2}n}, \text{ or}$$

$$= \overset{\frac{1}{2}(n+1)}{S_m} \frac{\underline{|n}_{\,m-1}}{\underline{|m-1}} \cdot (ab)^{m-1} \cdot (a^{n-2m+2} + b^{n-2m+2});$$

according as n is even or odd.

$$\text{For, } (a+b)^n = \overset{n+1}{S_m} \frac{\underline{|n}_{\,m-1}}{\underline{|m-1}} \cdot a^{n-m+1} \cdot b^{m-1}, \quad (86).$$

(1) Let n be even; then

$$(a+b)^n = \overset{\frac{1}{2}n}{S_m} \frac{\underline{|n}_{\,m-1}}{\underline{|m-1}} \cdot a^{n-m+1} \cdot b^{m-1} + \frac{\underline{|n}_{\,\frac{1}{2}n}}{\underline{|\tfrac{1}{2}n}} \cdot (ab)^{\frac{1}{2}n}$$

$$+ \overset{\frac{1}{2}n}{S_m} \frac{\underline{|n}_{\,m-1}}{\underline{|m-1}} \cdot a^{m-1} \cdot b^{n-m+1}, \quad (9), (8) \text{ and } (46).$$

$$=\overset{\frac{1}{2}n}{\mathrm{S}}_m \frac{\underline{|n}_{\,m-1}}{\underline{|m-1}} \cdot (ab)^{m-1}\left(a^{n-2m+2}+b^{n-2m+2}\right)+\frac{\underline{|n}_{\,\frac{1}{2}n}}{\underline{|\tfrac{1}{2}n}} \cdot (ab)^{\frac{1}{2}n}, \quad (5).$$

(2) Let n be odd, then

$$(a+b)^n=\overset{\frac{1}{2}(n+1)}{\mathrm{S}}_m \frac{\underline{|n}_{\,m-1}}{\underline{|m-1}} \cdot a^{n-m+1} \cdot b^{m-1}+\overset{\frac{1}{2}(n+1)}{\mathrm{S}}_m \frac{\underline{|n}_{\,m-1}}{\underline{|m-1}} \cdot a^{m-1} \cdot b^{n-m+1}, \quad (9),$$

(8) and (46);

$$=\overset{\frac{1}{2}(n+1)}{\mathrm{S}}_m \frac{\underline{|n}_{\,m-1}}{\underline{|m-1}} \cdot (ab)^{m-1}\left(a^{n-2m+2}+b^{n-2m+2}\right), \quad (5).$$

90. Cor. If n is even,

$$(a-b)^n=\overset{\frac{1}{2}n}{\mathrm{S}}_m (-1)^{m-1} \cdot \frac{\underline{|n}_{\,m-1}}{\underline{|m-1}} \cdot (ab)^{m-1}\left(a^{n-2m+2}+b^{n-2m+2}\right)$$

$$+(-1)^{\frac{1}{2}n} \cdot \frac{\underline{|n}_{\,\frac{1}{2}n}}{\underline{|\tfrac{1}{2}n}} \cdot (ab)^{\frac{1}{2}n};$$

and if n is odd,

$$(a-b)^n=\overset{\frac{1}{2}(n+1)}{\mathrm{S}}_m (-1)^{m-1} \cdot \frac{\underline{|n}_{\,m-1}}{\underline{|m-1}} \cdot (ab)^{m-1} \cdot \left(a^{n-2m+2}-b^{n-2m+2}\right).$$

91. *Theorem.* $(1+x)^{\pm\frac{n}{r}}=\overset{\infty}{\mathrm{S}}_m \underline{\left|\pm\frac{n}{r}\right.}_{\,m-1} \cdot \dfrac{x^{m-1}}{\underline{|m-1}}$; n and r being any positive integers, and x being less than unity.

For, $$\left(\overset{\infty}{\mathrm{S}}_m \underline{\left|\pm\frac{n}{r}\right.}_{\,m-1} \cdot \frac{x^{m-1}}{\underline{|m-1}}\right)^{\pm r}=\overset{\infty}{\mathrm{S}}_m \underline{|n}_{\,m-1} \cdot \frac{x^{m-1}}{\underline{|m-1}}, \quad (78) \text{ and } (82);$$

$$=\overset{n+1}{\mathrm{S}}_m \frac{\underline{|n}_{\,m-1}}{\underline{|m-1}} \cdot x^{m-1}, \quad (13);$$

$$=(1+x)^n, \quad (86).$$

$$\therefore\ (1+x)^{\pm\frac{n}{r}} = \overset{\infty}{\mathrm{S}}_m\, \underline{\left|\pm\frac{n}{r}\right.}_{\,m-1} \cdot \frac{x^{m-1}}{\underline{|m-1}}.$$

92. Cor. 1. $(1+x)^n = \overset{\infty}{\mathrm{S}}_m \dfrac{\underline{|n}_{\,m-1}}{\underline{|m-1}} \cdot x^{m-1}$, where n is any rational quantity; x being less than unity: (86), (13) and (91).

93. Cor. 2. $$(1+x)^{\frac{n}{r}} = \overset{\infty}{\mathrm{S}}_m\, \underline{\left|\frac{n}{r}\right.}_{\,m-1} \cdot \frac{x^{m-1}}{\underline{|m-1}}.$$

$$= \overset{\infty}{\mathrm{S}}_m \frac{\underline{|n}_{\,m-1,\,-r}}{\underline{|m-1}} \cdot \left(\frac{x}{r}\right)^{m-1}, \quad (39).$$

94. Cor. 3. $$(1+x)^{\frac{1}{n}} = \overset{\infty}{\mathrm{S}}_m\, \underline{\left|\frac{1}{n}\right.}_{\,m-1} \cdot \frac{x^{m-1}}{\underline{|m-1}}$$

$$= 1 + \overset{\infty}{\mathrm{S}}_m\, \underline{\left|\frac{1}{n}\right.}_{\,m} \cdot \frac{x^m}{\underline{|m}}, \quad (9);$$

But $$\underline{\left|\frac{1}{n}\right.}_{\,m} = \underline{|1}_{\,m,\,-n} \cdot \frac{1}{n^m}, \quad (39);$$

$$= \underline{|1-n}_{\,m-1,\,-n} \cdot \frac{1}{n^m}, \quad (41);$$

$$= \underline{|n-1}_{\,m-1,\,n} \cdot \frac{1}{(-1)^{m-1}.n^m}, \quad (39);$$

$$= (-1)^{m-1}.\underline{|n-1}_{\,m-1,\,n} \cdot \frac{1}{n^m}.$$

$$\therefore\ (1+x)^{\frac{1}{n}} = 1 + \overset{\infty}{\mathrm{S}}_m (-1)^{m-1}. \frac{\underline{|n-1}_{\,m-1,\,n}}{\underline{|m}} \cdot \left(\frac{x}{n}\right)^m.$$

95. Cor. 4. $(1+x)^{-\frac{1}{n}}=\overset{\infty}{S}_m \left\lfloor\underline{-\frac{1}{n}}\right._{m-1} \cdot \frac{x^{m-1}}{\lfloor\underline{m-1}}$

$$=\overset{\infty}{S}_m(-1)^{m-1}\cdot\frac{\lfloor\underline{1}_{m-1,\,n}}{\lfloor\underline{m-1}}\cdot\left(\frac{x}{n}\right)^{m-1}, \quad (39).$$

96. *Theorem.*

$$\left(\overset{\infty}{S}_m\frac{a^{m-1}\,x^{m-1}}{\lfloor\underline{m-1}}\right)\left(\overset{\infty}{S}_n\frac{b^{n-1}\,x^{n-1}}{\lfloor\underline{n-1}}\right)=\overset{\infty}{S}_m\frac{(a+b)^{m-1}\,.\,x^{m-1}}{\lfloor\underline{m-1}}.$$

For, $\left(\overset{\infty}{S}_m\frac{a^{m-1}\,x^{m-1}}{\lfloor\underline{m-1}}\right)\left(\overset{\infty}{S}_n\frac{b^{n-1}\,x^{n-1}}{\lfloor\underline{n-1}}\right)=\overset{\infty}{S}_m\, x^{m-1}\,.\,\overset{m}{S}_n\frac{a^{m-n}\,.\,b^{n-1}}{\lfloor\underline{m-n}\,\lfloor\underline{n-1}}$, (26);

$$=\overset{\infty}{S}_m\frac{(a+b)^{m-1}\,x^{m-1}}{\lfloor\underline{m-1}}, \quad (88).$$

97. Cor. 1. $\overset{n}{P}_r\left(\overset{\infty}{S}_m\frac{a_r^{\,m-1}\,.\,x^{m-1}}{\lfloor\underline{m-1}}\right)=\overset{\infty}{S}_m\frac{(\overset{n}{S}_r\,a_r)^{m-1}\,.\,x^{m-1}}{\lfloor\underline{m-1}}.$

98. Cor. 2. Put $a_r=a$,

then $\left(\overset{\infty}{S}_m\frac{a^{m-1}\,x^{m-1}}{\lfloor\underline{m-1}}\right)^n=\overset{\infty}{S}_m\frac{(n\,a)^{m-1}\,.\,x^{m-1}}{\lfloor\underline{m-1}}.$

99. Cor. 3. $\left\{\overset{\infty}{S}_m\left(\frac{a}{n}\right)^{m-1}\cdot\frac{x^{m-1}}{\lfloor\underline{m-1}}\right\}^n=\overset{\infty}{S}_m\frac{a^{m-1}\,.\,x^{m-1}}{\lfloor\underline{m-1}};$

$$\therefore\ \left(\overset{\infty}{S}_m\frac{a^{m-1}\,x^{m-1}}{\lfloor\underline{m-1}}\right)^{\frac{1}{n}}=\overset{\infty}{S}_m\left(\frac{a}{n}\right)^{m-1}\cdot\frac{x^{m-1}}{\lfloor\underline{m-1}}.$$

100. Cor. 4.

$$\left(\overset{\infty}{S}_m\frac{(a-b)^{m-1}\,.\,x^{m-1}}{\lfloor\underline{m-1}}\right)\left(\overset{\infty}{S}_n\frac{b^{n-1}\,.\,x^{n-1}}{\lfloor\underline{n-1}}\right)=\overset{\infty}{S}_m\frac{a^{m-1}\,.\,x^{m-1}}{\lfloor\underline{m-1}};$$

$$\therefore \left(\overset{\infty}{\mathrm{S}}_m \frac{a^{m-1} \cdot x^{m-1}}{\lfloor m-1}\right) \div \left(\overset{\infty}{\mathrm{S}}_n \frac{b^{n-1} \cdot x^{n-1}}{\lfloor n-1}\right) = \overset{\infty}{\mathrm{S}}_m \frac{(a-b)^{m-1} \cdot x^{m-1}}{\lfloor m-1}.$$

101. Cor. 5.

$$\left(\overset{\infty}{\mathrm{S}}_m \frac{(-a)^{m-1} \cdot x^{m-1}}{\lfloor m-1}\right)\left(\overset{\infty}{\mathrm{S}}_n \frac{a^{n-1} \cdot x^{n-1}}{\lfloor n-1}\right) = \overset{\infty}{\mathrm{S}}_m \frac{(a-a)^{m-1} \cdot x^{m-1}}{\lfloor m-1} = 1.$$

$$\therefore \left(\overset{\infty}{\mathrm{S}}_m \frac{a^{m-1} \cdot x^{m-1}}{\lfloor m-1}\right)^{-1} = \overset{\infty}{\mathrm{S}}_m \frac{(-a)^{m-1} \cdot x^{m-1}}{\lfloor m-1}.$$

102. Cor. 6.

$$\left\{\overset{\infty}{\mathrm{S}}_m \frac{a^{m-1} \cdot x^{m-1}}{\lfloor m-1}\right\}^{\pm p} = \left\{\overset{\infty}{\mathrm{S}}_m \frac{(\pm a)^{m-1} \cdot x^{m-1}}{\lfloor m-1}\right\}^{p}$$

$$= \overset{\infty}{\mathrm{S}}_m \frac{(\pm p\, a)^{m-1} \cdot x^{m-1}}{\lfloor m-1}, \quad (101);$$

and

$$\left(\overset{\infty}{\mathrm{S}}_m \frac{a^{m-1} \cdot x^{m-1}}{\lfloor m-1}\right)^{\pm \frac{p}{q}} = \left(\overset{\infty}{\mathrm{S}}_m \frac{(\pm p\, a)^{m-1} \cdot x^{m-1}}{\lfloor m-1}\right)^{\frac{1}{q}}$$

$$= \overset{\infty}{\mathrm{S}}_m \frac{(\pm \frac{p}{q} . a)^{m-1} \cdot x^{m-1}}{\lfloor m-1}.$$

103. Cor. 7. Put $x=1$, then $\left(\overset{\infty}{\mathrm{S}}_m \frac{a^{m-1}}{\lfloor m-1}\right)^{\pm \frac{p}{q}} = \overset{\infty}{\mathrm{S}}_m \frac{(\pm \frac{p}{q} . a)^{m-1}}{\lfloor m-1}.$

104. Cor. 8. Put $a=\sqrt{\pm 1}$, then

$$\left\{\overset{\infty}{\mathrm{S}}_m \frac{(\sqrt{\pm 1})^{m-1}}{\lfloor m-1}\right\}^{r} = \overset{\infty}{\mathrm{S}}_m \frac{(x\sqrt{\pm 1})^{m-1}}{\lfloor m-1}.$$

where x is any rational number.

The series $\overset{\infty}{S}_m \frac{1}{\lfloor m-1}$ occurs very frequently in algebraical investigations, and therefore we shall use the symbol ϵ to represent it; while $\epsilon^{\sqrt{-1}}$ will be used to denote $\overset{\infty}{S}_m \frac{(\sqrt{-1})^{m-1}}{\lfloor m-1}$. Hence the above equation may be written

$$\epsilon^{x\sqrt{\pm 1}} = \overset{\infty}{S}_m \frac{(x\sqrt{\pm 1})^{m-1}}{\lfloor m-1}.$$

105. Cor. 9. Let x be irrational; and suppose y and z are two rational numbers very nearly equal, such that $x > y$, and $x < z$.

Then ϵ^z, ϵ^x, and ϵ^y are in order of magnitude; that is,

$\overset{\infty}{S}_m \frac{z^{m-1}}{\lfloor m-1}$, ϵ^x, and $\overset{\infty}{S}_m \frac{y^{m-1}}{\lfloor m-1}$, are in order of magnitude.

But $\overset{\infty}{S}_m \frac{z^{m-1}}{\lfloor m-1}$, $\overset{\infty}{S}_m \frac{x^{m-1}}{\lfloor m-1}$, $\overset{\infty}{S}_m \frac{y^{m-1}}{\lfloor m-1}$, are also in order of magnitude however near the values of z, and y are taken to that of x;

$$\therefore \quad \epsilon^x = \overset{\infty}{S}_m \frac{x^{m-1}}{\lfloor m-1}.$$

106. Cor. 10. Hence, whatever the value of x may be, we shall have $\epsilon^x = \overset{\infty}{S}_m \frac{x^{m-1}}{\lfloor m-1}$.

107. *Theorem.* $a^x = \overset{\infty}{S}_m \frac{(x.\log_\epsilon.a)^{m-1}}{\lfloor m-1}$.

For, $a = \epsilon^{\log_\epsilon.a}$ *;

* By $\log_\epsilon a$ is denoted the logarithm of a in the system whose base is ϵ.

$$\therefore\ a^x=\epsilon^{x\,.\log_\epsilon .\, a}$$

$$=\overset{\infty}{\mathrm{S}}_m \frac{(x\,.\log_\epsilon .\, a)^{m-1}}{\lfloor m-1}, \quad (106).$$

108. *Theorem.* $(a\pm b)^2=a^2\pm 2ab+b^2$.

This will appear from actual multiplication.

109. *Theorem.* $(\overset{n}{\mathrm{S}}_m a_m)^2=\overset{n}{\mathrm{S}}_m a_m{}^2+2\,.\overset{n}{\mathrm{S}}_m a_m\,.\overset{n-m}{\mathrm{S}}_r a_{m+r}$.

For, $\overset{n-r+1}{\mathrm{S}}_m a_{r+m-1}=a_r+\overset{n-r}{\mathrm{S}}_m a_{r+m}$. (9);

$$\therefore\ (\overset{n-r+1}{\mathrm{S}}_m a_{r+m-1})^2=a_r{}^2+2\,a_r\overset{n-r}{\mathrm{S}}_m a_{r+m}+(\overset{n-r}{\mathrm{S}}_m a_{r+m})^2, \quad (108);$$

and $\overset{n}{\mathrm{S}}_r(\overset{n-r+1}{\mathrm{S}}_m a_{r+m-1})^2=\overset{n}{\mathrm{S}}_r a_r{}^2+2\overset{n}{\mathrm{S}}_r a_r\,.\overset{n-r}{\mathrm{S}}_m a_{r+m}+\overset{n}{\mathrm{S}}_r(\overset{n-r}{\mathrm{S}}_m a_{r+m})^2$, (4), (5), and (6);

$$\therefore\ (\overset{n}{\mathrm{S}}_m a_m)^2+\overset{n-1}{\mathrm{S}}_r(\overset{n-r}{\mathrm{S}}_m a_{r+m})^2=\overset{n}{\mathrm{S}}_r a_r{}^2+2\overset{n}{\mathrm{S}}_r a_r\,.\overset{n-r}{\mathrm{S}}_m a_{r+m}$$

$$+\overset{n-1}{\mathrm{S}}_r(\overset{n-r}{\mathrm{S}}_m a_{r+m})^2+(\overset{0}{\mathrm{S}}_m a_{n+m})^2, \quad (9);$$

$$\therefore\ (\overset{n}{\mathrm{S}}_m a_m)^2=\overset{n}{\mathrm{S}}_m a_m{}^2+2\overset{n}{\mathrm{S}}_m a_m\,.\overset{n-m}{\mathrm{S}}_r a_{m+r}.$$

CHAPTER V.

ON FINITE DIFFERENCES.

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110. If $\phi(u)$ is any function of u, then $\phi\phi(u)$ will denote the same function of $\phi(u)$. This last is expressed by $\phi^2(u)$; and, the same notation being extended, we get the equations

$$\phi\phi^m(u)=\phi^{m+1}(u), \text{ and } \phi^m\phi^n(u)=\phi^{m+n}(u).$$

111. Cor. 1. $\phi^0 . \phi^n(u)=\phi^{0+n}(u)=\phi^n(u)$.

$$\therefore \phi^0(u)=u.$$

112. Cor. 2.
$$\begin{aligned}\phi^{-n} . \phi^n(u)&=\phi^{-n+n}(u)\\ &=\phi^0(u)\\ &=u.\end{aligned}$$

113. *Definition.* If $\phi(u)$ is such a function of u that $\phi(u+v)=\phi(u)+\phi(v)$, then $\phi(u)$ is called a *distributive* function of u.

114. *Definition.* If $\phi(u)$, and $\psi(u)$ are such functions of u that $\phi\psi(u)=\psi\phi(u)$, then the functions $\phi(u)$ and $\psi(u)$ are said to be *commutative* with each other.

115. Instead of $\phi(u)+\psi(u)$, it is frequently convenient to write $(\phi+\psi)u$; in which case the latter expression must be carefully distinguished from the product $(\phi+\psi)\times u$, and must be considered merely as an abridgment of the full form $\phi(u)+\psi(u)$.

116. We shall express $\{(\phi+\psi)(\phi+\psi)\}u$, by $(\phi+\psi)_2 u$, and $\{(\phi+\psi)(\phi+\psi)(\phi+\psi)\}u$, by $(\phi+\psi)_3 u$:

that is, $(\phi+\psi)\, u=\phi(u)+\psi(u)$,

$$(\phi+\psi)_2 u=\phi\{(\phi+\psi)u\}+\psi\{(\phi+\psi)u\},$$

and, similarly, $(\phi+\psi)_n u=\phi(\phi+\psi)_{n-1}u+\psi(\phi+\psi)_{n-1}u$.

117. The symbol $\overset{n}{\underset{r}{\{}}(\phi_r+\psi_r)\underset{r+1}{\}}u$, will be equivalent to the expression

$$\{(\phi_1+\psi_1)(\phi_2+\psi_2)\ldots(\phi_n+\psi_n)\}u. \quad \text{(See Art. 49).}$$

118. *Theorem.* If $\phi_1(u)$, $\phi_2(u)$, $\phi_3(u)$, &c. and $\psi_1(u)$, $\psi_2(u)$, $\psi_3(u)$, &c. are all distributive functions, and commutative with each other, then shall

$$\overset{n}{\underset{r}{\{}}(\phi_r+\psi_r)\underset{r+1}{\}}u=\overset{n+1}{S_m}\{\overline{\overset{n-m+1,\, m-1}{}}\, C_{r,s}(\phi_r\cdot\psi_s)\}u.$$

For, $(\phi_1+\psi_1)u=\phi_1(u)+\psi_1(u)$, (115).

$$\overset{2}{\underset{r}{\{}}(\phi_r+\psi_r)\underset{r+1}{\}}u=(\phi_2+\psi_2)\{\phi_1(u)+\psi_1(u)\}, \quad (117);$$

$$=\phi_2\{\phi_1(u)+\psi_1(u)\}+\psi_2\{\phi_1(u)+\psi_1(u)\}, \quad (115);$$

$$=\phi_2\phi_1(u)+\phi_2\psi_1(u)+\psi_2\phi_1(u)+\psi_2\psi_1(u), \quad (113);$$

$$=\phi_1\phi_2(u)+\phi_1\psi_2(u)+\phi_2\psi_1(u)+\psi_1\psi_2(u), \quad (114).$$

$$\overset{3}{\underset{r}{\{}}(\phi_r+\psi_r)\underset{r+1}{\}}u=(\phi_3+\psi_3)\{\phi_1\phi_2(u)+\phi_1\psi_2(u)+\phi_2\psi_1(u)+\psi_1\psi_2(u)\}, \quad (117);$$

$$=\phi_3\{\phi_1\phi_2(u)+\phi_1\psi_2(u)+\phi_2\psi_1(u)+\psi_1\psi_2(u)\}$$
$$+\psi_3\{\phi_1\phi_2(u)+\phi_1\psi_2(u)+\phi_2\psi_1(u)+\psi_1\psi_2(u)\}, \quad (115);$$

$$=\phi_1\phi_2\phi_3(u)+\phi_1\phi_3\psi_2(u)+\phi_2\phi_3\psi_1(u)+\phi_3\psi_1\psi_2(u)$$

$$+\phi_1\phi_2\psi_3(u)+\phi_1\psi_2\psi_3(u)+\phi_2\psi_1\psi_3(u)+\psi_1\psi_2\psi_3(u),\quad (113)\ \&\ (114);$$

$$=(\phi_1\phi_2\phi_3+\phi_1\phi_2\psi_3+\phi_1\phi_3\psi_2+\phi_2\phi_3\psi_1+\phi_1\psi_2\psi_3+\phi_2\psi_1\psi_3+\phi_3\psi_1\psi_2$$

$$+\psi_1\psi_2\psi_3)u,\quad (115);$$

$$=\overset{4}{S}_m\{\overset{\overline{4-m,\ m-1}}{C}_{r,s}(\phi_r.\psi_s)\}u.$$

Suppose, therefore, $\overset{n}{\underset{r}{\{}}(\phi_r+\psi_r)\underset{r+1}{\{}u=\overset{n+1}{S}_m\{\overset{\overline{n-m+1,\ m-1}}{C}_{r,s}(\phi_r.\psi_s)\}u$;

then $\overset{n+1}{\underset{r}{\{}}(\phi_r+\psi_r)\underset{r+1}{\{}u=(\phi_{n+1}+\psi_{n+1})\overset{n+1}{S}_m\{\overset{\overline{n-m+1,\ m-1}}{C}_{r,s}(\phi_r.\psi_s)\}u,\quad (117);$

$$=\phi_{n+1}[\overset{n+1}{S}_m\{\overset{\overline{n-m+1,\ m-1}}{C}_{r,s}(\phi_r.\psi_s)\}u]+\psi_{n+1}[\overset{n+1}{S}_m\{\overset{\overline{n-m+1,\ m-1}}{C}_{r,s}(\phi_r.\psi_s)\}u],\ (115);$$

$$=\overset{n+1}{S}_m\phi_{n+1}[\{\overset{\overline{n-m+1,\ m-1}}{C}_{r,s}(\phi_r.\psi_s)\}u]+\overset{n+1}{S}_m\psi_{n+1}[\{\overset{\overline{n-m+1,\ m-1}}{C}_{r,s}(\phi_r.\psi_s)\}u],\ (113);$$

$$=\overset{n+1}{S}_m\{\phi_{n+1}\overset{\overline{n-m+1,\ m-1}}{C}_{r,s}(\phi_r.\psi_s)\}u+\overset{n+1}{S}_m\{\psi_{n+1}\overset{\overline{n-m+1,\ m-1}}{C}_{r,s}(\phi_r.\psi_s)\}u,\quad (115);$$

$$=\{\overset{n+1}{S}_m\phi_{n+1}\overset{\overline{n-m+1,\ m-1}}{C}_{r,s}(\phi_r.\psi_s)+\overset{n+1}{S}_m\psi_{n+1}\overset{\overline{n-m+1,\ m-1}}{C}_{r,s}(\phi_r.\psi_s)\}.u,\quad (115);$$

$$=\{\phi_{n+1}.\overset{\overline{n,\ 0}}{C}_{r,s}(\phi_r.\psi_s)+\overset{n}{S}_m\phi_{n+1}.\overset{\overline{n-m,\ m}}{C}_{r,s}(\phi_r.\psi_s)+\overset{n}{S}_m\psi_{n+1}\overset{\overline{n-m+1,\ m-1}}{C}_{r,s}(\phi_r.\psi_s)$$

$$+\psi_{n+1}.\overset{\overline{0,\ n}}{C}_{r,s}(\phi_r.\psi_s)\}u,\quad (9);$$

$$=\{\overset{\overline{n+1,\ 0}}{C}_{r,s}(\phi_r.\psi_s)+\overset{n}{S}_m[\phi_{n+1}.\overset{\overline{n-m,\ m}}{C}_{r,s}(\phi_r.\psi_s)+\psi_{n+1}.\overset{\overline{n-m+1,\ m-1}}{C}_{r,s}(\phi_r.\psi_s)]$$

$$+\overset{\overline{0,\ n+1}}{C}_{r,s}(\phi_r.\psi_s)\}u,\quad (5)\text{ and }(114);$$

$$=\{\overset{\overline{n+1,\ 0}}{C}_{r,s}(\phi_r.\psi_s)+\overset{n}{S}_m\overset{\overline{n-m+1,\ m}}{C}_{r,s}(\phi_r.\psi_s)+\overset{\overline{0,\ n+1}}{C}_{r,s}(\phi_r.\psi_s)\}u,\quad (61);$$

$$=\{\overset{n+2}{S}_m\overset{\overline{n-m+2,\ m-1}}{C}_{r,s}(\phi_r.\psi_s)\}u,\quad (9);$$

$$=\overset{n+2}{S}_m\{\overset{\overline{n-m+2,\ m-1}}{C}_{r,s}(\phi_r.\psi_s)\}u,\quad (115).$$

If, therefore, the law were true for n, it would also be true for $n+1$; but it *is* true for 3, and hence it *is* true for n.

119. Cor. 1. Let the functions $\phi_1(u)$, $\phi_2(u)$, &c. be all similar to each other, and to $\phi(u)$; also let $\psi_1(u)$, $\psi_2(u)$, &c. be all similar to $\psi(u)$; then

$$\overset{n}{\underset{r}{\{}}(\phi_r+\psi_r)\underset{r+1}{\}}u \text{ becomes } (\phi+\psi)_n u, \text{ and}$$

$$\overset{n+1}{\mathrm{S}}_m \left\{ \overline{\overset{n-m+1,\, m-1}{\mathrm{C}_{r,s}}}(\phi_r.\psi_s) \right\} u \text{ becomes } \overset{n+1}{\mathrm{S}}_m \left\{ \frac{\underset{m-1}{\lfloor n}}{\lfloor m-1}.\phi^{n-m+1}.\psi^{m-1} \right\} u,$$

(56) and (60);

$$\therefore\ (\phi+\psi)_n u = \overset{n+1}{\mathrm{S}}_m \frac{\underset{m-1}{\lfloor n}}{\lfloor m-1}.\phi^{n-m+1}\psi^{m-1}(u).$$

120. Cor. 2. But if ϕ, and ψ denoted quantities instead of functions, then would $(\phi+\psi)^n = \overset{n+1}{\mathrm{S}}_m \frac{\underset{m-1}{\lfloor n}}{\lfloor m-1}.\phi^{n-m+1}\psi^{m-1}$; and hence we may express the preceding result by the equation

$$(\phi+\psi)_n u = (\phi+\psi)^n.u.$$

This must by no means be considered as an identical equation; for the first side is merely an abridged expression of certain functional operations to be performed, while the second is a compendious method of denoting the expanded result of these operations. In fact these expressions will not generally be equivalent unless $\phi(u)$ and $\psi(u)$ are both *distributive* and *commutative with each other.*

121. Cor. 3. If $\psi(u) = \overset{n}{\mathrm{S}}_m a_{m-1}.\phi_{m-1}(u) + \chi_n(u)$, where $\chi_n(u)=0$ for some value of n and for all succeeding values; then we may put

$$\psi(u) = \overset{\infty}{\mathrm{S}}_m a_{m-1}.\phi_{m-1}(u), \quad (13);$$

and, if $\phi_1(u)$, $\phi_2(u)$,...$\phi_n(u)$ are distributive functions, and commutative both with each other, and with any constant factor, we shall have

$$\psi^n(u) = (\overset{\infty}{\text{S}}_m a_{m-1} . \phi_{m-1})^n u, \quad (120).$$

122. Cor. 4. If, in the same case, $a_{m-1} = a^{m-1}$, and $\phi_{m-1} = \phi^{m-1}$, then

$$\psi^n(u) = (\overset{\infty}{\text{S}}_m a^{m-1} \phi^{m-1})^n u$$

$$= \left(\frac{1}{1 - a\phi}\right)^n u, \quad (12) \text{ and } (13);$$

$$= (1 - a . \phi)^{-n} . u.$$

123. It will be readily seen that the preceding theorems of this chapter will hold not only when ϕ and ψ are symbols denoting functions of which the successive *orders* are deduced by a series of *substitutions*, but also when they denote functions of which the successive orders are deduced by performing a series of *operations* all of which are subject to any given law. An exception, however, must be made with respect to the theorem

$$\phi^{-n} . \phi^n(u) = u:$$

for, if ϕ_x denotes an operation such that

$$\phi_x(u + a) = \phi_x(u),$$

where a is independent of x; then

$$\phi_x^{-1} . \phi_x(u) = u + c_1,$$

$$\phi_x^{-1} . \phi_x^2(u) = \phi_x(u) + c_1,$$

$$\phi_x^{-2} . \phi_x^2(u) = u + \phi_x^{-1}(c_1) + c_2,$$

$$\text{and } \phi_x^{-n} . \phi_x^n(u) = u + \overset{n}{\text{S}}_m \phi_x^{-(n-m)} . c_m;$$

where c_m is some quantity independent of x, and is to be determined by the conditions of the problem.

124. If u is a function of any number of quantities, two of which are x and y; then $\phi_x(u)$ may be used to denote the result of an operation in which x only undergoes a change, and $\phi_y(u)$ a result similarly obtained on the supposition that y is the only variable; while $\phi_x . \phi_y(u)$ will denote $\phi_x . \{\phi_y . (u)\}$.

125. The symbols $(u)_{x=a}$, and $\phi_{x=a}(u)$ denote respectively the values of u, and $\phi_x(u)$, when x is put equal to a; this substitution, in the latter case, not being made till after the operation indicated by ϕ_x has been performed.

126. *Definition.* If in u, any function of x, we substitute $x+h$ for x, u will, in general, assume a new value, which is called the *New State (Etat) of u taken with respect to x*, and is denoted by the symbol $E_x(u)$.

127. *Definition.* The excess of the new value of u above its original value is called the *Difference of u taken with respect to x*, and is denoted by the symbol $D_x(u)$.

128. Cor. 1. $D_x(u)=E_x(u)-u$, $E_x(u)=u+D_x(u)$, and $u=E_x(u)-D_x(u)$.

129. Cor. 2. If $u=\phi(x)$, and $x+h$ is substituted for x, we shall get $E_x(u)=\phi(x+h)$, and $D_x(u)=\phi(x+h)-\phi(x)$; or, since h is the *difference* between the two values of x,

$$E_x(u)=\phi(x+Dx), \text{ and } D_x(u)=\phi(x+Dx)-\phi(x).$$

The case that most commonly occurs being that in which $Dx=1$, we shall denote the difference of u with respect to x, on this supposition, by the symbol $\Delta_x(u)$; that is

$$\Delta_x(u)=\phi(x+1)-\phi(x).$$

130. Cor. 3. $D_x^m . D_x^n(u)=D_x^{m+n}(u)$, and $D_x^0(u)=u$;

$$\text{also } E_x^m E_x^n(u)=E_x^{m+n}(u),$$

$E_x^0(u)=u$, and $E_x^{-n} . E_x^n(u)=u$: (110), (111), and (112).

131. *Theorem.* $E_x^n.\phi(x)=\phi(x+nDx)$.

For, $E_x\phi(x)=\phi(x+Dx)$.

$$E_x^2\phi(x)=E_x.E_x.\phi(x), \quad (110)$$
$$=E_x.\phi(x+Dx)$$
$$=\phi(x+2Dx).$$

&c. = &c.

$$E_x^n\phi(x)=\phi(x+nDx).$$

132. *Theorem.* $E_x^{-1}\phi(x)=\phi(x-Dx)$.

For, put $E_x^{-1}\phi(x)=\phi\psi(x)$; then

$$\phi(x)=E_x.E_x^{-1}\phi(x), \quad (130);$$
$$=E_x.\phi\psi(x)$$
$$=\phi\psi(x+Dx).$$
$$\therefore\ x=\psi(x+Dx), \text{ and}$$
$$x-Dx=\psi(x);$$
$$\therefore\ E_x^{-1}.\phi(x)=\phi(x-Dx).$$

133. Cor. $E_x^{-n}\phi(x)=\phi(x-nDx)$.

134. *Theorem.* $E_x^{\pm 1}(u+v)=E_x^{\pm 1}(u)+E_x^{\pm 1}(v)$.

For, put $u=\phi(x)$, and $v=\psi(x)$;

$$\text{then } E_x^{\pm 1}(u+v)=E_x^{\pm 1}\{\phi(x)+\psi(x)\}$$
$$=\phi(x\pm Dx)+\psi(x\pm Dx)$$
$$=E_x^{\pm 1}\phi(x)+E_x^{\pm 1}\psi(x)$$
$$=E_x^{\pm 1}(u)+E_x^{\pm 1}(v).$$

135. *Theorem.* $E_x^{\pm1}(au)=a.E_x^{\pm1}(u)$; a being independent of x.

For, put $u=\phi(x)$;

$$\begin{aligned} \text{then } E_x^{\pm1}(au) &= E_x^{\pm1}.\{a.\phi(x)\} \\ &= a.\phi(x\pm Dx) \\ &= a.E_x^{\pm1}\phi(x) \\ &= a.E_x^{\pm1}(u). \end{aligned}$$

136. *Theorem.* $D_x(u+v)=D_x(u)+D_x(v)$.

For, put $u=\phi(x)$, and $v=\psi(x)$;

$$\begin{aligned} \text{then } D_x(u+v) &= D_x\{\phi(x)+\psi(x)\} \\ &= \phi(x+Dx)+\psi(x+Dx)-\phi(x)-\psi(x) \\ &= \phi(x+Dx)-\phi(x)+\psi(x+Dx)-\psi(x) \\ &= D_x.\phi(x)+D_x.\psi(x) \\ &= D_x(u)+D_x(v). \end{aligned}$$

137. *Theorem.* $D_x(au)=a.D_x(u)$; a being independent of x.

For, put $u=\phi(x)$;

$$\begin{aligned} \text{then } D_x(au) &= D_x\{a.\phi(x)\} \\ &= a.\phi(x+Dx)-a.\phi(x) \\ &= a\{\phi(x+Dx)-\phi(x)\} \\ &= a.D_x.\phi(x) \\ &= a.D_x(x). \end{aligned}$$

138. *Theorem.* $D_x^{-n}.D_x^n(u)=u+\overset{n}{S}_m c_m.D_x^{-(n-m)}(1)$.

For, let a be any quantity independent of x, and put $u=\phi(u)$;

$$\text{then } D_x(u+a)=\{\phi(x+Dx)+a\}-\{\phi(x)+a\}$$

$$=\phi(x+Dx)-\phi(x)$$

$$=D_x(u).$$

$$\therefore\ D_x^{-n}.D_x^n(u)=u+\overset{n}{S}_m.D_x^{-(n-m)}.(c_m),\quad (123);$$

$$=u+\overset{n}{S}_m c_m.D_x^{-(n-m)}(1),\quad (137).$$

139. Cor. 1. $D_x^{-1}(u+v)=D_x^{-1}\{D_x.D_x^{-1}(u)+D_x.D_x^{-1}(v)\}$, (128);

$$=D_x^{-1}.D_x\{D_x^{-1}(u)+D_x^{-1}(v)\},\quad (136);$$

$$=D_x^{-1}(u)+D_x^{-1}(v),\quad (138).$$

The arbitrary constant must be added after the performance of the operations indicated in the second member of the equation.

140. Cor. 2. $D_x^{-1}(au)=D_x^{-1}\{a.D.D_x^{-1}(u)\}$, (130);

$$=D_x^{-1}.D_x\{a.D_x^{-1}(u)\},\quad (137);$$

$$=a.D_x^{-1}(u),\quad (138).$$

141. *Theorem.* $E_x.D_x(u)=D_x.E_x(u)$.

$$\text{For, } D_x(u)=E_x(u)-u.$$

$$\therefore\ E_xD_x(u)=E_xE_x(u)-E_x(u),\quad (134)$$

$$=D_x.E_x(u).$$

142. Cor. $E_x . D_x^{-1}(u) = D_x^{-1} E_x(u)$,

$$D_x E_x^{-1}(u) = E_x^{-1} . D_x(u),$$

and $E_x^{-1} D_x^{-1}(u) = D_x^{-1} . E_x^{-1}(u)$.

143. It follows from the last nine Articles that the functions denoted by the symbols $E_x^{\pm n}$, $D_x^{\pm n}$, are distributive, and commutative with each other and with any factor independent of x.

144. *Theorem.* $D_x^n(u) = \overset{n+1}{S}_m (-1)^{m-1} . \dfrac{\underline{|n}_{\,m-1}}{\underline{|m-1}} . E_x^{n-m+1}(u)$.

For, $D_x(u) = E_x(u) - u$

$$= (E_x - 1) . u, \quad (115).$$

$$\therefore\ D_x^n(u) = (E_x - 1)_n u, \quad (116);$$

$$= \overset{n+1}{S}_m (-1)^{m-1} . \dfrac{\underline{|n}_{\,m-1}}{\underline{|m-1}} . E_x^{n-m+1}(u), \quad (119)\ \&\ (143).$$

145. *Theorem.* $E_x^n(u) = \overset{n+1}{S}_m \dfrac{\underline{|n}_{\,m-1}}{\underline{|m-1}} . D_x^{m-1}(u)$.

For, $E_x(u) = u + D_x(u)$

$$= (1 + D_x) u, \quad (115);$$

$$\therefore\ E_x^n(u) = (1 + D_x)_n u, \quad (116);$$

$$= \overset{n+1}{S}_m \dfrac{\underline{|n}_{\,m-1}}{\underline{|m-1}} . D_x^{m-1}(u), \quad (119) \text{ and } (143).$$

146. *Theorem.* $D_x^n . x^n = |\underline{n} . h^n$; where $h = Dx$.

For, $D_x \; . x^n = (x+h)^n - x^n = n . x^{n-1} . h +$ inferior positive powers of x.

$$D_x^2 \; . x^n = |\underline{n}_{\,2} . x^{n-2} . h^2 + \text{inferior positive powers of } x.$$

$$D_x^m . x^n = |\underline{n}_{\,m} . x^{n-m} . h^m + \text{inferior positive powers of } x.$$

$$D_x^n \; . x^n = |\underline{n} . h^n.$$

147. When the quantity of which either the difference or new state is to be taken is a power of the independent variable, the index subscript of the letters D, Δ, or E may be omitted; and hence the above theorem will be expressed thus:

$$D^n . x^n = |\underline{n} . h^n.$$

148. *Theorem.* $\Delta^n . x^m = \overset{n+1}{S_r} (-1)^{r-1} . \dfrac{|\underline{n}_{\,r-1}}{|\underline{r-1}} . (x+n-r+1)^m$.

$$\text{For, } D^n . x^m = \overset{n+1}{S_r} (-1)^{r-1} . \frac{|\underline{n}_{\,r-1}}{|\underline{r-1}} . E_x^{n-r+1} . x^m, \quad (144).$$

$$\therefore \; \Delta^n . x^m = \overset{n+1}{S_r} (-1)^{r-1} . \frac{|\underline{n}_{\,r-1}}{|\underline{r-1}} . (x+n-r+1)^m, \quad (131).$$

149. Cor. 1. $\overset{n+1}{S_r} (-1)^{r-1} . \dfrac{|\underline{n}_{\,r-1}}{|\underline{r-1}} . (x+n-r+1)^n = \Delta^n . x^n$, (148);

$$= |\underline{n}, \quad (147).$$

150. Cor. 2. $\Delta^n . 0^m = \overset{n+1}{S_r} (-1)^{r-1} . \dfrac{|\underline{n}_{\,r-1}}{|\underline{r-1}} . (n-r+1)^m$.

The numbers comprehended under the symbol $\Delta^n . 0^m$ are of great utility in the expansion of various functions. The following values may be readily calculated by the theorem of this article:

	Δ^1	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6	Δ^7	Δ^8	Δ^9	Δ^{10}
0^1	1									
0^2	1	2								
0^3	1	6	6							
0^4	1	14	36	24						
0^5	1	30	150	240	120					
0^6	1	62	540	1560	1800	720				
0^7	1	126	1806	8400	16800	15120	5040			
0^8	1	254	5796	40824	126000	191520	141120	40320		
0^9	1	510	18150	186480	834120	1905120	2328480	1451520	362880	
0^{10}	1	1022	55980	818520	5103000	16435440	29635200	30240000	16329600	3628800

151. *Theorem.* $D_x^n . a^x = a^x . (a^h - 1)^n$;

For, $D_x . a^x = a^{x+h} - a^x$

$$= a^x (a^h - 1)$$

$$D_x^2 . a^x = (a^h - 1) . D_x . a^x, \quad (137);$$

$$= (a^h - 1)^2 . a^x;$$

and, similarly, $D_x^n . a^x = a^x . (a^h - 1)^n$.

152. *Theorem.* $D_x . \underline{|a+bx}_{\,n,\,bh} = bnh . \underline{|a+b.(x+h)}_{\,n-1,\,bh}$.

For, $D_x . \underline{|a+bx}_{\,n,\,bh} = \underline{|a+b.(x+h)}_{\,n,\,bh} - \underline{|a+bx}_{\,n,\,bh}$

$$= \{a + b(x+nh) - (a+bx)\} \underline{|a+b.(x+h)}_{\,n-1,\,bh}, \quad (41);$$

$$= bnh . \underline{|a+b.(x+h)}_{\,n-1,\,bh}.$$

153. $\Delta_x . \underline{|x}_{\,n} = n . \underline{|x}_{\,n-1}$.

For, $\Delta_x . \underline{|x}_{\,n} = \underline{|x+1}_{\,n} - \underline{|x}_{\,n}$

$$= (x+1) \underline{|x}_{\,n-1} - \underline{|x}_{\,n-1} . (x-n+1), \quad (41);$$

$$= n . \underline{|x}_{\,n-1}.$$

154. *Theorem.* $D_x \dfrac{1}{\lfloor\underline{\,a+bx}_{\,n,\,bh}} = \dfrac{-bnh}{\lfloor\underline{\,a+bx}_{\,n+1,\,bh}}$

For, $$D_x.\frac{1}{\lfloor\underline{\,a+bx}_{\,n,\,bh}} = \frac{1}{\lfloor\underline{\,a+b.(x+h)}_{\,n,\,bh}} - \frac{1}{\lfloor\underline{\,a+bx}_{\,n,\,bh}}$$

$$= \frac{a+bx}{\lfloor\underline{\,a+bx}_{\,n+1,\,bh}} - \frac{a+b.(x+nh)}{\lfloor\underline{\,a+bx}_{\,n+1,\,bh}}$$

$$= -\frac{bnh}{\lfloor\underline{\,a+bx}_{\,n+1,\,bh}}.$$

155. $\Delta_x.\dfrac{1}{\lfloor\underline{\,x}_{\,n}} = \dfrac{-n}{\lfloor\underline{\,x+1}_{\,n+1}}.$

For, $$\Delta_x.\frac{1}{\lfloor\underline{\,x}_{\,n}} = \frac{1}{\lfloor\underline{\,x+1}_{\,n}} - \frac{1}{\lfloor\underline{\,x}_{\,n}}$$

$$= \frac{x-n+1}{\lfloor\underline{\,x+1}_{\,n+1}} - \frac{x+1}{\lfloor\underline{\,x+1}_{\,n+1}}$$

$$= \frac{-n}{\lfloor\underline{\,x+1}_{\,n+1}}.$$

156. *Theorem.*

$$D_x.\overset{n}{\mathbf{P}}_r\phi\{x+(r-1).h\} = \{\phi(x+nh)-\phi(x)\}.\overset{n-1}{\mathbf{P}}_r\phi(x+rh)$$

For, $$D_x.\overset{n}{\mathbf{P}}_r\phi\{x+(r-1).h\} = \overset{n}{\mathbf{P}}_r\phi(x+rh) - \overset{n}{\mathbf{P}}_r\phi\{x+(r-1).h\}$$

$$= \{\phi(x+nh)-\phi(x)\}.\overset{n-1}{\mathbf{P}}_r\phi(x+rh), \ (32).$$

157. *Theorem.* $D_x.[\overset{n}{\mathrm{P}}_r\phi\{x+(r-1).h\}]^{-1}$

$$=-\{\phi(x+nh)-\phi(x)\}[\overset{n+1}{\mathrm{P}}_r\phi\{x+(r-1).h\}]^{-1}.$$

For, $D_x.[\overset{n}{\mathrm{P}}_r\phi\{x+(r-1).h\}]^{-1}$

$$=\{\overset{n}{\mathrm{P}}_r\phi(x+rh)\}^{-1}-[\overset{n}{\mathrm{P}}_r\phi\{x+(r-1).h\}]^{-1}$$

$$=-\{\phi(x+nh)-\phi(x)\}.[\overset{n+1}{\mathrm{P}}_r\phi\{x+(r-1).h\}]^{-1}.$$

158. *Theorem.* $D_x.\overset{m}{\mathrm{P}}_r(u_r)=\overset{m}{\mathrm{S}}_r\overset{\overline{m-r,\,r}}{\mathrm{C}}_{s,t}.(u_s.D_x.u_t).$

For, $E_x.\overset{m}{\mathrm{P}}_r(u_r)=\overset{m}{\mathrm{P}}_r(u_r+D_xu_r)$

$$=\overset{m+1}{\mathrm{S}}_r\overset{\overline{m-r+1,\,r-1}}{\mathrm{C}}_{s,t}(u_s.D_xu_t),\quad(67);$$

$$=\overset{m}{\mathrm{P}}_r(u_r)+\overset{m}{\mathrm{S}}_r\overset{\overline{m-r,\,r}}{\mathrm{C}}_{s,t}(u_s.D_xu_t),\quad(9).$$

$$\therefore\ D_x.\overset{m}{\mathrm{P}}_r(u_r)=\overset{m}{\mathrm{S}}_r\overset{\overline{m-r,\,r}}{\mathrm{C}}_{s,t}(u_s.D_xu_t),\quad(128).$$

159. *Theorem.* $D_x(uv)=u.D_xv+D_x(u).E_xv.$

For, $E_x(uv)=(u+D_xu)(v+D_xv)$

$$=uv+D_x(u).v+u.D_xv+D_x(u).D_xv$$

$$=uv+u.D_xv+D_x(u).E_xv;$$

$$\therefore\ D_x(uv)=u.D_xv+D_x(u).E_xv.$$

160. *Theorem.* $D_x\left(\frac{u}{v}\right) = \frac{D_x(u).v - u.D_x v}{v.E_x v}$.

For, $D_x\left(\frac{u}{v}\right) = \frac{u+D_x u}{v+D_x v} - \frac{u}{v}$

$$= \frac{uv + D_x(u).v - uv - u.D_x v}{v.E_x v}$$

$$= \frac{D_x(u).v - u.D_x v}{v.E_x v}.$$

161. *Theorem.* If $\phi(u)$, $\phi_1(u)$, $\psi(v)$, and $\psi_1(v)$ are distributive functions, commutative with each other and a constant factor, and if $\phi(u).\psi(v)+\phi_1(u).\psi_1(v)$ is denoted by $(\phi\psi+\phi_1\psi_1)uv$, then shall

$$(\phi\psi+\phi_1\psi_1)_n uv = \overset{n+1}{\mathrm{S}}_m \frac{\lfloor n_{m-1}}{\lfloor m-1} . \phi^{n-m+1} . \phi_1^{m-1}(u) . \psi^{n-m+1}\psi_1^{m-1}(v).$$

For, by proceeding precisely as in Art. 118, we shall find that

$$(\phi\psi+\phi_1\psi_1)_3 uv = \overset{4}{\mathrm{S}}_m \frac{\lfloor 3_{m-1}}{\lfloor m-1} . \phi^{4-m}\phi_1^{m-1}(u) . \psi^{4-m}\psi_1^{m-1}(v).$$

Suppose, therefore,

$$(\phi\psi+\phi_1\psi_1)_n uv = \overset{n+1}{\mathrm{S}}_m \frac{\lfloor n_{m-1}}{\lfloor m-1} . \phi^{n-m+1}\phi_1^{m-1}(u) . \psi^{n-m+1}\psi_1^{m-1}(v);$$

then

$$(\phi\psi+\phi_1\psi_1)_{n+1} uv = (\phi\psi+\phi_1\psi_1)\overset{n+1}{\mathrm{S}}_m \frac{\lfloor n_{m-1}}{\lfloor m-1} . \phi^{n-m+1}\phi_1^{m-1}(u) . \psi^{n-m+1} . \psi_1^{m-1}(v)$$

$$= \overset{n+1}{S_m} \frac{\underline{|n}_{\;m-1}}{\underline{|m-1}} \cdot \{\phi\cdot[\phi^{n-m+1}\phi_1^{m-1}(u)]\cdot\psi\cdot[\psi^{n-m+1}\psi_1^{m-1}(v)]$$

$$+\phi_1[\phi^{n-m+1}\cdot\phi_1^{m-1}(u)]\cdot\psi_1[\psi^{n-m+1}\cdot\psi_1^{m-1}(v)]\}, \quad (113);$$

$$= \overset{n+1}{S_m} \frac{\underline{|n}_{\;m-1}}{\underline{|m-1}} \{\phi^{n-m+2}\cdot\phi_1^{m-1}(u)\cdot\psi^{n-m+2}\cdot\psi_1^{m-1}(v)$$

$$+\phi^{n-m+1}\cdot\phi_1^{m}(u)\cdot\psi^{n-m+1}\cdot\psi_1^{m}(v)\}, \quad (114);$$

$$= \phi^{n+1}(u)\cdot\psi^{n+1}(v) + \overset{n}{S_m} \frac{\underline{|n}_{\;m}}{\underline{|m}} \cdot \phi^{n-m+1}\phi_1^{m}(u)\cdot\psi^{n-m+1}\cdot\psi_1^{m}(v)$$

$$+ \overset{n}{S_m} \frac{\underline{|n}_{\;m-1}}{\underline{|m-1}} \cdot \phi^{n-m+1}\cdot\phi_1^{m}(u)\cdot\psi^{n-m+1}\cdot\psi_1^{m}(v) + \phi_1^{n+1}(u)\cdot\psi_1^{n+1}(v),$$

(9);

$$= \phi^{n+1}(u)\cdot\psi^{n+1}(v) + \overset{n}{S_m} \frac{\underline{|n+1}_{\;m}}{\underline{|m}} \cdot \phi^{n-m+1}\cdot\phi_1^{m}(u)\cdot\psi^{n-m+1}\cdot\psi_1^{m}(v)$$

$$+\phi_1^{n+1}(u)\cdot\psi_1^{n+1}(v), \quad (5) \text{ and } (47);$$

$$= \overset{n+2}{S_m} \frac{\underline{|n+1}_{\;m-1}}{\underline{|m-1}} \cdot \phi^{n-m+2}\cdot\phi_1^{m-1}(u)\cdot\psi^{n-m+2}\cdot\psi_1^{m-1}(v), \quad (9).$$

If, therefore, the law were true for n it would be true for $n+1$; but it *is* true for 3, and therefore it *is* true for n.

162. Cor. 1. The equation just found may be written thus,

$$(\phi\psi+\phi_1\psi_1)_n uv = \{\overset{n+1}{S_m} \frac{\underline{|n}_{\;m-1}}{\underline{|m-1}} \cdot \phi^{n-m+1}\cdot\psi^{n-m+1}\cdot\phi_1^{m-1}\psi_1^{m-1}\} uv,$$

or $(\phi\psi+\phi_1\psi_1)_n uv = (\phi\psi+\phi_1\psi_1)^n uv$. See Art. 120.

It must be carefully observed that, in the expansion indicated by this last expression, the symbols ϕ and ϕ_1 are to be prefixed to u, while ψ and ψ_1 are to be prefixed to v.

163. Cor. 2. If $\phi_1, \phi_2, \ldots \phi_r$, and $\psi_1, \psi_2, \ldots \psi_r$, denote distributive functions, and commutative both with each other, and with a constant factor; then shall

$$(\overset{m}{S}_r \phi_r \psi_r)_n uv = (\overset{m}{S}_r \phi_r \psi_r)^n uv.$$

164. Cor. 3. With the same limitations,

$$(\overset{m}{S}_r {}^1\phi_r {}^2\phi_r \ldots {}^s\phi_r)_n u_1 . u_2 \ldots u_s = (\overset{m}{S}_r {}^1\phi_r {}^2\phi_r \ldots {}^s\phi_r)^n . u_1 . u_2 \ldots u_s;$$

where the symbols ${}^1\phi_r, {}^2\phi_r, \ldots {}^s\phi_r$, are to be prefixed to $u_1, u_2, \ldots u_s$, respectively.

165. *Theorem.*

$$D_x^n . (uv) = \overset{n+1}{S}_m \frac{\underline{|n}_{m-1}}{\underline{|m-1}} . D_x^{m-1}(u) . D_x^{n-m+1} . E_x^{m-1}(v).$$

For, $D_x . (uv) = u . D_x v + D_x(u) . E_x(v)$, (159);

$= (D_x + {}^1D_x . E_x) uv$, (161); where ${}^1D_x$ only belongs to u.

$\therefore\ D_x^n (uv) = (D_x + {}^1D_x E_x)_n . uv$

$$= \overset{n+1}{S}_m \frac{\underline{|n}_{m-1}}{\underline{|m-1}} . D_x^{m-1}(u) . D_x^{n-m+1} E_x^{m-1}(v), \quad (161).$$

166. *Theorem.*

$$D_x^n (uv) = \overset{n+1}{S}_m (-1)^{m-1} . \frac{\underline{|n}_{m-1}}{\underline{|m-1}} . E_x^{n-m+1}(u) . E_x^{n-m+1}(v).$$

For, $D_x(uv) = E_x(u) . E_x(v) - uv$

$= (E_x . E_x - 1) uv$, (161);

$\therefore\ D_x^n (uv) = (E_x . E_x - 1)_n . uv$

$$= \overset{n+1}{S}_m (-1)^{m-1} . \frac{\underline{|n}_{m-1}}{\underline{|m-1}} . E_x^{n-m+1}(u) . E_x^{n-m+1}(v), \quad (161).$$

167. Cor. 1. $D_x^n(u_1 u_2) = \{(1+{}^1D_x)(1+{}^2D_x)-1\}^n u_1 u_2$; and

$$D_x^n . \overset{m}{\mathrm{P}}_r(u_r) = \{(1+{}^1D_x)(1+{}^2D_x)\ldots(1+{}^mD_x)-1\}^n u_1 u_2 \ldots u_m$$

$$= \{\overset{m}{\mathrm{S}}_r \overset{r,m}{\mathrm{C}}_s({}^sD_x)\}^n \, u_1 u_2 \ldots u_m, \quad (68) \text{ and } (9).$$

168. *Theorem.* $E_x . E_y(u) = E_y . E_x(u)$.

For, put $u = \phi(x, y)$; then

$$E_y(u) = \phi(x, y+Dy)$$

$$E_x . E_y(u) = \phi(x+Dx, y+Dy)$$

$$= E_y \phi(x+Dx, y)$$

$$= E_y . E_x . \phi(x, y)$$

$$= E_y . E_x(u).$$

Hence, we may express either $E_x . E_y(u)$ or $E_y . E_x(u)$, by $E_{x,y}(u)$; while $D_{x,y}(u)$ will denote $E_{x,y}(u) - u$.

169. *Theorem.* $E_x E_y(u) = u + D_x(u) + D_y(u) + D_x D_y(u)$.

For, $E_y(u) = u + D_y(u)$, (128).

$$E_x . E_y(u) = E_x(u) + E_x . D_y(u), \quad (134);$$

$$= u + D_x(u) + D_y(u) + D_x D_y(u), \quad (128).$$

170. Cor. 1. Since $u + D_x(u) + D_y(u) + D_x D_y(u) = E_x E_y(u)$

$$= E_y E_x(u), \quad (168);$$

$$= u + D_y(u) + D_x(u) + D_y D_x(u).$$

$$\therefore \; D_x D_y(u) = D_y D_x(u).$$

171. Cor. 2. $E_{x,y}(u) = (1 + D_x + D_y + D_x D_y)u$, (115);

$$\therefore \; E_{x,y}^n(u) = (1 + D_x + D_y + D_x D_y)_n u$$

$$= (1 + D_x + D_y + D_x D_y)^n . u, \quad (121) \text{ and } (143).$$

172. Cor. 3. $D_{x,y}(u)=(D_x+D_y+D_xD_y)u,$ (168)

$$\therefore\ D^{n}_{x,y}(u)=(D_x+D_y+D_xD_y)_n u$$

$$=(D_x+D_y+D_xD_y)^n.u, \quad (121) \text{ and } (143);$$

$$=\{(1+D_x)(1+D_y)-1\}^n u.$$

173. *Theorem.* $D^{n}_{x,y}(u)=\overset{n+1}{\mathrm{S}}_m(-1)^{m-1}.\frac{\underline{|n}_{\,m-1}}{\underline{|m-1}}.E^{n-m+1}_{x,y}(u).$

For, $D_{x,y}(u)=E_{x,y}(u)-u,$ (168);

$$=(E_{x,y}-1)u.$$

$$\therefore\ D^{n}_{x,y}(u)=(E_{x,y}-1)_n u$$

$$=(E_{x,y}-1)^n u, \quad (121) \text{ and } (143);$$

$$=\overset{n+1}{\mathrm{S}}_m(-1)^{m-1}.\frac{\underline{|n}_{\,m-1}}{\underline{|m-1}}.E^{n-m+1}_{x,y}(u).$$

174. *Theorem.* $E^{n}_{x,y}(u)=\overset{n+1}{\mathrm{S}}_m\frac{\underline{|n}_{\,m-1}}{\underline{|m-1}}.D^{m-1}_{x,y}(u).$

For, $E_{x,y}(u)=u+D_{x,y}(u),$ (168);

$$=(1+D_{x,y})u;$$

$$\therefore\ E^{n}_{x,y}(u)=(1+D_{x,y})_n u$$

$$=(1+D_{x,y})^n.u, \quad (121) \text{ and } (143);$$

$$=\overset{n+1}{\mathrm{S}}_m\frac{\underline{|n}_{\,m-1}}{\underline{|m-1}}.D^{m-1}_{x,y}(u).$$

175. *Theorem.*

$$E_x^m \cdot E_y^n(u) = \overset{\infty}{\mathrm{S}}_r \overset{r}{\mathrm{S}}_s \frac{|\underline{m}_{r-s} \cdot |\underline{n}_{s-1}}{|\underline{r-s}\ |\underline{s-1}} \cdot D_x^{r-s} \cdot D_y^{s-1}(u).$$

For, $E_x^m(u) = \overset{\infty}{\mathrm{S}}_r \dfrac{|\underline{m}_{r-1}}{|\underline{r-1}} \cdot D_x^{r-1}(u)$, (145) and (13).

$$E_x^m \cdot E_y^n(u) = \overset{\infty}{\mathrm{S}}_r \frac{|\underline{m}_{r-1}}{|\underline{r-1}} \cdot D_x^{r-1} \cdot E_y^n(u), \quad (168), (134), (135) \ \& \ (141);$$

$$= \overset{\infty}{\mathrm{S}}_r \frac{|\underline{m}_{r-1}}{|\underline{r-1}} \cdot D_x^{r-1} \cdot \overset{\infty}{\mathrm{S}}_s \frac{|\underline{n}_{s-1}}{|\underline{s-1}} \cdot D_y^{s-1}(u), \quad (145) \text{ and } (13);$$

$$= \overset{\infty}{\mathrm{S}}_r \frac{|\underline{m}_{r-1}}{|\underline{r-1}} \cdot \overset{\infty}{\mathrm{S}}_s \frac{|\underline{n}_{s-1}}{|\underline{s-1}} \cdot D_x^{r-1} \cdot D_y^{s-1}(u), \quad (136) \text{ and } (137);$$

$$= \overset{\infty}{\mathrm{S}}_r \overset{r}{\mathrm{S}}_s \frac{|\underline{m}_{r-s} \cdot |\underline{n}_{s-1}}{|\underline{r-s}\ |\underline{s-1}} \cdot D_x^{r-s} \cdot D_y^{s-1}(u), \quad (6) \text{ and } (18).$$

176. *Theorem.*

$$D_x^{-1}(uv) = u \cdot D_x^{-1} u - D_x^{-1} \{ D_x(u) \cdot D_x^{-1} E_x v \}.$$

For, $D_x(u \cdot D_x^{-1} v) = uv + D(u) \cdot D_x^{-1} \cdot E_x(v)$, (159) and (142).

$$\therefore \ u \cdot D_x^{-1} v = D_x^{-1}(uv) + D_x^{-1} \{ D_x(u) \cdot D_x^{-1} E_x(v) \}, \quad (139);$$

and $D_x^{-1}(uv) = u \cdot D_x^{-1} v - D_x^{-1} \{ D_x(u) \cdot D_x^{-1} \cdot E_x v \}$.

177. *Theorem.*

$$D_x^{-1}(uv)=\overset{r}{\mathrm{S}}_m(-1)^{m-1}D_x^{m-1}(u).D_x^{-m}E_x^{m-1}v$$
$$+(-1)^r.D_x^{-1}\{D_x^r(u).D_x^{-r}E_x^rv\}.$$

For, $D_x^{-1}\{D_x^{m-1}(u).D_x^{-(m-1)}E_x^{m-1}v\}=D_x^{m-1}(u).D_x^{-m}E_x^{m-1}(v)$
$$-D_x^{-1}\{D_x^m(u).D_x^{-m}.E_x^mv\},\quad (176);$$

$$\therefore\ (-1)^{m-1}.D_x^{-1}\{D_x^{m-1}(u).D_x^{-(m-1)}.E_x^{m-1}v\}$$
$$=(-1)^{m-1}.D_x^{m-1}(u).D_x^{-m}.E_x^{m-1}v$$
$$+(-1)^m.D_x^{-1}\{D_x^m(u).D_x^{-m}.E_x^mv\}$$

$$\therefore\ \overset{r}{\mathrm{S}}_m(-1)^{m-1}.D_x^{-1}\{D_x^{m-1}(u).D_x^{-(m-1)}.E_x^{m-1}v\}$$
$$=\overset{r}{\mathrm{S}}_m(-1)^{m-1}.D_x^{m-1}(u).D_x^{-m}.E_x^{m-1}v$$
$$+\overset{r}{\mathrm{S}}_m(-1)^mD_x^{-1}\{D_x^m(u).D_x^{-m}E_x^mv\},\quad (5);$$

$$D_x^{-1}(uv)+\overset{r-1}{\mathrm{S}}_m(-1)^m.D_x^{-1}\{D_x^m(u).D_x^{-m}E_x^mv\}$$
$$=\overset{r}{\mathrm{S}}_m(-1)^{m-1}.D_x^{m-1}(u).D_x^{-m}E_x^{m-1}v$$
$$+\overset{r-1}{\mathrm{S}}_m(-1)^m.D_x^{-1}\{D_x^m(u).D_x^{-m}E_x^mv\}$$
$$+(-1)^r.D_x^{-1}\{D_x^r(u).D_x^{-r}.E_x^rv\},\quad (9);$$

$$\therefore\ D_x^{-1}(uv)=\overset{r}{\mathrm{S}}_m(-1)^{m-1}.D_x^{m-1}(u).D_x^{-m}E_x^{m-1}v$$
$$+(-1)^r.D_x^{-1}\{D_x^r(u).D_x^{-r}E_x^rv\}.$$

178. Cor. If, for some value of r, $D_x^ru=0$, then

$$D_x^{-1}(uv)=\overset{\infty}{\mathrm{S}}_m(-1)^{m-1}.D_x^{m-1}u.D_x^{-m}E_x^{m-1}v,\quad (13);$$
$$=\{D_x^{-1}(1+{}^1D_x.D_x^{-1}E_x)^{-1}\}uv,$$

where ${}^1D_x$ only belongs to u, (12);

$\therefore\ D_x^{-n}(uv)=\{D_x^{-n}.(1+{}^1D_x.D_x^{-1}.E_x)^{-n}\}\,uv,\quad (163);$

$$=\{D_x^{-n}.\overset{\infty}{S}_m(-1)^{m-1}.\frac{\underline{|n}_{m-1,1}}{|\underline{m-1}}.{}^1D_x^{m-1}.D_x^{-(m-1)}.E_x^{m-1}\}\,uv,$$

(92) and (39);

$$=\overset{\infty}{S}_m(-1)^{m-1}.\frac{\underline{|n}_{m-1,1}}{|\underline{m-1}}.D_x^{m-1}(u).D_x^{-(n+m-1)}.E_x^{m-1}v.$$

179. *Theorem.* $D_x^{-1}(a^x)=\dfrac{a^x}{a^h-1}+\text{const.}$

For, $a^x.(a^h-1)=D_x.a^x$, (151),

$$\therefore\ a^x=D_x.\frac{a^x}{a^h-1},\quad (137);$$

$$\text{and } D_x^{-1}.a^x=\frac{a^x}{a^h-1}+\text{const.}\quad (138).$$

180. Cor. $D_x^{-n}.a^x=a^x(a^h-1)^{-n}+\overset{n}{S}_m D_x^{-(n-m)}.c_m$, (138)

181. *Theorem.*

$$D_x^{-1}(a_x.u)=a^x.\overset{r}{S}_m(-1)^{m-1}.a^{(m-1)h}.(a^h-1)^{-m}.D_x^{m-1}u$$
$$+(-1)^r.D_x^{-1}\{D_x^r(u).a^{x+rh}.(a^h-1)^{-r}\}.$$

For, $D_x^{-1}.(a^x.u)=\overset{r}{S}_m(-1)^{m-1}.D_x^{m-1}(u).D_x^{-m}.E_x^{m-1}.a^x$
$+(-1)^r.D_x^{-1}\{D_x^r(u).D_x^{-r}.E_x^r.a^x\}$, (177);

$$=\overset{r}{S}_m(-1)^{m-1}.\frac{a^{x+(m-1)h}}{(a^h-1)^m}.D_x^{m-1}u+(-1)^r.D_x^{-1}\left\{D_x^r(u).\frac{a^{x+rh}}{(a^h-1)^r}\right\},$$

(180);

$$=a^x.\overset{r}{S}_m(-1)^{m-1}.a^{(m-1)h}.(a^h-1)^{-m}.D_x^{m-1}u$$
$$+(-1)^r.D_x^{-1}\{D_x^r(u).a^{x+rh}.(a^h-1)^{-r}\},\quad (6).$$

182. $D_x^{-1}.\underline{|a+bx}_{\,n,bh} = \dfrac{1}{bh(n+1)}.\underline{|a+b(x-h)}_{\,n+1,bh}.$

For, $bh(n+1).\underline{|a+bx}_{\,n,bh} = D_x.\underline{|a+b.(x-h)}_{\,n+1,bh}$, (152):

$$\therefore\ \underline{|a+bx}_{\,n,bh} = D_x.\left\{\frac{1}{bh(n+1)}.\underline{|a+b(x-h)}_{\,n+1,bh}\right\},\ (137);$$

and $D_x^{-1}\underline{|a+bx}_{\,n,bh} = \dfrac{1}{bh(n+1)}.\underline{|a+b.(x-h)}_{\,n+1,bh}.$

183. $\Delta_x^{-1}.\underline{|x}_{\,n} = \dfrac{\underline{|x}_{\,n+1}}{n+1}.$

For, $(n+1).\underline{|x}_{\,n} = \Delta_x.\underline{|x}_{\,n+1}$, (153).

$$\therefore\ \underline{|x}_{\,n} = \Delta_x.\frac{\underline{|x}_{\,n+1}}{n+1},\quad (137);\ \text{and}$$

$$\Delta_x^{-1}.\underline{|x}_{\,n} = \frac{\underline{|x}_{\,n+1}}{n+1}.$$

184. $D_x^{-1}.\dfrac{1}{\underline{|a+bx}_{\,n,bh}} = -\dfrac{1}{bh(n-1).\underline{|a+bx}_{\,n-1,bh}}.$

For, $\dfrac{-bh(n-1)}{\underline{|a+bx}_{\,n,bh}} = D_x.\dfrac{1}{\underline{|a+bx}_{\,n-1,bh}}$, (154).

$$\therefore\ \frac{1}{\underline{|a+bx}_{\,n,bh}} = D_x.\left\{-\frac{1}{bh(n-1).\underline{|a+bx}_{\,n-1,bh}}\right\},\quad (137);$$

and $D_x^{-1}.\dfrac{1}{\underline{|a+bx}_{\,n,bh}} = -\dfrac{1}{bh(n-1).\underline{|a+bx}_{\,n-1,bh}}.$

185. $$\Delta_x^{-1}.\frac{1}{\underline{|x}_{\,n}}=\frac{-1}{(n-1)\underline{|x-1}_{\,n-1}}.$$

For, $$\frac{-(n-1)}{\underline{|x}_{\,n}}=\Delta_x.\frac{1}{\underline{|x-1}_{\,n-1}},\quad (155).$$

$$\therefore\ \frac{1}{\underline{|x}_{\,n}}=\Delta_x\left\{-\frac{1}{(n-1)\underline{|x-1}_{\,n-1}}\right\},\quad (137);$$

and $$\Delta_x^{-1}.\frac{1}{\underline{|x}_{\,n}}=\frac{-1}{(n-1)\underline{|x-1}_{\,n-1}}.$$

186. $$\Delta^{-1}.x^n=\overset{x+1}{\mathrm{S}}_m\,\Delta^{m-1}0^n.\frac{\underline{|x}_{\,m}}{\underline{|m}};$$ x being any positive integer.

For, $$(y+x)^n=\overset{x+1}{\mathrm{S}}_m\frac{\underline{|x}_{\,m-1}}{\underline{|m-1}}.\Delta.^{m-1}y^n,\quad (145).$$

$$\therefore\ x^n=\overset{x+1}{\mathrm{S}}_m\frac{\underline{|x}_{\,m-1}}{\underline{|m-1}}.\Delta^{m-1}.0^n;$$

and $$\Delta^{-1}.x^n=\overset{x+1}{\mathrm{S}}_m\frac{\Delta^{m-1}.0^n}{\underline{|m-1}}.\Delta_x^{-1}.\underline{|x}_{\,m-1},\quad (139),\text{ and }(140);$$

$$=\overset{x+1}{\mathrm{S}}_m\,\Delta^{m-1}.0^n.\frac{\underline{|x}_{\,m}}{\underline{|m}},\quad (183).$$

187. *Theorem.* $\overset{n}{\mathrm{S}}_m a_m=(\Delta_n^{-1}-\Delta_{n=0}^{-1})\,a_{n+1}.$

For, $$\Delta_n.\overset{n}{\mathrm{S}}_m a_m=\overset{n+1}{\mathrm{S}}_m a_m-\overset{n}{\mathrm{S}}_m a_m$$

$$=a_{n+1}.$$

$$\therefore \overset{n}{\mathrm{S}}_m a_m = \Delta_n{}^{-1} . a_{n+1} + \text{constant}.$$

$$\text{and} \quad \overset{0}{\mathrm{S}}_m a_m = \Delta_{n=0}^{-1} a_{n+1} + \text{constant}.$$

$$\therefore \overset{n}{\mathrm{S}}_m a_m = \Delta_n{}^{-1} a_{n+1} - \Delta_{n=0}^{-1} a_{n+1};$$

or, as it may be conveniently expressed,

$$\overset{n}{\mathrm{S}}_m a_m = (\Delta_n{}^{-1} - \Delta_{n=0}^{-1}) a_{n+1}.$$

CHAPTER VI.

ON DIFFERENTIATION IN GENERAL.

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188. *Definition.* The quantity $\{(Dx)^{-n} D_x^n u\}_{Dx=0}$ is called *the* n^th^ *differential coefficient of* u, *taken with respect to* x, and is denoted by the symbol $d_x^n u$.

189. *Theorem.* $d_x^m d_x^n u = d_x^{m+n} u$.

For, put $Dx = h$; then

$$d_x^n u = (h^{-n} . D_x^n u)_{h=0},$$

and $d_x^m . d_x^n u = \{h^{-m} . D_x^m (h^{-n} . D_x^n u)_{h=0}\}_{h=0}$

$$= \{h^{-m} . D_x^m . (h^{-n} . D_x^n u)\}_{h=0}$$

$$= \{h^{-(m+n)} . D_x^{m+n} u\}_{h=0}, \quad (137) \text{ and } (130);$$

$$= d_x^{m+n} u.$$

190. Cor. $d_x^0 u = u$.

191. *Theorem.* $d_x . (u+v) = d_x u + d_x v$.

For, $d_x . (u+v) = \{h^{-1} . D_x (u+v)\}_{h=0}$

$$= (h^{-1} . D_x u + h^{-1} . D_x v)_{h=0}, \quad (136);$$

$$= (h^{-1} . D_x u)_{h=0} + (h^{-1} . D_x v)_{h=0}$$

$$= d_x u + d_x v.$$

192. *Theorem.* $d_x(u+a)=d_x u$; where a is independent of x.

For, $d_x.(u+a)=\{h^{-1}.D_x(u+a)\}_{h=0}$

$$=(h^{-1}.D_x u)_{h=0}, \quad (138);$$

$$=d_x u.$$

193. Cor. 1. $d_x.a=0$, (191).

194. Cor. 2. $d_x^{-n}.d_x^n u=u+\overset{n}{S}_m d_x^{-(n-m)}.c_m$, where c_m is independent of x, (123).

195. The symbol $\int_x^n u$ is equivalent to $d_x^{-n}u$, and is read *the* n[th] *integral of* u, *taken with respect to* x. Hence the equation just found may be written thus:

$$\int_x^n d_x^n u=u+\overset{n}{S}_m \int_x^{n-m} c_m.$$

196. *Theorem.* $d_x(au)=a\,d_x u$.

For, $d_x.(au)=\{h^{-1}.D_x(au)\}_{h=0}$

$$=\{h^{-1}a.D_x u\}_{h=0}, \quad (137);$$

$$=a\{h^{-1}.D_x u\}_{h=0}$$

$$=a\,d_x u.$$

197. *Theorem.* $d_x.x=1$.

For, $d_x.x=(h^{-1}.Dx)_{h=0}$

$$=(1)_{h=0}$$

$$=1.$$

198. *Theorem.* $\phi(x+h)=\overset{\infty}{\mathrm{S}}_m \frac{h^{m-1}}{\lfloor m-1} . d_x^{m-1} . \phi(x)$.

For, put $Dx=k$, and $h=nk$; then

$$\phi(x+nk)=\overset{n+1}{\mathrm{S}}_m \frac{\lfloor n_{m-1}}{\lfloor m-1} . D_x^{m-1} . \phi(x), \quad (145).$$

$$\therefore \ \phi(x+h)=\overset{hk^{-1}+1}{\mathrm{S}}_m \frac{\lfloor hk^{-1}_{m-1}}{\lfloor m-1} . D_x^{m-1} . \phi(x),$$

whatever the value of k may be;

$$=\left\{ \overset{hk^{-1}+1}{\mathrm{S}}_m \frac{\lfloor hk^{-1}_{m-1}}{\lfloor m-1} . D_x^{m-1} . \phi(x) \right\}_{k=0}$$

$$=\overset{\infty}{\mathrm{S}}_m \frac{h^{m-1}}{\lfloor m-1} . \left\{ k^{-(m-1)} . D_x^{m-1} . \phi(x) \right\}_{k=0}$$

$$=\overset{\infty}{\mathrm{S}}_m \frac{h^{m-1}}{\lfloor m-1} . d_x^{m-1} . \phi(x).$$

199. Cor. 1. $E_x u=\overset{\infty}{\mathrm{S}}_m \frac{h^{m-1}}{\lfloor m-1} . d_x^{m-1} u$,

and $D_x u=\overset{\infty}{\mathrm{S}}_m \frac{h^m}{\lfloor m} . d_x^m u$.

200. Cor. 2. $D_x u=(\epsilon^{hd_x}-1)\, u$, (106) and (115).

$\therefore \ D_x^n u=(\epsilon^{hd_x}-1)^n u$, (121).

201. Cor. 3. $d_x^m.\phi(x)$ is the coefficient of $\frac{h^m}{\lfloor m}$ in the expansion of $\phi(x+h)$.

202. *Theorem.* If u is a function of x, then

$$d_x.\phi(u)=d_u.\phi(u).d_x u.$$

For, $D_x.\phi(u)=\overset{\infty}{\text{S}}_m \frac{d_u^m.\phi(u)}{\lfloor m}.(D_x u)^m$, (199);

$$=\overset{\infty}{\text{S}}_m \frac{d_u^m.\phi(u)}{\lfloor m}.(\overset{\infty}{\text{S}}_n \frac{h^n}{\lfloor n}.d_x^n u)^m, \quad (199);$$

$$=h.\overset{\infty}{\text{S}}_m h^{m-1}.\frac{d_u^m.\phi(u)}{\lfloor m}.\left(\overset{\infty}{\text{S}}_n h^{n-1}.\frac{d_x^n u}{\lfloor n}\right)^m, \quad (6);$$

and $d_x.\phi(u)=\{h^{-1}.D_x.\phi(u)\}_{h=0}$

$$=d_u.\phi(u).d_x u.$$

203. *Theorem.* $d_x.(uv)=v\,d_x u+u\,d_x v$.

For, $D_x(uv)=(u+D_x u)(v+D_x v)-uv$

$$=v.D_x u+u.D_x v+D_x(u).D_x v.$$

$\therefore\ h^{-1}.D_x(uv)=v.h^{-1}.D_x u+u.h^{-1}.D_x v+h^{-1}.D_x(u).D_x v$;

and $\{h^{-1}.D_x(uv)\}_{h=0}=v.\{h^{-1}.D_x u\}_{h=0}+u\{h^{-1}.D_x v\}_{h=0}$

$+[\{h^{-1}.D_x u\}_{h=0}.(D_x v)_{h=0}$ or $(D_x.u)_{h=0}.\{h^{-1}.D_x v\}_{h=0}]$

$$\therefore\ d_x.(uv)=v\,d_x u+u\,d_x v+\{d_x u.0 \text{ or } 0.d_x v\}$$

$$=v\,d_x u+u\,d_x v.$$

204. Cor. $\frac{d_x.(uv)}{uv}=\frac{d_x u}{u}+\frac{d_x v}{v}$.

205. *Theorem.* $\dfrac{d_x . \overset{n}{\mathrm{P}}_r u_r}{\overset{n}{\mathrm{P}}_r u_r} = \overset{n}{\mathrm{S}}_m \dfrac{d_x u_m}{u_m}$.

For, $\dfrac{d_x . \overset{m+1}{\mathrm{P}}_r u_r}{\overset{m+1}{\mathrm{P}}_r u_r} = \dfrac{d_x . \overset{m}{\mathrm{P}}_r u_r}{\overset{m}{\mathrm{P}}_r u_r} + \dfrac{d_x . u_{m+1}}{u_{m+1}}$, (204).

$$\therefore \frac{d_x . \overset{n}{\mathrm{P}}_r u_r}{\overset{n}{\mathrm{P}}_r u_r} = \frac{d_x u_1}{u_1} + \overset{n-1}{\mathrm{S}}_m \frac{d_x u_{m+1}}{u_{m+1}}, \quad (24);$$

$$= \overset{n}{\mathrm{S}}_m \frac{d_x u_m}{u_m}, \quad (9).$$

206. Cor. $d_x . \overset{n}{\mathrm{P}}_r u_r = \overset{n}{\mathrm{S}}_m d_x u_m . \overset{n;m}{\mathrm{P}}_r u_r$.

207. *Theorem.* $d_x . u^n = n\, u^{n-1}\, d_x u$, for every rational value of n.

For, $\dfrac{d_x . \overset{n}{\mathrm{P}}_r u_r}{\overset{n}{\mathrm{P}}_r u_r} = \overset{n}{\mathrm{S}}_r \dfrac{d_x u_r}{u_r}$, (205).

Put $u_r = u$, then $\dfrac{d_x . u^n}{u^n} = \overset{n}{\mathrm{S}}_r \dfrac{d_x u}{u} = n . \dfrac{d_x u}{u}$.

$$\therefore\ d_x . u^n = n u^{n-1} . d_x u\,;$$

n being a positive integer.

Also, since $1 = u^n . u^{-n}$,

$$\therefore\ 0 = u^{-n} . n u^{n-1} d_x u + u^n . d_x . u^{-n}, \text{ (193) and (203)};$$

and $d_x . u^{-n} = -n u^{-n-1} d_x u$.

Again, $u^{\pm m} = (u^{\pm\frac{m}{n}})^n$; m and n being any positive integers.

$$\therefore\ \pm m u^{\pm m-1} d_x u = n.(u^{\pm\frac{m}{n}})^{n-1}.d_x.(u^{\pm\frac{m}{n}})$$

$$= n u^{\pm m \mp \frac{m}{n}} d_x.(u^{\pm\frac{m}{n}});$$

and $d_x.(u^{\pm\frac{m}{n}}) = \pm\frac{m}{n}.u^{\pm\frac{m}{n}-1}.d_x u.$

Hence, $d_x.u^n = n u^{n-1} d_x u$, for every rational value of n.

208. *Theorem.* $d_x.\left(\frac{u}{v}\right) = \frac{u}{v}.\left(\frac{d_x u}{u} - \frac{d_x v}{v}\right).$

For, $d_x.\left(\frac{u}{v}\right) = d_x.(uv^{-1})$

$$= v^{-1}.d_x u + u.(-1)v^{-2} d_x v,\ (203),\ \text{and}\ (207);$$

$$= \frac{u}{v}.\left(\frac{d_x u}{u} - \frac{d_x v}{v}\right).$$

209. Cor. $d_x.\left(\frac{\overset{m}{\mathrm{P}}_r u_r}{\overset{n}{\mathrm{P}}_r v_r}\right) = \frac{\overset{m}{\mathrm{P}}_r u_r}{\overset{n}{\mathrm{P}}_r v_r}.\left\{\frac{d_x.\overset{m}{\mathrm{P}}_r u_r}{\overset{m}{\mathrm{P}}_r u_r} - \frac{d_x.\overset{n}{\mathrm{P}}_r v_r}{\overset{n}{\mathrm{P}}_r v_r}\right\}$

$$= \frac{\overset{m}{\mathrm{P}}_r u_r}{\overset{n}{\mathrm{P}}_r v_r}.\left\{\overset{m}{\mathrm{S}}_r \frac{d_x u_r}{u_r} - \overset{n}{\mathrm{S}}_r \frac{d_x v_r}{v_r}\right\},\ (205).$$

210. *Theorem.* $d_x^m.x^n = \lfloor n_m . x^{n-m}.$

For, $d_x.x^n = n x^{n-1}$, (207) and (197);

$$d_x^2.x^n = n.d_x.x^{n-1},\ (196);$$

$$= \lfloor n_2 . x^{n-2};$$

&c. = &c.

$$d_x^m x^n = \lfloor n_m . x^{n-m}.$$

211. *Theorem.* $d_x^{\,n}.(uv) = \overset{n+1}{\mathrm{S}}_m \dfrac{\lfloor \underline{n}_{\,m-1}}{\lfloor \underline{m-1}} . d_x^{\,n-m+1}u . d_x^{\,m-1}v.$

For, $d_x.(uv) = v\,d_x u + u\,d_x v$, (203);

$$= (d_x + {}^1d_x)\,uv, \text{ where } {}^1d_x \text{ belongs to } v, \quad (161).$$

$$\therefore\ d_x^{\,n}.(uv) = (d_x + {}^1d_x)_n\,uv$$

$$= (d_x + {}^1d_x)^n\,uv, \quad (162);$$

$$= \overset{n+1}{\mathrm{S}}_m \frac{\lfloor \underline{n}_{\,m-1}}{\lfloor \underline{m-1}} . d_x^{\,n-m+1}u . d_x^{\,m-1}v.$$

Or thus:

$$E_1(uv) = \left(\overset{\infty}{\mathrm{S}}_n \frac{h^{n-1}}{\lfloor \underline{n-1}} . d_x^{\,n-1}u\right)\left(\overset{\infty}{\mathrm{S}}_m \frac{h^{m-1}}{\lfloor \underline{m-1}} . d_x^{\,m-1}v\right), \quad (199);$$

$$= \overset{\infty}{\mathrm{S}}_n h^{n-1} . \overset{n}{\mathrm{S}}_m \frac{d_x^{\,n-m}u . d_x^{\,m-1}v}{\lfloor \underline{n-m}\ \lfloor \underline{m-1}}, \quad (26).$$

$$\therefore\ \frac{d_x^{\,n}.(uv)}{\lfloor \underline{n}} = \overset{n+1}{\mathrm{S}}_m \frac{d_x^{\,n-m+1}u . d_x^{\,m-1}v}{\lfloor \underline{n-m+1}\ \lfloor \underline{m-1}}, \quad (201);$$

$$\text{and } d_x^{\,n}.(uv) = \overset{n+1}{\mathrm{S}}_m \frac{\lfloor \underline{n}_{\,m-1}}{\lfloor \underline{m-1}} . d_x^{\,n-m+1}u . d_x^{\,m-1}v.$$

212. Cor. $d_x . \overset{m}{\mathrm{P}}_r u_r = ({}^1d_x + {}^2d_x + \ldots + {}^md_x)\,u_1u_2\ldots u_m$, (206).

$$\therefore\ d_x^{\,n} . \overset{m}{\mathrm{P}}_r u_r = ({}^1d_x + {}^2d_x + \ldots + {}^md_x)^n\,u_1u_2\ldots u_m, \quad (164);$$

$$= (\overset{m}{\mathrm{S}}_r\,{}^rd_x)^n . \overset{m}{\mathrm{P}}_r u_r, \text{ where } {}^rd_x \text{ belongs to } u_r.$$

213. *Theorem.* If u is such a function of x as may be expanded in positive and integral powers of x, then shall

$$u = \overset{\infty}{\mathrm{S}}_m \frac{x^{m-1}}{\lfloor m-1} \cdot d^{m-1}_{x=0} \cdot u.$$

For, assume $u = \overset{\infty}{\mathrm{S}}_n a_{n-1} \cdot x^{n-1}$, where a_{n-1} is either zero or some finite quantity;

$$\text{then } d_x^m u = \overset{\infty}{\mathrm{S}}_n a_{n-1} \cdot \lfloor \underset{m}{n-1} \cdot x^{n-m-1}, \text{ (191), (196), and (210);}$$

$$= \overset{m}{\mathrm{S}}_n a_{n-1} \cdot \lfloor \underset{m}{n-1} \cdot x^{n-m-1} + a_m \cdot \lfloor m + \overset{\infty}{\mathrm{S}}_n a_{m+n} \lfloor \underset{m}{m+n} \cdot x^n, \quad (9);$$

$$= a_m \cdot \lfloor m + \overset{\infty}{\mathrm{S}}_n a_{m+n} \cdot \lfloor \underset{m}{m+n} \cdot x^n, \text{ since } \lfloor \underset{m}{n-1} = 0, \binom{n=1}{n=m}.$$

$$\therefore\ d^m_{x=0} \cdot u = a_m \cdot \lfloor m, \text{ and } a_m = \frac{1}{\lfloor m} \cdot d^m_{x=0} u, \binom{m=1}{m=\infty}.$$

Also $(u)_{x=0} = a_0$; and therefore, $a_{m-1} = \dfrac{1}{\lfloor m-1} \cdot d^{m-1}_{x=0} u, \binom{m=1}{m=\infty}$,

$$\text{and } u = \overset{\infty}{\mathrm{S}}_m \frac{x^{m-1}}{\lfloor m-1} \cdot d^{m-1}_{x=0} \cdot u.$$

214. *Theorem.* $d_x d_y u = d_y d_x u$; x and y being independent of each other.

For, put $Dx = h$, and $Dy = k$; then

$$d_y u = (k^{-1} \cdot D_y u)_{k=0};$$

$$\begin{aligned}
\text{and } d_x d_y u &= \{h^{-1} \cdot D_x (k^{-1} D_y u)_{k=0}\}_{h=0} \\
&= \{h^{-1} \cdot (k^{-1} \cdot D_x D_y u)_{k=0}\}_{h=0}, \quad (137); \\
&= \{h^{-1} \cdot k^{-1} \cdot D_x D_y u\}_{h=0,\, k=0} \\
&= \{k^{-1} \cdot h^{-1} \cdot D_y D_x u\}_{h=0,\, k=0}, \quad (170); \\
&= \{k^{-1} \cdot D_y (h^{-1} \cdot D_x u)\}_{h=0,\, k=0}, \quad (137); \\
&= \{k^{-1} \cdot D_y (h^{-1} \cdot D_x u)_{h=0}\}_{k=0} \\
&= \{k^{-1} \cdot D_y \cdot d_x u\}_{k=0} \\
&= d_y d_x u.
\end{aligned}$$

215. Cor. $d_x^{m}.d_y^{n}u=d_y^{n}.d_x^{m}u.$

216. *Theorem.*

$$\phi(x+h,\,y+k)=\overset{\infty}{\mathrm{S}}_m\,\overset{\infty}{\mathrm{S}}_n\,\frac{h^{m-1}k^{n-1}}{\lfloor m-1\,\lfloor n-1}\,.\,d_x^{\,m-1}.d_y^{\,n-1}\phi(x,y).$$

For, $\phi(x+h,\,y)=\overset{\infty}{\mathrm{S}}_m\,\frac{h^{m-1}}{\lfloor m-1}\,.\,d_x^{\,m-1}.\phi(x,y)$, (198);

and $\phi(x+h,\,y+k)=\overset{\infty}{\mathrm{S}}_m\,\frac{h^{m-1}}{\lfloor m-1}\,.\,d_x^{\,m-1}.\phi(x,\,y+k)$

$$=\overset{\infty}{\mathrm{S}}_m\,\frac{h^{m-1}}{\lfloor m-1}\,.\,d_x^{\,m-1}.\overset{\infty}{\mathrm{S}}_n\,\frac{k^{n-1}}{\lfloor n-1}\,.\,d_y^{\,n-1}.\phi(x,y),\quad (198);$$

$$=\overset{\infty}{\mathrm{S}}_m\,\overset{\infty}{\mathrm{S}}_n\,\frac{h^{m-1}.k^{n-1}}{\lfloor m-1\,\lfloor n-1}\,.\,d_x^{\,m-1}.d_y^{\,n-1}.\phi(x,y),\quad (191),$$

(196), and (6).

217. Cor.

$$\phi(x+h,\,y+k)=\overset{\infty}{\mathrm{S}}_m\,\overset{m}{\mathrm{S}}_n\,\frac{h^{m-n}k^{n-1}}{\lfloor m-n\,\lfloor n-1}\,.\,d_x^{\,m-n}.d_y^{\,n-1}.\phi(x,y),\quad (18).$$

218. *Theorem.* If u is such a function of x and y, that it may be expanded in positive integral powers of x, and y, then shall

$$u=\overset{\infty}{\mathrm{S}}_m\,\overset{\infty}{\mathrm{S}}_n\,\frac{x^{m-1}.y^{n-1}}{\lfloor m-1\,\lfloor n-1}\,.\,d_{x=0}^{m-1}.\,d_{y=0}^{n-1}.\,u.$$

For, assume $u=\overset{\infty}{\mathrm{S}}_m x^{m-1}.\overset{\infty}{\mathrm{S}}_n y^{n-1}.a_{m-1,\,n-1}$, where $a_{m-1,\,n-1}$ is not infinite;

then $\frac{d_{y=0}^{s-1}u}{\lfloor s-1}=\overset{\infty}{\mathrm{S}}_m x^{m-1}.a_{m-1,\,s-1}$, $\binom{s=1}{s=\infty}$, (213);

$\therefore\ \frac{d_{y=0}^{n-1}u}{\lfloor n-1}=\overset{\infty}{\mathrm{S}}_m x^{m-1}.a_{m-1,\,n-1}$, $\binom{n=1}{n=\infty}$.

And, $\dfrac{d_{x=0}^{r-1}.d_{y=0}^{n-1}u}{\lfloor r-1 . \lfloor n-1} = a_{r-1,\,n-1}$, $\binom{r=1}{r=\infty}$, $\binom{n=1}{n=\infty}$, (213);

$$\therefore \frac{d_{x=0}^{m-1}.d_{y=0}^{n-1}u}{\lfloor m-1 . \lfloor n-1} = a_{m-1,\,n-1}, \binom{m=1}{m=\infty}, \binom{n=1}{n=\infty};$$

$$\text{and } u = \overset{\infty}{\mathrm{S}}_m \overset{\infty}{\mathrm{S}}_n \frac{x^{m-1}.y^{n-1}}{\lfloor m-1 . \lfloor n-1} . d_{x=0}^{m-1}.d_{y=0}^{n-1}u.$$

219. *Theorem.* If z is a function of x and y,

then shall $d_x\{d_y z.\phi(z)\} = d_y\{d_x z.\phi(z)\}$.

For, $d_x.d_y\{\int_z \phi(z)\} = d_y.d_x\{\int_z \phi(z)\}$, (214);

$\therefore d_x\{d_y z.d_z.\int_z \phi(z)\} = d_y\{d_x z.d_z.\int_z \phi(z)\}$, (202);

or $d_x\{d_y z.\phi(z)\} = d_y\{d_z z.\phi(z)\}$.

220. If $y = \psi\{z + x.\phi(y)\}$, where z is independent of x and y; then shall

$$f(y) = f\psi(z) + \overset{\infty}{\mathrm{S}}_m \frac{x^m}{\lfloor m} . d_z^{m-1}\{\overline{\phi\psi(z)}|^m . d_z.f\psi(z)\}.$$

For, $f(y) = \{f(y)\}_{x=0} + \overset{\infty}{\mathrm{S}}_m \dfrac{x^m}{\lfloor m} . d_{x=0}^m f(y)$, (213).

Put $z + x.\phi(y) = u$;

$$\text{then } \frac{d_x.y}{d_z y} = \frac{d_x.\psi(u)}{d_z.\psi(u)}$$

$$= \frac{d_x u.d_u.\psi(u)}{d_z u.d_u.\psi(u)}, \quad (202);$$

$$= \frac{d_x u}{d_z u}$$

$$= \frac{\phi(y) + x.d_x.\phi(y)}{1 + x.d_z.\phi(y)}.$$

$$\therefore \ \{1+x.d_z.\phi(y)\}d_xy=\{\phi(y)+x.d_x.\phi(y)\}d_zy,$$

and, cancelling identical terms,

$$d_xy=d_zy.\phi(y), \text{ since } \frac{d_xy}{d_zy}=\frac{d_x.\phi(y)}{d_z.\phi(y)}, \quad (202).$$

Also, $d_x.f(y)=d_xy.d_yf(y)$, (202);

$$=d_zy.\phi(y).d_y.f(y);$$

$$d_x^2.f(y)=d_x\{d_zy.\phi(y).d_y.f(y)\}$$

$$=d_z\{d_xy.\phi(y).d_y.f(y)\}, \quad (219);$$

$$=d_z\{d_zy.\overline{\phi(y)}|^2.d_y.f(y)\};$$

and, similarly, $d_x^m.f(y)=d_z^{m-1}.\{d_zy.\overline{\phi(y)}|^m.d_y.f(y)\}$

$$=d_z^{m-1}.\{\overline{\phi(y)}|^m.d_z.f(y)\}, \quad (202).$$

$$\therefore \ d^m_{x=0}.f(y)=d_z^{m-1}.\{\overline{\phi\psi(z)}|^m.d_z.f\psi(z)\},$$

$$\text{and } f(y)=f\psi(z)+\overset{\infty}{\mathrm{S}}_m\frac{x^m}{\lfloor m}.d_z^{m-1}\{\overline{\phi\psi(z)}|^m.d_z.f\psi(z)\}.$$

221. Cor. If $y=z+x.\phi(y)$,

$$\text{then } f(y)=f(z)+\overset{\infty}{\mathrm{S}}_m\frac{x^m}{\lfloor m}.d_z^{m-1}.\{\overline{\phi(z)}|^m.d_z.f(z)\}.$$

222. *Theorem.*

$$\textstyle\int_x(uv)=\overset{n}{\mathrm{S}}_m(-1)^{m-1}.d_x^{m-1}u.\int_x^mv+(-1)^n.\int_x\{d_x^nu.\int_x^nv\}$$

For, $\int_x\{d_x^{m-1}u.\int_x^{m-1}v\}=d_x^{m-1}u.\int_x^mv-\int_x\{d_x^mu.\int_x^mv\}$, (203).

$$\therefore \ (-1)^{m-1}.\textstyle\int_x\{d_x^{m-1}u.\int_x^{m-1}v\}$$

$$\textstyle=(-1)^{m-1}d_x^{m-1}u.\int_x^mv+(-1)^m.\int_x\{d_x^mu.\int_x^mv\}$$

$$\overset{n}{\mathrm{S}}_m(-1)^{m-1}\cdot\textstyle\int_x\left\{d_x^{\,m-1}u\cdot\int_x^{m-1}v\right\}$$

$$=\overset{n}{\mathrm{S}}_m(-1)^{m-1}\cdot d_x^{\,m-1}u\cdot\textstyle\int_x^m v+\overset{n}{\mathrm{S}}_m(-1)^m\cdot\int_x\left\{d_x^{\,m}u\cdot\int_x^m v\right\},\ \text{(4) and (5);}$$

$$\textstyle\int_x(uv)+\overset{n-1}{\mathrm{S}}_m(-1)^m\int_x\left\{d_x^{\,m}u\cdot\int_x^m v\right\}$$

$$=\overset{n}{\mathrm{S}}_m(-1)^{m-1}\cdot d_x^{\,m-1}u\cdot\textstyle\int_x^m v+\overset{n-1}{\mathrm{S}}_m(-1)^m\int_x\left\{d_x^{\,m}u\cdot\int_x^m v\right\}$$

$$+(-1)^n\textstyle\int_x\left\{d_x^{\,n}u\cdot\int_x^n v\right\},\quad (9).$$

$$\therefore\ \textstyle\int_x(uv)=\overset{n}{\mathrm{S}}_m(-1)^{m-1}d_x^{\,m-1}u\cdot\int_x^m v+(-1)^n\cdot\int_x\left\{d_x^{\,n}u\cdot\int_x^n v\right\}.$$

223. Cor.

$$\textstyle\int_x u=\overset{n}{\mathrm{S}}_m(-1)^{m-1}\cdot\dfrac{x^m}{\lfloor m}\cdot d_x^{\,m-1}u+(-1)^n\int_x\left\{\dfrac{x^n}{\lfloor n}\cdot d_x^{\,n}u\right\},\ \text{(197) \& (210).}$$

CHAPTER VII.

ON POLYNOMIALS.

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224. The symbol $\overset{m,n}{\mathrm{S}}_{r,+s}({}^{s}a_r)$ denotes the sum of every term that can be formed with the following conditions: each term is the product of m quantities in which r has the values of the successive natural numbers, while s has any m values such that their sum shall be n, zero being admissible as a value of s, and repetitions of the same value of that letter being allowed in the same term. Thus:

$$\overset{3,4}{\mathrm{S}}_{r,+s}({}^{s}a_r) = {}^{4}a_1{}^{0}a_2{}^{0}a_3 + {}^{3}a_1{}^{1}a_2{}^{0}a_3 + {}^{3}a_1{}^{0}a_2{}^{1}a_3 + {}^{2}a_1{}^{2}a_2{}^{0}a_3 +$$

$${}^{2}a_1{}^{1}a_2{}^{1}a_3 + {}^{2}a_1{}^{0}a_2{}^{2}a_3 + {}^{1}a_1{}^{3}a_2{}^{0}a_3 + {}^{1}a_1{}^{2}a_2{}^{1}a_3 +$$

$${}^{1}a_1{}^{1}a_2{}^{2}a_3 + {}^{1}a_1{}^{0}a_2{}^{3}a_3 + {}^{0}a_1{}^{4}a_2{}^{0}a_3 + {}^{0}a_1{}^{3}a_2{}^{1}a_3 +$$

$${}^{0}a_1{}^{2}a_2{}^{2}a_3 + {}^{0}a_1{}^{1}a_2{}^{3}a_3 + {}^{0}a_1{}^{0}a_2{}^{4}a_3.$$

225. Cor. $\overset{m,n}{\mathrm{S}}_{r,+s}({}^{s}a_r) = \overset{n+1}{\mathrm{S}}_t{}^{n-t+1}a_m \overset{m-1,\,t-1}{\mathrm{S}}_{r,+s}({}^{s}a_r)$.

226. *Theorem.* $\dfrac{1}{\underline{|n}}(\overset{m}{\mathrm{S}}_r a_r)^n = \overset{m,n}{\mathrm{S}}_{r,+s}\dfrac{a_r^{\,s}}{\underline{|s}}$.

For, $\dfrac{1}{\underline{|n}}\cdot(\overset{2}{\mathrm{S}}_r a_r)^n = \overset{n+1}{\mathrm{S}}_t \dfrac{a_1^{\,n-t+1}\quad a_2^{\,t-1}}{\underline{|n-t+1}\;\;\underline{|t-1}}$, (88);

$$= \overset{2,n}{\mathrm{S}}_{r,+s}\frac{a_r^{\,s}}{\underline{|s}}.$$

Suppose, therefore,

$$\frac{1}{\lfloor n}\cdot(\overset{m-1}{\mathrm{S}_r} a_r)^n = \overset{m-1,\,n}{\mathrm{S}_{r,+s}}\frac{a_r^s}{\lfloor s};$$

then $\frac{1}{\lfloor n}\cdot(\overset{m}{\mathrm{S}_r} a_r)^n = \frac{1}{\lfloor n}(a_m + \overset{m-1}{\mathrm{S}_r} a_r)^n$

$$= \overset{n+1}{\mathrm{S}_t}\frac{a_m^{\,n-t+1}}{\lfloor n-t+1\,.\lfloor t-1}(\overset{m-1}{\mathrm{S}_r} a_r)^{t-1}, \quad (88);$$

$$= \overset{n+1}{\mathrm{S}_t}\frac{a_m^{\,n-t+1}}{\lfloor n-t+1}\cdot\overset{m-1,\,t-1}{\mathrm{S}_{r,+s}}\frac{a_r^s}{\lfloor s}$$

$$= \overset{m,\,n}{\mathrm{S}_{r,+s}}\frac{a_r^s}{\lfloor s}, \quad (225).$$

If, therefore, the law were true for $m-1$, it would be true for m; but it *is* true for 2, and therefore for m.

227. Cor. $\frac{1}{\lfloor n}\cdot d_x^{\,n}\cdot\overset{m}{\mathrm{P}_r} u_r = \overset{m,\,n}{\mathrm{S}_{r,+s}}\frac{d_x^{\,s} u_r}{\lfloor s}$, (212).

228. The symbol $\varpi_a^{\,m}\phi(a)$ denotes the coefficient of x^m in the development of $\phi(\overset{\infty}{\mathrm{S}_m} a_{m-1}x^{m-1})$, which coefficient may be called the m^{th} *polynomial coefficient of* $\phi(a)$ *taken with respect to* a. In this symbol the index subscript of ϖ is the letter according to the indices subscript of which the different powers of x ascend, and the quantity following the functional symbol is the term independent of x in the series $\overset{\infty}{\mathrm{S}_m} a_{m-1}x^{m-1}$*. If the index subscript of ϖ is omitted, that letter is understood which

* Throughout this Chapter a is put for a_0, for the sake of brevity.

immediately follows it, and if the function is a power of the polynomial, the parentheses including the first term of the polynomial may be omitted: thus

$\varpi^m \phi(a)$ denotes the coefficient of x^m in $\overset{\infty}{\mathrm{S}}_m \phi_{m-1}(a) x^{m-1}$,

$\varpi^m a^n$ .... $\{\overset{\infty}{\mathrm{S}}_m a_{m-1} x^{m-1}\}^n$,

$\varpi^m a_r^{\,m}$ .... $\{\overset{\infty}{\mathrm{S}}_m a_{r+m-1} x^{m-1}\}^n$.

229. Cor. $\varpi_a^{\,0} . \phi(a) = \phi(a)$.

230. *Theorem.* $\varpi^m a^n = \overset{m}{\mathrm{S}}_r \underline{|n}_r . a^{n-r} . \dfrac{\varpi^{m-r} a_1^{\,r}}{\underline{|r}}$, for every value of n.

For, $(\overset{\infty}{\mathrm{S}}_m a_{m-1} x^{m-1})^n = (a + x . \overset{\infty}{\mathrm{S}}_r a_r x^{r-1})^n$, (9) and (6);

$$= \overset{\infty}{\mathrm{S}}_m \frac{\underline{|n}_{m-1}}{\underline{|m-1}} . a^{n-m+1} . x^{m-1} . (\overset{\infty}{\mathrm{S}}_r a_r x^{r-1})^{m-1}, \quad (92);$$

$$= \overset{\infty}{\mathrm{S}}_m \frac{\underline{|n}_{m-1}}{\underline{|m-1}} . a^{n-m+1} . x^{m-1} . \overset{\infty}{\mathrm{S}}_r x^{r-1} . \varpi^{r-1} a_1^{\,m-1}, \quad (228);$$

$$= \overset{\infty}{\mathrm{S}}_m x^{m-1} \overset{m}{\mathrm{S}}_r \frac{\underline{|n}_{m-r}}{\underline{|m-r}} . a^{n-m+r} . \varpi^{r-1} . a_1^{\,m-r}, \quad (6) \text{ and } (18).$$

$$\therefore \ \varpi^m a^n = \overset{m+1}{\mathrm{S}}_r \frac{\underline{|n}_{m-r+1}}{\underline{|m-r+1}} a^{n-m+r-1} \varpi^{r-1} a_1^{\,m-r+1}, \quad (228);$$

$$= \overset{m}{\mathrm{S}}_r \frac{\underline{|n}_{m-r+1}}{\underline{|m-r+1}} . a^{n-m+r-1} \varpi^{r-1} a_1^{\,m-r+1}, \text{ since } \varpi^m . a_1^{\,0} = 0;$$

$$= \overset{m}{\mathrm{S}}_r \frac{\underline{|n}_r}{\underline{|r}} . a^{n-r} . \varpi^{m-r} . a_1^{\,r}, \quad (8).$$

231. Cor. 1. $\varpi^m a_s^n = \overset{m}{\mathrm{S}}_r \dfrac{\lfloor n}{\lfloor r} . a_s^{n-r} \varpi^{m-r} a_{s+1}^r .$

232. Cor. 2. If n is a positive integer,

$$\frac{\varpi^m . a_s^n}{\lfloor n} = \overset{m}{\mathrm{S}}_r \frac{a_s^{n-r} . \varpi^{m-r} a_{s+1}^r}{\lfloor n-r \quad \lfloor r}, \text{ and}$$

$$\frac{\varpi^m . a_s^n}{\lfloor n} = \overset{m}{\mathrm{S}}_r \frac{a_s^{n-m+r-1} . \varpi^{r-1} a_{s+1}^{m-r+1}}{\lfloor n-m+r-1 . \lfloor m-r+1}.$$

233 From this last value we may deduce any number of terms of the expansion of $\dfrac{\varpi^m a^n}{\lfloor n}$, much more readily than from the general expression for that expansion. We have

$$\frac{\varpi^m . a^n}{\lfloor n} = \frac{a^{n-m}}{\lfloor n-m} . \frac{\varpi^0 . a_1^m}{\lfloor m} + \frac{a^{n-m+1} \; \varpi^1 . a_1^{m-1}}{\lfloor n-m+1 \; \lfloor m-1}$$

$$+ \frac{a^{n-m+2} . \varpi^2 . a_1^{m-2}}{\lfloor n-m+2 . \lfloor m-2} + \&c.$$

Hence, putting $m=1$, and $n=m-1$,

$$\frac{\varpi^m a^n}{\lfloor n} = \frac{a^{n-m} . a_1^m}{\lfloor n-m \; \lfloor m} + \frac{a^{n-m+1}}{\lfloor n-m+1} . \frac{a_1^{m-2}}{\lfloor m-2} . a_2$$

$$+ \frac{a^{n-m+2} . \varpi^2 a_1^{m-2}}{\lfloor n-m+2 . \lfloor m-2} + \&c.$$

And, putting $m=2$, and $n=m-2$,

$$\frac{\varpi^m a^n}{\lfloor n} = \frac{a^{n-m} \; a_1^m}{\lfloor n-m . \lfloor m} + \frac{a^{n-m+1} . a_1^{m-2}}{\lfloor n-m+1 . \lfloor m-2} . a_2$$

$$+ \frac{a^{n-m+2}}{\lfloor n-m+2} . \left\{ \frac{a_1^{m-4} . a_2^2}{\lfloor m-4 \; \lfloor 2} + \frac{a_1^{m-3}}{\lfloor m-3} . a_3 \right\}$$

$$+ \frac{a^{n-m+3} . \varpi^3 . a_1^{m-3}}{\lfloor n-m+3 . \lfloor m-3} + \&c.$$

And, proceeding in the same manner, we shall obtain successively the following terms:

$$
\begin{aligned}
\frac{\varpi^m a^n}{\underline{|n}} &= \frac{a^{n-m} \,.\, a_1^{\,m}}{\underline{|n-m}\;\underline{|m}} + \frac{a^{n-m+1} \,.\, a_1^{\,m-2}}{\underline{|n-m+1}\;\underline{|m-2}} \,.\, a_2 \\
&+ \frac{a^{n-m+2}}{\underline{|n-m+2}} \left\{ \frac{a_1^{\,m-4} \,.\, a_2^{\,2}}{\underline{|m-4}\;\underline{|2}} + \frac{a_1^{\,m-3}}{\underline{|m-3}} \,.\, a_3 \right\} + \frac{a^{n-m+3}}{\underline{|n-m+3}} \,.\, \Big\{ \frac{a_1^{\,m-6} \,.\, a_2^{\,3}}{\underline{|m-6}\;\underline{|3}} \\
&+ \frac{a_1^{\,m-5}}{\underline{|m-5}} \,.\, a_2 a_3 + \frac{a_1^{\,m-4}}{\underline{|m-4}} \,.\, a_4 \Big\} + \frac{a^{n-m+4}}{\underline{|n-m+4}} \Big\{ \frac{a_1^{\,m-8} \,.\, a_2^{\,4}}{\underline{|m-8}\;\underline{|4}} + \frac{a_1^{\,m-7} \,.\, a_2^{\,3}}{\underline{|m-7}\;\underline{|3}} \,.\, a_3 \\
&+ \frac{a_1^{\,m-6}}{\underline{|m-6}} \left(\frac{a_3^{\,2}}{\underline{|2}} + a_2 a_4 \right) + \frac{a_1^{\,m-5}}{\underline{|m-5}} \,.\, a_5 \Big\} + \frac{a^{n-m+5}}{\underline{|n-m+5}} \Big\{ \frac{a_1^{\,m-10} \,.\, a_2^{\,5}}{\underline{|m-10}\;\underline{|5}} \\
&+ \frac{a_1^{\,m-9} \,.\, a_2^{\,3}}{\underline{|m-9}\;\underline{|3}} \,.\, a_3 + \frac{a_1^{\,m-8}}{\underline{|m-8}} \left(a_2 \,.\, \frac{a_3^{\,2}}{\underline{|2}} + \frac{a_2^{\,2}}{\underline{|2}} \,.\, a_4 \right) + \frac{a_1^{\,m-7}}{\underline{|m-7}} (a_3 a_4 + a_2 a_5) \\
&+ \frac{a_1^{\,m-6}}{\underline{|m-6}} a_6 \Big\} + \frac{a^{n-m+6}}{\underline{|n-m+6}} \Big\{ \frac{a_1^{\,m-12} \,.\, a_2^{\,6}}{\underline{|m-12}\;\underline{|6}} + \frac{a_1^{\,m-11} \,.\, a_2^{\,4}}{\underline{|m-11}\;\underline{|4}} \,.\, a_3 \\
&+ \frac{a_1^{\,m-10}}{\underline{|m-10}} \left(\frac{a_2^{\,2} \,.\, a_3^{\,2}}{\underline{|2}\;\underline{|2}} + \frac{a_2^{\,3}}{\underline{|3}} \,.\, a_4 \right) + \frac{a_1^{\,m-9}}{\underline{|m-9}} \left(\frac{a_3^{\,3}}{\underline{|3}} + a_2 a_3 a_4 + \frac{a_2^{\,2}}{\underline{|2}} \,.\, a_5 \right) \\
&+ \frac{a_1^{\,m-8}}{\underline{|m-8}} \left(\frac{a_4^{\,2}}{\underline{|2}} + a_3 a_5 + a_2 a_6 \right) + \frac{a_1^{\,m-7}}{\underline{|m-7}} \,.\, a_7 \Big\} + \frac{a^{n-m+7}}{\underline{|n-m+7}} \Big\{ \frac{a_1^{\,m-14} \,.\, a_2^{\,7}}{\underline{|m-14}\;\underline{|7}} \\
&+ \frac{a_1^{\,m-13} \,.\, a_2^{\,5}}{\underline{|m-13}\;\underline{|5}} \,.\, a_3 + \frac{a_1^{\,m-12}}{\underline{|m-12}} \left(\frac{a_2^{\,3} \,.\, a_3^{\,2}}{\underline{|3}\;\underline{|2}} + \frac{a_2^{\,4}}{\underline{|4}} \,.\, a_4 \right) \\
&+ \frac{a_1^{\,m-11}}{\underline{|m-11}} \left(a_2 \,.\, \frac{a_3^{\,3}}{\underline{|3}} + \frac{a_2^{\,2}}{\underline{|2}} \,.\, a_3 a_4 + \frac{a_2^{\,3}}{\underline{|3}} \,.\, a_5 \right) \\
&+ \frac{a_1^{\,m-10}}{\underline{|m-10}} \left(\frac{a_3^{\,2}}{\underline{|2}} \,.\, a_4 + a_2 \left[\frac{a_4^{\,2}}{\underline{|2}} + a_3 a_5 \right] + \frac{a_2^{\,2}}{\underline{|2}} \,.\, a_6 \right) \\
&+ \frac{a_1^{\,m-9}}{\underline{|m-9}} (a_4 a_5 + a_3 a_6 + a_2 a_7) + \frac{a_1^{\,m-8}}{\underline{|m-8}} \,.\, a_8 \Big\}
\end{aligned}
$$

$$+\frac{a^{n-m+8}}{\lfloor n-m+8}\left\{\frac{a_1^{m-16}\cdot a_2^8}{\lfloor m-16\ \lfloor 8}+\frac{a_1^{m-15}\cdot a_2^6}{\lfloor m-15\ \lfloor 6}\cdot a_3\right.$$

$$+\frac{a_1^{m-14}}{\lfloor m-14}\left(\frac{a_2^4\cdot a_3^2}{\lfloor 4\ \lfloor 2}+\frac{a_2^5}{\lfloor 5}\cdot a_4\right)+\frac{a_1^{m-13}}{\lfloor m-13}\left(\frac{a_2^2\cdot a_3^3}{\lfloor 2\ \lfloor 3}+\frac{a_2^3}{\lfloor 3}a_3a_4+\frac{a_2^4}{\lfloor 4}\cdot a_5\right)$$

$$+\frac{a_1^{m-12}}{\lfloor m-12}\left(\frac{a_3^4}{\lfloor 4}+a_2\cdot\frac{a_3^2}{\lfloor 2}\cdot a_4+\frac{a_2^2}{\lfloor 2}\left[\frac{a_4^2}{\lfloor 2}+a_3a_5\right]+\frac{a_2^3}{\lfloor 3}\cdot a_6\right)$$

$$+\frac{a_1^{m-11}}{\lfloor m-11}\left(a_3\cdot\frac{a_4^2}{\lfloor 2}+\frac{a_3^2}{\lfloor 2}\cdot a_5+a_2\left[a_4a_5+a_3a_6\right]+\frac{a_2^2}{\lfloor 2}\cdot a_7\right)$$

$$\left.+\frac{a_1^{m-10}}{\lfloor m-10}\left(\frac{a_5^2}{\lfloor 2}+a_4a_6+a_3a_7+a_2a_8\right)+\frac{a_1^{m-9}}{\lfloor m-9}\cdot a_9\right\}+\ \&c.$$

Whence, giving to m the values of the successive natural numbers,

$$\frac{\varpi\cdot a^n}{\lfloor n}=\frac{a^{n-1}}{\lfloor n-1}\cdot a_1.$$

$$\frac{\varpi^2\cdot a^n}{\lfloor n}=\frac{a^{n-2}\cdot a_1^2}{\lfloor n-2\ \lfloor 2}+\frac{a^{n-1}}{\lfloor n-1}\cdot a_2.$$

$$\frac{\varpi^3\cdot a^n}{\lfloor n}=\frac{a^{n-3}\cdot a_1^3}{\lfloor n-3\ \lfloor 3}+\frac{a^{n-2}}{\lfloor n-2}a_1a_2+\frac{a^{n-1}}{\lfloor n-1}\cdot a_3.$$

$$\frac{\varpi^4\cdot a^n}{\lfloor n}=\frac{a^{n-4}\cdot a_1^4}{\lfloor n-4\ \lfloor 4}+\frac{a^{n-3}\cdot a_1^2}{\lfloor n-3\ \lfloor 2}\cdot a_2+\frac{a^{n-2}}{\lfloor n-2}\left\{\frac{a_2^2}{\lfloor 2}+a_1a_3\right\}+\frac{a^{n-1}}{\lfloor n-1}\cdot a_4.$$

$$\frac{\varpi^5\cdot a^n}{\lfloor n}=\frac{a^{n-5}\cdot a_1^5}{\lfloor n-5\ \lfloor 5}+\frac{a^{n-4}\cdot a_1^3}{\lfloor n-4\ \lfloor 3}\cdot a_2+\frac{a^{n-3}}{\lfloor n-3}\left\{a_1\cdot\frac{a_2^2}{\lfloor 2}+\frac{a_1^2}{\lfloor 2}\cdot a_3\right\}$$

$$+\frac{a^{n-2}}{\lfloor n-2}\left\{a_2a_3+a_1a_4\right\}+\frac{a^{n-1}}{\lfloor n-1}\cdot a_5.$$

$$\frac{\varpi^6 \cdot a^n}{\underline{|n}} = \frac{a^{n-6} \cdot a_1^{\,6}}{\underline{|n-6}\;\underline{|6}} + \frac{a^{n-5} \cdot a_1^{\,4}}{\underline{|n-5}\;\underline{|4}} \cdot a_2 + \frac{a^{n-4}}{\underline{|n-4}} \left\{ \frac{a_1^{\,2} \cdot a_2^{\,2}}{\underline{|2}\;\underline{|2}} + \frac{a_1^{\,3}}{\underline{|3}} \cdot a_3 \right\}$$

$$+ \frac{a^{n-3}}{\underline{|n-3}} \left\{ \frac{a_2^{\,3}}{\underline{|3}} + a_1 a_2 a_3 + \frac{a_1^{\,2}}{\underline{|2}} \cdot a_4 \right\} + \frac{a^{n-2}}{\underline{|n-2}} \left\{ \frac{a_3^{\,2}}{\underline{|2}} + a_2 a_4 + a_1 a_5 \right\}$$

$$+ \frac{a^{n-1}}{\underline{|n-1}} \cdot a_6 .$$

$$\frac{\varpi^7 \cdot a^n}{\underline{|n}} = \frac{a^{n-7} \cdot a_1^{\,7}}{\underline{|n-7}\;\underline{|7}} + \frac{a^{n-6} \cdot a_1^{\,5}}{\underline{|n-6}\;\underline{|5}} \cdot a_2 + \frac{a^{n-5}}{\underline{|n-5}} \left\{ \frac{a_1^{\,3} \cdot a_2^{\,2}}{\underline{|3}\;\underline{|2}} + \frac{a_1^{\,4}}{\underline{|4}} \cdot a_3 \right\}$$

$$+ \frac{a^{n-4}}{\underline{|n-4}} \left\{ a_1 \cdot \frac{a_2^{\,3}}{\underline{|3}} + \frac{a_1^{\,2}}{\underline{|2}} \cdot a_2 a_3 + \frac{a_1^{\,3}}{\underline{|3}} \cdot a_4 \right\} + \frac{a^{n-3}}{\underline{|n-3}} \left\{ \frac{a_2^{\,2}}{\underline{|2}} \cdot a_3 \right.$$

$$\left. + a_1 \left(\frac{a_3^{\,2}}{\underline{|2}} + a_2 a_4 \right) + \frac{a_1^{\,2}}{\underline{|2}} \cdot a_5 \right\} + \frac{a^{n-2}}{\underline{|n-2}} \left\{ a_3 a_4 + a_2 a_5 + a_1 a_6 \right\}$$

$$+ \frac{a^{n-1}}{\underline{|n-1}} \cdot a_7 .$$

$$\frac{\varpi^8 \cdot a^n}{\underline{|n}} = \frac{a^{n-8} \cdot a_1^{\,8}}{\underline{|n-8}\;\underline{|8}} + \frac{a^{n-7} \cdot a_1^{\,6}}{\underline{|n-7}\;\underline{|6}} \cdot a_2 + \frac{a^{n-6}}{\underline{|n-6}} \left\{ \frac{a_1^{\,4}\, a_2^{\,2}}{\underline{|4}\;\underline{|2}} + \frac{a_1^{\,5}}{\underline{|5}} \cdot a_3 \right\}$$

$$+ \frac{a^{n-5}}{\underline{|n-5}} \left\{ \frac{a_1^{\,2}\, a_2^{\,3}}{\underline{|2}\;\underline{|3}} + \frac{a_1^{\,3}}{\underline{|3}} \cdot a_2 a_3 + \frac{a_1^{\,4}}{\underline{|4}} \cdot a_4 \right\}$$

$$+ \frac{a^{n-4}}{\underline{|n-4}} \left\{ \frac{a_2^{\,4}}{\underline{|4}} + a_1 \cdot \frac{a_2^{\,2}}{\underline{|2}} \cdot a_3 + \frac{a_1^{\,2}}{\underline{|2}} \left(\frac{a_3^{\,2}}{\underline{|2}} + a_2 a_4 \right) + \frac{a_1^{\,3}}{\underline{|3}} \cdot a_5 \right\}$$

$$+ \frac{a^{n-3}}{\underline{|n-3}} \left\{ a_2 \cdot \frac{a_3^{\,2}}{\underline{|2}} + \frac{a_2^{\,2}}{\underline{|2}} \cdot a_4 + a_1 (a_3 a_4 + a_2 a_5) + \frac{a_1^{\,2}}{\underline{|2}} \cdot a_6 \right\}$$

$$+ \frac{a^{n-2}}{\underline{|n-2}} \left\{ \frac{a_4^{\,2}}{\underline{|2}} + a_3 a_5 + a_2 a_6 + a_1 a_7 \right\} + \frac{a^{n-1}}{\underline{|n-1}} \cdot a_8 .$$

234. Since $\frac{1}{\lfloor n}(\overset{\infty}{\mathrm{S}}_m a_{m-1}x^{m-1})^n = \overset{\infty,n}{\mathrm{S}}_{r,+s}\frac{(a_{r-1}x^{r-1})^s}{\lfloor s}$, n being any positive integer, (226);

$$=\overset{\infty,n}{\mathrm{S}}_{r,+s}\frac{a_{r-1}^{\,s}}{\lfloor s}\cdot x^{s\cdot(r-1)}.$$

$\therefore\ \frac{\varpi^m a^n}{\lfloor n}$ = the sum of all those terms of $\overset{\infty,n}{\mathrm{S}}_{r,+s}\frac{a_{r-1}^{\,s}}{\lfloor s}$ in which $s(r-1)=m$.

This equation may be thus written:

$$\frac{\varpi^m . a^n}{\lfloor n} = \overset{\infty,n,m}{\mathrm{S}}_{r,+s,+s(r-1)}\frac{a_{r-1}^{\,s}}{\lfloor s}.$$

235. *Theorem.* $\varpi^m . a^{-1} = a^{-1} . \overset{m}{\mathrm{A}}_{+r}\left(\frac{-a_r}{a}\right)$.

For, $(\overset{\infty}{\mathrm{S}}_m a_{m-1}x^{m-1})^{-1} = \overset{\infty}{\mathrm{S}}_n x^{n-1}\varpi^{n-1}a^{-1}$, (228);

$$\therefore\ 1 = (\overset{\infty}{\mathrm{S}}_m a_{m-1}x^{m-1})(\overset{\infty}{\mathrm{S}}_n x^{n-1}\varpi^{n-1}a^{-1})$$

$$=\overset{\infty}{\mathrm{S}}_m x^{m-1} . \overset{m}{\mathrm{S}}_n a_{m-n} . \varpi^{n-1}a^{-1}, \quad (18);$$

$$=1+\overset{\infty}{\mathrm{S}}_m x^m . \overset{m+1}{\mathrm{S}}_n a_{m-n+1} . \varpi^{n-1}a^{-1}, \quad (9).$$

$$\therefore\ 0 = \overset{m+1}{\mathrm{S}}_n a_{m-n+1}\varpi^{n-1}a^{-1}$$

$$=\overset{m}{\mathrm{S}}_n a_{m-n+1} . \varpi^{n-1}a^{-1} + a . \varpi^m a^{-1}, \quad (9);$$

$$\therefore\ \varpi^m a^{-1} = -a^{-1} . \overset{m}{\mathrm{S}}_n a_{m-n+1} . \varpi^{n-1}a^{-1}$$

$$=\overset{m}{\mathrm{S}}_n\left(\frac{-a_n}{a}\right) . \varpi^{m-n}a^{-1}, \quad (8)$$

$$\therefore\ \varpi^m a^{-1} = (\varpi^0 . a^{-1}) . \overset{m}{\mathrm{A}}_{+r}\left(\frac{-a_r}{a}\right), \quad (65);$$

$$= a^{-1} . \overset{m}{\mathrm{A}}_{+r}\left(\frac{-a_r}{a}\right).$$

236. *Theorem.*

$$\frac{\overset{r}{\mathrm{S}}_m a_{m-1} x^{m-1}}{\overset{s}{\mathrm{S}}_n b_{n-1} x^{n-1}} = \overset{\infty}{\mathrm{S}}_m x^{m-1} . \overset{m}{\mathrm{S}}_n a_{m-n} b^{-1} . \overset{n-1}{\mathrm{A}}_{+t}\left(\frac{-b_t}{b}\right).$$

For, $\frac{\overset{r}{\mathrm{S}}_m a_{m-1} x^{m-1}}{\overset{s}{\mathrm{S}}_n b_{n-1} x^{n-1}} = (\overset{\infty}{\mathrm{S}}_m a_{m-1} x^{m-1})(\overset{\infty}{\mathrm{S}}_n b_{n-1} x^{n-1})^{-1}$, subject to the condition that all the coefficients after a_{r-1}, and b_{s-1} vanish;

$$= \overset{\infty}{\mathrm{S}}_m x^{m-1} . \overset{m}{\mathrm{S}}_n a_{m-n} . \varpi^{n-1} b^{-1}, \quad (228), \text{ and } (26);$$

$$= \overset{\infty}{\mathrm{S}}_m x^{m-1} . \overset{m}{\mathrm{S}}_n a_{m-n} . b^{-1} . \overset{n-1}{\mathrm{A}}_{+t}\left(\frac{-b_t}{b}\right), \quad (235).$$

237. $\frac{d_x^{\,n} . \phi(u)}{\lfloor n} = \overset{n}{\mathrm{S}}_m d_u^{\,m} . \phi(u) . \frac{\varpi^{n-m} a^m}{\lfloor m}$, where $a_{m-1} = \frac{d_x^{\,m} u}{\lfloor m}$.

For, $\phi(u + D_x u) = \overset{\infty}{\mathrm{S}}_n \frac{d_u^{\,n-1} . \phi(u)}{\lfloor n-1} . (D_x u)^{n-1}, \quad (198);$

$$= \overset{\infty}{\mathrm{S}}_n \frac{d_u^{\,n-1} . \phi(u)}{\lfloor n-1} (\overset{\infty}{\mathrm{S}}_m \frac{h^m}{\lfloor m} . d_x^{\,m} u)^{n-1}, \quad (199);$$

$$= \overset{\infty}{\mathrm{S}}_n \frac{d_u^{\,n-1} . \phi(u)}{\lfloor n-1} . h^{n-1} \left(\overset{\infty}{\mathrm{S}}_m h^{m-1} . \frac{d_x^{\,m} u}{\lfloor m}\right)^{n-1}, \quad (6);$$

$$=\overset{\infty}{\mathrm{S}}_n \frac{h^{n-1}}{\lfloor n-1} \cdot d_u^{n-1} \cdot \phi(u) \cdot \overset{\infty}{\mathrm{S}}_m h^{m-1} \cdot \varpi^{m-1} a^{n-1},$$

$$\text{where } a_{m-1} = \frac{d_x^{\,m} u}{\lfloor m}, \quad (228);$$

$$=\overset{\infty}{\mathrm{S}}_n h^{n-1} \cdot \overset{n}{\mathrm{S}}_m \frac{d_u^{\,n-m} \cdot \phi(u)}{\lfloor n-m} \cdot \varpi^{m-1} a^{n-m}, \quad (26).$$

$$\therefore \frac{d_x^{\,n} \cdot \phi(u)}{\lfloor n} = \overset{n+1}{\mathrm{S}}_m \frac{d_u^{\,n-m+1} \phi(u)}{\lfloor n-m+1} \cdot \varpi^{m-1} a^{n-m+1}, \quad (201);$$

$$= \overset{n}{\mathrm{S}}_m \frac{d_u^{\,n-m+1} \phi(u)}{\lfloor n-m+1} \cdot \varpi^{m-1} a^{n-m+1}, \text{ since } \varpi^n a^0 = 0;$$

$$= \overset{n}{\mathrm{S}}_m d_u^{\,m} \phi(u) \cdot \frac{\varpi^{n-m} a^m}{\lfloor m}, \quad (8).$$

238. *Theorem.*

$$\phi(\overset{\infty}{\mathrm{S}}_m a_{m-1} x^{m-1}) = \phi(a) + \overset{\infty}{\mathrm{S}}_n x^n \cdot \overset{n}{\mathrm{S}}_m d_a^{\,n-m+1} \phi(a) \cdot \frac{\varpi^{m-1} a_1^{\,n-m+1}}{\lfloor n-m+1}.$$

For, $\phi(\overset{\infty}{\mathrm{S}}_m a_{m-1} x^{m-1}) = \phi(a + x \cdot \overset{\infty}{\mathrm{S}}_m a_m x^{m-1})$

$$= \phi(a) + \overset{\infty}{\mathrm{S}}_n \frac{d_a^n \phi(a)}{\lfloor n} (x \cdot \overset{\infty}{\mathrm{S}}_m a_m x^{m-1})^n, \quad (198);$$

$$= \phi(a) + \overset{\infty}{\mathrm{S}}_n \frac{x^n}{\lfloor n} \cdot d_a^n \cdot \phi(a) \cdot \overset{\infty}{\mathrm{S}}_m x^{m-1} \cdot \varpi^{m-1} a_1^{\,n}, \quad (228);$$

$$= \phi(a) + \overset{\infty}{\mathrm{S}}_n x^n \overset{n}{\mathrm{S}}_m d_a^{\,n-m+1} \phi(a) \cdot \frac{\varpi^{m-1} a_1^{\,n-m+1}}{\lfloor n-m+1}, \quad (18).$$

239. *Theorem.* $\dfrac{1}{\epsilon^x-1}=x^{-1}-\frac{1}{2}+\overset{\infty}{\mathrm{S}}_m x^{2m-1}\,.\,\overset{2m}{\mathrm{A}}_{+r}\left(\dfrac{-1}{\underline{|r+1}}\right).$

For, $\dfrac{1}{\epsilon^x-1}=\left(\overset{\infty}{\mathrm{S}}_m \dfrac{x^m}{\underline{|m}}\right)^{-1}$, (106) and (9);

$=x^{-1}.\overset{\infty}{\mathrm{S}}_m x^{m-1}.\varpi^{m-1}a^{-1}$, when $a_{m-1}=\dfrac{1}{\underline{|m}}$, (6) and (228);

$=\overset{\infty}{\mathrm{S}}_m x^{m-2}.a^{-1}.\overset{m-1}{\mathrm{A}}_{+r}\left(\dfrac{-a_r}{a}\right)$, (6) and (235)

$=\overset{\infty}{\mathrm{S}}_m x^{m-2}.\overset{m-1}{\mathrm{A}}_{+r}\left(\dfrac{-1}{\underline{|r+1}}\right)$

$=\overset{\infty}{\mathrm{S}}_m x^{m-2}.b_{m-2}$, suppose.

But $-1=\dfrac{1-\epsilon^x}{\epsilon^x-1}$

$=\dfrac{1}{\epsilon^x-1}-\dfrac{1}{1-\epsilon^{-x}}$

$=\dfrac{1}{\epsilon^x-1}+\dfrac{1}{\epsilon^{-x}-1}$

$=\overset{\infty}{\mathrm{S}}_m b_{m-2}\{x^{m-2}+(-x)^{m-2}\}$

$=\overset{\infty}{\mathrm{S}}_m b_{2m-3}(x^{2m-3}-x^{2m+3})+\overset{\infty}{\mathrm{S}}_m b_{2m-2}(x^{2m-2}+x^{2m-2})$, (14);

$=2\overset{\infty}{\mathrm{S}}_m b_{2m-2}.x^{2m-2}$, (6)

$=2b_0+2\overset{\infty}{\mathrm{S}}_m b_{2m}x^{2m}$, (9).

$\therefore\ b_0=-\frac{1}{2}$, and $b_{2m}=0$.

Hence $\frac{1}{\epsilon^x-1}=x^{-1}-\frac{1}{2}+\overset{\infty}{\mathrm{S}}_m x^m.\overset{m+1}{\mathrm{A}}_{+r}\left(\frac{-1}{\lfloor r+1}\right)$, (9) and (64);

$$=x^{-1}-\tfrac{1}{2}+\overset{\infty}{\mathrm{S}}_m x^{2m-1}.\overset{2m}{\mathrm{A}}_{+r}\left(\frac{-1}{\lfloor r+1}\right), \quad (14).$$

240. Cor. 1. $\overset{2m+1}{\mathrm{A}}_{+r}\left(\frac{-1}{\lfloor r+1}\right)=0.$

241. The number $\overset{2m}{\mathrm{A}}_{+r}\left(\frac{-1}{\lfloor r+1}\right)$ is called the $(2m-1)^{\text{th}}$ number of Bernouilli. In order to deduce the successive values of these numbers, we have

$$\overset{m}{\mathrm{A}}_{+r}\left(\frac{-1}{\lfloor r+1}\right)=\overset{m}{\mathrm{S}}_n\frac{-1}{\lfloor m-n+2}.\overset{n-1}{\mathrm{A}}_{+r}\left(\frac{-1}{\lfloor r+1}\right), \text{ and}$$

$$\overset{2m}{\mathrm{A}}_{+r}\left(\frac{-1}{\lfloor r+1}\right)=\overset{m}{\mathrm{S}}_n\frac{-1}{\lfloor 2m-2n+3}.\overset{2n-2}{\mathrm{A}}_{+r}\left(\frac{-1}{\lfloor r+1}\right)$$

$$+\overset{m}{\mathrm{S}}_n\frac{-1}{\lfloor 2m-2n+2}.\overset{2n-1}{\mathrm{A}}_{+r}\left(\frac{-1}{\lfloor r+1}\right), \quad (14);$$

$$=\overset{m}{\mathrm{S}}_n\frac{-1}{\lfloor 2m-2n+3}.\overset{2n-2}{\mathrm{A}}_{+r}\left(\frac{-1}{\lfloor r+1}\right)+\frac{1}{2.\lfloor 2m8}, \quad (240);$$

and, putting, $\overset{2m}{\mathrm{A}}_{+r}\left(\frac{-1}{\lfloor r+1}\right)=\mathfrak{C}_{2m-1}$,

$$\mathfrak{C}_1=\frac{1}{2\lfloor 3}=,08\dot{3}$$

$$\mathfrak{C}_3=-\frac{1}{\lfloor 6}=-,0013\dot{8}$$

$$\mathfrak{C}_5 = \frac{1}{6\,\underline{|7}} = ,000033068783$$

$$\mathfrak{C}_7 = -\frac{3}{\underline{|10}} = -,000000826719$$

$$\mathfrak{C}_9 = \frac{10}{\underline{|12}} = ,000000020876$$

$$\mathfrak{C}_{11} = -\frac{691}{\underline{|15}} = -,000000000528$$

$$\mathfrak{C}_{13} = \frac{1}{12\,\underline{|13}} = ,000000000013$$

$$\mathfrak{C}_{15} = -\frac{3617}{30\,\underline{|17}} = -,000000000000.$$

242. *Theorem.*

$$D_x^{-1}u = h^{-1}.\int_x u - \frac{u}{2} + \overset{\infty}{\mathrm{S}}_m \mathfrak{C}_{2m-1} h^{2m-1}.d_x^{2m-1}u.$$

For, assume $D_x^{-1}u = \overset{\infty}{\mathrm{S}}_m h^{m-2}.a_{m-2}$, where a_{m-2}, is some function of x, and independent of h.

$$\therefore u = \overset{\infty}{\mathrm{S}}_m h^{m-2}.D_x.a_{m-2}, \quad (136) \text{ and } (137);$$

$$= \overset{\infty}{\mathrm{S}}_m h^{m-2}.\overset{\infty}{\mathrm{S}}_n \frac{h^n}{\underline{|n}}.d_x^n.a_{m-2}, \quad (199);$$

$$= \overset{\infty}{\mathrm{S}}_m h^{m-1}.\overset{m}{\mathrm{S}}_n \frac{d_x^n.a_{m-n-1}}{\underline{|n}}, \quad (6) \text{ and } (18);$$

$$= d_x.a_{-1} + \overset{\infty}{\mathrm{S}}_m h^m.\overset{m+1}{\mathrm{S}}_n \frac{d_x^n.a_{m-n}}{\underline{|n}}, \quad (9).$$

$$\therefore \; u = d_x . a_{-1}, \text{ or } a_{-1} = \int_x u$$

$$\text{and } 0 = \overset{m+1}{\mathrm{S}_n} \frac{d_x^{\,n} . a_{m-n}}{|\underline{n}}$$

$$\therefore \; d_x . a_{m-1} = -\overset{m}{\mathrm{S}_n} \frac{d_x^{\,n+1} . a_{m-n-1}}{|\underline{n+1}}, \quad (9).$$

A little consideration respecting the form of this development will convince us that $d_x^{\,m} u$ is a factor of $d_x . a_{m-1}$.

$$\text{Assume, therefore, } d_x . a_{m-1} = d_x^{\,m} u . \frac{(-1)^{m+1}}{|\underline{m+1}} . b_m;$$

$$\text{then } d_x . a_{-1} = u . (-1) . b_0, \text{ and } = u. \quad \therefore \; b_0 = -1.$$

$$\text{Also } d_x^{\,m} u . \frac{(-1)^{m+1}}{|\underline{m+1}} . b_m = -\overset{m}{\mathrm{S}_n} d_x^{\,n} . d_x^{\,m-n} u . \frac{(-1)^{m-n+1} . b_{m-n}}{|\underline{m-n+1} \; |\underline{n+1}},$$

by substitution;

$$= d_x^{\,m} u . \overset{m}{\mathrm{S}_n} \frac{(-1)^{m-n} . b_{m-n}}{|\underline{m-n+1} \; |\underline{n+1}}, \quad (6).$$

$$\therefore \; \frac{(-1)^m b_m}{|\underline{m+1}} = \overset{m}{\mathrm{S}_n} \frac{(-1)^{m-n} b_{m-n} . (-1)}{|\underline{m-n+1} \quad |\underline{n+1}},$$

$$\text{and } \frac{(-1)^m b_m}{|\underline{m+1}} = b_0 . \overset{m}{\mathrm{A}}_{+r} \left(\frac{-1}{|\underline{r+1}} \right), \quad (65);$$

$$= -\overset{m}{\mathrm{A}}_{+r} \left(\frac{-1}{|\underline{r+1}} \right).$$

$$\therefore \; d_x . a_{m-1} = d_x^{\,m} u . \overset{m}{\mathrm{A}}_{+r} \left(\frac{-1}{|\underline{r+1}} \right),$$

$$\text{and } a_{m-1} = d_x^{\,m-1} u . \overset{m}{\mathrm{A}}_{+r} \left(\frac{-1}{|\underline{r+1}} \right).$$

Hence $D_x^{-1}u=h^{-1}.\int_x u+\overset{\infty}{S}_m h^{m-1}.d_x^{\,m-1}u.\overset{m}{A}_{+r}\left(\frac{-1}{\lfloor r+1}\right)$

$$=h^{-1}.\int_x u-\frac{u}{2}+\overset{\infty}{S}_m h^{2m-1}.d_x^{\,2m-1}u.\overset{2m}{A}_{+r}\left(\frac{-1}{\lfloor r+1}\right),\quad (240),$$

$$=h^{-1}.\int_x u-\frac{u}{2}+\overset{\infty}{S}_m \mathfrak{C}_{2m-1}h^{2m-1}.d_x^{\,2m-1}u,\quad (241).$$

243. $\Delta_x^{-1}u=\int_x u-\frac{u}{2}+\overset{\infty}{S}_m \mathfrak{C}_{2m-1}.d_x^{\,2m-1}u.$

244. $\Delta^{-1}.x^n=\frac{x^{n+1}}{n+1}-\frac{x^n}{2}+\overset{\infty}{S}_m \mathfrak{C}_{2m-1}.\lfloor n_{2m-1}.x^{n-2m+1}+\text{const.}$

CHAPTER VIII.

ON THE DIFFERENTIATION OF EXPONENTIAL AND CIRCULAR FUNCTIONS.

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245. *Theorem.* $d_x . \epsilon^x = \epsilon^x$.

For, $\epsilon^x = 1 + \overset{\infty}{\text{S}}_m \dfrac{x^m}{\underline{|m}}$, (106) and (9).

$\therefore\ d_x . \epsilon^x = \overset{\infty}{\text{S}}_m \dfrac{x^{m-1}}{\underline{|m-1}}$, (193), (191), and (210);

$= \epsilon^x$, (106).

246. Cor. 1. $d_x^n . \epsilon^x = \epsilon^x$.

247. Cor. 2. $d_x . \epsilon^u = \epsilon^u . d_x u$, (202).

248. Cor. 3. $d_x^n . \epsilon^{ax} = \epsilon^{ax} . a^n$.

249. Cor. 4. $d_x^n . a^x = d_x^n . (\epsilon^{x . \log_\epsilon a})$

$= \epsilon^{x . \log_\epsilon a} . (\log_\epsilon a)^n$

$= a^x . (\log_\epsilon a)^n$.

250. *Theorem.* $d_x . (\log_\epsilon x) = \dfrac{1}{x}$.

For, $x = \epsilon^{\log_\epsilon x}$

$\therefore\ 1 = \epsilon^{\log_\epsilon x} . d_x (\log_\epsilon x)$, (197), and (247);

$= x . d_x . (\log_\epsilon x)$.

$\therefore\ d_x . (\log_\epsilon x) = \dfrac{1}{x}$.

251. Cor. 1. $d_x.(\log_\epsilon u) = \frac{d_x u}{u}$, (202).

252. Cor. 2. $d_x.(\log_a u) = d_x.\frac{\log_\epsilon u}{\log_\epsilon a}$

$$= \frac{1}{\log_\epsilon a}.d_x.(\log_\epsilon u), \quad (196);$$

$$= \frac{1}{\log_\epsilon a}.\frac{d_x u}{u}.$$

253. *Theorem.*

$$\log_\epsilon x = \frac{1}{n}.\overset{r}{\text{S}}_m(-1)^{m-1}.\frac{(x^n-1)^m}{m} + (-1)^r.\int_{\substack{x=x\\x=1}} \frac{(x^n-1)^r}{x}.$$

For, $d_x.(\log_\epsilon x) = \frac{1}{x}$, (250);

$$= \frac{x^{n-1}}{1+x^n-1}, \ n \text{ being any number};$$

$$= x^{n-1}\left\{\overset{r}{\text{S}}_m(-1)^{m-1}(x^n-1)^{m-1} + (-1)^r.\frac{(x^n-1)^r}{x^n}\right\}, \quad (12);$$

$$= \frac{1}{n}.\overset{r}{\text{S}}_m\frac{(-1)^{m-1}}{m}.d_x.(x^n-1)^m + (-1)^r.\frac{(x^n-1)^r}{x}, \quad (192), (210),$$

(191) and (207);

$$\therefore \log_\epsilon x = \frac{1}{n}.\overset{r}{\text{S}}_m(-1)^{m-1}.\frac{(x^n-1)^m}{m} + (-1)^r.\int_x \frac{(x^n-1)^r}{x} + \text{const.}$$

Hence

$$\log_\epsilon 1 = \frac{1}{n}.\overset{r}{\text{S}}_m(-1)^{m-1}.\frac{(1-1)^m}{m} + (-1)^r.\int_{x=1} \frac{(x^n-1)^r}{x} + \text{const.}$$

and $\log_\epsilon x = \frac{1}{n}.\overset{r}{\text{S}}_m(-1)^{m-1}.\frac{(x^n-1)^m}{m} + (-1)^r.\int_{\substack{x=x\\x=1}} \frac{(x^n-1)^r}{x}.$

254. Cor. 1. If $x^n \sim 1 < 1$,

then $\log_\epsilon x = \frac{1}{n} \cdot \overset{\infty}{\mathrm{S}}_m (-1)^{m-1} \cdot \frac{(x^n - 1)^m}{m}$, (12).

255. Cor. 2. If $x < 1$, then

$$\log_\epsilon (1+x) = \overset{\infty}{\mathrm{S}}_m (-1)^{m-1} \cdot \frac{x^m}{m},$$

$$\log_\epsilon (1-x) = -\overset{\infty}{\mathrm{S}}_m \frac{x^m}{m},$$

and $\log_\epsilon \frac{1+x}{1-x} = \log_\epsilon (1+x) - \log_\epsilon (1-x)$

$$= \overset{\infty}{\mathrm{S}}_m (-1)^{m-1} \cdot \frac{x^m}{m} + \overset{\infty}{\mathrm{S}}_m \frac{x^m}{m}$$

$$= \overset{\infty}{\mathrm{S}}_m \left\{ (-1)^{m-1} + 1 \right\} \frac{x^m}{m}, \quad (5);$$

$$= 2 \cdot \overset{\infty}{\mathrm{S}}_m \frac{x^{2m-1}}{2m-1}, \quad (14).$$

256. *Theorem.* $\log_\epsilon x = \frac{1}{n} \cdot \overset{\infty}{\mathrm{S}}_m \frac{x^{mn} - 1}{m(x^n+1)^m}$.

For, $\log_\epsilon x = \frac{1}{n} \cdot \left\{ \log_\epsilon \frac{x^n}{x^n+1} - \log_\epsilon \frac{1}{x^n+1} \right\}$

$$= \frac{1}{n} \cdot \left\{ \overset{\infty}{\mathrm{S}}_m \frac{(-1)^{m-1}}{m} \left(\frac{x^n}{x^n+1} - 1 \right)^m \right.$$

$$\left. - \overset{\infty}{\mathrm{S}}_m \frac{(-1)^{m-1}}{m} \left(\frac{1}{x^n+1} - 1 \right)^m \right\} \quad (255);$$

$$= \frac{1}{n} \cdot \left\{ \overset{\infty}{\mathrm{S}}_m \frac{(-1)^{2m-1}}{m(x^n+1)^m} + \overset{\infty}{\mathrm{S}}_m \frac{(-1)^{2m} \cdot x^{mn}}{m(x^n+1)^m} \right\}$$

$$= \frac{1}{n} \cdot \overset{\infty}{\mathrm{S}}_m \frac{x^{mn} - 1}{m(x^n+1)^m}, \quad (5).$$

257. *Theorem.* $\phi(\epsilon^x) = \overset{\infty}{S}_m \frac{x^{m-1}}{\lfloor m-1} \cdot \{\phi(1+\Delta)\} 0^{m-1}.$

For, $\phi(\epsilon^x) = \phi\{1 + \epsilon^x - 1\}$

$$= \overset{\infty}{S}_n \frac{d_{t=1}^{n-1} \cdot \phi(t)}{\lfloor n-1} (\epsilon^x - 1)^{n-1}, \quad (198);$$

$$= \overset{\infty}{S}_n \frac{d_{t=1}^{n-1} \cdot \phi(t)}{\lfloor n-1} \cdot \overset{n}{S}_r (-1)^{r-1} \cdot \frac{\lfloor n-1}{\lfloor r-1 \lfloor n-r} \cdot \epsilon^{(n-r)x}, \quad (86);$$

$$= \overset{\infty}{S}_n \frac{d_{t=1}^{n-1} \cdot \phi(t)}{\lfloor n-1} \cdot \overset{n}{S}_r (-1)^{r-1} \cdot \frac{\lfloor n-1}{\lfloor r-1 \lfloor n-r} \cdot \overset{\infty}{S}_m (n-r)^{m-1} \cdot \frac{x^{m-1}}{\lfloor m-1}, \ (106);$$

$$= \overset{\infty}{S}_m \frac{x^{m-1}}{\lfloor m-1} \cdot \overset{\infty}{S}_n \frac{d_{t=1}^{n-1} \cdot \phi(t)}{\lfloor n-1} \cdot \overset{n}{S}_r (-1)^{r-1} \cdot \frac{\lfloor n-1}{\lfloor r-1 \lfloor n-r} \cdot (n-r)^{m-1}, \quad (17);$$

$$= \overset{\infty}{S}_m \frac{x^{m-1}}{\lfloor m-1} \cdot \overset{\infty}{S}_n \frac{d_{t=1}^{n-1} \cdot \phi(t)}{\lfloor n-1} \cdot \Delta^{n-1} \cdot 0^{m-1}, \quad (150);$$

$$= \overset{\infty}{S}_m \frac{x^{m-1}}{\lfloor m-1} \cdot \{\phi(1+\Delta)\} 0^{m-1}, \quad (198).$$

258. *Theorem.* $(\epsilon^x - 1)^n = \overset{\infty}{S}_m \frac{x^{n+m-1}}{\lfloor n+m-1} \cdot \Delta^n \cdot 0^{n+m-1}.$

For, $(\epsilon^x - 1)^n = \overset{\infty}{S}_m \frac{x^{m-1}}{\lfloor m-1} \cdot (1 + \Delta - 1)^n \cdot 0^{m-1}, \quad (257);$

$$= \overset{\infty}{S}_m \frac{x^{m-1}}{\lfloor m-1} \cdot \Delta^n 0^{m-1}.$$

$$= \overset{n}{S}_m \frac{x^{m-1}}{\lfloor m-1} \cdot \Delta^n 0^{m-1} + \overset{\infty}{S}_m \frac{x^{n+m-1}}{\lfloor n+m-1} \cdot \Delta^n \cdot 0^{n+m-1}, \quad (9);$$

$$= \overset{\infty}{S}_m \frac{x^{n+m-1}}{\lfloor n+m-1} \cdot \Delta^n 0^{n+m-1}, \quad (146) \text{ and } (138).$$

259. *Theorem.* $\frac{1}{\epsilon^x+1} = \overset{\infty}{S}_m \frac{x^{m-1}}{\lfloor m-1} . \overset{m}{S}_n (-1)^{n-1} . \frac{\Delta^{n-1} . 0^{m-1}}{2^n}$.

For, $\frac{1}{\epsilon^x+1} = \overset{\infty}{S}_m \frac{x^{m-1}}{\lfloor m-1} . \left\{\frac{1}{1+\Delta+1}\right\} 0^{m-1}$, (257);

$$= \overset{\infty}{S}_m \frac{x^{m-1}}{\lfloor m-1} . \left(\frac{1}{2+\Delta}\right) 0^{m-1}$$

$$= \overset{\infty}{S}_m \frac{x^{m-1}}{\lfloor m-1} . \overset{\infty}{S}_n (-1)^{n-1} . \frac{\Delta^{n-1} . 0^{m-1}}{2^n}, \quad (12);$$

$$= \overset{\infty}{S}_m \frac{x^{m-1}}{\lfloor m-1} . \overset{m}{S}_n (-1)^{n-1} . \frac{\Delta^{n-1} . 0^{m-1}}{2^n}, \quad (146) \text{ and } (138).$$

260. *Theorem.* $\frac{x}{\epsilon^x-1} = \overset{\infty}{S}_m \frac{x^{m-1}}{\lfloor m-1} . \overset{m}{S}_n (-1)^{n-1} . \frac{\Delta^{n-1} . 0^{m-1}}{n}$

For, $\frac{x}{\epsilon^x-1} = \frac{\log_\epsilon \epsilon^x}{\epsilon^x-1}$

$$= \overset{\infty}{S}_m \frac{x^{m-1}}{\lfloor m-1} . \left\{\frac{\log_\epsilon (1+\Delta)}{1+\Delta-1}\right\} 0^{m-1}, \quad (257);$$

$$= \overset{\infty}{S}_m \frac{x^{m-1}}{\lfloor m-1} . \left\{\frac{\log_\epsilon (1+\Delta)}{\Delta}\right\} 0^{m-1}$$

$$= \overset{\infty}{S}_m \frac{x^{m-1}}{\lfloor m-1} . \overset{m}{S}_n (-1)^{n-1} . \frac{\Delta^{n-1} . 0^{m-1}}{n}, \quad (255), (146), \text{ and}$$

(138.)

261. Cor. Since $\frac{x}{\epsilon^x - 1} = 1 - \frac{x}{2} + \overset{\infty}{S}_m x^{2m} . \mathfrak{C}_{2m-1}$, (239) and (241);

$$\therefore \overset{2m+2}{S}_n (-1)^{n-1} . \frac{\Delta^{n-1} . 0^{2m+1}}{n} = 0,$$

$$\overset{2m+1}{S}_n (-1)^{n-1} . \frac{\Delta^{n-1} . 0^{2m}}{n} = \underline{|2m} . \mathfrak{C}_{2m-1}, \text{ and}$$

$$\frac{x}{\epsilon^x - 1} = 1 - \frac{x}{2} + \overset{\infty}{S}_m \frac{x^{2m}}{\underline{|2m}} . \overset{2m+1}{S}_n (-1)^{n-1} . \frac{\Delta^{n-1} . 0^{2m}}{n} .$$

262. *Theorem.* $D_x^{\ n} u = \overset{\infty}{S}_m \frac{\Delta^n 0^{n+m-1}}{\underline{|n+m-1}} . h^{n+m-1} . d_x^{\ n+m-1} u$

For, $D_x^{\ n} u = (\epsilon^{hd_x} - 1)^n . u$, (201);

$$= \overset{\infty}{S}_m \frac{\Delta^n 0^{n+m-1}}{\underline{|n+m-1}} . h^{n+m-1} . d_x^{\ n+m-1} u, \ (258).$$

263. *Theorem.* $\{\log_\epsilon (1 + \Delta)\}^n . 0^m = 0$, $(m \gtrless n)$; and

$$\{\log_\epsilon (1 + \Delta)\}^n . 0^n = \underline{|n}.$$

For, $\overset{\infty}{S}_m \frac{x^{m-1}}{\underline{|m-1}} . \{\log_\epsilon (1 + \Delta)\}^n . 0^{m-1} = (\log_\epsilon \epsilon^x)^n$, (257);

$$= x^n.$$

$\therefore \{\log_\epsilon (1 + \Delta)\}^n . 0^{m-1} = 0$, $(m \gtrless n)$, and $\frac{\{\log_\epsilon (1 + \Delta)\}^n . 0^n}{\underline{|n}} = 1$.

264. *Theorem.* $\{\log_\epsilon (1 + \Delta_x)\} u = d_x u$.

For, $\{\log_\epsilon (1 + \Delta_x)\} u = \overset{\infty}{S}_m (-1)^{m-1} . \frac{\Delta_x^{\ m} u}{m}$, (255);

$$= \overset{\infty}{S}_m \frac{(-1)^{m-1}}{m} . (\epsilon^{d_x} - 1)^m u, \quad (201);$$

$$=\overset{\infty}{\mathrm{S}}_m \frac{(-1)^{m-1}}{m} \cdot \overset{\infty}{\mathrm{S}}_n \frac{\Delta^m 0^{m+n-1}}{\lfloor m+n-1} \cdot d_x^{\,m+n-1} u, \quad (258);$$

$$=\overset{\infty}{\mathrm{S}}_m \frac{d_x^{\,m} u}{\lfloor m} \cdot \overset{m}{\mathrm{S}}_n \frac{(-1)^{m-n}}{m-n+1} \cdot \Delta^{m-n+1} 0^m, \quad (6) \text{ and } (18);$$

$$=\overset{\infty}{\mathrm{S}}_m \frac{d_x^{\,m} u}{\lfloor m} \cdot \overset{\infty}{\mathrm{S}}_n (-1)^{n-1} \cdot \frac{\Delta^n 0^m}{n}, \quad (8) \text{ and } (13);$$

$$=\overset{\infty}{\mathrm{S}}_m \frac{d_x^{\,m} u}{\lfloor m} \cdot \{\log_\epsilon (1+\Delta)\}\, 0^m, \quad (255);$$

$$=d_x u \quad (263).$$

265. Cor. $d_x^{\,n} u = \{\log_\epsilon (1+\Delta_x)\}^n u.$

266. *Theorem.* $\left(\frac{\sin x}{x}\right)_{x=0} = 1$, and $\left(\frac{\tan x}{x}\right)_{x=0} = 1$.

For, $\frac{\tan x}{x} = \frac{\sin x}{x} \cdot \frac{1}{\cos x}$.

$$\therefore \left(\frac{\tan x}{x}\right)_{x=0} = \left(\frac{\sin x}{x}\right)_{x=0}.$$

Also, for every finite value of x, $\tan x > x$, and $\sin x < x$.

And these two relations can only be made to agree by the equations

$$\left(\frac{\tan x}{x}\right)_{x=0} = 1 \text{ and } \left(\frac{\sin x}{x}\right)_{x=0} = 1.$$

267. *Theorem.* $d_x \cdot \sin x = \cos x$.

For, $d_x \cdot \sin x = \left\{\frac{\sin (x+h) - \sin x}{h}\right\}_{h=0}$

$$= \left\{\frac{\cos\left(x+\frac{h}{2}\right) \cdot \sin \frac{h}{2}}{\frac{h}{2}}\right\}_{h=0}$$

$$= \cos x, \quad (266).$$

268. Cor. 1. $d_x.\cos x = d_x.\sin\left(\frac{\pi}{2} - x\right)$

$$= -\cos\left(\frac{\pi}{2} - x\right), \quad (202)$$

$$= -\sin x.$$

269. Cor. 2. $d_x^{2n-1}.\sin x = (-1)^{n-1}\cos x,$

$$d_x^{2n}.\sin x = (-1)^n.\sin x,$$

$$d_x^{2n-1}.\cos x = (-1)^n.\sin x,$$

$$d_x^{2n}.\cos x = (-1)^n.\cos x.$$

270. *Theorem.* $d_x.\tan x = (\sec x)^2.$

For, $d_x.\tan x = d_x.\frac{\sin x}{\cos x}$

$$= \frac{(\cos x)^2 + (\sin x)^2}{(\cos x)^2}$$

$$= (\sec x)^2.$$

271. *Theorem.* $d_x.\sec x = \sec x.\tan x.$

For, $d_x.\sec x = d_x.(\cos x)^{-1}$

$$= (-1)(\cos x)^{-2}.(-\sin x) = \sec x.\tan x.$$

272. *Theorem.* $d_x.\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}.$

For, $x = \sin(\sin^{-1}x).$

$$\therefore\ 1 = \cos(\sin^{-1}x).d_x\sin^{-1}x, \quad (202);$$

$$= \sqrt{1-x^2}.d_x.\sin^{-1}x.$$

$$\therefore\ d_x.\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$$

273. *Theorem.* $d_x.\cos^{-1}x = \dfrac{-1}{\sqrt{1-x^2}}.$

For, $x = \cos.(\cos^{-1}x)$.

$$\therefore\ 1 = -\sin(\cos^{-1}x).d_x.\cos^{-1}x$$

$$= -\sqrt{1-x^2}.d_x.\cos^{-1}x.$$

$$\therefore\ d_x.\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}.$$

274. *Theorem.* $d_x.\tan^{-1}x = \dfrac{1}{1+x^2}.$

For, $x = \tan.(\tan^{-1}.x)$.

$$\therefore\ 1 = \{\sec.(\tan^{-1}x)\}^2.d_x.\tan^{-1}x$$

$$= (1+x^2).d_x.\tan^{-1}x.$$

$$\therefore\ d_x.\tan^{-1}x = \frac{1}{1+x^2}.$$

275. *Theorem.* $d_x.\sec^{-1}x = \dfrac{1}{x\sqrt{x^2-1}}.$

For, $x = \sec.(\sec^{-1}x)$.

$$\therefore\ 1 = \tan(\sec^{-1}x).\sec(\sec^{-1}x).d_x.\sec^{-1}x.$$

$$= \sqrt{x^2-1}.x.d_x.\sec^{-1}x.$$

$$\therefore\ d_x.\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}}.$$

CHAPTER IX.

ON THE EXPANSION OF CIRCULAR FUNCTIONS.

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276. *Theorem.*

$$\overset{n}{\mathrm{P}}_r(\cos x_r+\sqrt{-1}\,.\sin x_r)=\cos \overset{n}{\mathrm{S}}_r x_r+\sqrt{-1}\,.\sin \overset{n}{\mathrm{S}}_r x_r.$$

For, $(\cos x_1+\sqrt{-1}\,.\sin x_1)(\cos x_2+\sqrt{-1}\,.\sin x_2)$

$$=\cos(x_1+x_2)+\sqrt{-1}\,.\sin(x_1+x_2).$$

And the introduction on the first side of the equation of a new factor of the form $\cos x_r+\sqrt{-1}\,.\sin x_r$ will increase the arc on the second side by the quantity x_r. Hence the truth of the theorem is manifest.

277. Cor. 1. Put $x_r=x$, then

$(\cos x+\sqrt{-1}\,.\sin x)^n=\cos nx+\sqrt{-1}\,.\sin nx$, n being any positive integer.

Again, $(\cos x+\sqrt{-1}\,.\sin x)(\cos x-\sqrt{-1}\,.\sin x)=1$.

$$\therefore\ (\cos x+\sqrt{-1}\,.\sin x)^{-1}=\cos x-\sqrt{-1}\,.\sin x$$

$$=\cos(-x)+\sqrt{-1}\,.\sin(-x),$$

and $(\cos x+\sqrt{-1}\,.\sin x)^{-n}=\{\cos(-x)+\sqrt{-1}\,.\sin(-x)\}^n$

$$=\cos(-nx)+\sqrt{-1}\,.\sin(-nx).$$

Hence, $(\cos x+\sqrt{-1}.\sin x)^{\pm m}=\cos(\pm mx)+\sqrt{-1}.\sin(\pm mx)$, m being any positive integer;

$$=\cos\left\{n.\left(\pm\frac{m}{n}\right)x\right\}+\sqrt{-1}.\sin\left\{n\left(\pm\frac{m}{n}\right)x\right\},$$

n being any positive integer;

$$=\left\{\cos\left(\pm\frac{m}{n}.x\right)+\sqrt{-1}.\sin\left(\pm\frac{m}{n}.x\right)\right\}^n$$

$$\therefore\ (\cos x+\sqrt{-1}.\sin x)^{\pm\frac{m}{n}}=\cos\left(\pm\frac{m}{n}.x\right)+\sqrt{-1}.\sin\left(\pm\frac{m}{n}.x\right).$$

Consequently, whatever rational value n has,

$$(\cos x+\sqrt{-1}.\sin x)^n=\cos nx+\sqrt{-1}.\sin nx.$$

278. Cor. 2. $2\cos nx=(\cos x+\sqrt{-1}.\sin x)^n$

$$+(\cos x\pm\sqrt{-1}.\sin x)^{\mp n},$$

and $2\sqrt{-1}.\sin nx=(\cos x+\sqrt{-1}.\sin x)^n-(\cos x\pm\sqrt{-1}.\sin x)^{\mp n}$.

279. *Theorem.* $\cos \overset{n}{\mathrm{S}}_r x_r=\overset{\infty}{\mathrm{S}}_m(-1)^{m-1}.\overline{\mathrm{C}_{r,s}}^{\,n-2m+2,\,2m-2}(\cos x_r.\sin x_s)$,

and $\sin \overset{n}{\mathrm{S}}_r x_r=\overset{\infty}{\mathrm{S}}_m(-1)^{m-1}.\overline{\mathrm{C}_{r,s}}^{\,n-2m+1,\,2m-1}(\cos x_r.\sin x_s)$.

For, $\cos \overset{n}{\mathrm{S}}_r x_r+\sqrt{-1}.\sin \overset{n}{\mathrm{S}}_r x_r$

$=\overset{n}{\mathrm{P}}_r(\cos x_r+\sqrt{-1}.\sin x_r)$, (276);

$=\overset{\infty}{\mathrm{S}}_m\ \overline{\mathrm{C}_{r,s}}^{\,n-m+1,\,m-1}\{(\cos x_r)(\sqrt{-1}.\sin x_s)\}$, (67) and (13);

$=\overset{\infty}{\mathrm{S}}_m(-1)^{\frac{m-1}{2}}\ \overline{\mathrm{C}_{r,s}}^{\,n-m+1,\,m-1}(\cos x_r.\sin x_s)$, (55);

$=\overset{\infty}{\mathrm{S}}_m(-1)^{m-1}.\overline{\mathrm{C}_{r,s}}^{\,n-2m+2,\,2m-2}(\cos x_r.\sin x_s)$

$+\sqrt{-1}.\overset{\infty}{\mathrm{S}}_m(-1)^{m-1}.\overline{\mathrm{C}_{r,s}}^{\,n-2m+1,\,2m-1}(\cos x_r.\sin x_s)$, (14) and (6).

Whence, by equating possible and impossible parts, we obtain the above theorems.

280. *Theorem.* $\tan \overset{n}{S}_r x_r = \dfrac{\overset{\infty}{S}_m (-1)^{m-1} . \overset{2m-1,n}{C}_r (\tan x_r)}{\overset{\infty}{S}_m (-1)^{m-1} . \overset{2m-2,n}{C}_r (\tan x_r)}$.

For, $\cos \overset{n}{S}_r x_r + \sqrt{-1} . \sin \overset{n}{S}_r x_r$

$= \overset{n}{P}_r (\cos x_r) . \overset{n}{P}_r (1 + \sqrt{-1} . \tan x_r)$, (276);

$= \overset{n}{P}_r (\cos x_r) . \overset{\infty}{S}_m (-1)^{\frac{m-1}{2}} . \overset{m-1,n}{C}_r (\tan x_r)$, (68), (55) and (13);

$= \overset{n}{P}_r (\cos x_r) \left\{ \overset{\infty}{S}_m (-1)^{m-1} . \overset{2m-2,n}{C}_r (\tan x_r) + \sqrt{-1} . \overset{\infty}{S}_m (-1)^{m-1} . \overset{2m-1,n}{C}_r (\tan x_r) \right\}$, (14) and (6).

Hence, equating possible and impossible parts,

$$\cos \overset{n}{S}_r x_r = \overset{n}{P}_r (\cos x_r) . \overset{\infty}{S}_m (-1)^{m-1} . \overset{2m-2,n}{C}_r (\tan x_r), \text{ and}$$

$$\sin \overset{n}{S}_r x_r = \overset{n}{P}_r (\cos x_r) \overset{\infty}{S}_m (-1)^{m-1} . \overset{2m-1,n}{C}_r (\tan x_r);$$

and, by dividing the second of these equations by the first, we obtain the theorem sought.

281. *Theorem.*

$\cos nx = \frac{1}{2} n . \overset{\infty}{S}_m (-1)^{m-1} . \dfrac{\underline{|n-m}_{m-2}}{\underline{|m-1}} . (2 \cos x)^{n-2m+2}$; n being any integer.

For, $2 \cos nx = (\cos x + \sqrt{-1} . \sin x)^n$

$$+ (\cos x - \sqrt{-1} . \sin x)^n, \quad (278);$$

$$\therefore \; -2 . \overset{\infty}{S}_n \frac{\cos nx}{nt^n} = -\overset{\infty}{S}_n \frac{(\cos x + \sqrt{-1} . \sin x)^n}{nt^n}$$

$$-\overset{\infty}{S}_n \frac{(\cos x - \sqrt{-1} . \sin x)^n}{nt^n},$$

t being any quantity greater than unity, (4) and (6).

$$=\log_\epsilon\left(1-\frac{\cos x+\sqrt{-1}\,.\sin x}{t}\right)+\log_\epsilon\left(1-\frac{\cos x-\sqrt{-1}\,.\sin x}{t}\right),$$

(255);

$$=\log_\epsilon\left\{1-t^{-1}.(2\cos x-t^{-1})\right\}$$

$$=-\overset{\infty}{\mathrm{S}}_n\frac{(2\cos x-t^{-1})^n}{nt^n},\quad (255);$$

$$=-\overset{\infty}{\mathrm{S}}_n\frac{1}{nt^n}.\overset{\infty}{\mathrm{S}}_m\frac{\underline{|n}_{\,m-1}}{\underline{|m-1}}.\frac{(-1)^{m-1}}{t^{m-1}}.(2\cos x)^{n-m+1},\quad (86)\text{ and }(13);$$

$$=-\overset{\infty}{\mathrm{S}}_n\frac{1}{t^n}.\overset{n}{\mathrm{S}}_m(-1)^{m-1}.\frac{\underline{|n-m}_{\,m-2}}{\underline{|m-1}}.(2\cos x)^{n-2m+2},\quad (6)\text{ and }(18).$$

Hence, equating coefficients of t^{-n},

$$\frac{-2\cos nx}{n}=-\overset{n}{\mathrm{S}}_m(-1)^{m-1}.\frac{\underline{|n-m}_{\,m-2}}{\underline{|m-1}}.(2\cos x)^{n-2m+2};$$

$$\therefore\ \cos nx=\tfrac{1}{2}n.\overset{\infty}{\mathrm{S}}_m(-1)^{m-1}.\frac{\underline{|n-m}_{\,m-2}}{\underline{|m-1}}.(2\cos x)^{n-2m+2},\quad (13).$$

282. Cor. If n is even, every term will vanish after the $(\frac{1}{2}n+1)^{\text{th}}$, and

$$\cos nx=\tfrac{1}{2}n.(-1)^{\frac{1}{2}n}.\overset{\frac{1}{2}n+1}{\mathrm{S}}_n(-1)^{m-1}.\frac{\underline{|\tfrac{1}{2}n+m-2}_{\,\frac{1}{2}n-m}}{\underline{|\tfrac{1}{2}n-m+1}}.(2\cos x)^{2m-2},$$

(8) and (6);

$$\text{and } \tfrac{1}{2}n \cdot \frac{\lfloor\!\underline{\tfrac{1}{2}n+m-2}_{\,\frac{1}{2}n-m}}{\lfloor\!\underline{\tfrac{1}{2}n-m+1}} = \tfrac{1}{2}n \cdot \frac{\lfloor\!\underline{\tfrac{1}{2}n+m-2}}{\lfloor\!\underline{2m-2} \cdot \lfloor\!\underline{\tfrac{1}{2}n-m+1}}$$

$$= \tfrac{1}{2}n \cdot \frac{\lfloor\!\underline{\tfrac{1}{2}n+m-2}_{\,2m-3}}{\lfloor\!\underline{2m-2}}$$

$$= \frac{\tfrac{1}{2}n}{\lfloor\!\underline{2m-2}} \cdot \lfloor\!\underline{\tfrac{1}{2}n+m-2}_{\,m-2} \cdot \tfrac{1}{2}n \cdot \lfloor\!\underline{\tfrac{1}{2}n-1}_{\,m-2}$$

$$= \frac{(\tfrac{1}{2}n)^2}{\lfloor\!\underline{2m-2}} \cdot \lfloor\!\underline{\tfrac{1}{2}n+1}_{\,m-2,\,1} \cdot \lfloor\!\underline{\tfrac{1}{2}n-1}_{\,m-2}, \qquad (40);$$

$$= \frac{(\tfrac{1}{2}n)^2}{\lfloor\!\underline{2m-2}} \cdot \overset{m-2}{\mathrm{P}}_r \left\{(\tfrac{1}{2}n+r)(\tfrac{1}{2}n-r)\right\}$$

$$= \frac{(\tfrac{1}{2}n)^2}{\lfloor\!\underline{2m-2}} \cdot \overset{m-2}{\mathrm{P}}_r \left\{(\tfrac{1}{2}n)^2-r^2\right\}$$

$$= \frac{n^2}{\lfloor\!\underline{2m-2} \cdot 2^{2m-2}} \cdot \overset{m-2}{\mathrm{P}}_r (n^2-4r^2);$$

$$\therefore \cos nx = (-1)^{\frac{1}{2}n} \cdot n^2 \cdot \overset{\frac{1}{2}n+1}{\mathrm{S}}_m (-1)^{m-1} \cdot \frac{\overset{m-2}{\mathrm{P}}_r (n^2-4r^2)}{\lfloor\!\underline{2m-2}} \cdot (\cos x)^{2m-2}.$$

If n is odd, every term will vanish after the $\frac{1}{2}(n+1)^{\text{th}}$, and

$$\cos nx = \tfrac{1}{2}n \cdot (-1)^{\frac{1}{2}(n-1)} \cdot \overset{\frac{1}{2}(n+1)}{\mathrm{S}}_m (-1)^{m-1} \cdot \frac{\lfloor\!\underline{\tfrac{1}{2}(n-1)+m-1}_{\,\frac{1}{2}(n-1)-m}}{\lfloor\!\underline{\tfrac{1}{2}(n+1)-m}} \cdot (2\cos x)^{2m-1},$$

(8) and (6);

$$\text{and } \frac{\underline{|\tfrac{1}{2}(n-1)+m-1}_{\,\frac{1}{2}(n-1)-m}}{\underline{|\tfrac{1}{2}(n+1)-m}} = \frac{\underline{|\tfrac{1}{2}(n-1)+m-1}}{\underline{|2m-1}\,.\,\underline{|\tfrac{1}{2}(n+1)-m}} = \frac{\underline{|\tfrac{1}{2}(n-1)+m-1}_{\,2m-2}}{\underline{|2m-1}}$$

$$= \frac{\underline{|\tfrac{1}{2}(n-1)+m-1}_{\,m-1}\,.\,\underline{|\tfrac{1}{2}(n-1)}_{\,m-1}}{\underline{|2m-1}} = \frac{\underline{|\tfrac{1}{2}(n+1)}_{\,m-1,\,1}\,.\,\underline{|\tfrac{1}{2}(n-1)}_{\,m-1}}{\underline{|2m-1}}$$

$$= \frac{\overset{m-1}{\mathrm{P}}_r\left\{(\tfrac{1}{2}n+r-\tfrac{1}{2})(\tfrac{1}{2}n-r+\tfrac{1}{2})\right\}}{\underline{|2m-1}} = \frac{\overset{m-1}{\mathrm{P}}_r\left\{(\tfrac{1}{2}n)^2-(r-\tfrac{1}{2})^2\right\}}{\underline{|2m-1}}$$

$$= \frac{\overset{m-1}{\mathrm{P}}_r\left\{n^2-(2r-1)^2\right\}}{\underline{|2m-1}\,.\,2^{2m-2}};$$

$$\therefore\ \cos nx = (-1)^{\frac{1}{2}(n-1)}\,.\,n\,.\overset{\frac{1}{2}(n+1)}{\mathrm{S}_m}(-1)^{m-1}\,.\,\frac{\overset{m-1}{\mathrm{P}}_r\left\{n^2-(2r-1)^2\right\}}{\underline{|2m-1}}\,.\,(\cos x)^{2m-1}.$$

283. *Theorem.*

$$(\sin x)^n = (-1)^{\frac{1}{2}n}\,.\,2^{-n+1}\,.\overset{\frac{1}{2}n}{\mathrm{S}_m}(-1)^{m-1}\,.\,\frac{\underline{|n}_{\,m-1}}{\underline{|m-1}}\,.\cos(n-2m+2)x$$

$$+2^{-n}\,.\,\frac{\underline{|n}_{\,\frac{1}{2}n}}{\underline{|\tfrac{1}{2}n}}, \quad \text{or}$$

$$(\sin x)^n = (-1)^{\frac{1}{2}(n-1)}\,.\,2^{-n+1}\,.\overset{\frac{1}{2}(n+1)}{\mathrm{S}_m}(-1)^{m-1}\,.\,\frac{\underline{|n}_{\,m-1}}{\underline{|m-1}}\,.\sin(n-2m+2)x;$$

according as n is even or odd.

For, $2\sqrt{-1}\,.\,\sin x = (\cos x+\sqrt{-1}\,.\,\sin x)$

$$-(\cos x+\sqrt{-1}\,.\,\sin x)^{-1}.$$

(1) If n is even, then

$$2^n(-1)^{\frac{1}{2}n}.(\sin x)^n = \overset{\frac{1}{2}n}{\mathrm{S}}_m(-1)^{m-1}.\frac{\underline{|n}_{\,m-1}}{\underline{|m-1}}\left\{(\cos x+\sqrt{-1}.\sin x)^{n-2m+2}\right.$$

$$\left. + \frac{1}{(\cos x+\sqrt{-1}.\sin x)^{n-2m+2}}\right\} + (-1)^{\frac{1}{2}n}.\frac{\underline{|n}_{\,\frac{1}{2}n}}{\underline{|\tfrac{1}{2}n}}, \quad (90);$$

$$= \overset{\frac{1}{2}n}{\mathrm{S}}_m(-1)^{m-1}.\frac{\underline{|n}_{\,m-1}}{\underline{|m-1}}.2\cos(n-2m+2)x + (-1)^{\frac{1}{2}n}.\frac{\underline{|n}_{\,\frac{1}{2}n}}{\underline{|\tfrac{1}{2}n}}, \quad (278).$$

$\therefore\ (\sin x)^n$

$$= (-1)^{\frac{1}{2}n}.2^{-n+1}.\overset{\frac{1}{2}n}{\mathrm{S}}_m(-1)^{m-1}.\frac{\underline{|n}_{\,m-1}}{\underline{|m-1}}.\cos(n-2m+2)x + 2^{-n}.\frac{\underline{|n}_{\,\frac{1}{2}n}}{\underline{|\tfrac{1}{2}n}}.$$

(2) If n is odd, then

$$2^n.(-1)^{\frac{1}{2}n}.(\sin x)^n = \overset{\frac{1}{2}(n+1)}{\mathrm{S}}_m(-1)^{m-1}.\frac{\underline{|n}_{\,m-1}}{\underline{|m-1}}\left\{(\cos x+\sqrt{-1}.\sin x)^{n-2m+2}\right.$$

$$\left. - \frac{1}{(\cos x+\sqrt{-1}.\sin x)^{n-2m+2}}\right\}, \quad (90);$$

$$= \overset{\frac{1}{2}(n+1)}{\mathrm{S}}_m(-1)^{m-1}.\frac{\underline{|n}_{\,m-1}}{\underline{|m-1}}.2\sqrt{-1}.\sin(n-2m+2)x, \quad (278).$$

$$(\sin x)^n = (-1)^{\frac{1}{2}(n-1)}.2^{-n+1}.\overset{\frac{1}{2}(n+1)}{\mathrm{S}}_m(-1)^{m-1}.\frac{\underline{|n}_{\,m-1}}{\underline{|m-1}}.\sin(n-2m+2)x.$$

284. *Theorem.*

$$(\cos x)^n = 2^{-n+1}.\overset{\frac{1}{2}n}{\mathrm{S}}_m \frac{\underline{|n}_{\,m-1}}{\underline{|m-1}}.\cos(n-2m+2)x + 2^{-n}.\frac{\underline{|n}_{\,\frac{1}{2}n}}{\underline{|\tfrac{1}{2}n}}, \text{ or}$$

$$(\cos x)^n = 2^{-n+1}.\overset{\frac{1}{2}(n+1)}{\mathrm{S}}_m \frac{\underline{|n}_{\,m-1}}{\underline{|m-1}}.\cos(n-2m+2)x;$$ according as n is even or odd.

For, $2\cos x = (\cos x + \sqrt{-1}.\sin x) + (\cos x + \sqrt{-1}.\sin x)^{-1}$.

(1) If n is even, then

$$2^n(\cos x)^n = \overset{\frac{1}{2}n}{\mathrm{S}}_m \frac{\underline{|n}_{\,m-1}}{\underline{|m-1}} \left\{(\cos x + \sqrt{-1}.\sin x)^{n-2m+2} + \frac{1}{(\cos x + \sqrt{-1}.\sin x)^{n-2m+2}}\right\} + \frac{\underline{|n}_{\,\frac{1}{2}n}}{\underline{|\tfrac{1}{2}n}}, \quad (89);$$

$$= \overset{\frac{1}{2}n}{\mathrm{S}}_m \frac{\underline{|n}_{\,m-1}}{\underline{|m-1}}.2\cos(n-2m+2)x + \frac{\underline{|n}_{\,\frac{1}{2}n}}{\underline{|\tfrac{1}{2}n}}, \quad (278);$$

$$\therefore\ (\cos x)^n = 2^{-n+1}.\overset{\frac{1}{2}n}{\mathrm{S}}_m \frac{\underline{|n}_{\,m-1}}{\underline{|m-1}}.\cos(n-2m+2)x + 2^{-n}.\frac{\underline{|n}_{\,\frac{1}{2}n}}{\underline{|\tfrac{1}{2}n}}.$$

(2) If n is odd, then

$$2^n.(\cos x)^n = \overset{\frac{1}{2}(n+1)}{\mathrm{S}}_m \frac{\underline{|n}_{\,m-1}}{\underline{|m-1}} \left\{(\cos x + \sqrt{-1}.\sin x)^{n-2m+2} + \frac{1}{(\cos x + \sqrt{-1}.\sin x)^{n-2m+2}}\right\}, \quad (89);$$

$$= \overset{\frac{1}{2}(n+1)}{\mathrm{S}}_m \frac{\underline{|n}_{\,m-1}}{\underline{|m-1}}.2\cos(n-2m+2)x, \quad (278);$$

$$\therefore\ (\cos x)^n = 2^{-n+1}.\overset{\frac{1}{2}(n+1)}{\mathrm{S}}_m \frac{\underline{|n}_{\,m-1}}{\underline{|m-1}}.\cos(n-2m+2)x.$$

285. *Theorem.* $\sin x = \overset{\infty}{\mathrm{S}}_m (-1)^{m-1} . \dfrac{x^{2m-1}}{\underline{|2m-1}}$.

For, $\sin x = \overset{\infty}{\mathrm{S}}_m \dfrac{x^{m-1}}{\underline{|m-1}} . d^{m-1}_{x=0} . \sin x$, (213);

$$= \overset{\infty}{\mathrm{S}}_m \frac{x^{2m-2}}{\underline{|2m-2}} . d^{2m-2}_{x=0} . \sin x + \overset{\infty}{\mathrm{S}}_m \frac{x^{2m-1}}{\underline{|2m-1}} . d^{2m-1}_{x=0} . \sin x, \quad (14);$$

$$= \overset{\infty}{\mathrm{S}}_m \frac{x^{2m-2}}{\underline{|2m-2}} . (-1)^{m-1} . (\sin x)_{x=0} + \overset{\infty}{\mathrm{S}}_m \frac{x^{2m-1}}{\underline{|2m-1}} . (-1)^{m-1} . (\cos x)_{x=0},$$

(269);

$$= \overset{\infty}{\mathrm{S}}_m (-1)^{m-1} . \frac{x^{2m-1}}{\underline{|2m-1}}.$$

286. Cor. 1. $\cos x = d_x . \sin x$, (267)

$$= d_x . \overset{\infty}{\mathrm{S}}_m (-1)^{m-1} . \frac{x^{2m-1}}{\underline{|2m-1}}$$

$$= \overset{\infty}{\mathrm{S}}_m (-1)^{m-1} . \frac{x^{2m-2}}{\underline{|2m-2}}.$$

287. Cor. 2.

$$\epsilon^{x\sqrt{-1}} = \overset{\infty}{\mathrm{S}}_m (-1)^{m-1} . \frac{x^{2m-2}}{\underline{|2m-2}} + \sqrt{-1} . \overset{\infty}{\mathrm{S}}_m (-1)^{m-1} . \frac{x^{2m-1}}{\underline{|2m-1}}, \quad (106),$$

(14) and (6);

$$= \cos x + \sqrt{-1} . \sin x,$$

and $\epsilon^{-x\sqrt{-1}} = \cos(-x) + \sqrt{-1} . \sin(-x)$

$$= \cos x - \sqrt{-1} . \sin x;$$

$$\therefore \epsilon^{x\sqrt{-1}} + \epsilon^{-x\sqrt{-1}} = 2 \cos x,$$

and $\epsilon^{x\sqrt{-1}} - \epsilon^{-x\sqrt{-1}} = 2\sqrt{-1} . \sin x$.

288. Cor. 3. $\epsilon^{(2m-1)\frac{1}{2}\pi . \sqrt{-1}} = (-1)^{m-1} . \sqrt{-1}$,

and $\epsilon^{(m-1)\pi\sqrt{-1}} = (-1)^{m-1}$.

289. *Theorem.*

$$\tan x = \overset{\infty}{S}_n (-1)^{n-1} . x^{2n-1} . \overset{n}{S}_m \frac{1}{\underline{|2n-2m+1}} . \overset{m-1}{A}_{+r} \left(\frac{-1}{\underline{|2r}} \right) .$$

For, put $z = -x^2$, then

$$\tan x = \sin x . (\cos x)^{-1}$$

$$= x . \overset{\infty}{S}_n \frac{z^{n-1}}{\underline{|2n-1}} . \left(\overset{\infty}{S}_m \frac{z^{m-1}}{\underline{|2m-2}} \right)^{-1}, \quad (285) \text{ and } (286);$$

$$= x . \overset{\infty}{S}_n \frac{z^{n-1}}{\underline{|2n-1}} . \overset{\infty}{S}_m z^{m-1} . \varpi^{m-1} . a^{-1},$$

$$\text{where } a_{m-1} = \frac{1}{\underline{|2m-2}}, \quad (228);$$

$$= x . \overset{\infty}{S}_n z^{n-1} . \overset{n}{S}_m \frac{\varpi^{m-1} a^{-1}}{\underline{|2n-2m+1}}, \quad (18);$$

$$= \overset{\infty}{S}_n (-1)^{n-1} . x^{2n-1} . \overset{n}{S}_m \frac{1}{\underline{|2n-2m+1}} . a^{-1} . \overset{m-1}{A}_{+r} \left(\frac{-a_r}{a} \right), \quad (235);$$

$$= \overset{\infty}{S}_n (-1)^{n-1} . x^{2n-1} . \overset{n}{S}_m \frac{1}{\underline{|2n-2m+1}} . \overset{m-1}{A}_{+r} \left(\frac{-1}{\underline{|2r}} \right) .$$

290. *Theorem.* $\sec x = \overset{\infty}{S}_n (-1)^{n-1} . x^{2n-2} . \overset{n-1}{A}_{+r} \left(\frac{-1}{\underline{|2r}} \right) .$

For, $\sec x = (\cos x)^{-1}$

$$= \left(\overset{\infty}{S}_n \frac{z^{n-1}}{\underline{|2n-2}} \right)^{-1}, \quad (286);$$

$$= \overset{\infty}{S}_n z^{n-1} \varpi^{n-1} . a^{-1}, \text{ where } a_{n-1} = \frac{1}{\underline{|2n-2}}, \quad (228);$$

$$= \overset{\infty}{S}_n z^{n-1} . a^{-1} . \overset{n-1}{A}_{+r} \left(\frac{-a_r}{a} \right), \quad (235);$$

$$= \overset{\infty}{S}_n (-1)^{n-1} . x^{2n-2} . \overset{n-1}{A}_{+r} \left(\frac{-1}{\underline{|2r}} \right) .$$

291. *Theorem.*

$$\cot x = \overset{\infty}{\mathrm{S}}_n (-1)^{n-1} . x^{2n-3} . \overset{n}{\mathrm{S}}_m \frac{1}{\lfloor 2n-2m} . \overset{m-1}{\mathrm{A}}_{+r} \left(\frac{1}{\lfloor 2r+1} \right) .$$

For, $\cot x = \cos x . (\sin x)^{-1}$

$$= \overset{\infty}{\mathrm{S}}_n \frac{z^{n-1}}{\lfloor 2n-2} . \left(x . \overset{\infty}{\mathrm{S}}_m \frac{z^{m-1}}{\lfloor 2m-1} \right)^{-1}, \quad (285) \text{ and } (286);$$

$$= x^{-1} . \overset{\infty}{\mathrm{S}}_n \frac{z^{n-1}}{\lfloor 2n-2} . \overset{\infty}{\mathrm{S}}_m z^{m-1} . \varpi^{m-1} . a^{-1},$$

$$\text{where } a_{m-1} = \frac{1}{\lfloor 2m-1}, \quad (228);$$

$$= x^{-1} . \overset{\infty}{\mathrm{S}}_n z^{n-1} . \overset{n}{\mathrm{S}}_m \frac{\varpi^{m-1} . a^{-1}}{\lfloor 2n-2m}, \quad (18);$$

$$= \overset{\infty}{\mathrm{S}}_n (-1)^{n-1} . x^{2n-3} . \overset{n}{\mathrm{S}}_m \frac{1}{\lfloor 2n-2m} . a^{-1} . \overset{m-1}{\mathrm{A}}_{+r} \left(\frac{-a_r}{a} \right), \ (235);$$

$$= \overset{\infty}{\mathrm{S}}_n (-1)^{n-1} . x^{2n-3} . \overset{n}{\mathrm{S}}_m \frac{1}{\lfloor 2n-2m} . \overset{m-1}{\mathrm{A}}_{+r} \left(\frac{-1}{\lfloor 2r+1} \right) .$$

292. *Theorem.* $\operatorname{cosec} x = \overset{\infty}{\mathrm{S}}_n (-1)^{n-1} . x^{2n-3} . \overset{n-1}{\mathrm{A}}_{+r} \left(\frac{-1}{\lfloor 2r+1} \right) .$

For, $\operatorname{cosec} x = (\sin x)^{-1}$

$$= \left(x . \overset{\infty}{\mathrm{S}}_n \frac{z^{n-1}}{\lfloor 2n-1} \right)^{-1}, \quad (285);$$

$$= x^{-1} . \overset{\infty}{\mathrm{S}}_n z^{n-1} . \varpi^{n-1} . a^{-1}, \text{ where } a_{n-1} = \frac{1}{\lfloor 2n-1}, \ (228);$$

$$= x^{-1} . \overset{\infty}{\mathrm{S}}_n z^{n-1} . a^{-1} . \overset{n-1}{\mathrm{A}}_{+r} \left(\frac{-a_r}{a} \right), \quad (235);$$

$$= \overset{\infty}{\mathrm{S}}_n (-1)^{n-1} . x^{2n-3} . \overset{n-1}{\mathrm{A}}_{+r} \left(\frac{-1}{\lfloor 2r+1} \right) .$$

293. *Theorem.* $\sin x = x \cdot \overset{\infty}{\mathrm{P}}_r \left\{1 - \left(\frac{x}{r\pi}\right)^2\right\}$.

For, the roots of the equation

$$0 = \sin x$$

are $x=0$, and $x = \pm r\pi$, $\left(\begin{smallmatrix} r=1 \\ r=\infty \end{smallmatrix}\right)$.

$\therefore \sin x = a x \cdot \overset{\infty}{\mathrm{P}}_r \{(x - r\pi)(x + r\pi)\}$, where a is independent of x;

$$= a x \cdot \overset{\infty}{\mathrm{P}}_r (-r^2\pi^2) \cdot \overset{\infty}{\mathrm{P}}_r \left\{1 - \left(\frac{x}{r\pi}\right)^2\right\}$$

$$= a x \cdot \overset{\infty}{\mathrm{P}}_r (-r^2\pi^2) \cdot \{1 + \text{terms in } x^2\}.$$

But $\sin x = x + \text{terms in } x^3$, (285);

$$\therefore \; a \cdot \overset{\infty}{\mathrm{P}}_r (-r^2\pi^2) = 1,$$

and $\sin x = x \cdot \overset{\infty}{\mathrm{P}}_r \left\{1 - \left(\frac{x}{r\pi}\right)^2\right\}$.

294. *Theorem.* $\cos x = \overset{\infty}{\mathrm{P}}_r \left\{1 - \left(\frac{2x}{\overline{2r-1} \cdot \pi}\right)^2\right\}$.

For, the roots of the equation

$$0 = \cos x$$

are $x = \pm (2r-1)\pi$, $\left(\begin{smallmatrix} r=1 \\ r=\infty \end{smallmatrix}\right)$.

$\therefore \cos x = a \cdot \overset{\infty}{\mathrm{P}}_r \left\{x^2 - \left(\frac{2r-1}{2} \cdot \pi\right)^2\right\}$, where a is independent of x;

$$= a \cdot \overset{\infty}{\mathrm{P}}_r \left\{- \left(\frac{2r-1}{2} \cdot \pi\right)^2\right\} \cdot \overset{\infty}{\mathrm{P}}_r \left\{1 - \left(\frac{2x}{\overline{2r-1} \cdot \pi}\right)^2\right\}$$

$$= a \cdot \overset{\infty}{\mathrm{P}}_r \left\{- \left(\frac{2r-1}{2} \cdot \pi\right)^2\right\} \{1 + \text{terms in } x^2\}.$$

But $\cos x = 1 + \text{terms in } x^2$, (286);

$$\therefore\ a . \overset{\infty}{\mathrm{P}}_r \left\{ - \left(\frac{2r-1}{2} . \pi \right)^2 \right\} = 1,$$

$$\text{and } \cos x = \overset{\infty}{\mathrm{P}}_r \left\{ 1 - \left(\frac{2x}{\overline{2r-1} . \pi} \right)^2 \right\}.$$

295. *Theorem.* $\log_\epsilon \sin x = \log_\epsilon x - \overset{\infty}{\mathrm{S}}_m \left(\frac{x}{\pi} \right)^{2m} . \frac{1}{m} . \overset{\infty}{\mathrm{S}}_n n^{-2m}.$

For, $\sin x = x . \overset{\infty}{\mathrm{P}}_n \left\{ 1 - \left(\frac{x}{n\pi} \right)^2 \right\}$, (293);

$$\therefore\ \log_\epsilon \sin x = \log_\epsilon x + \overset{\infty}{\mathrm{S}}_n \log_\epsilon \left\{ 1 - \left(\frac{x}{n\pi} \right)^2 \right\}$$

$$= \log_\epsilon x - \overset{\infty}{\mathrm{S}}_n \overset{\infty}{\mathrm{S}}_m \left(\frac{x}{n\pi} \right)^{2m} . \frac{1}{m}, \quad (255);$$

$$= \log_\epsilon x - \overset{\infty}{\mathrm{S}}_m \left(\frac{x}{\pi} \right)^{2m} . \frac{1}{m} . \overset{\infty}{\mathrm{S}}_n n^{-2m}, \quad (17) \text{ and } (6).$$

296. *Theorem.* $\log_\epsilon \cos x = - \overset{\infty}{\mathrm{S}}_m \left(\frac{2x}{\pi} \right)^{2m} . \frac{1}{m} . \overset{\infty}{\mathrm{S}}_n (2n-1)^{-2m}.$

For, $\cos x = \overset{\infty}{\mathrm{P}}_n \left\{ 1 - \left(\frac{2x}{\overline{2n-1} . \pi} \right)^2 \right\}$, (294);

$$\therefore\ \log_\epsilon \cos x = \overset{\infty}{\mathrm{S}}_n \log_\epsilon \left\{ 1 - \left(\frac{2x}{\overline{2n-1} . \pi} \right)^2 \right\}$$

$$= - \overset{\infty}{\mathrm{S}}_n \overset{\infty}{\mathrm{S}}_m \left(\frac{2x}{\overline{2n-1} . \pi} \right)^{2m} . \frac{1}{m}, \quad (255) \text{ and } (6);$$

$$= - \overset{\infty}{\mathrm{S}}_m \left(\frac{2x}{\pi} \right)^{2m} . \frac{1}{m} . \overset{\infty}{\mathrm{S}}_n (2n-1)^{-2m}, \quad (17) \text{ and } (6).$$

297. *Theorem.*

$$\log_\epsilon \tan x = \log_\epsilon x + \overset{\infty}{S}_m \left(\frac{2x}{\pi}\right)^{2m} \cdot \frac{1}{m} \cdot \overset{\infty}{S}_n (-1)^{n-1} \cdot n^{-2m}.$$

For, $\log_\epsilon \tan x = \log_\epsilon \sin x - \log_\epsilon \cos x$

$$= \log_\epsilon x - \overset{\infty}{S}_m \left(\frac{2x}{\pi}\right)^{2m} \cdot \frac{1}{m} \cdot \overset{\infty}{S}_n (2n)^{-2m}$$

$$+ \overset{\infty}{S}_m \left(\frac{2x}{\pi}\right)^{2m} \cdot \frac{1}{m} \cdot \overset{\infty}{S}_n (2n-1)^{-2m}$$

$$= \log_\epsilon x + \overset{\infty}{S}_m \left(\frac{2x}{\pi}\right)^{2m} \cdot \frac{1}{m} \cdot \overset{\infty}{S}_n (-1)^{n-1} \cdot n^{-2m}, \quad (14).$$

298. *Theorem.*

$$\tan^{-1} x = \overset{n}{S}_m (-1)^{m-1} \cdot \frac{x^{2m-1}}{2m-1} + (-1)^n \cdot \int_{x=0}^{x=x} \frac{x^{2n}}{1+x^2}.$$

For, $d_x . \tan^{-1} x = \dfrac{1}{1+x^2}$, (274);

$$= \overset{n}{S}_m (-1)^{m-1} \cdot x^{2m-2} + (-1)^n \cdot \frac{x^{2n}}{1+x^2}, \quad (12);$$

$$\therefore \tan^{-1} x = \overset{n}{S}_m (-1)^{m-1} \cdot \frac{x^{2m-1}}{2m-1} + (-1)^n \cdot \int_{x=0}^{x=x} \frac{x^{2n}}{1+x^2}, \quad (191) \ \& \ (210).$$

299. Cor. If $x < 1$, $\tan^{-1} x = \overset{\infty}{S}_m (-1)^{m-1} \cdot \dfrac{x^{2m-1}}{2m-1}$.

300. *Theorem.*

$$\frac{\pi}{4} = ,8 . \overset{\infty}{\mathrm{S}}_m (-1)^{m-1} . \frac{(,04)^{m-1}}{2m-1} - \overset{\infty}{\mathrm{S}}_m (-1)^{m-1} . \frac{1}{(239)^{2m-1}(2m-1)} .$$

For, put $\tan^{-1}\frac{1}{5} = a$; then

$$\tan 2a = \frac{2 \tan a}{1-(\tan a)^2} = \frac{5}{12}, \quad \tan 4a = \frac{120}{119}, \text{ and}$$

$$\tan\left(4a - \frac{\pi}{4}\right) = \frac{\tan 4a - \tan\frac{\pi}{4}}{1 + \tan 4a . \tan\frac{\pi}{4}} = \frac{1}{239} .$$

$$\therefore \frac{\pi}{4} = 4a - \left(4a - \frac{\pi}{4}\right)$$

$$= 4 . \tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$$

$$= 4 . \overset{\infty}{\mathrm{S}}_m (-1)^{m-1} . \frac{1}{5^{2m-1} . (2m-1)}$$

$$- \overset{\infty}{\mathrm{S}}_m (-1)^{m-1} . \frac{1}{(239)^{2m-1}(2m-1)}, \quad (299).$$

But $\frac{4}{5^{2m-1}} = 4 \times (,2)^{2m-1} = ,8 \times (,04)^{m-1}$.

$$\therefore \frac{\pi}{4} = ,8 . \overset{\infty}{\mathrm{S}}_m (-1)^{m-1} . \frac{(,04)^{m-1}}{2m-1} - \overset{\infty}{\mathrm{S}}_m (-1)^{m-1} . \frac{1}{(239)^{2m-1} . (2m-1)} .$$

301. *Theorem.* $x^n+1=\overset{n}{\mathrm{P}}_m\left(x-\epsilon^{\frac{2m-1}{n}.\pi\sqrt{-1}}\right).$

For, put $0=x^n+1$; then $x^n=-1$

$$=\epsilon^{(2m-1)\pi\sqrt{-1}},\quad (288).$$

$$\therefore\ x=\epsilon^{\frac{2m-1}{n}.\pi\sqrt{-1}},$$

and the n different values of $\epsilon^{\frac{2m-1}{n}.\pi\sqrt{-1}}$ are the roots of the equation

$$0=x^n+1.$$

Hence the n different values of $(x-\epsilon^{\frac{2m-1}{n}.\pi\sqrt{-1}})$ are the simple factors of (x^n+1).

302. Cor. 1.

$$x^n+1=\overset{n}{\mathrm{P}}_m\left\{x-\left(\cos\frac{2m-1}{n}.\pi+\sqrt{-1}.\sin\frac{2m-1}{n}.\pi\right)\right\},\quad (287).$$

303. Cor. 2. If n is even, then

$$x^n+1=\overset{\frac{1}{2}n}{\mathrm{P}}_m\left\{\left(x-\epsilon^{\frac{2m-1}{n}.\pi\sqrt{-1}}\right)\left(x-\epsilon^{\frac{2n-2m+1}{n}.\pi\sqrt{-1}}\right)\right\},$$

by inverting the order of the latter factors, (31);

$$=\overset{\frac{1}{2}n}{\mathrm{P}}_m\left\{x^2-x\left(\epsilon^{\frac{2m-1}{n}.\pi\sqrt{-1}}+\epsilon^{-\frac{2m-1}{n}.\pi\sqrt{-1}}\right)+1\right\},\ \text{since } \epsilon^{2\pi\sqrt{-1}}=1,\ (288);$$

$$=\overset{\frac{1}{2}n}{\mathrm{P}}_m\left(x^2-2x.\cos\frac{2m-1}{n}.\pi+1\right),\quad (287).$$

If n is odd, then

$$x^n+1=\overset{\frac{1}{2}(n-1)}{\mathrm{P}}_m\left(x-\epsilon^{\frac{2m-1}{n}.\pi\sqrt{-1}}\right)(x+1).\overset{\frac{1}{2}(n-1)}{\mathrm{P}}_m\left(x-\epsilon^{\frac{2n-2m+1}{n}.\pi\sqrt{-1}}\right),$$

(32), (288), and by inverting the order of the latter factors, (31);

$$=(x+1).\overset{\frac{1}{2}(n-1)}{\mathrm{P}}_m\left\{x^2-x\left(\epsilon^{\frac{2m-1}{n}.\pi\sqrt{-1}}+\epsilon^{-\frac{2m-1}{n}.\pi\sqrt{-1}}\right)+1\right\},$$

since $\epsilon^{2\pi\sqrt{-1}}=1$, (288);

$$=(x+1).\overset{\frac{1}{2}(n-1)}{\mathrm{P}}_m\left(x^2-2x.\cos\frac{2m-1}{n}.\pi+1\right),\quad (287).$$

304. *Theorem.* $x^n-1=\overset{n}{\mathrm{P}}_m(x-\epsilon^{\frac{2m-2}{n}.\pi\sqrt{-1}})$.

For, put $0=x^n-1$; then $x^n=1$

$$=\epsilon^{\overline{2m-2}.\pi\sqrt{-1}}, \quad (288).$$

$$\therefore\ x=\epsilon^{\frac{2m-2}{n}.\pi\sqrt{-1}},$$

and the n different values of $\epsilon^{\frac{2m-2}{n}.\pi\sqrt{-1}}$ are the roots of the equation

$$0=x^n-1.$$

Hence the n different values of $(x-\epsilon^{\frac{2m-2}{n}.\pi\sqrt{-1}})$ are the simple factors of (x^n-1).

305. Cor. 1.

$$x^n-1=\overset{n}{\mathrm{P}}_m\left\{x-(\cos\frac{2m-2}{n}.\pi+\sqrt{-1}.\sin\frac{2m-2}{n}.\pi)\right\}.$$

306. Cor. 2. If n is even, then

$x^n-1=\overset{\frac{1}{2}n}{\mathrm{P}}_m\left\{(x-\epsilon^{\frac{2m-2}{n}.\pi\sqrt{-1}})(x-\epsilon^{\frac{2n-2m}{n}.\pi\sqrt{-1}})\right\}$, by inverting the order of the latter factors, (31);

$=(x-1).\overset{\frac{1}{2}n-1}{\mathrm{P}}_m\left\{(x-\epsilon^{\frac{2m}{n}.\pi\sqrt{-1}})(x-\epsilon^{\frac{2n-2m}{n}.\pi\sqrt{-1}})\right\}(x+1)$, (32), & (288);

$=(x^2-1).\overset{\frac{1}{2}n-1}{\mathrm{P}}_m\left\{x^2-x(\epsilon^{\frac{2m}{n}.\pi\sqrt{-1}}+\epsilon^{-\frac{2m}{n}.\pi\sqrt{-1}})+1\right\}$, since $\epsilon^{2\pi\sqrt{-1}}=1$;

$=(x^2-1).\overset{\frac{1}{2}n-1}{\mathrm{P}}_m(x^2-2x.\cos\frac{2m}{n}.\pi+1)$, (287).

If n is odd, then

$x^n-1=(x-1).\overset{\frac{1}{2}(n-1)}{\mathrm{P}}_m\left\{(x-\epsilon^{\frac{2m\pi}{n}.\sqrt{-1}})(x-\epsilon^{\frac{2n-2m}{n}.\pi\sqrt{-1}})\right\}$, by inverting the order of the latter factors, (31);

$=(x-1).\overset{\frac{1}{2}(n-1)}{\mathrm{P}}_m\left\{x^2-x(\epsilon^{\frac{2m\pi}{n}\sqrt{-1}}+\epsilon^{-\frac{2m\pi}{n}\sqrt{-1}})+1\right\}$, since $\epsilon^{2\pi\sqrt{-1}}=1$;

$=(x-1).\overset{\frac{1}{2}(n-1)}{\mathrm{P}}_m(x^2-2x.\cos\frac{2m\pi}{n}+1)$. (287).

307. *Theorem.* $(1+e.\cos x)^n = 1+\overset{\infty}{\mathrm{S}}_m \left\{ \dfrac{\underline{|n}_{2m}}{(\underline{|m})^2} \cdot \left(\tfrac{1}{2}e\right)^{2m} \right.$

$$\left. + 2\cos mx . \overset{\infty}{\mathrm{S}}_r \frac{\underline{|n}_{m+2r-2}}{\underline{|m+r-1} \, . \, \underline{|r-1}} \cdot \left(\tfrac{1}{2}e\right)^{m+2r-2} \right\}.$$

For, $(1+e.\cos x)^n = 1+\overset{\infty}{\mathrm{S}}_m \dfrac{\underline{|n}_m}{\underline{|m}} \cdot (e\cos x)^m$, (92) and (9);

$$= 1+\overset{\infty}{\mathrm{S}}_m \left\{ \frac{\underline{|n}_{2m-1}}{\underline{|2m-1}} \cdot (e\cos x)^{2m-1} + \frac{\underline{|n}_{2m}}{\underline{|2m}} \cdot (e\cos x)^{2m} \right\}, \quad (14) \text{ and } (5);$$

$$= 1+\overset{\infty}{\mathrm{S}}_m \left\{ \frac{\underline{|n}_{2m-1}}{\underline{|2m-1}} \cdot e^{2m-1} \cdot \frac{1}{2^{2m-2}} \cdot \overset{m}{\mathrm{S}}_r \frac{\underline{|2m-1}_{m-r}}{\underline{|m-r}} \cdot \cos(2r-1)x \right.$$

$$\left. + \frac{\underline{|n}_{2m}}{\underline{|2m}} \cdot e^{2m} \left(\frac{1}{2^{2m-1}} \cdot \overset{m}{\mathrm{S}}_r \frac{\underline{|2m}_{m-r}}{\underline{|m-r}} \cdot \cos 2rx + \frac{\underline{|2m}_m}{\underline{|m}} \cdot \frac{1}{2^{2m}} \right) \right\}, \quad (284) \text{ and } (8);$$

$$= 1+\overset{\infty}{\mathrm{S}}_m \frac{\underline{|n}_{2m}}{(\underline{|m})^2} \cdot \left(\tfrac{1}{2}e\right)^{2m}$$

$$+ \overset{\infty}{\mathrm{S}}_m \frac{\underline{|n}_{2m-1}}{\underline{|2m-1}} . 2 . \left(\tfrac{1}{2}e\right)^{2m-1} . \overset{m}{\mathrm{S}}_r \frac{\underline{|2m-1}_{m-r}}{\underline{|m-r}} . \cos(2r-1)x$$

$$+ \overset{\infty}{\mathrm{S}}_m \frac{\underline{|n}_{2m}}{\underline{|2m}} . 2 . \left(\tfrac{1}{2}e\right)^{2m} . \overset{m}{\mathrm{S}}_r \frac{\underline{|2m}_{m-r}}{\underline{|m-r}} . \cos 2rx$$

$$= 1+\overset{\infty}{\mathrm{S}}_m \frac{\underline{|n}_{2m}}{(\underline{|m})^2} \cdot \left(\tfrac{1}{2}e\right)^{2m}$$

$$+ \overset{\infty}{\mathrm{S}}_m \left\{ 2\cos(2m-1)x . \overset{\infty}{\mathrm{S}}_r \frac{\underline{|n}_{2m+2r-3} \cdot \underline{|2m+2r-3}_{r-1}}{\underline{|2m+2r-3} \, . \, \underline{|r-1}} . \left(\tfrac{1}{2}e\right)^{2m+2r-3} \right.$$

$$\left. + 2\cos 2mx . \overset{\infty}{\mathrm{S}}_r \frac{\underline{|n}_{2m+2r-2} \cdot \underline{|2m+2r-2}_{r-1}}{\underline{|2m+2r-2} \, . \, \underline{|r-1}} . \left(\tfrac{1}{2}e\right)^{2m+2r-2} \right\},$$

(19), (17) and interchanging m and r;

$$=1+\overset{\infty}{\mathrm{S}}_m\frac{\underline{|n}_{\,2m}}{(\underline{|m})^2}\cdot(\tfrac{1}{2}e)^{2m}$$

$$+\overset{\infty}{\mathrm{S}}_m 2\cos mx.\overset{\infty}{\mathrm{S}}_r\frac{\underline{|n}_{\,m+2r-2}\cdot\underline{|m+2r-2}_{\,r-1}}{\underline{|m+2r-2}\,.\,\underline{|r-1}}\cdot(\tfrac{1}{2}e)^{m+2r-2},\quad(14);$$

$$=1+\overset{\infty}{\mathrm{S}}_m\left\{\frac{\underline{|n}_{\,2m}}{(\underline{|m})^2}\cdot(\tfrac{1}{2}e)^{2m}\right.$$

$$\left.+2\cos mx.\overset{\infty}{\mathrm{S}}_r\frac{\underline{|n}_{\,m+2r-2}}{\underline{|m+r-1}\,.\,\underline{|r-1}}\cdot(\tfrac{1}{2}e)^{m+2r-2}\right\},\quad(5).$$

308. *Theorem.* If $y=z+x.\sin y$, where z independent of x, then shall

$$y=z$$

$$+2\overset{\infty}{\mathrm{S}}_m(-1)^{m-1}.\frac{(\frac{1}{2}x)^{2m-1}}{\underline{|2m-1}}.\overset{m}{\mathrm{S}}_r(-1)^{r-1}.\frac{\underline{|2m-1}_{\,m-r}}{\underline{|m-r}}.(2r-1)^{2m-2}.\sin(2r-1)z$$

$$+2\overset{\infty}{\mathrm{S}}_m(-1)^{m-1}.\frac{(\frac{1}{2}x)^{2m}}{\underline{|2m}}.\overset{m}{\mathrm{S}}_r(-1)^{r-1}.\frac{\underline{|2m}_{\,m-r}}{\underline{|m-r}}.(2r)^{2m-1}.\sin 2rz.$$

For, $y=z+\overset{\infty}{\mathrm{S}}_m\dfrac{x^{2m-1}}{\underline{|2m-1}}.d_z^{\,2m-2}.(\sin z)^{2m-1}$

$$+\overset{\infty}{\mathrm{S}}_m\frac{x^{2m}}{\underline{|2m}}.d_z^{\,2m-1}.(\sin z)^{2m},\quad(221)\text{ and }(14);$$

$$=z+\overset{\infty}{\mathrm{S}}_m\frac{x^{2m-1}}{\underline{|2m-1}}.d_z^{\,2m-2}\left\{\frac{1}{(-4)^{m-1}}.\overset{m}{\mathrm{S}}_r(-1)^{m-r}.\frac{\underline{|2m-1}_{\,m-r}}{\underline{|m-r}}.\sin(2r-1)z\right\}$$

$$+\overset{\infty}{\mathrm{S}}_m\frac{x^{2m}}{\underline{|2m}}.d_z^{\,2m-1}\left\{\frac{2}{(-4)^m}.\overset{m}{\mathrm{S}}_r(-1)^{m-r}.\frac{\underline{|2m}_{\,m-r}}{\underline{|m-r}}.\cos 2rz\right.$$

$$\left.+\frac{\underline{|2m}_{\,m}}{\underline{|m}}.4^{-m}\right\},\quad(283)\text{ and }(8);$$

$$=z+\overset{\infty}{\mathrm{S}}_m \frac{x^{2m-1}}{\underline{|2m-1}\,.\,(-4)^{m-1}}\,.\,\overset{m}{\mathrm{S}}_r(-1)^{m-r}\,.\,\frac{\underline{|2m-1}}{\overset{m-r}{\underline{|m-r}}}\,.\,(-1)^{m-1}\,.\,(2r-1)^{2m-2}\,.$$

$$\sin(2r-1)z$$

$$+\overset{\infty}{\mathrm{S}}_m \frac{x^{2m}\,.\,2}{\underline{|2m}\,.\,(-4)^{m}}\,.\,\overset{m}{\mathrm{S}}_r(-1)^{m-r}\,.\,\frac{\underline{|2m}}{\overset{m-r}{\underline{|m-r}}}\,(-1)^{m}\,.\,(2r)^{2m-1}\,.\,\sin 2rz,$$

(269) and (202);

$$=z+2\,.\,\overset{\infty}{\mathrm{S}}_m(-1)^{m-1}\,.\,\frac{(\frac{1}{2}x)^{2m-1}}{\underline{|2m-1}}\,.\,\overset{m}{\mathrm{S}}_r(-1)^{r-1}\,.\,\frac{\underline{|2m-1}}{\overset{m-r}{\underline{|m-r}}}\,.\,(2r-1)^{2m-2}\,.$$

$$\sin(2r-1)z$$

$$+2\,\overset{\infty}{\mathrm{S}}_m(-1)^{m-1}\,.\,\frac{(\frac{1}{2}x)^{2m}}{\underline{|2m}}\,.\,\overset{m}{\mathrm{S}}_r(-1)^{r-1}\,.\,\frac{\underline{|2m}}{\overset{m-r}{\underline{|m-r}}}\,.\,(2r)^{2m-1}\,.\,\sin 2rz.$$

309. *Theorem.* In the same case,

$$\cos y=\cos z-x\,.\,(\sin z)^2$$

$$+\overset{\infty}{\mathrm{S}}_m(-1)^{m}\,.\,\frac{(\frac{1}{2}x)^{2m}}{\underline{|2m}}\,.\,\overset{m+1}{\mathrm{S}}_n(-1)^{n-1}\,.\,\frac{\underline{|2m+1}}{\overset{m-n+1}{\underline{|m-n+1}}}\,.\,(2n-1)^{2m-1}\,.\,\cos(2n-1)z$$

$$+\overset{\infty}{\mathrm{S}}_m(-1)^{m}\,.\,\frac{(\frac{1}{2}x)^{2m+1}}{\underline{|2m+1}}\,.\,\overset{m+1}{\mathrm{S}}_n(-1)^{n-1}\,.\,\frac{\underline{|2m+2}}{\overset{m-n+1}{\underline{|m-n+1}}}(2n)^{2m}\,.\,\cos 2nz,\quad (6).$$

For, $\cos y=\cos z+\overset{\infty}{\mathrm{S}}_m \frac{x^m}{\underline{|m}}\,.\,d_z^{\,m-1}\,.\,\{(\sin z)^m d_z\,.\,\cos z\},\quad (221);$

$$=\cos z-\overset{\infty}{\mathrm{S}}_m \frac{x^m}{\underline{|m}}\,.\,d_z^{\,m-1}\,.\,(\sin z)^{m+1},\quad (268),\ (196)\text{ and }(6);$$

$$=\cos z-x\,.\,(\sin z)^2-\overset{\infty}{\mathrm{S}}_m \frac{x^{2m}}{\underline{|2m}}\,.\,d_z^{\,2m-1}\,.\,(\sin z)^{2m+1}$$

$$-\overset{\infty}{\mathrm{S}}_m \frac{x^{2m+1}}{\underline{|2m+1}}\,.\,d_z^{\,2m}\,.\,(\sin z)^{2m+2},\quad (9)\text{ and }(14);$$

$$=\cos z-x.(\sin z)^2$$

$$-\overset{\infty}{\mathrm{S}}_m\frac{x^{2m}}{\underline{\lvert 2m}}.d_z^{\,2m-1}\left\{\frac{1}{(-4)^m}.\overset{m+1}{\mathrm{S}}_n(-1)^{m-n+1}.\frac{\underline{\lvert 2m+1}_{\,m-n+1}}{\underline{\lvert m-n+1}}.\sin(2n-1)z\right\}$$

$$-\overset{\infty}{\mathrm{S}}_m\frac{x^{2m+1}}{\underline{\lvert 2m+1}}.d_z^{\,2m}.\left\{\frac{2}{(-4)^{m+1}}.\overset{m+1}{\mathrm{S}}_n(-1)^{m-n+1}.\frac{\underline{\lvert 2m+2}_{\,m-n+1}}{\underline{\lvert m-n+1}}.\cos 2nz\right.$$

$$\left.+\frac{\underline{\lvert 2m+2}_{\,m+1}}{\underline{\lvert m+1}}.4^{-(m+1)}\right\},\quad (283)\text{ and }(8);$$

$$=\cos z-x.(\sin z)^2$$

$$-\overset{\infty}{\mathrm{S}}_m\frac{x^{2m}}{\underline{\lvert 2m}}.\frac{1}{(-4)^m}.\overset{m+1}{\mathrm{S}}_n(-1)^{m-n+1}.\frac{\underline{\lvert 2m+1}_{\,m-n+1}}{\underline{\lvert m-n+1}}.(-1)^{m-1}.(2n-1)^{2m-1}.$$

$$\cos(2n-1)z$$

$$-\overset{\infty}{\mathrm{S}}_m\frac{x^{2m+1}}{\underline{\lvert 2m+1}}.\frac{2}{(-4)^{m+1}}.\overset{m+1}{\mathrm{S}}_n(-1)^{m-n+1}.\frac{\underline{\lvert 2m+2}_{\,m-n+1}}{\underline{\lvert m-n+1}}.(-1)^m.(2n)^{2m}.$$

$$\cos 2nz,\quad (196),\ (191),\ (269),\ (202),\text{ and }(193);$$

$$=\cos z-x.(\sin z)^2$$

$$+\overset{\infty}{\mathrm{S}}_m(-1)^m.\frac{(\frac{1}{2}x)^{2m}}{\underline{\lvert 2m}}.\overset{m+1}{\mathrm{S}}_n(-1)^{n-1}.\frac{\underline{\lvert 2m+1}_{\,m-n+1}}{\underline{\lvert m-n+1}}.(2n-1)^{2m-1}.\cos(2n-1)z$$

$$+\overset{\infty}{\mathrm{S}}_m(-1)^m.\frac{(\frac{1}{2}x)^{2m+1}}{\underline{\lvert 2m+1}}.\overset{m+1}{\mathrm{S}}_n(-1)^{n-1}.\frac{\underline{\lvert 2m+2}_{\,m-n+1}}{\underline{\lvert m-n+1}}(2n)^{2m}.\cos 2nz,\quad (6).$$

CHAPTER X.

ON THE INTEGRATION OF CERTAIN DEFINITE FUNCTIONS.

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310. *Theorem.* $\int_x \frac{1}{\sqrt{x^2 \pm 1}} = \log_\epsilon (x + \sqrt{x^2 \pm 1})$.

For, put $x^2 \pm 1 = u^2$;

then $x = u d_x u$,

and $x + u = u(1 + d_x u)$.

$$\therefore \frac{1}{u} = \frac{1 + d_x u}{x + u},$$

and $\int_x \frac{1}{u} = \int_x \frac{1 + d_x u}{x + u}$

$= \log_\epsilon (x + u)$, (251);

or $\int_x \frac{1}{\sqrt{x^2 \pm 1}} = \log_\epsilon (x + \sqrt{x^2 \pm 1})$.

311. *Theorem.* $\int_x \frac{1}{x\sqrt{1 \pm x^2}} = \log_\epsilon \frac{x}{1 + \sqrt{1 \pm x^2}}$.

For, $\int_x \frac{1}{x\sqrt{1 \pm x^2}} = \int_x \frac{x^{-2}}{\sqrt{x^{-2} \pm 1}}$

$$= -\int_x \frac{d_x . x^{-1}.}{\sqrt{x^{-2} \pm 1}}$$

$= -\log_\epsilon (x^{-1} + \sqrt{x^{-2} \pm 1})$, (310) and (202);

$$= \log_\epsilon \frac{1}{x^{-1} + \sqrt{x^{-2} \pm 1}}$$

$$= \log_\epsilon \frac{x}{1 + \sqrt{1 \pm x^2}}.$$

312. *Theorem.* $\int_x \frac{1}{\sin x} = \log_\epsilon . \tan \frac{x}{2}$.

For, $\frac{1}{\sin x} = \frac{\sin x}{1-(\cos x)^2}$

$$= \tfrac{1}{2} . \sin x \left\{ \frac{1}{1-\cos x} + \frac{1}{1+\cos x} \right\}$$

$$= \tfrac{1}{2} \left\{ \frac{d_x.(-\cos x)}{1-\cos x} - \frac{d_x.(\cos x)}{1+\cos x} \right\}, \quad (268);$$

$$\therefore \int_x \frac{1}{\sin x} = \tfrac{1}{2} \{ \log_\epsilon (1-\cos x) - \log_\epsilon (1+\cos x) \}, \quad (251);$$

$$= \log_\epsilon \sqrt{\frac{1-\cos x}{1+\cos x}}$$

$$= \log_\epsilon . \tan \frac{x}{2}.$$

313. *Theorem.* $\int_x \frac{1}{\cos x} = \log_\epsilon . \cot . \left(\frac{\pi}{4} - \frac{x}{2} \right)$.

For, $\frac{1}{\cos x} = \frac{\cos x}{1-(\sin x)^2}$

$$= \tfrac{1}{2} . \cos x \left\{ \frac{1}{1+\sin x} + \frac{1}{1-\sin x} \right\}$$

$$= \tfrac{1}{2} \left\{ \frac{d_x.\sin x}{1+\sin x} - \frac{d_x.(-\sin x)}{1-\sin x} \right\}, \quad (267);$$

$$\therefore \int_x \frac{1}{\cos x} = \tfrac{1}{2} \{ \log_\epsilon (1+\sin x) - \log_\epsilon (1-\sin x) \}, \quad (251);$$

$$= \log_\epsilon \sqrt{\frac{1+\sin x}{1-\sin x}}$$

$$= \log_\epsilon . \cot . \left(\frac{\pi}{4} - \frac{x}{2} \right)$$

314. *Theorem.*

$$\int_x \frac{x^n}{\sqrt{1-x^2}} = -\sqrt{1-x^2}\,.\,\overset{r}{\mathrm{S}}_m \frac{|\underline{n-1}_{\,m-1,-2}}{|\underline{n}_{\,m,-2}}\,.\,x^{n-2m+1} + \frac{|\underline{n-1}_{\,r,-2}}{|\underline{n}_{\,r,-2}}\,.\int_x \frac{x^{n-2r}}{\sqrt{1-x^2}}\,.$$

For, $\dfrac{x^n}{\sqrt{1-x^2}} = -x^{n-1}d_x.\sqrt{1-x^2}$

$$\therefore \int_x \frac{x^n}{\sqrt{1-x^2}} = -x^{n-1}.\sqrt{1-x^2} + \int_x (n-1)\,x^{n-2}.\sqrt{1-x^2}, \quad (222);$$

$$= -x^{n-1}\sqrt{1-x^2} + (n-1).\int_x \frac{x^{n-2}-x^n}{\sqrt{1-x^2}}, \quad (196).$$

$$\therefore n\int_x \frac{x^n}{\sqrt{1-x^2}} = -x^{n-1}\sqrt{1-x^2} + (n-1).\int_x \frac{x^{n-2}}{\sqrt{1-x^2}},$$

and $\displaystyle\int_x \frac{x^n}{\sqrt{1-x^2}} = -\frac{x^{n-1}}{n}\sqrt{1-x^2} + \frac{n-1}{n}.\int_x \frac{x^{n-2}}{\sqrt{1-x^2}}.$

$$\therefore \int_x \frac{x^n}{\sqrt{1-x^2}} = \overset{r}{\mathrm{S}}_m \left\{-\frac{x^{n-1-2(m-1)}}{n-2.(m-1)}\sqrt{1-x^2}\right\} \overset{m-1}{\mathrm{P}}_s \left\{\frac{n-1-2.(s-1)}{n-2.(s-1)}\right\}$$

$$+ \overset{r}{\mathrm{P}}_s \left\{\frac{n-1-2.(s-1)}{n-2.(s-1)}\right\}.\int_x \frac{x^{n-2r}}{\sqrt{1-x^2}}, \quad (51);$$

$$= -\sqrt{1-x^2}.\overset{r}{\mathrm{S}}_m \frac{|\underline{n-1}_{\,m-1,-2}}{|\underline{n}_{\,m,-2}}.\,x^{n-2m+1} + \frac{|\underline{n-1}_{\,r,-2}}{|\underline{n}_{\,r,-2}}.\int_x \frac{x^{n-2r}}{\sqrt{1-x^2}}, \quad (6).$$

315. Cor. If n is an even positive integer, then

$$\int_x \frac{x^n}{\sqrt{1-x^2}} = -\sqrt{1-x^2}.\overset{\frac{1}{2}n}{\mathrm{S}}_m \frac{|\underline{n-1}_{\,m-1,-2}}{|\underline{n}_{\,m,-2}}.\,x^{n-2m+1} + \frac{|\underline{1}_{\,\frac{1}{2}n,2}}{|\underline{2}_{\,\frac{1}{2}n,2}}.\sin^{-1}x, \quad (40) \text{ and}$$

(272);

and if n is an odd positive integer, then

$$\int_x \frac{x^n}{\sqrt{1-x^2}} = -\sqrt{1-x^2}\,.\,\overset{\frac{1}{2}(n+1)}{\mathrm{S}_m} \frac{\underline{|n-1}_{\,m-1,-2}}{\underline{|n}_{\,m,-2}}\,.\,x^{n-2m+1}.$$

316. *Theorem.*

$$\int_x \frac{x^{-n}}{\sqrt{1-x^2}} = -\sqrt{1-x^2}\,.\,\overset{r}{\mathrm{S}_m} \frac{\underline{|n-2}_{\,m-1,-2}}{\underline{|n-1}_{\,m,-2}}\,.\,\frac{1}{x^{n-2m+1}} + \frac{\underline{|n-2}_{\,r,-2}}{\underline{|n-1}_{\,r,-2}}\,.\int_x \frac{1}{x^{n-2r}\sqrt{1-x^2}}\,.$$

For, $n\,.\int_x \frac{x^n}{\sqrt{1-x^2}} = -x^{n-1}\sqrt{1-x^2}+(n-1)\,.\int_x \frac{x^{n-2}}{\sqrt{1-x^2}}$ (314).

$\therefore\ -n\,.\int_x \frac{x^{-n}}{\sqrt{1-x^2}} = -x^{-n-1}\sqrt{1-x^2}+(-n-1)\,.\int_x \frac{x^{-n-2}}{\sqrt{1-x^2}}$, and

$$(n-2)\,.\int_x \frac{x^{-(n-2)}}{\sqrt{1-x^2}} = x^{-(n-1)}\sqrt{1-x^2}+(n-1)\,.\int_x \frac{x^{-n}}{\sqrt{1-x^2}}\,.$$

$$\therefore\ \int_x \frac{x^n}{\sqrt{1-x^2}} = -\frac{x^{-(n-1)}}{n-1}\sqrt{1-x^2}+\frac{n-2}{n-1}\,.\int_x \frac{x^{-(n-2)}}{\sqrt{1-x^2}}\,.$$

$$\therefore\ \int_x \frac{x^n}{\sqrt{1-x^2}} = \overset{r}{\mathrm{S}_m}\left\{-\frac{x^{-(n-2\cdot\overline{m-1}-1)}}{n-2(m-1)-1}\,.\sqrt{1-x^2}\right\}\overset{m-1}{\mathrm{P}_s}\left\{\frac{n-2-2(s-1)}{n-1-2(s-1)}\right\}$$

$$+\overset{r}{\mathrm{P}_s}\left\{\frac{n-2-2.(s-1)}{n-1-2(s-1)}\right\}.\int_x \frac{x^{-(n-2r)}}{\sqrt{1-x^2}},\quad (51);$$

$$= -\sqrt{1-x^2}\,.\,\overset{r}{\mathrm{S}_m} \frac{\underline{|n-2}_{\,m-1,-2}}{\underline{|n-1}_{\,m,-2}}\,.\,\frac{1}{x^{n-2m+1}} + \frac{\underline{|n-2}_{\,r,-2}}{\underline{|n-1}_{\,r,-2}}\,.\int_x \frac{1}{x^{n-2r}\,.\sqrt{1-x^2}},\quad (6).$$

317. Cor. If n is an even positive integer, then

$$\int_x \frac{x^{-n}}{\sqrt{1-x^2}} = -\sqrt{1-x^2}\,.\,\overset{\frac{1}{2}n}{\mathrm{S}}_m \frac{\underline{|n-2}_{\,m-1,-2}}{\underline{|n-1}_{\,m,-2}}\,.\,\frac{1}{x^{n-2m+1}};$$

and if n is an odd positive integer, then

$$\int_x \frac{x^{-n}}{\sqrt{1-x^2}} = -\sqrt{1-x^2}\,.\,\overset{\frac{1}{2}(n-1)}{\mathrm{S}}_m \frac{\underline{|n-2}_{\,m-1,-2}}{\underline{|n-1}_{\,m,-2}}\,.\,\frac{1}{x^{n-2m+1}} + \frac{\underline{|1}_{\,\frac{1}{2}(n-1),2}}{\underline{|2}_{\,\frac{1}{2}(n-1),2}}\,.\log_\epsilon \frac{x}{1+\sqrt{1-x^2}},$$

(40) and (311).

318. *Theorem.*

$$\int_x (\sin x)^n = -\cos x\,.\,\overset{r}{\mathrm{S}}_m \frac{\underline{|n-1}_{\,m-1,-2}}{\underline{|n}_{\,m,-2}}\,.\,(\sin x)^{n-2m+1} + \frac{\underline{|n-1}_{\,r,-2}}{\underline{|n}_{\,r,-2}}\,.\int_x (\sin x)^{n-2r}.$$

For, $(\sin x)^n = -(\sin x)^{n-1}.d_x.\cos x$, (268).

$\therefore\ \int_x (\sin x)^n = -(\sin x)^{n-1}.\cos x + \int_x (n-1)(\sin x)^{n-2}.(\cos x)^2$, (222) and (267);

$$= -(\sin x)^{n-1}.\cos x + (n-1).\int_x \{(\sin x)^{n-2} - (\sin x)^n\}, \quad (196).$$

$\therefore\ n.\int_x (\sin x)^n = -(\sin x)^{n-1}.\cos x + (n-1).\int_x (\sin x)^{n-2}$, and

$$\int_x (\sin x)^n = -\frac{(\sin x)^{n-1}}{n}.\cos x + \frac{n-1}{n}.\int_x (\sin x)^{n-2}.$$

$$\therefore\ \int_x (\sin x)^n = \overset{r}{\mathrm{S}}_m \left\{-\frac{(\sin x)^{n-2(m-1)-1}}{n-2m+2}.\cos x\right\} \overset{m-1}{\mathrm{P}}_s \left\{\frac{n-2(s-1)-1}{n-2(s-1)}\right\}$$

$$+ \overset{r}{\mathrm{P}}_s \left\{\frac{n-2(s-1)-1}{n-2(s-1)}\right\}.\int_x (\sin x)^{n-2r}, \quad (51);$$

$$= -\cos x\,.\,\overset{r}{\mathrm{S}}_m \frac{\underline{|n-1}_{\,m-1,-2}}{\underline{|n}_{\,m,-2}}\,.\,(\sin x)^{n-2m+1} + \frac{\underline{|n-1}_{\,r,-2}}{\underline{|n}_{\,r,-2}}\,.\int_x (\sin x)^{n-2r}, \quad (6).$$

319. Cor. If n is an even positive integer, then

$$\int_x (\sin x)^n = -\cos x \,.\, \overset{\frac{1}{2}n}{\mathrm{S}_m} \frac{|\underline{n-1}_{\,m-1,-2}}{|\underline{n}_{\,m,-2}} \,.\, (\sin x)^{n-2m+1} + \frac{|\underline{1}_{\,\frac{1}{2}n,2}}{|\underline{2}_{\,\frac{1}{2}n,2}} \,.\, x, \quad (40) \text{ and } (197);$$

and if n is an odd positive integer, then

$$\int_x (\sin x)^n = -\cos x \,.\, \overset{\frac{1}{2}(n+1)}{\mathrm{S}_m} \frac{|\underline{n-1}_{\,m-1,-2}}{|\underline{n}_{\,m,-2}} \,.\, (\sin x)^{n-2m+1}.$$

320. *Theorem.*

$$\int_x (\sin x)^{-n} = -\cos x \,.\, \overset{r}{\mathrm{S}_m} \frac{|\underline{n-2}_{\,m-1,-2}}{|\underline{n-1}_{\,m,-2}} \,.\, \frac{1}{(\sin x)^{n-2m+1}} + \frac{|\underline{n-2}_{\,r,-2}}{|\underline{n-1}_{\,r,-2}} \,.\, \int_x (\sin x)^{-(n-2r)}.$$

For, $n \,.\, \int_x (\sin x)^n = -(\sin x)^{n-1} . \cos x + (n-1) \,.\, \int_x (\sin x)^{n-2}, \quad (318).$

$\therefore \; -n \,.\, \int_x (\sin x)^{-n} = -(\sin x)^{-n-1} . \cos x - (n+1) \int_x (\sin x)^{-n-2}$, and

$$n \,.\, \int_x (\sin x)^{-n} = (\sin x)^{-(n+1)} . \cos x + (n+1) \,.\, \int_x (\sin x)^{-(n+2)}.$$

$\therefore \; (n-2) \,.\, \int_x (\sin x)^{-(n-2)} = (\sin x)^{-(n-1)} . \cos x + (n-1) \,.\, \int_x (\sin x)^{-n}$, and

$$\int_x (\sin x)^{-n} = -\frac{\cos x}{(n-1)(\sin x)^{n-1}} + \frac{n-2}{n-1} \,.\, \int_x (\sin x)^{-(n-2)}.$$

$\therefore \; \int_x (\sin x)^{-n}$

$$= \overset{r}{\mathrm{S}_m} \left\{ -\frac{\cos x}{n-2(m-1)-1} \,.\, \frac{1}{(\sin x)^{n-2(m-1)-1}} \right\} \overset{m-1}{\mathrm{P}_s} \left\{ \frac{n-2-2 \,.\, (s-1)}{n-1-2(s-1)} \right\}$$

$$+ \overset{r}{\mathrm{P}_s} \left\{ \frac{n-2-2 \,.\, (s-1)}{n-1-2(s-1)} \right\} \,.\, \int_x (\sin x)^{-(n-2r)}, \quad (51);$$

$$= -\cos x \,.\, \overset{r}{\mathrm{S}_m} \frac{|\underline{n-2}_{\,m-1,-2}}{|\underline{n-1}_{\,m,-2}} \,.\, \frac{1}{(\sin x)^{n-2m+1}} + \frac{|\underline{n-2}_{\,r,-2}}{|\underline{n-1}_{\,r,-2}} \,.\, \int_x (\sin x)^{-(n-2r)}, \quad (6).$$

321. Cor. If n is an even positive integer, then

$$\int_x (\sin x)^{-n} = -\cos x \,.\, \overset{\frac{1}{2}n}{\mathrm{S}}_m \frac{|\underline{n-2}_{\,m-1,-2}}{|\underline{n-1}_{\,m,-2}} \,.\, \frac{1}{(\sin x)^{n-2m+1}};$$

and if n is an odd positive integer, then

$$\int_x (\sin x)^{-n} = -\cos x \,.\, \overset{\frac{1}{2}(n-1)}{\mathrm{S}}_m \frac{|\underline{n-2}_{\,m-1,-2}}{|\underline{n-1}_{\,m,-2}} \,.\, \frac{1}{(\sin x)^{n-2m+1}}$$

$$+ \frac{|\underline{1}_{\,\frac{1}{2}(n-1),2}}{|\underline{2}_{\,\frac{1}{2}(n-1),2}} \,.\, \log_\epsilon . \tan . \frac{x}{2}, \quad (40) \text{ and } (312).$$

322. *Theorem.*

$$\int_x (\cos x)^n = \sin x \,.\, \overset{r}{\mathrm{S}}_m \frac{|\underline{n-1}_{\,m-1,-2}}{|\underline{n}_{\,m,-2}} (\cos x)^{n-2m+1} + \frac{|\underline{n-1}_{\,r,-2}}{|\underline{n}_{\,r,-2}} \,.\, \int_x (\cos x)^{n-2r}.$$

For, $(\cos x)^n = (\cos x)^{n-1} \,.\, d_x . \sin x$, (267);

$\therefore\ \int_x (\cos x)^n = (\cos x)^{n-1} . \sin x + \int_x (n-1) . (\cos x)^{n-2} . (\sin x)^2$, (222) and (268);

$= (\cos x)^{n-1} . \sin x + (n-1) . \int_x \{(\cos x)^{n-2} - (\cos x)^n\}$, (196);

$\therefore\ n . \int_x (\cos x)^n = (\cos x)^{n-1} . \sin x + (n-1) . \int_x (\cos x)^{n-2}$, and

$$\int_x (\cos x)^n = \frac{(\cos x)^{n-1}}{n} . \sin x + \frac{n-1}{n} . \int_x (\cos x)^{n-2};$$

$$\therefore\ \int_x (\cos x)^n = \overset{r}{\mathrm{S}}_m \frac{(\cos x)^{n-2(m-1)-1} . \sin x}{n-2(m-1)} . \overset{m-1}{\mathrm{P}}_s \left\{\frac{n-2(s-1)-1}{n-2(s-1)}\right\}$$

$$+ \overset{r}{\mathrm{P}}_s \left\{\frac{n-2.(s-1)-1}{n-2(s-1)}\right\} . \int_x (\cos x)^{n-2r}, \quad (51);$$

$$= \sin x . \overset{r}{\mathrm{S}}_m \frac{|\underline{n-1}_{\,m-1,-2}}{|\underline{n}_{\,m,-2}} . (\cos x)^{n-2m+1} + \frac{|\underline{n-1}_{\,r,-2}}{|\underline{n}_{\,r,-2}} . \int_x (\cos x)^{n-2r}, \quad (6).$$

323. Cor. If n is an even positive integer, then

$$\int_x (\cos x)^n = \sin x \, . \overset{\frac{1}{2}n}{\mathrm{S}_m} \frac{\underline{|n-1}_{\,m-1,-2}}{\underline{|n}_{\,m,-2}} . (\cos x)^{n-2m+1} + \frac{\underline{|1}_{\,\frac{1}{2}n,2}}{\underline{|2}_{\,\frac{1}{2}n,2}} . x, \quad (40) \text{ and } (197);$$

and if n is an odd positive integer, then

$$\int_x (\cos x)^n = \sin x \, . \overset{\frac{1}{2}(n+1)}{\mathrm{S}_m} \frac{\underline{|n-1}_{\,m-1,-2}}{\underline{|n}_{\,m,-2}} . (\cos x)^{n-2m+1}.$$

324. *Theorem.*

$$\int_x (\cos x)^{-n} = \sin x \, . \overset{r}{\mathrm{S}_m} \frac{\underline{|n-2}_{\,m-1,-2}}{\underline{|n-1}_{\,m,-2}} . \frac{1}{(\cos x)^{n-2m+1}} + \frac{\underline{|n-2}_{\,r,-2}}{\underline{|n-1}_{\,r,-2}} . \int_x \frac{1}{(\cos x)^{n-2r}}.$$

For, $n . \int_x (\cos x)^n = (\cos x)^{n-1} . \sin x + (n-1) . \int_x (\cos x)^{n-2}$, (322).

$\therefore\ -n . \int_x (\cos x)^{-n} = (\cos x)^{-(n+1)} . \sin x - (n+1) . \int_x (\cos x)^{-(n+2)}$,

and $n . \int_x (\cos x)^{-n} = -(\cos x)^{-(n+1)} . \sin x + (n+1) . \int_x (\cos x)^{-(n+2)}$.

$\therefore\ (n-2) . \int_x (\cos x)^{-(n-2)} = -(\cos x)^{-(n-1)} . \sin x + (n-1) . \int_x (\cos x)^{-n}$,

and $\int_x (\cos x)^{-n} = \dfrac{(\cos x)^{-(n-1)}}{n-1} . \sin x + \dfrac{n-2}{n-1} . \int_x (\cos x)^{-(n-2)}$.

$\therefore\ \int_x (\cos x)^{-n}$

$$= \overset{r}{\mathrm{S}_m} \frac{\sin x}{n-2(m-1)-1} . \frac{1}{(\cos x)^{n-2.(m-1)-1}} . \overset{m-1}{\mathrm{P}_s} . \left\{ \frac{n-2(s-1)-2}{n-2(s-1)-1} \right\}$$

$$+ \overset{r}{\mathrm{P}_s} \left\{ \frac{n-2(s-1)-2}{n-2.(s-1)-1} \right\} . \int_x \frac{1}{(\cos x)^{n-2r}}, \quad (51);$$

$$= \sin x \, . \overset{r}{\mathrm{S}_m} \frac{\underline{|n-2}_{\,m-1,-2}}{\underline{|n-1}_{\,m,-2}} \frac{1}{(\cos x)^{n-2m+1}} + \frac{\underline{|n-2}_{\,r,-2}}{\underline{|n-1}_{\,r,-2}} . \int_x \frac{1}{(\cos x)^{n-2r}}, \quad (6).$$

325. Cor. If n is an even positive integer, then

$$\int_x (\cos x)^{-n} = \sin x \,.\, \overset{\frac{1}{2}n}{\mathrm{S}_m} \frac{|\underline{n-2}_{\,m-1,-2}}{|\underline{n-1}_{\,m,-2}} \,.\, \frac{1}{(\cos x)^{n-2m+1}};$$

and if n is an odd positive integer, then

$$\int_x (\cos x)^{-n} = \sin x \,.\, \overset{\frac{1}{2}(n-1)}{\mathrm{S}_m} \frac{|\underline{n-2}_{\,m-1,-2}}{|\underline{n-1}_{\,m,-2}} \,.\, \frac{1}{(\cos x)^{n-2m+1}}$$

$$+ \frac{|\underline{1}_{\,\frac{1}{2}(n-1),2}}{|\underline{2}_{\,\frac{1}{2}(n-1),2}} \,.\, \log_\epsilon \cot\left(\frac{\pi}{4} - \frac{x}{2}\right), \quad (40) \text{ and } (313).$$

326. *Theorem.* $\int_x (\sin x)^n$

$$= (-1)^{\frac{1}{2}n} \,.\, 2^{-n+1} \,.\, \overset{\frac{1}{2}n}{\mathrm{S}_m} (-1)^{m-1} \,.\, \frac{|\underline{n}_{\,m-1}}{|\underline{m-1}} \,.\, \frac{\sin(n-2m+2)x}{n-2m+2} + 2^{-n} \frac{|\underline{n}_{\,\frac{1}{2}n}}{|\underline{\tfrac{1}{2}n}} \,.\, x, \text{ or}$$

$$= (-1)^{\frac{1}{2}(n+1)} \,.\, 2^{-n+1} \,.\, \overset{\frac{1}{2}(n+1)}{\mathrm{S}_m} (-1)^{m-1} \,.\, \frac{|\underline{n}_{\,m-1}}{|\underline{m-1}} \,.\, \frac{\cos(n-2m+2)x}{n-2m+2};$$

according as n is an even or odd positive integer.

For, $$(\sin x)^n = (-1)^{\frac{1}{2}n} \,.\, 2^{-n+1} \,.\, \overset{\frac{1}{2}n}{\mathrm{S}_m} (-1)^{m-1} \,.\, \frac{|\underline{n}_{\,m-1}}{|\underline{m-1}} \,.\, \cos(n-2m+2)x$$

$$+ 2^{-n} \,.\, \frac{|\underline{n}_{\,\frac{1}{2}n}}{|\underline{\tfrac{1}{2}n}}, \text{ or}$$

$$= (-1)^{\frac{1}{2}(n-1)} \,.\, 2^{-n+1} \,.\, \overset{\frac{1}{2}(n+1)}{\mathrm{S}_m} \,.\, (-1)^{m-1} \,.\, \frac{|\underline{n}_{\,m-1}}{|\underline{m-1}} \,.\, \sin(n-2m+2)x,$$

according as n is even or odd, (283).

Hence, by (196), (191), (268), (202), (197), (267) and (6), the truth of the above theorems is manifest.

327. *Theorem.*

$$\textstyle\int_x(\cos x)^n = 2^{-n+1}.\overset{\frac{1}{2}n}{\mathrm{S}}_m \frac{\underline{|n}_{m-1}}{\underline{|m-1}}.\frac{\sin(n-2m+2)x}{n-2m+2} + 2^{-n}.\frac{\underline{|n}_{\frac{1}{2}n}}{\underline{|\frac{1}{2}n}}.x, \text{ or}$$

$$= 2^{-n+1}.\overset{\frac{1}{2}(n+1)}{\mathrm{S}}_m \frac{\underline{|n}_{m-1}}{\underline{|m-1}}.\frac{\sin(n-2m+2)x}{n-2m+2},$$

according as n is even or odd.

For, $$(\cos x)^n = 2^{-n+1}.\overset{\frac{1}{2}n}{\mathrm{S}}_m \frac{\underline{|n}_{m-1}}{\underline{|m-1}}.\cos(n-2m+2)x + 2^{-n}.\frac{\underline{|n}_{\frac{1}{2}n}}{\underline{|\frac{1}{2}n}}, \text{ or}$$

$$= 2^{-n+1}.\overset{\frac{1}{2}(n+1)}{\mathrm{S}}_m \frac{\underline{|n}_{m-1}}{\underline{|m-1}}.\cos(n-2m+2)x,$$

according as n is even or odd, (284).

Hence, by the same articles, the truth of the theorems is manifest.

328. *Theorem.* $\int_x(a^x.u)$

$$= a^x.\overset{n}{\mathrm{S}}_m(-1)^{m-1}.(\log_\epsilon a)^{-m}.d_x^{m-1}u + (-1)^n.(\log_\epsilon a)^{-n}.\textstyle\int_x(a^x.d_x^n u).$$

For, $\int_x u.a^x$

$$= \overset{n}{\mathrm{S}}_m(-1)^{m-1}.d_x^{m-1}u.\textstyle\int_x^m a^x + (-1)^n.\int_x\{(d_x^n u).\int_x^n a^x\}, \quad (222);$$

$$= \overset{n}{\mathrm{S}}_m(-1)^{m-1}.\frac{a^x}{(\log_\epsilon a)^m}.d_x^{m-1}u + (-1)^n.\int_x\left\{\frac{a^x}{(\log_\epsilon a)^n}.d_x^n u\right\}, \quad (249);$$

$$= a^x.\overset{n}{\mathrm{S}}_m(-1)^{m-1}.(\log_\epsilon a)^{-m}.d_x^{m-1}u + (-1)^n.(\log_\epsilon a)^{-n}.\textstyle\int_x(a^x.d_x^n u),$$
(6) and (196).

329. Cor. $\int_x(a^x.x^n) = a^x.\overset{n+1}{\mathrm{S}}_m(-1)^{m-1}.\underline{|n}_{m-1}.x^{n-m+1}.(\log_\epsilon a)^{-m}$;

n being a positive integer.

330. *Theorem.* $\int_x \frac{a^x}{x} = \log_\epsilon x + \overset{\infty}{S}_m \frac{(x \log_\epsilon a)^m}{m . \lfloor m}$.

For, $a^x = 1 + \overset{\infty}{S}_m . \frac{(\log_\epsilon a)^m}{\lfloor m} . x^m$, (107) and (9).

$\therefore \frac{a^x}{x} = \frac{1}{x} + \overset{\infty}{S}_m \frac{(\log_\epsilon a)^m}{\lfloor m} . x^{m-1}$, and

$\int_x \frac{a^x}{x} = \log_\epsilon x + \overset{\infty}{S}_m \frac{(x . \log_\epsilon a)^m}{m . \lfloor m}$, (250), (191), (196) and (210).

331. *Theorem.* $\int_x (a^x . u)$

$= a^x . \overset{n}{S}_m (-1)^{m-1} . (\log_\epsilon a)^{m-1} . \int_x^m u + (-1)^n . (\log_\epsilon a)^n . \int_x (a^x . \int_x^n u)$.

For, $\int_x a^x . u$

$= \overset{n}{S}_m (-1)^{m-1} . d_x^{m-1} . (a^x) . \int_x^m u + (-1)^n . \int_x \{ d_x^n . (a^x) . \int_x^n u \}$, (222);

$= \overset{n}{S}_m (-1)^{m-1} . (\log_\epsilon a)^{m-1} . a^x . \int_x^m u + (-1)^n . \int_x \{ (\log_\epsilon a)^n . a^x . \int_x^n u \}$, (249);

$= a^x . \overset{n}{S}_m (-1)^{m-1} . (\log_\epsilon a)^{m-1} . \int_x^m u + (-1)^n . (\log_\epsilon a)^n . \int_x (a^x . \int_x^n u)$, (6) and (196).

332. Cor. $\int_x \frac{a^x}{x^n} = a^x . \overset{n-1}{S}_m (-1)^{m-1} . (\log_\epsilon a)^{m-1} . (-1)^m . \frac{x^{-n+m}}{\lfloor n-1 \rfloor_m}$

$+ (-1)^{n-1} . (\log_\epsilon a)^{n-1} . \int_x \left(\frac{a^x (-1)^{n-1} . x^{-1}}{\lfloor n-1} \right)$, (210);

$= -a^x . \overset{n-1}{S}_m . (\log_\epsilon a)^{m-1} . \frac{x^{-n+m}}{\lfloor n-1 \rfloor_m} + \frac{(\log_\epsilon a)^{n-1}}{\lfloor n-1} . \int_x \frac{a^x}{x}$, (6) and (196);

$= -a^x . \overset{n-1}{S}_m (\log_\epsilon a)^{m-1} . \frac{x^{-n+m}}{\lfloor n-1 \rfloor_m} + \frac{(\log_\epsilon a)^{n-1}}{\lfloor n-1} . \left\{ \log_\epsilon x + \overset{\infty}{S}_m \frac{(x . \log_\epsilon a)^m}{m . \lfloor m} \right\}$,

(330).

333. *Theorem.* $\int_x \frac{1}{\log_\epsilon x} = \log_\epsilon^2 x + \overset{\infty}{\mathrm{S}}_m \frac{(\log_\epsilon x)^m}{m.\lfloor m}$.

For, put $\log_\epsilon x = u$;

then $x = \epsilon^u$, and $\int_x \frac{1}{\log_\epsilon x} = \int_x \frac{1}{u}$

$$= \int_u \frac{d_u x}{u}, \quad (202);$$

$$= \int_u \frac{\epsilon^u}{u}$$

$$= \log_\epsilon u + \overset{\infty}{\mathrm{S}}_m \frac{u^m}{m.\lfloor m}, \quad (330);$$

$$= \log_\epsilon^2 x + \overset{\infty}{\mathrm{S}}_m \frac{(\log_\epsilon x)^m}{m.\lfloor m}.$$

334. *Theorem.*

$$\int_x \log_\epsilon (1 + e.\cos x) = -x.\log_\epsilon \left\{2e^{-1}(e^{-1} - \sqrt{e^{-2}-1})\right\}$$
$$+ 2.\overset{\infty}{\mathrm{S}}_m (-1)^{m-1}.(e^{-1} - \sqrt{e^{-2}-1})^m.\frac{\sin mx}{m^2}.$$

For, put $e = \frac{2}{k+k^{-1}}$; then $k = e^{-1} - \sqrt{e^{-2}-1}$, and

$$\log_\epsilon (1 + e.\cos x) = \log_\epsilon \left(1 + \frac{2}{k+k^{-1}}.\cos x\right)$$

$$= -\log_\epsilon (1+k^2) + \log_\epsilon (1 + 2k.\cos x + k^2)$$

$$= -\log_\epsilon (1+k^2) + \log_\epsilon \left\{(1 + k.\epsilon^{x\sqrt{-1}})(1 + k.\epsilon^{-x\sqrt{-1}})\right\}, \quad (287)$$

$$= -\log_\epsilon(1+k^2) + \log_\epsilon(1+k.\epsilon^{x\sqrt{-1}}) + \log_\epsilon(1+k.\epsilon^{-x\sqrt{-1}})$$

$$= -\log_\epsilon(1+k^2) + \overset{\infty}{\mathrm{S}}_m(-1)^{m-1}.\frac{k^m}{m}(\epsilon^{mx\sqrt{-1}} + \epsilon^{-mx\sqrt{-1}}),\ \ (255) \text{ and } (5);$$

$$= -\log_\epsilon(1+k^2) + 2.\overset{\infty}{\mathrm{S}}_m(-1)^{m-1}.\frac{k^m}{m}.\cos mx, \quad (287) \text{ and } (6).$$

$$\therefore\ \int_x \log_\epsilon(1+e.\cos x) = -x.\log_\epsilon(1+k^2) + 2.\overset{\infty}{\mathrm{S}}_m(-1)^{m-1}.\frac{k^m}{m^2}\sin mx,$$

(267).

$$= -x\log_\epsilon\left\{2e^{-1}(e^{-1} - \sqrt{e^{-2}-1})\right\}$$

$$+2.\mathrm{S}_m(-1)^{m-1}.(e^{-1} - \sqrt{e^{-2}-1})^m.\frac{\sin mx}{m^2}.$$

CHAPTER XI.

ON GENERATING FUNCTIONS.

335. The symbol $\overset{n,r}{S}_m a_m$ will be used to denote the sum of the series formed by giving to m every integral value from n to r both inclusive; zero being also taken as a value if n is either zero or negative.

336. *Definition.* If $\phi(t) = \overset{-\infty,\infty}{S}_x u_x t^x$, then $\phi(t)$ is called *the generating function* of u_x, and is denoted by $G_t . u_x$.

337. *Theorem.* $G_t(u_x + v_x) = G_t . u_x + G_t . v_x$.

For, $G_t(u_x+v_x) = \overset{-\infty,\infty}{S}_x (u_x+v_x) t^x$

$$= \overset{-\infty,\infty}{S}_x u_x t^x + \overset{-\infty,\infty}{S}_x v_x . t^x, \quad (5);$$

$$= G_t . u^x + G_t . v_x.$$

338. *Theorem.* $G_t(a u_x) = a . G_t . u_x$; a being independent of t and x.

For, $G_t(au) = \overset{-\infty,\infty}{S}_x a u_x . t^x$

$$= a . \overset{-\infty,\infty}{S}_x u_x t^x, \quad (6),$$

$$= a . G_t . u_x.$$

339. *Theorem.* If $G_t.u_x = G_t.v_x$, then shall $u_x = v_x$.

For, $\overset{-\infty,\infty}{\mathrm{S}_x} u_x t^x = G_t.u_x$

$= G_t.v_x$, by hypothesis;

$= \overset{-\infty,\infty}{\mathrm{S}_x} v_x.t^x$.

$\therefore$ $u_x = v_x$, by equating coefficients of t^x.

340. *Theorem.* $t^{\pm n}.G_t.u_x = G_t.u_{x\mp n}$.

For, $t^{\pm n}.G_t.u_x = t^{\pm n}.\overset{-\infty,\infty}{\mathrm{S}_x} u_x.t^x$

$= \overset{-\infty,\infty}{\mathrm{S}_x} u_x.t^{x\pm n}$, (6);

$= \overset{-\infty,\infty}{\mathrm{S}_x} u_{x\mp n}.t^x$, (9).

$= G_t.u_{x\mp n}$.

341. *Theorem.* $(t^{-1}-1)^n.G_t.u_x = G_t.\Delta_x^n.u_x$.

For, $(t^{-1}-1).G_t.u_x = G_t.u_{x+1} - G_t.u_x$, (340);

$= G_t.\Delta_x u_x$, (337).

Hence, $(t^{-1}-1)^2.G_t.u_x = (t^{-1}-1).G_t.\Delta_x u_x$

$= G_t.\Delta_x^2 u_x$;

and, similarly, $(t^{-1}-1)^n.G_t.u_x = G_t.\Delta_x^n u_x$.

342. Cor. $t^m.(t^{-1}-1)^n.G_t.u_x = G_t.\Delta_x^n u_{x-m}$.

343. *Theorem.* $\left(\overset{n+1}{\mathrm{S}}_m \dfrac{a_{m-1}}{t^{m-1}}\right) . G_t . u_x = G_t . \overset{n+1}{\mathrm{S}}_m a_{m-1} . u_{x+m-1}.$

For, $\left(\overset{n+1}{\mathrm{S}}_m \dfrac{a_{m-1}}{t^{m-1}}\right) . G_t . u_x = \overset{n+1}{\mathrm{S}}_m \dfrac{a_{m-1}}{t^{m-1}} . G_t . u_x,$ (6);

$= \overset{n+1}{\mathrm{S}}_m a_{m-1} . G_t . u_{x+m-1},$ (340);

$= G_t . \overset{n+1}{\mathrm{S}}_m a_{m-1} . u_{x+m-1},$ (338), and (337).

344. Cor. 1. If $\nabla^r u_x = \overset{n+1}{\mathrm{S}}_m a_{m-1} . \nabla^{r-1} u_{x+m-1},$ and $\nabla^0 u_x = u_x,$

then will $\left(\overset{n+1}{\mathrm{S}}_m \dfrac{a_{m-1}}{t^{m-1}}\right)^r . G_t . u_x = G_t . \nabla^r u_x.$

345. Cor. 2.

$$(t^{-1}-1)^p . \left(\overset{n+1}{\mathrm{S}}_m \frac{a_{m-1}}{t^{m-1}}\right)^q . t^r . G_t . u_x = G_t . \Delta_x^p . \nabla^q . u_{x-r}.$$

346. *Theorem.* $\Delta_x^n u_x = \overset{n+1}{\mathrm{S}}_m (-1)^{m-1} . \dfrac{\underline{|n}_{\,m-1}}{\underline{|m-1}} . u_{x+n-m+1}.$

For, $G_t . \Delta_x^n u_x = (t^{-1}-1)^n . G_t . u_x,$ (341);

$= \overset{n+1}{\mathrm{S}}_m (-1)^{m-1} . \dfrac{\underline{|n}_{\,m-1}}{\underline{|m-1}} . t^{-(n-m+1)} . G_t . u_x,$ (86) and (6);

$= G_t . \overset{n+1}{\mathrm{S}}_m (-1)^{m-1} . \dfrac{\underline{|n}_{\,m-1}}{\underline{|m-1}} . u_{x+n-m+1},$ (6), (340), (338), and (337);

$\therefore \Delta_x^n u_x = \overset{n+1}{\mathrm{S}}_m (-1)^{m-1} . \dfrac{\underline{|n}_{\,m-1}}{\underline{|m-1}} . u_{x+n-m+1},$ (339).

347. *Theorem.* $u_{x+n} = \overset{n+1}{S}_m \dfrac{\underline{|n}_{\,m-1}}{\underline{|m-1}} . \Delta_x^{m-1} u_x .$

For, $G_t . u_{x+n} = t^{-n} . G_t . u_x$, (340);

$$= (1 + t^{-1} - 1)^n . G_t . u_x$$

$$= \overset{n+1}{S}_m \frac{\underline{|n}_{\,m-1}}{\underline{|m-1}} . (t^{-1} - 1)^{m-1} . G_t . u_x, \quad (86) \text{ and } (6);$$

$$= G_t . \overset{n+1}{S}_m \frac{\underline{|n}_{\,m-1}}{\underline{|m-1}} . \Delta_x^{m-1} . u_x, \quad (341), (338), \text{ and } (337).$$

$$\therefore \; u_{x+n} = \overset{n+1}{S}_m \frac{\underline{|n}_{\,m-1}}{\underline{|m-1}} . \Delta_x^{m-1} u_x, \quad (339).$$

348. *Theorem.* $u_{x+n} = u_x + n . \overset{\infty}{S}_m \dfrac{\underline{|n+mr-1}_{\,m-1}}{\underline{|m}} . \Delta_x^m u_{x-mr}$;

r being any integer.

For, put $t^{-1} = 1 + \dfrac{\theta}{t^r}$; then

$$t^{-n} = 1 + \overset{\infty}{S}_m \frac{\theta^m}{\underline{|m}} . d_{z=1}^{m-1} \{ z^{mr} . n . z^{n-1} \}, \quad (221), (207) \;\&\; (197);$$

$$= 1 + n . \overset{\infty}{S}_m \frac{\underline{|n+mr-1}_{\,m-1}}{\underline{|m}} . \theta^m, \quad (196), (210), \text{ and } (6);$$

$$= 1 + n . \overset{\infty}{S}_m \frac{\underline{|n+mr-1}_{\,m-1}}{\underline{|m}} . t^{mr} . (t^{-1} - 1)^m .$$

$$\therefore\ G_t.u_{x+n}=\left\{1+n.\overset{\infty}{\mathrm{S}}_m\,{}^{m-1}\frac{\underline{|n+mr-1}}{\underline{|m}}.t^{mr}(t^{-1}-1)^m\right\}G_t.u_x,\quad (340);$$

$$=G_t\left\{u_x+n.\overset{\infty}{\mathrm{S}}_m\,{}^{m-1}\frac{\underline{|n+mr-1}}{\underline{|m}}.\Delta_x{}^m u_{x-mr}\right\},\quad (6),\ (342),$$

(338), and (337).

$$\therefore\ u_{x+n}=u_x+n.\overset{\infty}{\mathrm{S}}_m\,{}^{m-1}\frac{\underline{|n+mr-1}}{\underline{|m}}.\Delta_x{}^m u_{x-mr},\quad (339).$$

349. *Theorem.*

$$u_{x+n}=(n+1)\overset{n+1}{\mathrm{S}}_m\frac{\overset{m-1}{\mathrm{P}}_r\{(n+1)^2-r^2\}}{\underline{|2m-1}}.\Delta_x{}^{2m-2}u_{x-m+1}$$

$$-n.\overset{n}{\mathrm{S}}_m\frac{\overset{m-1}{\mathrm{P}}_r(n^2-r^2)}{\underline{|2m-1}}.\Delta_x{}^{2m-2}u_{x-m}.$$

For, let θ and $\frac{\theta}{t}$ be any quantities less than unity; then

$$1+\overset{\infty}{\mathrm{S}}_n\frac{\theta^n}{t^n}=\frac{1}{1-\frac{\theta}{t}},\quad (12)\text{ and }(9);$$

$$=\frac{1-\theta t}{1-\theta t}\cdot\frac{1}{1-\frac{\theta}{t}}$$

$$=\frac{1-\theta t}{1-\theta(t^{-1}+t)+\theta^2}$$

$$=\frac{1-\theta t}{1-\theta(2+z)+\theta^2},\ \text{where } z=t(t^{-1}-1)^2;$$

$$=\frac{1-\theta t}{(1-\theta)^2-\theta z}$$

$$=(1-\theta t).\overset{\infty}{\mathrm{S}}_n\frac{\theta^{n-1}z^{n-1}}{(1-\theta)^{2n}},\quad (12);$$

$$=(1-\theta t).\overset{\infty}{\mathrm{S}}_n\theta^{n-1}.z^{n-1}.\overset{\infty}{\mathrm{S}}_m\frac{\underline{|2n}_{\,m-1,1}}{\underline{|m-1}}.\theta^{m-1},\quad (92)\text{ and }(39);$$

$$=(1-\theta t).\overset{\infty}{\mathrm{S}}_n\theta^{n-1}.\overset{n}{\mathrm{S}}_m\frac{\underline{|2n-2m+2}_{\,m-1,1}}{\underline{|m-1}}.z^{n-m},\quad (16)\text{ and }(26);$$

$$=(1-\theta t).\overset{\infty}{\mathrm{S}}_n\theta^{n-1}.\overset{n}{\mathrm{S}}_m\frac{\underline{|2m}_{\,n-m,1}}{\underline{|n-m}}.z^{m-1},\quad (8).$$

But $\dfrac{\underline{|2m}_{\,n-m,1}}{\underline{|n-m}}=\dfrac{\underline{|m+n-1}_{\,n-m}}{\underline{|n-m}},\quad (40);$

$$=\frac{\underline{|m+n-1}}{\underline{|2m-1}\,.\,\underline{|n-m}}$$

$$=\frac{\underline{|m+n-1}_{\,2m-1}}{\underline{|2m-1}}.$$

$$\therefore\ 1+\overset{\infty}{\mathrm{S}}_n\frac{\theta^n}{t^n}=(1-\theta t).\overset{\infty}{\mathrm{S}}_n\theta^{n-1}.\overset{n}{\mathrm{S}}_m\frac{\underline{|n+m-1}_{\,2m-1}}{\underline{|2m-1}}.z^{m-1}$$

$$=\overset{\infty}{\mathrm{S}}_n\theta^{n-1}.\overset{n}{\mathrm{S}}_m\frac{\underline{|n+m-1}_{\,2m-1}}{\underline{|2m-1}}.z^{m-1}-t.\overset{\infty}{\mathrm{S}}_n\theta^n\overset{n}{\mathrm{S}}_m\frac{\underline{|n+m-1}_{\,2m-1}}{\underline{|2m-1}}.z^{m-1},\quad (6);$$

$$=1+\overset{\infty}{\mathrm{S}}_n\theta^n.\left\{\overset{n+1}{\mathrm{S}}_m\frac{\underline{|n+m}_{\,2m-1}}{\underline{|2m-1}}.z^{m-1}-t.\overset{n}{\mathrm{S}}_m\frac{\underline{|n+m-1}_{\,2m-1}}{\underline{|2m-1}}.z^{m-1}\right\},\quad (9),\ (6)$$

and (5).

$$\therefore\ t^{-n}=\overset{n+1}{S}_m\frac{\lfloor\underline{n+m}_{2m-1}}{\lfloor\underline{2m-1}}.z^{m-1}-t.\overset{n}{S}_m\frac{\lfloor\underline{n+m-1}_{2m-1}}{\lfloor\underline{2m-1}}.z^{m-1}$$

$$=\overset{n+1}{S}_m\frac{\lfloor\underline{n+m}_{2m-1}}{\lfloor\underline{2m-1}}.t^{m-1}.(t^{-1}-1)^{2m-2}-\overset{n}{S}_m\frac{\lfloor\underline{n+m-1}_{2m-1}}{\lfloor\underline{2m-1}}.t^{m}(t^{-1}-1)^{2m-2},\quad (6).$$

$$\therefore\ u_{x+n}=\overset{n+1}{S}_m\frac{\lfloor\underline{n+m}_{2m-1}}{\lfloor\underline{2m-1}}.\Delta_x^{2m-2}u_{x-m+1}-\overset{n}{S}_m\frac{\lfloor\underline{n+m-1}_{2m-1}}{\lfloor\underline{2m-1}}.\Delta_x^{2m-2}u_{x-m},$$

(340), (342), (338), (337) and (339).

But $\lfloor\underline{n+m}_{2m-1}=\lfloor\underline{n+m}_{m-1}.(n+1).\lfloor\underline{n-m+2}_{m-1,\,1}$, (41) and (40);

$$=(n+1).\overset{m-1}{P}_r\{(n+m-r+1)(n-m+r+1)\}$$

$$=(n+1).\overset{m-1}{P}_r\{(n+1)^2-(m-r)^2\}$$

$$=(n+1).\overset{m-1}{P}_r\{(n+1)^2-r^2\},\quad (31).$$

$$\therefore\ u_{x+n}=(n+1).\overset{n+1}{S}_m\frac{\overset{m-1}{P}_r\{(n+1)^2-r^2\}}{\lfloor\underline{2m-1}}.\Delta_x^{2m-2}u_{x-m+1}$$

$$-n.\overset{n}{S}_m\frac{\overset{m-1}{P}_r(n^2-r^2)}{\lfloor\underline{2m-1}}.\Delta_x^{2m-2}u_{x-m},\quad (6).$$

350. Cor. Put $\frac{1}{t^n} = c_n - t . c_{n-1}$, where $c_n = \overset{n+1}{\mathrm{S}}_m \frac{|\underline{n+m}_{2m-1}}{|\underline{2m-1}} . z^{m-1}$;

then $\frac{1}{t^{n-1}} = c_{n-1} - t . c_{n-2}$; $\therefore$ $\frac{1}{t^n} = \frac{c_{n-1}}{t} - c_{n-2}$,

and $\frac{2}{t^n} = c_n - c_{n-2} + (t^{-1} - t) c_{n-1}$.

Now, $c_n - c_{n-2} = \overset{n-1}{\mathrm{S}}_m \frac{z^{m-1}}{|\underline{2m-1}} \left(|\underline{n+m}_{2m-1} - |\underline{n+m-2}_{2m-1} \right) + 2n . z^{n-1} + z^n$,

(9) and (5).

And, $|\underline{n+m}_{2m-1} - |\underline{n+m-2}_{2m-1}$

$= |\underline{n+m-2}_{2m-3} \{ (n+m)(n+m-1) - (n-m)(n-m+1) \}$, (41);

$= 2n . (2m-1) . |\underline{n+m-2}_{2m-3}$

$= 2n . (2m-1) . |\underline{n+m-2}_{m-2} . n . |\underline{n-m+2}_{m-2,\,1}$, (41) and (40);

$= 2n^2 . (2m-1) . \overset{m-2}{\mathrm{P}}_r \{ (n+m-r-1)(n-m+r+1) \}$

$= 2n^2 . (2m-1) . \overset{m-2}{\mathrm{P}}_r \{ n^2 - (m-r-1)^2 \}$

$= 2n^2 . (2m-1) . \overset{m-2}{\mathrm{P}}_r (n^2 - r^2)$, (31).

$$\therefore\ c_n - c_{n-2} = 2n^2 . \overset{n-1}{\mathrm{S}}_m \frac{z^{m-1}}{\underline{|2m-2}} . \overset{m-2}{\mathrm{P}}_r (n^2 - r^2) + 2n . z^{n-1} + z^n$$

$$= 2n^2 . \overset{n}{\mathrm{S}}_m \frac{z^{m-1}}{\underline{|2m-2}} . \overset{m-2}{\mathrm{P}}_r (n^2 - r^2) + z^n,$$

$$\text{since } \frac{n^2}{\underline{|2n-2}} . \overset{n-2}{\mathrm{P}}_r (n^2 - r^2) = n;$$

$$\therefore\ \frac{1}{t^n} = n^2 . \overset{n}{\mathrm{S}}_m \frac{z^{m-1}}{\underline{|2m-2}} . \overset{m-2}{\mathrm{P}}_r (n^2 - r^2)$$

$$+ \tfrac{1}{2} z^n + (1+t)(t^{-1}-1) . n . \overset{n}{\mathrm{S}}_m \frac{z^{m-1}}{\underline{|2m-1}} . \overset{m-1}{\mathrm{P}}_r (n^2 - r^2)$$

$$= n^2 . \overset{n}{\mathrm{S}}_m \frac{\overset{m-2}{\mathrm{P}}_r (n^2 - r^2)}{\underline{|2m-2}} . t^{m-1} . (t^{-1}-1)^{2m-2} + \tfrac{1}{2} . t^n (t^{-1}-1)^{2n}$$

$$+ n . \overset{n}{\mathrm{S}}_m \frac{\overset{m-1}{\mathrm{P}}_r (n^2 - r^2)}{\underline{|2m-1}} . (t^{-1}-1)^{2m-1} . \{t^{m-1} + t^m\}, \quad (6).$$

$$\therefore\ u_{x+n} = n^2 . \overset{n}{\mathrm{S}}_m \frac{\overset{m-2}{\mathrm{P}}_r (n^2 - r^2)}{\underline{|2m-2}} . \Delta_x^{2m-2} u_{x-m+1} + \tfrac{1}{2} . \Delta_x^{2n} . u_{x-n}$$

$$+ n . \overset{n}{\mathrm{S}}_m \frac{\overset{m-1}{\mathrm{P}}_r (n^2 - r^2)}{\underline{|2m-1}} . \{\Delta_x^{2m-1} u_{x-m+1} + \Delta_x^{2m-1} u_{x-m}\}.$$

351. *Theorem.*

$$u_{x+n} = \overset{\infty}{\mathrm{S}}_s \overset{m}{\mathrm{S}}_r \nabla^{s-1} u_{x+r-1} . \overset{m-r+1}{\mathrm{S}}_p a_{r+p-1} . \varpi^{n-ms+p} b^{-s};$$

where $\nabla^s u_x = \overset{m+1}{\mathrm{S}}_r a_{r-1} . \nabla^{s-1} u_{x+r-1}$, and $b_{r-1} = a_{m-r+1}$, $\binom{r=1}{r=m+1}$.

For, put $z = \overset{m+1}{\mathrm{S}}_r \dfrac{a_{r-1}}{t^{r-1}}$; then $a - z = -\overset{m}{\mathrm{S}}_r a_r t^{-r}$, and

$$1 + \overset{\infty}{\mathrm{S}}_n \theta^n t^{-n} = \frac{1}{1 - \theta t^{-1}}, \quad (12);$$

$$=\frac{1}{1-\theta t^{-1}}\cdot\frac{\overset{m}{\mathrm{S}}_r a_r\theta^{m-r}(1-\theta^r.t^{-r})}{\overset{m}{\mathrm{S}}_r a_r\theta^{m-r}(1-\theta^r.t^{-r})}$$

$$=\frac{1}{1-\theta t^{-1}}\cdot\frac{\overset{m}{\mathrm{S}}_r a_r\theta^{m-r}(1-\theta^r t^{-r})}{\overset{m}{\mathrm{S}}_r a_r\theta^{m-r}-\theta^m.\overset{m}{\mathrm{S}}_r a_r t^{-r}},\quad (5) \text{ and } (6);$$

$$=\frac{\overset{m}{\mathrm{S}}_r a_r\theta^{m-r}.\overset{r}{\mathrm{S}}_p\dfrac{\theta^{p-1}}{t^{p-1}}}{\overset{m+1}{\mathrm{S}}_r a_{r-1}\theta^{m-r+1}-z\theta^m},\quad (11) \text{ and } (9);$$

$$=\frac{P}{Q}, \text{ suppose.}$$

Then $\dfrac{1}{Q}=\overset{\infty}{\mathrm{S}}_s z^{s-1}.\theta^{m(s-1)}.(\overset{m+1}{\mathrm{S}}_r a_{m-r+1}\theta^{r-1})^{-s},\quad (12) \text{ and } (8);$

$$=\overset{\infty}{\mathrm{S}}_s z^{s-1}.\theta^{m(s-1)}.\overset{\infty}{\mathrm{S}}_n\theta^{n-1}.\varpi^{n-1}.b^{-s},\ b_{n-1}=a_{m-n+1}\binom{n=1}{n=m+1};$$

$$=\overset{\infty}{\mathrm{S}}_n\theta^{n-1}.\overset{\infty}{\mathrm{S}}_s z^{s-1}.\varpi^{n-m(s-1)-1}.b^{-s};$$

and $P=\overset{m}{\mathrm{S}}_r\theta^{m-r}.\overset{m-r+1}{\mathrm{S}_p}\dfrac{a_{r+p-1}}{t^{p-1}},\quad (22);$

$$=\overset{m}{\mathrm{S}}_r\theta^{r-1}.\overset{r}{\mathrm{S}}_p\frac{a_{m-r+p}}{t^{p-1}},\quad (8).$$

$$\therefore\ \frac{P}{Q}=\overset{\infty}{\mathrm{S}}_n\theta^{n-1}.\overset{m}{\mathrm{S}}_r\theta^{r-1}.\overset{r}{\mathrm{S}}_p\frac{a_{m-r+p}}{t^{p-1}}.\overset{\infty}{\mathrm{S}}_s z^{s-1}.\varpi^{n-m(s-1)-1}.b^{-s}$$

$$=\overset{\infty}{\mathrm{S}}_n\theta^{n-1}.\overset{m}{\mathrm{S}}_r\overset{r}{\mathrm{S}}_p\frac{a_{m-r+p}}{t^{p-1}}.\overset{\infty}{\mathrm{S}}_s z^{s-1}.\varpi^{n-r-m(s-1)}b^{-s},\quad (18),$$

since $\varpi^m a^n=0$, for every negative value of m;

$$\therefore\ t^{-n}=\overset{m}{S}_r\overset{r}{S}_p\frac{a_{m-r+p}}{t^{p-1}}.\overset{\infty}{S}_s z^{s-1}.\varpi^{n-r+1-m(s-1)}.b^{-s}$$

$$=\overset{m}{S}_r\frac{1}{t^{r-1}}.\overset{m-r+1}{S}_p a_{m-p+1}.\overset{\infty}{S}_s z^{s-1}.\varpi^{n-r-p+2-m(s-1)}.b^{-s},\quad (22);$$

$$=\overset{m}{S}_r\frac{1}{t^{r-1}}.\overset{m-r+1}{S}_p a_{r+p-1}\overset{\infty}{S}_s z^{s-1}.\varpi^{n-ms+p}.b^{-s},\quad (8);$$

$$=\overset{\infty}{S}_s\overset{m}{S}_r\frac{z^{s-1}}{t^{r-1}}.\overset{m-r+1}{S}_p a_{r+p-1}.\varpi^{n-ms+p}.b^{-s},\quad (6)\text{ and }(17).$$

$$\therefore\ u_{x+n}=\overset{\infty}{S}_s\overset{m}{S}_r\nabla^{s-1}u_{x+r-1}.\overset{m-r+1}{S}_p a_{r+p-1}.\varpi^{n-ms+p}.b^{-s},\quad (345).$$

352. *Theorem.* $G_s(u_x).G_t(v_x)=G_s.G_t(u_x.v_x)$, u_x being the coefficient of s^x, and v_x of t^x.

For, $G_sG_t(u_xv_x)$ denotes a series of the form

$$\overset{-\infty,\infty}{S}_m\ \overset{-\infty,\infty}{S}_n a_{m,n}.s^m.t^n,$$

such that the coefficient of $s^x.t^x$, shall be u_xv_x; and the product

$$(\overset{-\infty,\infty}{S}_x u_x s^x)(\overset{-\infty,\infty}{S}_x v_x t^x)$$

will be such a series.

$$\therefore\ G_sG_t(u_xv_x)=G_s(u_x).G_t(v_x).$$

353. Cor. 1. $s^{-m}.t^{-n}.G_s(u_x).G_t(v_x)=G_sG_t(u_{x+m}.v_{x+n})$.

354. Cor. 2. $(s^{-1}-1)^m(t^{-1}-1)^n.G_s(u_x).G_t(v_x)$

$$=G_s.G_t.\{\Delta_x^m(u_x).\Delta_x^n(v_x)\}.$$

355. Cor. 3. $(s^{-1}t^{-1}-1)^n.G_s(u_x).G_t(v_x)=G_sG_t.\Delta_x^n.(u_xv_x)$.

356. *Theorem.*

$$\Delta_x^{\,n}(u_x v_x) = \overset{n+1}{\mathrm{S}}_m \frac{|\underline{n}_{\,m-1}}{|\underline{m-1}} \cdot \Delta_x^{\,n-m+1}(u_{x+m-1}) \cdot \Delta_x^{\,m-1} \cdot v_x.$$

For, $G_s G_t \cdot \Delta_x^{\,n}(u_x v_x)$

$= (s^{-1}t^{-1}-1)^n \cdot G_s(u_x) \cdot G_t(v_x)$, (355);

$= \{(s^{-1}-1)+s^{-1}(t^{-1}-1)\}^n \cdot G_s(u_x) \cdot G_t(v_x)$

$= \overset{n+1}{\mathrm{S}}_m \dfrac{|\underline{n}_{\,m-1}}{|\underline{m-1}} \cdot (s^{-1}-1)^{n-m+1} \cdot s^{-(m-1)} \cdot (t^{-1}-1)^{m-1} \cdot G_s(u_x) \cdot G_t(v_x)$, (86);

$= \overset{n+1}{\mathrm{S}}_m \dfrac{|\underline{n}_{\,m-1}}{|\underline{m-1}} \cdot G_s \cdot (\Delta_x^{\,n-m+1} \cdot u_{x+m-1}) \cdot G_t \cdot \Delta_x^{\,m-1} v_x$, (6) and (342);

$= G_s \cdot G_t \cdot \overset{n+1}{\mathrm{S}}_m \dfrac{|\underline{n}_{\,m-1}}{|\underline{m-1}} \cdot \Delta_x^{\,n-m+1}(u_{x+m-1}) \cdot \Delta_x^{\,m-1} v_x$, (352), (338) and (337).

$$\therefore \ \Delta_x^{\,n}(u_x v_x) = \overset{n+1}{\mathrm{S}}_m \frac{|\underline{n}_{\,m-1}}{|\underline{m-1}} \cdot \Delta_x^{\,n-m+1}(u_{x+m-1}) \cdot \Delta_x^{\,m-1} v_x, \quad (339).$$

357. The symbol $G_{s,t} \cdot u_{x,y}$ will be used to denote the double series $\overset{-\infty,\infty}{\mathrm{S}}_x \cdot \overset{-\infty,\infty}{\mathrm{S}}_y s^x \cdot t^y \cdot u_{x,y}$.

358. Cor. 1. $s^{-m} \cdot t^{-n} \cdot G_{s,t} \cdot u_{x,y} = G_{s,t} \cdot u_{x+m,y+n}$.

359. Cor. 2. $(s^{-1}-1)^m \cdot (t^{-1}-1)^n \cdot G_{s,t} \cdot u_{x,y} = G_{s,t} \Delta_x^{\,m} \Delta_y^{\,n} u_{x,y}$.

360. Cor. 3. $(s^{-1}t^{-1}-1)^n \cdot G_{s,t} u_{x,y} = G_{s,t} \cdot \Delta_{x,y}^{\,n} u_{x,y}$.

361. *Theorem.*

$$u_{x+m,\,y+n}=\overset{m+1}{\mathrm{S}}_p\ \overset{n+1}{\mathrm{S}}_q\frac{\lfloor m_{p-1}\ .\ \lfloor n_{q-1}}{\lfloor p-1\ \lfloor q-1}\,.\,\Delta_x^{p-1}\,.\,\Delta_y^{q-1}u_{x,y}.$$

For, $G_{s,t}\,.\,u_{x+m,y+n}$

$$=\left\{1+s^{-1}-1\right\}^m\left\{1+t^{-1}-1\right\}^n.\,G_{s,t}\,.\,u_{x,y},\quad (358);$$

$$=\overset{m+1}{\mathrm{S}}_p\frac{\lfloor m_{p-1}}{\lfloor p-1}\,.\,(s^{-1}-1)^{p-1}\,.\,\overset{n+1}{\mathrm{S}}_q\frac{\lfloor n_{q-1}}{\lfloor q-1}\,.\,(t^{-1}-1)^{q-1}\,.\,G_{s,t}\,.\,u_{x,y},\quad (86);$$

$$=\overset{m+1}{\mathrm{S}}_p\ \overset{n+1}{\mathrm{S}}_q\frac{\lfloor m_{p-1}\ .\ \lfloor n_{q-1}}{\lfloor p-1\ \lfloor q-1}\,.\,G_{s,t}\,.\,\Delta_x^{p-1}\,\Delta_y^{q-1}\,u_{x,y},\quad (6)\text{ and }(359);$$

$$\therefore\ u_{x+m,y+n}=\overset{m+1}{\mathrm{S}}_p\ \overset{n+1}{\mathrm{S}}_q\frac{\lfloor m_{p-1}\ .\ \lfloor n_{q-1}}{\lfloor p-1\ \lfloor q-1}\,.\,\Delta_x^{p-1}\,.\,\Delta_y^{q-1}\,u_{x,y},\quad (338),\ (337),$$

and (339).

362. *Theorem.*

$$\Delta_{x,y}^{n}u_{x,y}=\overset{n+1}{\mathrm{S}}_m(-1)^{m-1}\,.\,\frac{\lfloor n_{m-1}}{\lfloor m-1}\,.\,u_{x+n-m+1,\,y+n-m+1}.$$

For, $G_{s,t}\,.\,\Delta_{x,y}^{n}u_{x,y}$

$$=(s^{-1}t^{-1}-1)^n.\,G_{s,t}u_{x,y},\quad (360);$$

$$=\overset{n+1}{\mathrm{S}}_m(-1)^{m-1}\,.\,\frac{\lfloor n_{m-1}}{\lfloor m-1}\,.\,(st)^{-(n-m+1)}\,.\,G_{s,t}u_{x,y},\quad (86);$$

$$=\overset{n+1}{\mathrm{S}}_m(-1)^{m-1}\,.\,\frac{\lfloor n_{m-1}}{\lfloor m-1}\,.\,G_{s,t}\,.\,u_{x+n-m+1,\,y+n-m+1},\quad (6)\text{ and }(358);$$

$$\therefore\ \Delta_{x,y}^{n}u_{x,y}=\overset{n+1}{\mathrm{S}}_m(-1)^{m-1}\,.\,\frac{\lfloor n_{m-1}}{\lfloor m-1}\,.\,u_{x+n-m+1,\,y+n-m+1},\quad (338),\ (337)$$

and (339).

NOTES AND ADDITIONS.

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25,1. *Theorem.* If $a_{m-1}=b_{m-1}-a_m$,

$$\text{then } a_0=\overset{n}{S}_m(-1)^{m-1}.b_{m-1}+(-1)^n.a_n.$$

For, $(-1)^{m-1}.a_{m-1}=(-1)^{m-1}.b_{m-1}+(-1)^m.a_m$,

$\therefore \overset{n}{S}_m(-1)^{m-1}.a_{m-1}=\overset{n}{S}_m(-1)^{m-1}.b_{m-1}+\overset{n}{S}_m(-1)^m.a_m$, (4) & (5);

and $a_0+\overset{n-1}{S}_m(-1)^m a_m=\overset{n}{S}_m(-1)^{m-1}.b^{m-1}+\overset{n-1}{S}_m(-1)^m.a_m+(-1)^n.a_n$, (9).

$$\therefore a_0=\overset{n}{S}_m(-1)^{m-1}.b_{m-1}+(-1)^n.a_n,$$

cancelling identical terms.

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42,1. Cor. This theorem being true for every value of a, and m, we may put $a=0$, and $m=-1$, and we shall have $\lfloor 0 =1$; which result will be found of perpetual occurrence.

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48,1. *Theorem.* If $a_{n+1,m+1}=a_{n,m+1}+a_{n,m}$,

$a_{n,1}=n$ and $a_{n,n+r}=0$, then shall

$$a_{n,m}=\frac{\underline{|n}_{m}}{\underline{|m}}.$$

For, $a_{r+1,m+1}=a_{r,m+1}+a_{r,m}$.

$$\therefore\ a_{n,m+1}=a_{1,m+1}+\overset{n-1}{\mathrm{S}}_r a_{r,m},\qquad (24);$$

$$=\overset{n-1}{\mathrm{S}}_r a_{r,m},\ \text{since}\ a_{1,m+1}=0.$$

Hence, putting $m=1$,

$$a_{n,2}=\overset{n-1}{\mathrm{S}}_r a_{r,1}=\overset{n-1}{\mathrm{S}}_r r=\frac{\underline{|n}_{2}}{\underline{|2}},\qquad (48).$$

Also, putting $m=2$,

$$a_{n,3}=\overset{n-1}{\mathrm{S}}_r a_{r,2}=\overset{n-1}{\mathrm{S}}_r \frac{\underline{|r}_{2}}{\underline{|2}}=\frac{\underline{|n}_{3}}{\underline{|3}},\qquad (48).$$

Suppose, therefore, $a_{n,m}=\dfrac{\underline{|n}_{m}}{\underline{|m}}$;

$$\text{then}\ a_{n,m+1}=\overset{n-1}{\mathrm{S}}_r a_{r,m}=\overset{n-1}{\mathrm{S}}_r \frac{\underline{|r}_{m}}{\underline{|m}}=\frac{\underline{|n}_{m+1}}{\underline{|m+1}},\qquad (48).$$

If, then, the law were true for m it would be so for $m+1$; but it is true for 3 and therefore for m.

48,2. Similarly it may be shewn that if

$a_{n+1,m}=a_{n,m}+a_{n,m-1}$, $a_{n,0}=1$, $a_{1,1}=1$, and $a_{n,n+r}=0$, then shall

$$a_{n,m-1}=\frac{\underline{|n}_{m-1}}{\underline{|m-1}}.$$

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54,1. *Problem.* To find the number of terms in $\overset{m,n}{\mathrm{C}}_r a_1$.

Put $b_{n,m}$ = the number sought; then

$$b_{n+1,m+1}=b_{n,m+1}+b_{n,m},\ \ (54),\ \ b_{n,1}=n,\ \text{ and }\ b_{n,n+r}=0.$$

$$\therefore\ b_{n,m}=\frac{\underline{|n}_{\,m}}{\underline{|m}},\quad (48,1).$$

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60,1. Cor. 2. If c is independent of s,

$$\text{then }\ \overset{\overline{m,\,n-m}}{\mathrm{C}}_{r,s}\,\{(a_r)(b_s c)\}=c^{n-m}.\overset{\overline{m,\,n-m}}{\mathrm{C}}_{r,s}(a_r b_s).$$

See Art. 55.

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74. The theorem of this Article may also be proved as follows:

It will be readily seen that we may assume,

$$\underline{|a+b}_{\,n,r}=\overset{n+1}{\mathrm{S}}_m c_{n,m-1}.\underline{|a}_{\,n-m+1,r}.\underline{|b}_{\,m-1,r},$$

where $c_{n,m-1}$ is independent of a and b;

$$\text{then }\ \underline{|a+b}_{\,n+1,r}=(a+b+nr).\underline{|a+b}_{\,n,r},$$

$$\text{or }\ \overset{n+2}{\mathrm{S}}_m c_{n+1,m-1}.\underline{|a}_{\,n-m+2,r}.\underline{|b}_{\,m-1,r}$$

$$=\overset{n+1}{\mathrm{S}}_m c_{n,m-1}.\underline{|a}_{\,n-m+1,r}.\underline{|b}_{\,m-1,r}\ \ (a+\overline{n-m+1}.r+b+\overline{m-1}.r).$$

$$\therefore\ c_{n+1,0}\cdot\lfloor\underline{a}_{\,n+1,r}+\overset{n+1}{\mathrm{S}}_m c_{n+1,m}\cdot\lfloor\underline{a}_{\,n-m+1,r}\cdot\lfloor\underline{b}_{\,m,r}$$

$$=\overset{n+1}{\mathrm{S}}_m c_{n,m-1}\left(\lfloor\underline{a}_{\,n-m+2,r}\cdot\lfloor\underline{b}_{\,m-1,r}+\lfloor\underline{a}_{\,n-m+1,r}\cdot\lfloor\underline{b}_{\,m,r}\right)$$

$$=c_{n,0}\cdot\lfloor\underline{a}_{\,n+1,r}+\overset{n}{\mathrm{S}}_m\left(c_{n,m}+c_{n,m-1}\right)\cdot\lfloor\underline{a}_{\,n-m+1,r}\cdot\lfloor\underline{b}_{\,m,r}+c_{n,n}\cdot\lfloor\underline{b}_{\,n+1,r};$$

$$\therefore\ c_{n+1,m}=c_{n,m}+c_{n,m-1};$$

also we have $c_{n,0}=1$, $c_{1,1}=1$, and $c_{n,n+r}=0$.

$$\therefore\ c_{n,m-1}=\frac{\lfloor\underline{n}_{\,m-1}}{\lfloor\underline{m-1}},\quad (48,2).$$

The theorems in Articles 86 and 161 may be proved in the same way.

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83,1. Cor. 7. $\left(\overset{\infty}{\mathrm{S}}_m\lfloor\underline{a}_{\,m-1,r}\cdot\dfrac{x^{m-1}}{\lfloor\underline{m-1}}\right)^n=\overset{\infty}{\mathrm{S}}_m\lfloor\underline{na}_{\,m-1,r}\cdot\dfrac{x^{m-1}}{\lfloor\underline{m-1}};$

for every rational value of n.

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102,1. Cor. 6,1. $\left(\overset{\infty}{\mathrm{S}}_m\dfrac{a^{m-1}x^{m-1}}{\lfloor\underline{m-1}}\right)^n=\overset{\infty}{\mathrm{S}}_m\dfrac{(na)^{m-1}\cdot x^{m-1}}{\lfloor\underline{m-1}};$

n being any rational quantity.

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177. Or thus:

Since, $D_x^{-1}\{D_x^{m-1}(u).D_x^{-(m-1)}E_x^{m-1}v\}$

$=D_x^{m-1}(u).D_x^{-m}E_x^{m-1}v-D_x^{-1}\{D_x^m(u).D_x^{-m}E_x^mv\},\quad(176);$

$$\therefore\ D_x^{-1}(uv)=\overset{r}{\mathrm{S}}_m(-1)^{m-1}.D_x^{m-1}(u).D_x^{-m}E_x^{m-1}v+(-1)^r.D_x^{-1}\{D_x^r(u).D_x^{-r}.E_x^r(v)\},\quad(25,1).$$

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152,1. *Theorem.* $D_x^n \underline{|x}_{\,m,-h} = \underline{|m}_{\,n} \cdot h^n \cdot \underline{|x}_{\,m-n,-h}$.

For, $D_x \cdot \underline{|x}_{\,m,-h} = \underline{|x+h}_{\,m,-h} - \underline{|x}_{\,m,-h}$

$$=(x+h)\,\underline{|x}_{\,m-1,-h} - \underline{|x}_{\,m-1,-h}\,(x-mh+h)$$

$$=mh \cdot \underline{|x}_{\,m-1,-h}.$$

$$D_x^2 \underline{|x}_{\,m,-h} = mh \cdot (m-1) \cdot h \cdot \underline{|x}_{\,m-2,-h}$$

$$= \underline{|m}_{\,2} \cdot h^2 \cdot \underline{|x}_{\,m-2,-h}.$$

Similarly, $D_x^n \underline{|x}_{\,m,-h} = \underline{|m}_{\,n} \cdot h^n \cdot \underline{|x}_{\,m-n,-h}$.

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155,1. *Theorem.* $u = \overset{\infty}{S}_m \dfrac{\underline{|x}_{\,m-1,-h}}{\underline{|m-1}} \cdot \dfrac{D^{m-1}_{x=0} u}{h^{m-1}}$; where $h = Dx$.

For, put $u = \overset{\infty}{S}_m \underline{|x}_{\,m-1,-h} \cdot a_{m-1}$, where a_{m-1} is some finite quantity independent of x; then

$$D_x^{n-1} u = \overset{\infty}{S}_m \underline{|m-1}_{\,n-1} \cdot h^{n-1} \cdot \underline{|x}_{\,m-n,-h} \cdot a_{m-1}, \quad (136), (137), \text{ and } (152,1);$$

$$= \overset{n-1}{S}_m \underline{|m-1}_{\,n-1} \cdot h^{n-1} \cdot \underline{|x}_{\,m-n,-h} \cdot a_{m-1} + \underline{|n-1} \cdot h^{n-1} \cdot a_{n-1}$$

$$+ \overset{\infty}{S}_m \underline{|n+m-1}_{\,n-1} \cdot h^{n-1} \cdot \underline{|x}_{\,m,-h} \cdot a_{n+m-1}, \quad (9) \text{ and } (42);$$

$$=\underline{|n-1}\,.\,h^{n-1}\,.\,a_{n-1}$$

$$+\overset{\infty}{\mathrm{S}}_m \underline{|n+m-1}_{\,n-1}\,.\,h^{n-1}\,.\,\underline{|x}_{\,m,-h}\,.\,a_{n+m-1}, \text{ since } \underline{|m-1}_{\,n-1}=0, \ \left(\begin{smallmatrix}m=1\\ m=n-1\end{smallmatrix}\right).$$

$$\therefore\ D^{n-1}_{x=0}u=\underline{|n-1}\,.\,h^{n-1}\,.\,a_{n-1},$$

$$\text{and } a_{m-1}=\frac{1}{\underline{|m-1}}\,.\,\frac{D^{m-1}_{x=0}u}{h^{m-1}}.$$

$$\therefore\ u=\overset{\infty}{\mathrm{S}}_m\frac{\underline{|x}_{\,m-1,-h}}{\underline{|m-1}}\,.\,\frac{D^{m-1}_{x=0}u}{h^{m-1}}.$$

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207,1. *Theorem.* $d_x.\overset{n}{\mathrm{P}}_r u_r^{a_r}=\overset{n}{\mathrm{P}}_r(u_r^{a_r-1}).\overset{n}{\mathrm{S}}_m a_m d_x u_m.\overset{n;m}{\mathrm{P}}_r u_r$; a_r being independent of x.

$$\text{For, } \frac{d_x.\overset{n}{\mathrm{P}}_r u_r^{a_r}}{\overset{n}{\mathrm{P}}_r u_r^{a_r}}=\overset{n}{\mathrm{S}}_m\frac{d_x.u_m^{a_m}}{u_m^{a_m}}, \quad (205);$$

$$=\overset{n}{\mathrm{S}}_m\frac{a_m d_x u_m}{u_m}, \quad (207).$$

$$\therefore\ d_x.\overset{n}{\mathrm{P}}_r u_r^{a_r}=\overset{n}{\mathrm{P}}_r(u_r^{a_r-1}).\overset{n}{\mathrm{S}}_m\frac{a_m d_x u_m}{u_m}.\overset{n}{\mathrm{P}}_r u_r, \quad (6);$$

$$=\overset{n}{\mathrm{P}}_r(u_r^{a_r-1}).\overset{n}{\mathrm{S}}_m a_m.d_x u_m.\overset{n;m}{\mathrm{P}}_r u_r.$$

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213. This theorem may also be deduced as follows:

$$\text{We have } u=\overset{\infty}{\mathrm{S}}_m\frac{\underline{|x}_{\,m-1,-h}}{\underline{|m-1}}\,.\,\frac{D^{m-1}_{x=0}u}{h^{m-1}}, \text{ where } h=Dx, \quad (155,1).$$

This being true for every value of h, we may put $h=0$;

$$\text{then } u=\overset{\infty}{\text{S}}_m \frac{x^{m-1}}{\underline{|m-1}} \cdot \left(\frac{D_{x=0}^{m-1} u}{h^{m-1}}\right)_{h=0}$$

$$=\overset{\infty}{\text{S}}_m \frac{x^{m-1}}{\underline{|m-1}} \cdot d_{x=0}^{m-1} u, \quad (188).$$

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222. Or thus:

$$\text{Since } \textstyle\int_x \left\{d_x^{m-1} u \,.\, \int_x^{m-1} v\right\} = d_x^{m-1} u \,.\, \int_x^m v - \int_x \left\{d_x^m u \,.\, \int_x^m v\right\}, \quad (203).$$

$$\therefore \textstyle\int_x(uv) = \overset{n}{\text{S}}_m (-1)^{m-1} d_x^{m-1} u \,.\, \int_x^m v + (-1)^n \,.\, \int_x (d_x^n u \,.\, \int_x^n v), \quad (25,1).$$

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253. The symbols $\int_{x \atop x=a} u$, $\int_{x=x \atop x=a} u$, and $(\int_x - \int_{x=a}) u$ are equivalent to $\int_x u - \int_{x=a} u$.

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270. Or thus:

$$\text{Since } \tan(x+h) - \tan x = \frac{\tan x + \tan h}{1 - \tan x \,.\, \tan h} - \tan x$$

$$= \frac{\tan h \,.\, (\sec x)^2}{1 - \tan x \,.\, \tan h}.$$

$$\therefore d_x \,.\, \tan x = \left\{\frac{\tan(x+h) - \tan x}{h}\right\}_{h=0}$$

$$= \left\{\frac{\tan h}{h} \cdot \frac{(\sec x)^2}{1 - \tan x \,.\, \tan h}\right\}_{h=0}$$

$$= (\sec x)^2, \quad (266).$$

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270,1. Cor. 1. $d_x.\cot x = d_x.(\tan x)^{-1}$

$$= (-1).(\tan x)^{-2}.(\sec x)^2$$

$$= -\left(\frac{\sec x}{\tan x}\right)^2 = -(\sin x)^{-2}$$

$$= -(\operatorname{cosec} x)^2.$$

271,1. *Theorem.* $d_x.\operatorname{cosec} x = -\operatorname{cosec} x.\cot x.$

For, $d_x.\operatorname{cosec} x = d_x.(\sin x)^{-1}$

$$= (-1).(\sin x)^{-2}.\cos x$$

$$= -\frac{1}{\sin x}.\frac{\cos x}{\sin x}$$

$$= -\operatorname{cosec} x.\cot x.$$

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281. Every term in this series will vanish in which zero is a factor of $\underline{|n-m}_{\,m-2}$, that is of

$$(n-m)(n-m-1)\ldots(n-2m+4)(n-2m+3).$$

(1) Let n be even; then $n-2m+4=0$ if $m=\frac{n}{2}+2$; and

$$\underline{|n-(\tfrac{1}{2}n+1+r)}_{\,\frac{1}{2}n+1+r-2} = \underline{|\tfrac{1}{2}n-1-r}_{\,\frac{1}{2}n-1+r} = 0, \text{ if } r \ngtr \tfrac{1}{2}n-1.$$

(2) Let n be odd; then $n-2m+3=0$, if $m=\frac{n+1}{2}+1$; and

$$\underline{|n-\{\tfrac{1}{2}(n+1)+r\}}_{\,\frac{1}{2}(n+1)+r-2} = \underline{|\tfrac{1}{2}(n-1)-r}_{\,\frac{1}{2}(n-1)+r-1} = 0, \text{ if } r \ngtr \tfrac{1}{2}(n-1).$$

Hence, the number of significant terms will be $\frac{1}{2}n+1$, or $\frac{1}{2}(n+1)$, according as n is even or odd.

It will also be observed that, although n appears as a multiplier of the whole series, yet the coefficient of the first term being $\dfrac{\lfloor\!\underline{n-1}_{\,-1}}{\lfloor\!\underline{0}} = \dfrac{1}{n}$, (43) and (42,1), the first term will become $\frac{1}{2}.(2\cos x)^n$.

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300,1. *Theorem.*

$$\log\frac{\pi}{2} = 2n\log 4 + 4\log\lfloor\!\underline{n} - 2\log\lfloor\!\underline{2n} - \log(2n+1), \quad (n=\infty).$$

For, $\sin x = x.\overset{\infty}{\mathrm{P}}_r\left\{\left(1+\dfrac{x}{r\pi}\right)\left(1-\dfrac{x}{r\pi}\right)\right\}$, (293);

$$\therefore\ 1 = \frac{\pi}{2}.\overset{\infty}{\mathrm{P}}_r\left\{\left(1+\frac{1}{2r}\right)\left(1-\frac{1}{2r}\right)\right\}$$

$$= \frac{\pi}{2}.\overset{\infty}{\mathrm{P}}_r\frac{(2r+1)(2r-1)}{4r^2}.$$

$$\therefore\ \frac{\pi}{2} = \overset{\infty}{\mathrm{P}}_r\frac{4r^2}{(2r-1)(2r+1)}$$

$$= \frac{4^n.(\lfloor\!\underline{n})^2}{(\lfloor\!\underline{1}_{\,n,2})^2.(2n+1)}, \quad (n=\infty);$$

$$\text{but } \lfloor\!\underline{1}_{\,n,2} = \frac{\lfloor\!\underline{2n}}{2^n.\lfloor\!\underline{n}}.$$

$$\therefore\ \frac{\pi}{2} = \frac{4^n.(\lfloor\!\underline{n})^2.4^n(\lfloor\!\underline{n})^2}{(\lfloor\!\underline{2n})^2.(2n+1)}$$

$$= \frac{4^{2n}.(\lfloor\!\underline{n})^4}{(\lfloor\!\underline{2n})^2.(2n+1)}.$$

$$\therefore\ \log\frac{\pi}{2} = 2n\log 4 + 4\log\lfloor\!\underline{n} - 2\log\lfloor\!\underline{2n} - \log(2n+1), \quad (n=\infty).$$

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301. That there are not more than n different values of $\epsilon^{\frac{2m-1}{n}.\pi\sqrt{-1}}$ may be shewn as follows:

For m put $nr+m$, when r is any integer and $m<n$; then

$$\epsilon^{\frac{2m-1}{n}.\pi\sqrt{-1}} \text{ becomes } \epsilon^{\frac{2nr+2m-1}{n}.\pi\sqrt{-1}} = \epsilon^{2r\pi\sqrt{-1}}.\epsilon^{\frac{2m-1}{n}.\pi\sqrt{-1}}$$

$$= \epsilon^{\frac{2m-1}{n}.\pi\sqrt{-1}}, \quad (288).$$

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304. For m put $nr+m$, when r is any integer and $m<n$;

$$\text{then } \epsilon^{\frac{2m-2}{n}.\pi\sqrt{-1}} \text{ becomes } \epsilon^{\frac{2nr+2m-2}{n}.\pi\sqrt{-1}} = \epsilon^{2r\pi\sqrt{-1}}.\epsilon^{\frac{2m-2}{n}.\pi\sqrt{-1}}$$

$$= \epsilon^{\frac{2m-2}{n}.\pi\sqrt{-1}}, \quad (288).$$

306,1. *Theorem.*

$$x^{2n} - 2\cos\theta.x^n + 1 = \overset{n}{\mathrm{P}}_m\left(x^2 - 2x.\cos\frac{\overline{m-1}.2\pi+\theta}{n} + 1\right).$$

For, put $x^{2n} - 2\cos\theta.x^n + 1 = 0$;

then $x^n = \cos\theta \pm \sqrt{-1}.\sin\theta$

$$= \cos(\overline{m-1}.2\pi+\theta) \pm \sqrt{-1}.\sin(\overline{m-1}.2\pi+\theta),$$

$$\text{and } x = \cos\frac{\overline{m-1}.2\pi+\theta}{n} \pm \sqrt{-1}.\sin\frac{\overline{m-1}.2\pi+\theta}{n}, \quad \binom{m=1}{m=n}, \quad (277).$$

$\therefore\ x^{2n} - 2\cos\theta.x^n + 1$

$$= \overset{n}{\mathrm{P}}_m\left[\left\{x - \left(\cos\frac{\overline{m-1}.2\pi+\theta}{n} + \sqrt{-1}.\sin.\frac{\overline{m-1}.2\pi+\theta}{n}\right)\right\}\times\right.$$

$$\left.\left\{x - \left(\cos\frac{\overline{m-1}.2\pi+\theta}{n} - \sqrt{-1}.\sin.\frac{\overline{m-1}.2\pi+\theta}{n}\right)\right\}\right]$$

$$= \overset{n}{\mathrm{P}}_m\left(x^2 - 2x.\cos\frac{\overline{m-1}.2\pi+\theta}{n} + 1\right).$$

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327,1. *Theorem.*

$$\int_x (\tan x)^n = \overset{r}{\mathrm{S}}_m (-1)^{m-1} \cdot \frac{(\tan x)^{n-2m+1}}{n-2m+1} + (-1)^r \cdot \int_x (\tan x)^{n-2r}.$$

For, $(\tan x)^n = (\tan x)^{n-2} \cdot (\overline{\sec x}|^2 - 1)$

$$= (\sec x)^2 \cdot (\tan x)^{n-2} - (\tan x)^{n-2}.$$

$$\therefore\ (\tan x)^n = (\sec x)^2 \cdot \overset{r}{\mathrm{S}}_m (-1)^{m-1} \cdot (\tan x)^{n-2m} + (-1)^r \cdot (\tan x)^{n-2r},$$

(51) and (6).

$$\therefore\ \int_x (\tan x)^n = \overset{r}{\mathrm{S}}_m (-1)^{m-1} \cdot \frac{(\tan x)^{n-2m+1}}{n-2m+1} + (-1)^r \cdot \int_x (\tan x)^{n-2r},\ (270).$$

327,2. Cor. 1.

$$\int_x (\tan x)^n = \overset{\frac{1}{2}n}{\mathrm{S}}_m (-1)^{m-1} \cdot \frac{(\tan x)^{n-2m+1}}{n-2m+1} + (-1)^{\frac{1}{2}n} \cdot x, \text{ or}$$

$$= \overset{\frac{1}{2}(n-1)}{\mathrm{S}}_m (-1)^{m-1} \cdot \frac{(\tan x)^{n-2m+1}}{n-2m+1} + (-1)^{\frac{1}{2}(n+1)} \cdot \log_\epsilon \cos x;$$

according as n is an even or an odd positive integer.

327,3. Cor. 2.

$$(\cot x)^n = (\operatorname{cosec} x)^2 \cdot \overset{r}{\mathrm{S}}_m (-1)^{m-1} \cdot (\cot x)^{n-2m} + (-1)^r \cdot (\cot x)^{n-2r}.$$

$$\therefore\ \int_x (\cot x)^n = \overset{r}{\mathrm{S}}_m (-1)^m \cdot \frac{(\cot x)^{n-2m+1}}{n-2m+1} + (-1)^r \cdot \int_x (\cot x)^{n-2r},\ (270,1).$$

327,4. Cor. 3.

$$\int_x (\cot x)^n = \overset{\frac{1}{2}n}{\mathrm{S}}_m (-1)^m \cdot \frac{(\cot x)^{n-2m+1}}{n-2m+1} + (-1)^{\frac{1}{2}n} . x, \text{ or}$$

$$= \overset{\frac{1}{2}(n-1)}{\mathrm{S}}_m (-1)^m \cdot \frac{(\cot x)^{n-2m+1}}{n-2m+1} + (-1)^{\frac{1}{2}(n-1)} . \log_\epsilon \sin x;$$

according as n is an even or an odd positive integer.

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* Stainville.

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* Newton's Binomial.

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* With the same limitations as in 118.

* Taylor's Theorem.

* Maclaurin's Theorem.

* Laplace's Theorem. † Lagrange's Theorem.

* Herschel's Theorem.

* Demoivre's Theorem.

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* Machin's Theorem.

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