THE RESOLUTION

OF

ALGEBRAIC EQUATIONS.

AN ESSAY

ON THE

RESOLUTION

OF

ALGEBRAIC EQUATIONS.

BY THE LATE

CHARLES JAMES HARGREAVE, LL.D., F.R.S.,

A JUDGE OF THE LANDED ESTATES COURT, IRELAND.

PRINTED FOR PRIVATE CIRCULATION.

DUBLIN:
HODGES, SMITH AND CO., GRAFTON STREET,
PUBLISHERS TO THE UNIVERSITY.

1866.

Presented to

M. Briot

With Mrs. Hargreave's Compliments.

PREFACE.

It is right to prefix to this Memoir some short account of its author, and of the circumstances under which his Essay is given to the world.

Charles James Hargreave was born in December, 1820, at Wortley, near Leeds, Yorkshire, and was educated at a local school, which, under the designation of Bramham College, maintains its repute, and under the same head-master. He entered University College at an early age, and having obtained the degree of LL.B., with honours, in the University of London, he was elected Professor of Jurisprudence at University College in 1843, which professorship he retained until his appointment in 1849 as one of the Commissioners of the Incumbered Estates Court, Ireland. In 1851 he received the rank of Q. C., and became one of the Benchers of the Inner Temple. In 1858 the Court was placed on its present footing, and the Commissioners became Judges. The following sketch of his professional career and legal acquirements is from the pen of one of his colleagues in the court, (Judge Longfield,) who had peculiar opportunities during sixteen years of estimating his worth.

"His intellect was especially adapted for Mathematics and for the more scientific branches of law. Accordingly, he selected the law of real property to be his peculiar business, and became, first, a pupil of Mr. Greening, and afterwards of of Mr. Duval, both eminent conveyancers in London. He was also for a short time in a solicitor's office in London. He was called to the bar by the Inner Temple in 1844, and very rapidly obtained a high reputation and increasing practice. The eminent conveyancer Mr. Christie had the highest opinion of him, and frequently described him as the best qualified person in England for the important office which at a very early age he was called upon to fill."

"He was appointed one of the Commissioners for sale of Incumbered Estates in Ireland, in 1849, before he had arrived at his twenty-ninth birth-day."

"He entered upon the duties of his office with great zeal, and devoted the summer with his colleagues to the preparation of a code of rules and practice for the new Court. Those rules were so well adapted to their object that they continue in force to this day, with very trifling alterations. It is to be borne in mind that Mr. Hargreave's practice had been confined to the chambers of a conveyancer, and that he had had no experience touching the rules and practice of Courts of Equity. Nevertheless, he showed the greatest readiness in comprehending the spirit and effects of every rule, and he exerted himself to ensure that no rule should be wanted that was necessary for the protection of the suitor or the public, but that no unnecessary rules should add delay and expense to the proceedings in the suit. His advice and opinion on every point relating to the practice of conveyance were always received with considerable deference."

"The proceedings of the Court after the rules were published consisted of two different branches. There was the perusal of abstracts of title, the settlement of deeds, and the determination of their construction and the rights of the parties. This was the conveyancer's proper department, and he executed it with great skill and industry. He was a neat and accurate draughtsman, and very felicitous in finding proper forms of expression. He omitted nothing, and repeated nothing. His forms were equally free from prolixity and ambiguity. The next great branch of business was the decision of causes after argument. Here he was without experience. It may be said that his first experience in a court of justice was to preside in it as judge. But no person could observe any deficiency. His patience, his learning and his impartiality quickly secured the respect and confidence of the practitioners in his court, and his unequalled sweetness of temper made him a general favorite. Although of a nervous and sensitive temperament, nothing seemed to irritate him. He acted on all occasions with great firmness, persisting in selling estates according to his own judgment, when he might have secured his own ease and safety and apparent popularity, by refusing to sell at the only prices that could then be obtained. But he was most in his element when an unusual combination of circumstances and complicated deeds seemed to produce inextricable confusion. His habits of order and his fine mathematical mind at once arranged the rights of the parties with a certainty approaching mathematical demonstration. He never seemed happier than when he was engaged in a subtle mathematical analysis, or in determining the rights arising from a deed when every event occurred except those contemplated by the conveyancer who drew the instrument. But he is no more. And for the good feeling and discernment of the Irish it must be said, that they loved and honoured him while he was living, and deeply lamented him when he was removed."

The following, from the pen of his friend and fellow-student Mr. J. Waley of Lincoln's Inn (who succeeded him in the Professorship at University College) will also be read with interest:—

"My recollections of Hargreave begin with a "prize-day" at University College, London, on which he—then very young—was singled out by Professor De Morgan for praise beyond all the rest of the class at University College; and on graduating at the University of London, he was a very successful student, distinguishing himself in Classics as well as Mathematics, though his special excellence lay in the latter.

"We were afterwards fellow-pupils in the chambers of the late Mr. Duval, who has had a share in the training of many distinguished lawyers of the past and present generations. The acute eye of the great conveyancer almost immediately discovered Hargreave's merit. He was, even before he was out of his pupilage, exact, ready and full of law. I remember, for example, that it was desired to confer on the tenant for life of a great estate, the power of accepting surrenders of leases for lives and granting in their place leases for terms of years certain. Hargreave maintained that the only equivalence possible between a lease for lives and a lease for years was equality in money value, while Mr. Duval thought that some comparison founded upon the probabilities of life, and expressed in terms of time, should be adopted. The point was mentioned to Professor De Morgan, whose view confirmed that taken by Hargreave.

"From the time that Hargreave went to Ireland, I of course ceased to be in habitual intercourse with him, though we remained on terms of friendly intimacy until his lamented death. To the eminent qualities which distinguished him as a judge, others more within the scope of their action can better than myself bear testimony. The patience, assiduity and learning, the moderation and love of justice, which invariably distinguished him, were, no doubt, important elements in the success of the great legal and social experiment which was worked out by means of the Incumbered Estates Court.

"Reverting to my own personal experience, if I were to state my impression of Hargreave as a lawyer, I should say that he was singularly clear, full and exact, quick and inventive, and, at the same time, very judicious and discriminating—and applying the great acuteness which was native to his mind, not in subtle reasonings or refined conclusions, but in aid of a sound common sense and spirit of fairness. I must also pay a passing tribute to his singularly amiable and equable temper, to the kindness and gentleness of his disposition. During the many years that we were in close association, never, I think, have I seen him for a moment irritated or vexed, never have I heard him say an angry or unkind word to or of anyone."

Although it is as a mathematician that Judge Hargreave is likely to be known to most of the readers of this Essay, precedence has been given in this sketch to his legal career, because, in truth, law was the business of his life, and mathematics the relaxation with which he filled up the intervals of official labour. And it is to be feared that his premature death must be imputed to his having chosen as his principal recreation, one which only substituted one form of mental labour for another, and which gave his brain scarely any rest from continuous exertion. One of his earliest memoirs is that by which he is best known, viz.: that on the Solution of Linear Differential Equations, which was published in the Philosophical Transactions for 1848, and to which was awarded one of the Royal Medals of the Royal Society. Shortly after the publication of this paper he was elected a

Fellow of the Royal Society. The other papers which he contributed to the Philosophical Transactions were, "On General Methods in Analysis for the Resolution of Linear Equations in Finite Differences and Linear Differential Equations," Phil. Trans. 1850, p. 261 ; and "On the Problem of Three Bodies," 1858. He also contributed the following papers to the London and Edinburgh Philosophical Magazine : "Notes on the Solution of Differential Equations," 1847, p. 8 ; "Analytical Researches concerning Numbers," 1849, p. 36 ; "On the Valuation of Life Contingencies," 1853, p. 39 ; "On the Application of the Calculus of Operations to Algebraical Expressions and Theorems," p. 351 ; "On the Law of Prime Numbers," 1854, p. 114 ; "On Differential Equations of the First Order," 1864, p. 355. The honorary degree of LL.D. was conferred on him by the University of Dublin in 1852, in company with Professor Boole, a mathematician also too early lost to science.

A few months ago Judge Hargreave's attention was attracted by the problem which forms the subject of the present Essay, and which, humanly speaking, may be regarded as the cause of his death. For some time previously he had not enjoyed robust health, and his friends had, on this account, frequently endeavoured to withdraw him from the mathematical investigations which seemed to form his favorite occupation during the intervals of court business. Their efforts, however, were in vain, and he sacrificed all relaxation, and even sleep, to the study of this problem, which strongly excited his interest. Over-exertion of the brain brought on an illness, of which he died at Bray, Co. Wicklow, on the 23rd April, 1866.

It remains to add a few words as to my own share in the publication of this Essay. I had been permitted by Judge Hargreave to read his memoir when the first draft of it was complete ; and my views being different from his, some controversy ensued, in which I had occasion to experience that sweetness of temper of which Judge Longfield speaks—for nothing could exceed the patience and good humour with which he bore my criticisms. The result of our discussion was, that while he admitted the validity of my objections to some non-essential parts of his memoir, he held that my principal difficulties arose from my not having thoroughly apprehended the spirit of his argument ; and he believed that by recasting his essay, he could state his argument in such a manner as to be more easily intelligible. Accordingly, he proceeded to re-write his memoir, and sent part of it to press. But before I had the opportunity of studying it in its new form, I was summoned to what proved to be the death-bed of its author. His medical advisers had then prescribed, as the only hope of saving his life, that he should completely abandon mathematical studies. And in order that he might be able to put from his thoughts the subject on which he had been labouring, I took away with me the manuscript of this Essay, the last appendix to which I had just written from his dictation, and undertook to see it through the press. It proved to be too late to save his life ; but it has been his widow's desire that I should proceed with the task I had undertaken, and submit the papers that had been placed in my hands to the judgment of the mathematical world. If there should be need to bespeak for them any indulgent consideration, it will readily be granted to a work which is under the disadvantage of not having received its author's own final revision. My own duty I have considered to be confined to correcting typographical errors, and obvious slips of the pen; and I have not thought myself at liberty to make any other alteration. I have to some extent gone over the numerical calculations, but only partially, as I considered that any error in these would not affect the validity of the main argument.

In conclusion, I have to bear testimony to the regard and esteem which Judge Hargreave's character inspired. My own intimacy with him can only be said to date from the time when he consulted me on the subject of this paper; yet I knew him long enough to learn how gentle and amiable, how thoroughly honest and sincere he was; and I feel that in his premature death, I have to mourn not only the public loss of an able mathematician and an efficient judge, but that of a valued friend.

GEORGE SALMON.

TRINITY COLLEGE, DUBLIN,
July 3rd, 1866.

INTRODUCTION.

1. The problem, "To resolve an algebraic equation," differs in a logical sense from the problems which occur in other branches of mathematical science. The only object which we have in view in this resolution is, to conceive and express a certain formula or combination of symbols. When the formula is so conceived or expressed, the problem is fully performed. We have not to predicate any proposition concerning it, or to prove any fact connected with it. The formula, so soon as it is conceived, proves itself to be what it professes to be—that is, an adequate representative of the root of the proposed equation. If any person, speculating upon combinations of two things or symbols x_1 and x_2, should, by chance, or by trial, or as the result of a method, conceive the combination

$$\frac{1}{2}\left\{x_1 + x_2 + \sqrt{x_1^2 + x_2^2 - 2x_1 x_2}\right\},$$

he would have thereby effected the resolution of the equation of the second degree. He is not called upon to say or prove anything about the formula, because by its mere form it represents *identically*—that is, by performance of the indicated operations only, the two things which it ought to represent. It is immaterial how the conception of the combination is arrived at. It is not necessary that it should be acquired as the result of a method, or of a process of reasoning. The mere existence of the formula proves its truth, and shows that the problem is in fact done.

We are not, however, very likely to be able to conceive or express the corresponding formulæ, or combinations of three or more things, as the mere result of chance, or even of unassisted trial.

To proceed with any well-founded hope of a successful result, it is necessary to reason on the subject; and on such a question, all reasoning must be in effect of an *à priori* character.

2. This problem, from the nature of it, therefore admits of being dealt with in a manner which in other branches of science could not be regarded as legitimate; that is to say, by processes of a tentative character, whether assisted or not by rules derived from general principles. Without applying any principle, a mathematician skilled in combinations, would readily light upon that combination of two symbols which represents the root of a quadratic equation. Obvious principles of symmetry and homogeneity, coupled with the knowledge that the result to be arrived at must undergo variations in multiplicity of value, when certain known combinations of the symbols vanish, would enable him to arrive tentatively at the formula which is capable of representing, at the same time, three independent symbols; or, in other words, to resolve algebraically the cubic equation. And if, in addition to the principles and knowledge above mentioned, he began to speculate successfully upon the possibility of finding combinations of several symbols which, on the one hand, are not symmetric, and which, on the other hand, do not possess what may be called their full complement of values, it is not difficult to imagine that he might, in a somewhat tentative manner, arrive even at the

solution of an equation of the fourth degree, by discovering such a combination of four symbols as should algebraically be a representation of each of them.

It is obvious, however, that the practical limit of this kind of mixed process would soon be arrived at; and the inquirer would probably then be led to consider, whether he could discover, *à priori*, all the possible combinations of symbols which enter into the expression of a root of an algebraic equation.

He would probably regard it as certain, that all those combinations of symbols, whose disappearance indicates a change in the *status* or condition of the equation, (such as the condition which denotes the equality of two of the symbols; the conditions which denote the equality of several symbols; the conditions which denote various systems of equalities between the symbols; the conditions which denote that whenever a particular system of equalities exists there also exists another system of equalities of a more extended nature ; as, for example, the condition that whenever an equation has two equal roots it will have three equal roots, or two pairs of equal roots, etc.), would in some shape appear in the expression which represents the root, and would so appear as to alter the multiplicity of value of the expression by their vanishing. He might even be led to suppose that the expression for the root would consist exclusively of this class of combinations; as the entry of a combination foreign to this class would enable him to make a change in the shape of the expression of the root, and perhaps also a change in its multiplicity of value, by causing this foreign combination to become zero; whilst, at the same time, the vanishing of the combination is not one of the conditions which produce a change in the *status* or condition of the equation or of its roots. A fuller consideration of the nature of algebraic expressions involving radicals, (such as in fact we are now treating of, under the name of multiplicity of value), would probably induce him to doubt the correctness of so general a conclusion, as he would become aware that there might be an expression in which such extraneous combinations as have been above mentioned appear, and are visible to the eye; and yet, that this expression might be equivalent to another in which these extraneous combinations do not appear; and this equivalence, or equality of ultimate value, might exist without our being able to reduce the one expression to the other by any algebraic process.

To illustrate this notion, let us consider a numerical cubic equation

$$x^3 + 4x^2 - 2x - 20 = 0$$

whose *arithmetical* roots are in fact 2, and $-3 \pm \sqrt{-1}$. The algebraic root is that formula which we obtain by substituting these numerical co-efficients in the known algebraic form of resolution. We know that this latter result must really be equivalent to the arithmetical roots; but we cannot by any algebraic process reduce it to these in point of form.

Bearing in mind this class of considerations, we are not to assume that all methods of resolving the same algebraic equation will conduct us to the same form, or even to several forms admitting of having their equivalence algebraically shown. It has, in fact, been made the subject of elaborate proof, in reference to the equations of the third and fourth degrees, that all the modes of solution hitherto discovered, are algebraically reducible to one and the same irreducible form; and it has even been attempted to be shown that every possible form of solution must be so reducible. I refer to this here not for the purpose of adopting or contesting these results, as they form the subject of consideration in the sequel of this essay, but merely to illustrate the necessity of not assuming that all modes of solution necessarily conduct to the same result, so far as relates to the *form* of the resolution, even when it is so conceived as to be irreducible.

3. Up to the present time, the net result of trial, *à priori* speculation, artificial processes, and systematic methods, applied to this subject, has been to produce formulæ for the equations of the second, third and fourth degree, which, (with some qualification), actually fulfil the conditions of algebraic resolution; and to produce, in respect to all higher equations, arguments, more or less conclusive, and not universally accepted, to prove that their algebraic resolution is impossible.

It is, I think, obvious that when we have failed by all the above ways to reach the resolution of the general algebraic equation, the only method left to us is to proceed by steps; that is, in effect, so soon as we have resolved the problem in any one case, say the lowest, to endeavour to make use of this step, in order, if possible, to mount to the next step; and to do this, as far as possible, upon some general principle, the applicability or non-applicability of which is shown by the success or failure of the attempt to apply it.

4. The position which this problem of algebraic resolution occupies in the existing state of the science is peculiarly unsatisfactory in a philosophic point of view. We seem to be stopped in our researches by an invisible barrier set up at an arbitrary point of the road on which we are travelling. If it be true that there is some number m such that the equation of the $(m-1)^{th}$ degree is algebraically resoluble while the equation of the m^{th} degree is not algebraically resoluble, it must be because the second problem is not *ejusdem generis* with the first; and consequently that there must be some simple and elementary way of stating the first problem which is not algebraically applicable to describe the

second. Yet nothing of this kind has manifested itself in all the infinite researches which have been made into this problem. The cases which have been resolved are known by that fact to be resoluble, but not very clearly in any other way. The cases which are not resoluble are demonstrated to be irresoluble not by anything really peculiar to these cases of such a nature as to distinguish them from the others, but by an elaborate enquiry into all the possible modes of algebraic expression, and an exhaustive proof that no one of them can qualify itself to be the expression of the root.

To proceed in this subject upwards by steps, to endeavour to feel consciously where the barrier intervenes, and to learn why it intervenes exactly there and not elsewhere, is the problem which I have proposed to myself in this essay, and to the discussion of which I now proceed.

NOTATION, DEFINITIONS, AND ELEMENTARY THEOREMS.

5. I consider the general algebraic equation, (which for shortness I shall call a *quantic*), in the form

$$x^n - na_1 x^{n-1} + \tfrac{1}{2}n(n-1)a_2 x^{n-2} - \ldots \pm na_{n-1}x \mp a_n = \phi_n = 0.$$

I denote by the symbol $(1p)$ the resultant of ϕ_1 and ϕ_p; that is, the result of substituting a_1 for x in ϕ_p, and changing the sign. The quantic may be placed under the form

$$t^n - (11)\,t^{n-1} - \tfrac{1}{2}n(n-1)(12)\,t^{n-2} - \ldots - n(1\,\overline{n-1})\,t - (1n) = 0,$$

in which t is written for $x - a_1$; and in which (11) is written only for the sake of symmetry, as it is zero. The values of (12), (13), &c. are

$$\begin{aligned}
(12) &= a_1^2 - a_2\,;\\
(13) &= 2a_1^3 - 3a_1a_2 + a_3\,;\\
(14) &= 3a_1^4 - 6a_1^2a_2 + 4a_1a_3 - a_4\,;\\
(15) &= 4a_1^5 - 10a_1^3a_2 + 10a_1^2a_3 - 5a_1a_4 + a_5\,;\\
&\vdots
\end{aligned}$$

For the above quantic, I propose to call x the *argument*; $a_1, a_2, \ldots a_n$, the coefficients; and the several values of x, viz., $x_1, x_2, \ldots\ldots x_n$, the roots.

If, in the second form of the quantic written above, we consider $(x - a_1)$ or t as a *new* argument, we have a series of new coefficients $(12), \ldots (1n)$, which are $(n - 1)$ in number. I shall speak of this throughout as "the linear transformation." The expressions $(12), \ldots, (1n)$ are therefore all symmetric functions of $(x_1 - a_1)$, $(x_2 - a_1)$, ... and $(x_n - a_1)$.

If we find occasion to use other resultants, we shall adopt a similar notation. Thus (23) will denote the resultant of ϕ_2 and ϕ_3, &c.

All functions employed are of necessity homogeneous; and the degree of a function is always estimated with reference to the roots $x_1, x_2, \ldots x_{n-1}, x_n$; or, which is the same thing, with reference to the coefficients $a_1, a_2, \ldots a_{n-1}, a_n$: all roots being of the same degree, and each coefficient of the degree denoted by its suffix. The resultants above given, for example, are respectively of the 2nd, 3rd, 4th, 5th, &c. degree.

The symbol Σ prefixed to any function of roots means the sum of all the values which the function can assume by interchanges of the roots.

The symbol S_p denotes some symmetric function of $x_1 \ldots x_n$ of the p^{th} degree. S_p therefore also is a rational function of the same degree of $a_1 \ldots a_n$.

The symbol R_p denotes some rational function of $x_1 \ldots x_n$ of the p^{th} degree, which may or may not be symmetric. These symbols however do not mean throughout the *same* symmetric or rational function. The letter N is appropriated to denote numerical factors; but other letters will sometimes be used.

6. We shall have much occasion to make use of the discriminant of a quantic. The linear transformation shows not only that the discriminant of a quantic, but that every combination of the roots or coefficients of a quantic whose vanishing denotes the condition or one of the conditions of some system or systems of equalities between the roots, (for example, all the leading Sturmian coefficients of the quantic), admits of being expressed in terms of the $(n-1)$ quantities (12), (13), . . . , $(1n)$.

For the purposes of this paper we need only consider the discriminant.

The discriminants of quantics up to the quintic are as follows:

Discriminant.

Quadratic	.... (12).
Cubic . .	.... $4\,(12)^3 - (13)^2$.
Quartic .	$81\,(12)^4\,(14) - 54\,(12)^3\,(13)^2 + 18\,(12)^2\,(14)^2$ $- 54\,(12)\,(13)^2\,(14) + 27\,(13)^4 + (14)^3$.
Quintic .	$3456\,(12)^5\,(15)^2 + 11520\,(12)^4\,(13)\,(14)\,(15)$ $- 6400\,(12)^4\,(14)^3 - 5120\,(12)^3\,(13)^3\,(15)$ $+ 3200\,(12)^3\,(13)^2\,(14)^2 - 1440\,(12)^3\,(14)\,(15)^2$ $+ 2640\,(12)^2\,(13)^2\,(15)^2 + 4480\,(12)^2\,(13)\,(14)^2\,(15)$ $- 2560\,(12)^2\,(14)^4 - 10080\,(12)\,(13)^3\,(14)\,(15)$ $+ 5760\,(12)\,(13)^2\,(14)^3 + 120\,(12)\,(13)\,(15)^3$ $- 160\,(12)\,(14)^2\,(15)^2 + 3456\,(13)^5\,(15)$ $- 2160\,(13)^4\,(14)^2 + 360\,(13)^2\,(14)\,(15)^2$ $+ 640\,(13)\,(14)^3\,(15) - 256\,(14)^5 + (15)^4$.

Each of these may be considered as affected by any numerical multiplier. The discriminant of the quartic admits of being placed in the form

$$27\Big((12)^3 + (12)\,(14) - (13)^2 \Big)^2 - \Big(3\,(12)^2 - (14)\Big)^3;$$

which, from its similarity to that of the cubic, leads to an interesting result which will hereafter be stated.

7. If the quantic admits of algebraic resolution through the medium of the linear transformation, the rational part of the root will always be a_1; and consequently the residue of the expression of the root will be functions of the $(n-1)$ quantities (12), (13), . . . $(1\,n)$. This proposition will be

understood as not excluding the possibility of resolving the original quantic by some method which does not include, or operate through, the linear transformation. We do not say that the rational part of the root is necessarily a_1, and that the irrational parts are functions of (12), . . . (1 n); but that if the rational part be a_1 in consequence of the resolution of the equation being effected through the medium of the linear transformation, the irrational parts will be functions exclusively of the $(n-1)$ resultants.

We always regard $x_1 \ldots x_n$ as n independent roots or symbols; and $a_1 \ldots . a_n$ as n independent coefficients or parameters. We do not recognize the possibility of relations existing between them, or of any character being attributed to them.

By the term "transformation" applied to ϕ_n, we mean its change into one or more equations of the same degree (n), in which there exist one or more relations between the coefficients and the roots. We shall use z as the argument of such conditioned equations, and the letter b for the coefficients.

Such an equation in z being called λ_n we shall denote the resultant of λ_1 and λ_p by (p 1), so as to distinguish them from the resultants of the equation in x.

In connection with ϕ_n we shall have occasion to discover and employ a subsidiary equation of a lower degree, which is ordinarily called the resolvent. The argument of this will be y; and when we find it necessary to use *its* resultants, we shall call them for distinction (I · II), (I · III), (I · IV) . . &c.

8. Many of the results arrived at in this paper will be merely indicated and not actually calculated. To make this indication succinct and expressive, I propose to denote certain symmetric functions of the roots by short symbols. The following will suffice, viz.: to denote

$(n-1)\,\Sigma(x^p)\,\Sigma(x^q) - n\,\Sigma(x^p x^q)$ by $\overline{p\,q}$, p and q being different, and

$(n-1)\left(\Sigma(x^p)\right)^2 - 2n\,\Sigma(x^p x^p)$ by $\overline{p\,p}$;

also to denote

$(n-1)(n-2)\,\Sigma(x^p)\,\Sigma(x^q)\,\Sigma(x^r) - n^2\,\Sigma(x^p x^q x^r)$ by $\overline{p\,q\,r}$, p, q, and r being different;

$(n-1)(n-2)\left(\Sigma(x^p)\right)^2\Sigma(x^q) - 2n^2\,\Sigma(x^p x^p x^q)$ by $\overline{p\,p\,q}$, p and q being different; and

$(n-1)(n-2)\left(\Sigma(x^p)\right)^3 - 6n^2\,\Sigma(x^p x^p x^p)$ by $\overline{p\,p\,p}$.

In the actual applications, p, q, and r will of course be numbers. The formulæ which are composed of these and similar expressions will consist wholly of symmetric functions of $x_1 \ldots x_n$. In most cases we shall have occasion to express such formulæ, both in the shape of functions of $x_1 \ldots x_n$ and in the shape of functions of $a_1 \ldots a_n$. The one is of course convertible into the other at pleasure.

CONSIDERATIONS RELATING TO THE OPERATIONS USED IN THIS SUBJECT.

9. The only operations which we have to consider in this purely algebraic theory, are addition, multiplication, and elevation to prime powers; with their respective inverses.

The first of these gives occasion to consider whether quantities are positive or negative; the second whether they are whole or fractional; the third whether they are rational or irrational.

It is evident, therefore, that in substance all our investigations will have reference simply to the rationality or the irrationality of expressions, as we can so conduct them as ordinarily to exhibit by the mere form of the expressions used whether they are positive or negative, and whether they are whole or fractional. If p be a prime number, and P an expression which cannot be obtained by the raising of any other algebraic expression to the pth power, then $\sqrt[p]{P}$ is an irrational or surd expression. It cannot be represented in any form which is free from the radical; and if we assume it to have p values, all these values are necessarily indistinguishable *inter se.* We attribute p values to the expression because we know that $\left\{\sqrt[p]{P}\right\}^p$ is the same as each of the p forms $\left\{a\sqrt[p]{P}\right\}^p$ where a is an unreal pth root of unity. Nevertheless, we do not really identify each root by putting before it any particular power of a, seeing that the symbol $\sqrt[p]{}$ still enters in all its ambiguity.

If by any process of reasoning, we arrive at expressions such as $\sqrt[p]{P}$ in reference to which there is an *à priori* necessity that its several values must be algebraically distinguishable, then we may conclude that $\sqrt[p]{P}$ is not a surd or irrational expression; but that it is of the form $\sqrt[p]{Q^p}$, in which case we can write down separately the several values $a\,Q$, and distinguish them one from another in such a manner that each separate value can always be identified.

In such a case as this, we can consider P as being the product of the p different quantities $a\,Q$, $a^2\,Q$, . . $a^p\,Q$.

In the contrary case no such expression as Q exists; and if we write $\sqrt[p]{P}$ in place of it, we merely reintroduce into each factor all the ambiguity of the original expression.

10. It may perhaps be supposed that this is a merely metaphysical distinction. Whether this be so or not, the distinction is a real one; and it constitutes probably the clearest mode of distinguishing between the two notions, a rational expression and an expression which is identically and necessarily surd.

We shall have frequent occasion to make use of this distinction in the course of our researches; and it may be convenient to exhibit an instance of it by way of anticipation.

In the case of a cubic equation, we shall find that a certain product of two rational functions of the roots, each of the third degree, is a perfect cube of a symmetric function of the same roots of the second degree; and we shall also be aware independently that the same proposition expressed in the same terms is equally applicable to two other cubics in which the symmetric function of the second degree is in the one case a times, and in the other case a^2 times, what it is in the original cubic, a being an unreal cube-root of unity. This last consideration conducts us to the conclusion, (which is a very important one), that *each* of the two rational functions whose product is a perfect cube, is itself also a perfect cube; for otherwise we could not divide each factor into three distinguishable subordinate factors.

To fix our ideas more closely, let it be considered as proved that two rational functions of the roots have for their product $(12)^3$, and that they would equally have had this expression for their product if in the cubic $a\,(12)$ or $a^2\,(12)$ had been written for (12), which is the same thing as writing $a\,(x-a_1)$ and $a^2\,(x-a_1)$ for each $x-a_1$. Then it follows that each of the two rational functions is a perfect cube, and therefore a perfect cube of a linear function of the roots. This is the only way in which each factor can be divisible into subfactors which differ only in the substitution of $a\,(x-a_1)$ and $a^2(x-a_1)$ for each $x-a_1$.

The only alternative mode of composing this cubic product is to suppose that one factor is of the form $R_1^2\,R'_1$, and the other $R_1\,R'^2_1$; and we could then only represent the sub-factors as $\sqrt[3]{R_1^2\,R'_1}$, and this expression multiplied by a and a^2. We should have no power of *distinguishing* the sub-factors so as to apply them in a proper manner to the *three* distinct cases to which we know from independent considerations that they ought to be applicable.

The principles here stated are of great importance in our method of dealing with algebraic equations, and they will be more developed as we proceed.

SYSTEMS OF EQUALITIES, CRITICAL FUNCTIONS.

11. Conditions existing among the roots of a quantic may expressed in terms of the roots, or in terms of the coefficients Thus if we wish to impose the linear condition

$$A_1 x_1 + A_2 x_2 + \ldots + A_n x_n = 0,$$

we multiply together the $1.2\ldots n$ equations which include this and all its changes by transpositions of the roots. The result so far as the roots enter, is a symmetric function of them, and may therefore be expressed rationally in terms of the coefficients. If $A_1 \ldots A_n$ are not all different, the degree of the condition is some submultiple of $1.2..n$.

The most important condition which we have occasion to consider is that whose existence causes the quantic to have two equal roots. This condition is expressed by equating to zero the product of all the values of $x_p - x_q$. These are $n(n-1)$ in number: and since for every factor $x_p - x_q$ there is also a factor $x_q - x_p$, this condition when expressed in the roots is a perfect square. When expressed in the coefficients, it is the resultant of ϕ_{n-1} and ϕ_n, *i.e.* the discriminant of ϕ_n.

In like manner the conditions of three equal roots are: first, the condition of two equal roots as above given; and secondly, the condition that the first derivative shall also have two equal roots, or that the quantic shall have a common root with its second derivative. Thus, in the cubic, the conditions of two equal roots may be represented as

$$(23) = 0, (12) = 0;$$

or as

$$(23) = 0, (13) = 0,$$

or since

$$(23) = (13)^2 - 4\,(12)^3,$$

as

$$(12) = 0,\ (13) = 0.$$

A little consideration will show that in the case of the cubic, the conditions of three equal roots can be presented by means of linear relations between the roots. For since (23) is a perfect square, and is of the 6th degree, we can write down the condition of two equal roots as a function of the third degree: and as the further condition (13) is also of that degree, we can by a suitable value of N present both conditions under the form

$$N\,(13) + \sqrt{(23)} = 0,$$

which expressed in the roots is a rational function having two values. The value of N must be such that these two conditions are equivalent to $(12) = 0$, $(23) = 0$; or to $(12) = 0$, $(13) = 0$.

They *are* equivalent to $(13) = 0$, as we see by adding them; and if we make N unity, they become equivalent to $(12)^3$ or $(12) = 0$; as we can see by multiplying them. We have therefore two rational functions of x_1, x_2 and x_3, whose product is the cube of a certain expression (12); and it follows from the considerations stated in (9) and (10) that each of these rational functions is a perfect cube of a linear function of the roots. We have, therefore, three linear functions of the roots, each of which when cubed gives $(13) + \sqrt{(23)}$; and three other linear functions of the roots each of which when cubed gives $(13) - \sqrt{(23)}$; and the sum of the cubes of one of each set is to be a symmetric function of the roots.

Under these conditions it is easy to see that, (a being a unreal cube root of unity),

$$A\,(x_1 + a\,x_2 + a^2x_3),\ A\,(x_1 + a^2x_2 + a\,x_3),$$

and the four forms derived from these by multiplying by a

and by a^2, are the only linear functions which will serve the purpose.

We may, therefore, say that, for the cubic, the conditions of three equal roots may be represented linearly by

$$x_1 + a\,x_2 + a^2\,x_3 = 0,\; x_1 + a^2\,x_2 + a\,x_3 = 0,$$

each of which is a six-valued function, whose cube is a two-valued function, of the roots.

12. We can apply a similar mode of reasoning to the quartic also. The condition of two equal roots is $(34) = 0$; and the further condition for three equal roots is $(23) = 0$. The first condition being of the 12th degree in the roots and a perfect square, its square-root is homogeneous with the other condition $(23) = 0$, or any other condition of the sixth degree.

Now, as (34) can be placed under the form

$$27\left((12)^3 + (12)(14) - (13)^2\right)^2 - \left(3(12)^2 - (14)\right)^3,$$

or

$$27\left((23) - (12)(3(12)^2 - (14))\right)^2 - \left(3(12)^2 - (14)\right)^3,$$

the conditions $(23) = 0$, $(34) = 0$, taken together, would be satisfied by $3(12)^2 - (14) = 0$, and $(12)^3 + (12)(14) - (13)^2 = 0$; and we may consider any two of the three
$3(12)^2 - (14) = 0,\ (12)^3 + (12)(14) - (13)^2 = 0,\ (34) = 0,$
as the conditions of two equal roots.

By a suitable value of N, therefore, the two conditions can be both expressed by

$$N\left((12)^3 + (12)(14) - (13)^2\right) \pm \sqrt{(34)};$$

for the sum of these is one condition, and the product becomes the other if N be $\sqrt{27}$. Repeating the argument of the last section, *mutatis mutandis*, we therefore find that the conditions for three equal roots in a quartic can be represented by some two functions, such as

$$R_2 + a\,R'_2 + a^2 R''_2$$
$$R_2 + a^2 R'_2 + a\,R''_2$$

being separately equated to zero, the R's being certain rational functions of the second degree of x_1, x_2, x_3, and x_4, which we should find by performing the calculations.

Other interesting results might probably be obtained by a systematic investigation of the various conditions of systems of equalities; but the foregoing will be sufficient for our purpose.

THE QUADRATIC.

13. THE simple application of the linear transformation constitutes, *ipso facto*, the resolution of the quadratic. It places it in the form

$$(x - a_1)^2 - (12) = 0 ;$$

from which we deduce

$$x = a_1 \pm \sqrt{(12)} ;$$

which is equivalent to stating that the combination

$$\tfrac{1}{2}\{ x_1 + x_2 + \sqrt{x_1^2 + x_2^2 - 2x_1x_2} \}$$

represents x_1 and x_2.

It is obvious that the quadratic admits of no transformation except the linear one, and consequently that the above is the only possible form of the root. The rational part being a_1, the irrational part contains nothing except a function of (12). The quadratic can undergo only one change of system in reference to equalities of roots; viz., the change from its general form to that of two equal roots. There can, therefore, only be one radical in its root, and that a square-root radical. Its solution is therefore unique.

Referring to the principles laid down in (9) and (10), and considering that the roots of the quadratic are expressible in the form $S_1 + \sqrt{S_2}$, we can immediately discern an *à priori* necessity that S_2 when expressed in terms of the roots is a perfect square. To solve the quadratic is to express the roots (which are arbitrary) in such a manner that we can always distinguish one from the other. Any result which should fail to give the two roots in a distinguishable form, would fail in one of the essential properties of a solution. Now, if S_2 were not a perfect square, there would be no possibility of distinguishing, algebraically, between the two forms of the expression, in consequence of the permanent ambiguity of the radical sign $\sqrt{\ }$. That radical must be expelled as the necessary condition of resolution.

In this consideration we see the first step towards the views embodied in Abel's Theorem.

The process by which we have arrived at the result, as well as the result itself, shows that we have been inverting an ordinary algebraic operation. If in the given quadratic we write $2a_1t$ for x, the equation presents itself in the form

$$t^2 - t = -\frac{a_2}{4a_1^2} = v \text{ (suppose)}$$

which we may call

$$\phi t = v.$$

The problem we have solved then is to find the form of the function ϕ^{-1} in

$$t = \phi^{-1} v$$

where ϕ has the special form here ascribed to it, containing no symbols but t, which is a multiple of x.

The algebraic statement of the problem is as definite as if we were to say, "Given $v = t^2$, what function of v is t."

THE CUBIC.

14. Having made the first step in the ascent by the resolution of the quadratic, we proceed to consider how we can from thence ascend to that of the cubic. The linear transformation suggests to us that it is at all events possible that the rational part of the root may be a_1 ; and it teaches us that in that case the irrational portions will be functions exclusively of the two quantities (12) and (13). There appears, therefore, some natural propriety in enquiring whether we can find functions of the roots which themselves shall be the roots of some equation of the second degree, whose coefficients are capable of rational expression in terms of those of the cubic.

Again, the general considerations stated in the introductory part of this essay suggest to us that if we can find such functions, there must exist some relations of a specific character between them and the roots of the cubic, if they are to be capable of being made of any use in aiding to resolve the cubic.

In reference to the algebraic expression of the roots of an equation of any degree, we can hardly avoid assuming that it must undergo a change in its form and its multiplicity of value, whenever any change takes place in the *status* of the equation itself in regard to the systems of equalities which may subsist between the roots. In enquiring for a quadratic whose resolution may aid us in solving a cubic, we shall evidently have occasion to make comparison of their respective roots ; and in so doing the idea readily suggests itself to us that any variation in the system of equalities subsisting between the roots of the one will be attended by some variation in the system of equalities subsisting among the roots of the other; and that in each case there will be a corresponding change in the constituents of the expressions which denote the roots of each. Thus, in the case under consideration, we observe that if the cubic have three equal roots, each root is a_1, and the irrational part of the root vanishes; so that any radicals which may enter into the expression of the root must wholly disappear when all the roots are made equal; such disappearance taking place *identically* when the expression is written down in terms of the roots; and taking place by means of the two *conditions* of three equal roots, when the expression is in terms of the coefficients. Consequently, in such case, the quadratic would cease to exist, as its coefficients are, on this hypothesis, functions of the irrational parts of the roots of the cubic. Again, when the cubic is made to have two equal roots and no more, the irrational parts of the roots undergo a variation in form, and are altered in multiplicity of value. We may expect a corresponding change to take place in the quadratic; and this can only be by reason of the quadratic undergoing such an alteration that the expression for its root is altered in multiplicity of value. Now a quadratic can only undergo one change in respect of the passage from one system of equalities to another; for it must have either two unequal roots or two equal roots. We have, therefore, no choice of critical functions to embarass us; and we are in effect reduced to the simple enquiry, Whether we can find a quadratic whose coefficients are rational functions of those of the cubic, and therefore of the two expressions (12) and (13), and whose discriminant is the same as that of the cubic.

In answering this question, we begin by observing that the discriminant of a cubic is of the sixth degree in its roots, and that of a quadratic of the second degree in its roots. It follows that if the discriminants are to be equal, the argument of the quadratic in y must be of the third degree, and its coefficients therefore of the degrees 0, 3, 6 respectively. The most general equation of such a form is

$$y^2 - 2N_1(13)y + N_2(12)^3 + N_3(13)^2 = 0\,;$$

and it is obvious that we may make N_3 equal to 0 without loss of generality. For, if N_3 were retained, it could be made to vanish by some linear transformation, (such as $y = t + N_4(13)$); and such a transformation affects neither the rationality of the coefficients, nor the expression for the discriminant. In the equation

$$y^2 - 2N_1(13)y + N_2(12)^3 = 0\,,$$

we have, therefore, to ascertain if the factors N_1 and N_2 admit of having such a relation between them as will make the discriminant of this equation the same, to a numerical factor, as that of the cubic. The discriminant of this quadratic is

$$N_1^2(13)^2 - N_2(12)^3\,;$$

and comparing this with the discriminant of the cubic as given in section (6), we observe that they become identical, provided we make N_2 equal to $4N_1^2$. The absolute value of N_1 is immaterial; we may call it unity; and we arrive at the result, that the quadratic

$$y^2 - 2(13)y + 4(12)^3 = 0$$

has the required connection with the cubic.

By solving this quadratic, two values for y are found in terms of (12) and (13), *i. e.*, of a_1, a_2, and a_3. These values are completely known.

These two values of y are also expressible in terms of x_1, x_2, and x_3 by substituting for (12) and (13), (or for a_1, a_2, a_3), their equivalents as symmetric functions of x_1, x_2, and x_3; so that, in fact, we have

$$y_1 = S_3 + \sqrt{S_6}$$
$$y_2 = S_3 - \sqrt{S_6}.$$

We perceive at once, without further proof, that these two expressions are rational; for we know that S_6 being the discriminant of the cubic, is a perfect square when expressed in terms of the roots. This rationality of y when expressed in terms of the x's may be clearly exhibited by comparing y as expressed in terms of y_1 and y_2, with y as expressed in terms of x_1, x_2, and x_3. The former is

$$\tfrac{1}{2}\left(y_1 + y_2 + \sqrt{y_1^2 - 2y_1 y_2 + y_2^2}\right);$$

and the latter is

$$\tfrac{2}{27}(\Sigma x)^3 - \tfrac{1}{3}\Sigma x \Sigma (xx) + x_1 x_2 x_3 + \sqrt{N\Big((x_1 - x_2)(x_1 - x_3)(x_2 - x_3)\Big)^2}$$

where N is a known numerical factor.

A comparison of these shows us that both $y_1 + y_2$ and $y_1 - y_2$ are expressible rationally in terms of the x's; so that y_1 and y_2 are separately so expressible. Consequently each y is a rational function of $x_1 - a_1$, $x_2 - a_1$, $x_3 - a_1$; and as the product of the two y's is a perfect cube of a rational function of the coefficients, we conclude as in (9) and (10), that each y expressed in the roots is a perfect cube; and therefore the cube of a linear function of $x_1 - a_1$, $x_2 - a_1$, $x_3 - a_1$. In short we have

$$y_1^{\frac{1}{3}} = A_1(x_1 - a_1) + A_2(x_2 - a_1) + A_3(x_3 - a_1),$$
$$y_2^{\frac{1}{3}} = B_1(x_1 - a_1) + B_2(x_2 - a_1) + B_3(x_3 - a_1);$$

where y_1 and y_2 are fully known, and the A's and B's develope themselves as the result of a calculation by an extraction of roots, which we have proved we can fully perform. These combined with

$$0 = (x_1 - a_1) + (x_2 - a_1) + (x_3 - a_1),$$

determine the roots.

The reader will not fail to see the analogy between this mode of resolving the cubic, and the discussion heretofore given relative to the rational and linear expression of the conditions of three equal roots.

15. I have called attention to the necessity which exists for calculating, or seeing our way to the calculation of, y both as a function of a_1, a_2, and a_3, and of x_1, x_2, and x_3. It *must* be calculated as a function of the a's in order to give us $y_1^{\frac{1}{3}}$ and y_2. It need not be actually calculated as a function of the x's; for the only *practical* result of such calculation is to find the A's and B's, which, as we have seen, can be inferred from some obvious considerations without actually forming and reducing $S_3 \pm \sqrt{S_6}$. My object is of a purely theoretical character; and if it were confined to the cubic, I might consider it in that point of view as fully accomplished, having succeeded in showing the resolubility of the cubic from considerations of an *à priori* character.

It is necessary, however, with a view to the extending of the theory to equations of higher degrees, to enter more fully into the character of this resolution, and the method by which we have proceeded.

If we direct our attention exclusively to the results which we have obtained, and which succeed in expressing certain known linear functions of the three $(x-a_1)$'s in terms of the coefficients, viz., in the forms $y_1^{\frac{1}{3}}$ and $y_2^{\frac{1}{3}}$, we find that we have not in fact arrived at a perfectly definite solution of the cubic. In forming $y_1^{\frac{1}{3}}$ and $y_2^{\frac{1}{3}}$ we have nothing to guide us as to which of the three cube-roots of unity is to be employed as the numerical factor. We merely know that the same cube-root must be used as the factor for both.

There is no method by which this question can be decided, so long as we confine ourselves strictly to algebraic resolution. The utmost we can do is to point out certain criteria which will be successful in appropriating the formulæ; but, from their nature, they cannot in general be applied except in an arithmetical manner. In some cases, indeed, and universally in the case of the cubic, we are able by attributing *characters* to the arbitrary symbols, to determine which set of formulæ is to be employed.

Thus if we choose to say that a_1, a_2, a_3 shall be real in character, we can by the theory of equations determine under what circumstances x_1, x_2, x_3, or one or more of them, shall be real or unreal; and we can thus learn which roots of unity must be used in the formation of $y_1^{\frac{1}{3}}$ and $y_2^{\frac{1}{3}}$, so as to adapt the solution to the conclusion thus arrived at. All this, however, is outside the province of strict algebraic resolution, which requires that coefficients and roots shall always remain arbitrary in value and general in character; that they shall be in fact mere symbols without attributes or meaning.

On consideration, it will be found that this difficulty or ambiguity is of the essence of any true theory of algebraic solution; and that a thorough enquiry into its origin and operation will open important views as to the connection between an equation and its algebraic root.

If, bearing this point in mind, we retrace our steps on the road we have traversed, we readily discover the origin of the above-mentioned circumstance, and we see that it is attended by a remarkable consequence.

On examining the quadratic in y, (which has proved by reason of the identity of discriminants to be what is generally called a resolvent for the cubic in x), we observe that a certain function of the coefficients, viz. (12), enters into it under the form $(12)^3$, and not in any other manner; so that the quadratic would have been precisely the same if the original cubic, instead of being, (as it is),

$$(x-a_1)^3 - 3(12)(x-a_1) - (13) = 0,$$

had been $$(x-a_1)^3 - 3\,\mathbf{a}\,(12)(x-a_1) - (13) = 0,$$

or $$(x-a_1)^3 - 3\,\mathbf{a}^2(12)(x-a_1) - (13) = 0;$$

which three equations, it will be observed, are essentially distinct from each other, having no root in common. So soon, therefore, as we find that the resolution of the cubic is to be effected (if at all) through the roots of this quadratic, we become aware that we must at this point necessarily lose all trace or means of distinguishing which of the three equations above written it is that we are solving.

Whatever solutions we arrive at must be just as much solutions of any one of those equations as of any other; and the fact that we actually obtained the quadratic from one cubic and not from another can have no bearing on the retrospective process.

But the connection between these three equations admits of being considered in another point of view. The first equation is changed into the second by substituting $\alpha\,(x-a_1)$ for each $x-a_1$, and it is changed into the third by substituting $\alpha^2\,(x-a_1)$ for each $(x-a_1.)$ Similarly the second may be changed into the first or into the third by changing each $x-a_1$ into a proper numerical multiple of it; and so also the third may be similarly changed into the second or first.

The form of the root of the cubic must, therefore, so far as it contains in it that of the quadratic, be capable of admitting of one kind of variation when each $x-a_1$ is changed into $\alpha\,(x-a_1)$ and of another kind of variation when each $x-a_1$ is changed into $\alpha^2\,(x-a_1)$. Consequently, although y itself is incapable of admitting of any such change, there must be some function of y entering into the expression of x which is capable of presenting this change on the face of it. This implies that y cannot be such a function that it must necessarily retain the surd form above given to it; for if it were, no function of y could undergo the change above indicated. In short, in order to fulfil the above conditions y must be rational in both its values; and the six values of $y^{\frac{1}{3}}$ must all be capable of linear expression in terms of the three values of $x - a_1$; and the connexion between the numerical coefficients must be such as will fulfil the above results of substitution. There is only one method in which this can be done, namely, by making the expression for one pair $y^{\frac{1}{3}}$ and $y_2^{\frac{1}{3}}$, α times these values for another, and α^2 times these values for the third, (as they must be since the cube of one set must be the same as that of the others); and creating such further relations between the numerical factors as will produce this result. Finally, each value of $y_1^{\frac{1}{3}}$ must be different from the corresponding value of $y_2^{\frac{1}{3}}$; and the result is that we are led to three systems,

$$\left\{\begin{aligned} y_1^{\frac{1}{3}} &= A\left((x_1 - a_1) + \alpha\,(x_2 - a_1) + \alpha^2\,(x_3 - a_1)\right) \\ y_2^{\frac{1}{3}} &= A\left((x_1 - a_1) + \alpha^2\,(x_2 - a_1) + \alpha\,(x_3 - a_1)\right) \end{aligned}\right\}$$

$$\left\{\begin{aligned} y_1^{\frac{1}{3}} &= A\left(\alpha\,(x_1 - a_1) + \alpha^2\,(x_2 - a_1) + (x_3 - a_1)\right) \\ y_2^{\frac{1}{3}} &= A\left(\alpha\,(x_1 - a_1) + (x_2 - a_1) + \alpha^2\,(x_3 - a_1)\right) \end{aligned}\right\}$$

$$\left\{\begin{aligned} y_1^{\frac{1}{3}} &= A\left(\alpha^2\,(x_1 - a_1) + (x_2 - a_1) + \alpha\,(x_3 - a_1)\right) \\ y_2^{\frac{1}{3}} &= A\left(\alpha^2\,(x_1 - a_1) + \alpha\,(x_2 - a_1) + (x_3 - a_1)\right) \end{aligned}\right\}$$

which amongst them resolve the three cubics

$$(x - a_1)^3 - 3\,(12)\,(x - a_1) - (13) = 0,$$
$$(x - a_1)^3 - 3\,\alpha\,(12)\,(x - a_1) - (13) = 0,$$
$$(x - a_1)^3 - 3\,(\alpha^2\,12)\,(x - a_1) - (13) = 0;$$

but we are left without the means (algebraically) of appropriating any one system to any one equation except in the limited manner above indicated.

This necessary ambiguity, so far from being a defect in the resolution, is an important element in making it perfect, provided we adhere to the strict notion of treating x_1, x_2, x_3, as symbols only.

16. It will now be readily observed, that if we had been

aware beforehand of this remarkable property, or rather necessity, of an algebraic solution, we should have been able at once to have predicated that any quadratic equation possessing the property of a resolvent for the cubic must have the same discriminant as the cubic.

It is an essential feature of this algebraic resolution of the cubic that the values of $y_1^{\frac{1}{3}}$ and $y_2^{\frac{1}{3}}$ shall be expressible rationally and linearly as functions of the x's or of the $(x-a_1)$'s.

The values of y_1 and y_2 must, therefore, be capable of rational expression in terms of the same symbols: and no form of $y^2-2\,N_1\,(13)\,y+N_2\,(12)^3=0$ will fulfil this condition, unless such a relation exists between N_1 and N_2 as makes $N_1^2\,(13)^2-N_2\,(12)^3$ a perfect square; and this, in fact, is the condition of equality of discriminants. Conversely, we also perceive that if the discriminants are made equal, the roots of the quadratic will then possess every property which is necessary to qualify the quadratic to perform the office of a resolvent, not only to the cubic in question, but to the conjugate cubics which cannot in the matter of resolution be separated from it.

17. The theory, however, of this resolution remains in some sense imperfect, unless we establish, *à priori*, that a quadratic with rational coefficients, and having a discriminant identical with that of the cubic, must possess the qualities of, and be, in fact, a resolvent. At present, it appears only as the result of a calculation. Now, what is the precise connection between the cubic and the quadratic which results from the identity of the discriminants? It consists in the following particulars:—that y is of the third degree in the x's: that the coefficients of the quadratic are symmetric functions of the x's of an appropriate degree: that whenever any two x's are equal, the values of y are also equal, so that the whole of the irrational part of the roots of the quadratic must also

present itself in the irrational part of the root of the cubic; and, lastly, that whenever all the x's are equal, the equation in y totally disappears, along with the irrational parts of the expression for x.

Now observing that S_2 and S_6 are functions of $x_1 - a_1$, $x_2 - a_1$, $x_3 - a_1$, or (by reason of the equation $\overline{x_1 - a_1} + \overline{x_2 - a_1} + \overline{x_3 - a_1} = 0$) of any two of them, we see, conversely, that the irrational parts of the root of the cubic are functions of y_1 and y_2 exclusively; and functions of them of such a nature that their irrational part always presents itself in them unaltered. It is also obvious that y_1 and y_2 must enter symmetrically, since there is no method of distinguishing one from the other. The x's, therefore, must be capable of being represented as symmetric irrational functions of y_1 and y_2 exclusively; or to speak more in detail, as an irrational function of one y combined with the same irrational function of the other y.

Moreover, from what we have heretofore proved, it results that y is a function of such a nature that its own value can undergo no alteration when each $x - a_1$ is changed into $\alpha\,(x - a_1)$, or each $(x - a_1)$ is changed into $\alpha^2(x - a_1)$, whilst, at the same time, there must exist some function of y which shall assume α times its existing value, when each $x - a_1$ is changed into $\alpha(x - a_1)$, and α^2 times its existing value when each $(x - a_1)$ is changed into $\alpha^2\,(x - a_1)$. This is the same thing as saying that y must be a function (ϕ) of some function ψ of y, say $\phi\,(\psi\,y)$, such that the distinctions produced by the changes above mentioned visibly exist in $\psi\,y$, but are obliterated in $\phi\,(\psi\,y)$; or that ϕ is a function which denotes an operation which obliterates all distinctions arising from the introduction of the factors α and α^2. Such a function can obviously only be the operation of cubing. Consequently we have $y = (y^{\frac{1}{3}})^3 = (\alpha\,y^{\frac{1}{3}})^3 = (\alpha^2 y^{\frac{1}{3}})^3$; or each y

is a perfect cube of some three-valued function of the $(x-a_1)$'s of the character above described. It follows that y must be rational; for if it were not rational, neither the cube-root of y nor any other function of y could be expressed in a form which should be capable of alteration by the change of each $(x-a_1)$ into $\alpha\,(x-a_1)$ or $\alpha^2(x-a_1)$. This is immediately seen from the form of y as $S_3 \pm \sqrt{S_6}$, in which both S_3 and S_6 are identically functions of the three values of $(x-a_1)^3$, and not otherwise functions of the three values of $x-a_1$; so that, if S_6 were an irreducible surd, no function of y could present itself otherwise than in a similar form.*

It is not, I think, with any sufficient degree of clearness pointed out in elementary works that the algebraic resolution of the cubic is necessarily of the ambiguous or halting character above described. The theory, in fact, has got mixed up with the general theory of equations in which the coefficients have a character of *reality* attributed to them; and speculations are gone into as to the character of the roots, such as whether they are positive or negative, real or unreal, commensurable or incommensurable, and so on. These speculations, although of great importance, have nothing to do with algebraic resolution, whose final object in the case of a cubic is to conceive a function of three symbols x_1, x_2 and x_3, which has identically, by the mere performance of the operations indicated in it, the three values x_1, x_2 and x_3.

This we have effected, in a peculiar manner.

We obtain the result in the form

$$S_1 + \sqrt[3]{S_3 + \sqrt{S_6}} + \sqrt[3]{S_3 - \sqrt{S_6}}$$

which has of necessity nine values; and these, properly

* Such forms as $R_1 + \sqrt{R_2}$ for $y^{\frac{1}{3}}$ are inadmissible, as they would lead to values of y from which the square-root radical sign could not be extirpated; and as y_1 and y_2 are arbitrary, this result would be inconsistent with (13).

assorted, solve three distinct cubics; the criterion being that we are to employ that cube-root of unity which makes the product of $y_1^{\frac{1}{3}}$ and $y_2^{\frac{1}{3}}$ that particular cube-root of $(12)^3$ which appears in the equation to be resolved.*

18. We may recapitulate our examination of the cubic by stating that we have discussed it in two modes.

The first mode is tentative, and gives us the solution merely as the result of an actual calculation. We find, as a matter of fact, that S_6 expressed in the roots is a perfect square, and that $S_3 \pm \sqrt{S_6}$ similarly expressed are perfect cubes. In the second mode, we show, *à priori*, that S_6 is a perfect square; and we then point out, from considerations of a general character, that the reducibility must extend also to the cube-root of the two-valued expression above given. This result is clearly seen to depend on the peculiar form of the resolvent in the two particulars, that it has the same discriminant as the cubic; and that by reason of the manner in which (12) enters into it, viz., only through its cube, it gives us necessarily also the solution of the two other and conjugate cubics above mentioned.

We have also discussed this second mode in a two-fold manner. We have shown from the identity of the critical functions of the cubic and the quadratic, that y must not only be rational, but that it must be a two-valued function, each value being a perfect cube of a linear function: and we observe incidentally from this circumstance that the resolvent in y must be just as applicable to resolve two certain other cubics as the one under consideration. We have also independently shewn that since (12) enters into the quadratic only through its cube, the quadratic, being known to be a resolvent to the cubic, would stand in the

* See, however, "Todhunter's Theory of Equations," pp 92-96. Also, Professor Cayley in Phil. Mag. Vol. xxi, p 213.

same relation (i. e., the relation of a resolvent) to the three conjugate cubics, whose roots are all different; and that this leads us also to the same conclusion, viz., that y is rational, and that the six values of $y^{\frac{1}{3}}$ are so connected that they must be linear in the x's, and be in other respects connected in the manner we have pointed out. It is to this last point of view that I desire to call particular attention, as we shall in the course of this Essay find this circumstance frequently occurring, and always in connection with two equations which have their critical functions identical.

19. Besides the linear transformation, there is only one other transformation which a cubic admits of without losing its form; viz., that transformation which consists in making (21) in the new equation to vanish. This is in effect the Tschirnhausen transformation ; which, we know, reduces the cubic to a binomial form by means of a quadratic, and gives, in effect, the same form of roots which we have obtained (Serret, p. 216). The solution of the cubic is, therefore, unique in form.

20. The question now suggests itself, whether the resolution of the cubic is an ordinary algebraic operation of an inverse kind, as was the case with that of the quadratic.

We can write the cubic in the form

$$(x-a_1)^3-3\,(12)\,(x-a_1)-(13)=0\,;$$

and if in this we make

$(x-a_1)=\sqrt{3\,(12)}\;t$, we have $t^3-t=\dfrac{(13)}{\{3(12)\}^{\frac{3}{2}}}=v$ (suppose).

This is of the form $\phi t=v$; and we have been finding what is the form of ϕ^{-1} in $t=\phi^{-1}\,v$.

The algebraic statement of the problem is therefore the same as in the case of the quadratic; and the root of a complete cubic is in effect a function of one single symbol v.

THE QUARTIC.

FIRST METHOD.

21. Proceeding to the quartic, we infer as before from the linear transformation that, if the quartic is resoluble *through that transformation*, the rational part of its root in such a resolution is a_1, and the irrational parts are functions of the three expressions (12), (13) and (14); and we are led to enquire whether we can find a cubic whose coefficients are rational functions of those of the quartic, and which has the same discriminant (to a numerical factor) as that of the quartic.

Having regard to the considerations heretofore developed, it will be readily seen that our object in looking for such a cubic is, that it may be a resolvent of the quartic; that is, that its roots may enter, as it were bodily, into the expression for the root of the quartic. Hence the necessity that the discriminants must be the same; for we know that in the root of any general equation its discriminant must enter.

The discriminant of a quartic is of the twelfth degree in its roots, and that of a cubic is of the sixth degree in its roots. The argument, therefore, of a cubic in y which is to have the same discriminant as the quartic in x, must be of the second degree, and its coefficients of the degrees 0, 2, 4, and 6 respectively. The most complete form of such a cubic is

$$y^3 - 3N_1(12)y^2 + 3\Big(N_2(12)^2 + N_2'(14)\Big)y - \Big(N_3(12)^2(14) + N_3'\,13^2\Big) = 0;$$

there being no need to introduce into the last term any multiple of $(12)^3$, since it could be made to vanish by a linear transformation. With these numerical constants at our disposal, or rather their ratios, we are to enquire whether such values can be given to them as will make the discriminant of this cubic the same as that of the quartic. The discriminant of this cubic is

$$4\,(\mathrm{I}\cdot\mathrm{II})^3-(\mathrm{I}\cdot\mathrm{III})^2;$$

where

$(\mathrm{I}\cdot\mathrm{II})=(N_1^2-N_2)\,(12)^2-N_2'\,(14)$, or $C\,(12)^2-N_2'\,(14)$;

$(\mathrm{I}\cdot\mathrm{III})=(2N_1^3-3N_1N_2)(12)^3-(3N_1N_2'-N_3)(12)(14)+N_3'(13)^2$,
or $D\,(12)^3-E\,(12)\,(14)+N_3'\,(13)^2$.

The discriminant of the quartic is given in (6).

Writing them side by side, we have

	Discriminant of Cubic.	Discriminant of Quartic.
$(12)^6$	$4C^3-D^2$	 0
$(12)^4\,(14)$.	$2D\,E-12C^2\,N_2'$ ...	 81
$(12)^2\,(14)^2$	$12C\,N_2'^{\,2}-E^2$	 18
$(14)^3$	... $-4N_2'^{\,3}$............	 1
$(12)^3\,(13)^2$	... $-2D\,N_3'$	.... -54
$(12)\,(13)^2\,(14)$.	... $2E\,N_3'$	.... -54
$(13)^4$	... $-\;N_3'^{\,2}$............	.... 27.

The comparison of these gives the six relations:

$$4C^3-D^2=0,$$
$$108\,N_2'^{\,3}-N_3'^{\,2}=0,$$
$$D\,N_3'+108\,N_2'^{\,3}=0,$$
$$E\,N_3'-108\,N_2'^{\,3}=0,$$
$$6C^2\,N_2'-D\,E-162\,N_2'^{3}=0,$$
$$12C\,N_2'^{\,2}-E^2+72\,N_2'^{\,3}=0.$$

The first of these equations implies

$$N_2=\tfrac{3}{4}N_1^2:\ \text{or}\ C=\tfrac{1}{4}N_1^2.$$

Therefore

$$N_2' = \tfrac{1}{3}C = \tfrac{1}{12}N_1^2;$$
$$N_3' = (108\,N_2'^{\,3})^{\frac{1}{2}} = \tfrac{1}{4}N_1^3;$$
$$E\ = 3N_1N_2' - N_3 = \tfrac{1}{4}N_1^3 - N_3;$$

But E must be equal to N_3 or $\frac{1}{4}N_1^3$; consequently $N_3 = 0$.

It thus appears, that, although we have to satisfy six equations with only four constants, yet the problem is possible; for the values here obtained verify all the six equations independently of whatever value may be assigned to N_1; and (making $N_1 = 6$, to avoid fractions) we find that the cubic in y which has the required properties, is

$$y^3 - 18\,(12)\,y^2 + 3\left(27\,(12)^2 + 3\,(14)\right)y - 54\,(13)^2 = 0.$$

We can, therefore, by resolving this cubic, *first*, express the three values of y explicitly in terms of (12), (13), and (14), *i.e.*, of a_1, a_2, a_3 and a_4; and *secondly*, express them in terms of symmetric functions of the x's under radicals, by substituting for each coefficient its corresponding value in the x's.

It is not necessary here actually to write down the first of these modes of expression; it is sufficient to see that they are known. The second may be represented by the formula

$$y = S_2 + \sqrt[3]{S_6 + \sqrt{S_{12}}} + \sqrt[3]{S_6 - \sqrt{S_{12}}}$$
$$= S_2 + \sqrt[3]{S_6 + R_6} + \sqrt[3]{S_6 - R_6};$$

$\sqrt{S_{12}}$ being, of course, reducible as being the square-root of the discriminant of the quartic expressed in its roots.

If our object were merely to solve the quartic by any process (tentative or otherwise) which should present itself as feasible, we should find now no further difficulty in this resolution. We should simply expand $S_6 \pm R_6$ in terms of the x's, and extract the cube-root of each of them. We should find, as a matter of calculation, that the extraction is possible so as to present y in a rational shape; and also that each value of y thus obtained would be a perfect square, and would therefore produce, by extraction of the square-root, a linear function of the x's; whose coefficients would be found by the process; and we should thus be conducted to the resolution of the quartic.

22. We desire, however, with a view to the construction of a general theory of resolution, or, if that cannot be effected, with a view to discover the circumstances which render algebraic resolution possible, or the reverse, to establish upon *à priori* grounds the fact that y is rational, and the square of a linear function of the x's. The proof of such fact must of course be based upon the nature of the connection which exists between the quartic in x and the cubic in y. This connection in substance is that the two equations have the same discriminant; for if we assume the rationality of the coefficients of the cubic, everything else in its form flows from the identity of the discriminants. The equations are so allied, that there must be at least one common radical in the expression of their roots; and this fact makes one step towards establishing the rationality of y by enabling us to say that S_{12} is a perfect square when expressed in the x's.

If we proceed to enquire further in this sense, we shall find another relation between the two equations. When the cubic has three equal roots, not only does its discriminant vanish, but the two components of the discriminant, the functions (I · II) and (I · III) are *each* zero; and we have

$$(14) - 3\,(12)^2 = 0,$$
$$(12)^3 + (12)\,(14) - (13)^2 = 0\,;$$

which give

$$4\,(12)^3 - (13)^2 = 0\,;$$

and this, with the vanishing of the discriminant, constitute the conditions of three equal roots for the quartic.

In short, *all* the conditions between the coefficients which affect the multiplicity of value of the roots of the cubic operate in a similar manner upon the roots of the quartic; from which we infer that *all* the radicals which enter into the root of the cubic must enter also without any alteration of shape into any algebraic expression for the root of the quartic which may result from this process. And this we may regard as the property, or definition, of a *resolvent.*

Now for a quartic we have already ascertained that the conditions of three equal roots can be represented as the product of six functions,

$$R_2 + aR'_2 + a^2R''_2; \qquad R_2 + a^2R'_2 + aR''_2;$$
$$a(R_2 + aR'_2 + a^2R''_2); \; a(R_2 + a^2R'_2 + aR''_2);$$
$$a^2(R_2 + aR'_2 + a^2R''_2); \; a^2(R_2 + a^2R'_2 + aR''_2);$$

in which the three R's are rational functions of the second degree of the roots of the quartic. We should thus learn indirectly that the values of y are rational.

It is material, however, for us to establish this rationality in a more direct manner, from the connection between the quartic in x and the cubic in y, coupled with the fact that the product of the roots of the cubic is the square of a rational function of the coefficients of the quartic.

Now the identity of the critical functions shows us (as in the case of the cubic) that y_1, y_2, and y_3 enter into the values of the four $(x-\alpha_1)$'s in such a manner as to preserve intact their irrational parts, so that the values of $x-\alpha_1$ being, from the nature of the case, symmetric in respect to y_1, y_2, and y_3 must be irrational functions of these symbols; and must therefore consist of irrational functions of y_1, y_2, and y_3 separately, or (which is in effect the same thing) combined only in pairs; and the other fact to which we have adverted (relative to the product of the roots of the cubic) shows us that, intimate as is the connection between the cubic and the quartic, there is also another quartic with which this cubic is allied in precisely the same manner; the two connections being so absolutely identical, that if we trace our steps backward we cannot discover from which of the two quartics the cubic was derived, so that whatever results we arrive at through this cubic must be common to the two quartics, although (as we shall see) they have no root in common.

Retracing our steps, then, we examine the cubic, and we find that the function (13) does not enter into it except through the form $(13)^2$; and consequently that the same cubic would have been derived in the same way, if the quartic, instead of being

$$(x-a)^4-6\,(12)\,(x-a_1)^2-4\,(13)\,(x-a_1)-(14)=0,$$

had been

$$(x-a_1)^4-6\,(12)\,(x-a_1)^2+4\,(13)\,(x-a_1)-(14)=0.$$

Consequently y is a function of the four values of $(x-a_1)$ only through the four values of $(x-a_1)^2$; as indeed we may make visible to the eye, if we express y in that form, since (12), $(13)^2$, and (14) can each be so expressed.

That being so, y is such a function that it cannot itself undergo any change when each of the four $(x-a_1)$'s is changed into the corresponding $-(x-a_1)$.

But, on the other hand, from the connections which we pointed out between the equations, it is easy to see that if the root of the quartic be algebraically expressible, it must be so expressible through some function of y which will admit of a change in value when each $(x-a_1)$ is changed into the corresponding $-(x-a_1)$; since this root is under that change to become the root of a new quartic which has no root in common with the quartic under consideration.

There must, then, be some mode of dealing with y, which will cause the result of such dealing to exhibit, what y itself cannot exhibit, the capacity of changing under the substitutions before mentioned.

In other words, there must be some function (Ψ) which operating on y explicitly exhibits a change when each $(x-a_1)$ is changed into the corresponding $-(x-a_1)$; but which faculty of changing is wholly obliterated when we perform upon Ψy the inverse operation Ψ^{-1}. We have then a function $\Psi^{-1}(\Psi y)$, in which Ψy exhibits the effect of the change, while in $\Psi^{-1}(\Psi y)$ or y, that effect is entirely obliterated. We must therefore conclude that Ψ^{-1} is the operation of squaring; and Ψ the operation of taking the square-root.

All this requires that y must be rational when expressed in terms of the x's; for if the cube-roots which enter into the expression of y were irreducible surds, any root, or power or function of y would also be of that character, and would contain y only through the four values of $(x-a_1)^2$.

The three values of y being rational and their product a perfect square, it follows from considerations similar to those stated in section (9) and (10), that *each* of the y's is a perfect square, and therefore each value of $y^{\frac{1}{2}}$ is a rational and linear function of the four values of $x-a_1$; and it is, of course, obvious that the three values which constitute one set will be the same as the three which constitute the other set, but with changed signs.

Speaking apart from the conditions of the problem, it is not, of course, absolutely necessary that each of these three rational quantities should be a square because their product is one. They might be of the form RR', $R'R''$ and $R''R$. But if they were, $(13)^2$ would be made up of three factors, each repeated twice; whereas, in order to satisfy the exigencies of the problem, (which are the same when $-(x-a_1)$ is written for $x-a_1$), it is necessary that for every factor there should be another factor differing from it in sign, and in sign only. And, finally, as the signs or square-roots of unity will enter in a symmetric connection with the roots, we must have

$$y_1^{\frac{1}{2}} = \pm A\left((x_1-a_1)+(x_2-a_1)+(x_3-a_1)+(x_4-a_1\right),$$

$$y_2^{\frac{1}{2}} = \pm A\left((x_1-a_1)-(x_2-a_1)+(x_3-a_1)-(x_4-a_1)\right),$$

$$y_3^{\frac{1}{2}} = \pm A\left((x_1-a_1)-(x_2-a_1)-(x_3-a_1)+(x_4-a_1)\right);$$

in which we retain a_1 merely because it is convenient to write down the two quartics which these forms indiscriminately solve, as

$$(x-a_1)^4-6(12)\,(x-a_1)^2-4(13)\,(x-a_1)-(14)=0$$
$$(x-a_1)^4-6(12)\,(x-a_1)^2+4(13)\,(x-a_1)-(14)=0$$

With that numerical form of the cubic which we have used, A is found by calculation to be $\sqrt{\frac{3}{8}}$.

23. In the arithmetical application of the formulæ to our original equation, we must use that square-root of unity which will make $(y_1\, y_2\, y_3)^{\frac{1}{2}}$ of the same sign as (13); and for the other form that square-root which makes $(y_1\, y_2\, y_3)^{\frac{1}{2}}$ of the same sign as $-(13)$.

The necessary ambiguity of the solution of the quartic is a well-known and admitted fact; and the criterion is given in elementary works, as it has not been found possible to evade it by collateral considerations, as was done in the cubic.

The function which expresses the algebraic solution by this method is therefore of the form

$$S_1 + \sqrt{S_2 + \sqrt[3]{S_6 + \sqrt{S_{12}}} + \sqrt[3]{S_6 - \sqrt{S_{12}}}}$$
$$+ \sqrt{S_2 + a\sqrt[3]{S_6 + \sqrt{S_{12}}} + a^2\sqrt[3]{S_6 - \sqrt{S_{12}}}}$$
$$+ \sqrt{S_3 + a^2\sqrt[3]{S_6 + \sqrt{S_{12}}} + a\sqrt[3]{S_6 - \sqrt{S_{12}}}}$$

which is evidently eight-valued, even when all ambiguity arising out of the solution of the resolvent cubic has been disposed of.

24. We do not find ourselves able in this case to express the problem in the simple algebraic form $\phi t = v$, so as to describe the process of resolution as the simple inverse of an ordinary algebraic operation; and, accordingly, we find that that multiple of $x - a_1$ which we call t is not in our expression of the root a function of any one symbol v.

At present, therefore, we must admit that the root of a general quartic, in the shape at least in which we have found it, is not capable of being represented by any function of a single symbol or parameter.

THE QUARTIC.

(Continued).

SECOND METHOD.

25. The success of the foregoing *à priori* method, as applied in the two instances of the cubic and the quartic, depends on the following circumstances:—

1st. That there exists an equation of the next inferior degree, whose coefficients are rational functions of those of the given quantic, and which has the same discriminant as the given quantic.

2nd. That the absolute term of the new equation, or the product of its roots, is a perfect power of a rational function of the original coefficients, the exponent of the power being the same as the degree of the argument of the auxiliary equation.

3rd. That the rational function last mentioned, enters the new equation only through the medium of that power of it above referred to.

In the course of this demonstration we ascertained the remarkable fact, that the whole process was equally applicable to a *set* of original quantics, as many in number as would be denoted by the exponent of the last term of the equation in y; the members of the set having a certain similarity *inter se*, but being so far distinct that they have no root in common.

A similar inference also we observed to be deducible from the equality of the discriminants, and the fact that in both cases the discriminants possessed a particular form, viz.

$S^2 + NS'^3$. Both these peculiarities conducted us in a nearly similar manner to the conclusion that y must be not only rational in the x's, but a power of a rational and linear function of them.

In the case of the cubic, these circumstances presented themselves in the course of the investigation, in a manner which was free from difficulty or peculiarity. In finding the auxiliary equation, so soon as its form was written down, we were able to see at once that such a ratio could be assigned between the numerical constants as would produce the desired result. It was otherwise in the case of the quartic. There, we had but four ratios at our disposal, whilst there were six equations to be satisfied by them. We could not predicate *à priori* that four ratios could be so assigned as to satisfy these six equations; and, therefore, we could not know beforehand that such an equation as we were looking for existed. This is a circumstance very worthy of attention, as it seems to indicate that the method which we pursued, if not imperfect, was yet not the most natural or simple in point of Theory.

The obvious remedy for this apparent imperfection is to endeavour to diminish the number of equations to be satisfied. In order to equate two general functions of the 12th degree containing (12), (13), and (14), we must satisfy six equations, there being seven terms possessing coefficients. Examining these terms as they appear in the first column of the comparison in section (21), we see that, if we strike out (12), we have only two terms left; and the ratio between their coefficients can be determined without any redundancy of equations or other chance of inconsistency.

If, then, we take a quartic whose coefficients are subjected to the condition (12) = 0, which is equivalent to

$$3\,(\Sigma\, z)^2 - 8\, \Sigma\, (zz) = 0,$$

it exists in the conditional form

$$z^4 - 4\,b_1 z^3 + 6\,b_1^2 z^2 - 4\,b_3 z + b_4 = 0;$$

and the cubic which has the same discriminant is

$$y^3 + 9\,(41)\,y - 54\,(31)^2 = 0.$$

We can resolve this cubic so as to determine explicitly the three values of y in terms of (41) and (31), or of b_1, b_3, and b_4; but we can no longer express y in any unique manner in terms of the z's. We may indeed express y by simply substituting $\frac{1}{4}\,\Sigma z$ for b_1, $\frac{1}{4}\,\Sigma\,(zzz)$ for b_3, and $z_1 z_2 z_3 z_4$ for b_4; but such an expression would admit of indefinite variation by means of the condition $3\,(\Sigma z)^2 = 8\,\Sigma\,(zz)$.

The problem is now before us in a form similar in many leading circumstances to those which we have already solved; but in one important respect, it occupies a different position. Whatever train of reasoning we may base on the intimate connection between the quartic in z and the cubic in y, and on the cardinal fact that (31) enters into the cubic only through its square, we have no means of making any application of our results unless we find some method of discovering a unique expression for y; and moreover any result we might arrive at would not bring us to that which we are seeking, the resolution of the *general quartic in x*.

Before proceeding further, then, with the considerations to which we are led by the points of similarity existing between this case and those which we have discussed, we must ascertain whether the general quartic in x, with its full complement of parameters, can be transformed into the species of conditioned quartic which is now under consideration; and, if so, in what manner such transformation must be made.

To those who are acquainted with the transformation of Tschirnhausen it will not fail to suggest itself that it is the one to which we must resort. We must therefore explain this transformation at length.

26. Taking the complete quartic in x

$$x^4 - 4a_1 x^3 + 6a_2 x^2 - 4a_3 x + a_4 = 0,$$

we require to transform it into a quartic in z; z being such a function (ψ) of x as will cause the result to appear in a form which is subjected to the required condition $(21) = 0$.

This, at least, is the problem in the form in which we are naturally led to propose it. We shall see in result that we ought to have contemplated the possibility of there being not merely one such quartic, but a system of such quartics.

The most general form of ψ for this purpose, is

$$z = \psi\, x = h\, x^3 + k\, x^2 + l\, x;$$

for if we should introduce any power of x higher than the third, the form would be reducible by means of the original quartic to the form just written down; and we need not introduce a constant or absolute term since that would incorporate itself with z. Let, then,

$$z = hx^3 + kx^2 + lx = \psi x;$$

then the transformed equation in z, is

$$z^4 - \Sigma\,(\psi x)\, z^3 + \Sigma(\psi x \cdot \psi x)\, z^2 - \Sigma\,(\psi x \cdot \psi x \cdot \psi x)\, z + \psi x_1\, \psi x_2\, \psi x_3\, \psi x_4 = 0;$$

and in order to satisfy the condition $(21) = 0$, it is necessary that the coefficients h, k, and l should be so taken that we shall have

$$3\left(\Sigma\,(\psi x)\right)^2 - 8\,\Sigma\,(\psi x \cdot \psi x) = 0;$$

or, in the notation given in (8).

$$\overline{33}\; h^2 + 2 \cdot \overline{32}\; hk + 2 \cdot \overline{31}\, hl + \overline{22}\; k^2 + 2 \cdot \overline{21}\, kl + \overline{11}\; l^2 = 0.$$

This relation may be satisfied in two distinct ways; the first of which admits of three varieties, and the second of two.

In the first place, as we have two constants at our disposal, and only one condition to satisfy, we may make one of the three h, k, and l, equal to zero; in which case the ratio subsisting between the other two is determined by resolving a quadratic equation. Thus we may make $h=0$, and determine the ratio of k to l by the quadratic

$$\overline{22}k^2+2\cdot\overline{21}kl+\overline{11}l^2=0.$$

We thus obtain two forms of ψ; and for each form a corresponding quartic in z which has the property which we require it to have. The result will be arrived at in a system or pair of quartics; in each of which the form of ψ, and the coefficients of the quartic in z, will contain functions of a_1, a_2, a_3, and a_4, of a known specific quadratic-irrational form. Similar modes of transformation would be obtained by making $k=0$, or $l=0$, in the first instance.

In the second place, we observe that the condition admits of the form

$$(\overline{33}h+\overline{32}k+\overline{31}l)^2+(\overline{33}\cdot\overline{22}-\overline{32}\cdot\overline{32})\,k^2$$
$$+\ 2\,(\overline{33}\cdot\overline{21}-\overline{32}\cdot\overline{31})\,k\,l+(\overline{33}\cdot\overline{11}-\overline{31}\cdot\overline{31})\,l^2=0;$$

which we can divide into two linear conditions

$$\overline{33}h+\overline{32}k+\overline{31}l=0,$$
$$A\,k+(B+\sqrt{B^2-A\,C})\,l=0;$$

A, B and C being the three symmetric functions of the 10th, 9th and 8th degrees above given. If we assign any convenient value to h, then k and l are determined by a linear process; and thus z again is represented in two ways as a function of x, whose coefficients are functions of a_1, a_2, a_3 and a_4, which contain another known specific quadratic-irrational, and the coefficients of the two quartics in z, are known in a similar manner.

The other method of effecting the reduction is similar to the above, and may be obtained from it mechanically, by interchanging 1 and 3 in the above formula, and, at the same time, interchanging h and l. A distinct form of irreducible irrational is obtained for each mode of elimination. The degrees of the new functions introduced are respectively, 14 and 18. All these methods appear to be quite distinct.

In every manner then of solving this problem, we arrive at a system of *two* quartics; each of which is turned into the other by changing the sign of the square-root radical which appears in every coefficient. We shall confine our future remarks to one of these possible transformations; and we select, as apparently the simplest, the form in which z is a quadratic in x, viz:

$$z = kx^2 + lx$$
$$= \overline{11}x^2 + \left(\overline{21} \pm \sqrt{\overline{21}\cdot\overline{21} - \overline{22}\cdot\overline{11}}\right)x$$

or

$$\tfrac{1}{16}z = 3(a_1^2 - a_2)x^2 + \left\{3(4a_1^3 - 5a_1a_2 + a_3) \pm \sqrt{9(4a_1^3 - 5a_1a_2 + a_3)^2 - 3(a_1^2 - a_2)(48a_1^4 - 72a_1^2a_2 + 9a_2^2 + 16a_1a_3 - a_4)}\right\}x.$$

The radical last written down is the essentially irrational form which will pervade the whole of our future results. Whether, when its subject is expressed in terms of the x's, it is actually irreducible or not, is not material. We shall always speak of it as irreducible, by which we mean that we never attempt to reduce it by extracting the square-root. There can however be no doubt that it is in fact irreducible.

With this value of z, b_1 is $\frac{1}{4}\Sigma z$; b_3 is $\frac{1}{4}\Sigma(zzz)$; and b_4 is $z_1 z_2 z_3 z_4$; which are all manifestly now expressible in terms of a_1, a_2, a_3, and a_4, these expressions all containing the irreducible radical, but being in other respects rational.

I may here call attention to the important fact that it is now evident that we are proceeding by a route, which, (if it leads to a result), will produce the result in such a form that the irrationals in it are *not* functions exclusively of (12), (13), and (14); a result, therefore, which will not, or at least may not, explicitly exhibit the rational part of the root of the quartic in x in the form of a_1. That the sum of the roots of the quartic in x in whatever form obtained, must be in fact equivalent to $4a_1$ is of course quite obvious; but we are not to regard it as indispensable that in each root there must appear a rational part which is simply a_1, and that the residue of the root will be a function of (12), (13), and (14). The result, then, which we have arrived at is, that the general quartic is equivalent to a system or pair of conditioned quartics of the form

$$z^4-4b_1 z^3+6b_1^2 z^2-4b_3 z+b_4=0,$$

with the relation $z=kx^2+lx$, where $l \div k$, b_1, b_3, and b_4 are quadratic-irrational functions of a_1, a_2, a_3, and a_4, all possessing precisely the same kind of irrationality; and each of them having two values, (one for each system), according as we consider the above quadratic-radical to be affected with one or other of the two square-roots of unity.

27. Now, confining our attention for the present to one of these conditioned quartics, and reverting to the cubic in y which we have previously found having its coefficients rational functions of those of the quartic in z, and having the same discriminant, viz.

$$y^3+9\,(41)\,y-54\,(31)^2=0,$$

we perceive that we have the means of expressing (31) and (41), $\left(\text{which are respectively } b_3-b_1^3 \text{ and } -(b_4-b_1^4)+4b_1\,(b_3-b_1^3)\right)$, in terms of a_1, a_2, a_3, and a_4; these expressions each containing the same quadratic-irrational, which we will call $\sqrt{I_6}$.

We can, therefore, express y completely in terms of a_1, a_2, a_3, and a_4, the expression involving irrationals, which, in Sir William Hamilton's notation, are of the third order.

And we can, of course, by substituting for each coefficient its corresponding equivalent as a symmetric function of the x's, exhibit y in terms of the x's in an expression containing the same order of irrationality.

So far as relates to the theory of the subject, it is of course immaterial whether we first exhibit y as a function of a_1, a_2, a_3, and a_4, and then substitute for each a its corresponding symmetric function of x_1, x_2, x_3, and x_4; or whether we first calculate y as a function of the x's, and then for each symmetric combination which appears, substitute the equivalent function of the a's.

Our theory, however, does require that both forms should be seen to admit of explicit calculation: and it is particularly to be observed that the expression whether in terms of the coefficients or of the roots of the quartic in x is no longer indefinite, but (for each separate mode of transformation) an absolutely unique expression.

We may therefore now resume the argument in proof of the rationality of y in the same plight as we had it in the cases previously considered, and with the same means of applying its results. We have, as before, y actually expressed in terms of x_1, x_2, x_3, and x_4 in a unique manner; and whatever results we arrive at by conceiving y expressed in terms of the z's may be transferred to the actual expression for y in terms of the x's provided we make the transfer with a due regard to the relation between each z and its corresponding x.

Let us then first examine the connection between the quartic in z or ψx, and the cubic in y, in order to ascertain if it is of such a nature as to admit of the application of the *à priori* argument as to the form of y; or in other words whether it is such a connection of critical functions as qualifies the cubic to be a resolvent for the quartic.

Now the two equations have the same discriminant, so that one radical entering into the expression of y in terms of the z's will also enter into the root of the quartic, if it is algebraically expressible; and in the conditioned form of the quartic, the discriminant is in reality the only critical function which exists.

The quartic is

$$(z-b_1)^4-4\,(31)\,(z-b_1)-(41)=0\,;$$

which has really only two parameters, so that we cannot introduce two conditions. In fact, we cannot suppose the equality of three or four roots, without causing the cubic to disappear, and the quartic to become simply $(z-b_1)^4=0$.

The connection between the quartic and the cubic is therefore as intimate as it was in the case of the general equation; and we may be assured that whatever affects the multiplicity of value of the expression for y will also affect in a similar manner the multiplicity of value of the expression for z or ψx; and therefore that all the radicals which enter into y enter also in an unaltered form into the root of the quartic as obtained by the process. These radicals are the three distinct forms of

$$\sqrt[3]{(31)^2+\sqrt{(31)^4+\left(\tfrac{1}{3}\,(41)\right)^3}}+\sqrt[3]{(31)^2-\sqrt{(31)^4+\left(\tfrac{1}{3}\,(41)\right)^3}},$$

which we can express uniquely in terms of x_1, x_2, x_3, and x_4, and also of a_1, a_2, a_3, and a_4.

It may perhaps be urged by way of objection that we are speaking of these functions as if they were really expressed in terms of the z's whilst we have no definite means of making such an expression. With reference however to the discriminant, this is not in fact the case, as we always know how to express that function, if not in a unique manner. yet in the manner which renders it a perfect square; and that is all we require. It will be observed that the discriminant of the quartic in x is a factor of that of the quartic in z when the latter is expressed in terms of the x's, so that the latter vanishes when the former does.

But independently of this, the objection is not of any weight, Properly speaking we have no equation in z; for the symbol z is merely used as an abbreviation of that form of ψx which in this transformation is equal to z. The equation which we are considering is still an equation in x, differing from an ordinary quantic in x in this respect, that its argument is not x but ψx (or in this transformation kx^2+lx). This should be borne in mind throughout, as we never attempt to solve any conditioned quartic, but deal entirely in the matter of resolution with the complete quartic in x, placing it for convenience in its trinomial form

$$(\psi x-b_1)^4-4(31)\ (\psi x-b_1)-(41)=0.$$

When the cubic has three equal roots, it is reduced to $y^3=0$, both (31) and (41) being zero; and both the radicals disappear from the root. The quartic degenerates simply into $(z-b_1)^4=0$, or $(\psi x-b_1)^4=0$; and has also three equal roots. Also whenever it has three equal roots, it has four equal roots, if considered as an equation in z. But if considered as an equation in x, then, when it has three equal roots, it is reduced to $\psi x-b_1$; and when it has four equal roots it becomes $x=a_1$. It is not necessary to consider the matter in any detail in this place; but of course it may happen that, in such extreme cases as these, the transformation may be nullified in consequence of relations existing between the original coefficients or roots which are inconsistent with the form of the equation in z.

Speaking generally, we find that the conditions of two equal roots and of three equal roots are the same for the quartic and for the cubic; and therefore that the two radicals which enter into the root of the latter enter in an unaltered shape into any algebraic expression for the root of the former, so that such expression must be in effect some function of y_1 combined with the same function of y_2 and also the same function of y_3. We are thus entitled to conclude that the connection between the quartic and the cubic continues, in this transformation in the same condition as we found it in the foregoing case; and we may proceed to argue upon the peculiar form of the cubic in a manner similar to that in which we proceeded before.

We observe then that the quadratic-irrational combination of coefficients or roots which we have designated (31) does not enter into the cubic in y except through its square; which leads us to the fact that this cubic resolvent is equally applicable to that other quartic which we shall obtain if in our original conditioned quartic we substitute —(31) for (31); or, which is the same thing, $-(z-b_1)$ for $(z-b_1)$. The quartic in z given by the transformation of the original quartic in x is

$$(z-b_1)^4-4(31)\,(z-b_1)-(41)=0;$$

the conjugate quartic therefore is

$$(z-b_1)^4+4(31)\,(z-b_1)-(41)=0;$$

and the quartic in x which would have produced this last form is that which we should obtain from the original quartic by substituting $-(kx^2+lx-b_1)$ for $(kx^2+lx^2-b_1)$; k, l, and b_1 being all supposed to be expressed in terms of a_1, a_2, a_3, and a_4.

The expression for y, therefore, must admit of being placed in such a form that some function of it shall be capable of variation by the above mentioned change of each $(z-b_1)$ or each (kx^2+l-b_1), while at the same time we know that y itself is incapable of such a change. As before we infer that this can only mean that $y^{\frac{1}{2}}$ exhibits this faculty of changing while $(y_{\frac{1}{2}})^2$ does not exhibit it; and we thence perceive the necessity that y when fully expressed in terms of the x's must be a rational function (not now of the x's but) of those expressions which represent z in terms of x; for if y were a function not reducible by extraction, any function of it would necessarily contain the four $(z-b_1)$'s only through the four $(z-b_1)^2$'s, inasmuch as (41) and $(31)^2$ are identically functions of the four $(z-b_1)^2$'s. This being the case, the square-root of each y must be a linear function of these same four expressions ψx_1, ψx_2, ψx_3, and ψx_4 for the same reason as in the last case. In short each y when expressed in terms of the x's, (and there is but one way in which it can be *so* expressed), will be actually reducible, by performance of all the operations indicated in it, (not, of course, including the square-root radical which we regard as *ipso facto* irreducible), to a whole rational function of ψx_1, ψx_2, ψx_3, and ψx_4 of the second degree; ψ being of that particular form kx^2+lx which represents the value of z in the transformation; and also each value of $y^{\frac{1}{2}}$ will be actually reducible to a linear function of ψx_1, ψx_2, ψx_3, and ψx_4. Performing the reductions, we obtain

$$A_1(\psi x-b_1)+A_2(\psi x_2-b_1)+A_3(\psi x_3-b_1)+A_4(\psi x_4-b_1)=y_1^{\frac{1}{2}},$$
$$B_1(\psi x_1-b_1)+B_2(\psi x_2-b_1)+B_3(\psi x_3-b_1)+B_4(\psi x_4-b_1)=y_2^{\frac{1}{2}},$$
$$C_1(\psi x_1-b_1)+C_2(\psi x_2-b_1)+C_3(\psi x_3-b_1)+C_4(\psi x_4-b_1)=y_3^{\frac{1}{2}};$$

which, combined with

$$(\psi x_1-b_1)+(\psi x_2-b_1)+(\psi x_3-b_1)+(\psi x_4-b_1)=0,$$

determine ψx_1, ψx_2, ψx_3, and ψx_4 completely. The A's, B's, and C's are numerical factors which have sprung up in the course of the reductions; and y_1, y_2, and y_3 are the known irrational functions of a_1, a_2, a_3, and a_4 previously determined.

This process being supposed to be gone through for one transformed quartic and its form of ψ, we may go through it again in precisely the same manner for the other transformed quartic and its corresponding form of ψ; and we thus obtain another set of values of ψx_1, ψx_2, ψx_3, and ψx_4, in which ψ represents the other value of z. That is, in the case under consideration, we first determine

$$kx_1^2+lx_1,\ kx_2^2+lx_2,\ kx_3^2+lx_3,\ kx_4^2+lx_4,$$

where l is of one value say $\overline{21} + \sqrt{\overline{21}\cdot\overline{21} - \overline{22}\cdot\overline{11}}$; and then determine

$$kx_1^2+l'x_1,\ kx_2^2+l'x_2,\ kx_3^2+l'x_3,\ kx_4^2+l'x_4$$

where l' has the other value, say,

$$\overline{21} - \sqrt{\overline{21}\cdot\overline{21} - \overline{22}\cdot\overline{11}}.$$

Now, if we form any symmetric function of $kx_1^2+lx_1$ and $kx_1^2+l'x_1$; this will be a function of x_1 all whose coefficients are rational functions of a_1, a_2, a_3, and a_4; then x_1 itself is absolutely determined by expelling all powers of x_1 higher than the first by means of the original quartic in x_1. For example, if we take the sum of $kx_1^2+lx_1$ and $kx_1^2+l'x_1$, we obtain

$$\overline{11}x_1^2+\overline{21}x_1 = V_a,$$

V_a denoting a known function of a_1, a_2, a_3, and a_4; and we obtain x_1 by extirpating x_1^4, x_1^3, and x_1^2, from $x_1^4-4a_1x_1^3+6a_2x_1^2-4a_3x_1+a_4=0$, and

$$(a_1^2-a_2)x_1^2+(4a_1^3-5a_1a_2+a_3)x_1 - V_a=0.$$

There is, therefore, no further extraction of roots than is implied in finding the double set of $y_1^{\frac{1}{2}}$, $y_2^{\frac{1}{2}}$ and $y_3^{\frac{1}{2}}$. It is, of course, obvious that the irrational functions of a_1, a_2, a_3, and a_4, which represent one set $y_1^{\frac{1}{3}}, y_2^{\frac{1}{3}}, y_3^{\frac{1}{3}}$, represent the other set also by simply changing throughout the sign of $\sqrt{I_6}$.

28. We thus acquire a new form of solution for the quartic, the essence of which consists in the introduction of a special irrational element pervading the whole process and the final result, and producing a material change in the *character* of the root, inasmuch as each term of it now contains only two constituents.

Had we taken ψ of the form hx^3+lx, we should have obtained another new form of solution of the quartic, whose distinction would have depended on the introduction of a known irreducible radical $\sqrt{I_8}$, producing a similar effect on the root; and had we taken ψ of the form hx^3+kx^2, we should have obtained another new form of solution of the quartic involving another known irreducible radical $\sqrt{I_{10}}$ with similar consequences. Moreover, if we had retained ψ in its full form hx^3+kx^2+lx, we should have got another distinct solution of the quartic, whose leading feature would have been the introduction of one or other of two known irreducible radicals of the respective degrees $\sqrt{I_{14}}$ and $\sqrt{I_{18}}$.

The ordinary solution of the quartic, being the one to which we were led in the former section, may be distinguished as the solution which does not introduce an irreducible radical. The possibility of arriving at such a solution depended upon a circumstance which we could not have anticipated *à priori*, viz. the power of finding certain rational combinations of the roots which were capable of being the roots of a cubic with coefficients rational functions of those of the quartic.

This means in effect that there exist rational functions of four symbols which have exactly three values.

The solutions given in this section do not involve this proposition. We do not deal with rational combinations of the roots; but with certain combinations which possess a specific kind of irrationality; to wit, that kind of irrationality which consists in their being of the form $A + B\sqrt{I}$ where A and B are rational, and I is rational, but not a perfect square, that is, not necessarily a perfect square. We have ascertained that there are several distinct forms of the irrational $\sqrt{I}$, the introduction of any one of which enables us to find a new mode of solution. In each case, we find that if we deal with functions of the roots which possess the specific irrationality under consideration and no other, we may assert that we can find a three-valued function of four symbols; and so arrive at the solution of the quartic.

If we had fallen upon the method of this chapter without knowing that of the last, we should have got a solution of the quartic without any knowledge of the fact that there exist three-valued *rational* functions of four symbols.

The method of this chapter and that of the last are quite independent of each other, as we have purposely avoided employing any results derived from the former method. If the first method had failed to produce a result, it would not have afforded any reason for thinking that the second would also fail to do so. It may very well happen, for anything we can know *à priori*, that an equation, not algebraically resoluble by a process which implies that the rational part of each root is a_1, and the irrational parts functions of (12), (13), (14), &c., may yet be algebraically resoluble by the introduction of some irrational function, (such as the I of this section), which alters the algebraic character of the proposed problem. It happens in the case of the quartic that the problem admits not only of statement in two ways, but also of resolution in both modes of statement.

In estimating the character and merits of the second method of resolving the quartic, it is necessary that we should not lose sight of the fact, that the values of the coefficients A, B, and C of the process are discoverable by the actual performance of the operations which we have pointed out, and that it is not necessary to call in aid any results of the former method.

Nevertheless, by collating the two processes, we may arrive at once at the conclusion that the A, B, and C of this process are the same as those of the last process; inasmuch as we may observe that the equation whose argument is z or ψx, is, in fact, a particular case of the general quartic. Now, as the general quartic is in fact soluble by the first process, we must ascribe to A, B, and C such numerical values as will be consistent with that solution. In other words, we need not in practice compute the values of y in this second process in explicit terms of the x's, seeing that we can find the coefficients by a simple reference to the former process.

In the new process, a rational function of each root is expressible ultimately in terms of *six* irrational functions (of the fourth Hamiltonian order) of the coefficients, besides the quasi-rational part, which is the sum of the two values of b_1, though it can obviously be also expressed by a smaller number of functions, but not so symmetrically.

The reader will be careful not to acquire the idea that we have been solving the general quartic by first finding the solution of the conditioned or trinomial quartic. We have not attempted the solution of any limited form, and do not here recognize any mode of solving a limited form except to take one of the solutions of the general quartic and to import into it the condition which represents the limitation.

29. It is perhaps scarcely necessary to point out that the new method of solution necessarily involves one or other of the five quadratic-irrationals which are introduced by the Tschirnhausen transformation. No other irrationals would

produce any effect towards obtaining a solution essentially distinct from the known linear solution. We might, for example, in the known linear solution when expressed in terms of the roots, introduce in place of each x any function of x with irrational coefficients, and then turn the results into functions of a_1, a_2, a_3, and a_4; which at first sight might appear to lead to some new form of resolution. This would obviously in general be nothing but a circuitous way of expressing the root, and in substance it would remain the same. But when we take the particular irrationals of the Tschirnhausen process in the way in which that process introduces them, we find that the result is represented in a very different manner by reason of the vanishing of one of the constituents out of the form or framework of the root, viz. S_2.

In order that the formula in section (23), containing S_2, S_6, and S_{12}, may represent the root of the general quartic, it is an essential feature of the formula that none of these quantities can vanish. S_2 for example could only vanish by reason of a condition existing between the coefficients and roots of the quartic, and we do not admit the existence of any condition. But if we substitute for each x in the expression of the root some function of x (say χx), it may happen that that function is such as to cause S_2, for example, to vanish identically without in any manner affecting the perfect generality of the root, or introducing any condition between the roots. What we have proved is, that this does take place whenever the function thus introduced is one of the quadratic-irrational forms proceeding from the Tschirnhausen transformation. Now, when this occurs, there is a discontinuity of form produced in the expression for the root, which precludes us from reducing that root back, by algebraic processes, to the form from which we are supposed to have derived it. If S_2

has once vanished by the effect of the process which we have performed upon it under the radicals, it cannot be restored by any process which operates only outside the radicals. Let the root be called ρ; then if we substitute χx for each x in that root, the result will be both equivalent to and algebraically reducible to $\chi\rho$ so long as the form of the root is not varied by the substitutions. But if χ be such a function, that the substitutions produce a variation of form, the result of them can no longer be represented explicitly by $\chi\rho$, although it may in an arithmetical sense be equivalent to it. The arithmetical equivalence remains, but the algebraic reducibility is put an end to by the sudden springing up of a discontinuity in the form.

The vanishing of this term for the particular irrationals in question is theoretically a matter of great importance. In an algebraic point of view, it presents the problem of resolution in another form. It shows that in fact the root of a quartic, like that of a quadratic or a cubic, can be represented as a function of a single parameter, $(31)^4 \div (41)^3$; a fact which did not flow from the linear mode of resolution, and could not be inferred from the form of the root which that process gave.

This, it will be understood, is the real distinction between the old and the new solutions; but I will add some observations which will more completely illustrate the distinctness of the two results.

30. Although the only new solutions which we obtain are those which we have described, yet it is competent for us to introduce any linear transformation into the original quartic, before we proceed to apply the Tschirnhausen process; and every different linear transformation would cause a difference in the irrational introduced by this method. These, however, would obviously be not really distinct solutions, but

merely linear variations of those which we have found. If we first make use of that linear transformation which causes the coefficient of the second term to vanish, we obtain a great simplification of all our formulæ without any loss of generality or the introduction of any condition.

It will be easier therefore in any calculations we make, to adopt this as the standard mode of transformation. It is effected by writing $x-a_1$ for x; making a_1 zero in all the formulæ, and then considering that a_2, a_3, and a_4 respectively mean $-(12)$, (13), and $-(14)$. We will compute them in this way, writing c_2, c_3, and c_4, in lieu of a_2, a_3, and a_4, to indicate that they mean the above combination of coefficients. This being so, we now have

$$\psi x = k\,(x-a_1)^2 + l\,(x-a_1),$$

in which k is $-3c_2$, and l is $-3c_3+\sqrt{3\,(9c_2^3-c_2\,c_4+3c_3^2)}$. We then obtain $b_1=9c_2^2$; and

$$4b_3 = l^3\,\Sigma\,(xxx) + l^2k\,\Sigma\,(x^2xx) + lk^2\,\Sigma\,(x^2x^2x) + k^3\,\Sigma\,(x^2x^2x^2),$$

or

$$b_3 = c_3\,l^3 + 3c_2\,c_1\,l^2 + 54c_2^3\,c_3\,l - 27c_2^3\,(4c_3^2 - 3c_2\,c_4)\,;$$

which, when divided out by means of

$$l^2 + 6c_3 l - 3c_2\,(9c_2^2 - c_4) = 0,$$

gives

$$b_3 = 81c_2^4\,c_4 - 513c_2^3\,c_3^2 - 9c_2^2\,c_4^2 + 162c_2\,c_3^2\,c_4 - 108c_3^4$$
$$+ (81c_2^3\,c_3 - 21c_2\,c_3\,c_4 + 36c_3^2)\,\sqrt{I_6}.$$

Similarly

$$b_4 = c_4\,(l^4 + 54c_2^3\,l^2 - 108c_2^3\,c_3\,l + 81c_2^4\,c_4)$$
$$= 9c_4\,\Big\{243c_2^6 - 36c_2^4\,c_4 + 324c_2^3\,c_3^2 + c_2^2\,c_4^2 - 24c_2\,c_3^2\,c_4 + 72c_3^4$$
$$- (72c_2^3\,c_3 - 4c_2\,c_3\,c_4 + 24c_3^3)\,\sqrt{I_6}\Big\}.$$

These give

$$(31) = -729c_2^6 + 81c_2^4c_4 - 513c_2^3c_3^2 - 9c_2^2c_4^2 + 162c_2c_3^2c_4 - 108c_3$$
$$+ (81c_2^3\,c_3 - 21c_2\,c_3\,c_4 + 36c_2^3)\,\sqrt{I_6}$$
$$= B_{12} + B_9\,\sqrt{I_6}\ \text{suppose};$$

$$\tfrac{1}{3}(41) = -6561c_2^8 + 243c_2^6 c_4 - 6156c_2^5 c_3^2 + 972c_2^3 c_3^2 c_4 - 1296c_2^2 c_3^4$$
$$- 3c_2^2 c_4^3 + 72c_2 c_3^2 c_4^2 - 216c_3^4 c_4$$
$$+ 12\,(81c_2^5 c_3 - 3c_2^3 c_3 c_4 + 36c_2^2 c_3^3 - c_2 c_3 c_4^2 + 6c_3^3 c_4)\sqrt{I_6}$$
$$= B_{16} + B_{13}\sqrt{I_6} \text{ suppose ;}$$

from which we have to form

$$y = 3\left\{\sqrt[3]{(31)^2 + \sqrt{(31)^4 + (\tfrac{1}{3}(41))^3}} + \sqrt[3]{(31)^2 - \sqrt{(31)^4 + (\tfrac{1}{3}(41))^3}}\right\}$$

$(31)^2$ being $B_{12}^2 + B_9^2 I_6 + 2B_9 B_{12}\sqrt{I_6}$;

and $(31)^4 + (\tfrac{1}{3}(41))^3$ being

$$B_{12}^4 + 6B_{12}^2 B_9^2 I_6 + B_9^4 I_6^2 + B_{16}^3 + 3B_{16} B_{13}^2 I_6$$
$$+\left(4B_{12}^3 B_9 + 4B_{12} B_9^3 I_6 + 3B_{16}^2 B_{13} + B_{13}^3 I_6\right)\sqrt{I}.$$

As a particular case, let us suppose the equation to have two equal roots, so that the values of y are

$$6\,(31)^{\frac{2}{3}}, -3\,(31)^{\frac{2}{3}}, -3\,(31)^{\frac{2}{3}},$$

whence

$$z_1 + z_2 - z_3 - z_4 = 4\,(31)^{\frac{1}{3}}$$
$$z_1 + z_2 + z_2 - z_4 = 2\sqrt{-2}\,(31)^{\frac{1}{3}}$$
$$z_1 - z_2 - z_3 + z_4 = 2\sqrt{-2}\,(31)^{\frac{1}{3}}$$
$$z_1 + z_2 + z_3 + z_4 = 36c_2^2;$$

and we have

$$z_1 = 9c_2^2 + \sqrt[3]{(31)}\,(1 + \sqrt{-2})$$
$$z_2 = 9c_2^2 + \sqrt[3]{(31)}\,(1 - \sqrt{-2})$$
$$z_3 = 9c_2^2 - \sqrt[3]{(31)}$$
$$z_4 = 9c_2^2 - \sqrt[3]{(31},$$

$$z \text{ being } -3c_2\,(x - a_1)^2 - (3c_3 - \sqrt{I_6})\,(x - a_1).$$

Now, in (31) change the sign of I_6, and we have similar values for

$$-3c_2\,(x - a_1)^2 - (3c_3 + \sqrt{I_6})\,(x - a_1);$$

and the addition of these gives

$$-6c_2(x-a_1)^2-6c_3(x-a_1), \text{ or } \Psi x,$$

which is represented by

$$\Psi x_1 = 18c_2^2 + (1+\sqrt{-2})\left\{\sqrt[3]{B_{12}+B_9\sqrt{I_6}} + \sqrt[3]{B_{12}-B_9\sqrt{I_6}}\right\}$$

$$\Psi x_2 = 18c_2^2 + (1-\sqrt{-2})\left\{\qquad do. \qquad\right\}$$

$$\Psi x_3 = 18c_2^2 \quad - \quad \left\{\qquad do. \qquad\right\}$$

$$\Psi x_4 = 18c_2^2 \quad - \quad \left\{\qquad do. \qquad\right\}$$

where B_{12}, B_9, and I_6 have the values before assigned.

The ordinary process would give a result, no doubt arithmetically equivalent to this; but in it the cube-root radicals would not appear at all, and the expression would be entirely free from surds. But the mere form of (31) shows that its cube-root cannot be algebraically taken, for it would have to be of some form $B_4+B_1\sqrt{I_6}$, which is an impossible form, seeing that (31) does not contain any symbol of one degree.

31. There is, however, one case in which the expression which we have obtained for the root assumes in effect the ordinary form, *i. e.*, what the ordinary form would be under the same restrictions; viz. when such a condition exists between the coefficients that I_6 itself is equal to zero. This circumstance requires attention.

I have already pointed out that the discriminant of the quartic in z or ψx, $\left(27\,(13)^4+(14)^3\right)$ contains in it as a factor the discriminant of the quartic in x. Consequently, whenever the latter discriminant is zero, the former is so also. But the converse of this is not the case. When the discriminant of the quartic in z vanishes, that circumstance shows us *either* that the equation in x has two equal roots, *or* that for some two x's the value of $k(x_1+x_2)+l$ is zero. In point of fact we have four cases. First, the equation in x has two equal roots; in that case the discriminants of *both* the conditioned quartics vanish.

Secondly, the condition $k(x_1+x_2)+l=0$ subsists; in that case the discriminant of the first conditioned quartic vanishes, and that of the other does not. Thirdly, the condition $k(x_1+x_2)+l'=0$ subsists, and then the vanishing is in the second quartic only. Fourthly, I_6 is zero, then neither discriminant vanishes; but the two quartics merge into one with rational coefficients, and z is simply a *rational* function of x, namely $3(a_1^2-a_2)x^2-3(4a_1^3-5a_1a_2+a_3)x$; and there ceases in effect to be in that particular case any real distinction between this solution and the ordinary one.

Another point of view which seems to indicate an essential distinctness of each of these new solutions as compared with the old solutions, (which I call the solutions by a linear transformation), is this—that we operate through a resolvent cubic, which is wholly distinct from the resolvent cubic of the existing modes of resolution.

If we take Euler's method of resolving the quartic, which leads to the same form as what I call the linear solution, and compare it (for example) with Simpson's method, we might at first be inclined to say that they are essentially different, because the radicals enter in another manner. But they are in fact the same in ultimate analysis, as has been recently shown in a very clear manner. (See Mr. Ball's Memoir in *Quarterly Journal of Mathematics*. No. 25, p. 6.)

Strictly speaking, in the new method of resolution by transformations through quadratic radicals, we do not arrive at a resolvent cubic, but rather at a resolvent sextic with rational coefficients, which can be divided into two cubics, each with quadratic-irrational coefficients.

This circumstance constitutes an essential difference, because so soon as we see that the two forms are not referable to the same cubic, we know that there is no other equation to which they can be referred except the quartic itself which we are solving; and it is a matter of course that any solutions, however different, must be capable of being connected through the quartic itself.

32. We may conclude this branch of the subject with this observation, that we now see that in final analysis the resolution of the quartic is an ordinary algebraic operation of an inverse kind, a fact which the former mode of solution did not enable us to prove.

The general quartic is reducible to the form

$$(z-b_1)^4-4\,(31)\,(z-b_1)-(41)=0;$$

or, making $$z-b_1=\sqrt[3]{4\,(31)}\,t,$$

$$t^4-t=\frac{(41)}{\sqrt[\frac{4}{3}]{4\,(31)}}=v;$$

that is, simply $\phi t=v$; from which we have been finding what is the form of ϕ^{-1} in $t=\phi^{-1}v$.

The results of this chapter give rise to observations in reference to the proposition supposed to have been established by Sir William R. Hamilton, as to the impossibility of finding new forms of resolving the quartic; but it will be convenient to postpone them, until we have ascertained how far these modes of resolution by irrationals can be pursued.

The Tschirnhausen transformation admits of another mode of application to the solution of the quartic, (commonly called Tschirnhausen's solution), as follows: transform the quartic in x into a quartic in t by means of the condition $(13)=0$, instead of $(12)=0$. This will be effected by means of a cubic; and the new equation in t will be in effect a quartic containing only even powers of t, and therefore resoluble by taking the square-root of the root of a quadratic equation, whose coefficients are such functions of a_1, a_2, a_3, and a_4 as are derived from the solution of a cubic.

This, in fact, leads to the same cubic resolvent as the linear method, and is one of the solutions already known.

We may therefore conclude that the solutions by quadratic-irrationals here given are the only distinct solutions which exist besides the linear one.

THE QUINTIC.

33. In discussing the quintic, we shall follow as closely as possible the methods which we have pursued for the lower quantics, so as to invite attention, by the form of the reasoning, to any circumstances which may present themselves differing from those which appeared in the case of the inferior degrees.

If the root of the quintic be algebraically expressible *through the medium of the linear transformation*, the rational part will be a_1, and the irrational parts will be functions of the four quantities (12), (13), (14), and (15). We therefore proceed to enquire whether we can find a quartic whose coefficients are rational functions of (12), (13), (14), and (15); and whose discriminant is equal to that of the quintic. If we find such a quartic, we shall know that its roots are functions of some kind of the five $(x-a_1)$'s.

The discriminant of a quintic is of the 20th degree in its roots: that of a quartic is of the 12th degree in its roots. No degree, therefore, can be assigned to the argument of a quartic which can make its discriminant actually identical with that of the quintic. We may, however, properly seek for a quartic whose discriminant shall be a power of that of the quintic; so that the principal critical function, or condition of two equal roots, shall still be same for both.

The least common multiple of 12 and 20 is 60: and consequently the simplest mode of effecting our object is to make the argument of the quartic of the *fifth* degree, in which case the discriminant of the quartic will be of the

60th degree, and must therefore be made equal to the *cube* of that of the quintic. The coefficients of such a quartic will be of the degrees 0, 5, 10, 15, 20.

34. There is no difficulty in writing down the most general form of such a quartic, (which, of course, involves a large number of numerical constants); and in forming its discriminant, and comparing it with the cube of the discriminant of the general quintic, as given in (6): and thus ascertaining whether the required quartic in y exists, and, if so, what are its coefficients. The process, however, is long; and we can arrive at a result in a more summary manner by availing ourselves of certain peculiarities which flow from the circumstance that the discriminant of the quartic is to be the *cube* of a rational function of (12), (13), 14), and (15).

Whenever the first derivative of a quartic has all its three roots equal, then it is plain that the discriminant of that quartic is a perfect cube. In other words, a quartic of the form

$$(y-l)^4+m=0$$

has a discriminant which is a perfect cube, viz., the cube of m.

If we denote the discriminant of the quintic as given at (6), by the symbol Δ, and write $N_3\Delta$ for m, and for l its most general form N_1 (12) (13) + N_2 (15), we have a quartic which fulfils all our requirements in the shape

$$\left\{y-\Big(N_1\,(12)\,(13)+N_2\,(15)\Big)\right\}^4-N_3\Delta=0.$$

I believe it will be found that we should come at some form similar to this by going through the process at length; and it might perhaps be interesting to ascertain whether the problem is quite definite, or whether there is any excess of equations or of constants. This enquiry, however, is not essential for our present purpose.

Solving the equation in y, we at once discover that the expression for y in terms of the x's is *ipso facto* irreducible; *i. e.*, not reducible to a rational form by extraction of roots. The root of this quartic contains $\sqrt[4]{\Delta}$ or $\sqrt{\sqrt{\Delta}}$. Now, of these two square-roots, the first in the order of calculation can always be performed, since Δ is by definition a perfect square when expressed in the x's. But the square-root of Δ similarly expressed cannot be the square of any rational function of the x's, since it is of the form $\Sigma x x x^3 x^4$ with half the terms negative, an expression which certainly is not a square. We cannot, therefore, determine y as a rational function of the five values of $x - a_1$.

If $\sqrt[4]{\Delta}$ had proved perfectly reducible, we should have got a rational expression for y in terms of the five $(x - a_1)$'s; or, in other words, we should have got a rational function of five independent symbols, possessing exactly four values: a result inconsistent with the known laws which govern the multiplicity of values of combinations of symbols; so that, in fact, we might have known beforehand, that we should fail in this search.

We must proceed, then, in some manner analogous to that which we adopted in the second method of dealing with the quartic; and ascertain if we can conduct the investigation through such a route as to introduce irreducible irrationals, and thus perhaps escape the necessity of passing through the linear transformation. We cannot find a four-valued rational combination of five symbols; but we may possibly find irrational functions (of known specific irrationality) of five symbols, the consideration of whose multiplicity of value may lead us to a solution of the quintic, corresponding in general character with those solutions of the quartic which were obtained by the use of certain quadratic-irrationals.

It may be noticed, *en passant*, that although the y's are not rational, yet that they admit of being divided into two pairs, so that the product of the two members of each pair is rational. Thus we have

$$y_1 y_2 = \left(N_1 (12)(13) + N_2 (15) \right)^2 - \sqrt{N_3 \Delta},$$

$$y_3 y_4 = \left(N_1 (12)(13) + N_2 (15) \right)^2 + \sqrt{N_3 \Delta};$$

which, when expressed in the roots, are rational.

This, in itself, is not a material circumstance, inasmuch as we know that for any quantic we can always find a two-valued function of the roots by introducing the square-root of the discriminant. We shall find, however, in the sequel, that it is worth while to bear this circumstance in mind.

DIGRESSION ON SOME SPECIAL FORMS.

35. Before proceeding further with the general problem, it will form a digression not without interest to ascertain if there are any conditioned forms of quintics in which $\sqrt[4]{\Delta}$ is totally reducible. It would probably be very difficult to determine generally under what conditions this reduction is possible; but the observation suggests itself, that if Δ, when expressed in terms of the coefficients, is an *ipso facto* square function, there is at least some probability that it may be made a perfect fourth power when expressed in the roots. Whether this should be so or not, the case which we are contemplating would have this peculiarity, that the products $y_1 y_2$ and $y_3 y_4$ would not only be rational, but would also be capable of symmetric expression in terms of the roots.

If we examine the terms of Δ as given in (6), it is not difficult to see that, by making (13) equal to zero, and introducing a relation between (12) and (14), we shall arrive at some cases in which Δ will be a natural square, and for which the relations between the coefficients will be of a simple character. Let then $(13)=0$, and $(14)-N(12)^2=0$; then the discriminant, (see 6), becomes

$$(6400N^3+2560N^4+256N^5)(12)^{10}$$
$$+(1440N+160N^2+3456)(12)^5(15)^2-(15)^4;$$

which will be a perfect square provided we have

$$N^5+35N^4+475N^3+3105N^2+9720N+11664=0;$$

or

$$(N+4)^2(N+9)^3=0;$$

that is $N=-4$, or $N=-9$.

The two equations represented by these conditions, are

$$(z-b_1)^5-10\,(21)\,(z-b_1)^3+20\,(21)\,(z-b_1)-(51)=0;$$
$$(z-b_1)^5-10\,(21)\,(z-b_1)^3+45\,(21)^2\,(z-b_1)-(51)=0;$$

which give respectively for the values of y

$$(51)+\sqrt[4]{\left((51)^2-128\,(21)^5\right)^2},$$

and

$$(51)+\sqrt[4]{\left((51)^2-1728(21)^5\right)^2};$$

so that we have

$$y_1\,y_2=128\,(21)^5,\text{ or }1728\,(21)^5;$$
$$y_3\,y_4=2\,(51)^2-128\,(21)^5,\text{ or }2\,(51)^2-1728\,(21)^5;$$

which are rational and capable of symmetric expression. One of them also is a perfect fifth-power; and in both (21) never enters except through $(21)^5$; so that they would have been derived equally from any one of the four other quintics which we should obtain by changing (21) successively into $a\,(21)$, $a^2\,(21)$, $a^3\,(21)$, and $a^4\,(21)$; or, what is the same thing, changing $z-b$ into $a^2(z-b_1)$, $a^4(z-b_1)$, $a(z-b_1)$, and $a^3(z-b_1)$.

Now if y_1 and y_2 were known to be rational functions of the $(z-b_1)$'s, it would follow that they would be each fifth-powers, inasmuch as their product is a fifth-power; for no other mode of obtaining a fifth power as a product would admit of being applicable to all the five quintics. Similarly, if they were known to be of an *irreducible* form—

$$(51)\pm(\sqrt{(51)^2-N^5\,(12)^5},$$

it would be necessary that this form should admit of having its fifth-root extracted so as to result in some form

$$R_1\pm\sqrt{R_2};$$

for otherwise no function of y_1 or y_2 could ever appear in a form admitting of variation, in passing from one to another of the five quintics above referred to.

In both cases the product of $y_1^{\frac{1}{5}}$ and $y_2^{\frac{1}{5}}$ is rational; and consequently the formula

$$a y_1^{\frac{1}{5}} + a^4 y_2^{\frac{1}{5}},$$

being equal to

$$a y_1^{\frac{1}{5}} + \frac{N(21)}{a y_1^{\frac{1}{5}}},$$

is a five-valued function of the roots. Expressing the five values thus—

$$y_1^{\frac{1}{5}} + y_2^{\frac{1}{5}},\ a y_1^{\frac{1}{5}} + a^4 y_2^{\frac{1}{5}},\ a^2 y_1^{\frac{1}{5}} + a^3 y_2^{\frac{1}{5}},\ a^3 y_1^{\frac{1}{5}} + a^2 y_2^{\frac{1}{5}},\ a^4 y_1^{\frac{1}{5}} + a y_2^{\frac{1}{5}},$$

and forming the equation in t of which these are the roots, viz.:

$$t^5 - 5N(21)\,t^3 + 5N^2(21)^2 t - 2(51) = 0;$$

then we know that t is a function of the $(z-b_1)$'s, having exactly five values.

When $N^5 = 128$, *i.e.* in the first of the two equations under consideration, we have

$$t^5 - 10 \cdot 2^{\frac{2}{5}}\,(21)t^3 + 20 \cdot 2^{\frac{4}{5}}\,(21)^2 t - 2\,(51) = 0;$$

or putting $t = 2^{\frac{1}{5}} v$,

$$v^5 - 10\,(21)\,v^3 + 20\,(21)^2\,v - (51) = 0;$$

so that, in fact, v or $2^{-\frac{1}{5}}\,t$, or $\left(\frac{y_1}{2}\right)^{\frac{1}{5}} + \left(\frac{y_2}{2}\right)^{\frac{1}{5}}$, is identical with $(z-b_1)$; and this is the solution of the equation; as the reader, I presume, has anticipated from the fact that the equation is in truth De Moivre's form. We thus see that in this case y_1 and y_2 admit of a rational form, and, accordingly, that t is a linear function of the $(z-b_1)$'s, having exactly five values by interchange of roots.

For the other quintic, the equation in t is

$$t^5 - 10\,(54)^{\frac{1}{5}}\,(21)\,t^3 + 20\,(54)^{\frac{2}{5}}\,(21)^2\,t - 2\,(51) = 0;$$

This circumstance might at first view lead us to conclude that the second of these two equations in z is reducible to the same form as that of the first by some transformation in which the new argument t would be a rational function of z with coefficients expressible in terms of (21) and (51). This, however, is not a legitimate deduction from what we have proved. We have shown that for each of these equations the formulæ which we call y_1 and y_2 are expressible as perfect fifth powers in terms of the z's respectively, but in both cases each expression is not unique but infinitely various; and we cannot compare one equation with the other without finding *unique* expressions for these resolvent functions; that is without going back to the complete quintic with five independent parameters of which each of the two equations in z may be regarded as a distinct transformation. Consequently what we have really proved is, that if the complete quintic in χ were reducible to the second form by the transformation $z = \psi x$ it would be reducible to the first form in t by a further transformation in which t would be expressible as a rational function of z, say χz. In such case it will be observed that the coefficients ψ and χ would be expressible only in terms of the coefficients of the general quintic in x, and could not be expressed in terms of (21) and (51).

THE QUINTIC.

(Resumed).

36. The course which we adopted in our second method of dealing with the quartic, supplies the clue to that which we are now to follow.

If we look at the discriminant of the quintic, and in it cause (12) and (13) both to vanish, we find that it assumes what we may now begin to regard as a canonical form of a discriminant. In the cubic, the discriminant is $(13)^2+N(12)^3$; in the conditioned quartic, it is $(14)^3+N(13)^4$; in the quintic conditioned as we now propose, it is $(15)^4+N(14)^5$; N being always $\pm(n-1)^{n-1}$. We shall thus be able to fulfil what appears to be an essential condition of resolubility; viz., we shall obtain an equation in y whose absolute term will be a perfect fifth power of a rational function of the coefficients, viz., $(41)^5$, and which will not contain (41) except through $(41)^5$. The first question for consideration, therefore, is whether we can find means to express the effect of the two conditions $(12)=0$, $(13)=0$ in terms of the roots.

The conditioned quintic with which we propose now to deal is

$$z^5-5b_1z^4+10b_1^2z^3-10b_1^3z^2+5b_4z-b_5=0;$$

and of the four quantities (21), (31), (41), (51), the two latter alone have any significant value. The most complete form of a quartic in y whose coefficients are rational functions of those of the quintic in z is, under these circumstances,

$$y^4-4N_1(51)y^3+6N_2(51)^2y^2-4N_3(51)^3y+N_4(41)^5=0;$$

in which we have purposely omitted from the last term any multiple of $(51)^4$, seeing that it would vanish under a proper linear transformation. We are now to ascertain what relations must exist between the five N's in order that the discriminant of this quartic may be the cube of that of the proposed quintic in z.

To find this discriminant, we have

$$(\text{I}\cdot\text{II}) = (N_1^2 - N_2)(51)^2 = C(51)^2 \text{ suppose,}$$
$$(\text{I}\cdot\text{III}) = (2N_1^3 - 3N_1N_2 + N_3)(51)^3 = D\,(51)^3 \text{ suppose,}$$
$$(\text{I}\cdot\text{IV}) = (3N_1^4 - 6N_1^2N_2 + 4N_1N_3)(51)^4 - N_4\,(41)^5$$
$$= E\,(51)^4 - N_4\,(41)^5, \text{ suppose.}$$

Consequently, it is, see (6.),

$$(81C^4E - 54C^3D^2 + 18C^2E^2 - 54CD^2E + 27D^4 + E^3)(51)^{12}$$
$$-(81C^4N_4 + 36C^2N_4E - 54CD^2N_4 + 3E^2N_4)(51)^8(41)^5$$
$$+(18C^2N_4^2 + 3EN_4^2)(51)^4(41)^{10}$$
$$-(N_4^3)(41)^{15};$$

which we have to compare with

$$\left\{(51)^4 - 256(41)^5\right\}^3;$$

so that, if we make N_4, (which is arbitrary), $=256$, we have the three equations

$$6C^2 + E = 1;$$
$$27C^4 + 12C^2E - 18CD^2 + E^2 = 1;$$
$$81C^4E - 54C^3D^2 + 18C^2E^2 - 54CD^2E + 27D^4 + E^3 = 1.$$

Deducting the second from the square of the first, we have

$$9C^4 + 18CD^2 = 0,$$

one solution of which is $C=0$; whence $E=1$, and $D=0$; that is $N_1=1$, $N_2=1$, $N_3=1$; and we find that the equation in y is

$$\Big(y - (51)\Big)^4 + 256(41)^5 - (51)^4 = 0;$$

and that it is arrived at without any redundancy of equations, or chance of inconsistency between them. The other solution, $C^3 + 2\,D^2 = 0$, gives the same result.

We can, therefore, express y in terms of (41) and (51), or

b_1, b_4, and b_5; and therefore also in terms of the z's; but we have no unique expression for y in this latter way, because the application of the conditions (12) $=0$ and (13) $=0$, (which are $2(\Sigma x)^2-5\,\Sigma\,(xx)=0$, and $2\,(\Sigma x)^3-25\,\Sigma\,(xxx)=0$), would cause it to assume an indefinite variety of forms.

Having thus placed the quintic in such a form that we have been able to find a quartic in y not only adequately connected with it in respect of identity of critical functions, but also such that the expression (41) enters into it only through its fifth-power, and that the product of the four values of y is a perfect fifth-power of (41), we may expect to be able to discuss the character of y as to its rationality or the nature of its irrationality, provided we can find some unique mode of representing y as a function of the coefficients or roots of a complete quintic whose root is connected with that of the quintic in z by a definite relation; that is, in other terms, provided we can find a method of adequately representing a complete quintic by means of one containing only two parameters; as we did in the case of the quartic.

If I have succeeded in carrying with me up to this point, the intelligent conviction of my readers, they will now become aware that the next step towards the resolution of the quintic, is one which will render it necessary for us to resort to the celebrated transformation of the quintic known as the Tschirnhausen-Jerrard transformation. In short, the principal question now remaining is, whether the complete quintic can be represented by a quintic which is, or by a system of quintics all of which are, subjected to the conditions (12) $=0$, (13) $=0$; and this is, in fact, the problem which the above-mentioned transformation enables us to resolve.

Taking the complete quintic in x,

$$x^5-5a_1x^4+10a_2x^3-10a_3x^2+5a_4x-a_5=0;$$

we are required to transform it into a quintic in z, z being such a function ψ, of x, as will cause the result to appear in a form possessing the required relations, which we now designate as $(21)=0$, $(31)=0$. The fullest form of ψ available for this purpose is

$$z=\psi x=hx^4+kx^3+lx^2+mx;$$

and the transformed equation in z then is

$$z^5-\Sigma(\psi x)z^4+\Sigma(\psi x\cdot\psi x)z^3-\Sigma(\psi x\cdot\psi x\cdot\psi x)z^2+\Sigma(\psi x\cdot\psi x\cdot\psi x\cdot\psi x)z$$
$$-\psi x_1\cdot\psi x_2\cdot\psi x_3\cdot\psi x_4\cdot\psi x_5=0.$$

In order to satisfy the conditions $(21)=0$, and $(31)=0$, it is necessary that h, k, l and m should be so taken that we shall have

$$4\left(\Sigma(\psi x)\right)^2-10\,\Sigma(\psi x\cdot\psi x)=0,$$

$$(12)\left(\Sigma(\psi x)\right)^3-150\,\Sigma(\psi x\cdot\psi x\cdot\psi x)=0.$$

In the notation above given, the first of these equations of condition is

$$h^2\cdot\overline{44}+2hk\cdot\overline{43}+2hl\cdot\overline{42}+2hm\cdot\overline{41}+k^2\cdot\overline{33}$$
$$+2kl\cdot\overline{32}+2km\cdot\overline{31}+l^2\cdot\overline{22}+2lm\cdot\overline{21}+m^2\cdot\overline{11}=0;$$

which admits, by means of the disposable constant, of division into two linear equations in different manners, of which the following appears to be the easiest.

It will readily appear that this condition admits of the following form:—

$$(\overline{11}\cdot\overline{22}-\overline{12}\cdot\overline{12})\,(m\,\overline{11}+l\,\overline{12}+k\,\overline{13}+h\,\overline{14})^2$$
$$+\left((\overline{11}\cdot\overline{22}-\overline{12}\cdot\overline{12})\,l+(\overline{11}\cdot\overline{23}-\overline{12}\cdot\overline{13})\,k+(\overline{11}\cdot\overline{24}-\overline{12}\cdot\overline{14})\,h\right)^2$$
$$+\left\{(\overline{11}\cdot\overline{22}-\overline{12}\cdot\overline{12})\,(\overline{11}\cdot\overline{33}-\overline{13}\cdot\overline{13})-(\overline{11}\cdot\overline{23}-\overline{12}\cdot\overline{13})^2\right\}k^2$$
$$+2\left\{(\overline{11}\cdot\overline{22}-\overline{12}\cdot\overline{12})\,(\overline{11}\cdot\overline{34}-\overline{13}\cdot\overline{14})-(\overline{11}\cdot\overline{23}-\overline{12}\cdot\overline{13})\right.$$
$$\left.(\overline{11}\cdot\overline{24}-\overline{12}\cdot\overline{14})\right\}h\,k$$
$$+\left\{(\overline{11}\cdot\overline{22}-\overline{12}\cdot\overline{12})\,(\overline{11}\cdot\overline{44}-\overline{14}\cdot\overline{14})-(\overline{11}\cdot\overline{24}-\overline{12}\cdot\overline{14})^2\right\}h^2=0.$$

This can be divided into two parts by means of the disposable constants; the first two terms being made equal to zero; and the last three terms (which contain only h and k) being also separately equated to zero. We thus have a quadratic in h and k with rational coefficients, (which I propose to call the first quadratic), whose solution gives a linear relation between h and k. The coefficient of this relation is a quadratic-irrational of the 30th degree.

Substituting for h in the equation formed by the sum of the first two terms above being made zero, we have another quadratic equation with quadratic-irrational coefficients, (which I call the second quadratic); and its solution gives a linear relation between m, l and k, whose coefficient is what we may call a quadratic-quadratic-irrational. We have thus *linear* values for k and h in terms of l and m. Substituting these in the cubic equation of condition, which, in this notation, is

$$\begin{aligned} h^3\,\overline{444}+3\,h^2k\,\overline{443}+3\,h^2l\,\overline{442}+3\,h^2m\,\overline{441} \\ +3\,hk^2\,\overline{433}+6\,hkl\,\overline{432}+6\,hkm\,\overline{431} \\ +3\,hl^2\,\overline{422}+6\,hlm\,\overline{421} \\ +3\,hm^2\,\overline{411} \\ +k^3\,\overline{333}+3\,k^2l\,\overline{332}+3\,k^2m\,\overline{331} \\ +3\,kl^2\,\overline{322}+6\,klm\,\overline{321} \\ +3\,km^2\,\overline{311} \\ +l^3\,\overline{222}+3\,l^2m\,\overline{221} \\ +3\,lm^2\,\overline{211} \\ +m^3\,\overline{111}=0; \end{aligned}$$

the result is produced in the form

$$A_1m^3+3\,A_2m^2l+3\,A_3ml^2+A_4l^3=0;$$

where A_1, A_2, A_3, and A_4 are quadratic-quadratic-irrational functions of $a_1 \ldots a_5$. Consequently, the relation between

m and l is determined, by the solution of this cubic, in the shape of an irrational function of the coefficients of the original quintic; which irrational function is of the specific kind which we have described; viz., such a function as proceeds from the resolution of a given cubic equation whose coefficients are themselves derived from the solution of a given quadratic equation whose coefficients are themselves irrational functions proceeding from the solution of another given quadratic equation whose own coefficients are rational functions of a_1, a_2, a_3, a_4, and a_5. We may call such an irrational a cubic-quadratic-quadratic-irrational. The form of ψx being thus completely determined, we ascertain b_1, b_4, and b^5 by substituting this form of ψ in the symmetric functions which compose the coefficients of the transformed equation. The labour of effecting these calculations would be very great; and I shall only go so far into them as is necessary to exhibit with some clearness the form of the result.

Now h in terms of k is formed from

$$A_{16}\,h = k\,(A_{15} + \sqrt{I_{30}}),$$

the suffix of each letter denoting the degree of the known rational function which the letter represents. Substituting for h in the cubic, it becomes

$$\begin{aligned}
m^3\,\overline{111}\,A^3_{16} &+ 3\,m^2 l\,\overline{112}\,A^3_{16} + 3\,m^2k\,(\overline{113}\,A_{16} + \overline{114}\,A_{15} \\
&+ \overline{114}\,\sqrt{I_{30}})\,A_{16}^2 + 3ml^2\,\overline{122}\,A_{16}^3 + 6\,mlk\,(\overline{123}\,A_{16} + \overline{124}\,A_{15} \\
&\qquad + \overline{124}\,\sqrt{I_{30}})A_{16}^2 + 3\,mk^2\Big(\overline{133}\,A_{16}^3 + \overline{134}\,A^2_{16}\,A_{15} \\
&\qquad\quad + \overline{144}\,A_{16}\,(A^2_{15} + I_{50}) + (\overline{134}\,A_{16}^2 + 2\cdot\overline{144}\,A_{16}\,A_{15})\,\sqrt{I_{30}}\Big) \\
&+ l^3\,\overline{222}\,A_{16}^3 + 3l^2\,k\,(\overline{223}\,A_{16}^3 + \overline{224}\,A_{16}^2\,A_{15} + \overline{224}\,A_{16}^2\,\sqrt{I_{30}}) \\
&\qquad + 3\,l\,k^2\Big(\overline{233}\,A_{16}^3 + \overline{234}\,A_{16}\,A_{15} + \overline{244}\,A_{16}\,(A_{15}^2 + I_{30}) \\
&\qquad\qquad + (\overline{234}\,A_{16}^2 + 2\cdot\overline{224}\,A_{16}\,A_{15})\,\sqrt{I_{30}}\Big)
\end{aligned}$$

$$+k^3\Big(\overline{333}\,A^3_{16} + 3\cdot\overline{334}\,A^2_{16}\,A_{15} + 3\cdot\overline{344}\,A_{16}\,(A^2_{15}+I_{30}) + \overline{444}\,(A^3_{15}+3\,A_{15}\,I_{30}) + (3\cdot\overline{334}\,A^2_{16} + 6\cdot\overline{344}\,A_{16}\,A_{15} + \overline{444}\,(3A^2_{15}+I_{30}))\sqrt{I_{30}}\Big).$$

Again k in terms of l proceeds from

$$\overline{11}\,A_{16}\,m + (\overline{12}+\sqrt{A_6})\,A_{16}\,l + (A_{16}\,\overline{13} + A_{15}\,\overline{14} + \sqrt{I_{30}}\,\overline{14})\,k = 0,$$

where $A_6 = -(\overline{11}\cdot\overline{22} - \overline{12}\cdot\overline{12})$; and, determining k so as to make denominators rational, we have

$$k\Big((\overline{13}A_{16}+\overline{14}A_{15})^2 - \overline{14}^2\,I_{30}\Big) + A_{16}\,(\overline{13}\,A_{16} + \overline{14}\,A_{15} + \overline{14}\,\sqrt{I_{30}})\Big(\overline{11}\,m + (\overline{12}+\sqrt{A_6})\,l\Big) = 0,$$

which we may call

$$A_{40}\,k + A_{16}\,(\overline{13}\,A_{16} + \overline{14}\,A_{15} + \overline{14}\,\sqrt{I_{30}})\Big(\overline{11}m + (\overline{12}+\sqrt{A_6})l\Big) = 0.$$

The complete expression for the cubic, therefore, will be

$$m^3\Big\{\overline{111}\,A_{40}^3 + 3\cdot\overline{11}\,A_{40}^2\,B_{41} + 3\cdot\overline{11}^2\,A_{40}\,B_{79} + \overline{11}^3\,B_{117} + (3\cdot\overline{11}\,A_{40}^2\,B_{26} + 3\cdot\overline{11}^2\,A_{40}\,B_{64} + \overline{11}^3\,B_{102})\sqrt{I_{30}}\Big\}$$

$$+3\,m^2\,l\Big\{\overline{112}\,A_{40}^3 + (\overline{12}+\sqrt{A_6})\,A^2_{40}\,B_{41} + 2\,\overline{11}\,A_{40}^2 B_{42} + 2\,\overline{11}\,(\overline{12}+\sqrt{A_6})A_{40}B_{79} + \overline{11}^2 A_{40} B_{80} + \overline{11}^2(\overline{12}+\sqrt{A_6})B_{117}$$
$$+\Big(A_{40}^2\,(\overline{12}+\sqrt{A_6})\,B_{26} + 2\,\overline{11}\,A_{40}^2\,B_{27} + 2\,\overline{11}\,(\overline{12}+\sqrt{A_6})\,A_{40}\,B_{64} + \overline{11}^2\,A_{40}\,B_{65} + \overline{11}^2\,(\overline{12}+\sqrt{A_6})\,B_{102}\Big)\sqrt{I_{30}}\Big\}$$

$$+3\,m\,l^2\Big\{\overline{122}\,A_{40}^3 + 2\,(\overline{12}+\sqrt{A_6})\,A_{40}^2\,B_{42} + \overline{11}\,A_{40}^2\,B_{43} + (\overline{12}+\sqrt{A_6})^2\,A_{40}\,B_{79} + 2\,\overline{11}\,(\overline{12}+\sqrt{A_6})\,A_{40}\,B_{80} + \overline{11}\,(\overline{12}+\sqrt{A_6})^2\,B_{117}$$
$$+\Big(2\,(\overline{12}+\sqrt{A})\,A_{40}^2\,B_{79} + \overline{11}\,A_{40}^2\,B_{28} + (\overline{12}+\sqrt{A_6})^2\,A_{40}\,B_{64} + 2\,\overline{11}\,(\overline{12}+\sqrt{A_6})\,A_{40}\,B_{45} + \overline{11}\,(\overline{12}+\sqrt{A_6})\,{}^2B_{10}\Big)\sqrt{{}_{30}I}\Big\}$$

$$+3\,l^3 \Big\{ \overline{222}\,A_{40}^3 + 3\,(\overline{12} + \sqrt{A_6})\,A_{40}^2\,B_{43} + 3\,(\overline{12} + \sqrt{A_6})^2 A_{40} B_{80}$$
$$+(\overline{12} + \sqrt{A_6})^3\,B_{117} + \Big(3(\overline{12} + \sqrt{A_6})A_{40}^2 B_{29} + 3(\overline{12} + \sqrt{A_6})^2$$
$$A_{40}\,B_{64} + (\overline{12} + \sqrt{A^6})^3\,B_{102}\Big)\sqrt{I_{30}}\Big\}\ ;$$

in which the B's are all rational, and not very difficult to express.

Before solving this cubic, it will be convenient to multiply it by the expression which corresponds with the coefficient of m^3 with $-\sqrt{I_{30}}$ written for $\sqrt{I_{30}}$ so as to rationalize it; and then if we make l equal to that product (which is rational), we shall be able to determine m, by solving the cubic, in a whole form.

If we conceive this to be done, we have then a tolerably accurate idea of the form of the coefficients of this cubic, and of the way in which the two kinds of quadratic-irrationality enter into them; and if we suppose these coefficients substituted in the root of a general cubic equation in its ordinary form, we obtain an idea of the kind of irrational to which this transformation leads us. This radical is the essentially irrational form which will pervade the whole of our future results. We shall call it irreducible, not so much because we know that none of the subordinate radicals which enter into it can be reduced by extraction, but because we do not desire to reduce it and never attempt to do so. We may be tolerably certain that in point of fact it would be irreducible when expressed in terms of the x's, but it is not necessary to prove this.

There is then no further difficulty in determining h and k.

The result which we have arrived at it is, that the general quintic is equivalent to a system of conditioned quintics, twelve in number of the form

$$z^5 - 5\,b_1 z^4 + 10\,b_1^2 z^3 - 10\,b_1^3 z^2 + 5\,b_4 z - b_5 = 0,$$

with the relation

$$z = hx^4 + kx^3 + lx^2 + mx;$$

where $k:h$, $l:h$, $m:h$, b_1, b_4, and b_5, are cubic-quadratic-quadratic-irrational functions of a_1, a_2, a_3, a_4, and a_5, all possessing the same kind of irrationality, and each of them having twelve values arising from the different square and cube roots of unity which we make use of in solving the equations which give rise to the irreducible function.

In the mode of elimination which I have now fully indicated, *i. e.*, eliminating h and k so as to obtain the cubic between l and m, we observe that the subjects of the two quadratic-radicals which are introduced are respectively of the sixth and thirtieth degrees, and that the highest coefficient in the cubic rises to the 126th degree, and would be of the 249th degree when one of the extreme coefficients is rationalized. This is however the simplest mode of proceeding.

If we had eliminated l and m first, and obtained the ultimate cubic between h and k, the subjects of the two quadratic-radicals would have been of the fourteenth and fiftieth degrees respectively; and the highest coefficient in the cubic would have been of the 192nd degree, and of twice this degree when rationalized.

Results of an intermediate character in degree would have been obtained by eliminating k and l first, and getting the cubic between h and m, and so on. There appear to be six methods of varying this elimination, so that there will be at least six apparently distinct modes of effecting the transformation by the introduction of different radicals; and it is possible that these may be capable of still further variation.

Every method, however, leads to the result in the shape of twelve conditioned quintics, each with its corresponding relation between z and x.

40. Let us, for the present, confine our attention to one of

these conditioned quintics; then, reverting to the quartic in y which we have found having its coefficients rational functions of those of the quintic in z, and having the same discriminant, we find that we have by its resolution a complete expression for y in terms of a_1, a_2, a_3, a_4, and a_5; the expression involving throughout it the peculiar irrationality above adverted to. And we can by substituting for each coefficient the symmetric function of the x's which is equivalent to it, express y in terms of the x's exclusively; and this expression of y is for each mode of transformation absolutely unique.

Now, the quintic in z or ψx and the quartic in y have the same critical functions, so that y is, as it were, capable of being a resolvent for the quintic. Neither equation, in reality, has any other critical function than the discriminant: for the quintic has but two parameters, and is therefore susceptible of only one condition. The equation in y answers to this peculiarity by the fact, that when it has two equal roots, it has all its roots equal.

The discriminant of the quintic in x is a factor of that of the quintic in z, so that the latter vanishes when the former does; and as the square-root of the discriminant is the only function which in the expression for y appears under a radical at all, we need not consider any other systems of equalities. As a matter of fact, however, it is obvious that when the quintic in z or ψx has more than two equal roots, it reduces itself to $(\psi x - b_1)^5 = 0$, and the quartic become $y^4 = 0$. But we must bear in mind, that if we suppose such conditions to exist between the coefficients of the original quintic as would be inconsistent with the form of the quartic in z, the transformation might require to be modified, or might even be nullified.

41. When, however, we come to look closely into the nature of the argument heretofore developed in proof of the

rationality of the y's, and to apply it in a case like the present where the quartic whose roots are these y's is of a limited form, we cannot avoid perceiving that there is a material variation in one of the leading circumstances between this case and those which have been already considered.

The substantial basis of the argument was, that in any algebraic expression for the root of the quantic, the roots of the resolvent must appear, *simpliciter* and unaltered. In this condition there is a tacit implication that these last-mentioned roots are independent of each other, and that no one of them can be expressed in direct terms of the others. Now in the quartic in y at present under consideration this is not the case. Any two of the roots may be separately expressed as linear functions of the other two, so that in substance there are only two independent roots of the resolvent. If for a moment we consider y_1 and y_2 as the two independent roots, then y_3 and y_4 are each of the form $N_1 y_1 + N_2 y_2$; and in any function of y_1, y_2, y_3 and y_4, it is evident that y_1 and y_2 would no longer enter *simpliciter*, but in combination; and thus the basis on which rests the applicability of the argument as to the rationality of y fails us.

We must therefore adopt another course, and, reflecting that we have not four but only two independent elements in the resolvent, we may reasonably conclude that if we require its root to be rational, it will result in the form of a quadratic.

If we appropriate the roots of the quartic thus,

$$\left.\begin{aligned} y_1 &= (51) + \sqrt[4]{\Delta}, \\ y_2 &= (51) - \sqrt[4]{\Delta}; \\ y_3 &= (51) + \sqrt{-1}\,\sqrt[4]{\Delta}, \\ y_4 &= (51) - \sqrt{-1}\,\sqrt[4]{\Delta}; \end{aligned}\right\}, \Delta \text{ being } (51)^4 - 256\,(41)^5,$$

we readily observe that $y_1\,y_2$ and $y_3\,y_4$ are rational (though not symmetric) functions of the z's; and that their product is $256\,(41)^5$ which is a fifth-power of a rational and symmetric function of the z's. If we call these products Y_1 and Y_2 respectively, they are the roots of the quadratic

$$Y^2 - 2\,(51)^2\,Y + 256\,(41)^5 = 0\,.$$

Now as each value of Y is a rational function of the z's, and their product is a perfect fifth-power of a rational function of the z's, viz. of (41); it follows that each Y is a perfect fifth-power of a rational function of the second degree of the z's; for we cannot suppose, consistently with the other requirements of the problem that they would be of the forms RR'^4 and R^4R', or $R^2R'^3$ and $R^3R'^2$, (which are the only alternatives); inasmuch as these forms would only produce the same two components or factors, each repeated five times; and we could not adapt them, as we must do, at the same time to the five quintics in z or ψx, which we obtain by writing $a\,(z-b_1)$, $a^2\,(z-b_1)$, $a^3\,(z-b_1)$, and $a^4(z-b_1)$ for $(z-b_1)$ in the original quintic.

42. The result, therefore, is that if each Y be calculated, (and the calculation involves only one extraction of a root, viz.: the square-root of the discriminant), the result of that calculation is to produce two expressions of the 10th degree which admit of having their fifth-roots taken; and these fifth-roots, therefore, will be of the form, (calling $z-b_1$ or $\psi x - b_1$ by the symbol t,)

$$\left.\begin{array}{l} \Sigma\,(A_1\,t^2) + \Sigma\,(B_1\,tt) = Y_1^{\frac{1}{5}} \\ \Sigma\,(A_2\,t^2) + \Sigma\,(B_2\,tt) = Y_2^{\frac{1}{5}} \end{array}\right\} \quad \ldots\ldots \quad Y.$$

in which the co-efficients A and B will present themselves as the result of the calculation; and Y_1 and Y_2 are known functions of a_1, a_2, a_3, a_4, and a_5.

Thus, instead of obtaining the values of four linear unsymmetric functions of the t's we obtain two unsymmetric functions of them of the *second* degree; which, although they do not so directly conduct us to the complete resolution as linear functions might have done, are yet sufficient to show that the quintic is resoluble.

We have here fallen upon a circumstance very similar to that which took place in the case of the cubic. We were prepared to prove from *à priori* condsiderations that the functions Y_1 and Y_2 must be rational when expressed in terms of the x's; that is, of course, rational functions of those functions of x which we have called z. But in fact we find that we do not require any such proof, inasmuch as the expressions are *identically* rational functions of the z's in consequence of the discriminant of any equation being always capable of being expressed as a perfect square in the roots. Nevertheless, as in the case of the cubic, we may, as a matter of theory, consider that it is possible to prove *à priori* the rationality of the Y's in this sense, as well as the fact, which probably cannot be proved otherwise, that they are perfect fifth-powers.

43. Having thus obtained these functions for one form of ψ we may write down the corresponding functions for all the other forms of ψ, and combine them in the following manner: let ψ with three suffixes attached, be employed to denote the origin of its different forms; thus ψ_{212} is used to denote that form of ψ which in the transformation is obtained by using the second root of the cubic, the first root of the second quadratic, and the second root of the first quadratic, and so on. Then if we form the functions Y for ψ_{111}, ψ_{211}, ψ_{311} and combine them in any symmetric manner, we shall obtain two other functions corresponding with Y; but from which that part of the irrationality of the ψ's which proceeds from the cubic, will have disappeared; and the new functions which we may call χ will contain only the irrationality proceeding from the two quadratics. If we denote these by χ_{11}, we can

then write down the same results, using throughout the second root of the second quadratic in place of the first, which will give χ_{21}: and any symmetric combination of these two, will give a new function, which we may call λ_1 which will contain only the irrationality which proceeds from the first quadratic. And if we then write down λ_2, (forming it from λ_1 by using the second root of the first quadratic in lieu of the first), and combine λ_1 and λ_2 symmetrically, we shall ultimately arrive at a form μ which will be a rational function of the x's, with rational coefficients. Thus, if we make all these symmetric functions simple sums, we shall obtain two rational unsymmetric functions of the roots of this form

$$\Sigma\left(A_1\,(\mu x)^2\right)+\Sigma\,(B_1\,\mu x\cdot\mu x)=S\,(Y_1{}^{\frac{1}{5}});$$

$$\Sigma\left(A_2\,(\mu x)^2\right)+\Sigma\,(B_2\,\mu x\cdot\mu x)=S\,(Y_2{}^{\frac{1}{5}}).$$

In these rational functions of the roots, we need not retain in evidence any powers of x higher than the fourth, for such powers could be eliminated by means of the original equation. So neither need we retain in these functions all the five x's; as we could eliminate some of them by means of the two values of $Y^{\frac{1}{5}}$ and the equations $\Sigma\, t=0$, $\Sigma\,(t\,t)=0$, and $\Sigma\,(t\,t\,t)=0$.

Having obtained these rational and unsymmetric functions of the roots, we are able by known processes to determine the roots themselves without introducing any other radicals.*

The roots are all thus completely determined, without any further extractions than are necessary to obtain the $(Y^{\frac{1}{5}})$'s. All the 24 values of $Y^{\frac{1}{5}}$ are of the same form, but with

* I refer especially to the methods given in the 11th and 12th Lessons of Serret. If I have in the text overstated the power of these methods, it does not affect my argument, as any deficiency in the system Y is fully supplied by the additional system in section 46^a.

different combinations of the square-roots and cube-roots of unity, as above specified.

44. Throughout the whole of this essay, I have been careful to point out the necessity, as a matter of theory, of, in every instance, calculating, or seeing our way to the calculation of, the values of y in two forms; first, in explicit irrational functions of $a_1 \ldots a_5$, and secondly, in terms of symmetric functions of $x_1 \ldots x_5$ under radicals; such last-mentioned functions being reducible, in one class of cases, to completely rational functions of the x's, and in another class of cases to the specifically irrational function which is introduced by the transformation employed. It is this reducibility which enables us to resolve the equation; and in actually effecting the reductions by extraction, we necessarily find the coefficients which we have called A's, B's, and C's ; and this is the natural method of finding them.

In the foregoing cases of the cubic and the quartic, we were able to infer the values (or at all events the ratios) of the corresponding numerical factors from considerations of a simple character, derived mainly from the fact, that the results would be applicable to the conjugate equations as well as to the equation under solution.

In the case of the quintic, if we had made all the necessary calculations for expressing Y in terms of x_1, x_2, x_3, x_4, and x_5, it would not be worth while to speculate on *à priori* grounds as to the value of these coefficients, as we could then detect them by mere inspection. Now, we are obliged to calculate Y in terms of a_1, a_2, a_3, a_4, and a_5; and as there are no extractions of roots to be performed, the additional labour of expressing it in terms of the x's would not be very great. The reason why I say that there are no extractions to be performed is, that we can form $\sqrt{\Delta}$ without forming Δ itself; since $\sqrt{\Delta}$ is the square-root of the discriminant of the original quintic multiplied by the 10 expressions of which $h\,(x_1^3+x_1^2x_2+x_1x_2^2+x_3^3)+k\,(x_1^2+x_1x_2+x_2^2)+l\,(x_1+x_2)+m$ is one, with some numerical factor.

45. The reader will be careful not to imagine that we have been solving the general quintic by first solving the trinomial form. We have made no attempt to solve the latter as a special form; nor could we, in the reasoning of this essay, recognize any mode of solving it, except to take the solution of the complete quintic and to introduce into it the conditions $a_1^2=a_2$, $a_1^3=a_3$. We do not allege or imply that the equation in z is solved by any such system as the above would become if z were written for each ψx. We must bear in mind that the y's in that system are expressed in terms of a_1, a_2, a_3, a_4 and a_5, and not in terms of b_1, b_4, and b_5. We know the b's in terms of the a's, but it is not possible to express the z's in a definite manner in terms of the b's.

46. The case of the quintic illustrates the principles and theory of resolution more completely than that of the other prime quantic which we have considered, namely, the cubic.

We might easily have been led to infer from the circumstances which presented themselves in the resolution of the cubic, that it is an essential feature of a resolvent that its roots should be expressible rationally in terms of the roots of the quantic. From the case of the quintic, however, we learn that this is not the fact; and we are conducted to the real principle, which is based upon the necessity that all our results in the way of resolution shall be equally applicable to a set of quantics of the same degree, in number equal to the exponent of the original quantic when that exponent is prime. This principle is, that of whatever order of irrationality (including rationality) the root of the resolvent is, when expressed in terms of the roots of the quantic, its prime-root will also be of exactly the same order of irrationality, or rational, as the case may be.

The value of y, in short, when reduced as far as possible, is always of such a form that, (n being the degree of the quantic, in this place supposed to be prime), its nth root is capable of being expressed by the n values of $1^{\frac{1}{n}} P$, P being of the same radical form as y, but having each member of it of a degree which is $(\frac{1}{n})$th of the degree of the corresponding member of y.

In the cubic, y was found to be rational, and of the third degree; therefore $y^{\frac{1}{3}}$ was rational and linear.

In the quintic y appears under the form

$$S_5 + \sqrt{R_{10}};$$

consequently $y^{\frac{1}{5}}$ will be expressible under the form

$$R_1 + \sqrt{R_2}$$

the S being a symmetric, and the R's rational functions of the ψx's or z's; or, to speak more accurately, of such functions of the x's as would be converted respectively into symmetric and rational functions of the ψx's, if for a moment we were to conceive them as so expressible; and there is, therefore, this connexion between the four values of $y^{\frac{1}{5}}$ that they admit of being divided into two couples, the product of whose respective members is a rational function of the five values of ψx, or rather such a function of the five x's as would be equivalent to some rational function of the ψx's.

The quartic resolvent, in short, is divisible into two quadratics; in each of which the fifth-root of the product of its roots is rational in the sense above indicated; and which are so connected that the product of this fifth-root in the one quadratic and the corresponding fifth-root in the other, is not only rational but symmetric in the same sense. We may therefore consider that we have two quadratic resolvents connected by the property last mentioned.

I propose to call by the name "De-Moivrian Equation" that quantic of prime degree (p) whose roots are expressible under the form $a\,P+a^{p-1}\,Q$, PQ being a symmetric or one-valued function of the roots. It is then easy to see that resolution by means of one quadratic resolvent is in effect reduction to a De-Moivrian form.

We may observe this in the case of the cubic; for its roots come out ultimately under the form $aP+a^2\,Q$, where PQ is symmetric in the roots, and rational in the coefficients.

Similarly, in the cases of the two particular quintics discussed in (35), each root, or a rational function of each root, appears under the form $aP+a^4Q$ where PQ is a rational function of the coefficients.

The peculiarity of such cases is that since $a\,P+a^{p-1}\,Q$ is in fact $a\,P+\dfrac{S}{a\,P}$, it is an expression which has only p values, and it must therefore be a rational function of some one root.

Now, consider a case which differs from the above in this respect only, that PQ is not symmetric in the roots, and not rational in the coefficients; but *demi*-symmetric in the roots, and involving a square-root radical when expressed in the coefficients.

This is what takes place in the general quintic when supposed to be expressed under the argument ψx, and it occurs in a two-fold manner.

For if we make the five values of $ay_1{}^{\frac{1}{5}}+a^4y_2{}^{\frac{1}{5}}$ the roots of the De-Moivrian quintic

$$t^5-5\,At^3+5\,A^2t-2\,(51)=0,$$

then the value of A is $(y_1\,y_2)^{\frac{1}{5}}$, which is a rational two-valued function of the ψx's, and of the form $P+\sqrt{Q}$ in the coefficients. Similarly, making the five values of $ay_3{}^{\frac{1}{5}}+a^4y_4{}^{\frac{1}{5}}$ the roots of

$$t^5-5\,Bt^3+5\,B^2t-2\,(51)=0,$$

the value of B is $(y_3 y_4)^{\frac{1}{5}}$ which is also a rational two-valued function of the ψx's, and of the form $P+\sqrt{Q}$ in the coefficients; and at the same time these two De-Moivrian forms are so connected that for each value of A of the form $P+\sqrt{Q}$ in the first of them, the corresponding value of B is $P-\sqrt{Q}$ in the second, P and Q being the same in both.

We thus learn that $a\,y_1^{\frac{1}{5}}+a^4y_2^{\frac{1}{5}}$ is a five-valued function by reason of the quadratic-radical which enters into it being restricted to one sign; and that $a\,y_3^{\frac{1}{5}}+a^4y_4^{\frac{1}{5}}$ is a similar five-valued form under the restriction that the radical has the other sign. In like manner we may observe that the five values of $a Y_1^{\frac{1}{5}}+a^4 Y_2^{\frac{1}{5}}$ are the roots of the De-Moivrian quintic, with coefficients symmetric functions of the z's,

$$t^5-5\cdot 256^{\frac{1}{5}}\,(41)\,t^3+5\cdot 256^{\frac{2}{5}}\,(41)^2\,t-2\,(51)^2=0.$$

46[a]. On further considering the theorems which profess to determine the roots of equations from the values of non-symmetric functions of those roots, I think it admits of some doubt whether the system Y in section (43), would, of itself, be sufficient for the complete ascertainment of the separate roots. But it is easy to see that the powers of the quartic resolvent are not exhaused by the results of that section, and that other equations can be procured by considering the individuals of each couple of the roots of the resolvent separately. Since $(y_1 y_2)^{\frac{1}{5}}$ is rational, when $y_1^{\frac{1}{5}}$ is of the form $R_1+\sqrt{R_2}$, then $y_2^{\frac{1}{5}}$ must be of the form $R_1-\sqrt{R_2}$, the R's being the same in both. Similarly, if $y_3^{\frac{1}{5}}$ is $R_1'+\sqrt{R_2'}$ then $y_4^{\frac{1}{5}}$ is $R_1'-\sqrt{R_2'}$.

Now, if we consider every possible value of $a^p\,y_1^{\frac{1}{5}}+a^q\,y_2^{\frac{1}{5}}$, we observe that there are five of them, and no more, which are rational and linear functions of the z's. These are $y_1^{\frac{1}{5}}+y_2^{\frac{1}{5}}$, and this expression multiplied by a, a^2, a^3, and a^4 successively.

A similar observation applies to the possible values of $a^p y_3^{\frac{1}{5}} + a^q y_4^{\frac{1}{5}}$.

We thus arrive at the following five systems of equations, writing t for $\psi x - b_1$,

$$\left.\begin{aligned} y_1^{\frac{1}{5}} + y_2^{\frac{1}{5}} &= \Sigma(At) \\ y_3^{\frac{1}{5}} + y_4^{\frac{1}{5}} &= \Sigma(Bt) \end{aligned}\right\}$$

$$\left.\begin{aligned} a\,(y_1^{\frac{1}{5}} + y_2^{\frac{1}{5}}) &= \Sigma(At) \\ a^4(y_3^{\frac{1}{5}} + y_4^{\frac{1}{5}}) &= \Sigma(Bt) \end{aligned}\right\}$$

$$\left.\begin{aligned} a^2(y_1^{\frac{1}{5}} + y_2^{\frac{1}{5}}) &= \Sigma(At) \\ a^3(y_{3\frac{1}{5}} + y_4^{\frac{1}{5}}) &= \Sigma(Bt) \end{aligned}\right\}$$

$$\left.\begin{aligned} a^3(y_1^{\frac{1}{5}} + y_2^{\frac{1}{5}}) &= \Sigma(At) \\ a^2(y_3^{\frac{1}{5}} + y_4^{\frac{1}{5}}) &= \Sigma(Bt) \end{aligned}\right\}$$

$$\left.\begin{aligned} a^4(y_1^{\frac{1}{5}} + y_2^{\frac{1}{5}}) &= \Sigma(At) \\ a\,(y_3^{\frac{1}{5}} + y_4^{\frac{1}{5}}) &= \Sigma(Bt) \end{aligned}\right\}$$

Let these systems be repeated for each of the twelve forms of t or $\psi x - b_1$ with the corresponding functions of y; and we shall then have 120 equations, which may be combined symmetrically in twelves, so as to produce the following systems, (S in this place denoting the sum of the twelve forms to which it applies): viz.,

$$\begin{aligned} S(y_1^{\frac{1}{5}}) + S(y_2^{\frac{1}{5}}) &= \Sigma(A\lambda x), \\ S(y_3^{\frac{1}{5}}) + S(y_4^{\frac{1}{5}}) &= \Sigma(B\lambda x); \qquad \cdots\cdots\cdots \quad y \end{aligned}$$

and the four derived from it by the use of the fifth-roots of unity as above. In this expression λx is a rational function of x of the fourth degree whose coefficients are rational functions of a_1, a_2, a_3, a_4, and a_5.

We cannot algebraically determine which of the five systems is to be employed for the given quintic. We only know that the five systems relate to the given quintic and the four conjugate quintics which, in point of resolution, necessarily accompany it.

The systems Y and y are clearly sufficient to determine the roots.

GENERAL REMARKS.

47. The number of values which a complete function of the roots of a quantic acquires by transposition of the roots is $1 \cdot 2 \cdot 3 \ldots n$; and in the solution of such a quantic we may expect to have ultimately that number of elements.

In our solution of the cubic, we have to employ as we have seen $2 \cdot 3$ equations, each of which contains one radical.

In our first, or imperfect solution of the quartic, we had $2 \cdot 3$ equations, each containing two radicals; in all twelve elements.

In our second, or complete solution of the quartic, we find $2 \cdot 2 \cdot 3$ equations, each containing two radicals, which produces 24 or the full complement of elements.

In our solution of the quintic, we have $2 \cdot 5$ equations, each containing $2 \cdot 2 \cdot 3$ radicals, which produces the proper number of elements, viz. 120.

In all our processes we have operated through a resolvent, *i. e.*, an equation whose roots enter into the algebraic expression of the root of the given quantic. We of course except the quadratic, as that is the lowest equation which can have a discriminant.

In the cubic, the resolvent is a quadratic with rational coefficients.

In the first method of dealing with the quartic, the resolvent is a cubic with rational coefficients.

In the the other method, it is ultimately a sextic with rational coefficients, capable of division into two cubics with quadratic-irrational coefficients.

In the case of the quintic, the resolvent proves to be of the 24th degree with rational coefficients, capable of separation into twelve quadratics having coefficients of the specifically irrational form which we have had to consider. In every case, the final term of the resolvent is the appropriate power, (the third in the cubic, the second in the quartic, and the fifth in the quintic), of a rational function of the coefficients of the original quantic; which enables us to infer that its roots will be equally applicable to the expression of the conjugate quantics as to that of the original quantic.

This consideration, whilst it limits or qualifies the meaning of the term resolution, is at the same time the very circumstance which renders resolution algebraically possible.

COMMENTS ON PRIOR RESEARCHES.

48. The results arrived at in this essay are at variance with those of Abel and Sir William R. Hamilton, in their elaborate investigations of this subject. The two results are not, however, diametrically opposed to each other, inasmuch as our conclusions rather verify theirs, provided we impose a certain condition. It must be admitted, however, that this condition is one of large import; and one which, in fact, we have not seen any reason for imposing at all. Abel's celebrated proposition is:—"If a root can be expressed as an "irreducible irrational function of the coefficients, every "radical which enters into the composition of the function "is expressible as a rational function of the roots; and "this rational function has the same multiplicity of value "by transposition of the roots, as the irrational has by "reason of the different roots of unity which are implied in "its radicals." Our results verify the truth of this proposition in every case in which the equation is resoluble and actually resolved through the medium of the linear transformation alone. In that case, the first radical in the order of calculation is the square-root of the discriminant, which is reducible to a *rational* form; and thereupon it follows that all the superior radicals in the root are also similarly reducible.

But we have seen, in reference to certain quantics, that there are methods of solution not operating exclusively through the medium of the linear transformation; and that in such cases the first radical in the composition of the root is the discriminant of an equation whose coefficients are not rational functions, but irrational functions of a certain specific character of irrationality, of the coefficients. In that case, the square-root of the discriminant can still be taken; but the result of such extraction is a function of the specific irrationality under consideration. Adopting this view, the first part of Abel's theorem should be; that of whatever order, degree and kind of irrationality, including rationality, the first radical that enters into the composition of the root is, when reduced as far as possible; of that same order, degree and kind of irrationality will also be all the other radicals which enter into the composition of the root, when similarly reduced; and with this qualification the second part of the theorem may be conceded, excluding from among the functional radicals which are supposed to have several values, those which we treat as essentially irreducible.

49. Sir William Hamilton's first important theorem is, in effect, that no method exists of resolving a quartic essentially different from those two methods which have been long known and universally adopted; which indeed he conceives to be (as in fact they are) in ultimate analysis substantially but one. Our results verify the theorem, but only on the assumption that we are dealing with such modes of resolution as can be arrived at through the linear transformation. We have found several essentially different modes of resolution, if we free ourselves from this restriction.

We have in fact found, as an equivalent for the general quartic, a system of conditioned quartics with irrational coefficients, through the discussion of which (without actually solving them) we can arrive at the solution of the original quartic.

50. The theorem that in any algebraic expression of a root, every radical is equivalent to a *rational* function of the

roots, is, I think, sufficiently disproved in the case of a quartic by the mere existence of the Tschirnhausen transformation, when applied to introduce the condition (12) = 0. That transformation teaches us that the two systems, one of which is

$$x^4 - 4\,a_1\,x^3 + 6\,a_2\,x^2 - 4\,a_3\,x + a_4 = 0,$$

and the other is

$$z^4 - 4\,b_1\ z^3 + 6\,b_1^2\ z^2 - 4\,b_3\ z + b_4 = 0\,;\ (z = kx^2 + lx),$$
$$z^4 - 4\,b_1'\,z^3 + 6\,b_1'^2\,z^2 - 4\,b_3'\,z + b_4' = 0\,;\ (z = kx^2 + l'x),$$

are equivalent, and therefore that we can express the resolution of the first system in the same radicals which we have to employ for the resolution of the two members of the second system, and that notwithstanding there may be another and simpler mode of resolution for the first system. But inasmuch as all quartics are resoluble by the ordinary process, (the first method of this essay), the two quartics in z are expressible by an algebraic formula in which the constituent is expressed in terms of b_1, b_3 and b_4, and is equivalent to a rational function of the z's. If then we substitute in this formula for b_1, b_3 and b_4, their respective equivalents in terms of a_1, a_2, a_3 and a_4, every function of b_1, b_3, and b_4 which appears in the root is turned into a corresponding function of the a's, each of the form $A + B\sqrt{I}$. Now, whatever may be the form of the radicals which enter into the solution of the quartics in z, we can always express the solution of the quartic in x, by means of them. The resolution of the quartic in x is therefore, independently of any process in this essay, shown to be expressible by means of radicals whose ultimate element is not a rational function of the a's or of the x's, but a rational function of the b's or of the z's, and therefore a quadratic-irrational function of the a's or the x's. It is proper, however, to observe, that we should not have known *à priori* that one form was not reducible to the other.

51. As it is essential to the due understanding of the theory of this essay that no ambiguity should be suffered to remain on this last mentioned point, I will at the risk of repetition, give some further explanations relative to the new solution, confining them for the present to the case of the quartic. In the ordinary mode of resolving the quartic, the root expressed in its simplest form, that is, after the removal of all parameters which admit of being removed, is of the following form:—

$$\frac{x-a_1}{\sqrt{S_2}} = \sqrt{1+\sqrt[3]{P+\sqrt{Q}}+\sqrt[3]{P-\sqrt{Q}}}$$
$$+\sqrt{1+a\sqrt[3]{P+\sqrt{Q}}+a^2\sqrt[3]{P-\sqrt{Q}}}$$
$$+\sqrt{1+a^2\sqrt[3]{P+\sqrt{Q}}+a\sqrt[3]{P-\sqrt{Q}}};$$

in which P and Q are two independent parameters, each being a rational but non-integral function of the coefficients, or a symmetric but non-integral function of the roots. They are non-integral because each of them contains a power of S_2 in its denominator.

Now, no doubt, in general, we do not alter the essential character of this form of resolution, by substituting throughout it for each x any function of x (say ψx) with or without irrational co-efficients. We merely obtain a more complicated representation of x, which, we may presume, must ordinarily be algebraically reducible to our original form. For to suppose otherwise, would be in effect to make the expression of the root almost arbitrary in form. But although this is so as a general proposition, yet we have found that there are certain special or singular forms of ψx, which operate in such a way as to alter what I have called the frame-work of the root, and which in this method of representing the root by means of two parameters has the effect of producing a zero in the denominators. The functions P and Q, the essential parameters of the root, cease to exist in that form; and we are obliged, and at the same time enabled, to represent the root in another and different manner, viz., as a function which contains only one parameter. By this method we obtain

$$\begin{aligned}\frac{\psi x - b_1}{\sqrt[3]{31}} &= \sqrt{\sqrt[3]{1+\sqrt{1+P}}+\sqrt[3]{1-\sqrt{1+P}}} \\ &+ \sqrt{a\sqrt[3]{1+\sqrt{1+P}}+a^2\sqrt[3]{1-\sqrt{1+P}}} \\ &+ \sqrt{a^2\sqrt[3]{1+\sqrt{1+P}}+a\sqrt[3]{1-\sqrt{1+P}}} \\ &= \chi P \text{ simply.}\end{aligned}$$

And although two parameters can thus be made to degenerate into one by an algebraic process, yet we cannot even conceive any reverse algebraic process, which can have the effect of carrying us back from the formula with one parameter to any formula which shall contain two independent parameters. For in whatever way we alter the constituents of P, χP is still a function of one symbol. These *singular* forms of ψ therefore conduct us to entirely new solutions of the quartic; solutions it must be observed more in analogy, (in a particular point of view,) with the known solution of the quadratic and cubic, than is the one which has been heretofore given; for we thus in all three cases reduce the problem to one of strict algebraic inversion.

We are therefore warranted in considering that Abel and Sir William Hamilton, either altogether omitted to weigh the possible effect of changes in the argument by means of transformation, or else too hastily concluded that all possible changes of the argument must necessarily lead to reducible results. We are also justified in introducing into Abel's theorem the qualification which I have suggested; a qualification which still leaves it a theorem of singular beauty and value, though it is no longer capable of being applied, in the manner in which it has been applied, to limit the power of algebraic expression in the resolution of equations. This limitation, if it exists, will depend upon and must be sought for upon other grounds.

52. The objection which I have established in the case of the quartic to Abel's theorem, applies equally in principle to that of the quintic, so soon as we learn, by means of the Tschirnhausen-Jerrard transformation, that there exists (in several forms) a special irrational function, the introduction of which in the manner pointed out in this essay brings the expression for the argument of the resolvent to a reducible form; *i. e.* reducible, not to rational functions of the x's or the a's, but to rational functions of the z's and the b's, and therefore to the specifically irrational functions of the x's and the a's which we have so introduced.

The first radical which enters into the root, so as to be capable of reduction is the square-root of the discriminant of the quintic in ψx expressed in terms of $x_1 \cdot\cdot x_5$; such square-root is not (as Abel's theorem implies) rational, but a cubic-quadratic-quadratic-irrational function of a given species.

The next set of radicals consists of the fifth-roots of the two values of Y; each of these is reducible in like manner. Thus Abel's theorem is verified except as to the form of the ultimate results. They are not rational, but they are all of the same irrationality.

53. If I have correctly apprehended the proof of Abel's

theorem in the form in which it is demonstrated by Sir W. Hamilton, it appears to me that the qualified mode of statement which I have suggested is the one to which the demonstration naturally conducts us. It is proved that the first reducible radical which enters into the expression of the root is the square-root of the discriminant, which square-root can always be extracted; and it is also proved that the superior radicals in the expression of the root are similarly reducible. The theorem demonstrated is, therefore, sufficiently stated by saying that all the radicals in the expression are reducible to the same form in point of rationality as the square-root of the discriminant. I do not find anything in the proof which necessarily implies that the discriminant and its square-root cannot be made irreducibly irrational even by means of transformations.

54. Further, it appears to me that Abel's proof as given in his own works, if read by the light of the Tschirnhausen-Jerrard transformation, would rather lead to the conjecture at least, if not to the conclusion, that the quintic has a quadratic resolvent. The argument of Abel proves, from very elaborate *à priori* considerations that the first radical operation to be performed must be a square-root, and the next a fifth-root. What he actually infers from this is, that there is a *reductio ad absurdum* in being obliged to equate such a function as $\left\{P+\sqrt{Q}\right\}^{\frac{1}{5}}$ which is ten-valued, to a linear function of the roots which is necessarily 120-valued. We now know that this was an erroneous conclusion, because we are able to reconcile this difference in multiplicity of value. We may *either* consider P and Q themselves as identically twelve-valued, in which case both sides have 120 values; *or*, we may consider that the function of the roots is reduced in multiplicity of value from 120 to 10, by reason of its being composed symmetrically of twelve different

forms of ψ, in which case both sides have 10 irrational values of the same kind of irrationality.

If Abel had been aware of the possibility of the transformation which we have employed, he would scarcely have failed to see that it gave an opportunity of escape from his *reductio ad absurdum;* and I should suppose that Sir Wm. Hamilton, who did know the transformation, was prevented from applying it in this manner in consequence of his argument having taken a rather different course from that of Abel in its latter part. The proper conclusions to draw from Abel's argument are: First, that if a quintic is algebraically resoluble, it has a quadratic resolvent; a fact which would at once have shown that the quintic, in order to be resoluble, must be capable of reduction without loss of generality to a quintic with two parameters; for the notion of a resolvent implies that the roots of the quantic are expressible in terms of those of the resolvent; and, secondly, that if the general quintic can be transformed into a quintic with two parameters and whose discriminant is (and therefore whose roots and coefficients are) susceptible of 12 values, then its resolution becomes possible.

CONCLUSION.

55. The leading ideas of this essay are the following :

1st. The method of proceeding by steps, based upon the notion that any algebraic expression for the root of a complete equation must contain in it the roots of an equation of some lower degree, being all rational functions of the roots of the proposed equation. This idea, which in its first conception is probably based on the fact, that the solution of a complete equation involves that of all equations of lower degree, suggests to us the *necessity* of discovering a resolvent, *i. e.* an equation whose critical functions correspond with those of the given quantic.

2nd. The special property of all effective resolvents, that they apply not exclusively to the given quantic, but to a set of quantics, which have a peculiar connection *inter se*, but no common roots. This circumstance, manifests itself in the fact that the root of the resolvent, when expressed in terms of the roots of the quantic, is a perfect power of another expression similar to it in point of rationality, the exponent of the power being the number of individuals in the set of quantics to which it must necessarily be applicable, if it is applicable to the given quantic. If we call this number p, we may say that the pth root of the root of the resolvent must be of the same form as the root itself, substituting for each symmetric function of the roots of the quantic of the degree mp, a rational function of the same roots of the degree m.

3rdly. The transformation of the quantic by means of the introduction of certain specifically irrational functions of the roots or coefficients of the quantic, which are of such a nature that they make an essential alteration in the form of the root, whilst at the same time they apply to the original quantic in its complete and unaltered form. This is the property which produces *uniqueness* in the expression which we obtain for the root of the resolvent when expressed in terms of the roots of the quantic,—a feature which could not exist if that root were expressed in terms of the coefficients or roots of any conditioned quantic whatever.

In general we can make no use of a resolvent unless we can express its roots in a perfectly unique manner; and expression in a unique manner is impossible except in terms of the roots or coefficients of some equation between whose roots or whose coefficients there exists no relation whatever. Thus, in the quintic we cannot express the root of the resolvent uniquely in terms of the x's; we must go back to the complete quintic in x before we obtain a unique expression.

We never, therefore, attempt to solve the trinomial form; we merely make use of the special properties which it possesses, *when it is so framed as to be a perfect equivalent to the complete quintic.*

4thly. The consideration that these transformations are in certain cases applicable in such a manner as to diminish the number of parameters of which the root is composed without affecting the generality of the problem; or, in other words, to alter the algebraic statement of the problem by reducing it from the form, "Given $\phi(u, v, w \,.\,.\,) = 0$, to find $u = \psi(v, w \,.\,.\,)$;" to the form of a simple inversion, "Given $u = \phi v$, to find ϕ^{-1} in $v = \phi^{-1} u$;" and to do this without qualifying its scope.

We have found that this can be done for all complete algebraic equations up to the fifth degree inclusive; or, in other words, that the problem of algebraic resolution can be fully disposed of in all cases in which it admits of a particular mode of statement. Beyond this point we cannot go in this direction; for we have arrived at the place where there is necessarily a fundamental change in the very statement and conditions of the problem. This I conceive is the real barrier to algebraic resolution: the impossibility of inventing any problem of two parameters, or any problem of simple inversion, which shall be an adequate representative of a complete algebraic equation of the sixth or higher degrees.

I therefore submit that this essay contains a theory of algebraic resolution which is complete in itself; and that the reason why we cannot proceed further, at all events in this direction, is, that the problem begins at this point to involve other and different considerations; and that so far as it admits of general algebraic treatment, the subject, as a problem of simple inversion, has been brought to its natural end.

SEQUEL, FURTHER EXPLAINING THE GENERAL PROCESS AND DEFINING THE EXTENT OF ITS THEORETICAL APPLICATION.

After the remarks made in section 35 in reference to the supposed reducibility of the particular equation there considered to the De-Moivrian form, it may perhaps appear unnecessary to apply to the result obtained in this section, observations of a similar purport. Nevertheless, the point is a very important one; and a correct appreciation of it is absolutely necessary to the full understanding of the methods developed in the essay.

It is of the essence of these methods that they must apply exclusively to the complete or general equation of the given degree. I say "exclusively" notwithstanding the fact that any *results* obtained must of course be applicable to *every* equation of that degree; for, although that is necessarily the case, yet the *process* of resolution is not intelligible in theory or possible in fact, except on the assumption that the equation proposed for resolution is complete of its kind.

We have indeed succeeded in solving one limited or conditioned quintic, viz: the De-Moivrian; but this resolution was in truth effected by the aid of an observation which may be called artificial, inasmuch as it depends upon an ocular inspection of two equations showing that they are identical. But speaking generally, we may say that the process of reasoning which we pursue compels us to discuss the equation to be resolved exactly as if it were the general equation of the same degree; or, in other words, as a case of a problem which in itself has the full number of parameters, and whose roots are wholly independent. The neglect of this consideration would lead to serious error. Thus it might be supposed that the result of sections 46 and 46^a would lead to the conclusion that the trinomial quintic $x^5+5a_4x-a_5=0$, is directly reducible to the De-Moivrian form. This, I need not say, is entirely out of the question. What then is it, that has been proved? It is this:—that if in any trinomial form $(z-b)^5+5\,(41)\,(z-b_1)-(51)=0$, b_1, (41) and (51), are such functions of the five parameters of the *general* quintic in x that the quintic in z is a transformation of, and therefore equivalent to, the general quintic in x, then we can also find an equation of the De-Moivrian form (as given at the end of section 46) whose roots have a specific connection with those of the quintic in z, and therefore with those of the general quintic in x.

The explanation of the difference between the false conclusion and the true one is derived from the circumstance that the process of transformation when applied to a general algebraic equation in such a manner as to diminish the number of parameters is a process which has no inverse either in fact or in conception. For example, if we transform the complete quartic with its four parameters to the form $z^4-4\,(31)\,z+(41)=0$, which has, considered by itself, only two parameters, we can express (31) and (41) in terms of a_1, a_2, a_3 *and* a_4. But if we were given a quartic $z^4-4b_3z+b_4=0$, we could not perform or even conceive any process which would turn it into a general quartic with four parameters; and it is the basis of all the reasoning of this essay that we are not to regard any conditioned equation except in connection with some general equation of the same degree of which it is a transformation; that is, in fact, *the* general equation of the degree under consideration.

The theory of transformation through which we operate implies that the equation to be transformed is the general equation of that degree; that is, we must deal with it as if it were quite general, even if in fact it is not. The reason of this is obvious. All the formulæ of transformation, the coefficients of the function which connects the new argument with the old one, and the coefficients of the new equation must be expressed primarily in terms of the roots of the given equation; by means of which we can then express them, if we desire it, in terms of the coefficients, whether such coefficients are in fact arbitrary or subject to conditions *inter se*. But the expression of these elements of transformation in terms of the roots requires that the given equation shall be regarded as general, for otherwise there would be no possibility of obtaining any *unique* expression for them. In other words, we cannot devise any theory or system for transforming one *conditioned* form of equation into another by means of one unique and universally applicable formula; but if we consider the given equation as being theoretically complete with its full complement of parameters, we can then deduce the necessary formulæ of transformation; which although unchangeable when expressed in the *roots* of the given equation would vary according to any conditions between the coefficients when expressed in terms of those coefficients.

There is one mode of expressing the root of any quantic (supposed to be algebraically resoluble) in terms of the roots which need not vary whatever conditions may exist between the coefficients. At the same time when conditions do exist between the coefficients the mode of expressing the root in terms of the roots may be varied indefinitely by means of these conditions. What I desire to point out clearly is that we are to resort to the former unique mode of expression, and not to any of the various forms of the latter; and thus we virtually treat the quantic, however it may be conditioned, as if it were complete, because we employ that form of its root which would equally serve for the complete quantic.

It will, now, I think, appear quite clearly that we cannot directly transform such an equation as $z^5+5b_4 z-b_5=0$, where b_4 and b_5 are two simple parameters, into a De-Moivrian quintic; but if b_4 and b_5 represent the complex forms which render the equation in z a transformation of, and equivalent to, the general quintic in x whose coefficients are the five arbitrary quantities a_1, a_2, a_3, a_4 and a_5, then we are able to express all further transformation of this quintic in a perfectly unique manner in terms of the coefficients or roots of the general quintic; and it is thus, and thus only, that we can succeed in proving that the general quintic, being reducible to Jerrard's form, is also reducible to De-Moivre's form; whilst at the same time Jerrard's form *simpliciter* is not so reducible.

I will give a specimen of the fallacious reasoning to which I refer, with a statement of the legitimate conclusion. Take the general trinomial equation

$$(z-b_1)^n+n\,b_{n-1}\,(z-b_1)-b_n=0,\ (n \text{ being prime}).$$

The discriminant of this is $b_n^{n-1}\,(n-1)^{n-1}\,b_{n-1}{}^n$; and it is easy, by following up the formula used in the quintic, to write down the De Moivrian $n-$tic which has the same discriminant. For if we take the quadratic in Y whose roots are

$$\left\{b_n^{\frac{n-1}{2}} \pm \sqrt{b^{n-1}-(n-1)^{n-1}\,b_{n-1}{}^n}\right\}^{\frac{1}{n}}$$

and then form the De-Moivrian whose roots are the n values of $aY_1+a^{n-1}\,Y_2$, (a being an unreal nth root of unity), we should then have a De-Moivrian in t; to which form, if we neglect the considerations put forward in this note, we might be apt to suppose the conditioned trinomial n – tic would be reducible.

The considerations here developed show that the reasoning leading to this conclusion is erroneous; and we should perceive the futility of the result, if we attempted to go through the actual process of transformation. Let us endeavor to form the values of Y, which ought to be rational functions of the roots of the quantic in $z - b_1$ of the degree $\frac{n-1}{2}$; and first we must find Y^n. This is, as we have seen, very readily expressible in terms of b_{n-1} and b_n; but what are the corresponding expressions for b_{n-1} and b_n in terms of the roots? The answer is, there is no unique expression for these coefficients in terms of the roots: for the $(n-2)$ conditions which exist between the roots would turn every function of them into an infinitely various expression. There is, therefore, no possibility of establishing a definite connection between Y or t and the z's.

Now let us consider it in the other point of view; and suppose that the trinomial quantic with $z - b_1$ for its argument is a transformation of, and therefore equivalent to, the complete quantic of the nth degree in x, with the n independent parameters $a_1, \ldots a_n$. We can then express Y_1 and Y_2 and therefore also the five values of t, in terms of the coefficients; for by hypothesis b_1, b_{n-1} and b_n are functions (complex indeed but still discoverable) of the n parameters $a_1 \ldots a_n$, and therefore expressible in a perfectly unique manner in terms of $x_1 \ldots x_n$.

We can then apply the reasoning of the essay to show that the square-root radical in the value of Y is actually reducible, as indeed we know *à priori* that it is so identically; and also that the nth root would be capable of extraction in the shape of functions of the $(\frac{n-1}{2})$th degree of that complex function of x which is denoted by z. And we could then form the De-Moivrian in t; which, even if it did not lead to resolution, would yet determine unsymmetric functions of the roots very likely to conduct to resolution.

Now, in point of theory, the general $n-$tic is reducible to the trinomial form by Tschirnhausen's process; though the practical limits of this reduction are very narrow, in consequence of the process requiring the algebraic solution of equations of which we cannot *à priori* say that they are resoluble. Neither, on the other hand, are we entitled to predicate that they are irresoluble; for although they are of higher degree than the given equation, yet they contain only a limited number of parameters, and may therefore be soluble in some artificial manner.

It will, perhaps, now be asked, what is the proper course to be adopted when the given quantic is not a complete quantic, but conditioned. Let us suppose, for example, that the quantic as proposed for solution is already of the above trinomial form. It may be urged, what is the use of reducing the quantic to the trinomial form by a complicated process, when it is already of that form to begin with. The answer to this question expresses the fallacy of any argument which would lead to the notion, that the trinomial form is directly reducible to De Moivre's form. It points out, in the clearest manner, the difference between $(z-b_1)^5+5b_4(z-b_1)-b_5=0$, considered as an individual case of a quintic, and the same form writing (41) for b_4, and (51) for b_5 where (41) and (51) are functions, such as we have heretofore employed, of the five parameters of a general quintic in x. The two equations are entirely dissimilar, even when in (41) and (51) we consider $a_1^2-a_2$ and $a_1^3-a_3$ as both zero, so as to make the original quintic trinomial. The one trinomial form is rational and one-valued; the other has coefficients which still continue to be cubic-quadratic-quadratic irrationals, and twelve-valued. The result therefore is, that even in this extreme case, it is necessary to go through the whole process of transformation precisely as if the given equation were a complete quintic. Of course, the expressions for Y and t in terms of the coefficients of the given quantic, would be very much simplified for such conditioned forms, but they would not be altered in multiplicity of value, nor would the expression of them in terms of the roots be in any away affected.

Probably the only form capable of solution without transformation is the De-Moivrian, in reference to which, as connected with the processes of this essay, we may say with truth, *solvitur ambulando*. We hardly regard it as an equation to be solved, but rather as a universal resolvent.

Although we cannot, in practice, reduce the general n – tic to the trinomial form above considered, it is of some interest to enquire how far the method of resolution developed in the essay would be applicable to it, if the reduction were really practicable. We are not to consider the subject as if our want of power to express the transformation arose from some *à priori* or theoretical impossibility. As a matter of *algebra*, the general quantic *is* reducible to the trinomial form; because we can find algebraic equations, the discovery of a root of which, in any way whatever, would produce a satisfactory result. Algebraically we have no need to go further, unless it should happen that the solution of the new equations arrived at would necessarily involve the solution of the given quantic, which, as a matter of fact, is obviously not the case; for we are entitled to regard a problem, which is not *per se* a problem of algebra as solved, when we have reduced it to a system of algebraic equations.

It may, perhaps, appear a strange thing to say, that the algebraic resolution of equations is not a problem of algebra. But, in point of fact, it is perfectly clear that such is the case. If we were to consider it as a problem of algebra, we could not propose it in any terms which would not *per se* imply that it was solved. It would be an extreme case of *solvitur ambulando.* If I take any algebraic problem, I have solved it when I have reduced it to an algebraic equation. Algebra goes no further, and when any one requires me to solve the equation, then he is giving me a *new* problem; and algebraically I have solved that as soon as I have reduced it to a system of algebraic equation, whose solution does not necessarily presuppose the solution of the given one.

I therefore say, that theoretically, the reduction of an $n-$tic to the trinomial form is possible; so that it becomes legitimate to enquire how far this would conduct us in the path of actual resolution. We, therefore, now suppose that the general $n-$tic in x is reduced to

$$(z-b_1)^n+(n-1)\,(n-1\ 1)\,(z-b_1)-(n\ 1)=0,$$

where z is of the form

$$hx^{n-1}+kx^{n-2}+lx^{n-3}+\ \ .\ \ .\ \ .\ \ \lambda x$$

and $\qquad h, k, l,\ .\ .\ .\ \lambda,\ (n-1, 1)$ and $(n\ 1)$

are capable of unique expression in terms of $x_1\ .\ .\ .\ x_n$ *exclusively.* We have scarcely the means of determining how many values each of these symbols would have. Theoretically, there appears no need to go beyond 1, 2, 3 . . . $(n-2)$ values. Probably, however, any practical method of effecting this object would produce twice this number of values.

The resolvent in y obviously is,

$$\left(y-(n\ 1)\right)^{n-1}-(n\ 1)^{n-1}+(n-1)^{n-1}\,(n-1,\ 1)^n=0,$$

and its roots can always be divided into $\frac{n-1}{2}$ sets of two, of the form,

$$(n\ 1)\pm 1^{\frac{n-1}{2}}\ \sqrt[n-1]{\Delta}$$

whose nth roots, according to the theory developed in the essay, must be of the form,

$$R_1 \pm 1^{\frac{n-1}{2}} \sqrt[n-1]{R_{n-1}}$$

multiplied respectively by the nth roots of unity.

Let, for example,

$$\begin{array}{ll} R_1' \pm \beta^1 \sqrt[n-1]{R'_{n-1}} & \text{be } y_1 \text{ and } y_2 \\ R_1'' \pm \beta^2 \sqrt[n-1]{R''_{n-1}} & \text{be } y_2 \text{ and } y_3 \\ R_1''' \pm \beta^3 \sqrt[n-1]{R'''_{n-1}} & \text{be } y_3 \text{ and } y_4 \\ \cdot \quad \cdot & \cdot \quad \cdot \\ \cdot \quad \cdot & \cdot \quad \cdot \\ \cdot \quad \cdot & \cdot \quad \cdot \\ R_1^{\frac{(n-1)}{2}} \pm \beta^{\frac{(n-1)}{2}} \sqrt[n-1]{R_{n-1}^{\frac{(n-1)}{2}}} & \text{be } y_{n-2} \text{ and } y_{n-1} \end{array}$$

then, if in reference to each of these we take every possible value of

$$a\, y_{2p-1} + a^{n-1}\, y_{2p}$$

for the n values of a, it is obvious that of these values which are n^2 in number there are n and n only which are rational and linear functions of the z's. These are $y_{2p-1} + y_{2p}$; and this expression multiplied successively by a, a^2, . . a^{n-1}. We therefore arrive at the following n system of equation, each system containing $\frac{n-1}{2}$ members.

$$\begin{aligned} y_1^{\frac{1}{n}} + y_2^{\frac{1}{n}} &= \Sigma\,(A\,t) \\ y_3^{\frac{1}{n}} + y_4^{\frac{1}{n}} &= \Sigma\,(B\,t) \\ \cdot \qquad \cdot \qquad & \cdot \\ y_{n-2}^{\frac{1}{n}} + y_{n-1}^{\frac{1}{n}} &= \Sigma\,(L\,t) \end{aligned}$$

and the same multiplied necessarily by a, . . a^{n-1}.

We then repeat this system for every possible form of t or $\psi x - b_1$, which we have from this reason supposed to be twice

1, 2 . . $(n-2)$; in order to produce in all (1, 2 . . n) equations. We then combine them systematically, in exactly the same manner as is pointed out in 46[a]; and we should arrive at n systems, solving indiscriminately the $n-$tic and the $(n-1)$ other $n-$tics, which in point of resolution necessarily accompany it.

We thus obtain the interesting result, that in order that a general quantic of prime degree should be algebraically resoluble, it is a necessary and sufficient condition that it should be capable of reduction to the trinomial form, a fact which is evidently intimately allied with Mr. Harley and Professor Boole's investigations as to the binomial form of the Differential Equations resulting from trinomial algebraic equations.

I think one may see an *à priori* reason for this necessity in the distinction to which I have alluded, between problems of algebra which are considered as solved when the proper algebraic equations are obtained, and problems of algebraic resolution. For a general quantic, its algebraic resolution is not a problem of algebra; for a trinomial quantic, the algebraic resolution is a problem of algebra: viz, the inversion of a function: finding t in terms of v from $v=\phi t$.

In the case of quantics of composite degree, the equation in y still holds good with a change of sign in one of the terms when n is even. Some variations would probably be required in the further investigations; but it is not necessary to go into their detail, partly because there is no probability of the reduction to the trinomial form being practically possible, and partly because the resolution of equations of prime degree implies that of all equations.

APPENDIX.

SINCE the foregoing essay has been put into type, I have obtained another method of discussing the quintic, which appears to open some new views on the general subject. I find that the quintic is capable of resolution, or some approach to resolution through the medium of a *cubic* resolvent, in the following manner:—

Inspecting the discriminant of the quintic at section (6), it will be easily seen that it can, in effect, be represented as the sum of a fifth power and a third power, under certain conditions existing among the elements (12), (13), (14) and (15). For if we make (12) = 0 and (14) = 0, it becomes

$$(15)\left((15)^3 - 3456\ (13)^5\right);$$

and as (15) cannot vanish except under very special conditions, we may consider that for such an equation the condition of two equal roots is

$$(15)^3 - 3456\ (13)^5 = 0.$$

Now we know that by an application of the Tschirnhausen-Jerrard transformation, the general quintic can be expressed in a form which contains only (13) and (15); or since the equation is a conditioned one, it will be better to write (31) and (51): (21) and (41) being each equated to zero.

We divide the condition (21) into two parts exactly as we did in section 44; and substituting the values in $(41)=0$, which by an extension of the notation in section 8 is

$$\overline{4444}\, h^4 + \overline{4443}\, h^3 k + \&c.,$$

we obtain the final ratios of h, k, l, and m by the solution of a quartic; so that altogether we determine them by the resolution of two quadratics and a quartic, and thus find that in order to effect the transformation completely it is necessary to have 16 forms of ψx; instead of the 12 which we had in section (44).

We have therefore 16 quintics of the form

$$(\psi x - b_1)^5 - 10\,(31)\,(\psi x - b_1)^2 - (51) = 0$$

in reference to each of which it is now easy to see that we can find a cubic of the form

$$y^3 - 3N_1\,(51)\,y^2 + 3N\,(51)^2\,y - N_3\,(31) = 0$$

which will have a discriminant vanishing simultaneously with that of the conditioned quintic. We do not in the absolute term insert a multiple of $(51)^3$ because it would vanish by a linear transformation.

Proceeding exactly as in section 36, but equating the discriminant of the cubic to the square of that of the quintic, we find $N_1=1$, $N_2=1$, $N_3=3456$: and the cubic is

$$\left(y - (51)\right)^3 + (51)^3 - 3456\,(31)^5 = 0,$$

or

$$y = (51) + \sqrt[3]{3456\,(31)^5 - (51)^3}.$$

We find then that this cubic so obtained has all the qualifications of a resolvent. Its last term is the fifth-power of a rational and symmetric function of the $(\psi x - b_1)$'s; and that function enters into the cubic only through that fifth-power; so that the equation in y must apply similarly to the five quintics obtained by using successively (31), $a(31)$, $a^2(31)$, $a^3(31)$, and $a^4(31)$ for the coefficients of the middle term; which is the same thing as employing successively $(\psi x - b_1)$, $a^3(\psi x - b_1)$, $a(\psi x - b_1)$, $a^4(\psi x - b_1)$, and $a(\psi x - b_1)$ for the argument of the quintic. Without repeating at length the argument so often employed, we infer, as before, that $y^{\frac{1}{5}}$ must be capable of actual extraction, in the form R_1 if the values of y be rational, and in the form $R_1 + \sqrt[3]{R_3}$ if the cube-root which appears in y is irreducible: one or other of which must be the case. The latter, no doubt, must be the correct view; it evidently tends in the direction of resolution, but I have not considered in what way it ought to be carried out.

APPENDIX No. II.

When I wrote this section in which Abel's theorem is represented as true in a qualified sense, and, as so qualified, still a theorem of importance, I was quite under the impression that my criticism upon it went as far as my own arguments and conclusions justified me in going. On further reflection, however, I think it quite clear that the theorem cannot be maintained in any sense as a statement of any real proposition connected with the resolution of equations. In short, so far as it is true, it is a truism, and amounts to nothing more than a circuitous definition of resolution. When we represent the root of a quantic in terms of symmetrical functions of the symbols which denote the roots, we learn nothing by being told that all the radical operations which enter into it can be actually performed by extraction; because if they could not be so performed, it would be impossible to reduce the expression to the linear unsymmetric functions of the roots denoted by $x_1, x_2 \ldots x_4$.

The formula $S_1 + \sqrt{S_2}$ cannot be x_1 and x_2 unless the square-root radical can be disposed of by extraction; the formula $S_1 + \sqrt[3]{S_3 + \sqrt{S_6}} + \sqrt[3]{S_3 - \sqrt{S_6}}$ cannot be x_1, x_2 and x_3, unless we can see that it is so identically by the extraction of the radicals, and so on. So far, therefore, as the theorem is true, it is a mere statement of an identical proposition.

APPENDIX No. III.

If we once adopt the definition of resolution in the sense which I have indicated on p. 105, we see an immediate necessity that all prime quantics, in order to be resoluble, must be reducible to a trinomial form. The trinomial form contains only one parameter really, and if we take the several forms by substituting successively ax, &c., for x, we arrive at the same results, though possibly not in the same order, as if we substitute a, &c., times the parameter for the parameter itself. No multinomial form, involving more parameters, can have this property. There is, however, one multinomial form of a prime $n$tic, which adapts itself to this theory of resolution, viz., the De Moivrian of that degree. This really possesses but one parameter, and has the same property of being accompanied by a system of other equations, all solved by the same formula. This affords a strong *à priori* reason for thinking that the trinomial form and the De Moivrian, have some conjugate relation between them; and, therefore, though I have not gone over the steps of Mr. Jerrard's attempt to prove the mutual reducibility of these forms, I feel strongly inclined to believe in the accuracy of results coinciding with views to which I have been led by a totally different process.

THE END.

R. CHAPMAN, PRINTER, DUBLIN.

Lectures on Quaternions: containing a Systematic Statement of a New Mathematical Method. With numerous Illustrative Diagrams, and some Geometrical and Physical Applications. By SIR WILLIAM ROWAN HAMILTON, LL.D., M.R.I.A., Andrews' Professor of Astronomy in the University of Dublin; Royal Astronomer of Ireland, &c. &c. 8vo, cloth boards, price 21s.

A Treatise on Heat. The Thermometer, Dilation, Change of State, and Laws of Vapours. By the Rev. R.V. DIXON, D.D., Rector of Clogherny, late Fellow of Trinity College, &c., &c. 8vo, cloth boards, price 12s. 6d.

The Elements of Plane and Spherical Trigo-NOMETRY, with the Nature and Properties of Logarithms, &c., for the use of Students in the University. By THOMAS LUBY, D.D., Senior Fellow of Trinity College, Dublin. Part I. Third Edition, corrected and enlarged, price 6s.

The Elements of Plane Astronomy. By JOHN BRINKLEY, D.D., late Bishop of Cloyne. Edited by THOMAS LUBY, D.D., M.R.I.A., Senior Fellow of Trinity College, Dublin. 8vo, cloth boards, price 12s.

Two Geometrical Memoirs of the General PROPERTIES OF CONES OF THE SECOND DEGREE AND OF SPHERICAL CONES. By M. CHASLES. Translated from the French, with Notes and additions, and an Appendix on the Application of Analysis to Spherical Geometry. By CHARLES GRAVES, D.D., Fellow of Trinity College, Dublin. 8vo, price 6s.

Elements of Optics. By HUMPHREY LLOYD, D.D., D.C.L., Vice-Provost of Trinity College, Dublin. 8vo, cloth boards, price 6s. 6d.

Tables for Facilitating the Calculation of Earthwork, in the cutting of Embankments of Railways, Canals and other Public Works. By Sir JOHN MACNEILL, LL.D., F.R.S., M.R.I.A., Professor of Practical Engineering, Trinity College, Dublin. 8vo, cloth boards, price 31s. 6d.

9