ESSAYS ON MATHEMATICAL EDUCATION

BY

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WITH AN INTRODUCTION BY

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INTRODUCTION

It has always been hard for people to judge with any accuracy the work of their own age, and it is hard for us to do so to-day. In spite of our optimism and of our certainty that we are progressing, what we conceive to be an era of great educational awakening may appear to the historian of the future as one in which noble ideals were sacrificed to the democratizing of the school, and the twentieth century may not rank with the sixteenth when the toll is finally taken.

It is, therefore, with some hesitancy that we should assert that we live in a period of remarkable achievement in all that pertains to education. That the period is one of advance is in harmony with the general principle of evolution, but that all that we do is uniformly progressive is not at all in accord with general experience. Certain it is that the present time is one of agitation, of the shattering of idols, and of the setting up of strange gods in their places. Nothing is sacred to the iconoclast, and he is found in the school as he is found in the church, in government, and in the social world.

Among the objects of attack in this generation is "the science venerable" that has come down to us from Pythagoras and Euclid, from Mohammed ben Musa and Bhaskara, and from Cardan, Descartes, and Newton. And yet it does not seem to be mathematics itself that is challenged so much as the way in which it has been presented to the youth in our schools, and to most of us the challenge seems justified. With all the excellence of Euclid, his work is not for the child; and with all the value of formal algebra, the science needs some other introduction than the arid one until recently accorded to it.

It is on this account that Mr. Carson's work in the English schools and before bodies of English teachers has great value. He is thoroughly trained as a mathematician, is a product of the college where Newton studied and taught, is a lover of the science in its purest form, and has had an unusual amount of experience in the technical applications of the subject; but he is a teacher by instinct and by profession, and is imbued with the feeling that mathematics can be saved to the school only through an improvement in our methods of teaching and in our selection of material. He stands for the principle that mathematics must be made to appeal to the learner as interesting and valuable, and he has shown in his own classes that, after this appeal has been successful, pupils need to be held back rather than driven forward in this branch of learning.

It is because of this feeling on the part of Mr. Carson that his essays on the teaching of mathematics have peculiar value at this time. They will encourage teachers to continue their advocacy of a worthy form of mathematics, at the same time seeking better lines of approach and endeavouring to relate the subject in a reasonable manner to the various other interests of the pupil. The problem is much the same everywhere, but the ties of a common language, a common spirit of freedom, and a common ancestry make it practically identical in English-speaking lands. On this account we, in the United States, feel that Mr. Carson's message is quite as much to us as to his own countrymen, and we shall appreciate it as we have appreciated the noteworthy work that he has already achieved in the teaching of mathematics in England.

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SOME PRINCIPLES OF MATHEMATICAL EDUCATION

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SOME PRINCIPLES OF MATHEMATICAL EDUCATION

Of all the problems which have perplexed teachers of mathematics in this generation, probably none has been more irritating and insistent than the choice of assumptions which must be made in each branch of the science. In geometry, in analysis, in mechanics, one and the same difficulty arises. Are we to prove that any two sides of a triangle are greater than the third? That the limit of the sum of a finite number of functions is equal to the sum of their limits? That the total momentum of two bodies is uninfluenced by their mutual action? And in every such case, on what is the proof to depend? A clear understanding of the answers to such questions, or, better still, a clear understanding of principles by which answers may be found, would go far to co-ordinate and simplify elementary teaching; the object of this paper is to state such principles and indicate their application.

Axiom, Postulate, Proof

It is first necessary to lay down definitions, as precise as may be possible, of the terms "axiom," "postulate," "proof." It is not implied that these definitions should be insisted on, or the terms used, in elementary teaching; nothing could be more likely to lead to failure. But a full comprehension of each is essential to every teacher of mathematics, and is too often lacking in current usage.

An axiom, or "common notion" in Euclid's language, is a statement which is true of all processes of thought, whatever be the subject matter under discussion. Thus the following are axioms: "If A is identical with B, and C is different from B, then C is different from A." "If B is a necessary consequence of A, and also C of B, then C is a necessary consequence of A." But "Two and two make four" and "The straight line is the shortest distance between two points" are not axioms, although they may be considered no less obvious. A statement is not an axiom because it is obvious, but because it concerns universal forms of thought, and not a particular subject matter such as arithmetic, geometry, and the like.

A postulate is a statement which is assumed concerning a particular subject matter; for example, "The whole is greater than a part" (subject matter, finite aggregates); "All right angles are equal" (subject matter, Euclidean space). It is essential to observe that, whereas an axiom is an axiom once for all, a postulate in one treatment of a science may not be a postulate in another. In Euclid's development of geometry, the statement that any two sides of a triangle are together greater than the third side is not a postulate, because it is deduced from other statements (postulates) which are avowedly assumed; but in many current developments it is adopted at once, without reference to other statements, and is therefore a postulate in such cases. To use an unconventional but expressive term, postulates are "jumping-off places" for the logical exploration of a subject. Their number and nature are immaterial; they may be readily acceptable, or difficult of credence. Their one function is to supply a basis for reasoning, which is conducted in accordance with the axioms. Postulates are thus doubly relative: they relate to one particular subject matter (number, space, and so on) and to one particular method of viewing that subject matter.

A statement which is deduced, by use of the axioms, from two or more postulates is said to be proved. There is thus no such thing as absolute proof. Proofs are related to the postulates on which they are based, and a demand for a proof must inevitably be met by a counter demand for a place to start from, that is, for some postulates. When a statement is said to have been proved, what is meant is that it has been shown to be a logical consequence of some other statements which have been accepted; if these statements are found to be incorrect, the statement which is said to be proved can no longer be accepted, though the logical character of the proof is in no way impugned. Thus the type of a proof is, "If *A*, then *B*"; relentless and final certainty surrounds "then"; but *A*, which is assumed in the "if," may nevertheless be utterly fantastic as viewed in the light of experience.

The Three Functions of Mathematics

The first application of mathematics to any domain of knowledge can now be explained. Starting from postulates, the truth of which is no concern of mathematics, sets of deductions are evolved by use of the axioms; agreement of the results with experience strengthens the evidence in favour of these postulates. If this evidence be deemed sufficient, as, for example, in geometry and mechanics, then deduction yields acceptable results which could not otherwise have been predicted or ascertained.

It is here that the prevailing concept of the power of mathematics ends; but such a concept presents a view of the subject so limited and distorted as to be almost grotesque. The process just described may be regarded as an upward development; a downward research is also possible, and no less valuable. It consists of a logical review of the set of postulates which have been adopted; in the result, either it is shown that some must be rejected, or the evidence in favour of all may be considerably enhanced. This review consists of two processes, which will be described in turn.

It is first necessary to ascertain whether the set of postulates is consistent; that is, whether some among them may not be logically contradictory of others. For example, Euclid defines parallel straight lines as coplanar lines which do not intersect, and proves in his twenty-seventh proposition that such lines can be drawn; for this purpose he uses his fourth postulate, which makes no allusion to parallels. If he had included among his postulates another, stating that every pair of coplanar lines intersect if produced sufficiently far, and had omitted his definition of parallel lines, his postulates would not have been consistent; for the twenty-seventh proposition proves that if the fourth postulate be granted, then the existence of non-intersecting coplanar lines must be admitted also. It is essential to realize that the contradiction implied in the term "inconsistent" is based on logic, not on experience; assumptions which are contrary to all experience are not thereby inconsistent. There is nothing in logic to veto the assumption that, for certain types of matter, weight and mass are inversely proportional; or that life may exist where there is no atmosphere, as on the moon. Such assumptions are not inconsistent with the other postulates of mechanics or biology; they are merely contrary to all experience gained up to the present time.

Here, then, is the second function of the mathematician — the investigation of the consistence of a set of postulates. And the task is not superfluous. Physical measurements are perforce inaccurate, and a set of inconsistent assumptions might well appear to be consistent with actual observations. More accurate measurements must, of course, expose the discrepancy, but these may for ever remain beyond our powers; logic renders them superfluous by demonstrating the consistence or otherwise of each set considered.

The next investigation concerns the redundance of a set of postulates. Such a set is said to be redundant if some of its members are logical consequences of others. For example, any ordinary adult will accept without difficulty the properties of congruent figures, the angle properties of parallel lines, and the properties of similar figures, as in maps or plans, regarding them as "in the nature of things." And electricians may, by experiment, convince themselves first, that Coulomb's law of attraction is very approximately true; and secondly, that within the limits of observation there is no electric force in the interior of a closed conductor. In neither the one case nor the other need there be the least suspicion that the statements are logically connected, so that they must stand or fall together. Yet so it is, and the fact is expressed by the statement that the assumptions are redundant.

The investigation of redundance, and the demonstration that sets of postulates are free therefrom, forms the third function of the mathematician. Its value, in connection with any subject matter to which it may be applied, may not at once be evident. It is based on the fact that all experiments, necessary and inevitable though they be, are nevertheless sources of uncertainty; it reduces this uncertainty to a minimum by removing the redundant assumptions into the category of propositions, and exposing the science in question as based on a minimum of assumption. And more; it can offer several alternative sets of assumptions for choice, that one being taken which is most nearly capable of verification. The labours of Faraday resulted in the offer of such a choice to electricians; either, they were told, you can base electrostatics (*inter alia*) on Coulomb's experiment, or on the absence of electric force inside a closed conductor; it is logically immaterial which course you adopt. The latter experiment, being far more capable of accurate demonstration in the laboratory, is chosen as the primary basis for faith in the deductions of electrostatics — a faith which is, of course, very much strengthened as such deductions are found to accord with our experience. But these considerations are for the physicist; the task of the mathematician is ended when he has put forward, for choice by the physicist, alternative sets of assumptions which are at once consistent with each other and free from redundance. In this way does he free the physicist, so far as may be, from the uncertainties of assumption, and assure him that no further increase of such freedom can be attained.[1]

[1] The antithesis between mathematician and physicist does not imply that the functions are of necessity performed by different individuals; it is used merely to enforce the argument.

Such is the range of application of mathematics to other sciences. When complete it reveals each science as a firmly knit structure of logical reasoning, based on assumptions whose number and nature are clearly exposed; of these assumptions it can be asserted that no one is inconsistent with the others, and that each is independent of the others. There is thus no fear that contradictions may in time emerge, and no false hope that one assumption may in time be shown to be a logical consequence of the others. Finality has been reached.

The acute critic may, of course, ask the mathematician whether his own house is in order. What is the precise statement of the axioms which are the basis of his science, and can they be shown to depend on a set of consistent mental postulates, free from redundance? Here it need only be said that the labours of the last generation have done much to answer these questions, and that their complete solution is certainly possible, if not actually achieved; to go further would be beyond the limits of this paper.

The Didactic Problem

The complete application of mathematics to any branch of knowledge being thus exhibited, the didactic problem can now be stated in explicit terms. In any given science—geometry, mechanics, and so on—what is the right point of entry to the structure, and in what order should its exploration be made? What results should be regarded as postulates, and should their consistence and possible interdependence one on another be investigated before upward deduction is undertaken? Should the minimum number be chosen on the ground that the pupil should at once be placed in possession of the ultimate point of view? Or should some larger number be taken, and if so, on what principles should they be chosen?

Bearing in mind that the pupils concerned are not presumed to be adults, it is easy to indicate principles from which answers to such questions may be deduced. One of the few really certain facts about the juvenile mind is that it revels in exploration of the unknown, but loathes analysis of the known. It is often said that boys and girls are indifferent to, and cannot appreciate, exact logic; that it is unwise to force detailed reasoning upon them. Few statements are farther from the truth. Logic, provided that it leads to a comprehensible goal, is not only appreciated, but demanded, by pupils whose instincts are normal. But the goal must be comprehensible; it must not be a result as easily perceived as the assumptions on which the proof is based. Let any one with experience in examining consider the types of answer given to two problems; one, an "obvious" rider on congruence, involving possibly the pitfall of the ambiguous case; the other, some simple but not obvious construction or rider concerning areas or circles. In the former, paper after paper exhibits fumbling uncertainty or bad logic; in the latter, there is usually success or silence, and more usually success; bad logic is hardly ever found. The phenomenon is too universal to be comfortably accounted for by abuse of the teachers; the abuse must be transferred to the crass methods which enforce the premature application of logic to analysis of the known, rather than to exploration of the unknown.

The natural order of exploration should now be evident. Let the leading results of the science under consideration be divided into two groups: one, those which are acceptable, or can be rendered acceptable by simple illustration, to the pupils under consideration; the other, those which would never be suspected and whose verification by experiment would at once produce an unreal and artificial atmosphere. Let the former group — which in geometry would include many of Euclid's propositions — be adopted as postulates, and let deductions be made from them with full rigour. Wherever possible, let the results of such deductions be tested by experiment, so as to give the utmost feeling of confidence in the whole structure. Later, when speculation becomes more natural, let it be suggested that gratuitous assumption is perhaps inadvisable, and let the meanings of the consistence and redundance of the set of postulates be explained. Finally, if it prove possible, let the postulates be analysed, their consistence and independence be demonstrated, and the science exposed in its ultimate form.

These second and third stages are even more essential to a "liberal education" than the first, for they exhibit scientific method and human knowledge in their true aspect. It is not suggested that they can be dealt with in schools, except perhaps tentatively in the last year of a long course. But it is definitely asserted that the general ideas involved should form part of the compulsory element of every University course, even though details be excluded, for they are of the very essence of the spirit of mathematics. The method of developing such ideas remains to be considered.

It may be presumed that the pupils concerned have some knowledge of arithmetic, geometry, the calculus, and mechanics, each subject having been developed from a redundant set of postulates. In which, then, of these four branches is it most natural to suggest the analysis of these assumptions?

Since analysis of the known may still be presumed to have its dangers, the branch chosen must be that one in which the investigation bears this aspect least prominently. Now the main ideas of arithmetic, geometry, and the calculus are so firmly held by boys and girls, that any attempt to discuss them in detail produces revolt or boredom. Such attempts account for much; the writer can well remember his feelings on first seeing a formal proof that the sum of a definite number of continuous functions is itself a continuous function; and at the same time he realized to the full that the proposition might well be untrue if the number of functions were not finite. Ground such as this is unfavourable for the development of this new analysis.

The same is by no means true of mechanics. Here the postulates, acceptable though they be, have been elucidated within the memory of the pupils, and they may reasonably be asked to examine the facility with which these assumptions were made, and to consider whether the evidence can in any way be strengthened. This being done, the ideas of consistence and redundance can be developed, and some idea of the structure of a science imparted. Even then it may probably be wise to lay little stress on analysis of the geometrical postulates; if the ideas are realized in connection with mechanics, we may well leave the seed to mature in minds to which it is congenial.

In the view of mathematics here taken, its various branches are regarded as structures with many possible entrances, and the discussion has been concerned with the choice of entrance and the route to be taken through the edifice. We cannot hope that our pupils will ever know more than the outline of each structure. Even we who are the guides cannot know each detail of any one; the labyrinth is too vast. But the best guide to a structure is he who knows its main outlines most completely, and a teacher who has clear ideas of mathematical principles can do much, in leading his pupils through such avenues of the structure as they can attain, to give them a view of the whole. Of the import and beauty of this view more need not be said.

INTUITION

(An address delivered to the Mathematical Association, and reprinted from *The Mathematical Gazette*, March, 1913)

INTUITION

If there be one duty more incumbent than any other upon mathematicians, it is to have a clear and common understanding of every term which they use. I do not say a formal definition, though that is most advisable if and when it can be obtained; but a class of entities must be known and recognised before it can be defined, and no term should be used unless it at least gives rise to definite, recognisable, and identical images in the minds of the speaker and listener. It cannot fairly be said that mathematicians are at fault in this respect, when dealing with their own special subjects; but I fear they cannot so easily be acquitted when discussing the didactic side of their work. Concrete and utilitarian, axiom and postulate, intuition and assumption; how many of us have definite meanings for these terms, and can feel certain that they represent the same meanings to others? The term which I have chosen as the title of this paper is one of the most commonly used and, as it seems to me, most often misunderstood; at the same time, the ideas and processes for which it stands lie at the root of all elementary teaching. I have therefore thought it worth while to discuss its meaning and to show the bearing of the process on mathematical education.

There is, I think, little doubt that to most of those who use the term "intuition," it connotes some peculiar quality of material certainty. Take, for example, the equality of all right angles, or the angle properties of parallel lines, and ask one who understands these statements with what degree of certainty he asserts their truth. It will be found almost invariably that he regards them as far more certain than statements such as "the sun will rise to-morrow morning" or "all men are mortal"; these, he admits, might be upset by some perversion of the order which he has regarded as customary, but the geometrical statements appear to be of the essential nature of things, eternal and invariable verities. So much, indeed, is this the case that the very idea of practical tests is grotesque. Who has ever experimented to ascertain whether, if two pieces of paper are folded, and the folds doubled again on themselves, the corners so formed are superposable? If the individual under examination be questioned as to the basis for this faith, he can only reply that it is the nature of things, or that he knows it intuitively; of the degree of his faith there is no doubt. It is to statements asserted in this manner that the term "intuition" is commonly applied; other facts, such as the mortality of all men, which are justified by the fact that all human experience points to them, are not classified under this heading nor, as I have said, are they accepted with the same faith.

These alleged certainties can of course be dissipated by purely philosophical considerations concerning the relations and differences between concepts and percepts; but "an ounce of practice is worth a ton of theory," and I propose here to show, mainly from historical considerations, that there is no ground for absolute faith in certain intuitions, however tenaciously they may be held. Take first the idea, still held by many, that a body in motion must be urged on by some external agent if its velocity is to be maintained. Until the time of Galileo this belief was held universally, even men of eminence who had considered the subject being convinced of its truth. Now this faith was of just such a kind, and just as strongly held, as the faith in geometrical statements which I have mentioned; it was, and still is by many, regarded as in the nature of things that a body should stop moving unless it is propelled by some external agency. And yet others, of whom Galileo was the forerunner, see the nature of things in a light wholly different. They regard it as utterly certain that a body can of itself neither increase nor retard its own motion. Ask a clever boy who has learnt some mechanics, or even a graduate who has not thought overmuch on the foundations of the subject, which he regards as more unlikely: that an isolated body should, contrary to Newton's first law, set itself in motion, or that the secret of immortality should be discovered. He will tell you that the second might happen, though personally he does not believe that it ever will; but that a body can *never* begin to move unless it has some other body "to lever against." We thus see two contradictory intuitions in existence, each held with equal strength.

Coming to more recent history, let me remind you of the development of the theory of parallels, and the rise of non-Euclidean geometries. Until the last century it may fairly be said that no one had ventured to doubt the so-called truth of the parallel postulate, though many eminent mathematicians had endeavoured to deduce it from the other postulates of geometry. The genius of Bolyai and Lobachewsky, however, put the matter in quite another light. They showed that a completely different theory of parallels was just as much in accord with the nature of things as that hitherto held; and that, to beings with more extended experience or finer perceptions than ours, this different theory might appear to correspond with observation while the current belief failed to do so. In other words, they showed that there are several ways of accounting for such space observations as we can make with our restricted opportunities; just as it was then well known that there were two theories which fitted the observations of astronomers, of which Newton's was the more simple and self-consistent.

It thus becomes clear that intuitions are no more than working hypotheses or assumptions; they are on the same footing as the primary assumptions concerning gravitation, electrostatics, or any other branch of knowledge based on sensation. They differ from these in that they are formed unconsciously, as a result of universal experience rather than conscious experiment; and they are so formed in regard to those experiences — space and motion — which are forced on all of us in virtue of our existence. It is not implied that their possessor is even fully conscious of them; ask some comparatively untrained adult how to test rulers for straightness, and he may be at a loss or give some ineffective reply; but suggest placing them back to back and then reversing one, and he at once assents. He regards this not as new information, but as something so simple and obvious that it had not occurred to him. It is to him the essential nature of things; he has held this view from so early an age, and it has remained so entirely free from challenge, that he revolts at the suggestion that things, viewed from another standpoint, may appear to have a different nature.

The formation of such working hypotheses is the normal method by which the mind investigates natural phenomena. After observation of a certain set of events, a theory is formed to fit them, the simplest being chosen if more than one be found to fit the facts equally well. This theory is developed, and its consequences compared with the results of further observations; so long as these are in accord, and so long as no simpler theory is found to accommodate the fact, the first theory holds the field. But, should either of these events occur, it is abandoned ruthlessly in favour of some better description of the recorded observations. There are famous historical cases of each event; Newton's corpuscular theory of light yielded deductions in actual disaccord with observation, and was therefore abandoned. The ancient theory of astronomy, wherein the stars were imagined to be fixed on a crystal sphere on which the planets travelled in epicycles, was abandoned in favour of the modern theory, not because it could not be modified to accord with observation, but because of its greater complexity. In every such case the question of absolute truth is irrelevant and beyond our reach; the problem is to find the simplest theory in accord with all the facts, abandoning in the quest each theory as a successor is found which better fulfils these requirements.

Shortly, then, we may say that intuitions are merely a particular class of assumptions or postulates, such as form the basis of every science. They are distinguished from other postulates first, in that they, with their subject matter — for example, space or motion — are common from an early age to every human being endowed with the ordinary senses; and secondly, in that no other assumptions fitting the sensations concerned ever occur to those who make them. Their formation is forgotten, and they are therefore regarded as eternal; they hold the field unchallenged, and are therefore regarded as inviolable.

Before passing to the consideration of the bearing of intuition on the teaching of mathematics, it may be well to illustrate what has been said by the consideration of a few particular cases.

First, suppose that one sees a jar on a shelf, and puts his hand up to find out whether it is empty. Is the act based on an intuition from the appearance of the jar? This is not the case; if asked before the act, one would not express any final certainty that the hand could enter the jar; it might have a lid or be a dummy. The individual can make more than one assumption which corresponds to the sight sensation; the first assumption made — that the hand can enter the jar — is merely the most likely as judged by experience.

Next suppose that a knock is heard in a room. The natural exclamation "What is that?" is based on intuition, for it expresses the now universal conviction that such a noise is an invariable accompaniment of some happening which, given opportunity, will also appeal to the other senses. Accumulation of human experience has led to the belief that such is invariably the case; but belief it is, and not certainty. If the reply were, "It is nothing; under no circumstances could you have correlated any sight or feeling with that sound," it would be received with complete incredulity.

Consider again the statement that, given a sufficient number of weights, no matter how small, one can with them balance a single weight, however large. No one would doubt this or treat it as anything but the most obvious of truisms, and yet it is a pure assumption, formed unconsciously as the result of general experience; it answers in every respect to our definition of an intuition. It may be thought by some that the statement can be proved arithmetically, but in every such alleged proof the assumption itself will be found somewhere concealed. We have, in fact, no warrant for assuming that the phenomenon called weight retains the same character, or even exists, for portions of matter which are so small as to be beyond our powers of subdivision.

Finally, consider the statement, "I knew intuitively that you would come to-day." In what respect do those who use it regard it as differing from, "I thought it was almost certain that you would come to-day"? It may fairly be said that the former expresses less basis of knowledge but more feeling of certainty than the latter; it means, "I don't in the least know how I knew it, but I did know beyond all doubt that you would come." Such ideas, with or without the use of the actual term "intuition," are common enough. They are here quoted to justify the statement, made above, that the term connotes to many of those who use it some peculiar degree of certainty. Such statements are not intuitions; they are mere superstitions, and those who are subject to them fail to realise how often they are unjustified by the event. Belief in the absolute truth of the angle properties of parallels or of the Laws of Motion is equally a superstition, though these are, until now, justified by the event. The truth is that they can never receive this absolute justification, for no material observation is beyond the possibility of error, nor can it be certain that some simpler theory will not be formed, accounting equally well for the observations; it is the belief in this impossible finality which constitutes the superstition.

Turning now to the more educational aspect of the subject, the first problem which confronts us is this: children, when they commence mathematics, have formed many intuitions concerning space and motion; are they to be adopted and used as postulates without question, to be tacitly ignored, or to be attacked? Hitherto teaching methods have tended to ignore or attack such intuitions; instances of their adoption are almost non-existent. This statement may cause surprise, but I propose to justify it by classifying methods which have been used under one or other of the two first heads, and I shall urge that complete adoption is the only method proper to a first course in mathematics.

Consider first the treatment of formal geometry, either that of Euclid or of almost any of his modern rivals; in every case intuition is ignored to a greater or less extent. Euclid, of set purpose, pushes this policy to an extreme; but all his competitors have adopted it in some degree at least. Deductions of certain statements still persist, although they at once command acceptance when expressed in non-technical form. For example, it is still shown in elementary text-books that every chord of a circle perpendicular to a diameter is bisected by that diameter. Draw a circle on a wall, then draw the horizontal diameter, mark a point on it, and ask any one you please whether he will get to the circle more quickly by going straight up or straight down from this point. Is there any doubt as to the answer?[1] And are not those who deduce the proposition just quoted, from statements no more acceptable, ignoring the intuition which is exposed in the immediate answer to the question? All that we do in using such methods is to make a chary use of intuition in order to reduce the detailed reasoning of Euclid's scheme; our attitude is that statements which are accepted intuitively should nevertheless be deduced from others of the same class, unless the proofs are too involved for the juvenile mind. We oscillate to and fro between the Scylla of acceptance and the Charybdis of proof, according as the one is more revolting to ourselves or the other to our pupils.

At this point I wish to suggest that a distinction should be drawn between the terms "deduction" and "proof." There is no doubt that proof implies access of material conviction, while deduction implies a purely logical process in which premisses and conclusion may be possible or impossible of acceptance. A proof is thus a particular kind of deduction, wherein the premisses are acceptable (intuitions, for example), and the conclusion is not acceptable until the proof carries conviction, in virtue of the premisses on which it is based. For example, Euclid *deduces* the already acceptable statement that any two sides of a triangle are together greater than the third side from the premiss (*inter alia*) that all right angles are equal to one another; but he *proves* that triangles on the same base and between the same

[1] There is often apparent doubt; but it will usually be found that this is due to an attempt to estimate the want of truth of the circle as drawn.

parallels are equal in area, starting from acceptable premisses concerning congruent figures and converging lines. The distinction has didactic importance, because pupils can appreciate and obtain proofs long before they can understand the value of deductions; and it has scientific importance, because the functions of proof and deduction are entirely different. Proofs are used in the erection of the superstructure of a science, deductions in an analysis of its foundations, undertaken in order to ascertain the number and nature of independent assumptions involved therein. If two intuitions or assumptions, *A* and *B*, have been adopted, and if we find that *B* can be deduced from *A*, and *A* from *B*, then only one assumption is involved, and we have so much the more faith in the bases of the science. Herein lies the value of deducing one accepted statement from another; the element of doubt involved in each acceptance is thereby reduced.

Next, to justify the statement that intuition has been attacked. Both Euclid and his modern rivals knew well enough that their schemes must be based on some set of assumptions; they differed only in the choice. Each agrees that intuitive assumptions are undesirable, but the modern school regards the extreme logic entailed by Euclid's principle of the minimum of assumption as impossible for young pupils. There is, however, a third school which pursues a different course; it professes to replace intuition by experimental demonstration. Pupils are directed to draw pairs of intersecting lines, measure the vertically opposite angles, and state what they observe; to perform similar processes for isosceles triangles, parallel lines, and so on. Instead of being asked, "Do you think that, if these lines were really straight, and you cut out the shaded pieces, the corners would fit?" they are told to find out, by a clumsy method, a belief which they had previously held, though it had never, perhaps, entered definitely into their consciousness. The question suggested is, in these homely terms, just sufficient to bring the idea before them, and it is at once recognised as according with the child's previous notions; he does not regard it as new, but merely as something of which he had not before thought so definitely.

It is this type of exercise in drawing and measurement which I regard as an attack upon intuition. It replaces this natural and inevitable process by hasty generalisation from experiments of the crudest type. Some advocates of these exercises defend them on the ground that they lead to the formation of intuitions, and that the pupils were not previously cognisant of the facts involved. But in the first place, a conscious induction from deliberate experiments is not an intuition; it lacks each of the special elements connoted by the term. And as to the alleged ignorance of the elementary idea of space, it appears to me to be a mistaken impression, based on undoubted ignorance of mathematical terminology. If you say to a child of twelve, "Are these angles equal?" he has to stop to think first, what an angle is, and next, when angles are equal; by the time he has done this his mind is incapable of grasping the peculiar relations of the angles in question, and he is labelled as ignorant of the answer. The real difficulty, and it is not a small one, is to lead the child to express familiar facts in precise mathematical terminology; to say "angles equal" rather than "corners fit." Until this terminology is thoroughly familiar, the effort of using it must absorb a large part of the child's attention, leaving little available for the matter in hand. This paper is not concerned with the methods or practice of teaching, but I would strongly urge all those who are concerned with young children to guard against this danger, by constant transition to and fro between common and technical phraseology, appealing at once to the former at the least sign of doubt or hesitation.[1] The learning of technical terms should not appear as part of the definite work, or it will inevitably be regarded as the major part; it should come incidentally and by gradual transition, as I have suggested.

The only alternative to this evasion or suppression of intuition is to accept it from the commencement as the natural basis for primary education. But to be of any avail, the acceptance must be unquestioned and complete; every intuition which can be formed by the pupils must, without suggestion of doubt, be adopted as a postulate, none being deduced from others which are themselves no more easy of acceptance. Such a course leads, it need hardly be said, to considerable simplification in the early treatment of any subject. For example, in geometry the angle properties of parallel lines, properties of figures evident from symmetry, and the theory of similar figures (excluding areas) appear as postulates; in the calculus it is not proved that the differential coefficient of the sum of a finite number of functions is equal to the sum of their differential coefficients; the statement is illustrated by, say, consideration

[1] It is no good to say, "Come now, what is an angle?" Appeal first to the tangible fact in the child's mind by saying, "Cannot you see that those corners *must* fit?" and then remind him that "equal angles" merely means the same thing.

of some expanding rods placed end to end, and at once commands acceptance. Here the question of terminology again arises; I have often been struck, in teaching schoolboys and students, by their slowness to accept this and similar results in the calculus; the clue was given to me by a boy who remarked that it was taking him all his time to remember what a rate of increase was, and he could not manage any more at the moment. Since that time I have avoided many seeming difficulties with elementary and advanced pupils by appeal from technical to familiar terms, always of course rephrasing the result in the proper form before leaving the matter in hand.

It will, I know, be thought by many that this adoption of all natural intuitions involves an appalling lack of rigour. But I would ask those who are of this opinion to do one thing before passing judgment, and that is, to define and exemplify with some care the meaning of the term "rigour." When they have done this, I think they may be disposed to agree with the answer to their accusation which I am now going to put forward. It is that the scheme suggested is perfectly rigorous, provided that every deduction made from the postulates adopted is logically sound; on the other hand, it is admitted that the mathematical training thus imparted is not complete, because no attempt has been made to analyse these intuitive postulates into their component parts, showing how many must perforce be adopted in the most complete system of deduction. In other words, we may be rigorous in regard to logical reasoning, or in regard to lessening the number of assumptions which form the basis of a science. The view for which I contend is, that in all stages of mathematical education, deductions from the assumptions made should be rigorous; but that in the earlier stages every acceptable statement or intuition should be taken as an assumption, the analysis of these, to show on how small an amount of assumption the science can be based, being deferred.

To avert misapprehension, let me say again that I propose that, when all intuitions are accepted as postulates, this should be done without question or discussion other than that necessary to give them some precision. To embark on a discussion of their nature, or to appear to cast doubt upon them, would be fatal, as fatal as has been the apparently futile process of deducing one accepted statement from another. The pupil is already in possession of a body of accepted truth; let us build on that and defer its analysis, or anything that pertains thereto, until he is sufficiently mature to appreciate the motive.

The first course of mathematics would, then, range from arithmetic and analysis through geometry to mechanics. In this last subject there is little scope for intuition. Most of the mechanical intuitions formed by the race as a whole have been mistaken, and it is just this fact which gives some indication of the proper commencement for the second course, in which the intuitive postulates are to be analysed and reduced as far as possible. Let the student learn something of the history of mechanics, realising that ideas which he regards as impossible and absurd were held, by men of great eminence, with faith just as strong as that which he places in his geometrical postulates. Then let it be suggested to him that this renders care in regard to assumption of vital importance, and so commence an analysis of the mechanical postulates, hitherto redundant, obtaining deductions of one from another to show their inter-connection. This completed, and the task is not a large one, it is natural to suggest that the postulates of geometry deserve some examination, and so, according to the time available and the ability of the pupil, we may pass backward through a review of the foundations of geometry to an examination of the foundations of analysis and arithmetic. It is not, of course, implied that every student of mathematics can reach this goal; few can ever get beyond some consideration of the foundations of geometry, with a clear understanding of the end to be attained in its general application to all sciences. But I do wish to put forward, with such emphasis as I can, this general scheme of mathematical education; namely, an upward progress, based on intuition, from arithmetic through geometry to mechanics, followed by consideration in the reverse order of the foundations of each branch, the upward progress constituting the first course, and the downward review the second course. It would, I believe, give an intelligible unity to the whole subject, and would do something to restore that purely intellectual appreciation which has so largely declined during the past generation.

Mathematics is a useful tool, but it is also something far greater, for it presents in unsullied outline that model after which all scientific thought must be cast. I have endeavoured to show how this outline may be developed, starting from those intuitions which are common to us all, and ending in an analysis demonstrating their true nature. The concrete illustrations, so necessary and illuminating in elementary teaching, are so many draperies, fashioned to render this outline visible to those who cannot otherwise appreciate it. Even the several branches—analysis, geometry, mechanics—serve the same end; behind them all is the one pure structure of mathematical thought. They who most appreciate the structure will best fashion the draperies, and so render it most clearly visible to those whom they instruct.

THE USEFUL AND THE REAL

THE USEFUL AND THE REAL

Among the many changes in mathematical education during the last twenty years, and among the many and often conflicting ideals which have directed these changes, one element at least appears throughout; a desire to relate the subject to reality, to exhibit it as a living body of thought which can and does influence human life at a multitude of points. The old scholastic ideal of development in the most abstract way, the realities being allowed to take care of themselves, is exploded for this as for most branches of education; it is recognised that the separated mediæval worlds of thought and action must be replaced by a single world wherein each exerts profound influences on the other. Our children must learn to think, and to think about the world as it now is and the manner of its evolution. Some few there may be who can with profit to us all devote themselves to one or other side of this world of thought and action, but the mass of men must be fitted to play their part between the two.

So far all are agreed, but community of pious opinion has before now been known to result in discord, and discord none the less acute because due to diversity of policy alone. Such has been the case with mathematical education; the community of ideals just described has not resulted in community of action; it is more nearly true that each man is a law unto himself in his method of forwarding them. Like most disorganised armies we have our shibboleths, and among the most prominent are "real," "useful," "concrete." An examination of what these do and should represent may not be without profit.

Starting from agreement that the world of thought is to be related to the world of action or reality (not thereby dependent upon or limited by that world), the natural course is to attempt to form some concept of the particular world of reality with which we are concerned. Suppose that one desires to explain the principles of the calculus to an assemblage of doctors; tables of population, mortality, and the like form one obvious world of reality from which thought can be developed; if to an assemblage of merchants, statistics of trade and finance would form such a world, and so on for other avocations. But suppose that the assembly consisted of men engaged in no one pursuit; the difficulty would be greatly enhanced, for there would be no obvious world of reality common and familiar to them all. So also in his dealings with young children must the teacher of mathematics determine fitting worlds of reality and develop his instruction for them.

But what is reality? What considerations determine the entities which have this attribute? For me, my hands, my furniture, this town, England, Cromwell, Macbeth, the binomial theorem, are all real; but Cromwell's hands, the furniture in a strange house, Hepscott (I take the name at random from a gazetteer), Fanning Island, Ben Jonson, Hedda Gabler, nitrates—all lack reality. Each one of the first is related to some definite recognisable sensation or concept of my own, but each of the second is (for me) a mere name which bears no relation to any such sensation or concept; I know nothing of Fanning Island, nor sufficient of Ben Jonson to distinguish him from other writers of his time; I have not read Ibsen, and I know little chemistry. The essence of reality is thus found in definite recognisable percepts or concepts, and is therefore a function of the individual and the time; what is real to me is not necessarily real to another, and much that was real to me in childhood is no longer so. It is for the teacher to determine the realities of his pupils and exemplify mathematical principles by as many as are suitable for the purpose. He will also find it necessary to enlarge their spheres of reality, but he must avoid confusion between a name and a thing; he must, for example, make sure that his pupils know what a parallelogram is before they use the name.

It is at this point that various policies have arisen, despite general agreement on ideals. One of these confuses the many worlds of reality, different for each individual, with some absolute world of reality supposed to be common to all. This absolute world is usually based on those applications of mathematics which have some commercial or scientific utility, such utility being considered to involve reality for the pupil. The result of this confusion of the useful with the real is seen in problems which deal with such mysteries as resistance in pounds per ton weight, the extension of helical springs, efficiency and load, ton-inches of twisting moment. To all children (and many adults) these phrases are as meaningless as the symbols of the purely scholastic algebra of thirty years ago; they are merely a cumbrous way of writing the x and y of that algebra and imply as little to those for whom they are intended. But their use may tend to impart an idea that realities are being dealt with — an idea thoroughly vicious in that it replaces entities by words. We may name entities which are direct sensations, and we may name entities which are pure creations of the imagination; but to imagine that a name which is co-related to neither sensation nor imagination possesses any sort of reality is the grossest of errors. Too many teachers are content to use words for which they have no definite meanings, and to allow their pupils to imagine that they have acquired something in learning such words; but we need not go out of our way to spread this error, the more so as we are concerned with the one subject which should suppress it most completely.

It is not, of course, suggested that any existing courses of mathematics are limited to such applications alone. But there is an obvious tendency to judge applications by such standards, attributing more and more importance to those which accord with them. An excellent instance of such judgments is the condemnation of the traditional problems dealing with tanks which are emptied and filled simultaneously by different pipes. It is argued that no adult ever deals with a cistern in this way, and that the problems should therefore be replaced by others having more reality. The term "reality" begs the whole question, for it has no absolute meaning for all people at all ages and must be defined by those who use it. And it is here confused with utility, a very different attribute. The essence of the contention is that no application should be used in education unless it is of actual use in some branch of science or walk of life. This is a far cry from the pupil's world of reality; the formalist attempted to transport him to a world of abstract thought wherein the entities are typified by letters x and y, but our utilitarian proposes to limit the play of his imagination to matters used by adults, no matter how far these may be removed from his cognisance or interest. There is at bottom little difference between the two, but the formalist is the more open in that he does not cloak his meaning under a mass of words which are full of sound and signify nothing.

A variant of this school may reply that they are being unjustly accused; that they are in entire accord with the rejection of matters such as voltage and twisting moment on the ground that they have no reality for the pupils concerned, but that there are plenty of applications which are real and also useful. This may be so, though examination of modern text-books hardly supports the claim; but in any case we cannot on such lines develop mathematical thought from any large portion of the pupil's world of reality; it is related to those parts only of that world which coincide with the worlds of various adults, and these may well be neither the most interesting nor the most familiar portions of his own world.

A second policy, exemplified for the most part in connection with geometry, interprets the child's world of reality as the world of his senses, and more particularly the senses of sight and touch, and so is allied with the concrete rather than the useful. It endeavours to develop thought from manipulations and measurements performed by the pupil himself, and is thus limited to the perceptory or concrete portion of his realities. In itself and so far as it goes this is an entire and most valuable gain as compared with the practice of thirty years ago; the pupils feel that they are dealing with matters within their own personal cognisance instead of abstractions which are evidently familiar only to men with whom, intellectually, they have and will have little in common.

Unfortunately, however, there has been a strong tendency to limit the work to this concrete domain, refusing any part to that world of imagination which is, especially in children, just as real and a great deal more vivid. Introductory courses of geometry consist of the construction and measurement by the pupil of figures whose dimensions are prescribed. They develop a detailed knowledge of perceptory space but make no use of that much larger and more important conceptual space wherein the creations of his imagination move and have their being. Travels, adventures, romances, history, and the hundred and one utterly useless but apparently practical things which interest a boy are situated in this space, and here, as well as in the smaller concrete space of the senses, should the world of thought be exemplified, for these things also are realities for him.

Three distinct policies have now been discussed: the first rejects all applications and insists throughout on development in abstract terms; the second insists that illustrations must be drawn from applications which are relevant to some branch of science, industry, or commerce; and the third insists that development must originate in the immediate evidence of the senses. Of course, no man or body of men holds one of these views to the entire exclusion of the other two, nor is the world of imagination entirely ignored in current practice. But most text-books and writings on mathematical education are influenced mainly by some one of them, and may be placed in a class which holds that particular policy as paramount. There is most in common between those who hold the second or third view, for they give a common allegiance to the use of reality and differ only in the scope of the term. Many teachers are, indeed, influenced by considerations of utility in algebra and considerations of concrete reality in geometry, their utterances on one subject often contradicting those on the other.

Now among the many uncertainties and conflicts which surround these (as all) questions of education, two statements at least stand out as certain beyond dispute. The first is that the operations and processes of mathematics are in practice concerned at least as much with creations of the imagination as with the evidences of the senses; it is enough to mention points, complex numbers, ether, electric charges, to make this plain. The second is that the purpose of mathematical education is to put the pupil in a "mathematical way"; to permeate his whole being with the elementary principles of the science so that he will apply them spontaneously in considering any matter to which they may be relevant. The formalists held that if principles were imparted in their utmost generality, each individual could and would make such applications as he might require, a statement not justified by experience and not in accord with such knowledge of the mind as we possess; the moderns believe that principles can only be seen by their exemplification throughout the world of reality of the pupil. The formalists thus seek unity of treatment for a class in generality of presentation, the moderns seek it among the experiences and concepts of the various pupils.

Fortunately for education in general, this modern search is certain to prove successful as regards children, because their experiences and imaginations run in grooves more or less alike ; they are interested in puzzles, hidden treasures, travels, railways, ships, and the like, and problems concerned with these entities are real to them no matter how absurd they appear from the standpoint of practical life. Their educational utility is not to be measured by their commercial or scientific value, but by their degree of reality for the pupils under instruction.

Putting the matter in more or less mathematical phraseology, we may say that the mathematical instruction of a beginner must be exemplified by a maximum number of his realities in order that the principles may permeate his whole being; in dealing with a class we must therefore find the greatest common measure of their realities and work from that. If the class is composed of adults having varying antecedents, this common measure may be small compared with the realities of any one member; but if the pupils are children, it is large in comparison with their individual realities, and the task of the teacher is correspondingly simplified.

Leaving generalities which may have appeared somewhat vague, we may now consider a few problems which are real for children but not directly useful to them or any one else.

First take the type already mentioned, which deals with the emptying and filling of a tank. There is no doubt that this is sufficiently real for any child ; he can visualise the whole process, and its value is increased because the entities are imagined and not perceived through the senses. The purpose served is the exemplification of the method of adding or comparing several rates by reducing all to a common unit, an idea sufficiently important in after life.

Those who attack such an illustration must find others which will serve the same purpose and satisfy their test of utility, and in doing this they will in all probability pass beyond the limits of reality. There is no doubt that the emptying of cisterns, the coincidence of clock hands, and other seeming trivialities do exemplify the handling of rates in ways which are more real to young children than others which have more actual utility, and they are therefore to be welcomed rather than condemned. The mistake in their treatment, and as gross a mistake as could well be made, has been their grouping by subject matter instead of principle. All questions which deal with one principle should be grouped together and the subject matter varied continually.

Next consider the Progressions, which have of late been attacked on the score that they are comparatively useless in mathematics or anywhere else. This is true, and advocates of their retention have done their cause no good by saying vaguely that they have their uses and then failing to give specific instances. They do, however, provide a number of problems which have reality for children, and they exemplify three most important matters: the concept of a series, the value of which extends far beyond mathematics; the insight which can be gained by a proper grouping of various entities; and the construction of a formula or law to cover any number of discrete cases.

Consider again the well-known problem in the calculus of a man who is on a common and wishes to reach a point on a straight road, along which he can walk more quickly than on open country, as soon as possible. If such a problem ever has practical utility, it is not for one man in ten thousand, and to regard it as in any way generally useful is obviously grotesque; but again it is real for those who study it, and it exemplifies the comparison of different modes of transition from one state to another, and the selection of the most suitable.

Another illustration is provided by the use of statistical graphs in the introduction of the calculus. Such graphs are of service in exemplifying the meaning of a differential coefficient and a definite integral by means which possess reality for the students, and their whole function is described in this statement. Now it may well happen that a set of statistics which have no practical use of any kind, or are even in actual disaccord with the results of some branch of science or industry, may serve these purposes better than others which have some direct use or are in accord with experience. For example, excellent problems can be made concerning the consumption of coal by locomotives, but they would never occur in the practice of any engineer, nor would the numbers which happen to give good graphs occur in the working of any conceivable locomotive. But this is in no way to the detriment of the problem for the purposes of instruction. The inaccuracy of the information contained in the figures is surely immaterial if students are told that actual numbers can be found in any handbook for engineers should they ever chance to need them, and no other objection seems relevant to the purpose of the problem. It exemplifies principles through illustrations which are real for the particular students,[1] and thus fulfils its aim.

These examples exhibit the tests by which applications should be judged. They must exemplify those leading ideas

[1] They would not be real for a class of locomotive engineers, and the example would not be used for such a class.

which it is desired to impart, and they must do so through media which are real to those under instruction. The reality is found in the students, the utility in their acquisition of principles.

The outcome of our discussion is, then, that illustrations must above all be real; they must be useful as well, if that be possible, and particularly with reference to other branches of study such as physics; but reality is the crucial test. And reality is a function of the individual and the time, so that no absolute schedule of the more and the less real can be devised; but there is sufficient community between children of the same age to handle them in groups, while adults might, on the other hand, require classification in regard to their realities before they could receive efficient instruction in groups. Many problems which interest and even excite children are to them hopelessly banal, and others must be used more in touch with their particular spheres of reality. Finally, each principle must be exemplified in as many ways as possible so that unity may be perceived in principle rather than subject matter.

We have travelled far from the useful applications of mathematics in our quest for fitting illustrations; we have been led to consider reality as the proper criterion, and to recognise that the term is essentially relative. But so also is "useful" a relative term; what is useful for one purpose is useless for another, and it may well be said that many applications of mathematics which are grotesquely useless in any branch of science or commerce are of the utmost use in education for their vivid illustration of ideas so abstract as to be otherwise vague or invisible.

SOME UNREALISED POSSIBILITIES OF MATHEMATICAL EDUCATION

(An address delivered to the Mathematical Association, and reprinted from *The Mathematical Gazette*, March, 1912)

SOME UNREALISED POSSIBILITIES OF MATHEMATICAL EDUCATION

The last half-century has seen a great and significant change in the popular estimation of mathematics. Formerly the subject was regarded as utterly unpractical and therefore useless in the narrow sense of this term, though it was recognised as providing a training unique in its character, in logical thought and in accurate expression. Now it is regarded, and correctly regarded, as having enormous practical importance in science and engineering. Most, if not all, of those discoveries and inventions which are so profoundly modifying civic and national life have found their origin, or development, or both, in the labours of mathematicians, and this fact is widely known. The mathematician is no longer regarded as a dreamer of dreams; he is classed with the doctor, the engineer, the chemist, and all those whose specialised labours have had immense import for the human race.

But simultaneously a change of no less magnitude has taken place in the mathematical world. The type of investigation which bore such fruit in the hands of Faraday, Clerk Maxwell, Kelvin, and many others no longer occupies the attention of those who are in the forefront of mathematical investigation. The theories of pure number, of space, of functions, and such names as Dedekind, Cantor, Grassmann, Klein, and, in our own country, Hobson, Whitehead, and Russell, have little or no connotation for the outer world. In so far as this outer world is cognisant of their existence, these theories, and the men to whom they are due, appear as chimerical and unpractical as would the labours of Clerk Maxwell have appeared to the Lancashire cotton spinner of 1850. And I fear that this view is too often shared, consciously or unconsciously, by mathematicians themselves, and especially by those who teach the subject. Here, they say or think, is a type of thought or investigation of great interest to those who can appreciate it, but it is utterly and permanently out of touch with the world at large. It can have no relevance or import for the ordinary boys and girls who learn mathematics at school, and can in no way assist them to become efficient citizens.

But is this really the case? He would be a bold man who would say with certainty that any branch of scientific investigation must be regarded, once for all, as having no bearing on the development of the individual or race. Is not the better answer that the practical import of these investigations has not yet been perceived; that it behoves all mathematicians, but especially those who are engaged in teaching, and therefore have some knowledge of the youthful mind, to do what they can to correlate this work with the outer world, and to examine to what extent it can now influence the manner or matter of teaching in our schools? The question will probably receive an affirmative answer from each one of you, but you may perhaps add that I am walking in the mists which hide from us the development of future centuries; that sufficient unto the day is the vision thereof; and that the ground to which I invite you is a morass which may conceivably be made firm by our great-grandchildren.

Nevertheless, I am going to ask you to bear with me while I endeavour to convince you that we can now commence to bridge the morass. I admit that it is one. Hesitating and imperfect our endeavours may be, but I am honestly convinced that the time is ripe for a commencement, and that the future of mathematics as a universal subject in the curricula of schools depends, in some part at least, on this commencement being made at once. My ground for this conviction is best stated tersely. I believe that the modern theories of pure mathematics are destined to illumine our understanding of the human mind and of cities and nations, just as the pure mathematics of fifty years ago has already illumined the previously dark and chaotic field of physical science; that modern mathematics is or will be to psychology, history, sociology, and economics as has been the older mathematics to electricity, heat, light, and other branches of physical science. For example, it may well be that the theory of sets of points or the theory of groups will find fruitful application in economics. You will see that I am suggesting that the range of applied mathematics may be widened far beyond its present scope. It was asserted recently at a meeting of head-masters that the reign of pure mathematics was closed. Would it not be more accurate to say that pure mathematics has of late extended and co-ordinated its dominions to an amazing extent, and that corresponding extensions of applied mathematics have yet to be found? If I am right, then our subject has an irresistible claim. We may trust our lives to engineers and scientists, just as we entrust our bodies to doctors and surgeons; but each member of a human society should, so far as he may, be competent to analyse and estimate for himself the workings of his own mind and the development of the society of which he is a unit. In the more detailed remarks which I am about to make, I will ask you to bear in mind that their main inspiration and justification lies in what I have just said — that they represent an individual attempt to relate mathematical education to human thought and social development.

Mathematics has been defined by Russell as the class of propositions, "If *A*, then *B*," and is applied to classes of entities concerning which certain propositions *A* are assumed; the truth of these is no concern of the subject. The entities form the universe of discourse. They can be ordered in respect of each of the attributes which characterise their class. This universe of discourse may be of any number of dimensions from one upwards; in arithmetic it is one-dimensional, and in geometry it should be three-dimensional, but is more often two-dimensional. I may remark in passing that some attempt to estimate the number of dimensions, that is, of quantities required for exact specification, of the entities discussed in such subjects as economics would often throw considerable light on these subjects. The abstract idea of entities and their dimensions is too often wanting. Hence arithmetic forms the basis of mathematics, since it explores the properties of one-dimensional fields. Any treatment of arithmetic which fails to explore the whole domain of such fields is *ipso facto* incomplete, and its victim is in possession of an imperfect instrument which cripples him alike in concrete and abstract applications. My first plea is, therefore, for a mathematical treatment of arithmetic from the earliest stages.

There is much which might be said concerning integers and fractions, and in particular scales of notation. My omission of these subjects is only to be interpreted as an admission that decimals and the theory of exact measurement are of more immediate importance, and must occupy such time as I can devote to arithmetic. To my thinking, young children are hurried on to fractions far too soon. There are many unexplored fields of concrete problems, possessing real interest for young pupils, the study of which would give a much firmer basis for future developments than is now obtained. And the proofs of such simple rules as "casting out the nines" may provide easy exercises in deduction, not without value.

To commence, then, with measurement. When, in actual practice, one measures a length, there are three distinct objects, any one of which may be in view. The purpose may be either (1) to state a length greater than that of the given object, but as little greater as may be, or (2) to state a length less than that of the given object, but as little less as may be, or (3) to state two lengths as close together as may be, between which the given object lies. I venture to suggest that training in measurement can only become of any value (other than manipulative) if it proceeds on these lines, phrases such as "nearly" and "exactly" being abolished as inexact, and therefore unscientific. "Nearly" is useless until we are told *how* near or within what nearness, and "exactly" only means "as nearly as I can see." By the use of a vernier — the theory of which should be included in every course of arithmetic — children should learn how nearly they *can* see, and then say, for example, 13·4 cm. within 0·2 mm.

We should thus sweep away all the loose statements which are, I honestly believe, responsible for much of that lack of accurate thought which is the subject of present complaint, and replace them by a training in the exact expression of practical measurements, the final form being of the type "between 7·38 and 7·39 cm."

The ground is now prepared for the extension of the idea of number, this being done, probably, in connection with mensuration, that is, by questions such as "Find the length of the side of a square whose area is 2 sq. in." Few trials are necessary in order to ensure conviction of the fact that the number of inches is not a fraction, and systematic approximation from above and below is attempted. By actual trial, using multiplication only, it is found that the following pairs of numbers are respectively smaller and greater than the number required: (1, 2), (1·4, 1·5), (1·41, 1·42), and so on. So far nothing more appears than can be realised by measurement, but it is at once seen that (1) this process can be continued indefinitely, given time and energy; and (2) that there is no limit to the closeness of the approximation. The human mind, by this systematic approach, has thus ridden roughshod over the imperfections of physical measurement. The latter leaves, and must always leave, an unexplored gap which cannot be diminished, but the method of successive approximation enables us to diminish the gap below any limit, however small.

Now this process, if carefully developed, is not beyond the comprehension of young pupils, and it may fairly be said to contain the germ of any proper study of functions and the calculus, whether this be undertaken on a graphical or analytical basis. In either case this method of inclusion between converging pairs is essential to any exact comprehension of the subject. And beyond this it develops the theory of pure number so far as to give the pupils — however unconsciously — an early example of a perfect mental structure, fashioned by extension from concrete experience, and it gives them the only true ideal for the exact estimation of any set of phenomena.

Shortly, then, I suggest the continuous development of the idea of a cut or *Schnitt* of the rational numbers, commencing it at an early age in connection with a scientific treatment of simple measurements, the purpose being to give a true concept of number in its relation to measurement.

I next make some reference to algebra, stating first that I am not to be taken as implying that the subject should be taught before geometry. On the contrary, I am convinced from actual experience that geometry should have been studied for two years at least before algebra is commenced.

At the risk of appearing to raise needlessly large issues, I must ask the question, What is an algebra for our present purpose, and what educational purpose may be served by its study? To my mind there are two essential steps in the development of an algebra: the first is the development of a symbolism which is usually suggested by certain combinations of entities, for example, $a+b=b+a$, $ab=ba$; and the second is the extension of this symbolism to cases which bear no interpretation in terms of these entities, and its subsequent application to other classes of entities. By this I mean the interpretation of symbols such as $3 - 5$, $x^{\frac{2}{3}}$, $3 + \sqrt{-7}$, in each of which the entities originally considered are found to form part of a larger class. I propose to allude shortly to each of these steps.

As regards the first I have little to say, for the unrealised possibilities with which I am concerned are here not conspicuous. But I do feel that the laws of algebra have received far too little attention in current and past teaching, in that their interpretation is so exclusively confined to the domain of pure number. Any ordinary boy or girl of 15 is able to realise that $a + b = b + a$ and $a + (b + c) = a + b + c$ are true when a, b, c are vectors, and to make simple deductions therefrom, as, for example, the proof of the median properties of a triangle. Such work, even if only a little time be devoted to it, gives a larger and truer view of algebra as a language with more than one interpretation. And it gives the idea of an algebra relevant to any field of human thought, an idea far more stimulating and fruitful for the ordinary man or woman than the narrower view of one absolute algebra, which is too often the only result of our teaching. But, when all is said and done, this first part of the subject only presents itself as the formation on methodical lines of a shorthand language; every step in the solution of equations, factorisation, or what you will, can be expressed in words whether the entities be numbers or vectors, and no new methods are involved.

But now take the second step, the interpretation of algebraic symbols such as $x^{\frac{p}{q}}$ or $\sqrt{-7}$, which have at first no meaning. The process involved is, or should be, purely logical. We assume that such laws of combination as $x^m \times x^n = x^{m+n}$ and $(x^m)^n = x^{mn}$ must also hold in cases which already bear interpretation, and then find that the one interpretation $x^{\frac{p}{q}} = \sqrt[q]{x^p}$ is consistent with each of these laws. It is too often assumed without proof that, because the one law $x^m \times x^n = x^{m+n}$ leads to this interpretation, the other laws, such as $(x^m)^n = x^{mn}$, must also be true in this case. I do not believe that complex exercises in the manipulation of fractional and negative indices can be of any profit, but I am convinced that a complete and logical interpretation of these indices, if only in particular numerical cases, can and should form part of every course in algebra. It is one of the best examples of constructive logic to be found in elementary mathematics, and it gives a sense of new methods for the discovery of hidden fields of entities which is hardly to be found elsewhere.

Passing now to imaginary expressions, I would suggest that the geometrical interpretation of these is not beyond the capacity of pupils of seventeen or eighteen years of age, and, further, that it provides a valuable link between the symbolism thus far developed and geometry of two dimensions. Not much knowledge of trigonometry is required in order to understand the expressions $a + bi$, $(a + bi)(c + di)$, nor is it necessary to plunge into useless elaborations. The pupils have ample scope for exercise in written descriptions of the processes; for example, in showing that this interpretation satisfies the laws $z_1 + z_2 = z_2 + z_1$, $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$. Work of this kind provides excellent material for short essays, a side of the work which has received scant recognition. The power of logical thought is a poor thing if its possessor is incapable of clear expression of his ideas, and this type of writing is well calculated to stimulate expression.

At this point the pupil may well review his experience of algebra. One after another apparent impossibilities of interpretation have been surmounted. Is there an end to the process, or can we go on in this manner indefinitely? The answer is, of course, that the performance of any algebraic operation on a quantity of the type $a + bi$ produces another quantity of the same type, and the process is closed. I would suggest that there is no inherent difficulty in the proof of this, granted a knowledge of elementary trigonometry, and that the view of algebra so gained is of real value as showing that the exploration of the field of entities under discussion has been completed.[1] If the boys and girls of the future can reach this point, they may, I admit, forget, and rightly forget, many of the details of their education, but this idea of the exploration of a field of entities, and the demonstration that this exploration is complete, may remain with them. If this be so, I do not think that you will question its value in dealing with the problems which present themselves — or should present themselves — to every citizen of a modern state.

Finally, I must make some reference to geometry. The primary value of the subject is, in my opinion at least, that it develops a power of dealing logically with manifolds of two and three dimensions. When we prove that, if A and B are fixed points, and the point P moves so that the angle APB is constant, then P must lie on one of two arcs of circles, we are selecting from all the points of the plane

[1] The entities are typified by the points of a plane, denoted by symbols such as $2, -\frac{1}{2}, \sqrt{3}, 7 + 4i$, and the exploration is not carried to three dimensions, as might have been expected after the extension from a line to a plane.

those which enjoy a certain property, and are showing that a certain other property is a necessary consequence of this principle of selection. And we develop the consequences in order to encourage the dormant faculty of selecting some set of a class of entities (the points in the plane) and examining their properties, not by the imperfect method of measurement, but with the relentless certainty of logical reasoning. But I will not dilate further on this aspect of geometry, as it can hardly be called an unrealised possibility of education.

My first suggestion in regard to geometry is that some simple idea of methods of transformation, such as projection and inversion, should form part of every course, at any rate for pupils who continue the subject until they are eighteen or nineteen years of age. Such transformations contain the idea, not illustrated so completely elsewhere in elementary mathematics, of a correlation between two sets of entities, such that to each entity of one set corresponds a definite entity of the other set; and from the known properties of one set we derive properties of the other set. It may, I know, be said that the study of graphs involves this idea, but graphs deal with one-dimensional sets only, and a general idea cannot be gained by one illustration. The problems which concern both the ordinary citizen and the workman in the trades must often involve sets of entities of several dimensions, and if he has attained to some idea of the correlation of such sets, and the examination of a new set in the light of known properties of an older set, he must thereby have more likelihood of forming some definite conclusions instead of floundering in vague uncertainties.

My last suggestion, and perhaps the most startling at first sight, is that older pupils should be given some idea of the nature of non-Euclidean geometry. One of the most vicious fallacies with which we are encumbered is the idea that our postulates of space, and in particular the parallel postulate, possess an absolute certainty which is denied to every other statement that is the result of experience. Most of us regard the parallel postulate as more obvious and certain than, say, the statement that all men must die some day, and we are utterly wrong in so doing. An outline of the idea and history of non-Euclidean geometries — I would refer especially to Poincaré's illustration and the recent paper by Carslaw [1] — is sufficient to dispel the idea, and to exhibit our space postulates as mere assumptions which fit our experience more simply and nearly than any others which can be made. I am not speaking at random; I have aroused keen interest in a form of classical specialists whose knowledge of geometry was distinctly limited. To what end, you may ask. In showing the true relation between thought and experience, the manner in which the mind deals with the sensations which reach it from the outer world. Far as we have progressed, the saying "Man, know thyself" still has force. No experience with which I am acquainted shows so conclusively the relation of each of us to the universe as the discovery that the supposed certainties of space are pure assumptions; as much so as Newton's laws of gravitation and motion, or Darwin's theory of evolution.

[1] See J.W.Young, *Fundamental Concepts of Algebra and Geometry;* also the article by Professor Carslaw in *Proceedings of the Edinburgh Mathematical Society*, Vol. XXVIII; W. B. Frankland, *Theories of Parallelism.*

You may, I fear, regard me as an unpractical visionary who has put before you a host of ludicrously impossible suggestions. But I would ask you to stop and consider whether they really are impossible, and I would remind you that I have suggested nothing that has not been attempted, in outline at any rate, with ordinary pupils, and with some measure of success. And I would ask you to remember one thing more. The whole world is going through a transformation, due in part to scientific and mechanical invention and in part to the growth of separate nations, each with its own methods and ideals, of which no man can see the outcome. Our function, the function of all teachers, is to produce men and women competent to appreciate these changes and to take their part in guiding them so far as may be possible. Mathematical thought is one fundamental equipment for this purpose, but mathematical teaching has not hitherto been devoted to it, because the need has but recently arisen. But now that it has arisen and is appreciated, we must meet it or sink, and sink deservedly. Neither the arid formalism of older days nor — I say it in no spirit of disrespect — the workshop reckoning introduced of late will save us. The only hope lies in grasping that inner spirit of mathematics which has in recent years simplified and co-ordinated the whole structure of mathematical thought, and in relating this spirit to the complex entities and laws of modern civilisation. Even though every suggestion that I have made be fallacious and impossible, this one statement remains, and the future lies with those who first achieve success in directing mathematical education to this end.

THE TEACHING OF ELEMENTARY ARITHMETIC

(An address delivered to the Southeastern Association of Teachers of Mathematics, and reprinted from the *Journal* of the Association, March, 1912)

THE TEACHING OF ELEMENTARY ARITHMETIC

I propose to commence my discussion of this subject by raising the question, a question joyous of sound to many a boy and girl wearied with obvious futilities — Why should we teach arithmetic at all? I raise it in no whimsical or revolutionary spirit, but in order that we may, if possible, agree upon the motives which determine the appropriation of so many valuable hours in the life of a child to this one subject. Our mission is not merely to occupy our pupils' time, nor to make them efficient but unintelligent beasts of burden; it is to educate them to take their places as efficient citizens of a free community. It is in the interpretation of this mission that subjects should be included in the curriculum, and in its furtherance should guidance as to matter and method of teaching be found.

But, some may say, what has this to do with teachers themselves? Are they not in the hands of those who draw up schemes and syllabuses, and can they with profit do more than carry out such instructions as they receive? The question is not unnatural, but it is based on grave misconception of the duties and privileges of each and every teacher. Schemes and syllabuses there must be, and by them all must be bound within reasonable limits, or anarchy will result; but their interpretation is in the hands of the teachers themselves. This interpretation can be performed either as a mechanical duty, or in free and willing co-operation, and those who ask the question I have suggested imply that they regard their duties as mechanical rather than co-operative; that they attend to the letter rather than the spirit. The better method is, surely, to endeavour to appreciate the motives of the schemes under which we work, and to shape them to the best advantage for our pupils. In so far as this is done, in so far will our profession acquire its proper influence in the general conduct of education, and associations such as this may do much to that end. Frankly, I am one of those who think that the body — I wish I could say corporate body — of teachers should have more voice in educational affairs than is now the case, and the remedy is largely in our own hands. It is by consideration of the why, as well as the how, of teaching that we shall best utilise our unique experience among the children themselves, and so gain our true position.

Why, then, do we teach arithmetic? First, of course, because a certain minimum knowledge is essential to the conduct of life. We must all be able to use money, keep our own simple accounts, and so on; but to how much does this amount? At most to simple operations in sums of money, lengths, and so on; certainly not to the cumbrous barbarisms which disfigure the pages of so many text-books.

"Find the cost of 17 tons 11 cwt. 7 qr. 14 lb. at £9 16s. $4\frac{1}{4}$d. per ton," or "Find the compound interest on £273 16s. 7d. for four years at $3\frac{3}{4}$ per cent. per annum paid half-yearly." Who on earth wants to do these sums in everyday life but a merchant's clerk, and what conceivable mental value can they have? If the need for such results does arise, may we not like the clerk use a ready reckoner as we use other time-saving devices made for us by the specialised labour of others? Every one should know what compound interest is, and why more frequent payments increase the amount, but simple examples and few of them suffice for this. If, then, utility of the narrow personal kind is the only reason for teaching arithmetic, let us ensure full proficiency in the operations of everyday life, show the use of ready reckoners when needed, and utilise the time so gained in teaching something more likely to assist in the production of capable citizens. Provided that children can perform simple calculations with fair speed and accuracy, they should learn the proper use of ready reckoners before they leave school, in accordance with the modern tendency to use labour-saving devices and so obtain greater efficiency. The individual is thus freed for the performance of other functions, and so increases his power of production. Those who deprecate this suggestion might as well deprecate the use of sewing-machines, and their introduction into girls' schools. In neither the one case nor the other is there a loss of independence; on the contrary, there is a gain.

But can arithmetic fulfil no other function in our schools? I am probably preaching to the converted when I say that there are two other aspects of the subject which not merely recommend but enforce its study in schools of all types. They are its application to the social life of cities and states (for example, to the intelligent consideration of schemes of insurance and pensions), and the concept of orderly and precise methods of thought which it may convey, in hardly less degree than the study of geometry. The two are indeed linked together, for these methods of thought find some of their best applications in the study of concrete problems which have some touch of reality for the children concerned.

To sum up, then, we base our teaching of arithmetic on three foundations: practical use, furtherance of the proper understanding of social and political problems, and development of power of independent thought; and we accept the use of labour-saving devices wherever possible, even though we are now compelled to waste time which will be better employed when our examinations are more rationally conducted.

First, then, for practical utility. We have to ensure the ready and accurate use of figures in concrete problems, and their combination by addition, subtraction, multiplication, and division; and the idea of a fraction must be gained — for its utility alone it is a necessary part of the equipment of every civilised being. At the base of all these things lies our scale of notation. Now I do not propose to enlarge upon the way in which this should be explained. I have never, unfortunately, taught it to young children. I would only commend the use of the abacus to those not familiar with it, as having historical sanction and being justified by modern experience. But I do wish to enlarge upon the importance of a correct understanding of the method of the scale, not only for its own sake, but for its applications also. It is the first example of orderly classification reached by the child, and as such deserves full elucidation, for if he once acquires the idea that things are taught to him which he need not and cannot understand, the impression will dog him and his teachers as an evil spectre for many a weary year.

The difficulty, such as it is, lies in the fact that only one scale of notation is presented as such, and the underlying principles cannot be grasped from a study of this or any one case. It is too little realised that our English systems of money, weights, etc., are also scales of notation. The notations

hundreds	tens	units
7	3	5

that is, $7 \times 10 \times 10 + 3 \times 10 + 5$ units; and

pounds	shillings	pence
7	3	5

that is, $7 \times 20 \times 12 + 3 \times 12 + 5$ pence

are exactly similar in method, though not in detail, for they each form groups of groups: tens of tens of units in the one case, twenties of twelves of pence in the other; and this is done merely to save time. Shillings and pounds are not necessary, but they are convenient as saving time and labour in speaking, in writing, and in carrying money. There is even a somewhat vague scale of notation in geography — hamlet, village, town, county, country, continent. I believe that such general considerations can and should be brought before children during their education, for they enlarge the mind and lead to the formation of general concepts from particular cases. They may invent examples for themselves, finding, for example, what 235 would mean for a race of beings who, having only eight fingers, counted in eights.

Beyond this we have to deal with money, weights, and measures, and simple sums concerned with them. Here again there is little of practical value to be said; methods of teaching and working have been thrashed out *ad nauseam*. The only plea one can make is for the utmost speed and accuracy in simple mental calculations, such as the cost of $2\frac{1}{2}$ lb. of tea at 1s. 7d. Frequent practice for short periods is the only way to ensure this, and such practice has its reward in an increase of general alertness and vigour.

I wish, however, to suggest a type of easy problem which has hitherto been neglected in elementary teaching. Take, first, an illustration. "Four boys *A*, *B*, *C*, *D* are to sit on one bench, and the teacher knows that *A* and *C*, if placed together, will talk. Show all the ways in which he can seat them." The solution should be systematic; *A* and *C* may be placed in six ways, thus:

A	—	*C*	—
C	—	*A*	—
—	*A*	—	*C*
—	*C*	—	*A*
A	—	—	*C*
C	—	—	*A*

and then *B* and *D* may be placed in two ways in each case, giving twelve ways altogether. The question can then be narrowed; *A* and *B* may have to share a book, and so on. Other problems are easily devised. "In how many ways can four people sit round a table?" "Three men are to be chosen out of five to perform a piece of work; *A* and *B* refuse to work with *C*. How many teams can be chosen?" Such questions not only give training in classification, they develop the idea that some things can be done in several ways, and that it may be worth while to reckon up all the ways and choose the best.

We now consider fractions, remarking first that they may and should be taught before long multiplication and division. The whole theory can be taught in concrete applications without the use of large numbers, and is only obscured by their introduction. Harder examples are accessible without further theory, once the fundamental processes are fully assimilated.

The first point is to develop the idea of a fraction, and the last way to do this is to commence with the notation $\frac{3}{5}$. This should come late — much later than is usually the case. The symbol means three-fifths of something, say of a pound or a line on the blackboard, and it should be regarded as three units of a new size. There is much difficulty in getting as far as this. A few concrete examples taken orally may suffice, written work, if there be any, being expressed with the denominator in words; thus, 3-fifths. It is essential that some unit be stated, 3-fifths of a pound or an apple; the abstract 3-fifths is far too general a concept at this stage.

Although there is not much difficulty in imparting the idea of a fraction, it is vital that this, as any other mathematical concept, should acquire living reality for the pupil, and not remain an arid tract of schoolroom formalism. The best safeguard against this danger is considerable practice in estimating one magnitude as a fraction of another — two lines drawn on the board, the areas of two pages, the sizes of two pieces of wood (to be tested by weighing), and so on. A little practice in this, the pupils being told which estimates were most nearly accurate, will soon induce that sense of proportion which is of the essence of fractions, and is so essential in practical life. And here we come to a method, which is two thousand years old, for the exact specification of one line or other magnitude as a fraction of another.

Suppose that two lines AB, PQ are to be compared, and let AB be the shorter. Lay off lengths equal to AB along PQ as far as possible, and let the remainder, if any, be RQ. Then lay off lengths RQ along AB and let the remainder, if any, be CB. Then lay off lengths CB along RQ, with remainder, if any, SQ, and so on. The remainders will soon become indistinguishably small, so that AB and PQ are expressed, with such accuracy as our instruments allow, as exact multiples of the smallest remainder visible, whence the fraction is at once obtained.

There is much more to be said of this process, but it pertains to the education of older pupils. Any one who can understand the process can, however, realise also that he has in this method a logical process which will continue until his instrument fails him; in other words, it beats the instrument every time and so illustrates the superiority of mental process over empirical measurement.

Next comes the addition and subtraction of fractions, still in concrete problems. For example: "A farmer wishes to sow two-thirds of his land with barley and one-quarter with wheat; what fraction is left for other purposes?" The best way to surmount the very considerable difficulty of this question for young children is to lay stress on the idea of change of unit. We may commence by saying, "Can you add 7 dollars to 4 francs and call it 11? No! What do you do? Convert them both to pence: 7×50 pence $+ 4 \times 10$ pence $= 390$ pence $= 39$ francs, or 7 dollars and 80 cents. In the same way we must convert 2-thirds and 1-quarter to the same kind of thing before we can add them." Now draw a line and demonstrate on it that 2-thirds = 8-twelfths and 1-quarter = 3-twelfths; we can then say 2-thirds + 1-quarter = 8-twelfths + 3-twelfths = 11-twelfths. It is unnecessary to enlarge upon this process; its nature is evident, and text-books contain many suitable examples in the chapters on ratio and proportion and in other parts. The essentials are constant verbal expression and continual illustration by division of a line or area until real comprehension is attained. There is no need to be particular about the lowest common dênominator; we may well allow our pupils to say:

1-quarter + 1-sixth = 6-twenty-fourths + 4-twenty-fourths
= 10-twenty-fourths
= 5-twelfths,

for at this stage the aim is clearness and accuracy, not brevity gained at their expense.

As soon as the pupils have become fluent in such statements as

2-thirds + 3-quarters = 8-twelfths + 9-twelfths
= 17-twelfths,

the fraction notation may be introduced on the ground that it saves time. Some stress should be laid upon this point, and it should be illustrated by analogies. Thus we save time by having one word "school," instead of saying "a place where children are taught"; "chair" instead of "a thing to sit upon." The fractional symbol thus assumes its proper aspect as a short expression of an idea already comprehended, and the child is receiving a valuable lesson on the meaning and use of language. Even when the notation is in use, frequent practice should be given in the verification, by division of lines or areas, of such statements as $\frac{3}{5} = \frac{12}{20}$. Unless they are understood they will inevitably be misapplied.

Next comes the multiplication of fractions. Here I wish to make a strong protest against the usual premature use of the term "multiply" in such statements as "$\frac{2}{3}$ multiplied by $\frac{4}{5}$." What does the child understand by multiplication? Surely nothing but repeated addition. If, then, we say to him "$\frac{2}{3}$ multiplied by 5," he can see that this means five times two of a certain thing (thirds), and is therefore ten of these things. But to tell him that "to multiply two fractions you multiply their numerators and denominators" confuses the term hopelessly. It has had one meaning, clearly comprehended, and now acquires a second which is apparently a mere juggle with figures. All sense of logic and exact use of language must depart with this step.

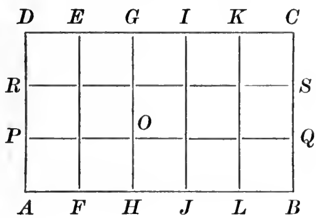
It is well to recognise that there is no obvious or easily apparent justification for the use of the same name for these two processes. Things receive the same name because they have something in common. We are all called human beings because, amidst much diversity, we have certain common attributes. Now the common element is not at all manifest, at first sight, in such statements as the following:

$$7 \times 3 = 7 + 7 + 7 = 14 + 7 = 21,$$

$$\frac{3}{8} \times \frac{2}{5} = \frac{3 \times 2}{8 \times 5} = \frac{6}{40}.$$

Unless we can see the common element we have no right to name them alike, and until the child perceives a common element it is absolutely pernicious to suggest a common name, for in so doing we debase that most wonderful creation of the human race—language as a clear expression of thought. So soon as we do this we may say farewell to clear thinking or exact expression on the part of the pupils. I honestly believe that this one step is responsible for most, if not all, of the doubt and haze which hang like a nightmare over many children in their dealings with fractions.

A good introduction is to consider questions such as, "A man left $\frac{2}{3}$ of his land to his children, and $\frac{2}{5}$ of this to his eldest son. What fraction of the land did the eldest son get?" Representing the land by *ABCD*, we divide it into thirds by *PQ*, *RS*, and then into fifths by *EF*, *GH*, *IJ*, *KL*. Then $\frac{2}{5}$ of $\frac{2}{3}$ of the land is seen to be *POGD*, and this is seen to be $\frac{4}{15}$ of the land. With many such questions the general idea that $\frac{2}{3}$ of $\frac{2}{5}$ of a thing is the same as $\frac{4}{15}$ of that thing is imparted, and the formal rule is seen to hold in all cases. The drawings can be discarded, except for revision, when this formal rule is grasped and not before. At no stage need they, or should they, be made with great accuracy. Freehand sketches are better than drawings with instruments, for they enforce the lesson that the process is essentially one of reasoning and not one of measurement.



Finally, we come to the division of one fraction by another. Here again, and for precisely the same reason, this term should be discarded until its application can be comprehended. By concrete questions the problem is raised, "Two-thirds of a thing is taken, and three-quarters of the same thing. What fraction is the former of the latter?" and the idea is evolved from such discussions.

It is not pretended that this work is easy, or that it can be learnt by rote; but experience shows that it is within the comprehension of ordinary boys and girls of twelve years of age or even less, and it has far more value, as a practical mental training, than purposeless juggling with numbers. A child who can perform these processes feels that he is using his own mind to answer definite questions with logical certainty. The mere appreciation of this fact raises him from drudgery and gives him an ideal of mental independence which he may, perchance, in some part retain in after years.

Before leaving the subject of fractions some further reference should be made to the premature use of the terms "multiplication" and "division" as applied to such numbers. First take the statement, "To multiply $\frac{7}{8}$ by $\frac{3}{4}$, do to $\frac{7}{8}$ what is done to unity to obtain $\frac{3}{4}$." This definition is hopelessly defective in that it omits to state exactly what is done to unity. Is $\frac{1}{4}$ subtracted from it, or is it increased by 2 and the result divided by 4, or which other of the innumerable ways of obtaining $\frac{3}{4}$ from it is meant?

The upholders of this definition will reply that this is splitting hairs; that every one knows it to mean that unity is to be divided into four equal parts and three of these parts taken. Certainly this is so, and the statement as thus amended loses its gross ambiguity. But what analogy is there between this process and the original view of multiplication as repeated addition to justify the same name for both processes? This question must still be answered before the definition can be accepted.

The justification is twofold. First, it can be shown that the latter definition includes the former as a particular case; that multiplication by $\frac{7}{1}$ according to the definition for fractions amounts to the same thing as multiplication by 7 according to the first definition. And secondly it can be shown that the formal laws such as $ba = ab$, $a(b + c) = ab + ac$, which are so easily seen to be consequences of the first definition, are consequences of the second also; that is, they are true when a, b, and c are fractions as well as when they are integers. The coincidence being complete, the use of the same term is justified, but it is evident that considerations of this nature can hardly be included in a first course of arithmetic. For pupils who are revising the subject for the second or third time they may be interesting and profitable, but before then the use of the term "of," as in $\frac{2}{3}$ *of* $\frac{3}{4}$, is preferable to the apparently ambiguous "multiply by."

The considerations above may serve to show the spirit which it is suggested should inform the treatment of the theory of arithmetic; we now pass to a discussion of some elementary applications which may go towards fitting the pupils to be capable citizens as well as efficient clerks.

Of all the various applications which appear in very elementary text-books of arithmetic, the theory of averages suffers perhaps more than any other from the banality of its treatment. And yet no other application possible in such books is possessed of equal ease and interest; nor are there many, if indeed any, others which have so direct a bearing on almost every question of national or civic importance. There are few such questions into which numbers or statistics do not enter in some shape or form, and their correct treatment by averaging is almost invariably essential to a proper view of the facts.

The one, and usually the only, thing which is taught in connection with averages is the rule for obtaining the average of a set of numbers; it is then applied without intelligence to problems fit and unfit for the purpose. Consider the two sets of numbers:

	10,	7,	12,	8,	9,	9,	8,	11,	14,	12;
and	0,	23,	2,	45,	6,	15,	0,	7,	1,	1.

Each has 10 for its average, but it can at once be seen that there is no real significance in this statement. In the first set the numbers are grouped closely round this average, but in the second they bear no special relation to it; it is nothing more than a levelling up of things widely diverse one from another, and has little or no other import.

Considering the first set a little more closely, we may make a table showing how many of the numbers come within different percentages of the average, thus:

Percentage of average	10	20	30	40
Number within	4	8	9	10
Percentage within	40	80	90	100

Such a table shows with what nearness the average represents the group of numbers, and enables us to compare the relative values of different averages. For example, the cricket averages of different players may be treated in this manner, when much information is gained as to their steadiness of play. Or the average age of each form and of the whole school can be so compared; the contrast between the irregularity of the distribution in many small forms and its uniformity for the whole school will convey its own lesson. The results can be exhibited graphically, laying off horizontally differences of 1, 2, 3, . . . per cent. from the average, and vertically the percentage of the whole number of observations which fall within each limit. All such graphs should be drawn to one uniform scale, so that a glance will indicate the relation of the average to the set of numbers, and oral or written statements of whatever can be seen from the work should be insisted on in every case.

A second application of averages concerns the "smoothing out" of a series of statistics which, though liable to large irregular variations, obey on the whole some definite law of change. Suppose, for example, that the shade temperature is observed each day at noon for a period of six months; the results will be very irregular but will show on the whole a steady increase, and the object is to eliminate the irregularities so far as may be possible and thus exhibit the general law of increase. This is done by taking the average for a number of consecutive days (say five) and assigning it to the middle of the period, this being done for every such period in the six months. The accidental irregularities, due mainly to the direction of the wind and the amount of cloud, are thus spread out and the increase corresponding to the change of season becomes apparent. In practice, periods of five days would be too short, as the wind often holds in one direction for a longer time, but they commence the smoothing process and longer intervals may be considered afterwards. The comparison of results for different periods is of interest as showing how the effect of a long period of high or low temperature is gradually eliminated.

Another method of obtaining the same final result, and one which is simpler in itself, is to take the average temperature over a series of years for each particular date and so smooth out the irregularities in a different way. But this method would be impossible in other cases, such as the study of the mortality from consumption or the price of corn in London, for these phenomena are not recurrent like the seasons and we cannot, therefore, eliminate accidental irregularities by reviewing several cycles of change. Moreover, the temperature averages obtained from a period of years assume that every season is the same apart from irregularities, and so conceal any possible change in the seasons from year to year; but a comparison of these averages with those obtained by the method of consecutive days will reveal such changes if they exist.

Other materials for the application of this method of smoothing out will be found in any book of reference and in most text-books of arithmetic. It can be applied to any set of statistics, but consideration of weather records is of special use, partly from their interest, but more from the repetition from year to year which has just been discussed.

Yet another important application of averages is the method for obtaining the mean value of a continuously varying quantity, such as the height of the barometer or the depth of a tidal river. In a certain town (there may be many such) the maximum and minimum temperatures for each day are recorded and the mean is taken to be their half-sum, which is solemnly written down as the mean temperature for the day. Now it is obvious that this method will give results which are too low in summer and too high in winter; for in summer the temperature stays in the neighbourhood of its maximum for the greater part of the day, and in winter it lingers near its minimum. This application will show how a proper estimate of the mean may be obtained.

It is clear that the truth would be shown more nearly by averaging readings taken every three hours, and that still better results would be gained if the readings were still more frequent. Such averages should be taken and the results compared with each other and with the mean of the maximum and minimum readings; statistics are easily obtained from the charts given in many newspapers[1] or from an instrument dealer who has recording instruments. With such charts we can, however, obtain an even better estimate of the mean by finding the height of a rectangle whose base is the horizontal width of the graph and whose area is equal to the area under the graph, for it is obvious that this height is the true average of the heights of all points on the curve. The area of the curve can be estimated in the usual way by counting squares, and the average height is then found by dividing by the base. This method of estimating mean values is of much importance in theory and practice, and examples of its use are not lacking in interest.

[1] The *London Daily Telegraph*, for example. In the United States, the reports distributed freely by the Weather Bureau may be used.

It has seemed worth while to discuss averages in some detail, even to the exclusion of other applications of arithmetic, for the work conflicts little, if at all, with the syllabi to which most schools are subject, and combines ease, interest, and value in exceptional degree. But I would mention also the understanding of insurance tables (not the formulæ from which they are constructed) and the meaning of the value of money and its variations in time and place as matters which should be considered by teachers of arithmetic; the fallacy of the thirty-shilling wage would find no wide acceptance if education were all that it should be.

I have endeavoured to suggest some simple and perhaps novel considerations concerned with the teaching of elementary arithmetic. It may perhaps be felt that they have some slight interest and value, but that it is hopeless to attempt the application of some at least, in view of prevailing custom and requirements. This frame of mind, excellent as a balancing factor, is nevertheless to be regarded with much caution, for salvation from our present difficulties can come only from the efforts and experiments of teachers themselves. Educational matters are in a ferment. Men are asking more and more insistently why this and that are done, and they are right in their insistence. Unless fitting answers are ready, our work will stand condemned; the degradation of our subject to the domain of purely immediate utility will surely follow, as also the loss of that higher mental training which is so essential to the formation of an efficient citizen. A man who has no power of intelligent numerical thought is to this extent a serf intellectually, and it is hard to believe that teachers as a whole will fail to point out the evil and insist upon its avoidance.

THE EDUCATIONAL VALUE OF GEOMETRY

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THE EDUCATIONAL VALUE OF GEOMETRY

"Every great study is not only an end in itself, but also a means of creating and sustaining a lofty habit of mind; and this purpose should be kept always in view throughout the teaching and learning of mathematics." — BERTRAND RUSSELL

The title of this paper has been chosen to indicate that the discussion will not be concerned with the value of geometry as applied to other sciences or to practical ends, nor even with its place and importance in schemes of mathematical education. The purpose is to state the reasons which appear to have led to the universal acceptance of the subject as a necessary element in education, to ascertain to what extent geometrical teaching in this country can find justification in them, and to give some slight account of experiments in teaching made on this basis by the writer and his colleagues at Tonbridge School. Lest it should be thought, however, that this avoidance of the practical importance of the subject and its relation to other branches of knowledge imposes unreasonable limitations, it may be well to state the reasons for it.

The danger of giving undue importance to considerations of practical utility need hardly be enlarged upon, since it is not proposed to consider geometry from this point of view. I am more concerned to point out that if the advocates of the subject rest any part of their case on such considerations, they at once enter into competition with a host of other interests, many of which have, on such grounds, much higher claims. The parents of a boy who is to adopt a business career will rightly prefer, if his education is guided by his future requirements, that he should spend his time on geography or economics, arguing that surveying and bridge building can have no relevance to his future interests; while those who take a wider but still utilitarian view will insist that subjects such as civics and the chemistry of food have stronger claims to a place in the education of every child.

Still more dangerous is the plea that every educated man should have some idea of a subject of such wide utility. Apart from the claims of many other branches of knowledge, it has a further demerit in that the object of teaching the subject is implied to be the acquisition of encyclopædic knowledge, rather than the development of the mental faculties. The old conception of education as the acquisition of information is dead, and it least becomes mathematicians to do anything to revive it. The use of justifications of this type, even though it be only in secondary positions, is likely to defeat the aims of those who advance them and to do much harm to educational ideals.

A discussion of the value of geometry in relation to other branches of science would be appropriate in a paper dealing with the co-ordination and relative importance of these branches. My object here is, however, to show that the subject has for its own sake a claim to a place in the education of every human being. Such a discussion could, therefore, give only a secondary and relatively weak support to this claim, a support which only becomes valid when the claims of these other subjects to a universal place in education have been admitted. If the view here taken is unjustified, geometry must then make its own place in such volume of scientific knowledge as may be found necessary to a liberal education. This place would be an important one, especially in view of the now almost universal teaching of natural science, but it is hoped that the considerations to be stated in support of the stronger view are such as will meet with agreement among mathematicians and convince laymen of its truth.

This is, of course, no new claim. Plato inscribed over the entrance to his Academy, "Let no one enter who is ignorant of geometry," and almost every university now imposes a similar condition. Such recognition of the subject by educationalists who are not mathematicians implies an inherent value which must be expressible in non-technical terms, and it behoves all those who teach it to assure themselves that they appreciate this value and that the education in schools is such as to realise it as fully as may be. In this country at the present time the duty is especially urgent. Educational systems and ideals are changing with some rapidity, and almost every subject in school curricula has been challenged to justify its place, geometry being one of the few exceptions, if indeed it still be one. It is almost certain that the motives for this forbearance are of utilitarian type, combined, perhaps, with some vague idea that the subject may train a boy to chop logic and hold his own in argument. Thus the lay advocates of the subject, on whom its continuance must depend, base their support on reasons which are open to successful attack from those who take a too material view of education, and are almost beneath attack from those who have higher ideals. Of this weakness they must become conscious;

signs are not wanting that this is in process, and unless mathematicians themselves take the initiative in defence they may find the attack developed with some suddenness. If the boy who specialises in science obtains exemption from the study of Greek, his fellow who specialises in classics or history will almost certainly claim exemption from mathematics, as also will those who intend to devote themselves to subjects such as law, medicine, or music.

The question for mathematicians is, then, whether they can convince others that the appropriation to the study of geometry of a portion of the school time of every boy and girl is really expedient. To do this it will almost certainly be necessary, even though those who are to be convinced have themselves had some portion of their school time so appropriated, to explain in some detail what geometry really is. The first element in the explanation must be that the subject is based on agreement as to a certain number of cardinal facts, this agreement resting on foundations of general experience common to every civilised human being.[1] The equality of vertically opposite angles, the angle properties of parallels, and those properties of a circle which can be perceived from considerations of symmetry are instances. It is essential to understand that these facts should not depend, even for their elucidation, on numerical experiments made in class-rooms or laboratories. Rough descriptive illustrations there may be, but their only purpose is to recall or intensify conceptions previously formed

[1] The latter part of this sentence defines the subject as taught in schools. It is, of course, an essentially vicious limitation from a more general point of view.

subconsciously, for it is this universal acceptance of postulates without conscious experiment which differentiates geometry from physics or chemistry. The necessary experiments and inductions are made in infancy without the aid of ruler, compasses, and protractor; the dog who follows his master by a "curve of pursuit" fails to describe a curve which is grasped by young children, although it is the one actually taken by infants who can only just crawl. They, in their apparently aimless wanderings, are in the true geometrical laboratory, performing experiments and making inductions.

Trite as this statement may appear to many, it is of some importance here. The appreciation of common conviction should, when possible, be the first aim of teacher and pupil, as of all those who would make a concerted effort, and here there is found a number of facts of which all pupils can say, "Yes, I *know* that these things are true." [1] They feel that they are not using arbitrary rules, as in the study of languages; nor dealing with facts asserted by others to be true, as in geography or history; nor dealing with experiments put before them by some one who knows what will happen, as in experimental science: here they rest on their own convictions. I need not enlarge upon the value of this consideration, but I would point out that its importance can be realised by the layman and should influence him considerably. Amid the many arbitrary rules and asserted facts which, perforce, find place in education, the presence of schemes of deduction based on statements

[1] The basis for and meaning of this assertion do not enter into the question. The fact that it is made so universally is the point of importance.

which find universal acceptance as descriptions of our space impressions must make for good in the child's development.

The peculiar nature of the premisses on which geometry is based having been explained, it might appear sufficient to complete the description of the subject by stating that it consists of a series of deductions from these premisses and therefore supplies a useful training in the art of deduction. But this statement would, if not amplified, be so bald as to mislead. The full sequence of processes involved is:

1. The separation of essential from irrelevant considerations involved in the appreciation of points, lines, and planes and their mutual relations.

2. The erection on this appreciation of continuous chains of reasoning, one result leading to another in such a way that each chain can be comprehended as an ordered whole and its construction realised as fully as that of each separate link.

3. A discussion of the interdependence of the various premisses and their precise statement.

Since some such sequence is common to every human construction, the educational value of a subject which provides a training in these processes is indisputable; for this purpose geometry stands alone in that its bases can be appreciated, and the deductions performed, at an age earlier than is possible in the study of any other subject having the same purpose.

There is yet one more consideration, and that not the least important, to be urged in favour of the subject. The appreciation of literary and artistic beauty has of late received increasing recognition as a necessary element in the training of the young. The study of the English language now includes literature; drawing and music advance in importance in schools of every type. But the element of intellectual beauty has not yet received general recognition, despite the supreme position ascribed to it by almost all schools of thought from the Greeks downward, and the loss is a great one. The contemplation of unassailable mental structures such as are found in mathematics cannot but raise ideals of perfection different in nature from those found in the more emotional creations of literature and art. It must induce an appreciation of intellectual unity and beauty which will play for the mind that part which the appreciation of schemes of shape and colour plays for the artistic faculties; or again, that part which the appreciation of a body of religious doctrine plays for the ethical aspirations. Fanciful though this may appear to many, I believe that it may be an important factor in determining the retention of geometry in schools. The conception of a body of truth invulnerable on all sides,[1] a conception which finds one of its best and most common expressions in the quotation, "Four square to all the winds that blow," appeals to most men, and they will welcome an effort to bring it to their children in one of its purest forms.

Such being the basis on which it is thought that the universal teaching of geometry may be justified, it remains to develop the leading principles of a scheme of education, and to examine to what extent teaching practice in this country accords with them.

[1] Euclidean geometry is still invulnerable in that it is based on the simplest known description of our space perceptions, and such descriptions form the foundation of most "truths."

The division of the processes of geometry into three classes implies a corresponding division of the period of education into three epochs. In the first, the imagination is stimulated and developed and some general power of reasoning should be acquired without any formal presentation; in the second, ordered systems of reasoning are developed from facts which are now within the scope of the imagination; and in the third, the true basis for the assertion of these facts is discussed and their interconnection investigated. It is at least doubtful whether this third stage would be a subject proper to a school course if it were within the pupils' grasp; and it is almost certain that its nature and the problem involved are as much as can be brought home to them. It must therefore be assumed that a school course should end with the second of these epochs, and the first two only need be considered in detail.

The increase in power of imagination, which is the main object in the first epoch, can only be effected by extension from those impressions in which this power is already developed to some extent. A child may know what is meant by a point and a line, and be able to recognise them at sight, and yet not be able to think of points and lines, just as an adult may recognise the whole of a tune when he hears it and yet not be able to reproduce a single note. It is therefore inadvisable to commence by drawing points, lines, and angles, and performing constructions and measurements, because this opens up a new field of unfamiliar ideas having no connection with any of that knowledge which has been so far assimilated as to be a possible subject for imagination. Houses, roads, mountains, islands, and the like can all be imagined by a child of ten years of age, and his geometrical imagination is developed by stating problems in such terms, but the construction of triangles having given sides leads nowhere at this age and is a mere gymnastic.

It is not hard to devise problems on these lines which stimulate the imagination, excite the spirit of research, and provide exercise in the simpler forms of geometrical reasoning. They may be divided into groups, in each of which one of the following methods is introduced:

1. Construction of triangles and polygons when lengths only are given.

2. Simple constructions for heights of buildings, ships' courses, and the like, depending on compass bearings and angles of elevation.

3. Construction of triangles and polygons when lengths and angles are given.

4. Extension of all the above to problems in more than one plane.

5. Determination of a point as the intersection of two loci, or its limitation to lie inside or outside two or more loci. These loci need not be lines or circles and are constructed by actual plotting.

It may be worth while to give a few specimen problems to show what is intended, and it should be stated that these and subsequent suggestions are the outcome of experience and have been tested in Tonbridge School. Each of these problems is well within the scope of a boy of eleven or twelve years of age, after he has received a reasonable amount of teaching in the shape of questions involving the same principles; the concrete terms should, of course, be varied constantly.

a (*Type* 1). A straight road runs east and west; a house is 95 yards north of the road and a well is 40 yards north-east of the house. Draw an accurate plan and locate upon it the position of a cow-shed which is to be built 65 yards from the well, within 70 yards of the road, and as far as possible from the house.

b (*Types* 2, 3, 4). The height of a tree or chimney may be found from measurements in one or in two planes, the measurements being made by the pupils themselves.

c (*Types* 2, 3, 4). A hill rises due north at a gradient of 1 in 10. Find the direction of a road which rises to the eastward on the hill at a gradient of 1 in 30.

d (*Type* 5). A lightship is 2 miles from a straight shore. A submarine, which is cruising in the neighbourhood, explodes and sinks, the only survivor being the man in the conning turret; he can only say that the wreck is equidistant from the lightship and shore, and the watch on the lightship says it was 3 miles away when the explosion happened. Show on an accurate plan where the wreck must lie.

e (*Type* 5). Two towns, *A*, *B*, are 5 miles apart, and a man who lives with his parents at a place 6 miles from *A* and 7 miles from *B*, bicycles to *A*, then to *B*, and then home every day. After a time he wishes to reduce his daily ride to 14 miles. Show on an accurate plan all the places where he can live, and indicate those places which are nearest to and farthest from his parents' home.

The first three types require no further explanation, but since the fourth and fifth involve departures from current practice, discussion of these may be of some value. The early introduction of solid geometry (examples *b*, *c*) may cause some surprise, especially in problems of such apparent difficulty. But I am convinced, from the experience of myself and my colleagues at Tonbridge, that boys of twelve or thirteen grasp these ideas with much more ease and rapidity than those who have deferred the work for two or three years, and that their whole outlook is improved in consequence. The first problems should concern things which can be seen; for example, the height of one corner of the class-room can be determined by measurements taken from the ends of a desk not in line with it, or the diagonal of a block of wood may be found by construction of two right-angled triangles. A problem such as *c* above is simply illustrated by using one corner of a book half open to represent the hill-side. These aids to imagination are soon found to be unnecessary, fairly complex constructions being undertaken without their assistance, and there is no lack of material from which examples may be constructed.

Since the method described in the fifth type and examples *d* and *e* is, except for the introduction of other loci, used in the preceding sections, its separate mention may appear redundant, but it is easily seen to involve principles which can hardly receive too much attention. The statement of the laws of selection which define two or more manifolds in such form as to exhibit the common elements (if any) of them all is one of the most familiar forms of mental activity, and experience has shown that children can understand and perform the process in cases such as those above. In so doing they apply it, not to new and unfamiliar abstractions, but to classes of objects so familiar as to be possible subjects for mental operations.

Such applications form the second of four stages into which education in logical processes may be divided. In the first they are applied to definite objects, as in the methodical selection from a number of tiles of all those which have given shape, size, and thickness; in the second, to mental images of classes of objects, as when "a house" is not thought of as any particular house; in the third, to abstractions, such as point, line, colour, sense, instinct; while in the last the processes themselves are considered in their utmost purity. For our purpose the first stage is dealt with in the kindergarten and in courses of practical measurement, and the third in formal geometry; the second does not, in my opinion, receive sufficient recognition in the teaching of geometry at the present time, and the exercises above are intended to remedy this deficiency.

The course may also be regarded as an introduction to the ideas of a manifold and a function. Their importance has of late been recognised and need not occupy us here, but their difficulty, especially in regard to manifolds, is hardly realised. If a child is directed to mark on paper a number of points such that the sum of their distances from two given points has a given value, he will do so without any idea that all such points form a continuous curve, and if he is told this he will not grasp the fact. But allow him to go on marking points and he will ultimately attain to a conception of the whole and their assemblage along a continuous curve. This cannot be taught; the teacher must wait for the child to reach it, and the value of the work is lost if mechanical means of describing the curve continuously are adopted. Such a process has its own value, but in this connection it obscures the idea of the curve as a manifold of points selected from the manifold which forms the plane.

The early formation of such habits of thought, even though subconscious and unsystematised, has a technical importance which deserves mention despite our general limitation to educational considerations. Recent research has exhibited the whole structure of mathematics as founded on the processes and conceptions which underlie them, and has gone far, with their aid, to perfect a co-ordination between the different branches of the subject. These fundamental ideas are now seen to be essential to the understanding of even elementary mathematics, and their absence is responsible for most of the difficulties which occur in the introduction to more advanced work. The difficulty of the child who in commencing algebra says, "Yes, but what *is* x?" of the boy who cannot solve a simple rider, and of the student who finds the calculus puzzling, are now all traced to this common origin. The moral for the teacher is obvious. Until the young child can in some way attain to such ideas in relation to matters within his own experience he has not grasped the groundwork of mathematical thought and so cannot erect the structure on proper foundations, and the present problem for teachers is to bring them to him in as many forms as possible. For this purpose some understanding of the philosophy of arithmetic, geometry, and algebra is essential, and I believe that the light which this philosophy throws on problems of elementary education constitutes by no means its least value.

The transition from the preliminary stage just described to a first course of formal geometry is marked by the introduction of connected reasoning leading from one result to another, and a gradual cessation of the aid to imagination involved in the use of houses, roads, and the like. The assumption of postulates as a basis for this reasoning should not be stated formally; unless the preliminary course has failed in its object the pupils will accept them without difficulty as they are required. But it is essential that the choice of these postulates should not be left to chance or dictated by convenience of presentation; it must be guided by the considerations which determine the adoption of the subject as a mental training.

If these considerations be such as are indicated above, there is no difficulty in specifying the nature and extent of the assumptions which should be adopted as facts in a first course of geometry. It must be possible to induce the child to accept them without the aid of numerical measurement of any kind, and every statement for which this is possible should be regarded as a postulate. This definition includes the following: (1) the equality of vertically opposite angles; (2) the angle properties of parallel lines; (3) properties of figures which are evident from symmetry;[1] (4) properties of figures which can be demonstrated by superposition, a method which should be once for all discarded as a proof.

The meaning and intention of this definition is best illustrated by reference to recent developments in the teaching of geometry. There has been a tendency in preliminary courses to associate such ideas with numerical measurement, if not, indeed, to profess demonstration by

[1] Demonstrations by folding are not proofs as the term is here used. They lack the essential element of deduction from two or more statements already admitted.

its aid. For example, children may be instructed to draw two straight lines, measure the vertically opposite angles, and state what they observe. It is hard to see what good can be derived from such exercises, and they may do much harm. If it be true that the education of the child should follow the development of the race, they are condemned at once, for it is inconceivable that these ideas were suggested by such processes, or that there was a geometrical Faraday who announced them to the ancient world. They can only be regarded as intuitions from rough experience; a person in whom they are wanting is ignorant of space, and a knowledge of (Euclidean) space is neither more nor less than their full comprehension — a comprehension resting on foundations wider by far than any schoolroom experiments. An attempt to aid its formation by numerical exercises may not unfitly be compared to an attempt to teach music by explaining the mechanism of a piano, and the relation between notation and keyboard, before the pupil has heard a single tune. It is a crippling of subjective growth at its most sensitive stage by the crudest form of materialism. A mind which has ranged over all its experience and has made these intuitions has gained a sense of power and accepted truth which cannot be induced to an equal extent by any substitute for the process.

The statement that all possible intuitions must be taken as postulates is hardly less important than the definition (for our purpose) of a postulate as any statement of common acceptance. The object of teaching geometry has been stated as twofold: the sense of logical proof is to be developed, and the conception of a chain of proofs is to be formed. There are many ways of doing this; they vary with the selection of the theorems used and the order of their sequence, but each should be subject to the condition that every theorem should prove some new fact which could not have been perceived by direct intuition, symmetry, or superposition. If this condition be violated the pupil's sense of a proof is confused; he appears to arrive with much painful labour at a result which he and every one else knew before to be true, and the impression of inevitable but unforeseen truth is not imparted. Take, for example, the theorems concerning bisected chords of a circle: if a boy cannot perceive the truth of these with, possibly, a little stimulus, his knowledge of space is so meagre that he ought not to have commenced formal geometry. If, on the other hand, he can see their truth, there can be few processes more destructive than the elaboration of deductions from other intuitions with the intention of imparting a sense of proof.

The details of the propositions and the order of their sequence is hardly germane to this paper, but the necessity for presenting them in sequences each of which can be grasped as a whole deserves further mention. If the Euclidean tradition be ignored, such sequences can still be found. For example, experience has shown that the angle properties of polygons, arcs of circles, and tangents to circles can be arranged in one sequence, so that the unity of the group can be appreciated by pupils to whom it is the first example of formal geometry. Another subject is found in the theory of the regular polyhedra; the limitation of their number, the investigation of their shapes and dimensions, and their construction, can be grasped by boys of ordinary ability at an early age. It will probably be agreed that a considerable element of solid geometry is essential; the tetrahedron, cone, and sphere are other themes which have been found possible and valuable.

It is, of course, true that Euclid presented geometry in a series of such sequences, and despite the logical deficiencies in his scheme his genius is probably still unrivalled. But his theme was the reduction of the number of postulates to a minimum and the introduction of each as late as possible. As soon as it is granted that all possible intuitions should, in a first course, be accepted as postulates, his sequence fails to have value for this purpose, and has, indeed, demerits. It is therefore a necessary consequence of the theory of geometrical education here developed that a deliberate effort should be made to replace this scheme by one more suitable. The need is not met by allowing propositions to be treated as intuitions when possible and retaining his sequence, for this destroys the whole meaning of that sequence; his motive enforces the introduction of solid geometry as late as possible, while ours demands it as early as possible, and numberless other contradictions arise.

This conclusion induces doubts whether even this residuum of Euclid's propositions, taken in any order, forms the best material for geometrical training of an educative value. Some of it undoubtedly deserves its place, both for its own value and its necessity in other parts of mathematics; but stripped of unessentials this is small in content and less than can be acquired in an ordinary school course. Before including more than this bare minimum it is advisable to ascertain whether, in the whole domain of geometrical knowledge, there may not be other matter of more educative value within the grasp of ordinary boys and girls. Here I am passing beyond the range of my own experience, but I have, as the outcome of a maturing conviction, made tentative experiments with individual pupils and small classes and deem it worth while to state the result.

If, without regard to the age of the pupils, one asks where the methods and ideals of geometry are presented in the greatest unity, simplicity, and beauty, the answer must be that geometry of position and projective geometry have no rivals. Estimated on such standards Euclid's work is dwarfed by these modern creations, and not least so in respect of the ease and generality of their conceptions. Must it be said, in spite of this, that they are to remain the property of professed mathematicians since they are beyond the grasp of the normal adolescent, or may it be that their power and beauty can be appreciated, even if only in some comparatively crude form? Even those, if there be any, who at once deny the possibility will not dispute the importance of the question and the value of success if it can be attained. My own conviction, fortified by such limited experience as I have indicated, is that the elementary concepts and methods of projective geometry can be grasped by ordinary pupils; that they would excite a greater interest and fuller spirit of enquiry than any form of Euclidean geometry, and that their educational value would be far greater. The amount of knowledge required is not so great as might be imagined. A pupil who has had some experience of three-dimensional work can grasp the relations between a figure in one plane and its projection in another plane, including the particular cases wherein a point or line in either plane have their corresponding element in the other at infinity. The theorem of Desargues, with its simpler consequences, is then within his grasp, and he may so gain some idea of the extent and variety of the results which can be deduced from the axioms of position only, and the manner in which they unite in one statement results which he had regarded as disconnected.

It is then natural to enquire whether there is any simple relation between corresponding segments in the two planes. The position ratio $AP : PB$, which defines the position of a point P on a line when two of its points A, B are given, should have become familiar in connection with earlier work, and it is now easy to prove that corresponding position ratios remain in a constant ratio as the points P, P' move along their respective lines. Hence the ratio $\frac{AP}{PB} : \frac{AQ}{QB}$, formed from two such ratios, is unchanged by projection. The metric properties of quadrangles and quadrilaterals (deduced by projection from a parallelogram), and the simple properties of the conic regarded as the projection of a circle, can then be investigated to such extent as may appear possible or desirable. I have myself, with ordinary boys of eighteen, reached Carnot's theorem without great difficulty.

It may be suggested that this work is unnecessary, and that when the minimum of geometry has been acquired the pupil should proceed to trigonometry or other branches of mathematics. As to this I can only say that this paper is in part an attempt to justify the teaching in schools of an amount of geometry much larger than has hitherto been thought possible, and this without increase of the time devoted to the subject. My own experience is that an early commencement of trigonometry can and should be made by some sacrifice of the large amount of time now devoted to algebra by pupils who are too young to understand the subject. Trigonometry is a valuable stimulus to geometrical thought, but is no substitute for it.

It only remains, in considering the scheme of geometrical education, to refer to the interdependence of the postulates which have been adopted. I do not believe that any detailed or systematic discussion of this is possible or advisable at the school age, but if towards the end of this period examples of deduction of some postulates from others were shown, it might be possible to lead the pupils to realise the ideal of a geometry based on a minimum number of assumptions concerning the nature of space. Such considerations would then acquire more reality for them in that they would have some acquaintance with mechanics and physical science and could therefore conceive the general ideal of a minimum of induction and a maximum of deduction. And I feel bound to state my conviction that every student of whatever subject, who proceeds to a university education worthy of the name, should gain some slight idea of the nature of non-Euclidean geometry. The simpler portions of the paper by Carslaw in the *Proceedings of the Edinburgh Mathematical Society for 1910*, or the description of Poincaré's well-known illustration given in Young's "Elementary Concepts of Geometry and Algebra," already mentioned on page 60, are within the grasp of any one who has even a slight acquaintance with the geometry of the circle; an appreciation of these ideas throws a light on the space-concept in particular and our so-called knowledge in general which can be gained in no other way. It may fairly be said that there are few portions of mathematical knowledge which have more educational value.

The nature of the current teaching of geometry in England is best understood by reference to recent history. Until a few years ago the use of Euclid's text in matter and sequence was universal owing to the regulations of examining bodies. Attempts to secure more freedom had not been wanting. The Association for the Improvement of Geometrical Teaching was formed with this object as early as 1871, and undoubtedly succeeded in awakening interest in the presentation of elementary geometry, though no tangible result appeared. In 1901, however, a paper read by Professor Perry before the British Association at Glasgow aroused fresh interest, and a committee was formed by the association to consider and report upon the teaching of elementary mathematics. In their report, issued a year later, they advocated a preliminary course of practical geometry, and stated that it was in their opinion unnecessary and undesirable that one text-book or one order of development should be placed in a position of authority.

Simultaneously the Mathematical Association, a development of the association founded in 1871, had formed a committee to consider the subject, and this body issued a report in May, 1902. In principle it proceeds on lines similar to the report of the British Association Committee, but it contains a definite statement that "it is not proposed to interfere with the logical order of Euclid's series of theorems." In effect it simplifies the development by introducing hypothetical constructions, changes the order of certain groups of propositions, and introduces an algebraic treatment (for commensurables only) of ratio and proportion. As a consequence of the work of these committees and their supporters the examining bodies yielded one by one, announcing that Euclid's order and methods would not be enforced, and setting questions involving numerical construction and calculation. Finally, the University of Cambridge issued in 1903 a syllabus indicating the amount of geometrical knowledge required for its entrance examination, and stated that the examiners would accept any proof which appeared to form part of a systematic treatment of the subject. This syllabus, which is still in force, omits portions of Euclid's text and introduces no additional matter.

Taking these changes in detail, we are first concerned with the postulates. The intention appears to have been the increase of their number in order to simplify certain proofs which had presented difficulty; proofs of facts which can be perceived by intuition are still retained, notably in regard to congruent figures and properties of the circle. The present scheme may therefore be not unfairly described as Euclid's ideal of the minimum number of assumptions, tempered by consideration for the age of the pupils — a consideration which renders the whole scheme meaningless from his point of view. It may be doubted whether such a compromise is likely to be successful. The broad lines of educational advance should be based on fixed principles and not on expediency. If the pupils can understand the development of a scheme based on a minimum of assumption, and can also understand this motive, let it be adopted as the highest ideal. But if not, as experience has shown with some certainty, the only logical alternative is to allow all possible intuitions as postulates. The educational advantages of this course have already been described, and it has the further advantage of lessening the amount of time and effort required, and so clearing the way for some study of more advanced geometry, which is not yet represented in any of the examinations referred to. Should any doubt be felt as to the persistence of Euclidean methods despite the abandonment of his ideal, consideration of the fact that even the elements of solid geometry are not included in school courses, except in preliminary work (to which reference will be made), may carry conviction. All examining bodies (the Civil Service Commission excepted) appear to imply by their schedules that sufficient training in schools can be obtained without it, the fact being, of course, that its omission is a survival of Euclid's order.

Thus it may perhaps be said that what is often called the abolition of Euclid must involve the complete abolition of his order and methods and the construction of another theory of development, and that in both respects the changes are incomplete. The structure, always of course fallacious in its foundations, has now been shattered and we are groping among the fragments.

Turning to educational methods, there are two important developments: the introduction of preliminary courses, and the use of numerical examples. The latter only requires brief reference; it must give greater reality and precision to the results which it is intended to illustrate, and there is common agreement that it has done this. The meaning and proofs of propositions are admittedly better comprehended, but the comprehension is too often of isolated results rather than structures of reasoning.

The object of the preliminary course is to enable the pupils to acquire some familiarity with the leading facts and concepts of the subject. For the most part such courses consist of exercises in measurement and construction, coupled (in some cases at any rate) with numerical introductions to or illustrations of the axioms. So long as such exercises are confined to the performance of constructions of known type there is little to be said for or against them in logic or philosophy, though they are, in the opinion of some, as deadening to the intellect as the excessive performance of algebraic simplifications. But when the angle properties of parallels and triangles are introduced by measurement with a protractor, instead of by turning a rod, and when we find an example such as: "Draw a triangle whose sides are 2, 3, and 4 inches and then draw the perpendiculars from each vertex to the opposite side. These lines should meet in a point; see that they do so," the matter becomes more serious, for numerical measurement has been substituted for intuition or demonstration and the impression is hard to eradicate. There is also some general introduction dealing with space-concepts, and here there is usually some allusion to objects in three dimensions; beyond this, solid geometry finds as a rule no place in such courses. Its persistent neglect by teachers, examining bodies, and writers of text-books is one of the most marked and regrettable features in the developments of recent years.

Apart from this, the main criticisms, from the point of view of this paper, to be directed against such courses are that they are not based on a gradual extension from previous experience and imagination, and that there is a distinct tendency, as has been said, to relate the postulates to numerical processes. Of the second I need not speak further; of the first it may be said that it violates the principle that the development of the powers of imagination, abstraction, and reasoning should be made continuously from experience and knowledge gained in daily life. To construct a preliminary course which consists of work concerning angles, lines, triangles, and circles, with perhaps a passing reference to a few surveying problems, is to place the child suddenly in a new world where things are replaced by abstractions, and to give him an occasional glimpse of his own sphere as from behind bars — bars which are not made thinner by assigning numerical measures to the lengths and angles with which he deals. In the alternative which I have suggested the endeavour is to lead him gradually to this new world of abstract thought and ideal truth, or, perhaps, to present it as an outcome of and one with that more limited world of which alone he is at first cognisant.

The result of this period of freedom has been summed up in a circular published by the Board of Education (No. 711, 1909), which contains luminous and practical suggestions to teachers, based on the experience of the Board's inspectors. For our present purpose the most striking statement made therein is that the time taken to acquire the matter of the first three books of Euclid varies from one to three years in different schools, and that it is where the work proceeds quickly that it is best, and nearly always where it proceeds slowly it is poor. The difference is ascribed to the manner of treating the earlier part of the work, with which the circular is mainly concerned. As to this, I will only say that, while its suggestions are far in advance of the matter contained in most text-books, it perhaps hardly lays sufficient stress on the need of development from the child's previous experience, nor does it suggest concrete problems requiring a considerable amount of imagination and reasoning. The importance of solid geometry is pointed out with some force, but no very definite suggestion is made as to the time or manner of its introduction in a deductive course, a point on which most teachers are in need of guidance, and especially those who now succeed in covering a matriculation course in two or three years.

The circular also deals with the general effect of the changes, stating that it has been beneficial. "Unintelligent learning by rote has practically disappeared, and classes, for the most part, understand what they are doing, though they often lack power of insight and have but a narrow extent of knowledge."

Had the changes already described been the only educational changes during this period, a fairly conclusive inference might be drawn. But it must not be forgotten that the modern secondary school, with graduates of teaching universities for its teachers (often trained) and a curriculum designed to develop all the pupils rather than to benefit the few of exceptional ability, has, during the same time, come into being, and the issue is thus confused. Greater comprehension, as due to improved teaching, would have been likely even though Euclid had not been dethroned. The older schools have for the most part been comparatively unaffected by such developments during this time. Speaking as one who has some experience, both as teacher and examiner, of these and other schools, I can but state my opinion that the improvement in the modern secondary schools is far greater than in the schools of other types, including those which have adopted the changes in geometry most fully. Devoting roughly an equal amount of time to the subject, they obtain better results at an earlier age. If I am right in this — and I believe that most men who have acquaintance with the various types of school in this country would confirm the statement — it follows that most of the recent improvement in modern secondary schools is due, not to any recent changes in the syllabus of geometry, but to the acquisition of teachers who not only understand the subject but also know how to teach it. Some confirmation of this view is given by an enquiry made some three years ago among the professors and examiners in the modern universities. They declared themselves as dissatisfied alike with the results of the older and more modern methods, the majority against the modern methods being the larger. The improvement in quality of knowledge was admitted in many cases; it was rather the material and its co-ordination that were criticised.

Finally, then, it may be said that improvements in teaching methods and in personnel of the teachers have produced their natural results, and to the teachers must be ascribed much of the admitted improvement. Schemes of geometrical education in this country are lacking in foundation, method, and extent, and this arises from the fact that Euclid's scheme — itself utterly unsuitable as an introduction to the subject—has been so far tampered with that hardly any scheme remains. So long as no attempt is made to devise a connected development based on the many intuitions which are common to all civilised beings before they reach maturity, so long will the subject realise a painfully small proportion of its potential value.

I have endeavoured in this paper to interpret the quotation at its head in its reference to my subject. I do not forget that children will have to do the work of the world and must be fitted for it, and I believe that a training such as is here described will assist them in this. But they will not be less fitted if their education provides them with widening and inspiring subjects for contemplation when they reach maturity, nor indeed is such fitness the sole end of life.

THE PLACE OF DEDUCTION IN ELEMENTARY MECHANICS

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THE PLACE OF DEDUCTION IN ELEMENTARY MECHANICS

A science consists of a definite class or of definite classes of entities, a set or sets of postulates relating them, and a series of deductions which are logical consequences of these postulates. In the earlier stages of its evolution it may be — it usually has been — that the sets of postulates contain redundancies. Only when the logical consistence of these sets has been investigated, and the number of independent postulates established, can the science be termed complete, for then only is it certain that the number of assumptions has been reduced to a minimum, and that no one of them conflicts with any other. Although the concept of a perfect science was attained by Euclid in connection with geometry, the first approximately successful presentation of a science in this form came as late as Newton, and then in connection with mechanics; geometry, on the other hand, has only been completed within the last generation, and this after struggles extending over two thousand years.

This historic distinction between geometry and mechanics implies a corresponding didactic distinction. It is the peculiarity of geometry, as opposed to other physical sciences, that its postulates, and many of the deductions which can be made from them, are and have long been the common property of civilised mankind. Who doubts, whether he has learnt geometry or not, that all right angles are equal, that only one parallel can be drawn through a point to a given line, or that any diameter divides a circle into two identical parts? Most, if not all, of the postulates of mechanics are, on the other hand, known only to those who have consciously adopted them. Indeed, many persons, otherwise educated, will dispute their truth. The problems before the teacher are therefore entirely different in the two cases. In geometry he steps into an inheritance of pre-acquired space concepts, crude perhaps, but formed beyond doubt; he can develop deductions with little trouble concerning postulates. But in mechanics he must set up the entities and develop the postulates from the commencement; he steps into no such inheritance as does the teacher of geometry.

With methods of exhibiting mechanical entities, and the choice of postulates for use in a first treatment of the subject, this paper is not concerned. It is assumed, however, in accordance with modern practice in most schools, that the number of postulates is more than the minimum. My first concern is to point out that, treat it how we may, the direct evidence which can be brought before a boy in support of any of these postulates is singularly narrow and unconvincing. And it is an essential part of the argument that this weakness should be exposed at the outset. Take, for instance, the triangle of forces; the pupil may fairly ask Within what degree of accuracy has it been demonstrated? Is it true whatever be the body on which the forces act? Is it true at all places and at all times? Is it true under any circumstances of motion? Is it true for forces of all kinds — electrical, magnetic, or any other?

Or again, take the proportionality between force and acceleration, if the subject be developed so that this is a postulate. The pupil may be convinced that, in his own locality and for some substances, the statement is approximately true. But does the functional relation between force and acceleration involve no other variables, for example, temperature? And is its form the same for all kinds of matter? May not the force be proportional to the square of the acceleration for some substances other than those used in the experiments? Pupils should be trained, and trained from the outset, to question in this manner the degree of accuracy of every measurement, and the generality of the circumstances under which each experiment is performed. It is scientific method, and education of the most practical and valuable character.

But, then, the pupils may say, what is the use of going further? Must not some better evidence be obtained? There are, of course, many reasons which can and should be given for going on in faith, but one of the most illuminating and interesting illustrations is contained in Faraday's verification of Coulomb's law of electrostatic attraction. Few boys fail to find interest in the picture of Faraday, basing highly complex calculations on Coulomb's crude experiments, testing them inside the highly charged iron box, and emerging from it with a demonstration that the law corresponds really closely with observed facts. The pupil thus realises the true importance of deduction as an aid to his very imperfect powers of observation. In place of building complacently on a foundation whose imperfections have been glossed over, too often with pulleys mounted on ball bearings and other viciously misleading trivialities, he has a sane idea of what he is doing. He sees that, on these meagre foundations, he is to erect a structure which will come into contact with practical experience at many points, and must be judged by its degree of accordance with such experience at all these points.

The structure having been erected on a number of these very dubious supports, it becomes necessary to examine their possible interconnection, to ascertain whether the truth or falsity of any one of these assumptions involves of logical necessity the truth or falsity of any others. To illustrate the process suggested, I have dealt with the postulates of statics, but the method is equally applicable to dynamics, or to any combination of mechanical postulates.

The customary assumptions in a first treatment of statics — suggested, and rightly so, by crude experiment — are three in number, namely, the triangle of forces, the principle of the lever, and the principle of moments for two forces acting along intersecting lines. All three have been adopted, but any or all may be true or false. Thus the possibilities are eight in number, as shown in the following table, and among them must the truth be sought:

	Triangle of Forces	Principle of Lever	Principle of Moments
1	true	true	true
2	false	true	true
3	true	false	true
4	true	true	false
5	true	false	false
6	false	true	false
7	false	false	true
8	false	false	true

Now it is possible to show, by logical deduction, that any two of these assumptions are necessary consequences of the third.[1] When this is done (and the proofs are well within the comprehension of a boy of seventeen), the alternatives 2 to 7 disappear, and the three assumptions have become one. The only possibility now being the simultaneous truth or falsity of all, the rough experimental results acquire greater import. If all three were false, it is trebly unlikely that they should every one accord fairly well with experiment, and the only alternative to the falsity of all is the truth of all. The probability of this truth has thus been strongly reinforced by processes purely logical in nature.

It is worthy of notice that the conventional deductive method does not give an equal amount of strength to the hypothesis. Postulating the triangle of forces, the principle of the lever and the principle of moments are obtained by logical deduction, the possible alternatives, five in number, being left thus:

Triangle of Forces	Principle of Lever	Principle of Moments
true	true	true
false	true	true
false	true	false
false	false	true
false	false	false

The proofs that *any* two of these postulates can be deduced from the third are not given, and the full power of deduction to reinforce assumption is not exhibited.

Treating other groups of postulates in the same manner, the structure is seen to be based not on weak, isolated

[1] If this is done, selecting any one assumption only, the consistence of the three assumptions follows at once, and the demonstration of consistence should not be overlooked.

supports, but on interlinked groups of such supports, the strength of the foundation being greatly increased by these interconnections. And finality is reached when the supports have been interlinked into groups, between which it can be demonstrated that no such logical interconnections are possible. This, I conceive, is the true aspect of Newton's achievement in the statement of his three laws of motion; he stated a number of consistent and independent hypotheses, and developed the whole subject from these by purely logical processes. I do not imagine that Newton had any such general concept of a science as is set out at the beginning of this paper, but he perceived, intuitively or sub-consciously, that mechanics could be based on three sets of postulates, each set referring to a different class of entities. In their statement his logic was defective, but this is trifling compared with the greatness of his achievement; he formed an ideal for mechanics similar to that of Euclid for geometry, and he attained practical success in its elaboration, in contrast with Euclid's decided failure.

We have now before us three distinct didactic treatments of mechanics — the old method, the method now current, and a development such as has here been suggested. The old method presents the science in its complete form, with no indication of its evolution. Three postulates are laid down, to be accepted in blind faith, and from them the subject is developed by logical processes; it is a course in applied deduction. The current method presents a number of mechanical assumptions, based on foundations whose strength is hardly discussed, and uses them in application to various problems. There is little or no attempt to discuss their logical interconnection, and certainly no suggestion of its scientific meaning; it is a course in applied computation. But if this current method were followed by a logical discussion, exhibiting mechanics as based on independent supports, each consisting of interlinked assumptions as has been described, I venture to suggest that it might fairly be called a course in applied mathematics.

The tendency of modern education, as it seems to me, is to lay undue stress on direct sensation as the one and only basis for faith. Undoubtedly education must find its origins, and these as widespread as possible, in direct sensation; but a false and very dangerous ideal is left, unless finally these origins are linked together by logical process, so as to give them their maximum strength and expose their ultimate weakness. Final contentment with a set of postulates which may or may not be inconsistent or redundant, and for which there has appeared little real justification, is vicious; vicious also is the attempt in a first course to develop any science from a minimum of hypothesis. The one method is an undue suppression of perception, the other an undue glorification. The first step of the teacher should be to develop a wide spirit of enquiry; the second should be to breed a "divine discontent" with the imperfections of perceptual evidence. Recognising its essential nature, our inevitable bondage to it, we may yet liberate ourselves, so far as may be, in each branch of knowledge. It is the peculiar function of mathematics to point the way to this freedom in each science, and it is here that modern developments of mathematical thought may yet find application in other sciences.

A COMPARISON OF GEOMETRY WITH MECHANICS

(A paper read before the Liverpool Association of Teachers of Mathematics and Physics)

A COMPARISON OF GEOMETRY WITH MECHANICS

The subject of this paper may at first sight seem unlikely to provide much opportunity for profitable discussion. When it has been said that the bodies which are the concern of mechanics move in the space which is the concern of geometry, and are therefore subject to the laws of that space, and when it is added that the same processes of arithmetic are applied to measurement in each case, it may be thought that the title is exhausted.

The comparison to which I invite you is not, however, directly concerned with the matter or the domain of either subject. My object is to compare them in their relation to mathematical education; to examine in how far each may fulfil the ideals of that education, and in how far each may supplement the deficiencies of the other. To do this, the basis of comparison must first be assured; that is, it is necessary that I should explain what are the ends which I conceive to be furthered by the teaching of mathematics. In the conflict of educational interests which has in the last fifteen years become so acute, mathematicians have borne their part, but they cannot be said to have spoken with one voice, whether in advocacy or defence. I do not know that this is to be regretted, for no progressive development is likely, except as the outcome of such differences; but the fact enforces, on any who would discuss mathematical education, a clear statement of their creed.

Even yet, however, the preliminary enquiry is unfinished. I have just said that differing views on the aims of mathematical teaching are held by those who are concerned with the subject. For the most part they are held consciously and can be explained at will. But an enquiry as to the nature of mathematics itself — the short question, "What is mathematics?" — is apt to produce no immediate or definite response. Too often, I fear, the only reply will be that mathematics consists of algebra, geometry, trigonometry, the calculus, and so on, arithmetic being, for some unaccountable reason, omitted from the category. Now this is a trifling with logic by just those people who ought, above all others, to be in this respect beyond suspicion. If they consider heat, light, sound, they can see why these are grouped under the common term "physics"; if they study atoms, decomposition, elements, they can defend the one term "chemistry" for these. But why the one term "mathematics" for algebra, geometry, calculus, and all the other branches? What are the common elements in these subjects which entitle them to a generic term? Failure to answer is a failure in logic, for to group entities in one class without cognisance of common elements among them is to offend the cardinal principles of classification and definition.

It may, I know, be said with truth that mathematics is concerned with reasoning — pure reasoning, if you will. But this is logic, and even those who are most uncertain as to the definition of mathematics are equally certain that there is some distinction between the study of mathematics and the study of logic. A necessary preliminary to the discussion which I have undertaken is, then, to define this distinction, and this must be my first task. After the explanation of the nature of mathematics, its relation to education must be discussed; our comparison of geometry and mechanics can then be developed.

The nature of mathematics may best be explained by showing its relation to the physical sciences. The study of any one of these commences by the assertion of a number of statements, on bases more or less uncertain. The assumption of laws of motion here on the earth, from observation of the planets, is a daring one, however exactly these laws describe the motions of the planets themselves; the direct evidence for the law of conservation of energy, or that for most other laws of physics and chemistry, is weak to a degree. To see general possibilities in a maze of results apparently unco-ordinated, to fashion various hypotheses to fit the facts as they are observed, is the function of the natural philosopher; it is no concern of the mathematician. The process is exemplified in the lives of any of the great natural philosophers — above all, perhaps, in the life of Newton, in that the development of his mind is known in such detail.

The natural philosopher, having thus fulfilled his first task, presents the hypotheses so formed to the mathematician, who accepts them for investigation without regard to the evidence for or against them. Let us call these hypotheses A, B, C, ... The mathematician does three things:

First, he makes deductions from the hypotheses; that is, he says to the natural philosopher: "If what you say is true (and that is no concern of mine) then must certain other statements P, Q, R, . . . also be true; further, if certain of these be true, then must certain of A, B, C, . . . also be true." The natural philosopher then tests the truth of P, Q, R, . . . and finds his hypotheses strengthened or destroyed as the case may be.

Next, the mathematician informs the natural philosopher that he has examined the hypotheses A, B, C, . . . and finds that they are consistent; that is, that there is nothing in any one of them to negative any other by the force of logic. It should here be borne in mind that all measurements are more or less inexact, and it is therefore possible, from actual observations, unknowingly to frame hypotheses which are logically contradictory one of another.

Finally, the mathematician informs the natural philosopher that some of his assumptions were unnecessary; that is, that some are logical consequences of others, and so need not have been assumed. Or he tells him that all were necessary, no one being deducible from some or all of the others. That is, he investigates the possible redundance of the hypotheses, and tells the natural philosopher to how many distinct assumptions he is really committed.

Shortly, then, the mathematician receives sets of hypotheses from the natural philosopher, tests their consistence, examines how many assumptions are in fact involved, and develops logical consequences from them. His function is entirely impartial — material truth is not his concern, but is that of the natural philosopher; but I would point out to you that the really great natural philosophers, from Archimedes downwards, have been men who in themselves combined both these functions. In no one have they been exemplified more nearly in their due proportions than in Newton; there have been greater mathematicians and there have been men whose power of speculation was more rapid and prolific, but no other man has so balanced the one with the other, and therein lay the secret of his pre-eminence.

I must not, however, leave an impression that the mathematician has no use for his imagination and performs no creative functions, though the slightest consideration of any branch of the subject suffices to show the fatuity of such an idea. The straight line at infinity, the circular points at infinity, complex numbers, lines of force, the ether, at once occur to the mind as commonplace instances of creations in which the natural philosopher has had no direct share. His entities are groups of sensations, and relations between them are suggested by further sensations; but the mathematician creates other entities which would forever remain beyond the vision of the natural philosopher, and has often, by their means, revealed unsuspected unities in his work. The creations of the mathematician which are opposed to the suggestions of the senses must forever rank among the most striking of all human creations, and the shortness of this allusion to them is only excused by their irrelevance to the matter under discussion.

It is now easy to explain the function of mathematics in a well-balanced education. The purpose of teaching natural science is to develop in combination the powers of observation and speculation; to train the pupil to use his senses and, from the material which they afford him, to frame hypotheses which accord with that material as nearly as may be possible. The purpose of teaching mathematics is to enable him to develop the consequences of these hypotheses, to test their consistence, and to reduce them to the minimum of pure assumption. I say to train him in these things, but it were perhaps better to speak of setting them before him as ideals for which he must strive in his dealings with things as he finds them. A man who has in his mind this chain of processes, observation, speculation, proof of consistence in speculation, rejection of redundant speculation, and finally the erection of deductions on this foundation, is in possession of an intellectual creation which, in beauty alone, is worthy to rank with the creations of poetry, music, or art; and beyond this, it is a possession which, in so far as it guides his life, will make of him a more efficient labourer and a better citizen.

We must now examine the relations of geometry and mechanics to the description of mathematics which has been given, and so ascertain which of the three processes which have been said to pertain to the mathematician are most clearly exemplified in each subject. First, however, I must point out that this description of mathematics consigns geometry, and even arithmetic, to the domain of applied mathematics. The one shows the application of mathematics to number, the other to space; in neither is the underlying essence seen, except through illustrations of one kind or the other. But the first thing to consider with respect to a machine is to see what it does, rather than to find out how it performs its functions, and we need not, therefore, cavil at the idea that our so-called branches of mathematics are really applications of mathematics, by examination and contrast of which we may perceive the underlying unity.

When a child commences geometry, what is his personal position in regard to space? Certainly it is far different from his position in regard to matter when he commences mechanics, or from his position in regard to light or heat when he commences physics. Personal experience of space — his space — has been forced on him from his earliest days in his every movement, and from this experience he has formed ideas or hypotheses concerning a space beyond, which he cannot reach with his own limbs, and is therefore not his space. Show him a triangle ABC cut out in cardboard, and make another by taking a tracing of the corner A and producing its sides until they are equal to AB and AC, and then ask him if these triangles are an exact fit. He will not have much doubt about this, nor will he question the equality of all right angles, if it be similarly suggested to him. Next draw a circle, rule a diameter, and ask him if one part of the curve so divided will fit the other. About this also he will not have much doubt. He will assert the truth of these things wherever and whenever the acts are performed, not merely in his own personal space; that is, he asserts them for the imagined space beyond, concerning which he has formed ideas fashioned from experience in his own space. Now as a matter of fact, as is well known to all of us here, the third statement concerning the circle is a necessary and inevitable consequence of the first two. These three assertions are in truth redundant, to use our technical phrase, as also are many others which the child will make with equal certainty concerning this imagined space beyond his personal space.

Thus, in commencing the study of geometry, we apply mathematics to a subject — space — concerning which the pupil is already in possession of a set of beliefs which are, as a matter of fact, interdependent one on another. And these beliefs are held with great tenacity; nothing will induce the pupil to doubt any of them, for they have, in truth, become a part of his very being. It is further to be remarked, for purposes of comparison, that we elders and experts have not recanted any of these beliefs; we hold them as does the child, though we know them to be but beliefs; our imagined space has the same properties as his.

Next let us consider the position of a pupil in regard to matter, when he commences the study of mechanics. It is true that here also experience has been forced on him from his earliest days, but it is of the narrowest kind, for it concerns little more than the sensation of lifting, and few assumptions are made. He will say, if two boxes are known to be exactly alike and one feels heavier than the other, that this one has something inside it; and, which is a consideration fully as important, if neither feels to him heavier than the other, he will refuse to say that each is empty, pointing out that the possible contents of one may be so light that he does not notice them. But he is unconscious of the concept of mass (or nearly so), of the triangle of forces, and of any of the mechanical concepts and laws which are to us so familiar. More, he only receives them with difficulty; explanation, illustration, and distinct effort are required before the concepts are grasped and the statements accepted as more or less nearly corresponding with the results of experience. Some of these statements, as, for example, Newton's first and third laws, are indeed received with incredulity; who has not heard it argued that the horse pulls the cart more than the cart pulls the horse?

At the outset, then, there is a sharp distinction between geometry and mechanics. In geometry the pupil commences with a number of ideas and beliefs concerning space—beliefs held so tenaciously that to question them in any way produces bewilderment. In mechanics he has but a few crude concepts and few or no beliefs, and he greets with disbelief some of those held by his elders. In the terms of our earlier discussion, he has, in regard to space, been his own natural philosopher—though all unconsciously—and has produced a set of beliefs which are ready for examination by the mathematician within him; but in regard to the matter the natural philosopher within him has yet to play his part before the mathematician can receive the materials for his task.

Returning to geometry, to which of the three processes performed by the mathematician should the pupil first be led in his study of this subject? Shall he find whether his beliefs concerning space are consistent with one another, or enquire whether some of them may not be logical consequences of others; or shall he, without any such analyses, pass on to make deductions from his spatial creed, deferring its analysis for the time at least? There can, I think, be little doubt as to the answer. To analyse the number of his beliefs, to question and examine their foundations, is repugnant to any normal child, though welcome to most educated adults; we should then avoid these processes and allow the study of geometry to centre round deductions from the pupils' spatial beliefs. In primary education this subject can only illustrate the deductive side of mathematics; it can do little or nothing to show the analytical processes which are concerned with consistence and redundance.

We have said that, in commencing mechanics, the child must play the natural philosopher before the mathematician can find scope for his efforts. This subject thus provides the first example of the methods of natural philosophy, and the part of the teacher is one of supreme importance. Shall he allow the child's fancy to roam whither it will, or set him down to prescribed tasks with definite ends, or endeavour to lead him to the examination of natural phenomena by those methods which history has shown to be most productive?

There is, I know, a strong movement in favour of relying upon the "interest" or "play" motive in primary education; and the vocal organs of this movement are highly developed. There is also, I believe, a perhaps stronger movement, whose vocal organs are as yet rudimentary, in favour of severe limitation of the use of these methods. To some extent each party misjudges the other. The upholders of interest accuse their opponents of enforcing meaningless drudgery, while these in their turn accuse their opponents of allowing education to degenerate into disconnected fripperies, and each is more or less unjust in so doing. For myself, and speaking only in regard to this present subject, I would say that the mere performance of prescribed tasks, which arise apparently from the brain of the teacher or the designer of apparatus for use in schools, can have nothing to do with the development of the spirit of natural philosophy. On the other hand, to allow education to be dominated by interest is to cast aside all hope of discipline — not discipline of class by teacher, but discipline of pupil by himself. To show the exact meaning of this statement, let me recall to you the behaviour of young boys learning to play football or cricket. Their interest is to toss the ball aimlessly from one to another, and to scamper about as irregularly as young horses; as much discipline is required to induce them to play the game as is used in many a class-room to ensure the due performance of work. The effect of this discipline is to replace casual interest in passing sensations by continued interest in definite achievements; in like manner, the effect of our teaching should be to leave the pupils with this desire for achievement, rather than with a mere craving for a tickling of the fancy. I do not mean to imply that interest is not to be considered; on the contrary, I will assert most emphatically that teaching which is met by a continued lack of interest must of necessity be at fault. But I assert also that the mere presence of interest is insufficient as a testimony to the value of education; and, further, that courses which are chosen on the basis of maximum immediate interest are, in all probability, thereby grievously at fault.

We must then, in commencing the study of mechanics, guide our pupils to the attitude of the natural philosopher towards the phenomena which he studies. But what is this attitude? How far is it concerned with matters of common experience, and how far with the more artificial happenings of the laboratory? And is there any order of priority as between common experience and laboratory investigation? Here there can be little doubt, whether the question be viewed in the light of history or of common sense. The foundation of this work must be common experience; it must be reviewed, questions must be asked to be answered in the laboratory, and hypotheses must be made to be tested there. Two illustrations may show my meaning more precisely.

The customary experimental introduction to the triangle of forces starts from nothing; bodies are suspended and balanced on pulleys without suggestion of a why or a wherefore. The attitude which seems more natural is to give to or draw from the pupils everyday illustrations of the balancing of two forces by a third, and to obtain from them as much information as possible about this third force; namely, that it always exists, that it lies between the two original forces, and is nearer to the larger of these. Then the question of the law of determination arises; this is a question to be answered in the laboratory, and the answer found must receive every possible verification.

Next, let us consider some hypothesis which may first be made and then be tested in the laboratory. Several such hypotheses are possible, but perhaps the best is found in the motion of a falling body. To surmise a rule for the composition of two forces is beyond any child; but speculation on the actual law of acceleration of a falling body, its dependence on distance or time, is natural from the outset, and should be encouraged before investigation is undertaken. It is not difficult to divide mechanical investigations into two classes; namely, those in which initial speculation is possible, and those in which the pupil can only ask a question with no power of suggesting a possible answer, and this division does much to simplify the early treatment of mechanics.

So much, then, for the part of the natural philosopher in mechanics; the pupil has now a body of statements which he is willing to accept, for the time at any rate, though he recognises them as assumptions which may be unwarranted. Observe that I call them statements, not beliefs, and do not refer to them as his mechanical creed; this because they are accepted in no such blind and unreasoning faith as the assumptions in geometry to which I gave such terms.

The mathematician in the child is now presented with these accepted statements; what is he to do with them? Can any or all of his three functions here be exercised with profit to the pupil in that he will so gain some idea of their nature and import? In particular, can we now develop the two questions of consistence and redundance in speculation, which were put on one side in the consideration of geometry?

First consider deduction. This process is possible and necessary, but its area of application is limited in comparison with geometry; the link polygon, the centre of mass and its motion, the path of a body under gravity, are instances, and others will occur to you. But there is no such wealth of propositions and riders as is found in geometry; most of the problems in mechanics are applications of principles rather than logical deductions.

Next take analysis. It must first be said that the application of such processes to mechanics is at least legitimate for pupils in schools. To question and investigate their geometrical creed at this age must lead to perplexity and boredom, as has already been said; but the accepted statements of mechanics lie under no such ban. They are assumptions made consciously on the best evidence obtainable, and investigations tending to their confirmation are at least acceptable; this in contrast to geometry, of which the reverse has just been said.

To discuss in detail the possibility of undertaking this analysis for mechanics would lead me too far. Here I can only say that for certain parts of mechanics, at any rate, it is not only possible but easy and interesting. For example, any boy can understand proofs that from any one of the three assumptions known as the triangle of forces, the principle of the lever, and the principle of moments, the other two can be deduced; he can thus see that only one of them need be assumed, of course that one for which the evidence is strongest; and he can see further that there need be no fear that, at some future time, one of these assumptions will be found in logical conflict with another. For the meaning of such conflict there is ample illustration in mechanics and physics.

This concludes our comparison of geometry and mechanics. What, in short, is the outcome? First, we have seen that there are three leading processes in mathematics,— deduction, analysis of consistence, and analysis of redundance, — these being exercised by the mathematician on material presented to him by the natural philosopher. Next, geometry was seen to be well suited for exercise in deduction, but not suited for illustration of the other processes; this is because the natural philosopher acted too early in regard to space, and will not now brook criticism of his results. Finally, this deficiency was seen to be remedied by mechanics, which should provide the first deliberate exercise in natural philosophy, and so present material better suited for illustration of these remaining processes of mathematics.

It may be asked whether it can be hoped that a pupil may end his school career with these ideas fully developed. Frankly, I do not think that he can; indeed, I am sure that he cannot. We may regard such ideas as mountain peaks, standing far above the mists of the particular applications, two of which we have in particular been discussing to-night. He who has scrambled longest among the mists sees these peaks most clearly; some indeed have pierced the clouds and seen them in their full beauty — have even scaled them and viewed one from another. The function of the teacher is to lead the child through the mists by such ways as will give him glimpses, even though they are but shadowy, of the higher ground beyond. These will remain and develop in minds to which they are suited — minds, I am convinced, far more common than is generally supposed.