Faculty Working Papers

ESTIMATION ERROR IN INCOME DETERMINATION

William Steve Albrecht

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College of Commerce and Business Administration
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The recent Trueblood Commission concluded that financial statements should:

1) Aid users in assessing risk

2) Separate information which is factual from information which is interpretative.

3) Disclose assumptions and judgments which enter into preparation of the statements.

4) Group and segregate resources and obligations according to the relative uncertainty of their realization of liquidation.

5) Disclose the inherent imprecision resulting from the necessity to use assumptions and estimates in many aspects of financial reporting.[1]

These objectives imply that we should consider the probabilistic nature of financial data reported. The commission recognized that financial statements attempt to assess past accomplishments in an environment in which the measure of that accomplishment is heavily dependent upon inherently uncertain future events.[2] Notwithstanding the recognition of this problem by many writers,[3] contemporary practice and theory embrace the traditional implicit

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[1] Objectives of Financial Statements, Report of the Study Group on the Objectives of Financial Statements, American Institute of Certified Public Accountants, October, 1973.

[2] For evidence concerning the seriousness of the estimation problem in financial reporting, see pages 15, 17, 22, 23, 34, 36 and 45 of the Trueblood Report. These excerpts imply that imprecision and uncertainty are not only serious problems in reporting but that current reporting practices which ignore their existence can even mislead headers of financial statements.

[3] See, for example, John K. Shank, "Income Determination Under Uncertainty: An Application of Markov Chains," The Accounting Review, January 1971, pp. 57-66; Richard P. Brief and Joel Owen, "The Estimation Problem in Financial Accounting," Journal of Accounting Research, Autumn 1970, pp. 167-77; Arthur L. Thomas, The Allocation Problem in Financial Accounting Theory, Studies in Accounting Research #3, American Accounting Association, 1969 and Henry B. Reiling and Russel A. Taussig, "Recent Liability Cases--Implications for Accountants," Journal of Accountancy, September 1970, pp. 39-53.

Reiling and Taussig even conclude that many lawsuits might be avoided if auditors would simply indicate that they are not certifying to deterministic facts, but are expressing an opinion on estimates from a probability distribution.

assumption of certainty.

The convincing arguments made by the Trueblood Commission and others present a strong case in favor of the explicit recognition of uncertainty in financial statements. In view of this support, the purpose of this study is to first provide a framework within which the uncertainty of financial statements can be interpreted and then illustrate, using a case study, the degree of uncertainty that can exist in financial statements. In particular, the uncertainty and resulting imprecision inherent in the net operating income of a test Company will be quantified and reported. This quantification requires several constraints which will be discussed later.

THE NATURE OF ACCOUNTING UNCERTAINTY

In accounting two variables are of considerable interest.[4] They are 1) the flow of earnings over a specified period and 2) the stock of wealth or well-off-ness at the end of that period. Traditionally, the flow variable has been measured via the income statement and the stock variable has been presented by the use of a balance sheet or statement of position. Over the lifetime of an enterprise, the earning or flow variable can be defined as:

$$E = SE - SB + D - C \quad \text{where}$$

SE = Stock of wealth at the end of a firm's life

SB = Stock of wealth at the beginning of a firm's life

D = Distributions of the stock to the owners

C = Contributions of stock to the firm by its owners

Assuming that there are no counting errors in these variables, the lifetime earnings can be measured with certainty. However, few financial statement

[4]It is recognized that readers are a heterogenous group and there may be many other variables of considerable interest. These two are isolated because, historically, they have been of prime importance.

readers are willing to wait until the termination of a firm's life to get an indication of the entity's performance. A partitioning of the entity's life with associated intermediate reports of enterprise goal accomplishment is demanded. But, since financial statements attempt to assess past accomplishments in an environment in which the measure of that accomplishment is heavily dependent upon inherently uncertain future events, such a partitioning results in estimates of many of the flow variables. Specifically, the forecast error resulting from estimates of revenue and expense allocations combine to subject the earnings measurement to a considerable amount of uncertainty. This type of uncertainty is labeled forecast or estimation error.

Another source of uncertainty in financial statements results from measurement or counting errors. There are always limits to the degree of accuracy with which properties can be measured. Sources of measurement error include differing conditions under which measurement takes place, limits imposed by the capacity of hand, eye and construct used as a measurement standard, and pragmatic limitations by the users such as the cost of measuring greater or smaller degrees of uncertainty.[5] Measurement error is most often related to the stock variables where physical counts or inventories are taken. There are flow variables such as the number of sales during a period or the number of hours worked by employees that are subject to measurement or counting error, but the amount of such error is usually rather minimal. Rather, it is forecasting error that accounts for most of the uncertainty in income determination.

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[5] For a more complete discussion of error sources, see Ackoff, Russel L., Scientific Method, New York: Wiley and Sons, 1962.

Forecasting error can arise from two sources. There can be forecasting error about the amount of an item (e.g., salvage value of fixed assets) and there can be forecasting error about the timing of an item (e.g., service lives of fixed assets). Forecasting error about the amount of an item is generally more serious because it affects the magnitude of the cash flows of a Company while timing uncertainties tend to wash out over several periods.

In summary, it is uncertainty about the future that gives rise to the problem of uncertainty in measuring period earnings.

MEASUREMENT OF UNCERTAINTY - AN ILLUSTRATION

The Constraints

The problem of illustrating the uncertainty inherent in income determination is a non-trivial task. Aside from the need of intimate access into the detailed financial records of a test Company, there are many complicated statistical problems in both assessing the uncertainty inherent in each determinant of net inocme and in aggregating the uncertainty variables to find the cumulative effect on net income. As a result of these difficulties, several limiting constraints must be invoked in this study. The first is that the quantification of uncertainty will focus on forecasting error only. The two reasons for this constraint are: 1) measurement error is rather minimal when compared to forecasting error and 2) the study is an ex post study and without actually simulating all conditions exactly as they were when the variables were measured, it is impossible to accurately quantify the measurement error.

A second constraint concerns the techniques used for measurement. Since both the relative frequency and subjective or personal interpretations of probability theory will be used to quantify uncertainty, it is necessary

to have sufficient data to construct probability distributions. In order to provide this repetitive data, monthly earnings were analyzed. Although the test Company does prepare monthly financial statements, it is recognized that the degree of uncertainty inherent in monthly calculations may differ markedly from that which exists in either quarterly or annual data.[6] This constraint does make two contributions, however. First, it allows the period of interest to be short enough so that an accurate determination of the ex post actual expenses and revenue can be determined, and second, it provides some guarantee that over the relatively short period of interest there are probably no significant changes in the corporate structure or the economy that would cause the amount of uncertainty to change substantially.

In all cases, accounting methods currently employed by the Company will be used to provide the estimates. Since the purpose of the study is to determine the extent or degree of uncertainty that actually exists, the appropriateness of their methods will not be questioned.

Finally, a stable or moderatley stable price level is assumed.

The Test Company

The test Company is an enterprise which operates several private vocational schools offering courses in computer programming, electronics, medical technology and hotel management. The Company was chosen for analysis because much of the uncertainty and imprecision inherent in the net operating income can be quantified by examining enrollment revenues,

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[6]Since it is the artificial partitioning of the entity's life that creates the uncertainty, the affect of the use of monthly data is to increase the amount of uncertainty over that which would exist if annual or quarterly data were used.

cancellations by students and the write-off of uncollectible accounts receivables. The Company experiences a significant number of cancellations after the start of each course which must be provided for at the time of enrollment. The Company is located in a large midwestern city and is audited by one of the "big eight" accounting firms.

The operations of the test Company can be summarized by the following expense and revenue classifications:

REVENUES
- Enrollment tuition
- Miscellaneous

EXPENSES
- Cancellations
- Bad Debts
- Amortization
- Depreciation
- Salary Expense
- Professional Services
- Employee Benefits
- Payroll taxes
- Utility Expenses
- Rental Expense
- Supplies Expense
- Insurance Expense
- Advertising Expense
- Miscellaneous Expense

(Salary Expense through Miscellaneous Expense: Cash Expenses)

Enrollment tuition is recognized on a quasi-installment basis and there is <u>timing</u> uncertainty concerning how much should be recognized on the contract date. Cancellations and bad debts both require estimates of <u>amounts</u> to be realized in future periods but which must be matched against current period's sales. Depreciation and amortization expenses require estimates of the <u>service lives</u> and <u>salvage values</u> of the fixed and intangible assets. None of the remaining expenses, although some are possibly subject to measurement error, require estimates or forecasts. Thus, in the analysis of uncertainty these expenses will be aggregated as cash expenditures and their distribution will be singular normal. The quanti-

fication of uncertainty in the test Company will thus focus on the timing uncertainties inherent in revenues, depreciation and amortization and the amount uncertainties inherent in the cancellation and bad debt expenses. Using both the subjective and relative frequency interpretations of probabilities, distributions will be calculated for the estimation error in each of these items. The variance or standard deviations of these distributions will then be aggregated to find the joint distribution of the estimation error in net operating income. The result will be a quantification of uncertainty for the September, 1974 net operating income.

Measurement Techniques

Two different techniques will be used to quantify the uncertainty of the income statement items. The first will involve the use of relative frequency probabilities and will be used on the revenues, cancellations and bad debt expense. In each of these accounts, the Company uses a model to estimate the amount that should be recognized in the current period. Since the models used by the Company provide estimates, estimation error can be assessed expost by determining the actual amounts of each of these items and subtracting them from the a priori estimates. The historical estimating error is calculated for the past thirty months and the distribution of the error provides a surrogate measure of uncertainty in the current period. This method assumes that the uncertainty in the current period does not differ significantly from that of recent periods. For this assumption to be valid and before it is possible to extend the variance of the historical estimating error to the current estimate, it is necessary that the errors are independent, normally distributed, have a mean of zero and a constant variance. In each case, several tests of these assumptions will be employed.

The second technique involves the use of subjective probabilities and will be used on the depreciation and amortization expenses. Since it is impossible in the test Company to view a time series of the historical estimating error of the lives and salvage values of the fixed and intangible assets, the probability distribution of the estimating error must be expressed as a degree of belief rather than a long-run frequency. The elicitation of subjective probabilities provide a convenient and theoretically correct method for extracting that degree of belief. In the sections that follow, variances or measures of uncertainty will be calculated for each of the income statement items.

Revenues

Except for its correspondence courses, it is the policy of the Company to record as revenue a percentage of the course tuition when the contract is signed and the down payment is received. The amount recognized immediately is intended to approximate the incurred costs of procuring the contract, including advertising, selling expenses and commissions. The unpaid balances of these contracts are reflected as tuition receivable in the balance sheet and the unearned portion of tuition receivable is classified as unearned income and included in current liabilities. The unearned tuition is recorded as revenue using the straight-line method over the length of the course. Provision is made for estimated losses on receivables that have been reflected as earned. Accordingly, revenue comes from three sources: 1) that amount recognized at the contract date to cover selling, advertising and commission costs, 2) the amortization of the balance on a straight-line basis over the life of the contract and 3) miscellaneous income.

The first component of revenue should theoretically equal the costs

of procuring the contract. These costs vary across periods and individuals and hence it is impossible to know a priori exactly what the amount of these costs will be. The consequence is that selling, advertising and commission costs associated with each period are not known with certainty for two or three months after the contract date. As a result, in order to properly match expenses with revenues, the Company is obligated to estimate the procurement costs in the enrollment period. In an effort to match these costs, the Company has, based on past experience, determined that a fixed percentage of the tuition receivable is a good approximation of these costs. Their approximation has varied from a low of twenty percent of tuition to the present rate of thirty percent of enrollment fees. Stated algebraically, their model is $.30(R_t)$, where R_t represents enrollment revenues. Since the theoretical justification for this model is to cover procurement costs, imprecision or uncertainty is introduced in all cases where their estimate does not exactly equal the actual procurement costs. Hence, uncertainty or imprecision is introduced when

$$.30(R_t) - (A_t + C_t + S_t) \neq 0 \text{ where}$$

A_t = Advertising costs incurred in period t

C_t = Commissions incurred in period t but not known until a later date

S_t = Selling costs incurred in period t but not known until a later date

A lag of at least three periods is needed to accurately identify all of the commission and selling costs that relate to period t.

The following data were used to calculate the variance of the historical estimating error:

Insert Exhibit 1 here

EXHIBIT 1

ESTIMATING ERROR IN REVENUES

000's

Period	1/72	2/72	3/72	4/72	5/72	6/72	7/72	8/72	9/72	10/72
$.30(R_t)$	33.9	32.4	54.8	35.6	37.9	41.9	26.0	45.9	40.6	24.2
$A_t + C_t = S_t$	27.3	31.9	29.0	31.8	34.8	32.3	30.8	33.6	32.0	30.3
Residual	6.6	.5	25.8	3.8	3.1	9.6	(4.8)	12.3	8.6	(6.1)

10/72	11/72	12/72	1/73	2/73	3/73	4/73	5/73	6/73	7/73
24.2	30.9	37.2	24.0	20.7	28.7	24.1	23.2	29.6	25.4
30.3	32.0	28.1	28.4	22.8	23.9	26.2	24.6	26.1	24.0
(6.1)	(1.1)	(9.1)	(4.4)	(2.1)	4.8	(2.1)	(1.4)	3.5	1.4

9/73	10/73	11/73	12/73	1/74	2/74	3/74	4/74	5/74	6/74
32.4	15.9	8.8	42.6	28.9	11.6	25.2	21.8	31.3	28.5
21.0	25.8	23.5	18.3	19.1	17.9	21.3	19.3	21.5	17.6
1.4	(9.9)	(14.7)	24.3	9.8	(6.3)	3.9	2.5	9.8	10.9

The mean of the residual series is approximately zero and the variance is $\$^2 79,912,612$ (Standard deviation of $8,940).

The second component of revenue has no variability or uncertainty. As indicated previously, when a contract is signed the unearned portion of tuition is classified as deferred income and amortized on the straight-line basis over the length of the contract. The time period is fixed and so there is no uncertainty with regard to the time dimension. The fact that a significant number of students drop out of school would introduce uncertainty if the income had been recognized, but since there is generally both an asset and a liability, the cancellation by a student has no effect on revenue. In those cases where the revenue is recognized prior to cash collection, the uncertainty in amount is accounted for by the cancellation and bad debt provisions.

The third component of revenue, miscellaneous income, is recognized on a cash basis.

The result is that total revenues can be characterised by the distribution of the first component and has a variance of $\$^2 79,912,512$.

<u>Cancellations</u>

Cancellation expense is the largest contra-revenue account of the test Company. It represents the reduction in revenues caused by students dropping out of school and not honoring their contracts. Over the past thirty months, approximately two out of every five enrollees subsequently withdrew from school. This massive disenrollment caused net revenues to decrease by forty-three percent. Because of the magnitude of this expense, a provision must be made for the estimated losses on receivables that have been reflected as earned. This provision subjects the account to uncertainty of <u>amount</u>. The

model used by the Company to estimate the losses from cancellations is:

$$.62(AR - UI) \quad \text{where}$$

AR = Accounts Receivables

UI = Unearned Income

The quantification of uncertainty is again based on the historical estimating error and the data are presented in Exhibit 2.

Insert Exhibit 2 here

The mean of the residual series is approximately zero and the variance is $\$^2 218{,}227{,}558$ (Standard deviation of $14,772).

Bad Debts

The bad debt expense represents that amount of the firm's receivables that are expected to become uncollectible in the future. It is associated only with students who do not drop out of school and is recognized as an expense prior to the actual write-off in order to properly match expenses with revenues. Because it is an estimate of an amount that will only become known with certainty at some future period, it is subject to uncertainty of amount at the time of recognition.[7] The model used by the Company to estimate the losses from bad debts is:

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[7]Because of the probabilistic nature of the bad debt expense account several researchers have suggested nondeterministic methods for treating it. See, for example, Schroderheim, Goran, "Using Mathematical Probability to Estimate the Allowance for Doubtful Accounts," The Accounting Review, July, 1964, pp. 679-684 or Benishay, Haskel, "A Stochastic Model of Credit Sales Debt," Journal of the American Statistical Association," December 1966, pp. 1010-28.

EXHIBIT 2

ESTIMATING ERROR IN CANCELLATIONS

000's

Period	1/72	2/72	3/72	4/72	5/72	6/72	7/72	8/72	9/72
Provision	37.4	42.9	43.1	50.5	56.0	55.5	55.5	55.5	59.5
Actual Canc.	39.4	62.5	49.1	30.3	45.2	52.6	51.3	65.0	67.3
Residual	(2.0)	(19.6)	(6.0)	19.8	10.8	2.9	4.2	(9.5)	(7.8)

10/72	11/72	12/72	1/73	2/73	3/73	4/73	5/73	6/73	7/73	8/73
55.0	62.5	54.5	54.5	53.5	51.0	51.0	51.5	49.0	41.0	41.0
43.5	49.3	86.0	58.9	38.1	34.2	37.3	60.8	40.9	76.8	44.9
11.5	13.2	(31.5)	(4.4)	15.4	16.8	13.7	(9.3)	8.1	(35.8)	(3.9)

9/73	10/73	11/73	12/73	1/74	2/74	3/74	4/74	5/74	6/74
41.5	42.5	46.0	46.5	42.5	41.5	37.5	40.5	40.5	43.0
63.4	42.2	28.8	41.9	52.9	44.6	56.8	24.2	27.9	24.7
(21.9)	.3	(17.2)	4.6	(10.4)	(3.1)	(19.3)	16.3	12.6	18.3

.08(AR - UI) where

AR = Accounts Receivable

UI = Unearned Income

The quantification of the uncertainty is again based on the historical estimating error and the data is:

Insert Exhibit 3 here

Again, the mean of the residual series is approximately zero and the variance and standard deviation are $\$^2 13,883,426$ and $3,726, respectively.

Tests of the Assumptions

Before it is possible to extend the distribution and associated parameters of the historical estimating error to the current estimate, it is necessary to insure that the residuals are independent, have a constant variance, a mean of zero and follow a normal distribution. If the models used by the Company have not been successful in removing all of the systematic and course errors, forecasting of the error distribution is impossible. Exhibit 4 presents the results of several tests of the assumptions.

Insert Exhibit 4 here

A close examination of the results of the residual tests suggest that it is plausible to assume that the Company's models are adequate in removing the systematic and course errors from the three series and it can be assumed

EXHIBIT 3

ESTIMATING ERROR IN BAD DEBTS

000's

Period	1/72	2/72	3/72	4/72	5/72	6/72	7/72	8/72	9/72
Provision	4.7	4.9	5.7	5.7	6.7	7.4	7.3	7.3	7.3
Actual B.D.	9.7	7.3	6.7	3.4	8.8	6.9	4.0	11.1	8.7
Residual	(5.0)	(2.4)	(1.0)	2.3	(2.1)	.5	3.3	(3.8)	(1.4)

10/72	11/72	12/72	1/73	2/73	3/73	4/73	5/73	6/73	7/73	8/73
7.9	7.3	8.3	7.2	7.2	7.1	6.8	6.8	6.8	6.5	5.4
12.4	7.4	2.0	15.8	5.0	7.0	7.4	3.5	4.4	6.7	8.3
(4.5)	(.1)	6.3	(8.6)	2.2	.1	(.6)	3.3	2.4	(.2)	(2.9)

9/72	10/73	11/73	12/73	1/74	2/74	3/74	4/74	5/74	6/74
5.4	5.5	5.6	6.1	6.2	5.6	5.5	5.0	5.4	5.4
4.9	16.1	3.0	2.6	4.8	1.9	5.1	5.7	1.0	.1
.5	(10.6)	2.6	3.5	1.4	3.5	.4	(.7)	4.4	5.3

EXHIBIT 4

RESIDUAL TESTS

	REVENUES	CANCELLATIONS	BAD DEBTS
Independence			
1. Plot of residuals vs. time	No significant trends among residuals	No significant trends among residuals.	No significant trends among residuals
2. Runs Up and Down--Non-parametric test for nonindependence	Actual number of runs is 20. Since expected number is 19.67 with σ of 5, no nonindependence.	Actual number of runs is again 20. Since expected number is 20.33 with σ of 5, no non-independence.	Actual number of runs is again 20. Since expected number is 20.33 with σ of 5, no non-independence.
3. Overall Chi Square Test	Chi Square Statistic is 11.856 with 24 degrees of freedom. Very insignificant.	Chi Square Statistic is 21.345 with 24 degrees of freedom. Still insignificant.	Chi Square Statistic is 9.741 with 24 degrees of freedom. Very insignificant.
Normality			
1. Overall Plot	Residuals cluster at center--typical of normal distribution.	Although somewhat spread out, still have normality characteristics--95% within $\pm 2\sigma$, etc.	Residuals cluster around mean--typical of normal distribution.
2. Plot on normal probability paper	Approximates a straight line--no reason to reject normality.	Causes some concern--but on basis of combined tests, can't reject normality.	Causes some concern--but on basis of combined tests, can't reject normality.
3. Chi-Square Goodness of Fit Test	Significance statistic is 11.07--Actual statistic is 6.0. Can't reject normality.	Significance statistic is 11.07--actual statistic is .55. Can't reject normality.	Significance statistic is 11.07--actual statistic is 4.97. Can't reject normality.
Mean of Zero			
1. Size of arithmetic average	Arithmetic average is within .3σ of 0.	Arithmetic average is within .02σ of zero.	Arithmetic average is zero.
Constant Var			
1. Plot of Residuals vs. time	Variance appears constant over time.	Variance appears constant over time.	Variance appears constant over time.

that the residuals come from normal populations, have zero means, are independent over time and have constant variances.

Depreciation and Amortization

The Company has a limited amount of long-term assets. Fixed assets, composed of instructional equipment, leasehold improvements, office furniture and fixtures and sales equipment, total \$106,200 while intangible assets, composed of book and course development costs, total \$109,860. In both cases, uncertainty is introduced because of the inability to know a priori the length of the service lives of these assets.[8] Since the Company is presently depreciating these assets on a straight-line basis with ten and five year lives, respectively, the method used to measure the uncertainty in the estimates will be to elicit a probability distribution around the mean estimates. Instead of calculating a depreciation expense for each probable service life and then averaging the lives using the associated probability distribution as has been suggested by several researchers,[9] that life with the highest probability will serve as the mean and a probability distribution will be assessed around that mean. This method allows the Company's estimates to remain essentially unchanged.[10]

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[8]The salvage values of the assets are assumed to be zero because any imprecision regarding a residual value is immaterial.

[9]See, for example, Ijiri, Yuji and Robert S. Kaplan, "Probabilisitc Depreciation and Its Implications for Group Depreciation," The Accounting Review, October 1969, pp. 746-756 or Friberg, Donald D., "Probabilistic Depreciation with a Varying Salvage Value," The Accounting Review, January 1973, pp. 50-60.

[10]Two alternative methods to obtain the mean of the depreciation distributions were considered. The first was to elicit the mean depreciation amount with its associated variance. The use of this technique, however, requires the subject to make a mental depreciation calculation. Although clearly an empirical question, it was concluded for this study that this type of decision model is not typical. Instead, it is likely that the decision maker would think in terms of the asset's life rather than the amount of depreciation. Hence, the procedure used was to elicit the distribution of the service lives instead of the amount. As will be seen in the depreciation calculations, this procedure results in depreciation and amortization expenses that differ from original Company estimates by \$17 and \$3, respectively.

The interview approach of the Schlaifer or direct estimating technique[11] was used to elicit the probabilities. The subject interviewed was the Company President, an engineer by training. The subject was asked the following seven questions about the service lives of the assets:

1) For what life do you think the asset (flow of benefits of the intangible assets) has an equal chance of living shorter or longer?

2) For what life do you think the probability of the asset living longer than that is negligible?

3) For what life do you think the probability of the asset living shorter than that is negligible?

4) For what life do you think the probability of the asset living shorter than that is 1/6?

5) For what life do you think the probability of the asset living longer than that is 1/6?

6) For what life do you think the probability of the asset living longer than that is 1/40?

7) For what life do you think the probability of the asset living shorter than that is 1/40?

The first question provided the mean service lives. The second and third provided $\pm$ three standard deviations. The next proviced $\pm$ one standard deviations and the final two provided $\pm$ two standard deviations. The questions were asked in this sequence to avoid bunching at either end.

As each question was asked a point was plotted to form a cumulative distribution. It was found that the service lives of both the fixed and intangible assets could be approximated very closely with the normal distribution. In the case of fixed assets, the mean and variance were 120 months

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[1]For a review of the elicitation of subjective probabilities, see G. R. Chesley, "Elicitation of Subjective Probabilities: A Review," The Accounting Review, April 1975, pp. 325-337.

and 24 months^2, respectively, and in the case of the intangible assets, the mean and variance were 60 months and 12 months^2, respectively.

The formula for calculating straight-line depreciation and amortization is

$$\text{Depreciation} = \frac{\text{Cost} - \text{Salvage Value}}{\text{Expected Lifetime}}$$

Since the results of the interviews suggested that the service lives of the fixed and intangible assets can be approximated by the normal distribution, the distributions for depreciation and amortization must necessarily be the inverse of the normal distribution. This argument follows:

Let y = depreciation
c = depreciable cost
x = service life

Then $y = \frac{c}{x}$ where $x = N(\mu,\sigma^2)$. Also $x = \frac{c}{y}$. Because the mean and variance of the inverse of the normal distribution do not exist,[12] it is necessary to use a partial Taylor Expansion to approximate the mean and variance for the depreciation and amortization expenses.[13] Using these results, the mean and variance are:

$$E(y) = \frac{c}{\mu} + 1/2(\sigma^2)(c/\mu^3)$$

$$Var(y) = \sigma^2 \; (c^2/\mu^4)$$

Substituting in the values of c, μ, and σ^2 for depreciation, the mean and variance are approximately:

$$E\text{ (Depreciation)} = \frac{106{,}200}{120\text{ mos.}} + 1/2(24)^2 \left(\frac{106{,}200}{120^3}\right) = 902.7$$

$$Var\text{ (Depreciation)} = (24)^2 \frac{(106{,}200)^2}{(120)^4} = \$^2 31{,}329$$

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[12] See appendix A for this argument.

[13] See appendix B for the partial Taylor Expansion.

Standard Deviation is 177.

Using the same methodology for the amortization expense gives:

$$E(\text{Amortization}) = \frac{109{,}860}{60} + 1/2(12)\ \frac{109{,}860}{60^3} \approx \$1{,}833$$

$$\text{Var(Amoritzation)} = 144\ \frac{(109{,}860)^2}{(60)^4} = \$^2 134{,}106$$

Standard deviation is $367.

Cash Expenditures

The distribution for the cash expenditures is singular normal.

AGGREGATION OF EXPENSES AND REVENUES

It is now possible to summarize the expense and revenue items examined individually and present the new probabilistic statement of income. The following summarizes the information to date:

	Mean Estimate	Variance	Standard Deviation
Revenues	$90,242	$\$^2$79,912,512	$ 8,940
Expenses			
Cancellations	40,812	218,227,558	14,772
Bad Debts	7,205	13,883,426	3,726
Amortization	1,833	134,102	367
Depreciation	903	31,329	177
Cash Expenses	46,461	0	0
Net Loss	$ 6,972	$\$^2$247,589,221	$15,735

The variance of the net loss is calculated by examining a linear combination[14] of the revenue and expenses. That linear combination is:

[14]The linear combination is only adequate in certain limited situations. See Appendix C for a more lengthy discussion of the aggregation issues.

$$(1)^2\ {}^{2}_{R} + (-1)^2\ {}^{2}_{C} = (-1)^2\ {}^{2}_{BD} + (-1)^2\ {}^{2}_{D} + (-1)^2\ {}^{2}_{A} +$$

$$+ (-1)^2\ {}^{2}_{CE} + -2Cov(R,BD) - 2Cov(R,D) -2Cov(R,C)$$

$$-2Cov(R,A) -2Cov(R,CE) = 2Cov(C,BD) + 2Cov(C,D)$$

$$+ 2Cov(C,A) + 2Cov(C,CE) + 2Cov(BD,D) + 2Cov(BD,A)$$

$$+ 2Cov(BD,CE) + 2Cov(D,A) + 2Cov(D,CE) + 2Cov(A,CE),$$

where R = Revenues
C = Cancellations
BD = Bad Debts
D = Depreciation
A = Amortization
CE = Cash Expenses

The correlation matrix for the income statement is[15]:

R	C	BD	D	A	CE	
.999999	.233908	.547424	.000333	.000666	.687637	Revenues
	1.000000	.318298	.000000	.000000	.047375	Cancellations
		1.000000	.000033	.000066	.137030	Bad Debts
			.000000	.000000	.000000	Depreciation
				.000000	.000000	Amortization
					1.000000	Cash Expenses

Substituting in the actul variances and covariances yields the following:

$$\$^2 79,912,512 + \$^2 218,227,558 + \$^2 13,883,426 + \$^2 134,329$$

$$+ \$^2 0 - 2(.233908)(8,940)(14,772) - 2(.547423)(8,940)(3,726)$$

$$- 2(.000333)(177)(8,940) - 2(.000666)(8,940)(367) -$$

$$2(.687637)(8,940)(0) + 2(318298)(14772)(3726) + 2(.000000)$$

$$(367)(14772) + 2(.047375)(0)(14772) + 2(.000033)(177)$$

$$(3726) + 2(.000066)(3726)(367) + 2(.137030)(3726)(0)$$

[15]The correlation matrix was computed over the last 30 periods.

+ 2(.000000)(367)(177) + 2(.000000)(177)(0) + 2(.000000)

(0)(367) + 2(.000000)(177)(14772).

These calculations result in a variance of $\$^2$247,589,221 and a standard deviation of $15,735. It is interesting to note that if all of the covariance terms had been zero, the variance of net operating income would be $\$^2$312,188,927. The effect of the covariances has been to reduce the variance by 64,599,706, or approximately twenty percent. (Reduction in the standard deviation was $8,038.)

Probability Statement About Net Operating Income

Before it is possible to make probability statements about net operating income it is necessary to know the nature of its distribution. In the case of the test Company, it was found that the distributions of revenues, cancellations, and bad debts were all approximately normal. The distributions of both amortization and depreciation expenses were the inverse of the normal. Since the total variance contribution of the three approximately normal distributions was .9995 percent of the total variance ($\$^2$312,023,498 vs. $\$^2$165,431 before the reduction caused by the covariance terms), it follows that the distribution of earnings must be normal. From this normal distribution it is possible to make probability statements.

CONCLUDING REMARKS

It has been shown in this illustration that the net loss reported by the test Company of $6,973 for the month of September, 1974 could actually have been between a loss of $38,444 and a net income of $24,497 with a ninety-five percent level of confidence. This variability was computed on the basis of forecasting error only. If the probabilities were expanded to include other variables such as measurement error, changing price levels,

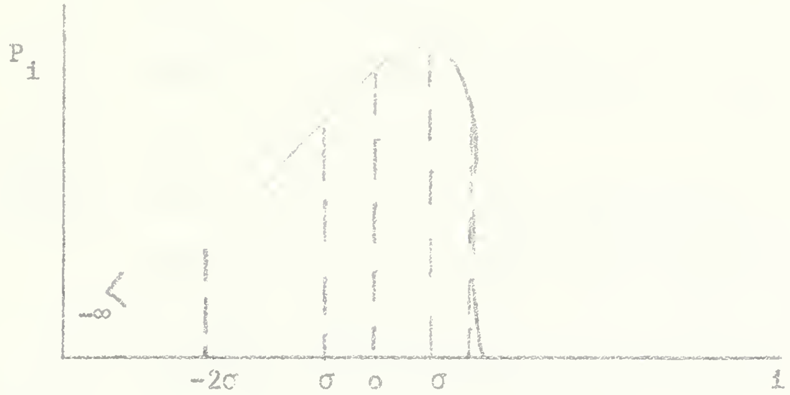
and alternative accounting methods allowed under historical costs, the uncertainty of income probably would have been much greater. The resulting variability is, nevertheless, significant. In fact, it may be so large as to obscure any information content of net operating income as now calculated by the test Company. It is even more critical that the largest contributors of uncertainty, the cancellation and bad debt expenses, affect the ultimate cash flows of the Company. The timing uncertainties in revenue recognition and fixed asset allocation may be less critical.

As outlined in the introduction, the purpose of this study was to provide a framework within which uncertainty of financial statements can be interpreted and then to illustrate the degree of uncertainty that can exist. The results have indicated that a very significant amount of uncertainty does exist, at least in the test Company. In attempting to measure that uncertainty, this study has also identified and dealt with several of the difficult issues of aggregating probability distributions.

It is recognized that this study is exploratory in nature and that the results can not be generalized beyond the test Company. However, if the quantification techniques employed in this study can be accepted, it appears that the cumulative affect of uncertainty on net income very seriously mitigates that numbers significance. This evidence is very important when one considers the consequences resulting to a firm (i.e., taxes, regulations, etc.) because of the magnitude of its reported net income.

APPENDIX A

If x has a normal distribution, $N(\mu, \sigma^2)$, then $y = \frac{c}{x}$ is such that $E(y) = \infty$, does not exist. The distributional form appears below:



The integral [illegible] exists, but the mean does not. Thus, in the case of determining the distributions of service lifetime, it must be remembered that the lifetimes are not normally distributed, but only approximations to the normal distribution. The normal is only used as a tool to provide guidelines for the specifications of μ_x and σ_x^2 which are used to determine the approximations of μ_y and σ_y^2.

APPENDIX B

The mean and variance are calculated via the Taylor Expansion as shown below:

$$y = f(x_i)$$

$$f(x) = f(\mu) + (c-\mu)f'(\mu) + (1/2)(x-\mu)^2\ f''\ (\mu)$$

$$E(f(x)) = f(\mu) + (1/2)\sigma^2 f''(\mu)$$

$$(f(x))^2 = f(\mu)^2 + 2(x-\mu)f(\mu)f'(\mu) + (x-\mu)^2 f'(\mu)^2$$

$$+ (x-\mu)^3 f''(\mu)f(\mu) + (1/4)(x-\mu)^4\ f''(\mu)^2$$

$$+ (x-\mu)^2\ f'(\mu)f(\mu)$$

$$E(f(x)^2) = f(\mu)^2 + \sigma^2 f'(\mu)^2 + \sigma^2 f'(\mu)f(\mu)$$

$$(e(f(x)))^2 = f(\mu)^2 + \sigma^2 f''(\mu)f(\mu)$$

$$\text{Variance } (f(x)) = \sigma^2(f'(\mu))^2 \ \ldots$$

In the present case, since $y = \frac{c}{x}$, substitution gives

$$E(y) = (c/\mu) + (1/2)\ \sigma^2\ (c/\mu^3)$$

$$\text{Variance } (y) = \sigma^2\ (c/\mu^2)^2 = \sigma^2\ (c^2/\mu^4)$$

APPENDIX C

AGGREGATION OF PROBABILITIES

The aggregation of the probability distributions in the test Company is not a difficult task. This is so because the joint distribution of the net income statement items is approximately multivariate normal. However, aggregation will often be much more complicated. Non-normality of distributions can combine with significant covariance terms to make aggregation very difficult. Figure 1 summarizes the aggregation issues than can result and suggests solutions to each.

We let $Y = \sum^{N} a_i X_i$, where the characterization of the distributions of the X's are given in the margins of Figure 1. In the figure, $E(X_i) = \mu_i$ and $Var(X_i) = \sigma_i^2$, which is assumed to exist.

FIGURE 1

SUMMARIZATION OF AGGREGATION ISSUES

Covariances \ Type of Distribution	ALL NORMAL	NON-NORMAL BUT ALL DISTRIBUTIONS ARE SAME TYPE	VARIED DISTRIBUTIONAL TYPES
ZERO or INSIGNIFICANT	$N(\mu,\sigma^2)$ where $\mu = a_1\mu_1 + a_2\mu_2 + \ldots + a_n\mu_n$. $\sigma^2 = a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + \ldots + a_n^2\sigma_n^2$. ALWAYS See footnote[16].	$N(\mu,\sigma^2)$ where μ and σ^2 are same as in the all normal case. This result is due to the Central Limit Theorem and holds even for a moderate number of X_i's.[17]	$N(\mu,\sigma^2)$ where μ and σ^2 are same as in the all normal case. This result is also due to the CLT and holds for only a large number of X_i's where each gives only a relatively insignificant contribution to total X.[18]
	$N(\mu,\sigma^2)$ where μ = same as above $\sigma^2 = a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + a_n^2\sigma_n^2 +$ $2Cov(\sigma_1\sigma_2) +$ $2Cov(\sigma_1\sigma_3) +$ $\ldots +$ $2Cov(\sigma n\text{-}1\sigma n)$	$N(\mu,\sigma^2)$ where μ and σ^2 are the same as in the all normal case with significant covariances. This is true only under somewhat restrictive conditions. In general terms, rigorous theorems are available which allow "some" dependence if the random variables are identically distributed; certain patterns of non-identical distributions are also allowed for more restricted patterns of dependence. Usually requires a larger number of observations to converge to normality.[19]	

It is important that the resulting distribution of net income be known, or at least a good approximation is available, so that probability statements can be made. Figure 1 suggested cases where the resulting distribution is normal or approximately normal.

A review of the causes of imprecision in the statements suggests that the most common aggregation situation faced would be the all normal--significant covariance type. Most of the uncertainty in the statements arises because of estimation or forecast errors. The errors result from the difference between the actual expense or revenue and that amount estimated by some model. (There is always some model--e.g. Bad Debt Expense is usually estimated using the model δ percent of sales or δ percent of receivables; likewise all estimates are calculated using a model.) It is assumed, by the theory of errors, that if the model is a good one, the errors should always be approximately normally distributed.[20] Also, they will be inde-

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[16] Anderson, T.W., Introduction to Multivariate Statistical Analysis, John Wiley and Sons, New York, 1958, pp. 19-27.

[17] Cramer, Harold, The Elements of Probability Theory, John Wiley and Sons, 1955, pp. 116-117.

[18] Dwass, Meyer, Probability Theory and Applications, W.A. Benjamin, Inc., New York, 1970, p. 333.

[19] Hillier, Frederick S., The Evaluation of Risky Interrelated Investments, North-Holland Publishing Company, Amersterdam-London, 1971, p. 29. For some of the rigorous theorems see Diananda, P.H., "The Central Limit Theorem for M-Dependent Variables," Proceedings of Cambridge Philosophical Society, 1955, pp. 92-95; Ibragimov, I.A., "A Central Limit Theorem for a Class of Dependent Random Variables," The Annals of Mathematical Statistics, August, 1968.

[20] Arley, Neils and Buch, K. Rander, Introduction to the Theory of Probabilities and Statistics, John Wiley and Sons, New York, 1950, p. 154

pendent, have a zero mean and a constant variance.[21] Obviously, if the model is not adequate in describing the behavior of the phenomenon of interest, there may be either course or systematic errors.[22]

Often, it is very difficult to be able to discern between insignificant and significant covariances, approximately normal vs. varied types of distributions and how large the number of observations must be before there is covergence to normality. Also, often there may be one or two large variances that dominate the resulting distribution of net income. These types of distribution problems must be examined individually and probably can only be solved using simulation. In particular, using an available covariance matrix, one can compare his results with the corresponding normal distribution using Monte Carlo Simulation.

Finally, there are some conclusions that can be drawn without any knowledge of the distribution other than the mean and variance. One such conclusion concerns upper bounds on the amount of uncertainty involved. The well-known Tchebycheff inequality yields:

$$\text{Prob } (|X - E(X)| \leq k \text{ Var } \sqrt{(X))} \leq k^{-2} \quad \text{for}$$

all k regardless of the distribution of X. Thus, for example, the probability is no more than 0.25, 0.04, and 0.01, respectively, that X will be less than $E(X) - 2\sqrt{Var\ (X)}$, $E(X) - 5\sqrt{Var\ (X)}$, and $E(X) - 10\sqrt{Var\ (X)}$, respectively.[23]

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[21] Draper, N.R. and H. Smith, Applied Regression Analysis, John Wiley and Sons, New York, 1966, p. 86.

[22] Arley, Neils and Rander K. Buch, Introduction to the Theory of Probabilities and Statistics, John Wiely and Sons, New York, 1950, p. 153.

[23] This example was taken from Hillier, op. cit., p. 23.

However, the Tchebycheff inequality usually is very conservative. For example, if the distribution of X actually is normal, the probability is only .0014 that X will be less than $E(X) - 3\sqrt{Var(X)}$. Hillier[24] suggests that because of this conservatism, the Tchebycheff inequaltiy should be tempered by comparison with calculations for the normal distribution. In fact, he suggests that the normal distribution be considered as a practical lower bound on the probability of a large deviation from the mean.

In summary, aggregation may be accomplished by the following procedures. First, if the problem is straight forward, a linear combination of the variables will suffice. (This will usually be adequate if the forecasting models employed by the company are successful in estimating the mean trend of the expenses and revenues.) Second, simulation can often provide us with some knowledge about the form of the resulting probability distribution in cases where a straight forward linear combination is questionable. Finally, it is always possible to make probability statements using various limit theorems or Tchebycheff inequalities.

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[24] Hillier, op.cit., p. 24