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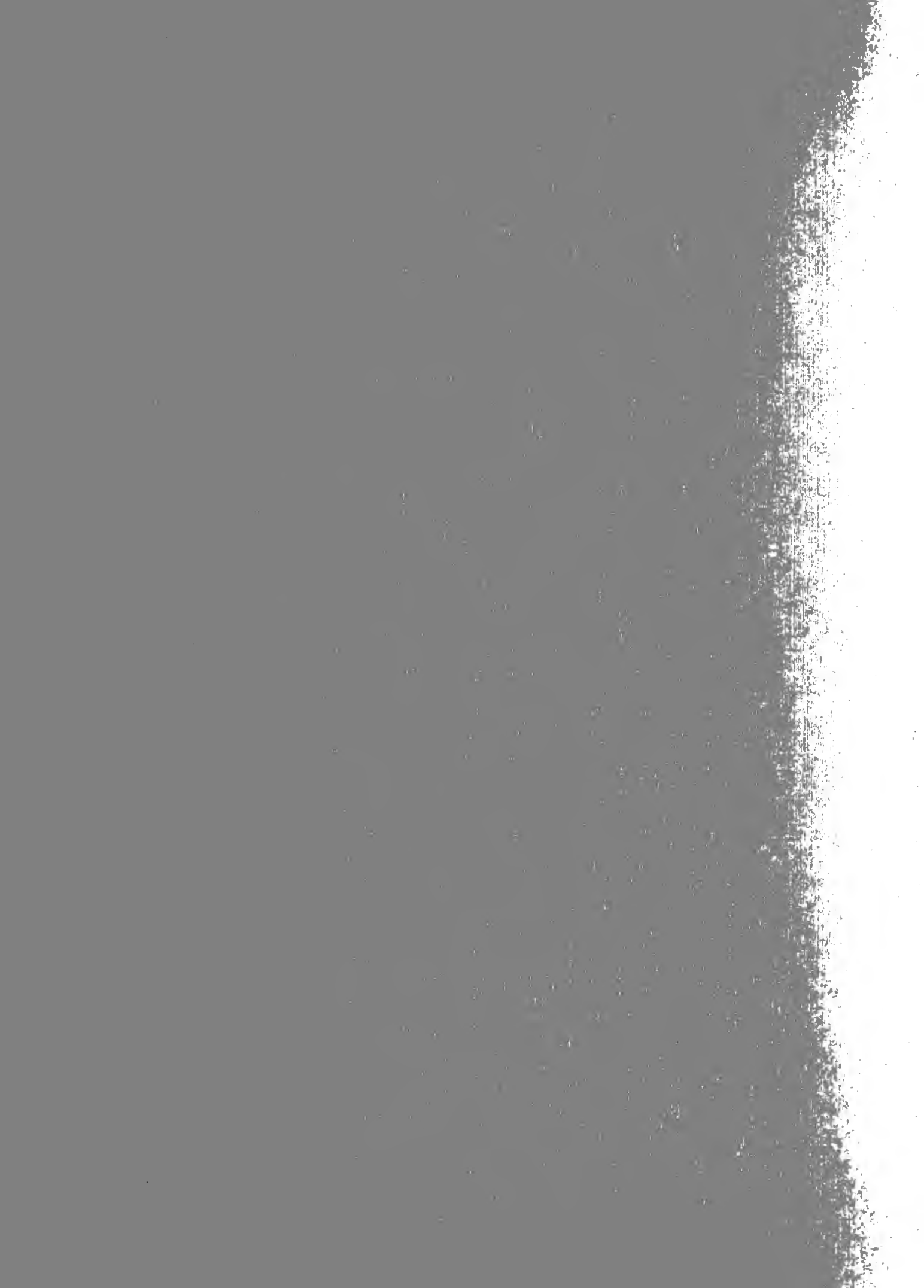
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Estimation of the Marginal Rate of Substitution in  
the Intertemporal Capital Asset Pricing Model

*Louis O. Scott*



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Intertemporal Capital Asset Pricing Model.

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ABSTRACT

The purpose of this paper is to use the intertemporal capital asset pricing model (CAPM) to develop empirical estimates of the marginal rate of substitution (MRS). We treat the MRS as an unobservable and develop a method of moments estimator which is consistent. We find that consistency depends on both a large number of time observations and a large number of securities. We use the MRS estimates to test restrictions implied by the intertemporal CAPM and the results generally support the model.

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## ESTIMATION OF THE MARGINAL RATE OF SUBSTITUTION IN THE INTERTEMPORAL CAPITAL ASSET PRICING MODEL

In this paper, we develop empirical estimates of the marginal rate of substitution (MRS) using the intertemporal capital asset pricing model (CAPM). The advantage of this approach is that we do not require a strong set of assumptions in order to estimate the MRS or to test the intertemporal CAPM. Tests of intertemporal CAPM's in the literature have followed two approaches. The most common one has been to use a consumption-based CAPM in which consumption data and a particular utility function are used to measure the marginal utility of real consumption. Examples of this approach can be found in Hansen and Singleton (1982, 1983), Dunn and Singleton (1983, 1986), Grossman and Shiller (1981), and Mankiw and Shapiro (1984). The empirical results have been generally negative: the models are rejected by the data on asset returns and the parameter estimates frequently result in implausible values. Mankiw and Shapiro find that consumption betas perform very poorly in the presence of betas estimated from the standard market model. A second approach has been to treat the MRS as an unobservable and impose additional assumptions on the joint distribution of asset returns and the MRS. Hansen and Singleton (1983) show that the joint lognormal distribution implies a restriction on the difference between the returns on two assets: specifically, expected excess returns are constant and excess returns should be unpredictable. Their tests with short-term interest rates and returns on large stock portfolios indicate rejection of these restrictions.

Several explanations for the poor performance of these empirical models have been mentioned in the literature. One argument is that we need to measure the instantaneous consumption rate and that temporal aggregation of the published consumption data poses a serious problem. Another argument is that the time-additive separable utility function is too restrictive and a more complicated utility function is needed for consumption-based models. Garber and King (1983), for example, have shown that estimates of utility function parameters are biased if there is a random shock in the representative agent's utility of consumption function. The empirical results from a variety of studies suggest that the investment opportunity set (conditional distributions of asset returns) changes over time; specifically, the conditional means and variances of asset returns, interest rates, and the MRS vary over time. In the first section, we develop the empirical model and the method of moments estimator for the MRS. In the second section we present the results of the model. We use both long time series and a large cross section of security returns to estimate the MRS series. After estimating the MRS, we use the estimates to test restrictions implied by the intertemporal CAPM; these second-stage tests are more in the spirit of a check on whether the model fits the data, and do not represent comprehensive tests of the intertemporal CAPM.

#### I. The Empirical Model and the Estimator of the MRS

Our approach to estimating and testing the intertemporal model is to treat the MRS as an unobservable variable and use both time series and cross-sectional data on returns to estimate the unobservable series. Let

$J_w(t)$  be the marginal utility of real wealth,  $p_{it}$  be the real asset price, and  $d_{it}$  be the real dividend or cashflow at the end of period  $t$ . As Breeden (1979) and others have shown, intertemporal asset pricing models imply the following asset pricing relation:

$$E_t [J_w(t)p_{it} - J_w(t+1)(p_{i,t+1} + d_{i,t+1})] = 0,$$

where  $E_t$  is the conditional expectations operator, conditional on information available at time  $t$ . Let  $\lambda_t$  equal the product of  $J_w(t)$  and the consumption price deflator at time  $t$ , and we have the following relationship:

$$E_t [\lambda_t P_{it} - \lambda_{t+1} (P_{i,t+1} + D_{i,t+1})] = 0,$$

where  $P_{it}$  and  $D_{it}$  are price and dividends in nominal terms (nominal \$), for security  $i$ . In an appendix we show that these asset pricing relations can be derived from a rather weak set of assumptions. We have the following relationship for nominal returns:

$$E_t \left[ \frac{\lambda_{t+1}}{\lambda_t} (1 + R_{i,t+1}) - 1 \right] = 0. \quad (1)$$

The asset pricing relationship also applies to short-term securities that are riskless in nominal terms. For one-period risk-free interest rates, we have

$$\frac{1}{1 + R_{F,t+1}} = E_t \left[ \frac{\lambda_{t+1}}{\lambda_t} \right], \quad (2)$$

where  $R_{F,t+1}$  is the return known at time  $t$  for a one-period discount bond that matures at  $(t+1)$ . This model is known in the literature as a MRS

model.  $\frac{\lambda_{t+k}}{\lambda_t}$  measures the ex post marginal rate of substitution for a \$ between  $t+k$  and  $t$ . Formally, this variable measures the ex post value of having an extra \$ at time  $t+k$  relative to time  $t$ . For convenience, we refer to  $\lambda_t$  as the marginal utility of wealth variable.

The asset pricing relation in (1) is a restriction on conditional moments, but the relationship implies the following restriction on unconditional moments;

$$E \left\{ \left[ \frac{\lambda_{t+1}}{\lambda_t} (1+R_{i,t+1}) - 1 \right] z_{it} \right\} = 0, \quad (3)$$

where  $z_{it}$  is a vector of information variables or instruments associated with security  $i$  known at time  $t$ . Using equation (2) and the observation that the marginal utility of wealth variable should be positive, we develop the following model for  $\frac{\lambda_t}{\lambda_{t-1}}$ :

$$\frac{\lambda_t}{\lambda_{t-1}} = \left( \frac{1}{1+R_{Ft}} \right) \eta_t,$$

where  $\eta_t > 0$  and  $E_{t-1}(\eta_t) = E(\eta_t) = 1$ . The  $\eta_t$  series is serially uncorrelated, but not necessarily serially independent. Substituting this into (3), we get

$$E \left\{ \left[ \eta_t \frac{(1+R_{it})}{(1+R_{Ft})} - 1 \right] z_{i,t-1} \right\} = 0.$$

One of the instruments can be a constant and we have the following sample moments with expected values of zero:

$$\frac{1}{T} \sum_{t=1}^T \left[ \eta_t \frac{(1+R_{it})}{(1+R_{Ft})} - 1 \right], \quad i = 1 \dots K$$

(4)

$$\frac{1}{K} \sum_{i=1}^K \left[ \eta_t \frac{(1+R_{it})}{(1+R_{Ft})} - 1 \right], \quad t = 1, \dots, T$$

where  $T$  is the number of time periods in the sample and  $K$  is the number of securities. The first set of sample moments consists of time-series moments, and the second set consists of cross-sectional moments. The time series moments converge in probability to zero as  $T$  gets large, but the cross-sectional moments do not necessarily converge in probability to zero as  $K$  gets large. Let  $u_{it} \equiv \eta_t \frac{(1+R_{it})}{(1+R_{Ft})} - 1$ . Each series  $u_{it}$  is serially uncorrelated, but in general there is contemporaneous correlation across the securities. Hence the variances of the time series moments go to zero as  $T$  gets large, but the variances of the cross-sectional moments do not go to zero as  $K$  gets large. One possibility for a consistent estimate of  $\underline{\eta}$ , where  $\eta' = (\eta_1, \dots, \eta_T)$ , is to set  $\underline{\eta}$  so that the time series moments are close to their expected values of zero. To identify  $\underline{\eta}$  in this estimator, one must have more securities or sample moments than time observations, but we are unable to show consistency for this estimator as the number of securities increases.

Our estimator proceeds as follows. First we note that if we have a cross-sectional moment that converges in probability to zero then we can develop a consistent estimator of each  $\eta_t$ . The problem with the cross-sectional moments above is that for each period the  $u_{it}$ 's will be systematically above or below the expected value of zero. Note that

$\eta_t$  should vary inversely with the market return. If the relative risk aversion parameter for the economy is greater than one, the variation of  $\eta_t$  over time should be greater than the variation of the market return. In this case  $u_{it}$  for most securities will tend to be negative during periods with positive surprises in the market return, and positive when there are negative surprises in the market return. If relative risk aversion for the economy is less than one, the variation of  $\eta_t$  will be less than that of the market return and  $u_{it}$  will tend to be positive when the market is up and negative when the market is down. To account for this contemporaneous covariation of  $u_{it}$ , we assume that we have a factor model of the following form:

$$u_{it} = \sum_{j=1}^L \beta_{ij} \xi_{jt} + e_{it},$$

where the  $\xi_{jt}$ 's represent common factors and  $e_{it}$  is the idiosyncratic error which is uncorrelated with the common factors. All of these variables,  $u_{it}$ ,  $\xi_{jt}$ , and  $e_{it}$ , represent unpredictable forecast errors. The common factors must be innovations that we can separately measure. A natural candidate that follows from our argument above is the innovation in the measured return for a large market portfolio. We also consider the innovations in other financial market variables such as short-term and long-term interest rates. We develop the estimator for a two-factor model and note that it is easy to incorporate more factors if necessary. The two factor model has the form:

$$u_{it} = \beta_{i1} \xi_{1t} + \beta_{i2} \xi_{2t} + e_{it}.$$

We now form cross-sectional moments as follows:

$$e_t \equiv \frac{1}{K} \sum_{i=1}^K \left[ \eta_t \frac{1+R_{it}}{1+R_{Ft}} - 1 - \beta_{i1} \xi_{1t} - \beta_{i2} \xi_{2t} \right], \quad t=1, \dots, T. \quad (5)$$

These sample moments converge in probability to zero if their variances go to zero as  $K$  gets large. A sufficient condition for this convergence is that each  $K \times K$  covariance matrix for  $e_{it}$ ,  $i=1, \dots, K$ , be diagonal, but this is not a necessary condition. From Chamberlain (1983) and Chamberlain and Rothschild (1983), we know that if the eigenvalues for this covariance matrix remain bounded as  $K$  gets large then the sample moment  $e_t$  converges in probability to zero. This weaker condition allows some of the covariances to be nonzero. For example, we can have nonzero covariances between firms in the same industry if this industry effect becomes negligible as  $K$  gets large.

Next we observe that the important parameters are  $\underline{\eta}$  and the average  $\beta$ 's, not  $\beta_{i1}$  and  $\beta_{i2}$  for all securities:

$$e_t = \eta_t \left( \frac{1}{K} \sum_{i=1}^K \frac{1+R_{it}}{1+R_{Ft}} \right) - 1 - \beta_1 \xi_{1t} - \beta_2 \xi_{2t} = 0 \quad t=1, \dots, T \quad (6)$$

where  $\beta_1 \equiv \frac{1}{K} \sum_{i=1}^K \beta_{i1}$  and  $\beta_2 \equiv \frac{1}{K} \sum_{i=1}^K \beta_{i2}$ . Once we have consistent estimates for  $\underline{\eta}$ ,  $\beta_1$ , and  $\beta_2$ , we can easily compute  $\beta_{i1}$  and  $\beta_{i2}$  by running an ordinary least squares (OLS) regression of  $\hat{u}_{it}$  on  $\xi_{1t}$  and  $\xi_{2t}$  for each security. Up to this point we have not used the time series moments in (4). We now set  $\beta_1$  and  $\beta_2$  so that the time series moments are close to zero. Formally we set  $\beta_1$  and  $\beta_2$  to minimize the sum of the squares of the time series moments. Our approach

is to use the moments that correspond to  $\eta_t$  and a subset of the securities. If we have more than two common factors in the model for  $u_{it}$ , there are more than enough time series moments available from which to estimate more average betas. From (6) for the two-factor model we have

$$\hat{\eta}_t = \frac{1 + \beta_1 \xi_{1t} + \beta_2 \xi_{2t}}{\left( \frac{1}{K} \sum_{i=1}^K \frac{1 + R_{it}}{1 + R_{Ft}} \right)} \quad t = 1, \dots, T \quad (7)$$

and we plug this into the time series moments, which we then set close to zero by minimizing the average of their sum of squares with respect to  $\beta_1$  and  $\beta_2$ :

$$\min_{\beta_1, \beta_2} \frac{1}{P} \sum_{j=1}^P \left\{ \frac{1}{T} \sum_{t=1}^T \frac{(1 + \beta_1 \xi_{1t} + \beta_2 \xi_{2t}) (1 + R_{jt})}{\left( \frac{1}{K} \sum_{i=1}^K \frac{1 + R_{it}}{1 + R_{Ft}} \right) (1 + R_{Ft})} - 1 \right\}^2$$

For  $j = 1$ , we use  $R_{jt} = R_{Ft}$  so that one of the sample moments includes  $(\hat{\eta}_t - 1)$ . We show consistency below with this unweighted general method of moments (GMM) estimator, but one can alternatively weight the moments by the inverse of the corresponding covariance matrix.

We now formally state the estimator for  $\underline{\eta}$ ,  $\beta_1$ , and  $\beta_2$  and show that this method of moments estimator is consistent. One cannot simply invoke the results of Hansen (1982) because in this application the parameter space is growing as we increase  $T$ . The estimator is formed so that the following sample moments equal their expected values of zero:



$$\frac{1}{K} \sum_{i=1}^K \eta_t \frac{1+R_{it}}{1+R_{Ft}} - 1 - \beta_1 \xi_{1t} - \beta_2 \xi_{2t} = 0 \quad t = 1, \dots, T$$

$$\frac{1}{P} \sum_{j=1}^P \left\{ \frac{1}{T} \sum_{t=1}^T \frac{(1+\beta_1 \xi_{1t} + \beta_2 \xi_{2t})}{C_t} \frac{(1+R_{jt})}{(1+R_{Ft})} - 1 \right\} \left\{ \frac{1}{T} \sum_{t=1}^T \frac{\xi_{1t}}{C_t} \frac{(1+R_{jt})}{(1+R_{Ft})} \right\} = 0 \quad (8)$$

$$\frac{1}{P} \sum_{j=1}^P \left\{ \frac{1}{T} \sum_{t=1}^T \frac{(1+\beta_1 \xi_{1t} + \beta_2 \xi_{2t})}{C_t} \frac{(1+R_{jt})}{(1+R_{Ft})} - 1 \right\} \left\{ \frac{1}{T} \sum_{t=1}^T \frac{\xi_{2t}}{C_t} \frac{(1+R_{jt})}{(1+R_{Ft})} \right\} = 0,$$

where  $C_t = \frac{1}{K} \sum_{i=1}^K \frac{1+R_{it}}{1+R_{Ft}}$ . Let  $\theta$  be a vector containing the parameters to be estimated:  $\theta' = (\eta', \beta_1, \beta_2)$ . In matrix form we have  $A\theta - \underline{b} = \underline{0}$

where  $A$  is a  $(T+2) \times (T+2)$  matrix and  $\underline{b}$  is a  $(T+2) \times 1$  vector. The elements of  $\underline{b}$  are

$$\underline{b}' = \{1, \dots, 1, \frac{1}{P} \sum_{j=1}^P [1 - \frac{1}{T} \sum_{t=1}^T \frac{(1+R_{jt})}{C_t (1+R_{Ft})}] \left[ \frac{1}{T} \sum_{t=1}^T \frac{\xi_{1t} (1+R_{jt})}{C_t (1+R_{Ft})} \right],$$

$$\frac{1}{P} \sum_{j=1}^P [1 - \frac{1}{T} \sum_{t=1}^T \frac{(1+R_{jt})}{C_t (1+R_{Ft})}] \left[ \frac{1}{T} \sum_{t=1}^T \frac{\xi_{2t} (1+R_{jt})}{C_t (1+R_{Ft})} \right] \}$$

$A$  has the following form:

$$A = \begin{bmatrix} a_1 & 0 & \dots & \dots & \dots & 0 & -\xi_{11} & -\xi_{21} \\ 0 & a_2 & 0 & \dots & \dots & 0 & -\xi_{12} & -\xi_{22} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & 0 & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & a_T & -\xi_{1T} & -\xi_{2T} \\ 0 & \vdots & \vdots & \vdots & \vdots & 0 & a_{T+1} & c \\ 0 & \vdots & \vdots & \vdots & \vdots & 0 & c & a_{T+2} \end{bmatrix}$$

$$\text{where } a_j = \frac{1}{K} \sum_{i=1}^K \frac{1+R_{jt}}{1+R_{Ft}} \quad j = 1, \dots, T$$

$$\frac{1}{P} \sum_{p=1}^P \left\{ \frac{1}{T} \sum_{t=1}^T \frac{\xi_{kt}(1+R_{pt})}{C_t(1+R_{Ft})} \right\}^2 \quad j = T+k \text{ and } k = 1, 2$$

$$\text{and } c = \frac{1}{P} \sum_{j=1}^P \left\{ \frac{1}{T} \sum_{t=1}^T \frac{\xi_{1t}(1+R_{jt})}{C_t(1+R_{Ft})} \right\} \left\{ \frac{1}{T} \sum_{t=1}^T \frac{\xi_{2t}(1+R_{jt})}{C_t(1+R_{Ft})} \right\}.$$

The resulting estimator is  $\hat{\theta} = A^{-1}\underline{b}$ ; it is linear and the inversion of  $A$  is trivial. The estimator can be computed as follows. Solve the last two sample moment conditions in (8) to get  $\beta_1$  and  $\beta_2$ . Plug these into (7) for  $\hat{\eta}_t$  which solves the first set of sample moments in (8).

To show consistency, we need to show that each term in  $(\hat{\theta} - \theta)$  converges in probability to zero. Note that  $\hat{\theta} - \theta = A^{-1}(\underline{b} - A\theta)$ . The vector in parentheses on the right hand side is simply the negative of the vector of sample moments in (8) evaluated at the true parameter values. By our previous arguments, each one of these terms converges in probability to zero as  $T$  and  $K$  get large. We now examine what happens to  $A^{-1}$  as  $T$  and  $K$  get large.  $A^{-1}$  has the following form:

$$A^{-1} = \begin{bmatrix} a_1^{-1} & 0 & \dots & 0 & \frac{\xi_{11} a_{T+2} - \xi_{21} c}{a_1 D} & \frac{\xi_{21} a_{T+1} - \xi_{11} c}{a_1 D} \\ 0 & a_2^{-1} & 0 & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & 0 & \frac{\xi_{1T} a_{T+2} - \xi_{2T} c}{a_T D} & \frac{\xi_{2T} a_{T+1} - \xi_{1T} c}{a_T D} \\ \vdots & \vdots & \vdots & a_T^{-1} & \vdots & \vdots \\ 0 & \dots & \dots & 0 & \frac{a_{T+2}}{D} & \frac{-c}{D} \\ 0 & \dots & \dots & 0 & -\frac{c}{D} & \frac{a_{T+1}}{D} \end{bmatrix}$$

where  $D = a_{T+1} a_{T+2} - c^2$ . As  $T$  and  $K$  get large, the nonzero terms in  $A^{-1}$  converge to either constants or random variables. For each term in  $(\hat{\theta} - \theta)$ , we have the products of no more than three terms in  $A^{-1}$  with sample moments that converge in probability to zero. By the results cited in Theil (1971, pp. 370-71), we can conclude that each term in  $(\hat{\theta} - \theta)$  converges in probability to zero as  $T$  and  $K$  get large. Hence  $\text{plim}_{K, T \rightarrow \infty} \hat{\theta} = \theta$  and  $\text{plim}_{K, T \rightarrow \infty} \hat{\eta} = \eta$ , and we have established the consistency of our estimator.

Deriving the asymptotic distribution for  $\hat{\eta}$  is much more difficult and we do not pursue it here. We do have an asymptotic distribution for the  $\beta$  estimates and we can easily compute the covariance matrix and standard errors for these estimates. To do this, observe that the estimator for  $\beta_1$  and  $\beta_2$  is a GMM estimator with a fixed parameter

space, and we can apply the results of Hansen. A more efficient estimator for  $\beta_1$  and  $\beta_2$  is the following weighted estimator:

$$\min_{\beta_1, \beta_2} \underline{x}' S^{-1} \underline{x},$$

where  $x_j = \frac{1}{T} \sum_{t=1}^T \frac{(1 + \beta_1 \xi_{1t} + \beta_2 \xi_{2t}) (1 + R_{jt})}{C_t (1 + R_{Ft})} - 1$  for  $j=1, \dots, P$ ,

and  $S$  is the estimated covariance matrix for  $\sqrt{T} \underline{x}$ . The resulting estimator is a linear two-step estimator: in the first stage we estimate  $\beta_1$  and  $\beta_2$  with the unweighted estimator and use the initial estimates to compute  $\hat{S}$ , then we re-estimate  $\beta_1$  and  $\beta_2$  using  $\hat{S}$  in the second stage. The asymptotic covariance matrix for  $\hat{\beta}_1$  and  $\hat{\beta}_2$  is

$$\frac{(G' S^{-1} G)^{-1}}{T}$$

where  $G = \begin{bmatrix} \frac{\partial \underline{x}}{\partial \beta_1} & \frac{\partial \underline{x}}{\partial \beta_2} \end{bmatrix}$ . We can develop approximate standard errors for  $\underline{\eta}$  by rewriting equation (7) as follows:

$$\hat{\eta}_t = \frac{1 + \hat{\beta}_1 \xi_{1t} + \hat{\beta}_2 \xi_{2t}}{C_t}$$

and noting that

$$\eta_t = \frac{1 + \beta_1 \xi_{1t} + \beta_2 \xi_{2t} + \bar{e}_t}{C_t} \tag{9}$$

where  $\bar{e}_t = \frac{1}{K} \sum_{i=1}^K e_{it}$ . The variance of  $(\hat{\eta}_t - \eta_t)$  depends on the variance of  $(\hat{\beta}_1 - \beta_1)$ ,  $(\hat{\beta}_2 - \beta_2)$ , and  $\bar{e}_t$ .  $\bar{e}_t$  is the average of cross-sectional errors and should be uncorrelated with  $\hat{\beta}_1$  and  $\hat{\beta}_2$  which are computed from time series data. In the next section we describe a calculation

for a rough approximation to the variance of  $\bar{e}_t$ . With the covariance matrix for  $\hat{\beta}_1$  and  $\hat{\beta}_2$  and an approximate variance for  $\bar{e}_t$ , we can calculate approximate standard errors for  $\hat{\eta}_t$ .

Finally we would like to have a method for checking whether we have included a sufficient number of common factors so that the cross-sectional moments will converge in probability to zero. There are no simple methods for doing this. One approach is to test for the significance of the average  $\beta$ 's on additional factors and to examine the  $\chi^2$  goodness-of-fit statistic for the time series moments:

$T \underline{x}' S^{-1} \underline{x}$ . Another approach is to check the sample covariance matrix for  $e_{it} = u_{it} - \beta_{i1} \xi_{1t} - \beta_{i2} \xi_{2t}$ , estimated over time. This sample covariance matrix provides an estimate of the unconditional covariance. One check is to compute the covariance matrix for  $N$  of the securities and look at the corresponding matrix of correlation coefficients. Some of these correlation coefficients may be large, but most should be close to zero. One can also compute the eigenvalues for the covariance matrix and examine the size of the eigenvalues as  $N$  increases.<sup>1</sup> As  $N$  increases, the eigenvalues should remain bounded and the ratio of the total variation to the number of securities should not increase.

Given estimates of  $\underline{\eta}$ , we can construct estimates of the MRS as follows:

$$\left( \frac{\hat{\lambda}_t}{\lambda_{t-1}} \right) = \frac{\hat{\eta}_t}{1 + R_{Ft}} . \quad (10)$$

By normalizing and setting  $\lambda_0$  equal to some arbitrary value, we can compute estimates of  $\lambda_1, \lambda_2, \dots, \lambda_T$ . These latter estimates are unique up to a scalar transformation. The estimates  $(\frac{\hat{\lambda}_t}{\lambda_{t-1}})$  are unique. One strategy for testing the asset pricing relation in (3) is to use the large cross section of securities to estimate  $\underline{\eta}$ , and then use the estimated values for  $\frac{\lambda_t}{\lambda_{t-1}}$  to test the relationship for a subset of securities. For each security, we have a sample moment vector  $\underline{u}_i$ :

$$\underline{u}_i = \frac{1}{T} \sum_{t=1}^T \left[ \frac{\lambda_t}{\lambda_{t-1}} (1+R_{it}) - 1 \right] \underline{z}_{i,t-1}, \quad i = 1, \dots, K$$

where  $\underline{z}_{i,t-1}$  represents a set of instruments for each security. Each vector  $\underline{u}_i$  has a covariance matrix,  $\frac{V_i}{T}$ , where  $V_i$  is

$$V_i = E \left\{ \left[ \left( \frac{\lambda_t}{\lambda_{t-1}} \right) (1+R_{it}) - 1 \right]^2 \underline{z}_{i,t-1} \underline{z}'_{i,t-1} \right\}$$

The matrix  $V_i$  can be estimated from the corresponding sample moments:

$$\hat{V}_i = \frac{1}{T} \sum_{t=1}^T \left[ \left( \frac{\hat{\lambda}_t}{\lambda_{t-1}} \right) (1+R_{it}) - 1 \right]^2 \underline{z}_{i,t-1} \underline{z}'_{i,t-1}.$$

This estimate allows for the possibility of conditional heteroskedasticity. In large samples, the distribution of  $\underline{u}_i$  is approximately normal with mean zero and variance  $\frac{V_i}{T}$ . We can compute standard errors and t-statistics for each of the sample moments and we can compute a  $\chi^2$  test statistic for each security,

$$\chi^2_{(k)} = T \underline{u}_i' V_i^{-1} \underline{u}_i,$$

where  $k$  is the degrees of freedom and is equal to the number of instruments in  $\underline{z}_{i,t-1}$ . One can also construct a  $\chi^2$  test for the time-series moments associated with a subset of securities, but the covariance matrix is not invertible if the number of sample moments exceeds  $T$ .

In terms of existing empirical models in the literature, this estimation is most closely related to the signal extraction problem. The standard problem is a linear model of the form  $y_t = x_t + e_t$ , where  $x_t$ , the variable of interest, is observed with error. A common example is extracting expected inflation rates from observed inflation rates with a model of how expectations are formed. Our problem is reversed because we observe interest rates, the conditional expectation of the MRS. The model from which we extract estimates of the MRS is nonlinear, but we have restrictions on a large number of moments and the resulting estimator is linear. The MRS estimator employs an asset pricing relation and does not require a complete description of how expectations on key variables are formed, even though we have implicitly used the assumption that expectations are formed rationally.

From the model in equation (1), we can derive the following risk-return relationship:

$$E_{t-1}(R_{i,t}) - R_{Ft} = - \text{Cov}_{t-1}[\eta_t, R_{i,t}], \quad (11)$$

which states that the risk premium on a security is negatively related to the conditional covariance of the security return and the MRS. With estimates of this covariance, one can perform cross-sectional

regression tests similar to those which have been used to test the standard CAPM and the APT as in Fama and MacBeth (1973) and Roll and Ross (1980). Estimating this conditional covariance could be difficult, and we suspect that the risk premia and conditional covariances change over time. For this reason we do not attempt a cross-sectional regression test of the model. From equation (3), we can develop a relationship between average returns and covariances with the MRS. Our procedure of testing the model on a subset of security returns incorporates this relationship when we include a constant in the set of instrumental variables.

## II. Empirical Results with the MRS Model

The first step in the empirical analysis is to estimate the MRS with the estimator developed in Section I. The data for the estimation stage include monthly returns for the period 1926-85. We use the returns on one-month Treasury bills, long-term Treasury bonds, and long-term corporate bonds computed by Ibbotson and Sinquefeld plus the returns on stocks taken from the CRSP tapes.<sup>2</sup> We have also split the sample into two subperiods for estimation: (1) February 1926 to December 1955 and (2) January 1956 to December 1985. For the variable  $C_t$  in equation (8), we use all of the returns included in the CRSP equally weighted return index and add three security returns (the one-month Treasury bill, long-term Treasury bonds, and long-term corporate bonds). The number of securities included is 503 for January 1926, 1056 for December 1955, and just over 1500 at the end of 1985. For the estimation of the average  $\beta$ 's on the common factors,



we use returns on one-month Treasury bills, long-term Treasury bonds, long-term corporate bonds, and the first 97 companies on the CRSP tapes that have complete return series for the sub-period.

For the common factors in  $u_{it}$ , we use the innovations in short-term interest rates (returns on one-month T bills), yields on long-term Treasury bonds, and the excess return on the NYSE-CRSP value weighted index.<sup>3</sup> Our strategy is to use low order autoregressive (AR) models to predict these variables, but we find that changes in long-term rates are useful in predicting short-term rates and lagged short-term rates and dividend yields are useful in predicting stock market returns. We also consider a first order AR model for the excess return on the stock market. In Table I, we present estimated prediction equations for the two sub-periods. Our calculations of the innovations, however, use a rolling regression method. For example, to predict the variables for January 1956, we run the regressions with 15 years of monthly data through December 1955 and use those parameter estimates to make the predictions. The innovations are simply the actual values minus the predicted values. Then for February 1956, we re-estimate our regression equations with 15 years of data through January 1956. At the beginning of the sample, we use the first 10 years of data, 1926-35, to estimate the regression equations and use the residuals as our innovations. Then beginning with January 1936, we initiate the rolling regression procedure and add data until we reach the point where we have 15 years of data for the regressions. After 1940, we use the most recent 15 years of data to estimate the prediction equations. We find that the regression coefficients which

correspond to equation (4) for excess stock market returns in Table I vary considerably throughout the sample periods. Using mean squared errors and mean absolute deviations, we find that this model does not predict as well as the first-order AR model. For this reason we use the first order AR model to compute innovations in the excess stock market return and note that the first order autocorrelation coefficients are typically small.

The results of the GMM estimation for the average  $\beta$ 's on the common factors are summarized in Table II. We estimate the  $\beta$ 's for the two sub-periods with a three-factor model and a two-factor model: in both periods the addition of a third factor is not significant. The  $\beta$  parameters are not estimated with a high level of precision as indicated by the size of the standard errors. For the period 1926-55, the t statistics are 1.48 and -1.50 for the second and third factors in the two-factor model. When we add the first factor (short-term interest rates), its t statistic is .17 and its standard error is very large. For the period 1956-85, the t statistics are significant on only two factors: the first one (short-term interest rates) and the third one (excess stock market returns). In all cases, the  $\chi^2$  goodness-of-fit statistics indicate acceptance of the models. For the estimation of  $\eta$ , we use the two-factor models because the standard errors are smaller and the coefficients for a third factor are not significantly different from zero. The innovations in long-term interest rates and the excess stock market return are used for the 1926-55 period. During this period there was very little variation in rates on short-term Treasury bills. The innovations in the short-term

rate and the excess stock market return are used for the 1956-85 period. In Section I, we have made a theoretical argument for including the market return innovation as a factor, and here we find that an additional interest rate innovation adds to the fit of the model, marginally in one case and significantly in the other.

With the  $\beta$  estimates, we then compute  $\hat{\eta}_t$  according to equation (7) and the MRS,  $(\hat{\lambda}_t/\lambda_{t-1})$ , from equation (10). The estimates for the MRS are plotted in Figures 1A and B. The estimates suggest considerable variability in the ex post MRS over time: most of the values fall between .75 and 1.3, but the range is from .2536 to 1.9860 and there are many outliers. Most of the outliers occur between 1929 and 1940, a period which includes the stock market crash and the Depression of the 1930s. Recall from Section I that  $\lambda_t$  is the product of the marginal utility of real wealth and the consumption price deflator. Large decreases in consumption prices would make  $\lambda_t/\lambda_{t-1}$  a number much greater than one. During this period there were shocks in the stock market and dramatic decreases in consumer prices. The large values ( $> 1.6$ ) for the MRS between 1929 and 1940 correspond to months when the stock market experienced returns between -13.4% and -23.6%. The low values ( $< .4$ ) for this period coincide with months when the stock market experienced returns between +21.1% and +38.3%. During the period 1956-85 there are fewer outliers. The high outlier in Figure 1B is for September 1974 when the stock market return was -11%. The low outlier is for November 1980 when the stock market return was 9.47% and short-term interest rates experienced their largest positive shock. The estimates of the MRS suggest considerable variability, and

the outliers coincide with periods of large shocks in financial markets.

Before moving to the empirical tests of the intertemporal CAPM relation, we present some diagnostic tests on the estimates. One implication of the model is that  $\eta_t$  should be randomly distributed about a mean of one. To check this randomness, we have computed first-order autocorrelation coefficients, and with sample sizes of 360 the standard error is .0527. The autocorrelation coefficients are .0488 for the 1926-55 period and .0328 for the 1956-85 period. The sample means for  $\eta_t$  are not significantly different from one. We have performed the same calculations on  $\hat{\eta}_t$ 's computed from one-factor models in which the innovation in the excess stock market return is the only factor and we find that the autocorrelation coefficients are .2959 and .1871, respectively. The  $\eta_t$ 's from these one-factor models also have means which are significantly different from one.

To check the covariation of  $e_{it}$  across securities we have computed sample covariance matrices and eigenvalues starting with 20 securities and increasing up to 200 securities.<sup>4</sup> These calculations are summarized in Table III. For both sub-periods the largest eigenvalue is still increasing as we approach 200 securities, but the ratio of the largest eigenvalue to the sum of the eigenvalues is decreasing. In both cases with 200 securities, the component with the largest variance (the one with the largest eigenvalue) explains less than 8% of the total variance. We also note that the ratio of the sum of the eigenvalues (the total variation) to the number of securities remains level in both cases.

From the numbers in the last column of Table III, we can compute approximate standard errors for  $\bar{e}_t$  in equation (9). If the ratio of the variation to the number of securities remains roughly constant as suggested by Table III, we can divide this constant by  $K$ , the number of securities used to compute  $G_t$  in order to approximate the variance of  $\bar{e}_t$ . The calculations imply that the standard error for  $\bar{e}_t$  varies between .0031 and .0044 for the 1926-55 period and between .0018 and .0022 for the 1956-85 period. We have also computed the variance from the  $\beta$  estimates,  $\underline{\xi}_t' \text{Var}(\hat{\beta}) \underline{\xi}_t$ , and when we add the variance of  $\bar{e}_t$  the increase is negligible in most cases. Using equations (9) and (10), we have calculated the approximate standard error for each MRS estimate: the average standard error is .0434 for the 1926-55 period and .0469 for the 1956-85 period. The ranges for these standard errors are .0032 to .3464 and .0021 to .2398, respectively, but only 51 (7% of the total) standard errors exceed .1. The large standard errors are associated with the outliers that we observe in Figures 1A and B. These calculations suggest that most of the MRS estimates have been estimated with a reasonable level of precision but there is substantial estimation error associated with the extreme values which occur during periods of large shocks to financial markets.

The second step in our empirical analysis is to use the estimated MRS series to test the intertemporal CAPM relation. We test the restriction  $E \left\{ \left[ \left( \frac{\lambda_t}{\lambda_{t-1}} \right) (1+R_{it}) - 1 \right] \underline{z}_{i,t-1} \right\} = 0$  by testing whether the time-series sample moments are close to zero. The results of these tests are contained in Tables IV-VI. The first tests are performed on the NYSE-CRSP value weighted return index, Treasury bonds,

corporate bonds, and the  $\eta_t$  series. In Table IV, we present the results for the entire period 1927-85, and in Table V we present results for a more recent period 1952-85, which is frequently used in empirical studies. For the NYSE index, we include the following four instruments: a constant,  $(1+R_{m,t-1})/(1+R_{Ft})$ ,  $(1+R_{Ft})$ , and  $\frac{\bar{D}_{m,t-1}}{P_{m,t-1}}$  where  $\bar{D}_{m,t-1}$  is the accumulation of cash dividends over the months  $(t-12)$  through  $(t-1)$ .<sup>5</sup> We have included the short-term interest rate and the dividend yield because several studies, including the regressions in Table I, have documented correlations of stock returns with short-term interest rates and lagged dividend yields. All of the  $t$  statistics in Table IV are small indicating that none of the sample moments are significantly different from zero. None of the  $\chi^2$  statistics are significant, and we conclude that these security returns satisfy the restrictions of the intertemporal CAPM. In Table V, the results for the period 1952-85 are mixed. The sample moments are small and none of the  $t$  statistics are significant, but the  $\chi^2$  statistic for the NYSE portfolio is significant at the 5% level. Each  $\chi^2$  test statistic is a test of the null hypothesis that all of the sample moments for the security are zero. In Table VI, we present a summary of the results of tests on 30 securities in the Dow Jones Industrial Average. We use four instruments for each security including the dividend yield.<sup>6</sup> None of the  $t$  statistics for the sample moments and none of the  $\chi^2(4)$  statistics are significant. The results for these individual stocks generally support the intertemporal CAPM restrictions.

In Table VII, we present the tests of the intertemporal CAPM restrictions in an alternative framework by using more conventional regression tests. The regressions have been computed with the NYSE index, and we estimate three different equations. An alternative method of testing the sample moments is to regress  $\frac{\lambda_t}{\lambda_{t-1}} (1+R_{mt})$  on a constant and the instrumental variables:

$$\frac{\lambda_t}{\lambda_{t-1}} (1+R_{mt}) = c_0 + c_1 \left( \frac{1+R_{m,t-1}}{1+R_{Ft}} \right) + c_2 (1+R_{Ft}) + c_3 \frac{\bar{D}_{m,t-1}}{P_{m,t-1}} + e_t$$

Under the null hypothesis of the intertemporal CAPM,  $c_0$  should equal one and  $c_1$ ,  $c_2$ , and  $c_3$  should equal zero. A regression of  $\frac{\lambda_t}{\lambda_{t-1}} (R_{mt} - R_{Ft})$  on a constant and lagged variables known at time  $t-1$  should produce a zero intercept and zero coefficients on all lagged variables. In the third regression, we regress  $(R_{mt} - R_{Ft})$  on a constant and lagged variables known as  $t-1$ . This third regression is not a test of the intertemporal CAPM, but it is a test of a conventional model that is used in empirical studies. By placing restrictions on the distributions of the MRS and security returns, Hansen and Singleton (1983) and others have derived a result that expected excess returns should be constant and excess returns should be unpredictable. The intercept in the third regression measures the risk premium and the coefficients on the lagged variables should be zero. In panel A of Table VII we have the regression results for the period 1927-85. The  $R^2$ 's for all three equations are small and all of the tests indicate acceptance of the intertemporal CAPM restrictions and the stronger restrictions implied by the excess return model. In Panel B, we present the same three

regressions for the more recent period 1952-85. In all three regression equations, the t statistics for some of the coefficients are significant and the joint  $\chi^2$  test statistics are significant. These regression tests indicate rejection of the intertemporal CAPM restrictions for the 1952-85 period. It is interesting to note that over the longer period all of the model restrictions are accepted. At this point, we conjecture that there may be something unique during the 1952-85 period which is averaged out or disappears over a longer time period. Given the results for the longer period, we conclude that the data generally support the restrictions of the intertemporal CAPM.

In Section I, we noted that the estimates and empirical tests allow for the possibility of conditional heteroscedasticity in the data. In Table VIII we present some evidence of autoregressive conditional heteroscedasticity in some of our key variables. We apply Engle's (1982) LaGrange multiplier test: in this application, we regress the square of the error term on three lagged values of itself and  $TR^2$  is approximately distributed as a  $\chi^2$  with three degrees of freedom. We have used several simple models for  $\eta_t$ , the market excess return, the short-term interest rate, and the long-term interest rate. The two models for  $\eta_t$  are

$$\eta_t = 1 + u_t$$
$$\ln \eta_t = \beta_0 + u_t.$$

We also consider two simple models for the market excess return:

$$R_m - R_{Ft} = \beta_0 + u_t$$
$$\ln \frac{(1+R_{mt})}{(1+R_{Ft})} = \beta_0 + u_t.$$



For the interest rate variables we use the models in Table I. We find evidence of conditional heteroscedasticity in the error terms for all but one of the models, the one for  $\ln \eta_t$ . Even though we have tested for only one form of autoregressive conditional heteroskedasticity, the results indicate that there is some conditional heteroskedasticity in these financial variables. These results indicate one possible explanation for the rejection of asset pricing relations in models which require additional restrictions on the distributions of security returns and the MRS.<sup>7</sup>

### III. Summary

In the first part of this paper, we develop a method for using security returns to extract consistent estimates of the MRS, which we treat as an unobservable. The estimator makes use of the large number of sample moments available on security returns to identify and estimate the underlying MRS series which is common across all securities. The estimates are then used to test the relationship which arises in the intertemporal CAPM by applying the test to a subset of securities. These second stage tests examine the ability of the estimates and the model to fit the data on security returns. We find that the results generally support the intertemporal CAPM, but the tests are not comprehensive tests of the intertemporal CAPM. With estimates of the MRS implicit in security returns, one can explore other important issues such as the relationship between security prices and market fundamentals and the relationship between aggregate consumption and the MRS in life-cycle models.

APPENDIX

From the budget constraints of intertemporal consumption-investment decisions, we have the following set of necessary conditions for an economy with  $N$  agents who are neither identical nor share the same information sets:

$$J_w^k(t) p_t - E[J_w^k(t+1) (p_{t+1} + d_{t+1}) | \phi_t^k] = 0, \quad k=1, \dots, N,$$

where prices and dividends are in real terms and we have suppressed the index on  $p_t$  and  $d_t$  for different securities. For examples, see Lucas (1978) and Breeden (1979). Restating the model in nominal terms, we have

$$\lambda_t^k P_t - E[\lambda_{t+1}^k (P_{t+1} + D_{t+1}) | \phi_t^k] = 0, \quad k=1, \dots, N \quad (A-1)$$

where  $\lambda_t^k$  is the marginal utility of real wealth times the consumption price deflator for individual  $k$ . These pricing equations are aggregated across all  $N$  investors and equilibrium prices are formed so that investors as a group are willing to hold all the shares outstanding. To avoid boundary conditions for some investors, we must assume unrestricted short selling. Given equilibrium market prices, the relationships in (A-1) should be satisfied. Private information plays a role in the price formation, but we do not investigate that issue here. Instead we consider the role of market information defined as follows:

$$\phi_t^m = \phi_t^1 \cap \phi_t^2 \cap \dots \cap \phi_t^N,$$

that is, market information includes information that is known by all agents. Agents may or may not know the marginal utility of wealth parameters,  $\lambda_t$ , for other agents. If they have this information, the

model is simplified. We consider the case in which agents do not, but instead we form an expectation about these preference parameters conditional on market information. Our next step is to take the expectation of each equation in (A-1) conditional on  $\phi_t^m$ , noting that  $\phi_t^m \subset \phi_t^k$  for  $k=1, \dots, N$ .

$$P_t E(\lambda_t^k | \phi_t^m) - E[\lambda_{t+1}^k (P_{t+1} + D_{t+1}) | \phi_t^m] = 0$$

Then we aggregate across the  $N$  investors.

$$P_t \sum_{k=1}^N E(\lambda_t^k | \phi_t^m) - \sum_{k=1}^N E[\lambda_{t+1}^k (P_{t+1} + D_{t+1}) | \phi_t^m] = 0$$

For the second term, we have

$$E[\sum_{k=1}^N \lambda_{t+1}^k (P_{t+1} + D_{t+1}) | \phi_t^m] = E[(P_{t+1} + D_{t+1}) (\sum_{k=1}^N \lambda_{t+1}^k) | \phi_t^m]$$

Let  $\lambda_t \equiv \sum_{k=1}^N E(\lambda_t^k | \phi_t^m)$ . By the law of iterated expectations,

$$E[(P_{t+1} + D_{t+1}) (\sum_{k=1}^N \lambda_{t+1}^k) | \phi_t^m] = E\{E[(P_{t+1} + D_{t+1}) (\sum_{k=1}^N \lambda_{t+1}^k) | \phi_{t+1}^m] | \phi_t^m\},$$

and it follows that

$$\lambda_t P_t - E[\lambda_{t+1} (P_{t+1} + D_{t+1}) | \phi_t^m] = 0, \quad (A-2)$$

where  $\lambda_{t+1} \equiv \sum_{k=1}^N E(\lambda_{t+1}^k | \phi_{t+1}^m)$ . If agents know current values of  $\lambda_t^k$

for all investors, then this result follows with  $\lambda_t \equiv \sum_{k=1}^N \lambda_t^k$ . (A-2)

implies the relationship studied in the paper with the interpretation that  $E_t$  is the expectation conditional on market information,  $\phi_t^m$ , and  $\lambda_t$  is an aggregation of the marginal utility of wealth variable across agents. It is not necessary to assume that agents are identical and share all information to derive this result.

Footnotes

<sup>1</sup>For  $N > T$ , the estimated covariance matrix is singular and  $N - T$  eigenvalues will equal zero.

<sup>2</sup>We have supplemented the series from Ibbotson and Sinquefeld (1982) with the corresponding data in R. G. Ibbotson Associates, Stocks, Bonds, Bills, and Inflation: 1986.

<sup>3</sup>We have used data for 1925 taken from the Federal Reserve's Banking and Monetary Statistics, 1943, so that our sample periods for the prediction equations and innovations can be initiated on January 1926. The yields on long-term Treasury bonds are from the DRI data base and the Federal Reserve's Banking and Monetary Statistics.

<sup>4</sup>We stop at 200 securities because at that point we hit the limits of the central memory available on our CDC Cyber computer. The covariance matrices are computed over time and are  $N \times N$ , where  $N$  is the number of securities.

<sup>5</sup>We use the NYSE returns calculated with dividends.

<sup>6</sup>The stock returns and dividend yields have been adjusted for stock splits and stock dividends.  $D_{i,t-1}$  is calculated in the same manner as  $D_{m,t-1}$ .

<sup>7</sup>For an example, see the tests on differences between two security returns in Hansen and Singleton (1983).

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FIGURE 1 A

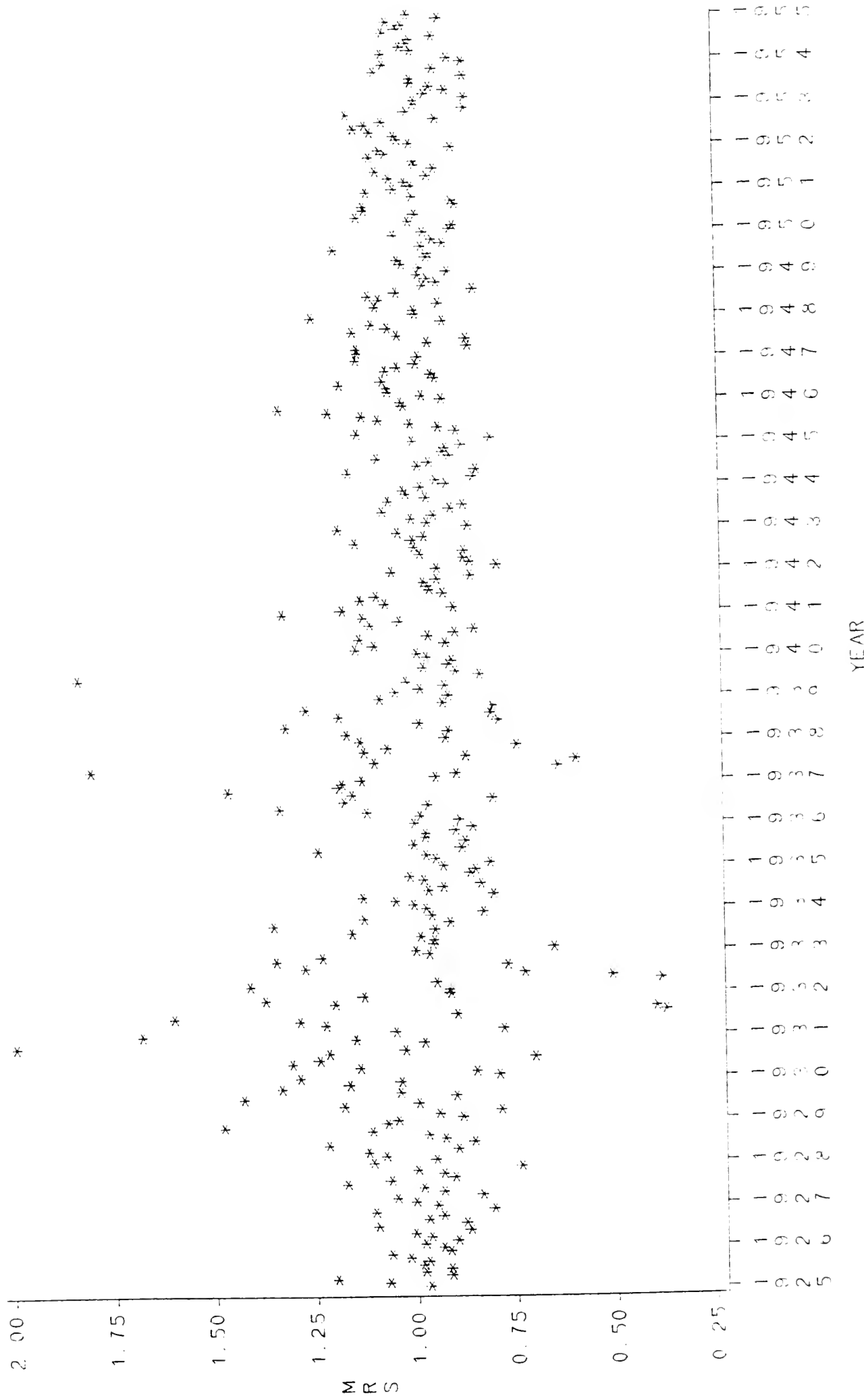




FIGURE 1 B

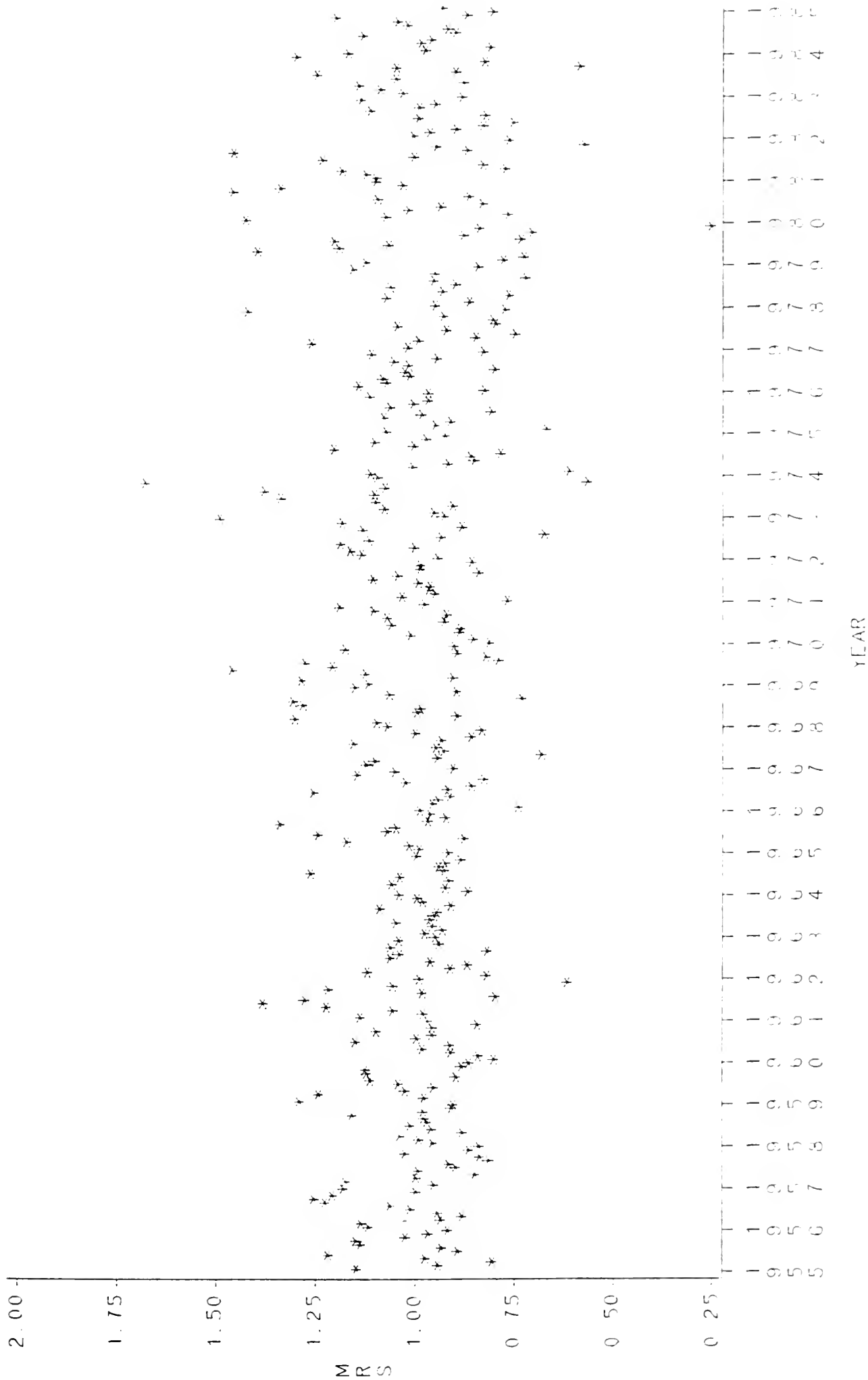


TABLE I

Formulating the Prediction Equations for the Common Factors

A. First Sub-Period, 1926-55

$$\begin{aligned}
 (1) \quad RS_t &= .00004507 + .2135 RS_{t-1} + .2414 R_{t-2} + .3242 RS_{t-3} \\
 &\quad (.00002908) (.0543) \quad (.0543) \quad (.0532) \\
 &\quad + .1871 RS_{t-4} - .01976 RS_{t-5} + .1011 \Delta RL_{t-1} \\
 &\quad (.0542) \quad (.0522) \quad (.0308) \\
 &\quad + .06455 \Delta RL_{t-2} \\
 &\quad (.0311)
 \end{aligned}$$

$$R^2 = .85 \quad D.W. = 1.99 \quad T = 360$$

$$\begin{aligned}
 (2) \quad \Delta RL_t &= -.00001966 + .3120 \Delta RL_{t-1} \\
 &\quad (.00003789) (.0526) \\
 &\quad - .1036 \Delta RL_{t-2} \\
 &\quad (.0526)
 \end{aligned}$$

$$R^2 = .09 \quad D.W. = 1.99 \quad T = 360$$

$$(3) \quad R_{mt} - RS_{t-1} = .007690 + .1295 (R_{m,t-1} - RS_{t-2}) + \xi_{3t} \\
 \quad \quad \quad (.003643) (.0524)$$

$$R^2 = .02 \quad D.W. = 2.00 \quad T = 360$$

$$\begin{aligned}
 (4) \quad R_{mt} - RS_{t-1} &= -.01673 + .1509 (R_{m,t-1} - RS_{t-2}) - .06362 RS_{t-1} \\
 &\quad (.01514) (.0549) \quad (3.7955) \\
 &\quad + .4693 \left( \frac{\bar{D}_{m,t-1}}{P_{m,t-1}} \right) + \xi_{3t} \\
 &\quad (.2634)
 \end{aligned}$$

$$R^2 = .03 \quad D.W. = 2.01 \quad T = 348$$

TABLE I (continued)

B. Second Sub-Period, 1956-85

$$(1) \quad RS_t = .0001676 + .7496 RS_{t-1} + .0211 RS_{t-2} + .2357 RS_{t-3} \\
\quad \quad \quad (.0000833) \quad (.0600) \quad (.0738) \quad (.0695) \\
\quad \quad \quad - .1921 RS_{t-4} + .1515 RS_{t-5} + .07476 \Delta RL_{t-1} \\
\quad \quad \quad (.0699) \quad (.0518) \quad (.01821) \\
\quad \quad \quad - .03256 \Delta RL_{t-2} + \xi_{1t} \\
\quad \quad \quad (.01867)$$

$$R^2 = .92 \quad D.W. = 2.00 \quad T = 354$$

$$(2) \quad \Delta RL_t = .0001677 + .4042 \Delta RL_{t-1} \\
\quad \quad \quad (.0001263) \quad (.0513) \\
\quad \quad \quad - .2856 \Delta RL_{t-2} \\
\quad \quad \quad (.0516)$$

$$R^2 = .17 \quad D.W. = 2.00 \quad T = 354$$

$$(3) \quad R_{mt} - RS_{t-1} = .003591 + .07439 (R_{m,t-1} - RS_{t-2}) + \xi_{3t} \\
\quad \quad \quad (.002212) \quad (.05317)$$

$$R^2 = .01 \quad D.W. = 2.00 \quad T = 354$$

$$(4) \quad R_{mt} - RS_{t-1} = -.03130 + .05629 (R_{m,t-1} - RS_{t-2}) - 5.1347 RS_{t-1} \\
\quad \quad \quad (.01031) \quad (.05197) \quad (1.0884) \\
\quad \quad \quad + 1.5654 \left( \frac{\bar{D}_{m,t-1}}{P_{m,t-1}} \right) + \xi_{3t} \\
\quad \quad \quad (.3316)$$

$$R^2 = .08 \quad D.W. = 1.97 \quad T = 354$$

NOTES: The conventional OLS standard errors are in parentheses.  $\bar{D}_{mt}$  is the accumulation of dividends over the periods  $t, t-1, \dots, t-11$ .  $RS_t \equiv R_{F,t+1}$ .

TABLE II

GMM Estimation of  $\underline{\beta}$ 

## A. First Sub-Period, 1926-55

Factors	<u>3 Factors</u>	<u>2 Factors</u>
	$\xi_{1t}, \xi_{2t}, \xi_{3t}$	$\xi_{2t}, \xi_{3t}$
$\beta_1$	37.3139 (215.9927)	--
$\beta_2$	119.4198 (92.8840)	65.9586 (44.6847)
$\beta_3$	-.8766 (.7540)	-1.0933 (.7297)
T	360	360
$T \underline{x}' \hat{S}^{-1} \underline{x}$	70.76	72.96

## B. Second Sub-Period, 1956-85

Factors	<u>3 Factors</u>	<u>2 Factors</u>
	$\xi_{1t}, \xi_{2t}, \xi_{3t}$	$\xi_{1t}, \xi_{3t}$
$\beta_1$	-106.2600 (40.4088)	-111.1016 (34.1643)
$\beta_2$	-5.0642 (23.5506)	--
$\beta_3$	-2.7610 (1.3763)	-2.6355 (1.3474)
T	360	360
$T \underline{x}' \hat{S}^{-1} \underline{x}$	67.09	67.04

NOTES: Standard errors are in parentheses.  $\xi_{1t}$  is the innovation in the short rate,  $\xi_{2t}$  is the innovation in the long rate, and  $\xi_{3t}$  is the innovation in the excess return on the value-weighted NYSE portfolio.

$$\underline{x}: x_i = \frac{1}{T} \sum_{t=1}^T \eta_t(\hat{\beta}) \frac{1+R_{it}}{1+R_{Ft}}, \quad i=1, \dots, 100 \text{ for the 100}$$

security returns used in the estimation.  $\hat{S}/T$  is the 100x100 estimated covariance matrix for  $\underline{x}$ .

TABLE III

Analysis of Eigenvalues for Sample Covariance Matrices of

$$e_{it} = \left( \frac{\lambda_t}{\lambda_{t-1}} \right) (1 + R_{it}) - 1 - \beta_{i1} \xi_{1t} - \beta_{i2} \xi_{2t}$$

## A. First Sub-Period, 1926-55

<u>Number of Securities</u>	<u>Largest Eigenvalue</u>	<u>Sum of the Eigenvalues</u>	<u>Ratio of the Largest Eigenvalue to the Sum</u>	<u>Ratio of the Sum to Number of Securities</u>
20	.02573	.1405	.1831	.0070
40	.1101	.4078	.2700	.0102
60	.1111	.5838	.1902	.0097
80	.1123	.7472	.1503	.0093
100	.1131	.8986	.1259	.0090
120	.1160	1.1209	.1035	.0093
140	.1189	1.3166	.0903	.0094
160	.1266	1.5022	.0843	.0094
180	.1448	1.7439	.0830	.0097
200	.1510	1.9223	.0785	.0096

## B. Second Sub-Period, 1956-85

<u>Number of Securities</u>	<u>Largest Eigenvalue</u>	<u>Sum of the Eigenvalues</u>	<u>Ratio of the Largest Eigenvalue to the Sum</u>	<u>Ratio of the Sum to Number of Securities</u>
20	.01355	.09590	.1413	.0048
40	.01735	.1862	.0932	.0047
60	.02981	.3043	.0980	.0051
80	.03908	.4300	.0909	.0054
100	.04240	.5158	.0822	.0052
120	.04666	.6079	.0768	.0051
140	.05253	.6951	.0756	.0050
160	.05660	.7891	.0717	.0049
180	.06183	.8846	.0699	.0049
200	.06766	.9605	.0704	.0048

NOTE: The ratio of the sum of the eigenvalues to the number of securities is also the total variation divided by the number of securities.

TABLE IV

## Tests on the Intertemporal CAPM

$$E(\underline{u}_i) = E \left\{ \frac{1}{T} \sum_{t=1}^T \left[ \frac{\lambda_t}{\lambda_{t-1}} (1+R_{it}) - 1 \right] z_{i,t-1} \right\} = 0$$

Sample Period: January 1927 to December 1985, T = 708

## A. NYSE-CRSP Value Weighted Return Index

<u>Instrument</u>	<u>Sample Moment</u>	<u>Standard Error</u>	<u>t Statistic</u>
Constant	-.001503	.004550	-.33
$\frac{(1+R_{m,t-1})}{(1+R_{Ft})}$	-.001979	.004590	-.43
$1+R_{Ft}$	-.001492	.004570	-.33
$\frac{\bar{D}_{m,t-1}}{P_{m,t-1}}$	-.0001978	.000247	-.80

$$\chi^2(4) = 3.53$$

## B. Long Term Treasury Bonds

<u>Instrument</u>	<u>Sample Moment</u>	<u>Standard Error</u>	<u>t Statistic</u>
Constant	.000957	.006377	.15
$\frac{(1+R_{B,t-1})}{(1+R_{Ft})}$	.000737	.006377	.12
$1+R_{Ft}$	.000982	.006399	.15

$$\chi^2(3) = 2.08$$

TABLE IV (continued)

C. Long Term Corporate Bonds

<u>Instrument</u>	<u>Sample Moment</u>	<u>Standard Error</u>	<u>t Statistic</u>
Constant	.001420	.006380	.22
$\frac{(1+R_{c,t-1})}{(1+R_{Ft})}$	.001245	.006363	.20
$1+R_{Ft}$	.001445	.006402	.23

$$\chi^2(3) = 1.65$$

$$D. \quad E(\underline{u}_i) = E \left\{ \frac{1}{T} \sum_{t=1}^T [\eta_t - 1] \underline{z}_{i,t-1} \right\} = 0$$

<u>Instrument</u>	<u>Sample Moment</u>	<u>Standard Error</u>	<u>t Statistic</u>
Constant	.000711	.006430	.11
$\frac{(1+R_{B,t-1})}{(1+R_{Ft})}$	.000482	.006432	.08
$1+R_{Ft}$	.000737	.006451	.11

$$\chi^2(3) = 2.26$$

TABLE V

## Tests of the Intertemporal CAPM

Sample Period: February 1952 to December 1985, T = 407

## A. NYSE-CRSP Value Weighted Return Index

<u>Instrument</u>	<u>Sample Moment</u>	<u>Standard Error</u>	<u>t Statistic</u>
Constant	-.002136	.006228	-.34
$\frac{(1+R_{m,t-1})}{(1+R_{Ft})}$	-.002455	.006189	-.40
$1+R_{Ft}$	-.002120	.006268	-.34
$\frac{\bar{D}_{m,t-1}}{P_{m,t-1}}$	-.0001714	.0002722	-.63

$$\chi^2(4) = 12.90$$

## B. Long Term Treasury Bonds

<u>Instrument</u>	<u>Sample Moment</u>	<u>Standard Error</u>	<u>t Statistic</u>
Constant	-.002569	.007921	-.32
$\frac{(1+R_{B,t-1})}{(1+R_{Ft})}$	-.002933	.007871	-.37
$1+R_{Ft}$	-.002531	.007969	-.32

$$\chi^2(3) = 5.54$$



TABLE V (continued)

Tests of the Intertemporal CAPM

Sample Period: February 1952 to December 1985, T = 407

C. Long Term Corporate Bonds

<u>Instrument</u>	<u>Sample Moment</u>	<u>Standard Error</u>	<u>t Statistic</u>
Constant	-.002405	.007848	-.31
$\frac{(1+R_{c,t-1})}{(1+R_{Ft})}$	-.002659	.007795	-.34
$1+R_{Ft}$	-.002367	.007897	-.30

$$\chi^2(3) = 4.72$$

$$D. \quad E(\underline{u}_i) = E \left\{ \frac{1}{T} \sum_{t=1}^T [\eta_t - 1] \underline{z}_{t-1} \right\} = 0$$

<u>Instrument</u>	<u>Sample Moment</u>	<u>Standard Error</u>	<u>t Statistic</u>
Constant	-.001917	.007878	-.24
$\frac{(1+R_{B,t-1})}{(1+R_{Ft})}$	-.002308	.007828	-.29
$1+R_{Ft}$	-.001877	.007926	-.24

$$\chi^2(3) = 5.49$$

TABLE VI

Summary of Results for 30 DJIA Stocks

$$E \left\{ \frac{1}{T} \sum_{t=1}^T \left[ \frac{\lambda_t}{\lambda_{t-1}} (1+R_{it})^{-1} \right] z_{i,t-1} \right\} = 0$$

t Statistics for Sample Moments

<u>Stock</u>	<u>1</u>	<u>2</u>	<u>3</u>	<u>4</u>	<u><math>\chi^2(4)</math></u>	<u>T</u>	<u>Sample Period</u>
Allied	-.31	-.97	-.95	-.71	1.04	704	1927-85
ALCOA	-.27	-.44	-.34	-.98	3.46	402	1951-85
American Brands	.39	.23	.22	.08	.20	708	1927-85
American Can	.06	-.45	-.44	-.30	.46	708	1927-85
AT&T	.25	-.17	-.17	-.21	.22	708	1927-85
Bethlehem Steel	-.39	-1.35	-1.36	-1.17	2.07	708	1927-85
Chevron	.12	-.37	-.37	-.30	.32	708	1927-85
DuPont	.14	-.79	-.76	-.63	1.28	708	1927-85
Kodak	.24	-.36	-.34	-.50	.48	708	1927-85
Exxon	.47	.04	.05	-.00	.44	708	1927-85
General Electric	-.05	-.50	-.47	-.57	.51	708	1927-85
General Foods	.42	.32	.33	.02	.22	706	1927-85
General Motors	.17	-.49	-.48	-.49	.60	708	1927-85
Goodyear	-.09	-.53	-.50	-.63	.52	687	1928-85
INCO	-.17	-1.34	-1.33	-1.09	2.33	708	1927-85
IBM	1.27	.33	.34	-.17	1.82	708	1927-85
International Harvester	-.45	-1.16	-1.15	-.83	1.44	708	1927-85

TABLE VI (continued)

## Summary of Results for 30 DJIA Stocks

<u>Stock</u>	t Statistics for Sample Moments					<u>T</u>	<u>Sample Period</u>
	<u>1</u>	<u>2</u>	<u>3</u>	<u>4</u>	<u><math>\chi^2(4)</math></u>		
International Paper	.67	-1.03	-.99	-.86	2.39	673	1930-85
Manville	-1.01	-1.70	-1.67	-1.40	3.05	682	1929-85
Merck	.60	.08	.13	-.31	2.09	463	1947-85
3M	.40	-.34	-.29	-.66	3.03	467	1947-85
Owens Illinois	-.00	-.79	-.77	-.72	.89	708	1927-85
Procter & Gamble	.22	-.04	-.02	-.22	.25	664	1930-85
Sears	.12	-.54	-.53	-.53	.55	708	1927-85
Texaco	.09	-.53	-.51	-.55	.53	708	1927-85
Union Carbide	-.14	-1.08	-1.05	-.85	1.75	705	1927-85
U.S. Steel	-.52	-.94	-.94	-.92	1.09	708	1927-85
United Technologies	.18	.05	.07	-.04	.11	667	1930-85
Westinghouse	-.17	-.50	-.50	-.43	.27	708	1927-85
Woolworth	-.12	-.41	-.40	-.31	.18	708	1927-85

NOTE: The instruments are (1) constant

$$(2) \frac{1 + R_{it}}{1 + R_{Ft}}$$

$$(3) 1 + R_{Ft}$$

$$(4) \frac{\bar{D}_{i,t-1}}{P_{i,t-1}}$$

TABLE VII

## Regression Tests

A. Sample Period: January 1927 to December 1985, T = 708

$$(1) \frac{\lambda_t}{\lambda_{t-1}} (1+R_{mt}) = .2357 - .1639 \frac{(1+R_{m,t-1})}{(1+R_{Ft})} - .9584 (1+R_{Ft}) - .7443 \frac{\bar{D}_{m,t-1}}{P_{m,t-1}} + e_t$$

(2.5648) (.1178) (2.5373) (.4925)

$$R^2 = .013$$

$$D.W. = 2.03$$

$$\text{Test of } \beta_1 = \beta_2 = \beta_3 = 0, \chi^2(3) = 4.22$$

$$\text{Test of } \beta_0 = 1 \text{ and } \beta_1 = \beta_2 = \beta_3 = 0, \chi^2(4) = 4.26$$

$$(2) \frac{\lambda_t}{\lambda_{t-1}} (R_{mt} - R_{Ft}) = .004447 + .06926 \frac{\lambda_{t-1}}{\lambda_{t-2}} (R_{m,t-1} - R_{F,t-1}) - 1.1333 R_{Ft} +$$

(.01109) (.05549) (.8319)

$$.07426 \frac{\bar{D}_{m,t-1}}{P_{m,t-1}} + e_t$$

(.2546)

$$R^2 = .008$$

$$D.W. = 2.00$$

$$\text{Test of } \beta_1 = \beta_2 = \beta_3 = 0, \chi^2(3) = 3.88$$

$$\text{Test of } \beta_0 = \beta_1 = \beta_2 = \beta_3 = 0, \chi^2(4) = 4.91$$

$$(3) R_{mt} - R_{Ft} = -.009310 + .1242 (R_{m,t-1} - R_{F,t-1}) - .9927 R_{Ft} + .3956 \frac{\bar{D}_{m,t-1}}{P_{m,t-1}} +$$

(.01362) (.07077) (.7382) (.3083)

$$R^2 = .026$$

$$D.W. = 2.00$$

$$\text{Test of } \beta_1 = \beta_2 = \beta_3 = 0, \chi^2(3) = 6.76$$

TABLE VII (continued)

B. Sample Period: February 1952 to December 1985, T = 407

$$(1) \frac{\lambda_t}{\lambda_{t-1}} (1+R_{mt}) = \frac{-3.9918}{(3.3031)} - \frac{.15283}{(.1767)} \frac{(1+R_{m,t-1})}{(1+R_{Ft})} + \frac{5.1872}{(3.2534)} (1+R_{Ft}) - \frac{1.6926}{(.6797)} \frac{\bar{D}_{m,t-1}}{P_{m,t-1}} + e_t$$

$$R^2 = .021$$

$$D.W. = 2.00$$

$$\text{Test of } \beta_1 = \beta_2 = \beta_3 = 0, \chi^2(3) = 11.04^*$$

$$\text{Test of } \beta_0 = 1 \text{ and } \beta_1 = \beta_2 = \beta_3 = 0, \chi^2(4) = 12.76^*$$

$$(2) \frac{\lambda_t}{\lambda_{t-1}} (R_{mt} - R_{Ft}) = \frac{-.01658}{(.00904)} + \frac{.08132}{(.06251)} \frac{\lambda_{t-1}}{\lambda_{t-2}} (R_{m,t-1} - R_{F,t-1}) - \frac{3.7545}{(.8748)} R_{Ft} + \frac{.8296}{(.2320)} \frac{\bar{D}_{m,t-1}}{P_{m,t-1}} + e_t$$

$$R^2 = .071$$

$$D.W. = 1.97$$

$$\text{Test of } \beta_1 = \beta_2 = \beta_3 = 0, \chi^2(3) = 30.16^{**}$$

$$\text{Test of } \beta_0 = \beta_1 = \beta_2 = \beta_3 = 0, \chi^2(4) = 32.01^{**}$$

$$(3) R_{mt} - R_{Ft} = \frac{-.01597}{(.008849)} + \frac{.04138}{(.05644)} (R_{m,t-1} - R_{F,t-1}) - \frac{3.2189}{(.7944)} R_{Ft} + \frac{.8900}{(.2407)} \frac{\bar{D}_{m,t-1}}{P_{m,t-1}} + e_t$$

$$R^2 = .058$$

$$D.W. = 1.97$$

$$\text{Test of } \beta_1 = \beta_2 = \beta_3 = 0, \chi^2(3) = 26.81^{**}$$

NOTE: Standard errors are in parentheses. We have allowed for conditional heteroskedasticity in computing the standard errors and  $\chi^2$  statistics.

\* Significant at the 5% level.

\*\* Significant at the 1% level.

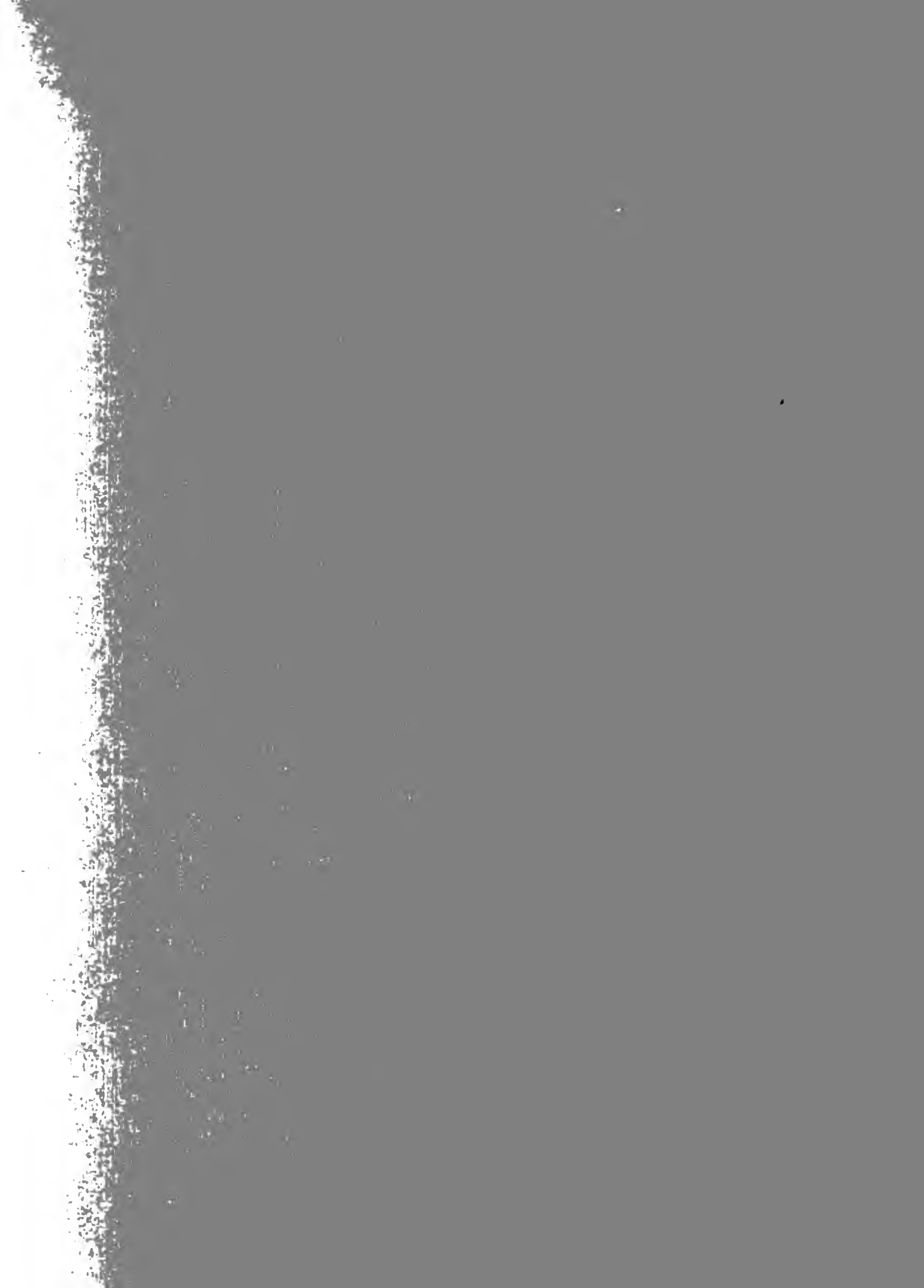
TABLE VIII

Tests for Autoregressive Conditional Heteroskedasticity  
 Sample Period: 1952-85

Models of the Form:  $u_t^2 = a_0 + a_1 u_{t-1}^2 + a_2 u_{t-2}^2 + a_3 u_{t-3}^2 + e_t$

	$\frac{R^2}{}$	$\frac{TR^2}{}$
1	.042	16.89**
$u_t + .01532114$	.009	3.56
$-R_{Ft} - .00529$	.052	20.89**
$\frac{+R_{mt}}{+R_{Ft}} - .004459$	.047	19.15**
$-. (.000131982 + .749709 RS_{t-1} + .018782 RS_{t-2} + .244077 RS_{t-3}$ $- .193055 RS_{t-4} + .151801 RS_{t-5} + .074729 \Delta RL_{t-1}$ $- .031869 \Delta RL_{t-2})$	.099	39.56**
$-. (.000155241 + .403823 \Delta RL_{t-1} - .283799 \Delta RL_{t-2})$	.124	49.44**

NOTE: \* indicates significance at the 5% level.  
 \*\* indicates significance at the 1% level.



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