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Every Finite Distributive Lattice Is a Set of Stable Matchings

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Abstract

We show that, given a lattice, a set of men and women with preferences can be constructed whose stable matchings are precisely that lattice. This is a converse of a result of J. H. Conway.

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by

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Suppose we have n men and n women. Each of the 2n people has a linear preference ordering on those of the opposite sex. We are interested in matchings to form n couples. A matching is stable if we cannot find a woman in one couple and a man in another who would prefer each other to their present partners.

Stable matchings were first defined by Gale and Shapley [1], who showed that for any preference orderings a stable matching always exists. In general, there will be several stable matchings. For example, if all the men happen to have different first preferences, giving each man his first choice will be stable, regardless of the women's preferences. Similarly, giving each woman her first choice (if possible) will be stable.

Conway [2, p. 87-92] defines a partial ordering on the set of stable matchings as follows: one matching is $\geq$ another if every man is at least as happy with his partner in the first matching as with his partner in the second. He shows that the set of stable matchings is always a finite distributive lattice. Knuth [2, p. 92] asks whether every finite distributive lattice can occur as the set of stable matchings generated by some set of men and women. We show this is the case.

We will require some preliminary facts about lattices. If L is a distributive lattice and $x \in L$ let $V = \{v_y | y \leq x\}$ be disjoint from L. L^x is the partial ordering on $L \cup V$ defined by (i) if $w, z \in L$ then $w \geq z$

in L^x iff $w \geq z$ in L. (ii) if $w \in L, v_z \in V$ $w \geq v_z$ iff $w \geq z$ in L. (iii) $v_w \geq v_z$ iff $w \geq z$. (iv) $v_w \not\geq z$ for any w,z. L^x is a distributive lattice. Intuitively, L^x is formed from L by making a copy of all the elements $\leq x$ and putting the copies immediately below the originals.

Lemma 1: If a set S of lattices includes a one-element lattice and includes a lattice isomorphic to L^x for every $L \in S$, $x \in L$ then every finite distributive lattice is isomorphic to a lattice in S.

Proof: Let M be a finite distributive lattice. We argue by induction on the size of M. If M has one element the result is immediate. Otherwise let z be the smallest member of M which is not the meet of two members different from z. Let w be the meet of all members > z. $N = \{y | y \not\leq z\}$ is a distributive sublattice in which meets and joins are preserved. The minimality property of z implies that if $y \leq z$ then $y = z \wedge u$ for some $u \in N$. Moreover, if $u_1 \wedge z = u_2 \wedge z \neq z$ then $u_1 \wedge w = (u_1 \wedge w) \wedge z = (u_2 \wedge w) \wedge z = u_2 \wedge w$. Hence M is isomorphic to N^w. By induction hypothesis, N is isomorphic to a lattice in S, so M is isomorphic to a lattice in S. Q.E.D.

To complete the proof we will show how to construct a set of men and women whose preferences yield L^x from a set whose preferences yield L.

Lemma 2: Let L be the set of stable matchings possible for women $w_1,\ldots,w_n$ and men $m_1,\ldots,m_n$. Suppose $x = (m_1w_1,\ldots,m_nw_n) \in L$. Then the set of stable matchings for the 2n men $m_1,\ldots,m_n$; $m_1',\ldots,m_n'$ and women $w_1,\ldots,w_n$; $w_1',\ldots,w_n'$ with the following preferences is isomorphic to L^x:

m_i: Use the original preferences of m_i in the n-couple situation for all women strictly preferred to (above) w_i. Replace w_i by w_i'. After w_i' put w_{i+1}' and finish the ordering arbitrarily.

m_i': First choice w_i', followed by w_i and the original preferences of m_i below w_i. Finish arbitrarily.

w_i: In the original preference ordering replace m_j by (m_j',m_j) for $j=i$ and all m_j above m_i. For m_j below m_i use (m_j,m_j'). Example: if the original ordering for w_2 is (best) $m_1m_2m_3$ new ordering is $m_1'm_1m_2'm_2m_3m_3'$.

w_i': First choice is m_{i-1}. Second choice is m_i, followed by m_i'. Finish arbitrarily.

In this definition all arithmetic is modulo n. We illustrate with an example after the proof.

Proof: We begin by observing that in any stable 2n-couple matching with these preferences (1) If for some i, m_i gets w_{i+1}' then w_i' (preferred by m_i) must get m_{i-1}, hence m_i must get w_{i+1}' for all i. (2) w_i' is the first choice of m_i', hence w_i' must get either m_{i-1} (and 1 applies), or m_i, or m_i'. (3) m_i must get somebody at least as good as w_{i+1}'. (4) If m_i does not get w_i' or w_{i+1}', then m_i' gets w_i'. (5) If m_i prefers w_j to w_i', m_i' does not get w_j. (Since x is stable w_j prefers m_i to m_i'. If m_i' got w_j (4) would imply m_i gets w_i' or w_{i+1}' so m_i and w_j would be happier together.)

These observations imply that nobody gets assigned to the arbitrary part of his or her ordering. Further if we are given a stable matching for the 2n couples we obtain a stable matching for the n-couple problem (i.e., a member of L) by giving each w_i her partner in the 2n-couple problem, deleting primes where necessary. Conversely if $y \in L$, there is a corresponding stable matching for the 2n-couple situation in which m_i is replaced by m_i' iff m_i gets w_i or somebody worse in y. If $y \leq x$ there are two 2n-couple matches corresponding to y--one in which each m_i gets

w'_i, and one in which each m_i gets w'_{i+1}. Those matches in which each m_i gets w'_{i+1} corresponds to V in the definition of L^x. Q.E.D.

<u>Example</u>: The four people with preferences given below have stable matching corresponding to a four-element lattice: (A) m_1 gets w_2, m_2 gets w_1, m_3 gets w_3, m_4 gets w_4 (abbreviated (2134)) (B) (1243) (C) (1234) (D) (2143).

m_1	m_2	m_3	m_4
w_2	w_1	w_3	w_4
w_1	w_2	w_4	w_3
		...	

w_1	w_2	w_3	w_4
m_1	m_2	m_4	m_3
m_2	m_1	m_3	m_4
		...	

(... = arbitrary)

$L^{(1234)}$ is a six-element lattice generated by the preferences:

m_1	m_2	m_3	m_4	m'_1	m'_2	m'_3	m'_4
w_2	w_1	w'_3	w'_4	w'_1	w'_2	w'_3	w'_4
w'_1	w'_2	w'_4	w'_1	w_1	w_2	w_3	w_4
w'_2	w'_3					w_4	w_3
	...						

w_1	w_2	w_3	w_4	w'_1	w'_2	w'_3	w'_4
m'_1	m'_2	m'_4	m'_3	m_4	m_1	m_2	m_3
m_1	m_2	m_4	m_3	m_1	m_2	m_3	m_4
m_2	m_1	m'_3	m'_4	m'_1	m'_2	m'_3	m'_4
m'_2	m'_1	m_3	m_4				
		...					

The stable matchings are (213'4'1'2'34), (1'2'3'4'1234), (1'2'3'4'1243), (213'4'1'2'43), (2'3'4'1'1234), and (2'3'4'1'1243). The last two are members of V.

The construction we have given does not use the smallest number of people needed to represent a given lattice. The six-element lattice can be represented using ten people as follows

m_1	m_2	m_3	m_4	m_5
w_1	w_2	w_3	w_4	w_5
w_3	w_3	w_2	w_5	w_4
	w_1			
	...			

w_1	w_2	w_3	w_4	w_5
m_2	m_3	m_1	m_5	m_4
m_1	m_2	m_2	m_4	m_5
		m_3		
		...		

The stable matches are (12345), (12354), (13245), (13254), (31245), and (31254). However, it is not possible in general to go from L to L^x by adding only one additional couple.

The structure of the set of matches is clearly reminiscent of the representation of a permutation by cycles. This theme will be explored in forthcoming work with Alvin Roth, whose recent work [3] motivated this note.

References

1. D. Gale and L. Shapley, "College Admissions and the Stability of Marriage," American Math Monthly 69 (1962), pp. 9-15.

2. D. Knuth, Marriage Stables, Montreal University Press 1976.

3. A. Roth, "The Economics of Matching: Stability and Incentives," to appear in Mathematics of Operations Research.

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