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An Example of Auction Design: A Theoretical Basis for 19th Century Modifications to the Port of New York Imported Goods Market

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Abstract

Over 150 years ago, with the express purpose of assuring the future prosperity of the Port, a New York auctioneer persuaded the State to lower the tax rate on goods imported through dockside auctions, and simultaneously, to extend the tax to goods offered for sale but not actually sold. He suggested that this would encourage the absolute sale of all goods offered in auctions, that the absolute sale of offered goods--possibly at bargain prices--would attract more buyers, and that, ultimately, the State would benefit. Neither the historical record nor the current theory of auctions provides much insight into these suggestions. Still, after the change, the Port prospered as never before.

This paper defines a model of auctions in which potential bidders join the auction so long as it is in their own best interest to do so, and the potential bidders presume that the seller will act in his own best interests--independent of any promises--in the auction itself. In an analytic example, taxing--at an appropriately lower rate--all goods offered for sale reduces the sellers' benefits from retaining the goods, lowers the anticipated reservation price, attracts more bidders, drives up the expected price, and ultimately benefits both the sellers and the tax collector. An examination of the example's underlying structure reveals the, rather general, factors driving this sequence of results. Finally, a more general argument suggests similar results for a wider class of models.

1. Introduction

Thousands of years ago, the Babylonians gathered annually to auction marriable maidens. The auctioneer awarded each damsel to the prospective bridegroom offering to pay the most to wed her, or, in the case of less attractive damsels, to the prospective bridegroom asking to be paid the least to wed her. (See Shubik (1983) for more details, and for a history of auctions in general.) Since then, an individual offering to pay the most, or asking to be paid the least, has been awarded just about any good or service imaginable. By now, the combined value of all the automobiles, houses, horses, farm machinery, farms, tobacco, antiques, paintings, financial instruments, miscellaneous junk, contracts of many forms, and--well you name it!--auctioned in a single day runs to many billions of dollars. Few, if any, market mechanisms rival the illustrious history and current prominence of auctions and related forms of competitive bidding in pricing goods and services.

In particular, auctions play an important role in the history of the Port of New York. Albion (1961, pp. 276-279, 410) reports that the War of 1812 cut off outside markets and created high surpluses of British goods. With the end of the war, markets reopened, and a flood of imported goods hit American ports. Importers anxious to quickly dispose of their goods turned to auctions. Soon, New York's dockside auctions alone handled one-fifth of the whole nation's imports.

Perhaps surprisingly for a field of such immense practical importance, the theory of auctions and competitive bidding in general, and the design of such mechanisms in particular, blossomed only very recently. As indicated by the survey of the field by Engelbrecht-Wiggans (1980), very little of the currently available theory predates 1960. Within two decades, the Stark and Rothkopf (1979) bibliography listed almost 500 works studying various aspects of auctions and competitive bidding; much of the current, formal mathematical theory postdates this bibliography. Still, even now, the theory still leaves many questions unanswered.

Without the current theory to guide design decisions, how did so many different forms of auctions and competitive bidding come about--progressive oral auctions of farm machinery, descending silent auctions of Dutch cut flowers, sealed bid auctions on Federal mineral rights, multi-stage auctions of defense system contracts, as well as many variations within each of these basic forms. Presumably, evolution played some role; an auctioneer might make a haphazard choice, but only the fittest mechanism for each type of situation would tend to survive. In addition, we suggest that conscious design, experimentation, and evaluation also played a role.

In fact, a timely, well conceived proposal appears to have helped New York establish itself as the chief American seaport and metropolis. Specifically, Albion reports that by 1817 the volume of imports auctioned in New York had grown to the point that the city's auctioneers feared the British might soon divert imports to less glutted markets. To forestall this possibility, a New York auctioneer by the name of Abraham Thompson proposed legislation that would, as he later boasted, "cause all of the Atlantic cities to become tributary to New York." The proposed legislation became law and reduced the tax rate on goods sold at dockside auctions by one-third to two-thirds.

At first glance, Thompson's proposal suggests a very simple argument. Reducing the tax rate makes New York's auctions relatively more attractive than before. By attracting more sellers--or by retaining a larger fraction of current sellers--the State could make up in volume for any taxes lost on individual goods.

Thompson, however, clearly had a much more profound argument in mind. In fact he later explained in Hunt's Merchant Magazine (as quoted by Albion):

> Every piece of goods offered at auction should positively be sold, and to encourage a sale, the duty should always be paid upon every article offered at auction... The truth was, that both in Boston and Philadelphia, the free and absolute sale of goods by auction was not encouraged. (It did not appear to be understood.) In Philadelphia, goods were allowed to be offered, and withdrawn, free from state duty, and the purchaser went to the auction rooms of that city with no certainty of making his purchases. He was not certain that the goods would be sold to the highest bidder.

Not only did Boston and Philadelphia apparently not understand the benefits to be gained from encouraging the absolute sale of all goods offered for sale, even the current theory of auctions and competitive bidding hardly addresses the subject! We shall take steps to develop the missing theory.

Clearly, Thompson makes two points. One, the proposed change in taxes would encourage the absolute sale of all goods offered. Although the record fails to record Thompson's thoughts on why the proposed changes should result in lower reservation prices, there can be little doubt that he felt that they would do so. Two, the State--and presumably also the auctioneers--would benefit from the absolute sale of all goods offered. Again, the record provides few details. However, it does reveal that the price at which goods sold varied considerably from sale to sale. Thus, a wiley seller might speculate; if the bidding on a particular good failed to reach some critical "reservation price," the seller could withhold the merchandise from sale in the hope of obtaining a better price for it later, possibly elsewhere. In fact, Myerson (1981) suggests that a strategic seller should set the reservation price strictly greater than his residual expected value for the goods if they fail to sell in this particular auction. So, at the time that a merchant was trying to decide whether to incur the cost of travelling to New York to bid in a particular good of interest to him, he was, according to Thompson, not even "certain that the goods would be sold." Apparently Thompson felt that this discouraged merchants from even attending the auctions, to the ultimate disadvantage of both the State and the auctioneers.

In short, Thompson felt that lowering the tax rate, but extending the tax to goods offered for sale even if not sold, would encourage lower reservation prices. Lower reservation prices--more specifically, the increased probability of advertised goods actually being sold, possibly at bargain prices--would attract more merchants to the auction. This increased number of bidders would firm up prices depressed by the glutted nature of the market, and ultimately the State would benefit, or so Thompson seems to have argued.

It worked! At the time of the change, Boston, New York, and Philadelphia handled roughly equal volumes of imports. By the completion of the Erie Canal in 1825, however, New York's volume of imports had grown to three times that of the other two ports. Perhaps Thompson deserves at least some of the credit so often accorded to the Erie Canal in establishing New York as the nation's chief seaport and metropolis.

This paper develops a theoretical basis for Thompson's suggestions. Section 2 presents an example in which things do work out as Thompson seems to have expected. Subsequent sections examine what underlies the example, and argue that the results illustrated by the example occur much more generally than just under the specific conditions of the example.

Specifically, two phenomona underlie the example. One, ex-ante (before potential bidders decide whether or not to attend and participate in the auction) the seller prefers a lower reservation price than ex-post (after bidders have committed themselves to participating); that means, as Thompson suspected, ex-ante, sellers would benefit from changes which ex-post encourage the absolute sale of goods. Two, Thompson's proposal reduces the ex-post optimal reservation price, thereby in effect assuring potential bidders of a reservation price closer to the (lower) reservation price that the seller would have like to have been able to commit to ex-ante; by setting the tax rates appropriately, both the seller and tax collector share in the resulting gain, a gain that comes not at the expense of the bidders, but rather as a result of a more efficient market.

Our investigation of these phenomona comprises three sections. Specifically, Section 3 defines a general model of oral auctions with bidders who privately know their respective values for the object being offered, and establishes that a reservation price equal to the seller's reservation value results in the socially efficient number of participants. An investigation of reservation prices and subsidies from the seller's perspective; reveals that the seller prefers a reservation price affecting the socially optimal number of participants, or a subsidy that in effect rounds the number of participants up to the next integer. Thus, roughly speaking, except for the discreteness in the number of bidders, in the oral auction model, the seller would prefer to commit ex-ante a reservation price equal to the reservation value; Engelbrecht-Wiggans (1987b) illustrated this for specific examples. We then argue more generally that the seller may prefer an ex-ante reservation price strictly less than the ex-post optimal reservation price. Next, Section 4 argues that Thompson's proposed change reduces the ex-post optimal reservation price, thereby if effect allowing the seller to commit to a lower--and closer to the ex-ante optimal--reservation price than previously possible. Finally, Section 5 summarizes the paper.

2. The Example

This example demonstrates that things can work out as Thompson suggested. In the example, taxing all goods offered for sale, but taxing them at an appropriately lower rate, drives down the seller's optimal reservation price; as Thompson suggested, the change in tax structure encourages the absolute sale of all goods offered for sale. Specifically, regardless of what promises may have been made, little prevents the seller from arbitrarily resetting the reservation price at any time throughout the auction. The potential bidders know this and correctly anticipate (on average) what reservation price the seller will ultimately settle on. This affects how many potential bidders decide to attend (and bid in) the auction; with everything else equal, lower reservation prices attract more bidders. The additional bidders, one of whom might value the goods offered more highly than any other bidder, push the expected price up more than enough to offset any losses attributable to the lowering of the reservation price. In the end, the final tax structure happens to split the increased revenues between the seller and the tax collector so that both parties benefit from the change, again just as Thompson seems to have expected.

In some ways, the example--and the subsequent models--resembles previously studied models. Most notably, the seller (in setting the reservation price) and the potential bidders (in deciding whether or not to attend and bid in the auction, and in deciding how they bid) act to maximize their own expected net profit conditional on what they know or presume about the state of Nature and subject to any restrictions placed on them by the model. Specifically, not only are the individual decision makers risk neutral, but nothing other than monetary profit enters into their utility functions. While this assumption simplifies the analysis significantly, it is not critical to the basic nature of the results.

Two aspects of the example--and of the subsequent models--distinguish it from most previously studied models. First, the tax structure appears explicitly in the model; this allows us to study how the outcome of the auction varies with the tax structure. Second, the expected number of bidders varies with the potential bidders' perceptions of how profitable it would be to attend and bid in the auction; so endogenizing the number of bidders recognizes that the auction's characteristics--say, for example, the specific reservation price that the seller ultimately settles on--may affect who attends and bids in the auction. Each of these two aspects will be discussed in some detail.

To parameterize the tax structure, imagine the tax to consist of two components. One, the seller pays a fraction α ($0 \leq \alpha < 1$) of the winning bid to the tax collector if someone other than an agent of the seller (or the seller himself) ends up with the goods. Second, the seller also pays a fraction β ($0 \leq \beta$, $\alpha+\beta<1$) of the winning bid regardless of who wins. In the example, imagine that the seller implements a reservation price of r by having an agent submit an opening bid of r; thus the "winning price" equals the reservation price when the seller retains the goods. In the subsequent models, however, other definitions of the winning price for the case when the seller retains the goods work just as well. Thus, in terms of this notation, Thompson proposed going from a system with α positive and β zero to one with α zero and β positive, but less than the original α.

The tax rates end up affecting the seller's choice of reservation price. In particular, the seller sets a reservation price r below which he will not sell a particular good; this reservation price may exceed the reservation value the seller derives from the good when he retains it. To avoid confounding the analysis with issues extraneous to our primary investigation, ignore the principal-agent problem by imagining that the seller acts as his own auctioneer; the seller directly sets the reservation price to maximize his expected net sales revenue. As the tax structure changes, the tradeoff between selling and retaining the goods changes, and therefore the seller's choice of reservation price changes.

The model's assumptions as to when the seller sets the reservation price critically affect the outcome. A seller might advertise that all goods will be sold without reserve, or otherwise try to commit to some specific reservation price before potential bidders decide whether or not to attend and bid in the auction. In practice though, little prevents the seller from implementing some higher reservation price in the auction itself. For example, the seller might have a "shill" bid on his behalf, or have the auctioneer pretend to observe bids when none occur if so doing is in the best interests of the seller. Given the difficulty of enforcing a pre-announced reservation price or of detecting deviations from it--especially in the case of many, many different individual foreign ship owners each importing goods to the Port of New York only every once in a while--we model the seller as setting the reservation price only after potential bidders have decided whether or not to attend and bid in the auction; as we will see, this results in higher reservation prices than when sellers can convincingly commit themselves to whatever reservation price they want before potential bidders decide whether or not to attend and bid in the auction.

Goods retained by the seller in one auction might be offered for sale again in another, later auction. Let v_o denote the expected value that the seller derives from the goods conditional in retaining them in some specific auction; in some cases, this reservation value v_o might be a constant, while in other cases it might depend on the reservation price and the number of bidders in the auction in which the goods failed to sell. To keep things as simple as possible, in the example the reservation value (as opposed to reservation price) equals zero; for instance, in the Dutch cut flower auctions, any goods not sold are destroyed and therefore of no value to the seller. (This assumption that v_o equals zero will be relaxed later.) In general, the seller need never consider reservation prices less than his reservation value, but, as Myerson (1981) suggests, may profit from setting the reservation price strictly greater than the reservation value.

Presumably, potential bidders have some idea as to what the reservation price will actually be. In practice, potential bidders may base their ideas on past experience. In our analysis, to keep things simple, we presume that potential bidders perfectly anticipate what reservation price the seller ultimately chooses to implement; perhaps our potential bidders are examples of the proverbial "perfectly rational player" and correctly analyze the seller's problem to deduce the "optimal" reservation price, or perhaps they have enough prior experience with similar auctions.

Given the potential bidder's ability to anticipate the actual reservation price, any changes in tax structure that affect the choice of reservation price also indirectly affect the number of bidders. In fact, for any fixed collection of bidders, lowering the reservation price both increases the probability that some bidder (as opposed to an agent of the seller) wins the good being auctioned, as well as decreases the expected price paid by a winning bidder; in short, decreasing the reservation price makes the auction more profitable to any fixed collection of bidders. This increased profitability makes the auction more attractive to those who would otherwise decide that the auction was not attractive enough to attend and bid in it. In fact, in our models, lowering the reservation price results in an increased expected number of bidders.

By endogenizing the setting of the number of bidders, we obtain results not possible from the more traditional models that exogenously specify who bids. In fact, this endogenization results in the tax structure affecting the expected number of bidders, and therefore also affecting the outcome of the auction. This seems to lie at the core of Thompson's arguments.

For the purpose of the example, consider a specific mechanism for determining the expected number of bidders. As already suggested, in the long term, the number of individuals in the business of retailing goods bought wholesale at Port of New York dockside auctions--the number of "merchants" in the greater New York area--varies with the expected profitability of the auction to the bidders. Specifically, if another individual could profitably enter the business, we presume that such an individual would have already entered the business. Conversely, if a merchant would be better off leaving the business, that individual would have already left. Thus, at least in the symmetric case, for any given reservation price, the number of merchants will be as large as possible without driving individual merchant's expected profits negative.

Not every merchant necessarily attends every auction. For instance, a merchant may travel to New York from time to time depending on his current inventory and on what the auction's advance notices list as to be offered for sale. Therefore, we view the number of merchants attending any particular auction and bidding in it--the "number of bidders"--as the outcome of some stochastic, possibly degenerate, process. Not only does this add realism to our model, but it also reduces the effect of any one additional merchant on the expected number of bidders; this allows small changes in the reservation price to be large enough to affect a discrete change in the number of merchants.

To define the distribution of the number of bidders, let P_k denote the probability that exactly k merchants decide to go attend the auction and bid in it. In the example, P_k equals the binomial probability $\binom{n}{k}p^k(1-p)^{n-k}$, where n denotes the number of merchants and p denotes the probability of any particular merchant participating in the auction; this distribution serves the purposes of the example particularly well in its being mathematically very tractable while still allowing us to vary the expected number of bidders in sufficiently small increments. Other distributions, including the degenerate case of a constant collection of bidders, produce results similar to those illustrated by the example.

The example focuses on the auctioning of a single object. On the one hand, this only crudely approximates the actual situation with its many, some simultaneous, auctions on any given day. On the other hand, this focusing on a single object allows us to isolate the effect that the change of tax structure has on the expected revenue from each unit of good offered from any more direct effect the lowered tax rates might have in attracting additional goods to the market. In fact, extending the tax--at an appropriately lower rate--to all goods offered for sale increases the seller's (as well as the tax collector's) expected revenue from each item offered, and this increases the attractiveness of the auction to sellers beyond the more direct effects of a lowered tax rate (and also assures that the tax collector benefits regardless of how many--or few--additional sellers the new rules attract).

Individuals' decisions whether to attend the auction and how to bid if they do attend depend on what they know about the goods to be offered. For our purposes, imagine that before bidders arrive at the auction site, they all have exactly the same information about the goods; perhaps this common public knowledge comes from the auction advertisement. Thus, the decision whether to attend the auction must be made before merchants have any unshared individual-specific information about the goods. This not only simplifies the analysis, but avoids confounding the results with any selection effects, and does so without changing the basic nature of the results.

Once at the auction site, however, a merchant inspects the merchandise and thereby gains some private insights about his own value for the goods, or possibly about how others might value the goods. The example presumes that each bidder inspects the merchandise carefully enough to remove any uncertainty about his own value (gross of any costs already incurred) for the merchandise. Moreover, the example models these values as being dependent draws from a random variable with cumulative distribution function $F(\cdot)$ and independent of the number k of attendees; specifically, the example models the values as being uniformly distributed on the unit interval. Of course, both the distribution of the number of attendees and of their values might vary with what is being offered for sale, or more generally, with the actual pre-sale common public information; we avoid this complication by simply holding what is offered--and the pre-sale information--fixed.

To attend the auction, a merchant incurs a known fixed cost of c. Perhaps this represents the cost of travelling between the merchant's retail location and the Port of New York. More generally, this may be viewed as the cost of private information, for example, the cost of collecting seismic data about an offshore tract (possibly) containing oil and other minerals. Alternatively, one might view this cost as some amoritization of the merchant's fixed cost of being in the business of reselling imported goods obtained at Port of New York auctions. In any case, merchants incur a fixed cost before obtaining any private information about the merchandise. However, once merchants incur this cost, only the winner of an item incurs any additional cost; thus we presume that all of the merchants attending an auction actually bid, and therefore refer to them simply as "bidders" throughout the paper.

Given the independent privately-known values nature of the example, the revenue equivalence results of Engelbrecht-Wiggans (1988) and Myerson (1981) assure quite generally that any pricing rule for an auction with a continuum of allowable bids--whether it be first price, second price, the outcome of an oral auction, or some other function of the equilibrium bids--generates the same expected selling price at equilibrium (for that pricing rule) for a fixed number of k bidders so long as the bidder with the highest value for the good wins it if and only if his value exceeds a fixed screening level. Clearly, this invariance of the expected selling price also assures that the bidders' expected profits, the seller's expected net revenue, and the tax collector's expected receipts be independent of the pricing rule. So, with little loss of generality, within the example, derive the expected profit and revenue expressions as if we conducted a sealed-bid second-price auction; specifically, the seller implements a reservation price of r by submitting a sealed bid of r himself, the high bidder wins, and the winning bidder (who might be the seller) pays the seller an amount equal to the winning bid; the seller must then pay the appropriate taxes. The equilibrium in this unrealistic auction mechanism generates the same expected revenues and profits in the example as would any equilibrium of any more realistic or commonly used high valuer wins auction mechanism with the same screening level r.

For this second price auction mechanism, Vickery (1961) established that each bidder has the dominant optimal bidding strategy of bidding equal to his own value. In this case, with everyone bidding equal to their value at equilibrium, the screening level coincides with the seller's reservation price. (If the pricing rule changes, the seller may have to change his reservation price in order to still effect the same screening level.) All this makes for relatively simple derivations.

In particular, start by looking the expected value of goods transferred by the auction to merchants. In independent private values auctions, this value equals

$$\sum_{k=0}^{k=\infty} P_k \int_{x=r}^{x=\infty} xkF^{k-1}(x)dF(x)$$

and, in our example with its Binomial distribution for the number of bidders and standard uniform distribution for each bidder's value, evaluates to

$$\sum_{k=0}^{k=\infty} \binom{n}{k} p^k(1-p)^{n-k}[\frac{k}{k+1}(1-r^{k+1})].$$

Now turn to the seller's and tax collector's revenues. Start by defining $G_1(r)$--to be interpreted as the expected payments to the seller by his agents when the seller retains the goods--to be

$$\sum_{k=0}^{\infty} P_k \int_{x=0}^{x=r} xkF^{k-1}(x)dF(x),$$

which, in our example, evaluates to

$$\sum_{k=0}^{k=n} \binom{n}{k} p^k(1-p)^{n-k}[\frac{k}{k+1} r^{k+1}].$$

Then, define $G_2(r)$--to be interpreted as the expected payments by merchants to the seller--to be

$$\sum_{k=0}^{k=\infty} P_k[rk(1-F(r))F^{k-1}(r) + \int_{x=r}^{x=\infty} xk(k-1)(1-F(x))F^{k-2}(x)dF(x),$$

which, in our example, evaluates to

$$\sum_{k=0}^{k=n} \binom{n}{k} p^k(1-p)^{n-k}[\frac{k-1}{k+1} + r^k - \frac{2k}{k+1} r^{k+1}].$$

Now, the expected net revenue to the seller after taxes may be written as

$$-\beta G_1(r) + (1-(\alpha+\beta))G_2(r)$$

while the tax collector's expected receipts may be written as

$$\beta G_1(r) + (\alpha+\beta)G_2(r)$$

Thus, the expected profit to the bidders collectively (net of payments and of participation costs) equals

$$\sum_{k=0}^{\infty} P_k[\int_{x=r}^{\infty} xkF^{k-1}(x)dF(x)-kc] - G_2(r)$$

which, our our example, evaluates to

$$\sum_{k=0}^{k=n} \binom{n}{k} p^k(1-p)^{n-k}[\frac{1}{k+1} - r^k + \frac{k}{k+1} r^{k+1} - kc].$$

To characterize the reservation price, differentiate the seller's expected net revenue with respect to r, set the resulting expression equal to zero, simplify, and remember the correspondence between reservation price and screening level to obtain the following necessary condition for a nontrivial optimal reservation price r*:

$$(1-(\alpha+\beta))(1-F(r^*)) = (1-\alpha)r^*dF(r^*)$$

For the standard uniform distribution, this necessary condition simplifies to

$$r^* = \frac{1-(\alpha+\beta)}{(1-(\alpha+\beta)) + (1-\alpha)}$$

and happens to be a sufficient condition.

Note that this condition is independent of the P_k's. In particular, for a binomially distributed number of actual bidders with independent private values, the optimal reservation price does not depend on the parameters p and n. Also note that for $\beta=0$, this condition is independent of α. However, for $\alpha=0$, the condition does involve β. While this dependence of the optimal reservation price on the tax rate might at first appear to be a drawback of Thompson's proposed change, we suggest that in practice sellers discover the optimal reservation price by some iterated trial and error process rather than by solving the above stated necessary condition; in practice, the optimal reservation price should be no more difficult to discover after the change in tax structure than before.

Given these expressions, consider what happens for specific choices of the parameters. In particular, start with $\beta=0$, $p=0.1$, c in the interval (0.0897, 0.0926], and $\alpha < 0.1449$; this illustrates the pre-Thompson situation. The first column of Table 1 summarizes the results.

Then consider the post-Thompson cases illustrated by $\beta=0.1$, $\alpha=0$, $p=0.1$, and c either in the interval (0.0882, 0.0912] or the interval (0.0912, 0.0944]; these two intervals for c result in different numbers of merchants n, but the two intervals together cover the interval for c in the pre-Thompson case. The second and third columns of Table 1 summarize the results for these cases. Notice that just as Thompson anticipated, changing from taxing only those goods sold to taxing--at an appropriately lower rate--all goods offered for sale increases the expected number of bidders; n increased from 10 to 12 or 13 depending on the exact value of c, and so the expected number of bidders increased from 1.0 to 1.2 or 1.3. The change in tax structure also resulted in the seller adopting a lower reservation price; again just as Thompson anticipated. In the end, the expected total revenue increased, and we restricted α so that both the seller and tax collector benefit from the change.

3. Ex-Post vs. Ex-Ante Optimal Reservation Prices

This section examines the basic structure of the previous example in attempt to understand what affects the amount of money available to be split between the seller and the top collector. In particular, we define "oral auctions with privately known (but not necessarily independent) values." In such auctions, the winner pays a price closely related to the nearest competitors estimated value for the object, and therefore the winner has an expected profit closely related to the increase in social value generated by the auction as a result of his participation. If the number of bidders increases continuously until

Table 1: Summary of Example Parameters and Numerical Results

Case:	Pre-Thompson I	Post-Thompson IIa	IIb
Parameter values:			
α	< 0.1449	0	0
β	0	0.1	0.1
p	0.1	0.1	0.1
c	(0.0897,0.0926]	(0.0882,0.0912]	(0.0912,0
Consequences:			
r*	1/2 (=0.5)	9/19(≅0.4737)	9/19(≅0
n	10	13	12
Expected Revenues:			
total	0.2160	0.2676	0.250
seller	0.2160 (1-α) ($\leq$ 0.2160 for all $\alpha \geq 0$	0.2346	0.219
taxes	0.2160α (≅0.0313 for α = 0.1449)	0.0330	0.031

(Note: All numbers rounded to four decimal places) none else could profitably enter, then bidders make zero profit, the seller and tax collector together capture the full net social value generated by the auction for any fixed number of bidders, and bidders enter until the social value is maximized as a function of the number of bidders. Therefore, as Engelbrecht-Wiggans (1987b) previously established for a specific example, if the seller could commit to a reservation price ex-ante, setting it equal to the seller's reservation value and letting the bidders then in effect set the number of bidders maximizes the expected total revenue. Given the integrality of bidders, total revenue may benefit from setting a higher reservation price, changing an entry fee, or providing a subsidy, but in any case only to the extent of in effect rounding the number of bidders up or down to the next integer.

We start by defining a model of oral auctions. In particular, the auctioneer starts by asking a price low enough so that at least two bidders (one of which may be a shill acting on behalf of the seller) would be willing to pay that price if offered the object on a take-it-or-leave-it basis. If someone "bids"--indicates a willingness to take the object at the current asking price--the auctioneer increases the asking price by some pre-specified increment. If no one bids, then the last bidder wins and pays the amount he last bid.

Look at the action from the viewpoint of the next to last bidder; we call this bidder the "price setter." Let p_2 denote the amount bid by the price setter, and let p_1 denote the amount paid by the winner (of course, $p_1 > p_2$). We presume that by bidding p_2, the price setter indicates that his expected value for the object conditional on everything he currently knows and conditional on the presumption (incorrect, as it turns out) that no one will outbid him equals at least p_2; in particular, we rule out the possibility that the price setter had such accurate information about what other bidders would do so that he bid up the price beyond his own value certain that someone else would eventually save him from winning the object. Furthermore, in not being willing to outbid the winner, we presume that each bidder other than the winner indicates that for each allowable bid level $p > p_1$, his expected value for the object conditional on what he now knows and conditional on no one else outbidding him if he were to bid p is strictly less than p.

Our example illustrates a special case of the oral auction model defined so far. In particular, the expected values just mentioned equal the expected value of the object to a bidder conditional on what he knew at the beginning of the auction and conditional on no one else bidding higher than the current asking price; that is, the expected value is functionally independent of anything a bidder learns about his own value for the object through the actions of other bidders. Since the bidder "knows" his (expected) value independent of what others reveal, we call this the case of "privately known values."

Also, in the example, the asking price in effect rises continuously. As a result, the bidder with the highest (privately known) value wins the object and pays an amount equal to the second highest (privately known) value. Raising the price continuously guarantees that the highest valuer wins, and that the second highest valuer becomes the price setter. In addition, raising the price continuously guarantees that the winner's price is exactly equal to the price setters' value, rather than the price setter's value rounded up the next allowable bid level. Taken together, this results in the winner paying an amount equal to the second highest (privately known) value.

For the remainder of this section, consider only oral auctions with privately known values in which the asking price rises continuously. This simplifies the statement of the results. The example satisfies these restrictions. And, as we relax these restrictions slightly, we expect the results to change only incrementally. Thus, adopting these restrictions simplifies the analysis without unduly limiting our insights into how the phenomona underlying the example affect the results of the example more generally.

To proceed, start by defining some notation. Let v_i denote the privately known value to bidder i of the object net of any amounts paid to individuals other than the auctioneer (e.g., unlike the uniformly distributed value in the example, v_i is now net of travel costs paid to attend the auction); v_o denotes the seller's privately known reservation value; assume $v_o \geq 0$. Then, as a function of the reservation price r below which the object will not be sold, and of the set of bidders N, let V(N,r) denote the social value $E[\max_{i:i\varepsilon N \,\&\, v_i \geq r} (v_i - v_o)]$ generated by the auction. (For the moment, think of N as deterministic--in our example, this would correspond to p=1; most of the results would (more or less obviously) carry over to the random case, but at a great cost in the complexity of the exposition.) Note that for fixed N, V(N,r) is concave in r and is maximized at $r = v_o$. If $\$_i(N,r,d)$ denotes the expected profit to bidder i from attending the auction as a function of the set N of bidders, the reservation price r that the seller ends up implimenting, and any entry fee d that each bidder must pay to the seller on entering the auction, then the total expected revenue $R(N,r,d,v_o)$ equals $V(N,r) - \Sigma_{i\varepsilon N} \$_i(N,r,d) + v_o$.

In the symmetric case, drop the subscript "i" and replace the set N by the number of its elements n. Assume that $\$(n,r,d)$ is continuous in r and a decreasing function of n; quite plausible, bidders' profits suffer as the number of competitors, or the competitiveness of the shill, increases. Finally, define $n^*(r,d)$ as the integer n such that $\$(n,r,d) \geq 0$, but $\$(n+1,r,d) < 0$. Note that $n^*(r,d)$ is a non-increasing function of r and d.

<u>Theorem 1</u>, In symmetric oral auctions with privately known values and continuously increasing asking prices, for any fixed n, r, and d, $\$(n,r,d) = (V(n,r) - V(n-1,r))/n - d$.

<u>Proof</u>: For any fixed N, r, and d, the fact that the winner pays an amount equal to the second highest privately known value--the highest value if the winner weren't present--implies that $\$_i(N,r,d) = V(N,r) - V(N\setminus i,r) - d$. In the symmetric case, i wins with probability 1/n, and conditional on i winning, $V(N,r) - V(N\setminus i,r) = V(n,r) - V(n-1,r)$; the remaining (n-1)/n of the time, i loses and conditional on i losing, $V(N,r) - V(N\setminus i,r) = 0$. Thus, $\$(n,r,d) = (1/n)(V(n,r) - V(n-1,r)) - d$ as claimed. Q.E.D.

This relationship between bidder profit and contribution to social value may be the extreme case of practical interest. In particular, in the contrasting "common values" case in which each bidder has the same, unknown value for the object, the social value is independent of n so long as $n \geq 1$, and, since bidders have a strictly positive expected profit in typical common values models, $\$(n,r,d) > (V(n,r) - V(n-1,r))/n - d$. Many, if not most, practical situations fall somewhere in between the common values and the privately known values extremes: this author knows of no practical auction model in which $\$(n,r,d) < (V(n,r) - V(n-1,r))/n - d$.

<u>Theorem 2</u>. If for each fixed n,r, and d bidders bid so that $\$(n,r,d) \geq (\leq) (V(n,r) - V(n-1,r))/n - d$, then $V(n,r) - V(n-1,r) < (\geq) 0$ for all $n > (\leq) n^*(r,0)$.

<u>Proof</u>: By hypothesis, $V(n,r) - V(n-1,r) \leq (\geq) n\$(n,r,d) + nd$ for all n,r,d. Since the left hand side of this inequality is independent of d, the right hand side must also be independent of d, and so must equal $n\$(n,r,0)$--the value obtained when $d=0$--for all n,r,d. But, by the definition of $n^*(r,d)$ and the monotonicity of $\$(n,r,d)$ in n, $\$(n,r,d)$ is $< (\geq) 0$ for all $n > (\leq) n^*(r,0)$, as claimed. Q.E.D.

<u>Corollary</u>. In the symmetric oral auctions with privately known values and continuously increasing asking prices, for each fixed r and d, $V(n,r)$ is maximized at $n = n^*(r,0)$.

Roughly speaking, this corollary states that if bidders enter and leave the auction in their own best interests, then the socially optimal number of bidders results. This result plays a crucial role throughout this section. In particular, the social value is an upper bound on the total revenue; bidders must make a non-negative profit, and that profit must come out of the social value generated by the auction. As subsequent theorems establish, deviations from the socially optimal number of bidders typically hurts the total revenue (presumably because of its relationship to the social value generated by the auction) more than any gains achieved by deviating from an ex-ante reservation price equal to the reservation value and/or deviating from an entry fee (subsidy) of zero. In fact, the seller should deviate from $r = v_o$ and $d = 0$ only to the extent that it has no effect on the number of bidders other than rounding the number to an integer if $n^*(r,d)$ would have been non-integer had we allowed non-integer numbers of bidders.

Theorem 3. In symmetric oral auctions with privately known values and continuously increasing asking prices, if bidders enter and leave until $n = n^*(r,d)$, then for any fixed v_o, $r = v_o$ and $d = (V(n,r) - V(n-1,r))/n$ maximizes the expected revenue $R(n,r,d,v_o)$ with respect to r and d.

Proof: By definition of d, $\$(n,r,d) = 0$ and $n^*(r,d) = n^*(r,0)$. But, $\$(n,r,d) = 0$ implies that $R(n,r,d,v_o) = V(n,r) + v_o$, and thus setting r and d as specified yields an expected revenue of $V(n^*(v_o,0),v_o) + v_o$. Since $r = v_o$ maximizes $V(n,r)$, $V(n,r) + v_o \leq V(n,v_o) + v_o$. By the corollary to Theorem 2, $V(n,v_o) + v_o \leq V(n^*(v_o,0),v_o) + v_o$. To summarize, $R(n,r,d,v_o) \leq V(n^*(v_o,0),v_o) + v_o$. Thus, for all n, r and d, $R(n,r,d,v_o)$ is at most the expected revenue actually achieved by setting r and d as specified in the hypothesis. Q.E.D

In effect, an appropriate entry fee adjusts for the integrality in the number of bidders, thereby allowing the seller and tax collector to capture any profit that the bidders would have otherwise obtained simply because our model did not allow a fractional number of bidders; in short, with an appropriate entry fee, this auction generates as much total revenue as any mechanism can. However, barring a positive entry fee, the revenue suffers, and adjusting for the integrality in the number of bidders calls for setting an appropriate reservation price strictly greater than v_o, or appropriately subsidizing bidders by setting $d < 0$. The next two theorems--and a subsequent example--examine these options.

To characterize the optimal ex-ante reservation price when $d = 0$, define r_o as the largest reservation price r such that $n^*(r,0) = n^*(v_o,0)$. Assume that $n^*(r,0) = n^*(v_o,0)$ for all r between v_o and r_o. Then, define r^* as the r that maximizes $R(n^*(r,0),r,0,v_o)$ subject to $v_o \leq r \leq r_o$.

<u>Theorem 4</u>. If 1) d restricted to be zero and r restricted to be no less than v_o; 2) all bidders bid such that $\$(n,r,0) \leq (V(n,r) - V(n-1,r))/n$; and 3) bidders enter/leave until $n = n^*(r,0)$, then r^* is an ex-ante optimal reservation price, and at any ex-ante optimal r, $n^*(r,0) = n^*(v_o,0)$.

<u>Proof</u>: Consider two cases; $v_o \leq r \leq r_o$ and $r > r_o$. First, if $v_o \leq r \leq r_o$, then by the definition of r^*, $R(n^*(r,0),r,0,v_o) \leq R(n^*(r^*,0),r^*,0,v_o)$. Second, if $r > r_o$, then the definitions of $R(n,r,d,v_o) \leq V(n^*(r,0),r) + v_o$. Since $V(n,r)$ is decreasing in r for

$r \geq v_o$, for $r > r_o$, $V(n^*(r,0),r) + v_o < V(n^*(r,0),r_o) + v_o$. Since $r > r_o$ implies that $n^*(r,0) \leq n^*(r_o,0)$, Theorem 2 establishes that $V(n^*(r,0),r_o) + v_o \leq V(n^*(r_o,0),r_o) + v_o$. By the definitions of r_o and $R(n,r,d,v_o)$, $V(n^*(r_o,0),r_o) + v_o = R(n^*(r_o,0),r_o,0,v_o)$. But by the definition of r^*, $R(n^*(r_o,0),r_o,0,v_o) \leq R(n^*(r^*,0),r^*,0,v_o)$. Summarizing, for $r > r_o$, $R(n^*(r,0),r,0,v_o) < R(n^*(r^*,0),r^*,0,v_o)$. This together with the first case gives the desired results. Q.E.D.

Theorem 4 says that in symmetric oral auctions with privately known values and continuously increasing asking prices, barring entry fees paid by the bidders to the auctioneer (or subsidies paid the other direction), the seller benefits from a reservation price in excess of v_o only because it in effect rounds off any fractional bidder that would occur if $r = v_o$ and fractional bidders were allowed. Roughly speaking, except for the discreteness of the number of bidders, an ex-ante reservation price equal to the seller's reservation value, maximizes expected revenue. This contrasts to the situation ex-post to bidders having committed themselves to attend--a situation similar to the model of Myerson (1981) with its exogenously fixed number of bidders--in which a reservation price strictly larger than the seller's reservation value maximizes revenue. Thus, for $d = 0$, the seller and tax collector together would benefit from committing to a lower reservation price ex-ante to bidders committing themselves than would be chosen ex-post.

The result that at the optimal reservation price r, $n^*(r,0) = n^*(v_o,0)$ requires some restriction on the assymetry of the model. To illustrate, consider a second-price sealed-bid auction with statistically independent privately known values; for each bidder, the value gross of the participation cost is either zero or one, each with probability one half, independent of the other bidders' values. The seller has a reservation value of zero, and pays no taxes. To introduce assymetry, let bidder i = 1 have a participation cost of $c_1 = e_1$, and each bidder $i > 1$ have a participation cost $c_i = 1/4 - e_2$, where we think of e_1 and e_2 as sufficiently small, but still strictly positive, quantities.

What happens at equilibrium? For a reservation price set to zero ex-ante, and appropriately small e_1 and e_2, the number of entrants n* equals two. In particular, at the dominant strategy Nash equilibrium each bidder bids his privately known value; the (or "any," in the case of ties) high bidder wins and pays an amount equal to the highest amount bid by any non-winning bidder. Thus, each bidder has an expected profit gross of the participation cost equal to the probability that his value for the object equals one times the probability that the value to all other bidders is zero. For two bidders, each has an expected gross profit of one-fourth; for $e_1 < 1/4$ and $e_2 > 0$, this leaves both bidders with a strictly positive expected profit net of the participated costs. For three or more bidders, the gross expected profit per bidder equals one-eighth, which is less than all but the first bidder's participation costs so long as $e_2 < 1/8$, and some bidder (other than the first bidder) should leave if $e_2 < 1/8$. Thus for an ex-ante reservation price of zero, $0 < e_1 < 1/4$ and $0 <$

$e_2 < 1/8$ results in two bidders. Note that for two bidders, the auctioneer has an expected revenue equal to one times the probability that at least two bidders have a value of one for the object; this equals one-fourth for the case of two bidders.

Now consider a reservation price equal to $1 - 3e_1$. Instead of the winner sometimes getting an object of value one for free, the winner must pay the reservation price. Thus, the gross expected profit has been reduced from one-fourth to one-fourth of $1 - r$, that is, to $(3/4)e_1$. No longer will the market support two bidders. However, any one bidder alone would have a gross expected profit of $(1-r)$ times the probability that he has a value of one for the object; in other words, the expected gross profit equals $(3/2)e_1$. Thus, the first bidder by himself would have a strictly positive expected net profit (as would any other bidder by themself under appropriate choices for e_1 and e_2). So, for $r = 1 - 3e_1$, only one bidder participates. But now, the expected revenue equals r times the probability that the lone bidder has a value of one for the object. That is, the expected revenue equals $(1-3e_1)/2$ which exceeds one-fourth for $e_1 < 1/6$.

Now pull the two cases together. For $0 < e_1 < 1/6$ and $0 < e_2 < 1/8$, and ex-ante r of zero results in two bidders and an expected revenue of one-fourth, while an ex-ante r of $1 - 3e_1$ results in one bidder and a (strictly greater) expected revenue of $(1-3e_1)/2$. In short, for this assymetric example, the seller benefits from using a reservation price enough larger than his reservation value of zero to drive away a bidder. Thus, we cannot hope to significantly weaken the symmetry presumed by Theorem 4 without affecting the results.

The restriction that $d = 0$ rather than $d \leq 0$ also affects the results of Theorem 4. The case of negative entry fees d corresponds to the auctioneer subsidizing bidders, something which seems to happen in some real world auctions. The next theorem examines optimal subsidies.

Theorem 5. If 1) the reservation price r is restricted to equal v_o and the entry fee d is restricted to be non-positive (that is, to be, in effect, a subsidy), 2) all bidders bid such that $\$(n,v_o,d) \geq (V(n,v_o) - V(n-1,v_o))/n - d$, and 3) bidders enter/leave until $n = n^*(r,d)$, then for the optimal $d \leq 0$, $n^*(v_o,0) \leq n^*(v_o,d) \leq n^*(v_o,0) + 1$. (In words, don't subsidize more than necessary to round up the number of bidders.)

Proof: For $k \geq 1$, define $d_k = \$(n^*(v_o,0) + k,\ v_o,d)$; in words, $|d_k|$ is the subsidy needed to just attract k more bidders than when $d = 0$. Since $\$(n,r,d)$ is non-increasing in n, d_k will be non-positive for all $k \geq 1$, and non-increasing in k. Also, as k goes to infinity, d_k must go to negative infinity.

Since for $d = d_1$, $n^*(v_o,d) = n^*(v_o,0) + 1$, and for $d_1 < d \leq 0$, $n^*(v_o,d) = n^*(v_o,0)$, it suffices to prove that any optimal d satisfies the condition $d_1 \leq d \leq 0$. We will do so in two steps, first showing that $R(n^*(v_o,d_k),v_o,d_k,v_o) < R(n^*(v_o,d_1),v_o,d_1,v_o)$ for all $k < 1$, and then showing that for each $k \geq 1$ $R(n^*(v_o,d),v_o,d,v_o) < R(n^*(v_o,d_k),v_o,d_k,v_o)$ for all d such that $d_{k+1} < d < d_k$.

To show the first part, note that by the definition of d_k, $R(n^*(v_o,d_k),\ v_o,d_k,v_o) = R(n^*(v_o,0)+k,v_o,d_k,v_o)$, which in turn equals

$V(n^*(v_o,0)+k,v_o) + (n^*(v_o,0)+k)d + v_o$ by the definition of $R(n,r,d,v_o)$. For all $k > 1$ and $d_k \leq d_1 \leq 0$, $V(n^*(v_o,0)+k,v_o) + (n^*(v_o,0)+k)d_k + v_o \leq V(n^*(v_o,0)+k,v_o) + (n^*(v_o,0)+1)d_1 + v_o$. By Theorem 2, the right hand side of this last inequality must be less than $V(n^*(v_o,0)+1,v_o) + (n^*(v_o,0)+1)d_1 + v_o$, which by the definition of d_1 equals $V(n^*(v_o,d_1),v_o) + n^*(v_o,d_1)d_1 + v_o$, and which is simply $R(n^*(v_o,d_1),v_o,d_1,v_o)$. In short, $R(n^*(v_o,d_1),v_o,d_k,v_o) < R(n^*(v_o,d_1),v_o,d_1,v_o)$ for all $k > 1$.

To show the second part, for $d_{k+1} < d < d_k$, $R(n^*(v_o,d),v_o,d,v_o) = R(n^*(v_o,d_k),v_o,d,v_o) = V(n^*(v_o,d_k),v_o) - n^*(v_o,d_k)\$(n^*(v_o,d_k),v_o) + n^*(v_o,d_k)d + v_o$, which in turn equals $V(n^*(v_o,d_k),v_o) - n^*(v_o,d_k)(d_k-d) + v_o$, which since $d < d_k$, must be strictly less than $V(n^*(v_o,d_k),v_o) + v_o$, which is simply $R(n^*(v_o,d_k),v_o,d_k,v_o)$. In short, for $d_{k+1} < d < d_k$, $R(n^*(v_o,d),v_o,d,v_o) < R(n^*(v_o,d_k),v_o,d_k,v_o)$.

Q.E.D

Theorem 5 says that the auctioneer should not subsidize more than needed to simply round up the number of bidders. In fact, given the corollary to Theorem 2, this seems intuitive; a subsidy of $|d_1|$ just attracts an additional bidder, thus reducing each bidders' expected net profit to zero, and giving a total revenue equal to the full social value generated by this slightly inefficient auction. Any larger subsidy either increases the bidders' profits, or attracts additional bidders thereby decreasing the social value generated in addition to decreasing the revenues by the amount of the subsidies. (This theorem also raises the question of just why do bid takers subsidize bidders to the extent that they appear to in certain actual auctions, auctions which though perhaps not symmetric oral auctions with privately known values, nonetheless seem to satisfy the conditions--condition 2 in particular--of the theorem.)

Depending on how close $n^*(r,d)$ would be to the next smaller or next larger integer if it were allowed to take non-integer values determines whether $d = 0$ and some appropriate $r > v_o$ out performs $r = v_o$ and some appropriate subsidy $-d$, or the other way around. Intuitively, the optimal subsidy (when $r=v_o$) and the optimal reservaton price (when $d=0$) in effect round the number of bidders. But any such rounding comes at a cost; increasing the reservation price decreases the social value, as does subsidizing to the point of increasing the number of bidders. Thus, as an example illustrates, we might expect the choice between subsidizing versus increased reservation price to depend, roughly speaking, on which direction requires less rounding.

To illustrate optimal reservation prices and subsidies in general, and more specifically, to illustrate that sometimes a reservation price should be preferred to a subsidy, and sometimes vice versa, again consider the case of independent private values (gross of participation costs) distributed uniformly on the unit interval. Assume that all potential participants have the same entry cost c, and that the reservation value is zero. Then with a reservation price and subsidy both equal to zero, the n bidders would have an expected equilibrium profit of $1/(n+1)$ gross of participation costs. Thus, for $1/12 < c \leq 1/6$, the equilibrium number of bidders will be two.

The bidding equilibrium to this example with a reservation price of r generates an expected price of $r^n + ((n+1)-2nr^{n+1})/(n+1)$, and an expected total profit the the n bidders (gross of participation costs) of $-r^n + (nr^{n+1}+1)/(n+1)$. We consider two different levels for the entry cost c. First, for a c just a hair above 1/12, on the one hand, a very small subsidy (and zero reservation price) would result in an equilibrium with three bidders, an expected price of 2/4 = 1/2, and an expected net revenue to the seller of just under 1/2. On the other hand, for zero subsidy and a reservation price of just under 1/2, two bidders would have a combined expected profit (gross of participation costs) of just over 1/6--just enough to cover the participation costs of two bidders. Thus, as c drops to 1/12, the optimal reservation price rises to 1/2. But even for reservation price of 1/2, the expected price from two bidders would be only 5/12--strictly less than the just under 1/2 that can be obtained from an appropriate subsidy. In short, as c drops to 1/12, an optimal subsidy together with a reservation price of zero results in a greater expected equilibrium revenue to the seller than that possible from an optimal reservation price and no subsidy.

Second, consider the case of c = 5/36. Now a reservation p price of 1/4 (and no subsidy) would give two bidders a combined expected profit 54/192 = 162/576--more than enough to cover their participation costs of 2(5/36) = 160/576. Thus, a reservation price of 1/4 (and no subsidy) would result in an equilibrium with two bidders and an expected price of 3/8. But, to get three bidders would require a subsidy of 3(5/36) - 1/4 (the bidders' expected profit when n=3) = 1/6. Three bidders would give rise to an expected equilibrium price of 1/2.

Net of the subsidy, the seller could expect a revenue of $1/2 - 1/6 = 1/3$--strictly less than the 3/8 possible with a reservation price of 1/4 and no subsidy. Here, when $c = 5/36$, even a suboptimal reservation price and no subsidy does better for the seller than optimal subsidy and zero reservation price; in fact this will be the case for $.1160256 < c \leq 1/6$, while the reverse is true for $1/12 < c < .1160255$. Thus a small change in one parameter of the model may swing the seller from preferring a subsidy over a reservation price to the other way around.

Despite this inconclusiveness, we can conclude something of interest from these last three theorems. In particular, even if the seller uses an entry fee, a reservation price, or a subsidy to custom tailor the basic oral auction to a specific situation, the resulting number of potential buyers need never be less than the original equilibrium number, nor need it ever exceed the original equilibrium by more than one. Thus, roughly speaking, in our oral auction model, the seller should set the reservation price equal to his reservation value; if the seller has a reservation value of zero, then as we previously quoted Thompson, "Every piece of goods offered at auction should be positively sold."

In fact, Engelbrecht-Wiggans (1987b) suggests an argument that quite generally, the ex-post optimal reservation price exceeds the ex-ante reservation price even if we can't show that the ex-ante optimal reservation price is essentially equal to the seller's reservation value. In particular, imagine that u parameterizes the distribution of the set of actual bidders; the example suggests thinking of u as the mean number of bidders--which, indeed, is how we will refer to it--even though it could be some other parameterization. Let $R_s(u,r,d,v_o,\alpha,\beta)$ denote the seller's expected net revenue as a function of the parameter u, the reservation price r, the entry fee d, the reservation value v_o, and the tax rates α and β; this revenue may be from a single auction, or from several auctions (each with the same reservation price $r \geq v_o$) of the same object if it was won back by the seller in all but at most one of them. Assume that the derivative of R_s with respect to u will be positive; plausibly, as u increases, so too does the probability of a bonifide bidder winning (as well as the expected price conditional on a bonifide bidder winning) and since bonifide bidders pay at least $r(r \geq v_o)$, an increase in their probability of winning increases the seller's expected revenue.

Ex-post to the auctioneer seeing u, for fixed d and v_o, the optimal reservaton price still depends on α and β as well as on u. Assume that the indicated derivatives exist and are well enough behaved so that for some function $\hat{r}(u,\alpha,\beta)$, $\frac{d}{dr} R_s(,r,d,v_o,\alpha,\beta)\Big|_{r=\hat{r}(u,\alpha,\beta)} = 0$ for all u, α and β, and $\frac{d^2}{dr^2} R_s(u,r,d,v_o,\alpha,\beta)\Big|_{r=\hat{r}(u,\alpha,\beta)} < 0$ for all u, α and β. Interpret this $\hat{r}$ as the ex-post optimal reservation price.

Now, ex-ante, u depends on r; therefore write u(r) when the dependence matters. Assume that the derivative of u(r) with respect to r exists and is negative. That is, as the reservation price increases, the mean number of bidders decreases. To characterize the ex-ante optimal reservation price, look at the derivative of $R_s(u(r),r,d,v_o,\alpha,\beta)$ with respect to r when $r = \hat{r}(u(r),\alpha,\beta)$. (This is an implicit equation for r.) In particular,

$$\frac{d}{dr} R_s(u(r),r,d,v_o,\alpha,\beta)$$

$$= \frac{d}{dr} R_s(u,r,d,v_o,\alpha,\beta)\Big|_{u=u(r)} + \frac{d}{du} R_s(u,r,d,v_o,\alpha,\beta) \frac{d}{dr} u(r)\Big|_{u=u(r)} .$$

When $r = \hat{r}(u(r),\alpha,\beta)$, the first right hand side term is zero by the definition of $\hat{r}$. The second right hand side term is the product of two quantities, both negative by assumption. Therefore, ex-ante, the seller's revenue decreases with r at the ex-post optimal r, and so the seller would prefer a smaller r ex-ante. While this argument presumes more continuity than is present in the original example, it does illuminate why we might reasonably expect that the ex-post optimal reservation price exceeds the ex-ante optimal reservation price even more generally than the symmetric oral auctions with privately known values. Thus, the seller and tax collector benefit quite generally from at least moving in the direction of encouraging the absolute sale of all goods offered.

4. Lowering the Expected Reservation Price

In practice, sellers do try to commit to a lower reservation price than might be ex-post optimal. Some auction notices advertise that all goods will be sold without reserve. Certain laws and auctioneers' codes of ethics prohibit shills. But does it work?

On several occasions, this author observed what appeared to be "cheating" by auctioneers who had promised to sell everything without reserve. On one occasion, an object which sold in one auction resurfaced a couple of weeks later in another auction by the same auctioneer. On another occasion, an auctioneer indicated receiving a bid from a part of the audience unlikely to have bid (this author knew that the individuals in question had never registered for the "bid numbers" required in order to bid). On yet other occasions, an auctioneer lost track of who made the current--and apparently final--bid and then backed up the bidding to a previous, lower level, in order to sell the object.

Whether or not auctioneers actually cheat is not the question. Rather, what matters is how much potential bidders expect an auctioneer to cheat, and what they expect the reservation price to be. As long as this author--and presumably other potential bidders as well--suspect certain auctioneers of implementing higher reservation prices than others, the number of actual bidders attending these auctioneers' auctions will be affected.

Thompson suggested that changing the tax rates would encourage the absolute sale of goods offered. In fact, the change does not directly provide the sellers with a mean for committing ex-ante to lower reservation prices. Rather, as will be shown in this section, the changes reduce the ex-post optimal reservation price. This reduces potential bidders' ex-ante expectations of the reservation price that a perfectly honest auctioneer will end up using, and reduces the incentive for--and therefore, possibly, the degree of--cheating by an auctioneer who promised to sell goods without reserve.

As before, let $R_s(u,r,d,v_o,\alpha,\beta)$ denote the seller's expected net revenue. Since attention focuses on how the tax rates α and β affect the ex-post optimal reservation price $\hat{r}(u,\alpha,\beta)$ (as defined before) think of u as being fixed. As before, restrict $\alpha \geq 0$, $\beta \geq 0$, and α +

$\beta < 1$. Since non-zero entry fees or subsidies d seemingly arose only in response to the original example's integrality of bidders, hold d fixed at zero throughout this section; in fact this appears to be the appropriate choice of d in modelling the Port of New York auctions.

Again, the seller acts as his own auctioneer and sells a single object. As before, the seller's utility depends only on money, and the seller is risk neutral. Also as before, the seller implements the reservation price through a (real or imagined) shill who bids so as to assure that no bonifide bidder wins the object at a price less than r; other than this effect on the selling price to bonifide bidders, we need not specify how the shill bids. Unlike before, the seller may re-auction (always with the same reservation price r) the object until a bonifide bidder wins it.

Now that the seller may repeatedly offer a single object for sale, the reservation value v_o must be defined more carefully. To do so, split the world into two markets--the market affected by changes in α and β, and the market not affected by such changes; in terms of the original example, the world consists of the Port of New York versus everything else. Then let v_o denote the expected net value of the object to the seller conditional on not selling it in the market affected by α and β, and let $F(u,r)$ denote the probability of the object not selling in the market affected by α and β; making F depend on only u and r implicitly assumes restrictions such as that the tax rates affect the probability only through their effects--directly or indirectly--on u and r.

The total tax paid breaks into two components. If $G_1(u,r)$ denotes the expected payments by the shills (who may win more than once) in the market affected by α and β, and $G_2(u,r)$ denotes the expected payments by bonifide bidders in the market affected by α and β, then the total tax

$$T(u,r,\alpha,\beta) = \beta G_1(u,r) + (\alpha+\beta)G_2(u,r).$$

Using the same notation,

$$R_s(u,r,0,v_o,\alpha,\beta) = -\beta G_1(u,r) + (1-\alpha-\beta)G_2(u,r) + v_oF(u,r).$$

As with F, making G_1 and G_2 depend only on r and u places implicit restrictions on these functions.

Now make five assumptions, each holding over the (unspecified) range of allowable u and r. One, the derivative of G_1 with respect to r exists and is positive. This seems plausible because as the reservation price increases, so does the likelihood of the shill winning, and so does the expected amount paid by the shill conditional on winning. Two, the derivative of G_1 with respect to u exists and is negative. This occurs if when the mean number of bidders increases, the shill's probability of winning drops rapidly enough to more than offset any increase (toward r) in the shill's expected payment conditional on winning. Three, the derivative of G_2 with respect to u exists and is positive. This seems plausible because as the number of bidders increases, so too does the probability of a bonifide bidder winning, and so too does the expected amount paid by the bonifide bidders conditional on winning. Four, both G_1 and G_2 are positive. This occurs if both the shill and the bonifide bidders have positive probability of winning, and each pays a positive expected amount conditional on winning. Finally, five, the derivative of F with respect to r exists and is non-negative; as the reservation price increases, the probability of selling the object (to a bonifide bidder) in the market affected by α and β doesn't increase.

(In the example, u = np, and as the integer n varies for fixed p, u varies discontinuously. Thus, the current section does not include the example as a special case; we feel that the previous section adequately deals with the effects of discrete changes in the (mean) number of bidders. Still, this section does include the example modified so that p varies with n fixed, and includes the example in the limiting case of u being the mean of a Poisson distribution.)

To see how α and β affect the ex-post optimal reservation price, examine the first order condition for $\hat{r}(u,\alpha,\beta)$. In particular,

$$0 = \left[-\beta \frac{d}{dr} G_1(u,r) + (1-\alpha-\beta) \frac{d}{dr} G_2(u,r) + v_o \frac{d}{dr} F(u,r)\right]\Big|_{r=\hat{r}(u,\alpha,\beta)} .$$

Notice, for use later, that by the assumptions on α, β, v_o, and F, the derivative of G_2 with respect to r evaluated at $r = \hat{r}(u,\alpha,\beta)$ must be non-positive. Differentiating the first order condition with respect to α and rearranging yields

$$\frac{d\hat{r}(u,\alpha,\beta)}{d\alpha} = \frac{\frac{d}{dr} G_2(u,r)}{\frac{d^2}{dr^2} R_s(u,r,0,v_o,\alpha,\beta)}\Bigg|_{r=\hat{r}(u,\alpha,\beta)} .$$

The second order condition defining $\hat{r}(u,\alpha,\beta)$, together with the above note on the derivative of G_2 at the ex-post optimal reservation price when β equals zero, implies that the derivative of $\hat{r}$ will be non-negative; when β equals zero, as α decreases, the ex-post optimal reservation price stays the same or decreases.

Similarly, differentiating the first order condition with respect to β and rearranging yields

$$\frac{dr(u,\alpha,\beta)}{d\beta} = \left.\frac{\frac{d}{dr} G_1(u,r) + \frac{d}{dr} G_2(u,r)}{\frac{d^2}{dr^2} R_s(u,r,0,v_o,\alpha,\beta)}\right|_{r=\hat{r}(u,\alpha,\beta)} .$$

To sign this second derivative requires establishing the relative sizes of the two derivatives in the numerator. Since G_1 depends on how the shill bids--does the shill initially bid r, does the shill bid only as needed until the price reaches r or he wins, or does the shill bid differently still--we have been unable to establish a general relationship. (Still, for the first two choices of shill behavior just mentioned, the numerator can be shown to be strictly positive in independently distributed, private values models.)

As an alternative approach to looking at specific models, hold the tax $T(u,r,\alpha,\beta)$ constant; that is, as β varies, vary α so that the tax remains the same. Thus, we now think of α as some function $\alpha(\beta)$ of β. Then, see how varying β affects the ex-post optima reservation price, by evaluating

$$\frac{d}{d\beta}\, \hat{r}(u,\alpha(\beta),\beta) = \left[\frac{d}{d\beta}\, \hat{r}(u,\alpha,\beta) + \frac{d}{d\alpha}\, \hat{r}(u,\alpha,\beta)\, \frac{d\alpha(\beta)}{d\beta}\right]\Bigg|_{\alpha=\alpha(\beta)} .$$

Differentiating $T(u,r,\alpha(\beta),\beta)$ implicitly with respect to β and rearranging yields

$$\frac{d\alpha(\beta)}{d\beta} = \frac{-G_1(u,r) + G_2(u,r)}{G_2(u,r)} .$$

Substituting this together with the two previously obtained expressions for the derivatives of $\hat{r}(u,\alpha,\beta)$ yields

$$\frac{d}{d\beta}\hat{r}(u,\alpha(\beta),\beta) = \left.\frac{\frac{d}{dr}G_1(u,r) + \frac{d}{dr}G_2(u,r)}{\frac{d^2}{dr^2}R_s(u,r,0,v_o,\alpha,\beta)}\right|_{r=\hat{r}(u,\alpha(\beta),\beta)}$$

$$+ \left.\frac{\frac{d}{dr}G_2(u,r)}{\frac{d^2}{dr^2}R_s(u,r,0,v_o,\alpha,\beta)}\;\frac{G_2(u,r) - G_1(u,r)}{G_2(u,r)}\right|_{r=\hat{v}(u,\alpha(\beta),\beta)}$$

which simplifies to

$$\frac{d}{d\beta}\hat{r}(u,\alpha(\beta),\beta) = \left.\frac{G_2(u,r)\frac{d}{dr}G_1(u,r) - G_1(u,r)\frac{d}{dr}G_2(u,r)}{G_2(u,r)\frac{d^2}{dr^2}R_s(u,r,0,v_o,\alpha,\beta)}\right|_{r=\hat{r}(u,\alpha(\beta),\beta}.$$

By assumption, G_1, $\frac{d}{dr}G_1$, and G_2 are strictly positive. By the definition of $\hat{r}$, $\frac{d^2}{dr^2}R$ is strictly negative when evaluated at $r = \hat{r}(u,\alpha,\beta)$. And, by a previous note, $\frac{d}{dr}G_2$ is non-negative when $\beta = 0$. Therefore, at $\beta = 0$, $\frac{d}{d\beta}\hat{r}(u,\alpha(\beta),\beta) < 0$.

In short, if we hold the total tax receipts constant, then initially as we raise β above zero (and correspondingly drop α toward zero), the ex-post optimal reservation price drops. That is, quite generally, moving at least somewhat in the direction of Thompson's proposal (to raise β while dropping α to zero) has the desired effect of lowering the reservation price that potential bidders can reasonably anticipate will be used in the auction. While this analysis says nothing once $\beta > 0$, the original example establishes that increasing β

until α drops all the way to zero decreases the ex-ante optimal reservation price.

5. Summary

In the early nineteenth century, an astute auctioneer proposed that rather than just taxing imported goods actually sold, the tax should be reduced, but at the same time extended to goods offered for sale but not sold. He felt it important to encourage the absolute sale of all goods offered, and claimed that his proposal would provide such an encouragement. Presumably, in making the proposal, the auctioneer felt that the auction houses--and therefore presumably also the sellers--would benefit; by adopting the proposal, the New York State Legislature presumably indicated it too expected to benefit.

This paper develops a theory to explain what may have happened. First, an analytically tractable example establishes that in at least certain specific models the proposal has the anticipated effects; lowering the tax rate while extending it to goods offered but not sold reduces the reservation price a wiley seller would wish to use, thereby making the auction more attractive to bidders and attracting more of them, thus raising the expected revenue, and in the end increasing both the seller and tax collector's receipts.

Subsequent sections attempt to understand what might drive the results more generally. The analysis breaks into two pieces. The first argues that for a variety of models, if the seller could convincingly commit to a reservation price ex-ante to potential bidders deciding whether or not to attend the auction, the seller would prefer a lower reservation price ex-ante than he would chose ex-post to bidders having committed themselves to attend. The second argues that moving in the direction of the proposed change at least initially decreases the ex-post optimal reservation price, thereby in effect enabling the seller to make the potential bidders expect that a lower--and more attractive to bidders--reservation price will be used than before, and as a result, increasing the revenue to be split between the seller and tax collector.

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