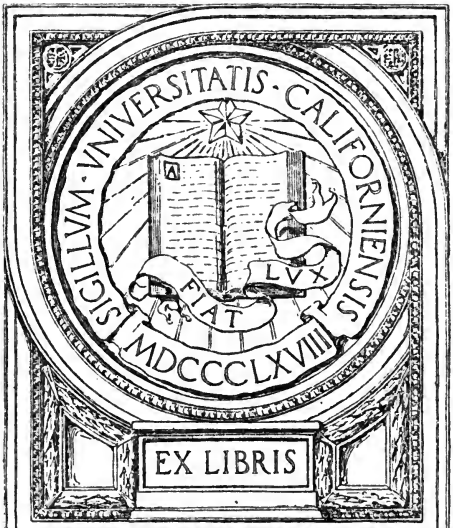
GIFT OF
Miss Emily Palmer



EDUCATION DEPT.

LANGE LIBRARY OF EDUCATION
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA

EXAMPLES IN ALGEBRA

A COLLECTION OF EXERCISES TO ACCOMPANY ANY TEXTBOOK

BY

ALBERT HARRY WHEELER

TEACHER OF MATHEMATICS IN THE HIGH SCHOOL OF COMMERCE
WORCESTER, MASSACHUSETTS

AND

AUTHOR OF "FIRST COURSE IN ALGEBRA"

BOSTON
LITTLE, BROWN, AND COMPANY
1915

Q A157
N5
Educ
Dept

Gift of Miss Emily Palmer
to education dept.

Copyright, 1914,
By Little, Brown, and Company.

All rights reserved

Set up and electrotyped by J. S. Cushing Co., Norwood, Mass., U.S.A.
Presswork by S. J. Parkhill & Co., Boston, Mass., U.S.A.

PREFACE

The best types of examples and applied problems to be found in American or Foreign Textbooks or collections of examples are included in this book.

All examples have been constructed by the author, and the subject matter covers standard courses of study and meets college-entrance requirements.

The collection is adapted for use with any textbook.

In presenting a new subject nothing is too simple for illustrative material, and accordingly a great number of very simple single-step process examples have been included.

The following new features are presented: New Mental Exercises; new types of very simple Process-Examples; new types of Blank Forms; new types of Preparatory Examples, and an exercise on the Functional Notation. Many arithmetic examples are found in the first parts of exercises; Decimal Fractions are used commonly; there is a new exercise on Incomplete Polynomial Squares, and a very complete exercise on Complex Fractions.

The exercise on Rationalization contains a great variety of forms and in particular has "confusing forms" grouped together.

There is an exercise on Exponential Equations and a set of Equations having Irrational Coefficients.

New types of Proportion Examples apply the Transformation Principles, the Determination of Specified Ratios, and Elimination.

The exercises on Proportion are so constructed that, if desired, the subject may be taken with fractional equations early in the course.

The treatment of Graphs is simple, and Intersecting Lines are so chosen that they meet near the origin.

632256

There are many new practical applied problems.

Under the head of Equivalent Equations a very simple introduction is given to Linear Equations.

The collection of examples on Fractional and Literal Equations is very complete and simple, and many forms occurring in later mathematical work are given.

The Completion of the Square with reference to a given Binomial is carefully worked out both in factoring and in quadratic equations.

There is an exercise on Special Irrational Equations in both Exponential and Radical Notation.

Special exercises for Synthetic Division and Applications of the Remainder Theorem are provided, and the treatment of Factoring and Quadratic Equations is very complete.

The review exercises contain many examples requiring distinctions to be made between Confusing Forms, and the examples are so chosen as to afford Cumulative Reviews at different points.

The Index cross-classifies the examples in such a way that principles may be quickly illustrated by choosing examples from different subject-headings.

Applied Problems, which have to do with questions or situations which would reasonably require or admit of algebraic solution, have been included. Questions which would naturally arise in the search for truth have been made the basis for practical problems, but there are no falsely practical problems which are concerned with unreal situations or which arise from juggling with statistics or which bring in information which may of itself be interesting in its proper place, but which has no reasonable or sensible place in illustrating algebraic principles.

The book is especially adapted for use in examinations because of the great variety of material and also because of the fact that no definitions or explanations are given.

The author wishes to acknowledge the enthusiastic reception which has been given his examples by teachers using his First Course in Algebra, and especially to thank the teachers of mathematics in his home city who have shown such a friendly attitude toward his work and whose constructive criticisms and recom-

mendations have been of great value in the preparation of the present collection of examples.

He would especially thank the Committee of Teachers from the four Worcester High Schools, Mr. Geo. H. Boyden, Miss Grace L. Hill, Miss A. Mae Lawrence, Mr. Wendell P. Parker, Miss Harriet R. Pierce, Mr. Chester T. Porter, and Miss Mary Trumbull, who inspired the present work.

He would extend thanks to the Superintendent of Schools, Mr. Homer P. Lewis, for his kind interest and encouragement, extending over many years.

He would also express sincere and grateful appreciation to the Assistant Superintendent of Schools, Mr. John F. Gannon, whose interest and enthusiasm in the undertaking have been a large contributing factor to its successful completion.

Digitized by the Internet Archive
in 2008 with funding from
Microsoft Corporation

http://www.archive.org/details/examplesinalgebr00wheerich

CONTENTS

PAGE

Numerical Substitutions 1
Parentheses 1

POSITIVE AND NEGATIVE NUMBERS

Classification of Quantities as being Positive or Negative . . . 2
Addition of Positive and Negative Numbers 4
Subtraction of Positive and Negative Numbers 4
Multiplication of Positive and Negative Numbers 5
Powers of Positive and Negative Numbers 6
Arithmetic Numbers 6
Division of Positive and Negative Numbers 6

LITERAL EXPRESSIONS

ADDITION AND SUBTRACTION

Addition of Monomials 7
Subtraction of Monomials 9
Addition and Subtraction of Polynomials 12
Parentheses 15

MULTIPLICATION

Multiplication 15
Products of Powers of Equal Bases 15
Multiplication of Monomials 16
Multiplication of Polynomials 18

SPECIAL PRODUCTS

Square of a Binomial Sum 21
Square of a Binomial Difference 22
Sum of Two Terms Multiplied by their Difference 22
Square of a Polynomial 23
Product of Two Binomials of the Forms $x + a$ and $x + b$ . . . 24
Product of Two Binomials of the Forms $ax + b$ and $cx + d$ . . . 24
Special Trinomial Products 25
Powers of a Binomial 26

DIVISION

PAGE

Quotients of Powers 26
Division of Monomials 27
Division of Polynomials 28
Synthetic Division 31
Remainder Theorem 32
Special Quotients 33
Functional Notation 33
Remainder Theorem 34

GRAPHS

Location of Points 34
Graphs of Equations 35
Solution of Equations by Graphs 36

EQUATIONS

Identical Equations 37
Equivalent Equations 38
Linear Equations 41
Algebraic Expression 44
Applied Problems 46
Science Problems 52

INTEGRAL FACTORS

Expressions all of whose Terms Contain a Common Monomial Factor . 53
Expressions in which Groups of Terms Have a Common Factor . . 55
Common Roots 56
Incomplete Trinomial Squares 56
Trinomial Squares 57
Incomplete Polynomial Squares 58
Polynomial Squares 59
Binomial Differences of Squares 59
Expressions which may be Factored as the Difference of Two Squares 61
Trinomials of the Type $x^2 + sx + p$ 63
Trinomials of the Type $x^{2m} + sx^m + p$ 64
Trinomials of the Type $ax^2 + bx + c$ 65
Binomial Sums and Differences of Like Powers 66
Polynomials of the Third Degree 67
Miscellaneous 67
Miscellaneous 70
Application of Factoring to the Solution of Equations 72

HIGHEST COMMON FACTOR

PAGE
Highest Common Factor of Monomials 74
Highest Common Factor of Polynomials 74

LOWEST COMMON MULTIPLE

Lowest Common Multiple of Monomials 76
Lowest Common Multiple of Polynomials 77
Review Exercise 78

FRACTIONS

Changes of Signs of the Terms of a Fraction 79
Identities Containing Fractions 80
Reduction to Lowest Terms 81
Reduction of Improper Fractions to Integral or Mixed Expressions . 83
Lowest Common Denominator 84
Addition of Fractions 85
Reduction to the Form of a Fraction Having a Specified Denominator . 86
Reduction of Mixed Expressions to the Forms of Improper Fractions . 87
Multiplication of Fractions 88
Powers of Fractions 90
Division of Fractions 91
Complex Fractions 93
Continued Fractions 95
Miscellaneous Exercise 96
Indeterminate Forms 97
Factors of Fractional Expressions 97
Review Exercise 98

FRACTIONAL EQUATIONS AND LITERAL EQUATIONS

Numerical Fractional Equations 99
Integral Literal Equations 102
Literal Fractional Equations 104
Applied Problems 107
General Problems 111
Physics Problems 112
Review 113

SIMULTANEOUS LINEAR EQUATIONS

Classification of Systems of Equations 114
Elimination by Substitution 115
Elimination by Comparison 116

PAGE
Elimination by Addition or Subtraction 116
Systems of Equations Containing Three or More Unknowns . . 118
Systems of Fractional Equations Solved like Equations of the First Degree 119
Applied Problems 121

EVOLUTION

Roots of Powers and Products of Monomials 127
Roots of Roots and Powers of Roots of Monomials 128
Roots of Quotients of Monomials 128
Square Roots of Polynomials 129
Cube Roots of Polynomials 129
Square Roots of Arithmetic Numbers 130
Cube Roots of Arithmetic Numbers 130

THEORY OF EXPONENTS

Negative Integral Exponents 130
Zero as an Exponent 131
Transformation of Exponents 131
Transference of Factors 132
Fractional Exponents 133
Translation from Exponential to Radical Notation 134
Translation from Radical to Exponential Notation 134
Products of Powers and Quotients of Powers 135
Powers of Powers 135
Powers of Products and Powers of Quotients 137
Products of Binomials whose Terms Have Fractional or Negative Exponents 137
Miscellaneous 138
Exponential Equations 139
Miscellaneous 140

SURDS

Reduction to Simplest Form 140
Addition and Subtraction of Surds 144
Reduction to Equivalent Forms 145
Reduction of Mixed Surds to Entire Surds 145
Change of Order 146
Comparison of Surds 146
Multiplication of Surds 146
Products of Two Binomial Surds 148
Products of Polynomial Surds 149

PAGE
Involution of Monomial Surds 150
Division of Monomial Surds 150
Rationalization 151
Expressions Having Surd Factors 154
Evolution of Surds 155
Square Root of a Binomial Surd 155

IMAGINARY NUMBERS

Reduction to Standard Form 156
Addition and Subtraction of Imaginary Numbers 157
Multiplication of Imaginary Numbers 158
Division of Imaginary Numbers 160

COMPLEX NUMBERS

Addition and Subtraction of Complex Numbers 161
Multiplication of Complex Numbers 162
Powers of Complex Numbers 163
Expressions Having Complex Factors 163
Square Roots of Complex Numbers 164
Simplification of Fractions Having Complex Denominators . . . 164
Review 165

QUADRATIC EQUATIONS

Incomplete Quadratic Equations 166
Complete Quadratic Equations 167
Completion of the Square 168
Solution by Completing the Square 169
Completing the Square by the Hindu Method 170
Solution by the Formula 170
Rational Fractional Equations Containing One Unknown . . . 171
Miscellaneous 174
Solution of Formulas for Specified Letters 176
Problems 177
Problems in Physics 181

THEORY OF QUADRATIC EQUATIONS

Nature of the Roots 183
Quadratic Equations Having a Parameter 184
Determination of One Root when the Other is Known 184
Formation of an Equation Having Specified Roots 186
Review 189

IRRATIONAL EQUATIONS

PAGE
Irrational Equations 191
Special Irrational Equations 194
Irrational Equations and Equations Having Irrational Coefficients . 196

SPECIAL EQUATIONS

Special Higher Equations and Equations Having Fractional Exponents 197
Binomial Equations and Reciprocal Equations 198
Problems in Physics 198
Review 199

SIMULTANEOUS QUADRATIC EQUATIONS

Elimination by Substitution 200
Reduction of Systems of Equations by Factoring 202
Systems of Two Homogeneous Equations of the Second Degree Containing Two Unknowns 203
Solution by Eliminating the Constant Terms and Factoring . . . 203
Solution by the "*vy*" Method 203
Reduction of Systems of Equations by Division 204
Systems of Symmetric Equations 204
Asymmetric Equations 206
Miscellaneous 206
Systems of Three or More Equations Containing Three or More Unknowns 209
Problems 211

RATIO

Reduction to Simplest Form 215
Compound Ratios 216
Comparison of Ratios 217

PROPORTION

Construction of Proportions 217
Mean Proportionals 217
Third Proportionals 218
Fourth Proportionals 218
Solution of Proportions 218
Determination of Specified Ratios 219
Construction of Proportions from Products 219
Transformation of Proportions by Inversion 220
Transformation of Proportions by Alternation 220

PAGE
Transformation of Proportions to Isolate Specified Letters . . . 220
Transformation of Proportions by Addition and Subtraction . . 221
Determination of Specified Ratios 222
Elimination 222
Reduction of Specified Terms to the First Degree 223
Problems 224
Problems in Physics 225

VARIATION

One Independent Variable 227
Two Independent Variables 229
Problems in Physics 230
Review 231

THE PROGRESSIONS

Sequences 233
Arithmetic Progression 234
Calculation of Specified Terms 234
Arithmetic Means 235
Sum of the Terms of an Arithmetic Progression 236
Problems in Physics 238
Harmonic Progression 238
Calculation of Specified Terms 238
Harmonic Means 239
Geometric Progression 240
Calculation of Specified Terms 240
Geometric Means 241
Sum of the Terms of a Geometric Progression 242
Sum of the Terms of an Infinite Decreasing Geometric Progression . 242
Repeating Decimal Fractions as Geometric Progressions . . . 243
Classification of Progressions 244
Review 245

THE BINOMIAL THEOREM

Expansion of Powers of Binomials 246
Binomial Theorem for Negative and Fractional Exponents . . . 247
Calculation of Specified Terms 247
Extraction of Roots of Numbers by the Binomial Theorem . . . 248
Review 248
Review 250

INDEX 253

EXAMPLES IN ALGEBRA

NUMERICAL SUBSTITUTIONS

EXERCISE 1

Find the values represented by each of the following expressions when the letters are replaced by the given numerical values.

If $a = 3$, $b = 2$, $c = 1$, $d = 4$, find the value of

1. $a + b$.	**16.** $4a$.	**31.** $4 + 2b$.	**46.** $a^3 - 2$.
2. $b + c$.	**17.** $6b$.	**32.** $2d - c$.	**47.** $c^3 + 3$.
3. $a + d$.	**18.** $2c$.	**33.** $5b + 2d$.	**48.** $d^2 + 9$.
4. $a - c$.	**19.** $8d$.	**34.** $3a - 7c$.	**49.** $a^2 + b^2$.
5. $d - b$.	**20.** $3ab$.	**35.** $8c + 5d$.	**50.** $b^2 + c^2$.
6. $a + 1$.	**21.** $5bd$.	**36.** a^2.	**51.** $(c^2 + d^2)^2$.
7. $b + 5$.	**22.** $3cd$.	**37.** b^2.	**52.** $(a^2 - c^2)^2$.
8. $c + 2$.	**23.** $6ab$.	**38.** c^2.	**53.** $a^3 + b$.
9. $d - 3$.	**24.** $7abc$.	**39.** d^2.	**54.** $d^3 - c^3$.
10. $4 - a$.	**25.** $4bcd$.	**40.** a^2b^2.	**55.** $b^2 + d^3$.
11. $7 - b$.	**26.** $2a + 1$.	**41.** $a^2 + 2$.	**56.** $2a^2 + b$.
12. $6 + c$.	**27.** $3b - 2$.	**42.** $b^2 - 1$.	**57.** $3b^2 + 2c$.
13. $d - 1$.	**28.** $4c + 6$.	**43.** $c^2 + 3$.	**58.** $5c^2 + b^2$.
14. $2 + b$.	**29.** $5d - 17$.	**44.** $5 + d^2$.	**59.** $a^3 + 2b^2$.
15. $6 - a$.	**30.** $7a - 2$.	**45.** $b^3 + 1$.	**60.** $b^3 + d^2$.

If $a = 2$, $b = 3$, $x = 1$, $y = 5$, find the value of

61. $a^2 + bx$.	**64.** $by - a^2$.	**67.** $ax^2 + y^2$.
62. $ab + y^2$.	**65.** $abx - y$.	**68.** $4ab - xy$.
63. $b^2 - xy$.	**66.** $3a^2 - 2bx$.	**69.** $5by - x^2$.

PARENTHESES

EXERCISE 2

Remove parentheses and find the numerical values of the following expressions.

1. $10 - (4 - 3)$.	**3.** $15 - (8 - 5)$.	**5.** $7 - (10 - 8)$.
2. $12 - (4 + 3)$.	**4.** $4 + (9 - 6)$.	**6.** $15 - 2(7 - 3)$.

7. $20 + 3(11 - 9)$.
8. $19 - 4(6 - 3)$.
9. $18 + 5(10 - 7)$.
10. $21 - 3(8 - 1)$.
11. $9 - [4 - (3 - 1)]$.
12. $11 - [10 - (8 - 5)]$.
13. $13 - 2[9 - 3(7 - 5)]$.
14. $10 + 4[30 - 4(10 - 4)]$.
15. $10 - 9[8 - 7(6 - 5)]$.
16. $-(-3)$.
17. $-(-(-2))$.
18. $-(-(-(-4)))$.
19. $5 - (-(-2))$.
20. $7 - (-(-(-1)))$.
21. $6 - (3 - 2(-1))$.
22. $3 - (-(-(-3))) - (-(-3))$.
23. $5 - (-(-(-2))) - (-(-(-1)))$.
24. $12 - 2(-2(-2))$.

POSITIVE AND NEGATIVE NUMBERS

EXERCISE 3

Classification of Quantities as Being either Positive or Negative

Classify the following changes in temperature as being both positive, both negative, or one positive and the other negative:

1. From 70° F. to 100° F. and to 60° F. respectively.
2. From 72° F. to 105° F. and to 212° F. respectively.
3. From 80° F. to 32° F. and to 0° F. respectively.
4. From 10° F. above zero to 3° F. below zero, and to 8° F. below zero respectively.
5. From 12° F. below zero to 18° F. below zero and to 22° F. below zero respectively.
6. From 40° F. above zero to 13° F. below zero and to 65° F. above zero respectively.
7. From 9° F. above zero to 31° F. below zero and to 40° F. below zero respectively.
8. From 15° F. below zero to 0° F. and to 8° F. below zero respectively.
9. From 2° F. below zero to 5° F. above zero and to 0° F. respectively.

Which of the following cities may be selected as points of reference in order that the distances to the remaining two may be classified as being both positive or both negative?

10. (*a*) Boston, Albany, Buffalo.
(*b*) Montreal, Buffalo, Chicago.
(*c*) Quebec, Washington, D.C., New Orleans.

Which of the following cities may be selected as points of reference in order that the distances to the remaining two may be classified as being one positive and the other negative?

11. (*a*) Edinburgh, London, Paris.
(*b*) Cairo, Lisbon, Vienna.
(*c*) Glasgow, Cairo, Cape Town.

Regarding a man's *income* as representing a positive quantity, classify the following items as positive or negative quantities wherever possible:

12. (*a*) Interest on money deposited in a bank.
(*b*) Interest paid on money borrowed on a note.
(*c*) Money paid for car fare.
(*d*) Cash prize won in an athletic contest.

Regarding money *on hand* as representing a positive quantity, wherever possible classify the following items as positive or negative quantities with reference to the depositor:

13. (*a*) Money deposited in a bank.
(*b*) Interest received on money deposited in the bank.
(*c*) Interest paid on a mortgage held by the bank.
(*d*) Money withdrawn from the bank.

14. Classify the above with reference to the bank.

With reference to the equator, classify the latitudes of the following places as being positive or negative quantities:

15. St. Louis, Richmond, Buenos Ayres, Berne, Athens, Zanzibar, Calcutta, Constantinople, Honolulu, Melbourne.

With respect to the meridian passing through Greenwich, classify the longitudes of the following places as being positive or negative quantities:

16. Dublin, Havana, San Francisco, Quito, Manila, Odessa.

ADDITION OF POSITIVE AND NEGATIVE NUMBERS

EXERCISE 4

Simplify the following:

1. $+3+{}+4$.
2. $+2+{}+6$.
3. $+4+{}+8$.
4. $+5+{}+2$.
5. $+7+{}+10$.
6. $+6+{}+1$.
7. $+8+{}+12$.
8. $+9+{}-2$.
9. $+10+{}-4$.
10. $+7+{}-3$.
11. $+1+{}-2$.
12. $-5+{}+9$.
13. $-11+{}-9$.
14. $-12+{}-12$.
15. $-13+{}-11$.
16. $+12+{}-14$.
17. $-3+{}+11$.
18. $-15+{}+13$.
19. $-14+{}+20$.
20. $+19+{}-15$.
21. $-16+{}-17$.
22. $+13+{}+16$.
23. $+11+{}-10$.
24. $-20+{}-19$.
25. $-1+{}+18$.
26. $+17+{}+6$.
27. $-10+{}+13$.
28. $+18+{}-8$.
29. $+14+{}-9$.
30. $-8+{}+15$.
31. $-18+{}+18$.
32. $-17+{}-16$.
33. $-4+{}+20$.
34. $+2+{}+3+{}+8$.
35. $+3+{}+5+{}-1$.
36. $+5+{}+6+{}-9$.
37. $-11+{}+13+{}-6$.
38. $+16+{}+14+{}-19$.
39. $+15+{}-18+{}-9$.
40. $-20+{}-13+{}-17$.
41. $-12+{}-11+{}-4$.
42. $+13+{}+19+{}+11$.

SUBTRACTION OF POSITIVE AND NEGATIVE NUMBERS

EXERCISE 5

Perform the following indicated subtractions:

1. From $+5$ subtract $+3$.
2. From $+8$ subtract $+2$.
3. From $+11$ subtract $+5$.
4. From $+14$ subtract $+10$.
5. $-12-{}-7$.
6. $-15-{}-11$.
7. $-7-{}-4$.
8. $-3-{}-5$.
9. $-6-{}-8$.
10. $-9-{}-6$.
11. $-4-{}+15$.
12. $+6-{}-9$.
13. $+3-{}-8$.
14. $+13-{}-14$.
15. $+16-{}+17$.
16. $-17-{}+13$.
17. $-10-{}+19$.
18. $+18-{}-22$.
19. $-19-{}+11$.
20. $-2-{}+7$.
21. $+21-{}-16$.
22. $+9-{}-12$.
23. $+15-{}-15$.
24. $-20-{}-20$.
25. $+12-{}-8$.
26. $+7-{}-13$.
27. $+17-{}-18$.
28. $+22-{}-19$.
29. $+15-{}-2$.
30. $+4-{}-17$.
31. $+19-{}-31$.
32. $+31-{}-29$.
33. $-28-{}+22$.
34. $+26-{}-34$.
35. $-37-{}+23$.
36. $+29-{}-11$.
37. $-23-{}+32$.
38. $-33-{}-39$.
39. $-35-{}+27$.
40. $+32-{}-28$.

41. $-38 - +32$.
42. $+30 - +37$.
43. $+39 - +35$.
44. $-27 - -40$.
45. $+23 - -21$.
46. $+34 - +47$.
47. $-31 - +49$.
48. $-45 - -35$.
49. $+43 - +26$.
50. $+41 - +19$.
51. $-24 - -41$.
52. $+28 - +42$.
53. $-47 - +31$.
54. $+49 - -44$.
55. $-50 - +46$.
56. $-48 - +41$.
57. $+42 + -48$.
58. $+36 + -43$.

Simplify the following:

59. $+1 - +2 + +6$.
60. $+2 - -10 + +8$.
61. $+10 - -10 + +13$.
62. $-1 - -19 + +12$.
63. $+20 + -13 - -18$.
64. $-16 + -21 - +30$.
65. $+23 + -27 - +33$.
66. $-21 + -29 - +30$.
67. $-26 - +34 + -40$.
68. $+35 - -40 - -25$.
69. $-31 + -32 - -33$.
70. $+27 + +27 - -16$.
71. $-41 - +43 - +50$.
72. $+44 - -46 - -33$.
73. $+40 - -42 - -43$.
74. $-39 - +41 + +37$.

MULTIPLICATION OF POSITIVE AND NEGATIVE NUMBERS

EXERCISE 6

Simplify the following:

1. $+2 \times +4$.
2. $+3 \times +6$.
3. $+5 \times -9$.
4. $+6 \times +2$.
5. $+8 \times +3$.
6. $+9 \times +8$.
7. $+7 \times +11$.
8. $+12 \times +7$.
9. $+4 \times -5$.
10. $+10 \times -13$.
11. $+14 \times -5$.
12. $+16 \times -3$.
13. $-8 \times +9$.
14. $-6 \times +12$.
15. -2×-18.
16. -5×-19.
17. -20×-17.
18. $+18 \times -10$.
19. $+15 \times -15$.
20. -16×-16.
21. $-9 \times +21$.
22. $-13 \times +20$.
23. $+19 \times -2$.
24. $+21 \times -7$.
25. -21×-6.
26. $-4 \times +14$.
27. $-3 \times +16$.
28. -12×-8.
29. $+9 \times -11$.
30. $+17 \times -20$.
31. $+20 \times +35$.
32. $-19 \times +40$.
33. -30×-22.
34. $+28 \times +50$.
35. -36×-3.
36. $+34 \times +30$.
37. $-23 \times +60$.
38. $-25 \times +70$.
39. $+40 \times +26$.
40. $+27 \times +30$.
41. $-20 \times +37$.
42. $+30 \times +45$.
43. $+22 \times -30$.
44. $+13 \times -40$.
45. $+25 \times -25$.

46. $+17 \times -17$.
47. $+18 \times +18$.
48. -21×-21.
49. $+2 \times -3 \times -1$.
50. $+4 \times -7 \times -2$.
51. $+7 \times +7 \times -1$.
52. $+6 \times +8 \times +3$.
53. $-9 \times -5 \times -4$.
54. $-10 \times -9 \times +8$.
55. $+12 \times -4 \times +2$.
56. $+20 \times -14 \times -1$.
57. $+5 \times -6 \times +7$.
58. $+11 \times -1 \times -12$.
59. $+16 \times -5 \times +4$.
60. $-18 \times -3 \times +2$.

POWERS OF POSITIVE AND NEGATIVE NUMBERS

EXERCISE 7

Find the values of the following indicated powers:

I. Arithmetic Numbers

1. 2^2.
2. 2^3.
3. 3^2.
4. 3^4.
5. 4^3.
6. 5^3.
7. 7^2.
8. 2^5.
9. 2^7.
10. 3^5.
11. 8^2.
12. 8^3.
13. 4^4.
14. 2^8.
15. 10^3.
16. 12^2.

II. Positive and Negative Numbers

17. $(+2)^3$.
18. $(-3)^2$.
19. $(-4)^3$.
20. $(-5)^2$.
21. $(-3)^3$.
22. $(-6)^3$.
23. $(+8)^2$.
24. $(-2)^5$.
25. $(+9)^3$.
26. $(-11)^2$.
27. $(-5)^4$.
28. $(+7)^3$.
29. $(-16)^2$.
30. $(-3)^5$.
31. $(-4)^4$.
32. $(-10)^5$.
33. $+2^2 + +3^2$.
34. $+5^2 - -4^2$.
35. $+3^3 - -5^2$.
36. $(-7)^2 - (-6)^2$.
37. $+8^2 - (-7)^2$.
38. $-10^2 - (-5)^3$.
39. $(-2)^5 + -2^5$.
40. $-3^4 - (-3)^4$.
41. $-6^2 - -2^5$.

DIVISION OF POSITIVE AND NEGATIVE NUMBERS

EXERCISE 8

Simplify the following:

1. $+6 \div +3$.
2. $+8 \div +2$.
3. $+10 \div +5$.
4. $+14 \div -7$.
5. $+18 \div -6$.
6. $-9 \div +1$.
7. $-12 \div +4$.
8. $-16 \div +8$.
9. $+15 \div -5$.
10. $+20 \div -10$.
11. $-22 \div -11$.
12. $+12 \div +6$.
13. $-7 \div +7$.
14. $+13 \div -13$.
15. $+19 \div -1$.
16. $+21 \div -3$.
17. $-27 \div -9$.
18. $+33 \div +11$.
19. $-30 \div -15$.
20. $+28 \div -14$.
21. $-38 \div -19$.
22. $+32 \div -16$.
23. $+60 \div -12$.
24. $-34 \div +17$.
25. $-39 \div -13$.
26. $-24 \div -4$.
27. $-56 \div -7$.

28. $+63 \div +9$.
29. $-75 \div +15$.
30. $+78 \div +13$.
31. $+74 \div -2$.
32. $+51 \div -17$.
33. $-90 \div +18$.
34. $-99 \div +11$.
35. $+108 \div -36$.
36. $-92 \div +4$.
37. $+84 \div +7$.
38. $-91 \div -7$.
39. $+132 \div +12$.
40. $+144 \div -36$.
41. $+196 \div +14$.
42. $-169 \div +13$.

43. $(+8 \times +3) \div +2$.
44. $(-10 \times +3) \div -5$.
45. $(-6 \times -4) \div -3$.
46. $(+12 \times +5) \div +4$.
47. $(-15 \div -5) \times +2$.
48. $(-16 \div +8) \times -3$.
49. $(+18 \div +2) \times -1$.
50. $(-21 \div -7) \times -3$.
51. $(+24 \times +3) \div -8$.
52. $(-36 \div -12) \times +4$.
53. $(-42 \div -7) \times -6$.
54. $(-48 \div -3) \times +5$.
55. $+3 \times (-10 \div -2)$.
56. $-5 \times (-12 \div +3)$.
57. $-18 \div (-2 \times -3)$.
58. $-15 \div (+20 \div -4)$.
59. $-11 \times (-14 \div -7)$.
60. $-7 \times (-24 \div -3)$.

LITERAL EXPRESSIONS

ADDITION OF MONOMIALS

MENTAL EXERCISE 9

Perform the following indicated additions:

1. a, a
2. $-b$, $-b$
3. $2c$, $3c$
4. $5d$, d
5. $-4g$, $-g$
6. $6h$, $4h$
7. $5x$, $-2x$
8. $-7x$, $11x$
9. $-8x$, $8x$
10. $-12y$, $-13y$
11. $9z$, $-19z$
12. $13w$, $-20w$
13. $8ab$, $2ab$
14. $6bc$, $9bc$
15. $-3ad$, $-11ad$
16. $16xy$, $-9xy$
17. $4c^2x$, $5c^2x$
18. $9by^2$, $8by^2$
19. $-17c^2d^2$, $21c^2d^2$
20. $-26x^3y^2$, $31x^3y^2$
21. ax, $2ax$, $3ax$
22. $5by$, $4by$, $2by$
23. $6c^2$, $2c^2$, $5c^2$
24. $7dw$, $-5dw$, $3dw$
25. $4xw$, $-2xw$, $-xw$
26. $11y^3$, $-3y^3$, $3y^3$

27. $12\,z^2w$, $4\,z^2w$, $-4\,z^2w$

28. $-20\,w^2$, $10\,w^2$, $-5\,w^2$

29. $-3\,abc$, $-5\,abc$, $-7\,abc$

30. $-13\,bxy$, $-7\,bxy$, $-9\,bxy$

31. $14\,cm^2$, $-16\,cm^2$, $18\,cm^2$

32. $-17\,dh^2$, $-13\,dh^2$, $29\,dh^2$

33. $22\,a^2bc$, a^2bc, $-3\,a^2bc$

34. $28\,ab^2c$, $-ab^2c$, $-5\,ab^2c$

35. $31\,xyz^2$, $-11\,xyz^2$, $-xyz^2$

36. $-43\,x^2y^2$, $-27\,x^2y^2$, x^2y^2

37. $7\,a^2b$, $5\,a^2b$, $8\,a^2b$, $6\,a^2b$

38. $9\,bc$, $-10\,bc$, $12\,bc$, $-14\,bc$

39. $-5\,x^2y$, $9\,x^2y$, $13\,x^2y$, $18\,x^2y$

40. $7\,xyz$, $-8\,xyz$, $-9\,xyz$, $10\,xgz$

41. $32\,ab$, $7\,ab$, $11\,ab$, $2\,ab$

42. $13\,xy$, $15\,xy$, $17\,xy$, $19\,xy$

43. $18\,x^2yz^2$, $12\,x^2yz^2$, $16\,x^2yz^2$, $14\,x^2yz^2$

44. $14\,x^3y^3z^3$, $14\,x^3y^3z^3$, $22\,x^3y^3z^3$, $25\,x^3y^3z^3$

45. $13\,abx$, $-15\,abx$, $27\,abx$, $12\,abx$

46. $-23\,b^2c^3d$, $37\,b^2c^3d$, $-6\,b^2c^3d$, $-b^2c^3d$

47. $-25\,a^3mw$, $-35\,a^3mw$, $-45\,a^3mw$, $95\,a^3mw$

48. $8\,mnx^2$, $17\,mnx^2$, $17\,mnx^2$, $-33\,mnx^2$

49. $\frac{1}{2}\,a$, $3\,a$

50. b, $\frac{1}{3}\,b$

51. $\frac{2}{5}\,c$, $-\frac{1}{5}\,c$

52. $\frac{2}{3}\,d$, $\frac{3}{4}\,d$

53. $\frac{6}{7}\,h$, $\frac{1}{2}\,h$

54. $\frac{3}{4}\,k$, $-\frac{4}{3}\,k$

55. $\frac{5}{9}\,m$, $\frac{3}{5}\,m$

56. $\frac{3}{10}\,n$, $\frac{3}{4}\,n$

57. $.2\,a$, $.3\,a$

58. $.04\,b$, $.05\,b$

59. $5.67\,x$, $-2.31\,x$

60. $2.001\,y$, $-.102\,y$

61. $0.7\,c + 0.02\,c$.

62. $8.1\,d + 0.05\,d$.

63. $1.1\,e + .11\,e$.

64. $10.1\,x - 1.01\,x$.

65. $.68\,y + .025\,y$.

66. $1.001\,z + 9.099\,z$.

67. $100.1\,w - 1.001\,w$.

68. $11.01\,a - 10.11\,a$.

Add the following with reference to the similar parts:

69. Find the sum of xz and yz.

70. ax, bx

71. cy, dy

72. mz, $2z$

73. $3w$, yw

74. ad, d

75. $6h$, bh

76. cx, $-dx$

77. $3k$, $-hk$

78. $2ab$, $3b$

79. a^2d, b^2d

80. x^2w, $-y^2w$

81. x, xy

82. bc, b

83. ab, bc

84. xy, yz

85. $6cd$, $5bd$

86. $2ax$, $3bx$

87. $5xy$, $-2xz$

88. $-9ab$, $-4ac$

89. $12abc$, $-5abd$

90. Find the sum of $a(x+y)$ and $b(x+y)$.

91. Find the sum of $3(x^2-y^2)$, $4a(x^2-y^2)$, and $-2b(x^2-y^2)$.

92. $x(c-d)$, $y(c-d)$

93. $a(b+c)$, $(b+c)$

94. $x(y-z)$, $-(y-z)$

95. $2a(m+n)$, $b(m+n)$

96. $2ab(g-k)$, $3cd(g-k)$

97. $a^2(x+y)$, $-b^2(x+y)$

98. $a(x^2-y^2)$, $b(x^2-y^2)$

99. $-(c^2+1)$, $2d(c^2+1)$

SUBTRACTION OF MONOMIALS

MENTAL EXERCISE 10

Perform the following indicated subtractions:

1. $6a$, $4a$

2. $9b$, $6b$

3. $11c$, $3c$

4. $-12d$, $-4d$

5. $-10e$, $-7e$

6. $-13m$, $5m$

7. $14n$, $-2n$

8. $-16q$, $-12q$

9. $5r$, $7r$

10. $8s$, $10s$

11. $-3x$, $-8x$

12. $-4y$, $-6y$

13. $-15z$, z

14. $-17w$, $17w$

15. $-18h$, $-18h$

16. $2a^2$, $-a^2$

17. $5\,b^3$ from $7\,b^3$

18. $4\,d^3$ from $-11\,d^3$

19. $-19\,x^2$ from $-23\,x^2$

20. $21\,y^4$ from $-19\,y^4$

21. $6\,ab$ from $27\,ab$

22. $31\,ac$ from $-26\,ac$

23. $-29\,bd$ from $41\,bd$

24. $33\,mw$ from $-27\,mw$

25. $24\,a^2b$ from $-14\,a^2b$

26. $-37\,cd^2$ from $-18\,cd^2$

27. $42\,m^2n^2$ from $-31\,m^2n^2$

28. $-58\,x^3y^2$ from $46\,x^3y^2$

29. $16\,abc$ from $-73\,abc$

30. $-25\,cdy$ from $71\,cdy$

31. $56\,bgm$ from $-17\,bgm$

32. $-43\,a^2bc$ from $28\,a^2bc$

33. $62\,ab^2c$ from $-47\,ab^2c$

34. $-77\,abc^2$ from $-38\,abc^2$

35. $76\,a^2b^2c$ from $-67\,a^2b^2c$

36. $-39\,ab^2c^2$ from $93\,ab^2c^2$

37. $45\,a^2bc^2$ from $54\,a^2bc^2$

38. $-56\,x^3y^2z$ from $67\,x^3y^2z$

39. $-101\,a^2xy$ from $-75\,a^2xy$

40. $123\,m^3nw$ from $95\,m^3nw$

41. $158\,syw$ from $85\,syw$

42. $-193\,gnq$ from $89\,gnq$

43. $-150\,chw$ from $91\,chw$

44. $-239\,bk^2$ from $167\,bk^2$

45. $2\,a$ from $\frac{1}{3}\,a$

46. $5\,b$ from $\frac{1}{6}\,b$

47. $\frac{1}{2}\,c$ from $\frac{1}{4}\,c$

48. $\frac{3}{5}\,d$ from $\frac{1}{2}\,d$

49. $\frac{1}{3}\,h$ from $\frac{1}{7}\,h$

50. $\frac{2}{3}\,k$ from $-\frac{1}{4}k$

51. $\frac{4}{5}\,m$ from $-\frac{5}{4}\,m$

52. $-\frac{4}{7}\,n$ from $-\frac{5}{8}\,n$

53. $.5\,x$ from $.2\,x$

54. $.7\,y$ from $.3\,y$

55. $.08\,z$ from $.04\,z$

56. $.11\,w$ from $.03\,w$

57. $1.6\,a$ from $.1\,a$

58. $2.28\,b$ from $.02\,b$

59. $.33\,c$ from $.03\,c$

60. $2.51\,d$ from $.25\,d$

61. $.36\,g$ from $3.64\,g$

62. $.345\,x$ from $-.655\,x$

63. $-.583\,y$ from $-5.417\,y$

64. $1.001\,z$ from $-.011\,z$

65. From ax subtract bx.

Perform the following subtractions with reference to similar parts:

66. ac / ac

67. dy / xy

68. aw / mw

69. $-ct$ / $-dt$

70. $-ay$ / xy

71. bw / $-nw$

72. bm^2 / hm^2

73. $-gy^3$ / $-my^3$

74. axy / bxy

75. m^2xw / $-n^2xw$

76. $-bnq$ / $-hnq$

77. $5\,axyz$ / $2\,bxyz$

78. $2\,ab$ / b

79. $-3\,cd$ / $-d$

80. $5\,x^2y$ / $-y$

81. $-7\,xy^2$ / y^2

82. ab / bc

83. xy / xz

84. x^2y^2 / $-y^2z^2$

85. xyz / yzw

86. From $a(x+y)$ subtract $-b(x+y)$.

87. $5(a+b)$ / $2(a+b)$

88. $8(c-a)$ / $-3(c-a)$

89. $-6(m-n)$ / $9(m-n)$

90. $-18(g+h)$ / $-18(g+h)$

91. $x(a+b)$ / $y(a+b)$

92. $c(m-n)$ / $d(m-n)$

93. $h(m-x)$ / $-c(m-x)$

94. $a(c+1)$ / $-b(c+1)$

95. $a(b+1)$ / $(b+1)$

96. $m(n-1)$ / $-(n-1)$

97. $(x-1)$ / $x(x-1)$

98. $2(c+g)$ / $3\,b(c+g)$

99. $xy(z-w)$ / $z(z-w)$

100. $ab(a-bc)$ / $c(a-bc)$

101. $x(xy-z)$ / $yz(xy-z)$

102. $ax(bx-cy)$ / $-by(bx-cy)$

EXERCISE 11

Reduce each of the following polynomials to simplest form:

1. $2\,a-4\,b+6\,c-a+5\,b-2\,c$.

2. $5\,x+7\,y-2\,z-4\,x-6\,y+z$.

3. $11\,a-2\,d+3\,b+4\,d-b+a$.

4. $12\,a-b+d+3\,c+2\,b-2\,c-a+3\,b$.

5. $13\,x-13\,y+14\,z-14\,w-2\,z+2\,y-2\,w+4\,z$.

6. $6\,x+y-10\,z-3\,y-2\,x+9\,z+4\,x-5\,y+8\,z$.

7. $3\,a+5\,b-c+2-2\,a-4\,b+6\,c+7-a$.

8. $5k-4m-8n+9-3k+4m+8n-9-2k$.

9. $4a^2-2ab+10c^2+3ab+a^2+2b^2-3c^2-ab+6b^2$.

10. $9x^2-xy+3-5x^2+8y^2+xy+5-4x^2-8y^2$.

ADDITION AND SUBTRACTION OF POLYNOMIALS

EXERCISE 12

Perform the following indicated additions:

1. $3a+2b$
$a+5b$

2. $6a-9b$
$8a-3b$

3. $7b-11c$
$4b-19c$

4. $6d+g$
$6d-12g$

5. $-10h-15y$
$-3h+4y$

6. $-16m-20q$
$-17m-11q$

7. $15ab-cd$
$-23ab+29cd$

8. $9a^2-14bc$
$-26a^2+33bc$

9. $2a+5b-7c$
$3a-11b+14c$

10. $4b-12c-23d$
$6b+12c-7d$

11. $9x-8y+3z$
$3x-8y+9z$

12. $10a-m+15w$
$-a+19m-28w$

13. $13a^2-17b^2+c^2$
$41a^2-21b^2-33c^2$

14. $11x^2+13xy+21y^2$
$11x^2-31xy-12y^2$

Perform the following indicated subtractions:

15. $5a+2b$
$3a+b$

16. $6c-3d$
$4c-2d$

17. $8m-11m$
$4m+14m$

18. $16x+21y$
$32x-23y$

19. $7a+b$
$2a+b$

20. $9b-3c$
$9b+8c$

21. $5x-7y$
$-5x+7y$

22. $3a-6b$
$3a-6b$

23. $2a+3b-4c$
$a-5b+7c$

24. $x-3y+5z$
$-2x+4y-6z$

25. $8a+6b-4c$
$-3a-5b+7c$

26. $10a-7b+12c$
$3a-7b-10c$

27. $3x-4y+5z$
$5x+4y-3z$

28. $4a^2+9b^2-16$
$5a^2+3b^2-1$

Find the sums of the following groups of expressions:

29. $a+2b+c$; $3a+b+c$; $a+b+4c$.

30. $2x+y+3z$; $x+4y+z$; $5x+6y+8z$.

31. $5b-c+2d$; $3b-2c+4d$; $b-3c+d$.

32. $6x-9y-2z$; $7x-10y-3z$; $4x-5y-8z$.

33. $7a-2b+6x$; $5a+8b-4x$; $a-9b-3x$.

34. $5b+9d-4w$; $8b-7d-3w$; $6b-2d-w$.

35. $g-4h+8k$; $7g-h-2k$; $3g+5h-6k$.

36. $m-3x+7y$; $5m-8x+2y$; $9m-4x+6y$.

37. $2b-5c-3d$; $-9b-c-4d$; $8b+6c+7d$.

38. $6a+3c-2z$; $-4a-7c-8z$; $-a+5c+9z$.

39. $3g-2h+4k-10$; $3h-2k+8$; $-2g-k+1$.

40. $2x^3-y^2+10$; y^2-w^2-9; x^3+w^2.

41. $4n^3+v^2-7$; $-6n^3+4v$; $7n^3-v^2-v+7$.

42. $a^3-4a^2b-2ab^2+2b^3$; $3a^2b+2ab^2$; a^2b-b^3.

43. $9b^2c-6bc^2-7b^2c^2-c^4$; $-8b^2c+6bc^2+7b^2c^3$.

44. $4xy-2xz+10xyz$; $-6xy-9xyz+2yz$; $3xy+4xz-xyz-3yz$.

45. $7k^3+8k-11$; $-4k^2+12$; $-5k^3-6k-2$; $-k^3+5k^2-k+2$.

EXERCISE 13

Perform the following indicated subtractions, reversing the signs mentally; the answers to the first sixteen examples may be obtained mentally:

1. $2x+y$ / $x-y$

2. $3x+4y$ / $2x-5y$

3. $4a-7b$ / $5a-6b$

4. $7y+w$ / $7y-2w$

5. $6h-k$ / $3h-k$

6. $2x+a$ / $-3x-2a$

7. $-2v-3$ / $-3v-4$

8. $2+x$ / $3-2x$

9. $-10a-b$ / $-11a+b$

10. $-xy-w$ / $2xy+2w$

11. $4a-5b$ / $-3a-6b$

12. $ab-cd$ / $-ab+cd$

13. $-xz+ad$ / $2xz-2ad$

14. $3x+y+z$ / $2x+2y-z$

15. $-4a-5b+6c$ / $5a+6b+7c$

16. $8x-7y+6z$ / $-6x+7y-8z$

17. From $6h - 10k - 7$ subtract $4h - 5k - 2$.

18. From $5h + 8k - 9$ subtract $2h + 3k - 4$.

19. Subtract $x^2 + 2xy + y^2$ from $2x^2 - 2xy + 2y^2$.

20. From $3a^2 + 4bc + 8$ subtract $4a^2 - 2bc - 3$.

21. Subtract $a^2 + 3ab + b^2$ from $4a^2 + 10ab + 13b^2$.

22. Subtract $b^3 + b^2c + bc^2 + c^3$ from $b^3 + c^3$.

23. From $x^3 + 3x^2y + 3xy^2 + y^3$ subtract $x^3 + y^3$.

24. Subtract $h^3 - 3h^2k + 3hk^2 - k^3$ from $h^3 - k^3$.

25. From $x^3 + x^2 + x$ subtract $-2x^3 - 3x^2 + x$.

26. From $xy + yz + zx$ subtract $-xy + yz - zx$.

27. Subtract $2m^2 - mn - 5n^2$ from $3m^2 + mn - 4n^2$.

28. Subtract $6a^2 - 7ab - 8b^2$ from $7a^2 - 8ab - 7b^2$.

29. Subtract $5b^2c^2 - 10bc^3 + 16c^4$ from $5b^2c^2 + 10bc^3 + 16c^4$.

30. Subtract $m^3 + 3m^2n + 3mn^2 + n^3$ from $m^3 - 3m^2n + 3mn^2 - n^3$.

31. From $h^4 + 5h^3k + 6h^2k^2 + 3hk^3 + k^4$ subtract $5h^3k + 6h^2k^2 + 3hk^3$.

32. Subtract $a^3 - a^2b + b^3$ from $3a^3 - a^2b + 4ab^2 + 5b^3$.

33. From $5x^5 + 6x^3y^2 + 3x^2y + 7y^5$ subtract $3x^5 + 6x^3y^2 - 5y^5$.

34. Subtract $4m^4 + 7m^3n + 6m^2n^2 + n^4$ from $4m^4 + 10m^3n - 15m^2n^2 + 9n^4$.

35. From $12x^3 + 19x^2y + 17xy^2 + 21y^3$ subtract $5x^3 + 17x^2y - xy^2 - 20y^3$.

36. From $13g^5 - 23g^4h + 11g^3h^2 + 14h^5$ subtract $13g^5 - 24g^4h + 10g^3h^2 + 14h^5$.

37. From the sum of $x + 2y + 3z$ and $3x - y + 4z$, subtract $2x - y - 5z$.

38. From the sum of $5a - b + 4c$ and $4a - 2b - 3c$, subtract $3a + 3b - 5c$.

39. Subtract $3g + 2h + 2k - 3r$ from the sum of $g - h + k - r$ and $2g - h + 2k - r$.

40. Subtract $7x - 3y - z - w$ from the sum of $6x - y + 4z + w$ and $3x + 4y - 2z + w$.

41. From the sum of $m^2 + mn + n^2$ and $3m^2 - 2mn - 4n^2$, subtract $2m^2 - 3mn - 3n^2$.

42. Subtract $xy + 3yz - 4zw$ from the sum of $6xy - 4yz + zw$ and $-4xy + 2yz - 3zw$.

43. From the sum of $ax+xy$, $xy-yz$, and $za-yz$, subtract $ax+xy+yz+za$.

44. From the sum of $3ab+ax$, $-4ax+xy$ and $2by-2ax$, subtract $2ab-2ax+xy+by$.

PARENTHESES

EXERCISE 14

Remove the following parentheses and simplify:

1. $a-\{7a-(3y-2x)+(x-2y)\}$.

2. $[2x-(3y-x)]-[-(4y-x)-2x]$.

3. $3a-[a-(2a-3b)+3a]-(5b-4)$.

4. $2x+[3x-(4x-3y)]+(4y-3x)$.

5. $3x-2y-[\{+2z-8x-(y+5z)-7x\}+z]$.

6. $5x-[4x-2y-\{4x-2y-(2x-\overline{y-2x}+x\}-4y]$.

7. $x-[x-\{x-(x-\overline{x+x})\}]$.

8. $x-y-[x+y-\{x-y+(x+y-\overline{x-y})\}]$.

MULTIPLICATION

PRODUCTS OF POWERS OF EQUAL BASES

EXERCISE 15

Express each of the following products of different powers of the same base as a single power of the given base.

1. $2^4\times 2^3$.

2. $4^2\times 4$.

3. $3^5\times 3$.

4. $5^4\times 5$.

5. $2^5\times 2^2$.

6. $a^2\times a^4$.

7. $b^3\times b^7$.

8. $c^4\times c^6$.

9. $d^5\times d^2$.

10. $e^7\times e^4$.

11. $g^8\times g^3$.

12. $h^6\times h^9$.

13. $k^5\times k^8$.

14. $m^7\times m^{10}$.

15. $n^9\times n^8$.

16. $x^8\times x^7$.

17. $a^2\times a^3\times a^4$.

18. $b^3\times b^5\times b^6$.

19. $c^5\times c^4\times c^7$.

20. $d^8\times d^{10}\times d^4$.

21. $h^6\times h^7\times h$.

22. $m^2\times m^8\times m^{10}$.

23. $n^6\times n^8\times n^3$.

24. $x^8\times x^9\times x^{10}$.

25. $a^{2m}\times a^{3m}$.

26. $b^{4n}\times b^{2n}$.

27. $c^{5r}\times c^r$.

28. $d^{n-2}\times d^2$.

29. $h^{n+5}\times h^{n-5}$.

30. $x^{a+b+1}\times x^{a-b-1}$.

31. $x^a\times x^b\times x^c$.

32. $y^{2n}\times y^{3n}\times y^{4n}$.

33. $z^{2r}\times z^{3s}\times z^{5t}$.

Express each of the following powers of powers as a single power of the given base:

34. $(2^2)^3$. **40.** $(a^2)^2$. **46.** $(g^4)^5$. **52.** $(q^9)^7$. **58.** $(d^4)^n$.
35. $(3^3)^2$. **41.** $(b^2)^3$. **47.** $(h^2)^6$. **53.** $(r^8)^{10}$. **59.** $(h^m)^m$.
36. $(5^2)^2$. **42.** $(c^3)^2$. **48.** $(k^6)^2$. **54.** $(s^{12})^{12}$. **60.** $(k^n)^n$.
37. $(6^2)^2$. **43.** $(d^2)^4$. **49.** $(m^5)^5$. **55.** $(a^2)^m$. **61.** $(m^{2x})^x$.
38. $(10^2)^3$. **44.** $(e^3)^4$. **50.** $(n^6)^7$. **56.** $(b^n)^2$. **62.** $(n^{3y})^{2y}$.
39. $(7^2)^2$. **45.** $(g^2)^5$. **51.** $(p^8)^9$. **57.** $(c^n)^3$. **63.** $(x^{4a})^{4b}$.

Express each of the following powers of products as products of powers:

64. $(3a)^2$. **73.** $(3bc)^4$. **82.** $(x^2y^3)^7$. **91.** $(bc)^d$.
65. $(4b)^3$. **74.** $(2xy)^6$. **83.** $(ab^2c)^2$. **92.** $(a^2b)^n$.
66. $(6c)^2$. **75.** $(-3yz)^3$. **84.** $(a^2bc)^2$. **93.** $(cd^3)^r$.
67. $-(2d)^4$. **76.** $-(5yw)^3$. **85.** $-(ab^2c^3)^4$. **94.** $-(xy^2)^n$.
68. $(3m)^4$. **77.** $-(-3ac)^3$. **86.** $(a^5b^4c^3)^2$. **95.** $(a^mb)^m$.
69. $(-7x)^2$. **78.** $(a^2b)^3$. **87.** $(-2x^3y^4z^6)^2$. **96.** $(x^ay^b)c$.
70. $(-4y)^3$. **79.** $(bc^2)^4$. **88.** $-(7x^5y^7z^2)^2$. **97.** $-(a^xb^y)^c$.
71. $(-2z)^5$. **80.** $(c^2d^2)^5$. **89.** $-(-4x^4)^4$. **98.** $(a^{2x}b^y)^{2z}$.
72. $-2ab^5)$. **81.** $(m^nn)^6$. **90.** $(ab)^c$. **99.** $(2x^{3y})^{4z}$.

Express the following products of powers as powers of products:

100. $2^3 \times z^3$. **105.** $b^3 \times c^3$. **110.** $25x^2y^2z^2$.
101. $5^2 \times 2^2$. **106.** $m^4 \times n^4$. **111.** $32a^5b^5x^5$.
102. $3^2 \times 7^2$. **107.** $a^2 \times b^2 \times c^2$. **112.** $64a^6x^6z^6$.
103. $4^2 \times 3^2$. **108.** $x^3 \times y^3 \times z^3$. **113.** $(-a)^4b^4$.
104. $a^2 \times b^2$. **109.** $8a^3b^3$. **114.** $(-x)^3y^3z^3$.

PRODUCTS OF TWO OR MORE MONOMIALS

EXERCISE 16

Perform the following indicated multiplications:

1. $a \times 2$ **3.** $-c \times 4$ **5.** $2n \times 3$ **7.** $-7y \times 5$ **9.** $-11w \times -8$
2. $3 \times b$ **4.** $d \times -5$ **6.** $4x \times 6$ **8.** $9z \times -6$ **10.** $c \times d^2$

11. $-h$ / $-x^2$

12. ab / $-c$

13. $-xz$ / y

14. $-bc$ / $-a$

15. ab / cd

16. ay / bx

17. mw / nx

18. a^2b / c^2

19. xy^2 / z

20. b^3y / $-n^2$

21. $2a^2$ / a

22. $3b^4$ / b^2

23. $5c^5$ / $-c^6$

24. $-7d^5$ / $-d^7$

25. $3h^2$ / $2h$

26. $5m^3$ / $6m$

27. $8c^4$ / $3c^2$

28. $9d^3$ / $8d^4$

29. $10g^5$ / $-4g^3$

30. $11h^4$ / $-7h^6$

31. $13k^5$ / $3k^2$

32. $7ab$ / $4a$

33. $14bc$ / $5bc$

34. $4ab$ / $11bc$

35. $-6xy$ / $8xz$

36. $12mn$ / $12mx$

37. $15a^2b$ / $-4bc$

38. $11bc^2$ / $9bd$

39. $16m^2n$ / $-6mn^2$

40. $17a^3b$ / $11ab^2$

41. $-7x^2yz$ / $13yz^2$

42. $19a^2bc$ / $-5abc^2$

43. $18x^3y^2z$ / $6xy^2z^3$

44. $12a^2b^4c^6$ / $11a^5b^3c$

45. a^mb^n / ab

46. x^ay^b / xy

47. x^ny^r / $-xy$

48. $-a^{2n}x^r$ / $-ay$

49. $a^{3n}b^{5x}$ / $a^{2n}b^x$

50. $32x^my^{2m}z^{3m}$ / $-3xyz$

51. $14a^{2m+1}b^{n-1}$ / $-5a^3b^3$

52. $a^{n-1}b^nc^{n+1}$ / $a^{n+1}b^nc^{n-1}$

53. $ab \times bc \times ca$.

54. $xy \times xz \times xw$.

55. $a^2b \times b^2c \times c^2a$.

56. $abc \times ab \times a$.

57. $a^2b^3 \times b^2c^4 \times c^2d^5$.

58. $-a^3x \times b^3y \times c^3x^2y^2$.

59. $a^2bc \times ab^3c \times abc^4$.

60. $x^2yz^3 \times xy^4z^2 \times y^8z^5w^6$.

61. $(8a^2b)(2ac^4)(6b^3c)$.

62. $-(9m^3x)(3m^2x)(2mnx)$.

63. $(3a^3b^3)(4b^2c^3)(-5c^2a^3)$.

64. $(7x^2z^3)(-3x^5y^4)(5y^3z^2)$.

65. $-(6a^3bc)(2a^2c^2)(3b^4c^5)$.

66. $(-8x^6y^2z)(-3xy^7)(4x^3z^8)$.

67. $(-2a^3bc)(-4ab^2d)(-3bc^2d^3)$.

68. $(-5x^4yz^5)(2x^3z^2w)(-6yzw^2)$.

69. $(-7a^2b^2x^3)(2a^3b^2y^2)(-6b^2x^3y^2)$.

70. $(-9a^2b^3c^4)(-5a^3b^4d^5)(-2b^4c^5d^6)$.

POLYNOMIAL MULTIPLIED BY A MONOMIAL

EXERCISE 17

Perform the following indicated multiplications:

1. $a+b$ by 2
2. $b-c$ by 3
3. $x+y$ by z
4. $y-z$ by w
5. $2m+3n$ by 4
6. $6h-8k$ by 5
7. $2x+3y$ by -4
8. $7m-n$ by -8
9. $4a+b$ by $3c$
10. $d-5h$ by $4x$
11. $6m-5n$ by $-2y$
12. $ab+cd$ by xy
13. $ab+ac$ by bc
14. $xy-xz$ by xy
15. $ab-bc$ by ca
16. a^2-b^2 by ab
17. x^2-y^2 by $-xy$
18. $2mn+3bc$ by $4ax$
19. $7ax+5by$ by $3mn$
20. $8cd+9bh$ by $-2kr$
21. $4ax^2+3by^2$ by $2cd$
22. $3a^2b+5ab^2$ by $4ab$
23. $6xy^2-7y^2z$ by $-2xyz$
24. $4ax^2+3by^2$ by $-3abh$
25. $6xyz^2+5x^2yz$ by $4xyz$
26. $9a^2b^2c+12ab^2c^2$ by $3abc$
27. $11x^4y^2z^5-6x^3y^6$ by $7x^7z^8$
28. a^3+a^2+a+1 by $2a$
29. b^3-b^2+b-1 by $4b^2$
30. $a^2-2ab+b^2$ by $3ab$
31. $m^2-2mn+n^2$ by $-2mn$
32. $ab+bc+ca$ by abc
33. $abc+abd+bcd$ by $abcd$
34. $(a+b+c)abc$.
35. $(x^2+y^2+z^2+xyz)xyz$.
36. $(1+a+a^2+a^3)a^4$.
37. $(x^4-x^3+x^2-x+1)x^2$.
38. $(a^3+a^2b+ab^2+b^3)a^4$.
39. $(x^3-x^2y+xy^2-y^3)xy$.
40. $(a^3+b^2+c)ab^2c^3$.
41. $(xyz-yzw-zwx)xyzw$.
42. $(x^4+x^3y+x^2z+xw)xyzw$.
43. $(2a^2y-3b^2z-4c^2w)5abc$.

44. $(1 - a^4b^2 + a^3b^3c - a^2b^4cd)\,abcd.$

45. $(4\,a^2x - 3\,b^2y + 2\,x^2y^2)\,5\,cxy.$

46. $(8\,a^2b^3 + 13\,b^2c^3 + 6\,c^2d^3)\,(-4\,abcd).$

47. $(a^n + a^{n+1} + a^{n+2} + a^{n+3})\,a.$ **48.** $(b^{4n} + b^{3n} + b^{2n} + b^n)\,b.$

49. $(c^{n-3} + c^{n-2} + c^{n-1} + c^n)\,c.$ **50.** $(d^{n-1} + d^n + d^{n+1} + d^{n+2})\,d^n.$

51. $(n^{2r-3} + n^{2r-1} + n^{2r+1} + n^{2r+3})\,n^2.$

52. $(m^{4r-3} - m^{3r-4} + m^{2r-1} - m^{r-2})\,m^5.$

53. $(a^{n-2} + a^{n-1}b^n + a^nb^{n+1} + b^{n+2})\,ab.$

54. $(x^{n-5} - x^{n-3}y^{n-1} + x^{n-1}y^{n-3} - y^{n-5})\,x^4y^4.$

55. $(a^{2x+2} + a^{2x+1}b^{2x} - a^{2x}b^{2x+1} - b^{2x+2})\,a^xb^x.$

56. $(m^{a+1} - m^{a+2}n^2 + m^{a+3}n^3 - n^4)\,m^{a-1}n^{a+1}.$

POLYNOMIAL MULTIPLIED BY A POLYNOMIAL

EXERCISE 18

Perform the following indicated multiplications checking all results numerically:

1. $a + b$
$x + y$

2. $b + c$
$b + d$

3. $c - d$
$d - e$

4. $f + g$
$f - g$

5. $k^2 + k$
$k + 1$

6. $r^2 - r^4$
$r^3 + r^5$

7. $2\,m + 3\,n$
$4\,m + 5\,n$

8. $6\,t - 10\,n$
$7\,t + 3\,n$

9. $a + b + c$
$a + b$

10. $x + y + z$
$x + z$

11. $y - z + w$
$z - w$

12. $2\,a - 2\,b - 3\,c$
$3\,a + 2\,b$

13. $5\,x - 4\,y - 32$
$2\,x + y$

14. $6\,m - 8\,n + 10\,q$
$n - 3\,q$

15. $5\,h - 7\,k + 9\,w$
$2\,h - 4\,w$

16. $3\,a^2 + 4\,ab + 5\,b^2$
$2\,a - 3\,b$

17. $a + b + c$
$a + b + c$

18. $b - c + d$
$b - c + d$

19. $a + x + y$
$a - x - y$

20. $x - y + 2$
$x + y - 2$

21. $a + b + 5$
$a - b + 4$

22. $\begin{array}{r} a^2+a+1 \\ \underline{a-1} \end{array}$

23. $\begin{array}{r} a^2+2\,ab+b^2 \\ \underline{a+b} \end{array}$

24. $(a^2-ab+b^2)(a+b)$.

25. $(2\,x^2+5\,xy+7\,y^2)(2\,x-3\,y)$.

26. $(2\,c^2+3\,cd+4\,d^2)(6\,c-5\,d)$.

27. $(w^4+w^2+1)(w^4-w^2+1)$.

28. $(a^2+2\,a-3)(a^2+a-6)$.

29. $(5\,r^2+4\,r-2)(3\,r^2-2\,r-5)$.

30. $(a^2-3\,ab-b^2)(a^2+3\,ab+b^2)$.

31. $(g^3+g^2y^2+y^3)(g^3-g^2y^2+y^3)$.

32. $(k^2-ky+y^2)(2\,k^2-3\,ky+y^2)$.

33. $(h^3+h^2y+hy^2+y^3)(h-y)$.

34. $(d^3-4\,d^2+3\,d+1)(d^2-2\,d+5)$.

35. $(y^3-5\,y^2+1)(2\,y^3+5\,y+1)$.

36. $(s^6-3\,s^4+2\,s^2-1)(s^3-s+1)$.

37. $(6\,k^5+4\,k^3+2\,k+1)(k^4-k^2-1)$.

38. $(a^2-ab+b^2+a+b+1)(a+b-1)$.

39. $(1-2\,m+2\,m^2-m^3)(1+2\,m+2\,m^2+m^3)$.

Multiply

40. $x^2+y^2+z^2-xy-zx-zy$ by $x+y+z$.

41. $x^5+x^4y+x^3y^2+x^2y^3+xy^4+y^5$ by $x-y$.

42. $x^8-x^6y^3+x^4y^6-x^2y^9+y^{12}$ by x^2+y^3.

43. $a^2b-b^2c^3-c^3d^2+d^2a$ by $a^2bc^2-ab^2c$.

44. $5\,a^3b^3-4\,b^3c^3+5\,c^3a^3$ by $4\,ab-5\,bc+4\,ca$.

45. $a^2bc-ab^2c+abc^2$ by $ab^2c^2-a^2bc^2+a^2b^2c$.

46. $ab^2c^3-ab^3c^2+a^2bc^3$ by $a^2b^3c-a^3bc^2+a^3b^2c$.

47. $\frac{1}{2}\,a^2+\frac{1}{3}\,a+\frac{1}{4}$ by $\frac{1}{3}\,a+\frac{1}{4}$.

48. $\frac{1}{4}\,x^2-\frac{1}{3}\,x+\frac{1}{2}$ by $\frac{1}{3}\,x-\frac{1}{2}$.

49. $\frac{1}{2}\,a^2-\frac{1}{3}\,ab+\frac{1}{4}\,b^2$ by $\frac{1}{2}\,a+\frac{1}{3}\,b$.

50. $\frac{2}{3}\,m^2+mn+\frac{3}{2}\,n^2$ by $\frac{3}{2}\,m+\frac{2}{3}\,n$.

51. $\dfrac{a^3}{2}-\dfrac{5\,a^2}{6}+\dfrac{3\,a}{2}-1$ by $6\,a-\dfrac{1}{15}$.

52. $\frac{1}{2}-\frac{2}{3}\,a+\frac{3}{4}\,a^2-\frac{4}{5}\,a^3$ by $2-5\,a$.

EXERCISE 19

Perform the following multiplications:

1. $(3\,x^{n+4}-4\,x^{n+3}+6\,x^{n+2})(2\,x^{n+1}-3\,x^n)$.

2. $(2\,x^{n+2}+5\,x^{n+1}+3\,x^n+2)(3\,x-1)$.

3. $(4\,a^{4n}+2\,a^{3n}+5\,a^{2n}+6\,a^n)(2\,a+3)$.

4. $(a^{n-3}b^3+a^{n-2}b^2+a^{n-1}b)(a^3b^n-a^nb^3)$.

5. $(a^{2n} - a^n + 1)(a^{2n} + a^n - 1)$.
6. $(a^{n-1}b^{n+1} + a^{n+1}b^{n-1})(a^{n-1}b^{n+1} - a^{n+1}b^{n-1})$.
7. $(a^{2x+1} + a^{2x} + a^{2x-1})(a^{2x-1} - a^{2x} + a^{2x+1})$.
8. $(x^{m+n} + x^m + 1)(x^{m-n} - x^m + 1)$.
9. $(x^{n+2} - x^{n+1}y + x^n y^{n+1} - y^{n+2})(x - y)$.
10. $(x^{3n} - 3x^{2n} + 3x^n - 1)(x^{3n} + 3x^{2n} - 3x^n + 1)$.

EXERCISE 20

Simplify each of the following expressions:

1. $1 + 2\{1 + 3[1 + 4(1 + 5x)]\}$.
2. $x - \{(y - z) - [x + y - z - 2(x - y + z)]\}$.
3. $2 + 2\{2 - 2[2 + 2(2 - 2x)]\}$.
4. $(x + 1) - 2\{(x + 2) + 3[(x + 3) - 4(x + 4)]\}$.
5. $a - \{b + c[d - e(f + g)]\}$.
6. $5\{4[3(2 + a)]\} - 5\{-4[-3(2 - a)]\}$.
7. $7\{b - 4[b - 4(b + y)]\} - 6\{b - 4[b - 2(b - y)]\}$.
8. $a\{1 + b[1 + c(1 + d)]\} - d\{1 + c[1 + b(1 + a)]\}$.
9. $a\{b - c[a - b(a + b + c) - (b + a)]\} - c - b\}$.
10. $b^2 - b\{b + c[a(b - c) + b(c - a) + c(a - b)]\}$.

SPECIAL PRODUCTS

SQUARE OF A BINOMIAL SUM

EXERCISE 21

Expand each of the following:

1. $(a + 3)^2$.
2. $(b + 4)^2$.
3. $(c + 6)^2$.
4. $(c + 7)^2$.
5. $(5 + d)^2$.
6. $(8 + h)^2$.
7. $(k + 9)^2$.
8. $(m + 10)^2$.
9. $(11 + n)^2$.
10. $(x + 12)^2$.
11. $(13 + y)^2$.
12. $(z + 14)^2$.
13. $(15 + w)^2$.
14. $(2a + 1)^2$.
15. $(3b + 1)^2$.
16. $(4c + 1)^2$.
17. $(5d + 1)^2$.
18. $(4e + 3)^2$.
19. $(5h + 4)^2$.
20. $(3k + 5)^2$.
21. $(6g + 7)^2$.
22. $(5 + 8h)^2$.
23. $(9 + 6k)^2$.
24. $(7 + 9m)^2$.
25. $(10x + 3)^2$.
26. $(4a + 5b)^2$.
27. $(3c + 7d)^2$.
28. $(6h + 10k)^2$.
29. $(5m + 13n)^2$.
30. $(2r + 17y)^2$.
31. $(16s + 3w)^2$.
32. $(12x + 5y)^2$.
33. $(7t + 20z)^2$.
34. $(18k + 10n)^2$.
35. $(19a + 2c)^2$.
36. $(3b + 18d)^2$.
37. $(20c + 5g)^2$.
38. $(10b + 16k)^2$.
39. $(14d + 5n)^2$.
40. $(12h + 11r)^2$.
41. $(21s + 2v)^2$.
42. $(8ab + 5)^2$.

43. $(9xy + 10)^2$.
44. $(11a + 4bc)^2$.
45. $(7x + 4yz)^2$.
46. $(6ab + 5cd)^2$.
47. $(10xw + 9yz)^2$.
48. $(15ac + 5bd)^2$.

SQUARE OF A BINOMIAL DIFFERENCE

EXERCISE 22

Expand each of the following:

1. $(a - 3)^2$.
2. $(b - 7)^2$.
3. $(c - 8)^2$.
4. $(4 - d)^2$.
5. $(5 - e)^2$.
6. $(6 - f)^2$.
7. $(g - 9)^2$.
8. $(10 - h)^2$.
9. $(h - 11)^2$.
10. $(12 - m)^2$.
11. $(n - 14)^2$.
12. $(16 - r)^2$.
13. $(s - 17)^2$.
14. $(18 - t)^2$.
15. $(x - 19)^2$.
16. $(2a - 1)^2$.
17. $(7b - 1)^2$.
18. $(9c - 1)^2$.
19. $(1 - 4d)^2$.
20. $(1 - 5e)^2$.
21. $(8g - 1)^2$.
22. $(3k - 5)^2$.
23. $(4h - 7)^2$.
24. $(6v - 5)^2$.
25. $(7w - 9)^2$.
26. $(8 - 6s)^2$.
27. $(11 - 4t)^2$.
28. $(13r - 3)^2$.
29. $(2s - 15)^2$.
30. $(4q - 8)^2$.
31. $(9a - 3b)^2$.
32. $(6c - 7d)^2$.
33. $(5h - 12r)^2$.
34. $(9p - 7q)^2$.
35. $(8x - 7z)^2$.
36. $(14t - 4y)^2$.
37. $(10q - 8z)^2$.
38. $(6r - 11y)^2$.
39. $(15m - 5s)^2$.
40. $(12n - 20y)^2$.
41. $(17g - 5r)^2$.
42. $(2ab - 18)^2$.
43. $(5 - 4cq)^2$.
44. $(5ab - 10cd)^2$.
45. $(9xw - 4yz)^2$.
46. $(3mx - 8ny)^2$.
47. $(2ay - 9bx)^2$.
48. $(7cd - 6hk)^2$.

MULTIPLICATION OF THE SUM OF TWO TERMS BY THEIR DIFFERENCE

EXERCISE 23

Obtain the following products:

1. $(x + y)(x - y)$.
2. $(c + k)(c - k)$.
3. $(r + w)(r - w)$.
4. $(m + q)(m - q)$.
5. $(a + z)(a - z)$.
6. $(x + 3)(x - 3)$.
7. $(y + 4)(y - 4)$.
8. $(5 + z)(5 - z)$.
9. $(6 + w)(6 - w)$.
10. $(k + 9)(k - 9)$.
11. $(11 + g)(11 - g)$.
12. $(h + 13)(h - 13)$.
13. $(14 + k)(14 - k)$.
14. $(m + 17)(m - 17)$.

15. $(16+n)(16-n)$.
16. $(2a+1)(2a-1)$.
17. $(3b+1)(3b-1)$.
18. $(c+2d)(c-2d)$.
19. $(c+7d)(c-7d)$.
20. $(a+5b)(a-5b)$.
21. $(a+8c)(a-8c)$.
22. $(4a+3b)(4a-3b)$.
23. $(5c+7q)(5c-7q)$.
24. $(8k+3w)(8k-3w)$.
25. $(9n+11z)(9n-11z)$.
26. $(12m+8r)(12m-8r)$.
27. $(7g+15t)(7g-15t)$.
28. $(13n+9s)(13n-9s)$.
29. $(15q+12z)(15q-12z)$.
30. $(17d+6v)(17d-6v)$.
31. $(18n+3s)(18n-3s)$.
32. $(10p+16t)(10p-16t)$.
33. $(9q+20r)(9q-20r)$.
34. $(19a+5h)(19a-5h)$.
35. $(6c+12g)(6c-12g)$.
36. $(4s+14z)(4s-14z)$.
37. $(3ab+5)(3ab-5)$.
38. $(2cd+9)(2cd-9)$.
39. $(7+6my)(7-6my)$.
40. $(8+9nz)(8-9nz)$.
41. $(4ab+5c)(4ab-5c)$.
42. $(10x+7yz)(10x-7yz)$.
43. $(11bk+9w)(11bk-9w)$.
44. $(8gx+10hy)(8gx-10hy)$.
45. $(12am+13bn)(12am-13bn)$.
46. $(7abc+8d)(7abc-8d)$.
47. $(14x+11yzw)(14x-11yzw)$.
48. $(15ab+10cd)(15ab-10cd)$.
49. $(16bc+15mn)(16bc-15mn)$.
50. $(17mn+18pq)(17mn-18pq)$.

SQUARE OF A POLYNOMIAL

EXERCISE 24

Expand each of the following:

1. $(a+2b+3c)^2$.
2. $(4c+d+5e)^2$.
3. $(6g+3n+w)^2$.
4. $(4x+7y+2z)^2$.
5. $(5a-b+2c)^2$.
6. $(4x+3y-z)^2$.
7. $(6a-2b-3c)^2$.
8. $(9a+2b+1)^2$.
9. $(7x-3y+2)^2$.
10. $(a+b+c+1)^2$.
11. $(x+y-z-1)^2$.
12. $(a-b+c-2)^2$.
13. $(3x-y+2z-w)^2$.
14. $(a-2b+c-5d)^2$.
15. $(7-a-b-c)^2$.
16. $(2a+3b+4c+5d)^2$.
17. $(a-b-c-d-e)^2$.
18. $(a-2b+c-3d+e)^2$.
19. $(4x+2y-z+3w-5)^2$.
20. $(x-2y+3z-2w+1)^2$.

PRODUCT OF TWO BINOMIALS IN WHICH THE FIRST TERMS ARE EQUAL AND THE SECOND TERMS ARE UNEQUAL

EXERCISE 25

Obtain each of the following indicated products:

1. $(a+2)(a+3)$.
2. $(b+4)(b+6)$.
3. $(c+5)(c+8)$.
4. $(d+6)(d+1)$.
5. $(h+7)(h+4)$.
6. $(k+3)(k+7)$.
7. $(g+8)(g+9)$.
8. $(m+10)(m+7)$.
9. $(x-9)(x-6)$.
10. $(y-7)(y-8)$.
11. $(z-12)(z-5)$.
12. $(w-10)(w-13)$.
13. $(x-1)(x-17)$.
14. $(y-2)(y-19)$.
15. $(r-3)(r-14)$.
16. $(s-7)(s-11)$.
17. $(a+9)(a-5)$.
18. $(b+13)(b-2)$.
19. $(c+14)(c-5)$.
20. $(d-15)(d+6)$.
21. $(r+12)(r-6)$.
22. $(h+17)(h-2)$.
23. $(g-19)(g+2)$.
24. $(q-1)(q+32)$.
25. $(v-23)(v+1)$.
26. $(t+18)(t-3)$.
27. $(w-5)(w+14)$.
28. $(z+12)(z-10)$.
29. $(h-15)(h+10)$.
30. $(k-13)(k+20)$.
31. $(y-20)(y-7)$.
32. $(g+10)(g+19)$.
33. $(s-17)(s-3)$.
34. $(ab+2)(ab+5)$.
35. $(cd+3)(cd+2)$.
36. $(mn+6)(mn-5)$.
37. $(kv-7)(kv-6)$.
38. $(abc+4)(abc-3)$.
39. $(xyz-3)(xyz-5)$.
40. $(abx+4)(abx-7)$.
41. $(mnv-10)(mnv+9)$.
42. $(2a+1)(2a+3)$.
43. $(2b+5)(2b+4)$.
44. $(3c-4)(3c-8)$.
45. $(2m-5)(2m+7)$.
46. $(3n-1)(3n+4)$.
47. $(4n+1)(4n-5)$.
48. $(5x-2)(5x-3)$.
49. $(6x-5)(6x+7)$.
50. $(7x+3)(7x-4)$.

PRODUCT OF TWO BINOMIALS OF THE FORMS $ax+b$ AND $cx+d$

EXERCISE 26

Obtain each of the following indicated products:

1. $(3a+2)(a+1)$.
2. $(2b+1)(3b+4)$.
3. $(4c+1)(2c+3)$.
4. $(5d+7)(d+1)$.

5. $(g+2)(6g+5)$.
6. $(7h+3)(2h+1)$.
7. $(8k+5)(k-1)$.
8. $(9m-2)(m+1)$.
9. $(6n-1)(4n+3)$.
10. $(5r-4)(2r-3)$.
11. $(7s-9)(s-4)$.
12. $(11x-8)(3x+2)$.
13. $(10y-7)(5y-4)$.
14. $(12z+5)(7z-3)$.
15. $(13w+6)(5w-2)$.
16. $(8s-15)(3s+5)$.
17. $(9t+11)(4t-5)$.
18. $(15v+14)(5v-4)$.
19. $(6k+17)(2k-5)$.
20. $(10a+19)(2a+1)$.
21. $(16b-9)(5b+3)$.
22. $(14d+11)(4d+3)$.
23. $(20h+17)(5h-4)$.
24. $(13a-18)(2a-3)$.
25. $(7q+12)(4q-7)$.
26. $(6g+7)(11g-10)$.
27. $(12c-13)(5c+7)$.
28. $(8w+9)(7w-8)$.
29. $(19k+7)(3k-1)$.
30. $(17c-4)(3c-10)$.
31. $(13b+9)(5b-4)$.
32. $(18h+7)(8h-3)$.
33. $(15g-11)(6g+5)$.
34. $(16m+9)(5m+3)$.
35. $(14n+13)(7n-6)$.
36. $(5r+3s)(4r+7s)$.
37. $(9p-4q)(5p-2q)$.
38. $(12b+7d)(8b-5d)$.
39. $(15c-13g)(4c+3g)$.
40. $(14h+17q)(3h-4q)$.
41. $(16m-9w)(8m+5w)$.
42. $(13r+9z)(7r-4z)$.

EXERCISE 27

Obtain each of the following products:

1. $(m+n+q)(m+n-q)$.
2. $(a+c+4)(a+c-4)$.
3. $(b+d+2)(b+d-2)$.
4. $(a+b+1)(a-b+1)$.
5. $(2+c+d)(2-c+d)$.
6. $(a+b+c)(a-b-c)$.
7. $(x+y+z)(x-y-z)$.
8. $(m+n+1)(m-n-1)$.
9. $(b+c+3)(b-c-3)$.
10. $(a+b-c)(a-b+c)$.
11. $(x+y-z)(x-y+z)$.
12. $(m+n-5)(m-n+5)$.
13. $(a+b+3)(a+b+2)$.
14. $(x-y+7)(x-y-4)$.
15. $(m+n+2)(m+n+1)$.
16. $(x+y+4)(x+y+5)$.
17. $(a+b+6)(a+b+3)$.
18. $(b+c+7)(b+c+2)$.
19. $(x-y+4)(x-y+1)$.
20. $(y-z+5)(y-z+3)$.
21. $(a-b+6)(a-b-2)$.
22. $(b-c+9)(b-c-5)$.
23. $(c-x-3)(c-x-4)$.
24. $(d-y-7)(d-y-2)$.
25. $(m-q-10)(m-q-6)$.
26. $(m-w-11)(m-w-7)$.
27. $(a+b+c+d)(a+b-c-d)$.
28. $(x+y+z+w)(x+y-z-w)$.

29. $(a+b+c+1)(a+b-c-1)$.
30. $(x+y+z+3)(x+y-z-3)$.
31. $(a-b+c-d)(a-b-c+d)$.
32. $(x-y+z-w)(x-y-z+w)$.
33. $(a-b+c-2)(a-b-c+2)$.

POWERS OF A BINOMIAL

EXERCISE 28

Obtain the indicated powers of the following binomials:

1. $(a+b)^3$.
2. $(c+1)^3$.
3. $(1+d)^3$.
4. $(x-y)^3$.
5. $(x-1)^3$.
6. $(1-x)^3$.
7. $(m+n)^4$.
8. $(m+1)^4$.
9. $(1+n)^4$.
10. $(g-h)^4$.
11. $(g-1)^4$.
12. $(1-g)^4$.
13. $(a+b)^5$.
14. $(a+1)^5$.
15. $(1+b)^5$.
16. $(b-c)^5$.
17. $(b-1)^5$.
18. $(1-b)^5$.

Find the first three terms in the expansions of each of the following binomials:

19. $(a+b)^6$.
20. $(a+1)^6$.
21. $(b-c)^6$.
22. $(c+d)^7$.
23. $(m+n)^8$.
24. $(n+x)^9$.
25. $(a+w)^{10}$.
26. $(a+b)^{11}$.
27. $(c+z)^{14}$.
28. $(d+x)^{15}$.
29. $(a+y)^{16}$.
30. $(m+w)^{20}$.

DIVISION

QUOTIENTS OF INTEGRAL POWERS

EXERCISE 29

Express the following quotients of different powers of the same bases as single powers of positive bases:

1. $3^5 \div 3^2$.
2. $2^7 \div 2^4$.
3. $8^5 \div 3^2$.
4. $6^3 \div 6$.
5. $5^7 \div 5^4$.
6. $4^9 \div 4^6$.
7. $7^{10} \div 7^9$.
8. $2^{11} \div 2^5$.
9. $3^{12} \div 3^8$.
10. $4^{10} \div 4^8$.
11. $6^6 \div 6^4$.
12. $(-3)^8 \div (-3)^5$.
13. $(-6)^9 \div (-6)^7$.
14. $(-2)^8 \div (-2)$.
15. $(-2)^{10} \div (-2)^5$.
16. $a^8 \div a^6$.
17. $b^{13} \div b^{10}$.
18. $c^{12} \div c^7$.
19. $h^{11} \div h^3$.
20. $g^{12} \div g^{10}$.
21. $m^{14} \div m^{13}$.
22. $k^{16} \div k^{12}$.
23. $(-n)^{17} \div (-n)^7$.
24. $(-r)^{15} \div (-r)^9$.

25. $(-x)^{18} \div (-x)^{11}$.
26. $(-y)^{13} \div (-y)^{11}$.
27. $(-z)^{19} \div (-z)^{5}$.
28. $a^m \div a$.
29. $b^n \div b^2$.
30. $c^{2r} \div c$.
31. $d^{3s} \div d^4$.
32. $m^a \div m^b$.
33. $a^m \div a^b$.
34. $c^r \div c^x$.
35. $-d^x \div d^5$.
36. $(-g)^k \div (-g)^6$.
37. $(-h)^y \div (-h)^z$.
38. $a^{m+1} \div a$.
39. $b^{n+2} \div b$.
40. $c^{2n} \div c^n$.
41. $d^{4r} \div d^{2r}$.
42. $g^{5x} \div g^{2x}$.
43. $a^{2b+1} \div a$.
44. $b^{3c+2} \div b$.
45. $c^{4d+5} \div c^2$.
46. $d^{3m+6} \div d^3$.
47. $h^{a+b} \div h^a$.
48. $k^{m+n} \div k^n$.
49. $a^{2m+5} \div a^m$.
50. $b^{4n+7} \div b^{2n}$.
51. $c^{5r-2} \div c^{3r}$.
52. $d^{3x+2} \div d^{2x}$.
53. $a^{3r+2} \div a^{r+1}$.
54. $b^{4s+6} \div b^{2s+4}$.
55. $c^{5n+6} \div c^{4n+5}$.
56. $d^x \div d^{x-1}$.
57. $g^y \div g^{y-2}$.
58. $h^r \div h^{r-3}$.
59. $k^{r+1} \div k^{r-1}$.
60. $m^{2n} \div m^{n-1}$.
61. $n^{3r+2} \div n^{2r-2}$.
62. $q^{4x+3} \div q^{4x-5}$.
63. $x^{2m+n} \div x^{2n-1}$.
64. $y^{3n+r} \div y^{r-2}$.
65. $z^{4x+y} \div z^{4x-y}$.
66. $a^{3b+5c} \div a^{b+2c}$.
67. $b^{4x+7y} \div b^{3x+5y}$.
68. $c^{6m+5n} \div c^{4m+2n}$.
69. $d^{2d+3} \div d^{d+3}$.
70. $x^{a+b+c} \div x^a$.
71. $y^{m+n+3} \div y^n$.
72. $z^{r+s+5} \div z^5$.
73. $a^{m+n+r} \div a^{m+n}$.
74. $b^{x+y+z} \div b^{x+z}$.
75. $c^{r+s+4} \div c^{s+4}$.
76. $a^{x+y+7} \div a^{x+y+1}$.
77. $x^{a+b+c} \div x^{a+5}$.
78. $y^{m+n+8} \div y^{n+6}$.
79. $z^{a+b+4} \div z^{a+2}$.
80. $a^{2m+n+1} \div a^{m+n+2}$.
81. $b^{3r+2s+4} \div b^{3r+s+3}$.
82. $x^{4a+3b+2c} \div x^{3a+2b+c}$.
83. $y^{5a+6b+7c} \div y^{4a+3b+2c}$.

DIVISION OF ONE MONOMIAL BY ANOTHER

EXERCISE 30

Obtain the following quotients of monomials:

1. $2\,b) 6\,b^3$
2. $3\,c) 12\,c^4$
3. $5\,d) 25\,d^6$
4. $4\,h) -24\,h^2$
5. $6\,m) -30\,m^6$
6. $-7\,n) 28\,n^8$.
7. $-8\,x) -16\,x^5$
8. $2\,a^2) 8\,a^3$
9. $3\,b^2) 15\,b^4$
10. $4\,c^3) 20\,c^6$
11. $-9\,d^4) -18\,d^{10}$
12. $7\,h^5) 14\,h^5$
13. $6\,ab) -12\,a^2b^3$
14. $4\,cd) 8\,c^2d$
15. $-3\,gx) 21\,gx^6$
16. $-11\,a^2m) -22\,a^3m^4$
17. $12\,b^2k^7) 36\,b^7k^9$
18. $9\,xyz) -27\,x^2y^3z^4$
19. $15\,a^2b^3c) 30\,a^3b^3c^2$
20. $-13\,ac^3y^5) 52\,a^2c^4y^6$

21. $14\,a^2b^3c^2)56\,a^3b^4c^5d$

22. $16\,x^5y^2z)-96\,x^6y^2z^2w^3$

23. $-15\,a^2b^4y^3)75\,a^3b^4x^2y^4$

24. $-9\,b^6c^4z^5)-72\,b^7c^4y^6z^5$

25. $42\,a^8 \div 6\,a^2$.

26. $39\,b^7 \div 13\,b^4$.

27. $72\,a^4b^6 \div 12\,a^3b^2$.

28. $56\,b^3d^7 \div 8\,b^2d^5$.

29. $44\,x^4y^5 \div -11\,x^3y^5$.

30. $75\,m^2w^6 \div (-15\,m^2w^5)$.

31. $(-90\,n^6z^7) \div (-10\,n^3z^3)$.

32. $(-96\,g^6h^7) \div (-16\,g^4h^1)$.

33. $-72\,h^4k^3 \div 18\,hk^3$.

34. $80\,a^3b^3c^3 \div 16\,a^2b^2c^2$.

35. $63\,x^4y^2z^3 \div 7\,x^2y^2z^2$.

36. $(-60\,a^3b^2d) \div (-15\,abd)$.

37. $-45\,m^4n^3w^2 \div 9\,m^2n^3w^2$.

38. $50\,g^5h^4k^5 \div 25\,g^5k^5$.

39. $51\,x^6y^6z^6 \div (-17\,x^4y^4z^4)$.

40. $(-57\,a^4b^5y^6) \div (-19\,a^3b^4y^5)$.

41. $a^nb^2 \div ab$.

42. $c^3d^r \div cd$.

43. $m^xn^y \div mn$.

44. $x^{2a}y^3 \div x^ay$.

45. $a^{3m}x^{2n} \div a^mx^n$.

46. $b^{4r}y^{2x} \div b^2y^x$.

47. $c^{x+1}w^{y+1} \div cw$.

48. $b^{n+2}x^{n+1} \div b^2x^n$.

49. $d^{r+4}q^{r+2} \div d^{r+1}q^2$.

50. $a^{2m+3}b^{3n+2} \div a^2b^2$.

51. $c^{3x+5}d^{2x+4} \div c^{x+2}d^{2x+3}$.

52. $x^{4a+2b}y^{3a+4c} \div x^{2a+b}y^{2a+3c}$.

DIVISION OF A POLYNOMIAL BY A MONOMIAL

EXERCISE 31

Perform the following indicated divisions:

1. $2\,a)6\,ab+8\,ac$

2. $3\,b)12\,bc+18\,bx$

3. $6\,d)30\,dxy+42\,d$

4. $5\,m)5\,m-10\,mn$

5. $7\,n)14\,n^2+21\,nx$

6. $4\,c)8\,cy-28\,cw$

7. $8\,x)24\,xy+16\,x^2$

8. $9\,y)27\,y^2z+45\,y^4\,w$

9. $6\,w)30\,x^2w^2+48\,y^3w^4$

10. $3\,a^2)6\,a^3b+12\,a^2c$

11. $\dfrac{8\,ab^4+14\,b^5c^2}{2\,b^3}$.

12. $\dfrac{4\,c^5-8\,c^4d^2}{4\,c^4}$.

13. $\dfrac{4\,abc+10\,abd}{2\,ab}$.

14. $\dfrac{15\,x^2y+18\,xy^2}{3\,xy}$.

15. $\dfrac{32\,m^3n^2-40\,m^2n^3}{4\,mn}$.

16. $\dfrac{18\,a^2b^2-12\,a^3bc}{6\,a^2b}$.

17. $\dfrac{14\,xy^3-35\,x^2y^2z}{7\,xy^2}$.

18. $\dfrac{16\,m^3x^2y-44\,m^2x^3z}{4\,m^2x^2}$.

19. $\dfrac{10\,a^3b^5c-15\,a^2bc^2+5\,a^2bc}{5\,a^2bc}$.

20. $\dfrac{9\,a^2bc^4+12\,a^4b^2c+24\,ab^4c^2}{3\,abc}$.

21. $\dfrac{21\,a^3b^4c^2 - 7\,a^2b^3c^4 - 28\,a^4b^2c^3}{7\,a^2b^2c^2}$.

22. $\dfrac{30\,x^3y^5z^4 + 12\,x^5y^4z^3 - 18\,x^4y^3z^5}{-\,6\,x^3y^3z^3}$.

23. $\dfrac{25\,m^6n^3z^5 - 30\,m^3n^5z^6 - 45\,m^5n^6z^3}{-\,5m^3n^3z^3}$.

24. $\dfrac{28\,r^5s^2w^6 + 32\,r^6s^3w^7 - 44\,r^3s^4w^8}{4\,r^3s^2w^5}$.

25. $\dfrac{40\,a^3bd^5 + 24\,a^2b^2d^3 + 32\,a^4b^3d^4}{8\,a^2bd^3}$.

26. $\dfrac{27\,m^4n^5z^6 - 18\,m^5n^6z^7 - 54\,m^6n^7z^8}{-\,9\,m^3n^4z^6}$.

27. $\dfrac{33\,x^7y^7z^7 - 44\,x^5y^6z^5 - 22\,x^6y^4z^6}{11\,x^5y^4z^5}$.

28. $\dfrac{49\,g^4h^7k^5 + 42\,g^6h^5k^7 + 35\,g^5h^6k^4}{-\,7\,g^4h^5k^4}$.

29. $\dfrac{-24\,a^6c^5h^7 - 12\,a^7c^6h^5 - 48\,a^5c^7h^6}{-\,12\,a^5c^5h^5}$.

30. $\dfrac{-\,26\,b^3h^5n^7 - 13\,b^2h^4n^6 - 39\,b^4h^6n^8}{-\,13\,b^2h^4n^6}$.

31. $(\frac{4}{5}\,a^3bc + \frac{2}{3}\,ab^3c + \frac{4}{15}\,abc^3) \div 2\,abc$.

32. $(\frac{3}{4}\,a^2b^3c^2 + \frac{3}{5}\,a^3b^2c^3 + \frac{3}{7}\,a^2b^3c^3) \div 3\,a^2b^2c^2$.

33. $(\frac{1}{2}\,xy^2z^2 - 2\,x^2y^3z - \frac{1}{4}\,x^3yz^2) \div \frac{1}{2}\,xyz$.

34. $(\frac{5}{7}\,m^5n^3w^4 - \frac{10}{17}\,m^3n^4w^5 - \frac{15}{19}\,m^4n^5w^3) \div 5\,m^3n^3w^3$.

35. $(\frac{2}{3}\,a^4bd^3 - \frac{5}{6}\,a^3b^4d - \frac{1}{2}\,a^2b^2d^2) \div (-\frac{1}{3}\,a^2bd)$.

36. $[3(a+b)^3 + 9(a+b)^4 + 6(a+b)^5] \div 3(a+b)^2$.

37. $[5(x+y)^6 - 15(x+y)^5 - 30(x+y)^4] \div 5(x+y)^3$.

38. $[4(b-d)^5 + 20(b-d)^6 + 16(b-d)^7] \div 4(b-d)^5$.

39. $[6(g^2-h^2) + 18(g-h)^2 + 12(g-h)^3] \div 6(g-h)$.

40. $(a^{m+6} + a^{m+5} + a^{m+4}) \div a$.

41. $(b^{n+7} + b^{n+5} + b^{n+3}) \div b^2$.

42. $(c^{2n+3} + c^{2n+2} + c^{2n+1}) \div c^{2n}$.

43. $(d^{a+2} - d^{a+4} + d^{a+6}) \div d^{a+1}$.

44. $(x^{3n+4} - x^{3n+3} - x^{3n+2}) \div x^{2n+2}$.

45. $(y^a + y^{2a} + y^{3a} + y^{4a}) \div y^a$.

46. $(z^{5n} + z^{4n} + z^{3n} + z^{2n}) \div z^2$.

47. $(a^{4x+2y} + a^{3x+2y} + a^{2x+3y} + a^{x+4y}) \div a^{x+y}$.

48. $(b^{m+4n} - b^{2m+3n} + b^{3m+2n} - b^{4m+n}) \div b^{m+n}$.

49. $(a^{2m+1}b^3 + a^{2m+2}b^5 + a^{2m+3}b^7) \div a^2mb^2$.

DIVISION OF ONE POLYNOMIAL BY ANOTHER

EXERCISE 32

When performing the following divisions, check all results numerically. Divide:

1. $a^2 + 6a + 8$ by $a + 2$.

2. $6b^2 + 5b - 6$ by $3b - 2$.

3. $5c^2 + c - 6$ by $5c + 6$.

4. $21d^2 + 38d + 16$ by $7d + 8$.

5. $1 + 2c + 2c^2 + c^3$ by $1 + c$.

6. $6a^2 + 8a + 28$ by $3a + 7$.

7. $6a^2 + 13ab + 6b^2$ by $2a + 3b$.

8. $8x^2 - 22xy + 15y^2$ by $2x - 3y$.

9. $a^6 + b^6$ by $a^2 + b^2$.

10. $m^3 + 2n^3 + 3m^2n + 4mn^2$ by $m + n$.

11. $k^3 + k^2m - km^2 - m^3$ by $k - m$.

12. $a^4 - 2a^3b + 2ab^3 - b^4$ by $a^2 - b^2$.

13. $16x^4 - 1$ by $2x - 1$.

14. $32m^5 + 1$ by $2m + 1$.

15. $3a^6 - 24$ by $a^2 - 2$.

16. $6d^3 - d^2 - 12d + 4$ by $3d^2 + 4d - 1$.

17. $z^4 + z^2w^2 + w^4$ by $z^2 + zw + w^2$.

18. $q^4 - q^3 - q^2 + q$ by $q^2 + q + 1$.

19. $h^4 - 6h^3z + 9h^2z^2 - 4z^4$ by $h^2 + 3hz - 2z^2$.

20. $c^4 - c^3 - 8c^2 + 10c - 10$ by $c^2 + 2c - 2$.

21. $15a^4 - a + 8a^2 - 1 - 19a^3$ by $5a^2 - 3a - 1$.

22. $6y^4 - 13xy^3 + 13x^2y^2 - 13x^3y - 5x^4$ by $2y^2 - 3xy - x^2$.

23. $c^5 - 6c^4 + 16c^3 - 25c^2 + 13c + 5$ by $c^3 - 4c^2 + 3c + 1$.

24. $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$ by $x^2 - 2xy + y^2$.

25. $s^5 + 3s^4 - 20s^3 - 60s^2 + 64s + 192$ by $s^3 + 9s^2 + 26s + 24$.

26. $3x^5 - 8x^4 - 5x^3 + 26x^2 - 28x + 24$ by $x^3 - 2x^2 - 4x + 8$.

27. $a^5 + 10a^3b^2 + b^5 + 10a^2b^3 + 5a^4b + 5ab^4$ by $a^2 + 2ab + b^2$.

28. $3x^5 + 7x^4y - 11x^3y^2 - 11x^2y^3 + 6xy^4 - 18y^5$ by $x + 3y$.

29. $1 + 2a^2 - 7a^4 - 16a^6$ by $1 + 2a + 3a^2 + 4a^3$.

30. $v^6 - 2v^3 + 1$ by $v^2 - 2v - 1$.

31. $3x^6 + 43x^2 + 6x^3 - 30x + 80 - 32x^4 + 20x^5$ by $x + 8$.

32. $15a^6 + 16a^5 + 8a^4 - 9a^3 - 7a^2 + 19a - 42$ by $5a^2 + 2a - 7$.

33. $c^7 - 6\,c^6d^3 + 14\,c^5d^6 - 12\,c^4d^9$ by $c^3 - 2\,c^2d^3$.

34. $x^3 + y^3 + z^3 - 3\,xyz$ by $x + y + z$.

EXERCISE 33

Divide:

1. $\frac{x^3}{6} - \frac{5\,x^2y}{36} + \frac{xy^2}{6} + \frac{2\,y^3}{9}$ by $\frac{x}{2} + \frac{y}{3}$.

2. $\frac{2}{3}\,b^{15} - \frac{5\,0\,3}{1\,6\,0}\,b^{12} + \frac{8\,0\,3}{2\,4\,0}\,b^9 - \frac{8\,0\,3}{3\,6\,0}\,b^6 + \frac{5\,0\,3}{3\,0\,0}\,b^3 - \frac{3}{2}$ by $\frac{8}{1\,5}\,b^3 - \frac{1\,5}{8}$.

3. $\frac{1}{6}\,k^8 - \frac{1\,1}{6\,0}\,k^7 + \frac{2\,2\,3}{1\,2\,6\,0}\,k^6 - \frac{2\,9}{4\,2\,0}\,k^5 + \frac{1}{4\,2}\,k^4$ by $\frac{1}{3}\,k^2 - \frac{1}{5}\,k + \frac{1}{7}$.

4. $a^5 - \frac{1\,0}{3}\,a^4 + 10\,a^3 - 30\,a^2 + 90\,a - 27$ by $81\,a - 27$.

5. $\frac{t^2}{8} - \frac{5}{4}\,t^3 + \frac{5}{2}\,t^4 - 5\,t^5 + 8\,t^6$ by $\frac{1}{16}\,t - \frac{1}{2}\,t^2$.

6. $m^7 - \frac{7}{6}\,m^6 + \frac{9}{3\,5}\,m^5 + \frac{4}{3}\,m^4 - \frac{2\,6}{3\,5}\,m^3 + \frac{1\,2}{7}\,m^2 + \frac{6\,1}{3\,0}\,m - 1$ by $\frac{1}{7}\,m^2 - \frac{1}{6}\,m + \frac{1}{5}$.

7. $a^5 - \frac{6\,7}{9\,0}\,a^4 + \frac{8\,3}{2\,1\,0}\,a^3 + \frac{1\,1\,1}{1\,4\,0}\,a^2 - \frac{1\,2\,3}{3\,5\,0}\,a + \frac{2}{5}$ by $\frac{2}{3}\,a^2 - \frac{1}{5}\,a + \frac{2}{7}$.

8. $\frac{3}{5}\,x^7 - \frac{4\,3}{2\,4}\,x^6 + \frac{8\,1}{2\,0}\,x^5 - \frac{4\,4\,3}{9\,0}\,x^4 + \frac{5\,9}{1\,2}\,x^3 - 4\,x^2$ by $\frac{2}{5}\,x^2 - \frac{3}{4}\,x + \frac{4}{3}$.

9. $a^{m+1} + a^mb + ab^m + b^{m+1}$ by $a^m + b^m$.

10. $2\,a^{3m} - 6\,a^{2m}b^m + 6\,a^mb^{2m} - 2\,b^{3m}$ by $2\,a^m - 2\,b^m$.

11. $a^{4n} + a^{2n}b^{2n} + b^{4n}$ by $a^{2n} + a^nb^n + b^{2n}$.

SYNTHETIC DIVISION

EXERCISE 34

Find the quotient and remainder when

1. $x^2 + 5\,x + 7$ is divided by $x + 3$.

2. $y^2 + 7\,y + 11$ is divided by $y + 4$.

3. $z^2 - 17\,z + 39$ is divided by $z + 3$.

4. $w^2 - 48\,w + 97$ is divided by $w + 10$.

5. $a^3 + 11\,a^2 - 19\,a + 117$ is divided by $a + 4$.

6. $b^3 - 19\,b^2 + 18\,b - 17$ is divided by $b - 5$.

7. $c^3 - 14\,c^2 + 28\,c - 42$ is divided by $c - 6$.

8. $d^4 - 5\,d^3 + 6\,d^2 - 10\,d + 1$ is divided by $d + 1$.

9. $k^4 + 2\,k^3 + 3\,k^2 + 4\,k + 5$ is divided by $k - 5$.

10. $3\,x^5 - 5\,x^4 + 6\,x^3 - 2\,x^2 + x - 1$ is divided by $x - 2$.

Employing the Remainder Theorem, find the numerical value of each of the following expressions when x is given the value indicated:

11. x^3+3x^2+4x+3, when $x=1$.
12. $2x^3+x^2+7x+4$, when $x=-1$.
13. x^3+2x^2+3x+5, when $x=2$.
14. x^3+2x^2+5x+7, when $x=2$.
15. $x^3-8x^2+17x-10$, when $x=1$.
16. x^3+5x^2+3x+1, when $x=2$.
17. x^3+9x^2-7x+5, when $x=1$.
18. $x^3-7x^2+13x-6$, when $x=2$.
19. $3x^3+2x^2+x+5$, when $x=2$.
20. $2x^3-5x^2+4x+7$, when $x=3$.
21. $3x^3+x^2-7x+9$, when $x=1$.
22. $4x^3-9x^2-x+6$, when $x=1$.
23. $2x^3+x^2+3x+2$, when $x=2$.
24. $4x^3-3x^2+2x+5$, when $x=3$.
25. $3x^3+7x^2-4x+3$, when $x=-3$.
26. $4x^3-5x^2+3x-4$, when $x=-2$.
27. $2x^3-9x^2+4x+5$, when $x=5$.
28. $3x^3+11x^2-7x+5$, when $x=-1$.
29. $2x^3+11x^2-3x+10$, when $x=-5$.
30. $6x^3-3x^2+7x+4$, when $x=-2$.
31. $11x^3-9x^2+7x+1$, when $x=-1$.
32. $x^4+2x^3+3x^2+x+1$, when $x=1$.
33. $x^4-x^3-7x^2+x-6$, when $x=1$.
34. $x^4-3x^3+x^2+2x+5$, when $x=3$.
35. x^3+2x^2+7, when $x=2$.
36. x^5+1, when $x=3$.
37. x^3+4x^2+12, when $x=4$.
38. $x^3+15x+3$, when $x=5$.
39. $2x^3+7x+20$, when $x=2$.
40. $2x^3+5x^2-33x+12$, when $x=3$.
41. $3x^3-6x^2+5x+21$, when $x=2$.
42. $2x^4+5x^3+2x+1$, when $x=1$.
43. $3x^4+5x^2+4x+1$, when $x=-1$.

SPECIAL QUOTIENTS

MENTAL EXERCISE 35

Obtain each of the following quotients mentally, stating in each case the general principle applied:

1. $\frac{x^3+y^3}{x+y}$.
2. $\frac{a^7+b^7}{a+b}$.
3. $\frac{c^3-d^3}{c-d}$.
4. $\frac{m^9-n^9}{m-n}$.
5. $\frac{a^6-b^6}{a+b}$.
6. $\frac{a^6-b^6}{a-b}$.
7. $\frac{a^{11}+b^{11}}{a+b}$.
8. $\frac{x^{10}-y^{10}}{x-y}$.
9. $\frac{a^3+8}{a+2}$.
10. $\frac{x^3+1}{x+1}$.
11. $\frac{y^3-27}{y-3}$.
12. $\frac{1+x^5}{1+x}$.
13. $\frac{1-x^9}{1-x}$.
14. $\frac{a^{12}-b^{12}}{a-b}$.
15. $\frac{a^8-b^8}{a+b}$.

FUNCTIONAL NOTATION

EXERCISE 36

Find the values of the following functions:

1. If $f(x)=x+5$, find $f(1), f(2), f(0), f(-5), f(\frac{1}{3})$.
2. If $f(x)=x-1$, find $f(0), f(1), f(2), f(-1)$.
3. If $f(x)=x+b$, find expressions for $f(a), f(-x), f(2), f(b)$.
4. If $f(x)=x^2$, find $f(1), f(3), f(5), f(0), f(\frac{1}{4})$.
5. If $f(x)=x^2$, find expressions for $f(a), f\left(\frac{1}{b}\right), f(-x), f\left(\frac{1}{x}\right)$.
6. If $f(x)=x+\frac{1}{x}$, find $f(1), f(2), f(-1), f(3)$.
7. If $f(a)=a^2+a+1$, find $f(1), f(2), f(3), f(-4)$.
8. If $f(b)=b^2-b+1$, find $f(1), f(2), f(-1), f(-3), f(4)$.
9. If $f(x)=x^2+3x+2$, find $f(0), f(1), f(5), f(-1)$.
10. If $f(d)=2d^2+3d-5$, find $f(0), f(2), f(-1), f(5)$.
11. If $f(h)=3h^2-7h+9$, find $f(0), f(1), f(-1), f(-2)$.
12. If $f(x, y)=x+y$, find $f(x, a), f(b, y), f(c, d), f(2, 3), f(x, 0), f(0, y), f(0, 0), f(1, 2), f(5, -3)$.
13. If $f(y, z)=y-z$, find $f(y, 0), f(0, z), f(0, 0), f(y, 1), f(2, z), f(8, 5), f(2, 7)$.
14. If $f(x, y)=x^2+y$, find $f(x, -y), f(-x, y), f(2, 1), f(3, 5), f(4, -7)$.

15. If $f(a, b) = a^2 - 2ab + b^2$, find $f(a, 1)$, $f(1, b)$, $f(a, 0)$, $f(0, b)$, $f(0, 0)$, $f(6, 2)$, $f(3, 7)$.

16. If $f(b) = b^3 + b^2 + b$, find $f(c)$, $f(x)$, $f(-b)$, $f(1)$, $f(-1)$, $f(0)$, $f(-2)$.

17. If $f(a, b, c) = a + b + c$, find $f(x, y, z)$, $f(a, a, a)$, $f(1, 2, 3)$, $f(2x, 3y, z)$, $f(a, 0, c)$, $f(a, b, 1)$.

18. If $f_1(a) = 3a + 1$, and $f_2(a) = 2a + 5$, find $f_1(a) + f_2(a)$; $f_1(a) - f_2(a)$; $f_1(1) \times f_2(1)$.

19. If $f_1(x) = x^2 + 5x + 6$ and $f_2(x) = x + 3$, find $f_1(x) + f_2(x)$; $f_1(x) \div f_2(x)$; $f_1(-1) + f_2(-1)$.

20. If $f_1(x) = x + 4$ and $f_2(x) = x + 6$, find $f_1(x) - f_2(x)$; $f_1(x) \times f_2(x)$; $f_1(1) \div f_2(1)$.

EXERCISE 37

Find the numbers represented by the following functions for the values specified by applying the Remainder Theorem:

1. If $f(x) = x^3 + 4x^2 + 2x + 9$, find $f(2), f(3), f(4)$.

2. If $f(x) = x^3 + 6x^2 - 5x + 10$, find $f(-1), f(2), f(3)$.

3. If $f(x) = x^4 + 2x^3 + 3x^2 + 4x - 8$, find $f(1), f(2), f(3)$.

4. If $f(x) = 2x^3 + 4x^2 - 7x + 9$, find $f(1), f(-2), f(3)$.

5. If $f(x) = 2x^4 + 5x^3 + 3x^2 - 4x + 1$, find $f(1), f(2), f(-3)$.

6. If $f(x) = 3x^4 + 5x^3 + 2x^2 + 6x + 9$, find $f(1), f(2), f(-4)$.

7. If $f(x) = 4x^4 - 5x^3 - 11x^2 + 2x + 4$, find $f(1)$, $f(-2)$, $f(3)$.

8. If $f(x) = 6x^4 + 15x^3 + 7x^2 + 8x + 12$, find $f(-1)$, $f(-2)$, $f(-3)$.

9. If $f(x) = 2x^5 - 8x^4 + 10x^3 - 17x^2 + 19x + 13$, find $f(1)$, $f(2)$, $f(3)$.

10. If $f(x) = 3x^5 + 11x^4 + 3x^3 - 14x^2 - 21x + 12$, find $f(1)$, $f(2)$, $f(-3)$.

GRAPHS

EXERCISE 38

Plot the points whose coördinates are

1. (2, 4).
2. (3, 5).
3. (1, 6).
4. (4, 3).
5. (7, 1).
6. (9, 9).
7. (5, 0).
8. (6, 2).
9. (2, −1).

10. $(3, -4)$.
11. $(1, -5)$.
12. $(4, -6)$.
13. $(5, -5)$.
14. $(6, -8)$.
15. $(7, 0)$.
16. $(0, -9)$.
17. $(-1, 3)$.
18. $(-2, 4)$.
19. $(-3, 5)$.
20. $(-6, 0)$.
21. $(-8, 6)$.
22. $(-5, -2)$.
23. $(-4, -1)$.
24. $(-6, -2)$.
25. $(-9, -7)$.
26. $(0, -4)$.
27. $(3, 0)$.
28. $(-3, 0)$.
29. $(0, 3)$.
30. $(0, -3)$.
31. $(5, 4)$.
32. $(5, -4)$.
33. $(-5, 4)$.
34. $(-5, -4)$.
35. $(0, 0)$.
36. $(3, \frac{1}{2})$.
37. $(4, \frac{1}{2})$.
38. $(-5, 2\frac{1}{2})$.
39. $(-6, -3\frac{1}{3})$
40. $(7, -4.5)$.
41. $(8, -6.6)$.
42. $(3.5, 2.3)$.
43. $(4.4, 6.6)$.
44. $(-3.9, -9.1)$.
45. $(0.7, 2.8)$.
46. $(5, 1)$.
47. $(5, 0)$.
48. $(5, -1)$.

EXERCISE 39

Plot portions of the graphs of the following equations:

1. $y = x$.
2. $y = 2x$.
3. $y = 4x$.
4. $y = -x$.
5. $y = -3x$.
6. $y = 2$.
7. $y = 5$.
8. $y = -3$.
9. $y = 0$.
10. $x = 4$.
11. $x = 6$.
12. $x = -7$.
13. $y = x + 1$.
14. $y = x + 3$.
15. $y = x - 2$.
16. $y = 2x + 1$.
17. $y = 3x + 4$.
18. $y = 4x - 2$.
19. $y = 5x + 3$.
20. $y = -2x + 4$.
21. $y = -3x + 5$.
22. $y = -4x - 1$.
23. $y = x^2$.
24. $y = 2x^2$.
25. $y = 3x^2 + 1$.
26. $y = 4x^2 - 2$.
27. $y = 5x^2 - 15$.
28. $y = x^2 + x$.
29. $y = x^2 + 3x$.
30. $y = x^2 - 4x$.
31. $y = 2x^2 + x$.
32. $y = 2x^2 - x$.
33. $y = 3x^2 + x + 1$.
34. $y = 4x^2 + x + 3$.
35. $y = 2x^2 + 5x + 2$.
36. $y = x^2 + 5x - 6$.
37. $y = x^2 + 7x - 8$.
38. $y = x^2 - x - 12$.
39. $y = x^2 + x - 20$.
40. $y = x^2 - 9x + 8$.
41. $y = x^2 - 12x + 20$.
42. $y = x^2 + 11x + 10$.
43. $y = -x^2$.
44. $y = -x^2 + 5x$.
45. $y = -x^2 - 2x + 3$.
46. $y = -2x^2 + 3x - 5$.
47. $y = \dfrac{1}{x}$.
48. $y = -\dfrac{1}{x}$.
49. $y = \dfrac{2}{x}$.
50. $y = \dfrac{1}{x - 1}$.
51. $y = \dfrac{1}{x - 3}$.
52. $y = \dfrac{1}{x + 4}$.
53. $y = \dfrac{6}{x - 2}$.

54. $y = \dfrac{7}{2x+1}$.
55. $y = (x+1)(x-3)$.
56. $x = (y+1)(y-3)$.
57. $y = x^2 - 9$.
58. $x = y^2 - 9$.
59. $xy = 24$.
60. $xy = 36$.
61. $xy = -20$.
62. $xy = -1$.
63. $y = x^3$.
64. $y = x^3 + 1$.
65. $y = x^3 - 4x$.
66. $y = x^3 + 2x^2$.
67. $y = x^3 + x^2 - x$.
68. $y = x^3 + x^2 + x + 1$.
69. $y = x^3 + 3x^2 + 2x + 1$.
70. $y = x^3 - 3x^2 + 2x - 1$.
71. $y = -x^3 + 3x^2 - 2x + 1$.
72. $y = x^3 + 3x^2 + 6x + 11$.
73. $y = x^3 - 3x^2 + 6x - 11$.
74. $y = x^3 + 4x^2 + 2x + 5$.
75. $y = x^3 - 4x^2 + 2x - 5$.
76. $y = x^4 - 10x^2 + 12$.
77. $y = x^4 - 13x^2 + 36$.

EXERCISE 40

Plot the graphs of the following equations:

1. $3x + 5y = 15$.
   $3x + 4y = 12$.
3. $4x - 7y = 28$.
4. $7x + 3y = -21$.
5. $2x + 3y = 6$.
6. $2x + 3y = 10$.
7. $2x - 3y = 10$.
8. $3x - 2y = 6$.
9. $2x + 5y = 10$.
10. $4x + 10y = 20$.
11. $6x + 15y = -30$.
12. $x - y = 10$.
13. $y - x = 10$.
14. $x + y = 10$.
15. $x + y = -10$.

SOLUTION OF EQUATIONS BY GRAPHS

EXERCISE 41

Plot the graphs of the following equations and locate the points in which they intersect:

1. $x + 2y = 8$ and $3x - y = 3$.
2. $x + y = 12$ and $x - y = 6$.
3. $x + y = 7$ and $x - y = 1$.
4. $5x - 4y = -22$ and $x + 2y = 4$.
5. $2x + 3y = 9$ and $3x + y = -4$.
6. $4x - 5y = -15$ and $2x + y = -11$.
7. $5x + 6y = 12$ and $3x - y = 21$.
8. $2x - 3y = 14$ and $3x - y = -7$.
9. $x - 10y = -10$ and $x = 10$.
10. $3x + 4y = 36$ and $x = y$.

11. $y = x^2$ and $y = x + 2$.
12. $y = x^2 - 3x - 10$ and $16x + 7y = 8$.
13. $y = x^2 - 9$ and $y = 7$.
14. $y = x^2 - 9x + 8$ and $x = y^2 - 8y + 12$.
15. $y = x^2 - 9$ and $x = y^2 - 4$.
16. $y = x^2 - 25$ and $x = y^2 - 16$.
17. $xy = 24$ and $x = y$.
18. $xy = 3$ and $x + y = 4$.
19. $xy = 6$ and $x + y = 5$.
20. $xy = 16$ and $x - y = 6$.
21. $xy = 12$ and $5x - 8y = -14$.

IDENTICAL EQUATIONS

MENTAL EXERCISE 42

Show that each of the following equalities is an identity:

1. $2a + 3a \equiv 5a$.
2. $4b + 6b \equiv 10b$.
3. $8c - 2c \equiv 6c$.
4. $11d \equiv 6d + 5d$.
5. $13e \equiv 20e - 7e$.
6. $6m + 9m \equiv 4m + 11m$.
7. $7n + 10n \equiv 20n - 3n$.
8. $12x - 4x \equiv 17x - 9x$.
9. $2(3a + 4b) \equiv 6a + 8b$.
10. $3(5b + 7c) \equiv 15b + 21c$.
11. $4(6c - 5) \equiv 24c - 20$.
12. $13a + 7a \equiv (5 \times 4)a$.
13. $16b + 3b \equiv 38b \div 2$.
14. $19a^2 + 5a^2 \equiv 8a \times 3a$.
15. $23b^2 + 5b^2 \equiv 7b + 4b$.
16. $31c^2 - 9c^2 \equiv 11c \times 2c$.
17. $(x + 2)(x - 2) \equiv x^2 - 4$.
18. $(y + 3)(y - 3) \equiv y^2 - 9$.
19. $(z + 8)(z - 8) \equiv z^2 - 64$.
20. $(1 + m)(1 - m) \equiv 1 - m^2$.
21. $(a + 3)^2 \equiv a^2 + 6a + 9$.
22. $(6 + b)^2 \equiv 36 + 12b + b^2$.
23. $(9 - c)^2 \equiv 81 - 18c + c^2$.
24. $a(a + 4) + 4 \equiv a^2 + 4(a + 1)$.
25. $b^2 + 16(b + 4) \equiv b(b + 16) + 64$.
26. $x(x - 20) + 100 \equiv x^2 - 20(x - 5)$.
27. $x(x - 6) + 3(2x - 3) \equiv x^2 - 9$.
28. $x(x - 12) + 12(x - 3) \equiv x^2 - 36$.
29. $x(x + 18) - 9(2x + 9) \equiv x^2 - 81$.
30. $x(x - 22) + 11(2x - 11) \equiv x^2 - 121$.
31. $x(x + 24) - 24(x + 6) \equiv x^2 - 144$.
32. $28(x + 7) - x(x + 28) \equiv 196 - x^2$.

33. $(a+1)^2-4a \equiv (a-1)^2$.
34. $(b+2)^2-8b \equiv (b-2)^2$.
35. $(a+3)^2-12a \equiv (a-3)^2$.
36. $(c+5)^2-20c \equiv (c-5)^2$.
37. $(d-4)^2+16d \equiv (d+4)^2$.
38. $(m-6)^2+24m \equiv (m+6)^2$.
39. $(7-n)^2+28n \equiv (7+n)^2$.
40. $(2a+b)^2-8ab \equiv (2a-b)^2$.
41. $(3b-4c)^2+48bc \equiv (3b+4c)^2$.
42. $(a+1)^2-(a-1)^2 \equiv 4a$.
43. $(b+2)^2-(b-2)^2 \equiv 8b$.
44. $(c+5)^2-(c-5)^2 \equiv 20c$.
45. $(7-x)^2-(7+x)^2 \equiv -28x$.
46. $(9-y)^2-(9+y)^2 \equiv -36y$.
47. $(a-5b)^2-(a+5b)^2 \equiv -20ab$.
48. $(a+b)^2+(a-b)^2 \equiv 2a^2+2b^2$.
49. $(b+c)^2+(b-c)^2 \equiv 2b^2+2c^2$.
50. $(m-n)^2+(m+n)^2 \equiv 2(m^2+n^2)$.
51. $(1-x)^2+(1+x)^2 \equiv 2+2x^2$.

EQUIVALENT EQUATIONS

MENTAL EXERCISE 43

From each of the following conditional equations derive an equivalent equation by transposing to the first member, the terms containing x, and to the second member, all other terms. Then simplify the members separately:

1. $x-5=0$.
2. $x+7=0$.
3. $x-a=0$.
4. $x+b=0$.
5. $0=9-x$.
6. $0=12-x$.
7. $0=-x+14$.
8. $0=-x-17$.
9. $x-2=1$.
10. $x-8=2$.
11. $x-9=13$.
12. $x+3=7$
13. $x+7=8$.
14. $6+x=11$.
15. $8+x=14$.
16. $x+9=2$.
17. $x+11=2$.
18. $x+5=5$.
19. $x-4=4$.
20. $x+7=-7$.
21. $3=4-x$.
22. $4=9-x$.
23. $5=11-x$.
24. $6=4-x$.
25. $12=5-x$.
26. $8=-3-x$.
27. $-15=11-x$.
28. $-5=-x-2$.
29. $-12=-x-18$.
30. $x-1-a=0$.
31. $x+2-b=0$.
32. $x-4+c=0$.
33. $x+9+d=0$.
34. $2x=13+x$.
35. $4x=3x+5$.
36. $6x=8+5x$.
37. $9x=8x-15$.
38. $23x=22x-24$.
39. $3x-25=2x$.

40. $12x - 7 = 11x$.
41. $15x + 4 = 14x$.
42. $-3x - 10 = -4x$.
43. $17 + 30x = 29x$.
44. $14 - 8x = -9x$.
45. $2x + 5 = x + 7$.
46. $3x - 1 = 2x + 6$.
47. $4x - 7 = 3x - 2$.
48. $5x + 8 = 4x + 5$.
49. $7x + 3 = 6x - 5$.
50. $10x + 11 = 9x + 7$.
51. $13x - 6 = 12x - 5$.
52. $6x + 17 = 23 + 5x$.
53. $15x - 1 = 11 + 14x$.
54. $13 + 12x = 11x + 21$.
55. $4 - 15x = 11 - 16x$.
56. $16 - 17x = -18x - 19$.
57. $5 - 19x = -6 - 20x$.
58. $-19x - 33 = -20x - 31$.
59. $-28x - 37 = -40 - 29x$.
60. $-31x - 11 = 29 - 32x$.
61. $21x + \frac{1}{3} = 1 + 20x$.
62. $26x + \frac{1}{5} = 25x + 1$.
63. $27x + \frac{1}{3} = 2 + 26x$.
64. $23x - \frac{1}{5} = 22x + 1$.
65. $34x + \frac{5}{9} = 33x + 1$.
66. $\frac{3}{2}x + 1 = \frac{1}{2}x + 6$.
67. $\frac{7}{4}x + 2 = \frac{3}{4}x + 9$.
68. $\frac{8}{3}x - 7 = 9 + \frac{5}{3}x$.
69. $\frac{4}{7}x - 13 = 4 - \frac{3}{7}x$.
70. $\frac{3}{8}x - 10 = 3 - \frac{5}{8}x$.
71. $\frac{5}{9}x - \frac{1}{7} = \frac{6}{7} - \frac{4}{9}x$.
72. $\frac{4}{11}x - \frac{1}{3} = \frac{5}{3} - \frac{7}{11}x$.
73. $\frac{1}{2}x - \frac{15}{11} = \frac{18}{11} - \frac{1}{2}x$.
74. $\frac{7}{12}x + 18 = -\frac{5}{12}x - 18$.
75. $\frac{13}{9}x - 23 = 23 + \frac{4}{9}x$.
76. $35x + 1 = a + 34x$.
77. $37x - 2 = 36x + b$.
78. $40x + c = 39x + 5$.
79. $42x - d = 41x - 6$.
80. $m - 48x = 4 - 49x$.
81. $50x + \frac{1}{2} = 49x + a$.
82. $54x + b = \frac{1}{3} + 53x$.
83. $61x + b = a + 60x$.
84. $63x - c = 62x + b$.

From each of the following conditional equations derive an equivalent equation by omitting the identical terms from both members:

85. $x + a = a + c$.
86. $x + b = a + b$.
87. $x - d = b - d$.
88. $m + d = m + x$.
89. $a - b = x - b$.
90. $ax + x = ax + 2$.
91. $bx + x = 3 + bx$.
92. $mx - 1 = x + mx$.
93. $x + a = a$.
94. $x - cx = -d - cx$.

MENTAL EXERCISE 44

From each of the following conditional equations derive an equivalent equation in which the coefficient of x shall be unity:

1. $2x = 8$.
2. $3x = 15$.
3. $4x = 12$.
4. $5x = 30$.
5. $6x = 42$.
6. $7x = 56$.

7. $9x = 81$.
8. $14 = 2x$.
9. $18 = 3x$.
10. $25 = 5x$.
11. $33 = 11x$.
12. $42 = 14x$.
13. $8x = -24$.
14. $9x = -54$.
15. $11x = -77$.
16. $-16 = 8x$.
17. $-7 = 7x$.
18. $-22 = 2x$.
19. $3x = 1$.
20. $4x = 1$.
21. $5x = -1$.
22. $1 = 6x$.
23. $1 = 8x$.
24. $-1 = 9x$.
25. $2x = 0$.
26. $0 = 3x$.
27. $5x = 2$.
28. $7x = 3$.
29. $8x = 5$.
30. $9x = 2$.
31. $3 = 11x$.
32. $4 = 13x$.
33. $-5 = 14x$.
34. $3x = 4$.
35. $2x = 5$.
36. $4x = 7$.
37. $5x = 13$.
38. $6x = 19$.
39. $7x = 29$.
40. $8x = 43$.
41. $9x = -28$.
42. $12x = -25$.
43. $24 = 7x$.
44. $32 = 9x$.
45. $43 = 11x$.
46. $-49 = 13x$.
47. $-52 = 12x$.
48. $6x = 3$.
49. $8x = 2$.
50. $18x = 6$.
51. $42x = 7$.
52. $65x = 5$.
53. $34x = 2$.
54. $50x = -10$.
55. $3x = a$.
56. $4x = b$.
57. $5x = c$.
58. $6x = d$.
59. $7x = 2m$.
60. $8x = 3n$.
61. $9x = 5k$.
62. $6a = 11x$.
63. $12b = 17x$.
64. $10x = 8a$.
65. $12x = 14b$.
66. $16x = 18c$.
67. $21d = 14x$.
68. $33k = 6x$.
69. $3x = \frac{1}{2}$.
70. $4x = \frac{1}{3}$.
71. $5x = \frac{1}{7}$.
72. $6x = \frac{1}{4}$.
73. $8x = \frac{1}{8}$.
74. $9x = -\frac{1}{9}$.
75. $2x = \frac{4}{5}$.
76. $3x = \frac{12}{7}$.
77. $4x = \frac{20}{21}$.
78. $5x = \frac{25}{6}$.
79. $\frac{1}{8}x = 2$.
80. $\frac{1}{2}x = 3$.
81. $\frac{1}{4}x = 5$.
82. $\frac{1}{6}x = 4$.
83. $\frac{1}{7}x = 9$.
84. $\frac{1}{11}x = 1$.
85. $\frac{1}{12}x = 12$.
86. $\frac{1}{5}x = -10$.
87. $\frac{1}{10}x = -3$.
88. $\frac{1}{13}x = -13$.
89. $\frac{1}{2}x = 0$.
90. $6 = \frac{1}{7}x$.
91. $12 = \frac{1}{5}x$.
92. $-14 = \frac{1}{3}x$.
93. $\frac{1}{2}x = \frac{1}{3}$.
94. $\frac{1}{3}x = \frac{1}{5}$.
95. $\frac{1}{4}x = \frac{1}{7}$.
96. $\frac{1}{6}x = \frac{1}{9}$.
97. $\frac{1}{8}x = \frac{1}{14}$.
98. $\frac{1}{10}x = \frac{1}{10}$.
99. $\frac{1}{21}x = \frac{1}{28}$.
100. $\frac{1}{25}x = \frac{1}{30}$.
101. $\frac{1}{5}x = \frac{1}{4}$.
102. $\frac{1}{7}x = \frac{1}{2}$.
103. $\frac{1}{9}x = \frac{1}{5}$.
104. $\frac{1}{10}x = \frac{1}{8}$.
105. $\frac{1}{4}x = \frac{1}{2}$.
106. $\frac{1}{12}x = \frac{1}{3}$.
107. $\frac{1}{32}x = \frac{1}{4}$.
108. $\frac{1}{48}x = \frac{1}{8}$.
109. $\frac{1}{3} = \frac{1}{27}x$.
110. $\frac{1}{5} = \frac{1}{55}x$.
111. $\frac{1}{6} = \frac{1}{6}x$.
112. $\frac{4}{3}x = 1$.
113. $\frac{5}{2}x = 1$.
114. $\frac{6}{5}x = 1$.
115. $\frac{9}{5}x = 1$.
116. $\frac{2}{7}x = 1$.
117. $\frac{3}{10}x = 1$.

118. $\frac{2}{3}x = 0$.
119. $1 = \frac{12}{7}x$.
120. $1 = \frac{3}{8}x$.
121. $1 = \frac{5}{11}x$.
122. $\frac{9}{4}x = 2$.
123. $\frac{13}{3}x = 4$.
124. $\frac{5}{2}x = 3$.
125. $\frac{4}{3}x = 5$.
126. $\frac{6}{5}x = 4$.
127. $\frac{9}{10}x = 8$.
128. $\frac{7}{8}x = 9$.
129. $\frac{5}{6}x = 6$.
130. $\frac{7}{5}x = 7$.
131. $\frac{8}{9}x = 6$.
132. $\frac{4}{9}x = \frac{3}{7}$.
133. $\frac{3}{5}x = \frac{4}{7}$.
134. $\frac{3}{4}x = \frac{8}{11}$.
135. $\frac{5}{4}x = \frac{4}{5}$.
136. $\frac{5}{2}x = \frac{8}{3}$.
137. $\frac{8}{3}x = \frac{2}{3}$.
138. $\frac{5}{6}x = \frac{5}{7}$.
139. $\frac{7}{8} = \frac{10}{9}x$.
140. $\frac{10}{13} = \frac{3}{2}x$.
141. $\frac{7}{10} = \frac{2}{3}x$.
142. $\frac{11}{13} = \frac{3}{4}x$.
143. $\frac{1}{2}x = \frac{1}{3}c$.
144. $\frac{1}{3}x = \frac{1}{4}b$.
145. $\frac{1}{5}x = \frac{1}{8}c$.
146. $\frac{1}{8}x = \frac{1}{11}d$.
147. $\frac{1}{9}x = \frac{1}{4}a$.
148. $\frac{1}{15}x = \frac{1}{7}b$.
149. $\frac{1}{6}x = \frac{1}{3}m$.
150. $\frac{1}{20}x = \frac{1}{5}a$.

LINEAR EQUATIONS

EXERCISE 45

Solve each of the following equations regarding x as representing the unknown number:

1. $2x = 6$.
2. $3x = 12$.
3. $10 = 5x$.
4. $15 = 3x$.
5. $7x = 7$.
6. $6x = -18$.
7. $8x = -32$.
8. $-x = 5$.
9. $-6x = 30$.
10. $-9x = 36$.
11. $-11x = -22$.
12. $-12x = -48$.
13. $3x = 4$.
14. $5x = 2$.
15. $6x = 1$.
16. $3 = 5x$.
17. $13 = -6x$.
18. $17 = -2x$.
19. $ax = b$.
20. $bx = b^2$.
21. $cx = 1$.
22. $m = nx$.
23. $-a = bx$.
24. $2 = dx$.
25. $3x = \frac{1}{2}$.
26. $6x = \frac{2}{3}$.
27. $8x = \frac{8}{9}$.
28. $mx = \frac{1}{n}$.
29. $\frac{1}{2}x = 3$.
30. $\frac{1}{4}x = 5$.
31. $\frac{1}{6}x = 8$.
32. $\frac{1}{3}x = \frac{4}{5}$.
33. $\frac{1}{4}x = \frac{9}{10}$.
34. $\frac{1}{6}x = \frac{1}{6}$.
35. $\frac{1}{4}x = \frac{1}{4}$.
36. $\frac{3}{4}x = \frac{2}{5}$.
37. $\frac{2}{3} = \frac{1}{3}x$.
38. $\frac{5}{7} = \frac{1}{5}x$.
39. $\frac{7}{3}x = 2$.
40. $\frac{2}{7}x = 3$.
41. $\frac{3}{5}x = \frac{1}{10}$.
42. $\frac{4}{9}x = \frac{7}{3}$.
43. $(a+b)x = a$.
44. $(m-n)x = 1$.
45. $ax = b + c$.
46. $mx = m + 1$.
47. $ax = c - d$.
48. $x + 1 = 3$.
49. $x + 4 = 9$.
50. $x - 2 = 6$.
51. $x - 7 = 3$.
52. $1 + x = 5$.
53. $3 + x = 2$.
54. $5 = x + 1$.
55. $12 = x - 2$.
56. $14 = 3 + x$.
57. $20 = 7 + x$.
58. $15 = 18 + x$.
59. $9 = x - 10$.

60. $x+a=b$.
61. $c-x=d$.
62. $m=n+x$.
63. $a=c-x$.
64. $2x+1=7$.
65. $3x+2=14$.
66. $4x-5=3$.
67. $5x+6=1$.
68. $6x-1=17$.
69. $7x-9=19$.
70. $3x-8=-20$.
71. $4+3x=13$.
72. $6+8x=38$.
73. $9-2x=1$.
74. $12-5x=2$.
75. $14+6x=-10$.
76. $8=3x+2$.
77. $9=5x+29$.
78. $17=5x+2$.
79. $24=2x+4$.
80. $29=5x-6$.
81. $31=6x-5$.
82. $14=20+3x$.
83. $3x=a+2$.
84. $5x=7-b$.
85. $9x=1+c^2$.
86. $.1x=2$.
87. $.2x=3$.
88. $.4x=.8$.
89. $.9x=9$.
90. $.7x=5.6$.
91. $6x=3.6$.
92. $.3x=.09$.
93. $.2x=.08$.
94. $.01x=1$.
95. $.02x=10$.
96. $.01x=.001$.
97. $.2x=20$.
98. $.05x=.5$.
99. $.002x=.06$.

EXERCISE 46

Solve the following equations, verifying all results by substitution; the first sixty-four examples may be solved mentally:

1. $4x+5=3x+7$.
2. $6x+11=5x+17$.
3. $10x-7=9x+8$.
4. $7x-10-6x=0$.
5. $5x+11-4x=0$.
6. $11x-6-9x=0$.
7. $3x+1=x-1$.
8. $9x+1=3x+5$.
9. $13x+4=11x+10$.
10. $6x-11=2x+9$.
11. $13x-18=x+6$.
12. $8x-13=3x-53$.
13. $22x+15=19x-12$.
14. $15x+37=3x+13$.
15. $12x+1=28+3x$.
16. $19+17x=59-3x$.
17. $15x-13=29+8x$.
18. $21+22x=8x-35$.
19. $7x+2=4x+7$.
20. $17-18x=87-25x$.
21. $15+11x=79-5x$.
22. $2x+23=5x+2$.
23. $4x-23=1-4x$.
24. $2x-37=7x+3$.
25. $3x+5=9x+59$.
26. $5x+1+2x=15$.
27. $6x-2+3x=25$.
28. $7x+10-4x=40$.
29. $9x-2-4x=-57$.
30. $12x=8+30-x$.
31. $5x-4+6x-7=0$.
32. $13x-50-2x=x$.
33. $19x-5+2x=39$.
34. $18x-33-7x=3x-1$.
35. $4x+5+6x=7+8x+4$.
36. $5x+7+3x=x+10+6x$.
37. $8x+3+6x=4x+11+9x$.
38. $9x-7-5x=11x+5-8x$.

39. $9x-7-4x=10x+5-7x$.
40. $15-7x+6=12-2x+19$.
41. $5x+12-8x=19-13x+2$.
42. $22x-9-6x=5x-6+x$.
43. $7x-5+2x=3x+8+x$.
44. $4x+9-7x=8x-11+3x$.
45. $11+7x-18-3x=9+x+5$.
46. $12+11x+3=5x-2-4x$.
47. $19x+9-12x+6=2x+35+3x$.
48. $25+12x-23+14x=25x+12-23x+2$.
49. $8-4x-2+9x=7+2x-19-6x$.
50. $7x+9-3x+5=4x-11+2x+45$.
51. $3x+13+5x-7=x+7+2x+2$.
52. $14x-1+3x+5=7x+2-4x-8$.
53. $5x-(2x+3)=12$.
54. $3x-(13-x)=61$.
55. $12-(4x+7)=13$.
56. $17x+5(2-3x)=18$.
57. $9x-2(1+4x)=3$.
58. $17-3(x+11)=-7x$.
59. $5(x-4)=4(x-3)$.
60. $3(x-6)-2(4-x)=0$.
61. $7(3-x)-4(7-2x)=0$.
62. $6(x+5)-12=3(3x-1)+4x$.
63. $22-5(3-2x)=x-4(x+8)$.
64. $8(x-7)-6(x-5)=5(x-4)-4(x-3)$.
65. $(x+1)^2=x^2+5$.
66. $(x-3)^2=x^2-21$.
67. $(x+4)^2=x(x+3)$.
68. $(3x+1)^2-2x=9x^2+13$.
69. $(x+1)(x+2)=x^2+11$.
70. $(x+3)(x+5)=x^2+31$.
71. $(x+1)(x+5)=(x+2)(x+3)$.
72. $(x-10)(x-7)=(x-9)(x-6)$.
73. $(x+2)(x+4)=(x+3)(x+1)+1$.
74. $(x-4)(x+1)-(x-5)(x-2)=0$.
75. $(x-6)(x-1)-(x+7)(x+3)=0$.
76. $(2x+1)(3x+1)=(6x-1)(x+2)$.
77. $(16x-5)(3x+4)=(12x-1)(4x+3)$.
78. $(2x+5)(5x-4)-5x=(10x-3)(x+1)+8$.
79. $\frac{2}{3}x=8-\frac{1}{3}x$.
80. $\frac{3}{5}x=1-\frac{2}{5}x$.
81. $\frac{3}{2}x=8+\frac{1}{2}x$.
82. $\frac{2}{3}x=4+\frac{1}{3}x$.
83. $\frac{5}{2}x-7=\frac{3}{2}x$.
84. $\frac{5}{6}x-1=1-\frac{1}{6}x$.
85. $\frac{1}{8}x+\frac{1}{4}x=2$.
86. $\frac{1}{2}x-\frac{1}{5}x=4$.
87. $\frac{1}{4}x-\frac{1}{7}x=12$.
88. $\frac{1}{2}x+\frac{1}{3}x=\frac{1}{4}$.
89. $\frac{2}{3}x+\frac{2}{5}x=\frac{5}{6}$.
90. $\frac{4}{3}x-5=\frac{3}{4}x-4$.
91. $\frac{1}{2}x-\frac{1}{3}x+\frac{1}{4}x=\frac{1}{5}$.
92. $\frac{1}{3}x+\frac{1}{4}x+\frac{1}{5}x=x-13$.

93. $\frac{1}{2}(x+3)=4.$ **96.** $\frac{1}{2}(x+4)-\frac{1}{6}x=8.$

94. $\frac{1}{3}(x-4)=1.$ **97.** $\frac{1}{3}(x+7)=\frac{1}{15}(x+6).$

95. $\frac{4}{5}(x+1)-2=0.$ **98.** $\frac{1}{7}(5x-1)-8=\frac{1}{3}(4x-2).$

99. $\frac{1}{8}(1-x)-\frac{1}{10}(2-x)=\frac{1}{12}(3+x).$

100. $\frac{1}{8}(x-15)-\frac{1}{18}(9x-2)=\frac{1}{4}x+\frac{1}{9}.$

101. $\frac{1}{5}(x+\frac{1}{2})-\frac{1}{2}(x-\frac{1}{5})=10.$

102. $\frac{3}{4}(4x-\frac{1}{3})+\frac{4}{3}(3x-\frac{1}{4})=\frac{1}{12}.$

103. $\frac{5}{6}(12x-\frac{6}{5})-\frac{6}{7}(14x+\frac{7}{6})=84.$

104. $\frac{x}{4}-1\frac{1}{4}+x=\frac{1}{3}(6x-9)-\frac{2}{3}x.$

Equations in which decimal fractions appear among the coefficients.

105. $.5x=.015.$ **108.** $.5x=1.$ **111.** $.25x=1.25.$

106. $.01x=200.$ **109.** $.3x=3.$ **112.** $.2x=48-.04x.$

107. $.7x=.07.$ **110.** $.2x=4.$ **113.** $x-.1=1-.1x.$

114. $.2x+3-.04x=3.8.$ **115.** $.2x+.04=.25x-.26.$

116. $.093-.1x=.02x-.13x+.01.$

117. $x-10+.1x=1100-.01x.$

118. $3-.2x+30=.02x-300+.002x.$

ALGEBRAIC EXPRESSION

MENTAL EXERCISE 47

1. By how much does a exceed b?

2. By how much does m exceed n?

3. By how much does a exceed 16?

4. By how much does 25 exceed x?

5. By how much does b exceed 2?

6. What number must be added to y to obtain b?

7. By what number must y be diminished to equal z?

8. What number is less than 12 by b?

9. What number is greater than 16 by a?

10. If x represents an integer, how may the next greater integer be represented? The next less?

11. If m represents an odd integer, how may the next greater odd integer be represented? The next less?

12. If $2x$ represents an even integer, how may the next greater even integer be represented?

13. Find an expression for three consecutive integers of which x is the least.

14. Find an expression for three consecutive integers of which y is the greatest.

15. Find an expression for three consecutive integers of which z is the one between the other two.

16. If a number represented by n is separated into two parts one of which is 4, what is the other part?

17. If a number represented by n is separated into two parts one of which is a, what is the other part?

18. Find an expression for the greater of two numbers if the less is l and the difference is d.

19. A man sold a horse for $\$h$ and gained $\$c$ on the cost. Find an expression for the cost of the horse.

20. A boy is 16 years old now. Find an expression for his age x years ago. How old will he be in y years?

21. If a boy is y years old now, how old will he be 4 years from now? How old was he 5 years ago?

22. A man has $\$a$ and spends $\$x$. Find an expression for the sum remaining.

23. In uniform motion Distance = Rate $\times$ Time.

How far can a person walk in h hours at the rate of m miles per hour?

24. How long will a train require to move uniformly a distance of x miles at the rate of r miles per hour?

25. If a man's expenses are $\$w$ per week, how much will they be for a year?

26. How much will a man whose wages are $\$d$ per day earn in n days? In ab days?

27. Express $\$d$ in terms of cents.

28. Express c cents in terms of dollars.

29. Express d dimes in terms of dollars.

30. Express $\$x$ and y dimes in terms of cents.

31. Express h half dollars and q quarters in terms of cents.

32. Express y yards in terms of feet.

33. Express f feet in terms of inches.

34. Express i inches in terms of yards.

35. Express m inches in terms of feet.

36. Express a feet plus b inches in terms of inches.

37. Express y yards plus f feet plus i inches in terms of inches.

38. Find an expression for g gallons in terms of quarts. In terms of pints.

39. Find an expression for p pounds in terms of ounces and of t tons in terms of pounds.

40. Find an expression for g gallons plus q quarts in terms of quarts.

41. Find an expression for p pounds plus y ounces in terms of ounces.

42. Find an expression for a acres in terms of square rods.

43. Find an expression for x acres plus y square rods in terms of square rods.

44. Express h hours in terms of minutes.

45. Express m minutes in terms of hours. In terms of seconds.

46. Express d days plus h hours in terms of hours.

47. Express x hours plus y minutes in terms of seconds.

48. What is the cost of b books at c cents each?

49. If b books cost $\$d$, find an expression for the cost of one book.

50. If one book costs c cents, how many cents do b books cost? How many dollars?

51. If the interest on $\$1$ for one year is r cents, what will be the interest on $\$d$ for one year at the same rate?

52. Find an expression for one per cent of x; five per cent of y; seventy-five per cent of z; thirty-seven and one-half per cent of w.

53. Find an expression for x per cent of b.

54. Find an expression for x per cent of x.

55. Find an expression in square feet for the area of a rectangular room which is l feet long and w feet wide. In square yards.

56. What is the area in square feet of a square room each of whose sides is x feet in length? In square yards?

57. Find an expression in square yards for the area of a rectangular room which is a yards long and b yards wide. In square feet.

PROBLEMS

PROBLEMS SOLVED BY MEANS OF LINEAR EQUATIONS

EXERCISE 48

Solve the following problems, and examine all of the solutions obtained to see if they satisfy the conditions of the given problems:

1. Find two numbers whose sum is 33 and whose difference is 5.

2. Find two numbers whose sum is 18 and whose difference is 8.

3. Find two numbers whose sum is 30 and whose difference is 4.

4. Find two numbers whose sum is 55 and whose difference is 7.

5. Find two numbers whose sum is 60 and whose difference is 8.

6. Find two numbers, the greater of which exceeds the less by one, whose sum is 75.

7. Find two numbers, such that the greater exceeds the less by one, whose sum is 49.

8. Find two consecutive numbers whose sum is 67.

9. Find three consecutive numbers whose sum is 36.

10. Find three consecutive numbers whose sum is 60.

11. Find four consecutive numbers whose sum is 198.

12. Find two consecutive odd numbers whose sum is 44.

13. Find two consecutive even numbers whose sum is 70?

14. What number is it whose double is 49 more than its fourth part?

15. What number is it whose double is 30 more than its third part?

16. What number is it whose double is 36 more than its fifth part?

17. What number is it whose double is 77 more than its sixth part?

18. Find two numbers differing by 18 whose sum is four times their difference.

19. Find two numbers whose difference is 18 and whose sum is twice their difference.

20. Find two numbers whose difference is 28 and whose sum is twice their difference.

21. Find two numbers differing by 16 whose sum is twice their difference.

22. Find two numbers differing by 8 whose sum is seven times their difference.

23. Find two numbers differing by 23 whose sum is five times their difference.

24. Separate 85 into two parts such that one part shall exceed the other by 11.

25. Separate 74 into two parts such that one part shall be greater than the other part by 2.

26. Separate 72 into two parts such that one part shall exceed the other part by 18.

27. Separate 125 into two parts such that one part shall be less than the other by 23.

28. Separate 147 into two parts such that one part shall be less than the other by 37.

29. If $2x + 9$ represents 23, what number will $x + 3$ represent?

30. If $3x + 5$ represents 35, what number will $x + 2$ represent?

31. If $4x - 1$ represents 43, what number will $x - 4$ represent?

32. If $x - 10$ represents 3, what number will $10 - x$ represent?

33. If $4x - 7$ represents 25, what number will $5x - 8$ represent?

34. If $2x - 7$ represents 17, what number will $3x - 5$ represent?

35. At what time between one and two o'clock are the hands of a watch together?

36. At what time between 6 and 7 o'clock are the hands of a watch together?

37. At what time between 5 and 6 o'clock are the hands of a watch together?

38. At what time between 12 and 1 o'clock are the hands of a watch opposite each other?

39. At what time between 7 and 8 o'clock are the hands of a watch opposite each other?

40. At what time between 9 and 10 o'clock are the hands of a watch opposite each other?

41. At what time between 3 and 4 o'clock are the hands of a watch at right angles to each other?

42. What number when subtracted from 85 will produce the same result as when added to 37?

43. What number when subtracted from 50 will produce the same result as when added to 12?

44. What number when subtracted from 75 will produce the same result as when added to 35?

45. What number when subtracted from 63 will produce the same result as when added to 27?

46. Find the number whose double increased by 5 is equal to 19.

47. Find the number whose double increased by 14 is equal to 40.

48. Five times a certain number diminished by 7 is equal to four times the number increased by 3. Find the number.

49. What number exceeds 6 by as much as three times the number exceeds 28?

50. What number is that from which if 14 be subtracted, $\frac{2}{5}$ of the difference will be equal to 20?

51. Find A's age if in 34 years he will be three times as old as he is now.

52. Find A's age, if in 24 years he will be four times as old as he is now.

53. Find B's age, if in 45 years he will be four times as old as he is now.

54. Eight years hence a man will be twice as old as he was 10 years ago. How old is he now?

55. A's present age is four times B's. Find the ages of A and B if in 20 years A's age will be double B's age.

56. Divide \$75 between A and B in such a way that A shall receive \$29 more than B.

57. Divide \$96 between A and B in such a way that A shall receive twice as much as B.

58. Divide \$154 between A and B in such a way that B shall receive \$24 more than A.

59. Two men starting from towns 80 miles apart travel toward each other, one at the rate of 6 miles an hour, and the other at the rate of 4 miles an hour. In how many hours will they meet?

60. Two men starting from the towns of (name here two towns in your State) which are (give here the distance between the towns in miles) apart and travel toward each other, one at the rate of (name some number of

miles per hour which you think a man could walk) and the other at the rate of (name the number of miles per hour which you think the other man could walk) per hour. In how many hours will the men meet? Vary the problem by choosing different cities or towns and different rates at which the men are supposed to walk.

61. Two ships starting from (name two cities) which are about (state here your estimate of the distance from one city to the other) miles apart, sail toward each other, one at the rate of (name some rate in miles per hour for one ship) miles per hour, and the other at the rate of (name some rate in miles per hour for the other ship). In how many hours will the ships meet? Vary the problem by choosing different cities and different rates for the ships.

62. A rectangular field is five times as long as it is wide. The length of the fence about the field is 180 yards. What are the length and the width of the field?

63. A rectangular field is twice as long as it is wide. The length of the fence about it is 216 yards. What are the dimensions of the field?

64. A rectangular field is three times as long as it is wide and the distance around it is 320 yards. Find its dimensions.

65. Separate 65 into two parts such that two times one part shall equal three times the other part.

66. Separate 95 into two parts such that two times one part shall equal three times the other.

67. Separate 147 into two parts such that four times one part shall equal three times the other part.

68. Separate 36 into two parts such that one part shall be 6 greater than two times the other part.

69. What number is that to which if 3 be added, $\frac{2}{5}$ of the sum will equal 16?

70. What number is that to which if 8 be added, $\frac{2}{3}$ of the sum will equal 16?

71. What number is that to which if 25 be added, $\frac{3}{4}$ of the sum will equal 30?

72. One-fourth of a certain number exceeds $\frac{1}{3}$ of the number by 5. Find the number.

73. The excess of a number over 12 equals $\frac{1}{3}$ of that number. Find the number.

74. The seventh part of a number is 5 less than $\frac{1}{2}$ of the number. What is the number?

75. One-fourth of a number added to four times the number is equal to 34. Find the number.

76. One-fifth of a number increased by twice the number is equal to 22. Find the number.

77. Find a number such that its fifth part shall exceed its twelfth part by 7.

78. One-fifth part of a certain number exceeds 6 by as much as 6 exceeds one-seventh of the number. Find the number.

79. Find a number which when multiplied by 7 exceeds 27 by as much as 27 exceeds twice the number.

EXERCISE 49

Solve the following problems and examine the solutions to see if they satisfy the given conditions:

1. The sum of the ages of A and B is 42 years, and four years hence A's age will be four times that of B's. Find their present ages.

2. In a company of 30 persons a resolution is carried by a majority of 6, all voting. How many voted for the measure?

3. In an informal ballot a resolution was adopted by a majority of six votes, but in a formal vote one-third of those who had before voted for it voted against it and the resolution was lost by a majority of four votes. How many voted each way in the formal ballot?

4. A grocer estimated that his supply of sugar would last eight weeks. He sold on an average 50 pounds a day more than he expected. It lasted him six weeks. How much did he have?

5. Determine how an amount of $135 must be divided among three persons in such a way that the share of the first shall be three times that of the second, and the share of the second twice that of the third.

6. A sum of $7924 was bequeathed to three persons with the stipulation that the first was to receive twice as much as the second and one-half as much as the third. Determine the amounts.

7. A man wishes to divide the sum of $99 into five parts in such a way that the first part shall exceed the second by $3, be less than the third part by $10, greater than the fourth part by $9, and less than the fifth part by $16. Find the parts.

8. A paymaster wishing to use $25,662 on pay day requested the paying teller to make up the amount in the following way: A certain number of $100 bills, three times as many fifties, four times as many twenties as fifties, twice as many tens as fifties, three times as many fives as tens, as many twos as tens, as many ones as twos.

How many bills of each denomination were given?

9. At two stations A and B on a line of railway the prices of coal are $3.50 per ton and $4 per ton, respectively. If the distance between A and B be 150 miles and coal can be shipped for one-half a cent per ton per mile, find the place on the railway between A and B at which it will be indifferent to a customer whether he buys coal from A or from B.

10. A farmer estimated that his supply of feed for his 50 cows would last only 12 weeks. How many cows must he sell in order that the supply may last 20 weeks?

11. A contractor undertakes to put in a concrete foundation in a given time. By the terms of the contract he is to receive $12 for each day's work during the given time, and is to forfeit $5 for each day taken beyond that time. If the total amount received was $167 for 21 days' work, find the time for the original contract.

12. It was estimated that a certain amount of earth could be excavated by a steam shovel alone in 12 days, or by a gang of laborers alone in 28 days. After being used a certain number of days the shovel was disabled and the work was then completed by the men, who worked 2 days less than the time during which the shovel had been used. During how many days was the shovel used?

13. Sixty laborers were engaged to remove an embankment. Some of them were engaged at the rate of $1.10 a day, and the others at the rate of $1.60 a day. The memorandum having been lost, it is required to find how many worked at each rate if the total amount paid was $80.

14. The help of a certain factory numbering 316 consists of men and boys. If the weekly pay of each man is $12 and that of each boy $4, find the number of each, if the weekly pay roll amounts to $2688.

15. It is observed that a square room requires one and one-ninth square yards less of carpeting than a rectangular room whose length is one yard longer and width two feet less than the side of the square room. Find the area of each of the rooms.

16. A man has $5600 in four banks. He has twice as much in the second bank as in the first, as much in the third as in the first and second together, and twice as much in the fourth as in the first and third together. How much money has he in the fourth bank?

17. A man invests $\frac{1}{3}$ of his capital at 4 per cent., $\frac{1}{4}$ at 3 per cent., and the remainder at $3\frac{1}{2}$ per cent., and thus secures an annual income of $1390. What is his capital?

18. A man invests $\frac{5}{6}$ of his capital at 3 per cent. and the rest at $3\frac{1}{2}$ per cent., and thus receives an annual income of $55.50. What is his capital?

19. A man desires to invest his capital of $20,000 partly at 5 per cent. and partly at 3 per cent. How must he invest the amounts so that his yearly income shall be at the yearly rate of $4\frac{1}{4}$ per cent.?

20. How may $10,800 be invested partly in 5 per cent. stock whose market value is eighty cents on a dollar, and the remainder in 6 per cent. stock selling for one dollar and twenty cents on a dollar, in order that the income from the two sources may be the same?

21. How long will it take an investment of $5730 to amount to $6589.50 at 3 per cent. simple interest?

22. At what time between ten and eleven o'clock are the hands of a watch together?

23. At what time between 9 and 10 o'clock are the hands of a watch opposite each other?

24. At what times between 7 and 8 o'clock will the hands of a clock be at right angles to each other?

25. At what time between 4 and 5 o'clock is the minute hand of a watch two minutes in advance of the hour hand?

26. Two trains start at the same time from different stations 400 miles apart. One travels at the rate of 48 miles an hour and the other at the rate of 32 miles an hour. How far does the faster train travel before meeting the slower one?

27. Two hours after a train left a certain station a second train was dispatched, and it overtook the first train in four hours. To accomplish this it was necessary for the second train to run 15 miles an hour faster than the first. How many miles per hour did the trains run?

28. Where must a side track be placed on a single-track railway in order that an express train, traveling at the rate of 46 miles an hour, may not be delayed by an accommodation train traveling toward it at the rate of 29 miles an hour, the two trains starting at the same time from two places 60 miles apart?

29. If an outward trip of an excursion train is made at the rate of 20 miles an hour and the return trip at 16 miles an hour, the whole time being 9 hours, what is the distance?

30. Two ships start from a given port at the same time, one going north at the rate of 11 miles per hour and the other going south at the rate of 7 miles per hour. How long after starting will they at these rates be exactly 108 miles apart?

31. Two ferry boats, whose rates are 15 miles and 12 miles an hour, respectively, start simultaneously from opposite shores of a river, three-fourths of a mile wide. Where will they meet?

EXERCISE 50

Problems in Science

1. A certain grade of sulphuric acid is known to be 96 per cent. pure. How much distilled water must be added to a gallon of this sulphuric acid in order that the mixture may be 80 per cent. pure?

2. How much water must be added to a pint of alcohol 85 per cent. pure in order to make the mixture three-fourths water?

3. How much 12 per cent. solution of a certain chemical must be added to a gallon of 4 per cent. solution to raise it to a 6 per cent. solution?

4. How much water must be added to 6 quarts of acid which is 10 per cent. of full strength to make the mixture $8\frac{1}{2}$ per cent. of full strength?

5. How much water must be added to a gallon of three per cent. solution of a certain chemical to reduce it to a one per cent. solution?

6. How many ounces of pure silver must be melted with 300 ounces of silver 600 fine in order to make a bar of metal 800 fine?

By 600 fine is meant the number of parts in 1000 which are pure metal.

7. How many ounces each of two bars of silver which are 800 fine and 725 fine, respectively, must be melted together to make a bar of 60 ounces which shall be 775 fine?

8. How many pounds of pure copper must be melted with 500 pounds of gold $\frac{7}{12}$ pure in order to make the composition $\frac{9}{10}$ pure gold?

9. It has been said that the crown of Hiero of Syracuse, which was part gold and part silver, weighed 20 pounds in the air and $18\frac{3}{4}$ pounds when weighed while immersed in water. Find how much gold and how much silver it contained, knowing that $19\frac{1}{4}$ pounds of gold and $10\frac{1}{2}$ pounds of silver each lose one pound when weighed immersed in water.

10. The pendulum of a clock swings 364 times in 5 minutes while that of a second clock swings 233 times in 4 minutes. After how long will the second have swung 582 times less than the first?

INTEGRAL FACTORS

PREPARATORY EXERCISE 51

Perform the following indicated multiplications:

1. $3(a+b)$.
2. $5(x+2)$.
3. $2x(b-c)$.
4. $4a(a^2+1)$.
5. $a^2(a-b)$.
6. $b^2(b^2-b+1)$.
7. $xy(x^2-xy+y^2)$.
8. $a^2bc(a+b+c)$.
9. $2a^mb^m(x-3y+4z)$.
10. $a^mb^mc^m(a+b+c)$.

EXPRESSIONS ALL OF WHOSE TERMS CONTAIN A COMMON MONOMIAL FACTOR

EXERCISE 52

Factor each of the following expressions:

1. $3a+3b$.
2. $5x-5y$.
3. $7m+14n$.
4. $12x-4y$.
5. $15a-18b$.
6. $16x-36y$.
7. $8c+22de$.
8. $24ab-42cd$.
9. $36xy-63$.
10. $5m-10$.
11. $12+6x$.
12. $7-14y$.
13. $ab+ac$.
14. $bx+xy$.
15. $ac-cd$.
16. a^2-ab.
17. b^2+bc.
18. $ax-x^2$.
19. $by+b^2z$.
20. $15m+18mn$.
21. $5xy-20x^2$.

22. $12\,ab + 54\,ac$.
23. $21\,x^2y - 49\,yz^2$.
24. $x^2y + xy^2$.
25. $a^2b - ab^2$.
26. $abc + abd$.
27. $bcx - bcy$.
28. $xyz - yzw$.
29. $a^2bc - ab^2c$.
30. $x^2yz + xyz^2$.
31. $bc^2d + b^2cd^2$.
32. $x^2yz^2 + x^2y^2z$.
33. $d^2h^2k - dh^2k^2$.
34. $a^3b^2c + ab^2c^3$.
35. $a^5b^3c + a^6b^4c^2$.
36. $ab + ac + ad$.
37. $abc + bcd + cde$.
38. $x^2y^2 + x^2y + xy^2$.
39. $a^2bc + ab^2c + abc^2$.
40. $x^2y^3z^4 + x^4y^2z^3 + x^3y^4z^2$.
41. $a^2bc^2d + ab^2cd^2 + a^2bcd^2$.
42. $6\,a^2b + 12\,ab^2 + 18\,a^2b^2$.
43. $21\,x^2y - 7\,xy^2z + 14\,xyz$.
44. $13\,x^4y - 39\,x^3z - 26\,x^2w$.
45. $60\,ag^2m + 84\,by^3m + 36\,cy^4m$.
46. $38\,x^3y^4 + 57\,x^4y^3 - 19\,x^3$.
47. $34\,a^3b^3 - 51\,a^3bc + 85\,a^2bc^2$.
48. $32\,x^2y^3z^2 - 48\,xy^4z^2 + 64\,x^2y^4z$.
49. $77\,m^5nr - 99\,m^5n + 88\,mrs^2$.
50. $13\,a^3qx^2 + 15\,aq^2x^2 - 2\,a^2q^2x - 9\,agx$.
51. $ab^2c^2d^2 + a^2bc^2d^2 + a^2b^2cd^2 + a^2b^2c^2d$.
52. $x^3y^3z^2 + x^2y^3z^3 + x^3y^2z^3$.
53. $a^2b^2c^3d^4 + ab^3c^3d^4 + ab^2c^4d^4$.
54. $3\,x^3y^2z - 6\,xy^3z^2 + 3\,x^2yz^3$.
55. $ax^my^m - bx^my^m + cx^my^m - dx^my^m$.
56. $a^{m+1}b^{m+1}c^m - a^mb^{m+1}c^{m+1} + a^{m+1}b^mc^{m+1}$.

PREPARATORY EXERCISE 53

Perform the following indicated multiplications:

1. $(a + b)(x + y)$.
2. $(m + n)(a + b)$.
3. $(x - y)(2\,a + b)$.
4. $(m - n)(3\,m + 4)$.
5. $(x^2 + 1)(x + 1)$.
6. $(2\,a - 3\,b)(a + 4\,b)$.
7. $(a + b)(x + y + z)$.
8. $(2\,x - y)(m - n + 4)$.
9. $(a + b + c)(x + y + z)$.
10. $(x - y + z)(r + s - t)$.

PREPARATORY EXERCISE 54

Write each of the following expressions as the product of two factors:

1. $3(a + b) + x(a + b)$.
2. $5(x + y) + y(x + y)$.
3. $2(m - n) + a(m - n)$.
4. $a(b + c) + d(b + c)$.
5. $x(a - c) + y(a - c)$.
6. $3\,m(x + 2) + (x + 2)$.
7. $4\,x(y - 1) - 3\,z(y - 1)$.
8. $6(x - y) - b(y - x)$.

9. $ab(x^2-2)+c(2-x^2)$.
10. $a(x+y)+b(x+y)+c(x+y)$.
11. $x(a-b)-y(a-b)+z(a-b)$.
12. $m(a+1)-n(a+1)+(a+1)$.
13. $5x(2a-b)-2y(b-2a)+(2a-b)$.

EXPRESSIONS IN WHICH GROUPS OF TERMS HAVE A COMMON FACTOR

EXERCISE 55

Factor each of the following expressions:

1. $ax+ay+bx+by$.
2. $mx+my+nx+ny$.
3. $bm-bn+cm-cn$.
4. $ax+2a+bx+2b$.
5. $cy+3c+dy+3d$.
6. $ax+ay+5x+5y$.
7. $bm-bn-cm+cn$.
8. $ab+ay+bx+xy$.
9. $bc+by-cx-xy$.
10. $mn-mz-ny+yz$.
11. a^3+a^2+a+1.
12. $1-x+x^2-x^3$.
13. $1+a+b+ab$.
14. a^3+2a^2+3a+6.
15. $21+3x+7x^2+x^3$.
16. $a^2+4a+ab+4b$.
17. $x^2+x+y+xy$.
18. $y^2-yz+xy-xz$.
19. a^3+a^2-5a-5.
20. $6+6y+y^2+y^3$.
21. $ax+ay+az+bx+by+bz$.
22. $5a+5b+5c+ax+bx+cx$.
23. $ag+ak+am+2g+2k+2m$.
24. $cx-cy+cz+dx-dy+dz$.
25. $2mn-2ny-mx+xy+2n^2-nx$.
26. $21b-5a+3ab-2ac-14c-35$.
27. $xz+x-5yz-5y-6z-6$.
28. $ax+ay-bx-by+bz-az$.
29. $ax-bx+x-ay+by-y$.
30. $cx-dx+cy-dy+cz-dz$.
31. $am-bn+an-bm+cm+cn$.
32. $abx+bcx+cax+aby+bcy+cay$.
33. $a^2c+b^2e+a^2d+b^2d+a^2e+b^2c$.

Fill in the following blank forms with numbers and letters such that the resulting expressions can be factored:

34. ( )+( )+( )+( ).
35. ( )−( )+( )−( ).

PREPARATORY EXERCISE 56

1. Commit to memory the squares of all of the integral numbers from 1 to 25 inclusive.

2. Commit to memory the cubes of all of the integral numbers from 1 to 12 inclusive.

3. Commit to memory the following powers:

$$2^4,\ 3^4,\ 4^4,\ 5^4,\ 6^4,\ 2^5,\ 3^5,\ 2^6,\ 3^6.$$

Find the following indicated roots:

4. $\sqrt{4}$.
5. $\sqrt{25}$.
6. $\sqrt{81}$.
7. $\sqrt[4]{81}$.
8. $\sqrt{169}$.
9. $\sqrt{196}$.
10. $\sqrt{324}$.
11. $\sqrt{361}$.
12. $\sqrt{576}$.
13. $\sqrt{a^2}$.
14. $\sqrt{b^4}$.
15. $\sqrt{c^6}$.
16. $\sqrt{4\,a^2}$.
17. $\sqrt{9\,b^2}$.
18. $\sqrt{25\,c^2}$.
19. $\sqrt[3]{8\,x^3}$.
20. $\sqrt[5]{27\,y^6}$.
21. $\sqrt{36\,c^4}$.
22. $\sqrt{25\,d^8}$.
23. $\sqrt{16\,x^{16}}$.
24. $\sqrt[4]{16\,x^{16}}$.

PREPARATORY MENTAL EXERCISE 57

In the following exercise find the missing terms which will when supplied make the expressions trinomial squares of the form $a^2 + 2\,ab + b^2$:

1. $a^2 + (\ \ ) + 9$.
2. $b^2 + (\ \ ) + 25$.
3. $c^2 + (\ \ ) + 64$.
4. $d^2 + (\ \ ) + 100$.
5. $x^2 - (\ \ ) + 16$.
6. $y^2 - (\ \ ) + 49$.
7. $z^2 - (\ \ ) + 81$.
8. $d^2 + (\ \ ) + 121$.
9. $e^2 - (\ \ ) + 144$.
10. $4\,a^2 + (\ \ ) + 1$.
11. $9\,b^2 + (\ \ ) + 1$.
12. $16\,c^2 + (\ \ ) + 1$.
13. $36\,d^2 - (\ \ ) + 1$.
14. $25\,g^2 - (\ \ ) + 1$.
15. $9\,x^2 + (\ \ ) + 4$.
16. $16\,y^2 + (\ \ ) + 25$.
17. $49\,z^2 - (\ \ ) + 4$.
18. $9\,w^2 - (\ \ ) + 9$.
19. $16\,a^2 - (\ \ ) + 16$.
20. $b^2 + (\ \ ) + c^2$.
21. $a^2 - (\ \ ) + x^2$.
22. $c^2 - (\ \ ) + y^2$.
23. $4\,d^2 - (\ \ ) + x^2$.
24. $9\,h^2 - (\ \ ) + k^2$.
25. $y^2 - (\ \ ) + 64\,z^2$.
26. $z^2 - (\ \ ) + 81\,w^2$.
27. $1 + (\ \ ) + x^2$.
28. $1 - (\ \ ) + y^2$.

29. $1 + (\quad) + 169\,a^2$.
30. $1 - (\quad) + 196\,b^2$.
31. $4\,a^2 + (\quad) + 9\,n^2$.
32. $1 + (\quad) + 121\,c^2$.
33. $1 - (\quad) + 225\,d^2$.
34. $1 + (\quad) + 256\,k^2$.
35. $16\,b^2 + (\quad) + 25\,c^2$.
36. $9\,d^2 - (\quad) + 16\,k^2$.
37. $25\,x^2 + (\quad) + 64\,y^2$.
38. $49\,x^2 + (\quad) + 16\,y^2$.
39. $36\,a^2 - (\quad) + 4\,b^2$.
40. $64\,b^2 - (\quad) + 9\,c^2$.
41. $49\,x^2 + (\quad) + 49\,y^2$.
42. $81\,a^2 + (\quad) + 25\,b^2$.
43. $121\,a^2 - (\quad) + 16\,b^2$.
44. $9\,d^2 + (\quad) + 100\,m^2$.
45. $4\,y^2 + (\quad) + 169\,z^2$.
46. $81\,c^2 - (\quad) + 121\,w^2$.
47. $100\,h^2 - (\quad) + 64\,q^2$.
48. $121\,k^2 - (\quad) + 49\,m^2$.
49. $225\,r^2 - (\quad) + 36\,v^2$.
50. $25\,s^2 - (\quad) + 256\,t^2$.
51. $196\,n^2 + (\quad) + 25\,r^2$.
52. $a^2 + 2\,ab + (\quad)$.
53. $c^2 + 2\,cd + (\quad)$.
54. $a^2 + 2\,ac + (\quad)$.
55. $b^2 + 2\,bm + (\quad)$.
56. $c^2 - 2\,ck + (\quad)$.
57. $d^2 + 4\,d + (\quad)$.
58. $x^2 + 6\,x + (\quad)$.
59. $y^2 + 10\,y + (\quad)$.
60. $y^2 - 14\,y + (\quad)$.
61. $z^2 + 18\,z + (\quad)$.
62. $a^2 - 24\,a + (\quad)$.
63. $4\,a^2 + 12\,a + (\quad)$.
64. $9\,b^2 + 18\,b + (\quad)$.
65. $25\,c^2 + 40\,c + (\quad)$.
66. $49\,d^2 - 28\,d + (\quad)$.
67. $(\quad) + 2\,m + 1$.
68. $(\quad) + 2\,x + 1$.
69. $(\quad) + 4\,y + 1$.
70. $(\quad) + 10\,z + 1$.
71. $(\quad) + 16\,w + 1$.
72. $(\quad) + 4\,a + a^2$.
73. $(\quad) + 6\,b + b^2$.
74. $(\quad) + 12\,c + c^2$.
75. $(\quad) + 14\,d + d^2$.
76. $(\quad) + 2\,ak + k^2$.
77. $(\quad) + 2\,mn + n^2$.
78. $(\quad) - 2\,ay + y^2$.
79. $(\quad) - 4\,az + z^2$.
80. $(\quad) - 10\,by + y^2$.
81. $(\quad) + 6\,x + (\quad)$.
82. $(\quad) + 12\,y + (\quad)$.
83. $(\quad) - 10\,z + (\quad)$.
84. $(\quad) - 24\,w + (\quad)$.

TRINOMIAL SQUARES

EXERCISE 58

Factor each of the following expressions:

1. $x^2 + 2\,x + 1$.
2. $m^2 - 2\,m + 1$.
3. $z^2 - 2\,zw + w^2$.
4. $a^2 + 2\,ak + k^2$.
5. $r^2 - 2\,rv + v^2$.
6. $q^2 - 2\,qx + x^2$.
7. $a^2 + 4\,a + 4$.
8. $b^2 + 8\,b + 16$.
9. $c^2 + 12\,c + 36$.
10. $d^2 - 18\,d + 81$.
11. $25 + 10\,a + a^2$.
12. $49 - 14\,y + y^2$.

13. $64 - 16z + z^2$.
14. $100 + 20g + g^2$.
15. $4a^2 + 4a + 1$.
16. $9b^2 + 6b + 1$.
17. $36c^2 + 12c + 1$.
18. $81n^2 - 18n + 1$.
19. $16a^2 + 24ab + 9b^2$.
20. $9c^2 + 30cd + 25d^2$.
21. $25k^2 + 70kw + 49w^2$.
22. $121s^2 + 44st + 4t^2$.
23. $49h^2 - 140hy + 100y^2$.
24. $16g^2 + 72gm + 81m^2$.
25. $121k^2 - 242ks + 121s^2$.
26. $36m^2 + 108mn + 81n^2$.
27. $81d^2 - 126de + 49e^2$.
28. $64h^2 + 320hw + 400w^2$.
29. $225r^2 - 120rs + 16s^2$.
30. $169t^2 - 260tw + 100w^2$.
31. $100a^2 - 280ab + 196b^2$.
32. $256c^2 + 96cd + 9d^2$.
33. $64a^2b^2 + 80abc + 25c^2$.
34. $16x^2 - 56xyz + 49y^2z^2$.
35. $4c^2 - 36cde + 81d^2e^2$.
36. $25a^2b^2 + 40abcd + 16c^2d^2$.
37. $36a^2x^2 - 84abxy + 49b^2y^2$.
38. $9b^2y^2 + 48bcyz + 64c^2z^2$.
39. $49a^2y^2 + 42abxy + 9b^2x^2$.
40. $4a^2z^2 - 20acmz + 25c^2m^2$.
41. $25a^2b^2c^2 - 60abcd + 36d^2$.
42. $81x^2 + 180xyzw + 100y^2z^2w^2$.
43. $(x+y)^2 + 2(x+y) + 1$.
44. $(a-b)^2 - 2(a-b) + 1$.
45. $(m+n)^2 + 6(m+n) + 9$.
46. $(x-y)^2 + 10(x-y) + 25$.
47. $16 + 8(a+b) + (a+b)^2$.
48. $36 - 12(a-x) + (a-x)^2$.
49. $(a+b)^2 + 2(a+b)(x+y) + (x+y)^2$.
50. $(a-b)^2 - 2(a-b)(x-y) + (x-y)^2$.
51. $(b+c)^2 - 8(b+c)(x+y) + 16(x+y)^2$.
52. $9(a+b)^2 + 12(a+b)(c+d) + 4(c+d)^2$.
53. $49(d+k)^2 - 70(d+k)(m+w) + 25(m+w)^2$.

PREPARATORY EXERCISE 59

Supply the terms necessary to make the following polynomials complete squares:

1. $a^2 + b^2 + c^2 + 2ab + ? + ?$.
2. $x^2 + y^2 + z^2 + 2xy - ? - ?$.
3. $b^2 + h^2 + m^2 - ? + 2bm - ?$.
4. $k^2 + l^2 + m^2 - ? - ? + 2lm$.
5. $w^2 + x^2 + y^2 - 2wx + ? - ?$.
6. $c^2 + d^2 + h^2 + ? - 2ch - ?$.
7. $4x^2 + 9y^2 + z^2 + ? + ? + ?$.
8. $9a^2 + x^2 + 16y^2 + ? + ? + ?$.
9. $b^2 + 25y^2 + n^2 + ? - ? - ?$.
10. $a^2 + 36w^2 + 16 - ? - ? + ?$
11. $x^2 + 49y^2 + 4 + ? - ? - ?$.
12. $25a^2 + 4b^2 + 64 - ? + ? - ?$.
13. $49a^2 + 9b^2 + ? - ? + 28a - 12b$.
14. $9x^2 + ? + ? + 6xy + 30x + 10y$.
15. $4a^2 + b^2 + ? + ? + 12a + ?$.
16. $a^2 + ? + ? - 2ac + 2ay - ?$.

17. $25\,x^2 + ? + 9 - ? + 30\,x - 6\,y$.
18. $h^2 + ? + y^2 + ? - ? - 2\,ny$.
19. $? + ? + ? + 2\,bx + 2\,bm + 2\,mx$.
20. $? + ? + ? - 2\,xy + 2\,xz - 2\,yz$.

EXERCISE 60

Obtain factors of the following expressions, checking all results numerically:

1. $a^2 + 2\,ab + b^2 + 2\,a + 2\,b + 1$.
2. $m^2 + 2\,mn + n^2 + 10\,m + 10\,n + 25$.
3. $a^2 + b^2 + x^2 + 2\,ab + 2\,ax + 2\,bx$.
4. $x^2 + 6\,xy + 9\,y^2 + 2\,x + 6\,y + 1$.
5. $c^2 + d^2 + e^2 - 2\,cd - 2\,ce + 2\,de$.
6. $c^2 + d^2 + 4 - 2\,cd + 4\,c - 4\,d$.
7. $x^2 + y^2 + z^2 + 2\,zx - 2\,zy - 2\,xy$.
8. $4\,a^2 + b^2 + 1 + 4\,ab + 4\,a + 2\,b$.
9. $9\,a^2 + 4\,b^2 + c^2 - 12\,ab - 6\,ac + 4\,bc$.
10. $a(a + 2\,k) + w(w + 2\,a) + k(k + 2\,w)$.
11. $b(b + 2\,c) + c(c + 2\,d) + d(d + 2\,b)$.
12. $a(a - 2\,b) + b(b + 2\,c) + c(c - 2\,a)$.
13. $(x^2 + 2\,yz) + (y^2 + 2\,xy) + (z^2 + 2\,xz)$.
14. $a^2 + b^2 + c^2 - 2(ab - ac + bc)$.
15. $4\,x^2 + y^2 + 9\,z^2 + 2(2\,xy - 6\,xz - 3\,yz)$.

PREPARATORY EXERCISE 61

Obtain each of the following indicated products:

1. $(x + 3)(x - 3)$.
2. $(y + 5)(y - 5)$.
3. $(z + 6)(z - 6)$.
4. $(2\,a + 1)(2\,a - 1)$.
5. $(3\,x + 4)(3\,x - 4)$.
6. $(5\,a + 2\,b)(5\,a - 2\,b)$.
7. $(7\,xy + z)(7\,xy - z)$.
8. $(a^n + 8)(a^n - 8)$.
9. $(x^3 + 1)(x^3 - 1)$.
10. $(y^4 + 4)(y^4 - 4)$.

THE DIFFERENCE OF TWO SQUARES

EXERCISE 62

Factor each of the following expressions:

1. $a^2 - 4$.
2. $b^2 - 16$.
3. $c^2 - 25$.
4. $d^2 - 49$.
5. $h^2 - 81$.
6. $25 - x^2$.

7. $36 - y^2$.
8. $64 - z^2$.
9. $81\,a^2 - 1$.
10. $49\,b^2 - 1$.
11. $100\,a^2 - b^2$.
12. $4\,c^2 - 25\,d^2$.
13. $9\,r^2 - 16\,w^2$.
14. $16\,q^2 - 49\,t^2$.
15. $25\,g^2 - 64\,m^2$.
16. $1 - 121\,n^2$.
17. $1 - 169\,x^2$.
18. $25\,a^2 - 144\,b^2$.
19. $9\,b^2 - 100\,c^2$.
20. $25 - 64\,a^2b^2$.
21. $64 - 49\,x^2y^2$.
22. $81 - 36\,m^2w^2$.
23. $36 - 81\,c^2t^2$.
24. $25\,p^2 - 144\,x^2w^2$.
25. $100\,a^2b^2 - 81\,x^2y^2$.
26. $196\,c^2d^2 - 100\,m^2y^2$.
27. $225\,a^2x^2 - 169\,b^2y^2$.
28. $256\,a^2d^2 - 144\,b^2c^2$.
29. $289\,a^2x^2 - 49\,b^2y^2$.
30. $324\,a^2n^2 - 64\,c^2w^2$.
31. $121\,x^4 - 1$.
32. $64\,x^6 - 9$.
33. $25\,y^8 - 16$.
34. $36\,z^{10} - 49$.
35. $64\,k^{12} - 100$.
36. $81 - 121\,a^{14}$.
37. $144\,b^2 - 169\,x^4$.
38. $16\,y^2 - 81\,z^6$.
39. $49\,a^8 - 25\,b^{12}$.
40. $9\,a^{2n} - 25$.
41. $x^{2m} - 64\,y^{2r}$.
42. $49\,a^{2s} - 100\,b^{2t}$.
43. $100\,a^{2n} - 121\,n^2$.
44. $64\,a^{2r} - 49\,r^2$.
45. $x^{2x} - 1$.
46. $y^{2y} - 1$.
47. $4\,a^2b^4c^6 - 1$.
48. $9\,x^8y^4z^2 - 1$.
49. $16\,a^{10}b^6c^4 - d^2$.
50. $25\,a^{10}b^8c^6 - d^4e^2$.
51. $m^2a^{2n} - 1$.
52. $r^2s^{2r} - t^2$.
53. $a^{2b} - b^{2a}$.
54. $4\,x^{2x} - 1$.
55. $9\,a^{2a} - b^2$.
56. $16\,x^{2y} - y^2$.
57. $4\,a^{4y} - 9$.

Fill in the following blank forms and factor:

58. $(\quad)^2 - (\quad)^2$.
59. $(\quad)^4 - (\quad)^2$.

PREPARATORY EXERCISE 63

Remove the inner parentheses only:

1. $[x + (y - z)][x - (y - z)]$.
2. $[a + (b - c)][a - (b - c)]$.
3. $[a + (x - y + z)][a - (x - y + z)]$.
4. $[(a - b) + (c - d)][(a - b) - (c - d)]$.
5. $[(x - y) + (a + 5)][(x - y) - (a + 5)]$.
6. $[(a - 4) + (b - c)][(a - 4) - (b - c)]$.

EXERCISE 64

Obtain factors of the following expressions, checking all results numerically:

1. $(a + b)^2 - 1$.
2. $(a + b)^2 - 4\,c^2$.
3. $(x - y + z)^2 - 1$.
4. $(a + b - c)^2 - 9\,d^2$.
5. $(a + b)^2 - 81\,c^2$.
6. $(c - n)^2 - 100\,q^2$.
7. $36\,h^2 - 49(m + r)^2$.
8. $25\,n^2 - 64(a + d)^2$.

9. $4a^2 - 9b^2(c+d)^2$.
10. $16x^2 - 25y^2(z+w)^2$.
11. $49c^2(x+y)^2 - 1$.
12. $81d^2(x-m)^2 - 4$.
13. $(a+b)^2 - (c+d)^2$.
14. $(x+y)^2 - (z+w)^2$.
15. $(a+x)^2 - (b-y)^2$.
16. $(b-m)^2 - (c+n)^2$.
17. $(a-b)^2 - (c-d)^2$.
18. $(a+b+c)^2 - d^2$.
19. $(x-y+z)^2 - w^2$.
20. $a^2 - (b+c+d)^2$.
21. $b^2 - (x-y+w)^2$.
22. $c^2 - (m-t-w)^2$.
23. $(a+b)^2 - (x-y+z)^2$.
24. $(a-b)^2 - (x+y-z)^2$.
25. $(x-y-z)^2 - (a+b)^2$.
26. $(a+b-c)^2 - 9(x-y)^2$.
27. $9(x+y)^2 - 16(z+w)^2$.
28. $9(a-b)^2 - 49(c-d)^2$.
29. $4x^2 - 49(a+b+c)^2$.
30. $64(g+h)^2 - 121(k+r)^2$.
31. $36(g-x)^2 - 169(h-y)^2$.
32. $225(a+b)^2 - 225(c+d)^2$.
33. $49(a^2+b)^2 - 144(a+b^2)^2$.
34. $16(a^2+b^2)^2 - 121(a+b)^2$.
35. $64(a+b+c)^2 - 169d^2$.
36. $121(a-b+c)^2 - 225d^2$.

EXPRESSIONS THE TERMS OF WHICH MAY BE SO GROUPED AS TO APPEAR AS THE DIFFERENCE OF TWO SQUARES

EXERCISE 65

Factor each of the following expressions:

1. $x^2 + 2xy + y^2 - z^2$.
2. $m^2 + 2mn + n^2 - w^2$.
3. $a^2 - 2ab + b^2 - y^2$.
4. $c^2 - 2cd + d^2 - k^2$.
5. $a^2 + 2ab + b^2 - 4$.
6. $c^2 - 2cd + d^2 - 9$.
7. $a^2 + 2ab + b^2 - 1$.
8. $g^2 - 2gk + k^2 - 4m^2$.
9. $h^2 - 2hx + x^2 - 16y^2$.
10. $a^2 + 6ab + 9b^2 - c^2$.
11. $x^2 - 10xy + 25y^2 - z^2$.
12. $m^2 - 12mn + 36n^2 - t^2$.
13. $1 - a^2 - 2ab - b^2$.
14. $1 - x^2 - 2xy - y^2$.
15. $4 - c^2 - 2cd - d^2$.
16. $9 - m^2 + 2mn - n^2$.
17. $25a^2 - 10ab + b^2 - s^2$.
18. $c^2 - 18c + 81 - d^2$.
19. $h^2 - 20h + 100 - k^2$.
20. $16a^2 - 8ab + b^2 - 16$.
21. $49d^2 - 14dr + r^2 - 49$.
22. $81p^2 + 36px + 4x^2 - 1$.
23. $1 - 9a^2 - 30ab - 25b^2$.
24. $49a^2 - 28ab + 4b^2 - 9c^2$
25. $36d^2 + 60dz + 25z^2 - 121w^2$.
26. $64c^2 - 48cn + 9n^2 - 16v^2$.
27. $a^2b^2 + 28ab + 196 - 169c^2$.
28. $c^2g^2 - 24cg + 144 - 121k^2$.
29. $a^2 + 2ab + b^2 - c^2 - 2cd - d^2$.

30. $c^2 - 2ch + h^2 - m^2 - 2mr - r^2$.
31. $k^2 + 2kn + n^2 - r^2 + 2rs - s^2$.
32. $h^2 - 2ht + t^2 - n^2 + 2ny - y^2$.
33. $9a^2 + 6ab + b^2 - c^2 - 4cd - 4d^2$.
34. $x^2 - 10xy + 25y^2 - 36z^2 - 12zw - w^2$.
35. $a^2 - 14ab + 49b^2 - x^2 - 16xy - 64y^2$.
36. $100b^2 - 20bd + d^2 - y^2 + 18yz - 81z^2$.
37. $48a^2 - 3b^2$.
38. $7x^2 - 63y^2$.
39. $45 - 125a^2b^2$.
40. $28x^2y^2 - 7(c+d)^2$.
41. $ax^3y - axy^3$.
42. $x^2 - y^2 + xz + yz$.
43. $x^2 - y^2 + x - y$.
44. $a^3b - ab^3 + a^2b + ab^2$.

EXERCISE 66

Factor each of the following expressions:

1. $a^4 + a^2 + 1$.
2. $b^4 - 6b^2 + 1$.
3. $c^4 - 7c^2 + 1$.
4. $d^4 - 14d^2 + 1$.
5. $4x^4 - 21x^2 + 1$.
6. $16y^4 - 17y^2 + 1$.
7. $25z^4 + 9z^2 + 1$.
8. $36w^4 - 37w^2 + 1$.
9. $m^4 - 8m^2 + 4$.
10. $n^4 + 5n^2 + 9$.
11. $x^4 - x^2 + 16$.
12. $y^4 - 14y^2 + 25$.
13. $4a^4 + 3a^2b^2 + 9b^4$.
14. $16b^4 + 4b^2c^2 + 25c^4$.
15. $36r^4 + 83r^2s^2 + 49s^4$.
16. $25c^4 + 35c^2y^2 + 36y^4$.
17. $121m^4 + 116m^2w^2 + 36w^4$.
18. $81d^4 + 41d^2w^2 + 25w^4$.
19. $121g^4 + 101g^2h^2 + 25h^4$.
20. $81a^4 + 143a^2b^2 + 64b^4$.
21. $49x^4 + 64y^4 + 87x^2y^2$.
22. $25m^4 + 36n^4 + 35m^2n^2$.
23. $25a^8 + 81b^8 + 41a^4b^4$.
24. $x^{12} - 34x^6y^6 + 81y^{12}$.

PREPARATORY EXERCISE 67

Find two numbers having given their product and also their sum:

1. Product 2, sum 3.
2. Product 6, sum 5.
3. Product 3, sum 4.
4. Product 8, sum 6.
5. Product 10, sum 7.
6. Product 16, sum 8.
7. Product 6, sum 7.
8. Product 21, sum 10.
9. Product 16, sum 10.
10. Product 30, sum 11.
11. Product 20, sum 9.
12. Product 54, sum 15.
13. Product 55, sum 16.
14. Product 84, sum 19.
15. Product 4, sum -5.
16. Product 12, sum -8.

17. Product 35, sum -12.
18. Product 48, sum -14.
19. Product -24, sum $+2$.
20. Product -24, sum -2.
21. Product -42, sum $+1$.
22. Product -40, sum -3.
23. Product -63, sum -2.
24. Product -4, sum 0.
25. Product -65, sum -8.

TRINOMIALS OF THE TYPE $x^2 + sx + p$

EXERCISE 68

Factor each of the following expressions:

1. $a^2 + 4a + 3$.
2. $b^2 + 6b + 5$.
3. $c^2 + 6c + 8$.
4. $d^2 + 5d + 6$.
5. $x^2 + 7x + 12$.
6. $g^2 + 10g + 21$.
7. $h^2 + 8h + 15$.
8. $m^2 + 12m + 32$.
9. $m^2 + 10m + 24$.
10. $y^2 + 12y + 35$.
11. $z^2 + 15z + 50$.
12. $w^2 + 15w + 56$.
13. $s^2 - 12s + 27$.
14. $a^2 - 14a + 48$.
15. $t^2 - 16t + 55$.
16. $g^2 - 19g + 90$.
17. $r^2 - 16r + 48$.
18. $a^2 - 18a + 45$.
19. $x^2 - 16x + 63$.
20. $y^2 - 18y + 80$.
21. $z^2 - 4z - 12$.
22. $b^2 + 5b - 24$.
23. $k^2 - 3k - 54$.
24. $m^2 - 4m - 45$.
25. $t^2 - 8t - 33$.
26. $a^2 + 5a - 66$.
27. $b^2 + 10b - 39$.
28. $n^2 + 10n - 56$.
29. $c^2 - c - 56$.
30. $d^2 + d - 72$.
31. $h^2 + 23h + 130$.
32. $s^2 - 20s + 91$.
33. $r^2 + 2r - 99$.
34. $v^2 - 4v - 140$.
35. $h^2 + 22h + 96$.
36. $a^2 - 23a + 90$.
37. $b^2 + 16b - 57$.
38. $a^2 - 13a - 68$.
39. $c^2 + 12c - 64$.
40. $d^2 + 23d + 132$.
41. $x^2 - 18x + 77$.
42. $y^2 + 4y - 96$.
43. $z^2 + 22z + 85$.
44. $r^2 - 7r - 120$.
45. $s^2 + 6s - 160$.
46. $b^2 - 32b + 240$.
47. $c^2 - 29c + 190$.
48. $d^2 + 27d + 170$.
49. $h^2 + 6h - 280$.
50. $k^2 - 18k + 180$.
51. $a^2b^2 + 5ab + 6$.
52. $x^2y^2 - 5xy + 4$.
53. $k^2w^2 - 3kw - 28$.
54. $c^2y^2 + cy - 30$.

55. $c^2d^4 - 4\,cd^2 - 5$.
56. $k^4r^2 + 3\,k^2r - 10$.
57. $(a + b)^2 + 6(a + b) + 8$.
58. $(c + d)^2 + 10(c + d) + 21$.
59. $(x - y)^2 + 11(x - y) + 24$.
60. $(r + s)^2 - 12(r + s) + 27$.
61. $(t - v)^2 - 13(t - v) + 36$.

Fill in the following Blank Forms in different ways and then factor:

62. $(\quad)^2 + 5(\quad) + 6$.
63. $(\quad)^2 - 7(\quad) + 12$.
64. $a^2 + 6\,a + (\quad)$.
65. $b^2 + 6\,b - (\quad)$.
66. $c^2 - 6\,c + (\quad)$.
67. $d^2 - 6\,d - (\quad)$.
68. Write binomials in the above Blank Forms and then factor.

TRINOMIALS OF THE TYPE $x^{2m} + sx^m + p$

EXERCISE 69

Factor each of the following expressions:

1. $a^4 + 11\,a^2 + 30$.
2. $b^4 + 12\,b^2 + 20$.
3. $x^4 + 15\,x^2 + 56$.
4. $a^4 + 9\,a^2 + 18$.
5. $x^4 + 2\,x^2 - 15$.
6. $c^4 - c^2 - 90$.
7. $c^6 + 5\,c^3 + 4$.
8. $d^6 + 8\,d^3 + 7$.
9. $a^6 + 19\,a^3 + 90$.
10. $b^6 + 6\,b^3 - 55$.
11. $g^6 - 7\,g^3 - 120$.
12. $x^6 - 18\,x^3 + 72$.
13. $c^8 + 5\,c^4 + 6$.
14. $v^8 + 7\,v^4 - 170$.
15. $s^{10} - 7\,s^5 + 10$.
16. $k^{10} - 13\,k^5 + 42$.
17. $h^{10} - 20\,h^5 + 99$.
18. $x^{10} + 10\,x^5 - 96$.
19. $m^{10} + 22\,m^5 + 40$.
20. $y^{12} + 9\,y^6 + 20$.
21. $z^{14} - 17\,z^7 + 70$.
22. $c^{18} + 19\,c^9 + 84$.
23. $n^{20} + 2\,n^{10} - 35$.
24. $a^{2c} + 20\,a^c + 96$.
25. $d^{2m} + 16\,d^m + 63$.
26. $b^{2m} - 8\,b^m - 65$.
27. $a^{2d} - 10a^d - 39$.
28. $r^{2n} + 11\,r^n + 30$.
29. $y^{2n} - 14\,y^n + 48$.
30. $z^{2b} - 2\,z^b - 35$.
31. $4\,d^2 + 16\,d + 7$.
32. $4\,c^2 + 20\,c + 9$.
33. $4\,x^2 - 28\,x + 13$.
34. $9\,d^2 - 18\,d + 8$.
35. $25\,a^2 - 35\,a + 12$.
36. $49\,y^2 + 14\,y - 3$.
37. $16\,v^2 + 40\,v + 9$.
38. $81\,h^2 - 45\,h - 14$.
39. $36\,z^2 + 24\,z - 5$.
40. $64\,x^2 - 80\,x - 11$.
41. $c^2 + 15\,cd + 26\,d^2$.
42. $b^2 - 12\,bx + 27\,x^2$.

43. $a^2 - 19\,ad + 34\,d^2$.
44. $y^2 - 21\,yz + 38\,z^2$.
45. $x^2 - 12\,xy - 28\,y^2$.

Fill in the following Blank Forms and factor:

46. $(\ )^4 + 7(\ )^2 + 6$.
47. $(\ )^4 - 2(\ )^2 - 24$.
48. $(\ )^6 + (\ )^3 - 20$.
49. $(\ )^8 - 12(\ )^4 - 28$.

TRINOMIALS OF THE TYPE $ax^2 + bx + c$.

EXERCISE 70

Factor each of the following expressions:

1. $2\,x^2 + 3\,x + 1$.
2. $3\,y^2 + 4\,y + 1$.
3. $4\,z^2 + 5\,z + 1$.
4. $2\,a^2 + 5\,a + 2$.
5. $2\,b^2 + 7\,b + 3$.
6. $3\,c^2 + 7\,c + 2$.
7. $3\,d^2 + 10\,d + 3$.
8. $2\,g^2 + 5\,g + 3$.
9. $4\,h^2 + 13\,h + 3$.
10. $5\,k^2 + 13\,k + 6$.
11. $3\,r^2 + 11\,r + 6$.
12. $4\,s^2 + 11\,s + 6$.
13. $6\,w^2 + 13\,w + 2$.
14. $6\,a^2 - 13\,a + 7$.
15. $7\,b^2 - 15\,b + 2$.
16. $7\,c^2 - 16\,c + 4$.
17. $6\,c^2 + 17\,c + 12$.
18. $8\,d^2 + 18\,d + 9$.
19. $9\,g^2 + 19\,g + 2$.
20. $9\,h^2 + 21\,h + 10$.
21. $7\,z^2 - 20\,z + 13$.
22. $8\,w^2 - 25\,w + 17$.
23. $19\,k^2 - 21\,k + 2$.
24. $11\,h^2 - 30\,h + 19$.
25. $22\,s^2 + 29\,s + 7$.
26. $10\,q^2 + 17\,q + 7$.
27. $14\,t^2 - 19\,t + 5$.
28. $5\,y^2 + 6\,y - 11$.
29. $15\,a^2 + 8\,a - 7$.
30. $14\,b^2 - 3\,b - 11$.
31. $13\,d^2 - 7\,d - 6$.
32. $4\,g^2 + 5\,g - 9$.
33. $5\,x^2 + 13\,x - 6$.
34. $17\,y^2 + 6\,y - 11$.
35. $14\,m^2 + 73\,m + 15$.
36. $14\,a^2 - 33\,a + 18$.
37. $54\,a^2 - 15\,a - 50$.
38. $36\,m^2 + 27\,m + 2$.
39. $30\,m^2 - 47\,m - 5$.
40. $8\,b^2 - 14\,b - 39$.
41. $7\,x^2 - 10\,x + 3$.
42. $30\,c^2 - 31\,c + 8$.
43. $56\,x^2 + 65\,x - 9$.
44. $40\,y^2 + 14\,y - 33$.
45. $7\,x^2 + 13 + 20\,x$.
46. $28\,x^2 + 17 - 48\,x$.
47. $48\,x^2 - 23\,x - 13$.
48. $18\,x^2 + 37\,x + 19$.
49. $20\,a^2 - 17\,a - 3$.
50. $8\,a^2b^2 + 103\,ab - 13$.
51. $17\,a^2x^2 - 69\,ax + 4$.
52. $20\,m^2 + 19\,mn + 3\,n^2$.
53. $7\,b^2 + 41\,bc - 6\,c^2$.
54. $40\,x^2 - 83\,xy + 42\,y^2$.
55. $9\,x^2 - 4\,xy - 13\,y^2$.
56. $9\,x^2 + 55\,xy - 56\,y^2$.
57. $6\,a^2b^2 + 7\,abc - 5\,c^2$.
58. $3(4\,x^2 - 21) - x$.
59. $13\,m^2 - (m + 14)$.
60. $15 + w(19\,w - 34)$.
61. $h(22\,h - 19) - 15$.
62. $b(1 + 15\,b) - 16$.
63. $2(3\,y^2 - 28\,z^2) + 41\,yz$.
64. $7\,c(29\,d + 14\,c) + 14\,d^2$.

Fill in the following Blank Forms and factor:

65. $2\,x^2 + (\quad)x + 7$.
66. $3\,y^2 - (\quad)y + 8$.
67. $8\,z^2 + (\quad)z + 10$.
68. $10\,w^2 - (\quad)w - 21$.

BINOMIAL SUMS AND DIFFERENCES OF LIKE POWERS

EXERCISE 71

Factor each of the following binomials:

1. x^3+y^3.
2. y^3-z^3.
3. d^3+1.
4. $1+b^3$.
5. c^3-1.
6. $1-a^3$.
7. a^3+8.
8. $8a^3+1$.
9. $125+a^3$.
10. $27c^3+1$.
11. a^3+8b^3.
12. $8c^3+d^3$.
13. b^3-64.
14. $27-d^3$.
15. m^3+216.
16. $216-n^3$.
17. a^3+343.
18. $343a^3-1$.
19. c^3+512.
20. $512-d^3$.
21. c^4-1.
22. a^5+b^5.
23. c^5-d^5.
24. a^5+1.
25. $1+b^5$.
26. b^5-1.
27. $1-a^5$.
28. a^5+32.
29. $32-b^5$.
30. a^5+1.
31. $32a^5+1$.
32. a^6-b^6.
33. a^6-1.
34. $1-b^6$.
35. x^6-64y^6.
36. x^7-y^7.
37. b^7-1.
38. a^7+1.
39. m^7+128.
40. $128a^7-1$.
41. $81a^4-b^4$.
42. $125+27a^3$.
43. $16a^4-1$.
44. $16a^4-625b^4$.
45. $81x^4-256y^4$.
46. $1296-a^4$.
47. $2401-x^4$.
48. $32a^5-243b^5$.
49. $1024a^5+b^5$.
50. $3125+c^5$.
51. $64a^6+1$.
52. $64a^6+b^3$.
53. $1000m^3-z^6$.
54. $1728a^3-b^{12}$.
55. $32a^{10}+1$.
56. $a^{12}+b^{12}$.

Fill in the following Blank Forms and factor:

57. $x^3+(\ \ )$.
58. $y^3-(\ \ )$.
59. $z^4-(\ \ )$.
60. $w^6+(\ \ )$.

POLYNOMIALS OF THE THIRD DEGREE

PREPARATORY EXERCISE 72

Obtain the following products:

1. $(a+1)(a+2)(a+3)$.
2. $(b+2)(b+3)(b+5)$.
3. $(c+1)(c+4)(c+6)$.
4. $(a+2)(a+4)(a+7)$.
5. $(x+1)(x+5)(x+6)$.
6. $(y-2)(y-3)(y-4)$.
7. $(b-3)(b-5)(b-7)$.
8. $(a-1)(a-6)(a-9)$.
9. $(m+1)(m+2)(m-3)$.
10. $(a+1)(a-2)(a+3)$.
11. $(d-2)(d-3)(d+5)$.
12. $(b-5)(b+6)(b-8)$.
13. $(a+1)(a+6)(a+6)$.
14. $(x+2)(x+2)(x+9)$.
15. $(a+4)(a+4)(a+4)$.

EXERCISE 73

Factor each of the following polynomials:

1. $x^3 - 6x^2 + 11x - 6$.
2. $x^3 - 7x^2 + 14x - 8$.
3. $x^3 - 8x^2 + 19x - 12$.
4. $x^3 - 11x^2 + 31x - 21$.
5. $x^3 - 10x^2 + 31x - 30$.
6. $x^3 - 14x^2 + 59x - 70$.
7. $x^3 - 13x^2 + 47x - 35$.
8. $x^3 - 12x^2 + 41x - 42$.
9. $x^3 - 10x^2 + 27x - 18$.
10. $x^3 - 12x^2 + 29x - 18$.
11. $x^3 - 8x^2 + 21x - 18$.
12. $x^3 - 8x^2 + 11x + 20$.
13. $x^3 - 7x^2 - 28x - 20$.
14. $x^3 - 9x^2 + 24x - 20$.
15. $x^3 - 7x^2 + 15x + 9$.
16. $x^3 + 16x^2 + 41x + 26$.
17. $x^3 + 5x^2 + 8x + 4$.
18. $x^3 + 10x^2 - 13x - 22$.
19. $x^3 - 5x^2 - 33x - 27$.
20. $x^3 - 9x^2 + 15x + 25$.

Determine either by the Remainder Theorem or by the process for long division whether or not $x + 2$ is a factor of each of the following expressions:

21. $x^3 + 4x^2 + 2x + 4$.
22. $x^3 - x^2 - 5x + 2$.
23. $x^3 - 2x^2 + 4x - 6$.
24. $x^3 + 2x^2 - 29x - 30$.
25. $x^3 + 3x^2 + 3x + 2$.
26. $x^3 + x^2 - 10x + 8$.
27. $x^3 - 10x^2 + 27x - 18$.
28. $x^3 + 10x^2 + 29x + 20$.
29. $x(x^2 - 9) - 2(3x^2 - 7)$
30. $x(x^2 - 10) - 3(x^2 - 8)$.
31. $x^3 + 6x^2 - x - 18$.
32. $x^3 - 25x^2 + 71x - 46$.

Determine whether or not $x - 2$ is a factor of each of the following expressions:

33. $x^3 + 9x^2 + 8x - 60$.
34. $x^3 + 7x^2 - 3x - 30$.
35. $x^3 - 5x^2 + 10x - 8$.
36. $x^3 + 10x^2 - 25x + 2$.
37. $x^3 + 3x^2 + 4x - 24$.
38. $x^3 - 9x^2 - 3x + 40$.
39. $x^3 + 6x^2 - 21x - 8$.
40. $x^3 - 14x^2 + 27x - 6$.

Fill in the following Blank Forms and factor:

41. $a^3 + 14a^2 + (\quad) + 40$.
42. $b^3 + 13b^2 + (\quad) + 36$.
43. $c^3 + (\quad)c^2 + (\quad)c - 48$.
44. $d^3 - (\quad)d^2 - (\quad)d - 30$.

EXERCISE 74

Miscellaneous

Factor each of the following expressions:

1. $2a + 2b$.
2. $15c - 5cd$.
3. $cx + cy + dx + dy$.
4. $c^2 - 16c + 64$.
5. $a^2 - 16$.
6. $b^2 - 64$.

7. $x^2 + 2xy + y^2 - w^2$.
8. $11mx - 11my$.
9. $x^2 + 4x - 5$.
10. $mn + n + m + 1$.
11. $b^2 - 7b + 10$.
12. $5x^2 + 6x + 1$.
13. $7d^2 - 14d + 7$.
14. $h^3 + 1$.
15. $a^3 + a^2$.
16. $3q^2 - 3$.
17. $25 - m^2$.
18. $a^4 - 9$.
19. $b^4 - 49$.
20. $x^4 - 25y^2$.
21. $4 - x^4$.
22. $81g^4 - 1$.
23. $.3cd - .12de$.
24. $x^5 + 1$.
25. $y^6 - 1$.
26. $5c^2 - 10cd + 5d^2$.
27. $49x^2 - 9y^2$.
28. $81 - 18g + g^2$.
29. $c^2 - 9c + 20$.
30. $3b^2 + 5b + 2$.
31. $6gh + 18gk$.
32. $5r^2 - 5s^2$.
33. $3c^2 + 6cd + 3d^2$.
34. $4a^2 + 12ab + 9b^2$.
35. $27b^2 + 36bc$.
36. $d^2 + 2d - 24$.
37. $2h^2 + h - 3$.
38. $4c^2 - 9$.
39. $c^2d^2 - 9$.
40. $100 - b^2k^2$.
41. $25d^2 - 81$.
42. $x^2 - 3x - 28$.
43. $100z^2 - 121w^2$.
44. $144g^2 - 25k^2$.
45. $y^2 + 27y + 50$.
46. $9c^2d^2 + 6cdk + k^2$.
47. $x^3 + x^2 + x$.
48. $x^3 + x^2 + x + 1$.
49. $27b^2 - 3d^2$.
50. $g^2h^2 - k^2t^2$.
51. $36z^2w^2 - 12zw + 1$.
52. $169m^2n^2 - 1$.
53. $c^2 + 2cd + d^2 - 4h^2$.
54. $x^2yz + xy^2z$.
55. $z^2 - 30z + 81$.
56. $3c^2 + 13c + 4$.
57. $a^3 + a^2 - a - 1$.
58. $k^3 - 8$.
59. $8m^3 - 1$.
60. $x^4 + x^2 + 1$.
61. $w^2 - 18w - 63$.
62. $5d^2 + 11d + 2$.
63. $bh^2k^2 - b^2h^2k$.
64. $d^3 - 3d^2 - 4d + 12$.
65. $4 + 4b + b^2 + b^3$.
66. $7a^2 + 8a + 1$.
67. $h^2 - 9h - 70$.
68. $5x + 5 - x^3 - x^2$.
69. $9m^2n^2 + 30mn + 25$.
70. $x^2yz + xy^2z + xyz^2$.
71. $121x^2y^2 - 4z^2w^2$.
72. $169y^8 - 100z^{10}$.
73. $27 + n^3$.
74. $2a^3 + 54$.
75. $m^7 + n^7$.
76. $(a+b)^2 + 8(a+b) + 15$.
77. $(c-d)^2 - 6(c-d) - 72$.
78. $29c^2 - 26cd - 3d^2$.
79. $2d^2 + 33dh + 31h^2$.
80. $a^2c + 2abc + b^2c$.

81. $a^4 + a^2 + 25$.

82. $a^4 + a^2b^2 + b^4$.

83. $12\,cd^2 + 24\,c^2d - 36\,c^2d^2$.

84. $26\,m^3n - 13\,m^2n^2 + 39\,mn^3$.

85. $a^2b^2 + 21\,ab + 80$.

86. $c^4 - 3\,c^2d^2 + d^4$.

87. $49\,a^2 - 28\,ab + 4\,b^2$.

88. $81\,x^2y^2 + 90\,xyzw + 25\,z^2w^2$.

89. $c^2d^2 + 20\,cd + 96$.

90. $m^2n^2 + 13\,mn - 48$.

91. $a^3b^3 + 1$.

92. $(m+n)^2 + 12\,(m+n) + 36$.

93. $9\,(x-y)^2 - 42\,(x-y) + 49$.

94. $(g+h)^2 + 2\,(g+h)\,(k+w) + (k+w)^2$.

95. $(a-b)^2 + 8\,(a-b)\,(c-d) + 16\,(c-d)^2$.

96. $8\,m^2 - 11\,m + 3$.

97. $7\,n^2 - 5\,n - 2$.

98. $11\,h^2 + 14\,h + 3$.

99. $13\,c^2 - 18\,c + 5$.

100. $x^4 + 4\,x^2 - 21$.

101. $k^8 - 5\,k^4 - 36$.

102. $16 - x^{16}$.

103. $144\,a^6 - 49\,b^8$.

104. $36\,z^{2n} - 1$.

105. $81\,w^{2m} - 4$.

106. $c^{2m} - d^{2n}$.

107. $x^{2m} + 14\,x^m + 45$.

108. $r^{2m} + 2\,r^m s^m + s^{2m}$.

109. $4\,d^{2n} - 4\,d^n k^n + k^{2n}$.

110. $cx + cy + cz + dx + dy + dz$.

111. $ar + as + at - 3\,r - 3\,s - 3\,t$.

112. $(h-k)^2 + 5\,(h-k) - 84$.

113. $5\,m^2 - 38\,mn - 43\,n^2$.

114. $9\,h^4 - 66\,h^2 + 25$.

115. $4\,g + 4\,h + 4\,k + cg + ch + ck$.

116. $a^2 - 2\,ab + b^2 - 9\,c^2$.

117. $9 - x^2 - 2\,xy - y^2$.

118. $4 - r^2 + 2\,rs - s^2$.

119. $8 - c^3d^3$.

120. $216 + a^3$.

121. $5\,a^3 + 5$.

122. $7\,x^3 - 7$.

123. $2\,ab - 1.5\,ac - .5\,ad$.

124. $(x+y)^2 + 24\,(x+y) + 95$.

125. $121\,g^2 - 66\,gh + 9\,g^2 - 9\,k^2$.

126. $a^2 + 2\,ab + b^2 - c^2 - 2\,cd - d^2$.

127. $4\,b^4 + 16\,b^2 + 25$.

128. $16\,x^4 - 49\,x^2y^2 + 9\,y^4$.

129. $d^{10} + d^5 - 42$.

130. $a^4 + 6\,a^2b^3 + 8\,b^6$.

131. $13\,x^2 - 4\,x - 17$.

132. $19\,w^2 + 4\,w - 23$.

133. $17\,a^2b^2 - 52\,ab + 3$.

134. $15\,x^2 + 31\,xy + 16\,y^2$.

135. $4^{2x} - x^4$.

136. $169\,a^{2m} - 121\,b^{2m}$.

137. $8\,c^3 - 27\,d^3$.

138. $(x+y)^2 - 1$.

139. $(a+b-c)^2 - 9$.

140. $(a+b)^2 - 4\,c^2d^2$.

141. $(m+n)^2 - (r+s)^2$.

142. $9\,b^2 + 6\,bc + c^2 - 25\,k^2$.

143. $49\,h^2 - 14\,hk + k^2 - 81\,y^2$.

144. $16\,x^2 + 40\,xy + 25\,y^2 - 9\,z^2$.

145. $b^4 + 10\,b^2c^3 + 16\,b^6$.

146. $(z-w)^2 - 29(z-w) + 100$

147. $81\,a^2b^2c^2 - 100\,(d+h)^2$.

148. $16\,(d-h)^2 - 49\,(t-v)^2$.

149. $4\,(c-d-h)^2 - 9\,(x+y+z)^2$.

150. $25\,c^2 + 20\,cd + 4\,d^2 - g^2 - 12\,gh - 36\,h^2$.

EXERCISE 75

Miscellaneous

Obtain factors of the following, checking all results numerically:

1. $a^3 + a^2 + a$.
2. $xy + 4x - 3y - 12$.
3. $a^2 + 22a + 121$.
4. $x^2 - 144$.
5. $x^2 + 3x + 2$.
6. $2x^2 + 11x + 12$.
7. $3a^2 - 12$.
8. $3k^2 + 33k + 72$.
9. $km + 2lm - sk - 2ls$.
10. $x^2 + 5x - 6$.
11. $49h^2 - 36y^2$.
12. $x^{2m} - y^{2m}$.
13. $x^2 - 22x + 105$.
14. $x^3 + x^2 - x - 1$.
15. $x^2 + cx + 2dx + 2cd$.
16. $(a-b)^2 - 11(a-b) - 12$.
17. $a^3 - 343$.
18. $2a^3 + 54$.
19. $7a^7 - 7a$.
20. $5m + 6m^2 + 1$.
21. $m^2 + m - 2$.
22. $2a^3 + 28a^2 + 66a$.
23. $3a^2 + 30a + 27$.
24. $13x^2 + 25x - 2$.
25. $2x^2 + 12x + 18$.
26. $2a^2 + 3a + 1$.
27. $6a^2 - 3a - 3$.
28. $169c^2 - 9d^2$.
29. $56 - 15x + x^2$.
30. $15x^2 + 2xy - 24y^2$.
31. $75a^2 - 3b^2$.
32. $5x^2 + 20x + 20$.
33. $15a^2 + 41a + 14$.
34. $a + a^6$.
35. $x^6 + 19x^3 + 88$.
36. $a^2b^2 + 30ab + 104$.
37. $a^2x^2 + 3abx + 2b^2$.
38. $14 - 21m - 14m^2$.
39. $128m^4 - 18n^2$.
40. $c^4 - 5c^2d^2 + 4d^4$.
41. $27c^2 + 18c + 3$.
42. $x^2y^2z^2 - 4xyz - 12$.
43. $8a^2b^2c^2 - 18c^4$.
44. $x^4 - 21x^2 + 80$.
45. $25a^{10} - 26a^5 + 1$.
46. $144x^2 - 625y^2$.
47. $16a^4 - 41a^2c^2 + 25c^4$.
48. $64a^5 + 2$.
49. $8a^9 + 729$.
50. $b^2 + 288 - 34b$.
51. $x^6 + 25x^3 + 24$.
52. $50 - 20x + 2x^2$.
53. $147a^2 - 75$.
54. $x^8 - 38x^4 + 105$.
55. $8 - 9x^4 + x^8$.
56. $64 - a^6$.
57. $6a^2 + 150 - 60a$.
58. $3x^2 + 36xy + 108y^2$.
59. $4x^2 - 28xy + 49y^2$.
60. $80a^2 - 20a^2b^2$.
61. $64r^2 + 80rs + 25s^2$.
62. $x^3 - 7x^2 + 14x - 8$.
63. $(a+b)^3 - 1$.
64. $9a^2 + 24ab + 16b^2$.
65. $(a+b)^2 + 8(a+b) + 15$.
66. $(a+b)^2 + 5(a+b) + 6$.

67. $3\,c^2 - 14\,cy + 8\,y^2$.

68. $a^{2m} + 2\,a^m b^m + b^{2m}$.

69. $30\,a^2 - 154\,a + 20$.

70. $x^{2m} - y^{2m}$.

71. $(c+d)^2 + 12(c+d) + 20$.

72. $a^2 - 2\,ab + b^2 - 11\,a + 11\,b - 12$.

73. $4\,x^4 - 13\,x^2y^2 + 9\,y^4$.

74. $a^6 - 50\,a^3 + 49$.

75. $x^3 + 9\,x^2 + 26\,x + 24$.

76. $a^3 + 9\,a^2b + 23\,ab^2 - 15\,b^3$.

77. $(a+b)^2 - 2(a+b) + 1$.

78. $d^4 + 11(2\,d^2 + 11)$.

79. $18\,h^2 - 31\,hk + 6\,k^2$.

80. $16\,a^2 - (2\,b + 3\,c)^2$.

81. $a^3 - 13\,a + 12$.

82. $12\,a^2 - 12\,b(2\,a - b)$.

83. $(c+d)^2 + 10(c+d) + 21$.

84. $2\,a(a+6) + 18$.

85. $x^2y^2z^2 - 8\,xyzw - 20\,w^2$.

86. $4\,m(m+3) + 9$.

87. $2\,a(4\,a - 19) + 35$.

88. $(2\,x + 3\,y)^2 - (3\,a + b)^2$.

89. $x^2 - y^2 - 2\,yz - z^2$.

90. $49\,m^4 - 65\,m^2n^2 + 16\,n^4$.

91. $a^2 - 7(2\,a - 7)$.

92. $a^7 + 8\,a$.

93. $y^{2m} - 68\,y^m - 140$.

94. $4\,x(x+y) + y^2$.

95. $3\,r^2 + 4(2\,r + 1)$.

96. $x^3 - 3\,xy^2 + 2\,y^3$.

97. $a^2 + 49\,b^2 - 1 + 14\,ab$.

98. $c^2 + d^2 + 2\,cd - a^2 - b^2 - 2\,ab$.

99. $c^2 - 2\,cd + d^2 - 2(c-d)e + e^2$.

100. $x^2 + (c + 2\,d)x + 2\,cd$.

101. $a^4 - a^3 - 7\,a^2 + a + 6$.

102. $m^2 + n^2 + 2\,mn - a^2 - b^2 - 2\,ab$.

103. $1 + 2\,cd - c^2 - d^2$.

104. $h^2 + 2\,hm + m^2 + 2\,hy + 2\,my + y^2$.

105. $16\,x^4 - 81\,y^4$.

106. $9 + 49\,x^2 - 58\,x^4$.

107. $x^7 - 9\,x$.

108. $a^2b^3 - 16\,b^3c^2$.

109. $x^3 + y^3 + x + y$.

110. $x^4 - y^4 - (x+y)(x-y)$.

111. $a^2b + 3\,ab^2 - 3\,a^3 - b^3$.

112. $a^2 + b^2 - c^2 - m^2 - 2\,ab + 2\,mc$.

113. $4\,x^{2n} - 32\,x^ny^n + 64\,y^{2n}$.

114. $x^2 + 2\,xy + y^2 + 8\,xz + 8\,yz + 15\,z^2$.

115. $4\,a^2 - 25\,b^2 + 2\,a + 5\,b$.

116. $x^2y^2z^2 - 8\,axyz - 20\,a^2$.

117. $a^4 + 9\,a^2b^3 + 18\,b^6$.

118. $y^{2m} - 168\,y^m - 340$.

119. $a^3b^2 - c^3b^2 - a^3d^2 + c^3d^2$.

120. $a^{12} + 4\,b^{12} - 5\,a^6b^6$.

121. $ab(x^2+1) + x(a^2+b^2)$.

122. $1 + b - 56\,b^2$.

123. $1 - 17\,x^2 + 16\,x^4$.

124. $4 - 52\,x^7 + 169\,x^{14}$.

125. $abc^3 + 3\,abc^2 - abc - 3\,ab$.

126. $4(a^2+c^2)(a^2-c^2)+3\,b^3(4\,a^2+3\,b^3)$.

127. $a^3-b^3-a(a^2-b^2)+b(a-b)^2$.

128. $9\,a^6+71\,a^3-8$.

129. $(x+y)(x+y+7)+10$.

130. $a^4-b^2(11\,a^2-b^2)$.

131. $na^{2n}-19\,na^n+34\,n$.

132. $a^{10}+18\,a^5b^5-144\,b^{10}$.

133. $bcx^2+(ac+bd)x+ad$.

134. $x^{2m}+(a+b)x^m+ab$.

135. $25\,a^8+81\,b^8+41\,a^4b^4$.

136. $a^2(a+1)+b^2(b+1)+2\,ab$.

137. $x^2(x+1)-2\,xy-y^2(y-1)$.

138. $a^2(a+3\,b)+b^2(b+3\,a)$.

139. $x^2+2\,ax+a^2-b^2$.

140. $8\,a^8-8$.

141. $16\,a^{16}-16$.

142. $12\,a^{12}+12$.

143. $6\,a^2+5\,ax+x^2$.

144. $14\,a^2-109\,ab-24\,b^2$.

145. $7\,x^2+47\,xy+30\,y^2$.

146. $a^2-d^2-b(2\,a-b)+c(2\,d-c)$.

147. $c^2+d^2-e^2-f^2+2(ef-cd)$.

148. $x^4-2\,x^3+2\,x-1$.

149. $a^2x+b^2y+b^2x+a^2y+2\,(abx+aby)$.

150. $(a+b)^2+(b+d)^2-(c+d)^2-(c+a)^2$.

APPLICATION OF FACTORING TO THE SOLUTION OF EQUATIONS

EXERCISE 76

Solve the following equations, regarding the letters appearing in them as unknowns, and verify all solutions by substituting for the letters in the original equations the particular values found:

1. $x^2-4\,x+3=0$.

2. $y^2-5\,y+4=0$.

3. $a^2-6\,a-7=0$.

4. $b^2-7\,b+10=0$.

5. $c^2-8\,c+12=0$.

6. $g^2-9\,g+18=0$.

7. $h^2+5\,h+6=0$.

8. $k^2+6\,k+8=0$.

9. $m^2+4\,m-5=0$.

10. $n^2+7\,n-8=0$.

11. $z^2+11\,z+30=0$.

12. $w^2+11\,w+24=0$.

13. $c^2+11\,c+10=0$.

14. $g^2-10\,g+25=0$.

15. $r^2+14\,r+49=0$.

16. $s^2-18\,s+81=0$.

17. $x^2+12\,x+36=0$.

18. $y^2-16\,y+64=0$.

19. $z^2-2\,z=0$.

20. $w^2-3\,w=0$.

21. $c^2+4\,c=0$.

22. $d^2=6\,d$.

23. $m^2=7\,m$.

24. $5\,n=n^2$.

25. $8\,p=p^2$.

26. $q^2=-9\,q$.

27. $h^2 = -10h$.
28. $k^2 - 4 = 0$.
29. $a^2 - 9 = 0$.
30. $b^2 - 16 = 0$.
31. $v^2 = 25$.
32. $m^2 = 36$.
33. $n^2 = 49$.
34. $d^2 = d$.
35. $k^2 - k = 6$.
36. $h^2 - h = 12$.
37. $r^2 + r = 20$.
38. $s^2 - s = 42$.
39. $x^2 + x = 56$.
40. $t^2 - t = 72$.
41. $y^2 + 2y = 15$.

42. $z^2 + 3z = 28$.
43. $w^2 + 4w = 45$.
44. $a^2 + 6a = 16$.
45. $b^2 - 5b = 50$.
46. $c^2 - 7c = 18$.
47. $d^2 - 8d = 48$.
48. $h^2 + 12 = 7h$.
49. $k^2 + 40 = 13k$.
50. $m^2 + 32 = 12m$.
51. $n^2 + 13 = 14n$.
52. $r^2 - 14 = 5r$.
53. $s^2 - 21 = 4s$.
54. $t^2 - 26 = 11t$.
55. $v^2 - 35 = 2v$.
56. $b^2 - 38 = 17b$.
57. $k^2 - 48 = 13k$.
58. $h^2 - 33 = -8h$.
59. $m^2 - 34 = -15m$.
60. $r^2 - 27 = -6r$.
61. $2z^2 - 7z + 3 = 0$.
62. $3g^2 - 7g + 2 = 0$.
63. $13n^2 - 14n + 1 = 0$.
64. $5d^2 - 21d + 4 = 0$.
65. $7w^2 - 15w + 2 = 0$.
66. $11x^2 - 34x + 3 = 0$.
67. $5y^2 - 7y + 2 = 0$.
68. $7z^2 - 10z + 3 = 0$.
69. $2w^2 - 5w + 3 = 0$.
70. $3a^2 + 4a + 1 = 0$.
71. $5b^2 + 6b + 1 = 0$.
72. $7c^2 + 22c + 3 = 0$.
73. $11d^2 + 23d + 2 = 0$.
74. $5g^2 + 7g + 2 = 0$.
75. $6k^2 + 5k + 1 = 0$.
76. $10m^2 + 7m + 1 = 0$.
77. $15n^2 + 8n + 1 = 0$.
78. $20r^2 + 12r + 1 = 0$.
79. $20s^2 + 9s + 1 = 0$.
80. $21t^2 + 10t + 1 = 0$.
81. $18x^2 - 11x + 1 = 0$.
82. $26y^2 - 15y + 1 = 0$.
83. $30z^2 - 13z + 1 = 0$.
84. $w^3 = 64w$.
85. $a^3 = 81a$.
86. $b^3 = 100b$.
87. $c^3 = c^2$.
88. $c^3 - 6c^2 - 7c = 0$.
89. $d^3 - 7d^2 - 8d = 0$.
90. $h^3 + 3h^2 - 10h = 0$.
91. $k^3 + 4k^2 - 12k = 0$.
92. $m^3 + 6m^2 - 7m = 0$.
93. $n^3 - 8n^2 - 9n = 0$.
94. $p^3 + 10p^2 - 11p = 0$.
95. $q^3 + 4q^2 - 21q = 0$.
96. $r^4 - 11r^2 + 28 = 0$.
97. $x^4 - 12x^2 + 35 = 0$.
98. $y^4 - 13y^2 + 36 = 0$.
99. $z^4 + 9z^2 + 14 = 0$.
100. $w^4 + 10w^2 + 16 = 0$.

HIGHEST COMMON FACTOR

EXERCISE 77

Find the H. C. F. of the expressions in each of the following groups:

1. ab, ac.
2. xy, yz.
3. $2a$, $2b$.
4. $5x$, $7x$.
5. $4m$, $6n$.
6. $3ab$, $3ac$.
7. 25, $5a$.
8. $12x$, 8.
9. $6xy$, $8xy$.
10. abc, acd.
11. xyz, yzw.
12. abc, acd.
13. $3adk$, $6adh$.
14. a^2, ab.
15. bc, c^2.
16. x^2y, x^2z.
17. a^2c, ac^2.
18. mn^2, n^2q.
19. a^2bc, abc^2.
20. $a^3b^3c^3$, $a^2b^2c^2$.
21. ab^2c, a^2bc^2.
22. x^2y^2z, xy^2z^2.
23. a^3b^2c, ab^2c^3.
24. $a^2b^4c^6$, ab^3c^5.
25. $10abc$, $18acx$.
26. $9a^2b$, $21b^2c$.
27. $8xy^2$, $36xz^2$.
28. $18a^3b^2$, $51a^2b^3$.
29. $33x^2y^3z^2$, $44x^3y^2z^3$.
30. $26a^7b^5c^3$, $65a^3b^5c^7$.
31. $63bc^2d^3$, $72b^3c^2d$.
32. $48x^2yz^2$, $64xy^2z$.
33. $56abc^2$, $70a^2bc$.
34. $45abc^2d$, $75a^2bcd$.
35. $51a^2b^2c^2d$, $68abcd^2$.
36. $54a^2b^3c^3d^2$, $90a^3b^2c^2d^3$.
37. a^5, a^3, a^2.
38. x^3y^2, x^2y, x^2y^3.
39. m^5n^3, m^2n^4, m^2n^6.
40. x^2yz, xy^2z, xyz^2.
41. a^3bc^2, a^2b^3cd, ab^2c^3.
42. a^3b^2xy, a^2b^3y, a^2b^2zw.
43. $a^4b^2c^3e$, $b^3c^4d^2$, $a^2b^4c^2d^2$.
44. a^8b^5c, a^6b^6d, a^3b^4de.
45. $x^4y^3z^2$, $x^3y^2z^4$, $x^2y^4z^3$.
46. $a^5b^4c^3$, $a^3b^5c^4$, $a^4b^3c^5$.
47. $19kmz$, $57gkm$, $38agk$.
48. $28m^3t^5x$, $7m^2n^3y$, $35m^5z$.
49. $32a^5b^5c^4$, $16a^4b^4c^5$, $24a^3b^3c^4$.
50. $34a^2b^2c^4$, $85a^3b^3c^5$, $51a^7b^6c^6$.
51. $27x^2y^3z^2$, $18x^3y^4z^2$, $36x^2y^3$, $81x^4y^4$.
52. $42a^5bx$, $56a^5b$, $14a^4ck$, $21a^3bc$.
53. $24x^3yz$, $48x^3z$, $18x^2yw$, $36xyz$.
54. $18abc^2$, $36a^2bcd$, $27ab^2c$, $45abcd^2$.
55. $36a^2bc^2$, $48ab^2c$, $72abc^3$, $60ab^3c$.

EXERCISE 78

Find the H. C. F. of the expressions in each of the following groups:

1. $(a+b)$, $(a+b)^2$.
2. $(x-y)^3$, $(x-y)^2$.
3. m^2-n^2, $m+n$.
4. x^2-1, $x-1$.
5. a^2+2a+1, $a+1$.
6. a^3-b^3, a^2-b^2.

7. $a^2 - 5a + 6$, $a^2 - 4a + 4$.
8. $x^2 + 7x + 12$, $x^2 + 4x + 3$.
9. $a^3 + b^3$, $a^2 - ab + b^2$.
10. $m^3 - n^3$, $m^2 - n^2$.
11. $c^2 - 9d^2$, $(c - 3d)^2$.
12. $(x - 2)^2$, $x^2 - 2x$.
13. $m^2 + 9m + 20$, $m^2 + 10m + 25$.
14. $a^2 + 3a + 2$, $a^2 + a - 2$.
15. $m^2 + 6m + 5$, $m^2 + 5m + 4$.
16. $t^2 - 13t + 30$, $t^2 - t - 6$.
17. $w^2 - w - 56$, $w^2 + 9w + 14$.
18. $b^2 + 9b - 22$, $b^2 + 14b + 33$.
19. $c^2 + c - 12$, $c^2 + 6c + 8$.
20. $d^2 + 11d + 30$, $d^2 + 3d - 10$.
21. $a^2 - 5a - 36$, $a^2 - 18a + 81$.
22. $b^2 + 11b + 24$, $b^2 + 9b + 8$.
23. $x^2 - 3x - 28$, $x^2 - 49$.
24. $y^2 + 7y - 30$, $y^2 + 8y - 20$.
25. $ax + bx + ay + by$, $a + b$.
26. $a^4 - b^4$, $a^2 - b^2$, $a - b$.
27. $a - 2$, $a^2 - 3a + 2$, $a^2 - 4$.
28. $c + d$, $c^2 + 2cd + d^2$, $c^2 + 3cd + 2d^2$.
29. $a^2 - b^2$, $a - b$, $a^2 - 2ab + b^2$.
30. $2x - 4y$, $x^2 - 4y^2$, $x^2 - 4xy + 4y^2$.
31. $3ab + 3b$, $2ax + 2x$, $8a^2z + 8az$.
32. $a + b$, $a^2 + 2ab + b^2$, $a^3 + b^3$.
33. $x^2 + 8x + 15$, $x^2 - x - 12$, $x^2 + 6x + 9$.
34. $xz + xw - yz - yw$ and $x^2 - y^2$.
35. $x^4 - y^4$, $x^3 + y^3$, $x + y$.
36. $4a^2 + 20a + 25$, $2a + 5$, $8a^3 + 125$.
37. $3a^2 + 7ab - 20b^2$, $a^2 - 16b^2$, $a + 4b$.
38. $2ac + 3ad - 2bc - 3bd$ and $4c^2 - 9d^2$.
39. $a^2 + 3a - 10$, $a^2 + 6a + 5$, $a^2 + 2a - 15$.
40. $2x^2 + x - 6$, $x^2 + 3x + 2$, $x^2 - 4$.

EXERCISE 79

Find the H. C. F. of each of the following groups of expressions:

1. $x^3 + 2x^2 - 8x - 16$, $x^3 + 3x^2 - 8x - 24$.
2. $z^2 - 5z + 4$, $z^3 - 5z^2 + 4$.

3. a^3-a^2-5a-3, $a^3-4a^2-11a-6$.
4. $b^3+2b^2-13b+10$, $b^3+b^2-10b+8$.
5. $4c^3-3c^2-24c-9$, $8c^3-2c^2-53c-39$.
6. $2d^2-5d+2$, $12d^3-8d^2-3d+2$.
7. g^3+2g^2+2g+1, g^3-2g-1.
8. h^4-2h^2+1, $h^4-4h^3+6h^2-4h+1$.
9. $2s^3+3s^2t-t^3$, $4s^3+st^2-t^3$.
10. w^4-2w^3+w, $2w^4-2w^3-2w-2$.
11. $2x^2-5x+2$ and $2x^3-3x^2-8x+12$.
12. $4a^3-a^2b-ab^2-5b^3$ and $7a^3+4a^2b+4ab^2-3b^3$.
13. $3y^4-4y^3+2y^2+y-2$ and $3y^4+8y^3-5y^2-6y$.
14. b^3+3b^2+4b+2 and $2b^3+b^2+1$.
15. $2c^4+9c^3+14c+3$ and $3c^4+15c^3+5c^2+10c+2$.
16. $3d^3-3d^2-33d+9$ and $6d^4+6d^3-42d^2-18d$.
17. $3h^3-9h^2+21h-63$ and $2h^4+19h^2+35$.
18. $6a^3-6a^2b+2ab^2-2b^3$ and $4a^2-5ab+b^2$.
19. $m^4+7m^3+7m^2-15m$ and $2m^3-4m^2-26m+220$.
20. $4n^5+14n^4+20n^3+70n^2$ and $8n^7+28n^6-8n^5-12n^4+56n^3$.
21. $2w^4-2w^3+4w^2+2w+6$ and $3w^4+6w^3-3w-6$.
22. $x^4+x^3-9x^2-3x+18$ and $x^5+6x^2-49x+42$.
23. $8z^4-6z^3+3z^2-3z+1$ and $18z^3-3z^2-15z+6$.
24. $64g^5-3g^2h^3+5gh^4$ and $20g^4-3g^3h+h^4$.
25. $2s^4-7s^3+9s^2-8s-5$ and $6s^4-11s^3-16s^2+15s$.

LOWEST COMMON MULTIPLE

EXERCISE 80

Find the L. C. M. of the expressions in each of the following groups:

1. x, y.
2. $5a$, $7a$.
3. ab, ac.
4. x^2, x^3.
5. y^8, y^3.
6. xy, yz.
7. a^2b, ab^2.
8. x^2yz^2, xy^2z.
9. a^2b^2c, a^3bc^3.
10. $m^2n^4z^6$, m^5n^3w.
11. $a^2b^3c^4d^5$, $a^6b^5c^2d$.
12. $a^3b^2x^4y^5$, $a^4b^5y^2z^3$.
13. a, ab, abc.
14. ab, bc, ca.
15. xy, yz, zw.
16. a, a^2, a^3.
17. x^4, x^2, x.
18. a, a^2b, a^3c.
19. abx, bcy, caz.
20. a^2b, b^2c, c^2a.
21. x^2y^3, y^2z^3, z^2x^3.

22. a^3bx, ab^3y, a^2b^2z.
23. a^2xy, b^2yz, c^2zx.
24. ab, bc, cd, da.
25. xy, xz, xw, yz, yw, zw.
26. bcd, acd, abd, abc.
27. xyz, yzw, zwx, wxy.
28. $a^2b^3c^4$, $a^3b^4c^2$, $a^4b^2c^3$.
29. $x^2y^4z^6$, $x^3y^3z^5$, $x^4y^2z^7$.
30. $3\,ab$, $6\,bc$, $8\,abc$.
31. $15\,xy$, $5\,xz$, $6\,xw$.
32. $9\,a^2x$, $12\,b^2y$, $18\,c^2z$.
33. $5\,ax^2$, $15\,by^2$, $4\,abxy$.
34. $14\,ax^2y$, $6\,bxy^2$, $7\,cxyz$.
35. $3\,acx$, $4\,bdy$, $5\,cdz$.
36. $6\,ax$, $8\,bcy$, $10\,abcw$.
37. $4\,x^2yw$, $5\,xy^2z$, $7\,yz^2w$.
38. $2\,ab$, $3\,ac$, $4\,acd$, $5\,bcd$.
39. a^2bcd, ab^2cd, abc^2d, $abcd^2$.
40. $3\,x^4y^2z$, $6\,x^3y^5z^2$, $x^2y^4z^3$, $4\,xy^3z^4$.
41. $9\,b^3cw$, $2\,ac^2x$, $4\,a^2bc$, $3\,cx^2w$.
42. $2\,a^5bx^2$, $5\,a^4bx$, $6\,b^4x^3y$, $3\,b^5xz$.
43. $2\,x^8yz^2$, $4\,x^4y^3z$, $8\,x^2y^5z$, $16\,xy^7z^3$.
44. $2\,a^2b^2c^3d^4$, $7\,ab^3cd^3$, $3\,a^3b^2c^5d$, $6\,a^4bcd^2$.

45. a, $b+c$.
46. $x+y$, z.
47. $m+n$, x^2.
48. y^2, $y-z$.
49. a^2, $a+b$.
50. c^2-d^2, c.
51. $a+b$, ab.
52. x^2-y^2, xy.
53. cd, $c-d$.
54. 2, $2x-1$.
55. $3x-1$, 4.
56. $2m+2n$, 3.
57. 6, $3a+3b$.
58. 8, $6x-6y$.
59. $5a+5$, 10.
60. $9a-9$, 12.
61. a^2+ab, b.
62. x^2, x^2-y^2.
63. a, b, $a+b$.
64. $c-d$, c, d.
65. $x+y$, x, y.
66. $a+b$, $a-b$.
67. $x-y$, $x+y$.
68. $a+2$, $a+3$.
69. $b-4$, $b-1$.
70. $c+5$, $c+6$.
71. $d+8$, $d+10$.
72. $x-12$, $x-3$.
73. $y+11$, $y-2$.
74. $z-15$, $z-4$.
75. $w+1$, $w-17$.
76. $x+y$, $x-y$, x, y.
77. $5(a+b)$, $3(a+b)$.
78. $x(x-y)$, $y(x-y)$.
79. $a(b-c)$, $b(b-c)$.
80. $6(x+1)$, $4(x+2)$.
81. $5(x-4)$, $10(x-5)$.
82. $3(x-2)$, $4(x-1)$.

EXERCISE 81

Find the L. C. M. of the expressions in each of the following groups:

1. $a^4+5\,a^3+5\,a^2-5\,a-6$ and $a^3+6\,a^2+11\,a+6$.
2. $b^3+3\,b^2-b-3$ and $b^3+4\,b^2+b-6$.
3. $c^3+c^2-8\,c-6$ and $2\,c^3-5\,c^2-2\,c+2$.
4. $6\,x^3y-7\,ax^2y-20\,a^2xy$ and $6\,x^2+2\,ax-8\,a^2$.

5. $2d^4 - 2d^3 - 2d - 2$ and $d^4 - 2d^3 + d$.
6. $y^3 - 6y^2 + 11y - 6$ and $y^3 - 8y^2 + 19y - 12$.
7. $14s^3 - 4s^2 - 10$ and $14s^3 + 24s^2 + 20s + 10$.
8. $2r^3 + 12r^2 + 22r + 12$ and $r^4 + r^3 - 4r^2 - 4r$.
9. $6z^3 - 8z^2 - 17z - 6$ and $12z^3 - z^2 - 21z - 10$.
10. $w^3 + 3w^2 + 4w + 2$ and $2w^3 + w^2 + 1$.
11. $h^3 - h^2 + h + 3$ and $h^4 + h^3 - 3h^2 - h + 2$.
12. $6k^3 + 15k^2 - 6k + 9$ and $9k^3 + 6k^2 - 51k + 36$.
13. $a^3 + a^2 - 10a + 8$; $a^2 - 3a + 2$ and $a^3 - 4a^2 + 5a - 2$.
14. $m^3 + 2am^2 + 4a^2m + 8a^3$; $m^3 - 2am^2 + 4a^2m - 8a^3$ and $m^3 + 4a^2m + am^2 + 4a^3$.
15. $2n^2 + 3n - 20$; $6n^3 - 25n^2 + 21n + 10$ and $2n^3 - 5n^2 + 6n - 15$.
16. $x^2 - 3bx + 2b^2$; $x^2 - 5bx + 4b^2$ and $3x^2 - 19bx + 28b^2$.

REVIEW

MENTAL EXERCISE 82

Simplify each of the following:

1. $(x + y)(x^2 - xy + y^2)$.
2. $(a - b)(a^2 + ab + b^2)$.
3. $(b^5 + c^5) \div (b + c)$.
4. $(c^6 - d^6) \div (c^2 - d^2)$.

Distinguish between

5. $a + bc$ and $(a + b)c$.
6. $x + yz + w$, $(x + y)z + w$, and $(x + y)(z + w)$.

Perform the following multiplications:

7. $(a + b + x)(a + b - x)$.
8. $(a + b + 7)(a + b - 7)$.
9. $(x - y + 1)(x + y - 1)$.
10. $(a + b + m + n)(a + b - m - n)$.
11. Find the continued product of $x + y$, $x^2 - y^2$, $x^2 + y^2$, and $x^4 + y^4$.
12. Show that $x^4 - 2x^3 + x^2$ is the square of a binomial.

Are the following expressions conditionally or identically equal?

13. $3x$ and $2x + x$.
14. $3x$ and $2 + 1$.
15. $(a + b)^2$ and $a^2 + 2ab + b^2$.
16. $a + b$ and $b + a$.
17. $a + b$ and $b + d$.
18. $\frac{x}{2}$ and 1.

Are the following expressions identical?

19. a^3+b^3 and $(a+b)^3$.

20. $(a+b)^2(a-b)^2$ and $(a^2-b^2)^2$.

21. a^2-b^2 and $(a-b)^2$.

22. $(x-1)^2$ and x^2-1.

Of the following equations select those which are equivalent to the equation $2x-3y=5$:

23. $4x-6y=10$; $6x-9y=15$; $8x+12y=30$; $10x-15y=25$.

FRACTIONS

EXERCISE 83

Change of Signs

Express each of the following negative fractions as an equivalent positive fraction:

1. $-\dfrac{-x}{2}$.

2. $-\dfrac{y}{-3}$.

3. $-\dfrac{z}{-4}$.

4. $-\dfrac{-m}{5}$.

5. $-\dfrac{6}{n}$.

6. $-\dfrac{7}{-x}$.

7. $-\dfrac{-9}{-y}$.

8. $-\dfrac{1}{1-a}$.

9. $-\dfrac{2}{b-3}$.

10. $-\dfrac{4}{6-c}$.

11. $-\dfrac{d-1}{2}$.

12. $-\dfrac{k-2}{5}$.

13. $-\dfrac{-a-b}{2}$.

14. $-\dfrac{a+3}{-7}$.

15. $-\dfrac{-8}{b+1}$.

16. $-\dfrac{-a+b}{c}$.

17. $-\dfrac{-x+1}{2}$.

18. $-\dfrac{-y+2}{4}$.

19. $-\dfrac{6}{-z+7}$.

20. $-\dfrac{a-1}{a+1}$.

21. $-\dfrac{1+x}{x-1}$.

22. $-\dfrac{2+y}{2-y}$.

23. $-\dfrac{z-3}{z+3}$.

24. $-\dfrac{3}{-x-y}$.

25. $-\dfrac{2}{-m-n}$.

26. $-\dfrac{-3}{a-b}$.

27. $-\dfrac{-5}{-d+6}$.

28. $-\dfrac{1}{(b-a)(a+b)}$.

29. $-\dfrac{3}{(x+y)(y-x)}$.

30. $-\dfrac{a}{(-b-c)(c+a)}$.

31. $-\dfrac{x}{(y+z)(-z-x)}$.

32. Write the expressions in each of the first 27 examples in three other ways by changing signs and decide which one of the four ways is the best in each case.

EXERCISE 84

Show that each of the following identities is true:

1. $-\dfrac{a}{b-a} \equiv \dfrac{a}{a-b}$.

2. $-\dfrac{a-b}{-b} \equiv \dfrac{a-b}{b}$.

3. $-\dfrac{b-a}{b-c} \equiv \dfrac{a-b}{b-c}$.

4. $-\dfrac{x-y}{z-x} \equiv \dfrac{x-y}{x-z}$.

5. $\dfrac{a}{c-b} \equiv -\dfrac{a}{b-c}$.

6. $\dfrac{x-1}{2} \equiv -\dfrac{1-x}{2}$.

7. $\dfrac{-x}{y-x-z} \equiv \dfrac{x}{x-y+z}$.

8. $\dfrac{-b}{c-a-b} \equiv \dfrac{b}{a+b-c}$.

9. $\dfrac{b-a-c}{c-a-b} \equiv \dfrac{a-b+c}{a+b-c}$.

10. $\dfrac{z-x-y}{z-x+y} \equiv \dfrac{x+y-z}{x-y-z}$.

11. $\dfrac{b}{a(c-b)} \equiv -\dfrac{b}{a(b-c)}$.

12. $\dfrac{1}{(x-y)(z-y)} \equiv -\dfrac{1}{(x-y)(y-z)}$.

13. $\dfrac{1}{(a-b)(c-b)(a-c)} \equiv \dfrac{1}{(a-b)(b-c)(c-a)}$.

14. $\dfrac{1}{(y-x)(z-y)(z-x)} \equiv \dfrac{1}{(x-y)(y-z)(z-x)}$.

15. $\dfrac{1}{(b-a)(b-c)(a-c)} \equiv \dfrac{1}{(a-b)(b-c)(c-a)}$.

16. $\dfrac{a-b}{(c-b)(a-c)} \equiv \dfrac{a-b}{(b-c)(c-a)}$.

17. $\dfrac{y-x}{(z-y)(z-x)} \equiv \dfrac{x-y}{(y-z)(z-x)}$.

18. $\dfrac{a-1}{(2-a)(3-a)} \equiv \dfrac{a-1}{(a-2)(a-3)}$.

19. $\dfrac{4-b}{(c-4)(d-4)} \equiv \dfrac{4-b}{(4-c)(4-d)}$.

20. $\dfrac{b-a}{(c-b)(d-c)(a-d)} \equiv \dfrac{a-b}{(b-c)(c-d)(d-a)}$.

21. $\dfrac{1}{(y-x)(y-x)} \equiv \dfrac{1}{(x-y)(x-y)}$.

22. $\dfrac{1}{(b-a)^2} \equiv \dfrac{1}{(a-b)^2}$.

23. $\dfrac{1}{b^2-a^2} \equiv -\dfrac{1}{a^2-b^2}$.

24. $\dfrac{1}{(y-x)^3} \equiv -\dfrac{1}{(x-y)^3}$.

25. $-\dfrac{1}{(c-b)^5} \equiv \dfrac{1}{(b-c)^5}$.

26. $\dfrac{-a}{(c-b)^3} \equiv \dfrac{a}{(b-c)^3}$.

27. $\dfrac{2}{y^3-x^3} \equiv -\dfrac{2}{x^3-y^3}$.

REDUCTION OF FRACTIONS TO LOWEST TERMS

EXERCISE 85

Reduce each of the following fractions to lowest terms:

1. $\dfrac{-14}{-28}$.
2. $\dfrac{-12}{-18}$.
3. $\dfrac{21}{-35}$.
4. $\dfrac{-34}{-51}$.
5. $\dfrac{-18}{90}$.
6. $\dfrac{-26}{65}$.
7. $\dfrac{-38}{-57}$.
8. $\dfrac{-56}{70}$.
9. $\dfrac{-42}{-105}$.
10. $\dfrac{-.2}{4}$.
11. $\dfrac{.5}{-5}$.
12. $\dfrac{.3}{-9}$.
13. $\dfrac{-.2}{-12}$.
14. $\dfrac{-.3}{-1.5}$.
15. $\dfrac{-.4}{2.4}$.
16. $\dfrac{.08}{-1.6}$.
17. $\dfrac{-2.5}{50}$.
18. $\dfrac{3.6}{-.09}$.
19. $\dfrac{-.09}{3.6}$.
20. $\dfrac{3.6}{10.8}$.
21. $\dfrac{1.08}{3.6}$.
22. $\dfrac{xy}{xz}$.
23. $\dfrac{ab}{a^2}$.
24. $\dfrac{2x}{6}$.
25. $\dfrac{3y}{12}$.
26. $\dfrac{16}{8a}$.
27. $\dfrac{7}{21b^2}$.
28. $\dfrac{20z}{5w}$.
29. $\dfrac{28a^2}{14a^3}$.
30. $\dfrac{x^4}{x^6}$.
31. $\dfrac{x^2y}{xy^2}$.
32. $\dfrac{7a^2}{14a}$.
33. $\dfrac{25h^2}{5h^3}$.
34. $\dfrac{8bc}{24c^2}$.
35. $\dfrac{18cd}{9d^2}$.
36. $\dfrac{10abc}{35bcd}$.
37. $\dfrac{4c^2d}{4cd^2e}$.
38. $\dfrac{a^2bc}{ab^2c}$.
39. $\dfrac{x^2yz}{xyz^2}$.
40. $\dfrac{r^2st}{rs^2t}$.
41. $\dfrac{a^2mw}{a^2m^2w^2}$.
42. $\dfrac{a^4b^3c^2}{a^2b^3c^4}$.
43. $\dfrac{a^m}{a^{m+1}}$.
44. $\dfrac{b^{m-1}}{b^m}$.
45. $\dfrac{c^{n-1}}{c^{n+1}}$.
46. $\dfrac{x^{n+2}}{x^{n+5}}$.
47. $\dfrac{x^{r-5}}{x^{r-3}}$.
48. $\dfrac{a^{2n}}{a^{2n+1}}$.

49. $\frac{x^{n-r}}{x^{n-r+1}}$.

50. $\frac{a^n b^{n+1}}{a^{n+1} b^n}$.

51. $\frac{x^{n+2} y^{n-1}}{x^{n-3} y^{n+5}}$.

52. $\frac{ab}{ac+ad}$.

53. $\frac{xy}{yz-yw}$.

54. $\frac{m}{m^2+m}$.

55. $\frac{2a^2}{2a+2}$.

56. $\frac{am+an}{ax+ay}$.

57. $\frac{ab-bc}{ab+bc}$.

58. $\frac{4a+8b}{12c+16d}$.

59. $\frac{5x-15y}{10x+20z}$.

60. $\frac{6c+12d}{3m-9n}$.

61. $\frac{a+b}{(a+b)^2}$.

62. $\frac{a-b}{a^2-b^2}$.

63. $\frac{x+1}{x^2-1}$.

64. $\frac{2x+1}{4x^2-1}$.

65. $\frac{(c+d)^2}{c^2-d^2}$.

66. $\frac{m^2-n^2}{(m-n)^2}$.

67. $\frac{2a+3b}{4a^2-9b^2}$.

68. $\frac{5x-8y}{25x^2-64y^2}$.

69. $\frac{a+b}{a^3+b^3}$.

70. $\frac{b-c}{b^3-c^3}$.

71. $\frac{c^2-d^2}{c^4-d^4}$.

72. $\frac{d^2-1}{d^3+1}$.

73. $\frac{x^2-1}{x^3-1}$.

74. $\frac{a^2-b^2}{a^3+b^3}$.

75. $\frac{x-1}{x^2-2x+1}$.

76. $\frac{a+b}{a^2+2ab+b^2}$.

77. $\frac{y-4}{y^2-8y+16}$.

78. $\frac{a+1}{a^2+3a+2}$.

79. $\frac{x-5}{x^2-7x+10}$.

80. $\frac{a^3-3a^2}{a^2-6a+9}$.

81. $\frac{m^4+2m^3}{m^2+4m+4}$.

82. $\frac{a^2+2ab+b^2}{a^3+b^3}$.

83. $\frac{c^2-2cd+d^2}{c^3-d^3}$.

84. $\frac{x^2-xy+y^2}{x^3+y^3}$.

85. $\frac{a^2+ab+b^2}{a^3-b^3}$.

86. $\frac{a^2+2a+1}{a^2+3a+2}$.

87. $\frac{b^2-5b+6}{b^2-4b+4}$.

88. $\frac{c^2+6c+9}{c^2+c-6}$.

89. $\frac{x^2-3x-10}{x^2-6x+5}$.

90. $\frac{y^2-8y+12}{y^2-9y+18}$.

91. $\frac{1+5x+6x^2}{1+6x+8x^2}$.

92. $\frac{a^2+6a-55}{a^2-9a+20}$.

93. $\frac{2a^2+5a+3}{5a^2+12a+7}$.

94. $\frac{a^2+3a+2}{2a^2+5a+2}$.

95. $\frac{k^2-7k+12}{k^2-5k+6}$.

96. $\frac{(a+b)^2-c^2}{a^2-(b+c)^2}$.

97. $\frac{x^2-(y-z)^2}{(x+y)^2-z^2}$.

98. $\frac{a^2-(b+c)^2}{(a+c)^2-b^2}$.

99. $\frac{(x-y)^2-z^2}{x^2-(y+z)^2}$.

Fill in the following Blank Forms with such numbers that the fractions can be simplified:

100. $\dfrac{b^2+5b+6}{b^2+7b+(\ )}$. 101. $\dfrac{c^2+7c+12}{c^2+(\ )+(\ )}$. 102. $\dfrac{d^2-7d+12}{d^2-7d-(\ )}$.

103. $\dfrac{3a^2+7a+4}{a^2+8a+(\ )}$. 104. $\dfrac{(\ +\ )^2-(\ )^2}{(\ )^2-(\ -\ )^2}$.

REDUCTION OF IMPROPER FRACTIONS TO INTEGRAL OR MIXED EXPRESSIONS

EXERCISE 86

Reduce each of the following improper fractions to the form of an integral or mixed expression:

1. $\dfrac{-16}{-12}$. 5. $\dfrac{-42}{26}$. 9. $\dfrac{-1.5}{.12}$. 13. $\dfrac{a+b}{a}$.

2. $\dfrac{-18}{-15}$. 6. $\dfrac{-56}{35}$. 10. $\dfrac{-5}{2.5}$. 14. $\dfrac{c-d}{d}$.

3. $\dfrac{24}{-14}$. 7. $\dfrac{-.8}{.6}$. 11. $\dfrac{-3}{.15}$. 15. $\dfrac{x^2+1}{x}$.

4. $\dfrac{-28}{20}$. 8. $\dfrac{-.12}{.08}$. 12. $\dfrac{2.4}{-.15}$. 16. $\dfrac{ab+c}{b}$.

17. $\dfrac{x^2-yz}{x}$. 21. $\dfrac{m^4+1}{m^2}$. 25. $\dfrac{a^2-ab+b^2}{a}$.

18. $\dfrac{2x^2+4yz}{2x}$. 22. $\dfrac{ab^2-bc}{ab}$. 26. $\dfrac{x^2+xy+y^2}{y}$.

19. $\dfrac{6a^3-5b}{3a}$. 23. $\dfrac{x^2y+yz}{xy}$. 27. $\dfrac{a^2+a+1}{a}$.

20. $\dfrac{x^3+2}{x^2}$. 24. $\dfrac{abc+cd^2}{ac}$. 28. $\dfrac{x^3-x^2+x-1}{x^2}$.

29. $\dfrac{12a^2+8a+1}{4a}$. 32. $\dfrac{20w^2-15w-11}{5w}$.

30. $\dfrac{15b^2-12b+9}{3b}$. 33. $\dfrac{24a^3-12a^2+5}{6a^2}$.

31. $\dfrac{18x^2-9x-3}{9x}$. 34. $\dfrac{36a^2b^2+12abc-1}{12ab}$.

35. $\dfrac{a^3+1}{a+1}$. 36. $\dfrac{b^3-1}{b-1}$. 37. $\dfrac{m^3-8}{m-2}$.

38. $\frac{8+x^3}{2+x}$.

39. $\frac{a^3-27}{a-3}$.

40. $\frac{a^5-b^5}{a-b}$.

41. $\frac{a^2-9}{a+3}$.

42. $\frac{b^3-1}{b-1}$.

43. $\frac{x^2+1}{x}$.

44. $\frac{4\,d^4+2\,d^2+1}{2\,d^2}$.

45. $\frac{h^3-3}{h-3}$.

REDUCTION OF TWO OR MORE FRACTIONS TO EQUIVALENT FRACTIONS HAVING A LOWEST COMMON DENOMINATOR.

PREPARATORY EXERCISE 87

Reduce each of the following groups of fractions to a group of equivalent fractions having a lowest common denominator:

1. $\frac{1}{2}, \frac{1}{3}$.
2. $\frac{1}{3}, \frac{1}{5}$.
3. $\frac{1}{4}, \frac{1}{7}$.
4. $\frac{1}{2}, \frac{1}{8}$.
5. $\frac{2}{3}, \frac{1}{6}$.
6. $\frac{4}{5}, \frac{7}{8}$.
7. $\frac{3}{10}, \frac{5}{12}$.
8. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}$.
9. $\frac{1}{3}, \frac{1}{9}, \frac{1}{12}$.
10. $\frac{2}{3}, \frac{4}{5}, \frac{7}{10}$.
11. $\frac{8}{9}, \frac{1}{10}, \frac{5}{12}$.

EXERCISE 88

Reduce each of the following groups of fractions to a group of equivalent fractions having a lowest common denominator:

1. $\frac{a}{2}, \frac{a}{3}$.
2. $\frac{b}{5}, \frac{b}{7}$.
3. $\frac{c}{9}, \frac{c}{12}$.
4. $\frac{d}{5}, \frac{d}{20}$.
5. $\frac{1}{a}, \frac{1}{b}$.
6. $\frac{2}{x^2}, \frac{2}{y^2}$.
7. $\frac{5}{x^2}, \frac{6}{x}$.
8. $\frac{2}{a^2}, \frac{7}{a}$.
9. $\frac{3}{x}, \frac{4}{y}$.
10. $\frac{6}{z}, \frac{8}{w}$.
11. $\frac{5}{2\,a}, \frac{6}{7\,a}$.
12. $\frac{8}{ab}, \frac{10}{bc}$.
13. $\frac{a}{b}, \frac{b}{c}$.
14. $\frac{x}{y}, \frac{y}{x}$.
15. $\frac{a}{bc}, \frac{b}{cd}$.
16. $\frac{x^2}{yz}, \frac{y^2}{zw}$.
17. $\frac{1}{3\,ab}, \frac{2}{4\,bc}$.
18. $\frac{a}{5\,bc}, \frac{a^2}{10\,cd}$.
19. $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$.
20. $\frac{x}{2\,y}, \frac{x}{4\,y}, \frac{x}{6\,y}$.
21. $\frac{b}{a^3}, \frac{b^2}{a^2}, \frac{b^3}{a}$.
22. $\frac{1}{2\,x}, \frac{1}{3\,y}, \frac{1}{4\,z}$.
23. $\frac{x}{y}, \frac{y}{z}, \frac{z}{x}$.
24. $\frac{a}{bc}, \frac{b}{ca}, \frac{c}{ab}$.

25. $\frac{2}{a}, \frac{1}{2a}, \frac{2}{a^2}$.

26. $\frac{x}{y+z}, \frac{y}{z+x}$.

27. $\frac{1}{x+1}, \frac{1}{x^2-1}$.

28. $\frac{1}{a-b}, \frac{1}{a^2-b^2}$.

ADDITION AND SUBTRACTION OF FRACTIONS

EXERCISE 89

Simplify each of the following:

1. $\frac{2}{5}+\frac{1}{5}$.
2. $\frac{3}{7}+\frac{1}{7}$.
3. $\frac{5}{9}-\frac{2}{9}$.
4. $\frac{14}{11}-\frac{3}{11}$.
5. $\frac{1}{6}+\frac{1}{2}$.
6. $-\frac{1}{4}-\frac{1}{8}$.
7. $\frac{1}{6}+\frac{1}{12}$.
8. $\frac{1}{2}+\frac{1}{3}$.
9. $-\frac{1}{4}-\frac{1}{6}$.
10. $\frac{3}{5}+\frac{1}{7}$.
11. $\frac{3}{4}+\frac{5}{7}$.
12. $-\frac{2}{3}-\frac{3}{2}$.
13. $\frac{2}{5}-\frac{1}{2}$.
14. $-\frac{3}{8}-\frac{1}{5}$.
15. $\frac{a}{2}+\frac{a}{3}$.
16. $\frac{b}{4}+\frac{b}{5}$.
17. $\frac{c}{6}+\frac{c}{8}$.
18. $\frac{d}{3}+\frac{d}{9}$.
19. $\frac{e}{12}+\frac{e}{6}$.
20. $\frac{2h}{5}+\frac{h}{2}$.
21. $\frac{3x}{4}+\frac{4x}{3}$.
22. $\frac{5y}{8}-\frac{y}{4}$.
23. $\frac{2y}{9}-\frac{y}{3}$.
24. $\frac{z}{2a}+\frac{z}{a}$.
25. $\frac{3w}{4y}-\frac{w}{2y}$.
26. $\frac{1}{a}+\frac{1}{b}$.
27. $\frac{2}{a}+\frac{3}{a}$.
28. $\frac{a}{2}+\frac{a}{3}$.
29. $\frac{5}{x}+\frac{2}{x}$.
30. $\frac{x}{5}+\frac{x}{2}$.
31. $\frac{x}{3}+\frac{y}{3}$.
32. $\frac{3}{x}+\frac{3}{y}$.
33. $\frac{3}{m}-\frac{4}{m}$.
34. $\frac{m}{3}-\frac{m}{4}$.
35. $\frac{a}{c}+\frac{b}{c}$.
36. $\frac{c}{a}+\frac{c}{b}$.
37. $\frac{1}{c}-\frac{1}{d}$.
38. $\frac{2}{x}+\frac{3}{y}$.
39. $\frac{4}{z}-\frac{5}{w}$.
40. $\frac{2}{c}+\frac{c}{2}$.
41. $\frac{1}{a}+\frac{1}{a^2}$.
42. $\frac{2}{b^2}-\frac{3}{b}$.
43. $\frac{a}{c^2}+\frac{a^2}{c}$.
44. $\frac{1}{x^2y}+\frac{1}{xy^2}$.
45. $\frac{a}{bc}+\frac{b}{ca}$.
46. $\frac{xy}{z}+\frac{y}{xz}$.
47. $\frac{a^2}{b^2c}+\frac{a^2}{bc^2}$.
48. $\frac{xy}{wz}+\frac{xz}{wy}$.
49. $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}$.
50. $\frac{a}{4}-\frac{a}{6}+\frac{a}{8}$.
51. $\frac{x}{3}+\frac{x}{4}-\frac{x}{5}$.
52. $\frac{2a}{3}+\frac{3a}{4}+\frac{4a}{5}$.
53. $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$.

54. $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}$.

55. $\frac{x}{y}+\frac{y}{z}+\frac{z}{w}$.

56. $\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}$.

57. $\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}$.

58. $\frac{1}{x^2yz}+\frac{1}{xy^2z}+\frac{1}{xyz^2}$.

59. $\frac{1}{a+2}+\frac{1}{a+3}$.

60. $\frac{1}{a+b}+\frac{1}{a+c}$.

61. $\frac{1}{x-4}+\frac{1}{x-5}$.

62. $\frac{1}{y-3}-\frac{1}{y-2}$.

63. $\frac{a}{a+b}+\frac{b}{a-b}$.

64. $\frac{a+b}{2}+\frac{a-b}{3}$.

65. $\frac{x-y}{4}+\frac{x+y}{5}$.

66. $\frac{a+b}{a}+\frac{b+c}{b}$.

67. $\frac{x-y}{5}-\frac{x-y}{3}$.

68. $\frac{a+2}{2}+\frac{a-2}{2}$.

69. $\frac{x+5}{5}+\frac{x+6}{6}$.

70. $\frac{y-5}{5}-\frac{y-4}{4}$.

71. $\frac{1}{a+b}+\frac{1}{a-b}$.

72. $\frac{2}{c-d}-\frac{2}{c+d}$.

73. $\frac{x}{x+y}-\frac{x}{x-y}$.

74. $\frac{a+b}{2}+\frac{2}{a+b}$.

75. $\frac{x-3}{3}-\frac{3}{x-3}$.

76. $\frac{1}{x-y}-\frac{1}{(x-y)^2}$.

77. $\frac{1}{a+b}-\frac{b}{(a+b)^2}$.

78. $\frac{1}{x^2-4}+\frac{1}{x+2}$.

79. $\frac{3}{x^2-9}-\frac{1}{x+3}$.

80. $\frac{x}{x^2-16}-\frac{1}{x-4}$.

81. $\frac{a+b}{a-b}+\frac{a-b}{a+b}$.

82. $\frac{x-1}{x+1}+\frac{x+1}{x-1}$.

83. $\frac{c+d}{c-d}-\frac{c-d}{c+d}$.

REDUCTION OF A GIVEN EXPRESSION TO THE FORM OF A FRACTION HAVING A SPECIFIED DENOMINATOR

EXERCISE 90

Express each of the following as a fraction having the denominator specified:

1. 3 with the denominator 4.
2. 2 with the denominator 5.

3. -4 with the denominator 7.
4. -5 with the denominator 8.
5. 6 with the denominator -6.
6. 8 with the denominator -11.
7. $\frac{2}{5}$ with the denominator 10.
8. $\frac{3}{7}$ with the denominator -21.
9. $-\frac{4}{6}$ with the denominator 12.
10. $\frac{7}{8}$ with the denominator 40.
11. $-\frac{9}{16}$ with the denominator 48.
12. $\frac{5}{12}$ with the denominator 72.
13. 2 with the denominator a.
14. -3 with the denominator b.
15. 4 with the denominator $5c$.
16. 6 with the denominator $7d$.
17. -8 with the denominator $5x$.
18. a with the denominator b.
19. b with the denominator x.
20. c with the denominator c.
21. $-d$ with the denominator y.
22. ab with the denominator b.
23. xy with the denominator yz.
24. xyz with the denominator xz.
25. $a+b$ with the denominator c.

REDUCTION OF MIXED EXPRESSIONS TO THE FORMS OF IMPROPER FRACTIONS

EXERCISE 91

Reduce each of the following mixed expressions to the form of an improper fraction:

1. $1+\frac{a}{2}$.
2. $3+\frac{b}{5}$.
3. $2+\frac{b}{3}$.
4. $\frac{c}{4}+7$.
5. $\frac{d}{6}+8$.
6. $2-\frac{a}{3}$.
7. $3-\frac{a}{2}$.
8. $3+\frac{2a}{5}$.
9. $2-\frac{x^2}{2}$.

10. $a-\frac{b^2}{a}$.
11. $\frac{x^2}{y}-y$.
12. $6+\frac{3b}{7}$.
13. $8-\frac{5b}{2}$.
14. $7-\frac{2}{3a}$.
15. $5-\frac{4}{5a}$.
16. $\frac{3b}{10}-1$.
17. $x+\frac{y}{z}$.
18. $a-\frac{m}{x}$.
19. $\frac{m}{n}-1$.
20. $\frac{c}{d}+x+y$.
21. $a+2+\frac{1}{a}$.
22. $x-2+\frac{1}{x}$.
23. $a+b+\frac{b^2}{a}$.
24. $m-n+\frac{n^2}{m}$.
25. $a+b+\frac{1}{a+b}$.
26. $\frac{n^2}{m-n}+m+n$.
27. $\frac{6}{a+3}+a-2$.
28. $\frac{6}{x+4}+x-3$.
29. $\frac{9}{b+2}+b+4$.

MULTIPLICATION OF FRACTIONS

EXERCISE 92

Simplify each of the following:

1. $\frac{2}{3}\times 5$.
2. $\frac{3}{5}\times 8$.
3. $\frac{7}{10}\times(-3)$.
4. $(-\frac{5}{9})\times 4$.
5. $(-\frac{6}{7})\times 8$.
6. $\frac{7}{12}\times(-2)$.
7. $\frac{8}{15}\times(-5)$.
8. $\frac{3}{16}\times(-4)$.
9. $(-\frac{5}{14})\times(-7)$.
10. $4\times\frac{2}{3}$.
11. $5\times\frac{3}{7}$.
12. $6\times\frac{3}{4}$.
13. $(-8)\times\frac{5}{6}$.
14. $(-12)\times\frac{7}{8}$.
15. $20\times(-\frac{2}{15})$.
16. $24\times(-\frac{7}{16})$.
17. $(-25)\times(-\frac{4}{7})$.
18. $(-36)\times(-\frac{3}{14})$.
19. $\frac{2}{3}\times\frac{5}{7}$.
20. $\frac{3}{8}\times\frac{9}{11}$.
21. $\frac{3}{4}\times\frac{5}{8}$.
22. $(-\frac{4}{6})\times(-\frac{5}{7})$.
23. $(-\frac{3}{4})\times(-\frac{4}{5})$.
24. $(-\frac{6}{7})\times(-\frac{5}{6})$.
25. $\frac{11}{12}\times\frac{16}{17}$.
26. $(-\frac{8}{15})\times\frac{9}{10}$.
27. $\frac{7}{9}\times\frac{18}{35}$.
28. $\frac{49}{51}\times\frac{9}{77}$.
29. $(-\frac{3}{5})\times(-\frac{25}{9})$.
30. $(-\frac{81}{16})\times(-\frac{4}{9})$.
31. $\frac{a}{c}\times d$.
32. $\frac{x}{y^2}\times y$.
33. $\frac{2a}{b}\times 3c$.
34. $\frac{9}{10x}\times 2x^2$.
35. $\frac{ab}{c}\times c^2d$.
36. $x\times\frac{y}{2}$.
37. $2\times\frac{a}{b}$.
38. $4\times\frac{3}{m}$.
39. $a^2\times\frac{bc}{a}$.
40. $16x^2\times\frac{-y}{8x}$.
41. $18a^3x\times\left(-\frac{1}{6a}\right)$.
42. $4b\times\left(-\frac{1}{20ab^2}\right)$.

43. $\frac{a}{2}\times\frac{2}{3}$.

44. $\frac{b}{3}\times\frac{4}{5}$.

45. $\frac{5}{6}\times\frac{c}{10}$.

46. $\frac{8}{11}\times\frac{2z}{d}$.

47. $\frac{a}{d}\times\frac{b}{c}$.

48. $\frac{x^2}{y}\times\frac{y}{z}$.

49. $\frac{a}{b^2}\times\frac{b}{c}$.

50. $\frac{b^2}{c}\times\frac{c}{b^3}$.

51. $\frac{m^3}{n^2}\times\frac{n^3}{m^4}$.

52. $\frac{ab}{x}\times\frac{xy}{bc}$.

53. $\frac{ax^2}{y}\times\frac{b}{xy^2}$.

54. $\frac{2a}{3b}\times\frac{4c}{5d}$.

55. $\frac{1}{xy}\times\frac{y^2}{z}$.

56. $\frac{1}{ab}\times\frac{1}{ac}$.

57. $\frac{3}{x^2y}\times\frac{3}{yz^2}$.

58. $\frac{x}{y}\times\left(-\frac{a}{b}\right)$.

59. $\frac{m^2}{16}\times\left(-\frac{8}{m}\right)$.

60. $\left(-\frac{12}{ab^3}\right)\times\frac{abc}{3}$.

61. $\left(-\frac{x^2}{2y}\right)\left(\frac{4y^4}{x^3}\right)$.

62. $\left(\frac{abx}{cy}\right)\left(-\frac{acy}{bx}\right)$.

63. $\left(-\frac{a}{bcy}\right)\left(-\frac{by}{acz}\right)$.

64. $\left(\frac{a}{b}\right)\left(\frac{b}{c}\right)\left(\frac{c}{a}\right)$.

65. $\left(\frac{x}{5}\right)\left(\frac{10}{y}\right)\left(\frac{z}{2}\right)$.

66. $\left(\frac{2a}{3}\right)\left(\frac{b}{4c}\right)\left(\frac{6d}{e}\right)$.

67. $\left(\frac{a}{bc}\right)\left(\frac{bc}{a}\right)\left(\frac{c}{ab}\right)$.

68. $\frac{a+b}{2}\times\frac{6}{a^2-b^2}$.

69. $\frac{x^2-y^2}{10}\times\frac{5}{x-y}$.

70. $\frac{x}{x+y}\times\frac{y}{x-y}$.

71. $\frac{a+b}{c+d}\times\frac{a-b}{c-d}$.

72. $\frac{x^3+1}{14}\times\frac{2}{x+1}$.

73. $\frac{51}{x^3-1}\times\frac{x-1}{17}$.

74. $\frac{a+1}{x^2-1}\times\frac{x-1}{a+1}$.

75. $\frac{a^2+2a+1}{3xy}\times\frac{6xy}{a^2-1}$.

76. $\frac{x+3y}{2a+1}\times\frac{4a^2-1}{x^2-9y^2}$.

77. $\frac{4a-b}{m-5n}\times\frac{m^2-25n^2}{16a^2-b^2}$.

78. $\frac{x^2+7x+12}{abc}\times\frac{bcd}{x+3}$.

EXERCISE 93

Express the following products as single fractions in lowest terms, checking all results numerically:

1. $\frac{6a^2b}{5xy^2}\times\frac{25x^2y}{18ab}$.

2. $\frac{5a^2b^2}{7xy^2}\times\frac{14xy^3}{15a^3b}$.

3. $\frac{12abc}{7yzw}\times\frac{9xyz}{28bcd}$.

4. $\frac{13akt}{6rs^2m}\times\frac{6r^2sm}{11k^2tx}$.

5. $\frac{9a^3x^2}{4bcd}\times\frac{2b^2}{3acx}\times\frac{2cd}{3abx}$.

6. $\frac{5xz^3}{21y^2}\times\frac{3yz^2}{8x^2}\times\frac{7z^4}{30x^2y^2}$.

7. $\frac{a+b}{2} \times \frac{6}{a^2-b^2}$.

8. $\frac{x^2-y^2}{10} \times \frac{5}{x-y}$.

9. $\frac{x}{x+y} \times \frac{y}{x-y}$.

10. $\frac{a+b}{c+d} \times \frac{a-b}{c-d}$.

11. $\frac{x^3+1}{14} \times \frac{2}{x+1}$.

12. $\frac{51}{x^3-1} \times \frac{x-1}{17}$.

13. $\frac{a+1}{x^2-1} \times \frac{x-1}{a+1}$.

14. $\frac{a^2+2a+1}{3xy} \times \frac{6xy}{a^2-1}$.

15. $\frac{x+3y}{2a+1} \times \frac{4a^2-1}{x^2-9y^2}$.

16. $\frac{4a-b}{m-5n} \times \frac{m^2-25n^2}{16a^2-b^2}$.

17. $\frac{x^2+7x+12}{abc} \times \frac{bcd}{x+3}$.

18. $\frac{(a+2)^2}{a^2+8a+15} \times \frac{a+5}{a+2}$.

19. $\frac{a^2+3a}{3bc} \times \frac{3abc}{a+3}$.

20. $\frac{a^2+ab+b^2}{x+y} \times \frac{(x+y)^3}{a^3-b^3}$.

21. $\frac{a^2+a}{x^2-xy+y^2} \times \frac{x^3+y^3}{ab+b}$.

22. $\frac{x+z}{x-y} \times \frac{x^2-y^2}{x^2+xz}$.

23. $\frac{(a+b)^2}{a^2-b^2} \times \frac{a-b}{a+b}$.

24. $\frac{m^2-n^2}{x+y} \times \frac{x(x+y)}{m-n}$.

25. $\frac{x^2+xy}{abcx} \times \frac{3a^2bdy}{3x+3y}$.

26. $\frac{x+5}{a-3} \times \frac{5a-15}{10+2x}$.

27. $\frac{2ax+a^2}{b^2+ab} \times \frac{a^2-b^2}{a^2-4x^2}$.

28. $\frac{m^2+2mn+n^2}{m^2+mn} \times \frac{m^2}{m^2-n^2}$.

29. $\frac{x^2+x-6}{x^2-x-6} \times \frac{x^2+x-12}{x^2-x-12}$.

30. $\frac{(a-b)b}{a^2-2ab+b^2} \times \frac{xy(a^2-b^2)}{a^2+2ab+b^2}$.

31. $\frac{(a-b)b}{a^2+b^2+2ab} \times \frac{a(b+a)}{a^2+b^2-2ab} \times \frac{xz(a^2-b^2)}{a^2b^2y}$.

32. $\frac{a^2-4a+3}{a^2-6a+8} \times \frac{a^2-4a+4}{2a+a^2} \times \frac{a^2-2a-8}{a^2-5a+6}$.

POWERS OF FRACTIONS

EXERCISE 94

Express the following powers of quotients as quotients of powers:

1. $(\frac{2}{3})^2$.
2. $(\frac{3}{5})^2$.
3. $(-\frac{2}{3})^4$.
4. $(-\frac{4}{5})^3$.
5. $(\frac{1}{6})^2$.
6. $(\frac{7}{8})^2$.
7. $(-\frac{5}{3})^3$.
8. $-(\frac{4}{5})^4$.
9. $-(\frac{1}{2})^5$.
10. $(\frac{3}{2})^6$.
11. $(-\frac{5}{6})^3$.
12. $-(-\frac{3}{4})^3$.

13. $\left(\frac{a}{b}\right)^2$.
14. $\left(\frac{x}{3}\right)^2$.
15. $\left(\frac{2}{y}\right)^3$.
16. $\left(-\frac{c}{5}\right)^2$.
17. $\left(-\frac{3}{w}\right)^3$.
18. $\left(\frac{2a}{3b}\right)^2$.
19. $\left(-\frac{4x}{5y}\right)^3$.
20. $-\left(\frac{2}{m}\right)^5$.
21. $\left(-\frac{3}{2a}\right)^4$.
22. $\left(\frac{1}{ab}\right)^6$.
23. $\left(\frac{1}{a^2}\right)^4$.
24. $\left(\frac{ab}{x^3}\right)^5$.

Express each of the following quotients of powers as powers of fractions:

25. $\frac{25}{36}$.
26. $\frac{9}{25}$.
27. $\frac{16}{49}$.
28. $\frac{25}{81}$.
29. $-\frac{8}{125}$.
30. $\frac{16}{81}$.
31. $-\frac{1}{32}$.
32. $-\frac{27}{64}$.
33. $-\frac{125}{64}$.
34. $\frac{144}{169}$.
35. $\frac{81}{625}$.
36. $\frac{289}{400}$.
37. $\frac{196}{225}$.
38. $\frac{121}{256}$.
39. $\frac{49}{324}$.
40. $-\frac{729}{1000}$.

DIVISION OF FRACTIONS

EXERCISE 95

Express each of the following quotients in simplest form:

1. $\frac{6}{7} \div 2$.
2. $\frac{8}{11} \div (-4)$.
3. $(-\frac{9}{10}) \div 3$.
4. $(-\frac{15}{16}) \div (-5)$.
5. $\frac{1}{3} \div 4$.
6. $(-\frac{1}{9}) \div 10$.
7. $6 \div \frac{1}{2}$.
8. $8 \div \frac{1}{4}$.
9. $(-10) \div (-\frac{1}{3})$.
10. $7 \div (-\frac{1}{7})$.
11. $3 \div \frac{3}{4}$.
12. $4 \div (-\frac{4}{5})$.
13. $\frac{1}{4} \div (-\frac{1}{2})$.
14. $(-\frac{1}{12}) \div (-\frac{1}{6})$.
15. $\frac{9}{10} \div \frac{3}{5}$.
16. $\frac{8}{15} \div (-\frac{2}{3})$.
17. $(-\frac{16}{21}) \div (-\frac{8}{3})$.
18. $\frac{15}{28} \div (-\frac{3}{14})$.
19. $(-\frac{5}{12}) \div (-\frac{5}{2})$.
20. $(-\frac{12}{35}) \div (-\frac{3}{7})$.
21. $\frac{3}{4} \div \frac{5}{7}$.
22. $\frac{4}{9} \div \frac{3}{2}$.
23. $\frac{11}{13} \div (-\frac{4}{9})$.
24. $(-\frac{10}{39}) \div (-\frac{5}{12})$.
25. $\frac{20}{33} \div \frac{7}{3}$.
26. $\frac{26}{27} \div \frac{4}{9}$.
27. $(-\frac{1}{28}) \div (-\frac{1}{4})$.
28. $(-\frac{1}{30}) \div (-\frac{1}{15})$.
29. $\frac{1}{16} \div (-\frac{3}{4})$.
30. $(-\frac{1}{15}) \div (-\frac{1}{6})$.
31. $\frac{a^2}{b} \div a$.
32. $\frac{5m}{n} \div m$.
33. $\frac{12x}{y} \div 2x$.
34. $\frac{15c}{y} \div (-3c)$.
35. $\frac{2}{3} \div x$.
36. $\frac{4}{5} \div (-y)$.
37. $\frac{1}{2} \div z$.
38. $\frac{9}{10} \div 3a$.
39. $\frac{7}{8} \div 2b$.
40. $\frac{2a}{5} \div 3$.
41. $\frac{4b}{7} \div (-5)$.
42. $\frac{2a}{x} \div 2y$.
43. $\frac{3b}{2z} \div 6$.
44. $x \div (-\frac{2}{3})$.
45. $y \div \frac{1}{2}$.
46. $z \div \frac{5}{6}$.
47. $2w \div (-\frac{4}{5})$.
48. $3a \div \frac{3}{4}$.
49. $\frac{a}{b} \div \frac{2}{3}$.
50. $\frac{x}{y} \div \frac{4}{5}$.
51. $\frac{7}{8} \div \frac{m}{n}$.

52. $\frac{9}{10} \div \frac{d}{c}$.

53. $\frac{1}{2} \div \frac{x}{y}$.

54. $\frac{1}{3} \div \frac{a}{b}$.

55. $\frac{m}{n} \div \frac{1}{4}$.

56. $\frac{z}{w} \div \left(-\frac{1}{5}\right)$.

57. $\frac{a}{b} \div \frac{d}{c}$.

58. $\frac{x}{y} \div \frac{y}{x}$.

59. $\frac{m^2}{n^2} \div \frac{m}{n}$.

60. $\frac{a^2}{b^2} \div \frac{b}{a}$.

61. $\frac{ab}{c} \div \frac{b}{c}$.

62. $\frac{x}{yz} \div \frac{yz}{x}$.

63. $\frac{abc}{xy} \div \frac{ab}{xyz}$.

64. $\frac{7\,a}{3\,b} \div \frac{9\,b}{14\,a}$.

65. $\frac{2\,a^2}{5\,b} \div \frac{8\,a^4}{15\,b^4}$.

66. $\frac{9\,ab^2}{7\,x^2y} \div \frac{12\,a^2b}{35\,xy^2}$.

67. $\frac{a+1}{5} \div \frac{a^2-1}{10\,a}$.

68. $\frac{a+1}{2} \div \frac{3}{a+1}$.

69. $\frac{x-2}{3} \div \frac{5}{x+2}$.

70. $\frac{m+n}{8} \div \frac{6}{m+n}$.

71. $\frac{b-6}{10} \div \frac{3}{b+1}$.

72. $\frac{a+b}{x+y} \div \frac{x-y}{a-b}$.

73. $\frac{10\,a^2}{(a+b)^2} \div \frac{5\,a}{a+b}$.

74. $\frac{9\,x^2}{x^2-9} \div \frac{3\,x}{x-3}$.

75. $\frac{18\,x}{16\,x^2-1} \div \frac{6\,x}{4\,x-1}$.

76. $\frac{2}{2\,x+3\,y} \div \frac{4}{4\,x^2-9\,y^2}$.

77. $\frac{c^2-9}{c^2-c-2} \div \frac{c-3}{c^2+c-6}$.

78. $\frac{m^2-m-2}{m^2-m-6} \div \frac{m^2-2\,m}{2\,m+m^2}$.

79. $\frac{(x+y)^2}{x+3\,y} \div \frac{(x^2-y^2)^2}{x^3+27\,y^3}$.

80. $\frac{33}{a^3+b^3} \div \frac{11}{a^2-ab+b^2}$.

81. $\frac{x^3-y^3}{14} \div \frac{x^2+xy+y^2}{7}$.

82. $\frac{a^3+1}{x^2-4\,y^2} \div \frac{a^2-a+1}{x-2\,y}$.

83. $\frac{(a+b)^2-c^2}{abc} \div \frac{a+b+c}{bc}$.

84. $\frac{(x-y)^2-z^2}{xy^2z} \div \frac{x-y-z}{xyz}$.

85. $\left(\frac{x}{a}+\frac{y}{b}\right) \div \left(\frac{a}{x}+\frac{b}{y}\right)$.

86. $\left(1+\frac{x}{y}\right) \div \left(1-\frac{y^2}{x^2}\right)$.

87. $\left(4-\frac{3}{a+1}\right) \div \left(6+\frac{5}{a^2-1}\right)$.

88. $\left(1-\frac{y}{x}\right) \div \left(x^2-\frac{y^3}{x}\right)$.

89. $\frac{3\,a^2+a-24}{6\,ax-20\,x} \times \frac{6\,a^2}{a^2-9} \times \frac{a-3}{3\,a-8} \div \frac{3\,a^2+9\,a}{3\,a^2-a-30}$.

90. $\frac{(a+b)^2-y^2}{a^2-ay+ab} \times \frac{a}{(a+y)^2-b^2} \div \frac{ab-by-b^2}{(a-b)^2-y^2}$.

COMPLEX FRACTIONS

EXERCISE 96

Simplify each of the following complex fractions:

1. $\dfrac{\frac{5}{6}}{5}$.
2. $\dfrac{5}{\frac{6}{7}}$.
3. $\dfrac{\frac{7}{9}}{\frac{5}{6}}$.
4. $\dfrac{\frac{2}{5}}{\frac{7}{5}}$.
5. $\dfrac{-\frac{2}{3}\frac{7}{2}}{\frac{9}{1}\frac{}{0}}$.
6. $\dfrac{\frac{x}{y}}{2}$.
7. $\dfrac{\frac{g}{k}}{5}$.
8. $\dfrac{x}{\frac{1}{2}}$.
9. $\dfrac{\frac{z}{w}}{4}$.
10. $\dfrac{\frac{2z}{3}}{2}$.
11. $\dfrac{a}{-\frac{b}{2}}$.
12. $\dfrac{-\frac{x}{5}}{6}$.
13. $\dfrac{\frac{1}{2}}{3m}$.
14. $\dfrac{\frac{a}{b}}{c}$.
15. $\dfrac{\frac{a}{b}}{c}$.
16. $\dfrac{-a}{-\frac{9}{a}}$.
17. $\dfrac{-b}{\frac{5}{-b}}$.
18. $\dfrac{\frac{a}{b}}{\frac{c}{d}}$.
19. $\dfrac{\frac{4}{5}}{\frac{5}{x}}$.
20. $\dfrac{\frac{a^2}{b^2}}{\frac{a}{b}}$.
21. $\dfrac{\frac{x}{2}}{\frac{y}{3}}$.
22. $\dfrac{\frac{1}{a}}{\frac{1}{a^2}}$.
23. $\dfrac{\frac{1}{2}}{\frac{3}{4a}}$.
24. $\dfrac{\frac{2}{b}}{\frac{b}{2}}$.
25. $\dfrac{\frac{2x}{3y}}{\frac{4z}{9w}}$.
26. $\dfrac{\frac{a^3}{b^3}}{\frac{a^4}{b^4}}$.
27. $\dfrac{\frac{c}{d}}{\frac{d}{a}}$.
28. $\dfrac{\frac{x^2}{y^2}}{\frac{x}{y}}$.
9. $\dfrac{\frac{a^2}{b}}{\frac{b^2}{a}}$.

30. $\dfrac{x-\frac{y}{z}}{w}$.
31. $\dfrac{\frac{1}{a}+\frac{1}{b}}{c}$.
32. $\dfrac{\frac{a}{b}-\frac{b}{a}}{a-b}$.
33. $\dfrac{x+y}{\frac{1}{x}+\frac{1}{y}}$.
34. $\dfrac{\frac{1}{a}-\frac{1}{b}}{\frac{1}{a}+\frac{1}{b}}$.
35. $\dfrac{\frac{1}{x}-\frac{1}{y}}{\frac{x-y}{xy}}$.
36. $\dfrac{\frac{x}{y}+\frac{y}{z}}{\frac{x}{yz}}$.
37. $\dfrac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{c}}$.
38. $\dfrac{\frac{a}{b}+\frac{a}{c}}{\frac{a}{bc}}$.
39. $\dfrac{a+\frac{b}{c}}{a-\frac{b}{c}}$.

40. $\dfrac{a+\dfrac{1}{b}}{b+\dfrac{1}{a}}$.

41. $\dfrac{a-\dfrac{b}{3}}{\dfrac{a}{3}-b}$.

42. $\dfrac{a-\dfrac{1}{a}}{1+\dfrac{1}{a}}$.

43. $\dfrac{a+\dfrac{1}{a}}{a}$.

44. $\dfrac{\dfrac{a}{b}+1}{\dfrac{b}{a}+1}$.

45. $\dfrac{\dfrac{x}{y}-1}{1-\dfrac{y}{x}}$.

46. $\dfrac{1+\dfrac{1}{x}}{\dfrac{1+x}{x}}$.

47. $\dfrac{\dfrac{a+b}{cd}}{\dfrac{a^2-b^2}{cd}}$.

48. $\dfrac{\dfrac{m^2-1}{m}}{\dfrac{m+1}{m^2}}$.

49. $\dfrac{1-\dfrac{1}{a}}{a-2+\dfrac{1}{a}}$.

50. $\dfrac{a-3+\dfrac{1}{a+3}}{a+3+\dfrac{1}{a-3}}$.

51. $\dfrac{1+\dfrac{1}{a-1}}{\dfrac{1}{a-1}}$.

52. $\dfrac{a-2-\dfrac{14}{a+3}}{a-1-\dfrac{21}{a+3}}$.

53. $\dfrac{\dfrac{b+c}{c}+\dfrac{c}{b+c}}{\dfrac{1}{b}+\dfrac{1}{c}}$.

54. $\dfrac{a+b+\dfrac{b^2}{a}}{a+b+\dfrac{a^2}{b}}$.

55. $\dfrac{\dfrac{1}{1-x}+\dfrac{1}{1+a}}{\dfrac{1}{1-x}-\dfrac{1}{1+a}}$.

56. $\dfrac{1-\dfrac{2b-2c}{a+b-c}}{1+\dfrac{2c}{a-b-c}}$.

57. $\dfrac{\dfrac{a+b}{c-d}+\dfrac{a-b}{c+d}}{\dfrac{a+b}{c+d}+\dfrac{a-b}{c-d}}$.

58. $\dfrac{c-3-\dfrac{1}{c-3}}{c-2-\dfrac{3}{c-4}} \times \dfrac{c-4-\dfrac{1}{c-4}}{c-5+\dfrac{3}{c-1}}$.

59. $\dfrac{\dfrac{c^2+y^2}{y}-c}{\dfrac{1}{y}-\dfrac{1}{c}} \div \dfrac{c^3+y^3}{c^2-y^2}$.

60. $\dfrac{\dfrac{1+2k}{1-2k}-\dfrac{1-2k}{1+2k}}{\dfrac{1-2k}{1+2k}+\dfrac{1+2k}{1-2k}}$.

61. $\dfrac{\dfrac{1}{xy}+\dfrac{1}{xz}+\dfrac{1}{yz}}{\dfrac{x^2-(y+z)^2}{xy}}$.

62. $\dfrac{\dfrac{a}{b}-\dfrac{b}{a}}{\dfrac{a}{b}+\dfrac{b}{a}} \div \dfrac{\dfrac{a}{b}-\dfrac{b}{a}}{\dfrac{1}{a}+\dfrac{1}{b}}$.

CONTINUED FRACTIONS

EXERCISE 97

Simplify each of the following:

1. $\dfrac{1}{1+\dfrac{1}{2}}$.

2. $\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{2}}}$.

3. $\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{2}}}}$.

4. $\dfrac{1}{1+\dfrac{2}{3+\dfrac{1}{4}}}$.

5. $\dfrac{1}{2+\dfrac{1}{3+\dfrac{1}{4+\dfrac{1}{5}}}}$.

6. $\dfrac{a}{b+\dfrac{c}{d}}$.

7. $x+\dfrac{y}{z+\dfrac{a}{b}}$.

8. $\dfrac{a}{b+\dfrac{c}{d+\dfrac{e}{f}}}$.

9. $\dfrac{a}{b-\dfrac{b}{a-\dfrac{a}{b}}}$.

10. $\dfrac{a}{b-\dfrac{c}{d-\dfrac{e}{f}}}$.

11. $1-\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{x}}}$.

12. $\dfrac{x}{1-\dfrac{x}{2-\dfrac{x}{3-\dfrac{x}{4}}}}$.

13. $\dfrac{a-2}{a-2-\dfrac{a}{a-\dfrac{a-1}{a-2}}}$.

14. $\dfrac{1}{m-\dfrac{m^2-1}{m+\dfrac{1}{m-1}}}$.

EXERCISE 98

Miscellaneous

Simplify the following fractional expressions, checking all results numerically:

1. $1-a+a^2-\dfrac{a^3}{1+a}$.
2. $x^2+xy+y^2+\dfrac{y^3}{x-y}$.
3. $\dfrac{a^2b-b^3}{a}\times\dfrac{3a}{2ab-2b^2}$.
4. $\dfrac{m^2-a^2}{az}\times\dfrac{m^2+a^2}{a-m}$.
5. $\dfrac{x^4-y^4}{x+y}\times\dfrac{x^2}{xy-y^2}$.
6. $\dfrac{y+y^2}{3a^2}\div\dfrac{2ay+2ay^2}{7}$.
7. $\dfrac{m^2-n^2}{m^2-9x^2}\times\dfrac{3mx+m^2}{m^2+mn}$.
8. $\dfrac{d^2-4}{d^2-1}\times\dfrac{d^2-1}{2d}\times\dfrac{d-2}{2+d}$.
9. $\left(\dfrac{1}{a^2}-\dfrac{1}{b^2}\right)\left(\dfrac{ab}{a+b}\right)$.
10. $\left(x^2-\dfrac{y^3}{x}\right)\left(\dfrac{x}{x-y}\right)$.
11. $\left(\dfrac{a^2}{x}-\dfrac{x^2}{a}\right)\left(\dfrac{a}{x^2}+\dfrac{x}{a^2}\right)$.
12. $\dfrac{4}{(a-1)(a-3)}+\dfrac{2}{(a-2)(3-a)}-\dfrac{2}{(2-a)(1-a)}$.
13. $\dfrac{2}{a}-\dfrac{a+b-c}{ab}+\dfrac{2}{b}-\dfrac{a+c-b}{ac}+\dfrac{2}{c}-\dfrac{b+c-a}{bc}$.
14. $\dfrac{k^2-2k+1}{k^2-5k+6}\times\dfrac{k^2-4k+4}{k^2-4k+3}\times\dfrac{k^2-6k+9}{k^2-3k+2}$.
15. $\dfrac{3a^2+a-24}{6ax-20x}\times\dfrac{6a^2}{a^2-9}\times\dfrac{a-3}{3a-8}\times\dfrac{3a^2+9a}{3a^2-a-30}$.
16. $\left(\dfrac{a^2}{x}-8x+\dfrac{12x^3}{a^2}\right)\div\left(a-\dfrac{2x^2}{a}\right)$.
17. $\left(x-a-\dfrac{x^2+a^2}{x+a}\right)\left(x+a-\dfrac{x^2+a^2}{x-a}\right)$.
18. $\left(\dfrac{12}{m-3}+1-\dfrac{4}{m-1}\right)\left(\dfrac{4}{m+1}+1-\dfrac{12}{m+3}\right)$.
19. $\left(\dfrac{a}{2+a}-\dfrac{2-a}{a}\right)\div\left(\dfrac{a}{a+2}+\dfrac{2-a}{a}\right)$.
20. $\dfrac{m^2+mn}{m^2+n^2}\times\left(\dfrac{m}{m-n}-\dfrac{n}{m+n}\right)$.
21. $\left(2+\dfrac{m}{m-3}\right)\times\dfrac{9-m^2}{4-m^2}\times\dfrac{m+2}{m^2+m-6}-\dfrac{2}{m+2}$.

22. $\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\times\frac{abcd}{ac+ab+bc}$.

23. $\frac{a^2bc}{(a-b)(a-c)}+\frac{b^2ca}{(b-c)(b-a)}+\frac{c^2ab}{(c-a)(c-b)}$.

24. $\frac{r+1}{(r-s)(r-t)}+\frac{s+1}{(s-t)(s-r)}+\frac{t+1}{(t-r)(t-s)}$.

25. $\frac{c^2}{(c^2-d^2)(c^2-e^2)}+\frac{d^2}{(d^2-e^2)(d^2-c^2)}+\frac{e^2}{(d^2-c^2)(e^2-d^2)}$.

INDETERMINATE FORMS

EXERCISE 99

Find the limiting values of the following expressions for the values specified:

1. $\left.\frac{x^2-1}{x-1}\right]_1$.
2. $\left.\frac{a^2-25}{a+5}\right]_{-5}$.
3. $\left.\frac{x^3+1}{x+1}\right]_{-1}$.
4. $\left.\frac{b^5-1}{b-1}\right]_1$.
5. $\left.\frac{x^2-x-2}{x-2}\right]_2$.
6. $\left.\frac{x^3-4x^2+x+6}{x^2-5x+6}\right]_3$.
7. $\left.\frac{x^3-3x^2+3x-1}{x-1}\right]_1$.
8. $\left.\frac{x^3-6x^2+12x-8}{x^2-2x+4}\right]_2$.

FACTORS OF FRACTIONAL EXPRESSIONS

EXERCISE 100

Factor the following expressions, checking all results numerically:

1. $x^2-\frac{1}{4}$.
2. $\frac{1}{y^2}-9$.
3. $\frac{1}{a^2}-\frac{1}{b^2}$.
4. $\frac{1}{a^2}+\frac{2}{a}+1$.
5. $\frac{1}{m^2}+\frac{2}{mn}+\frac{1}{n^2}$.
6. $x^2+2+\frac{1}{x^2}$.
7. $\frac{a^2}{b^2}-2+\frac{b^2}{a^2}$.
8. $\frac{a^2}{b^2}+\frac{b^2}{a^2}+1$.
9. $\frac{a^2}{b^2}+\frac{5a}{b}+6$.
10. $\frac{x^2}{y^2}+\frac{x}{y}-12$.
11. $\frac{6x^2}{y^2}+\frac{13x}{y}+5$.
12. $\frac{8a^2}{b^2}+\frac{6a}{b}+1$.
13. $x^3-\frac{1}{x^3}$.
14. $a^3-\frac{1}{a^3}+a-\frac{1}{a}$.

15. $x^3+\frac{1}{x^3}+x+\frac{1}{x}$.

16. $x^2+x+2+\frac{1}{x}+\frac{1}{x^2}$.

17. $\left(\frac{x^2+4}{x}\right)^2-7\left(\frac{x^2+4}{x}\right)+12$.

18. $\left(\frac{5}{a}+a\right)^2+\frac{5}{a}+a-42$.

19. $\left(x+\frac{1}{x}\right)^2-\left(x+\frac{1}{x}\right)-12$.

20. $\left(\frac{x^2-8}{x}\right)^2+3x-\frac{24}{x}-10$.

REVIEW

MENTAL EXERCISE 101

Review

Obtain the following products:

1. $(x-1)(1-x)$.
2. $(2-b)(b-2)$.
3. $(4+d)(d-4)$.
4. $(c+5)(-c-5)$.

Simplify each of the following:

5. $1\div\frac{a}{b}$.
6. $a\div\frac{1}{b}$.
7. $1\div\frac{1}{xy}$.
8. $xy\div\frac{x}{y}$.

Distinguish between

9. $\frac{1}{x}+\frac{1}{y}$ and $\frac{1}{x+y}$.
10. $\frac{1}{a}-\frac{1}{b}$ and $\frac{1}{a-b}$.
11. $\frac{a+1}{2}$ and $\frac{a}{2}+1$.

Simplify each of the following:

12. $\frac{a}{b+c}+\frac{a}{b-c}$.
13. $\frac{x}{y-z}-\frac{x}{y+z}$.

Supply the terms which make the following expressions trinomial squares:

14. $a^2+8a+(\ )$.
15. $b^2-10b+(\ )$.
16. $c^4+2c^2+(\ )$.
17. $d^6+6d^3+(\ )$.
18. $m^4+2m^3+(\ )$.
19. $m^6+4m^4+(\ )$.
20. $9h^4+6h^3+(\ )$.
21. $16x^8-24x^7+(\ )$.
22. $25y^{10}-30y^7+(\ )$.

Are the following expressions conditionally or identically equal?

23. $6+2$ and $5+3$.
24. $6x+2x$ and $5+3$.
25. $6x+2$ and $5x+3$.
26. $6x+2x$ and $5x+3x$.

27. To which of the equations $5x+2y=7$, $3x-4y=8$, $4x-3y=5$ are the following equations equivalent?

$10x+4y=14$; $15x+6y=21$; $6x-8y=16$; $9x-12y=24$; $16x-12y=20$.

Are the following expressions identical?

28. $\frac{bh}{2}$, $\frac{1}{2}bh$, $\frac{b}{2}h$, and $b\frac{h}{2}$. **29.** $\frac{x+y}{2}$, $\frac{x}{2}+\frac{y}{2}$, and $\frac{1}{2}(x+y)$.

Show that the following identities are true:

30. $\frac{1}{(b-a)(c-b)} \equiv \frac{1}{(a-b)(b-c)}$. **32.** $\frac{1}{(b-a)^2} \equiv \frac{1}{(a-b)^2}$.

31. $-\frac{1}{(d-c)^3} \equiv \frac{1}{(c-d)^3}$. **33.** $\frac{(a-x)^2}{a^2} \equiv \left(1-\frac{x}{a}\right)^2$.

FRACTIONAL EQUATIONS

EXERCISE 102

Solve the following fractional equations for the letters appearing in them:

1. $\frac{5}{x}=1$. **2.** $\frac{4}{y}=1$. **3.** $1=\frac{2}{z}$. **4.** $1=\frac{3}{w}$. **5.** $\frac{4}{a}=-1$. **6.** $\frac{-5}{m}=1$.

7. $\frac{1}{x}=3$. **8.** $\frac{1}{z}=4$. **9.** $2=\frac{1}{x}$. **10.** $3=\frac{1}{a}$. **11.** $\frac{1}{3x}=1$. **12.** $\frac{1}{2y}=1$.

13. $\frac{1}{2z}=-1$. **14.** $\frac{3}{4a}=1$. **15.** $\frac{3}{5b}=1$. **16.** $1=\frac{6}{7c}$. **17.** $1=\frac{-5}{11d}$. **18.** $\frac{1}{2x}=3$.

19. $\frac{1}{3y}=5$. **20.** $-8=\frac{7}{3w}$. **21.** $5=\frac{1}{5k}$. **22.** $2=\frac{3}{4x}$. **23.** $6=\frac{5}{2y}$. **24.** $10=\frac{13}{3z}$.

25. $\frac{5}{x}-1=0$. **26.** $\frac{12}{y}-1=0$. **27.** $\frac{17}{z}-1=0$. **28.** $1-\frac{15}{w}=0$. **29.** $1-\frac{13}{a}=0$. **30.** $\frac{1}{x}-6=0$.

31. $\frac{1}{y}-5=0.$

32. $\frac{1}{z}-8=0.$

33. $14-\frac{1}{a}=0.$

34. $17-\frac{1}{b}=0.$

35. $\frac{2}{c}-3=0.$

36. $\frac{4}{d}+5=0.$

37. $\frac{9}{h}+13=0.$

38. $\frac{1}{x}-2=3.$

39. $\frac{1}{y}-4=7.$

40. $\frac{1}{z}-1=1.$

41. $\frac{1}{a}+3=6.$

42. $\frac{1}{b}+7=5.$

43. $\frac{1}{c}+8=2.$

44. $5+\frac{1}{w}=8.$

45. $9+\frac{1}{a}=2.$

46. $10+\frac{1}{b}=6.$

47. $4-\frac{1}{x}=3.$

48. $5-\frac{1}{y}=7.$

49. $6-\frac{1}{z}=10.$

50. $7-\frac{1}{w}=12.$

51. $\frac{3}{x+2}=1.$

52. $\frac{5}{y+3}=1.$

53. $\frac{2}{z-3}=1.$

54. $\frac{4}{w-6}=1.$

55. $\frac{5}{1-a}=1.$

56. $\frac{6}{3-b}=1.$

57. $5=\frac{1}{5+c}.$

58. $1=\frac{7}{d-7}.$

59. $1=\frac{8}{8-g}.$

60. $\frac{2}{2x+1}=1.$

61. $\frac{3}{3y+1}=1.$

62. $\frac{4}{x+4}=1.$

63. $\frac{3}{h+4}=2.$

64. $\frac{4}{y+2}=5.$

65. $\frac{1}{x+2}=2.$

66. $\frac{1}{z-3}=4.$

67. $\frac{2}{m-5}=7.$

68. $\frac{1}{n}=\frac{1}{3}.$

69. $\frac{1}{y}=\frac{1}{5}.$

70. $\frac{1}{z}=-\frac{1}{7}.$

71. $\frac{1}{9}=\frac{1}{w}.$

72. $\frac{1}{x+2}=\frac{1}{3}.$

73. $\frac{1}{y-4}=\frac{1}{4}.$

74. $\frac{1}{z+6}=\frac{1}{5}.$

75. $\frac{1}{2a+1}=\frac{1}{a+3}.$

76. $\frac{1}{3b-2}=\frac{1}{2b+2}.$

77. $\frac{2}{5c-1}=\frac{2}{3c+5}.$

78. $\frac{1}{x}=\frac{2}{3}.$

79. $\frac{1}{y}=\frac{6}{5}.$

80. $\frac{1}{z}=-\frac{8}{9}.$

81. $\frac{4}{7}=\frac{1}{x}.$

82. $\frac{3}{8}=\frac{2}{y}$.

83. $\frac{1}{5}=\frac{5}{z}$.

84. $\frac{1}{5}=\frac{3}{2w}$.

85. $\frac{2}{7a}=\frac{5}{9}$.

86. $\frac{3}{7b}=-\frac{7}{3}$.

87. $\frac{4}{5}=-\frac{7}{6c}$.

88. $\frac{2}{x}+\frac{3}{x}=5$.

89. $\frac{4}{y}+\frac{6}{y}=9$.

90. $\frac{8}{z}-\frac{3}{z}=4$.

91. $\frac{9}{a}-\frac{5}{a}=2$.

92. $\frac{7}{b}+1=10-\frac{2}{b}$.

93. $\frac{4}{c}+3=\frac{3}{c}+4$.

94. $\frac{2x+3}{x+4}=1$.

95. $\frac{5x+1}{2x+7}=1$.

96. $\frac{3m-8}{m}=1$.

97. $\frac{4n-7}{n}=3$.

98. $\frac{5q-9}{2q}=2$.

99. $\frac{7h}{3h+4}=2$.

100. $\frac{13k}{4k-5}=3$.

101. $\frac{w+7}{w+1}=3$.

102. $\frac{a+1}{a-2}=4$.

103. $\frac{b-1}{b+3}=5$.

104. $\frac{1}{2x}+\frac{1}{x}=3$.

105. $\frac{1}{d}+\frac{2}{3d}=2$.

106. $\frac{1}{3x}+\frac{1}{2x}=1$.

107. $\frac{3}{2y}-\frac{5}{4y}=2$.

108. $\frac{3}{5z}+\frac{1}{2z}=3$.

109. $\frac{2}{x+5}=\frac{1}{x+2}$.

110. $\frac{3}{x+4}=\frac{2}{x+2}$.

111. $\frac{2}{x-3}=\frac{1}{x-2}$.

112. $\frac{7}{x+3}=\frac{6}{x-2}$.

113. $\frac{5}{x-1}=\frac{4}{x+1}$.

114. $\frac{3}{2x+1}=\frac{1}{x-1}$.

115. $\frac{7}{3x-1}=\frac{2}{x-1}$.

116. $\frac{3}{5x-1}=\frac{1}{2x-1}$.

117. $\frac{5}{3x+2}=\frac{7}{5x-2}$.

118. $\frac{6}{3x+4}=\frac{3}{2x-1}$.

119. $\frac{4}{7x+3}=\frac{3}{6x+2}$.

120. $\frac{6}{2x-3}=\frac{11}{4x-5}$.

121. $\frac{x+1}{x-2}=\frac{x-3}{x+5}$.

122. $\frac{4x-3}{2x-1}=\frac{4x-7}{2x-5}$.

123. $\frac{6x-2}{3x+4}=\frac{2x+1}{x+3}$.

124. $\dfrac{1}{(x+1)(x+4)} = \dfrac{1}{(x+2)(x+5)}$.

125. $\dfrac{1}{(x+14)(x-7)} = \dfrac{1}{(x-13)(x-6)}$.

126. $3 - \dfrac{12}{x+1} = 7 - \dfrac{4x+30}{x+3}$.

127. $\dfrac{1}{x-2} + \dfrac{2}{x-3} = \dfrac{3}{x-4}$.

128. $\dfrac{7}{x-7} - \dfrac{3}{x-3} = \dfrac{4}{x-4}$.

LITERAL EQUATIONS

MENTAL EXERCISE 103

Regarding x, y, z, and w as unknowns, solve the following literal equations which are integral with reference to x, y, z, and w:

1. $x - b = a$.
2. $x + d = c$.
3. $n = m - z$.
4. $k + h = w$.
5. $x + 1 = c$.
6. $y + 4 = d$.
7. $z - 6 = k$.
8. $w - g = 7$.
9. $ax = b$.
10. $by = c$.
11. $az = -h$.
12. $a = bw$.
13. $-c = dx$.
14. $ax - b = 1$.
15. $ax + c = 1$.
16. $by - 2 = c$.
17. $3 + mz = n$.
18. $ax + b = c$.
19. $by - c = a$.
20. $cz - b = -a$.
21. $ax + b = ac$.
22. $a = bx + c$.
23. $r = sx - t$.
24. $m = n - qy$.
25. $ax = b + c$.
26. $2\,abx = a + b$.
27. $a^2x = a + 1$.
28. $(m+n)x = 2\,mn$.
29. $(a + 5)y = 5\,a$.
30. $(3 - m)z = 3\,m$.
31. $ax = a^2$.
32. $b^2x = b$.
33. $abx = bc$.
34. $ab^2cx = a^2bc^2$.
35. $-abx = a - b$.
36. $(a + b)x = c + d$.
37. $(c - 2)y = c + 3$.
38. $(a + b)y = a^2 - b^2$.
39. $(m - n)z = m^2 - n^2$.
40. $(a^2 - 4)x = a - 2$.
41. $(b^2 - 1)y = b + 1$.
42. $\dfrac{x}{a} = b$.
43. $\dfrac{y}{b} = c$.
44. $\dfrac{y}{r+s} = rs$.
45. $\dfrac{z}{m-n} = mn$.
46. $\dfrac{x}{a+3} = 3\,a$.
47. $\dfrac{x}{a+1} = a^2$.
48. $\dfrac{x}{c} = \dfrac{1}{b}$.

49. $\frac{x}{a}=\frac{b}{c}$.

50. $\frac{x}{c}=\frac{c}{d}$.

51. $\frac{x}{2}=\frac{2}{a}$.

52. $\frac{y}{b}=\frac{b}{5}$.

53. $\frac{my}{n}=1$.

54. $\frac{dw}{h}=-1$.

55. $\frac{bx}{c}=\frac{1}{d}$.

56. $\frac{ax}{b}=\frac{b}{a}$.

57. $x-m=n-x$.

58. $a-x=x-a$.

59. $b-x=x-c$.

60. $dy-1=1-dy$.

61. $ax-b=b-ax$.

62. $hz-3=3-hz$.

63. $a-bx=bx-a$.

64. $b-cx=cx-b$.

65. $1-ax=ax-1$.

66. $2-cy=cy-2$.

67. $ax+bx=a+b$.

68. $h+mx=hx+m$.

69. $bx-cx=b-c$.

70. $ky-k=ry-r$.

71. $by-b=ny-n$.

72. $mx+n^2=nx+m^2$.

73. $ax=bx+1$.

74. $dy=1-ny$.

75. $2\,ay=3\,by+4$.

76. $abx+bcx=ca$.

77. $ax-bx-1=cx$.

78. $ay+by=d-cy$.

79. $a(y-b)=c$.

80. $a(x+1)=b$.

81. $c(by+1)=d$.

82. $b(b-y)=a$.

83. $a(1-bx)=b$.

84. $\frac{x}{a}-1=b$.

85. $\frac{y}{d}+1=c$.

86. $\frac{x}{a+b}-b=a$.

87. $\frac{ax}{b}+\frac{bx}{a}=2$.

EXERCISE 104

Solve the following literal equations for x, verifying all results either by substituting the literal solutions directly, or by making proper numerical substitutions:

1. $a(x+b)=b(x+2a)$.

2. $ax+bc=d(b+x)$.

3. $(x+a)(x+b)=x(x+c)$.

4. $m=a+(n-1)x$.

5. $(m-n)x-m^2=(m+n)x$.

6. $a(1+x)+b(1+x)=x(a+b+1)$.

7. $a(x-1)+(a-1)x=a+x$.

8. $a(a-2x)+b(b-2x)+2ab=0$.

9. $3(3x-b)+2b=b(bx-3)+6$.

10. $hk(x^2-1)=(h+kx)(k+hx)$.

11. $(x+a)(x-b)=(x+a-b)^2$.

12. $(b-c)(x-b)=(b-a)x$.

13. $(x-a)(b-c)=(a-c)(x-b)$.

14. $(a+b)^2+(a-x)(b-x)=(x+a)(x+b)$.

15. $(a-x)(x+b)-c(a+c)=(c-x)(x+c)+ab$.

16. $\frac{x}{a+b}+b=a$.

17. $\frac{x}{c+d}-d=c$.

18. $\frac{3x}{a}+\frac{2x}{b}=3$.

19. $\frac{cx}{d}+\frac{dx}{c}=1$.

20. $\frac{x}{a}+b=\frac{x}{b}+a$.

21. $\frac{x}{g}-h=\frac{x}{h}-g$.

22. $\frac{x}{m}+\frac{a-x}{n}=b$.

23. $\frac{x}{a}-\frac{m-x}{b}=n$.

24. $\frac{x}{b}=x-b+\frac{1}{b}$.

25. $bx+b=\frac{x}{b}+\frac{1}{b}$.

26. $\frac{x-a}{b}=\frac{x-b}{a}$.

27. $\frac{d+x}{k}=\frac{x}{d+k}$.

28. $\frac{x-a}{b}+\frac{x-b}{a}=\frac{a^2+b^2}{ab}$.

29. $\frac{b^2-ax}{b}=b-\frac{a^2-bx}{a}$.

30. $\frac{x}{b}-2-\frac{b}{a}+\frac{x}{a}-\frac{a}{b}=0$.

31. $\frac{cx}{c-d}=c-x$.

32. $bx-ac=\frac{c^2x}{c-b}$.

33. $\frac{a+x}{a+b}+\frac{ax}{b}=\frac{b}{a}$.

34. $\frac{ax}{bc}+\frac{bx}{ac}+\frac{cx}{ab}=1$.

LITERAL EQUATIONS WHICH ARE FRACTIONAL WITH REFERENCE TO THE UNKNOWNS

MENTAL EXERCISE 105

Solve the following equations for x, y, z, and w:

1. $\frac{b}{x}=1$.

2. $\frac{a}{y}=1$.

3. $1=\frac{d}{w}$.

4. $1=\frac{a}{x}$.

5. $\frac{1}{x}=b$.

6. $\frac{3}{y}=n$.

7. $\frac{5}{z}=n$.

8. $\frac{8}{x}=b$.

9. $\frac{d}{y}=2$.

10. $\frac{m}{y}=4$.

11. $5=\frac{r}{x}$.

12. $6=\frac{t}{w}$.

13. $\frac{a}{y}=b.$

14. $\frac{bc}{y}=a.$

15. $\frac{d}{y}=fg.$

16. $\frac{b}{x}=\frac{4}{5}.$

17. $\frac{b}{y}=\frac{5}{9}.$

18. $\frac{m}{x}=-1.$

19. $\frac{n}{y}=-2.$

20. $9=-\frac{b}{y}.$

21. $\frac{1}{ax}=1.$

22. $\frac{2}{by}=1.$

23. $\frac{3}{cy}=-1.$

24. $4=\frac{3}{mz}.$

25. $\frac{mn}{x}=q.$

26. $\frac{r}{sx}=t.$

27. $\frac{a+b}{x}=a-b.$

28. $\frac{c-d}{y}=c+d.$

29. $m+n=\frac{m-n}{z}.$

30. $k-1=\frac{k+1}{w}.$

31. $\frac{1}{x}=\frac{a}{b}.$

32. $\frac{d}{h}=\frac{1}{x}.$

33. $\frac{g}{n}=\frac{1}{y}.$

34. $\frac{1}{x}=-\frac{c}{d}.$

35. $\frac{3}{x}=\frac{4}{a}.$

36. $\frac{5}{y}=\frac{b}{6}.$

37. $\frac{m}{2}=\frac{3}{z}.$

38. $\frac{a}{x}=\frac{b}{a}.$

39. $\frac{5}{h}=\frac{h}{z}.$

40. $\frac{b}{c}=\frac{c}{y}.$

41. $\frac{a}{2}=\frac{2}{ax}.$

42. $\frac{b}{3}=\frac{3}{by}.$

43. $\frac{6}{cz}=\frac{c}{6}.$

44. $\frac{8}{dw}=\frac{d}{8}.$

45. $\frac{a}{b}=\frac{b}{ax}.$

46. $\frac{m}{ny}=\frac{n}{m}.$

47. $\frac{2a}{bx}=\frac{c}{2a}.$

48. $\frac{c}{w-1}=1.$

49. $\frac{b}{y-1}=1.$

50. $\frac{1}{x-a}=1.$

51. $\frac{1}{x-c}=\frac{1}{b}.$

52. $\frac{1}{y-b}=\frac{1}{c}.$

53. $\frac{1}{z+d}=\frac{1}{m}.$

54. $\frac{1}{w-g}=\frac{1}{g}.$

55. $\frac{1}{a}=\frac{1}{x-a}.$

56. $\frac{1}{b}=\frac{1}{y-b}.$

57. $\frac{1}{z+a}=\frac{1}{3a}.$

58. $\frac{1}{w+2b}=\frac{1}{5b}.$

59. $\frac{1}{x-a}=\frac{1}{a-b}.$

60. $\frac{1}{x+c}=\frac{1}{c+d}.$

61. $\frac{1}{z-g}=\frac{1}{g-h}.$

62. $\frac{1}{2x-m}=\frac{1}{m-2}.$

63. $\frac{1}{3y-t}=\frac{1}{t-3}.$

64. $\frac{1}{x-a}=\frac{1}{b+c}$.

65. $\frac{1}{x+b}=\frac{1}{a+c}$.

66. $\frac{1}{x+c}=\frac{1}{a+b}$.

67. $\frac{1}{ax+b}=\frac{1}{b+c}$.

68. $\frac{a}{x-b}=1$.

69. $\frac{c}{y+d}=1$.

70. $\frac{2k}{z+h}=1$.

71. $\frac{4m}{w+m}=1$.

72. $\frac{a}{x-a}=2$.

73. $\frac{b}{y-b}=3$.

74. $\frac{1}{x-b}=2$.

75. $\frac{1}{y-c}=3$.

76. $\frac{1}{z+d}=4$.

77. $\frac{1}{x-2}=a$.

78. $\frac{1}{z+5}=h$.

79. $5=\frac{c}{x-2}$.

80. $\frac{a}{x-1}=2$.

81. $\frac{a}{x-b}=c$.

82. $\frac{a}{bx+c}=d$.

EXERCISE 106

Regarding x as the unknown, solve the following literal equations:

1. $g-\frac{h}{x}=\frac{g}{x}-h$.

2. $\frac{n}{x}=m(n-t)+\frac{t}{x}$.

3. $a=\frac{b}{c+dx}$.

4. $b=\frac{c-x}{ax}$.

5. $n=\frac{e-a}{x}+1$.

6. $m=\frac{a+3(x-m)}{x}$.

7. $\frac{dh}{dx+hx}=1$.

8. $r=\frac{x-a}{x-s}$.

9. $\frac{x+b}{x-b}=\frac{3}{4}$.

10. $\frac{x-a}{x-b}=\frac{a^2}{b^2}$.

11. $\frac{b}{a}=\frac{x-a^2}{x-b^2}$.

12. $\frac{c-dx}{cx-d}=\frac{c}{d}$.

13. $\frac{bx^2}{c-ax}+b+\frac{bx}{a}=0$.

14. $\frac{x}{x-a}=\frac{a+b}{b}$.

15. $\frac{g+h}{g-x}=\frac{g}{g-h}$.

16. $\frac{x+p}{x-q}=\frac{p-q}{p+q}$.

17. $\frac{e-x}{x-d}=\frac{d-x}{x-e}$.

18. $\frac{m-x}{n-x}=\frac{m-1}{n-1}$.

19. $\frac{x-b}{x-4}=\frac{x-c}{x-5}$.

20. $\frac{a-x}{a+x}=\frac{x}{a-x}$.

21. $\frac{x-a}{3}-\frac{3}{x+a}=\frac{x+a}{3}$.

22. $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=\frac{1}{x}$.

23. $\frac{x-a}{2bc}+\frac{x-b}{2ac}+\frac{x-c}{2ab}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$.

24. $\frac{a}{b-c}+\frac{b-c}{x}=\frac{a}{b+c}+\frac{b+c}{x}$.

PROBLEMS

PROBLEMS SOLVED BY MEANS OF FRACTIONAL EQUATIONS

EXERCISE 107

Solve the following problems, and examine the solutions obtained, to see if they satisfy the conditions of the problem as stated:

1. What must be the value of a in order that $\frac{(2a-5)}{(a+6)}$ shall equal $\frac{5}{3}$?

2. What must be the value of a in order that $\frac{(3a-9)}{(a+1)}$ shall equal $\frac{3}{5}$?

3. Find two numbers whose sum is 20, such that the quotient obtained by dividing the greater number by the less is 3.

4. Find two numbers whose sum is 15, such that the quotient obtained by dividing the greater number by the less is 4.

5. Find two numbers whose sum is 12, such that the quotient obtained by dividing the less number by the greater is $\frac{1}{5}$.

6. Find two numbers whose sum is 14, such that the quotient obtained by dividing the greater number by the less is $\frac{4}{3}$.

7. The sum of two numbers is 22. If the greater be divided by the less, the quotient is 3 and the remainder 2. Find the numbers.

8. The sum of two numbers is 16. If the greater be divided by the less the quotient is 4 and the remainder 1. Find the numbers.

9. The difference of two numbers is 8 and the quotient obtained by dividing the greater by the less is 3 and the remainder 2. What are the numbers?

10. The sum of two numbers is 21. If the greater be divided by the less number increased by 3, the quotient is 2. Find the numbers.

11. The sum of two numbers is 19, and if the greater be divided by the less increased by 1, the quotient is 3. Find the numbers.

12. The sum of two numbers is 25, and if the greater be divided by the less diminished by 1, the quotient will be 2. Find the numbers.

13. The difference of two numbers is 18. If the greater be divided by the less diminished by 2, the quotient will be 5. Find the numbers.

14. What number must be added to the numerator and denominator of the fraction $\frac{2}{5}$ in order that the resulting fraction shall equal $\frac{2}{3}$?

15. What number must be added to the numerator and denominator of the fraction $\frac{2}{3}$ in order that the resulting fraction shall equal $\frac{3}{4}$?

16. What number must be added to the numerator and denominator of the fraction $\frac{1}{4}$ in order that the resulting fraction shall equal $\frac{1}{2}$?

17. What number must be subtracted from the numerator and denominator of the fraction $\frac{3}{8}$ in order that the resulting fraction shall equal $\frac{1}{6}$?

18. What number must be added to the numerator and subtracted from the denominator of the fraction $\frac{5}{7}$ in order that the result may be the reciprocal of the given fraction?

19. The reduced value of a certain fraction is $\frac{3}{7}$ and its denominator exceeds its numerator by 20. Find the fraction.

20. The reduced value of a certain fraction is $\frac{2}{3}$ and its denominator exceeds its numerator by 4. Find the fraction.

21. The reduced value of a certain fraction is $\frac{4}{5}$ and the denominator exceeds the numerator by 3. Find the fraction.

22. The reduced value of a certain fraction is $\frac{5}{8}$ and the denominator exceeds the numerator by 12. Find the fraction.

23. Find a number whose reciprocal is equal to five times the reciprocal of the number increased by 32.

24. Find a number whose reciprocal is equal to 3 divided by the number increased by 14.

25. Find a number whose reciprocal is equal to 6 divided by the number increased by 45.

26. A man can clear a wood lot in 6 days and his son can do it in 9 days. How many days will be required to do the work if they work together?

27. A can do a piece of work in 5 days and B can do it in 4 days. How many days will be required if both work together?

28. A carpenter can build a fence in 3 days and his brother can do it in 7 days. How many days will be required if both work together?

29. A can shingle a barn in 2 days and B can do it in 6 days. How many days will be required if both work together?

30. Find the time which A, B, and C working together will require to repair a bridge if A working alone can do the work in 3 days, B alone in 5 days, and C alone in 6 days.

31. An electrician can by working alone wire a house for electric lighting in 7 days and working with his helper can do the same work in 5 days. How many days would the helper require to do the work alone?

32. A can by working alone paint a house in 10 days and working with B can do the same work in 4 days. How many days would B require to do the work alone?

33. A cistern is supplied by two pipes. The first can fill it in 4 hours and the second in 6 hours. How long will it take both together to fill it?

34. A tank is supplied by three pipes. The first can fill it in 2 hours, the second in 3 hours, and the third in 4 hours. How long will it take all three together to fill it?

35. A swimming tank can be filled by two inlet pipes in 3 hours and 4 hours respectively, and emptied by an outlet pipe in 5 hours. In how many hours will it be filled if all three pipes are open together?

EXERCISE 108

Problems Solved by Fractional Equations

Solve each of the following problems, examining the solutions to see if they satisfy the conditions of the problems as stated:

1. What must be the value of a in order that $\frac{(4a+8)}{(11a-3)}$ shall have the value $\frac{2}{3}$?

2. Find two numbers whose sum is 85, which are such that the quotient obtained by dividing the greater by the less is 4 and the remainder 5.

3. Find two numbers whose sum is 52, which are such that the quotient obtained by dividing the less by the greater is $\frac{5}{8}$.

4. The sum of two numbers is 39, and the quotient obtained by dividing the less by the greater is $\frac{6}{7}$. Find the numbers.

5. The sum of two numbers is 72, and if the greater be divided by the less, the quotient is 3 and the remainder is 4. Find the numbers.

6. The sum of two numbers is 111, and if the greater be divided by the less, the quotient is 7 and the remainder 7. Find the numbers.

7. The difference between two numbers is 70; and if the greater is divided by the less, the quotient is 9 and the remainder 6. Find the numbers.

8. What number must be added to the numerator and also to the denominator of the fraction $\frac{49}{65}$ in order that the resulting fraction shall equal $\frac{13}{17}$?

9. What number must be added to the numerator and subtracted from the denominator of the fraction $\frac{8}{15}$ in order that the result shall be equal to the reciprocal of the given fraction?

10. When four is subtracted from the numerator of a fraction, of which the numerator is two less than its denominator, the value of the fraction becomes $\frac{1}{4}$. What is the original fraction?

11. The reduced value of a certain fraction is $\frac{2}{5}$ and its denominator exceeds its numerator by 24. Find the fraction.

12. The value of a fraction is $\frac{1}{10}$. If its numerator is increased by 6 and its denominator by 5, the resulting fraction will be equal to $\frac{1}{5}$. Find the fraction.

13. Separate 580 into two parts such that when the greater part is divided by the less part, the quotient is 21 and the remainder is 7.

14. The reciprocal of a number is equal to four times the reciprocal of the sum of the number and 18. Find the number.

15. The figure in units' place of a number expressed by two figures exceeds the figure in tens' place by 7. If the number increased by 13 is divided by the sum of the figures in units' and tens' places, the quotient is 5. What is the number?

16. The figure in tens' place of a number of two figures exceeds the figure in units' place by 7, and if the number increased by 7 is divided by the sum of its figures, the quotient is 9. Find the number.

17. A can do a piece of work in 20 days, and B in 15 days. How many days will be required if both work together?

18. A and B together can paint a house in 15 days, A and C together in 20 days, and A alone in 25 days. In what time can B and C together paint it? In what time can A, B, and C working together paint it?

19. A can do a certain amount of work in 20 days, B in 12 days, and A, B, and C together in 5 days. In how many days can C do the work?

20. A can do a piece of work in 12 days, B in 8 days, and C in 6 days. How many days will be required if all work together?

21. A can hoe a field in $3\frac{1}{2}$ days, B in $4\frac{1}{3}$ days, and C in $5\frac{1}{4}$ days. If A, B, and C work together, how long will it take them to do the work?

22. A sum of \$1500 was to be divided equally among a certain number of persons. If there had been 5 more persons, each would receive $\frac{2}{3}$ as much. How many persons were there?

23. A gasoline tank may be emptied by any one of three taps. It can be emptied by the first alone in 20 minutes, by the second alone in 24 minutes, and by the third alone in 32 minutes. What time would be required to empty the tank by using all three together?

24. A vat in a paper mill can be filled by one pipe in $1\frac{1}{2}$ hours, by a second in $2\frac{1}{3}$ hours, and by a third in 3 hours. What time would be required to fill it when all are running together?

25. A train runs 168 miles in a given time. If it were to run 2 miles an hour faster, it would go 8 miles farther in the same time. Find the train's rate of speed in miles per hour.

26. It is observed that a steamer can run 60 miles with the current in the same time that it can run 36 miles against the current. Find the rate of the current in miles per hour, knowing that the steamer can run 12 miles an hour in still water.

27. A can row 5 miles and B 4 miles an hour in still water. A is 12 miles farther up stream than B and they row toward each other until they meet 4 miles above B's starting-place. Find the rate of the current in miles per hour.

EXERCISE 109

Problems Involving Literal Equations

Find the general solution of each of the following problems:

1. Separate the number n into two parts such that a times the first part shall exceed b times the second part by c.

2. Separate a into two parts such that b times the first part shall equal c times the second part.

3. The sum of two numbers is s and their difference is d. What are the numbers?

4. Separate n into three parts such that the first part shall be x times the second part, and the second part y times the third part.

5. Separate n into two parts such that when one part is divided by the other, the quotient shall be q and the remainder r.

6. What number must be added to each term of the fraction $\frac{n}{d}$ in order that the resulting fraction shall be equal to $\frac{a}{b}$?

7. A can do a certain amount of work in a hours, and B can do the same amount of work in b hours. How many hours will be required if both work together?

8. One pipe can fill a tank in a hours, and a second pipe can fill it in b hours. If a third pipe can empty it in c hours, how many hours will be required to fill the tank when the three pipes are open?

Discuss the problem for $a + b \gtreqless c$.

9. A tank can be filled from three taps. By using the first alone it is filled in a minutes, by using the second alone, in b minutes, and by using the third alone, in c minutes. In how many minutes would it be filled if the taps were all open at the same time?

10. In how many years will P dollars amount to A dollars at r per cent simple interest per year?

11. What principal at r per cent interest per year will amount to A dollars in t years?

12. An alloy of two metals is composed of a parts of one to b parts of the other. How many pounds of each are required to make c pounds of the alloy?

13. Two men A and B on motor cycles, d miles apart, start at the same time and travel toward each other at the rates of a miles per hour and b miles per hour respectively. How far will each have traveled when they meet?

14. In a certain time a train ran a miles. If it had run b miles an hour faster, it would have gone c miles farther in the same time. Find the rate of the train in miles per hour.

15. If A and B can travel at the rates of a and b miles an hour respectively, how far must A travel to overtake B, if both move in the same direction, and B be given a start of s miles?

16. A naphtha launch can run a miles an hour in still water. If it can run b miles against the current in the same time that it can run c miles with the current, what is the rate of the current in miles per hour?

17. A crew can row a certain distance up a stream in a hours and can row back again in b hours. If the rate of the crew in still water is s miles an hour, find the velocity of the stream in miles per hour.

18. A dealer mixes a pounds of tea worth x cents a pound with b pounds of tea worth y cents a pound and with c pounds of tea worth z cents a pound. Find the value v of the mixture in cents per pound.

19. Pieces of money of one denomination are of such value that a pieces are equal in value to one dollar, and pieces of money of another denomination are of such value that b pieces are equal in value to one dollar. Find how many pieces of each denomination must be taken on condition that c pieces of money shall be equal in value to one dollar.

PROBLEMS IN PHYSICS

EXERCISE 110

Solve each of the following problems:

1. How heavy a stone can a man by exerting a force of 150 pounds lift with a crowbar 6 feet in length, if the fulcrum be 1 foot from the stone (neglecting the weight of the crowbar)?

2. A wheelbarrow is loaded with 60 bricks, each weighing 6 pounds. What lifting force must be applied at the handles to raise the load (neglecting the weight of the wheelbarrow), provided that the center of gravity of the load is 2 feet from the center of the wheel and the hands are placed at a distance of 4 feet from the center of the wheel?

3. A beam 18 feet in length and weighing 45 pounds is supported at a point 3 feet from one end. What force must be applied at the end farthest from the point of support to keep the beam in equilibrium? What force must be applied at the end nearer the point of support?

4. A board 18 feet in length and weighing 24 pounds is supported at a point 3 feet from the center. If the board is kept in equilibrium by a stone placed on it at a point 4 feet from the fulcrum, find the weight of the stone.

5. A horizontal bar 20 inches in length is in equilibrium when forces of 3 pounds and 2 pounds respectively are acting downward at its ends. Find the position of the point of support.

6. A basket weighing 100 pounds is suspended at a point two feet from the end of a stick which is 8 feet in length and which weighs 3 pounds. If the stick is being carried by two boys, one at each end, how many pounds does each boy lift?

7. Two boys, one at each end of a stick 10 feet in length which weighs 5 pounds, raise a certain weight which is suspended from the stick. How heavy is the weight, and at what point does it hang, if one boy lifts 28 pounds and the other lifts 24 pounds?

8. A safety valve having an area of 4 square inches is held down by a lever which is hinged at one end. If the lever is 10 inches in length and the point of application of the valve is 2 inches from the hinged end of the lever and a weight of 12 pounds is placed on the free end, find the pressure per square inch on the valve which will lift the safety valve, disregarding the weight of the lever.

9. A dogcart carrying a load of 576 pounds is found when on a level road to exert a pressure of only 8 pounds on the horse's back. If the distance from the point of support on the horse's back to the axle be 6 feet, find the distance of the center of gravity of the load from the axle.

10. A uniform board weighing 30 pounds is balanced when supported at a point 4 feet from one end and a weight of 70 pounds is placed one foot from this end. Find the length of the board.

11. A uniform beam 20 feet in length is supported at a point 8 feet from one end. If the beam is balanced when a weight of 80 pounds is placed on the end nearer the fulcrum, and a weight of 30 pounds is placed on the end farther from the fulcrum, what is the weight of the beam?

REVIEW

EXERCISE 111

Review

Simplify each of the following:

1. $(a^2+a+1)\left(\frac{1}{a^2}-\frac{1}{a}+1\right)$.

2. $\left(\frac{a^2}{b^2}+\frac{a}{b}+1\right)\left(\frac{a^2}{b^2}-\frac{a}{b}+1\right)$.

3. $(a^n-b^n)(a^n+b^n)(a^{2n}+b^{2n})$.

4. $\left(\frac{a}{b}+\frac{c}{d}\right)(ad-bc)$.

5. $(x+y)\left(\frac{1}{x}+\frac{1}{y}\right)$.

6. Find the remainder when $6x^4 - 7x^3 + 5x^2 - 2x + 3$ is divided by $x - 5$.

7. Factor $10(a^2 + 1) - 29a$.

8. Factor $x^2 + 2xy + y^2 + x + y$.

9. Factor $(a - b)^3 - 8$.

10. Find the L. C. M. of $a^4 + a^2 + 1$ and $a^4 - a^2 + 1$.

Simplify each of the following:

11. $(a^3 + \frac{1}{27}) \div (a + \frac{1}{3})$.

12. $\left(a^3 - \frac{1}{125}\right) \div \left(a^2 + \frac{a}{5} + \frac{1}{25}\right)$.

13. $\dfrac{a^2 + b^2}{a - b} + \dfrac{a^3 - b^3}{a^2 + b^2}$.

14. $\dfrac{(5 + x)^2}{(5 - x)^2} - \dfrac{5 + x}{5 - x}$.

15. $\dfrac{a - \frac{1}{b}}{c} + \dfrac{b - \frac{1}{c}}{a} + \dfrac{c - \frac{1}{a}}{b}$.

16. Show that

$$(x^2 + 3x + 2)(x^2 + 7x + 12) \equiv (x^2 + 4x + 3)(x^2 + 6x + 8).$$

SIMULTANEOUS LINEAR EQUATIONS

CLASSIFICATION OF SYSTEMS OF EQUATIONS

EXERCISE 112

Which of the following systems of equations contain equations which are consistent, inconsistent, independent, equivalent?

1. $5x + 3y = 18$,
   $2x + 7y = 13$.

2. $3x + 8y = 10$,
   $3x + 8y = 15$.

3. $2x + y = 14$,
   $x + 3y = 17$.

4. $4x + 3y = 22$,
   $8x + 6y = 44$.

5. $2x + 5y = 12$,
   $3x - y = 1$.

6. $7x + 3y = 13$,
   $7x - 3y = 1$.

7. $5x + 6y = 7$,
   $5x - 7y = 7$.

8. $\dfrac{x}{2} + y = 1$,
   $x + 2y = 2$.

9. $2x + \dfrac{y}{4} = 7$,
   $2x - \dfrac{y}{4} = 8$.

10. $\dfrac{x}{5} + y = 3$,
    $x + 5y = 12$.

11. $x + \dfrac{y}{3} = 2$,
    $3x + y = 6$.

12. $x - y = 0$,
    $x + y = 1$.

13. $2x - 4y = 6$,
    $x - 2y = 2$.

14. $2x + 8y = 10$,
    $8x + 2y = 10$.

15. $3x + 2y = 4$,
    $6x + 5y = 8$.

16. $3x + y = x + 9$,
    $2x + y = 9$.

17. $4x - 9y = 11 + y$,
    $3x - 10y = 11 - x$.

18. $6x-7y=6y+8,$
$5x-9=1-4y.$

19. $4x+3y=2x-y+3,$
$2x+4y=3.$

20. $3x-10y+1=x+5,$
$3x-7y=4+3y.$

ELIMINATION BY SUBSTITUTION

EXERCISE 113

Solve each of the following systems of equations, eliminating the unknowns by the method of substitution:

1. $3x+2y=15,$
$x=y.$

2. $4x+7y=33,$
$x=y.$

3. $6x+y=35,$
$x=y.$

4. $8x-3y=30,$
$x=y.$

5. $9x-5y=56,$
$x=-y.$

6. $10x+3y=52,$
$x-y=0.$

7. $11x-2y=72,$
$x-y=0.$

8. $13x-7y=60,$
$x+y=0.$

9. $12x-5y=77,$
$x-y=0.$

10. $15x-11y=3,$
$x-y=0.$

11. $5x+7y=34,$
$x=2y.$

12. $9x-5y=62,$
$x=4y.$

13. $12x+7y=23,$
$x=-5y.$

14. $14x+3y=60,$
$2x=y.$

15. $9x+8y=60,$
$x-y=1.$

16. $4x+3y=43,$
$x-y=2.$

17. $11x+7y=47,$
$x+y=5.$

18. $9x-5y=13,$
$x-y=-3.$

19. $3x+17y=35,$
$x+2y=8.$

20. $5x-12y=21,$
$x-4y=1.$

21. $x+11y=0,$
$x+y=10.$

22. $3x-2y=4,$
$5x+y=11.$

23. $2x=12,$
$3x+5y=53.$

24. $3x-y=6,$
$x+9y=86.$

25. $2x+y=21,$
$2y+x=12.$

26. $4x=3y,$
$7x=5y+1.$

27. $7x = 4y,$
$10x = 3y + 19.$

28. $12x - 17y - 2 = 0,$
$x - y - 1 = 0.$

29. $x = 3y - 2,$
$y = 3x + 2.$

30. $x - y = 0,$
$2x + 3y = 5.$

31. $5x - y - 5 = 0,$
$7x + 3y - 24 = 0.$

32. $x + y = 2a,$
$(a - b)x = (a + b)y.$

33. $cx - by = 0,$
$bx - cy = a.$

34. $ax + y = b,$
$x + cy = d.$

ELIMINATION BY COMPARISON

EXERCISE 114

Solve the following systems of equations, eliminating the unknown numbers by the method of comparison:

1. $x = 3y - 4,$
$x = 4y - 7.$

2. $5x - 2y = 11,$
$2x - 3y = 0.$

3. $3x + 8y = 19,$
$7x - 2y = 1.$

4. $8x = 6y,$
$10x = 27y - 4.$

5. $3x + 7y = 42,$
$5x + 6y = 53.$

6. $2x - 5y = -23,$
$3x - 4y = -3.$

7. $y = 2x + 1,$
$y = 3x - 5.$

8. $3x - 4y - 19 = 0,$
$7x + 2y - 50 = 0.$

9. $5x + 6y = 7,$
$8x + 9y = 10.$

10. $11x - 9y = 7,$
$9x - 10y = 11.$

11. $bx + ay = 2ab,$
$ax + by = a^2 + b^2.$

12. $x + ay + a^2 = 0,$
$x + by + b^2 = 0.$

ELIMINATION BY ADDITION OR SUBTRACTION

EXERCISE 115

Solve each of the following systems of equations, regarding the letters appearing in them as unknowns:

1. $x + y = 5,$
$x - y = 1.$

2. $x + 2y = 9,$
$x - 2y = 1.$

3. $3a + b = 10,$
$3a - b = 2.$

4. $b + c = 9,$
$5b - c = 3.$

5. $c + 7d = 23,$
$c + 3d = 15.$

6. $4m + n = 14,$
$6m + n = 16.$

7. $2p+3q=18,$
$2p-3q=6.$

8. $r+3s=7,$
$r+\ \ s=5.$

9. $z+5w=11,$
$z-\ \ w=\ \ 5.$

10. $g+3k=\ \ 8,$
$g+4k=10.$

11. $7a-4d=\ \ 5,$
$3a+4d=25.$

12. $5b+2z=22,$
$3b+2z=18.$

13. $d+4k=7,$
$d-\ \ k=2.$

14. $3c+4s=24,$
$3c+5s=27.$

15. $m+4r=17,$
$m+2r=11.$

16. $3n+2t=12,$
$6n+\ \ t=15.$

17. $10a+\ \ b=18,$
$5a-2b=-11.$

18. $2x+3y=17,$
$x+9y=16.$

19. $12c-7h=27,$
$3c-5h=-3.$

20. $6b+5k=\ \ 7,$
$3b-7k=13.$

21. $4r-3w=25,$
$3r-4w=\ \ 0.$

22. $2s-5t=-1,$
$3s-2t=15.$

23. $3x+4y+5=0,$
$5x+4y+3=0.$

24. $8x-\ \ 9y+\ \ 1=0,$
$16x+27y-17=0.$

25. $3(x+y)=57,$
$5(x-y)=15.$

26. $ax+by=(a-b)^2,$
$ax-by=a^2-b^2.$

27. $x-y=m-n,$
$mx-ny=2(m^2-n^2).$

28. $rx+sy=2rs,$
$sx+ry=r^2+s^2.$

29. $k^2x+m^2y=0,$
$kx\ +my\ =k+m.$

30. $x+ay+a^2=0,$
$x+by+b^2=0.$

31. $(p+q)x-(p-q)y=3,$
$(p-q)x+(p+q)y=3.$

32. $0.8x+0.1y=0.19,$
$0.6x+0.9y=0.39.$

33. $0.5x+0.4y=0.13,$
$0.7x+0.3y=0.13.$

34. $0.3x+0.2y=\ \ 9.5,$
$0.2x+0.3y=10.5.$

35. $x-\dfrac{7y}{3}=-21,$

$x+\ \dfrac{y}{5}=17.$

36. $\dfrac{x}{3}+3y=15,$

$\dfrac{y}{4}+4x=37.$

37. $x+\dfrac{7y}{3}=\dfrac{27}{2},$

$\dfrac{17y}{5}+x=\dfrac{15}{2}.$

38. $\dfrac{x+3}{2}+5y=9,$

$\dfrac{y+9}{10}-\dfrac{x-2}{3}=0.$

SYSTEMS OF SIMULTANEOUS EQUATIONS CONTAINING THREE OR MORE UNKNOWNS

EXERCISE 116

Solve the following systems of simultaneous linear equations:

1. $x+y=5,$
$z+x=7,$
$y+z=8.$

2. $x+y=8,$
$y+z=-9,$
$z+x=3.$

3. $x+y=2a,$
$x+z=2b,$
$y+z=2c.$

4. $x+y-z=7,$
$y+z-x=-1,$
$z+x-y=5.$

5. $x+y-z=6,$
$y+z-x=-2,$
$z+x-y=4.$

6. $x+y-z=-1,$
$y+z-x=7,$
$z+x-y=3.$

7. $x+y-z=a,$
$y+z-x=b,$
$z+x-y=c.$

8. $2x+y+z=17,$
$x+2y+z=16,$
$x+y+2z=15.$

9. $x+3y+z=10,$
$x+y+3z=12,$
$3x+y+z=8.$

10. $x+4y+z=33,$
$y+4z+x=36,$
$z+4x+y=21.$

11. $2x+5y-3z=23,$
$3x+2y+7z=41,$
$5x-4y+6z=35.$

12. $5x+4y+3z=35,$
$4x+3y+2z=25,$
$3x+2y-z=15.$

13. $x+y+z=19,$
$y=2x-3,$
$z=y-10.$

14. $3x+4y+5z=-68,$
$2x+y=-2,$
$4y-z=-14.$

15. $x-y-2z=-4,$
$x-2y-z=-4,$
$2x-y-z=0.$

16. $6x-y=3z-36,$
$y-3z=3x-39,$
$z-2x=3y+2.$

17. $2x-3y=4z-21,$
$2y-3z=4x-14,$
$2z-3x=4y-10.$

18. $2x-3y+2z=\frac{7}{36},$
$3x-2y+3z=\frac{1}{2},$
$2x+3y-2z=\frac{17}{36}.$

19. $2x+3y+4z+5=0,$
$2x+3y-4z+6=0,$
$2x-3y+4z+7=0.$

20. $10x-8y+z=40,$
$x+y+2z=5,$
$3x-z+5=0.$

21. $0.3\,x + 0.5\,y = 0.8,$
$0.4\,x + 0.7\,z = 1.8,$
$0.1\,y + 0.1\,z = 0.3.$

22. $0.1\,x + 0.3\,y = 1.9,$
$0.2\,x + 0.4\,z = 3.2,$
$0.5\,y + 0.1\,z = 3.1.$

23. $x + y = c,$
$y + z = b,$
$z + x = a.$

24. $x + y = a + b,$
$y + z = b + c,$
$z + x = c + a.$

25. $x + 3\,y = a,$
$y + 3\,z = b,$
$z + 3\,x = c.$

26. $x + y - z = a,$
$x - y + z = b,$
$z + y - x = c.$

27. $bx + ay + cz = a,$
$cx + by + az = b,$
$ax + cy + bz = c.$

28. $2\,x - 3\,y = 6,$
$4\,y - 5\,z = 7,$
$2\,z - 3\,u = 8,$
$4\,u - 5\,x = 9.$

29. $2\,x - y = 8,$
$3\,y - z = 13,$
$4\,z - w = 16,$
$5\,w - x = 13.$

30. $3\,x + 4\,y - 2\,z = 20,$
$2\,x - 7\,y + 5\,u = -9,$
$8\,x + 2\,z + 3\,u = 27,$
$2\,y - 3\,z + 4\,u = 17.$

31. $7\,x + y - 4\,z = w,$
$x + w - y = 0,$
$2\,z - w + 3\,y = 15,$
$3\,y - 7\,x - 2\,z = 3\,w - 8.$

32. $x + y + z = 3,$
$y + z + u = 4,$
$z + u + x = 5,$
$u + x + y = 6.$

33. $x + y + z + u = 12,$
$x + y + z + v = 14,$
$x + y + u + v = 16,$
$x + z + u + v = 18,$
$y + z + u + v = 20.$

SYSTEMS OF FRACTIONAL EQUATIONS SOLVED LIKE EQUATIONS OF THE FIRST DEGREE

EXERCISE 117

Solve the following systems of equations which are fractional in the unknowns, verifying all results obtained:

1. $\dfrac{1}{x} - \dfrac{1}{y} = m,$
$\dfrac{1}{x} + \dfrac{1}{y} = n.$

2. $\dfrac{9}{x} - \dfrac{10}{y} = 1,$
$\dfrac{12}{x} + \dfrac{15}{y} = 7.$

3. $\dfrac{1}{x} - \dfrac{2}{y} + \dfrac{3}{4} = 0,$
$\dfrac{2}{x} + \dfrac{3}{y} - \dfrac{4}{5} = 0.$

4. $\frac{4}{x}+\frac{7}{y}=\frac{5}{6}$,
$\frac{6}{x}-\frac{14}{y}=-\frac{1}{2}$.

5. $\frac{10}{x}+\frac{9}{y}=8$,
$\frac{8}{x}+\frac{3}{y}=5$.

6. $\frac{3}{x}+\frac{4}{y}=\frac{7}{24}$,
$\frac{8}{x}-\frac{6}{y}=\frac{1}{12}$.

7. $\frac{1}{2x}+\frac{1}{3y}=8$,
$\frac{1}{4x}+\frac{1}{9y}=\frac{10}{3}$.

8. $\frac{1}{5x}-\frac{1}{12y}=\frac{9}{2}$,
$\frac{1}{15x}+\frac{1}{6y}=5$.

9. $\frac{1}{2x}+\frac{2}{3y}=15$,
$\frac{4}{5x}+\frac{5}{6y}=20$.

10. $\frac{8}{ax}+\frac{3}{by}=\frac{1}{c}$,
$\frac{4}{ax}+\frac{9}{by}=\frac{1}{2d}$.

11. $\frac{a}{x}+\frac{b^2}{y}=c$,
$\frac{a^2}{x}-\frac{b}{y}=c^2$.

12. $11x-\frac{4}{y}=3$,
$10x-\frac{3}{y}=4$.

13. $\frac{a}{x}+\frac{b}{y}=1$,
$\frac{b}{x}-\frac{a}{y}=3$.

14. $\frac{a}{x}+\frac{b}{y}=\frac{c}{d}$,
$\frac{a}{x}-\frac{b}{y}=\frac{d}{c}$.

15. $\frac{3}{x}+\frac{1}{3y}-3=0$,
$\frac{1}{2x}-\frac{2}{y}+\frac{1}{2}=0$.

16. $\frac{3}{x}+\frac{7}{y}=\frac{43}{20}$,
$2y-6x=-\frac{7xy}{10}$.

17. $\frac{a}{bx}+\frac{b}{ay}=a+b$,
$\frac{b}{x}+\frac{a}{y}=a^2+b^2$.

18. $\frac{5}{y}-\frac{5}{x}=4$,
$\frac{1}{y}+\frac{1}{x}=\frac{5}{6}$.

19. $\frac{4}{x+1}+\frac{5}{y-2}=9$,
$\frac{5}{x+1}-\frac{3}{y-2}=2$.

20. $\frac{2}{x+2}+\frac{3}{y+3}-4=0$,
$\frac{3}{x+2}-\frac{2}{y+3}+4=0$.

21. $\frac{1}{x}+\frac{1}{y}=3$,
$\frac{1}{x}+\frac{1}{z}=4$,
$\frac{1}{y}+\frac{1}{z}=5$.

22. $\frac{1}{x}-\frac{2}{y}+3=0$,
$\frac{1}{y}-\frac{3}{z}+4=0$,
$\frac{1}{z}-\frac{4}{x}+5=0$.

23. $\frac{1}{x}+\frac{1}{y}=\frac{1}{a}$,
$\frac{1}{y}+\frac{1}{z}=\frac{1}{b}$,
$\frac{1}{z}+\frac{1}{x}=\frac{1}{c}$.

24. $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{2},$

$\frac{1}{x}-\frac{1}{y}+\frac{1}{z}=\frac{1}{3},$

$\frac{1}{x}+\frac{1}{y}-\frac{1}{z}=\frac{1}{4}.$

25. $\frac{2}{x}+\frac{3}{y}+\frac{1}{z}=4,$

$\frac{3}{x}+\frac{1}{y}+\frac{2}{z}=5,$

$\frac{1}{x}+\frac{2}{y}-\frac{3}{z}=6.$

26. $\frac{1}{x}+\frac{2}{y}+\frac{3}{z}=20,$

$\frac{2}{x}+\frac{3}{y}+\frac{1}{z}=17,$

$\frac{3}{x}+\frac{1}{y}+\frac{2}{z}=17.$

27. $\frac{xy}{x+y}=a,$

$\frac{yz}{y+z}=b,$

$\frac{zx}{z+x}=c.$

PROBLEMS

PROBLEMS SOLVED BY MEANS OF SIMULTANEOUS LINEAR EQUATIONS

EXERCISE 118

1. Find two numbers whose sum is 41 and whose difference is 15.

2. Find two numbers whose sum is 59 and whose difference is 9.

3. Find two numbers whose sum is 104 and whose difference is 22.

4. Find two numbers whose sum is 60 and whose difference is 6.

5. Find two numbers whose sum is 45 and whose difference is 7.

6. Separate 77 into two parts such that the greater part shall exceed the less by 11.

7. Separate 56 into two parts such that the greater part shall exceed the less part by 22.

8. Separate 60 into two parts such that twice the less part shall exceed the greater part by 6.

9. Separate 49 into two parts such that the greater part shall exceed three times the less part by one.

10. Separate 72 into two parts such that the greater part shall exceed four times the less part by 7.

11. What is the price of peaches and plums per dozen if 9 peaches and 6 plums cost \$0.39, and 12 peaches and 12 plums cost \$0.60?

12. What is the price per dozen of apples and pears if 6 apples and 6 pears cost \$0.18, and 18 apples and 12 pears cost \$0.42?

13. What must be the values of x and y if $x+2y$ represents 19 and $2x+y$ represents 23?

14. What must be the values of x and y if $3x - y$ represents 21 and $x + 2y$ represents 14 ?

15. Find the values of x and y in order that $2x + 3y$ shall represent 57 and $2x - y$ shall represent 14.

16. Find the values of x and y on condition that $7x - 2y$ shall represent 64 and $5x + 4y$ shall represent 62.

17. A can row 5 miles an hour downstream and 3 miles an hour upstream. Find his rate in miles per hour in still water and the rate of the stream in miles per hour.

18. B can row 7 miles an hour downstream and 2 miles an hour upstream. Find B's rate in miles per hour in still water and the rate of the stream in miles per hour.

19. A can row 6 miles an hour downstream and 3 miles an hour upstream. Find his rate in miles per hour in still water and the rate of the stream in miles per hour.

20. A steamboat can run 15 miles per hour upstream and 9 miles per hour downstream. Find the rate of the steamboat in miles per hour in still water and the rate of the stream in miles per hour.

21. The greater of two numbers is three times the less, and the sum of the numbers is 124. Find the numbers.

22. The greater of two numbers is four times the less, and the sum of the numbers is 45. Find the numbers.

23. The greater of two numbers is nine times the less, and the sum of the numbers is 50. Find the numbers.

24. The greater of two numbers is three times the less, and the sum of the numbers is 36. Find the numbers.

25. The greater of two numbers is six times the less, and the difference of the numbers is 20. Find the numbers.

26. The greater of two numbers is five times the less, and the difference of the numbers is 24. Find the numbers.

27. Find two numbers such that if four times the first is increased by seven times the second the result will be 19, and if six times the first is diminished by seven times the second the result will be 11.

28. Find two numbers such that if five times the first is added to the second the sum will be 19, but if seven times the first is diminished by the second the remainder will be 17.

29. Find two numbers such that if twice the first be added to three times the second the sum will be 46, but if eight times the first be diminished by three times the second the remainder will be 4.

30. Find two numbers such that if three times the first is added to five times the second the sum will be 42, but if nine times the first is diminished by five times the second the remainder will be 6.

31. If four times the greater of two numbers be divided by the less, the quotient is 5 and the remainder is 4, and if three times the less be divided by the greater, the quotient is 2 and the remainder is 12. What are the numbers?

32. What is that fraction which equals $\frac{1}{5}$ when 1 is added to the numerator, and equals $\frac{1}{6}$ when 1 is added to the denominator?

33. Find a fraction which is equal to $\frac{1}{5}$ when its numerator and denominator are each diminished by 2, and is equal to $\frac{1}{3}$ when its terms are increased by 3.

34. If 1 is added to the numerator of a certain fraction, its value becomes $\frac{5}{7}$, and if 1 is added to the denominator, the value becomes $\frac{3}{5}$. What is the fraction?

35. Find a fraction such that if 1 be added to both numerator and denominator the value becomes $\frac{1}{2}$, while if 1 be subtracted from both numerator and denominator the value becomes $\frac{7}{16}$.

36. If 3 be added to both numerator and denominator of a certain fraction, its value becomes $\frac{7}{9}$; if 3 be subtracted from both numerator and denominator, its value becomes $\frac{1}{3}$. Find the fraction.

37. If the numerator of a certain fraction be multiplied by 3 and its denominator be increased by 9, the value of the fraction becomes $\frac{3}{4}$; if the denominator be multiplied by 2 and the numerator be increased by 12, the value of the fraction becomes one-half. Find the numerator and denominator of the fraction.

38. The numerator and denominator of a certain proper fraction each consists of the same two digits whose sum is 9, written in different orders. If the value of the fraction be $\frac{4}{7}$, find the numerator and denominator.

39. The numerator and denominator of a certain improper fraction each consists of the same two figures, whose sum is 6, written in different orders. If the value of the fraction be $\frac{5}{17}$, find the numerator and denominator.

40. Separate 100 into three parts such that if the second part be divided by the first the quotient is 4 and the remainder 2; and if the third be divided by the second the quotient is 3 and the remainder is 7.

41. A number expressed by two figures is equal to 6 times the sum of its figures. If 9 be subtracted from the number, the figures in tens' and units' places are interchanged. Find the number.

42. Separate the two numbers 75 and 70 into two parts each such that the sum of one part of the first and one part of the second shall equal 100, and the difference of the remaining parts shall equal 25.

43. Separate the two numbers 60 and 50 into two parts each such that the sum of one part of the first and one part of the second shall equal 75, and the difference of the remaining parts shall equal 5.

44. Separate a into two parts such that $1/m$th of the greater part shall exceed $1/n$th of the less by b.

45. A number is composed of two figures whose sum is 14. If the figures in tens' and units' places are interchanged, the number is increased by 18. Find the number.

46. A farmer bought 100 acres of land for $3304. If part of it cost him $50 an acre and the remainder $18 an acre, find the number of acres bought at each price.

47. If five pounds of sugar and ten pounds of coffee together cost $3.80, and at the same price ten pounds of sugar and five pounds of coffee cost $2.35, what is the price of each per pound?

48. A man invested $5000, part at 5 per cent and the remainder at 4 per cent interest. If the annual income from both investments was $235, what were the separate amounts invested?

49. There are two pumps drawing water from a tank. When the first works three hours and the second five hours, 1350 cubic feet of water are withdrawn. When the first works four hours and the second three hours, 1250 cubic feet of water are withdrawn. How many cubic feet of water can each pump discharge in one hour?

50. A plumber and his helper together receive $4.80. The plumber works 5 hours and the helper 6 hours. At another time the plumber works 8 hours and the helper $9\frac{1}{2}$ hours, and they receive $7.65. What are the wages of each per hour?

51. Three men and three boys can do in 4 days a certain amount of work which can be done in 6 days by one man and 5 boys. How long would it require one man alone or one boy alone to do the work?

52. A certain amount of work can be completed by 3 men and 6 boys in 2 days. At another time it is observed that an equal amount of work is performed in 3 days by 1 man and 8 boys. Find the length of time required for one man alone or for one boy alone to do the given amount of work.

53. Two persons A and B can complete a certain amount of work in l days; they work together m days, when A stops and B finishes it in n days. Find the time each would require to do it alone.

54. A steamer makes a trip of 70 miles up a river and down again in 24 hours, allowing 5 hours for taking on a cargo. It is observed that it requires the same time to go $2\frac{1}{2}$ miles up the river as 7 miles down the river. Find the number of hours required for the up trip and for the down trip respectively.

55. A train ran a certain distance at a uniform rate. If the rate had been increased by 4 miles an hour, the journey would have required 16 minutes less, but if the rate had been diminished by 4 miles an hour, the journey would have required 20 minutes more. Find the length of the journey and the rate of the train in miles per hour.

56. A steamer runs a miles up a river and back again in t hours. It is observed that it requires the same time to go b miles with the stream as it does to go c miles against it. Find expressions for the number of hours required for the up and down trips respectively, and also for the velocity of the stream in miles per hour.

57. Three trains start for a certain city, the second h hours after the first and the third k hours after the first. The second and third run at the rates of a and b miles an hour respectively. If all three arrive together, find an expression for the distance and for the rate of the first train in miles per hour.

58. A marksman fires at a target 600 yards distant. He hears the bullet strike 4 seconds after he fires. An observer standing 525 yards from the target and 300 yards from the marksman hears the bullet strike 3 seconds after he hears the report of the rifle. Find the velocity of the sound in yards per second and also the velocity of the bullet, supposing each to be uniform.

59. Having given two alloys of the following composition: A, composed of 4 parts (by weight) of gold and 3 of silver; B, 2 parts of gold and 7 of silver; how many ounces of each must be taken to obtain 6 ounces of an alloy containing equal amounts (by weight) of gold and silver?

60. Two alloys A and B contain; A, 2 parts (by weight) of tin and 9 parts of copper; B, 7 parts of tin and 3 parts of copper. To obtain 1000 pounds of alloy containing (by weight) 5 parts of tin and 16 parts of copper, how many pounds of each must be taken and melted together?

61. A bar of metal contains 20.625 per cent pure silver, and a second bar 12.25 per cent. How many ounces of each bar must be used if when the parts taken are melted together a new bar weighing 50 ounces is obtained, of which 15 per cent is pure silver?

62. A and B run two quarter-mile races. In the first race A gives B a start of 2 seconds and beats him by 20 yards. In the second race A gives B a start of 6 yards and beats him by 4 seconds. Find the rates of A and B in yards per second.

63. A and B run a race of 500 yards. In the first trial A gives B a start of 7 yards and wins by 10 seconds. In the second trial A gives B a start of 56 yards and wins by 2 seconds. Find the rates of A and B in yards per second.

64. In a race of 100 yards A beats B by $\frac{1}{3}$ of a second. In the second trial A gives B a start of 3 yards, and B wins by $1\frac{1}{11}$ yards. Find the time required for A and B each to run 100 yards.

65. A and B run a race of 440 yards. In the first trial A gives B a start of 65 yards and wins by 20 seconds. In the second trial A gives B a start of 34 seconds and B wins by 8 yards. Find the rates of A and B in yards per second.

Problems containing Three or More Unknowns

66. If 270 is added to a certain number of three figures the figures in tens' and hundreds' places are interchanged. When 198 is subtracted from the number the figures in hundreds' and units' places are interchanged. The figure in hundreds' place is twice that in units' place. Find the number.

67. A number is expressed by three figures whose sum is 11. The sum of the figures in hundreds' and units' places is less by one than the figure in tens' place, and if the figures in units' and tens' places are interchanged the resulting number is less by 36 than the original number. Find the original number.

68. Find three numbers such that the sum of the reciprocals of the first and second is $\frac{1}{2}$; of the second and third, $\frac{1}{3}$; and of the third and first, $\frac{1}{4}$.

69. Separate 400 into 4 parts such that if the first part be increased by 9, the second diminished by 9, the third multiplied by 9, and the fourth divided by 9, the results will all be equal.

70. A and B together can dig a cellar in $7\frac{1}{5}$ days, A and C in 6 days. All three work together for 2 days when A stops and B and C finish the work in $2\frac{4}{7}$ days. How long would it require each man alone to do the work?

71. A and B can plaster a house in r days, A and C can do the same work in s days, and B and C can do it in t days. Find in how many days each can do the work alone.

72. In a mile race A can beat B by 60 yards and can beat C by 230 yards. By how much can B beat C?

73. In a race of 500 yards A can beat B by 20 yards, and C by 30 yards. By how many yards can B beat C?

74. Having given three bars of metal, the first containing (by weight) 6 parts of gold, 2 parts of silver, and 1 part of lead; the second, 3 parts of gold, 4 parts of silver, and 2 parts of lead; the third, 1 part of gold, 3 parts of silver, and 5 parts of lead; find how many ounces of each must be taken to obtain 12 ounces of an alloy containing equal amounts (by weight) of gold, silver, and lead.

75. Of three bars of metal, the first contains 13 parts (by weight) of silver, 5 parts of copper, and 2 parts of tin; the second, 35 parts of silver, 4 parts of copper, and 1 part of tin; the third, 8 parts of silver, 7 parts of copper, and 5 parts of tin. How many ounces of each bar must be used if when the parts taken are melted together a bar is obtained which weighs 10 ounces, of which 5 ounces are silver, 3 ounces are copper, and 2 ounces are tin.

EVOLUTION

ROOTS OF MONOMIALS

Roots of Powers and Products

EXERCISE 119

Simplify each of the following expressions:

1. $\sqrt{4}$.
2. $\sqrt{25}$.
3. $\sqrt[3]{8}$.
4. $\sqrt{64}$.
5. $\sqrt[3]{512}$.
6. $\sqrt[4]{16}$.
7. $\sqrt{121}$.
8. $\sqrt[3]{-27}$.
9. $\sqrt[5]{-32}$.
10. $\sqrt{49}$.
11. $\sqrt[5]{243}$.
12. $\sqrt[7]{128}$.
13. $\sqrt[8]{256}$.
14. $\sqrt[3]{-125}$.
15. $\sqrt[6]{64}$.
16. $\sqrt{169}$.
17. $\sqrt[3]{216}$.
18. $\sqrt[3]{343}$.
19. $\sqrt[3]{729}$.
20. $\sqrt{196}$.
21. $\sqrt{289}$.
22. $\sqrt{a^2}$.
23. $\sqrt{b^4}$.
24. $\sqrt{c^6}$.
25. $\sqrt[3]{a^3}$.
26. $\sqrt[3]{-b^3}$.
27. $\sqrt[3]{c^6}$.
28. $\sqrt{d^{10}}$.
29. $\sqrt[4]{d^8}$.
30. $\sqrt[5]{a^{25}}$.
31. $\sqrt[3]{-m^9}$.
32. $\sqrt{4\,x^2}$.
33. $\sqrt{9\,y^2}$.
34. $\sqrt[3]{8\,z^3}$.
35. $\sqrt{36\,w^4}$.
36. $\sqrt{49\,n^6}$.
37. $\sqrt{16\,a^8}$.
38. $\sqrt{64\,x^6}$.
39. $\sqrt[3]{64\,y^6}$.
40. $\sqrt[6]{64\,z^6}$.
41. $\sqrt{m^2n^4}$.
42. $\sqrt[3]{-x^3y^6}$.
43. $\sqrt[3]{c^3d^9}$.
44. $\sqrt{a^6b^8}$.
45. $\sqrt[3]{-x^6y^9z^3}$.
46. $\sqrt[5]{-a^5b^{10}c^{15}}$.
47. $\sqrt[6]{x^{12}y^6z^{18}}$.
48. $\sqrt[4]{a^4b^{12}c^8}$.
49. $\sqrt[5]{x^5y^{10}z^{15}}$.
50. $\sqrt[6]{a^{12}b^{18}c^6}$.
51. $\sqrt[3]{-64\,a^6b^3}$.
52. $\sqrt[4]{16\,m^{16}n^4}$.
53. $\sqrt[7]{a^7b^{14}c^{21}}$.
54. $\sqrt{400\,a^{20}}$.
55. $\sqrt[3]{-125\,x^{12}y^{15}}$.
56. $\sqrt[7]{128\,a^7b^7c^{14}}$.
57. $\sqrt[n]{a^{2n}}$.
58. $\sqrt[3]{a^{3n}}$.
59. $\sqrt[n]{b^{3n}}$.
60. $\sqrt[n]{x^ny^{2n}}$.
61. $\sqrt[n]{b^nc^{2n}d^{3n}}$.
62. $\sqrt[n]{x^{2n}y^{4n}z^{6n}}$.
63. $\sqrt[n]{a^{5n}b^{3n}c^n}$.

ROOTS OF ROOTS AND POWERS OF ROOTS

EXERCISE 120

Simplify each of the following expressions:

1. $\sqrt{\sqrt{a^4}}$.
2. $\sqrt{\sqrt[3]{b^6}}$.
3. $\sqrt[3]{\sqrt[3]{c^{18}}}$.
4. $\sqrt[4]{\sqrt[2]{d^8}}$.
5. $\sqrt{\sqrt[4]{x^8y^4}}$.
6. $\sqrt[3]{\sqrt{64\,a}}$.
7. $\sqrt[3]{\sqrt[4]{x^{12}y^{24}}}$.
8. $\sqrt{\sqrt[5]{a^{10}b^{20}}}$.
9. $\sqrt[4]{\sqrt[3]{b^{12}c^{18}}}$.
10. $\sqrt[5]{\sqrt{2^{10}x^{20}y^{10}}}$.
11. $\sqrt[6]{\sqrt{x^{12}y^6}}$.
12. $\sqrt[3]{\sqrt[6]{a^{18}b^{36}}}$.
13. $\sqrt{\sqrt{16\,a^4b^8}}$.
14. $\sqrt[3]{\sqrt[3]{3^9a^9b^{27}}}$.
15. $\sqrt{\sqrt{81\,a^4b^{16}}}$.
16. $(\sqrt{3})^4$.
17. $(\sqrt{2})^6$.
18. $(\sqrt{5})^8$.
19. $(\sqrt{3})^{10}$.
20. $(\sqrt[3]{4})^{12}$.
21. $(\sqrt[5]{3})^{15}$.
22. $(\sqrt{2})^{14}$.
23. $(\sqrt{ab})^4$.
24. $(\sqrt[3]{xy})^6$.
25. $(\sqrt[3]{a^2})^6$.
26. $(\sqrt{2\,a})^8$.
27. $(\sqrt{3\,x})^{10}$.
28. $(\sqrt[3]{x^2y})^6$.
29. $(\sqrt[4]{ab^2c^3})^8$.
30. $(\sqrt[5]{xy^2z^3})^{10}$.
31. $(\sqrt{a})^{2n}$.
32. $(\sqrt[3]{b})^{3n}$.
33. $(\sqrt[n]{x})^{4n}$.
34. $(\sqrt[n]{y^2})^{5n}$.
35. $(\sqrt[2n]{a})^{4n}$.
36. $(\sqrt[n]{n})^{n2}$.

ROOTS OF QUOTIENTS

EXERCISE 121

Simplify each of the following:

1. $\sqrt{\frac{1}{4}}$.
2. $\sqrt{\frac{1}{25}}$.
3. $\sqrt[3]{-\frac{1}{8}}$.
4. $\sqrt{\frac{4}{9}}$.
5. $\sqrt{\frac{49}{81}}$.
6. $\sqrt[3]{\frac{27}{64}}$.
7. $\sqrt[3]{-\frac{8}{27}}$.
8. $\sqrt[4]{\frac{16}{81}}$.
9. $\sqrt{\frac{144}{169}}$.
10. $\sqrt{\frac{121}{256}}$.
11. $\sqrt[3]{\frac{64}{125}}$.
12. $\sqrt[4]{\frac{81}{16}}$.
13. $\sqrt{\frac{a^2}{9}}$.
14. $\sqrt{\frac{16}{b^2}}$.
15. $\sqrt{\frac{c^2}{25}}$.
16. $\sqrt{\frac{36}{d^2}}$.
17. $\sqrt[3]{\frac{8}{x^3}}$.
18. $\sqrt{\frac{9\,x^2}{25}}$.
19. $\sqrt[3]{\frac{y^3}{125}}$.
20. $\sqrt{\frac{x^{16}}{16}}$.
21. $\sqrt[4]{\frac{16}{a^4b^4}}$.
22. $\sqrt[5]{\frac{a^5b^{10}}{32}}$.
23. $\sqrt{\frac{36\,a^4}{25\,b^6}}$.
24. $\sqrt{\frac{64\,a^2}{b^6c^4}}$.
25. $\sqrt[3]{\frac{x^6y^3}{64}}$.
26. $\sqrt[6]{\frac{64}{m^6n^6}}$.

27. $\sqrt[3]{-\frac{8\,a^3b^3}{24^3}}$.
28. $\sqrt[n]{\frac{a^n}{b^n}}$.
29. $\sqrt[m]{\frac{c^{2m}}{d^{3m}}}$.
30. $\sqrt[n]{\frac{x^n y^{2n}}{z^{4n}}}$.
31. $\sqrt[n]{\frac{a^{mn}b^n}{c^{5n}}}$.
32. $\sqrt[a]{\frac{x^{3a}}{y^{5a}z^{2a}}}$.
33. $\sqrt[r]{\frac{x^{rs}y^{rt}}{z^r}}$.
34. $\sqrt{.04}$.
35. $\sqrt[3]{.008}$.
36. $\sqrt{.25}$.
37. $\sqrt[3]{.001}$.
38. $\sqrt{.36\,a^2}$.
39. $\sqrt{.49\,b^2}$.
40. $\sqrt[3]{.027\,c^2}$.
41. $\sqrt[3]{.125\,x^3}$.
42. $\sqrt[3]{.064\,y^3}$.

SQUARE ROOTS OF POLYNOMIALS

EXERCISE 122

Find the square roots of the following expressions:

1. $a^4+2\,a^3-a^2-2\,a+1$.
2. $16\,x^4-24\,x^3+25\,x^2-12\,x+4$.
3. $81\,x^4+54\,x^3+81\,x^2+24\,x+16$.
4. $9\,x^6+24\,x^5+22\,x^4+38\,x^3+41\,x^2+10\,x+25$.
5. $25\,b^6-30\,b^4-20\,b^3+9\,b^2+12\,b+4$.
6. $36\,a^2+48\,ab+12\,ac+16\,b^2+8\,bc+c^2$.
7. $a^4+\frac{11a^2b^2}{12}+\frac{b^4}{9}-a^3b-\frac{ab^3}{3}$.
8. $x^4+3+\frac{1}{x^4}-2\,x^2-\frac{2}{x^2}$.
9. $4-4\,y+13\,y^2+16\,y^6+17\,y^4-22\,y^3-24\,y^5$.
10. $4\,x^{4n}+12\,x^{3n}+29\,x^{2n}+30\,x^n+25$.

CUBE ROOTS OF POLYNOMIALS

EXERCISE 123

Find the cube roots of the following expressions.

1. $27\,x^3+54\,x^2+36\,x+8$.
2. $8\,x^3+84\,x^2y+294\,xy^2+343\,y^3$.
3. $512\,a^3-1344\,a^2b+1176\,ab^2-343\,b^3$.
4. $8\,a^6-36\,a^5b+66\,a^4b^2-63\,a^3b^3+33\,a^2b^4-9\,ab^5+b^6$.
5. $125\,a^9-225\,a^7+150\,a^6+135\,a^5-180\,a^4+33\,a^3+54\,a^2-36\,a+8$.
6. $343\,a^6+441\,a^5b+777\,a^4b^2+531\,a^3b^3+444\,a^2b^4+144\,ab^5+64\,b^6$.

7. $729\,x^6+972\,x^5y+918\,x^4y^2+496\,x^3y^3+204\,x^2y^4+48\,xy^5+8\,y^6$.
8. $a^{12}-3\,a^{10}-3\,a^9+6\,a^7+8\,a^6+3\,a^5-3\,a^4-7\,a^3-6\,a^2-3\,a-1$.
9. $\frac{a^6}{27}+\frac{a^5}{6}+\frac{a^4}{3}+\frac{3\,a^3}{8}+\frac{a^2}{4}+\frac{3\,a}{32}+\frac{1}{64}$.
10. $\frac{x^3}{8}+\frac{3\,x^2}{4}+\frac{3\,x}{2}+\frac{5}{2}+\frac{6}{x}+\frac{6}{x^2}+\frac{6}{x^3}+\frac{12}{x^4}+\frac{8}{x^6}$.

SQUARE ROOTS OF ARITHMETIC NUMBERS

EXERCISE 124

Find the square roots of the following numbers.

1. 2209.
2. 6241.
3. 26244.
4. 64009.
5. 643204.
6. 6625.96.
7. 64.1601.
8. 4.008004.
9. 4096.256004.
10. 4.141225.
11. .00009409.
12. 1.00020001.
13. 49.61511844.
14. .3603841024.

To four decimal places.

15. 1.00001.
16. 10000.00001.
17. 59.

CUBE ROOTS OF ARITHMETIC NUMBERS

EXERCISE 125

Find the cube root of each of the following numbers:

1. 32768.
2. 42875.
3. 68921.
4. 21952.
5. 140608.
6. 1061208.
7. 28652616.
8. 94818816.
9. 568722789.
10. 967361669.
11. 448399762.264.
12. 1.003003001.
13. 10.
14. 34903.588968101.
15. 4.
16. $\frac{4}{7}$.
17. $\frac{3}{8}$.

To four decimal places.

THEORY OF EXPONENTS

NEGATIVE INTEGRAL EXPONENTS

EXERCISE 126

Find the numerical value of each of the following expressions:

1. 2^{-1}.
2. 3^{-2}.
3. 5^{-2}.
4. 2^{-4}.
5. 3^{-3}.
6. -8^{-3}.
7. 3^{-4}.
8. 2^{-5}.

9. -2^{-2}.
10. -3^{-3}.
11. $-(-2)^{-2}$.
12. $-(-2)^{-3}$.
13. $-(-3)^{-3}$.
14. $-(-2)^{-5}$.
15. $\frac{1}{2^{-1}}$.
16. $\frac{1}{3^{-2}}$.
17. $\frac{1}{4^{-3}}$.
18. $\frac{1}{5^{-4}}$.
19. $\frac{1}{(-2)^{-3}}$.
20. $-\frac{1}{(-3)^{-2}}$.
21. $\frac{1}{-(-4)^{-3}}$.
22. $\frac{3}{2^{-1}}$.
23. $\frac{5}{2^{-3}}$.
24. $\frac{4^{-1}}{3}$.
25. $\frac{5^{-2}}{2}$.
26. $\frac{5^{-3}}{4}$.
27. $-\frac{2^{-3}}{4}$.
28. $(\frac{1}{2})^{-1}$.
29. $(\frac{1}{3})^{-2}$.
30. $(\frac{1}{4})^{-3}$.
31. $(\frac{2}{3})^{-1}$.
32. $(\frac{3}{4})^{-3}$.
33. $(\frac{4}{5})^{-2}$.
34. $(-\frac{2}{5})^{-3}$.
35. $-(-\frac{3}{5})^{-4}$.
36. $.2^{-1}$.
37. $.5^{-1}$.
38. $.1^{-1}$.
39. $.1^{-2}$.
40. $.01^{-1}$.
41. $.2^{-2}$.

ZERO AS AN EXPONENT

EXERCISE 127

Find the value of each of the following expressions:

1. 2^0.
2. 5^0.
3. 1^0.
4. $\frac{7^0}{3}$.
5. $\frac{4}{3^0}$.
6. $\frac{3}{6^0}$.
7. $\frac{7^0}{3^0}$.
8. 3×8^0.
9. $3 + 8^0$.
10. $5^0 + 9^0$.
11. $10^0 - 6^0$.
12. $2^0 - 8^0$.
13. $(\frac{2}{3})^0$.
14. $\frac{1}{4^0}$.
15. $\frac{2^0}{3^0 + 4^0}$.
16. $(a + b)^0$.
17. $x^0 - (x - y)^0$.
18. $\frac{a^0 - b^0}{a^0 + b^0}$.

TRANSFORMATION OF EXPONENTS

EXERCISE 128

Express each of the following terms with positive exponents:

1. x^{-2}.
2. y^{-3}.
3. $-ab^{-1}$.
4. $c^{-2}d$.
5. $x^{-1}y$.
6. $a^{-3}b^2$.
7. $-x^4y^{-5}$.
8. $m^{-2}n^{-2}$.
9. $-x^{-3}y^{-4}$.
10. $a^{-1}b^2c^{-3}$.
11. $dx^{-4}y$.
12. $3a^{-1}$.
13. $2^{-1}b$.
14. $3^{-2}x^{-2}$.
15. $5x^{-2}y$.

16. $-7mx^{-3}$.
17. $9a^{-4}b^{-5}$.
18. $4x^{-1}yz^{-2}$.
19. $6^{-2}a^{3}b^{-2}$.
20. $2a^{-3}b^{-2}c^{-1}$.
21. $5^{-2}a^{-3}b^{-4}$.
22. $\frac{1}{a^{-2}}$.
23. $\frac{2}{b^{-3}}$.
24. $\frac{4x}{c^{-1}}$.
25. $\frac{6a}{b^{-1}c^{-1}}$.
26. $\frac{a^{-1}b}{c}$.
27. $\frac{dx^{-2}}{y}$.
28. $-\frac{m^{-1}x}{n^{-2}}$.
29. $\frac{ky^{-2}}{z^{-3}}$.
30. $\frac{b^{-2}}{cd^{-2}}$.
31. $\frac{a^{-1}b^{-2}}{c}$.
32. $\frac{-xy}{z^{-3}}$.
33. $-\frac{x^{-2}}{y^{-2}}$.
34. $\frac{xy^{-3}}{z^{-1}w}$.
35. $\frac{a^{-2}b}{cd^{-3}}$.
36. $\frac{xy^{-4}}{zw^{-2}}$.
37. $\frac{b^{-1}cd^{-1}}{3}$.
38. $\frac{a^{-2}b^{-3}}{c^{-3}d^{-4}}$.
39. $\frac{2x^{-3}y^{4}}{z^{-5}w^{6}}$.
40. $\frac{4m^{-2}b^{-2}}{5na^{-4}}$.
41. $(a+b)^{-2}$.
42. $(x-y)^{-3}$.
43. $4(m+n)^{-2}$.
44. $\frac{2}{(a+b)^{-2}}$.
45. $\frac{(x+y)^{-2}}{(a+b)^{3}}$.
46. $\frac{(m-n)^{2}}{(x-y)^{-2}}$.
47. $\frac{3(a+b)^{-4}}{4(x-y)^{-2}}$.
48. $\frac{6(c+d)^{-1}}{7(a-b)^{-2}}$.
49. $\frac{1}{(a-b)^{-2}}$.
50. $-(x-y)^{-1}$.
51. $a^{-1}+b^{-1}$.
52. $x^{-2}+x^{-1}$.
53. $m^{-3}-n^{-3}$.
54. $x^{2}+2+x^{-2}$.
55. $a^{-2}+2a^{-1}b^{-1}+b^{-2}$.

EXERCISE 129

In each of the following expressions transfer the factors from the denominator to the numerator:

1. $\frac{x}{y^{2}}$.
2. $\frac{2}{z}$.
3. $\frac{a^{2}}{b^{3}}$.
4. $\frac{c}{d^{-2}}$.
5. $\frac{a^{-2}}{b}$.
6. $\frac{1}{x^{-2}}$.
7. $-\frac{1}{y^{2}}$.
8. $-\frac{3}{a^{3}}$.
9. $\frac{3xy}{z}$.
10. $-\frac{1}{a^{-2}}$.
11. $\frac{5}{a^{-2}b^{3}}$.
12. $\frac{6}{x^{3}y^{-4}}$.
13. $\frac{8}{a^{-3}b^{-2}}$.
14. $\frac{a^{2}b^{2}}{c^{2}}$.
15. $\frac{xyz}{w}$.
16. $\frac{ace}{b^{-1}d^{-1}}$.
17. $\frac{1}{a^{-2}b^{-3}c}$.
18. $-\frac{1}{x^{2}y^{2}z^{2}}$.

FRACTIONAL EXPONENTS

EXERCISE 130

Find the value of each of the following expressions:

1. $4^{\frac{1}{2}}$.
2. $8^{\frac{1}{3}}$.
3. $9^{\frac{1}{2}}$.
4. $-16^{\frac{1}{4}}$.
5. $32^{\frac{1}{5}}$.
6. $27^{\frac{1}{3}}$.
7. $4^{\frac{3}{2}}$.
8. $-8^{\frac{2}{3}}$.
9. $9^{\frac{3}{2}}$.
10. $16^{\frac{3}{4}}$.
11. $-32^{\frac{3}{5}}$.
12. $-25^{\frac{1}{2}}$.
13. $49^{-\frac{1}{2}}$.
14. $36^{-\frac{1}{2}}$.
15. $8^{-\frac{4}{3}}$.
16. $100^{-\frac{3}{2}}$.
17. $169^{-\frac{1}{2}}$.
18. $125^{\frac{2}{3}}$.
19. $25^{\frac{3}{2}}$
20. $27^{-\frac{2}{3}}$.
21. $128^{\frac{2}{7}}$.
22. $4^{\frac{7}{2}}$.
23. $32^{\frac{2}{5}}$.
24. $81^{\frac{3}{4}}$.
25. $81^{\frac{5}{4}}$.
26. $343^{\frac{2}{3}}$.
27. $256^{\frac{1}{2}}$.
28. $256^{\frac{3}{8}}$.
29. $.04^{\frac{1}{2}}$.
30. $.16^{\frac{1}{2}}$.
31. $.027^{\frac{1}{3}}$.
32. $.001^{\frac{1}{3}}$.
33. $.125^{\frac{1}{3}}$.
34. $1.44^{\frac{1}{2}}$.
35. $1.96^{\frac{1}{2}}$.
36. $.064^{\frac{2}{3}}$.
37. $.008^{\frac{2}{3}}$.
38. $.027^{\frac{2}{3}}$.
39. $6.25^{\frac{1}{2}}$.
40. $.001^{\frac{2}{3}}$.
41. $.008^{\frac{2}{3}}$.
42. $.04^{\frac{3}{2}}$.
43. $.01^{\frac{3}{2}}$.
44. $.001^{\frac{2}{3}}$.

TRANSLATION FROM THE EXPONENTIAL TO THE RADICAL NOTATION

MENTAL EXERCISE 131

Express each of the following in the radical notation:

1. $a^{\frac{1}{2}}$.
2. $b^{\frac{1}{3}}$.
3. $c^{\frac{3}{4}}$.
4. $d^{\frac{1}{4}}$.
5. $x^{-\frac{1}{2}}$.
6. $y^{-\frac{1}{3}}$.
7. $e^{\frac{2}{5}}$.
8. x^3.
9. $y^{\frac{5}{3}}$.
10. $z^{\frac{2}{7}}$.
11. $a^{\frac{m}{2}}$.
12. $b^{\frac{3}{n}}$.
13. $c^{\frac{m}{n}}$.
14. $a^{\frac{b}{a}}$.
15. $b^{\frac{c}{a}}$.
16. $c^{\frac{e}{d}}$.
17. $d^{\frac{5}{a}}$.
18. $z^{\frac{6}{5r}}$.
19. $y^{-\frac{5}{2n}}$.
20. $z^{\frac{2m}{9}}$.
21. $w^{\frac{3m}{2n}}$.
22. $3^{\frac{m}{3}}$.
23. $2^{\frac{2}{r}}$
24. $x^{\frac{n}{n+1}}$.
25. $a^{\frac{1}{2}}b^{\frac{1}{2}}$.
26. $m^{\frac{1}{2}}n^{\frac{1}{3}}$.

27. $b^{\frac{3}{4}}c^{\frac{4}{5}}$.
28. $x^{\frac{1}{m}}y^{\frac{1}{n}}$.
29. $a^{\frac{2}{n}}x^{\frac{n}{2}}$.
30. $a^{\frac{1}{b}}b^{\frac{1}{a}}$.
31. $x^{\frac{x}{y}}y^{\frac{y}{z}}z^{\frac{z}{x}}$.
32. $a^{\frac{1}{2}}b^{-\frac{1}{3}}c^{-\frac{1}{4}}$.
33. $a^{\frac{b}{c}}b^{\frac{c}{a}}c^{\frac{a}{b}}$.
34. $3\,x^{\frac{1}{2}}$.
35. $5\,y^{\frac{1}{3}}$.
36. $4\,z^{-\frac{1}{4}}$.
37. $7\,w^{\frac{2}{5}}$.
38. $2\,c^{\frac{c}{2}}$.
39. $10\,m^{\frac{1}{10}}$.
40. $12\,a^{\frac{1}{3}}b^{\frac{3}{4}}$.
41. $-6\,a^2b^{\frac{1}{2}}$.
42. $7\,c^{\frac{1}{3}}d^2$.
43. $ab^{\frac{a}{b}}$.
44. $(ab^2)^{\frac{1}{3}}$.
45. $(cd^3)^{\frac{1}{5}}$.
46. $(x^2y^4)^{\frac{1}{7}}$.
47. $(z^2w^4)^{\frac{1}{5}}$.
48. $1 \div a^{\frac{1}{2}}$.
49. $2 \div b^{\frac{1}{3}}$.
50. $(a+b)^{\frac{1}{2}}$.

TRANSLATION FROM THE RADICAL TO THE EXPONENTIAL NOTATION

EXERCISE 132

Express the following roots indicated by radical signs by the notation of fractional exponents:

1. $\sqrt{a}$.
2. $\sqrt[3]{b}$.
3. $\sqrt[4]{c}$.
4. $\sqrt[5]{x}$.
5. $\sqrt[5]{d^2}$.
6. $-\sqrt[4]{x^3}$.
7. $\sqrt[5]{-x^3}$.
8. $\sqrt[n]{y}$.
9. $\sqrt[n]{n}$.
10. $\sqrt[n]{z^{n-1}}$.
11. $3\sqrt{x}$.
12. $2\sqrt[3]{y}$.
13. $4\sqrt[3]{z^2}$.
14. $-5\sqrt[3]{y^2}$.
15. $7\,a\sqrt[5]{bc}$.
16. $6\sqrt[7]{x^5y^3}$.
17. $\sqrt{1 \div a}$.
18. $-\sqrt{1 \div c}$.
19. $\sqrt[3]{1 \div b^2}$.
20. $-\sqrt[3]{1 \div d^2}$.
21. $\sqrt[3]{2 \div c^n}$.
22. $\sqrt[m]{1 \div m}$.
23. $\sqrt{a^{-1}}$.
24. $-\sqrt{x^{-1}}$.
25. $\sqrt[3]{b^{-2}}$.
26. $\sqrt[5]{c^{-3}}$.
27. $\sqrt[n]{x^{-m}}$.
28. $\sqrt[3]{1 \div x^{-2}}$.
29. $5\sqrt[3]{ab^2c}$.
30. $m\sqrt[4]{m^2n^3w}$.
31. $-\sqrt[5]{ab^3c^4}$.
32. $\sqrt[n]{a^{n-1}b}$.
33. $2\sqrt[a]{a^2}$.
34. $-a\sqrt[b]{b^a}$.
35. $\sqrt{\sqrt{m}}$.
36. $\sqrt[3]{\sqrt{n}}$.
37. $\sqrt[3]{a\sqrt{b}}$.
38. $-a\sqrt{b\sqrt{c}}$.
39. $x^3\sqrt[3]{y^2\sqrt{z}}$.
40. $2\sqrt[3]{3\sqrt[4]{5\,n}}$.

PRODUCTS OF POWERS AND QUOTIENTS OF POWERS

EXERCISE 133

Simplify the following products and quotients:

1. $2^{\frac{1}{2}} \cdot 2^{\frac{1}{3}}$.
2. $3^{\frac{1}{4}} \cdot 3^{\frac{1}{3}}$.
3. $5^{\frac{1}{4}} \cdot 5^{-1}$.
4. $6^{\frac{1}{2}} \cdot 6^{-\frac{1}{3}}$.
5. $-7^{\frac{2}{3}} + 7^{\frac{4}{5}}$.
6. $10^{-\frac{1}{4}} \cdot 10^{-2}$.
7. $4^{\frac{1}{2}} \div 2^{3}$.
8. $25^{\frac{1}{2}} \div 5$.
9. $3^{4} \div 9^{\frac{3}{2}}$.
10. $9^{\frac{1}{2}} \cdot 3^{-1}$.
11. $8^{\frac{1}{3}} \cdot 2^{-2}$.
12. $25^{\frac{1}{2}} \cdot 5^{2}$.
13. $36^{\frac{1}{2}} \cdot 6^{-3}$.
14. $49^{-\frac{1}{2}} \cdot 7^{-2}$.
15. $32^{\frac{3}{5}} \cdot 2^{3}$.
16. $-27^{\frac{2}{3}} \cdot 3^{-5}$.
17. $81^{-\frac{3}{4}} \cdot 3^{3}$.
18. $a^{-2} \cdot a^{-3}$.
19. $b^{-5} \cdot b^{-7}$.
20. $c^{7} \cdot c^{-7}$.
21. $d^{-\frac{1}{2}} \cdot d^{-2}$.
22. $g^{3} \cdot g^{-\frac{1}{3}}$.
23. $h^{6} \cdot h^{-6}$.
24. $-k^{3} \cdot k^{0}$.
25. $2\,a^{-5} \cdot 5\,a^{7}$.
26. $3\,b^{-8} \cdot 6\,b^{-3}$.
27. $-5\,c^{-9} \cdot 8\,c^{-4}$.
28. $7\,d^{-\frac{1}{2}} \cdot 9\,d^{\frac{1}{3}}$.
29. $11\,k^{-6} \cdot 8\,k^{-\frac{6}{a}}$.
30. $12\,g^{-\frac{2}{3}} \cdot 3\,g^{\frac{3}{2}}$.
31. $a^{\frac{1}{2}} \cdot a^{\frac{1}{3}} \cdot a^{\frac{1}{4}}$.
32. $b^{\frac{1}{4}} \cdot b^{\frac{1}{5}} \cdot b^{-1}$.
33. $c^{\frac{2}{3}} \cdot c^{-2} \cdot c^{-3}$.
34. $d^{\frac{4}{5}} \cdot d^{-\frac{1}{4}} \cdot d^{-\frac{1}{5}}$.
35. $m^{a} \cdot m^{-b} \cdot m^{c}$.
36. $n^{x} \cdot n^{-y} \cdot n^{-z}$.
37. $x^{\frac{a}{2}} \cdot x^{\frac{b}{3}} \cdot x^{-1}$.
38. $y^{2n} \cdot y^{-3n} \cdot y^{-n}$.
39. $z^{-3} \cdot z^{-\frac{1}{3}} \cdot z$.
40. $w^{\frac{a}{2}} \cdot w^{-\frac{2}{a}} \cdot w^{\frac{1}{2a}}$.
41. $a^{x-y} \cdot a^{y-z} \cdot a^{z-x}$.
42. $x^{\frac{a}{2}} \cdot y^{\frac{b}{4}} \cdot z^{\frac{c}{6}}$.
43. $3\,y^{\frac{m}{3}} \cdot 5\,y^{\frac{m}{5}}$.
44. $a^{5} \div a^{-7}$.
45. $b^{-8} \div b^{-3}$.
46. $c^{-2} \div c^{-9}$.
47. $-d^{\frac{1}{2}} \div d^{\frac{1}{3}}$.
48. $h^{\frac{2}{3}} \div h^{\frac{1}{2}}$.
49. $k^{-\frac{4}{5}} \div k^{-2}$.
50. $-m^{-3} \div m$.
51. $n^{a-3} \div n^{-3}$.
52. $a^{x-y} \div a^{y-x}$.
53. $8\,x^{\frac{1}{3}} \div (-2\,x^{-1})$.
54. $12\,y^{-\frac{5}{6}} \div 4\,y^{\frac{1}{2}}$.
55. $-16\,z^{a-3} \div 2\,z^{-3}$.
56. $4\,a \div a^{-4}$.
57. $-5\,b \div b^{-5}$.
58. $-c \div c^{-1}$.
59. $-2\,d^{-3} \div (-4\,d^{-5})$.
60. $\dfrac{-a^{-2}}{-a^{-5}}$.

POWERS OF POWERS

EXERCISE 134

Find the values of the following expressions:

1. $(2^{-1})^{2}$.
2. $(3^{2})^{-1}$.
3. $(5^{-2})^{-2}$.
4. $(4^{\frac{1}{2}})^{-2}$.
5. $(5^{\frac{1}{3}})^{6}$.
6. $(6^{4})^{-\frac{1}{2}}$.
7. $(7^{-2})^{-1}$.
8. $-(8^{\frac{1}{3}})^{2}$.

9. $(9^{-\frac{1}{3}})^{\frac{3}{2}}$.
10. $(16^{-3})^{-\frac{1}{4}}$.
11. $(25^{-\frac{3}{2}})^{-\frac{1}{3}}$.
12. $(36^{-\frac{5}{8}})^{-\frac{4}{5}}$.
13. $(49^{\frac{3}{10}})^{-\frac{5}{3}}$.
14. $(27^{-\frac{4}{3}})^{\frac{1}{2}}$.
15. $(32^{-2})^{\frac{2}{5}}$.
16. $(81^{3})^{-\frac{1}{4}}$.
17. $(125^{-\frac{1}{2}})^{-\frac{2}{3}}$.
18. $(64^{-5})^{-\frac{1}{6}}$.
19. $-(100^{7})^{-\frac{1}{7}}$.
20. $((2^{\frac{1}{2}})^{\frac{1}{3}})^{12}$.
21. $((3^{\frac{1}{2}})^{\frac{1}{5}})^{20}$.
22. $(c^{2})^{-3}$.
23. $(d^{-4})^{2}$.
24. $(a^{-2})^{-4}$.
25. $(b^{-5})^{-3}$.
26. $(x^{-3})^{0}$.
27. $(y^{0})^{-2}$.
28. $(z^{0})^{0}$.
29. $(w^{\frac{1}{2}})^{0}$.
30. $(g^{10})^{\frac{1}{2}}$.
31. $(h^{\frac{1}{3}})^{12}$.
32. $(k^{-6})^{\frac{1}{3}}$.
33. $-(m^{8})^{\frac{1}{4}}$.
34. $(n^{-\frac{2}{3}})^{\frac{1}{2}}$.
35. $-(t^{\frac{3}{4}})^{\frac{1}{3}}$.
36. $(x^{\frac{4}{5}})^{\frac{3}{2}}$.
37. $(y^{-\frac{1}{2}})^{-4}$.
38. $(z^{-\frac{2}{5}})^{-5}$.
39. $-(w^{\frac{1}{2}})^{\frac{1}{3}}$.
40. $-(m^{-1})^{-1}$.
41. $-(m^{-2a})^{3}$.
42. $(h^{2r})^{2}$.
43. $(k^{3n})^{2}$.
44. $(d^{-5x})^{2}$.
45. $(n^{4r})^{-5}$.
46. $-(q^{2n})^{-4}$.
47. $(x^{3a})^{-6}$.
48. $(a^{2})^{b}$.
49. $-(b^{3})^{c}$.
50. $(c^{4})^{-d}$.
51. $(d^{5})^{-x}$.
52. $(h^{-2})^{y}$.
53. $(k^{-3})^{-z}$.
54. $-(m^{2})^{3a}$.
55. $(n^{3})^{2b}$.
56. $(x^{-1})^{-3x}$.
57. $(a^{b})^{c}$.
58. $-(l^{x})^{y}$.
59. $(c^{m})^{-y}$.
60. $(d^{-r})^{z}$.
61. $(h^{-m})^{-n}$.
62. $(k^{a})^{\frac{1}{b}}$.
63. $(m^{\frac{1}{n}})^{m}$.
64. $(n^{\frac{1}{x}})^{\frac{1}{y}}$.
65. $(x^{a})^{-\frac{1}{n}}$.
66. $(y^{-m})^{\frac{1}{r}}$.
67. $(z^{-\frac{1}{a}})^{\frac{1}{b}}$.
68. $(b^{3x})^{-\frac{1}{6}}$.
69. $(a^{b})^{b}$.
70. $(b^{n})^{n}$.
71. $(c^{-m})^{m}$.
72. $(d^{r})^{-r}$.
73. $(x^{-a})^{-a}$.
74. $(y^{2x})^{x}$.
75. $-(z^{a})^{3a}$.
76. $(w^{4n})^{-n}$.
77. $(a^{2x})^{2x}$.
78. $(b^{-3x})^{3x}$.
79. $(c^{-4n})^{-4n}$.
80. $((x^{a})^{b})^{c}$.
81. $((y^{m})^{-n})^{-r}$.
82. $((a^{m})^{-\frac{1}{n}})^{-n}$.
83. $(a^{m+1})^{2}$.
84. $(b^{n-2})^{3}$.
85. $(d^{2})^{n+1}$.
86. $(h^{3})^{r-1}$.
87. $(k^{4})^{s+2}$.
88. $(m^{r+1})^{\frac{1}{2}}$.
89. $(x^{a+b})^{2}$.
90. $(z^{2n+1})^{3}$.
91. $(w^{3r-2})^{4}$.
92. $(m^{5a-b})^{-1}$.
93. $(a^{4x+2})^{\frac{1}{2}}$.
94. $(b^{6x-9})^{\frac{1}{3}}$.
95. $(h^{\frac{1}{3}})^{6r+3}$.
96. $(k^{a+2})^{a-2}$.
97. $(a^{m+n})^{m-n}$.
98. $(x^{a+1})^{a+2}$.
99. $(y^{b+3})^{b+5}$.
100. $(z^{m-2})^{m+3}$.

POWERS OF PRODUCTS AND POWERS OF QUOTIENTS

MENTAL EXERCISE 135

Simplify each of the following expressions applying either the index law $(ab)^n \equiv a^n b^n$, or the index law $(a \div b)^n \equiv a^n \div b^n$:

1. $(a^{\frac{1}{2}}b^3)^4$.
2. $(b^5c^{-2})^3$.
3. $(c^{-1}d^4)^{-2}$.
4. $(d^{-2}h^{-3})^5$.
5. $(m^5n^{-4})^{-1}$.
6. $(x^{-2}y^{-6})^{-2}$.
7. $-(a^{\frac{1}{2}}x^2)^4$.
8. $-(b^3y^{\frac{1}{3}})^6$.
9. $-(c^{-\frac{1}{3}}z^4)^{\frac{1}{2}}$.
10. $(d^5w^{-\frac{1}{5}})^{-10}$.
11. $(a^{-\frac{1}{3}}n^{-\frac{1}{4}})^{\frac{1}{5}}$.
12. $(b^{\frac{1}{6}}x^{-\frac{1}{4}})^{-8}$.
13. $(a^{\frac{1}{3}}b^{\frac{1}{2}})^6$.
14. $(b^{\frac{1}{4}}c^{\frac{1}{6}})^{12}$.
15. $(c^{-2}d^{-\frac{1}{4}})^{-\frac{1}{2}}$.
16. $(k^{-\frac{2}{3}}z^{\frac{3}{2}})^{-\frac{1}{6}}$.
17. $(a^{-2} \div b^3)^2$.
18. $(b^5 \div c^4)^{-1}$.
19. $(c^{-2} \div d^3)^{-2}$.
20. $(a^{-3} \div x^4)^{-3}$.
21. $(b^{-1} \div c^{-2})^3$.
22. $(c^{-2} \div y^{-1})^{-4}$.
23. $(a^{-1} \div b^{-1})^{-1}$.
24. $(a^{\frac{1}{4}} \div b^{\frac{1}{6}})^2$.
25. $-(b^{\frac{1}{3}} \div x^{\frac{1}{2}})^6$.
26. $(c^{\frac{1}{4}} \div y^{\frac{1}{2}})^{-4}$.
27. $(d^{-\frac{1}{3}} \div q)^{-3}$.
28. $(h \div m^{-2})^{-\frac{1}{2}}$.
29. $(a^{2b} \div x^{-1})^{\frac{1}{2}}$.
30. $(b^{-3c} \div d^{\frac{1}{3}})^{\frac{1}{3}}$.
31. $(2\,a)^{-2}$.
32. $(4\,b^{-1})^3$.
33. $(2\,c^{-2})^3$.
34. $(3\,d^3)^{-2}$.
35. $(4\,x)^{\frac{1}{2}}$.
36. $-(9\,y^2)^{\frac{1}{2}}$.
37. $(16\,z^2)^{-\frac{1}{2}}$.
38. $(25\,z^{-4})^{-\frac{1}{2}}$.
39. $(36^{-1}ab^{-2})^{-\frac{1}{2}}$.
40. $(9^{-3}m^{-4}n^{-6})^{-\frac{1}{2}}$.

PRODUCTS AND QUOTIENTS OF BINOMIALS WHOSE TERMS HAVE FRACTIONAL OR NEGATIVE EXPONENTS

EXERCISE 136

Obtain each of the following powers and products of binomials:

1. $(a^{-1} + 2)^2$.
2. $(b^{-2} + 3)^2$.
3. $(c^{-4} - 4)^2$.
4. $(2d^{-2} + 3)^2$.
5. $(3\,h^{-3} - 5)^2$.
6. $(x^{-1} - y^{-1})^2$.
7. $(a^{-2} - b^{-3})^2$.
8. $(3x^{-3} - 4y^{-4})^2$.
9. $(a^{-x} + b^{-y})^2$.
10. $(x^{-a} - y^{-b})^2$.
11. $(x^{-3} + 4)(x^{-3} + 7)$.
12. $(a^{-1} + 2)(a^{-1} + 3)$.
13. $(b^{-2} - 5)(b^{-2} - 4)$.
14. $(c^{-3} + 6)(c^{-3} + 1)$.
15. $(d^{-4} - 7)(d^{-4} + 3)$.
16. $(m^{-5} - 5)(m^{-5} - 8)$.
17. $(n^{-2} - 11)(n^{-2} + 4)$.
18. $(x^{-6} - 9)(x^{-6} + 2)$.
19. $(y^{-8} - 8)(y^{-8} + 7)$.
20. $(z^{-5} + 10)(z^{-5} - 6)$.
21. $(w^{-4} - 9)(w^{-4} - 5)$.

22. $(a^{\frac{1}{2}}+1)^2$.
23. $(a^{\frac{1}{3}}+1)^2$.
24. $(b^{\frac{1}{5}}+1)^2$.
25. $(c^{\frac{1}{4}}+1)^2$.
26. $(d^{\frac{2}{3}}+3)^2$.
27. $(m^{\frac{3}{4}}+4)^2$.
28. $(n^{\frac{1}{2}}-2)^2$.
29. $p^{\frac{1}{3}}-4)^2$.
30. $(x^{\frac{3}{4}}+3)^2$.
31. $(y^{\frac{2}{5}}-5)^2$.
32. $(2a^{\frac{1}{3}}-3)^2$.
33. $(c^{\frac{1}{2}}-d)^2$.
34. $(5x^{\frac{1}{3}}-y^{\frac{1}{3}})^2$.
35. $(b^{\frac{2}{3}}+c^{\frac{1}{5}})(b^{\frac{2}{3}}-c^{\frac{1}{5}})$.
36. $(a^{\frac{1}{2}}+1)(a^{\frac{1}{2}}-1)$.
37. $(b^{\frac{1}{3}}+2)(b^{\frac{1}{3}}-2)$.
38. $(c^{\frac{1}{4}}+5)(c^{\frac{1}{4}}-5)$.
39. $(d^{\frac{2}{3}}+6)(d^{\frac{2}{3}}-6)$.
40. $(2m^{\frac{1}{3}}+7)(2m^{\frac{1}{3}}-7)$.
41. $(x^{\frac{1}{3}}+5)(x^{\frac{1}{3}}+8)$.
42. $(a^{\frac{1}{2}}+1)(a^{\frac{1}{2}}+2)$.
43. $(b^{\frac{1}{3}}+4)(b^{\frac{1}{3}}+3)$.
44. $(c^{\frac{1}{5}}+6)(c^{\frac{1}{5}}+5)$.
45. $(d^{\frac{1}{7}}+8)(d^{\frac{1}{7}}-3)$.
46. $(m^{\frac{1}{2}}-7)(m^{\frac{1}{2}}-2)$.
47. $(n^{\frac{1}{3}}-10)(n^{\frac{1}{3}}+4)$.
48. $(x^{\frac{1}{4}}+11)(x^{\frac{1}{4}}-3)$.
49. $(y^{\frac{1}{7}}-9)(y^{\frac{1}{7}}-4)$.
50. $\dfrac{x-y}{x^{\frac{1}{2}}-y^{\frac{1}{2}}}$.
51. $\dfrac{m^{\frac{2}{5}}-n^{\frac{2}{5}}}{m^{\frac{1}{5}}+n^{\frac{1}{5}}}$.
52. $\dfrac{a-b}{a^{\frac{1}{2}}+b^{\frac{1}{2}}}$.
53. $\dfrac{c-d}{c^{\frac{1}{2}}-d^{\frac{1}{2}}}$.
54. $\dfrac{z-w}{z^{\frac{1}{3}}-w^{\frac{1}{3}}}$.
55. $\dfrac{m-n}{m^{\frac{1}{4}}-n^{\frac{1}{4}}}$.
56. $\dfrac{a^{-2}-b^{-2}}{a^{-1}-b^{-1}}$.
57. $\dfrac{c^{-3}+d^{-3}}{c^{-1}+d^{-1}}$.
58. $\dfrac{x^{-3}-y^{-3}}{x^{-1}-y^{-1}}$.

EXERCISE 137

Simplify the following expressions:

Multiply

1. $a^{\frac{3}{4}}+b^{\frac{5}{6}}$ by $a^{\frac{1}{4}}+b^{\frac{2}{3}}$.
2. $x^{\frac{1}{2}}-y^{\frac{1}{3}}$ by $x^{\frac{1}{3}}-y^{\frac{1}{2}}$.
3. $x^{\frac{1}{3}}-y^{\frac{3}{2}}$ by $x^{\frac{2}{3}}-y^{-\frac{1}{2}}$.
4. $x^{-\frac{1}{3}}+y^{-\frac{1}{2}}$ by $x^{\frac{4}{3}}-y^{\frac{2}{3}}$.
5. $a^{-\frac{1}{6}}-b^{\frac{4}{5}}$ by $a^{-\frac{5}{6}}-b^{\frac{1}{5}}$.
6. $b^{-\frac{1}{4}}+c^{\frac{3}{2}}$ by $b^{\frac{5}{4}}+c^{-\frac{1}{2}}$.
7. $a^{\frac{4}{5}}+x^{\frac{1}{3}}$ by $a^{\frac{1}{5}}+x^{\frac{2}{3}}$.
8. $x^{-3}-3x^{-2}+1$ by $x^{-4}+2x^{-1}$.

9. $2x^{\frac{3}{4}} - 3x^{\frac{1}{4}} - 4 + x^{-\frac{1}{4}}$ by $3x^{\frac{5}{4}} + x - 2x^{\frac{3}{4}}$.

10. $a^{-3}b^{-2} + 2a^{-1}b^{2} - 3a^{2}b^{-4}$ by $a^{2}b^{-3} + 3a^{4}b$.

Divide

11. $x^{-5} - 3x^{-3} + 2x^{-2}$ by $x^{-2} - 2x^{-1} + 1$.

12. $a^{\frac{5}{4}} - a^{\frac{1}{2}}b^{\frac{5}{6}} - a^{\frac{3}{4}}b^{\frac{2}{3}} + b^{\frac{3}{2}}$ by $a^{\frac{1}{2}} - b^{\frac{2}{3}}$.

13. $x - y$ by $x^{\frac{1}{2}} - y^{\frac{1}{2}}$.

14. $x^{-6} - 3x^{-4}y^{-2} + 3x^{-2}y^{-4} - y^{-6}$ by $x^{-2} - y^{-2}$.

15. $a + b$ by $a^{\frac{1}{3}} + b^{\frac{1}{3}}$.

16. $x^{-1}y - 5xy^{-1} + 4x^{3}y^{-3}$ by $x^{2}y^{-1} + x^{3}y^{-2} - 2x^{4}y^{-3}$.

17. $a - b$ by $a^{\frac{1}{4}} - b^{\frac{1}{4}}$.

18. $2a^{-\frac{8}{3}}b^{3} - 4a^{-\frac{4}{3}}b^{\frac{3}{2}} + 2$ by $2a^{-2}b^{\frac{9}{4}} - 4a^{-\frac{4}{3}}b^{\frac{3}{2}} + 2a^{-\frac{2}{3}}b^{\frac{3}{4}}$.

EXPONENTIAL EQUATIONS

EXERCISE 138

Solve the following exponential equations for x:

1. $3^{x} = 81$.
2. $2^{x} = 128$.
3. $2^{3x} = 8$.
4. $2^{2x} = 16$.
5. $3^{4x} = 81$.
6. $2^{3x} = 64$.
7. $2^{7x} = 128$.
8. $3^{4x} = 9$.
9. $5^{6x} = 25$.
10. $6^{3x} = 36$.
11. $4^{8x} = 256$.
12. $8^{5x} = 4$.
13. $25^{5x} = 125$.
14. $16^{7x} = 128$.
15. $3^{x} = \frac{1}{9}$.
16. $2^{x} = \frac{1}{8}$.
17. $2^{4x} = \frac{1}{16}$.
18. $3^{3x} = \frac{1}{27}$.
19. $8^{8x} = \frac{1}{8}$.
20. $5^{\frac{x}{2}} = 125$.
21. $3^{\frac{x}{5}} = 243$.
22. $3^{\frac{8}{x}} = 81$.
23. $128^{\frac{2}{x}} = 32$.
24. $2^{\frac{2x}{3}} = 64$.
25. $4^{\frac{3x}{2}} = 32$.
26. $9^{\frac{2x}{3}} = \frac{1}{27}$.
27. $16^{3x} = \frac{1}{8}$.
28. $2^{x+1} = 16$.
29. $3^{x-1} = 27$.
30. $2^{2x+1} = 128$.
31. $5^{2x-1} = 125$.
32. $2^{5x-1} = 512$.
33. $6^{2x-3} = 216$.
34. $2^{6x+1} = 8$.
35. $3^{3x+1} = 27$.
36. $4^{3x+2} = 128$.
37. $3^{3x-2} = 9^{x}$.
38. $2^{3x} = 4^{x+1}$.
39. $5^{4x+1} = 125^{x+1}$.
40. $9^{2x} = 27^{x+1}$.
41. $4^{3x} = 32^{x+2}$.
42. $16^{2x-1} = 64^{x+1}$.
43. $8^{3x-2} = 16^{2x-3}$.
44. $27^{2x-5} = 81^{x-3}$.
45. $64^{3x-1} = 128^{2x+2}$.

EXERCISE 139

Miscellaneous

Simplify each of the following expressions:

1. $4^{\frac{1}{2}} \times 16^{-\frac{3}{4}} \times 64^{\frac{1}{3}}$.

2. $\dfrac{16^{\frac{1}{2}} \times 125^{-\frac{1}{3}}}{25^{\frac{1}{2}} \times 32^{\frac{2}{5}}}$.

3. $9^0 + 9^{\frac{1}{2}} - 9^{-\frac{1}{2}} + 9^{\frac{3}{2}}$.

4. $32^0 + 32^{\frac{1}{5}} + 32^{\frac{2}{5}} + 32^{\frac{3}{5}} + 32^{\frac{4}{5}} + 32$.

5. $2(a^{-6})^{-4} - (a^{-12})^{-2} - (a^{-8})^{-3}$.

6. $\dfrac{2^{n+4} - 2 \cdot 2^n}{2 \cdot 2^{n+3}}$.

7. $\dfrac{4(2^{n-1})^n \cdot 2^{-1}}{(2^{n+1})^{n-1} \cdot 2^{-n}}$.

8. $2^{\frac{2}{3}} \cdot 6^{\frac{1}{3}} - 9^{-\frac{2}{3}} \cdot 3^{\frac{5}{3}} - \dfrac{576^{\frac{1}{6}}}{2}$.

9. $\left(\dfrac{1}{a+b}\right)^{-2}$.

10. $(x \div x^{\frac{1}{a}})^a$.

11. $\left(\dfrac{1}{1 \div a^{\frac{5}{4}}}\right)^{\frac{4}{5}}$.

12. $\left(\dfrac{a^0 + b^0}{2}\right)^2$.

13. $\dfrac{a^{-1} - b^{-1}}{a^{-1}b^{-1}}$.

14. $[(x^a)^{a-\frac{1}{a}}]^{\frac{1}{a-1}}$.

15. $(x^{\frac{a}{a-1}})^{a^2-1} \div \dfrac{(x^{\frac{n}{2}})^2}{(x^2)^{\frac{1}{2}}}$.

16. $\dfrac{3^n \times (3^{n-1})^n}{3^{n+1} \times 3^{n-1} \times 9^{-n}}$.

17. $\dfrac{1 - a^{-2}b^2}{a^{-1} - a^{-2}b} \times \dfrac{a^{-1}b^{-1}}{a^{-1} + b^{-1}}$.

18. $\left(\dfrac{a^{-1} - b^{-1}}{a^{-1}b - ab^{-1}}\right)\left(\dfrac{b - a}{ab^{-1}}\right) \div \left(\dfrac{a^{-1}b^{-1}}{1 + ab^{-1}}\right)$.

SURDS

REDUCTION TO SIMPLEST FORM

MENTAL EXERCISE 140

Reduce each of the following surds to simplest form:

1. $\sqrt{8}$.
2. $\sqrt{12}$.
3. $\sqrt{20}$.
4. $\sqrt{28}$.
5. $\sqrt{40}$.
6. $\sqrt{52}$.
7. $\sqrt{60}$.
8. $3\sqrt{24}$.
9. $5\sqrt{44}$.
10. $6\sqrt{56}$.
11. $\sqrt{18}$.
12. $\sqrt{45}$.

13. $\sqrt{54}$.
14. $\sqrt{90}$.
15. $\sqrt{99}$.
16. $2\sqrt{27}$.
17. $3\sqrt{63}$.
18. $\sqrt{32}$.
19. $\sqrt{48}$.
20. $\sqrt{80}$.
21. $2\sqrt{96}$.
22. $3\sqrt{160}$.
23. $\sqrt{75}$.
24. $\sqrt{125}$.
25. $3\sqrt{50}$.
26. $\sqrt{175}$.
27. $4\sqrt{250}$.
28. $\sqrt{72}$.
29. $\sqrt{108}$.
30. $2\sqrt{180}$.
31. $3\sqrt{98}$.
32. $5\sqrt{128}$.
33. $6\sqrt{162}$.
34. $7\sqrt{200}$.
35. $\sqrt[3]{16}$.
36. $\sqrt[3]{24}$.
37. $\sqrt[3]{32}$.
38. $6\sqrt[3]{40}$.
39. $3\sqrt[3]{48}$.
40. $5\sqrt[3]{56}$.
41. $4\sqrt[3]{72}$.
42. $7\sqrt[3]{80}$.
43. $8\sqrt[3]{88}$.
44. $\sqrt[3]{54}$.
45. $4\sqrt[3]{81}$.
46. $2\sqrt[3]{108}$.
47. $\sqrt[4]{32}$.
48. $5\sqrt[4]{80}$.
49. $3\sqrt[4]{96}$.
50. $\sqrt[5]{64}$.
51. $\sqrt[5]{96}$.
52. $3\sqrt{242}$.

53. $4\sqrt{288}$.
54. $10\sqrt{1000}$.
55. $3\sqrt[3]{250}$.
56. $5\sqrt[3]{600}$.
57. $9\sqrt[3]{128}$.
58. $6\sqrt{500}$.
59. $12\sqrt{700}$.
60. $5\sqrt{243}$.
61. $4\sqrt{343}$.
62. $2\sqrt{512}$.
63. $3\sqrt{450}$.
64. $3\sqrt[3]{375}$.
65. $\sqrt{a^3}$.
66. $\sqrt{b^5}$.
67. $\sqrt{c^7}$.
68. $a\sqrt{b^3}$.
69. $b\sqrt{c^5}$.
70. $c\sqrt{c^3}$.
71. $d\sqrt{d^5}$.
72. $\sqrt{x^2y^3}$.
73. $\sqrt{ab^3}$.
74. $\sqrt{c^3d^3}$.
75. $x\sqrt{y^2z^9}$.
76. $m\sqrt{m^2n}$.
77. $\sqrt{h^5x^4}$.
78. $\sqrt{b^7y^6}$.
79. $\sqrt{m^8n^7}$.
80. $\sqrt{a^2b^3c^4}$.
81. $x\sqrt{x^4yz^3}$.
82. $\sqrt{4\,a^5b}$.
83. $\sqrt{9\,c^3d^2}$.
84. $5\,c\sqrt{8\,ab^2}$.
85. $3\,b\sqrt{12\,c^3d^4}$.
86. $4\,a\sqrt{27\,a^4b^3c^2}$.
87. $3\,c\sqrt[3]{48\,a^5b^3c^5}$.
88. $2\sqrt[4]{32\,g^6h^4k^5}$.
89. $\sqrt{(a+b)^2c}$.
90. $\sqrt{(b-c)^2b^3c^3}$.
91. $\sqrt{ac^2+bc^2}$.
92. $\sqrt[m]{a^mb}$.
93. $\sqrt[n]{b^{n+1}}$.
94. $\sqrt[r-1]{c^r}$.
95. $\sqrt[n+1]{a^{n+2}}$.
96. $\sqrt[n-2]{b^{n-1}}$.
97. $\sqrt[n]{c^{2n}d}$.
98. $\sqrt[2n]{x^{2n}y^{4n}z}$.
99. $\sqrt[n-r]{a^{n-r+1}}$, $n>r$.
100. $\sqrt[r]{a^{r+1}b^{r+2}c^{r+3}}$, $r>1$.

EXERCISE 141

Reduce each of the following surds to simplest form:

1. $\sqrt{\frac{1}{2}}$.
2. $\sqrt{\frac{1}{3}}$.
3. $\sqrt{\frac{1}{5}}$.
4. $5\sqrt{\frac{1}{6}}$.
5. $4\sqrt{\frac{1}{7}}$.
6. $11\sqrt{\frac{1}{11}}$.
7. $\sqrt{\frac{1}{8}}$.
8. $\sqrt{\frac{1}{12}}$.

9. $\sqrt{\frac{1}{18}}$.
10. $30\sqrt{\frac{1}{20}}$.
11. $13\sqrt{\frac{1}{13}}$.
12. $4\sqrt{\frac{5}{8}}$.
13. $\frac{1}{2}\sqrt{\frac{1}{3}}$.
14. $\frac{1}{4}\sqrt{\frac{1}{6}}$.
15. $\frac{1}{8}\sqrt{\frac{1}{8}}$.
16. $\frac{1}{9}\sqrt{\frac{1}{27}}$.
17. $\sqrt[3]{\frac{1}{2}}$.
18. $\sqrt[3]{\frac{1}{3}}$.
19. $\sqrt[3]{\frac{1}{5}}$.
20. $\sqrt[3]{\frac{1}{6}}$.
21. $\sqrt[3]{\frac{1}{4}}$.
22. $3\sqrt[3]{\frac{1}{9}}$.
23. $5\sqrt[3]{\frac{1}{25}}$.
24. $4\sqrt[3]{\frac{1}{16}}$.
25. $\frac{1}{9}\sqrt[3]{\frac{1}{9}}$.
26. $\frac{1}{10}\sqrt[3]{\frac{1}{10}}$.
27. $12\sqrt[3]{\frac{1}{36}}$.
28. $\frac{1}{2}\sqrt[4]{\frac{1}{2}}$.
29. $9\sqrt[4]{\frac{1}{3}}$.
30. $4\sqrt[4]{\frac{1}{8}}$.
31. $8\sqrt[5]{\frac{1}{4}}$.
32. $\sqrt[5]{\frac{1}{16}}$.
33. $\sqrt[6]{\frac{7}{32}}$.
34. $\sqrt{\frac{4}{7}}$.
35. $\sqrt{\frac{9}{10}}$.
36. $\sqrt{\frac{16}{3}}$.
37. $\sqrt{\frac{25}{32}}$.
38. $\frac{1}{6}\sqrt{\frac{36}{5}}$.
39. $\sqrt[3]{\frac{8}{9}}$.
40. $\sqrt[3]{\frac{27}{16}}$.
41. $\sqrt[3]{\frac{125}{36}}$.
42. $\frac{1}{3}\sqrt[3]{\frac{27}{4}}$.
43. $\sqrt[3]{\frac{-5}{4}}$.
44. $\sqrt[5]{-\frac{32}{81}}$.

45. $-7\sqrt[3]{-\frac{1}{49}}$.
46. $\sqrt{\frac{1}{a}}$.
47. $a\sqrt{\frac{1}{x}}$.
48. $\sqrt{\frac{1}{c^3}}$.
49. $\sqrt{\frac{1}{d^5}}$.
50. $\sqrt{\frac{1}{h^7}}$.
51. $\sqrt[3]{\frac{1}{m}}$.
52. $\sqrt[3]{\frac{1}{x^2}}$.
53. $\frac{1}{x}\sqrt[3]{\frac{1}{y^7}}$.
54. $\frac{1}{z}\sqrt[3]{\frac{1}{z^4}}$.
55. $\sqrt[4]{\frac{1}{b}}$.
56. $\sqrt[4]{\frac{1}{c^3}}$.
57. $\sqrt[5]{\frac{1}{d}}$.
58. $\sqrt[5]{\frac{1}{d^2}}$.
59. $\sqrt{\frac{4a}{b}}$.
60. $\sqrt{\frac{c}{9d}}$.
61. $\sqrt{\frac{16}{25x}}$.
62. $\sqrt{\frac{a^2}{x}}$.
63. $\sqrt{\frac{c^3}{z}}$.
64. $\sqrt{\frac{a}{bc}}$.
65. $\sqrt[3]{\frac{ab}{cd}}$.
66. $\sqrt{\frac{m}{n^2x}}$.
67. $\sqrt{\frac{a^2}{bc^2}}$.
68. $\sqrt{\frac{2}{3x^2y}}$.
69. $\sqrt[3]{\frac{a}{b^2c}}$.
70. $\sqrt{\frac{1}{9a^2b}}$.
71. $\sqrt{\frac{c^3}{d^2x}}$.
72. $\sqrt[3]{\frac{1}{a^2b}}$.
73. $ab\sqrt{\frac{b}{a}}$.
74. $\frac{a}{b}\sqrt{\frac{c}{d}}$.
75. $\frac{x}{y}\sqrt{\frac{1}{xy}}$.
76. $\frac{a^2}{b}\sqrt{\frac{c}{a^3}}$.
77. $\frac{b}{a}\sqrt[3]{\frac{a}{b}}$.

78. $\frac{x^2}{y^2}\sqrt[3]{\frac{y}{x}}$.

79. $\sqrt[4]{\frac{5}{8x^3y^4}}$.

80. $\sqrt{\frac{1}{a+b}}$.

81. $\sqrt{\frac{a+1}{a+2}}$.

82. $\sqrt{\frac{x-3}{x+1}}$.

83. $\sqrt[n]{\frac{1}{a}}$.

84. $\sqrt[x]{\frac{1}{b^2}}$, $x>2$.

85. $\sqrt[n+1]{\frac{1}{a^n}}$.

86. $\sqrt[n]{\frac{1}{a^{n-1}}}$.

87. $\sqrt[n]{\frac{1}{x^{n-2}}}$, $n>2$.

88. $\sqrt[m]{\frac{1}{a^n}}$, $m>n$.

89. $\sqrt[2n]{\frac{1}{x^n}}$.

90. $\sqrt[3n]{\frac{1}{a^{2n}}}$.

91. $\sqrt[a+2]{\frac{1}{x^a}}$.

92. $\sqrt{\frac{1}{x^{2n+1}}}$.

93. $\sqrt[2n]{\frac{1}{a^{n+2}}}$.

94. $\sqrt[3]{\frac{1}{w^{3n+1}}}$.

95. $\sqrt[3]{\frac{1}{y^{3n+2}}}$.

96. $n\sqrt[n]{\frac{1}{n^{n-1}}}$.

97. $\sqrt[n-2]{\frac{1}{x^{n-3}}}$.

98. $x\sqrt[n]{\frac{y^n}{x^{n-1}}}$.

99. $\frac{1}{x}\sqrt[a]{\frac{1}{x^{a-b}}}$.

100. $\sqrt[8]{\frac{1}{x^n}}$, $n<8$.

101. $\sqrt[6]{\frac{1}{y^{n-1}}}$, $n<7$.

MENTAL EXERCISE 142

Simplify each of the following:

1. $\sqrt[4]{25}$.
2. $\sqrt[4]{36}$.
3. $\sqrt[6]{4}$.
4. $\sqrt[6]{9}$.
5. $\sqrt[9]{27}$.
6. $\sqrt[8]{16}$.
7. $\sqrt[6]{a^3b^3}$.
8. $\sqrt[4]{a^2b^2}$.
9. $\sqrt[4]{49\,a^4}$.
10. $\sqrt[4]{x^2y^4}$.
11. $\sqrt[6]{125\,a^3}$.
12. $\sqrt[8]{a^2b^4c^6}$.
13. $\sqrt[10]{x^5y^5}$.
14. $\sqrt[10]{100\,a^2}$.
15. $\sqrt[14]{a^7b^{14}}$.
16. $\sqrt[10]{a^6b^8}$.
17. $\sqrt[8]{x^6y^4z^2}$.
18. $\sqrt[8]{16\,a^4x^4}$.
19. $\sqrt[12]{a^3b^6c^9}$.
20. $\sqrt[12]{x^4y^6z^8}$.
21. $\sqrt[12]{m^6n^6}$.
22. $\sqrt[2n]{a^2}$.
23. $\sqrt[3n]{b^3}$.
24. $\sqrt[4n]{c^2}$.
25. $\sqrt[3a]{x^3}$.
26. $\sqrt[5n]{y^n}$.
27. $\sqrt[ax]{y^x}$.
28. $\sqrt[nr]{z^n}$.
29. $\sqrt[an]{x^{ab}y^{ac}}$.
30. $\sqrt[2n]{x^{2r}y^{2s}}$.
31. $\sqrt[4]{\frac{1}{4}}$.
32. $\sqrt[4]{\frac{1}{9}}$.
33. $\sqrt[4]{\frac{1}{25}}$.
34. $\sqrt[4]{\frac{1}{a^2}}$.
35. $\sqrt[6]{\frac{1}{c^3}}$.
36. $\sqrt[8]{\frac{1}{d^4}}$.
37. $\sqrt[2n]{\frac{1}{a^n}}$.
38. $\sqrt[3n]{\frac{1}{b^n}}$.
39. $\sqrt[4n]{\frac{1}{c^{3n}}}$.
40. $\sqrt[6n]{\frac{1}{d^{4n}}}$.

ADDITION AND SUBTRACTION OF SURDS

EXERCISE 143

Simplify each of the following:

1. $\sqrt{3}+4\sqrt{3}$.
2. $2\sqrt{5}+3\sqrt{5}$.
3. $6\sqrt{7}-2\sqrt{7}$.
4. $8\sqrt{10}-9\sqrt{10}$.
5. $4\sqrt{13}-7\sqrt{13}$.
6. $\sqrt{3}+\sqrt{12}$.
7. $\sqrt{2}+\sqrt{18}$.
8. $\sqrt{5}+\sqrt{20}$.
9. $2\sqrt{6}+\sqrt{24}$.
10. $3\sqrt{7}+\sqrt{28}$.
11. $4\sqrt{3}+2\sqrt{48}$.
12. $\sqrt{45}-2\sqrt{5}$.
13. $2\sqrt{72}-5\sqrt{2}$.
14. $9\sqrt{6}-4\sqrt{54}$.
15. $\sqrt{50}+\sqrt{8}$.
16. $\sqrt{75}+\sqrt{27}$.
17. $3\sqrt{80}-2\sqrt{20}$.
18. $2\sqrt{28}-3\sqrt{63}$.
19. $5\sqrt{72}-2\sqrt{32}$.
20. $3\sqrt{125}-4\sqrt{80}$.
21. $3\sqrt[3]{16}+\sqrt[3]{54}$.
22. $\sqrt[3]{24}-\sqrt[3]{81}$.
23. $5\sqrt[3]{108}-\sqrt[3]{32}$.
24. $5\sqrt[4]{2}-\sqrt[4]{32}$.
25. $\sqrt[4]{3}+\sqrt[4]{243}$.
26. $\sqrt[4]{2}+\sqrt[4]{512}$.
27. $\sqrt{10}+\sqrt{40}+\sqrt{90}$.
28. $\sqrt{2}+\sqrt{50}-\sqrt{72}$.
29. $\sqrt{3}-\sqrt{12}-\sqrt{27}$.
30. $\sqrt{6}-\sqrt{24}+\sqrt{54}$.
31. $2\sqrt{2}+\sqrt{18}+\sqrt{32}$.
32. $4\sqrt{3}+2\sqrt{12}+\sqrt{75}$.
33. $2\sqrt{11}+5\sqrt{44}-3\sqrt{99}$.
34. $2\sqrt{x}+3\sqrt{x}$.
35. $4\sqrt{y}-\sqrt{y}$.
36. $a\sqrt{2}+b\sqrt{2}$.
37. $a\sqrt{m}+\sqrt{m}$.
38. $\sqrt{n}-x\sqrt{n}$.
39. $\sqrt{4x}+\sqrt{9x}$.
40. $2\sqrt{64z}-\sqrt{49z}$.
41. $\sqrt{a}+\sqrt{a^3}$.
42. $\sqrt{b}+\sqrt{b^5}$.
43. $\sqrt{c^3}-\sqrt{c}$.
44. $\sqrt[3]{a}+\sqrt[3]{a^4}$.
45. $\sqrt[3]{b^5}-\sqrt[3]{b^2}$.
46. $\sqrt[3]{8x}+\sqrt[3]{x^4}$.
47. $a\sqrt{a^3}+\sqrt{a^5}$.
48. $m\sqrt[4]{m^5}+\sqrt[4]{m^9}$.
49. $a\sqrt{b^2c}+\sqrt{c}$.
50. $\sqrt{yz^2}+\sqrt{yw^2}$.
51. $\sqrt{a^2bc}+\sqrt{b^3c}$.
52. $\sqrt{ab^2c^2}+\sqrt{ax^2y^2}$.
53. $\sqrt{(a+b)^3}+\sqrt{a+b}$.
54. $\sqrt{x-y}-\sqrt{(x-y)^3}$.
55. $\sqrt{2}+\sqrt{2a^2}+\sqrt{8b^2}$.
56. $\sqrt{x^3}+\sqrt{x^5}+\sqrt{x^7}$.
57. $\sqrt{a}-2\sqrt{a^3}+\sqrt{a^5}$.
58. $a\sqrt{xy}+\sqrt{b^2xy}-\sqrt{c^2xy}$.
59. $a\sqrt{a^3b}+\sqrt{4a^3b^3}+\sqrt{ab^5}$.
60. $\sqrt[3]{m^2n}+\sqrt[3]{m^5n}+\sqrt[3]{m^2n^4}$.
61. $\sqrt{2}+\sqrt{3}+\sqrt{18}+\sqrt{8}$.
62. $\sqrt{5}+\sqrt{20}+\sqrt{6}+\sqrt{24}$.

REDUCTION TO EQUIVALENT FORMS

EXERCISE 144

Reduce to the forms of surds of the orders indicated:

1. 3, second order.
2. 2, third order.
3. 3, fourth order.
4. $\frac{1}{4}$, second order.
5. $\frac{1}{2}$, fifth order.
6. $\frac{2}{3}$, fourth order.
7. $\frac{1}{5}$, third order.
8. a, second order.
9. b, third order.
10. c, fourth order.
11. d, fifth order.
12. $2m$, second order.
13. $4n$, third order.
14. $5xy$, second order.
15. $6x^2$, second order.
16. $7a^2b$, second order.
17. $5x^2y^3$, third order.
18. a, nth order.
19. b^n, second order.
20. c^x, third order.
21. d^2, rth order.

REDUCTION OF MIXED SURDS TO ENTIRE SURDS

EXERCISE 145

Transform each of the following mixed surds into an entire surd:

1. $2\sqrt{3}$.
2. $3\sqrt{5}$.
3. $4\sqrt{2}$.
4. $2\sqrt[3]{2}$.
5. $3\sqrt[3]{3}$.
6. $5\sqrt[3]{2}$.
7. $4\sqrt[3]{3}$.
8. $6\sqrt[3]{10}$.
9. $\frac{1}{2}\sqrt{2}$.
10. $\frac{1}{3}\sqrt{6}$.
11. $\frac{1}{5}\sqrt{5}$.
12. $\frac{1}{6}\sqrt{7}$.
13. $3\sqrt{\frac{1}{3}}$.
14. $5\sqrt{\frac{1}{10}}$.
15. $\frac{1}{10}\sqrt{5}$.
16. $\frac{1}{2}\sqrt{\frac{1}{3}}$.
17. $\frac{1}{4}\sqrt{\frac{1}{5}}$.
18. $\frac{2}{3}\sqrt{\frac{3}{2}}$.
19. $\frac{1}{3}\sqrt[3]{\frac{1}{3}}$.
20. $a\sqrt{b}$.
21. $b\sqrt{c}$.
22. $c\sqrt[3]{d}$.
23. $m\sqrt[4]{n}$.
24. $x\sqrt{x}$.
25. $y\sqrt[3]{y}$.
26. $ab\sqrt{c}$.
27. $xy\sqrt{yz}$.
28. $x^2\sqrt{y}$.
29. $\frac{1}{a}\sqrt{a}$.
30. $b\sqrt{\frac{1}{b}}$.
31. $\frac{1}{b}\sqrt{\frac{1}{b}}$.
32. $\frac{a}{2}\sqrt{\frac{2}{a}}$.
33. $\frac{3}{b}\sqrt{\frac{b}{3}}$.
34. $\frac{m}{n}\sqrt[3]{\frac{m}{n}}$.
35. $a\sqrt[n]{b}$.
36. $b\sqrt[n]{c}$.
37. $c\sqrt[n]{c}$.
38. $d\sqrt[n]{d^2}$.
39. $x\sqrt[b]{x^a}$.
40. $a^n\sqrt[r]{a}$.

CHANGE OF ORDER

EXERCISE 146

Express as equivalent surds of the same order:

1. $\sqrt{2}$ and $\sqrt[3]{5}$.
2. $\sqrt{3}$ and $\sqrt[3]{4}$.
3. $\sqrt{5}$ and $\sqrt[3]{10}$.
4. $\sqrt{6}$ and $\sqrt[3]{14}$.
5. $\sqrt{7}$ and $\sqrt[4]{50}$.
6. $\sqrt[3]{2}$ and $\sqrt[4]{3}$.
7. $\sqrt[3]{3}$ and $\sqrt[4]{5}$.
8. $\sqrt[5]{2}$ and $\sqrt[6]{3}$.
9. $\sqrt{3}$ and $\sqrt[5]{15}$.
10. $\sqrt[3]{9}$ and $\sqrt[6]{80}$.
11. $\sqrt{3}$ and $\sqrt[8]{75}$.
12. $\sqrt{2}$ and $\sqrt[7]{12}$.
13. $\sqrt{a}$ and $\sqrt[3]{a^2}$.
14. $\sqrt[3]{b^2}$ and $\sqrt[4]{b^3}$.
15. $\sqrt{x}$ and $\sqrt[5]{x^4}$.
16. $\sqrt[n]{y}$ and $\sqrt[2n]{y}$
17. $\sqrt[2n]{z}$ and $\sqrt[3n]{z}$.
18. $\sqrt[n]{w}$ and $\sqrt[n+1]{w}$.

Which is the greater?

19. $3\sqrt{5}$ or $4\sqrt{3}$.
20. $4\sqrt{6}$ or $5\sqrt{3}$.
21. $5\sqrt{7}$ or $8\sqrt{3}$.
22. $\sqrt[3]{36}$ or $2\sqrt[3]{5}$.
23. $\sqrt[4]{3}$ or $\sqrt{2}$.
24. $\sqrt{5}$ or $\sqrt[4]{20}$.
25. $\sqrt{6}$ or $2\sqrt[4]{2}$.
26. $\sqrt[3]{14}$ or $2\sqrt[6]{3}$.
27. $\sqrt{3}$ or $\sqrt[6]{29}$.
28. $\sqrt{6}$ or $\sqrt[6]{200}$.
29. $\sqrt{a}$ or $\sqrt[3]{a^2}$, for $a > 1$.
30. $\sqrt[3]{b^2}$ or $\sqrt[5]{b^3}$, for $b > 1$.

MULTIPLICATION OF SURDS

EXERCISE 147

Write each of the following products in simplest form:

1. $\sqrt{2} \times \sqrt{3}$.
2. $\sqrt{3} \times \sqrt{7}$.
3. $\sqrt{5} \times \sqrt{11}$.
4. $\sqrt{6} \times \sqrt{2}$.
5. $\sqrt{7} \times \sqrt{14}$.
6. $\sqrt{7} \times \sqrt{35}$.
7. $\sqrt{10} \times \sqrt{6}$.
8. $\sqrt{11} \times \sqrt{22}$.
9. $\sqrt{3} \times \sqrt{15}$.
10. $2\sqrt{2} \times \sqrt{5}$.
11. $4\sqrt{3} \times \sqrt{10}$.
12. $\sqrt{13} \times 5\sqrt{7}$.
13. $\sqrt{17} \times 8\sqrt{2}$.
14. $5\sqrt{6} \times 2\sqrt{11}$.
15. $2\sqrt{14} \times 2\sqrt{7}$.
16. $5\sqrt{3} \times 4\sqrt{21}$.
17. $3\sqrt{2} \times 7\sqrt{10}$.
18. $\sqrt[3]{4} \times \sqrt[3]{6}$.
19. $\sqrt[3]{9} \times \sqrt[3]{12}$.
20. $\sqrt[3]{50} \times \sqrt[3]{5}$.
21. $\sqrt[3]{12} \times \sqrt[3]{20}$
22. $-2\sqrt[3]{9} \times \sqrt[3]{-15}$.
23. $\sqrt[4]{8} \times \sqrt[4]{6}$.
24. $3\sqrt[4]{2} \times \sqrt[4]{24}$.
25. $\sqrt{2} \times \sqrt[3]{4}$.
26. $\sqrt{3} \times \sqrt[3]{9}$.
27. $\sqrt{5} \times \sqrt[4]{5}$.

28. $\sqrt{3} \times \sqrt[6]{3}$.
29. $\sqrt{\frac{1}{3}} \times \sqrt{\frac{1}{2}}$.
30. $\sqrt{\frac{1}{2}} \times \sqrt{\frac{1}{5}}$.
31. $\sqrt{\frac{6}{5}} \times \sqrt{\frac{7}{5}}$.
32. $\sqrt{\frac{5}{7}} \times \sqrt{\frac{7}{5}}$.
33. $\sqrt{\frac{2\frac{1}{2}}{2}} \times \sqrt{\frac{3\frac{5}{2}}{2}}$.
34. $\sqrt{\frac{1\frac{4}{5}}{5}} \times \sqrt{\frac{2\frac{1}{5}}{5}}$.
35. $\sqrt[3]{\frac{1}{2}} \times \sqrt[3]{\frac{1}{3}}$.
36. $\sqrt{x} \times \sqrt{y}$.
37. $\sqrt{ab} \times \sqrt{c}$.
38. $\sqrt{mx} \times \sqrt{ny}$.
39. $x\sqrt{a} \times \sqrt{c}$.
40. $\sqrt{d} \times y\sqrt{z}$.
41. $a\sqrt{x} \times b\sqrt{y}$.
42. $\sqrt{ab} \times \sqrt{a}$.
43. $\sqrt{bc} \times \sqrt{c}$.
44. $\sqrt{xy} \times \sqrt{yz}$.
45. $\sqrt{abc} \times \sqrt{bc}$.
46. $\sqrt{xyz} \times \sqrt{yzw}$.
47. $\sqrt[3]{a^2} \times \sqrt[3]{ab}$.
48. $\sqrt[3]{bc^2} \times \sqrt[3]{cd}$.
49. $\sqrt[4]{x^3y} \times \sqrt[4]{xy}$.
50. $\sqrt{2\,a} \times \sqrt{2\,b}$.
51. $\sqrt{5\,c} \times \sqrt{7\,d}$.
52. $\sqrt{2\,x} \times \sqrt{2\,y}$.
53. $\sqrt{3\,m} \times \sqrt{6\,n}$.
54. $\sqrt{7\,k} \times \sqrt{21\,w}$.
55. $5\sqrt{10\,b} \times \sqrt{2\,c}$.
56. $4\sqrt{6\,ab} \times \sqrt{2\,ac}$.
57. $\sqrt[3]{abc} \times \sqrt[3]{a^2}$.
58. $\sqrt[3]{xyz} \times \sqrt[3]{y^2z^2}$.
59. $\sqrt[3]{abc} \times \sqrt[3]{a^2b^2c}$.
60. $\sqrt[4]{ab^2c^3} \times \sqrt[4]{a^3bc}$.
61. $\sqrt[5]{xy^2z^3w^4} \times \sqrt[5]{y^3w}$.
62. $\sqrt{\frac{a}{c}} \times \sqrt{\frac{b}{c}}$.
63. $\sqrt{\frac{x}{y}} \times \sqrt{\frac{x}{z}}$.
64. $\sqrt{\frac{1}{ab}} \times \sqrt{\frac{a}{b}}$.
65. $\sqrt{\frac{a}{bc}} \times \sqrt{\frac{c}{ab}}$.
66. $\sqrt[n]{a} \times \sqrt[n]{a^{n-1}b}$.
67. $\sqrt[n]{c^{n-4}} \times \sqrt[n]{c^5d}$.
68. $n\sqrt[n]{n^{n-1}} \times \sqrt[n]{n^2}$.

MULTIPLICATION OF A SURD POLYNOMIAL BY A SURD MONOMIAL MULTIPLIER

EXERCISE 148

Simplify each of the following expressions:

1. $(\sqrt{11} - \sqrt{13} + \sqrt{7})\sqrt{3}$.
2. $(\sqrt{2} + \sqrt{7} - \sqrt{3})\sqrt{5}$.
3. $(\sqrt{7} - \sqrt{11} + \sqrt{13})\sqrt{6}$.
4. $(\sqrt{5} - \sqrt{10} - \sqrt{15})\sqrt{7}$.
5. $(\sqrt{2} - \sqrt{5} + \sqrt{10})\sqrt{10}$.
6. $(\sqrt{2} + \sqrt{7} + 2\sqrt{14})\sqrt{14}$.

7. $(\sqrt{3}-\sqrt{5}+\sqrt{10})\sqrt{15}$.
8. $(\sqrt{10}+2\sqrt{3}-\sqrt{5})\sqrt{5}$.
9. $(3\sqrt{2}-2\sqrt{3}+\sqrt{12})\sqrt{6}$.
10. $(\sqrt{5}-\sqrt{21}+4\sqrt{27})2\sqrt{3}$.
11. $(2\sqrt{5}-4\sqrt{10}-\sqrt{30})3\sqrt{5}$.
12. $(\sqrt{a}+\sqrt{b}+\sqrt{c})\sqrt{abc}$.
13. $(\sqrt{xy}+\sqrt{yz}+\sqrt{zx})\sqrt{xyz}$.
14. $(\sqrt[3]{2}+\sqrt[3]{6}-\sqrt[3]{10})\sqrt[3]{4}$.
15. $(\sqrt[3]{9}-2\sqrt[3]{3}-\sqrt[3]{18})\sqrt[3]{3}$.
16. $(\sqrt[3]{4}+\sqrt[3]{9}+\sqrt[3]{36})\sqrt[3]{6}$.
17. $(\sqrt[3]{100}-\sqrt[3]{25}+\sqrt[3]{4})\sqrt[3]{10}$.
18. $(\sqrt[3]{a^2bc}+\sqrt[3]{ab^2c}+\sqrt[3]{abc^2})\sqrt[3]{abc}$.
19. $(\sqrt[3]{a^2c}-\sqrt[3]{b^2d}+\sqrt[3]{c^2d})\sqrt[3]{abcd}$.
20. $(\sqrt[n]{a^{n-1}}+\sqrt[n]{b^{n-1}}+\sqrt[n]{c^{n-1}})\sqrt[n]{abc}$.

PRODUCT OF TWO BINOMIAL SURDS

EXERCISE 149

Simplify each of the following:

1. $(\sqrt{6}+\sqrt{2})(\sqrt{6}-\sqrt{2})$.
2. $(\sqrt{7}+\sqrt{5})(\sqrt{7}-\sqrt{5})$.
3. $(\sqrt{10}+\sqrt{3})(\sqrt{10}-\sqrt{3})$.
4. $(\sqrt{11}+\sqrt{6})(\sqrt{11}-\sqrt{6})$.
5. $(\sqrt{13}+\sqrt{2})(\sqrt{13}-\sqrt{2})$.
6. $(\sqrt{10}+1)(\sqrt{10}-1)$.
7. $(\sqrt{13}+2)(\sqrt{13}-2)$.
8. $(\sqrt{15}+8)(\sqrt{15}-8)$.
9. $(\sqrt{a}+3)(\sqrt{a}-3)$.
10. $(5+\sqrt{b})(5-\sqrt{b})$.
11. $(1+\sqrt{y})(1-\sqrt{y})$.
12. $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})$.
13. $(2\sqrt{3}+\sqrt{5})(2\sqrt{3}-\sqrt{5})$.
14. $(3\sqrt{2}+\sqrt{6})(3\sqrt{2}-\sqrt{6})$.

Expand the following:

15. $(3\sqrt{5}+\sqrt{7})^2$.
16. $(\sqrt{3}+\sqrt{2})^2$.
17. $(\sqrt{10}-\sqrt{7})^2$.
18. $(\sqrt{6}+\sqrt{3})^2$.
19. $(\sqrt{15}-\sqrt{5})^2$.
20. $(2\sqrt{3}+5)^2$.
21. $(7-3\sqrt{6})^2$.
22. $(4\sqrt{2}+3\sqrt{3})^2$.
23. $(5\sqrt{6}-2\sqrt{7})^2$.

Fill in the following Blank Forms and expand:

24. $(\sqrt{(\ \ )}+\sqrt{(\ \ )})^2$.
25. $(\sqrt{(\ \ )}-\sqrt{(\ \ )})^2$.
26. $[(\ \ )\sqrt{(\ \ )}+(\ \ )\sqrt{(\ \ )}]^2$.
27. $[(\ \ )\sqrt{(\ \ )}-(\ \ )]^2$.

Simplify each of the following:

28. $(\sqrt{5}+2)(\sqrt{5}+4)$.
29. $(\sqrt{7}+3)(\sqrt{7}+1)$.
30. $(\sqrt{6}+8)(\sqrt{6}-3)$.
31. $(\sqrt{10}-5)(\sqrt{10}-9)$.
32. $(\sqrt{13}+4)(\sqrt{13}-8)$.
33. $(2\sqrt{3}+1)(2\sqrt{3}+5)$.
34. $(3\sqrt{5}+4)(3\sqrt{5}+6)$.
35. $(4\sqrt{2}-7)(4\sqrt{2}-1)$.
36. $(5\sqrt{6}-4)(5\sqrt{6}-6)$.
37. $(\sqrt{3+\sqrt{5}})(\sqrt{3-\sqrt{5}})$.
38. $(\sqrt{4+\sqrt{7}})(\sqrt{4-\sqrt{7}})$.
39. $(\sqrt{5+\sqrt{21}})(\sqrt{5-\sqrt{21}})$.
40. $(\sqrt{5+2\sqrt{6}})(\sqrt{5-2\sqrt{6}})$.
41. $(\sqrt{6+2\sqrt{5}})(\sqrt{6-2\sqrt{5}})$.
42. $(\sqrt{6+3\sqrt{3}})(\sqrt{6-3\sqrt{3}})$.

PRODUCTS OF POLYNOMIAL SURDS

EXERCISE 150

Simplify each of the following expressions:

1. $(\sqrt{2}+\sqrt{3}-\sqrt{5})(\sqrt{2}-\sqrt{3}+\sqrt{5})$.
2. $(\sqrt{2}-\sqrt{5}+\sqrt{7})(\sqrt{2}+\sqrt{5}-\sqrt{7})$.
3. $(\sqrt{3}+\sqrt{7}+\sqrt{10})(\sqrt{3}-\sqrt{7}-\sqrt{10})$.
4. $(\sqrt{2}+\sqrt{5}+\sqrt{10})(\sqrt{2}-\sqrt{5}-\sqrt{10})$.
5. $(\sqrt{a}+\sqrt{b}-\sqrt{c})(\sqrt{a}-\sqrt{b}+\sqrt{c})$.
6. $(\sqrt{x}-\sqrt{y}-\sqrt{z})(\sqrt{x}+\sqrt{y}+\sqrt{z})$.
7. $2\sqrt{3}+3\sqrt{5}+\sqrt{7})(2\sqrt{3}-3\sqrt{5}+\sqrt{7})$.
8. $(3\sqrt{2}+2\sqrt{5}-8\sqrt{6})(2+5\sqrt{2}-3\sqrt{12})$.
9. $(\sqrt{\frac{1}{6}}-\sqrt{\frac{1}{7}}+\sqrt{\frac{1}{8}})(\sqrt{6}+\sqrt{7}-\sqrt{8})$.
10. $(\sqrt{\frac{5}{6}}-\sqrt{\frac{6}{7}}-\sqrt{\frac{7}{5}})(\sqrt{\frac{6}{5}}-\sqrt{\frac{7}{6}}-\sqrt{\frac{5}{7}})$.
11. $(\sqrt{2}-\sqrt{3})(\sqrt{3}-\sqrt{5})(\sqrt{5}-\sqrt{7})$.
12. $(a\sqrt{b}-\sqrt{ab}+b\sqrt{a})(\sqrt{a}-\sqrt{b})$.
13. $(\sqrt{ab}+\sqrt{bc}+\sqrt{ca})(\sqrt{a}+\sqrt{b}+\sqrt{c})$.
14. $(\sqrt{a}+\sqrt{b}+\sqrt{c})(\sqrt{a}+\sqrt{b}-\sqrt{c})(\sqrt{a}-\sqrt{b}+\sqrt{c})$
$(\sqrt{a}-\sqrt{b}-\sqrt{c})$.

INVOLUTION OF MONOMIAL SURDS

EXERCISE 151

Reduce the following indicated powers to simplest form:

1. $(\sqrt[3]{2})^2$.
2. $(\sqrt[3]{5})^2$.
3. $(\sqrt[4]{3})^3$.
4. $(\sqrt{2})^3$.
5. $(\sqrt{5})^3$.
6. $(\sqrt[5]{8})^2$.
7. $(2\sqrt{3})^3$.
8. $(3\sqrt{5})^3$.
9. $(4\sqrt[3]{4})^2$.
10. $(2\sqrt[6]{2})^5$.
11. $(-2\sqrt{7})^3$.
12. $(\sqrt[3]{-4})^4$.
13. $(\sqrt{a})^3$.
14. $(\sqrt{b})^3$.
15. $(\sqrt{x})^5$.
16. $(\sqrt{y})^5$.
17. $(\sqrt[3]{c})^2$.
18. $(\sqrt[3]{d})^4$.
19. $(\sqrt[3]{h})^5$.
20. $(\sqrt[3]{-a})^2$.
21. $(\sqrt[3]{-b^2})^2$.
22. $(-\sqrt[3]{-x})^4$.
23. $(a\sqrt{x})^3$.
24. $(b\sqrt{y})^4$.
25. $(c\sqrt[3]{m})^2$.
26. $(d\sqrt[3]{n})^4$.
27. $(a\sqrt[3]{b^2c})^2$.
28. $(2\,x\sqrt{x})^3$.
29. $(3\,y\sqrt[3]{y^2})^2$.
30. $(\sqrt[n]{a})^2$, $n>2$.
31. $(\sqrt[n]{b})^3$, $n>3$.
32. $(x\sqrt[n]{y})^4$, $n<4$.

DIVISION OF MONOMIAL SURDS

EXERCISE 152

Simplify each of the following quotients:

1. $\sqrt{6}\div\sqrt{2}$.
2. $\sqrt{10}\div\sqrt{5}$.
3. $\sqrt{14}\div\sqrt{2}$.
4. $\sqrt{21}\div\sqrt{3}$.
5. $\sqrt{15}\div\sqrt{5}$.
6. $3\sqrt{22}\div\sqrt{11}$.
7. $5\sqrt{26}\div\sqrt{13}$.
8. $8\sqrt{30}\div 2\sqrt{2}$.
9. $\sqrt{39}\div 2\sqrt{13}$.
10. $\sqrt{51}\div 3\sqrt{17}$.
11. $\sqrt[3]{4}\div\sqrt[3]{2}$.
12. $\sqrt[3]{9}\div\sqrt[3]{3}$.
13. $\sqrt[3]{12}\div\sqrt[3]{4}$.
14. $5\sqrt[3]{18}\div\sqrt[3]{3}$.
15. $\sqrt{27}\div\sqrt[4]{3}$.
16. $\sqrt[4]{50}\div\sqrt[4]{2}$.
17. $\sqrt{48}\div\sqrt[6]{6}$.
18. $\sqrt{7}\div\sqrt{2}$.
19. $\sqrt{3}\div\sqrt{5}$.
20. $\sqrt{10}\div\sqrt{3}$.
21. $\sqrt{13}\div\sqrt{7}$.
22. $\sqrt{10}\div\sqrt{6}$.
23. $\sqrt{15}\div\sqrt{10}$.
24. $\sqrt{21}\div\sqrt{14}$.
25. $\sqrt[3]{2}\div\sqrt[3]{4}$.
26. $\sqrt[3]{10}\div\sqrt[3]{12}$.
27. $\sqrt[3]{3}\div\sqrt[3]{2}$.
28. $\sqrt[3]{5}\div\sqrt{4}$.
29. $5\sqrt{2}\div\sqrt{10}$.
30. $3\sqrt{5}\div\sqrt{15}$.
31. $6\sqrt{7}\div\sqrt{42}$.
32. $10\sqrt{3}\div\sqrt{6}$.
33. $14\sqrt{2}\div\sqrt{14}$.
34. $\sqrt{7}\div 3\sqrt{21}$.
35. $\sqrt{11}\div 2\sqrt{22}$.
36. $\sqrt{3}\div 4\sqrt{15}$.

37. $\sqrt{2} \div 5\sqrt{6}$.
38. $\sqrt[3]{a^2} \div \sqrt[3]{a}$.
39. $\sqrt[3]{b^2c} \div \sqrt[3]{c}$.
40. $\sqrt[3]{a^2bc} \div \sqrt[3]{ab}$.
41. $\sqrt[3]{xy^2z} \div \sqrt[3]{xyz}$.
42. $\sqrt[4]{a^3} \div \sqrt[4]{a}$.
43. $\sqrt[4]{x^2y^3} \div \sqrt[4]{xy}$.
44. $\sqrt[4]{ab^2c^3} \div \sqrt[4]{ab^2c}$.
45. $\sqrt{a} \div \sqrt{b}$.
46. $\sqrt{c} \div \sqrt{x}$.
47. $\sqrt{d} \div \sqrt{y}$.
48. $\sqrt[3]{a} \div \sqrt[3]{b^2}$.
49. $\sqrt[3]{b^2} \div \sqrt[3]{c}$.
50. $a\sqrt{y} \div \sqrt{a}$.
51. $a\sqrt{b} \div b\sqrt{a}$.
52. $\sqrt{xyz} \div xy\sqrt{z}$.
53. $ab\sqrt{c} \div a\sqrt{bc}$.
54. $\sqrt{xyz} \div \sqrt{yzw}$.
55. $\sqrt{5\,a} \div \sqrt{2\,b}$.
56. $\sqrt{7\,x} \div \sqrt{14\,y}$.
57. $6\sqrt{3\,xy} \div 3\sqrt{6\,xz}$.
58. $\sqrt[n]{a^2} \div \sqrt[n]{a}$, $n > 2$.
59. $\sqrt[n]{a^{n-1}} \div \sqrt[n]{ab^{n-1}}$, $n > 1$.
60. $(\sqrt{30} + \sqrt{42}) \div \sqrt{6}$.
61. $(12\sqrt{35} - \sqrt{45}) \div 3\sqrt{5}$.
62. $(8\sqrt[3]{51} - 5\sqrt[3]{33}) \div (-2\sqrt[3]{3})$.
63. $(\sqrt{15} - \sqrt{6} + \sqrt{2} - \sqrt{3}) \div \sqrt{3}$.
64. $(\sqrt{2} + \sqrt{8} - \sqrt{21}) \div \sqrt{2}$.

RATIONALIZATION

EXERCISE 153

Rationalize the denominators of each of the following:

1. $\frac{1}{\sqrt{2}}$.
2. $\frac{1}{\sqrt{3}}$.
3. $\frac{1}{\sqrt{5}}$.
4. $\frac{1}{\sqrt[3]{2}}$.
5. $\frac{1}{\sqrt[3]{4}}$.
6. $\frac{1}{\sqrt[3]{9}}$.
7. $\frac{1}{\sqrt[4]{27}}$.
8. $\frac{2}{\sqrt{3}}$.
9. $\frac{8}{\sqrt{7}}$.
10. $\frac{10}{\sqrt[3]{3}}$.
11. $\frac{1}{\sqrt{a}}$.
12. $\frac{2}{\sqrt{b}}$.
13. $\frac{4}{\sqrt{c}}$.
14. $\frac{a}{\sqrt{b}}$.
15. $\frac{1}{\sqrt[3]{d}}$.
16. $\frac{1}{\sqrt[3]{m^2}}$.
17. $\frac{1}{\sqrt[4]{y^3}}$.
18. $\frac{1}{\sqrt[5]{w}}$.
19. $\frac{n}{\sqrt[3]{y^2}}$.
20. $\frac{2}{\sqrt{5\,c}}$.
21. $\frac{3}{\sqrt{7\,d}}$.
22. $\frac{1}{\sqrt{3\,b}}$.
23. $\frac{1}{3\sqrt{b}}$.
24. $\frac{1}{2\sqrt{11}}$.
25. $\frac{1}{5\sqrt{7}}$.
26. $\frac{9}{\sqrt[3]{2\,x}}$.
27. $\frac{1}{3\sqrt[3]{2}}$.
28. $\frac{1}{2\sqrt[3]{4}}$.
29. $\frac{1}{a\sqrt{x}}$.
30. $\frac{2}{\sqrt{2}}$.
31. $\frac{5}{\sqrt[3]{5}}$.
32. $\frac{3}{\sqrt[4]{3}}$.

33. $\frac{2}{\sqrt[5]{2}}$.

34. $\frac{10}{\sqrt{5}}$.

35. $\frac{9}{\sqrt[3]{3}}$.

36. $\frac{5}{\sqrt[3]{25}}$.

37. $\frac{9}{\sqrt[3]{9}}$.

38. $\frac{49}{\sqrt[3]{49}}$.

39. $\frac{8}{\sqrt[4]{8}}$.

40. $\frac{a^2}{\sqrt{a}}$.

41. $\frac{b^3}{\sqrt{b}}$.

42. $\frac{c}{\sqrt[3]{c^2}}$.

43. $\frac{d^3}{\sqrt[3]{d}}$.

44. $\frac{6}{\sqrt{12}}$.

45. $\frac{3}{\sqrt{15}}$.

46. $\frac{4}{\sqrt{6}}$.

47. $\frac{18}{\sqrt{15}}$.

48. $\frac{28}{\sqrt{21}}$.

49. $\frac{2}{\sqrt{2m}}$.

50. $\frac{3}{\sqrt{3w}}$.

51. $\frac{6}{\sqrt[3]{3c}}$.

52. $\frac{8}{\sqrt[3]{2m}}$.

53. $\frac{\sqrt{2}}{\sqrt{5}}$.

54. $\frac{\sqrt{7}}{\sqrt{6}}$.

55. $\frac{\sqrt{2}}{\sqrt{37}}$.

56. $\frac{\sqrt{13}}{\sqrt{3}}$.

57. $\frac{\sqrt{5}}{\sqrt{19}}$.

58. $\frac{\sqrt[3]{3}}{\sqrt[3]{10}}$.

59. $\frac{\sqrt{a}}{\sqrt{b}}$.

60. $\frac{\sqrt{x}}{\sqrt{y}}$.

61. $\frac{\sqrt{m}}{\sqrt{2}}$.

62. $\frac{\sqrt[3]{a}}{\sqrt[3]{b}}$.

63. $\frac{\sqrt[4]{x}}{\sqrt[4]{y}}$.

64. $\frac{\sqrt[3]{b}}{\sqrt[3]{c^2}}$.

65. $\frac{2}{3\sqrt{5}}$.

66. $\frac{5}{6\sqrt{7}}$.

67. $\frac{10}{3\sqrt{11}}$.

68. $\frac{2\sqrt{3}}{\sqrt{2}}$.

69. $\frac{3\sqrt{5}}{\sqrt{3}}$.

70. $\frac{5}{7\sqrt{5}}$.

71. $\frac{6\sqrt{7}}{\sqrt{6}}$.

72. $\frac{4}{3\sqrt{2}}$.

73. $\frac{ab}{\sqrt{c}}$.

74. $\frac{a}{\sqrt{bc}}$.

75. $\frac{a}{b\sqrt{c}}$.

76. $\frac{xy}{\sqrt{zw}}$.

77. $\frac{a\sqrt{y}}{\sqrt{a}}$.

78. $\frac{b\sqrt{k}}{\sqrt{b}}$.

79. $\frac{2a}{\sqrt{2b}}$.

80. $\frac{5c}{\sqrt{5d}}$.

81. $\frac{ab}{\sqrt{a}}$.

82. $\frac{bc}{\sqrt{c}}$.

83. $\frac{x}{\sqrt{xy}}$.

84. $\frac{2ab}{\sqrt{3cd}}$.

85. $\frac{a\sqrt{c}}{c\sqrt{b}}$.

86. $\frac{1}{\sqrt[3]{4a}}$.

87. $\frac{1}{\sqrt[3]{5b^2}}$.

88. $\frac{1}{\sqrt[n]{x^{n-1}}}$.

89. $\frac{1}{\sqrt{x+1}}$.

90. $\frac{1}{\sqrt{z-4}}$.

91. $\frac{2}{\sqrt{a+3}}$.

92. $\frac{5}{\sqrt{x+5}}$.

93. $\frac{a}{\sqrt{a+1}}$.

94. $\frac{y}{\sqrt{y-3}}$.

95. $\dfrac{a^2-4}{\sqrt{a-2}}$.

96. $\dfrac{b^2-9}{\sqrt{b+3}}$.

97. $\dfrac{x+5}{\sqrt{x^2-25}}$.

98. $\dfrac{a-7}{\sqrt{7-a}}$.

99. $\dfrac{1}{\sqrt{3}+\sqrt{2}}$.

100. $\dfrac{1}{\sqrt{2}-\sqrt{7}}$.

101. $\dfrac{1}{\sqrt{7}-\sqrt{6}}$.

102. $\dfrac{1}{\sqrt{5}-\sqrt{6}}$.

103. $\dfrac{1}{\sqrt{a}+\sqrt{b}}$.

104. $\dfrac{1}{\sqrt{b}-\sqrt{c}}$.

105. $\dfrac{1}{\sqrt{x}+1}$.

106. $\dfrac{1}{\sqrt{z}-4}$.

107. $\dfrac{1}{2\sqrt{3}+1}$.

108. $\dfrac{2}{\sqrt{11}-\sqrt{10}}$.

109. $\dfrac{3}{\sqrt{6}+\sqrt{5}}$.

110. $\dfrac{5}{\sqrt{7}-\sqrt{2}}$.

111. $\dfrac{7}{\sqrt{10}+\sqrt{3}}$

112. $\dfrac{10}{\sqrt{5}+\sqrt{3}}$.

113. $\dfrac{15}{\sqrt{7}-\sqrt{2}}$.

114. $\dfrac{11}{2\sqrt{5}-3}$.

115. $\dfrac{5}{3\sqrt{5}+5}$.

116. $\dfrac{36}{3\sqrt{2}-2\sqrt{3}}$.

117. $\dfrac{a}{\sqrt{b}+\sqrt{c}}$.

118. $\dfrac{b}{\sqrt{c}-\sqrt{a}}$.

119. $\dfrac{2x}{\sqrt{2}+\sqrt{x}}$.

120. $\dfrac{\sqrt{5}-\sqrt{7}}{\sqrt{3}}$.

121. $\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{2}}$.

122. $\dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{c}}$.

123. $\dfrac{\sqrt{x}+\sqrt{y}}{\sqrt{y}}$.

124. $\dfrac{\sqrt{a}-\sqrt{b}}{\sqrt{ab}}$.

125. $\dfrac{\sqrt{ab}+\sqrt{bc}}{\sqrt{abc}}$.

126. $\dfrac{1}{5\sqrt{3}-6\sqrt{2}}$.

127. $\dfrac{2}{5\sqrt{2}-4\sqrt{3}}$.

128. $\dfrac{a}{\sqrt{2a}+\sqrt{a}}$.

129. $\dfrac{b}{\sqrt{b}-\sqrt{2b}}$.

130. $\dfrac{\sqrt{3}+2}{\sqrt{3}-2}$.

131. $\dfrac{3+\sqrt{7}}{3-\sqrt{7}}$.

132. $\dfrac{5+\sqrt{3}}{5-\sqrt{3}}$.

133. $\dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{c}+\sqrt{d}}$.

134. $\dfrac{3-x}{\sqrt{3y}+\sqrt{xy}}$.

135. $\dfrac{1}{\sqrt{x+2}+1}$.

136. $\dfrac{1}{\sqrt{z-2}-1}$.

137. $\dfrac{1}{1+\sqrt{1-x}}$.

138. $\dfrac{y}{1+\sqrt{1-y}}$.

139. $\dfrac{z}{5+\sqrt{4-z}}$.

140. $\dfrac{1}{\sqrt{x+m}+\sqrt{x}}$.

141. $\dfrac{b}{\sqrt{a+b}+\sqrt{a}}$.

142. $\dfrac{b^2}{a+\sqrt{a^2-b^2}}$.

143. $\dfrac{1}{\sqrt{x+4}+\sqrt{x+3}}$.

144. $\dfrac{1}{\sqrt{x+a}+\sqrt{x+b}}$.

145. $\dfrac{1}{\sqrt{x+m}+\sqrt{m-x}}$.

146. $\dfrac{\sqrt{x+3}+\sqrt{x+2}}{\sqrt{x+3}-\sqrt{x+2}}$.

147. $\dfrac{\sqrt{a+5}+\sqrt{a-6}}{\sqrt{a+5}-\sqrt{a-6}}$.

148. $\dfrac{5\sqrt{15}}{\sqrt{5}+\sqrt{3}}$.

149. $\dfrac{6\sqrt{6}}{3\sqrt{2}-2\sqrt{3}}$.

150. $\dfrac{ab\sqrt{ab}}{a\sqrt{b}+b\sqrt{a}}$.

Fill in the following Blank Forms and rationalize:

151. $\dfrac{(\ )}{\sqrt{(\ )}}$.

152. $\dfrac{(\ )}{\sqrt[3]{(\ )}}$.

153. $\dfrac{(\ )}{\sqrt{(\ )}+(\ )}$.

154. $\dfrac{(\ )}{\sqrt{(\ )}-(\ )}$.

155. $\dfrac{(\ )}{\sqrt{(\ )}+\sqrt{(\ )}}$.

156. $\dfrac{(\ )}{\sqrt{(\ )}-\sqrt{(\ )}}$.

157. $\dfrac{(\ )}{\sqrt{(\ )}+(\ )}$.

158. $\dfrac{(\ )}{(\ )-\sqrt{(\ )}}$.

159. $\dfrac{\sqrt{(\ )}+\sqrt{(\ )}}{\sqrt{(\ )}}$.

160. $\dfrac{\sqrt{(\ )}}{\sqrt{(\ )}+\sqrt{(\ )}}$.

161. $\dfrac{\sqrt{(\ )}+\sqrt{(\ )}}{\sqrt{(\ )}-\sqrt{(\ )}}$.

162. $\dfrac{\sqrt{(\ )}-\sqrt{(\ )}}{\sqrt{(\ )}+\sqrt{(\ )}}$.

163. $\dfrac{(\ )}{\sqrt{(\ )}+\sqrt{(\ )}+\sqrt{(\ )}}$.

164. $\dfrac{(\ )}{\sqrt{(\ )}-\sqrt{(\ )}-\sqrt{(\ )}}$.

EXPRESSIONS HAVING SURD FACTORS

EXERCISE 154

Obtain factors containing surds for each of the following:

1. x^2-2.
2. y^2-3.
3. z^2-8.
4. a^2-50.
5. $4x^2-5$.
6. $9y^2-13$.
7. $11-m^2$.
8. a^2+8a+1.
9. b^2-6b+2.
10. $c^2-12c-3$.
11. $d^2-10d+20$.
12. x^2+2x-1.
13. $y^2-14y-18$.
14. $9a^2-12a+1$.
15. $b^2+16b+19$.

16. $c^2+18c+57$.
17. $d^2-10d-100$.
18. a^2+a+1.
19. b^2-b-1.
20. c^2-3c-5.
21. k^2-5k-7.
22. x^3+2.
23. y^3+5.
24. z^3-7.
25. $2a^3-8$.

EVOLUTION OF SURDS

EXERCISE 155

Simplify each of the following:

1. $\sqrt{\sqrt{a}}$.
2. $\sqrt[3]{\sqrt{b}}$.
3. $\sqrt{\sqrt[3]{c}}$.
4. $\sqrt[3]{\sqrt[3]{d}}$.
5. $\sqrt{\sqrt[3]{9}}$.
6. $\sqrt[3]{\sqrt{8}}$.
7. $\sqrt[n]{\sqrt{5}}$.
8. $\sqrt[3]{\sqrt[n]{6}}$.
9. $\sqrt{\sqrt[5]{x^2}}$.
10. $\sqrt{\sqrt[3]{y^2}}$.
11. $\sqrt{\sqrt[3]{32}}$.
12. $\sqrt[3]{\sqrt[4]{81}}$.
13. $\sqrt{\sqrt{\sqrt{x}}}$.
14. $\sqrt{\sqrt[3]{\sqrt{y}}}$.
15. $\sqrt{8\sqrt{8}}$.
16. $\sqrt{2\sqrt{2}}$.
17. $3\sqrt[3]{3\sqrt[3]{3}}$.
18. $\sqrt{a\sqrt{a\sqrt{a}}}$.
19. $\sqrt[3]{a\sqrt[3]{b\sqrt[3]{c}}}$.
20. $\sqrt[m]{x\sqrt[n]{y\sqrt[p]{z}}}$.

SQUARE ROOT OF A BINOMIAL QUADRATIC SURD

EXERCISE 156

Find the square root of each of the following binomial quadratic surds:

1. $5-2\sqrt{6}$.
2. $7-2\sqrt{10}$.
3. $9-2\sqrt{14}$.
4. $8-2\sqrt{15}$.
5. $3-2\sqrt{2}$.
6. $6-2\sqrt{5}$.
7. $4-2\sqrt{3}$.
8. $8+2\sqrt{7}$.
9. $11+2\sqrt{10}$.
10. $11+2\sqrt{30}$.
11. $10+2\sqrt{21}$.
12. $13+2\sqrt{22}$.
13. $15+2\sqrt{26}$.
14. $14+2\sqrt{33}$.
15. $12-2\sqrt{35}$.
16. $19-2\sqrt{34}$.
17. $16-2\sqrt{39}$.
18. $17-2\sqrt{52}$.
19. $18+2\sqrt{77}$.
20. $7-\sqrt{24}$.
21. $7+\sqrt{40}$.
22. $12-\sqrt{44}$.
23. $14+\sqrt{52}$.
24. $12-4\sqrt{5}$.
25. $11-4\sqrt{6}$.
26. $15-10\sqrt{2}$.
27. $18+6\sqrt{5}$.
28. $9+3\sqrt{8}$.
29. $24+3\sqrt{28}$.
30. $20-5\sqrt{12}$.
31. $51+7\sqrt{8}$.
32. $88-9\sqrt{28}$.
33. $6+3\sqrt{3}$.
34. $3+\sqrt{5}$.
35. $4+\sqrt{7}$.
36. $5+\sqrt{21}$.

Fill in the following Blank Forms and simplify:

37. $\sqrt{(\ )+2\sqrt{10}}$.
38. $\sqrt{(\ )-2\sqrt{15}}$.
39. $\sqrt{12+2\sqrt{(\ )}}$.
40. $\sqrt{7-2\sqrt{(\ )}}$.
41. $\sqrt{(\ )-2\sqrt{(\ )}}$.
42. $\sqrt{(\ )+\sqrt{(\ )}}$.

IMAGINARY NUMBERS

REDUCTION TO STANDARD FORM

EXERCISE 157

Reduce each of the following powers of i to one of the numbers $1, -1, i$, or $-i$:

1. i^{21}.
2. i^{23}.
3. i^{34}.
4. i^{36}.
5. i^{39}.
6. i^{43}.
7. i^{50}.
8. $-i^{59}$.
9. $-i^{60}$.
10. i^{73}.
11. i^{82}.
12. i^{101}.
13. $(\sqrt{-1})^{9}$.
14. $(\sqrt{-1})^{12}$.
15. $(\sqrt{-1})^{14}$.
16. $(\sqrt{-1})^{15}$.
17. $(\sqrt{-1})^{25}$.
18. $(\sqrt{-1})^{32}$.
19. $(\sqrt{-1})^{27}$.
20. $(\sqrt{-1})^{53}$.
21. $-(\sqrt{-1})^{62}$.
22. $(\sqrt{-1})^{84}$.
23. $(\sqrt{-1})^{99}$.
24. $(\sqrt{-1})^{102}$.

Express each of the following imaginary numbers as a multiple of the unit of imaginaries $(\sqrt{-1}) \equiv i$:

25. $\sqrt{-4}$.
26. $4\sqrt{-25}$.
27. $a\sqrt{-b^2}$.
28. $\sqrt{-9}$.
29. $\sqrt{-16}$.
30. $\sqrt{-25}$.
31. $2\sqrt{-36}$.
32. $-3\sqrt{-49}$.
33. $5\sqrt{-64}$.
34. $-7\sqrt{-81}$.
35. $9\sqrt{-100}$.
36. $4\sqrt{-121}$.
37. $\sqrt{-a^2}$.
38. $\sqrt{-b^2}$.
39. $2\sqrt{-c^2}$.
40. $-3\sqrt{-d^2}$.
41. $x\sqrt{-y^2}$.
42. $-a\sqrt{-z^2}$.
43. $\sqrt{-4\,a^2}$.
44. $\sqrt{-49\,b^2}$.
45. $x\sqrt{-x^2}$.
46. $y\sqrt{-y^2}$.
47. $-(c\sqrt{-c^2}$.
48. $\sqrt{-a^4}$.
49. $\sqrt{-b^6}$.
50. $\sqrt{-c^8}$.
51. $\sqrt{-d^{10}}$.
52. $\sqrt{-h^{12}}$.
53. $\sqrt{-4\,a^4}$.
54. $\sqrt{-9\,b^4}$.

55. $\sqrt{-16\,c^6}$.
56. $\sqrt{-25\,d^8}$.
57. $2\sqrt{-36\,x^4}$.
58. $3\sqrt{-49\,y^4}$.
59. $5\sqrt{-64\,z^8}$.
60. $(\sqrt{-a})^{16}$.
61. $(\sqrt{-b})^{19}$.
62. $(\sqrt{-c})^{29}$.
63. $(\sqrt{-x})^{34}$.
64. $(\sqrt{-y})^{37}$.
65. $-(\sqrt{-m})^{46}$.
66. $(\sqrt{-n})^{51}$.
67. $(\sqrt{-h})^{76}$.
68. $(\sqrt{-a})^{97}$.
69. $(a\sqrt{-a})^4$.
70. $(b\sqrt{-b})^6$.
71. $-(x\sqrt{-x})^7$.
72. $(-y\sqrt{-y})^{10}$.
73. $(-z\sqrt{-z})^{13}$.
74. $-(w\sqrt{-w})^{16}$.
75. $-(a\sqrt{-a})^8$.

ADDITION AND SUBTRACTION OF IMAGINARY NUMBERS

EXERCISE 158

Simplify each of the following:

1. $2\sqrt{-1}+3\sqrt{-1}$.
2. $\sqrt{-4}+\sqrt{-1}$.
3. $\sqrt{-16}+\sqrt{-9}$.
4. $\sqrt{-49}-\sqrt{-25}$.
5. $\sqrt{-64}+\sqrt{-100}$.
6. $\sqrt{-121}-\sqrt{-100}$.
7. $2\sqrt{-81}+3\sqrt{-1}$.
8. $4\sqrt{-9}-6\sqrt{-1}$.
9. $\sqrt{-169}-2\sqrt{-36}$.
10. $\sqrt{-196}-4\sqrt{-9}$.
11. $3\sqrt{-49}-2\sqrt{-64}$.
12. $\sqrt{-8}+\sqrt{-18}$.
13. $\sqrt{-12}+\sqrt{-48}$.
14. $2\sqrt{-27}+3\sqrt{-75}$.
15. $4\sqrt{-32}-2\sqrt{-98}$.
16. $\sqrt{-a^2}+4\sqrt{-a^2}$.
17. $5\sqrt{-a^4}-2\sqrt{-a^4}$.
18. $\sqrt{-64\,x^2}+\sqrt{-25\,x^2}$.
19. $7\sqrt{-y^2}+2\sqrt{-49\,y^2}$.
20. $\sqrt{-225\,b^4}-3\sqrt{-9\,b^4}$.
21. $5\sqrt{-81\,c^2}-9\sqrt{-25\,c^2}$.
22. $2\sqrt{-121\,d^2}+3\sqrt{-100\,d^2}$.
23. $3\sqrt{-144\,a^2}-7\sqrt{-16\,a^2}$.
24. $\sqrt{-25}+\sqrt{-16}+\sqrt{-1}$.
25. $\sqrt{-144}-\sqrt{-36}-\sqrt{-4}$.
26. $9\sqrt{-1}+2\sqrt{-81}-12\sqrt{-4}$.
27. $2\sqrt{-36}+3\sqrt{-25}-4\sqrt{-9}$.
28. $5\sqrt{-81}-12\sqrt{-1}-11\sqrt{-4}$.
29. $\sqrt{-12}+\sqrt{-48}+\sqrt{-75}$.
30. $\sqrt{-50}+\sqrt{-8}+\sqrt{-72}$.
31. $3\sqrt{-20}-2\sqrt{-45}+5\sqrt{-125}$.
32. $4\sqrt{-96}+2\sqrt{-54}+\sqrt{-150}$.

33. $\sqrt{-a^2}+\sqrt{-4a^2}+\sqrt{-9a^2}$.
34. $\sqrt{-a^4}+\sqrt{-25a^4}-\sqrt{-16a^4}$.
35. $\sqrt{-36x^2}+\sqrt{-9x^2}-\sqrt{-64x^2}$.
36. $\sqrt{-49b^2}+\sqrt{-81b^2}-\sqrt{-49b^2}$.
37. $2\sqrt{-c^4}+3\sqrt{-25c^4}-\sqrt{-144c^4}$.
38. $3\sqrt{-4a^4}+4\sqrt{-9a^4}-2\sqrt{-81a^4}$.

Find the numerical value of each of the following:

39. $2+i^2$.
40. $4-i^2$.
41. $3+i^4$.
42. $3-i^4$.
43. $4+i^6$.
44. $6-i^6$.
45. $7+i^8$.
46. $8-i^8$.
47. $1+i^2$.
48. $9+2i^2$.
49. $10-3i^2$.
50. $11+4i^4$.
51. $2+2i^2$.
52. i^4+i^8.
53. i^4-i^2.
54. i^2+i^4.
55. i^9+i^7.
56. $2i^2+i^4$.
57. $i^2+i^4+i^8$.
58. $i^{16}-i^{14}-i^{10}$.
59. $8i^8-4i^4+2i^2$.
60. $i^3+i^4+i^5+i^6$.
61. $i^{20}-i^{18}+i^{16}-i^{14}$.
62. $2i^2-3i^4+4i^8-5i^{10}$.

MULTIPLICATION OF IMAGINARY NUMBERS

EXERCISE 159

Simplify each of the following:

1. $2i\times i$.
2. $3i\times i$.
3. $i\times 6i$.
4. $4i\times 5i$.
5. $8i\times 3i$.
6. $-7i\times i$.
7. $4i\times(-3i)$.
8. $-9i\times(-i)$.
9. $-10i\times(-2i)$.
10. $\sqrt{3}\times\sqrt{-2}$.
11. $\sqrt{5}\times\sqrt{-4}$.
12. $\sqrt{-2}\times\sqrt{-1}$.
13. $\sqrt{-1}\times\sqrt{-3}$.
14. $\sqrt{-2}\times\sqrt{-3}$.
15. $\sqrt{-5}\times\sqrt{-6}$.
16. $-\sqrt{-1}\times\sqrt{-2}$.
17. $-\sqrt{-1}\times(-\sqrt{-3})$.
18. $-2\sqrt{-3}\times(-3\sqrt{-2})$.
19. $2\sqrt{-4}\times 3\sqrt{-9}$.
20. $-\sqrt{-16}\times\sqrt{-1}$.
21. $\sqrt{-2}\times\sqrt{-6}$.
22. $\sqrt{-5}\times\sqrt{-10}$.
23. $\sqrt{-6}\times\sqrt{-3}$.
24. $\sqrt{-14}\times\sqrt{-7}$.
25. $\sqrt{-8}\times\sqrt{-3}$.
26. $ia\times i$.
27. $ib\times i$.
28. $ix\times iy$.
29. $-ia\times ib$.
30. $-ic\times(-iw)$.
31. $2ix\times im$.

32. $3\,ia \times iy$

33. $ic \times 2\,id.$

34. $ix \times ix.$

35. $3\,ig \times 3\,ik.$

36. $-6\,ik \times 3\,iy.$

37. $-2\,ia \times 8\,ib.$

38. $ix^2 \times ix.$

39. $iab \times iac.$

40. $iabc \times ia.$

41. $iabc \times ibcd.$

42. $\sqrt{-d} \times \sqrt{-1}.$

43. $-\sqrt{-h} \times \sqrt{-1}.$

44. $\sqrt{-x} \times \sqrt{-y}.$

45. $\sqrt{-m} \times \sqrt{-n}.$

46. $\sqrt{-a^2} \times \sqrt{-1}.$

47. $\sqrt{-2\,a} \times \sqrt{-y}.$

48. $\sqrt{-3\,b} \times \sqrt{-c}.$

49. $2\sqrt{-c} \times \sqrt{-3\,c}.$

50. $5\sqrt{-5\,d} \times \sqrt{-2\,h}.$

51. $\sqrt{-3\,a} \times \sqrt{-3\,b}.$

52. $\sqrt{-5\,c} \times \sqrt{-6\,c}.$

53. $\sqrt{-10\,a} \times \sqrt{-2\,a}.$

54. $\sqrt{-3\,b} \times \sqrt{-6\,b}.$

55. $\sqrt{-21\,d} \times \sqrt{-7\,d}.$

56. $\sqrt{-xy} \times \sqrt{-x}.$

57. $\sqrt{-xyz} \times \sqrt{-xz}.$

58. $x\sqrt{-yz} \times xy\sqrt{-z}.$

59. $x\sqrt{-x^2} \times x^3\sqrt{-x^4}.$

60. $3\,i \times 2\,i \times i.$

61. $5\,i \times 3\,i \times i.$

62. $6\,i \times 4\,i \times i.$

63. $-7\,i \times 5\,i \times i.$

64. $-8\,i \times 3\,i \times 2\,i.$

65. $ia \times ib \times ic.$

66. $ix \times iy \times iz.$

67. $2\,ix \times 4\,ix \times 6\,ix.$

68. $ia^3 \times ia^2 \times ia.$

69. $\sqrt{-3} \times \sqrt{-2} \times \sqrt{-5}.$

70. $\sqrt{-9} \times \sqrt{-4} \times \sqrt{-1}.$

71. $\sqrt{-25} \times \sqrt{-49} \times \sqrt{-16}.$

72. $\sqrt{-64} \times \sqrt{-36} \times \sqrt{-9}.$

73. $\sqrt{-12} \times \sqrt{-3} \times \sqrt{-1}.$

74. $\sqrt{-14} \times \sqrt{-7} \times \sqrt{-2}.$

75. $\sqrt{-15} \times \sqrt{-5} \times \sqrt{-3}.$

76. $\sqrt{-a} \times \sqrt{-b} \times \sqrt{-c}.$

77. $\sqrt{-2\,c} \times \sqrt{-3\,d} \times \sqrt{-k}.$

78. $\sqrt{-3\,x} \times \sqrt{-4\,y} \times \sqrt{-5\,z}.$

79. $\sqrt{-3\,abc} \times \sqrt{-2\,ab} \times \sqrt{-a}.$

80. $\sqrt{-x^2} \times \sqrt{-y^2} \times \sqrt{-z^2}.$

81. $(-\sqrt{-a^2})(-\sqrt{-b^2})(-\sqrt{-c^2}).$

82. $ia \times ib \times ic \times id.$

83. $ib \times id \times ix \times iy.$

84. $2\,ia \times 3\,ih \times im \times i.$

85. $iabc \times iab \times ia \times i.$

86. $5\,ib \times 4\,ib \times ic \times i.$

87. $\sqrt{-2} \times \sqrt{-3} \times \sqrt{-4} \times \sqrt{-1}.$

88. $\sqrt{-5} \times \sqrt{-6} \times \sqrt{-2} \times \sqrt{-1}$.

89. $\sqrt{-6} \times \sqrt{-2} \times \sqrt{-3} \times \sqrt{-1}$.

90. $\sqrt{-10} \times \sqrt{-5} \times \sqrt{-2} \times \sqrt{-1}$.

91. $i^3 \times i$.

92. $i^4 \times i^2$.

93. $i^5 \times i^4$.

94. $2\,i^6 \times 4\,i^3$.

95. $-7\,i^7 \times 3\,i^3$.

96. $i^4 \times i^3 \times i^2$.

97. $i^5 \times i^6 \times i^8$.

98. $-i^3 \times i^5 \times i^7$.

99. $3\,i^4 \times 4\,i^5 \times i^2$.

100. $5\,i^3 \times 4\,i^4 \times 3\,i^5$.

101. $-7\,i^8 \times 5\,i^6 \times 3\,i^4$.

102. $i^2 \times i^4 \times i^6 \times i^3$.

103. $2\,i^5 \times 3\,i^4 \times 4\,i^3 \times 5\,i^2$.

DIVISION OF IMAGINARY NUMBERS

EXERCISE 160

Simplify each of the following:

1. $\sqrt{-22} \div \sqrt{11}$.

2. $\sqrt{-39} \div \sqrt{13}$.

3. $\sqrt{-30} \div \sqrt{3}$.

4. $\sqrt{-6} \div -\sqrt{2}$.

5. $\sqrt{-14} \div -\sqrt{7}$.

6. $\sqrt{-15} \div -\sqrt{3}$.

7. $-\sqrt{-26} \div \sqrt{-13}$.

8. $\sqrt{-28} \div (-\sqrt{-7})$.

9. $\sqrt{-20} \div \sqrt{-5}$.

10. $12\,i \div 3\,i$.

11. $18\,i \div 6\,i$.

12. $24\,i \div 8\,i$.

13. $35\,i \div 5\,i$.

14. $\sqrt{-7} \div \sqrt{-2}$.

15. $\sqrt{-6} \div \sqrt{-5}$.

16. $\sqrt{-1} \div \sqrt{-10}$.

17. $\sqrt{-11} \div \sqrt{-3}$.

18. $2\sqrt{-5} \div \sqrt{-4}$.

19. $3\sqrt{-6} \div \sqrt{-9}$.

20. $7\sqrt{-14} \div \sqrt{-2}$.

21. $9\sqrt{-16} \div 16\sqrt{-9}$.

22. $49\sqrt{-25} \div 25\sqrt{-49}$.

23. $\sqrt{-a} \div \sqrt{-1}$.

24. $-\sqrt{-b} \div \sqrt{-1}$.

25. $\sqrt{-c} \div (-\sqrt{-1})$.

26. $-\sqrt{-d} \div (-\sqrt{-1})$.

27. $\sqrt{-abc} \div \sqrt{-a}$.

28. $-\sqrt{-x} \div \sqrt{-y}$.

29. $\sqrt{-a^2} \div \sqrt{-b^2}$.

30. $-\sqrt{-x^2} \div \sqrt{-y^2}$.

31. $\sqrt{-a^2} \div \sqrt{-a}$.

32. $\sqrt{-b} \div \sqrt{-b^2}$.

33. $\sqrt{-abc} \div \sqrt{bcd}$.

34. $x\sqrt{-y} \div y\sqrt{-x}$.

35. $a\sqrt{-a} \div b\sqrt{-b}$.

36. $i\sqrt{7} \div i\sqrt{10}$.

37. $i\sqrt{2} \div i\sqrt{5}$.

38. $-i\sqrt{11} \div i\sqrt{3}$.

39. $\dfrac{2}{\sqrt{-3}}$.
40. $\dfrac{4}{\sqrt{-5}}$.
41. $\dfrac{6}{\sqrt{-2}}$.
42. $\dfrac{12}{\sqrt{-6}}$.
43. $\dfrac{11}{\sqrt{-11}}$.
44. $\dfrac{-9}{\sqrt{-3}}$.
45. $\dfrac{1}{i}$.
46. $\dfrac{2}{i^2}$.
47. $\dfrac{3}{i^3}$.
48. $\dfrac{4}{i^4}$.
49. $\dfrac{5}{i^5}$.
50. $\dfrac{8}{i^8}$.
51. $\dfrac{9}{i^9}$.
52. $\dfrac{11}{i^{11}}$.
53. $\dfrac{13}{i^{10}}$.
54. $-\dfrac{15}{i^{14}}$.
55. $\dfrac{6}{i^7}$.
56. $-\dfrac{1}{i^6}$.

COMPLEX NUMBERS

ADDITION AND SUBTRACTION OF COMPLEX NUMBERS

EXERCISE 161

Simplify each of the following expressions:

1. $(3+4i)+(7+2i)$.
2. $(4+2i)+(6+3i)$.
3. $(9+3i)+(8+5i)$.
4. $(6+7i)+(10-11i)$.
5. $(8-3i)+(4-2i)$.
6. $(5-9i)+(10-i)$.
7. $(4+11i)-(2+8i)$.
8. $(12+14i)-(15+7i)$.
9. $(19-8i)-(4+5i)$.
10. $(18-6i)-(6-18i)$.
11. $(13+4i)-(13+6i)$.
12. $(9-5i)+(-3+5i)$.
13. $(7-7i)-(7+7i)$.
14. $(12-13i)-(14-15i)$.
15. $(2+\sqrt{-1})+(3+\sqrt{-1})$.
16. $(4+3\sqrt{-1})+(-7+5\sqrt{-1})$.
17. $(6-2\sqrt{-1})-(-3-8\sqrt{-1})$.
18. $(9-19\sqrt{-1})-(10-20\sqrt{-1})$.
19. $(20-\sqrt{-1})-(1-20\sqrt{-1})$.
20. $(25-8\sqrt{-1})-(8-25\sqrt{-1})$.
21. $(2+\sqrt{-9})+(-5+\sqrt{-4})$.
22. $(6+\sqrt{-9})+(4+\sqrt{-4})$.
23. $(3+\sqrt{-16})+(-2+\sqrt{-25})$.
24. $(9+\sqrt{-81})+(8+\sqrt{-64})$.

25. $(12+\sqrt{-81})+(7-\sqrt{-49})$.
26. $(9+\sqrt{-4})+(-4+\sqrt{-9})$.
27. $(16-\sqrt{-25})-(25-\sqrt{-16})$.
28. $(36-\sqrt{-49})-(49-\sqrt{-36})$.
29. $(12-\sqrt{-121})-(13+\sqrt{-144})$.
30. $(11-\sqrt{-169})-(22-\sqrt{-196})$.

MULTIPLICATION OF COMPLEX NUMBERS

EXERCISE 162

Simplify each of the following expressions:

1. $(2+i)(2-i)$.
2. $(4+i)(4-i)$.
3. $(5+i)(5-i)$.
4. $(6+i)(6-i)$.
5. $(7+2i)(7-2i)$.
6. $(8+3i)(8-3i)$.
7. $(9+8i)(9-8i)$.
8. $(2+7i)(2-7i)$.
9. $(-3+5i)(-3-5i)$.
10. $(-4+9i)(-4-9i)$
11. $(1+\sqrt{-1})(1-\sqrt{-1})$.
12. $(2+\sqrt{-1})(2-\sqrt{-1})$.
13. $(4+\sqrt{-1})(4-\sqrt{-1})$.
14. $(3+\sqrt{-2})(3-\sqrt{-2})$.
15. $(1+\sqrt{-5})(1-\sqrt{-5})$.
16. $(2+\sqrt{-7})(2-\sqrt{-7})$.
17. $(4+3\sqrt{-1})(4-3\sqrt{-1})$.
18. $(5+4\sqrt{-1})(5-4\sqrt{-1})$.
19. $(\sqrt{2}+\sqrt{-1})(\sqrt{2}-\sqrt{-1})$.
20. $(\sqrt{3}+\sqrt{-1})(\sqrt{3}-\sqrt{-1})$.
21. $(\sqrt{5}+\sqrt{-1})(\sqrt{5}-\sqrt{-1})$.
22. $(\sqrt{10}+\sqrt{-5})(\sqrt{10}-\sqrt{-5})$.
23. $(\sqrt{5}+\sqrt{-7})(\sqrt{5}-\sqrt{-7})$.
24. $(2\sqrt{3}+\sqrt{-1})(2\sqrt{3}-\sqrt{-1})$.
25. $(3\sqrt{2}+\sqrt{-3})(3\sqrt{2}-\sqrt{-3})$.
26. $(4\sqrt{5}+\sqrt{-4})(4\sqrt{5}-\sqrt{-4})$.
27. $(5\sqrt{6}+2\sqrt{-3})(5\sqrt{6}-2\sqrt{-3})$.
28. $(2\sqrt{5}+3\sqrt{-6})(2\sqrt{5}-3\sqrt{-6})$.
29. $(2\sqrt{3}+4\sqrt{-5})(2\sqrt{3}-4\sqrt{-5})$.
30. $(3\sqrt{2}+5\sqrt{-4})(3\sqrt{2}-5\sqrt{-4})$.
31. $(9\sqrt{2}+2\sqrt{-9})(9\sqrt{2}-2\sqrt{-9})$.

POWERS OF COMPLEX NUMBERS

EXERCISE 163

Expand each of the following:

1. $(2+i)^2$.
2. $(3+i)^2$.
3. $(4+i)^2$.
4. $(5-i)^2$.
5. $(1+i)^2$.
6. $(1-i)^2$.
7. $(6-i)^2$.
8. $(7-i)^2$.
9. $(1+2\,i)^2$.
10. $(1+3\,i)^2$.
11. $(1+4\,i)^2$.
12. $(1-5\,i)^2$.
13. $(1-6\,i)^2$.
14. $(2+3\,i)^2$.
15. $(3+5\,i)^2$.
16. $(4-3\,i)^2$.
17. $(5-6\,i)^2$.
18. $(7-2\,i)^2$.
19. $(5+\sqrt{-1})^2$.
20. $(8+\sqrt{-1})^2$.
21. $(9-\sqrt{-1})^2$.
22. $(6+\sqrt{-2})^2$.
23. $(7-\sqrt{-3})^2$.
24. $(8-\sqrt{-5})^2$.
25. $(\sqrt{2}+\sqrt{-1})^2$.
26. $(\sqrt{3}+\sqrt{-1})^2$.
27. $(\sqrt{5}+\sqrt{-1})^2$.
28. $(\sqrt{6}-\sqrt{-1})^2$.
29. $(\sqrt{7}-\sqrt{-1})^2$.
30. $(\sqrt{10}-\sqrt{-1})^2$.
31. $(5+\sqrt{-2})^2$.
32. $(3+\sqrt{-5})^2$.
33. $(6-\sqrt{-3})^2$.
34. $(\sqrt{11}+\sqrt{-2})^2$.
35. $(\sqrt{13}+\sqrt{-3})^2$.
36. $(\sqrt{17}-\sqrt{-5})^2$.
37. $(\sqrt{21}-\sqrt{-2})^2$.
38. $(2\sqrt{3}+\sqrt{-1})^2$.
39. $(3\sqrt{5}+\sqrt{-1})^2$.
40. $(a+i)^2$.
41. $(b+i)^2$.
42. $(c-i)^2$.
43. $(x-i)^2$.
44. $(x+iy)^2$.
45. $(m+\sqrt{-1})^2$.
46. $(n-\sqrt{-1})^2$.
47. $(a+\sqrt{-2})^2$.
48. $(b+\sqrt{-3})^2$.
49. $(c-\sqrt{-5})^2$.
50. $(d-\sqrt{-6})^2$.
51. $(x+2\sqrt{-1})^2$.
52. $(y+3\sqrt{-1})^2$.
53. $(z-5\sqrt{-1})^2$.
54. $(w+3\sqrt{-2})^2$.
55. $(a-2\sqrt{-3})^2$.
56. $(b+4\sqrt{-5})^2$.
57. $(c-6\sqrt{-7})^2$.
58. $(a+2\,i)^2$.
59. $(b+3\,i)^2$.
60. $(c-4\,i)^2$.
61. $(d-5\,i)^2$.

EXPRESSIONS HAVING COMPLEX FACTORS

EXERCISE 164

Find complex factors for each of the following:

1. x^2+y^2.
2. m^2+n^2.
3. a^2+4.
4. b^2+9.
5. c^2+1.
6. d^2+2.
7. $x^2+4\,x+5$.
8. $y^2+6\,y+10$.
9. $z^2+8\,z+19$.
10. $w^2-2\,w+2$.
11. $a^2-8\,a+21$.
12. $4\,x^2+12\,x+15$.

13. $9y^2+30y+32$.
14. $16a^2-8a+17$.
15. $2a^2+3a+4$.
16. $9b^2-8b-7$.
17. x^4+y^4.
18. x^2+3y^2.

SQUARE ROOTS OF COMPLEX NUMBERS

EXERCISE 165

Express as complex numbers the square roots of the following complex numbers:

1. $1+2\sqrt{-2}$.
2. $-1+2\sqrt{-2}$.
3. $-1-\sqrt{-8}$.
4. $3+4\sqrt{-1}$.
5. $4+2\sqrt{-12}$.
6. $2+\sqrt{-60}$.
7. $14+\sqrt{-60}$.
8. $41-14\sqrt{-8}$.
9. $-36+9\sqrt{-48}$.
10. $1-6\sqrt{-10}$.
11. $-1-2i\sqrt{2}$.
12. $11+4i\sqrt{3}$.
13. $-5-4i\sqrt{6}$.
14. $13+8i\sqrt{3}$.
15. $-11-8i\sqrt{5}$.

SIMPLIFICATION OF FRACTIONS HAVING COMPLEX DENOMINATORS

EXERCISE 166

Express each of the following fractions as an equivalent fraction with a real denominator:

1. $\dfrac{3}{3+\sqrt{-2}}$.
2. $\dfrac{5}{2+\sqrt{-1}}$.
3. $\dfrac{10}{3+\sqrt{-1}}$.
4. $\dfrac{17}{4-\sqrt{-1}}$.
5. $\dfrac{13}{5+\sqrt{-1}}$.
6. $\dfrac{74}{6-\sqrt{-1}}$.
7. $\dfrac{12}{1+\sqrt{-3}}$.
8. $\dfrac{18}{1-\sqrt{-5}}$.
9. $\dfrac{15}{5+\sqrt{-2}}$.
10. $\dfrac{16}{3+\sqrt{-3}}$.
11. $\dfrac{5}{5-\sqrt{-5}}$.
12. $\dfrac{2+\sqrt{-1}}{2-\sqrt{-1}}$.
13. $\dfrac{3+\sqrt{-1}}{3-\sqrt{-1}}$.
14. $\dfrac{5-\sqrt{-1}}{5+\sqrt{-1}}$.
15. $\dfrac{4+\sqrt{-3}}{4-\sqrt{-3}}$.
16. $\dfrac{2+\sqrt{-5}}{2-\sqrt{-5}}$.
17. $\dfrac{a+ib}{a-ib}$.
18. $\dfrac{x-iy}{x+iy}$.

REVIEW

MENTAL EXERCISE 167

1. Show that $x^6 + 6x^5 + 9x^4$ is the square of a binomial.

2. Find the continued product of $x^2 - y^2$, $x^2 + y^2$, $x^4 + y^4$, and $x^8 + y^8$.

Obtain each of the following quotients:

3. $(a - b) \div (b - a)$.
4. $(c - 4) \div (4 - c)$.
5. $(5 - d) \div (d - 5)$.
6. $(x^2 - y^2) \div (y - x)$.

Square each of the following:

7. a.
8. b^2.
9. c^3.
10. $\frac{1}{2}$.
11. $\frac{1}{3}$.
12. $-\frac{1}{4}$.
13. $\frac{2}{5}$.
14. .2.
15. .3.
16. .5.
17. .06.
18. $a^{\frac{1}{2}}$.
19. $b^{\frac{2}{3}}$.
20. c^n.
21. $d^{\frac{1}{n}}$.
22. .01.
23. $c^{-\frac{1}{2}}$.
24. $d^{-\frac{2}{3}}$.
25. -3.
26. $-x^3$.

Find the value of

27. $\frac{2^2}{3}$.
28. $\left(\frac{2}{3}\right)^2$.
29. $\frac{2}{3^2}$.

Express as a single power of 5:

30. 25^3; 125^2; 625^4.

Find the value of

31. $3^2 - 2^3$.
32. $2^3 + 2^{-3}$.

Express as a power of a base:

33. $(2^x)^x$.
34. $(y^y)^y$.
35. $(z^2)^{2z}$.

36. Find the values of $(\frac{1}{2})^{-2}$ and -2^{-2}.

Express the following with positive exponents:

37. $a^{-1} \div b^{-\frac{1}{2}}$.
38. $x^{-1} \div y^{-2}$.
39. $\left(\frac{1}{m}\right)^{-1} \div \left(\frac{1}{n}\right)^{-1}$.

Show that the following identities are true:

40. $\frac{y - x}{z - y} \equiv \frac{x - y}{y - z}$.
41. $\frac{(a - b)^2}{a^2} \equiv 1 - 2\left(\frac{b}{a}\right) + \left(\frac{b}{a}\right)^2$.
42. $(x - y)(z - y)(x - z) \equiv (x - y)(y - z)(z - x)$.

Simplify each of the following:

43. $\sqrt[6]{8\,a^3b^6}$. **44.** $\sqrt[8]{16\,a^6y^4}$. **45.** $\sqrt[9]{27\,m^3n^6}$.

46. Regarding x as the unknown, solve $\frac{a}{x}+\frac{b}{x}+\frac{c}{x}=d$.

Distinguish between

47. $\frac{3\,a}{2}$ and $a^{\frac{3}{2}}$.

48. x^{-1} and $-x$.

49. $(\frac{2}{3})-1$, $(\frac{2}{3})^{-1}$, and $(\frac{2}{3})(-1)$.

50. $a^{-1}+b^{-1}$ and $\frac{1}{a+b}$.

51. $-1\div a$ and $1\div a^{-1}$.

52. $\frac{0}{a}$ and $\frac{1}{a^0}$.

53. $0\cdot 5$, 5^0, 0^5, $\frac{5}{0}$, and $\frac{0}{5}$.

54. $(-a)^{-2}$ and $-a^{-2}$.

55. $(-b)^{-3}$ and $-b^{-3}$.

QUADRATIC EQUATIONS

INCOMPLETE QUADRATIC EQUATIONS

EXERCISE 168

Solve the following incomplete quadratic equations, regarding x, y, z, and w as representing unknown numbers:

1. $x^2-4=0$.
2. $x^2-9=0$.
3. $x^2-25\,x=0$.
4. $y^2=36$.
5. $y^2=100$.
6. $y^2=121\,y$.
7. $144=z^2$.
8. $196=z^2$.
9. $z^2=3$.
10. $x^2=5$.
11. $x^2=12$.
12. $x^2=18\,x$.
13. $y^2-7=0$.
14. $y^2-10=0$.
15. $y^2-20\,y=0$.
16. $z^2-27=0$.
17. $z^2-24=0$.
18. $z^2=m^2$.
19. $w^2=a$.
20. $w^2=bw$.
21. $w^2=a+b$.
22. $w^2-k^2=0$.
23. $x^2+n=m$.
24. $x^2+1=a$.
25. $y^2=c-d$.
26. $y^2=\frac{1}{4}$.
27. $z^2=\frac{1}{81}$.
28. $z^2=\frac{1}{100}$.
29. $x^2=\frac{1}{a^2}$.
30. $x^2=\frac{a^2}{b^2}$.
31. $169-y^2=0$.
32. $225-y^2=0$.
33. $2\,z^2-64=z^2$.
34. $9\,z^2=1$.
35. $49\,w^2=1$.
36. $4\,x^2=9$.
37. $9\,x^2=25$.
38. $36\,y^2=121$.
39. $16\,y^2=25$.
40. $a^2y^2=b^2$.
41. $2\,z^2=1$.
42. $3\,z^2=4$.
43. $5\,w^2=49+w^2$.
44. $36-w^2=8\,w^2$.
45. $64+w^2=17\,w^2$.
46. $x^2-a=b-x^2$.
47. $x^2-3\,a^2=5\,a^2-x^2$.
48. $x^2+a^2=5\,a^2$.
49. $y^2-c^2=8\,c^2$.

50. $2y^2 = 18$.
51. $3z^2 = 27$.
52. $cw^2 = 1$.
53. $mw^2 = n$.
54. $\frac{x^2}{49} = 1$.
55. $\frac{y^2}{5} = 20$.
56. $\frac{y^2}{25} = 4$.
57. $\frac{w^2}{a} = a$.
58. $\frac{x^2}{d^2} = 1$.
59. $\frac{c^2}{x^2} = 1$.
60. $\frac{x^2}{5} - 5 = 0$.
61. $\frac{4}{z^2} = 1$.
62. $\frac{16}{w^2} = 1$.
63. $\frac{1}{x^2} = 25$.
64. $\frac{1}{y^2} = 81$.
65. $\frac{m^2}{z^2} = 4$.
66. $\frac{1}{z^2} - 1 = 0$.
67. $w = \frac{a^2}{w}$.
68. $\frac{1}{4x^2} = 1$.
69. $\frac{1}{25y^2} = 1$.
70. $3y^2 = \frac{1}{3}$.
71. $\frac{1}{8w^2} = 2$.
72. $\frac{1}{x^2} = \frac{1}{4}$.
73. $\frac{1}{y^2} = \frac{1}{a^2}$.
74. $\frac{a^2}{b^2} = \frac{1}{z^2}$.
75. $\frac{1}{4z^2} = \frac{1}{9}$.
76. $\frac{w}{2} = \frac{2}{w}$.
77. $\frac{5}{w} = \frac{w}{5}$.
78. $\frac{6}{x} - \frac{x}{6} = 0$.
79. $\frac{x}{7} - \frac{7}{x} = 0$.
80. $\frac{x}{a} = \frac{a}{x}$.
81. $\frac{c}{y} = \frac{y}{c}$.
82. $\frac{d}{z} - \frac{z}{d} = 0$.
83. $\frac{2w}{5} - \frac{5}{2w} = 0$.
84. $\frac{7}{3w} - \frac{3w}{7} = 0$.
85. $\frac{x^2+1}{5} = 2$.
86. $\frac{x^2+2}{2} = 3$.
87. $\frac{x^2+2}{9} = 2$.

COMPLETE QUADRATIC EQUATIONS

EXERCISE 169

Solve the following complete quadratic equations by the method of factoring:

1. $x^2 - 2x + 1 = 0$.
2. $x^2 - 2x - 3 = 0$.
3. $x^2 + x - 6 = 0$.
4. $x^2 + x = 12$.
5. $x^2 + x = 42$.
6. $x^2 - 7x = -10$.
7. $x^2 + 2x = 8$.
8. $x^2 + 2x = 15$.
9. $x^2 - 13x = 48$.
10. $x^2 - 20 = x$.
11. $x^2 - 72 = x$.
12. $x^2 - 21 = 4x$.
13. $x^2 + 28 = 11x$.
14. $x^2 - 90 = x$.
15. $x^2 - 63 = -2x$.
16. $x(x+7) = 18$.
17. $x(x-3) = 70$.
18. $x(x+4) = 45$.

19. $x(x-8)=33$.
20. $x^2+2(3x+4)=0$.
21. $x^2+4(x-3)=0$.
22. $x^2+9(x+2)=0$.
23. $x^2+2(x-12)=0$.
24. $2x^2-3x+1=0$.
25. $3x^2+x-2=0$.
26. $2x^2+3x-5=0$.
27. $5x^2-11x+2=0$.
28. $7x^2-4x-3=0$.
29. $(x-2)(x-5)=x-5$.
30. $(x-4)(x-7)=x-4$.
31. $(x-3)(x-10)=x-10$.
32. $(x-2)(x+5)=x+5$.
33. $(x+1)(x-6)=x+1$.
34. $(x+8)(x+9)=x+9$.
35. $(x+11)(x-5)=x+11$.
36. $(x-7)(x+1)=2(x+1)$.
37. $(x-4)(x+2)=3(x+2)$.
38. $(x-10)(x-5)=4(x-10)$.

COMPLETION OF THE SQUARE

MENTAL EXERCISE 170

Complete the square with respect to each of the following binomials:

1. x^2+2x.
2. y^2+4y.
3. z^2+8z.
4. a^2+12a.
5. b^2-14b.
6. c^2-16c.
7. d^2+18d.
8. g^2-20g.
9. $81h^2-18h$.
10. $121c^2-22c$.
11. b^2+2bc.
12. x^2+22xy.
13. a^2-24ab.
14. $4x^2-12xy$.
15. $9h^2+60hk$.
16. $4b^2+28by$.
17. $25z^2-60zw$.
18. $64a^2+48ab$.
19. $9a^2+66ab$.
20. $121c^2-44cd$.
21. x^2y^2-2xy.
22. c^2d^2+2cd.
23. $25g^2k^2-40gk$.
24. $81c^2d^2-36cd$.
25. $49b^2c^2+70bc$.
26. $144c^2x^2+120cx$.
27. $16b^4+8b^2$.
28. $9c^4-12c^2d^2$.
29. $4h^4k^4+36h^2k^2$.
30. $36c^6-12c^3d^3$.
31. $25h^8+30h^4k^4$.
32. $49d^6+84d^3h^2$.
33. $81x^{2m}+36x^m$.
34. $100x^{2m}y^{2m}-60x^my^m$.
35. $c^{6x}y^{4x}+20^{3x}y^{2x}$.
36. $121x^{4n}y^{2n}+44x^{2n}y^n$.
37. x^2+3x.
38. z^2-7z.
39. y^2+5y.
40. a^2+a.
41. $4a^2+5a$.
42. $9b^2+8b$.
43. $16c^2-10c$.
44. $25d^2-5d$.
45. $36h^2-13h$.
46. $9m^2+m$.
47. $16n^2-n$.
48. $a^2+\frac{a}{2}$.
49. $x^2+\frac{x}{3}$.
50. $y^2+\frac{y}{4}$.
51. $z^2-\frac{z}{5}$.
52. $w^2+\frac{2w}{3}$.
53. $a^2+\frac{4a}{5}$.
54. $b^2+\frac{3b}{2}$.

55. $c^2 - \frac{5c}{3}$.
56. $d^2 - \frac{7d}{5}$.
57. $4x^2 + \frac{x}{3}$.
58. $9y^2 + \frac{y}{2}$.
59. $16z^2 + \frac{z}{5}$.
60. $4w^2 - \frac{2w}{5}$.
61. $9a^2 + \frac{6a}{7}$.
62. $25b^2 - \frac{20b}{3}$.

SOLUTION BY COMPLETING THE SQUARE

EXERCISE 171

Solve each of the following quadratic equations by the method of completing the square:

1. $x^2 + 2x = 3$.
2. $x^2 + 2x = 8$.
3. $x^2 - 4x = 5$.
4. $x^2 + 6x = 7$.
5. $x^2 - 6x = -5$.
6. $x^2 - 8x = 9$.
7. $x^2 + 8x = 48$.
8. $x^2 + 10x = 24$.
9. $x^2 - 12x = -11$.
10. $x^2 + 14x = -24$.
11. $x^2 - 16x = 17$.
12. $x(x - 10) = 11$.
13. $x(x + 12) = 13$.
14. $x(x - 14) = 15$.
15. $x(x + 16) = -15$.
16. $x^2 + 2bx = a^2 - b^2$.
17. $x^2 + 2dx = c^2 + 2cd$.
18. $x^2 + 4bx = 5b^2$.
19. $x^2 - 8cx = -7c^2$.
20. $x^2 + 18dx = 19d^2$.
21. $4x^2 + 12x = 7$.
22. $4x^2 + 20x = -9$.
23. $9x^2 - 30x = -16$.
24. $25x^2 - 40x = 9$.
25. $x^2 + 3x = \frac{7}{4}$.
26. $x^2 - 5x = \frac{11}{4}$.
27. $x^2 - \frac{3x}{2} = \frac{27}{16}$.
28. $x^2 + \frac{2x}{3} = \frac{8}{9}$.
29. $x^2 - \frac{4x}{5} = \frac{21}{25}$.
30. $x^2 - \frac{6x}{5} = \frac{7}{25}$.
31. $x^2 + 2x = 13$.
32. $x^2 - 4x = 9$.
33. $x^2 - 6x = 10$.
34. $x^2 + 8x = 7$.
35. $x^2 + 10x = 8$.
36. $x^2 + 16x = 5$.
37. $x^2 + 18x = -50$.
38. $x^2 - 20x = -105$.
39. $x^2 - 22x = -100$.
40. $x^2 + 24x = -109$.
41. $x^2 - 8x = -27$.
42. $x^2 - 10x = -59$.
43. $x^2 - 12x = -43$.
44. $x^2 - 14x = -59$.
45. $x^2 + 22x = -127$.
46. $x^2 - 24x = -149$.

47. $x^2 - 2\,dx = c^2 - d^2$.
48. $x^2 - 4\,cx = 12\,c^2$.
49. $x^2 - 3\,a^2 = 2\,ax$.
50. $x^2 + 2\,ax + b^2 = 0$.
51. $a^2x^2 - 2\,ax = b - a^2$.
52. $11\,mx = 6\,x^2 - 7\,m^2$.
53. $4(3\,x^2 - 5\,d^2) = dx$.
54. $2(6\,x^2 + 5\,h^2) + 23\,hx = 0$.
55. $x^2 - (a + b)x + 5\,ab = 2(a^2 + b^2)$.

EXERCISE 172

Solve each of the following equations by the Hindu Method, verifying all exact rational results. Whenever surds appear in the *exact* solutions, *approximate* values correct to four places of decimal should be obtained.

1. $3\,x^2 + 10\,x + 8 = 0$.
2. $8\,x^2 + 26\,x + 15 = 0$.
3. $5\,x^2 - 16\,x + 121 = 0$.
4. $3\,x^2 + 50 = 25\,x$.
5. $15\,x^2 + 16\,x = 15$.
6. $10\,x^2 - 9 = 43\,x$.
7. $25\,x^2 + 20\,x = 33$.
8. $x - 1 = 20\,x^2$.
9. $(2\,x - 7)^2 = 6\,x$.
10. $(x + 1)(2\,x + 3) = x^2 - 11$.
11. $3(x+1)(x-1) = 2(27 - 5\,x)$.
12. $(2\,x + 1)(x + 2) = (x - 1)^2$.
13. $\dfrac{x^2}{20} - \dfrac{1}{4} = \dfrac{x}{5}$.
14. $\dfrac{x^2}{3} - \dfrac{x}{7} = \dfrac{20}{21}$.
15. $\dfrac{x^2}{11} + \dfrac{1}{11} = \dfrac{x}{5}$.
16. $\dfrac{x(2\,x - 5)}{30} + \dfrac{1}{6} = \dfrac{x}{5}$.

SOLUTION BY THE FORMULA

EXERCISE 173

Solve each of the following quadratic equations by substituting the values represented by a, b, and c in the general solution $x = \dfrac{-b \pm \sqrt{b^2 - 4\,ac}}{2\,a}$ of the standard quadratic equation $ax^2 + bx + c = 0$:

1. $x^2 - 3\,x - 2 = 0$.
2. $x^2 + 3\,x + 1 = 0$.
3. $x^2 - 3\,x - 1 = 0$.
4. $x^2 - 5\,x - 2 = 0$.
5. $x^2 + 5\,x - 7 = 0$.
6. $x^2 - 5\,x + 5 = 0$.
7. $x^2 - 11\,x - 1 = 0$.
8. $x^2 - 3\,x - 7 = 0$.
9. $x^2 + 3\,x - 3 = 0$.
10. $x^2 - 7\,x + 11 = 0$.
11. $x^2 + 7\,x + 8 = 0$.
12. $x^2 - 7\,x - 2 = 0$.
13. $x^2 + 9\,x + 17 = 0$.
14. $x^2 - 9\,x - 2 = 0$.

15. $x^2+x+1=0$.
16. $x^2+x+2=0$.
17. $x^2+3x+3=0$.
18. $x^2+5x+7=0$.
19. $2x^2+3x-1=0$.
20. $3x^2+5x+1=0$.
21. $5x^2-3x+1=0$.
22. $11x^2+7x+1=0$.
23. $7x^2+5x+1=0$.
24. $3x^2-3x+1=0$.
25. $5x^2-7x-1=0$.
26. $x^2+mx+1=0$.
27. $x^2-nx+1=0$.
28. $x^2+mx+n=0$.
29. $x^2+dx+2=0$.
30. $x^2-dx+3=0$.
31. $x^2-hx-5=0$.
32. $x^2+3hx+k=0$.
33. $x^2+5dx+k=0$.
34. $x^2-7gx-h=0$.
35. $x^2+7mx+2n=0$.
36. $x^2-9nx-3k=0$.

RATIONAL FRACTIONAL EQUATIONS CONTAINING ONE UNKNOWN

EXERCISE 174

Solve the following equations, rejecting extra solutions, and verifying all others:

1. $\frac{x}{2}+\frac{2}{x}=\frac{x}{8}+\frac{8}{x}$.
2. $\frac{7}{6x^2}-\frac{6}{7x^2}=\frac{13}{42}$.
3. $\frac{2}{x^2}+\frac{3}{5x^2}=\frac{13}{5}$.
4. $\frac{x-6}{8}=\frac{8}{x+6}$.
5. $\frac{x+3}{4}=\frac{4}{x-3}$.
6. $\frac{x-6}{x-4}=x$.
7. $2-\frac{2x+1}{3}=\frac{1}{2x-1}$.
8. $\frac{40}{x-5}+\frac{27}{x}=13$.
9. $\frac{4x-3}{1-x}=3+\frac{x^2}{1-x}$.
10. $x+\frac{3x-8}{x-3}+\frac{2x-7}{x-3}=9$.
11. $x-\frac{5x-7}{x-4}=\frac{2x-21}{x-4}-1$.
12. $\frac{1}{x}+\frac{1}{13}=\frac{1}{x+13}$.
13. $\frac{x}{x-1}=\frac{2}{3}+\frac{x-1}{x}$.
14. $x-\frac{7x-30}{x-5}=\frac{3x-20}{x-5}-2$.
15. $\frac{4}{x-6}-\frac{24}{x(x-6)}=1$.
16. $x-\frac{7x+23}{x+2}=\frac{8x+7}{x+2}-11$.
17. $\frac{x+5}{x-5}+\frac{x-5}{x+5}=\frac{37}{6}$.
18. $\frac{x+4}{x-4}+\frac{x-4}{x+4}=\frac{82}{9}$.

19. $\frac{4x-6}{x^2-9}-\frac{1}{x-3}-1=0.$

20. $\frac{6}{x-1}-1=\frac{12}{x^2-1}.$

21. $\frac{6+x}{1+x}+\frac{x-5}{2x}=\frac{15}{8}.$

22. $\frac{2x}{x-2}+\frac{3x-14}{x(x-2)}=3.$

23. $\frac{4}{x-2}+\frac{x-18}{x^2-4}=1.$

24. $\frac{1}{x-1}+\frac{2}{x+2}=\frac{1}{x+1}.$

25. $\frac{5x-6}{x-2}+\frac{x-18}{x^2-4}=6.$

26. $\frac{9}{x+2}+\frac{7}{x}=\frac{20}{x+3}.$

27. $\frac{1}{3}+\frac{1}{3+x}+\frac{1}{4+x}=0.$

28. $\frac{1}{x-1}+\frac{2}{x-2}=\frac{6}{x-3}.$

29. $\frac{1}{x-4}-\frac{3}{(x-1)(x-4)}=\frac{8}{x+6}.$

30. $\frac{1}{x-5}-\frac{11}{(x+6)(x-5)}=\frac{1}{3(x-6)}.$

31. $\frac{2x-7}{x-1}+1=\frac{2x+3}{x}-\frac{17}{20}.$

32. $\frac{4(x+5)}{x-3}=\frac{7x-10}{7}.$

33. $\frac{2x+3}{x-5}=\frac{7x-6}{x+7}.$

34. $\frac{2}{x+3}+\frac{12}{x^2-9}=1.$

35. $\frac{5}{x^2-1}-\frac{2}{x+1}=\frac{1}{8}.$

36. $\frac{7}{x-5}+\frac{17x-155}{x^2-25}=2.$

37. $\frac{x}{x-1}+\frac{x-3}{x^2-1}=\frac{5}{4}.$

38. $\frac{5}{x-6}+\frac{4}{5}=\frac{17x-42}{x^2-36}.$

39. $\frac{1}{x-7}-\frac{14}{x^2-49}=\frac{1}{17}.$

40. $\frac{x}{x-3}-\frac{9(x-1)}{x^2-9}=\frac{2}{5}.$

41. $\frac{1}{1-x}-\frac{1}{1-2x}=\frac{1}{1-3x}.$

42. $\frac{10}{x}-\frac{9}{x+1}=\frac{8}{x+2}.$

43. $\frac{4}{4-x}-\frac{3}{3-x}=\frac{1}{1-4x}.$

44. $\frac{5}{5-x}-\frac{3}{3-x}=\frac{16}{x-2}.$

45. $\frac{x-1}{x-3}-\frac{5}{2}=\frac{5-x}{x^2-5x+6}.$

46. $\frac{x-5}{x-7}+\frac{x+4}{2x+5}=\frac{25}{7}.$

47. $\frac{1}{x-2}-\frac{x+11}{(2x+9)(x-2)}=\frac{1}{3(2x-5)}.$

48. $\frac{1}{3x-2}-\frac{x+1}{(4x-1)(3x-2)}=\frac{3}{5(2x+1)}.$

49. $\dfrac{x}{2(x+2)}+\dfrac{3x+2}{x^2-4}=\dfrac{9}{10}$.

50. $\dfrac{2}{x-2}+\dfrac{x-4}{(x-1)(x-2)}=\dfrac{5}{x+1}$.

51. $\dfrac{x-4}{x-9}-\dfrac{3x-12}{x+4}=\dfrac{5}{x-9}$.

52. $\dfrac{2}{x+5}-\dfrac{x-3}{(x+1)(x+5)}=\dfrac{5}{x+9}$.

53. $\dfrac{1}{x+4}-\dfrac{7}{(x-3)(x+4)}=\dfrac{2}{x+2}$.

54. $\dfrac{2}{x+5}-\dfrac{x+3}{(x+4)(x+5)}=\dfrac{1}{4(x-8)}$.

55. $\dfrac{2}{x+2}-\dfrac{x-8}{x^2-x-6}=\dfrac{2}{x+1}$.

56. $\dfrac{1}{x-5}+\dfrac{1}{x^2-11x+30}=\dfrac{5}{x+10}$.

57. $\dfrac{1}{x+6}-\dfrac{1}{2(x-3)}=\dfrac{1}{x^2+13x+42}$.

58. $\dfrac{1}{2x-1}-\dfrac{3}{(4x+1)(2x-1)}=\dfrac{1}{8x-1}$.

59. $\dfrac{1}{x-4}+\dfrac{2}{3x-7}=\dfrac{5}{(2x-3)(x-4)}$.

60. $\dfrac{5}{5x-1}-\dfrac{26}{(x+5)(5x-1)}=\dfrac{1}{3x-5}$.

61. $\dfrac{3}{6x-1}+\dfrac{26}{(5x-9)(6x-1)}=\dfrac{3}{x+3}$.

62. $\dfrac{7}{3x-1}+\dfrac{2x-10}{(x+1)(3x-1)}=\dfrac{4}{3x+1}$.

63. $\dfrac{2}{3x-2}+\dfrac{7x-16}{(4x+3)(3x-2)}=\dfrac{2}{6x-1}$.

64. $\dfrac{15}{5x-2}+\dfrac{4(5x-11)}{(x+2)(5x-2)}=\dfrac{15}{3x+2}$.

65. $\dfrac{1}{2x-5}-\dfrac{2(5x+2)}{(12x-1)(2x-5)}=\dfrac{1}{4x+5}$.

66. $\dfrac{1}{2x+3}-\dfrac{4(x+1)}{(6x+7)(2x+3)}=\dfrac{3}{6x+11}$.

67. $\dfrac{x}{x-6}-\dfrac{5x-6}{(x-2)(x-6)}=\dfrac{x^2-7}{(x-3)(x-1)}-\dfrac{3}{x-1}$.

68. $\dfrac{x-5}{x-7}-\dfrac{x+17}{(x+5)(x-7)}=\dfrac{x^2-5x-15}{(x-8)(x+1)}-\dfrac{1}{x+1}$.

69. $\dfrac{x}{x-1}+\dfrac{2}{(x-3)(x-1)}=\dfrac{x}{x+1}+\dfrac{2x-1}{(x-2)(x+1)}$.

70. $\dfrac{x}{x-4}-\dfrac{13x+24}{(x+11)(x-4)}=\dfrac{x}{x-2}-\dfrac{7x+8}{(x+9)(x-2)}$.

71. $\dfrac{x}{x-3}-\dfrac{2(x-9)}{(x-7)(x-3)}=\dfrac{x}{x-2}-\dfrac{x-8}{(x-5)(x-2)}$.

72. $\dfrac{x}{x-6}-\dfrac{3(5x+8)}{(x+13)(x-6)}=\dfrac{x}{x+4}-\dfrac{5x+4}{(x+8)(x+4)}$.

73. $\dfrac{x}{x+3}+\dfrac{12(x+2)}{(x-1)(x+3)}=\dfrac{x}{x-2}+\dfrac{7x-20}{(x+1)(x-2)}$.

74. $\dfrac{x}{x-4}+\dfrac{4(x-5)}{(x-3)(x-4)}=\dfrac{x}{x-2}+\dfrac{6(x-3)}{(x+1)(x-2)}$.

75. $\dfrac{x}{x+2}+\dfrac{2(3x+8)}{(x+4)(x+2)}=\dfrac{x}{x+3}+\dfrac{7x+15}{(x+1)(x+3)}$.

MISCELLANEOUS

EXERCISE 175

Solve the following equations, verifying such solutions as are neither irrational nor imaginary:

1. $x^2-15x+54=0$.
2. $2x^2-7x=15$.
3. $3(x^2-1)=8x$.
4. $(x+4)^2=9x^2$.
5. $55(x^2-x)=11x$.
6. $(x+2)(x+3)=20$.
7. $(x+1)(x-4)=50$.
8. $\dfrac{2}{x-1}-10x=9$.
9. $6=\dfrac{x}{2x^2-1}$.
10. $x+\dfrac{1}{2}=2+\dfrac{1}{x}$.
11. $x+\dfrac{1}{5}=5+\dfrac{1}{x}$.
12. $10(x+2)(x-2)=41x$.
13. $x^2+x=d^2+d$.
14. $x+\dfrac{1}{b}=b+\dfrac{1}{x}$.
15. $x+\dfrac{mn}{x}=m+n$.
16. $x^2+n^2=m^2+2nx$.
17. $x(2x-a)+x(x-a)=bx$.
18. $\dfrac{x}{2}+\dfrac{2}{x}=\dfrac{3}{2}+\dfrac{2}{3}$.

19. $\frac{x}{a}+\frac{a}{x}=\frac{b}{a}+\frac{a}{b}$.

20. $\frac{x}{x-3}+\frac{8}{x+2}=3$.

21. $\frac{3}{x-5}-5=-\frac{2x}{x-3}$.

22. $\frac{1}{x}-\frac{1}{2}=\frac{1}{x-2}$.

23. $\frac{7x}{8}+\frac{8}{7x}=\frac{9}{10}$.

24. $\frac{5x}{7}+\frac{1}{x}=\frac{7}{5}$.

25. $\frac{x}{5}-\frac{5}{6}=\frac{6}{5}-\frac{5}{x}$.

26. $\frac{3x}{2}+\frac{2}{x}=\frac{2x}{3}$.

27. $\frac{1}{x}+x=3+\frac{3}{x}$.

28. $\frac{2x}{3}-4x^2=\frac{1}{2}$.

29. $\frac{15x}{x-1}=11(x-1)$.

30. $\frac{1}{x+1}-\frac{1}{x+2}-\frac{1}{x+3}=0$.

31. $\frac{x+16}{8}-\frac{64-x}{x-4}=\frac{x}{4}-13$.

32. $\frac{x}{x+1}+\frac{x+1}{x}=\frac{13}{6}$.

33. $\frac{1}{x}-\frac{1}{6}=\frac{1}{x-6}$.

34. $\frac{1}{x}+\frac{1}{8-x}=\frac{1}{8}$.

35. $\frac{x}{x+1}+\frac{x+1}{x}=\frac{113}{56}$.

36. $\frac{x+2}{x+3}+\frac{x+3}{x+2}=\frac{41}{20}$.

37. $\frac{1}{x-2}+1=\frac{6-x}{x^2-4}+\frac{1}{x+2}$.

38. $(2x+1)(2x-1)\div(x+2)(x-2)=x(x+3)$.

39. $(x+2)(x-3)+(x+9)(x+3)=3x+15$.

40. $\frac{1}{x}-\frac{1}{d}=\frac{1}{x-d}$.

41. $\frac{a}{x-b}+\frac{b}{x-a}=2$.

42. $\frac{a}{x}+\frac{x}{a}=\frac{33a^2-x^2}{ax}$.

43. $\frac{x}{a^2}+\frac{a^2}{x}=\frac{x}{b^2}+\frac{b^2}{x}$.

44. $x^2+8x=16g+2gx$.

45. $x^2+2bx=a^2+2ab$.

46. $(x+b)^2+6(x+b)+9=0$.

47. $x^2+4ab=2x(b+a)$.

48. $2x(7x-a)=(a+x)(a-x)$.

49. $\frac{m^2}{x^2}=\frac{m+1}{x+1}$.

50. $\frac{x-4m+n}{n-m}=\frac{9n+x}{x}$.

51. $\frac{(x+1)(x-2)}{2}-\frac{(x-1)(x+2)}{3}=4$.

52. $(a-x)^2+(b-x^2)=(b-a)^2$.

53. $(x-g)^2+(x-h)^2=g^2+h^2$.

54. $(c-a)x^2+(a-b)x+b-c=0$.

55. $\dfrac{1}{a+b+x}=\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{x}$.

56. $\dfrac{x+c}{x-d}+\dfrac{x-c}{x+d}=1+\dfrac{c^2+2d^2}{x^2-d^2}$.

57. $\dfrac{1}{x}+\dfrac{1}{x+4}=\dfrac{1}{x+1}+\dfrac{7}{7x+3}$.

58. $\dfrac{x}{b+x}+\dfrac{x}{c+x}=\dfrac{a}{a+b}+\dfrac{a}{a+c}$.

SOLUTION OF FORMULAS FOR SPECIFIED LETTERS

EXERCISE 176

In the following formulas find the expressed values of the letters specified in terms of those remaining:

1. Solve for R, $S=\pi R^2$.
2. Solve for E, $w=\dfrac{E^2t}{R}$.
3. Solve for t, $S=\frac{1}{2}at^2$.
4. Solve for s, $F=\dfrac{ms^2}{r}$.
5. Solve for r, $f=\dfrac{mm_1}{r^2}$.
6. Solve for a, $S=\dfrac{a^2}{4}\sqrt{3}$.
7. Solve for n, $d=\dfrac{2(ln-s)}{n(n-1)}$.

Find the value of n, if $d=2$, $l=19$, and $s=96$.

8. Solve for R, $V=\frac{1}{3}\pi R^2h$.

Find the value of R, if $V=132$, and $h=14$.

9. Solve for R, $T=2\pi R(H+R)$.

Find the value of R, if $T=352$ and $H=10$.

10. Solve for t, $ab=t^2+pq$.

Find the value of t, if $a=16$, $b=14$, $p=7$, and $q=8$.

11. Solve for c, $a^2=b^2+c^2+2cp$.

Find the value of c, if $a=18$, $b=11$, and $p=\frac{11}{15}$.

12. Solve for m, $2a^2+2b^2=c^2+4m^2$.

Find the value of m, if $a=12$, $b=16$, and $c=20$.

13. Solve for n, $s=\dfrac{n}{2}[2a+(n-1)d]$.

Find the value of n, if $s=272$, $a=6$, and $d=8$.

14. Solve for a, $s = \frac{l+a}{2} + \frac{l-a^2}{2d}$.

Find the value of a, if $s = 70$, $l = 16$, and $d = 2$.

15. Solve for l, $d = \frac{l^2 - a^2}{2s - l - a}$.

Find the value of l, if $d = 15$, $a = 18$, and $s = 1206$.

PROBLEMS

PROBLEMS SOLVED BY MEANS OF QUADRATIC EQUATIONS CONTAINING A SINGLE UNKNOWN

EXERCISE 177

Solve the following problems, and examine the solutions obtained to see if they satisfy the conditions of the given problems:

1. Find two numbers whose sum is 19, and whose product is 84.

2. Find two numbers whose sum is 14, and whose product is 33.

3. Find two numbers whose sum is 20, and whose product is 75.

4. Find two numbers whose sum is 14, and whose product is 45.

5. Separate 26 into two parts, whose product is 165.

6. Separate 22 into two parts, whose product is 117.

7. Separate 20 into two parts, whose product is 96.

8. Separate 18 into two parts, whose product is 77.

9. The difference of two numbers is 3 and their product is 40. Find the numbers.

10. The difference of two numbers is 9 and their product is 90. Find the numbers.

11. The difference of two numbers is 13 and their product is 68. Find the numbers.

12. The difference of two numbers is 12 and their product is 160. Find the numbers.

13. Separate 30 into two parts, such that one part is the square of the other.

14. Separate 42 into two parts, such that one part is the square of the other.

15. Separate 56 into two parts, such that one part is the square of the other.

16. Separate 72 into two parts, such that one part is the square of the other.

17. Find two consecutive integers, the greater of which exceeds the less by unity, and the product of which is 20.

18. Find two consecutive integers whose product is 72.

19. Find two consecutive integers whose product is 90.

20. Find two consecutive integers whose product is 132.

21. Find two consecutive even integers whose product is 48.

22. Find two consecutive even integers whose product is 120.

23. Find two consecutive even integers whose product is 224.

24. Find two consecutive even integers whose product is 440.

25. Find two consecutive odd integers whose product is 143.

26. Find two consecutive odd integers whose product is 63.

27. Find two consecutive odd integers whose product is 255.

28. Find two consecutive integers, the sum of whose squares is 25.

29. Find two consecutive integers, the sum of whose squares is 61.

30. Find two consecutive integers, the sum of whose squares is 181.

31. Find two consecutive even integers, the sum of whose squares is 52.

32. Find two consecutive even integers, the sum of whose squares is 164.

33. Find two consecutive even integers, the sum of whose squares is 221.

34. Find two numbers in the ratio, $\frac{2}{3}$ the sum of whose squares is 208.

35. Find two numbers in the ratio, $\frac{1}{2}$ the sum of whose squares is 45.

36. Find two numbers in the ratio, $\frac{3}{4}$ the sum of whose squares is 225.

37. Find two numbers in the ratio, $\frac{3}{2}$ the difference of whose squares is 20.

38. Find two numbers in the ratio, $\frac{4}{3}$ the difference of whose squares is 63.

39. Find two numbers in the ratio, $\frac{5}{4}$ the difference of whose squares is 81.

40. Find a number such that the product of one third the number by one fourth the number shall equal 108.

41. Find a number such that the product of one fifth the number by one sixth the number shall equal 120.

42. Find a number such that the product of one half the number by one third the number shall equal 150.

43. The product of a number and one half the number is 72. Find the number.

44. The product of a number and one third the number is 75. Find the number.

45. The product of a number and one fifth the number is 80. Find the number.

46. Find a number whose square exceeds 34 by as much as the number exceeds 4.

47. Find a number whose square exceeds 13 by as much as the number exceeds 1.

48. Find a number whose square exceeds 44 by as much as the number exceeds 2.

49. Find a number such that the sum of the number and its reciprocal shall be $1\frac{7}{4}$.

50. Find a number such that the sum of the number and its reciprocal shall be $\frac{122}{11}$.

51. Find a number whose square diminished by 6 is equal to five times the number.

52. Find a number whose square diminished by 16 is equal to six times the number.

53. Find a number whose square diminished by the number is equal to four times the number.

54. Find a number whose square increased by the number is equal to eight times the number.

55. Find a number whose square increased by 14 is equal to nine times the number.

56. Find a number whose square increased by 8 is equal to six times the number.

57. Find a number whose square diminished by 6 is equal to the number increased by 6.

58. Find a number whose square diminished by 15 is equal to eleven times the excess of the number over 3.

59. Find a number whose square diminished by 9 is equal to nine times the excess of the number over 3.

60. Find a number whose square diminished by 20 is equal to eight times the number.

61. Separate a number represented by a into two parts, such that one part shall be the square of the other.

62. Find a number, such that if 66 be subtracted from it, the remainder equals the quotient obtained by dividing 280 by the number.

63. Find a positive fraction, such that two times its square is 10 more than the fraction.

64. The denominator of a certain fraction exceeds its numerator by 4, and the reciprocal of the fraction exceeds the fraction by $\frac{56}{45}$. Find the fraction.

65. The numerator of a certain improper fraction exceeds the denominator by 5, and the fraction exceeds its reciprocal by $\frac{55}{24}$. Find the fraction.

66. Find a number such that one half its square shall exceed the square of one half the number by one half the number.

67. If the eighth part and the ninth part of a certain number are multiplied together, and the product is divided by 6, the quotient is 108. Find the number.

68. It is found that when a number which is the product of three consecutive integral numbers is divided separately by each of these three factors, the sum of the quotients thus obtained is 107. What are the numbers?

69. I have thought of a number. I multiply it by $2\frac{1}{7}$, then add 5 to the product; I then multiply the result by two times the number thought of, and

finally divide by 7 and subtract from the quotient four times the number originally thought of, obtaining thus 12. What was the original number?

70. It is necessary to construct a coal bin to hold 6 tons of coal. Allowing 40 cubic feet of space per ton of coal, what must be the dimensions if, the depth being 6 feet, the length is equal to the sum of the width and the depth?

71. A crew can row 8 miles, down a stream and back again, in 3 hours and 40 minutes. If the rate of the stream is $2\frac{1}{2}$ miles an hour, find the rate of the crew in still water in miles per hour.

72. A man bought two farms for $3600 each. The larger contained 15 acres more than the smaller, but $8 more per acre was paid for the smaller than for the larger. How many acres did each contain?

73. If $3000 amounts to $3213.675 when put at compound interest for 2 years, interest being compounded annually, what is the rate per cent per year?

74. If $4250 amounts to $4508.825 when placed at compound interest for 2 years, interest being compounded annually, find the rate per cent per year.

75. Telegraph poles are placed at equal intervals along a certain railway. In order that there should be two less poles per mile it would be necessary to increase the space between every two consecutive poles by 24 feet. Find the number of poles to a mile.

76. A man bought a certain number of shares of a railway stock for $6600. The next day they declined in value $12 a share, when he could have bought five shares more for the same amount. Find the price paid per share.

77. A broker purchased a certain number of shares of stock for $2560. After reserving 10 shares, those remaining were sold for $2450, at an advance of $3 a share on the cost price. How many shares did he buy?

78. It is desired to carpet a floor in the form of a rectangle 15 feet long by 12 feet wide, with a carpet having a plain color border of uniform width. Allowing $1.44 per square yard for the center and $0.45 per square yard for the border, determine the width of the border in order that the entire expense may be $18.68.

79. Two steamers ply between two ports, a distance of 475 miles. One goes half a mile an hour faster than the other, and requires $2\frac{1}{2}$ hours less for the voyage. Find the rates of the steamers in miles per hour.

80. An engineer of a freight train on a trip of 72 miles found it necessary at the twenty-fourth milestone to decrease his speed to a rate of 6 miles an hour less, with the result that he was on the road 16 minutes longer than would have been the case had no alteration in speed been made. Find the rate in miles per hour before the speed was changed, and also the time required to make the entire trip.

81. In answering a false alarm, a fire engine traveled a distance of $\frac{3}{5}$ of a mile at the rate of 5 miles an hour faster than when returning. If it returned immediately on reaching the "alarm box" and was gone from the station $16\frac{1}{2}$ minutes in all, what was its rate at first in miles per hour?

82. A and B run a half-mile race. A, who is faster than B by $\frac{1}{2}$ a yard a second, allows B a head start of $\frac{1}{4}$ of a minute, and beats him by 5 yards. Find their respective rates in yards per second.

83. A warship which is approaching a port is discovered when it is 12 miles away. A flotilla of torpedo boats, the maximum speed of which is known to exceed that of the warship by 7 miles an hour, is sent out $5\frac{3}{5}$ minutes later to meet the warship. They intercept it when it has covered half the distance to the port. Find the rate of the warship in miles per hour.

84. After having gone 40 miles of his trip at a uniform rate, an engineer found that his train was behind time. He immediately increased the speed of the engine to a rate of 4 miles an hour more, and completed the trip of 62 miles, arriving at the terminus 3 minutes earlier than would have been the case if no change in rate had been made. Find the rate of the train in miles per hour.

85. A man travels 24 miles by an accommodation train and returns by an express which runs 10 miles an hour faster. Find the rates of the two trains in miles per hour, provided that the time occupied for the two trips was one hour and twenty-four minutes.

86. It is found that two steam fire engines can, by working together, pump all of the water out of a partly filled cellar in 24 minutes. The more powerful one alone would have been able to perform the work in 20 minutes less than the other one alone. Find the time required by each one working alone.

87. After traveling 8 miles in an automobile, a man found that on account of an accident to the machine it was necessary to walk back. If the rate of the automobile exceeded the man's rate when walking by 17 miles an hour, and he was 2 hours 16 minutes longer in returning than going, find the rate of the machine in miles per hour.

PROBLEMS IN PHYSICS

EXERCISE 178

Solve the following problems relating to moving bodies:

1. What velocity, expressed in feet per second, will a body acquire by falling 6 seconds?

2. In what time will a falling body acquire a velocity of 192 feet per second?

3. What is the height of a tower if a stone dropped from the top requires 3 seconds to reach the ground ?

4. A stone dropped from the top of a cliff is observed to reach the bottom in 4 seconds. Find the height of the cliff.

5. A balloon is moving horizontally at a height of one mile above the ground. How long will it take a bag of ballast to reach the ground ?

6. What velocity, expressed in feet per second, will a body acquire by falling a distance of 625 feet ?

7. What velocity, expressed in feet per second, would a body acquire in falling a distance of 800 feet ?

8. A stone thrown vertically upward strikes the ground after an interval of 8 seconds. With what velocity was it thrown upward and to what height did it rise ?

9. How high is a tree if it requires three seconds for a stone which is thrown over it to reach the ground ?

10. With what velocity, expressed in feet per second, must an arrow be shot vertically upward to reach the top of a tower which is 196 feet high ?

11. Find the distance passed over in four seconds by a body which is thrown vertically downward with a velocity of 24 feet per second.

12. Find the time, expressed in seconds, required for a body which is thrown vertically downward with an initial velocity of 6 feet per second to move a distance of 500 feet.

13. A balloon is 2 miles from the ground and is descending at the rate of 10 feet per second when a sand bag is dropped. Find the number of seconds required for the sand bag to reach the ground.

14. A balloon is 1000 feet from the ground and ascending at the rate of 10 feet per second when a sand bag is dropped. How many seconds will be required for the sand bag to reach the ground ?

15. If a body is projected upward with an initial velocity of 160 feet per second, what is the height to which it will rise ?

16. A rifle bullet is shot vertically upward with an initial velocity of 400 feet per second. Find the height to which it will rise.

17. A bullet is fired vertically upward with a velocity of 100 feet per second. Find the time required for it to reach a point 156 feet above the ground, and also the velocity with which it passes this point.

18. How high is a tower if an arrow shot over the top reaches the ground in 8 seconds from the time it left the bow ?

THEORY OF QUADRATIC EQUATIONS

NATURE OF THE ROOTS

EXERCISE 179

Determine, without solving, the nature of the roots of the following equations:

1. $x^2 - 6x + 9 = 0$.
2. $x^2 + 4x + 4 = 0$.
3. $x^2 + 3x + 2 = 0$.
4. $x^2 + 5x + 6 = 0$.
5. $x^2 + 7x + 6 = 0$.
6. $x^2 + 14x + 49 = 0$.
7. $x^2 + 4x - 5 = 0$.
8. $x^2 + 5x + 4 = 0$.
9. $x^2 + x - 110 = 0$.
10. $x^2 - 3x - 6 = 0$.
11. $x^2 + 2x - 35 = 0$.
12. $x^2 + 6x + 5 = 0$.
13. $x^2 + 13x + 36 = 0$.
14. $25x^2 + 10x + 1 = 0$.
15. $x^2 - 16x + 64 = 0$.
16. $x^2 - x - 42 = 0$.
17. $2x^2 - 3x - 2 = 0$.
18. $x^2 + 6x - 1 = 0$.
19. $3x^2 + 10x + 3 = 0$.
20. $6x^2 - 13x + 5 = 0$.
21. $9x^2 - 12x + 4 = 0$.
22. $4x^2 + 9x + 5 = 0$.
23. $8x^2 + 2x - 3 = 0$.
24. $x^2 - 5x + 3 = 0$.
25. $x^2 - x - 7 = 0$.
26. $6x^2 + 11x + 5 = 0$.
27. $5x^2 - 2x - 3 = 0$.
28. $x^2 + 3x + 4 = 0$.
29. $x^2 - 3x + 5 = 0$.
30. $x^2 - 7x + 8 = 0$.
31. $2x^2 + 7x + 6 = 0$.
32. $x^2 + 5x - 2 = 0$.
33. $x^2 - 5x - 5 = 0$.
34. $x^2 + 5x + 7 = 0$.
35. $x^2 - 4x + 7 = 0$.
36. $10x^2 + 11x + 3 = 0$.
37. $x^2 - 7x + 14 = 0$.
38. $2x^2 - 3x - 1 = 0$.
39. $3x^2 - 3x - 1 = 0$.
40. $x^2 - 9x + 21 = 0$.
41. $x^2 - 13x + 44 = 0$.
42. $5x^2 + 7x - 1 = 0$.
43. $x^2 + 11x + 32 = 0$.
44. $2x^2 + 5x + 4 = 0$.
45. $15x^2 - 7x - 2 = 0$.
46. $3x^2 - 7x + 5 = 0$.
47. $9x^2 - 13x + 4 = 0$.
48. $7x^2 - 5x + 2 = 0$.
49. $x^2 + 2mx + m^2 = 0$.
50. $x^2 - 2dx + d^2 = 0$.
51. $x^2 + 3mx - 4m^2 = 0$.
52. $x^2 + nx + n^2 = 0$.
53. $x^2 - 6mx + 9m^2 = 0$.
54. $x^2 - 9mx - m^2 = 0$.
55. $x^2 - 6wx + 5w^2 = 0$.
56. $x^2 - 8hx + 16h^2 = 0$.
57. $x^2 - kx + k^2 = 0$.
58. $5x^2 + 6nx + n^2 = 0$.
59. $x^2 + 7mx + 5m^2 = 0$.
60. $m^2x^2 + 4mx + 5 = 0$.

61. $3x^2 - 6gx + g^2 = 0$.
62. $2n^2x^2 - 3nx + 2 = 0$.
63. $5x^2 + 8gx + 5g^2 = 0$.
64. $4k^2x^2 + 4kx + 1 = 0$.
65. $d^2x^2 + 2dghx + g^2h^2 = 0$.
66. $m^2x^2 + 2mnx + n^2 = 0$.
67. $mx^2 + (m+n)x + n = 0$.
68. $gx^2 + (g-h)x - h = 0$.
69. $x^2 + (k-3)x - 3k = 0$.
70. $5mx^2 + (5+m)x + 1 = 0$.
71. $7x^2 - (7-n)x - n = 0$.
72. $mx^2 + (3m+n)x + 3n = 0$.
73. $mx^2 + (2m+n)x + 2n = 0$.
74. $5mx^2 + (3m+5n) + 3n = 0$.
75. $3mx^2 + (2m-3n)x - 2n = 0$.

76. Prove that the roots of $x^2 - 2ax + a^2 = b^2 + c^2$ are real.

77. Prove that the roots of $2ax^2 + (2a+3b)x + 3b = 0$ are real for all real values of a and b.

78. Prove that $3cx^2 - (2c+3d)x + 2d = 0$ has rational roots.

79. Show that the roots of $5x^2 + 4ax + a^2 = 0$ are imaginary.

80. Show that the roots of $(a+b)^2x^2 - 2(a^2-b^2)x + (a-b)^2 = 0$ are neither irrational nor imaginary.

Determine the value which k must have in order that the following equations shall have equal roots:

81. $x^2 + (k-3)x + k = 0$.
82. $x^2 = 2k(x-4) + 15$.
83. $x^2 + 2(k+2)x + 9k = 0$.
84. $x^2 - 2kx + 6x + 4k = 0$.
85. $x^2 + k(2x-8) = 15$.
86. $x^2 = (k-1)x - 2(k-1)$.
87. $(2k+1)x^2 + 3kx + k = 0$.
88. $x^2 + 3kx + 4k + 1 = 0$.
89. $12kx^2 - 2x + 3k = 0$.
90. $x^2 - (2k-3)x + 2k = 0$.
91. $(8k+5)x^2 - 12kx + 1 = 0$.
92. $(k+2)x^2 + 3k + 4 = 5(k-1)x$.
93. $x^2 + (6k+7)x + 6k + 22 = 0$.
94. $(k+6)x^2 - 3(k-2)x = 1 - k$.
95. $(k-11)x^2 + 3k + 4 = 2(k-1)x$.

DETERMINATION OF ONE ROOT WHEN THE OTHER ROOT IS KNOWN

EXERCISE 180

Find, without solving, the remaining root of each of the following equations when one of the roots is given:

1. $x^2 - 18x + 77 = 0$, one root being 11.
2. $x^2 - 19x + 78 = 0$, one root being 6.

3. $x^2 - 20x + 96 = 0$, one root being 12.
4. $x^2 - 29x + 210 = 0$, one root being 14.
5. $x^2 - 36x + 323 = 0$, one root being 17.
6. $x^2 + 34x + 288 = 0$, one root being -16.
7. $x^2 + 46x + 525 = 0$, one root being -25.
8. $x^2 + 61x + 924 = 0$, one root being -33.
9. $x^2 + 79x + 1554 = 0$, one root being -37.
10. $x^2 + 106x + 2793 = 0$, one root being -57.
11. $x^2 - 18x - 319 = 0$, one root being 29.
12. $x^2 + 16x - 132 = 0$, one root being 6.
13. $x^2 + 45x - 144 = 0$, one root being 48.
14. $x^2 - 4x - 1440 = 0$, one root being 40.
15. $x^2 + 14x - 1472 = 0$, one root being 32.
16. $x^2 + 20x - 861 = 0$, one root being 21.
17. $x^2 + 9x - 972 = 0$, one root being 27.
18. $x^2 - 15x - 496 = 0$, one root being 31.
19. $x^2 + 16x - 561 = 0$, one root being 17.
20. $x^2 - 22x - 1323 = 0$, one root being 49.
21. $x^2 - 24x - 1081 = 0$, one root being 47.
22. $x^2 - 5x - 456 = 0$, one root being -19.
23. $x^2 + 7x - 368 = 0$, one root being -23.
24. $x^2 - 15x - 406 = 0$, one root being -14.
25. $x^2 + 20x - 629 = 0$, one root being -37.
26. $x^2 + 13x - 1680 = 0$, one root being -48.
27. $x^2 - 24x - 817 = 0$, one root being -19.
28. $25x^2 - 20x + 3 = 0$, one root being $\frac{1}{5}$.
29. $49x^2 - 35x + 6 = 0$, one root being $\frac{3}{7}$.
30. $81x^2 - 54x + 8 = 0$, one root being $\frac{2}{9}$.
31. $16x^2 - 8x + 1 = 0$, one root being $\frac{1}{8}$.
32. $100x^2 - 100x + 21 = 0$, one root being $\frac{3}{10}$.
33. $36x^2 - 36x + 5 = 0$, one root being $\frac{1}{6}$.
34. $25x^2 - 25x + 4 = 0$, one root being $\frac{1}{5}$.
35. $121x^2 + 66x + 8 = 0$, one root being $-\frac{2}{11}$.
36. $144x^2 + 72x + 5 = 0$, one root being $-\frac{1}{12}$.
37. $81x^2 - 36x - 5 = 0$, one root being $\frac{5}{9}$.
38. $100x^2 - 20x - 3 = 0$, one root being $\frac{3}{10}$.
39. $16x^2 + 2x - 3 = 0$, one root being $\frac{3}{8}$.

40. $48x^2 - 8x - 5 = 0$, one root being $\frac{5}{12}$.
41. $27x^2 + 3x - 2 = 0$, one root being $-\frac{1}{3}$.
42. $64x^2 - 16x - 3 = 0$, one root being $-\frac{1}{8}$.
43. $2x^2 - 15x + 7 = 0$, one root being $\frac{1}{2}$.
44. $3x^2 - 25x + 8 = 0$, one root being $\frac{1}{3}$.
45. $4x^2 - 45x + 11 = 0$, one root being $\frac{1}{4}$.
46. $5x^2 - 61x + 12 = 0$, one root being $\frac{1}{5}$.
47. $6x^2 - 85x + 14 = 0$, one root being $\frac{1}{6}$.
48. $3x^2 - 26x + 16 = 0$, one root being $\frac{2}{3}$.
49. $4x^2 - 39x + 27 = 0$, one root being $\frac{3}{4}$.
50. $5x^2 + 8x - 4 = 0$, one root being $\frac{2}{5}$.
51. $7x^2 + 19x - 6 = 0$, one root being $\frac{2}{7}$.
52. $9x^2 + 32x - 16 = 0$, one root being $\frac{4}{9}$.
53. $6x^2 + 7x - 10 = 0$, one root being $\frac{5}{6}$.
54. $5x^2 - 11x - 12 = 0$, one root being $-\frac{4}{5}$.
55. $7x^2 - 11x - 6 = 0$, one root being $-\frac{3}{7}$.
56. $8x^2 - 3x - 5 = 0$, one root being $-\frac{5}{8}$.
57. $7x^2 - 16x - 15 = 0$, one root being 3.
58. $9x^2 - 11x - 14 = 0$, one root being 2.
59. $11x^2 - 19x - 6 = 0$, one root being 2.
60. $10x^2 - 7x + 1 = 0$, one root being $\frac{1}{5}$.
61. $6x^2 - 5x + 1 = 0$, one root being $\frac{1}{3}$.
62. $12x^2 - 7x + 1 = 0$, one root being $\frac{1}{4}$.
63. $35x^2 - 12x + 1 = 0$, one root being $\frac{1}{5}$.
64. $24x^2 - 11x + 1 = 0$, one root being $\frac{1}{3}$.
65. $36x^2 - 13x + 1 = 0$, one root being $\frac{1}{4}$.
66. $20x^2 - 13x + 2 = 0$, one root being $\frac{1}{4}$.
67. $24x^2 - 22x + 3 = 0$, one root being $\frac{1}{6}$.
68. $54x^2 - 57x - 10 = 0$, one root being $\frac{2}{9}$.

FORMATION OF AN EQUATION HAVING SPECIFIED ROOTS

EXERCISE 181

Construct integral equations having rational coefficients whose roots are

1. 2 and 1.
2. 1 and 3.
3. 2 and 5.
4. 3 and 7.
5. 5 and 4.
6. 7 and 2.

7. 6 and 8.
8. 4 and 9.
9. 8 and 7.
10. 9 and 5.
11. 4 and 4.
12. 7 and 7.
13. -3 and -1.
14. -5 and -3.
15. -10 and -4.
16. -7 and -6.
17. -11 and -9.
18. -12 and -8.
19. -13 and -7.
20. -6 and -10.
21. -7 and -12.
22. -8 and -15.
23. -9 and -9.
24. -12 and -12.
25. 5 and -2.
26. 6 and -4.
27. -8 and $+9$.
28. $+10$ and -11.
29. -3 and $+13$.
30. $+4$ and -16.
31. -14 and $+5$.
32. $+13$ and -10.
33. $+6$ and -6.
34. $+9$ and -9.
35. 16 and 0.
36. -17 and 0.
37. 0 and $+6$.
38. 0 and -9.
39. m and m.
40. $2m$ and m.
41. $3n$ and n.
42. $2g$ and $2g$.
43. $4d$ and d.
44. $4h$ and $2h$.
45. $3k$ and $2k$.
46. $5g$ and $4g$.
47. $6t$ and $8t$.
48. $-n$ and $-n$.
49. $-5g$ and $-g$.
50. $-4k$ and $-7k$.
51. $-9m$ and $-3m$.
52. $-10n$ and $-5n$.
53. $+6d$ and $-2d$.
54. $+7g$ and $-4g$.
55. $-8h$ and $+3h$.
56. $-10k$ and $+12k$.
57. k and $-k$.
58. n and $-n$.
59. $2d$ and $-2d$.
60. $3g$ and $-3g$.
61. $4h$ and $-4h$.
62. mn and $-mn$.
63. hk and $-hk$.
64. m^2 and $-m^2$.
65. n^3 and $-n^3$.
66. 1 and m.
67. 2 and n.
68. 3 and d.
69. 4 and g.
70. 5 and k.
71. 6 and $-m$.
72. 7 and $-n$.
73. 2 and $-t$.
74. 8 and $-h$.
75. 9 and $-k$.
76. -1 and $-m$.
77. -2 and $-n$.
78. $-d$ and -3.
79. $-g$ and -5.
80. m and 0.
81. n and 0.
82. 0 and $-d$.
83. 0 and $-g$.
84. 0 and 0.

85. $\frac{2}{3}$ and $\frac{1}{2}$.
86. $\frac{2}{5}$ and $\frac{1}{5}$.
87. $\frac{2}{7}$ and $\frac{3}{7}$.
88. $\frac{4}{9}$ and $\frac{1}{9}$.
89. $\frac{3}{8}$ and $\frac{1}{8}$.
90. $\frac{1}{2}$ and $\frac{1}{3}$.
91. $\frac{1}{4}$ and $\frac{1}{5}$.
92. $\frac{1}{3}$ and $\frac{1}{7}$.
93. $\frac{1}{2}$ and $\frac{1}{4}$.
94. $\frac{1}{6}$ and $\frac{1}{3}$.
95. $\frac{1}{6}$ and $\frac{1}{9}$.
96. $-\frac{1}{3}$ and $-\frac{1}{5}$.
97. $-\frac{1}{4}$ and $-\frac{1}{6}$.
98. $\frac{1}{4}$ and 1.
99. 1 and $\frac{1}{3}$.
100. $\frac{1}{2}$ and 2.
101. 3 and $\frac{1}{3}$.
102. 4 and $\frac{1}{2}$.
103. -5 and $\frac{1}{4}$.
104. -6 and $\frac{1}{2}$.
105. -4 and $\frac{1}{5}$.
106. -3 and $-\frac{1}{2}$.
107. -2 and $-\frac{1}{3}$.
108. $\frac{1}{2}$ and 0.
109. 0 and $\frac{1}{3}$.
110. 0 and $-\frac{1}{4}$.
111. 0 and $-\frac{1}{5}$.
112. $\frac{2}{3}$ and 0.
113. 0 and $\frac{3}{4}$.
114. 0 and $-\frac{4}{5}$.
115. 0 and $-\frac{2}{7}$.
116. $\frac{1}{m}$ and $\frac{2}{m}$.
117. $\frac{3}{n}$ and $\frac{5}{n}$.
118. $\frac{2}{k}$ and $\frac{3}{k}$.
119. $\frac{4}{h}$ and $\frac{5}{h}$.
120. $-\frac{6}{m}$ and $-\frac{3}{m}$.
121. $-\frac{7}{n}$ and $-\frac{2}{n}$.
122. $\frac{8}{g}$ and $-\frac{3}{g}$.
123. $\frac{5}{d}$ and $-\frac{9}{d}$.
124. m and $\frac{1}{m}$.
125. n and $\frac{1}{n}$.
126. $\frac{1}{m}$ and $\frac{1}{2m}$.
127. $\frac{1}{3n}$ and $\frac{1}{n}$.
128. 0 and $\frac{m}{3}$.
129. 0 and $\frac{n}{4}$.
130. 0 and $-\frac{d}{2}$.
131. 0 and $-\frac{g}{8}$.
132. 0 and $\frac{2m}{7}$.
133. 0 and $\frac{2}{m}$.
134. 0 and $\frac{3}{n}$.
135. 0 and $\frac{3}{8n}$.
136. 0 and $-\frac{4}{5m}$.
137. $3+\sqrt{2}$ and $3-\sqrt{2}$.
138. $2+\sqrt{3}$ and $2-\sqrt{3}$.
139. $3+\sqrt{5}$ and $3-\sqrt{5}$.
140. $4+\sqrt{6}$ and $4-\sqrt{6}$.
141. $5+\sqrt{2}$ and $5-\sqrt{2}$.
142. $5+\sqrt{3}$ and $5-\sqrt{3}$.
143. $6+\sqrt{10}$ and $6-\sqrt{10}$.
144. $1+\sqrt{7}$ and $1-\sqrt{7}$.
145. $8+\sqrt{6}$ and $8-\sqrt{6}$.
146. $2+\sqrt{10}$ and $2-\sqrt{10}$.
147. $3+\sqrt{11}$ and $3-\sqrt{11}$.
148. $5+2\sqrt{3}$ and $5-2\sqrt{3}$.

149. $4+2\sqrt{2}$ and $4-2\sqrt{2}$.

150. $7+3\sqrt{2}$ and $7-3\sqrt{2}$.

151. $6+2\sqrt{5}$ and $6-2\sqrt{5}$.

152. $2+3\sqrt{7}$ and $2-3\sqrt{7}$.

153. $3+4\sqrt{3}$ and $3-4\sqrt{3}$.

154. $\sqrt{2}$ and $-\sqrt{2}$.

155. $\sqrt{3}$ and $-\sqrt{3}$.

156. $\sqrt{5}$ and $-\sqrt{5}$.

157. $2\sqrt{3}$ and $-2\sqrt{3}$.

158. $3\sqrt{5}$ and $-3\sqrt{5}$.

159. $4\sqrt{6}$ and $-4\sqrt{6}$.

160. $\frac{1}{2}\sqrt{2}$ and $-\frac{1}{2}\sqrt{2}$.

161. $\frac{1}{2}\sqrt{3}$ and $-\frac{1}{2}\sqrt{3}$.

162. $\dfrac{\sqrt{5}}{5}$ and $\dfrac{-\sqrt{5}}{5}$.

163. $m+\sqrt{n}$ and $m-\sqrt{n}$.

164. $d+\sqrt{h}$ and $d-\sqrt{h}$.

165. $-h+\sqrt{k}$ and $-h-\sqrt{k}$.

166. $2+i$ and $2-i$.

167. $3+i$ and $3-i$.

168. $1+i$ and $1-i$.

169. $4+i$ and $4-i$.

170. $-5+i$ and $-5-i$.

171. $-6+i$ and $-6-i$.

172. $3+2\,i$ and $3-2\,i$.

173. $2+3\,i$ and $2-3\,i$.

174. $3+5\,i$ and $3-5\,i$.

175. $-2+7\,i$ and $-2-7\,i$.

176. $2+\sqrt{-3}$ and $2-\sqrt{-3}$.

177. $3+\sqrt{-5}$ and $3-\sqrt{-5}$.

178. $4+\sqrt{-2}$ and $4-\sqrt{-2}$.

179. $5+\sqrt{-6}$ and $5-\sqrt{-6}$.

180. $4+3\sqrt{-2}$ and $4-3\sqrt{-2}$.

181. $3+4\sqrt{-5}$ and $3-4\sqrt{-5}$.

REVIEW

MENTAL EXERCISE 182

Cube each of the following:

1. x.	**5.** $-b^2$.	**9.** $x^{\frac{1}{2}}$.	**13.** a^{-1}.
2. y^2.	**6.** $-\dfrac{c}{2}$.	**10.** $y^{\frac{1}{3}}$.	**14.** $b^{-\frac{2}{3}}$.
3. z^3.	**7.** $\dfrac{d}{-3}$.	**11.** $z^{\frac{2}{3}}$.	**15.** $c^{-\frac{3}{4}}$.
4. $-a$.	**8.** $-\frac{2}{4}$.	**12.** $\dfrac{3\,m^{\frac{3}{2}}}{2}$.	**16.** $d^{-\frac{5}{3}}$.

Express each of the following as a power of 2:

17. $(4^3)^2$. **18.** $(2^3)^4$. **19.** $(16^{\frac{1}{2}})^5$. **20.** $(32^{\frac{2}{5}})^3$.

Solve each of the following equations:

21. $\dfrac{x^2}{9}=1$. **22.** $25\,y^2=1$. **23.** $\dfrac{9\,z^2}{16}=1$. **24.** $\dfrac{w^2}{36}=4$.

Factor each of the following expressions:

25. $a^2 - 10ab + 25b^2 - 36$.
26. $x^2 + 18xy + 45y^2$.
27. $9m^4 + 23m^2n^2 + 16n^4$.
28. $15x^2 + 13xy + 2y^2$.

Solve each of the following equations:

29. $x^{\frac{1}{2}} = 3$.
30. $y^{\frac{1}{3}} = 2$.
31. $z^{\frac{1}{2}} = \frac{1}{2}$.
32. $w^{\frac{1}{5}} = -2$.
33. $x^{\frac{1}{3}} = -1$.
34. $\sqrt{\frac{x}{2}} = 3$.
35. $\frac{\sqrt{z}}{2} = 3$.
36. $x^{-1} = -1^{-1}$.
37. $\sqrt[n]{x} = n$.

Express the following as positive fractions:

38. $-\frac{1}{1-a}$.
39. $-\frac{b-2}{3}$.
40. $-\frac{y-x}{x+z}$.

Express the following without negative exponents:

41. $(-a)^{-2}$.
42. $(-b)^{-3}$.
43. $(-c^2)^{-5}$.

Simplify each of the following:

44. $-2a \cdot a^{-2}$.
45. $\frac{ab^c}{cb^a}$.
46. $a(a^a)^a$.
47. $\frac{b^{-b}}{b}$.
48. $(2a \div b^2)^{-2}$.
49. $-(x^{-1} \div y^{-1})^{-1}$.
50. $(m^{\frac{1}{n}}n^{\frac{1}{m}})^{mn}$.

Distinguish between

51. $a^{-\frac{1}{b}}$ and $\frac{1}{a^b}$.
52. $-a^{\frac{1}{x}}$ and $a^{-\frac{1}{x}}$.

53. Express $\left(\frac{a}{b}\right)^{\frac{b}{a}}$ as a quotient of powers.

Find the value of

54. $.2^{-1}$.
55. $.04^{-1}$.
56. $.5^{-2}$.
57. $(.02)^{-2}$.

Rationalize the denominators of each of the following:

58. $\frac{1}{\sqrt{2}}$.
59. $\frac{1}{\sqrt[3]{4}}$.
60. $\frac{1}{\sqrt[3]{a^2}}$.
61. $\frac{2}{\sqrt{2a}}$.
62. $\frac{1}{\sqrt[3]{5b}}$.
63. $\frac{2}{\sqrt{2c}}$.

What roots may possibly be introduced by squaring both members of each of the following equations?

64. $x=5$.
65. $y+2=3$.
66. $z-1=6$.
67. $x-2=0$.
68. $y-3=7$.
69. $z+8=3$.

Distinguish between

70. $(-\sqrt{2})^2$ and $(\sqrt{-2})^2$.
71. $-\sqrt[3]{8}$ and $\sqrt[3]{-8}$.
72. $\sqrt{6}-\sqrt{6}$ and $\sqrt{6}+\sqrt{-6}$.

Simplify each of the following:

73. $(\sqrt{5}+\sqrt{-5})^2$.
74. $(\sqrt{-7}-\sqrt{7})^2$.
75. $(2\,a^{\frac{1}{2}}+\frac{1}{2})^2$.

Solve each of the following equations:

76. $0=2\sqrt{7}\sqrt{x-7}$.
77. $\sqrt{x-a}=0$.
78. $(x-3)(9\cdot 7\cdot \frac{1}{2}+32\cdot 16-5)=0$.

Show that the following identities are true:

79. $-i\equiv\frac{1}{i}$.
80. $-i\equiv i^{-1}$.
81. $-i^{-2}\equiv 1$.
82. $(a-b)(c-b)(d-c)(d-a)\equiv(a-b)(b-c)(c-d)(d-a)$.
83. Rationalize $\frac{1}{\sqrt{2}}$.
84. Realize $\frac{1}{\sqrt{-2}}$.
85. Find the values of $(a^0+b^0)^2$ and $(a^2+b^2)^0$.

Simplify each of the following:

86. $-(a^{-1}b^{-2}c^{-3})^{-4}$.
87. $-(x^{-1}\div y^{-1})^{-2}$.
88. $(n\sqrt[n]{n})^n$.

IRRATIONAL EQUATIONS

EXERCISE 183

Solve the following irrational equations, verifying integral or fractional results and rejecting "extra roots":

1. $\sqrt{7+x^2}-4=0$.
2. $\sqrt{x^2+20}-6=0$.
3. $\sqrt{4x+1}+\sqrt{3x+4}=1$.
4. $\sqrt{16+x}+\sqrt{1-x}=5$.
5. $\sqrt{3x+7}=\sqrt{5x-6}-1$.
6. $\sqrt{x+7}=\sqrt{3x-5}-2$.
7. $\sqrt{25-x}+\sqrt{16+x}=9$.
8. $\sqrt{x+13}+\sqrt{3x-65}=8$.
9. $\sqrt{2x+1}+\sqrt{3x+4}=7$.
10. $\sqrt{5x-4}-\sqrt{2x-4}=2$.

11. $\sqrt{x^2+20}+\sqrt{x^2+9}=11.$

12. (*a*) $3+\sqrt{2x-6}-\sqrt{5x}=0.$
(*b*) $3-\sqrt{2x-6}+\sqrt{5x}=0.$
(*c*) $3-\sqrt{2x-6}-\sqrt{5x}=0.$
(*d*) $3+\sqrt{2x-6}+\sqrt{5x}=0.$

13. (*a*) $\sqrt{x+1}+\sqrt{x-14}+\sqrt{x+10}=0.$
(*b*) $\sqrt{x+1}+\sqrt{x-14}-\sqrt{x+10}=0.$
(*c*) $\sqrt{x+1}-\sqrt{x-14}+\sqrt{x+10}=0.$
(*d*) $\sqrt{x+1}-\sqrt{x-14}-\sqrt{x+10}=0.$

14. $\sqrt{x+11}+\sqrt{x-13}-\sqrt{x+22}=0.$

15. $\sqrt{5-x}+\sqrt{2+x}=\sqrt{14}.$

16. $\sqrt{6-x}+\sqrt{9-x}=\sqrt{15-2x}.$

17. $\sqrt{2x+9}-\sqrt{x-4}=\sqrt{x+1}.$

18. $\sqrt{5-2x}=\sqrt{6+x}+\sqrt{3+x}.$

19. $\sqrt{14+x}=\sqrt{6+x}+\sqrt{9+x}.$

20. $\sqrt{3x+1}+\sqrt{x-4}=\sqrt{57-x}.$

21. $\sqrt{3x+4}+\sqrt{x-3}=\sqrt{x+42}.$

22. $\sqrt{2x+4}-\sqrt{2x-7}=\sqrt{x-15}.$

23. $\sqrt{(x-6)(x-5)}+\sqrt{(x-4)(x-3)}=\sqrt{2}.$

24. $\sqrt{(4+x)(x+1)}+\sqrt{(4-x)(x-1)}=4\sqrt{x}.$

25. $\sqrt{b-x}+\sqrt{b+x}=\sqrt{b}.$

26. $\sqrt{a^2-x}-\sqrt{b^2+x}=a-b.$

27. $\sqrt{a-x}+\sqrt{b-x}=\sqrt{a+b-2x}.$

28. $\sqrt[3]{x-a}-\sqrt[3]{x-b}=\sqrt[3]{b-a}.$

29. $\dfrac{3}{\sqrt{x-4}}+\sqrt{x-4}=4.$

30. $\dfrac{3x+\sqrt{3x-3}}{3x-\sqrt{3x-3}}=1.$

31. $\dfrac{1}{x+\sqrt{x^2-1}}-\dfrac{1}{x-\sqrt{x^2-1}}=2.$

32. $\dfrac{\sqrt{2x-3}}{\sqrt{3x-2}}-\dfrac{\sqrt{3x-2}}{\sqrt{2x-3}}=-\dfrac{7}{12}.$

33. $2\sqrt{2x}+\sqrt{2x+9}=\dfrac{65}{\sqrt{2x+9}}.$

34. $\sqrt{x+2}+\dfrac{1}{\sqrt{x+2}}=x+3.$

35. $\dfrac{x-2}{\sqrt{x}}=\dfrac{a-2}{\sqrt{a}}.$

36. $\dfrac{a-x}{\sqrt{a-x}}+\dfrac{x-b}{\sqrt{x-b}}=\sqrt{a-b}.$

37. $\dfrac{1}{k-\sqrt{k^2-x^2}}-\dfrac{1}{k+\sqrt{k^2-x^2}}=\dfrac{k}{x^2}.$

38. $\sqrt{\dfrac{m}{x}}-\sqrt{\dfrac{n}{x}}=\sqrt{\dfrac{x}{n}}-\sqrt{\dfrac{x}{m}}.$

39. $\dfrac{\sqrt{a^2+x^2}+\sqrt{a^2-x^2}}{\sqrt{a^2+x^2}-\sqrt{a^2-x^2}}=\dfrac{\sqrt{a}+\sqrt{c}}{\sqrt{a}-\sqrt{c}}.$

40. $x^2+8x-\sqrt{x^2+8x}-6=0.$

41. $x^2+3x+\sqrt{x^2+3x}-6=0.$

42. $\sqrt{10-x^2-x}=8-x^2-x.$

43. $2x^2+x+\sqrt{2x^2+x}-42=0.$

44. $2x^2+6x+\sqrt{x^2+3x}=10.$

45. $2x^2+x-3\sqrt{2x^2+x+4}=6.$

46. $2x^2-10x+12-2\sqrt{x^2-5x+8}=0.$

47. $3x^2-4x+\sqrt{3x^2-4x-6}=18.$

Fill in the following Blank Forms in such a way that the resulting equations may be solved for a single unknown x:

48. $\sqrt{(\ \ )}+\sqrt{(\ \ )}=(\ \ ).$

49. $\sqrt{(\ \ )}-\sqrt{(\ \ )}=(\ \ ).$

50. $\sqrt{(\ \ )}-(\ \ )=\sqrt{(\ \ )}.$

51. $\sqrt{(\ \ )}+\sqrt{(\ \ )}=\sqrt{(\ \ )}.$

52. $\sqrt{(\ \ )}-\sqrt{(\ \ )}=\sqrt{(\ \ )}.$

SPECIAL IRRATIONAL EQUATIONS

MENTAL EXERCISE 184

Obtain one or more solutions of each of the following equations regarding x, y, z, and w as unknowns and all other letters as representing known numbers:

1. $x^{\frac{1}{2}} = 2$.
2. $x^{\frac{1}{3}} = 3$.
3. $x^{\frac{1}{4}} = 2$.
4. $x^{\frac{1}{3}} = -3$.
5. $y^{\frac{1}{3}} = -6$.
6. $-y^{\frac{1}{3}} = 4$.
7. $-y^{\frac{1}{3}} = 5$.
8. $\sqrt{z} = 5$.
9. $\sqrt{z} = 7$.
10. $\sqrt[3]{z} = 4$.
11. $\sqrt[3]{w} = 3$.
12. $\sqrt[3]{w} = -4$.
13. $\sqrt[4]{x} = 3$.
14. $\sqrt[4]{x} = 2$.
15. $\sqrt[5]{x} = 1$.
16. $\sqrt[5]{y} = -2$.
17. $\sqrt[3]{z} - \frac{1}{2}$.
18. $x^{\frac{1}{2}} = b^3$.
19. $x^{\frac{1}{4}} = c^4$.
20. $y^{\frac{1}{3}} = ab^2$.
21. $y^{\frac{1}{4}} = h^2m^3$.
22. $z^{\frac{1}{2}} = ab^2$.
23. $w^{\frac{1}{3}} = -bc^3$.
24. $w^{\frac{2}{3}} = 4$.
25. $x^{\frac{3}{4}} = 27$.
26. $x^{\frac{3}{4}} = 8$.
27. $y^{\frac{3}{2}} = 27$.
28. $y^{\frac{2}{3}} = 8$.
29. $z^{\frac{2}{3}} = 16$.
30. $z^{\frac{2}{3}} = 9$.
31. $\sqrt[3]{x^2} = 36$.
32. $\sqrt[5]{y^2} = 16$.
33. $x^{\frac{3}{2}} = 64$.
34. $x^{\frac{5}{6}} = 32$.
35. $x^{\frac{5}{3}} = -1$.
36. $x^{\frac{3}{2}} = 8$.
37. $x^{\frac{1}{3}} = a^3$.
38. $x^{\frac{3}{2}} = a^3$.
39. $x^{\frac{2}{3}} = a^6$.
40. $y^{\frac{2}{5}} = b^4$.
41. $z^{\frac{2}{3}} = a^{-2}$.
42. $y^{\frac{3}{2}} = m^{-9}$.
43. $x^{\frac{3}{2}} = \dfrac{a^3}{8}$.
44. $w^{\frac{2}{3}} = \dfrac{a^6}{9}$.
45. $\sqrt{2x} = 4$.
46. $\sqrt{3y} = 6$.
47. $\sqrt{5z} = 10$.
48. $\sqrt{2w} = 8$.
49. $\sqrt[3]{4x} = -2$.
50. $\sqrt[3]{2x} = 2$.
51. $\sqrt[3]{4y} = 4$.
52. $\sqrt{6z} = 2$.
53. $\sqrt{10z} = 5$.
54. $\sqrt{3w} = 4$.
55. $\sqrt{5w} = 6$.
56. $x^{\frac{1}{2}} = \frac{1}{3}$.
57. $y^{\frac{1}{3}} = \frac{1}{2}$.
58. $z^{\frac{1}{2}} = \frac{1}{2}$.
59. $x^{\frac{1}{2}} = \dfrac{1}{a}$.
60. $x^{\frac{1}{2}} = \dfrac{a}{b}$.
61. $\sqrt{w} = \frac{4}{5}$.
62. $\sqrt{w} = \frac{1}{5}$.
63. $\sqrt{x} = \frac{3}{2}$.
64. $\sqrt[3]{x} = \frac{3}{4}$.
65. $\sqrt[3]{x} = -\frac{1}{2}$.
66. $2x^{\frac{1}{2}} = 3$.
67. $3x^{\frac{1}{2}} = 4$.
68. $2x^{\frac{1}{3}} = 5$.
69. $3x^{\frac{1}{4}} = 2$.
70. $2x^{\frac{1}{4}} = 5$.
71. $\frac{1}{3}x^{\frac{1}{2}} = \frac{1}{4}$.
72. $3\sqrt{y} = 2$.
73. $3\sqrt{z} = 4$.

74. $4\sqrt{w}=1.$
75. $5\sqrt{x}=6.$
76. $2\sqrt{3\,y}=5.$
77. $3\sqrt{2\,z}=2.$
78. $4\sqrt{5\,w}=5.$
79. $\sqrt{ax}=b.$
80. $\sqrt{cx}=d.$
81. $\sqrt{by}=1.$
82. $a\sqrt{w}=b.$
83. $c\sqrt{z}=d.$
84. $d\sqrt{w}=1.$
85. $a\sqrt{x}=2.$
86. $\sqrt{\frac{y}{m}}=n.$
87. $\sqrt{\frac{x}{a}}=b.$
88. $\sqrt{z}=\frac{a}{b}.$
89. $\sqrt{aw}=\frac{1}{k}.$
90. $\frac{\sqrt{ax}}{a}=1.$
91. $\sqrt[n]{y}=2.$
92. $\sqrt[n]{y}=3.$
93. $\sqrt[n]{2\,z}=1.$
94. $\sqrt[n]{2\,w}=3.$
95. $\sqrt[n]{3\,x}=3.$
96. $\sqrt[n-1]{y}=2.$
97. $x^{\frac{1}{2}}=a+b.$
98. $y^{\frac{1}{2}}=a-b.$
99. $z^{\frac{1}{3}}=c+d.$
100. $w^{\frac{1}{3}}=m-n.$
101. $\sqrt{\frac{x}{2}}=3.$
102. $\frac{\sqrt{x}}{2}=3.$
103. $\sqrt{\frac{y}{3}}=5.$
104. $\frac{\sqrt{y}}{3}=5.$
105. $2\sqrt{\frac{z}{3}}=7.$
106. $\frac{1}{4}\sqrt{\frac{x}{3}}=\frac{1}{3}.$
107. $\frac{1}{3}\sqrt{\frac{x}{2}}=\frac{1}{4}.$
108. $\frac{2}{3}\sqrt{\frac{x}{5}}=1.$
109. $\frac{\sqrt{3\,x}}{3}=1$
110. $\frac{\sqrt{5\,y}}{5}=2.$
111. $y^{-\frac{1}{2}}=6.$
112. $z^{-\frac{1}{3}}=3.$
113. $z^{-\frac{1}{4}}=2.$
114. $\frac{1}{x^{\frac{1}{2}}}=\frac{1}{2}.$
115. $\frac{1}{x^{\frac{1}{3}}}=\frac{1}{3}.$
116. $x^{-\frac{1}{3}}=\frac{1}{5}.$
117. $y^{-\frac{1}{3}}=\frac{1}{4}.$
118. $z^{-\frac{1}{4}}=\frac{1}{2}.$
119. $z^{\frac{1}{2}}=3^{\frac{1}{4}}.$
120. $\sqrt{x}=\sqrt{6}.$
121. $z^{\frac{1}{3}}=2^{\frac{1}{6}}.$
122. $\sqrt{y}=\sqrt[3]{5}.$
123. $w^{\frac{2}{3}}=5^{\frac{1}{3}}.$
124. $\sqrt[3]{x}=\sqrt{6}.$
125. $\sqrt{y}=2\sqrt[3]{5}.$
126. $\sqrt{z}=3\sqrt[3]{7}.$
127. $\sqrt[3]{w}=2\sqrt{5}.$
128. $\sqrt[3]{w}=3\sqrt[3]{4}.$
129. $\sqrt[3]{x}=5\sqrt[4]{2}.$
130. $\sqrt[4]{x}=2\sqrt{2}.$
131. $\sqrt[4]{y}=3\sqrt{3}.$
132. $\sqrt{x+1}=2.$
133. $\sqrt{x}+1=2.$
134. $\sqrt{y+2}=5.$
135. $\sqrt{y}+2=5.$
136. $\sqrt{z-4}=3.$
137. $\sqrt{z}-4=3.$
138. $\sqrt{x-1}=4.$
139. $\sqrt{x}-1=4.$
140. $\sqrt{y-3}=3.$
141. $\sqrt{z+1}=1.$
142. $\sqrt{w-2}=2.$
143. $\sqrt{x+4}=2.$
144. $\sqrt{y+9}=3.$
145. $\sqrt[3]{z+1}=4.$
146. $\sqrt[3]{w-2}=3.$
147. $\sqrt{x+5}=4.$
148. $\sqrt{y-6}=2.$
149. $\sqrt{z-7}=7.$
150. $\sqrt{2\,x+1}=3.$
151. $\sqrt{3\,y-2}=5.$
152. $\sqrt{4\,z+5}=7.$
153. $2\sqrt{x-1}=3.$

154. $3\sqrt{y+1}=2.$

155. $\sqrt{\frac{x}{2}+1}=2.$

156. $\sqrt{\frac{y}{3}-1}=4.$

157. $\sqrt{\frac{z}{2}-2}=2.$

158. $3+\sqrt{w}=4.$

159. $5+\sqrt{x}=6.$

160. $4-\sqrt{y}=3.$

161. $6-\sqrt{z}=1.$

162. $2+\sqrt{w}=12.$

IRRATIONAL EQUATIONS AND EQUATIONS HAVING IRRATIONAL COEFFICIENTS

EXERCISE 185

Solve each of the following equations:

1. $(\sqrt{3}+\sqrt{2})x=1.$
2. $(\sqrt{7}-\sqrt{5})y=2.$
3. $(\sqrt{3}-\sqrt{2})z=\sqrt{6}.$
4. $(\sqrt{5}+3)a-13=7\sqrt{5}.$
5. $(\sqrt{2}-1)x-\sqrt{2}=1.$
6. $(5+\sqrt{2})y-17=8\sqrt{2}.$
7. $(4-\sqrt{3})x-5=2\sqrt{3}.$
8. $(6+\sqrt{7})x+9=-16\sqrt{7}.$
9. $\sqrt{x}+\sqrt{3\,x}=1.$
10. $\sqrt{x}-\sqrt{2\,x}=1.$
11. $\sqrt{5\,x}-\sqrt{3\,x}=2.$
12. $2\sqrt{x}+\sqrt{2\,x}=2-\sqrt{2}.$
13. $3\sqrt{x}-\sqrt{3\,x}=3+\sqrt{3}.$
14. $5\sqrt{x}+\sqrt{3\,x}=5+\sqrt{3}.$
15. $\sqrt{ax}+\sqrt{bx}=c.$
16. $\sqrt{ax}-\sqrt{bx}=d.$
17. $1+\sqrt{x}=2(1-\sqrt{x}).$
18. $x+1=\sqrt{2}(3-x).$
19. $\sqrt{x}+\sqrt{2\,x}+\sqrt{3\,x}=2\sqrt{2}.$
20. $\sqrt{ax}+\sqrt{bx}+\sqrt{cx}=1.$
21. $2\,x+\sqrt{3\,x+1}=14.$
22. $3\,x+\sqrt{2\,x-5}=24.$
23. $5\,x-\sqrt{4\,x+9}=43.$
24. $5\,x-7=\sqrt{3}(x+3).$
25. $\sqrt{2\,x+1-2\sqrt{2\,x-3}}=2.$
26. $\sqrt{x+7-\sqrt{5(x-2)}}=3.$
27. $x^2+2\,x-6=4\sqrt{3}.$
28. $x^2-4\,x+2(3\sqrt{5}-5)=0.$
29. $x^2-(5+\sqrt{3})x+\sqrt{3}=0.$
30. $x^2-5\sqrt{3}\,x+13+\sqrt{15}=0.$
31. $x^2-(3\sqrt{5}+\sqrt{3})x+4+3\sqrt{5}=0.$

Solve for both x and y:

32. $\sqrt{5}\,x-y=6.$
$7\,x-2\sqrt{5}\,y=3.$

33. $\sqrt{2}\,x+\sqrt{3}\,y=1.$
$\sqrt{3}\,x-\sqrt{2}\,y=1.$

34. $\sqrt{5}\,x+\sqrt{3}\,y=8.$
$\sqrt{3}\,x-\sqrt{5}\,y=4.$

SPECIAL EQUATIONS

EXERCISE 186

Find one or more solutions of each of the following equations, verifying all integral and fractional solutions:

1. $x^4 - 5x^2 + 4 = 0$.
2. $y^4 - 13y^2 + 36 = 0$.
3. $z^4 - 29z^2 + 100 = 0$.
4. $x^2(x^2 - 40) + 144 = 0$.
5. $x^4 - (x-2)^2 = 0$.
6. $x^4 = (x-6)^2$.
7. $16(x^2 - 3) = x^4$.
8. $(x-20)^2 - x^4 = 0$.
9. $3x^3 + 40 = x^6$.
10. $x^6 - 28x^3 + 27 = 0$.
11. $x - 3x^{\frac{1}{2}} + 2 = 0$.
12. $x + 2x^{\frac{1}{2}} - 15 = 0$.
13. $2x^{\frac{1}{2}} - 5x = -39$.
14. $x^{\frac{1}{2}} + 4x^{\frac{1}{4}} = 21$.
15. $x^{\frac{1}{2}}(3x^{\frac{1}{2}} - 2) = 8$.
16. $2x^{\frac{2}{3}} - 2 = 3x^{\frac{1}{3}}$.
17. $4x^{\frac{2}{3}} - 12x^{\frac{1}{3}} + 5 = 0$.
18. $3x^{\frac{3}{2}} = 88 - x^3$.
19. $3x^{\frac{3}{2}} - 4x^{\frac{3}{4}} = 160$.
20. $3x^{\frac{3}{2}} - 7 = 4x^{\frac{3}{4}}$.
21. $x^{\frac{6}{5}} + x^{\frac{3}{5}} = 756$.
22. $5 - 3x^{-1} = 2x^{-2}$.
23. $x^{-1} + x^{-\frac{1}{2}} = 12$.
24. $4x^{-2} - 32 + x^{-4} = 0$.
25. $20x^{-\frac{2}{5}} - x^{-\frac{4}{5}} = 64$.
26. $x^3 - 5x^2 = 14x$.
27. $x(x^2 - 16) = 45(x-4)$.
28. $x^{\frac{5}{2}} - 6x^{\frac{3}{2}} - 40x^{\frac{1}{2}} = 0$.

EXERCISE 187

Solve the following equations, verifying all integral and fractional solutions:

1. $(x^2 - 3x)^2 - 8(x^2 - 3x) = 20$.
2. $(x^2 + x)^2 - 26(x^2 + x) + 120 = 0$.
3. $3x^2 + 2x + 1 = \dfrac{30}{3x^2 + 2x}$.
4. $x^2 - 4x - 26 + \dfrac{105}{x^2 - 4x} = 0$.
5. $x^4 + 2x^3 - 6x^2 - 7x - 60 = 0$.
6. $x^4 - 2x^3 + 6x^2 - 5x = 14$.
7. $x^4 + 6x^3 + 14x^2 + 15x + 6 = 0$.
8. $x^4 - 10x^3 + 14x^2 + 55x + 30 = 0$.
9. $(x^2 - x - 4)(x^2 - x - 3) - 6 = 0$.

10. $(x^2+x+1)(x^2+x+3)=63.$

11. $\dfrac{x}{x^2-1}+\dfrac{x^2-1}{x}=\dfrac{73}{24}.$

12. $\dfrac{x^2+2}{x-3}+\dfrac{35}{6}=\dfrac{x-3}{x^2+2}.$

13. $\left(x+\dfrac{1}{x}\right)^2+4\left(x+\dfrac{1}{x}\right)=\dfrac{65}{4}.$

14. $\left(\dfrac{5}{x}+x\right)^2+\left(\dfrac{5}{x}+x\right)=42.$

15. $\left(\dfrac{x^2+1}{x}\right)^2-\left(\dfrac{x^2+1}{x}\right)=12.$

16. $\left(x-\dfrac{8}{x}\right)^2+3x-\dfrac{24}{x}=10.$

BINOMIAL EQUATIONS AND RECIPROCAL EQUATIONS

EXERCISE 188

Solve the following equations, verifying all integral and fractional solutions:

1. $x^3-8=0.$ 3. $x^3=1.$ 5. $x^3+8=0.$
2. $x^3-64=0.$ 4. $x^3+1=0.$ 6. $x^3+125=0.$
7. $(x+5)^3=8.$ 8. $(x-5)^4-81=0.$
9. $6x^4-35x^3+62x^2-35x+6=0.$
10. $10x^4-77x^3+150x^2-77x+10=0.$
11. $6x^4-49x^3+86x^2-49x+6=0.$
12. $6x^4-25x^3+38x^2-25x+6=0.$
13. $12x^4-91x^3+194x^2-91x+12=0.$
14. $15x^3-49x^2+49x-15=0.$
15. $8x^5-46x^4+47x^3+47x^2-46x+8=0.$

PROBLEMS IN PHYSICS

EXERCISE 189

Solve each of the following problems relating to the single pendulum:

1. Find the length in feet of a pendulum which vibrates once in a second at a place at which $g=32.16$.

2. Find the value of g at a certain place if a pendulum which is 10 feet in length makes 20 vibrations in 35 seconds.

3. If a certain pendulum vibrates once in a second, find the time required for a pendulum which is twice as long to vibrate once.

4. Find the length of a pendulum which makes 80 vibrations per minute at a place at which the value of g is 32.16 per second per second.

5. Find the length of a pendulum which vibrates once per second at a place at which the value of g is 32.19.

6. If at a certain place a pendulum 39 inches in length vibrates once in a second, find the length of a pendulum which at the same place will make one vibration in one minute.

7. If a ball suspended by a fine wire makes 88 vibrations in 15 minutes, find the length of the wire.

8. If a pendulum which is 39.1 inches in length vibrates once in a second at a certain place, find the length of a pendulum which will vibrate once in 5 seconds.

REVIEW

EXERCISE 190

1. If $a=1$, $b=3$, $c=2$, find the value of $(a+b)(b+c)(c+a) + a^b + b^c + c^a$.

2. Factor $1 + 10\,x - 11\,x^2$.

3. Factor $x^2(y+1)+y^2(x+1)+x+y+2\,xy$.

4. Factor $a^3 - x^3 - a(a^2 - x^2) + x(a-x)^2$.

5. Find the prime factors of $(a-a^2)^3 + (a^2-1)^3 + (1-a)^3$.

Simplify each of the following expressions:

6. $\left(\frac{1}{x^2}+\frac{1}{x}+1\right)(x^2-x+1)$.

7. $\left(\frac{a}{x}-\frac{b}{y}\right)\left(\frac{x}{a}-\frac{y}{b}\right)$.

8. $(x^5 - \frac{1}{32}) \div (x - \frac{1}{2})$.

9. $\dfrac{(a^4-b^4)(a^3-b^3)}{(a^6-b^6)(a-b)}$.

10. $\left(\frac{a-1}{a+1}\right)^2 - \left(\frac{a-2}{a+2}\right)^2$.

11. $(a^2-b^2-c^2-2\,bc) \div \dfrac{a+b+c}{a+b-c}$.

12. $\dfrac{a^2-b^2}{\frac{1}{a}-\frac{1}{b}}$.

13. $(x^{-1}+y^{-1}) \div (x^{-\frac{1}{3}}+y^{-\frac{1}{3}})$.

14. $\dfrac{27^{-2} \times 9 \times \frac{1}{3^{-1}} - 3^{-3}}{27^2 \times 3^3}$.

15. Express $(-a-b) \div (-a^{-1}-b^{-1})$ with the minimum number of minus signs.

16. Find the value of $\dfrac{x+2m}{2n-x}+\dfrac{x-2m}{2n+x}+\dfrac{4mn}{x^2-4n^2}$, if $x=\dfrac{mn}{m+n}$.

17. Show that $(a^2-b^2)^2 \equiv a^3-ab+b^3$, if $a+b=1$.

Simplify each of the following expressions:

18. $(3-\sqrt{-2})(2-\sqrt{-3})$.

19. $ab+\sqrt{ab}+(a-\sqrt{b})(\sqrt{a}-b)$.

20. $(\sqrt{-3}-\sqrt{-2})^2+(\sqrt{3}-\sqrt{2})^2$.

21. $(\sqrt{5}-\sqrt{7}+2)(\sqrt{5}+\sqrt{7}-2)$.

22. $(\sqrt{7}+\sqrt{5}-\sqrt{3})(\sqrt{7}-\sqrt{5}+\sqrt{3})$.

23. $(\sqrt{11}-\sqrt{6}+5)(\sqrt{11}-\sqrt{6}-5)$.

24. $\left(\dfrac{\sqrt{a}}{\sqrt{b}}-\dfrac{\sqrt{b}}{\sqrt{a}}\right)\div\left(\dfrac{1}{\sqrt{b}}-\dfrac{1}{\sqrt{a}}\right)$.

Simplify:

25. $\sqrt{R^2-\left(\dfrac{R}{2}\sqrt{5}-\dfrac{R}{2}\right)^2}$.

26. $\left[\left(R-\dfrac{R}{2}\sqrt{3}\right)^2+\left(\dfrac{R}{2}\right)^2\right]^{\frac{1}{2}}$.

27. $\left[R^2-\left(\dfrac{R\sqrt{2-\sqrt{2}}}{2}\right)^2\right]^{\frac{1}{2}}$.

28. $\dfrac{\dfrac{nx^{2a}\cdot x^{n-1}-3ax^n\cdot x^{2a-1}}{(x^{2a})^2}}{\dfrac{x^n}{x^{2a}}}$.

29. $\dfrac{\dfrac{y^{3a}ny^{n-1}-y^n 3ay^{3a-1}}{(y^{3a})^2}}{\dfrac{y^n}{y^{3a}}}$.

SIMULTANEOUS QUADRATICS

ELIMINATION BY SUBSTITUTION

EXERCISE 191

Solve the following systems of simultaneous quadratic equations:

1. $xy=18$, $x=2y$.

2. $xy=48$, $3x=y$.

3. $xy=-9$, $x=-y$.

4. $xy=-75$. $x=-3y$.

5. $x-4y=0$, $xy=16$.

6. $5x-y=0$, $xy=20$.

7. $x+6y=0$, $xy=-150$.

8. $xy=6$, $3x=2y$.

9. $xy=20$, $5x=4y$.

10. $xy - 15 = 0,$
$3x = 5y.$

11. $xy - 80 = 0,$
$4x = 5y.$

12. $x^2 + xy = 18,$
$x = y.$

13. $x^2 - xy = 32,$
$x = -y.$

14. $xy + y^2 = 8,$
$x = y.$

15. $x^2 + y^2 = 98.$
$x = y.$

16. $2x^2 - xy = 36,$
$x = y.$

17. $3x^2 - 2xy = 45,$
$x = -y.$

18. $2xy - y^2 = 49,$
$x - y = 0.$

19. $3x^2 - y^2 = 162,$
$x = y.$

20. $2xy + 3y^2 = 45,$
$x - y = 0.$

21. $5x^2 - 3xy = 128,$
$x - y = 0.$

22. $4x^2 + 3xy = 7,$
$x - y = 0.$

23. $11x^2 + 7xy = 36,$
$x = -y.$

24. $x^2 + xy = 6,$
$x = 2y.$

25. $x^2 + y^2 = 40,$
$x = 3y.$

26. $x^2 + y^2 = 5,$
$2x - y = 0.$

27. $x^2 + xy + y^2 = 75,$
$x = y.$

28. $x^2 - xy + y^2 = 16,$
$x - y = 0.$

29. $x^2 + xy + y^2 = 21,$
$x - 4y = 0.$

30. $5x^2 - 8y + 3 = 0,$
$x - y = 0.$

31. $x + y^2 = 12,$
$x - y = 0.$

32. $x^2 + 4y + 1 = 0,$
$x = 2y.$

33. $10x + y^2 + 25 = 0,$
$x = y.$

34. $2x^2 - 7y + 5 = 0,$
$x - y = 0.$

35. $3y^2 + 5x + 2 = 0,$
$x = y.$

36. $xy - 10y + 7 = 0,$
$x = 3y.$

37. $xy + 8x - 15 = 0,$
$7x = y.$

38. $3xy - 5y + 1 = 0,$
$x = 2y.$

39. $x^2 - y^2 = 40,$
$2x + y = 17.$

40. $x^2 + y^2 = 17,$
$x - 3y = 1.$

41. $x^2 + 4y^2 = 32,$
$5x + 6y = 8.$

42. $x^2 + 2y^2 = 73,$
$3x - y = 3.$

43. $5x^2 - xy = 15,$
$2x + 3y = 36.$

44. $x + 8y = xy,$
$x - y = 5.$

45. $x^2 + xy + y^2 = 7,$
$x + 4y = -1.$

46. $x - 5y = \frac{1}{2}$,
$3x - 2y + 4y^2 = 9$.

47. $x + y + 2xy = \frac{25}{6}$,
$5x - 2y = \frac{1}{3}$.

48. $2x^2 + 3xy + 4y^2 = 64$,
$x + y = -2$.

49. $2x - 3y = 2$,
$4x^2 - 3xy + y^2 = 44$.

50. $\frac{x+y}{y} = 6$,
$xy = 45$.

51. $\frac{x}{y} + \frac{y}{x} = 2$,
$6x - 5y = 1$.

52. $\frac{1}{x} - \frac{1}{y} = -\frac{1}{2}$,
$x - 3y = -1$.

53. $\frac{x}{y} + \frac{y}{x} = \frac{5}{2}$,
$x - y = -2$.

54. $\frac{x}{4} + \frac{y}{5} = 6$,
$\frac{4}{x} + \frac{5}{y} = \frac{6}{5}$.

55. $x - y - \frac{5}{6} = 0$,
$\frac{1}{x} + \frac{1}{y} - 5 = 0$.

56. $\frac{1}{x} + \frac{y}{2} = 2$,
$\frac{1}{y} + \frac{x}{2} = -\frac{1}{3}$.

57. $\frac{x+1}{y+1} = \frac{6}{5}$,
$\frac{x^2+y}{x+y^2} = \frac{65}{46}$.

58. $\frac{x}{x+y} + \frac{y}{x-y} = \frac{61}{11}$,
$2x + 3y = 54$.

REDUCTION OF SYSTEMS OF EQUATIONS BY FACTORING

EXERCISE 192

Reduce each of the following systems of equations to equivalent groups of separate systems of equations and solve:

1. $(x-4)(y-7) = 0$,
$x + y = 12$.

2. $4x^2 - xy = 0$,
$5x - y = 1$.

3. $(x-3)(x+y-5) = 0$,
$(x-y-3)(y-4) = 0$.

4. $(x-y)(x-y-1) = 0$,
$(x+3)(x+4) = 0$.

5. $x^2 + 6xy = 16y^2$,
$x + y = 6$.

6. $x^2 + 4y^2 = 20$,
$x^2 - 3xy = 4y^2$.

7. $(x-y)^2 - 16 = 0$,
$(x+y)^2 = 36$.

8. $y^2 - x^2 = 2y - 1$,
$x^2 - 10y^2 = 3xy$.

9. $xy - 7y + 4x = 28$,
$x + y = 10$.

10. $xy - x + y = 25$,
$x(x-y) = 0$.

11. $x^2 - xy = 66$,
$5x^2 - 16xy + 11y^2 = 0$.

12. $x^2 - y^2 = x + y$,
$x^2 - 5xy = 5x - 25y$.

13. $x^2 + xy + x - y = 50$,
$5x^2 + 2xy - y^2 = 0$.

14. $6x^2 - 5xy + 4x - 3y = 12$,
$x^2 - xy = 0$.

SYSTEMS OF TWO HOMOGENEOUS EQUATIONS OF THE SECOND DEGREE CONTAINING TWO UNKNOWNS

SOLUTION BY FACTORING

EXERCISE 193

Solve the following systems of homogeneous equations:

1. $x^2 + xy = 5$,
$xy + y^2 = 3$.

2. $x^2 + xy = 4$,
$3x^2 + 2xy + y^2 = 18$.

3. $x^2 + 3xy = 40$,
$x^2 + 2y^2 = 24$.

4. $x^2 + xy - 7y^2 = 35$,
$x^2 - 5xy + 9y^2 = 15$.

5. $x^2 + xy + y^2 = 52$,
$x^2 - xy + y^2 = 28$.

6. $5x^2 - 10xy + 7y^2 = 28$,
$3x^2 - 7xy + 4y^2 = 8$.

7. $(2x + y)(2y + x) = 500$,
$(x + y)(x - y) = 75$.

8. $13xy - 2x^2 - 18y^2 = 12$,
$2x^2 - 5xy - 3y^2 = 16$.

9. $4x(x + 3y) + 11y^2 = 3$,
$x^2 + 3y(x + 3y) = 7$.

10. $3x(x + 2y) + 5y^2 = 21$,
$x^2 + 2y(x + 2y) = 28$.

11. $(3x + y)(3y + x) = \frac{35}{36}$,
$(x + y)(x - y) = \frac{1}{12}$.

12. $x + y = \dfrac{20}{x}$,
$x - y = \dfrac{3}{y}$.

SOLUTION BY EXPRESSING THE VALUE OF ONE UNKNOWN AS A MULTIPLE OF THE OTHER

EXERCISE 194

Solve the following systems of homogeneous equations:

1. $5x^2 + 2xy + 3y^2 = 91$,
$3x^2 - 2xy + 4y^2 = 44$.

2. $x^2 + 5xy + y^2 = -21$,
$2x^2 - 3y^2 = 38$

3. $4x^2 - 2xy + 3y^2 = 71$,
$2x^2 + 5xy + 3y^2 = 133$.

4. $x^2 + xy + y^2 = 37$,
$x^2 - 3xy = 72$.

5. $4xy + y^2 = 68$,
$2x^2 - xy = 112$.

6. $2x^2 + 11xy + 3y^2 = 96$,
$x^2 - y^2 = 5$.

7. $x^2 - 4xy = 9$,
$7xy - 14y^2 = 70$.

8. $x^2 + 3xy + y^2 = 59$,
$9x^2 - y^2 = 11$.

9. $3x(x - 3y) + y^2 = -11$,
$x^2 + 2y(x - 3y) = 21$.

10. $2x^2 - 7y(x - y) = 143$,
$3x(x - 2y) - y^2 = 2$.

REDUCTION OF SYSTEMS OF EQUATIONS BY DIVISION

EXERCISE 195

Solve each of the following systems of equations:

1. $x^3 = y(x + y)$,
$x^2 = x + y$.

2. $1 + y = x$,
$1 + y^3 = x^3$.

3. $x(x - 3) = 4 - y^2$,
$x = 2 + y$.

4. $x(y + 3) = 9y^2 - 1$,
$x = 3y - 1$.

5. $x(x - a) = b^2 - y^2$,
$x - a = b + y$.

6. $x^2 - y^2 = 65$,
$x - y = 5$.

7. $x + y = 11$,
$x^2 - y^2 = 55$.

8. $x^3 + y^3 = 126$,
$x + y = 6$.

9. $x^3 - y^3 = 26$,
$x - y = 2$.

10. $x^3 - y^3 = a^3 - b^3$,
$x - y = a - b$.

11. $x^3 + y^3 = 91$,
$x^2 - xy + y^2 = 13$.

12. $x^3 - y^3 = 56$,
$x^2 + xy + y^2 = 28$.

13. $27x^3 + y^3 = 35$,
$9x^2 - 3xy + y^2 = 7$.

14. $x^4 + x^2y^2 + y^4 = 21$,
$x^2 - xy + y^2 = 3$.

SYSTEMS OF SYMMETRIC EQUATIONS

EXERCISE 196

Solve each of the following systems of symmetric equations:

1. $x + y = 7$,
$xy = 10$.

2. $x + y = 8$,
$xy = 12$.

3. $x + y = -3$,
$xy = -28$.

4. $x + y = 4$,
$xy = -5$.

5. $x + y = 0$,
$xy = -1$.

6. $x^2 + y^2 = 29$,
$xy = 10$.

7. $x^2 + y^2 = 17,$
$xy = 4.$

8. $x^2 + y^2 = 29,$
$xy = -10.$

9. $x^2 + xy + y^2 = 19,$
$xy = 6.$

10. $x^2 + xy + y^2 = 52,$
$xy = 12.$

11. $x^2 - xy + y^2 = 19,$
$xy = 10.$

12. $x^2 - xy + y^2 = 21,$
$xy = 4.$

13. $x^2 + y^2 = 82,$
$x + y = 10.$

14. $x^2 + y^2 = 25,$
$x + y = -7.$

15. $x^2 - xy + y^2 = 57,$
$x + y = 9.$

16. $x^2 + xy + y^2 = 79,$
$x + y = 10.$

17. $x^2 + xy + y^2 = 7,$
$x + y = -3.$

18. $x^2 + y^2 + x + y = 24,$
$x = y.$

19. $xy + x + y = 35,$
$x - y = 0.$

20. $xy - x - y = 48,$
$x - y = 0.$

21. $x^2 + xy + y^2 = 13,$
$x^2 - xy + y^2 = 7.$

22. $x^2 + xy + y^2 = 52,$
$x^2 - xy + y^2 = 28.$

23. $x^2 + xy + y^2 = 49,$
$x + y = -8.$

24. $x^2 + y^2 = b,$
$x + y = a.$

25. $x + y = a,$
$xy = b.$

26. $\frac{1}{x} + \frac{1}{y} = 9,$
$\frac{1}{x^2} + \frac{1}{y^2} = 41.$

27. $\frac{1}{x} + \frac{1}{y} = 10,$
$\frac{1}{xy} = 24.$

28. $\frac{1}{x} - \frac{1}{y} = 1,$
$xy = \frac{1}{42}.$

29. $\frac{1}{x^3} + \frac{1}{y^3} = 91,$
$\frac{1}{x} + \frac{1}{y} = 7.$

30. $\frac{1}{x^2} + \frac{1}{y^2} = \frac{13}{36},$
$xy = 6.$

31. $xy(x + y) = 30,$
$\frac{x + y}{xy} = \frac{5}{6}.$

32. $x^4 + x^2y^2 + y^4 = 21,$
$x^2 - xy + y^2 = 3.$

33. $x^2 + xy + y^2 = 13,$
$x^4 + x^2y^2 + y^4 = 91.$

34. $x^2 + xy + y^2 = 84,$
$x + \sqrt{xy} + y = 14.$

35. $x^2 + xy + y^2 = 133,$
$x - \sqrt{xy} + y = 7.$

36. $(x + 1)(y + 1) = 42\frac{1}{2},$
$xy = 1.$

37. $\frac{1}{x} + \frac{1}{y} = \frac{5}{4},$
$\frac{1}{x + 1} + \frac{1}{y + 1} = \frac{7}{10}.$

SYSTEMS OF ASYMMETRIC EQUATIONS

EXERCISE 197

Solve the following equations by the methods employed for the solution of systems of symmetric equations:

1. $x - y = 4,$
   $xy = 2.$
2. $x - y = 6,$
   $xy = 16.$
3. $x - y = 9,$
   $xy = -14.$
4. $x - y = 11,$
   $xy = -24.$
5. $x - y = 5,$
   $xy = 0.$
6. $x - y = 0,$
   $xy = -9.$
7. $x^2 + y^2 = 45,$
   $x - y = 3.$
8. $x^2 - xy + y^2 = 12,$
   $x - y = -2.$
9. $x^2 + xy + y^2 = 39,$
   $x - y = 3.$
10. $x^2 + xy + y^2 = 49,$
    $x - y = -2.$

EXERCISE 198. MISCELLANEOUS

Solve the following systems of equations:

1. $x = 5 + y,$
   $y^2 = 7 - xy.$
2. $9x^2 - y^2 = 11,$
   $x + 2y = 12.$
3. $x + y = x^2,$
   $3y - x = y^2.$
4. $x^2 + y^2 = 173,$
   $x - y = 11.$
5. $(x + y)^2 = 64,$
   $(x - y)^2 = 36.$
6. $x^2 + 2xy + y^2 = 49,$
   $x^2 - 2xy + y^2 = 9.$
7. $(x - 1)(y + 1) = 42,$
   $x - y = 13.$
8. $x - 4y = 11,$
   $xy = 4x^2 - 1.$
9. $\frac{x}{y} = 3,$
   $xy = 12.$
10. $xy = 20,$
    $\frac{y}{x} = 5.$
11. $xy = a^2,$
    $\frac{x}{y} = b^2.$
12. $\frac{x + y}{x} = 4,$
    $xy = 75.$
13. $\frac{x + y}{y} = a,$
    $xy = b.$
14. $x + \frac{1}{y} = 1,$
    $y + \frac{1}{x} = 4.$
15. $9x + \frac{1}{y} = 4,$
    $15x - \frac{2}{y} = 3.$

16. $\dfrac{25x}{y}=\dfrac{4y}{x}$,
$xy+x+y=17$.

17. $\dfrac{1}{x}-\dfrac{1}{y}=3$,
$\dfrac{1}{x-1}+\dfrac{1}{y-1}=-\dfrac{5}{3}$.

18. $\dfrac{1}{x^2}+\dfrac{1}{y^2}=\dfrac{65}{64}$,
$\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{9}{8}$.

19. $x^{-1}-y^{-1}=2$,
$x^{-2}-y^{-2}=12$.

20. $y(x+y)=x^2$,
$x(x-y)=y$.

21. $(x+1)(y+1)=48$,
$x+y=12$.

22. $x^2+xy=28$,
$xy+y^2=-12$.

23. $x^2-y^2=24$,
$xy=35$.

24. $x(x+y)-20=0$,
$y(y+x)-16=0$.

25. $x^2+xy=56$,
$y(x-y)=6$.

26. $\dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}=\dfrac{10}{3}$,
$x^2+y^2=45$.

27. $x^4+x^2y^2+y^4=3$,
$x^2-xy+y^2=1$.

28. $x^2+xy+y^2=84$,
$x+\sqrt{xy}+y=14$.

29. $3x^2y^2-xy=14$,
$x+y=-1$.

30. $x^2y^2+xy=30$,
$x+y=6$.

31. $x^2+y^2=189-xy$,
$x+y=9+\sqrt{xy}$.

32. $x^3+y^3=152$,
$x^2y+xy^2=120$.

33. $x^3+y^3=126$,
$xy(x+y)=30$.

34. $x^3-y^3=26$,
$x^2y-xy^2=6$.

35. $x^3-y^3=56$,
$x-y=\dfrac{16}{xy}$.

36. $\dfrac{x-y}{x}-\dfrac{y}{x+y}=0$,
$x+y=1$.

37. $y(x+y)+x(y+x)=4xy$,
$y(x+y)+x+y=24$.

38. $\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}+\dfrac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}=\dfrac{26}{5}$,
$x^2+y^2=97$.

39. $x+y\sqrt{x}=22$,
$x^2+xy^2=340$.

40. $(4x-y)(x-4y)=19$,
$(4y+x)(y+4x)=189$.

41. $x^3+y^3=152$,
$xy=15$.

42. $x+y+2\sqrt{x+y}=24$,
$x^2+y^2=130$.

43. $x^2+y^2-8(x+y)=13$,
$x+y+xy=-9$.

44. $\dfrac{y+1}{x+1}=\dfrac{2}{3}$,
$\dfrac{y^2+y+1}{x^2+x+1}=\dfrac{31}{13}$.

45. $(x+y)(x^2+y^2)=-20$,
$(x-y)(x^2-y^2)=-32$.

46. $ax = by,$
$x^2 + y^2 = c.$

47. $px = qy,$
$(p+q)x - (p-q)y = r.$

48. $a(x+y) = b(x-y) = xy,$

49. $x^2 - y^2 = a,$
$x^4 - y^4 = b.$

50. $\frac{a^2}{x^2} + \frac{y^2}{b^2} = 6,$
$\left(\frac{a}{x}\right)\left(\frac{b}{y}\right) = 1.$

51. $ax + by = 2\,ab,$
$\frac{a}{y} + \frac{b}{x} = 2.$

52. $bx + by = 2,$
$abxy = 1.$

53. $\frac{x+y}{y} = ax,$
$xy = b.$

54. $\frac{x-y}{\sqrt{x} - \sqrt{y}} = 8,$
$\sqrt{xy} = 12.$

55. $\frac{x-y}{\sqrt{x} - \sqrt{y}} = a,$
$\sqrt{xy} = b.$

56. $x^2 = 6\,x + 4\,y,$
$y^2 = 4\,x + 6\,y.$

57. $x^2 = cx + dy,$
$y^2 = dx + cy.$

58. $x^5 + y^5 = 244,$
$x + y = 4.$

59. $x^4 + y^4 = 97,$
$x + y = 5.$

60. $x^4 + y^4 = 257,$
$x - y = 3.$

61. $\frac{1}{x} - \frac{1}{y} = \frac{1}{x-y},$
$\frac{1}{x^2} - \frac{1}{y^2} = \frac{1}{a^2}.$

62. $\frac{x^2}{y^2} + \frac{y^2}{x^2} + \frac{x}{y} - \frac{y}{x} = \frac{127}{36},$
$xy = \frac{1}{6}.$

Solve the following systems of equations and find approximate values of the unknowns which are correct to three places of decimals:

63. $x + y = 2.8,$
$xy = 1.87.$

64. $x + y = .051,$
$xy = .000518.$

65. $10\,x - 10\,y = .5,$
$10\,xy = .126.$

66. $.5\,x - .17 = .1\,y,$
$10\,xy = 1.2.$

67. $.001\,xy = 1.075,$
$.1\,x - .1\,y = 1.8.$

68. $x^2 + y^2 = .89,$
$10\,xy = 4.$

69. $100\,x^2 + 100\,y^2 = 65,$
$xy = .28.$

70. $.01\,x + .01\,y = .0015,$
$.1\,x^2 + .1\,y^2 = .00125.$

71. $x^2 + y^2 = 11.3,$
$x + y = 4.4.$

72. $x^2 + y^2 = 9.0625,$
$x + y = 3.25.$

73. $4x^2 - y^2 = 2.03$,
$x + y = 2$.

74. $.2x^2 - xy = -.742$,
$x + .1y = .82$.

75. $2.5x + .3y = 7$,
$.5xy = 6$.

76. $.7x + y = 8$,
$xy = 2.4$.

77. $x^2 + 10y^2 = 14.49$,
$x + y = 1.5$.

78. $5x^2 + y^2 = 9.2$,
$xy = .6$.

79. $x^3 - y^3 = .056$,
$x - y = .2$.

80. $3x^2 - y^2 = 299.99$,
$xy = 1$.

SYSTEMS OF THREE OR MORE EQUATIONS CONTAINING THREE OR MORE UNKNOWNS

EXERCISE 199

Find sets of values which satisfy each of the following systems of equations:

1. $xy = 24$,
$yz = 48$,
$xz = 32$.

2. $xy = a^2$,
$yz = b^2$,
$zx = c^2$.

3. $x^2y = 1$,
$y^2z = 128$,
$z^2x = 32$.

4. $x^2yz = a$,
$xy^2z = b$,
$xyz^2 = c$.

5. $yz = 2y + 4z$,
$zx = 4z + x$,
$xy = x + 2y$.

6. $x(y + z) = 8$,
$y(x + z) = 18$,
$z(x + y) = 20$.

7. $yz = bc$,
$\frac{x}{a} + \frac{y}{b} = 1$,
$\frac{x}{a} + \frac{z}{c} = 1$.

8. $x^2 + y^2 = 20$,
$y^2 + z^2 = 52$,
$z^2 + x^2 = 40$.

9. $y + z = \frac{1}{x}$,
$z + x = \frac{1}{y}$.
$x + y = \frac{1}{z}$.

10. $xy^2z^2 = -54$,
$\frac{yz^2}{x} = \frac{9}{2}$,
$\frac{x^2y}{z} = -\frac{4}{3}$.

11. $\frac{xy}{x+y} = \frac{12}{5}$,
$\frac{yz}{y+z} = \frac{24}{7}$,
$\frac{zx}{z+x} = \frac{8}{3}$.

12. $\frac{xyz}{x+y} = 8$,
$\frac{xyz}{y+z} = \frac{24}{5}$,
$\frac{xyz}{z+x} = 6$.

13. $\frac{xyz}{x+y} = a$,
$\frac{xyz}{y+z} = b$,
$\frac{xyz}{x+z} = c$.

14. $\dfrac{x^2+y^2}{xyz}=\dfrac{5}{8},$
$\dfrac{y^2+z^2}{xyz}=\dfrac{5}{2},$
$\dfrac{z^2+x^2}{xyz}=\dfrac{17}{8}.$

15. $(x+1)(y+1)=6,$
$(y+1)(z+1)=21,$
$(z+1)(x+1)=14.$

16. $x(2-y)=16,$
$y(2-z)=9,$
$z(2-x)=4.$

17. $x^2=yz,$
$x+y+z=28,$
$xyz=512.$

18. $xy+xz+yz=4,$
$x-y=3,$
$y-z=1.$

19. $x(x+y+z)=22,$
$y(x+y+z)=44,$
$z(x+y+z)=55.$

20. $(y+z)(x+y+z)=6,$
$(z+x)(x+y+z)=10,$
$(x+y)(x+y+z)=-8.$

21. $xy=c(x+y+z),$
$yz=a(x+y+z),$
$xz=b(x+y+z).$

22. $(x+z)(x+y)=a^2,$
$(x+y)(y+z)=b^2,$
$(y+z)(x+z)=c^2.$

23. $x+y+z=a,$
$xy=b,$
$xyz=c.$

24. $x+y=4,$
$y+z=6,$
$z+w^2=9,$
$x^2+z=14.$

25. $x+y+z=10,$
$xy+yz+zx=31,$
$x^2+y^2-z^2=20.$

26. $x^2-(y-z)^2=1,$
$y^2-(z-x)^2=9,$
$z^2-(x-y)^2=16.$

27. $x(y+z)+2=0,$
$y(z-2x)+12=0,$
$z(2x-y)=0.$

28. $x+xy+y=5,$
$y+yz+z=9,$
$x+xz+z=14.$

29. $x^2-yz=9,$
$y^2-zx=3,$
$z^2-xy=1.$

30. $x^2-yz=58,$
$y^2-zx=8,$
$z^2-xy=73.$

31. $yz+x+y=-3,$
$xz+y=-1,$
$xy=2.$

32. $x^2+xy+y^2=3,$
$y^2+yz+z^2=7,$
$z^2+zx+x^2=7.$

33. $x+y+z=1\frac{3}{3},$
$\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=\dfrac{13}{3},$
$xyz=1.$

PROBLEMS

PROBLEMS WHICH MAY BE SOLVED BY MEANS OF SIMULTANEOUS QUADRATIC EQUATIONS

EXERCISE 200

Solve the following problems, and examine the solutions obtained to see if they satisfy the conditions of the given problems:

1. The sum of two numbers is 9 and their product is 20. Find the numbers.

2. The sum of two numbers is 14 and their product is 45. Find the numbers.

3. The sum of two numbers is 15 and their product is 54. Find the numbers.

4. The sum of two numbers is 18 and their product is 80. Find the numbers.

5. The sum of two numbers is 28 and their product is 195. Find the numbers.

6. The difference of two numbers is 4 and their product is 60. Find the numbers.

7. The difference of two numbers is 5 and their product is 84. Find the numbers.

8. The difference of two numbers is 6 and their product is 91. Find the numbers.

9. The difference of two numbers is 6 and their product is 187. Find the numbers.

10. The sum of two numbers is 8 and the sum of their squares is 34. Find the numbers.

11. The sum of two numbers is 11 and the sum of their squares is 65. Find the numbers.

12. The sum of two numbers is 16 and the sum of their squares is 136. Find the numbers.

13. The sum of two numbers is 17 and the sum of their squares is 169. Find the numbers.

14. The difference of two numbers is 5 and the sum of their squares is 73. Find the numbers.

15. The difference of two numbers is 3 and the sum of their squares is 149. Find the numbers.

16. The difference of two numbers is 4 and the sum of their squares is 170. Find the numbers.

17. The difference of two numbers is 5 and the sum of their squares is 227. Find the numbers.

18. The sum of two numbers is 6 and the difference of their squares is 24. Find the numbers.

19. The sum of two numbers is 8 and the difference of their squares is 32. Find the numbers.

20. The sum of two numbers is 11 and the difference of their squares is 33. Find the numbers.

21. The sum of two numbers is 13 and the difference of their squares is 91. Find the numbers.

22. The difference of two numbers is 5 and the difference of their squares is 65. Find the numbers.

23. The difference of two numbers is 3 and the difference of their squares is 63. Find the numbers.

24. The difference of two numbers is 6 and the difference of their squares is 96. Find the numbers.

25. The perimeter of a rectangular field is 260 rods and its area is 4200 square rods. Find the dimensions of the field.

26. The perimeter of a rectangular field is 190 rods and its area is 2200 square rods. Find the dimensions of the field.

27. The length of the fence about a rectangular field is 180 rods. Find the dimensions of the field if the area is 2000 square rods.

28. The product of two numbers is 75, and the quotient obtained by dividing the greater number by the less is 3. Find the numbers.

29. If the sum of two numbers is divided by the less number, the quotient is 3 and the product of the numbers is 32. Find the numbers.

30. Find two numbers, the sum of which is 25, such that the sum of the quotients obtained by dividing each number by the other is $1\frac{7}{4}$.

31. Find two numbers such that, if each be increased by 1, the product is 144, and the product obtained by multiplying the first number by a number less than the second number by one is 70.

32. Find two numbers, the sum of which is three times their difference, and the difference of the squares of which is 600.

33. The product of two numbers is 48, and the sum of their squares is ten times the sum of the numbers. Find the numbers.

34. Find two numbers, of which the sum is 18, which are such that the product of the first and the reciprocal of the second, increased by the product of the second and the reciprocal of the first is $\frac{4}{2}\frac{1}{0}$.

35. The sum of two numbers is 36, and the sum of the quotients resulting from dividing each number by the other is $\frac{6}{8}\frac{5}{}$. Find the numbers.

36. The first of two numbers is eighteen times the reciprocal of the second, and the sum of the second number and thirty-six times the reciprocal of the first is equal to the square of the second number. Find the numbers.

37. Find two fractions, such that the sum of the first fraction and the reciprocal of the second is equal to 2, and the sum of the second fraction and the reciprocal of the first is $1\frac{8}{5}$.

38. Find two numbers, the sum of which is 39, and half the product of which is equal to the cube of the less number.

39. Find a fraction, the value of which is $\frac{4}{5}$, and the product of the numerator and denominator of which is 80.

40. If a certain two-figure number, the sum of the figures of which is 14, be multiplied by the units' figure, the product is 516. What is the number?

41. A number expressed by two figures is equal to four times the sum of the figures. The number formed by writing the figures in reversed order exceeds three times the product of the figures by the square of the figure in tens' place of the given number. Find the number.

42. If it requires 260 rods of fence to inclose a rectangular field of 25 acres, what are the dimensions of the field?

43. A rectangular field contains 30 acres. By increasing its length by 40 rods and diminishing its width by 4 rods the area is increased by 6 acres. What are its dimensions?

44. The length of the fence around a rectangular field is 274 yards, and the distance measured diagonally from corner to corner is 97 yards. What is the area?

45. Thirty-two yards of the fence about a rectangular field which is 184 yards long and 76 yards wide are destroyed. What must be the dimensions of a rectangular field in order that the length of fence remaining shall inclose the same area as before?

46. A property owner wishes to use the material from a stone wall inclosing a field which has the form of a rectangle 80 rods long and 60 rods wide to build another wall greater by 16 rods, which shall inclose a second tract of land which has the form of a rectangle having the same area as the first. Find the dimensions of the second tract of land.

47. In widening a street a strip of land 6 feet in width was removed from the entire frontage of a tract containing 28,800 square feet. By increasing the frontage of the reduced lot by 8 feet the entire area became the same as before. Find the original dimensions of the land.

48. It is observed that if a guy rope which is attached to a stake 7 feet from the foot of a derrick were lengthened by 15 feet, it would reach to a stake 32 feet from the foot of the derrick. Find the height of the derrick and the length of the rope.

49. A tract of 10 acres of land is inclosed by a certain length of fence in such a way that there are two separate lots, each in the form of a square, so situated that the side of the smaller lot forms a part of the side of the larger lot. It is observed that the fence may be rebuilt to inclose a single lot in

the form of a square containing $5\frac{5}{8}$ acres more than the original lots. Find the dimensions of the original lots.

50. At an entertainment \$750 was realized from the sale of seats. For each reserved seat twenty-five cents more was charged than for an unreserved seat, but the sale of the unreserved seats yielded the same total amount as that of the reserved seats. Find the total number of seats sold, if the number of unreserved seats exceeded the number of reserved seats by 125.

51. Three stone crushers working together can crush a certain amount of stone in a week. The first machine has a capacity twice as great as that of the second, but working alone would require one week more than the third machine to perform the work. What time would each require, working alone?

52. The sum of a fraction and its reciprocal is equal to the numerator increased by the reciprocal of twice the denominator; and the difference between the reciprocal of the numerator and the reciprocal of the denominator is equal to the reciprocal of twice the denominator. Required the fraction.

53. Find a fraction, such that if its numerator be increased by 3 and its denominator diminished by 3, the result is the reciprocal of the fraction; but if the denominator be increased by 3 and the numerator diminished by 3, the result will be $1\frac{3}{20}$ less than the reciprocal of the fraction.

54. Find a fraction the value of which is $\frac{2}{3}$, such that if the numerator be diminished by the reciprocal of the denominator and the denominator be increased by the reciprocal of the numerator, the value of the fraction will be multiplied by $\frac{149}{154}$.

55. Find two numbers, the sum of the cubes of which is 152 and the sum of the squares of which diminished by their product is 19.

56. Find two numbers of which the sum multiplied by the product is equal to 48 and the sum of the cubes of which is 72.

57. A number is expressed by three figures, the sum of which is 11. The middle figure exceeds the sum of the other two by 1, and the sum of the squares of the separate figures is eight times the figure in tens' place increased by the sum of the other two. Find the number.

58. Two boats leave simultaneously the opposite shores of a river which is $2\frac{1}{4}$ miles wide, and pass each other in 15 minutes. The faster boat completes the trip $6\frac{3}{4}$ minutes before the other reaches the opposite shore. Find the rates of the boats in miles per hour.

59. A train starts from a certain station to make a trip of 180 miles, traveling uniformly. Forty-five minutes later a faster train, also traveling uniformly, starts from the same station, and after traveling two hours and fifteen minutes reaches the station which the first train had passed thirty-six minutes previously. The speed of the second train is now increased by four miles an hour, with the result that the trains reach the terminus at the same time. Find the rates in miles per hour at which they started.

60. Fifteen hours after an ocean steamship leaves the American shore, to make a voyage of 3300 miles at a certain average uniform rate, a second ship starts for America from the opposite shore. The two ships meet after the second ship has been out $71\frac{3}{7}$ hours, and they complete their trips at the same instant. Find the rates of the ships in miles per hour.

61. An express train, an electric car, and an automobile all leave a given place for a certain destination. The express train travels 9 miles an hour faster than the electric car, and the automobile 13 miles an hour faster than the express. The express starts one hour after the electric and 39 minutes before the automobile. If all arrive at the end of their journeys at the same time, find the distance and the rates of traveling in miles per hour.

62. Sighting an enemy's war vessel at a distance of 10 miles, a submarine boat starts toward it, running on the surface at a certain uniform rate which exceeds its speed when submerged by 6 miles an hour. At a certain point in its course it dives beneath the surface, and when submerged at a distance of one-half mile from the battleship, a torpedo is discharged which travels at the rate of a mile in two minutes. It is observed from the shore that the time, measured from the instant the submarine starts until the explosion takes place, is exactly $44\frac{1}{3}$ minutes. Returning immediately after delivering its torpedo, and traveling the entire distance under water, the time required is 1 hour $1\frac{1}{3}$ minutes. Find the rate of the submarine on the surface in miles per hour and also the distance from the starting place of the point at which it sank beneath the surface.

63. A torpedo boat on being discovered $1\frac{1}{2}$ miles from port immediately turns and tries to escape. One minute later a torpedo-boat destroyer is sent out and this overtakes it after a run of $9\frac{1}{2}$ miles. If the rates of the torpedo boat and destroyer could have been increased by 6 miles an hour, and 2 miles an hour respectively, when the distance between the two boats was reduced to 1 mile the torpedo boat would have escaped to its squadron $10\frac{1}{2}$ miles away. Find the rates of the boats in miles per hour.

RATIO

REDUCTION TO SIMPLEST FORM

EXERCISE 201

Express the values of the following ratios in terms of the smallest possible integral numbers:

1. 4 : 6.	**5.** 10 : 16.	**9.** 48 : 84.	**13.** 125 : 375.
2. 9 : 12.	**6.** 25 : 45.	**10.** 34 : 85.	**14.** 143 : 99.
3. 6 : 15.	**7.** 32 : 48.	**11.** 84 : 144.	**15.** 38 : 209.
4. 8 : 18.	**8.** 24 : 96.	**12.** 39 : 169.	**16.** 42 : 98.

17. $28 : 70$.
18. $\frac{1}{2} : \frac{1}{3}$.
19. $\frac{1}{5} : \frac{1}{4}$.
20. $\frac{1}{6} : \frac{1}{8}$.
21. $\frac{1}{10} : \frac{1}{2}$.
22. $\frac{2}{3} : \frac{1}{5}$.
23. $\frac{3}{4} : \frac{1}{7}$.
24. $\frac{1}{8} : \frac{3}{5}$.
25. $\frac{5}{6} : \frac{1}{2}$.
26. $\frac{3}{4} : \frac{3}{5}$.
27. $\frac{5}{6} : \frac{5}{8}$.
28. $\frac{2}{3} : \frac{5}{3}$.
29. $\frac{3}{7} : \frac{2}{7}$.
30. $1 : \frac{1}{2}$.
31. $2 : \frac{1}{3}$.
32. $\frac{1}{4} : 3$.
33. $\frac{1}{5} : 5$.
34. $\frac{1}{6} : 3$.
35. $7 : \frac{1}{7}$.
36. $3 : \frac{3}{4}$.
37. $\frac{5}{7} : 5$.
38. $.2 : .5$.
39. $.7 : .3$.
40. $.2 : .4$.
41. $.6 : .8$.
42. $.1 : .01$.
43. $.2 : .02$.
44. $.5 : 5$.
45. $6 : .6$.

Express each of the following ratios in simplest form:

46. $ab : ac$.
47. $xy : yz$.
48. $a^2 : ab$.
49. $ac : c^2$.
50. $abc : b^2$.
51. $x^2 : xyz$.
52. $a^2b : ab^2$.
53. $a^2b^3 : a^3b^2$.
54. $a^2b : abc$.
55. $xyz : x^2y$.
56. $2a : 3a$.
57. $4b : 5b$.
58. $6c : 10c$.
59. $8a^2 : 8b^2$.
60. $14a^3 : 21a^2$.
61. $16b^2 : 32b^3$.
62. $24c^3 : 32c$.
63. $35d^4 : 40d^6$.
64. $16a^4 : 4a^8$.
65. $5b^6 : 25b^3$.
66. $10a^{10} : 6a^6$.
67. $12x^4 : 4x^{12}$.
68. $\frac{a}{b} : \frac{c}{d}$.
69. $\frac{x}{y} : \frac{w}{z}$.
70. $\frac{1}{m} : \frac{1}{n}$.
71. $a : \frac{1}{a}$.
72. $\frac{1}{b} : b$.
73. $\frac{1}{c} : d$.
74. $a^n : a^{n+1}$.
75. $a^{n+2} : a^{n+3}$.
76. $a^{n-1} : a^n$.
77. $a^{2n} : a^{3n}$.
78. $a^n : a^{2n}$.
79. $a + b : a^2 - b^2$.
80. $a - b : a^2 - b^2$.
81. $a + b : a^3 + b^3$.
82. $a - b : a^3 - b^3$.
83. $a^2 - ab + b^2 : a^3 + b^3$.
84. $a^2 + ab + b^2 : a^3 - b^3$.

COMPOUND RATIOS

EXERCISE 202

Find the ratio compounded of the ratios:

1. $\frac{2}{3}$ and $\frac{4}{5}$.
2. $\frac{1}{7}$ and $\frac{3}{4}$.
3. $\frac{5}{6}$ and $\frac{3}{2}$.
4. $\frac{2}{5}$ and $\frac{5}{8}$.
5. $\frac{12}{14}$ and $\frac{7}{6}$.
6. $\frac{4}{5}$ and $\frac{5}{6}$.
7. $\frac{7}{8}$ and $\frac{8}{9}$.
8. $\frac{2}{4}$ and $\frac{6}{8}$.
9. $\frac{a}{3}$ and $\frac{4}{b}$.
10. $\frac{5}{x}$ and $\frac{6}{y}$.
11. $\frac{m}{7}$ and $\frac{10}{m}$.
12. $\frac{a}{b}$ and $\frac{b}{a}$.
13. $\frac{x^2}{y}$ and $\frac{y^2}{x^3}$.
14. $\frac{2}{a^n}$ and $\frac{n}{a^n}$.
15. $\frac{a^n}{b}$ and $\frac{b}{a^{n+1}}$.

16. $\frac{a^{n+1}}{c^3}$ and $\frac{c}{a^{n-1}}$.

17. $\frac{a^{n-2}}{x}$ and $\frac{x^2}{a^{n-1}}$.

18. $\frac{n^2}{a^n}$ and $\frac{2\,a^2}{n}$.

Find the duplicate ratio of

19. $2/3$.
20. $3/4$.
21. $1/2$.
22. a/b.
23. c/d.
24. x/y.
25. m^2/n^2.
26. $5/x^3$.
27. $y^2/4$.
28. a^n/b^n.
29. $2/c^{2n}$.
30. $3/d^{3n}$.

Find the triplicate ratio of

31. $2/3$.
32. $3/5$.
33. $6/7$.
34. $a/4$.
35. $5/b$.
36. c/d.
37. x^2/y^2.
38. m^3/n^3.
39. a^2/b^3.
40. a^n/b^n.
41. n/x^n.
42. a^{2n}/b^{3n}.

Which is the greater ratio

43. $\frac{2}{3}$ or $\frac{5}{6}$?
44. $\frac{3}{4}$ or $\frac{5}{8}$?
45. $\frac{3}{11}$ or $\frac{5}{22}$?
46. $\frac{2}{3}$ or $\frac{3}{4}$?
47. $\frac{5}{6}$ or $\frac{4}{5}$?
48. $\frac{5}{8}$ or $\frac{14}{24}$?

PROPORTION

EXERCISE 203

Which of the following pairs of ratios can be used to form proportions?

1. $\frac{2}{3}$ and $\frac{4}{6}$.
2. $\frac{6}{9}$ and $\frac{8}{12}$.
3. $\frac{10}{12}$ and $\frac{15}{18}$.
4. $\frac{6}{10}$ and $\frac{9}{15}$.
5. $\frac{6}{8}$ and $\frac{9}{12}$.
6. $\frac{4}{10}$ and $\frac{6}{15}$.
7. $\frac{6}{21}$ and $\frac{4}{14}$.
8. $\frac{9}{21}$ and $\frac{30}{70}$.
9. $\frac{5}{15}$ and $\frac{4}{12}$.
10. $\frac{15}{10}$ and $\frac{18}{12}$.
11. $\frac{1}{2} : \frac{1}{3}$ and $6 : 4$.
12. $\frac{1}{4} : \frac{1}{3}$ and $9 : 12$.
13. $\frac{2}{3} : 4$ and $2 : 12$.
14. $\frac{4}{5} : 6$ and $4 : 30$.
15. $\frac{4}{7} : \frac{2}{5}$ and $\frac{2}{7} : \frac{1}{5}$.
16. $\frac{2}{3} : \frac{3}{4}$ and $\frac{4}{3} : \frac{6}{4}$.
17. $\frac{3}{8} : \frac{1}{4}$ and $\frac{1}{2} : \frac{1}{3}$.
18. $\frac{1}{2} : \frac{5}{3}$ and $\frac{3}{4} : \frac{5}{2}$.

MEAN PROPORTIONALS

EXERCISE 204

Find the mean proportional between each of the following pairs of numbers:

1. 2 and 8.
2. 3 and 27.
3. 4 and 25.
4. 5 and 20.
5. 1 and 64.
6. 2 and 32.

7. 12 and 3.
8. 49 and 1.
9. 16 and 4.
10. 1 and 100.
11. $\frac{1}{2}$ and 8.
12. $\frac{1}{3}$ and 27.
13. $\frac{1}{5}$ and 20.
14. $\frac{1}{2}$ and $\frac{1}{8}$.
15. $\frac{2}{9}$ and $\frac{1}{2}$.
16. $\frac{2}{25}$ and $\frac{1}{2}$.
17. $\frac{4}{5}$ and $\frac{5}{9}$.
18. $\frac{7}{25}$ and $\frac{16}{7}$.
19. $\frac{81}{8}$ and $\frac{2}{9}$.
20. a and a^3.
21. x^5 and x.
22. ab and $\frac{b}{a}$.
23. $\frac{x}{y}$ and $\frac{y^3}{x^3}$.
24. $\frac{ab}{c}$ and $\frac{ac}{b}$.

Find the third proportional to

25. 2 and 4.
26. 1 and 3.
27. 8 and 4.
28. 2 and 8.
29. 1 and 7.
30. 4 and 6.
31. 1 and 9.
32. 2 and 10.
33. 36 and 12.
34. 27 and 9.
35. 18 and 12.
36. 25 and 10.

FOURTH PROPORTIONALS

EXERCISE 205

Find the fourth proportional to the three numbers in the order as given in each of the following sets:

1. 2, 3, 4.
2. 4, 6, 8.
3. 5, 7, 10.
4. 2, 4, 4.
5. 1, 3, 6.
6. 3, 1, 6.
7. 4, 6, 8.
8. 2, 4, 8.
9. 2, 8, 4.
10. 4, 2, 8.
11. 4, 8, 2.
12. 8, 2, 4.
13. 8, 4, 2.
14. 12, 4, 3.
15. 3, 4, 36.
16. 3, 36, 4.
17. 4, 3, 36.
18. 4, 36, 3.
19. 36, 3, 4.
20. 36, 4, 3.
21. 48, 6, 4.

SOLUTION OF PROPORTIONS

EXERCISE 206

Find the value of x, y, z, or w in terms of the remaining letters or numbers in each of the following proportions :

1. $\frac{x}{4}=\frac{5}{2}$.
2. $\frac{x}{6}=\frac{8}{3}$.
3. $\frac{x}{4}=\frac{4}{8}$.
4. $\frac{x}{9}=\frac{4}{9}$.
5. $\frac{x}{8}=\frac{-5}{4}$.
6. $\frac{x}{3}=\frac{4}{7}$.
7. $\frac{x}{7}=\frac{7}{5}$.
8. $\frac{x}{2}=\frac{a}{3}$.
9. $\frac{x}{5}=\frac{b}{6}$.
10. $\frac{x}{4}=\frac{2}{c}$.
11. $\frac{x}{3}=\frac{m}{3}$.
12. $\frac{x}{5}=\frac{5}{n}$.

13. $\frac{x}{3}=\frac{a}{b}$. **22.** $\frac{x}{a}=\frac{b}{c}$. **31.** $\frac{1}{y}=\frac{a}{b}$. **40.** $\frac{m}{a}=\frac{z}{n}$.

14. $\frac{x}{2}=\frac{m}{n}$. **23.** $\frac{x}{bc}=\frac{1}{a}$. **32.** $\frac{1}{ab}=\frac{c}{x}$. **41.** $\frac{5c}{d}=\frac{z}{3b}$.

15. $\frac{x}{5}=\frac{a}{2b}$. **24.** $\frac{2}{y}=\frac{3}{5}$. **33.** $\frac{2a}{y}=\frac{3b}{c}$. **42.** $\frac{2}{3}=\frac{5}{w}$.

16. $\frac{x}{a}=\frac{3a}{b}$. **25.** $\frac{3}{y}=\frac{4}{7}$. **34.** $\frac{a^2}{y}=\frac{b}{a}$. **43.** $\frac{8}{9}=\frac{2}{w}$.

17. $\frac{x}{a^2}=\frac{b}{a}$. **26.** $\frac{4}{y}=\frac{3}{11}$. **35.** $\frac{b^3}{y}=\frac{b^2}{c}$. **44.** $\frac{2}{5}=\frac{2}{w}$.

18. $\frac{x}{c^3}=\frac{1}{c}$. **27.** $\frac{10}{y}=\frac{2}{5}$. **36.** $\frac{2}{11}=\frac{z}{5}$. **45.** $\frac{a}{b}=\frac{1}{w}$.

19. $\frac{x}{b^2}=\frac{1}{bc}$. **28.** $\frac{6}{y}=\frac{6}{7}$. **37.** $\frac{3}{4}=\frac{z}{4}$. **46.** $\frac{1}{a}=\frac{ab}{x}$.

20. $\frac{x}{g}=\frac{k}{3}$. **29.** $\frac{a}{y}=\frac{2}{3}$. **38.** $\frac{5}{6}=\frac{z}{-12}$. **47.** $\frac{3x}{a}=\frac{2}{b}$.

21. $\frac{x}{m}=\frac{n}{4}$. **30.** $\frac{3}{y}=\frac{b}{c}$. **39.** $\frac{a}{b}=\frac{z}{a}$. **48.** $\frac{x}{ab}=\frac{1}{c}$.

49. $\frac{2x}{3}=\frac{8}{9}$. **50.** $\frac{2x}{a}=\frac{b}{3c}$.

DETERMINATION OF SPECIFIED RATIOS

EXERCISE 207

Find the value of the ratio $\frac{x}{y}$ from each of the following equations:

1. $bx=ay$.
2. $2x=3y$.
3. $xw=yz$.
4. $2x=wy$.
5. $xz=4y$.
6. $(a+b)x=(a-b)y$.
7. $x^2=yz$.
8. $mx=y$.

Construct four or more proportions from each of the following equations in such a way that each of the two factors of either member of a given equation shall appear as a single term of every proportion derived from it:

9. $2a=3b$.
10. $5x=6y$.
11. $4x=3y$.
12. $2a=bc$.
13. $ab=5c$.
14. $mx=ny$.

15. $ab = xy$.
16. $x^2 = yz$.
17. $ab = c^2$.
18. $3x = y^2$.
19. $ab = (a+b)^2$.
20. $a^2 - b^2 = ab$.
21. $(a+b)b = (c+d)d$.
22. $(x-y)y = (z-w)w$.
23. $(x+2)3 = (y+3)2$.
24. $a^2 - b^2 = c^2 - d^2$.

TRANSFORMATIONS OF PROPORTIONS

EXERCISE 208

Transform the following proportions by inversion:

1. $\frac{2}{x} = \frac{3}{4}$.
2. $\frac{5}{y} = \frac{10}{11}$.
3. $\frac{5}{6} = \frac{x}{7}$.
4. $\frac{8}{13} = \frac{y}{9}$.
5. $\frac{a}{x} = \frac{b}{c}$.
6. $\frac{a}{b} = \frac{y}{c}$.
7. $\frac{b}{a} = \frac{c}{b}$.
8. $\frac{y}{x} = \frac{z}{y}$.
9. $\frac{x}{4} = \frac{9}{x}$.
10. $\frac{y}{2} = \frac{32}{y}$.
11. $\frac{1}{z} = \frac{2}{b}$.
12. $\frac{a}{x} = \frac{1}{m}$.

EXERCISE 209

Transform the following proportions by alternation:

1. $\frac{a}{4} = \frac{b}{6}$.
2. $\frac{x}{10} = \frac{y}{8}$.
3. $\frac{b}{2} = \frac{c}{4}$.
4. $\frac{a}{6} = \frac{b}{3}$.
5. $\frac{16}{a} = \frac{20}{b}$.
6. $\frac{14}{x} = \frac{18}{y}$.
7. $\frac{m}{a^2} = \frac{n}{b^2}$.
8. $\frac{a}{x^2} = \frac{b}{x^3}$.
9. $\frac{x}{ab} = \frac{y}{bc}$.
10. $\frac{a}{a+b} = \frac{c}{c+d}$.
11. $\frac{x+y}{y} = \frac{z+w}{w}$.
12. $\frac{a-b}{b} = \frac{c-d}{d}$.
13. $\frac{a-2}{2} = \frac{b-7}{7}$.
14. $\frac{a+b}{c+d} = \frac{a-b}{c-d}$.

EXERCISE 210

Transform each of the following proportions so that one term shall consist of x, and x shall be found in no other term:

1. $\frac{3}{2} = \frac{10-x}{x}$.
2. $\frac{3}{4} = \frac{14-x}{x}$.
3. $\frac{4}{5} = \frac{18-x}{x}$.

4. $\frac{7}{2}=\frac{27-x}{x}$.
5. $\frac{m}{n}=\frac{a-x}{x}$.
6. $\frac{2}{3}=\frac{x-6}{6}$.
7. $\frac{4}{3}=\frac{x-12}{12}$.
8. $\frac{5}{4}=\frac{x-8}{8}$.
9. $\frac{3}{5}=\frac{x-15}{15}$.
10. $\frac{a}{b}=\frac{x-c}{c}$.
11. $\frac{5}{6}=\frac{10}{x-10}$.
12. $\frac{5}{3}=\frac{15}{x-15}$.
13. $\frac{a}{b}=\frac{c}{x-c}$.
14. $\frac{5}{3}=\frac{20}{20-x}$.
15. $\frac{11}{8}=\frac{22}{22-x}$.
16. $\frac{a}{b}=\frac{c}{c-x}$.
17. $\frac{x}{20-x}=\frac{7}{3}$.
18. $\frac{x}{22-x}=\frac{8}{3}$.
19. $\frac{x}{a-x}=\frac{b}{c}$.
20. $\frac{17}{7}=\frac{20+x}{x}$.
21. $\frac{a}{b}=\frac{x+c}{x}$.
22. $\frac{16}{9}=\frac{x}{x-21}$.
23. $\frac{a}{b}=\frac{x}{x-c}$.
24. $\frac{x}{x-24}=\frac{11}{5}$.
25. $\frac{x}{x-34}=\frac{20}{3}$.
26. $\frac{x}{x-a}=\frac{b}{c}$.

EXERCISE 211

Transform each of the following proportions so as to obtain the value of the ratio $\frac{x+y}{x-y}$:

1. $\frac{x}{y}=\frac{7}{4}$.
2. $\frac{x}{y}=\frac{10}{3}$.
3. $\frac{x}{y}=\frac{12}{7}$.
4. $\frac{x}{y}=\frac{11}{5}$.
5. $\frac{x}{y}=\frac{15}{7}$.
6. $\frac{x}{y}=\frac{14}{9}$.
7. $\frac{x}{y}=\frac{17}{10}$.
8. $\frac{x}{y}=\frac{20}{11}$.
9. $\frac{x}{y}=\frac{a}{2}$.
10. $\frac{x}{y}=\frac{3}{b}$.
11. $\frac{x}{y}=\frac{2d}{3b}$.
12. $\frac{x}{y}=\frac{4z}{5w}$.

Find the value of $\frac{m-n}{m+n}$ if

13. $\frac{m}{n}=\frac{3}{2}$.
14. $\frac{m}{n}=\frac{6}{5}$.
15. $\frac{m}{n}=\frac{8}{3}$.
16. $\frac{16}{9}=\frac{m}{n}$.
17. $\frac{24}{17}=\frac{m}{n}$.
18. $\frac{28}{15}=\frac{m}{n}$.

EXERCISE 212

Transform each of the following proportions by addition and subtraction and simplify one of the resulting ratios so that one term shall consist of x, and x shall be found in no other term:

1. $\frac{15}{7}=\frac{22+x}{22-x}$.
2. $\frac{23}{5}=\frac{42+x}{42-x}$.
3. $\frac{15}{2}=\frac{68+x}{68-x}$.
4. $\frac{16}{3}=\frac{38+x}{38-x}$.
5. $\frac{a}{b}=\frac{c+x}{c-x}$.
6. $\frac{13}{4}=\frac{a+x}{a-x}$.
7. $\frac{a}{5}=\frac{b+x}{b-x}$.
8. $\frac{x+8}{x-8}=\frac{7}{3}$.
9. $\frac{x+15}{x-15}=\frac{13}{3}$.
10. $\frac{x+18}{x-18}=\frac{25}{7}$.
11. $\frac{x+13}{x-13}=\frac{29}{3}$.
12. $\frac{x+a}{x-a}=\frac{b}{c}$.
13. $\frac{x+m}{x-m}=\frac{2a}{7b}$.
14. $\frac{x+b}{x-b}=\frac{3c}{2d}$.

DETERMINATION OF SPECIFIED RATIOS AND ELIMINATION

EXERCISE 213

1. If $\frac{a}{b}=\frac{c}{d}=\frac{2}{5}$ show that $\frac{a+c}{b+d}=\frac{2}{5}$.
2. If $\frac{x}{y}=\frac{z}{w}=\frac{3}{7}$ show that $\frac{x+3}{y+7}=\frac{z}{w}$.
3. If $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$ show that $\frac{a+b}{b+c}=\frac{b+c}{c+d}$.
4. Show that from $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{5}{6}$ we may derive $\frac{a+5}{b+6}=\frac{c+5}{d+6}$.
5. If $\frac{x}{y}=\frac{z}{w}=\frac{3}{7}$ show that $\frac{x+3}{y+7}=\frac{z}{w}$.
6. If $\frac{a}{x}=\frac{b}{y}=\frac{c}{z}$ show that $\frac{a+c}{x+z}=\frac{b+c}{y+z}$.

Combine the following pairs of proportions in such a manner as to eliminate x, and simplify the resulting proportions as far as possible:

7. $\frac{x}{b}=\frac{3}{4}$ and $\frac{a}{x}=\frac{5}{7}$.
8. $\frac{x}{n}=\frac{7}{8}$ and $\frac{m}{x}=\frac{3}{2}$.

9. $\frac{x}{d}=\frac{1}{2}$ and $\frac{c}{x}=\frac{9}{11}$.

10. $\frac{x}{c}=\frac{3}{7}$ and $\frac{b}{x}=\frac{4}{3}$.

11. $\frac{x}{d}=\frac{7}{5}$ and $\frac{c}{x}=\frac{4}{7}$.

12. $\frac{x}{n}=\frac{a}{c}$ and $\frac{m}{x}=\frac{c}{b}$.

13. $\frac{m}{x}=\frac{2}{3}$ and $\frac{x}{n}=\frac{5}{7}$.

14. $\frac{a}{x}=\frac{m}{n}$ and $\frac{x}{b}=\frac{c}{d}$.

15. $\frac{c}{x}=\frac{7}{6}$ and $\frac{x}{5}=\frac{d}{7}$.

16. $\frac{x}{2}=\frac{a}{b}$ and $\frac{2}{x}=\frac{c}{d}$.

17. $\frac{2}{3}=\frac{x}{m}$ and $\frac{5}{7}=\frac{m}{x}$.

18. $\frac{m}{2}=\frac{x}{5}$ and $\frac{3}{n}=\frac{7}{x}$.

19. $\frac{3}{4}=\frac{x}{b}$ and $\frac{5}{11}=\frac{a}{x}$.

20. $\frac{2}{a}=\frac{x}{d}$ and $\frac{3}{b}=\frac{c}{x}$.

21. $\frac{a}{b}=\frac{x}{n}$ and $\frac{c}{d}=\frac{m}{x}$.

22. $\frac{4}{5}=\frac{x}{d}$ and $\frac{5}{9}=\frac{c}{x}$.

23. $\frac{b}{13}=\frac{x}{4}$ and $\frac{13}{c}=\frac{9}{x}$.

24. $\frac{x}{6}=\frac{b}{d}$ and $\frac{x}{5}=\frac{c}{a}$.

25. $\frac{x}{11}=\frac{a}{b}$ and $\frac{x}{8}=\frac{c}{d}$.

26. $\frac{x}{9}=\frac{a}{c}$ and $\frac{x}{4}=\frac{a}{b}$.

27. $\frac{x}{n}=\frac{7}{6}$ and $\frac{x}{m}=\frac{11}{6}$.

28. $\frac{x}{15}=\frac{m}{13}$ and $\frac{x}{10}=\frac{n}{17}$.

29. $\frac{2}{5}=\frac{x}{b}$ and $\frac{3}{7}=\frac{x}{a}$.

30. $\frac{3}{4}=\frac{x}{d}$ and $\frac{5}{6}=\frac{x}{c}$.

REDUCTION OF SPECIFIED TERMS TO THE FIRST DEGREE

EXERCISE 214

In the following proportions raise the terms to such powers or extract such roots as will lead to proportions in which x appears to the first power:

1. $\frac{\sqrt{x}}{2}=\frac{3}{5}$.

2. $\frac{\sqrt{x}}{3}=\frac{4}{7}$.

3. $\frac{\sqrt{x}}{7}=\frac{1}{8}$.

4. $\frac{5}{2}=\frac{9}{\sqrt{x}}$.

5. $\frac{\sqrt[3]{x}}{3}=\frac{2}{5}$.

6. $\frac{\sqrt[3]{x}}{6}=\frac{1}{7}$.

7. $\frac{1}{4}=\frac{5}{\sqrt[3]{x}}$.

8. $\frac{\sqrt[4]{x}}{2}=\frac{1}{3}$.

9. $\frac{\sqrt[4]{x}}{4}=\frac{1}{5}$.

10. $\frac{\sqrt[5]{x}}{2}=\frac{a}{b}$.

11. $\frac{\sqrt[5]{x}}{m}=\frac{3}{n}$.

12. $\frac{\sqrt[n]{x}}{a}=\frac{b}{c}$.

13. $\frac{x^2}{4}=\frac{9}{25}$.

14. $\frac{x^2}{9}=\frac{16}{49}$.

15. $\frac{x^2}{25}=\frac{1}{36}$.

16. $\frac{x^2}{100}=\frac{144}{169}$.

17. $\frac{x^3}{8}=\frac{1}{27}$.

18. $\frac{x^3}{216}=\frac{27}{343}$.

19. $\frac{x^4}{16}=\frac{81}{625}$.

20. $\frac{x^4}{256}=\frac{1}{625}$.

PROBLEMS

EXERCISE 215

Solve the following problems, applying the principles of Ratio and Proportion, and examine the solutions found to see if they satisfy the conditions of the given problems:

1. Divide $100 between A and B so that A shall receive $2 to every $3 that *B* receives.

2. Divide $555 between A and B so that A shall receive $4 to every $1 that B receives.

3. Divide $1000 between A and B so that A shall receive $7 to every $1 that B receives.

4. Divide $1400 between A and B so that A shall receive $4 to every $3 that B receives.

5. Separate 45 into two parts, such that one part shall be to the other part as 1 is to 2.

6. Separate 50 into two parts, such that one part shall be to the other part as 2 is to 5.

7. Separate 42 into two parts, such that one part shall be to the other part as 2 is to 5.

8. Separate 63 into two parts, such that one part shall be to the other part as 3 is to 4.

9. Separate 77 into two parts, such that one part shall be to the other part as 5 is to 6.

10. The sum of two numbers in the ratio 2/5 is 24. Find the numbers.

11. The sum of two numbers in the ratio 5/7 is 24. Find the numbers.

12. The sum of two numbers in the ratio 3/4 is 28. Find the numbers.

13. The sum of two numbers in the ratio 4/5 is 32. Find the numbers.

14. The sum of the squares of two numbers in the ratio 2/3 is 117. Find the numbers.

15. The sum of the squares of two numbers in the ratio 1/2 is 20. Find the numbers.

16. The sum of the squares of two numbers in the ratio 1/4 is 153. Find the numbers.

17. The sum of the squares of two numbers in the ratio 2/5 is 261. Find the numbers.

18. The difference of the squares of two numbers in the ratio 3/2 is 45. Find the numbers.

19. The difference of the squares of two numbers in the ratio 4/3 is 28. Find the numbers.

20. The difference of the squares of two numbers in the ratio 5/2 is 189. Find the numbers.

The areas of two similar plane figures have the same ratio as the squares of any two corresponding dimensions.

21. The area of a triangle is 80 square inches and the base is 16 inches. What is the area of a similar triangle provided that the base is 20 inches?

22. Of two similar polygons the area of the first is 162 square inches and the area of the second is 250 square inches. If one side of the first polygon is 18 inches, find the corresponding side of the second polygon.

The volumes of two similar solids have the same ratio as the cubes of any two corresponding dimensions.

23. Of two bottles which are of similar shape, the diameter of the first is two times that of the second. If the first holds 3 ounces, how much does the second hold?

24. If a sphere which is 2 inches in diameter weighs 6 lbs., what is the weight of a sphere of the same substance which is 4 inches in diameter?

PROBLEMS IN PHYSICS

EXERCISE 216

Solve the following problems:

1. Find the force which must be exerted to draw a sled weighing 250 lbs. up a hill which is 400 feet long and 40 feet high.

2. What is the weight of a body if a force of 120 lbs. exerted along a smooth inclined plane which is 90 feet in length and 30 feet in height prevents the body from sliding down the plane?

3. A boy who is able to exert a maximum force of 75 lbs. is able to keep a barrel from rolling down a plank which is 15 feet in length and the upper end of which is 3 feet from the ground. Find the weight of the barrel.

4. A porter who can exert a maximum force of 250 lbs. undertakes to roll a cask weighing 600 lbs. up a board which is 12 feet long. How high can the upper end of the board be placed in order that he may be able to keep the cask from rolling down the board?

5. A car weighing 2000 lbs. is held at rest on a smooth inclined plane by a force of 50 lbs. applied parallel to the plane. If the length of the plane is 600 feet, find the height of the plane.

6. A boy is able to exert a maximum force of 90 lbs. How long an inclined plane must he use to push a truck weighing 360 lbs. up to a doorway which is 4 feet above the ground ?

7. If when confined with a pressure of 30 lbs. per square inch a mass of gas occupies a volume of one cubic foot, find the volume of the gas when the pressure becomes 60 lbs. per square inch.

8. A gas bag containing 4 cubic feet of gas under a pressure of 16 lbs. per square inch must be subjected to what pressure to reduce the volume to $\frac{1}{2}$ cubic foot ?

9. Eight cubic feet of gas under a pressure of 65 lbs. per square inch will have what volume if the pressure is reduced to 15 lbs. per square inch ?

10. A bladder holds 50 cubic inches of air under a pressure of 15 lbs. per square inch. What is the size of the bladder when the pressure is reduced to 10 lbs. per square inch ?

11. When under a pressure of 75 lbs. per square inch the volume of a mass of gas is 160 cubic inches. What is the pressure when the volume becomes 300 cubic inches ?

12. If 100 cubic inches of air at a pressure of 27 lbs. per square inch be admitted to a vessel the volume of which is 450 cubic inches, what will be the pressure ?

13. The volume of a certain quantity of gas is 150 cubic centimeters at 0° C. At what temperature will the volume become 300 cubic centimeters, assuming that the pressure remains constant ?

14. If the volume of a certain mass of gas is 400 cubic centimeters at 20° C. find the volume of the gas at 80° C., assuming that the pressure remains constant.

15. A mass of gas occupying a volume of 200 cubic centimeters at a temperature of 45° C. is cooled to a temperature of 15° C. Find the volume at the lower temperature.

16. A certain mass of gas occupying a volume of 300 cubic centimeters at 15° C. is raised in temperature to 60° C. Find the volume at the higher temperature.

17. Six hundred cubic centimeters of a gas at a temperature of 25° C. are cooled to 5° C., and at the same time the external pressure upon the gas is changed from 74 centimeters of mercury to 76 centimeters of mercury. What does the volume of the gas become ?

18. Sixty liters of gas are generated at a temperature of 18° C. and a pressure of 68 centimeters of mercury. Find the volume of the gas at 0° C. when the pressure is 76 centimeters of mercury.

19. The volume of a certain quantity of gas is found to be 400 cubic centimeters at a temperature of 0° C. and pressure of 75 centimeters of mercury. What must be the temperature of the gas in order that the volume may be 450 cubic centimeters when the pressure is 76 centimeters of mercury ?

20. The volume of a certain mass of gas is found to be 700 cubic centimeters at a pressure of 75 centimeters of mercury and temperature of 9° C. What must be the pressure in centimeters of mercury if the volume of the gas becomes 900 cubic centimeters when the the temperature is 27° C. ?

21. An iron steam pipe is 80 feet in length at 0° C. What does its length become when steam at a temperature of 112° C. is passed through it, provided that the coefficient of linear expansion of iron is .000012 ?

22. The distance between two graduations on a brass bar is exactly one meter at 30° C. What is the distance between the graduations at 60° C., provided that the coefficient of linear expansion of brass is .000018 ?

23. A lightning rod which is made of copper is 50 feet in length when at a temperature of 0° C. Find the length of the rod when the temperature is 32° C., provided that the coefficient of linear expansion of copper is .000017.

24. What allowance should be made for expansion in a 1700 foot span of a steel bridge assuming that the highest summer temperature is 40° C. and the lowest winter temperature is 20° C., provided that the coefficient of linear expansion of steel is .000011 ?

25. If a steel rule is exactly one foot in length at a temperature of 0° C., find the error in the rule expressed as a fraction of an inch at a temperature of 24° C., provided that the coefficient of linear expansion of steel is .000011.

VARIATION

ONE INDEPENDENT VARIABLE

EXERCISE 217

From each of the following relations find the value of the ratio x/y:

1. If $x \propto y$ and $x = 10$ when $y = 5$.
2. If $x \propto y$ and $x = 12$ when $y = 2$.
3. If $x \propto y$ and $x = 20$ when $y = 5$.
4. If $x \propto y$ and $x = 15$ when $y = 3$.
5. If $x \propto y$ and $x = 14$ when $y = -2$.
6. If $x \propto y$ and $x = -27$ when $y = 3$.
7. If $y \propto y$ and $x = -30$ when $y = -10$.
8. If $x \propto y$ and $x = 2$ when $y = 3$.
9. If $x \propto y$ and $x = 5$ when $y = 2$.
10. If $x \propto y$ and $x = 1/2$ when $y = 3$.

11. If $x \propto y$ and $x = 5$ when $y = 1/4$.
12. If $x \propto y$ and $x = 6$ when $y = 3/4$.
13. If $x \propto y$ and $x = 3$ when $y = 4/5$.
14. If $x \propto y$ and $x = 1/2$ when $y = 1/3$.
15. If $x \propto y$ and $x = 2/3$ when $y = 4/5$.
16. If $x \propto y$ and $x = 5/6$ when $y = 6/7$.
17. If $x \propto y$, find x when $y = 9$ if $x = 16$ when $y = 8$.
18. If $x \propto y$, find x when $y = 2$ if $x = 15$ when $y = 3$.
19. If $x \propto y$, find x when $y = 3$ if $x = 20$ when $y = 5$.
20. If $x \propto y$, find x when $y = 4$ if $x = 21$ when $y = 7$.
21. If $x \propto y$, find x when $y = 5$ if $x = 48$ when $y = 8$.
22. If $x \propto y$, find x when $y = 6$ if $x = 35$ when $y = 5$.
23. If $x \propto y$, find x when $y = 9$ if $x = 12$ when $y = 4$.
24. If $x \propto y$, find x when $y = 10$ if $x = 14$ when $y = 7$.
25. If $x \propto y$, find x when $y = 8$ if $x = 30$ when $y = 3$.

EXERCISE 218

1. If $x \propto \frac{1}{y}$, find x when $y = 3$ if $x = 9$ when $y = 2$.
2. If $x \propto \frac{1}{y}$, find x when $y = 8$ if $x = 6$ when $y = 4$.
3. If $x \propto \frac{1}{y}$, find x when $y = 15$ if $x = 3$ when $y = 10$.
4. If $x \propto \frac{1}{y}$, find x when $y = 12$ if $x = 6$ when $y = 6$.
5. If $x \propto \frac{1}{y}$, find x when $y = 5$ if $x = 10$ when $y = 4$.
6. If $x \propto \frac{1}{y}$, find x when $y = 8$ if $x = 8$ when $y = 6$.
7. If $x \propto \frac{1}{y}$, find x when $y = -10$ if $x = 12$ when $y = 5$.
8. If $x \propto \frac{1}{y}$, find x when $y = 14$ if $x = -8$ when $y = -7$.
9. If $x \propto \frac{1}{y}$, find x when $y = -5$ if $x = -25$ when $y = -3$.
10. If $x \propto \frac{1}{y}$, find x when $y = 1$ if $x = 9$ when $y = 9$.

TWO INDEPENDENT VARIABLES

EXERCISE 219

If $x \propto yz$, find the conditional equation restricting the values of x, y and z:

1. If $x = 40$ when $y = 2$ and $z = 5$.
2. If $x = 24$ when $y = 4$ and $z = 3$.
3. If $x = 16$ when $y = 2$ and $z = 4$.
4. If $x = 30$ when $y = 3$ and $z = 2$.
5. If $x = 60$ when $y = 5$ and $z = 4$.
6. If $x = 72$ when $y = 9$ and $z = 8$.
7. If $x = 84$ when $y = 3$ and $z = 4$.
8. If $x = 96$ when $y = 4$ and $z = -4$.
9. If $x = 100$ when $y = -2$ and $z = 5$.
10. If $x = -108$ when $y = -4$ and $z = -3$.

EXERCISE 220

If $x \propto \frac{y}{z}$, find the conditional equation restricting the values of x, y and z:

1. If $x = 5$ when $y = 2$ and $z = 8$.
2. If $x = 6$ when $y = 3$ and $z = 4$.
3. If $x = 14$ when $y = 7$ and $z = 2$.
4. If $x = 8$ when $y = 2$ and $z = 4$.
5. If $x = 11$ when $y = 1$ and $z = 3$.
6. If $x = 7$ when $y = 14$ and $z = 6$.
7. If $x = 3$ when $y = -9$ and $z = 12$.
8. If $x = 6$ when $y = 3$ and $z = -8$.
9. If $x = -4$ when $y = 8$ and $z = 14$.
10. If $x = -9$ when $y = 21$ and $z = -7$.
11. If $x = -16$ when $y = -10$ and $z = 5$.
12. If $x = -15$ when $y = -9$ and $z = -6$.

EXERCISE 221

1. If $x \propto y$ and when $y = 6$, $x = 48$, find x when $y = 1$.
2. If $x \propto \frac{1}{y}$ and $x = 4$ when $y = 6$, find x when $y = 8$.
3. If $x \propto \frac{1}{y}$ and $x = \frac{1}{3}$ when $y = 36$, find y when $x = 3$.

4. If x varies jointly as y and z, and $x = 36$ when $y = 9$ and $z = 1$, find x when $y = 3$ and $z = 3$.

5. If x varies directly as y and inversely as z, and $x = 26$ when $y = 30$ and $z = 15$, find x when $y = 42$ and $z = 14$.

6. Find x when $y = 2$ if $x \propto \frac{1}{y^2}$ and $x = 4$ when $y = 3$.

7. Find y when $x = 3$ and $w = 5$, if y varies as x and w jointly and $y = 3$ when $x = 2$ and $w = 6$.

8. If $x \propto y$ show that $ax \propto ay$ when a is either a constant or a variable.

9. If $x \propto \frac{1}{y}$ and $y \propto \frac{1}{z}$ show that $x \propto z$.

The area of a circle varies as the square of its diameter.

10. If the area of a circle, the radius of which is 21 feet, is 1386 square feet, find the area of a circle the radius of which is 16 feet.

11. Show that the area of a circle, the diameter of which is 15 inches, is equal to the sum of the areas of two circles, the diameters of which are 12 inches and 9 inches respectively.

12. The volume of a sphere varies as the cube of its radius, and the volume of a sphere of which the radius is 6 inches is $905\frac{1}{7}$ cubic inches. Find the volume of a sphere, the radius of which is 7 inches.

13. Prove that the sum of the volumes of 3 spheres, the radii of which are 3, 4, and 5 inches respectively, is equivalent to the volume of a sphere, the radius of which is 6 inches.

Problems in Physics

14. If a body falls 16 feet in 1 second, how far will it fall in 10 seconds?

15. A stone is dropped from the top of a cliff and strikes the bottom of the cliff in $3\frac{1}{4}$ seconds, nearly. What is the approximate height of the cliff?

16. From what height must a body fall from a state of rest to reach the earth after 12 seconds?

It has been found by experiment that the velocity acquired by a body falling freely from a state of rest varies directly as the time.

17. If the velocity of a falling body is 180 feet per second at the end of 5 seconds, what will be its velocity at the end of 10 seconds?

18. If the velocity of a falling body is 128 feet per second at the end of 4 seconds, what will be its velocity at the end of 8 seconds?

19. A candle is placed at a distance of 1 foot from a cardboard screen, and a second candle is placed at a distance of 8 feet from the screen on the other side. Compare the intensity of illumination on the two sides of the screen.

20. A gas jet which is 16 feet from a photometer and a candle which is 4 feet from the photometer are found to illuminate it equally. Compare the intensity of light from the two sources.

21. A "standard" 16-candle-power lamp, when placed at a distance of 51 centimeters from a screen, is found to illuminate it with the same intensity as an incandescent light when placed at a distance of 49 centimeters from the screen. What is the candle power of the incandescent light?

22. If a certain wire $\frac{1}{10}$ of an inch in diameter and 48 inches in length stretches 4 inches under a force of 20 lbs., how much will it stretch under a force of 25 lbs.?

23. A certain wire, the diameter of which is $\frac{1}{10}$ of an inch, and the length of which is 5 feet, is increased in length by 4 inches by a force of 24 lbs. Find the length of a second wire of the same material and diameter if a force of 40 lbs. increases it in length by 7 inches.

24. If a wire which is $\frac{1}{16}$ of an inch in diameter and 25 feet in length stretches 3 inches under a force of 15 lbs., how long is a wire, the diameter of which is $\frac{1}{20}$ of an inch, if a force of 40 lbs. produces an increase in length of 2 inches?

REVIEW

MENTAL EXERCISE 222

Solve each of the following equations:

1. $(x-6)^2=16$.

2. $(x-8)^2=4x^2$.

3. $x+\frac{1}{x}=9+\frac{1}{9}$.

4. $\sqrt{x}=0$.

5. $\sqrt[3]{y}=0$.

6. $\sqrt[n-1]{y}=0$.

7. Show that $x^2-1=0$, if $x-1=0$.

8. Show that $x^2-4=0$, if $x-2=0$.

9. Show that $x^2-9=0$, if $x+3=0$.

10. Show that $a^2-ab+b^2=0$, if $a+b=0$.

Simplify each of the following:

11. $\frac{\sqrt{a+b}}{\sqrt{a}+\sqrt{b}}$.

12. $\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x-y}}$.

13. $(3+\sqrt{-2})(3-\sqrt{-2})$.

14. $(2+\sqrt{-3})(2-\sqrt{-3})$.

15. $(2\sqrt{-3}-3\sqrt{2})^2$.

16. $(5\sqrt{-2}-2\sqrt{-5})^2$.

Distinguish between

17. $-2\div\sqrt{-4}$ and $-4\div\sqrt{-2}$.

18. $-5\div\sqrt{-25}$ and $-25\div\sqrt{-5}$.

19. Simplify $\sqrt{250\,a^3}$; $\sqrt[3]{\frac{x^3y}{2z^2}}$; $\sqrt[6]{16\,a^6b^2}$.

Express the following as entire surds:

20. $2\sqrt[a]{a^2}$.

21. $a\sqrt[b]{b^a}$.

22. $3x\sqrt[3]{x}$.

23. $\frac{1}{2}\sqrt{3a}$.

24. $\frac{a}{3}\sqrt[3]{9b}$.

25. $\frac{1}{2c}\sqrt[4]{8c}$.

Simplify and express with positive exponents:

26. $x^2\left(\frac{x}{y}\right)^{-2}$.

27. $a^{-2}\left(\frac{b}{a}\right)^{-1}$.

28. $\left(\frac{a^{-1}}{b^{-2}}\right)^{-2}$.

Solve each of the following equations:

29. $\frac{1}{3}=\frac{64}{x^2}$.

30. $x^{\frac{2}{3}}=4$.

31. $x^{\frac{2}{3}}=9$.

32. $x^{\frac{3}{2}}=1$.

33. $x^{\frac{3}{2}}=-1$.

34. $\frac{\sqrt[3]{x}}{3}=1$.

35. Rationalize the denominators of $\frac{a^2-1}{\sqrt{a}+1}$ and $\frac{\sqrt{x-1}}{\sqrt{x}-1}$.

From each of the following conditional equations find the ratio of x to y.

36. $2x=3y$.

37. $5x=2y$.

38. $7x=4y$.

39. $8x=y$.

40. $x=\frac{y}{5}$.

41. $x=\frac{6}{11}y$.

Distinguish between

42. $\frac{a}{2}+\frac{b}{2}$ and $a^{\frac{1}{2}}+b^{\frac{1}{2}}$.

43. $(x-y)^{-1}$ and $x^{-1}-y^{-1}$.

Find the value of each of the following expressions:

44. $(\frac{1}{2})^{-3}$.

45. $\left(\frac{1}{2^3}\right)^{-2}$.

46. $\left(\frac{1}{2^{-2}}\right)^{-2}$.

47. 3^{-2}.

48. 2^{-2}.

49. -2^2.

50. $2^{-2}-2^2$.

51. $3^{-3}-3$.

52. $(\sqrt{-2})^3+(-\sqrt{2})^3$.

Show that

53. $12^3\cdot 8^2=2^5\cdot 3^3$.

54. $6^3\cdot 18^5=2^8\cdot 3^{13}$.

Simplify each of the following:

55. $a^{\frac{x}{y}}\div a^{\frac{y}{x}}$.

56. $x^{\frac{3}{2}}\div x^{\frac{2}{3}}$.

57. $a^{\frac{3}{4}}\div a^{-\frac{3}{4}}$.

58. $\dfrac{a^2}{\sqrt{a}}$.

59. $\dfrac{b^2}{\sqrt{b}}$.

60. $\dfrac{c^3}{\sqrt[3]{c^2}}$.

61. $\dfrac{d}{\sqrt[3]{d}}$.

62. $(a \div a^{\frac{1}{n}})^n$.

63. $\dfrac{7\,ab}{5\,a^0 + 2\,b^0}$.

64. $\left(\dfrac{5\,a^{\frac{1}{4}}}{3} + 2\right)\left(\dfrac{5\,a^{\frac{1}{4}}}{3} - 2\right)$.

65. $(\sqrt{a} - \sqrt{-a})^2$.

Express the following proportions in the least numbers possible without altering the terms containing x:

66. $\dfrac{24}{40} = \dfrac{36}{x}$.

67. $\dfrac{24}{15} = \dfrac{6}{x}$.

68. $\dfrac{30}{4} = \dfrac{35}{x}$.

69. $\dfrac{12}{42} = \dfrac{x}{6}$.

70. $\dfrac{x}{10} = \dfrac{20}{75}$.

71. $\dfrac{4}{36} = \dfrac{x}{24}$.

72. $\dfrac{16}{15} = \dfrac{12}{x}$.

73. $\dfrac{x}{21} = \dfrac{8}{60}$.

74. $\dfrac{4}{90} = \dfrac{x}{10}$.

75. $\dfrac{4}{88} = \dfrac{x}{6}$.

76. $\dfrac{54}{24} = \dfrac{21}{x}$.

77. $\dfrac{8}{80} = \dfrac{x}{24}$.

THE PROGRESSIONS

SEQUENCES

EXERCISE 223

Construct the first five terms of each of the sequences whose general terms are given as follows:

1. $n + 1$.
2. $n + 3$.
3. $n - 5$.
4. $2\,n$.
5. $3\,n + 1$.
6. $4\,n - 1$.
7. $5\,n + 2$.
8. $7\,n - 4$.
9. $n^2 + 1$.
10. $n^2 - 1$.
11. $n^2 + n$.
12. $\dfrac{n}{2}$.
13. $\dfrac{1}{n}$.
14. $\dfrac{1}{2\,n}$.
15. $\dfrac{1}{5\,n}$.
16. $\dfrac{1}{n^2}$.
17. $\dfrac{n+1}{n}$.
18. $\dfrac{n-1}{n+1}$.
19. $\dfrac{n}{2\,n+1}$.
20. $\dfrac{1}{n^2+1}$.
21. $\dfrac{n^2}{n+1}$.

22. $\frac{n-1}{n^2}$.

23. $\frac{n^2-1}{n}$.

24. $\frac{n}{(n+1)^2}$.

25. $\frac{(n+1)^2}{2n+1}$.

26. $\frac{n}{(n+1)(n+3)}$.

27. $\frac{1}{n(n+1)(n+2)}$.

ARITHMETIC PROGRESSION

CALCULATION OF SPECIFIED TERMS

EXERCISE 224

1. Find the 9th term of 5, 7, 9, ···.
2. Find the 10th term of 1, 5, 8, ···.
3. Find the 17th term of 2, 6, 10, ···.
4. Find the 8th term of 1, 7, 13, ···.
5. Find the 11th term of 1, 6, 11, ···.
6. Find the 13th term of 12, 17, 22, ···.
7. Find the 14th term of 1, 11, 21, ···.
8. Find the 19th term of 7, 15, 23, ···.
9. Find the 25th term of 11, 15, 19, ···.
10. Find the 31st term of 12, 15, 18, ···.
11. Find the 9th term of $-6, -3, 0, \cdots$.
12. Find the 11th term of $-9, -3, +3, \cdots$.
13. Find the 16th term of $-4, 0, 4, \cdots$.
14. Find the 15th term of $-5, -1, 3, \cdots$.
15. Find the 12th term of $-3, 7, 17, \cdots$.
16. Find the 18th term of $-2, 4, 10, \cdots$.
17. Find the 27th term of $-9, -2, 5, \cdots$.
18. Find the 35th term of $-7, -1, 5, \cdots$.
19. Find the 41st term of $-6, -1, 4, \cdots$.
20. Find the 21st term of 205, 201, 197, ···.
21. Find the 31st term of 150, 145, 140, ···.
22. Find the 42d term of 126, 123, 120, ···.
23. Find the 17th term of $a, 5a, 9a, \cdots$.
24. Find the 12th term of $-a, 3a, 7a, \cdots$.
25. Find the 10th term of $a, a+1, a+2, \cdots$.
26. Find the 25th term of $x, x-2x, -4, \cdots$
27. Find the 14th term of $a, a+2b, a+4b, \cdots$.

28. Find the 15th term of $a+b$, $2a+2b$, $3a+3b$, ...

29. Find the 16th term of $a+b$, $3a+3b$, $5a+5b$, ...

30. Find the 7th term of $a-2b$, $a-4b$, $a-6b$, ...

31. Find the 10th term of $2y$, $2y+2z$, $2y+4z$, ...

32. Find the 9th term of $3a$, $3a+b$, $3a+2b$, $3a+3b$, ...

33. Find the 8th term of $5x$, $5x-y$, $5x-2y$, ...

34. Find the 7th term of $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, ...

35. Find the 12th term of $\frac{7}{6}$, $\frac{5}{6}$, $\frac{1}{2}$, ...

36. Find the 19th term of $-\frac{2}{5}$, $\frac{1}{5}$, $\frac{4}{5}$, ...

37. Find the 22d term of $\frac{4}{9}$, 1, $\frac{14}{9}$, ...

38. Find the 26th term of $\frac{1}{6}$, $\frac{2}{3}$, $\frac{7}{6}$, ...

39. Find the 31st term of $\frac{a}{2}$, a, $\frac{3a}{2}$, ...

40. Find the 13th term of $\frac{1}{a}$, $\frac{3}{a}$, $\frac{5}{a}$, ...

41. Find the 34th term of $\frac{a+b}{3}$, $a+b$, $\frac{5a+5b}{3}$, ...

42. Find the 12th term of $a+\frac{b}{2}$, $a+b$, $a+\frac{3b}{2}$, ...

43. Find the ath term of $\frac{a-1}{a}$, $\frac{a-2}{a}$, $\frac{a-3}{a}$, ...

ARITHMETIC MEANS

EXERCISE 225

Find the arithmetic mean of

1. 2 and 12.

2. 3 and 17.

3. 5 and 21.

4. 7 and 33.

5. 11 and 21.

6. 15 and 45.

7. 8 and -6.

8. 10 and -10.

9. -16 and -20.

10. 5 and -5.

11. $\frac{1}{4}$ and $\frac{3}{4}$.

12. $\frac{1}{3}$ and $\frac{2}{3}$.

13. $\frac{4}{5}$ and $\frac{6}{5}$.

14. $\frac{1}{2}$ and $\frac{1}{3}$.

15. $\frac{1}{2}$ and $\frac{2}{3}$.

16. a and $3a$.

17. $2b$ and $10b$.

18. x and $\frac{1}{x}$.

19. $2x$ and $-2y$.

20. $m+n$ and $m-n$.

21. $(x+y)^2$ and $(x-y)^2$.

22. $\frac{x+y}{x-y}$ and $\frac{x-y}{x+y}$.

EXERCISE 226

1. Find the two arithmetic means of 1 and 10.
2. Find the two arithmetic means of 2 and 14.
3. Find the three arithmetic means of 5 and 21.
4. Find the three arithmetic means of 9 and 37.
5. Find the four arithmetic means of 7 and 27.
6. Find the four arithmetic means of 8 and 40.
7. Find the five arithmetic means of 2 and 20.
8. Find the five arithmetic means of 3 and 33.
9. Find the six arithmetic means of 6 and 34.
10. Find the six arithmetic means of 3 and 45.
11. Find the ten arithmetic means of 4 and 37.
12. Find the twelve arithmetic means of 1 and 27.
13. Find the fourteen arithmetic means of 10 and 70.
14. Find the nineteen arithmetic means of 6 and 106.

SUM OF THE TERMS OF AN ARITHMETIC PROGRESSION

EXERCISE 227

Find the sum of the terms of each of the following A. P.'s:

1. 1, 3, 5, ··· to 8 terms.
2. 6, 8, 10, ··· to 20 terms.
3. 15, 13, 11, ··· to 40 terms.
4. 2, 5, 8, ··· to 10 terms.
5. 10, 8, 6, ··· to 12 terms.
6. $-3, -1, +1,$ ··· to 16 terms.
7. $-5, -1, 3,$ ··· to 18 terms.
8. 1, 6, 11, ··· to 14 terms.
9. 2, 7, 12, ··· to 20 terms.
10. 10, 8, 6, ··· to 10 terms.
11. $-15, -13, -11,$ ··· to 12 terms.
12. $-5, 0, 5,$ ··· to 14 terms.
13. 1, 11, 21, ··· to 40 terms.
14. 1, 10, 19, ··· to 12 terms.
15. 3, 7, 11, ··· to 16 terms.
16. 4, 9, 14, ··· to 20 terms.
17. 2, 8, 14, ··· to 12 terms.

18. $3, 11, 19, \cdots$ to 14 terms.

19. $1, 12, 23, \cdots$ to 10 terms.

20. $1, 11, 21, \cdots$ to 30 terms.

21. $3a, 6a, 9a, \cdots$ to 16 terms.

22. $2a, 6a, 10a, \cdots$ to 18 terms.

23. $10a, 5a, 0, \cdots$ to 16 terms.

24. $-a, 0, a, \cdots$ to 20 terms.

25. $(a+b), a, (a-b), \cdots$ to 12 terms.

26. $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \cdots$ to 8 terms.

27. $\frac{2}{3}, 1, \frac{4}{3}, \cdots$ to 10 terms.

28. $\frac{1}{5}, 1, \frac{9}{5}, \cdots$ to 12 terms.

29. $2, \frac{5}{2}, 3, \cdots$ to 14 terms.

30. Find the sum of all of the even numbers from 30 to 90 inclusive.

31. If $a_1 = 24$, $d = 3$, and $s_n = 860$, find n.

32. Find d, knowing that $a_1 = 7$ and $a_{13} = 31$.

33. If $s_{11} = 66$ and $a_{11} = 23$, find a_8.

34. Find a_{20}, having given $a_6 = 2$ and $a_{15} = -2$.

35. Find the sum of the terms the values of which lie between 0 and 50 of the series the fourth term of which is 13 and the common difference of which is -7.

36. The product of three numbers in arithmetic progression is 120, and the first number is five times the last. Find the numbers.

37. Of three numbers which are in arithmetic progression the third is eleven times the first. Find the numbers if the sum of the three numbers is equal to the eighteenth term of the arithmetic progression $-18, -16, -14, \cdots$.

38. The sum of three numbers in arithmetic progression is 24 and the sum of their squares is 224. Find the numbers.

39. Find the number of terms in the arithmetic progression $1, 8, 15, \cdots$, the sum of which approximates most closely to 1200.

40. Find the sum of all of the multiples of 6 which lie between zero and 300.

41. Show that the sum of the first n odd numbers is equal to n^2.

PROBLEMS IN PHYSICS

42. A car starting from a state of rest moves down an inclined track, passing over distances of 1 foot the first second, 3 feet the second second, 5 feet the third second, etc. Find the distance passed over in one minute.

43. A ball starting from a state of rest rolls down an inclined board, passing over distances of 4 inches, 12 inches, 20 inches, etc., in succesive seconds. Find the number of seconds required for the ball to pass over a distance of 20 feet.

44. If a ball starting up an inclined plane passes over 40 feet the first second, 36 feet the second second, 32 feet the third second, etc., find the number of seconds required by the ball to pass over a distance of 196 feet.

45. It is found that when a ball is thrown vertically upward the force of gravity diminishes the distance passed over in successive seconds by 32 feet per second (nearly). Find the distance passed over in 4 seconds by a ball which, when thrown vertically upward, rises to a height of 128 feet during the first second.

46. If the force of gravity increases the space passed over by a falling body in successive seconds by 32 feet per second, find the distance passed over in 6 seconds by a falling body which, when thrown downward, passes over a distance of 24 feet during the first second.

HARMONIC PROGRESSION

CALCULATION OF SPECIFIED TERMS

EXERCISE 228

1. Find the 9th term of $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$.
2. Find the 11th term of $\frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \cdots$.
3. Find the 9th term of $\frac{1}{6}, \frac{1}{10}, \frac{1}{14}, \cdots$.
4. Find the 11th term of $\frac{1}{3}, \frac{1}{9}, \frac{1}{15}, \cdots$.
5. Find the 16th term of $\frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \cdots$.
6. Find the 11th term of $\frac{2}{3}, \frac{1}{2}, \frac{2}{5}, \cdots$.
7. Find the 8th term of $\frac{3}{4}, \frac{3}{7}, \frac{3}{10}, \cdots$.
8. Find the 10th term of $\frac{3}{2}, 1, \frac{3}{4}, \cdots$.
9. Find the 11th term of $2, \frac{1}{2}, \frac{2}{7}, \cdots$.
10. Find the 11th term of $2, \frac{4}{3}, 1, \cdots$.
11. Find the 9th term of $1, \frac{3}{5}, \frac{3}{7}, \cdots$.
12. Find the 8th term of $6, \frac{3}{2}, \frac{6}{7}, \cdots$.
13. Find the 7th term of $1/a, 1/3\,a, 1/5\,a, \cdots$.
14. Find the 9th term of $1/3\,a, 1/5\,a, 1/7\,a, \cdots$.

15. Find the 8th term of $1/a$, $1/(a+1)$, $1(a+2)$, ...
16. Find the 10th term of $1/(a+1)$, $1/(a+3)$, $1/(a+5)$, ...
17. Find the 10th term of $3/a$, $3/(a+2)$, $3/(a+4)$, ...

HARMONIC MEANS

EXERCISE 229

Find the harmonic mean of

1. 2 and 5.
2. 3 and 8.
3. 4 and 5.
4. 2 and 6.
5. 3 and 5.
6. 4 and 12.
7. 5 and 20.
8. -2 and -5.
9. -4 and -7.
10. 5 and -8.
11. $\frac{1}{2}$ and 3.
12. $\frac{1}{3}$ and 3.
13. $\frac{2}{3}$ and $\frac{3}{2}$.
14. $\frac{3}{5}$ and $\frac{5}{3}$.
15. $\frac{1}{2}$ and $\frac{1}{6}$.
16. $\frac{1}{3}$ and $\frac{1}{5}$.
17. $-\frac{1}{4}$ and $-\frac{1}{6}$.
18. $\frac{1}{2}$ and $-\frac{1}{5}$.
19. Find the two harmonic means of 1 and $\frac{1}{4}$.
20. Find the two harmonic means of $\frac{1}{5}$ and $\frac{1}{11}$.
21. Find the three harmonic means of $\frac{1}{4}$ and $\frac{1}{24}$.
22. Find the three harmonic means of $\frac{1}{10}$ and $\frac{1}{34}$.
23. Find the five harmonic means of $\frac{1}{5}$ and $\frac{1}{23}$.
24. Find the seven harmonic means of $\frac{1}{3}$ and $\frac{1}{19}$.
25. Find the eight harmonic means of $\frac{1}{2}$ and $\frac{1}{29}$.
26. Find the two harmonic means of 2 and 8.
27. Find the two harmonic means of 3 and 6.
28. Find the two harmonic means of 2 and 5.
29. Find the three harmonic means of 5 and 9.
30. Find the three harmonic means of 2 and 3.
31. Find the three harmonic means of 3 and 7.
32. Find the three harmonic means of 4 and 8.
33. Find two numbers the sum of which is 24 and the harmonic mean of which is 9.
34. The difference between the arithmetic and harmonic means of two numbers is $\frac{40}{3}$ and one of the numbers is five times the other. Find the numbers.
35. If $x+y$, $y+z$, and $z+x$ form a harmonic progression, show that y^2, x^2 and z^2 form an arithmetic progression.

GEOMETRIC PROGRESSION

CALCULATION OF SPECIFIED TERMS

EXERCISE 230

1. Find the 5th term of $1, 2, 4, \cdots$.
2. Find the 6th term of $3, 6, 12, \cdots$.
3. Find the 6th term of $3, 9, 27, \cdots$.
4. Find the 9th term of $-1, -2, -4, \cdots$.
5. Find the 7th term of $2, 1, \frac{1}{2}, \cdots$.
6. Find the 6th term of $9, 3, 1, \cdots$.
7. Find the 6th term of $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \cdots$.
8. Find the 6th term of $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots$.
9. Find the 7th term of $-6, -3, -\frac{3}{2}, \cdots$.
10. Find the 7th term of $-3, 6, -12, \cdots$.
11. Find the 7th term of $1, -\frac{1}{3}, \frac{1}{9}, \cdots$.
12. Find the 8th term of $2, -1, \frac{1}{2}, \cdots$.
13. Find the 5th term of $\frac{1}{2}\frac{1}{5}, \frac{1}{5}, 1, \cdots$.
14. Find the 9th term of $\sqrt{7}, \sqrt{21}, 3\sqrt{7}, \cdots$.
15. Find the 9th term of $\sqrt{2}, \sqrt{10}, 5\sqrt{2}, \cdots$.
16. Find the 9th term of $\sqrt{2}, \sqrt{6}, 3\sqrt{2}, \cdots$.
17. Find the 13th term of $\sqrt[3]{3}, \sqrt[3]{6}, \sqrt[3]{12}, \cdots$.
18. Find the 10th term of $\sqrt[3]{5}, \sqrt[3]{15}, \sqrt[3]{45}, \cdots$.
19. Find the 8th term of $a, a^2, a^3, \cdots$.
20. Find the 9th term of $x^3, x^5, x^7, \cdots$.
21. Find the 12th term of $a, ab, ab^2, \cdots$.
22. Find the 14th term of $a, abc, ab^2c^2, \cdots$.
23. Find the 5th term of $a, 3a^2, 9a^3, \cdots$.
24. Find the 7th term of $2a, 4a^2, 8a^3, \cdots$.
25. Find the 6th term of $a, a/2, a/4, \cdots$.
26. Find the 7th term of $-a/3, -2a/3, -4a/3, \cdots$.
27. Find the 6th term of $a/9, a/3, a, \cdots$.
28. Find the 8th term of $4a, 2a, a, \cdots$.

GEOMETRIC MEANS

EXERCISE 231

Find the geometric mean of

1. 1 and 4.
2. 1 and 9.
3. 16 and 1.
4. 2 and 8.
5. 9 and 4.
6. 3 and 12.
7. 18 and 2.
8. 16 and 4.
9. 2 and 32.
10. 3 and 27.
11. 5 and 20.
12. 25 and 4.
13. 2 and 50.
14. 100 and 1.
15. 2 and 72.
16. 4 and 36.
17. 18 and 8.
18. 16 and 9.
19. 24 and 6.
20. 48 and 3.
21. 28 and 7.
22. 98 and 2.
23. 45 and 5.
24. 75 and 3.
25. 32 and 8.
26. 64 and 4.
27. 2 and 128.
28. 36 and 9.
29. 54 and 6.
30. 200 and 2.
31. 80 and 5.
32. 4 and 100.
33. 10 and 40.
34. $\frac{1}{2}$ and 32.
35. $\frac{1}{3}$ and 27.
36. $\frac{1}{3}$ and 48.
37. $\frac{1}{4}$ and 64.
38. $\frac{1}{4}$ and 100.
39. $\frac{1}{2}$ and $\frac{1}{8}$.
40. $\frac{1}{3}$ and $\frac{1}{27}$.
41. $\frac{2}{3}$ and $\frac{1}{6}$.
42. $\frac{3}{4}$ and $\frac{1}{3}$.
43. $\frac{4}{7}$ and $\frac{9}{7}$.
44. $\frac{2}{5}$ and $\frac{1}{10}$.
45. $\frac{2}{9}$ and $\frac{1}{8}$.
46. $\frac{9}{11}$ and $\frac{11}{4}$.
47. $\frac{3}{8}$ and $\frac{3}{2}$.
48. $\frac{27}{16}$ and $\frac{3}{4}$.
49. a and a^3.
50. b^2 and b^4.
51. c^3 and c^5.
52. d and d^7.
53. ab and abc^2.
54. a^2b and bc^2.
55. ab^2c^3 and a^3b^2c.
56. a^b and a^{3n}.
57. b^{3n} and b^{5n}.
58. c^{n+1} and c^{n-1}.
59. x^ny^3 and $x^{3n}y$.
60. $x^{n+2}y^nz^2$ and $x^ny^nz^2$.
61. Insert 2 geometric means between 2 and 16.
62. Insert 2 geometric means between 1 and 27.
63. Insert 2 geometric means between 5 and 625.
64. Insert 2 geometric means between 3 and 24.
65. Insert 2 geometric means between 5 and 40.
66. Insert 2 geometric means between 4 and 108.
67. Insert 2 geometric means between 2 and 250.
68. Insert 3 geometric means between 1 and 16.
69. Insert 3 geometric means between 1 and 81.
70. Insert 3 geometric means between 1 and 256.
71. Insert 3 geometric means between 2 and 162.

SUM OF THE TERMS OF A GEOMETRIC PROGRESSION

EXERCISE 232

1. Find the sum of 8, 16, 32, ··· to 7 terms.
2. Find the sum of 5, 10, 20, ··· to 8 terms.
3. Find the sum of 7, 14, 28, ··· to 6 terms.
4. Find the sum of 40, 80, 160, ··· to 7 terms.
5. Find the sum of 100, 200, 400, ··· to 8 terms.
6. Find the sum of 4, 12, 36, ··· to 5 terms.
7. Find the sum of 8, 24, 72, ··· to 6 terms.
8. Find the sum of 3, 12, 48, ··· to 5 terms.
9. Find the sum of 3, 6, 12, 24, 48, 96, 192, 384, 768, 1536.

Find the sum of the terms of each of the following progressions:

10. 1, 3, 9, 27, 81, 243, 729.
11. 2, 4, 8, 16, 32, 64, 128, 256.
12. 2, 8, 32, 128, 512, 2048.
13. 2, 6, 18, 54, 162, 486, 1458.
14. 3, 12, 48, 192, 768, 3072.
15. 1, 5, 25, 125, 625, 3125, 15,625.
16. $-2, -10, -50, -250, -1250, -6250, -31{,}250$.
17. $\frac{1}{3}$, 1, 3, 9, 27, 81, 243.
18. $\frac{1}{4}$, 1, 4, 16, 64, 256, 1024, 4096.
19. 7, 14, 28, 56, 112, 224, 448, 896.
20. 9, 18, 36, 72, 144, 288, 576, 1152.
21. 10, 20, 40, 80, 160, 320, 640, 1280.
22. 10, 30, 90, 270, 810, 2430, 7290.
23. 11, 22, 44, 88, 176, 352, 704, 1408.

SUM OF THE TERMS OF AN INFINITE DECREASING GEOMETRIC PROGRESSION

EXERCISE 233

Find the sum of an infinite number of terms of each of the following progressions:

1. 8, 4, 2, ···.
2. 12, 6, 3, ···.
3. 15, 5, $\frac{5}{3}$, ···.
4. 16, 8, 4, ···.
5. 18, 6, 2, ···.
6. 24, 12, 6, ···.

7. 100, 20, 4, ···.
8. 48, 12, 3, ···.
9. $1, \frac{1}{5}, \frac{1}{25}, \cdots$.
10. $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \cdots$.
11. $\frac{5}{3}, \frac{25}{9}, \frac{125}{27}, \cdots$.
12. $\frac{3}{4}, \frac{15}{8}, \frac{75}{16}, \cdots$.
13. $1, \frac{2}{3}, \frac{4}{9}, \cdots$.
14. $1, \frac{3}{4}, \frac{9}{16}, \cdots$.
15. 18, 12, 8, ···.
16. 32, 24, 18, ···.
17. 100, 60, 36, ···.
18. 48, 36, 27, ···.
19. 64, 48, 36, ···.
20. 81, 54, 36, ···.
21. $\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$.
22. $1 - \frac{1}{3} + \frac{1}{9} - \cdots$.
23. $\frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots$.
24. $600 + 120 + 24 + \cdots$.
25. $5 - \frac{1}{2} + \frac{1}{20} - \frac{1}{200} + \cdots$.
26. $2 - 4 + 8 - 16 + \cdots$.
27. $1 - x + x^2 - x^3 + \cdots$, for $x < 1$.
28. $1 - \frac{1}{3} + \frac{1}{9} - \cdots$.
29. $1 + \frac{1}{x} + \frac{1}{x^2} + \cdots$, for $x > 1$.

30. Find r having given $a_2 = 12$ and $a_8 = 768$.

31. Find two numbers the difference of which is 63 and whose geometric mean is 8.

32. Find s_{10} having given $a_4 = 72$ and $a_7 = -\frac{64}{3}$.

33. Find a_1 provided that $a_5 = \frac{1}{64}$ and $a_8 = \frac{1}{512}$.

34. Find a_9 knowing that $a_3 = .008$ and $a_6 = .000064$.

35. The difference between two numbers is 70 and their arithmetic mean exceeds their geometric mean by 25. Find the numbers.

36. Find three numbers in geometric progression such that their sum shall be 13 and the sum of their squares 91.

REPEATING DECIMAL FRACTIONS CONSIDERED AS INFINITE DECREASING GEOMETRIC PROGRESSIONS

EXERCISE 234

Find the generating fraction of each of the following repeating decimal fractions:

1. $.\dot{2}$.
2. $.\dot{3}$.
3. $.\dot{4}$.
4. $.\dot{5}$.
5. $.\dot{6}$.
6. $.\dot{7}$.
7. $.\dot{8}$.
8. $.0\dot{2}$.
9. $.0\dot{3}$.
10. $.0\dot{6}$.
11. $.0\dot{9}$.
12. $.1\dot{2}$.
13. $.1\dot{3}$.
14. $.1\dot{5}$.
15. $.2\dot{7}$.
16. $.4\dot{8}$.

17. $.7\dot{5}$.
18. $.\dot{0}0\dot{2}$.
19. $.\dot{0}0\dot{4}$.
20. $.\dot{0}0\dot{3}$.
21. $.\dot{0}0\dot{6}$.
22. $.\dot{0}1\dot{2}$.
23. $.\dot{0}1\dot{5}$.
24. $.\dot{3}0\dot{3}$.
25. $.\dot{6}0\dot{9}$.
26. $.\dot{6}1\dot{5}$.
27. $.\dot{1}0\dot{8}$.
28. $.\dot{1}4\dot{4}$.

CLASSIFICATION OF PROGRESSIONS

MENTAL EXERCISE 235

Classify each of the following as Arithmetic, Harmonic or Geometric Progressions:

1. 1, 4, 7, ...
2. 12, 16, 20, ...
3. 23, 17, 11, ...
4. 4, 12, 36, ...
5. 5, 20, 80, ...
6. $-15, -12, -9$, ...
7. $-7, -11, -15$, ...
8. $2, -4, 8$, ...
9. $-3, 6, -12$, ...
10. $2, \frac{5}{2}, 3$, ...
11. $2, \frac{2}{5}, \frac{2}{9}$, ...
12. $\frac{1}{3}, \frac{1}{6}, \frac{1}{9}$, ...
13. $\frac{3}{7}, \frac{3}{5}, 1$, ...
14. $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}$, ...
15. $\frac{1}{20}, \frac{1}{10}, \frac{1}{5}$, ...
16. $\frac{1}{4}, \frac{1}{2}, 1$, ...
17. $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}$, ...
18. $\frac{1}{3}, 1, 3$, ...
19. $\frac{1}{5}, 2, 20$, ...
20. $15, 3, \frac{3}{5}$, ...
21. $21, 28, 35$, ...
22. $\frac{1}{2}, \frac{2}{5}, \frac{1}{3}$, ...
23. $\frac{1}{2}, \frac{2}{5}, \frac{8}{25}$, ...
24. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, ...
25. $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$, ...
26. $2, 3, 6$, ...
27. $6, 3, 2$. ....
28. $\frac{1}{2}, \frac{1}{3}, \frac{2}{9}$, ...
29. $-1, 1, 3$, ...
30. $-1, 1, \frac{1}{3}$, ...
31. $1, 3, 9$, ...
32. $1, \frac{1}{3}, \frac{1}{9}$, ...
33. $5, 1, -3$, ...
34. $5, 1, \frac{1}{5}$, ...
35. $5, 1, \frac{5}{9}$, ...
36. $4, 6, 12$, ...
37. $3, 5, 15$, ...
38. $6, 9, 18$, ...
39. $6, 4, 3$, ...
40. 20, 15, 12.
41. $1, x, x^2$, ...
42. $x, 2x, 3x$, ...
43. a^2, a^4, a^6, ...
44. b^2, b^5, b^8, ...
45. c^2, c^6, c^{10}, ...
46. $b/2\ a, b/4\ a, b/6\ a$, ...
47. d, d^8, d^{15}.
48. a^3, a^3b, a^3b^2, ...
49. a, ab^2, ab^4, ...
50. $x/5, x/10, x/15$, ...
51. $1, 1/x, 1/x^2$, ...
52. $1/b^2, 1/b^4, 1/b^6$, ...
53. $1/x^2, 4/x^2, 7/x^2$, ...
54. $c, 1, 1/c$, ...
55. $a/2, a/3, a/4$, ...
56. $a, b, b^2/a$, ...
57. $3/x, 4/x, 5/x$, ...
58. x^{-1}, x^{-2}, x^{-3}, ...
59. $a/bc, a/c, ab/c$, ...
60. $x, -y, y^2/x$, ...
61. $a, a/b, a/b^2$, ...
62. $a, a+b, a+2b$, ...
63. $x, x-y, x-2y$, ...
64. $a+2, a-2, a-6$, ...
65. $x-6, x-3, x$, ...

66. $a-2b,\ a-b,\ a,\ \cdots.$

67. $a+2b,\ a,\ a-2b,\ \cdots.$

68. $m,\ 2m+n,\ 3m+2n,\ \cdots.$

69. $a^2,\ a^2-b^2,\ a^2-2b^2,\ \cdots.$

70. $a,\ a+\frac{1}{b},\ a+\frac{2}{b},\ \cdots.$

71. $\frac{a-1}{a},\ \frac{a-2}{a},\ \frac{a-3}{a},\ \cdots.$

72. $\frac{a-b}{c},\ \frac{a-b}{2c},\ \frac{a-b}{4c},\ \cdots.$

73. $a+b,\ a^2+ab,\ a^3+a^2b,\ \cdots.$

REVIEW

EXERCISE 236

Simplify each of the following:

1. $\left(\frac{x^2}{y^2}+1+\frac{y^2}{x^2}\right)\left(\frac{x}{y}-\frac{y}{x}\right).$

2. $\left(a^2+\frac{1}{a^2}\right)\left(a+\frac{1}{a}\right)\left(a-\frac{1}{a}\right).$

3. $\frac{a^2+ab}{a^2+b^2}\times\frac{a^4-b^4}{ab+b^2}\times\frac{b}{a}.$

4. $\left[\frac{a}{1+\frac{1}{a}}+1-\frac{1}{a+1}\right]\div\left[\frac{a}{1-\frac{1}{a}}-a-\frac{1}{a-1}\right].$

5. $(a^{-2}-b^{-2})\div(a-b).$

6. $\frac{a^{-1}-b^{-1}}{a^{-1}+b^{-1}}-\frac{a^{-1}+b^{-1}}{a^{-1}-b^{-1}}.$

7. $\left(\frac{x^b}{x^c}\right)^{b+c}\left(\frac{x^c}{x^a}\right)^{c+a}\left(\frac{x^a}{x^b}\right)^{a+b}.$

8. If $a=-1,\ b=-2,\ c=-3$, find the value of $\frac{a^a}{bc}+\frac{b^b}{ca}+\frac{c^c}{ab}.$

9. Simplify $[(x+y)\sqrt{x-y}][(x-y)\sqrt{x+y}].$

10. Simplify $(\sqrt{x^2-y^2})(\sqrt{x+y})(\sqrt{x-y}).$

11. Simplify $(\sqrt{a}+\sqrt{b})(\sqrt[4]{a}+\sqrt[4]{b})(\sqrt[4]{a}-\sqrt[4]{b}).$

12. Show that $(a+3)(a+4)(a+5)(a+6)+1\equiv(a^2+9a+19)^2.$

13. Show that $(x+2)(x+3)(x+4)(x+5)+1$ is a square.

14. Show that $\sqrt{2m+2\sqrt{m^2-n^2}}\equiv\sqrt{m+n}+\sqrt{m-n}.$

15. Show that $\frac{\sqrt{a-b}}{\sqrt{a}-\sqrt{b}}\equiv\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a-b}}.$

16. Show that $\frac{\sqrt[n]{\sqrt[n+1]{a}}}{\sqrt[n+1]{\sqrt[n+2]{a}}}\equiv\frac{\sqrt[n]{a}\cdot\sqrt[n+2]{a}}{\sqrt[n+1]{a^2}}.$

17. Rationalize the denominator of $\frac{5+3\sqrt{5}}{3+5\sqrt{3}}.$

Solve each of the following equations:

18. $x^6 + 7x^3 = 8.$

19. $\dfrac{1}{\sqrt{x}+1} + \dfrac{1}{\sqrt{x}-1} = \dfrac{1}{x-1}.$

20. $\sqrt{x+5} + \sqrt[4]{x+5} = 12.$

21. $\sqrt{7x-6} - \sqrt{17x-2} + \sqrt{3x-2} = 0.$

22. $\sqrt{3x+10} + \sqrt{5x} = \sqrt{19x+5}.$

23. $\dfrac{2(x-a)}{x-b} + \dfrac{3(x-b)}{x-a} = 5.$

24. $\dfrac{x}{x+8} + \dfrac{5x-56}{(x-4)(x+8)} = \dfrac{x}{x+10} + \dfrac{7x+90}{(x-6)(x+10)}.$

25. $\dfrac{x}{x-3} - \dfrac{2(x+9)}{(x+5)(x-3)} = \dfrac{x}{x+6} + \dfrac{7x-6}{(x-2)(x+6)}.$

THE BINOMIAL THEOREM

EXPANSION OF POWERS OF BINOMIALS

EXERCISE 237

Write the expansion of each of the following powers of Binomials:

1. $(a+x)^3.$
2. $(b+c)^4.$
3. $(d+y)^5.$
4. $(a+c)^7.$
5. $(a+b)^{10}.$
6. $(b-c)^6.$
7. $(b-x)^9.$
8. $(m-y)^9.$
9. $(r-s)^4.$
10. $(a-b)^{11}.$
11. $(a-1)^4.$
12. $(b-1)^5.$
13. $(c-4)^3.$
14. $(x-5)^4.$
15. $(2-x)^6.$
16. $(3+y)^5.$
17. $(2-g)^6.$
18. $(1+m)^7.$
19. $(2x-y)^8.$
20. $(2a+b)^9.$
21. $(x+2y)^{12}.$

Write the first five terms of

22. $(a+b)^{15}.$
23. $(b+c)^{14}.$
24. $(y+z)^{17}.$
25. $(a+d)^{18}.$
26. $(h+x)^{20}.$
27. $(b-y)^{16}.$

Write the first six terms of

28. $(a-x)^{12}.$
29. $(b-y)^{20}.$
30. $(c-z)^{21}.$

Write the first and last three terms of

31. $(a+d)^{19}.$
32. $(x+y)^{24}.$
33. $(b+c)^{17}.$

Write the expansion of each of the following:

34. $(3a + 5b)^3$.
35. $(5a - 6b)^3$.
36. $(4m - 3n)^4$.
37. $(7x - 10y)^2$.
38. $(3a - 2b)^5$.
39. $(5x - yz)^6$.
40. $(x^2 + y)^3$.
41. $(y + z^2)^4$.
42. $(a^2 + b^3)^4$.
43. $(b^3 + c^2)^5$.
44. $(a^2 - b^4)^5$.
45. $(b^3 - c)^6$.
46. $(c^2 - 2)^6$.
47. $(d^2 - 1)^7$.
48. $(1 - x^2)^7$.
49. $(2x + y^2)^8$.
50. $(a^2 - 2b)^8$.

BINOMIAL THEOREM FOR NEGATIVE AND FRACTIONAL EXPONENTS

EXERCISE 238

Write the expansions of each of the following:

1. $(a^{-3} + b^{-2})^3$.
2. $(2x^{-1} - y)^2$.
3. $(3xy^{-2} - 4z^{-5})^3$.
4. $(x^{-\frac{1}{2}} + y^{\frac{1}{3}})^3$.
5. $(a^{-\frac{1}{3}} - b^{-\frac{1}{4}})^4$.
6. $(x^{-2} + xn^2)^4$.
7. $(a^{-1} - b^{-2})^5$.
8. $(2x^{-3} - y^{-1})^6$.
9. $(2^{-1}x + y^{-2})^7$.
10. $(a^{\frac{1}{4}} - b^{-\frac{1}{2}})^8$.

Expand to four terms each of the following:

11. $(1 + x)^{-1}$.
12. $(1 - a)^{-1}$.
13. $(1 - b)^{-2}$.
14. $(1 - c)^{-3}$.
15. $(1 - d)^{-4}$.
16. $(1 - e)^{-5}$.
17. $(1 - 2n)^{-3}$.
18. $(a - b^2)^{-5}$.
19. $(1 + x)^{\frac{1}{2}}$.
20. $(1 + y)^{\frac{1}{3}}$.
21. $(1 + z)^{\frac{1}{4}}$.
22. $(1 - m)^{-\frac{1}{2}}$.
23. $(x - 2)^{-\frac{1}{2}}$.
24. $(1 - 2x)^{-\frac{1}{2}}$.
25. $(2 + 3x)^{-\frac{1}{3}}$.
26. $(3x - 4y)^{-\frac{1}{4}}$.

CALCULATION OF SPECIFIED TERMS

EXERCISE 239

Write the indicated terms in the expansion of the following powers of binomials:

1. The 6th term of $(a + b)^{12}$.
2. The 5th term of $(b + c)^{14}$.
3. The 8th term of $(c + d)^{10}$.
4. The 9th term of $(x + y)^{17}$.
5. The 10th term of $(y + z)^{20}$.
6. The 12th term of $(a - x)^{19}$.
7. The 15th term of $(a - b)^{22}$.
8. The 20th term of $(c - z)^{25}$.
9. The 16th term of $(x - y)^{16}$.
10. The 18th term of $(b - x)^{18}$.

11. The 14th term of $(c-d)^{27}$.
12. The 20th term of $(a+b)^{23}$.
13. The 17th term of $(x+y)^{26}$.
14. The 13th term of $(b-c)^{29}$.
15. The 15th term of $(a-m)^{30}$.
16. The 10th term of $(x-y)^{32}$.
17. The 12th term of $(b-y)^{36}$.
18. The 19th term of $(x-y)^{29}$.

EXERCISE 240

Write the specified terms in the expansions of the following powers of binomials:

1. The 5th term of $(x^{-2}+y^{-1})^{12}$.
2. The 7th term of $(a^{-1}+b^{-3})^{15}$.
3. The 6th term of $(x^{-2}+x^{2})^{18}$.
4. The 11th term of $(x^{-3}-x^{2}y^{-1})^{16}$.
5. The term of $(x^{3}+2\,x^{-\frac{1}{3}})^{15}$ which contains x^{15}.
6. The middle term of $(x^{-2}+2\,x^{\frac{1}{2}}y)^{20}$.
7. The term of $(x^{\frac{1}{2}}-y^{\frac{1}{3}})^{15}$ which contains $x^{3}y^{3}$.
8. The term of $(3\,x+2\,x^{-\frac{1}{3}})^{20}$ which is free from x.
9. The 6th term of $(1-2\,x)^{\frac{1}{2}}$.
10. The 9th term of $(1-y^{2})^{\frac{3}{2}}$.
11. The 6th term of $(1-3\,x)^{\frac{1}{3}}$.
12. The 8th term of $(1-2\,y)^{\frac{1}{4}}$.

EXTRACTION OF ROOTS OF NUMBERS BY THE BINOMIAL THEOREM

EXERCISE 241

Find to four places of decimals:

1. $\sqrt{17}$.
2. $\sqrt{37}$.
3. $\sqrt{103}$.
4. $\sqrt[3]{9}$.
5. $\sqrt[3]{26}$.
6. $\sqrt[3]{124}$.
7. $\sqrt[4]{17}$.
8. $\sqrt[5]{33}$.
9. $\sqrt[4]{626}$.
10. $\sqrt[6]{65}$.
11. $\sqrt[6]{730}$.
12. $\sqrt[7]{2188}$.

REVIEW

MENTAL EXERCISE 242

Solve each of the following equations:

1. $\frac{1}{a}=\frac{b^{2}}{x^{2}}$.
2. $\sqrt[n]{nx}=n$.
3. $(x+4)^{2}=9\,x^{2}$.

4. $\sqrt{9x^2 - b^2} = 0$.

5. $\sqrt{x + b} = a$.

6. $\frac{x^2}{b} = a^2 b$.

7. $\frac{mx^2}{n} = mn$.

8. Show that $x^3 + b^3 = 0$ if $x + b = 0$.

Simplify the following:

9. $-\frac{y-3}{\sqrt{y-3}}$.

10. $\frac{2-x}{\sqrt{x-2}}$.

11. $(\sqrt{5} + \sqrt{-3})(\sqrt{5} - \sqrt{-3})$.

12. $(\sqrt{-5} + \sqrt{-3})(\sqrt{-5} - \sqrt{-3})$.

13. $(a^{\frac{1}{2}} + \frac{1}{2})^2$.

14. $(b^{\frac{1}{3}} - \frac{1}{3})^2$.

15. $(2x^{\frac{1}{2}} + \frac{1}{2})^2$.

16. $(3\sqrt{-3} - 3\sqrt{3})^2$.

17. $\frac{a\sqrt{b}}{b\sqrt{a}}$.

18. $\frac{xa^x}{ax^a}$.

19. $\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)\left(\frac{x^2yz}{xy + yz + zx}\right)$.

20. $-(-x\sqrt{-x})^4$.

Find the value of:

21. $(x^0 + y^0)^2$.

22. $\frac{6}{a^0 b^0}$.

23. $\frac{10}{2x^0 + 3y^0}$.

24. $(-\sqrt{2})^2 - (\sqrt{-2})^2$.

25. Show that $-3\sqrt[3]{-3} \equiv 3\sqrt[3]{3}$ while $-2\sqrt{-2} \not\equiv 2\sqrt{2}$.

26. Simplify $\frac{1}{a}\left(\frac{b}{a}\right)^{-1}$ and express the result with positive exponents.

27. Show that a, b, c, and d are proportional if $\frac{a}{b} \div \frac{d}{c} = 1$.

28. Express $2\sqrt{\frac{1}{2}\sqrt{\frac{1}{2}}}$ in simplest form.

Simplify

29. $(n\sqrt[n]{n^{n-1}})(n-1)\sqrt[n]{n}$.

30. $\left(\frac{a^{-1}}{b^{-1}} + c^{-1}\right)\left(\frac{a^{-1}}{b^{-1}} - c^{-1}\right)$.

31. Find the geometric mean of a and $\frac{1}{a}$.

32. Find the mean proportional between a and $\frac{1}{a}$.

Classify each of the following progressions as being arithmetic, harmonic, or geometric:

33. 2, 4, 6, ····. **34.** 2, 4, 8, ····. **35.** $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{6}$, ····.

Find the values of:

36. 2! 3! **37.** $\frac{3!}{2!}$. **38.** $\frac{8!}{7!}$.

Show that the following identities are true:

39. $4! \equiv 3!\ 2!\ 2!$

40. $2 \cdot 3! - 3 \cdot 2! \equiv 3!$

41. $\frac{5!}{2} + 2 \cdot 3! \equiv 3 \cdot 4!$

42. $\frac{25}{5!} = \frac{1}{4!} + \frac{1}{3!}$.

REVIEW

EXERCISE 243

Simplify each of the following:

1. $6abc + (c+a)^3 + (b+c)^3 + (a+b)^3 - (a+b+c)^3$.

2. $(x+y+z)^2 + (x+y-z)^2 - (y+z-x)^2 - (z+x-y)^2$.

3. $\dfrac{1}{x^2 - \dfrac{x^2-1}{x+\dfrac{1}{x+1}}}$.

4. $\left(a+\frac{1}{a}\right)^2 + \left(b+\frac{1}{b}\right)^2 + \left(ab+\frac{1}{ab}\right)^2 - \left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)\left(ab+\frac{1}{ab}\right)$.

5. $\dfrac{a-b}{c^2-(a-b)^2} + \dfrac{b-c}{a^2-(b-c)^2} + \dfrac{c-a}{b^2-(c-a)^2}$.

6. Show that $\left(\dfrac{x}{x-a} + \dfrac{x}{x-b} + \dfrac{x}{x-c}\right) - \left(\dfrac{a}{x-a} + \dfrac{b}{x-b} + \dfrac{c}{x-c}\right) \equiv 3$, if $x \neq a$, $x \neq b$, $x \neq c$.

7. Show that $[(a-b)^2 + (b-c)^2 + (c-a)^2]^2 \equiv 2[(a-b)^4 + (b-c)^4 + (c-a)^4]$.

8. Simplify $\dfrac{x^{-2} - y^{-2}}{x^{-3} - y^{-3}}$, using the minimum number of negative signs.

9. Simplify $\dfrac{(x^a)^2}{x^{b+c}} \times \dfrac{(x^b)^2}{x^{c+a}} \times \dfrac{(x^c)^2}{x^{a+b}}$.

10. Simplify $\dfrac{4^n \cdot 2 \times \dfrac{1}{2^{-3n}} - 32^n}{2^{5n} \times 4}$.

11. Simplify $\left[\sqrt[5]{\dfrac{a^{-\frac{1}{2}}b^{\frac{1}{3}}}{a^2b^{-\frac{1}{2}}}} \times \sqrt[3]{\dfrac{a^{\frac{1}{2}}b^{-\frac{1}{2}}}{a^{-\frac{1}{4}}b}}\right]^{-12}$.

12. Show that $\dfrac{\left(a+\dfrac{1}{b}\right)^a\left(a-\dfrac{1}{b}\right)^b}{\left(b+\dfrac{1}{a}\right)^a\left(b-\dfrac{1}{a}\right)^b} \equiv \left(\dfrac{a}{b}\right)^{a+b}$.

13. Show that $\dfrac{\left(1+\dfrac{a}{b}\right)^m\left(1-\dfrac{b}{a}\right)^n}{\left(1+\dfrac{b}{a}\right)^n\left(1-\dfrac{a}{b}\right)^m} \equiv (-1)^m\left(\dfrac{a+b}{a-b}\right)^{m-n}$.

14. For what value of n is $x^{n+1}y^{\frac{n}{5}+1} - y^{2n}z^{n-\frac{1}{2}}$ a homogeneous binomial?

15. For what value of n is $x^{n+4}y^{\frac{n}{3}} + y^nz^{\frac{n}{2}+3}$ a homogeneous binomial?

16. Simplify $\dfrac{x^2+2x\sqrt{y}+y}{x+\sqrt{y}}$.

17. Square the complex number $-\frac{1}{2} - \frac{1}{2}\sqrt{-2}$.

18. Simplify $(\sqrt{m+n}+\sqrt{m-n})^6 - (\sqrt{m+n}-\sqrt{m-n})^6$.

19. Simplify $\dfrac{\left[\sqrt{\dfrac{3+\sqrt{-3}}{3}} - \sqrt{\dfrac{3-\sqrt{-3}}{3}}\right]^2}{3-2\sqrt{3}}$.

20. Rationalize the denominator of $\dfrac{\sqrt{30}}{\sqrt{5}+\sqrt{3}-\sqrt{2}}$.

Solve each of the following equations:

21. $\sqrt{x+2}+\sqrt{4x-3}-\sqrt{9x+1}=0$.

22. $\left(x+\dfrac{9}{x}\right)^2 - 4\left(x+\dfrac{9}{x}\right) = 60$.

23. $\dfrac{x}{x+7} + \dfrac{8x-77}{(x-12)(x+7)} = \dfrac{x}{x+10} + \dfrac{11x-70}{(x-8)(x+10)}$.

24. $\dfrac{x}{x-5}+\dfrac{8x-35}{(x-6)(x-5)}=\dfrac{x}{x-9}+\dfrac{4x-99}{(x-2)(x-9)}$.

25. $\dfrac{x}{x-1}-\dfrac{5}{(x+4)(x-1)}=\dfrac{x}{x-3}-\dfrac{2(x+9)}{(x+5)(x-3)}$.

26. $\dfrac{x}{x-2}+\dfrac{x-12}{(x+3)(x-2)}=\dfrac{x}{x-5}-\dfrac{2(x+5)}{(x-1)(x-5)}$.

27. $\dfrac{x}{x-3}-\dfrac{2x-15}{(x-6)(x-3)}=\dfrac{x}{x-4}-\dfrac{3x-16}{(x-5)(x-4)}$.

28. $\dfrac{x}{x-9}+\dfrac{3(2x-33)}{(x-4)(x-9)}=\dfrac{x}{x-15}-\dfrac{270}{(x+3)(x-15)}$.

Solve the following systems of equations:

29. $18x+12y=7xy$,
$12x+18y=8xy$.

30. $x=\dfrac{y+m}{2}+\dfrac{n}{3}$,
$y=\dfrac{x+n}{2}+\dfrac{m}{3}$.

31. $x^2+xy+y=21$,
$x+xy+y^2=9$.

32. $x^2-xy-y=22$,
$x-xy+y^2=-2$.

33. Expand $(3x^{-\frac{1}{4}}-2y^3)^5$ by the binomial theorem.

34. Expand $(1-\frac{1}{3}x^3)^6$ by the binomial theorem.

35. Find the middle term in the expansion of $\left(x-\dfrac{1}{x}\right)^8$.

36. Find the middle term in the expansion of $\left(x^2+\dfrac{1}{x^2}\right)^{12}$.

37. Find the middle term in the expansion of $\left(\dfrac{a}{x}+\dfrac{x}{a}\right)^8$.

38. In the expansion of $\left(x+\dfrac{1}{x}\right)^{14}$ write the term which is free from x.

39. Find the term free from x in the expression of $\left(x-\dfrac{1}{x}\right)^{18}$.

40. Find the coefficient of $a^{12}b^5$ in $(a-b)^{17}$.

INDEX

(Numbers Refer to Pages.)

Addition, 7–9, 9–11, 12–15.
  complex numbers, 161–162.
  fractions, 85–86.
  imaginary numbers, 157–158.
  surds, 144.
Algebraic expression, 44–46.
Alternative Forms, 80.
Arithmetic Numbers, 1, 2, 6, 56, 62, 63, 81, 83, 84, 85, 86–87, 88, 90, 91, 93, 95, 127, 128, 130, 131, 133, 135, 136, 140–156, 165, 189, 190, 199, 215, 216, 217, 218, 234, 235, 236, 238, 239, 240, 241, 242, 243, 244, 250.
Arithmetic Progression, 234–238.

Binomials, Special Products of, 21–26, 98, 137–138.
  having negative or fractional exponents, 137–138.
  powers of, 26.
Binomial Theorem, 246–248, 252.
  calculation of a specified term, 247–248, 252.
  negative and fractional exponents, 247, 248.
Blank Forms, 55, 56–57, 60, 64, 65, 66, 67, 83, 98, 148, 154, 156, 193.

Complex Numbers, 161–164, 200, 231, 251.
  addition, 161–162.
  division, 164.
  factors, 163–164.
  fractions having complex denominators, 163–164.
  multiplication, 162, 200, 231, 249.
  powers, 163, 200.
  square root, 164.
  subtraction, 161–162.
Confusing Forms, 78–79, 98, 99, 165, 166, 190, 191, 231, 232.
Cyclic Differences, 80, 81, 97, 99, 165, 250.

Division, 26–33.
  complex numbers, 164.
  fractions, 91–92.
  imaginary numbers, 160–161.
  integral powers, 26–27.
  special quotients, 33.
  surds, 150–151, 151–154.
  synthetic, 31–32.

Elimination,
  addition, 116–118.
  comparison, 116.
  constant terms, 203.
  substitution, 115–116, 200–202.
  subtraction, 116–118.
  "*vy*" method, 203–204.
Equations,
  asymmetric, 206.
  binomial, 198.
  biquadratic, 197–198.
  classification, 99, 114–115.
  consistent, 114, 115.
  equivalent, 38–41, 114–115.
  exponential, 139.
  extra roots, 171–174, 175, 191–193, 246, 251, 252.
  factoring, 72–73, 167–168, 202, 203.
  formation of having specified roots, 186–189.
  fractional, 99–103, 104–107, 119–121, 166, 171–176, 197, 198, 218–219, 246, 251, 252.
  graphs, 35–37.
  graphical solution, 36–37.
  higher, 197, 197–198.
  homogeneous quadratic, 203–204.

Equations, — *Continued.*
identical, 37–38, 78, 79, 80–81, 98–99, 114, 232, 245, 249, 250, 251.
inconsistent, 114–115.
independent, 114–115.
irrational, 191–193, 194–196, 197.
irrational coefficients, 196.
linear, 41–44, 103–104, 114–119.
literal, 102–107.
quadratic, 72–73, 166–215, 231, 246, 248, 251, 252.
complete, 72–73, 167–171, 174–176, 231.
homogeneous, 203–204.
incomplete, 72–73, 166–167, 189.
solution by "*vy*" method, 203–204.
reciprocal, 198.
simultaneous linear, 114–121.
simultaneous quadratics, 200–215.
solution by factoring, 72–73, 167–168, 203.
symmetric, 204–205, 207, 208.
systems having three or more unknowns, 118–119, 120–121.
theory, 183–189.
Evolution, 56, 127–130.
complex numbers, 164.
surds, 155.
Exponents,
fractional, 133–134, 135–139, 140, 166, 189, 199, 232, 233, 247, 248, 251.
integral, 15–16, 26–27, 139, 165, 189, 190, 245, 250, 251.
negative, 130–131, 131–132, 135–139, 140, 165–166, 189–191, 197, 199, 207, 232, 233, 245, 247, 248, 249, 250.
zero, 131, 140, 166, 191, 233, 249.

Factorial Notation, 250.
Factors, 53–73, 97–98, 114, 154–155, 163–164, 190, 199.
complex, 163–164, 245, 250, 251.
fractional expressions, 97–98.
monomial, 53–55, 67–72.
polynomial, 55, 68, 69, 70–72.
reduction of quadratic systems by, 202–203.
solution of equations by, 72–73, 167–168, 203.
surd, 154–155.
Formula, solution by, 170–171.
Formulas, 176–177.
Fractions, 79–98.
See Ratio.
addition, 85–86.
change of sign, 79–81, 165, 190.
common denominators, 84–85.
complex, 93–95, 200.
continued, 95.
decimal, 42, 44, 81, 83, 117, 119, 208–209, 243–244.
division, 91–92, 96.
improper, 86–87.
indeterminate, 97.
lowest common denominator, 84–85.
lowest terms, 81–83.
mixed expressions, 83–84, 87–88.
multiplication, 88–90.
powers, 90–91.
subtraction, 85–86.
Fractional Equations, 99–103, 104–107, 119–121, 171–176, 197–198.
See Proportion.
Fractional Form, 86–87.

Geometric Progression, 240–244, 250.
Geometric Means, 241–249.
Graphs, 34–37.
equations, 34–37.
points, 34–35.
solution of equations by, 36–37.

Harmonic Progression, 238–239, 244, 250.
Highest Common Factor, 74–76.
Hindu Method, 170.

Identities, 37–38, 78, 79, 80–81, 98–99, 114, 232, 245, 249, 250, 251.
Imaginary Numbers, 156–161, 189, 191, 231, 232, 233, 249, 251.
addition, 157–158.
division, 160–161.
multiplication, 158–160, 200.
powers, 156–157, 158, 160, 161.
standard form, 156–157.
subtraction, 157–158.
unit of, 156–157.
Indeterminate Forms, 97.
Involution, 16, 21–22, 23, 26, 245, 246–247, 249, 252.
binomials, 26, 137–138, 246–248, 252.

Involution, — *Continued.*
complex numbers, 163.
fractions, 90–91.
imaginary numbers, 156–157, 158, 160.
surds, 150.

Lowest Common Multiple, 76–78, 114.

Mean Proportional, 217–218, 249.
Means,
arithmetic, 235–236, 239, 243.
geometric, 241, 243, 249.
harmonic, 239.
Multiplication, 15–26.
complex numbers, 162, 200, 231, 249.
fractions, 88–90.
imaginary numbers, 158–160.
powers, 15–16.
surds, 146–149.

Notation,
exponential, 133–134, 139.
factorial, 250.
functional, 33–34.
radical, 134.
Numbers,
arithmetic, 1–2, 6, 56, 62, 63, 81, 83, 84, 85, 86–87, 88, 90, 91, 93, 95, 127, 128, 130, 131, 133, 135, 136, 140–156, 165, 189, 190, 199.
complex, 161–164, 200, 231, 249, 251.
imaginary, 156–161.
positive and negative, 2–7.
addition, 4.
division, 6–7.
multiplication, 5–6.
powers, 6.
subtraction, 4–5.

Parentheses, 1–2, 15, 21, 60.
Points, 34, 35.
Polynomials,
addition, 12–15.
cube roots of, 129.
division, 28–31.
multiplication, 18–21.
reduction, 11–12.
special products, 25–26.
subtraction, 12–15.
square of, 23.
square roots of, 129.
third degree, 67, 68, 69–72.
Powers, 15–16, 26–27, 135–137, 165, 166, 189, 190, 191, 232.
binomials, 26, 246–248, 252.
binomial theorem, 26, 246–248, 252.
fractions, 90–91.
products, 16, 137.
Preferred Forms, 79, 80, 81.
Problems,
accelerated motion, 181, 182, 238.
age, 48, 50.
alligation, 52, 53, 112.
alloy, 53, 111, 125, 126.
area, 49, 51, 212, 213, 225, 230.
clock, 47–48, 52–53.
elasticity, 231.
expansion, 227.
falling bodies, 181–182, 230, 238.
gas, 226–227.
general, 111–112.
inclined plane, 225–226, 238.
industrial, 50–51, 108–110, 121, 124.
lever, 112–113.
money, 48, 50, 51, 110, 111, 112, 124, 180, 224.
number, 46–50, 107–108, 109–111, 121–124, 126, 177–180, 211–214, 224–225.
pendulum, 198–199.
photometric, 230–231.
physics, 181–182, 198–199, 225–227, 230–231, 238.
pressure, 226–227.
race, 125–126, 181.
science, 52–53, 112–113.
solution, 52, 53.
tank, 109, 110, 111, 124.
uniform motion, 48, 49, 50, 52, 110, 111, 112, 122, 124, 125, 126, 180–181, 214–215.
volume, 180, 225, 226–227, 230.
work, 51, 108–109, 110, 111, 124, 126, 181, 214.
Products, special, 21–26.
Progressions, 233–245, 250.
arithmetic, 234–238.
arithmetic means, 235–236.
calculation of specified terms, 234–235.
sum of terms of, 236–238.
classification of, 244–245, 250.

Progressions, — *Continued.*
geometric, 240–244, 250.
calculation of specified terms of, 240.
infinite decreasing, 242–244.
means, 241, 243, 249.
repeating decimal fractions as progressions, 243–244.
sum of the terms of, 242–244.
harmonic, 238–239.
calculation of specified terms of, 238.
means, 239.
Proportions, 217–227.
construction from products, 219–220.
determination of specified ratios, 219, 221, 222.
elimination, 222.
formation of, 217, 219–220, 232.
fourth proportionals, 218.
fractional equations as, 100–101, 105–106.
isolation of specified letters in, 220–221.
mean proportionals, 217–218.
problems solved by proportions, 224–227.
reduction of specified terms to the first degree, 223–224.
simplification of, 223–224, 233.
solution of, 218–219.
third proportionals, 218.
transformations of, 220–224.
addition or subtraction, 221–222.
alternation, 220.
evolution, 223–224.
inversion, 220.
involution, 223–224.
subtraction, 221–222.

Quadratic Equations, 72–73, 166–215, 231, 246, 248, 251, 252.
complete, 72–73, 167–171.
elimination, 200–206.
constant terms, 203.
substitution, 200–202.
symmetric equations, 204–205.
"*vy*" method, 203–204.
Hindu Method, 170.
homogeneous, 203–204, 206–209.
incomplete, 72–73, 166–167, 189.
nature of roots, 183–184.
one root known, 184–186.
parameters in, 184.
problems, 177–182, 198–199, 211–215.
reduction of systems by division, 204, 207, 208, 209.
reduction of systems by factoring, 202–203, 207.
simultaneous, 200–215, 252.
solution by completing the square, 169–170, 174–176.
solution by factoring, 72–73, 167–168.
solution by formula, 170–171.
solution by "*vy*" method, 203–204.
theory, 183–189.
three or more unknowns, 209–210.
Quotients, Special, 33, 165.

Ratio, 215–217, 232.
comparison of, 217.
compound, 216–217.
determination of specified, 219, 221–223, 232.
duplicate, 217.
reduction to simplest form, 215–216.
triplicate, 217.
Rationalization, 151–154, 190, 191, 231, 232, 233, 245, 249, 251.
Remainder Theorem, 32, 34, 67, 114.
Reviews, 78–79, 98–99, 113–114, 165–166, 189–191, 199–200, 231–233, 245–246, 248–250, 250–252.
Roots, 56, 127–130.
complex numbers, 164.
cube, 129–130.
extra, 171–174, 175, 191–193, 246, 251, 252.
extraction by Binomial Theorem, 248.
fractions, 128–129, 141–143, 224.
quadratic surds, 155, 156.
roots, 128, 155.
square, 129, 130, 155–156, 164.

Sequences, 233–234.
Signs, Change of, in Fractions, 79–81, 165, 190.
minimum number of negative, 79–81, 165, 190, 199, 250.

Squares, 21, 22, 24, 57–58, 59, 68–69, 70–71, 78, 98, 165, 191, 231, 245, 250, 251.
completing the square, 56–57, 58–59, 168–169.
Hindu Method, 170.
solution by, 169–171, 174–176.
difference of, 59–62, 199.
incomplete, 56–57, 58–59.
polynomial, 23, 59.
trinomial, 57.
Substitution, numerical, 1, 97, 199.
Subtraction, 9–11, 12–15.
complex numbers, 161–162.
fractions, 85–86.
imaginary numbers, 157–158.
surds, 144.
Surds, 140–156, 190, 200, 231, 232, 233, 245, 248, 249, 251.
addition, 144.
change of order, 146.
division, 150–151.
entire, 145.
equivalent forms, 145.
evolution, 155.
factors involving, 154–155.
involution, 150.
mixed, 145.
multiplication, 146–149, 200.
rationalization, 151–154, 190, 191.
simplest form, 140–143.
square root of quadratic surds, 155–156.
subtraction, 144.
Synthetic Division, 31–32.

Variation, 227–231.
one independent variable, 227–228.
problems, 230–231.
two independent variables, 229–230.

Zero, exponents, 131, 166, 191, 233, 249.
roots, 72, 73, 166, 187–188, 191–193, 197, 204, 208.

RETURN TO the circulation desk of any
University of California Library

or to the

NORTHERN REGIONAL LIBRARY FACILITY
Bldg. 400, Richmond Field Station
University of California
Richmond, CA 94804-4698

ALL BOOKS MAY BE RECALLED AFTER 7 DAYS

- 2-month loans may be renewed by calling (510) 642-6753
- 1-year loans may be recharged by bringing books to NRLF
- Renewals and recharges may be made 4 days prior to due date

DUE AS STAMPED BELOW

MAR 0 5 2005

DD20 15M 4-02

YB 35939

632256

QA157
W5
Educ. Dept.

UNIVERSITY OF CALIFORNIA LIBRARY