EXPONENTIALS MADE EASY

OR

THE STORY OF 'EPSILON'

BY

M. E. J. GHEURY DE BRAY

'Surely, all men should be Road-menders.'
MICHAEL FAIRLESS

MACMILLAN AND CO., LIMITED
ST. MARTIN'S STREET, LONDON

1921

COPYRIGHT

GLASGOW: PRINTED AT THE UNIVERSITY PRESS
BY ROBERT MACLEHOSE AND CO. LTD.

TO

THE MEMORY OF

DR. S. P. THOMPSON

IN REMEMBRANCE OF HAPPY MOMENTS SPENT IN DISCUSSING THIS LITTLE BOOK, I DEDICATE THESE PAGES AS A TOKEN OF MY DEEP REGRET.

M. GHEURY DE BRAY.

CONTENTS

INTRODUCTION

SOME time ago the author came across a certain little book, and although he was supposed to know all about the things explained in it, he found a great delight in reading it. In so doing he re-learned several things he had forgotten and learned a few others he had not chanced to meet before. But the most useful knowledge he derived from reading this truly delightful little book—*Calculus Made Easy*—was that, indeed, it is possible to make the study of such mathematical processes as those of the Calculus so easy that one may learn them by oneself without the help of a teacher, provided one has in one's hand the necessary guide and faithfully follows it from beginning to end.

There are few branches of mathematics which seem more puzzling to beginners than the study of imaginaries and hyperbolics; indeed, many students who are no longer looking askance at $\frac{dy}{dx}$ or at the sign $\int$ confess that the appearance of i in a mathematical expression gives them a nerve-shattering shock, while the sight of sinh or cosh is the signal for undignified retreat. It has been suggested to the author that there is no more difficulty in exorcising the evil spirit lurking in i and in the members of the hyperbolic tribe, and in rendering these impotent to scare anyone approaching them with the proper talisman in his hands, than there was in taming $\frac{dy}{dx}$ and $\int$ and rendering them docile. Trial showed that this was indeed true.

While gathering material for this purpose, the fact became evident that if various secondary stumbling-blocks could be preliminarily removed from the path of the unwary, the treatment of the more unwieldy material would greatly gain in homogeneity and continuity. Also, several interesting and elementary properties of "epsilon," not usually met with in text books, were encountered on the way and deemed to be likely to bring to sharper focus the conceptions a beginner's mind might have formed concerning this remarkable mathematical constant.

The outcome of this preparatory prospecting raid into the field of "imaginaries" and "hyperbolics" is the birth of this little brother to *Calculus Made Easy*. This newcomer has no pretension to equal its elder, but it is setting forth with the desire to be worthy of its kinship, and it certainly could not choose a better example to emulate.

The author gladly acknowledges his grateful indebtedness to Mr. Alexander Teixeira de Mattos for his kind permission to borrow the matter of the preliminary pages from one of Henri Fabre's most charming chapters.

PRELIMINARY.

As an introduction to this little book, the writer will, for a first chapter, yield the pen to another and merely assume the humble part of a translator—a translator whose task is far from easy if he is to retain some of the captivating quaintness of style and of the combined wealth and simplicity of phraseology of the French original. Henri Fabre, that most remarkable personality in the army of Truth seekers, shall tell you here how, in his studies of the insect world, he came to meet the ubiquitous ϵ dangling on a spider's web, and how he was compelled awhile to let the mathematician in him step into the entomologist's shoes; for—luckily for us—he was both.

* "I am now confronted with a subject which is at the same time highly interesting and somewhat difficult: not that the subject is obscure, but it postulates in the reader a certain amount of geometrical lore, substantial fare which one is apt to pass untasted. I do not address myself to geometricians, who are generally indifferent to

* Quoted by permission of Mr. Alexander Teixeira de Mattos, the holder of the English copyright, from the *Souvenirs entomologiques* of J. Henri Fabre (Paris: Librairie Delagrave; London: Hodder & Stoughton; New York: Dodd, Mead & Co.). The full text of Mr. Teixeira's translation will be found in the Appendix to the volume entitled "The Life of the Spider."

facts appertaining to instinct. I do not write either for entomologists, who as such are not concerned with mathematical theorems; I seek to interest any mind which can find pleasure in the teachings of an insect.

"How can I manage this? To suppress this chapter would be to leave untouched the most remarkabl feature of the spider's industry; to give it the fuln of treatment it deserves, with an array of learn formulae, would be a task beyond the pretension these modest pages. We will take a middle cour avoiding alike abstruse statements and extreme ign ance.

"Let us direct our attention to the webs of *Epeira* preferably to those of the silky *Epeira* and the stripe *Epeira*, numerous in autumn in my neighbourhood, ar so noticeable by their size. We shall first observe th the radial threads are equidistant, each making equ angles with the two threads situated on either side it, despite their great number, which, in the work of th silky *Epeira* exceeds two score. We have seen * by wha strange method the spider attains its purpose, which is to divide the space where the net is to be woven into a grea number of equiangular sectors, a number which is nearl always the same for each species: disorderly evolution suggested, one might believe, by wild fancy alone, resul in a beautiful rose pattern worthy of a draughtsman' compass.

"We shall also observe that in each sector the variou steps or elements of each turn of the spiral, are parall

* See *Souvenirs Entomologiques*, IXeme Serie.

Epeira a genus of spiders which includes the common garden spider to one another, and close gradually upon one another as they near the centre. They make, with the two radii which limit them at either end, an obtuse and an acute angle, on the side away from, and towards the centre, respectively, and these angles are the same throughout the same sector, because of the parallel disposition of these elements of the spiral.

"More than this: in different sectors these obtuse and acute angles are the same, as far as one can rely on the testimony of the eye unaided by any measuring instrument. As a whole, the funicular structure is therefore a series of transverse threads which cut obliquely the various radii at an angle of invariable magnitude.

"This is the characteristic feature of the *logarithmic spiral.* Geometricians give this name to the curve which cuts obliquely, at a constant angle, all the straight lines radiating from a centre, called the *pole.* The web of *Epeira* is nothing else but a polygonal line inscribed in a logarithmic spiral. It would coincide with this spiral if the radii were unlimited in number, so that the rectilinear elements were indefinitely short and the polygonal line modified into a curve.

"To give an insight into the reasons which make this spiral a favourite subject for the speculations of scientific minds, let us confine ourselves to a few statements, the demonstration of which may be found in treatises on advanced geometry.

"The logarithmic spiral describes an infinite number of circumvolutions about its pole, which it always approaches without ever reaching it. The central point, nearer at every turn, remains for ever inaccessible. It goes without saying that this property does not belong to the realm of facts of which our senses are cognisant. Even with the help of the most precise instrument, our sight could not follow the spiral's endless circuits, and would speedily refuse to pursue farther the subdivision of the invisible. It is a volute to which the mind conceives no limit. Alone, cultivated reason, more acute than our retina, sees clearly that which defies the eye's power of perception.

"The *Epeira* obeys as faithfully as possible this law of endless winding. The spires of its web close up more and more as they approach the pole. At a certain distance from it, they stop suddenly, but there, continuing the spiral, is the thread which was woven during the first stages of the construction of the web, as a scaffolding to support the spider in the elaboration of its net, and, as such, destroyed as the work progresses, but allowed to subsist in the vicinity of the pole which it approaches, like the rest of the spiral, in circuits which become closer and closer together and hardly distinguishable from one another. It is not, evidently, of rigorous mathematical accuracy, but, nevertheless, it is a very close approximation to it. The *Epeira* winds its thread nearer and nearer to the pole of its web as closely as it is enabled to do so by the imperfection of its tools, which, like ours, are inadequate to the task; one would think that it is deeply versed in the properties of the spiral.

"Without entering into explanations, let us mention a few other properties of this curious curve. Imagine a flexible thread coiled upon the logarithmic spiral. If we uncoil it, keeping it tight the while, its free end will describe a spiral in every respect similar to the first, but merely shifted to another position.

"Jacques Bernouilli, to whom Geometry is indebted for this beautiful theorem, caused the parent spiral and its offspring, generated by the unwinding thread, to be engraved upon his tomb, as one of his greatest titles to fame, together with the motto *Eadem mutata resurgo* (I rise again, changed but the same). With difficulty could Geometry find anything better than this inspiring flight towards the Great Problem of the Beyond.

"Another geometrical epitaph is no less widely celebrated: Cicero, when questor in Sicily, sought under the veil of oblivion, cast by brambles and wild grasses, the tomb of Archimedes, and recognised it amongst the ruins by the geometrical figure engraved upon the stone: a cylinder circumscribing a sphere. Archimedes was the first to know the approximate ratio of the circumference to the diameter, and from it he deduced the perimeter and surface area of the circle, together with the surface area and the volume of the sphere. He demonstrated also that the latter has, for surface area and volume, two thirds of the surface area and volume of the circumscribing cylinder. Disdaining a pretentious inscription, the Syracusan geometer relied upon his theorem alone as an epitaph to transmit his name to posterity.

The geometrical figure proclaimed the identity of the remains underneath as clearly as alphabetical characters.

"To bring our description to a close, let us mention one more property of the logarithmic spiral. Cause the curve to roll upon a straight path, and its centre will describe a straight line. The endless winding leads to the rectilinear trajectory; perpetual variation engenders uniformity.

"Is this logarithmic spiral, with its curious properties, merely a conception of the geometers, who combine number and space at will to open a field wherein to practise mathematical methods? Is it but a dream in the night of the intricate, an abstract enigma intended to feed our understanding? Not at all. . . . It is a reality in the service of life . . . a plan of which animal architecture makes frequent use. The mollusc, in particular, never shapes the volutes of its shell without reference to this transcendent curve. The first-born of the series knew it, and copied it, as perfect in primaeval times as it is to-day.

"Consider the ammonites, ancient relics of what was once the highest expression of living things, when the abysmal slime separated itself from the deep, and dry ground appeared on the face of the earth. When they are cut along a median plane, the fossils exhibit a magnificent logarithmic spiral as the general scheme of the building, which has been a mother-of-pearl palace with multiple chambers intercommunicating by a narrow canal. . . .

"To-day, the last representative of the cephalopods with multicellular shells, the Nautilus of the Indian

Ocean, remains faithful to the antique design. It has not discovered anything better than its distant ancestor. It has only modified the position of the communication canal, and placed it at the centre instead of in its former dorsal situation, but it still winds its spire logarithmically, as the ammonites did in the first ages of the world.

"We must not, however, entertain the belief that these highly developed molluscs have the monopoly of the elegant curve. In the stagnant waters of our sedge-lined ditches, the flattened shells, the humble Planorbes or water-snails, sometimes scarcely larger than lentils, are the rivals of the ammonite and the Nautilus in high geometry. One of them, for instance—the *Planorbis Vortex*—is a marvel of logarithmic winding.

"In the shells assuming an elongated shape, the structure becomes more complex, although still governed by the same fundamental laws. I have before my eyes some species of the genus *Terebra*, originating from New Caledonia. They are very tapering cones, almost as long as the hand. Their surface is smooth, quite bare, without any of the usual ornaments, folds, knots, or strings of beads. The spiraliform structure is superb, with its simplicity for sole ornament. I count a score of whorls which gradually diminish and are lost in the delicate details of the point. A fine groove delineates them.

"I trace with a pencil any generating line of this cone, and, relying merely on the evidence of my eyes, somewhat trained in geometrical measurements, I find that the spiral groove cuts this generatrix at a constant inclination.

"The consequence is an easy deduction: by projection on a plane normal to the axis of the shell, the line generating the cone, in each of its various positions, becomes a radius, and the groove which whirls in ascending from the base to the apex is converted into a plane curve which, meeting these radii with an invariable inclination, is therefore nothing else but the logarithmic spiral. Inversely, the groove on the shell may be considered as the projection of the logarithmic spiral on a conical surface.

"We can even go a step further: conceive a plane normal to the axis of the shell, and passing through the apex. Imagine also a thread wound in the spiral-shaped groove. If we unwind it, keeping it tight without slipping off the groove, and to this end maintaining it normal to the line, generating the cone, which passes by the point where the thread leaves the surface of the shell, the extremity of the thread will remain on this plane and describe in it a logarithmic spiral. It is, with a greater complexity, a variation of the *eadem mutata resurgo* of Bernouilli: The conical logarithmic spiral changes itself into a plane logarithmic curve.

"A similar geometry is found in the construction of the other shells whether affecting the shape of an elongated or that of a flattened cone. The shells coiled in globular volutes are no exception to the rule . . . all, down to the humble snail, are constructed on a logarithmic pattern. The spiral, famous among geometricians, is the general plan copied by the mollusc in coiling its stone sheath.

. . . .

"Of this celebrated curve, the spider elaborates but an elementary frame, which nevertheless proclaims the principle of the ideal edifice. The spider works on the same lines as the mollusc having a convoluted shell.

"The latter, to construct its spire, takes whole years, and attains in its coiling an exquisite perfection. The spider, to fashion her web, takes only one hour at most, so that the swiftness of execution entails greater simplicity of construction. It abbreviates, so to speak, limiting itself to the sketch of the curve which the other describes in its full perfection. It is therefore learned in the geometrical secrets known to the ammonite and the Nautilus, and merely simplifies, in putting them in practice, the logarithmic lines beloved by the snail.

"What is its guide? Necessarily, the animal must have in itself the virtual design of its spiral. Never could chance, however fecund in surprises we suppose it to be, have taught it the high geometry where our mind goes astray without a preliminary training.

. . . .

"Can it be premeditated combination on its part? Is there calculation, mensuration of angles, verification of parallelism, by sight or otherwise? I incline to believe that there is nothing of all that . . . nothing but an innate propensity of which the animal has not to regulate the effect, no more than the flower has to regulate the disposition of its petals. The spider practises advanced geometry without knowing, without caring. . . . The process goes by itself, the initial impulse having been given by an instinct conferred at the origin.

"The pebble thrown by the hand, in returning to the ground, describes a certain curve; the dead leaf, detached and carried away by the wind, in performing its journey from the tree to the soil, follows a similar curve. In either case no influence of the moving body regulates the fall . . . nevertheless, the descent is performed according to a scientific trajectory, the parabola, of which the section of a cone by a plane has furnished the prototype for the meditation of geometers. A figure, the fruit of a speculative concept, has become tangible by the fall of a stone out of the vertical.

"The same speculations take up the parabola once more and suppose it to roll on an indefinite straight line, and enquire into the nature of the path followed by the focus of the curve. The reply is that the focus of the parabola describes a *catenary*, a line of very simple shape, but for the algebraical expression of which we must have recourse to a cabalistic number, at variance with all systems of numeration, and which digits refuse to express exactly, however far one may pursue their orderly array. This number is called *epsilon*, being represented by the Greek letter ϵ. Its value is the following series indefinitely continued:

$$\epsilon = 1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \dots.$$

" If the reader has the patience to perform the calcu- ation of the first few terms of this series, which has no mit, since the sequence of natural numbers is itself ndless, he will find $\epsilon = 2{\cdot}7182818 \ldots$.

" With this strange number, are we now restricted to he rigid domain of the mind ? Not at all : the catenary ppears in the realm of reality whenever gravitation nd flexibility act jointly. This name is given to the urve formed by a chain suspended by two points not ituated on the same vertical. It is the shape naturally aken by a flexible tape the two ends of which are held n one's hands ; it is the outline of a sail inflated by the vind ; it is the form of the milk-bag of the goat re- urning from the pasture where its udder became illed . . . and all these things involve the number psilon.

" What a lot of abstruse science for a bit of string ! ,et us not be surprised. A pellet of lead swinging at he end of a thread, a drop of dew trembling at the end f a straw, a puddle ruffled by ripples under a puff of ir, a mere nothing, after all, requires a titanic scaffolding vhen we wish to examine it with the eye of the calculus. . . We need the club of Hercules to crush a midget !

" Surely our methods of mathematical investigation re full of ingenuity . . . one cannot admire too much he powerful brains which have invented them . . . but ow slow and painstaking when facing the least realities ! hall it ever be given to us to investigate the truth in a nore simple fashion ? Shall mind be able some time to lo without the heavy arsenal of formulae ? Why not ?

"Here the occult number epsilon reappears inscribed on a spider's thread. On a misty morning, look at the web which has just been constructed during the night. Owing to its hygroscopic nature the sticky network has become laden with droplets, and, bending under the weight, the threads are now as many catenaries, as many rosaries of limpid gems, graceful rows of beads, arranged in exquisite order, and hanging in elegant curves. Let the sun pierce the mist . . . and lo! the whole becomes iridescent with adamantine fire and, in lovely garlands of fairy lights, the number ϵ appears in all its glory!

"Geometry, that is, the science of harmony in space, presides over all things. It is in the arrangement of the scales of a fir-cone, as in the disposition of the Epeira's web; it is in the shell of a snail as in the rosary of a spider's dewladen thread, as in the orbit of a planet; it is everywhere, as majestic in an atom as in the world of immensities. . . .

"And this Universal Geometry speaks to us of a Universal Geometer, whose divine compass has measured all things. . . . As an explanation of the logarithmic curve of the Ammonite and of the Epeira, it is perhaps not in agreement with the teachings of to-day . . . but how much loftier is its flight! . . ."

PART I.

'HE SIMPLE MEANING OF SOME AWE-INSPIRING NAMES AND OF SOME TERRIBLE-LOOKING, BUT HARMLESS, SIGNS.

CHAPTER I.

THE TRUTH ABOUT SOME SIMPLE THINGS CALLED FUNCTIONS.

IATHEMATICIANS are very fond of the word "function," nd indeed they are to be excused, for every time they se it they avoid a long sentence. The expression, in act, is so convenient that we shall certainly use it often urselves, therefore we must make sure of its exact ignificance so as to be quite clear what is meant by it.

A "function" of a certain thing is simply something which varies when that thing varies. The veekly pay of a workman is a function of the number f hours he works per week, since his pay varies with he number of working hours he "puts in"; similarly, he unburned length of a candle is a function of the ime elapsed since it was lit, since it is different for various intervals of time during which the candle has been burning; the weight of a healthy child is a function of his age, since his weight alters as the child becomes older; the cost of an engine is a function of the size of the engine, etc. As a matter of fact it would be difficult to find something which is *not* a function of something else: the length of the pencil with which you write is a function of the number of words you have written with it (supposing no breakage)—the more numerous the words the shorter the pencil; the value of the clothes you wear is a function of the wear to which they are subjected, since as wear takes place their value will steadily diminish. Everything practically is a function of the time, since it is bound to change—possibly in an imperceptible manner—as time goes on.

So, when we say that y is a function of x, for instance, we mean that the value of y varies when the value of x varies, that is, that the value of y depends upon the value taken by x. This fact we express by the notation $y=$ function of x, or one of its many abbreviations: $y=f(x)$, $y=F(x)$, $y=\phi(x)$, $y=\psi(x)$, etc. All these are read "y equals a function of x." You know now what a function is; you see it is quite a simple thing despite its imposing name.

One must first notice that the thing in terms of which a function is expressed must necessarily be capable of taking different values, that is, it must be what mathematicians call a "variable quantity" or simply a "variable," to distinguish it from a "constant," or a quantity which has always the same value. In the examples stated above the number of working hours "put in" by the workman, the time elapsed since the candle was lit, the age of the child, the size of the engine, the number of words written by the pencil, the wear of the clothes, are quantities the value of which is liable to considerable variation. In the last case it is not so easy to conceive how the value may be expressed numerically, but the means by which the variability is expressed matters little, the main facts are: first, a variability; second, this variability must cause a corresponding variation of the thing which is stated to be a function of the variable quantity, and this thing is therefore also necessarily a "variable."

Since, when we write $y =$ a function of x, we wish to express that the value of y depends upon the value of x, y is called the "dependent variable" and x is called the "independent variable," that is, the variable which can take any suitable arbitrary value. In the examples above, the number of working hours, the time elapsed since the candle was lit, the age of the child, etc., may have any likely value according to the case: these quantities are independent variables; on the other hand, the weekly pay, the length of unburnt candle, the weight of the child, etc., depend on the value given to the former quantities respectively, and these latter quantities are therefore the dependent variables.

But, although the statement that "y is a function of x" conveys the important information that the variation of y depends upon that of x, it does not tell us anything about the *manner* in which y varies when x changes in value. It does not tell us even whether y

gets larger or smaller when x increases, and so it is really a very vague statement. Yet, in many cases, it is useful to be able to write a relation such as $y=f(x)$ between two variables x and y, but little can be done as a rule unless we know *how* y varies when x varies, that is, unless we know the "form" of the function, because there is an infinite number of ways in which y can respond to changes in the value of x.

In the examples above, for instance, the weekly pay P of the workman is the product of the number N of working hours put in and his hourly wages, p, P and p being expressed in terms of the same unit, say in shillings, so that not only can we write $P=f(N)$ but we have $P=pN$ for the "form" of the function, P being the dependent and N the independent variable. We can now find the value of P corresponding to any value of N, provided we know the value of the constant p. In the case of the candle, the length l left unburnt decreases as the time t of burning increases; if l is in inches and t in minutes, and if L inches was the initial length of the candle, while a is the length in inches consumed in one minute—this depends on the diameter of the candle, size and trimming of wick, material of which the candle is made, supply of air, etc., and is a constant for that particular candle as long as the external conditions affecting the combustion remain the same—then it is evident that $l=L-at$, and this is the form of the function enabling us to find l for any value of t, provided we know the value of the constants L and a. A similar expression, namely, $l=L-an$, will express the

ngth l of the pencil as a function of the number n of
ıousand words written, say, if a is the shortening—*i.e.*
ıe wear—corresponding to the writing of one thousand
ords.

In many cases it is impossible to state mathematically ıe form of a function. If W is the weight of a child, ıd A his age in weeks or months or years, there is no ‹pression which will exactly represent the numerical ·lation between W and A, simply because W depends ı so many other things besides A—food, state of health, :c.—that its variation is altogether erratic, that is, it impossible to calculate the value of W for any given alue of A. It is likewise difficult if not impossible to ive a form to the value of a suit of clothes as a function f the wear to which it is subjected.

Even when we know the form of a function, however, 'e are not able to calculate the value of the dependent ariable from given values of the independent variable nless we know the *numerical* value of the constants ccurring in the expression of the function. We cannot nd the workman's weekly pay P when he works 30, 5 or 40 hours per week, say, until we know what is p, is rate of pay per hour ; as soon as we know that p is ıy 3 shillings per hour, we know that his weekly pay orresponding to the above number of hours is 90/- or $4\frac{1}{2}$, 105/- or £$5\frac{1}{4}$, and 120/- or £6 respectively. Like-'ise, it is only when we know that the candle was ıitially 8 inches long and burns 2·4 inches per hour or ·04 inch per minute that we can say that the length emaining after 10, 20, 30 minutes is $8-0{\cdot}04\times10=7{\cdot}6$

inches, $8-0{\cdot}04\times 20=7{\cdot}20$ inches, $8-0{\cdot}04\times 30=6{\cdot}8$ inches respectively; similarly for the length of the pencil. In these last two cases, a, the length burned per minute or the length used per thousand words is to be found by actual experiment.

It is also evident that if we give a definite value to a dependent variable, expressed as a function of another (independent) variable with numerical coefficients, the value of this independent variable cannot be anything we please, but must be such as to correspond to the value given to the dependent variable. The value of the independent variable, therefore, depends on the value of the dependent variable just as much as the value of the latter depends on that of the former. The dependence is reciprocal, and the function can generally be expressed in such a way that the independent variable becomes the dependent variable and *vice versa*. For instance, in the case of the candle mentioned above, instead of $l=L-at$ we may write $t=\frac{L-l}{a}$. The new function is then said to be the "inverse function" of the original one. It is usual to represent the independent variable by x and the dependent variable by y, so that $y=f(x)$, the inverse function being $x=f_1(y)$, f_1 simply indicating that the function has another form than the form denoted by f.

In all the above cases where the form of the function could be stated, one of the quantities, y for instance has been directly expressed in terms of the other quantity x. Which of the two variables is the independent and which is the dependent one is therefore clearly or *explicitly* stated, and the variable y is said to be an "explicit function" of the (independent) variable x. For instance, in $t=\frac{L-l}{a}$, t is an explicit function of l, while in $l=L-at$, l is an explicit function of t. If, however, it is not clear from the form of the expression which of the two quantities is the independent variable, although their interdependence is *implied*, as when we write $l-L=at$, then one of the variables is an "implicit function" of the other.

A quantity may be a function of several different variables. In the examples given, the workman's weekly pay P is a function not only of the number N of hours he works in the week, but also, as we have seen, of the hourly pay p he receives. The length of the candle after burning t minutes is a function, not only of t, but of the original length L—if several values are possible for L—and of the rate of burning a—if this rate can take several values. The existence of this complex dependence on several variables is evident from the form of the functions considered, since a change in the value of p in the first case, or of L or a in the second, causes a change in the value of P or of l respectively. The relationship would then be expressed in a general manner by the notations $P=f(p, N)$ and $l=f(t, L, a)$ respectively. Similarly, in the case of the pencil, $l=f(n, L, a)$ would be the general form of the function.

A very interesting and useful exercise consists in "plotting"—that is, in drawing on squared paper—the graph of a function of which the form and numerical constants are known. Giving various suitable values to the independent variable x, one calculates the corresponding values of the dependent variable y, and by plotting the successive pairs of values one obtains (generally) a curved line which represents the variation in value of the dependent variable y, that is, of the value of the function itself, as the independent variable x varies through the range of the possible values it can assume.

CHAPTER II.

THE MEANING OF SOME QUEER-LOOKING EXPRESSIONS.

WHEN we write a^3, we mean $a \times a \times a$, that is, the product of 3 factors, each equal to a, or, better still, $1 \times a \times a \times a$, the product of unity by three factors each equal to a. Similarly, when we write a^n we mean the product of n factors each equal to a or unity multiplied by n factors each equal to a. Logically, one would think that, when we find before us such an expression as a^0, which is read "a to the power zero," it means the product of zero factors equal to a, namely, zero, and a beginner is therefore invariably puzzled when told that $a^0 = 1$. According to our second definition above, however, a^0 means unity multiplied zero times by a, and this is obviously unity, so this definition holds good in this particular case. As a matter of fact, *anything* raised to the power zero gives unity as the result of the operation. This follows directly from the rule we have seen in algebra for the division of powers of the same quantity, namely: $x^m/x^n = x^{m-n}$, for, if we apply this rule to the case $x^m/x^m = 1$, we get $x^{m-m} = x^0 = 1$. This is true whatever is the value of x, so that, as a certain humorous continental teacher used to tell his pupils to impress them with this fact, $(\text{cow})^0=1$ or $(\text{slipper})^0=1$.

So, following a well-established rule, we found a result which has no meaning if we try to explain it by the definition hitherto available. Starting again at the beginning, however, and following another method, we find another result about which there is no possible doubt. The two results are evidently equivalent, one being merely a different way of writing the other, so that, in the case just considered, x^0 is only another way of expressing unity. Such expressions, the meaning of which can only be found by independent investigation, are often met with in mathematics. For instance, $\frac{0}{a}=0$, since nothing, divided by any number you like, gives always a result equal to nothing; also $\frac{a}{0}=$ an infinitely great number, since the smaller the denominator of a fraction, the larger the value of the fraction, so that if the denominator be very small, the value of the fraction is very large, and when the denominator is so small that we can write a zero in its stead without perceptible error, the value of the fraction is greater than any number we can conceive, that is, it is infinitely large. Once the meaning has been found, we can either substitute its proper and simpler meaning to it or we can use it whenever convenient. Henceforward, for instance, whenever we shall come across such expressions as a^0, $(x+3)^0$, $(\sin\theta-3\tan\phi)^0$, etc., we shall know that each is equivalent to unity and may be replaced by 1, so that

$$3a^0=3, \quad \sin\theta(x+3)^0=\sin\theta,$$
$$\frac{(\sin\theta-3\tan\phi)^0}{\cos^2\theta}=\frac{1}{\cos^2\theta};$$

also, $a=a\times$(any expression we like)^0, for instance

$$a=a\left(7\sin x+\frac{\sqrt[3]{\tan\phi}}{x^2}-\log_\epsilon\phi\right)^0.$$

The same rule of algebra leads us yet to another curious expression; consider a^2/a^5, this is equal to $a^{2-5}=a^{-3}$ or a "to the power minus three," another puzzling result for, according to our definitions a^{-3} means the product of minus three factors each equal to a—this is quite another thing than the product of three factors each equal to $-a$—or unity multiplied minus three times by a, both of which are meaningless. If, however, we consider that $a^2/a^5=1/a^3$, we see that a^{-3} is only another way of writing $1/a^3$, that, similarly, a^{-4} is the same as $1/a^4$ and so on; and the "minus multiplication" above is explained as meaning a division: unity divided three times by a, so that the second definition is here still susceptible of an interpretation. It follows that we can always replace such expressions as $1/x^n$, $1/(a+x)^2$, $1/(\sin\theta+\theta)$ by x^{-n}, $(a+x)^{-2}$, $(\sin\theta+\theta)^{-1}$. The sign $-$ in front of an index therefore means simply: "put the expression to which this index is affixed, exactly as it is, but without this $-$ sign, as denominator to a fraction the numerator of which is 1." Such an expression as $1/x^n$ is called the "reciprocal" of x^n, so that x^{-n} is the "index form" of the reciprocal of x^n.

We also often meet in mathematics expressions such as $a^{2/3}$, $x^{m/n}$, etc., which one reads "a to the power two-thirds," x to the power m over n," etc. Evidently these expressions cannot be explained by either of the definitions of a^n quoted above. $x^{m/n}$ is not obtained by applying the rule $x^m \times x^n = x^{m+n}$ or $x^m/x^n = x^{m-n}$ to any particular case, so we must try to find how we arrive at such expressions, in order to discover their meaning. Now, we have seen in algebra that to raise to any power, say the fifth power, a power of a quantity, such as x^2, one must multiply the two indices together, so that $(x^2)^5 = x^{2\times 5} = x^{10}$. More generally $(x^m)^n = x^{m\times n}$. If we apply this rule to $a^{2/3}$ we get $(a^{2/3})^3 = a^{\frac{2}{3}\times 3} = a^2$. But we also get $(\sqrt[3]{a^2})^3 = a^2$. It follows that $a^{2/3}$ is the same thing as $\sqrt[3]{a^2}$ since, by raising both to the cube or third power we get the same result, namely a^2. More generally, $x^{m/n}$ is only another way of writing $\sqrt[n]{x^m}$; it is, in fact, what is called the "index form" of $\sqrt[n]{x^m}$—which is called the "radical form." Note that the "order" of the root, 3 and n in the above examples, appears as a *denominator* to the index.

We have, up to now, made acquaintance with three queer-looking expressions, and we know now exactly what they mean, so that we shall not be puzzled by them any more; they are:

x^0, "x to the power zero," the value of which is 1.

x^{-n}, "x to the power minus n," which is exactly the same as $1/x^n$.

$x^{m/n}$, "x to the power m over n," which is exactly the same as $\sqrt[n]{x^m}$.

Moreover these equivalent expressions are absolutely general, instead of x we can put anything we please, so that, for example,

$$[3x^3-(\log_\epsilon \theta)+5]^{3/5}=\sqrt[5]{(3x^3-\log_\epsilon \theta+5)^3}.$$

It is very useful to be able to pass readily from one form to the other, as in this way some complicated expressions can sometimes be simplified considerably. The following worked examples will help you to see how this is done.

Example 1. Simplify $3a^{-2}/5$ by expressing it with a positive index.

Remember that an index only affects the letter (or bracket) to which it is affixed, and nothing else; for instance, $3a^2$ means $3\times a^2$ and not $9\times a^2$, while $(3a)^2$ means $9a^2$. Here the index -2 only affects the letter a, and has nothing to do with the coefficient 3 in front of a, so that $3a^{-2}/5$ is the same as $\frac{3}{5}a^{-2}$. But $a^{-2}=1/a^2$, hence the given expression becomes $3/5a^2$. That's all!

Example 2. Simplify a^0/x^{-3}.

We know that $a^0=1$ and that $x^{-3}=1/x^3$, so that

$$a^0/x^{-3}=1\Big/\frac{1}{x^3}=x^3.$$

Example 3. Simplify $3(x+1)^{-1}\times\sqrt[3]{(x+1)^2}$.

We know that $\sqrt[3]{(x+1)^2}=(x+1)^{2/3}$.

We get therefore

$$3\times(x+1)^{-1}\times(x+1)^{2/3}=3(x+1)^{\frac{2}{3}-1}=3(x+1)^{-1/3}$$
$$=3/\sqrt[3]{x+1}.$$

Example 4. Simplify $(2x)^{-3}/3x^{-1}$.

It becomes successively

$$\frac{1}{(2x)^3} \div \frac{3}{x} = (1/8x^3) \div 3/x = (1/8x^3) \times x/3 = 1/24x^2.$$

Example 5. Simplify $x^{-2}\sqrt{x^3} \times (3x^2)^0$.

We write

$$x^{-2}\sqrt{x^3}(3x^2)^0 = x^{-2} \times x^{3/2} \times 1 = x^{\frac{3}{2}-2} = x^{-1/2} \quad \text{or} \quad 1/\sqrt{x},$$

whichever is most convenient.

Example 6. Simplify $\dfrac{3x^{-2}a^{2/3}}{2(2a)^{-3}\sqrt{x^3}}$.

Proceeding as above we get successively

$$\frac{3(1/x^2)\,a^{2/3}}{2(1/(2a)^3 \times x^{3/2}} = \frac{3 \times a^{2/3} \times 8a^3}{2 \times x^2 \times x^{3/2}} = \frac{24a^{\frac{2}{3}+3}}{2x^{2+\frac{3}{2}}} = 12a^{3\frac{2}{3}}/x^{3\frac{1}{2}}$$

or preferably $12a^{11/3}/x^{7/2}$.

Example 7. Simplify $(3m^{-2}k^{-1/2})^4$.

We get
$$\begin{aligned} 3^4(m^{-2})^4 \times (k^{-1/2})^4 &= 81\,(1/m^2)^4 \times 1/(k^{1/2})^4 \\ &= 81 \times (1/m^8) \times (1/k^2) \\ &= 81/m^8k^2. \end{aligned}$$

We could also have proceeded as follows after the first transformation :

$$3^4(m^{-2})^4 \times (k^{-1/2})^4 = 81 \times m^{-8} \times k^{-2} = 81m^8k^2.$$

Example 8. Simplify $(2\sqrt{m} \times x^{-2})^{-2}$.

We get

$$2^{-2} \times (m^{1/2 \times -2}) \times (1/x^{-4}) = (1/4)(1/m)\left(1 \Big/ \frac{1}{x^4}\right) = x^4/4m$$

or, by another way $1 \Big/ \left(2 \times m^{1/2} \times \dfrac{1}{x^2}\right)^2 = x^4/4m$ as before.

Example 9. Simplify $(2x^{-2}a^{-1/3})^{-a} \div \left(\dfrac{3\sqrt{x^a}}{a^{-1/2}}\right)^2$.

Deal with each bracket separately.

$$(2x^{-2}a^{-1/3})^{-a} = 1/(2x^{-2}a^{-1/3})^a = 1/2^a(1/x^{2a})(1/a^{a/3})$$
$$= x^{2a} \times a^{a/3}/2^a.$$

Also $$\left(\frac{3\sqrt{x^a}}{a^{-1/2}}\right)^2 = (3x^{a/2}a^{1/2})^2 = 9x^a \times a.$$

So that the whole expression becomes

$$\frac{x^{2a} \times a^{a/3}}{2^a \times 9x^a \times a} = \frac{x^{2a-a}a^{\frac{a}{3}-1}}{9 \times 2^a} = \frac{x^a a^{\frac{a}{3}-1}}{9 \times 2^a}.$$

Both transformations should proceed simultaneously, as in the next example.

Example 10. Simplify

$$\sqrt{\left(\frac{2x^{-3p}m^{a-1}}{m^{-1}\sqrt[p]{x}}\right)} \times \sqrt[3]{(m^{-a}\sqrt{x^{-p}})}.$$

We get $$\frac{2^{1/2}x^{-3p/2}m^{(a-1)/2}}{m^{-1/2}x^{1/2p}} \times m^{-a/3}x^{-p/6}$$

$$= \frac{2^{1/2} \times (1/x^{3p/2}) \times m^{(a-1)/2} \times (1/m^{a/3})(1/x^{p/6})}{(1/m^{1/2})x^{1/2p}}$$

$$= \frac{2^{1/2}m^{(a-1)/2}m^{1/2}}{x^{3p/2} \times x^{1/2p} \times x^{p/6} \times m^{a/3}}$$

$$= \frac{\sqrt{2} \times m^{\frac{a-1}{2}+\frac{1}{2}-\frac{a}{3}}}{x^{\frac{3p}{2}+\frac{1}{2p}+\frac{p}{6}}} = \frac{\sqrt{2}m^{a/6}}{x^{\frac{10p^2+3}{6p}}}.$$

This may be written : $$\frac{\sqrt{2}\sqrt[6]{m^a}}{\sqrt[6p]{x^{10p^2+3}}}.$$

Try now the following exercises. They are not important in themselves, but they will help you to get quite familiar with all sorts of indices.

Exercises I. (See p. 244 for Answers.)

Express with positive indices and simplify:

(1) a^{-1}. (2) x^{-a}. (3) $2m^{-2}$. (4) $\frac{1}{3}ax^{-1}$.
(5) $2^{-1}x$. (6) $(\frac{1}{2})^{-1}a^3$. (7) $(3^{-2}x)^2$. (8) $(2a^{-1})^3$.
(9) $\dfrac{x^{-1}}{4^{-2}}$. (10) $2^{-3}a^{-x}$. (11) $(3a^2)^{-1}$.
(12) $2^{-1}a^x/a^{-3}$. (13) $a^{1/2}$. (14) $8^{1/3}a^2$.
(15) $2a/a^{1/3}$. (16) $4^{1/2}x^{-1}/3x^{1/2}$. (17) $3^2a^{-1/2}$.
(18) $2x^{-2/3}$. (19) $4^{-1/2}a^{-x/2}$. (20) $2^{-1}a^2/a^{-x}$
(21) $(\frac{1}{5})^{-1}x^{1/3}$. (22) $3m^{-1/5}$. (23) $7x^0a^{-1}$.
(24) 12^0x^{-1}. (25) $3^{-1}x^2a^{-x}$. (26) $3^{1/2}/x^{-3}$.
(27) $2^{-2}/a^mx^{-3}$. (28) $2a^2/a^{-3}x^{-1}$.
(29) $(2^{-1}a^xm^{-2})^0$. (30) $(3^{-2}a^{1/2})^{-1}$.
(31) $3(2^{-1}a^{-x})^{-2}$. (32) $(3^{-1}/x^{2m})^{-a}$.
(33) $4^0x^{-3}a^{1/2}/ax^{1/3}$. (34) $4^{1/2}x^{-2}(3x^2-2x^{-1/3})$.
(35) $a^{x+1}x^{-a}/a^{-2}x^{2a}(3ax)^0$.
(36) $3^xa^{-a}(3^{-2}a^x+3^{-1}a^{-3/2})$.
(37) $\sqrt[3]{(m^2a^{-3})}\div a^2\sqrt{m^{-3}}$.
(38) $m^{-2}k^{x+1}\div 3k^{x-1}(mk)^{1/2}$.
(39) $2x^{-2a}\div(x^{3a})^{1/2}$. (40) $[(a^{-3})^x\div(a^{-x})^{3/2}]^{1/2}$.
(41) $(3a^{-2}/x^{2/3})\div(a^{3/2}/2x^{-3})$.
(42) $(2x^{-1/3})^{-2/a}\div(2^{-a}x^{a/3})^{1/2}$.

CHAPTER III.

EXPONENTIALS, AND HOW TO TAME THEM.

N exponential function—also called simply an " exonential "—is simply an expression in which one of ie variable quantities, usually the independent one, ppears in the index or exponent of some power of nother quantity ;

$$y=5^x, \quad y=a^{3x}, \quad y=(a-1)^{1/x}, \quad y=k^{-x}+x^{-k},$$

re explicit exponential functions of x, in which x is ie independent and y the dependent variable, as it evident that if x is given various values, y will ike corresponding values. The above expressions are ead : " five to the power x," " a to the power x," " a iinus one to the power one over x," " k to the power iinus x plus x to the power minus k " respectively, and o on.

Exponentials are not quite so easy to deal with as ther expressions simply because if we are told that n unknown quantity x is to be raised to a *known* power, quare or cube, say, we know exactly what to do with his unknown quantity, as $x^2=x\times x$, $x^3=x\times x\times x$, and o on ; but if we are told that a known quantity has to be raised to an *unknown* power, there is no way of expressing the question in a definite manner, and 3^x, say, has to remain 3^x. If we have $2=x^3$ we have at once $x=\sqrt[3]{2}$ and the value of x is obtainable at once; but if we have $2=3^x$, and try to proceed along similar lines, we get $\sqrt[x]{2}=3$, and x has only been shifted to an even more awkward place.

It is, however, possible to bring an exponential function to a simpler algebraical form by the use of logarithms. Nowadays, every schoolboy knows how to use a table of logarithms, and he knows that the logarithm of the power of a number is found by multiplying the logarithm of the number by the index of the power, so that $\log(3^x)=x\times\log 3$, the exponential becoming an ordinary product, since $\log 3=0{\cdot}4771$, a mere number.

If $2=3^x$ then $\log 2=x\times\log 3$ or $0{\cdot}3010=0{\cdot}4771\times x$, and $x=\frac{0{\cdot}3010}{0{\cdot}4771}$. We can use logarithms to perform the division in the usual way, so that

$$\log x=\log 0{\cdot}3010-\log 0{\cdot}4771 \quad \text{and} \quad x=0{\cdot}6310.$$

"What! These are *already* logarithms!" I hear you exclaim. "Shall we take the logarithm of a logarithm?" Why not? A logarithm is only a number. Treat it as a number and go ahead! In mathematics rules are general.

Whenever we have an exponential function we can always state it as an expression containing logarithms and this will generally be found easy to deal with

$y=5^x$ becomes $\log y=x\times\log 5$, that is, $\log y=0{\cdot}699x$; $y=a^x$ becomes $\log y=x\times\log a$; $y=(a-1)^{1/x}$ becomes $\log y=\frac{1}{x}\log(a-1)$ and so on. As will be seen in the examples below, a wicked-looking exponential often becomes most tame at the mere sight of a logarithmic table.

It may be stated here that logarithms are closely related to exponentials, for if we write $10^x=3$, say, then x is, by definition, the common logarithm of 3. That is, the common logarithm of any number is merely the index indicating to which power the number 10 must be raised in order to obtain the first number. For instance, the common logarithm of 7·2 is the value of x for which $10^x=7{\cdot}2$. It follows that since $10^1=10$, the common logarithm of 10 is unity.

That, if $10^x=7{\cdot}2$, then $x=\log 7{\cdot}2$, is evident, for, since the expression $10^x=7{\cdot}2$ is an exponential, we have, from what we have seen above, $x\times\log 10=\log 7{\cdot}2$, and as $\log 10=1$, $x=\log 7{\cdot}2$. Similarly, if $0{\cdot}00183=10^m$, then $m=\log 0{\cdot}00183$. In fact, whenever we are given such an expression as $10^k=N$, we can always write at once $k=\log N$.

You will notice that the number which is raised to the power x, or m, or k, is always 10. 10 is selected because it is the basis of our system of numeration, and the logarithms used in connection with it are therefore called "common logarithms," 10 being called the "base" of the system of common logarithms. These are the ones given in any ordinary table of logarithms.

In such a table, the logarithm of 10 will be found to be 1, that is, to have 0·0000 for its decimal part or mantissa.

Instead of 10 we could have any other constant number. For instance, if $7^x=13$, then x is the logarithm of 13 in a system of logarithms the base of which is 7; to avoid this long sentence, we use the notation $x=\log_7 13$ the number 7 placed in this way after the name logarithm (or its abbreviation) means that the base of the system is 7. We should have written above $\log_{10} 7{\cdot}2$ for "the common logarithm of 7·2," but, for common logarithms, it is understood that the number 10 needs not be appended to the abbreviation "log." Similarly $k=\log_a N$ means: "k is the logarithm of N in a system the base of which is a, that is, it is the same statement exactly as $a^k=N$. The fact that these two statements $k=\log_a N$ and $a^k=N$ are always simultaneous, so that one necessarily implies the other, is absolutely general, and is of great importance in dealing with exponentials. It holds good whatever are the symbols used. For instance, if

$$(1+x)^{\sin 2\theta/\sqrt{\theta}}=x^\theta/\sqrt{(\tan\theta)},$$

then $\sin 2\theta/\sqrt{\theta}$ is the logarithm of $x^\theta/\sqrt{(\tan\theta)}$ in a system the base of which is $(1+x)$, that is:

$$\sin 2\theta/\sqrt{\theta}=\log_{(1+x)}(x^\theta/\sqrt{(\tan\theta)}).$$

Or, to be again incongruous, if $\text{cat}^{\text{cow}}=\text{dog}$, then $\text{cow}=\log_{\text{cat}}\text{dog}$.

It follows that, in every system of logarithms, since $1=(\text{base})^0$, $\log 1=0$, whatever the base may be. Also the logarithm of a number smaller than 1 is negative if the ase is greater than 1, since, if $N<1$ and $a>1$, $N=a^x$ ecessitates x to be negative, so that $N=\frac{1}{a^x}$, where x is positive power of a number greater than 1. A system f logarithms can then be conceived the base of which any number we like. Such a system could be used for alculations just like common logarithms, provided we ave first calculated a table of logarithms in this system. here would be, however, no particular advantages in sing such a system, and some disadvantages, and n practice common logarithms are always used for alculating.

There is, however, another system of logarithms, of ven greater importance in mathematics than the ystem of logarithms in the base 10. Its base, strangely nough, is not an easy whole number, but an awkward ndless decimal: 2·7182818284596...; note that it is ot a recurring decimal, as one might think from a glance t the first nine decimals. This number occurs so requently in mathematics that it is represented by he Greek letter "epsilon," ϵ, just as the number ·1415926535 ... is represented by the Greek letter Pi," π.[1] Why this particular number was selected we hall see later. Logarithms in this system are called apierian logarithms, from the name of the mathema-ician John Napier, who is generally credited with their nvention. They are also called Natural or Hyperbolic ogarithms, for reasons which we shall soon under-tand.

[1] In many text-books the letter e is used instead of ϵ.

The Natural logarithm m of a number N is therefore represented by $m = \log_\epsilon N$, and, as we have seen, this is equivalent to $\epsilon^m = N$, ϵ standing for 2·718, neglecting the other decimals.

Of course, Napierian logarithms can be used for calculations just like common logarithms, or logarithms in any other system. They have, besides, many important properties with which we shall become better acquainted later on.

When the common and Napierian logarithms are used together, the common logarithms should be denoted fully as shown above, $\log_{10} N$, say, to avoid confusion. In calculating, when only common logarithms are used, the notation may be simplified by omitting the suffix 10, so that $\log N$ means the same as $\log_{10} N$. The Napierian logarithms are of such importance in mathematics, however, that whenever the notation $\log N$ is employed without a suffix, except in actual calculation, the Napierian logarithm is always intended.

The Napierian logarithm of any number can easily be calculated from its common logarithm, as follows: suppose we want $\log_\epsilon 4{\cdot}8$; if $x = \log_\epsilon 4{\cdot}8$, then, $\epsilon^x = 4{\cdot}8$ or $2{\cdot}718^x = 4{\cdot}8$, that is, $x \times \log_{10} 2{\cdot}718 = \log_{10} 4{\cdot}8$. But $\log_{10} 2{\cdot}718 = 0{\cdot}4343$—an easy number to remember—so that $0{\cdot}4343x = \log_{10} 4{\cdot}8$ and $x = \log_\epsilon 4{\cdot}8 = 0{\cdot}6812/0{\cdot}4343 = 1{\cdot}5686$. As multiplication is quicker than division, and since $1/0{\cdot}4343 = 2{\cdot}3026$, the same result can be more readily obtained by performing the operation $2{\cdot}3026 \times 0{\cdot}6812 = 1{\cdot}5686$. Hence the familiar rule: to get the Napierian logarithm of a number, multiply the ommon logarithm of the number by 2·3026. Inversely, $\log_{10} N = 0·4343 \times \log_\epsilon N$, so that if $\log_\epsilon 4·8 = 1·5686$ be iven, then $\log_{10} 4·8 = 0·4343 \times 1·5686 = 0·6812$.

You are advised to work through the following xamples so as to become quite familiar with the process f reducing exponentials to a harmless condition.

Example 1. Given $\epsilon = 2·718$, find $\log_\epsilon 13·2$.

If $x = \log_\epsilon 13·2$, then $\epsilon^x = 13·2$ or $2·718^x = 13·2$.

Hence $\quad x \times \log 2·718 = \log 13·2$,

$$x = \log 13·2/\log 2·718 = 1·1206/0·4343 = 2·580;$$

ence $\quad \log_\epsilon 13·2 = 2·580$.

Example 2. Find x if $0·31^x = 0·0048$.

We have $\quad x \times \log 0·31 = \log 0·0048$.

$$x \times \bar{1}·4914 = \bar{3}·6812.$$

$\bar{1}·4914$ may be written $-1 + 0·4914$, that is, $-0·5086$; imilarly, $\bar{3}·6812 = -3 + 0·6812 = -2·3188$, so that

$$x = -2·3188/-0·5086 = 4·559.$$

Example 3. Find x if $3^x = 7$, and hence find $\log_3 7$.

We get $x \times \log_{10} 3 = \log_{10} 7$, $\quad x = 0·8451/0·4771 = 1·772$.

Hence, since $\quad 3^{1·772} = 7, \quad 1·772 = \log_3 7$.

Example 4. Solve the equation $1·5^{(x+1)} = 2·4$.

We get $\quad (x+1) \log 1·5 = \log 2·4$

or
$$x + 1 = \log 2·4/\log 1·5$$
$$= 0·3802/0·1761 = 2·16.$$

Hence $x = 1·16$.

Example 5. If $1{\cdot}46^{3\theta^2}=12$, find the value of θ. We ge

$$3\theta^2\times\log 1{\cdot}46=\log 12,$$

$$\theta^2=\log 12/3\log 1{\cdot}46=1{\cdot}0792/3\times 0{\cdot}1644=2{\cdot}19$$

and $$\theta=1{\cdot}48.$$

Try now the following exercises :

Exercise II. (For Answers see p. 244.)

(1) Find the value of x if $12=5^x$.

(2) Find the value of x if $3=1{\cdot}5^x$.

(3) Find the value of y if $3^y=2^{(y+3)}$.

(4) Find the value of m if $7=2^{m+1}$.

(5) Find x and y, if $y=3x-1$ and $1{\cdot}8^y=5{\cdot}3^x$.

(6) Find k if $3{\cdot}45=1{\cdot}18^{k^2}$.

(7) If $1{\cdot}5^{-t}=0{\cdot}2$ find t.

(8) Solve the equation $146=5{\cdot}2^{1/x}$.

(9) If $3\times 4{\cdot}3^{-x}=1$, find x.

(10) If $3{\cdot}2^m/5{\cdot}7^n=1$, and $m+n=3$, find m and n.

(11) Find x if $12{\cdot}4^{1/x}=1{\cdot}6^{2x^2}$.

(12) Solve the equation $0{\cdot}4^{(x-1)}=1{\cdot}2^{-1/x}$.

(13) If $y^x=x^{(\log y+1)}$, find y when $x=1{\cdot}72$, and als when $\log_{10}x=\frac{x}{1000}$.

(14) Find the angle θ if $(3/8)^{\sin\theta}-2=0$.

(15) If $7{\cdot}42^{\frac{3\log k}{x}}=10$ and $k^x=100$, find k and x.

(16) Given $\epsilon=2{\cdot}718$, calculate $\log_\epsilon 2$, $\log_\epsilon 5$, $\log_\epsilon 1$ and verify that $\log_\epsilon 10=\log_\epsilon 2+\log_\epsilon 5$.

(17) Given $\epsilon=2{\cdot}718$, calculate $\log_\epsilon 3{\cdot}2$ and $\log_\epsilon 0{\cdot}11$.

(18) Calculate $\frac{74 \cdot 3 \times 1 \cdot 808}{10 \cdot 95}$, using Napierian logarithms alculate the logarithms if no table is available).

(19) From $10=5^x$ calculate x, and hence find the value the logarithm of 10 in the system the base of which 5.

(20) Calculate $x=\log_{10}6/\log_6 10$.

(21) From $\log_\epsilon 3=1{\cdot}0986$ derive the value of ϵ.

(22) Find the base of the system of logarithms in ich the logarithm of 24·8 is 0·8.

(23) Calculate $\log_7 3$, $\log_7 4$, $\log_7 9$, $\log_7 12$ and $\log_7 27$, d with these verify experimentally that a system of garithms to the base 7 can be used exactly in the me way as common logarithms to calculate products ×4), quotients (27÷9), powers (3^2) and roots ($\sqrt[3]{27}$).

(24) In what system is the number 5 equal to its own garithm ?

(25) In what system of logarithms is the number 100 ual to 20 times its own logarithm ?

(26) Calculate $1{\cdot}5^3$ using logarithms whose system has for base.

(27) Solve the equation $21{\cdot}7^{\left(\frac{x^2}{3}-1\right)}=0{\cdot}4^{(3x+2)}$.

(28) If $y^{-3x}=y\times 0{\cdot}3^{1/3x}$ find the value of y when $=11{\cdot}1$ and the value of x when $y=0{\cdot}00111$.

CHAPTER IV.

A WORD ABOUT TABLES OF LOGARITHMS.

CONSIDER the three lines below :

	$\ldots\epsilon^{-3}=1/\epsilon^3$	$\epsilon^{-2}=1/\epsilon^2$	$\epsilon^{-1}=1/\epsilon$	ϵ^0	ϵ^1	ϵ^2	$\epsilon^3\ldots.$
Numbers	…0·0498	0·1354	0·3679	1	2·7183	7·3876	20·0793…
Indices = Logarithms	…−3	−2	−1	0	1	2	3….

The second line gives the numerical value of tl terms in the first line ; the third line consists of tl indices of ϵ in the first line, that is, of the Napieria logarithms of the numbers in the second line. Tl second and third line together constitute, in fact, small bit of a table of Napierian logarithms, only v have but powers of epsilon among the numbers, and t natural sequence of numbers among the logarithms.

Note that, in the second line, each number is obtain by multiplying the number immediately to the l of it by a constant number, here epsilon ; such sequence of numbers is called a geometrical progressi Note also that, in the third line, each number (here logarithm) is obtained by adding the same number this case unity) to the number immediately to the l of it ; such a sequence of numbers is called an arithmetical progression.

Well, in any system of logarithms, whatever may be the base, we shall always find these features :

(1) The sequence of numbers form a geometrical progression ;

(2) The sequence of logarithms form an arithmetical progression ;

(3) The term of the former corresponding to zero in the latter is unity ; (base $^0 = 1$.)

(4) The term of the former corresponding to unity in the latter is the base of the system of logarithms itself.

Whenever these conditions are fulfilled, the two progressions form a system of logarithms.

In the bit of Napierian Table above, we only have powers of epsilon and the sequence of natural numbers. The gaps can be filled up easily if we keep in view the two first conditions stated above. For instance, if we want to place a number between ϵ^2 and ϵ^3, if x is the constant factor by which each term of the *new* geometrical progression is to be multiplied in order to get the one immediately following, we must have :

$$\epsilon^2 \times x = N \quad \text{and} \quad N \times x = \epsilon^3 \quad \text{or} \quad N = \epsilon^3/x,$$

hence $\epsilon^2 \times x = \epsilon^3/x, \quad x^2 = \epsilon, \quad \sqrt{\epsilon} = 1{\cdot}6487$

and $N = \epsilon^2 \times 1{\cdot}6487 = 12{\cdot}1850.$

Its logarithm is evidently 2·5.

Similarly, putting a number between ϵ^0 and ϵ^1 and between ϵ^1 and ϵ^2, we get $\log 1{\cdot}6487 = 0{\cdot}5000$ and $\log 4{\cdot}4817 = 1{\cdot}5000$. We can in this way put a number and a logarithm in the middle of each gap of our bit of table, then in the middle of each gap of the table so obtained, and so on, until we get a table of numbers advancing by such a small step each time that it will contain the sequence of the natural numbers.

Clearly this is not practical. The method has only been outlined to illustrate an important difference between Common and Napierian logarithms.

If we deal as explained above with common logarithms, we have

	... 10^{-3}	10^{-2}	10^{-1}	10^0	10^1	10^2	10^3....
Numbers	... 0·001	0·01	0·1	1	10	100	1000....
Indices = Logarithms	... -3	-2	-1	0	1	2	3....

Introducing a number in each gap by the method explained above, we get $\sqrt{10} = 3{\cdot}1623$ for the constant factor.

Numbers									
... 0·01	0·316228	0·1	0·316228	1	3·16228	10	31·6228	100....
Logarithms									
... -2	$-1{\cdot}5$	-1	$-0{\cdot}5$	0	0·5	1	1·5	2....

which may be written

... $\bar{2}{\cdot}0000$ $\bar{2}{\cdot}5000$ $\bar{1}{\cdot}0000$ $\bar{1}{\cdot}5000$ 0 0·5000 1·0000 1·5000 2·0000....

where the figures under the minus sign are negative and the decimal parts (mantissae) are positive.

A fact is evident at first sight: the numbers 0·0316228 0·316228, 3·16228, 31·6228 ..., which only differ by the position of the decimal point, have the same decimal namely, 0·5000, in their logarithms, the integer alone of the latter being different. This characteristic is moreover easily found, as every schoolboy knows, being positive and one unit less than the number of integers if the number is greater than unity, and negative and numerically equal to unity added to the number of noughts immediately on the right of the decimal point if the number is smaller than unity. It follows that, in a system of common logarithms, we need not tabulate the logarithms of all the numbers, but only the decimals of the logarithms of the numbers from say 1000 to 10000, or 10000 to 100000, according to the size of the table. If we have in the table the decimal of the logarithm of 76835, for instance, we can get the logarithm of any number consisting of these figures, whether it be 76,835,000,000 or 0·00000076835 ; it is only a matter of giving the proper characteristic to the tabulated decimal.

Nothing like this is to be found in the system of Napierian logarithms. We have seen that

$$\log_e 12{\cdot}1850 = 2{\cdot}5000$$
$$\log_e 4{\cdot}4817 = 1{\cdot}5000$$
$$\log_e 1{\cdot}6487 = 0{\cdot}5000$$

and it is evident that no such single relation exists. It follows that in a table of Napierian logarithms there must be as many logarithms as there are numbers, from the greatest integer imaginable down to the smallest decimal fraction we can think of. For this reason, Napierian logarithms are not used for performing calculations, as it would be impossible to make a complete table, and even for a restricted range of numbers consistent with general usefulness, its size would be prohibitive. The great importance of Napierian logarithms resides in their intimate connection with important series and mathematical functions which causes them to appear in many mathematical investigations, as we shall see later.

CHAPTER V.

A LITTLE CHAT ABOUT THE RADIAN.

In the pages which follow we shall often deal with angles, and it is necessary that you should be quite familiar with the measure used by mathematicians when they want to ascertain the magnitude of an angle. This measure is the *radian*. You are accustomed to form an idea of the magnitude of an angle by stating how many *degrees* and fractions of a degree—minutes and seconds—it contains. These units are called "sexagesimal" units, because the principal unit, the degree—which is defined as the $\frac{1}{90}$ part of a right angle, is subdivided into sixty—six times ten—equal parts or "minutes"—*i.e.* "smalls," each minute being divided into sixty "second minutes," as they were originally called, that is, "smalls of the second order," later called "seconds" for shortness; the seconds are subdivided decimally. It may be noted here that minutes and seconds of arc should always be denoted by the symbols $'$ and $''$ respectively, *never* by m and s, these being used for minutes and seconds *of time*.

Now, whenever we want to combine together several quantities of the same kind, it is convenient—and often even necessary—that they should be expressed in terms of the same unit. For instance, when finding the area of a rectangular room, the length and breadth must both be expressed in feet or yards or some other unit; similarly, when finding the capacity of a cylindrical vessel, the height and the radius must both be expressed in feet or inches or any other suitable unit. Now, in trigonometry, we continually use such ratios as $\sin\theta=\frac{BN}{OB}$, $\tan\theta=\frac{AP}{OA}$ (Fig. 1), etc., and $\frac{BN}{OB}$, $\frac{AP}{OA}$, etc., are merely the measure of the length of BN, AP, etc., when OB or OA, namely the *radius* of the arc BQA, is used as a unit. In other words, when we deal with such ratios—which we call "circular functions" because their value depends on the magnitude of lines drawn in a certain definite way with respect to the circumference of a circle—the unit of length which we use is the radius. Now, it is not only logical, but essential to correct mathematical reasoning and to the results derived from it, that the same unit should be used for measuring the length of all the lines connected with the given angle θ, and among these is the arc BQA itself.

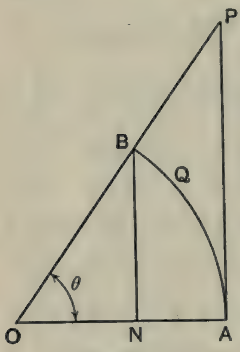


FIG. 1.

How can we measure the length of this arc, since it is a curved line? Very easily, if we remember that the length of any arc is exactly proportional to the magnitude of the angle it subtends at the centre of the circle. Suppose, for instance, that we have ascertained the magnitude of the angle subtended at the centre of the circle by an arc of the circumference the length of which is exactly the same as the length of the radius of the circle, that is, by an arc of unit length. Then, as many times as this angle is contained in any given angle, as many times the length of the unit of arc will be contained in the length of the arc corresponding to the given angle. This angle, corresponding to unit arc, is taken as unit angle and is called the *radian*. The radian is called the "circular unit" of angular measurement because it is derived from the measurement of an arc of circle. It is divided decimally.

Mathematicians *always* express angles in radians, so that it is superfluous to note the unit. By "an angle θ," they mean always "an angle of θ radians," and this is the same as

$$\frac{\text{length of arc corresponding to angle } \theta}{\text{length of radius of the circumference to which this arc belongs}},$$

so that $\theta = \dfrac{AQB}{OA}$ (see Fig. 1) or, if l be the length of an arc and r be the length of the radius of the circle to which it belongs, while θ is the angle subtended at the centre of the circle by the arc l, then $\theta = \dfrac{l}{r}$ or $l = r\theta$, the angle θ being expressed in radians.

It follows that an angle of four right angles $= \dfrac{2\pi r}{r} = 2\pi$ radians, while an angle of two right angles is an angle of π radians; likewise, one right angle is an angle of $\frac{\pi}{2}$ radians. The notations 2π, π, and $\frac{\pi}{2}$, the radian being the unit implied, of course, are therefore used instead of 360°, 180° and 90° respectively.

It is worth while noticing here that $\sin\theta$, $\tan\theta$, etc., retain exactly the same value, whether θ is expressed in radians or in sexagesimal units, as this value depends solely on the actual magnitude of the angle θ, and, being a *ratio* of two lines, is quite independent of the unit employed in measuring these lines or the angle itself. It follows that the tables of trigonometrical functions, which give the values of $\sin\theta$, $\tan\theta$, etc., and in which the angles are usually expressed in sexagesimal units, degrees and minutes, can be used with angles expressed in circular units provided one can readily pass from one system of units to the other. Now, this is very easily done, as we have just seen that 4 right angles $=360°=2\pi$ or 6·283184 ... radians, from which we find at once that 1 radian $=\dfrac{360}{6{\cdot}283184\ldots}$ degrees or 57°·29577 ... that is, 57°·30 approximately.

The use of the radian as a unit of angle simplifies considerably all problems involving the length of an arc. For instance, let it be required to find the length of an arc of the circumference of a circle of radius $7\frac{1}{4}$ inch, corresponding to an angle θ. If we first suppose the angle θ to be given in sexagesimal units, say 41° 15′, then

$$\frac{\text{length of circumference}}{\text{length of arc}}=\frac{360°}{41°\ 15'},$$

that is $$\frac{2\pi r}{l}=\frac{360}{41{\cdot}25},$$

hence $$l=\frac{6{\cdot}2832\times7{\cdot}25\times41{\cdot}25}{360}=5{\cdot}22 \text{ inches.}$$

The operation by which the result is obtained involves four numbers.

Now suppose the angle θ to be given in circular units, say 0·72 radians; then $l=\theta r=0{\cdot}72\times7{\cdot}25=5{\cdot}22$ inches. The simplification is so great that it is often quicker to convert angles expressed in term of the degree and its subdivisions into radians before proceeding with the required calculations. There are tables which allow of this conversion being made by mere inspection.*

The following worked examples will help to clear any haziness yet lingering in the beginner's mind.

Example 1. Express in radians an angle of 68° 26′, and express in sexagesimal units an angle of 0·36 radian. Find in each case the length of the subtending arc on a circumference of 20 inches radius.

$$(a)\ 68^\circ\ 26'=68^\circ+26^\circ/60=68^\circ{\cdot}4333.$$

Since 1 radian$=57^\circ{\cdot}30$, the number of radians in the angle is $68{\cdot}4333/57{\cdot}30=1{\cdot}1943$ radian, hence the length of the arc is $1{\cdot}1943\times20=23{\cdot}886$ inches. (The more exact value 57·29577 gives 1·1944 radian.)

* Such a table is given in Cargill G. Knott's *Four-Figure Mathematical Tables* (W. & R. Chambers, Ltd.). This cheap little book of tables contains also, besides the usual tables, tables of exponential and hyperbolic functions, which the reader will find very useful.

(*b*) 0·86 radian $=0{\cdot}86\times57{\cdot}30$ degrees

$$=49^\circ{\cdot}278=49^\circ\ 16'\ 40''{\cdot}8.$$

The length of the arc is $0{\cdot}86\times20=17{\cdot}2$ inches. (The more exact value 57·29577 gives $49^\circ\ 16'\ 27''{\cdot}7$.)

Example 2. On a circumference of radius 3 ft. 3 in., an arc is taken equal in length to that of an arc of 38° on a circumference of 5 ft. 6 in. radius. Find the angle subtended at the centre by the arc taken on the first circumference.

The simplest way to do this is as follows:

Length of arc of second circumference

$$=38\times5{\cdot}5/57{\cdot}30=3{\cdot}648 \text{ ft.}$$

This is also the length of the arc on the first circumference, hence the angle in radians subtended at the centre is $3{\cdot}648/3{\cdot}25=1{\cdot}123$ radian, that is

$$1{\cdot}123\times57{\cdot}30=64^\circ{\cdot}348=64^\circ\ 20'\ 52''{\cdot}8.$$

Example 3. What is the radius of the circumference on which a length of 10 inches subtends at the centre an angle of 153°?

$$153^\circ=153/57{\cdot}30 \text{ radians}=2{\cdot}67 \text{ radians.}$$

Hence $2{\cdot}67\times$radius $=10$ inches and

$$\text{radius}=10/2{\cdot}67=3{\cdot}745 \text{ inches.}$$

Example 4. Find the value of $y=\sin\theta+3\theta$ when $\theta=20^\circ$. Remember that although, if convenient, one can take the angle in degrees when using the trigonometrical tables, yet in any other case the angle is *always in radians*, so that we have:

$$y=\sin 20^\circ+3\times20/57{\cdot}30=0{\cdot}3420+1{\cdot}0472=1{\cdot}3892.$$

Example 5. A cyclist riding on a circular track could ach the centre of the track in $2^m\ 17^s$ at the speed at ich he follows the track. What angle does he turn rough in 5 minutes ?

Expressing the times in seconds, the length of the arc, radians, is evidently

$\frac{5\times 60}{137}=2{\cdot}190$ radian $=2{\cdot}190\times 57^\circ{\cdot}30=125^\circ\ 29'\ 13''{\cdot}2.$

By working through the following exercises you will alize the simplification brought about by the use of e radian.

Exercises III. (For Answers, see p. 245.)

(1) Find the length of an arc of 0·6 radian in a circle 7·3 inch radius.

(2) Express 71° 15′ in radians and 0·715 radians in xagesimal units.

(3) Express in radians angles of 1°, 1′, 1″ respecvely, and find the length of the corresponding arcs in circle of 1 foot radius. (Take 1 radian $=57^\circ{\cdot}296$.)

(4) If the smallest subdivision possible in the graduaon of a protractor is 0·01 inch, find the radius of the nallest protractor to read (*a*) to degrees, (*b*) to minutes, ıd (*c*) to seconds of arc.

(5) One angle of a triangle is 0·576 radian and ıother angle is 79° 34′. Find the third angle in circular ıd in sexagesimal units.

(6) Find the angle, in radian, at the centre of a rcle of 8 inches radius, corresponding to an arc 5·6 inch length.

(7) Find the radius of the circle in which an angle 1·2 radian at the centre subtends an arc the length which is 10 inches.

(8) Find the radius of the circle in which an ang of 3° is subtended at the centre by an arc equal in leng to an arc of 48° 12′ on a circumference of radius 3 feet

(9) Find the value of $x=\frac{3\theta}{\sin\theta}$ and of $y=\theta^2\sqrt{\tan$

with $\theta=50°$. (Do *not* replace 3θ by 150 !)

(10) Find the value of

$$y=\frac{\sin^2\theta}{\theta+1}+\frac{\sqrt{3\theta}}{\tan(\theta+\frac{1}{2})},$$

with $\theta=42°$.

(11) Find in sexagesimal units the value of the angle if $3^\omega=5$.

(12) The coordinates x, y of a point on the cycloid a given by the formulae $x=R(\theta-\sin\theta)$, $y=R(1-\cos$ where θ is the angle turned through by the generati circle. Find the value of x and y when $R=10$ incl and $\theta=45°$; hence find the distance of the correspond point of the curve from the point for which $\theta=0°$, origin of the curve.

CHAPTER VI.

'PREADING OUT ALGEBRAICAL EXPRESSIONS.

'HEN a complicated thing is difficult to understand, it ın often be grasped if it can be taken to pieces, so to)eak, and each piece examined separately. Obviously, l the little pieces together occupy more space than the riginal thing they made up, which has been *expanded* ı the process. Similarly, a great many mathematical xpressions which are found too difficult to deal with s they are can be quite easily tackled by splitting them own into many smaller bits, the sum of which makes p the original expression exactly, or to a given pproximation which usually can be made as close as ne cares to have it. This process is called *expanding*, nd the result of the process is called the *expansion* of he expression. This process of expanding an expression s so important that there is hardly any mathematical nalysis of some importance in which one does not esort to it.

It is a curious fact that beginners are very much fraid of expansions. They look in dismay at the rray of terms, and foolishly think that their number vill make the whole thing absolutely unmanageable, when, as a matter of fact, it is exactly the reverse. On cannot swallow a plum pudding without cutting it int slices, and the slices into spoonfuls, but once the sub division has been performed, see how easily it disappears The difficulty is absolutely imaginary, as you will agre yourself when we get to the end of the chapter.

First of all, let us remember two expansions you kno already; they are short ones:

$$(a+x)^2=a^2+2ax+x^2.$$

$$(a+x)^3=a^3+3a^2x+3ax^2+x^3.$$

Note that we arrange the terms in the expansion i a definite order: the first term does not contain x—w can say it contains x to the power zero, since

$$a^2=a^2\times 1=a^2\times x^0,$$

and similarly $a^3=a^3\times x^0$;

the second term contains x to the power 1, or x merel the third term contains x to the power two, or x^2, an so on, the index of x in successive terms increasing b unity each time.

Did you ever wonder what $(a+x)^4$ would be like It is easy to calculate, it is $(a+x)^3\times(a+x)$. Performir the operation in the ordinary way, we get

$$(a+x)^4=a^4+4a^3x+6a^2x^2+4ax^3+x^4.$$

Similarly,

$$(a+x)^5=(a+x)^4\times(a+x)$$

or $(a+x)^5=a^5+5a^4x+10a^3x^2+10a^2x^3+5ax^4+x^5$

Note also that the terms become more and mo numerous as we expand higher powers of $(a+x)$. I ch expansion, the number of terms is one more than e index, and we shall find this is always the case as ng as the index is a positive integer. Note again at these terms are perfectly simple terms like those alt with at school by small boys just beginning gebra. In other words, each power of the quantity thin the brackets has been split into many simple tle terms. The higher the power—and therefore the ore complicated the expression—the greater the mber of terms, but these terms remain quite simple : complexity is introduced by their greater number.

However, their number might be a source of trouble we had a great many, but—and this is the beauty of is process—the expansion of functions is usually done in ses where the terms get gradually smaller and smaller, that after a few terms, sometimes as few as three, even two, all the following terms can be entirely nored, as they are too small to affect the result appreci- ly. It follows that the useful expansion of a function generally limited to a very few quite simple terms, that the fact that the actual number of terms is very eat does not matter in the least. In most cases, in fact, e number of terms is indefinitely large.

Let us make sure that we understand this most im- rtant feature of the use of expansions by an example : us return to the expansion of

$$(a+x)^5=a^5+5a^4x+10a^3x^2+10a^2x^3+5ax^4+x^5.$$

Suppose x is very small compared to a, for instance ppose $a=1$ and $x=0{\cdot}01$.

Then to find the value of $(a+x)^5$ we have $1{\cdot}01^5$ to calculate. It is easy—but tedious—to calculate this by ordinary calculation; we get 1·0510100501 with 10 places of decimals. But most probably we did not want 10 places of decimals, so a portion of the labour is wasted. Suppose we only wanted three decimals and that we have no table of logarithms at hand. Since the expansion is equivalent to the given expression, we may use the expansion, by replacing in it a by 1 and x by 0·01 respectively. We get

$$1^5+5\times1^4\times0{\cdot}01+10\times1^3\times0{\cdot}01^2+10\times1^2\times0{\cdot}01^3+5\times1\times0{\cdot}01^4+0{\cdot}01^5$$

or $1+0{\cdot}05+0{\cdot}001+0{\cdot}00001+0{\cdot}00000005+0{\cdot}0000000001$

Note how the terms are rapidly dwindling down to a negligible value. Since we only want three places of decimals, we can neglect all the terms after the third and write, using the sign $\lessgtr$ instead of $=$ to mean "approximately equal to":

$$(1{\cdot}01)^5 \lessgtr 1+0{\cdot}05+0{\cdot}001 \quad\text{or}\quad (1{\cdot}01)^5 \lessgtr 1{\cdot}051.$$

Had we wanted $(1{\cdot}01)^{23}$ correct to three places of decimals, the expansion would have contained $23+1$ = 24 terms, but we should not even need to write them down, for we would need only the three or four first ones.

Yes, you will say, but how shall we get the three or four first ones, for $(1{\cdot}01)^{23}$ or $(a+x)^{23}$ is obtained from multiplying $(1{\cdot}01)^{22}\times1{\cdot}01$ or $(a+x)^{22}\times(a+x)$, so we need first calculate the 23 terms in the expansion of $(1{\cdot}01)^{22}$ or $(a+x)^{22}$.

If it were so, the prospect before us would not be a ·ight one, but, luckily, it is not so. There is a most ·autiful law that runs through the whole domain of athematics—a law called the Principle of Mathematical ıduction, which is this : If a certain process is applied · a certain quantity, and yields a certain result, and exactly the same process, applied to several graduated ight modifications of that quantity, yields results hich are gradually modified in some regular manner, ıen, as the quantity is further modified in exactly the ıme way, the results obtained by submitting it to this ıme process will continue to be modified according to ıe regular manner which has become evident in the ırlier stages, however far the modifications of the uantity are pushed.

Let us take an easy example to illustrate this ;atement.

Consider the number of integers (that is, of whole umbers) of one figure, of two figures, of three, four, ve figures, etc.

We can easily form the following little table :

Number of integers of	1 figure :	9 or 9×10^0.
,, ,,	2 figures :	90 or 9×10^1.
,, ,,	3 ,,	900 or 9×10^2.
,, ,,	4 ,,	9000 or 9×10^3.

The number of figures in the integer is the quantity vhich is regularly modified, as it takes the values , 2, 3, 4, ... etc. ; the process to which it is submitted s the finding of the number of integers having that particular number of figures; the results, if rewritten as shown on the right, evidence at first sight a law of formation: the number of integers is 9 multiplied by a power of 10 whose index is equal to the number of figures in the integer less one. We can now without any more thinking write down at once the continuation of the table:

Number of integers of 5 figures:		$9\times10^4=90{,}000.$
,, ,, 6 ,,		$9\times10^5=900{,}000.$

and so on.

When finally we get to an unknown number of figures, say x, we still have:

Number of integers of x figures $=9\times10^{x-1}$.

All we have done is this: we have assumed that the law of formation of the successive results remains the same throughout; we have found that this law is true for the first few cases which we could easily calculate and then we have applied it to cases which were of less easy calculation, or which we could not calculate at all because letters were used instead of numbers. These results are found to be correct.

This principle is often used to obtain further terms of a sequence—or series—when enough terms are given to show what is called the law of formation of the successive terms. For instance, if it is required to continue the series

$$1+\frac{3x}{2}+\frac{9x^2}{2\times4}+\frac{27x^3}{2\times4\times6}+\ldots.$$

and to find the term of position n.

The series can be written

$$\frac{3^0x^0}{1}+\frac{3^1x^1}{1\times 2}\times\frac{3^2x^2}{(1\times 2)\times(2\times 2)}+\frac{3^3x^3}{(1\times 2)\times(2\times 2)\times(3\times 2)}+\dots$$

The next term is evidently

$$\frac{3^4x^4}{(1\times 2)\times(2\times 2)\times(3\times 2)\times(4\times 2)}=\frac{81x^4}{2\times 4\times 6\times 8},$$

and the next is $\quad \dfrac{243x^5}{2\times 4\times 6\times 8\times 10}.$

The rank of any term is one more than the index of x in the numerator, or than the highest multiplier of two in the denominator, hence the term of rank n is

$$\frac{3^{n-1}x^{n-1}}{2\times 4\times 6\times\dots\times[(n-1)\times 2]}.$$

The term of rank $n+1$ will be more convenient to write ; it is

$$\frac{3^n x^n}{2\times 4\times 6\times\dots\times 2n}.$$

Now, if we could find, in some similar manner, the law of formation of the expansions of the expressions $(a+x)^2$ $(a+x)^3$ ... etc., successively, we would be able to write down the expansion of, say, $(a+x)^{23}$ without the extremely tedious process of multiplying out, merely by following that law of formation; we could also write the expansion of $(a+x)^n$, which we cannot get otherwise, since we get it by working out the product of n factors each equal to $(a+x)$, an operation which of course we cannot perform actually, since we do not know what number n represents.

Note that in seeking the way in which the number of integers of n figures may be found, we rearranged the expressions so as to put in evidence the analogy of their features, for instance, since $90=9\times10^1$, for 9 we wrote 9×10^0, and so on.

Let us try if we cannot do anything similar with the expansions so far obtained. We get

$$(a+x)^2=a^2+2ax+x^2=a^2+\frac{2}{1}ax+\frac{2\times1}{1\times2}x^2;$$

$$(a+x)^3=a^3+3a^2x+3ax^2+x^3$$

$$=a^3+\frac{3}{1}a^2x+\frac{3\times2}{1\times2}ax^2+\frac{3\times2\times1}{1\times2\times3}x^3;$$

$$(a+x)^4=a^4+4a^3x+6a^2x^2+4ax^3+x^4$$

$$=a^4+\frac{4}{1}a^3x+\frac{4\times3}{1\times2}a^2x^2$$

$$+\frac{4\times3\times2}{1\times2\times3}ax^3+\frac{4\times3\times2\times1}{1\times2\times3\times4}x^4.$$

How we came to write $\frac{2}{1}$ instead of 2 in $2ax$, $\frac{2\times1}{1\times2}x^2$ instead of x^2, etc., does not matter at all. The important fact is that if we do the multiplications shown and simplify the coefficients, we fall back on the correct expansion, showing that these coefficients have been correctly split into their various factors and divisors. If you like, it is only a certain way, found by trial, to arrange these factors and divisors so as to get the required symmetry, just as there is a certain way to arrange the pieces of a jig-saw puzzle to make a picture.

By so doing we find that, while the value of the expansions remains unchanged, the way in which the various terms change as the index of the power of $(a+x)$ increases gradually is now quite clear. We see now that the first terms of the successive expansions are respectively a^2, a^3, a^4, a^5, a^6, ... a^n, that the second terms are respectively

$$\frac{2}{1}ax,\ \frac{3}{1}a^2x,\ \frac{4}{1}a^3x,\ \frac{5}{1}a^4x,\ \frac{6}{1}a^5x,\ \ldots,\ \frac{n}{1}a^{n-1}x,$$

that the third terms are respectively,

$$\frac{2\times 1}{1\times 2}a^0x^2,\ \frac{3\times 2}{1\times 2}ax^2,\ \frac{4\times 3}{1\times 2}a^2x^2,\ \frac{5\times 4}{1\times 2}a^3x^2,$$

$$\frac{6\times 5}{1\times 2}a^4x^2,\ \ldots,\ \frac{n(n-1)}{1\times 2}a^{n-2}x^2,$$

and so on.

We can make a table of these as shown on next page.

Starting from the third line, the columns may be completed both upwards and downwards as shown.

In each column the law of formation is manifest, and we can see now how the successive terms of any expansion are found. The first term is *always* a raised to the same power as $(a+x)$, the second term has always for coefficient the index of that power, and consists of the letter a to a power indicated by the index of $(a+x)$

Expressions.	1st terms.	2nd terms.	3rd terms.	4th terms.	5th terms.	
$(a+x)^0=1$	$a_0=1$	$\frac{0}{1}a^{-1}x=0$	all other	terms are likewise zero		
$(a+x)^1$	a^1	$\frac{1}{1}a^0x=x$	$\frac{1\times 0}{1\times 2}a^{-1}x^2=0$	all other terms are	likewise zero.	
$(a+x)^2$	a^2	$\frac{2}{1}a^1x$	$\frac{2\times 1}{1\times 2}a^0x^2=x^2$	$\frac{2\times 1\times 0}{1\times 2\times 3}a^{-1}x^3=0$ all	other terms are zero.	
$(a+x)^3$	a^3	$\frac{3}{1}a^2x$	$\frac{3\times 2}{1\times 2}a^1x^2$	$\frac{3\times 2\times 1}{1\times 2\times 3}a^0x^3=x^3$	$\frac{3\times 2\times 1\times 0}{1\times 2\times 3\times 4}a^{-1}x^4=0$	} all other terms=0.
$(a+x)^4$	a^4	$\frac{4}{1}a^3x$	$\frac{4\times 3}{1\times 2}a^2x^2$	$\frac{4\times 3\times 2}{1\times 2\times 3}a^1x^3$	$\frac{4\times 3\times 2\times 1}{1\times 2\times 3\times 4}a^0x^4=x^4$	
$(a+x)^5$	a^5	$\frac{5}{1}a^4x$	$\frac{5\times 4}{1\times 2}a^3x^2$	$\frac{5\times 4\times 3}{1\times 2\times 3}a^2x^3$	$\frac{5\times 4\times 3\times 2}{1\times 2\times 3\times 4}a^1x^4$	etc., ...
$(a+x)^6$	a^6	$\frac{6}{1}a^5x$	$\frac{6\times 5}{1\times 2}a^4x^2$	$\frac{6\times 5\times 4}{1\times 2\times 3}a^3x^3$	$\frac{6\times 5\times 4\times 3}{1\times 2\times 3\times 4}a^2x^4$	
⋮						
$(a+x)^n$	a^n	$\frac{n}{1}a^{n-1}x$	$\frac{n(n-1)}{1\times 2}a^{n-2}x^2$	$\frac{n(n-1)(n-2)}{1\times 2\times 3}a^{n-3}x^3$	$\frac{n(n-1)(n-2)(n-3)}{1\times 2\times 3\times 4}a^{n-4}x^4.$	

ess one, and of the first power of x, the third term has or coefficient the index of $(a+x)$ multiplied by this ndex less one, and divided by 1×2, and consists of a aised to a power indicated by the index of $(a+x)$ less wo, and of the second power of x; and once we have he three first terms of the series, it is easy to continue t, as we have seen a few pages above.

It is useful to note that if the indices of the powers of a and of x in any one term are added together, the same number is obtained, namely the index of the power of $(a+x)$, whatever the term may be. Mathematicians express this by saying that the expansion is a "homogeneous" expression.

Let us try it on $(a+x)^5$; we get

$$a^5+\frac{5}{1}a^{5-1}x+\frac{5(5-1)}{1\times 2}a^{5-2}x^2$$

$$+\frac{5(5-1)(5-2)}{1\times 2\times 3}a^{5-3}x^3+\frac{5(5-1)(5-2)(5-3)}{1\times 2\times 3\times 4}a^{5-4}x^4$$

$$+\frac{5(5-1)(5-2)(5-3)(5-4)}{1\times 2\times 3\times 4\times 5}a^{5-5}x^5$$

$$+\frac{5(5-1)(5-2)(5-3)(5-4)(5-5)}{1\times 2\times 3\times 4\times 5\times 6}a^{5-6}x^6+\dots,$$

or

$$(a+x)^5=a^5+\frac{5}{1}a^4x+\frac{5\times 4}{1\times 2}a^3x^2+\frac{5\times 4\times 3}{1\times 2\times 3}a^2x^3$$

$$+\frac{5\times 4\times 3\times 2}{1\times 2\times 3\times 4}ax^4+\frac{5\times 4\times 3\times 2\times 1}{1\times 2\times 3\times 4\times 5}a^0x^5$$

$$+\frac{5\times 4\times 3\times 2\times 1\times 0}{1\times 2\times 3\times 4\times 5\times 6}a^{-1}x^6+\dots.$$

The last term written is $0\times\frac{x^6}{6a}=0$, and clearly all the following terms are zero, since zero is a factor of their coefficients. We have therefore finally:

$$(a+x)^5=a^5+5a^4x+10a^3x^2+10a^2x^3+5ax^4+x^5,$$

which is exactly what we have obtained above by long and tiresome multiplications.

In a similar way, in order to expand $(a+x)^{23}$ we would merely write

$$(a+x)^{23}=a^{23}+\frac{23}{1}a^{22}x+\frac{23\times 22}{1\times 2}a^{21}x^2+\dots,$$

and so on.

Is it not easy? Well, there is no other difficulty lurking behind it!

You can now expand anything you like, for instance:

$$(3+\theta)^7=3^7+\frac{7}{1}3^6\theta+\frac{7\times 6}{1\times 2}3^5\theta^2+\frac{7\times 6\times 5}{1\times 2\times 3}3^4\theta^3$$
$$+\frac{7\times 6\times 5\times 4}{1\times 2\times 3\times 4}3^3\theta^4+\dots\text{ etc.},\dots$$
$$=2187+5103\,\theta+5103\,\theta^2+2835\,\theta^3$$
$$+945\,\theta^4+\dots,\text{ etc.}\dots.$$

(You notice that, if θ is very small, the terms decrease rapidly.)

Or again this:

$$\left(\frac{1}{x}+2\right)^5=\left(\frac{1}{x}\right)^5+\frac{5}{1}\left(\frac{1}{x}\right)^4\times 2+\frac{5\times 4}{1\times 2}\left(\frac{1}{x}\right)^3\times 2^2$$
$$+\frac{5\times 4\times 3}{1\times 2\times 3}\left(\frac{1}{x}\right)^2\times 2^3+\dots$$
$$=\frac{1}{x^5}+\frac{10}{x^4}+\frac{40}{x^3}+\frac{80}{x^2}+\dots.$$

Is it now quite clear ? Very well, let us see if we can now find the expansion when the index of the power is no more 2, or 3, or any other integer, but a letter, for nstance n. We follow exactly the same method, and ve get

$$(a+x)^n = a^n + \frac{n}{1} a^{n-1}x + \frac{n(n-1)}{1\times 2} a^{n-2}x^2 + \frac{n(n-1)(n-2)}{1\times 2\times 3} a^{n-3}x^3 + \dots .$$

This equality is called the *Binomial Theorem.* It is the statement of the most general case, and from it we can derive all others by replacing a, x and n by their respective values.

When $n=1$ we get

$$(a+x)^1 = a^1 + \frac{1}{1} a^0 x + \frac{1(1-1)}{1\times 2} a^{-1}x^2 = a+x.$$

When $n=0$ we get

$$(a+x)^0 = a^0 + \frac{0}{1} a^{-1}x + \frac{0\times -1}{1\times 2} a^{-2}x^2 + \dots = a^0 = 1,$$

and so on for any other value of n, or of a, or of x ... .

What makes the Binomial Theorem a thing of such great importance, however, is the fact that it is absolutely general, that is, the equality

$$(a+x)^n = a^n + \frac{n}{1} a^{n-1}x + \frac{n(n-1)}{1\times 2} a^{n-2}x^2 + \dots$$

holds good whatever we put for a, or x, or n.

You will perhaps jokingly ask now what is the expansion of $(\text{cow}+\text{book})^{\text{pin}}$, thinking you are going to score .... But nothing is easier to expand; here it is:

$$(\text{cow}+\text{book})^{\text{pin}}=\text{cow}^{\text{pin}}+\frac{\text{pin}}{1}\text{cow}^{\text{pin}-1}\,\text{book}$$
$$+\frac{\text{pin}(\text{pin}-1)}{1\times 2}\text{cow}^{\text{pin}-2}\,\text{book}^2+\dots,$$

and so on as long as you like, for we shall never reach a zero coefficient. Of course, the expression is meaningless and appears incongruous, because we do not know the real meaning of "cow," "book," or "pin." It is really not more incongruous than $(a+x)^n$. It is merely intended here as a quaint way of bringing home to you the fact that the expansion *can* always be written.

Since the Binomial Theorem is true for any value of a, x, or n, it is true if $a=1$. This gives

$$(1+x)^n=1^n+\frac{n}{1}\times 1^{n-1}x+\frac{n(n-1)}{1\times 2}1^{n-2}x^2$$
$$+\frac{n(n-1)(n-2)}{1\times 2\times 3}1^{n-3}x^3+\dots$$
$$=1+nx+\frac{n(n-1)}{1\times 2}x^2+\frac{n(n-1)(n-2)}{1\times 2\times 3}n^3+\dots,$$

since all powers of 1 are equal to unity.

It is also true if we have $-x$ instead of x, then

$$[a+(-x)]^n=(a-x)^n=a^n+\frac{n}{1}a^{n-1}(-x)$$
$$+\frac{n(n-1)}{1\times 2}a^{n-2}(-x)^2+\frac{n(n-1)(n-2)}{1\times 2\times 3}a^{n-3}(-x)^3+\dots.$$

This is how one must always write the expansion rst; it is then easy to write it, without mistakes, in its nal form:

$$(a-x)^n = a^n - na^{n-1}x + \frac{n(n-1)}{1\times 2}a^{n-2}x^2 - \frac{n(n-1)(n-2)}{1\times 2\times 3}a^{n-3}x^3 + \ldots.$$

It will be noticed that in *all* the expansions there gures in the denominators the product of consecutive actors the first one of which is unity, such as 1×2, $\times 2\times 3$, $1\times 2\times 3\times 4$ ... etc.; each of these products s called a "factorial," 1×2 is read "factorial two," $\times 2\times 3$ is read "factorial three" .... and so on. They re represented by the notation $\underline{|2}$ or $2!$ and $\underline{|3}$ or $3!$ espectively. For instance, "factorial five" is

$$1\times 2\times 3\times 4\times 5,$$

and is represented by $\underline{|5}$ or $5!$, "factorial n" or $\underline{|n}$ or $n!$ is $1\times 2\times 3\times 4 \ldots \times (n-2)(n-1)n$.

The Binomial Theorem can therefore be written

$$(a+x)^n = a^n + \frac{n}{1}a^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 + \ldots.$$

Do you begin to realize the real significance of the fact that this is true for all values of a, or x, *or of* n? It is true, for instance, if n has the value -1. Some think that the expansion of $(a+x)^{-1}$ is less easy; but it is just as easy. Do not go too quickly, write it step by step; then you can re-write it again in its final form. In this case, a remains a, x remains x, but whenever we have n we must put -1. We have then :

$$(a+x)^{-1}=a^{-1}+\frac{-1}{1}a^{-1-1}x+\frac{-1\times(-1-1)}{1\times 2}a^{-1-2}x^2$$
$$+\frac{-1\times(-1-1)(-1-2)}{1\times 2\times 3}a^{-1-3}x^3+\dots$$
$$=a^{-1}-a^{-2}x+a^{-3}x^2-a^{-4}x^3+\dots,$$

or $$\frac{1}{a+x}=\frac{1}{a}-\frac{x}{a^2}+\frac{x^2}{a^3}-\frac{x^3}{a^4}+\dots,$$

the number of terms being indefinitely great.

And we can write almost at sight the quotient of unity by any binomial expression, or even by any one of its powers, since

$$\frac{1}{(a+x)^n}=(a+x)^{-n}$$
$$=a^{-n}+\frac{-n}{1}a^{-n-1}x+\frac{-n(-n-1)}{1\times 2}a^{-n-2}x^2+\dots$$
$$=a^{-n}-na^{-(n+1)}x+\frac{n(n+1)}{1\times 2}a^{-(n+2)}x^2+\dots$$
$$=\frac{1}{a^n}-\frac{nx}{a^{n+1}}+\frac{n(n+1)x^2}{1\times 2\times a^{n+2}}-\dots.$$

But even this is not all. The Theorem is true if n has any fractional value, 1/2, 1/3, 1/7, 3/11, etc., ... so that we can also write almost at sight the result of extracting the corresponding roots :

$$(a+x)^{1/2} \text{ or } \sqrt{(a+x)},\quad (a+x)^{1/3} \text{ or } \sqrt[3]{(a+x)},$$
$$(a+x)^{3/11} \text{ or } \sqrt[11]{(a+x)^3}\dots \text{etc.}, \dots.$$

This is done just as easily as before :

$$\sqrt{(a+x)}=(a+x)^{1/2}=a^{1/2}+\tfrac{1}{2}a^{\frac{1}{2}-1}x$$

$$+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{1\times2}a^{\frac{1}{2}-2}x^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{1\times2\times3}a^{\frac{1}{2}-3}x^3+\ldots$$

$$=a^{\frac{1}{2}}+\tfrac{1}{2}a^{-\frac{1}{2}}x-\frac{\frac{1}{2}\times\frac{1}{2}}{1\times2}a^{-\frac{3}{2}}x^2+\frac{\frac{1}{2}\times\frac{1}{2}\times\frac{3}{2}}{1\times2\times3}a^{-\frac{5}{2}}x^3-\ldots$$

$$=a^{\frac{1}{2}}+\frac{x}{2a^{1/2}}-\frac{x^2}{8a^{3/2}}+\frac{x^3}{16a^{5/2}}-\ldots \text{ etc., } \ldots$$

Similarly,

$$(a+x)^{2/3}=a^{2/3}+\tfrac{2}{3}a^{\frac{2}{3}-1}x+\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)}{1\times2}a^{\frac{2}{3}-2}x^2+\ldots$$

$$=a^{2/3}+\frac{2x}{3a^{1/3}}-\frac{x^2}{9a^{4/3}}+\ldots \text{ etc. } \ldots .$$

One can get just as easily $1/\sqrt[3]{(a+x)^2}$, for this is the same as $(a+x)^{-2/3}$, that is :

$$a^{-2/3}+\left(-\frac{2}{3}\right)a^{-\frac{2}{3}-1}x+\frac{\left(-\frac{2}{3}\right)\left(-\frac{2}{3}-1\right)}{1\times2}a^{-\frac{2}{3}-2}x^2+\ldots$$

$$=a^{-2/3}-\tfrac{2}{3}a^{-5/3}x+\frac{\frac{2}{3}\times\frac{5}{3}}{1\times2}a^{-8/3}x^2-\ldots$$

$$=\frac{1}{a^{2/3}}-\frac{2x}{3a^{5/3}}+\frac{5x^2}{9a^{8/3}}-\ldots,$$

and so on ; one is almost as easy as another.

When dealing with a less simple binomial expression the method is exactly the same; try $\dfrac{1}{\sqrt[3]{\left(\frac{1}{m}-kz^2\right)^5}}$.

$$\left(\frac{1}{m}-kz^2\right)^{-3/5}=\left(\frac{1}{m}\right)^{-3/5}+\left(-\frac{3}{5}\right)\left(\frac{1}{m}\right)^{-\frac{3}{5}-1}(-kz^2)$$

$$+\frac{-\frac{3}{5}\left(-\frac{3}{5}-1\right)}{1\times 2}\left(\frac{1}{m}\right)^{-\frac{3}{5}-2}(-kz^2)^2+\dots$$

$$=\frac{1}{m^{-3/5}}+\frac{3}{5}\frac{1}{m^{-8/5}}kz^2+\frac{3}{5}\times\frac{8}{5}\times\frac{1}{1\times 2}$$

$$\times\frac{1}{m^{-13/5}}k^2z^4+\dots$$

$$=m^{3/5}+\tfrac{3}{5}kz^2m^{8/5}+\tfrac{12}{25}k^2z^4m^{13/5}+\dots,$$

and so on.

Try also this:

$$(1-\cos\theta)^\theta=1^\theta+\theta\times 1^{\theta-1}(-\cos\theta)$$

$$+\frac{\theta(\theta-1)}{1\times 2}1^{\theta-2}(-\cos\theta)^2+\dots$$

$$=1-\theta\cos\theta+\frac{\theta(\theta-1)}{1\times 2}\cos^2\theta$$

$$-\frac{\theta(\theta-1)(\theta-2)}{1\times 2\times 3}\cos^3\theta+\dots,\text{ etc.}$$

and this:

$$(1+a)^{kx}=1^{kx}+kx\times 1^{kx-1}a+\frac{kx(kx-1)}{1\times 2}1^{kx-2}a^2+\dots$$

$$=1+kx\,a+\frac{kx(kx-1)\,a^2}{1\times 2}+\dots,\text{ etc.}\dots$$

You notice that the last expansions we have con-dered can be continued indefinitely, just as the decimals f π or of ϵ can be continued indefinitely, the exact alue being never reached. Also, you notice that the ıccessive terms contain gradually increasing powers of ıe second term in the binomial; for instance, in the xpansion of $(a+x)^n$, the successive terms contain or x^0, x^1, x^2, x^3 ... etc. .... It follows that, if the second erm of the binomial is small compared with the first erm, the successive terms become smaller and smaller s we advance along the expansion. This is evident ince $(a+x)^n=\left[a\left(1+\frac{x}{a}\right)\right]^n=a^n\left(1+\frac{x}{a}\right)^n$, and since x s smaller than a, x/a is a fraction, and the powers f a/x get smaller and smaller in value as the indices ncrease. It follows that after a certain number of erms have been calculated we get finally a term small nough to be neglected for our purpose; and as all the erms that follow are smaller, we can generally neglect hem also. The expansion is then said to be *convergent.* When, on the other hand, the terms become larger and larger as we proceed, the expansion is said to be *divergent.*

The skilful mathematician tries always to arrange his expansions so that the terms converge rapidly, that is, become negligibly small very soon, so that he needs to calculate only two or three, or four, according to the degree of accuracy required.

The following examples will make this clear, and will also serve to show various uses to which the Binomial Theorem can be put.

Example 1. Calculate $(1{\cdot}005)^7$ to 5 places of decimals. We get

$$(1{\cdot}005)^7 = \left(1+\frac{5}{1000}\right)^7 = 1^7 + 7\times 1^6 \times \frac{5}{1000} + \frac{7\times 6}{1\times 2}$$

$$\times 1^5\left(\frac{5}{1000}\right)^2 + \frac{7\times 6\times 5}{1\times 2\times 3}1^4\left(\frac{5}{1000}\right)^3 + \ldots$$

$$= 1 + \frac{35}{1000} + \frac{21\times 25}{1000000} + \frac{35\times 125}{1000000000} + \ldots$$

$$= 1 + {\cdot}035 + {\cdot}000525 + {\cdot}000004375 + \ldots .$$

We stop when we arrive in the expansion at a term the first five, or better six, decimals of which are zeros. Here, clearly, the fourth term is the last one to take; so that, adding these

$$(1{\cdot}005)^7 \approx 1{\cdot}035529 \quad \text{or} \quad 1{\cdot}03553 \text{ to 5 places.}$$

The result could be obtained by continued multiplication, a very tedious process, or by logarithms if tables are available.

Example 2. Calculate $\sqrt[5]{1{\cdot}07}$ to 5 places of decimals.

$$\sqrt[5]{1{\cdot}07} = \left(1+\frac{7}{100}\right)^{1/5} = 1^{1/5} + \frac{1}{5}1^{(\frac{1}{5}-1)}\frac{7}{100}$$

$$+ \frac{\frac{1}{5}\left(\frac{1}{5}-1\right)}{1\times 2}1^{(\frac{1}{5}-2)}\left(\frac{7}{100}\right)^2 + \ldots .$$

Of course it is not necessary to write the factors $1^{(\frac{1}{5}-1)}$, $1^{(\frac{1}{5}-2)}$, etc. ..., but you are advised to write them as shown until you "get your sea legs," as it will help you to get the general expression well in your mind, and save you from omitting terms.

This then becomes

$$1+\frac{7}{5\times 10^2}-\frac{2\times 7^2}{5^2\times 10^4}+\frac{6\times 7^3}{5^3\times 10^6}-\frac{21\times 7^4}{5^4\times 10^8}+\dots.$$

The calculation can be simplified by multiplying both terms of the fractions by 2, 2^2, 2^3 ... respectively, so as to leave only powers of 10 in the denominators. We get then

$$1+\frac{7\times 2}{10^3}-\frac{2\times 7^2\times 2^2}{10^6}+\frac{6\times 7^3\times 2^3}{10^9}-\frac{21\times 7^4\times 2^4}{10^{12}}+\dots$$

$$=1+0{\cdot}014-0{\cdot}000392+0{\cdot}000016464 \\ -0{\cdot}000000806736\dots$$

We need only the four first terms; these give

$$\sqrt[5]{1{\cdot}07}\approx 1{\cdot}01362.$$

You have been taught how to extract square roots and cube roots of numbers, and you found the latter to be a much more complicated operation than the former. Have you ever wondered how complicated would be an extraction of, say, a fifth root? With the Binomial theorem, the method is the same for roots of all orders, and, as you see, it is quite an easy method.

Example 3. Calculate $\sqrt[7]{2205}$ to 5 places of decimals. When extracting square and cube roots by the ordinary method, we first sought the highest square or cube contained in the given number. In this case, too, we proceed in a similar manner, and find the highest seventh

power contained in 2205, this is $3^7=2187$, so tha
$2205=2187+18$, and

$$\sqrt[7]{2205}=\sqrt[7]{(3^7+18)}=(3^7+18)^{1/7}$$
$$=\left[3^7\left(1+\frac{18}{3^7}\right)\right]^{1/7}=3\left(1+\frac{18}{3^7}\right)^{1/7}=3\left(1+\frac{2}{3^5}\right)^{1/7}.$$

Now, expanding we get

$$\sqrt[7]{2205}=3\left[1+\frac{1}{7}\times\frac{2}{3^5}+\frac{\frac{1}{7}\left(\frac{1}{7}-1\right)}{1\times 2}\left(\frac{2}{3^5}\right)^2 + \frac{\frac{1}{7}\left(\frac{1}{7}-1\right)\left(\frac{1}{7}-2\right)}{1\times 2\times 3}\left(\frac{2}{3^5}\right)^3+\dots\right]$$

(dropping all the factors such as $1^{\frac{1}{7}-1}$, etc. ...)

$$=3\left[1+\frac{2}{7\times 3^5}-\frac{3\times 2^2}{7^2\times 3^{10}}+\frac{13\times 2^3}{7^3\times 3^{15}}-\dots\right]$$
$$=3\,[1+0{\cdot}001175-0{\cdot}00000415\dots],$$

as, clearly, the fourth term is negligible; so that

$$\sqrt[7]{2205}\approx 3{\cdot}00351.$$

Example 4. Calculate $\sqrt[7]{2180}$ to 5 places of decimal
The highest seventh power contained in 2180 is $2^7=128$
since $3^7=2187$. So that

$$2180=2^7+2052=2^7(1+2052/128).$$

Then

$$\sqrt[7]{2180}=2\left(1+\frac{1026}{64}\right)^{1/7}$$
$$=2\left[1+\frac{1}{7}\times\frac{1026}{64}+\frac{\frac{1}{7}\left(\frac{1}{7}-1\right)}{1\times 2}\left(\frac{1026}{64}\right)^2+\dots\right]$$
$$=2(1+2{\cdot}290178\div 15{\cdot}734753+\dots).$$

What is the meaning of this? The next term will be vidently larger still. In fact, the terms grow in-efinitely instead of gradually diminishing. The series diverging! Is the method going to fail us then?

When we expanded $(a+x)^n$ in a converging series, e stipulated that x was small compared to a. In he above case, $a=1$, $x=2052/128=16$ nearly; not t all small, but indeed large compared to 1. We only btain what we should expect, namely, a divergent xpansion.

What shall we do then? Note that

$$2180=2187-7=3^7-7=3^7\left(1-\frac{7}{3^7}\right);$$

ence

$$\sqrt[7]{2180}=\left[3^7\left(1-\frac{7}{3^7}\right)\right]^{1/7}=3\left(1-\frac{7}{3^7}\right)^{1/7}=3\left[1-\frac{1}{7}\times\frac{7}{3^7}\right.$$

$$\left.+\frac{\frac{1}{7}\left(\frac{1}{7}-1\right)}{1\times 2}\left(\frac{7}{3^7}\right)^2-\frac{\frac{1}{7}\left(\frac{1}{7}-1\right)\left(\frac{1}{7}-2\right)}{1\times 2\times 3}\left(\frac{7}{3^7}\right)^3+\ldots\right]$$

$$=3\left(1-\frac{1}{3^7}-\frac{1}{3^{13}}-\frac{13}{3^{21}}-\ldots\right)$$

$$\approx 3(1-0{\cdot}000457-0{\cdot}0000006)$$

$$\approx 3\times 0{\cdot}999543=2{\cdot}998629.$$

The diverging expansion obtained above shows that in some cases the expansion is not the true mathematical equivalent of the indexed form of the binomial. We shall find that, when this occurs, if the expression is put in the form $(1+x)^n$, then x is greater than 1. In other words, such an expansion as, say,

$$(1-x)^{-2}=1+2x+3x^2+4x^3+\dots$$

is only arithmetically true if x is smaller than 1.

This seems to throw a doubt on the generality of the equality $(1+x)^n=1+nx+\frac{n(n-1)}{1\times 2}x^2+\dots$, generality upon which so much stress has been laid in the previous pages. It is worth while investigating this more fully.

Take

$$(1-x)^n=1-nx+\frac{n(n-1)}{1\times 2}x^2-\frac{n(n-1)(n-2)}{1\times 2\times 3}x^3+\dots.$$

If n has the value -1, say, then

$$(1-x)^{-1}=1+x+x^2+x^3+x^4+\dots.$$

This is not numerically true if $x>1$; for instance, let $x=2$.

$$(1-2)^{-1}=(-1)^{-1}=\frac{1}{-1}=-1=1+2+4+8+16+\dots,$$

an equality which obviously will never be satisfied whatever is the number of terms taken.

Now, if S is the sum of all the terms up to the one of rank m in the expansion of $(1-x)^{-1}$, then

$$S=1+x+x^2+x^3+\dots+x^{m-1}.$$

Multiply by x,

$$x\times S=x+x^2+x^3+\dots+x^{m-1}+x^m.$$

Hence, taking the difference,

$$xS-S=S(x-1)=x^m-1 \quad\text{and}\quad S=\frac{x^m-1}{x-1}=\frac{1-x^m}{1-x},$$

$$S=\frac{1}{1-x}-\frac{x^m}{1-x}.$$

If $x>1$ the second term increases indefinitely as m creases, that is, as we take more terms of the expanon. If $x<1$, as m increases x^m decreases, and the cond term becomes negligible when x is very great. hen $S=\frac{1}{(1-x)}=(1-x)^{-1}$, that is, the equality is veried. In other words, the expansion of $(1+x)^n$ is lways arithmetically correct when $x<1$.

The foregoing examples, accessible to the very beinner, give an idea of the usefulness of the Binomial heorem. The following example is of a more advanced ind, although quite simple for readers of *Calculus Made asy*. It can be skipped without inconvenience by others who have not yet overcome their terror of the calculus.

Example 5. Obtain an expression with which one can alculate easily the length θ of an arc, given its trigononetric tangent x; that is, expand $\theta=\text{arc}\tan x$.

If $\theta=\text{arc}\tan x$, then $x=\tan\theta$.

Hence $\frac{dx}{d\theta}=\sec^2\theta=1+\tan^2\theta=1+x^2$ (see *Calculus Made Easy*, p. 168).

And therefore $\frac{d\theta}{dx}=1\Big/\frac{dx}{d\theta}=\frac{1}{1+x^2}=(1+x^2)^{-1}$.

Expanding this by the binomial theorem we get

$$\frac{d\theta}{dx}=1-x^2+x^4-x^6+x^8. \quad \ldots\ldots\ldots\ldots\ldots(1)$$

Binomial expressions are not the only ones which can be expanded in a series of terms similar to the ones we have obtained by means of the Binomial Theorem. For instance, let us suppose that θ can be expanded in such a series; the expansion of θ will be an expression such as

$$\theta=A_0+A_1x+A_2x^2+A_3x^3+A_4x^4+\dots, \quad \dots\dots\dots(2)$$

where A_0, A_1, A_2 ... are numerical coefficients, some of which may be zero, the corresponding terms being then missing in the series.

If we differentiate the above expansion with respect to x we get:

$$\frac{d\theta}{dx}=A_1+2A_2x+3A_3x^2+4A_4x^3+\dots. \quad \dots\dots(3)$$

We have two different expressions for $\frac{d\theta}{dx}$, (1) and (3); these two expressions are necessarily identically equal, so that

$$1-x^2+x^4-x^6+x^8+\dots$$
$$=A_1+2A_2x+3A_3x^2+4A_4x^3 \dots\dots\dots(4)$$

Now, when two such expressions in x are identically equal, and do not contain x either in denominator or under the sign indicating the extraction of a root, the coefficients of the same powers of x are identically equal. Here we have, therefore,

$$A_1=1,\ A_2=0,\ 3A_3=-1 \text{ or } A_3=-\frac{1}{3},$$

$$A_4=0,\ 5A_5=+1 \text{ or } A_5=\frac{1}{5}, \text{ etc. ...}$$

Replacing in (2) we get

$$\theta = A_0 + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9}, \dots,$$

in which A_0 is still unknown.

But when $x=0$, $\theta=0$ obviously, hence $0=A_0+0$, and $A_0=0$, so that

$$\theta = \frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots,$$

which is the required expression, θ being, of course, in radians (see p. 45).

This expansion is convenient for small arcs, as it converges then rapidly. For arcs near 45° it converges too slowly to be of any use. For arcs larger than 45°, $x = \tan\theta$ is larger than unity, and the expansion is divergent, as explained in the last example.

For instance, find to 5 places of decimals the arc the tangent corresponding to which is 0·3. Here $x=0{\cdot}3$.

$$\theta = {\cdot}3 - \frac{{\cdot}3^3}{3} + \frac{{\cdot}3^5}{5} - \frac{{\cdot}3^7}{7} + \frac{{\cdot}3^9}{9}, \dots,$$

or $\theta \approx {\cdot}3 - {\cdot}009 + {\cdot}000486 - {\cdot}000031 + {\cdot}000002 = {\cdot}29146$ radians.

To convert in degrees, multiply by 57·29577. (See p. 46.) We get $16°42'$ very nearly.

You can now work through the following exercises:

Exercises IV. (For Answers, see p. 245.)

Expand to 4 terms:

(1) (*a*) $(1+2x)^7$; (*b*) $\left(2x+\frac{y}{2}\right)^4$;

(*c*) $\left(ax+\frac{y}{a}\right)^9$ (*d*) $(1-2y)^5$;

(e) $\left(2-\frac{x}{2}\right)^6$; (f) $\left(a-\frac{3}{b}\right)^7$;

(g) $(1+x)^{1/3}$; (h) $(1+x)^{3/4}$; (k) $(1+x)^{3/5}$.

(2) $(1+3x)^{-2}$. (3) $(1-x)^{-5}$. (4) $(1-x^2)^{-3}$.

(5) $(2+x)^{-3}$. (6) $\sqrt{(1+2\theta)}$. (7) $1/\sqrt{(1+2x)}$.

(8) $1/\sqrt{(1-x)}$. (9) $1/\sqrt{(a-2x)^3}$. (10) $\sqrt[3]{\left(\epsilon+\frac{x}{2}\right)^5}$.

(11) $\sqrt[2]{\left(\frac{1}{\theta}-\theta^x\right)}$. (12) $1/(1-\sin\theta)^2$.

Calculate, correct to 5 places of decimals :

(13) $\sqrt[4]{613}$. (14) $\sqrt[3]{324}$. (15) $\sqrt[5]{3145}$. (16) $\sqrt{3443}$.

Expand to 4 terms :

(17) $(1+\cos x)^{\theta}$. (18) $(1-\epsilon^2)^{\tan x}$.

(19) $\sqrt{\left(x-\frac{1}{\cos\theta}\right)}$. (20) $(1+\epsilon)^{\epsilon/2}$.

(21) Expand $\theta=\text{arc}(\sin x)$, and find θ when $x=0{\cdot}2$. (Remember that $\frac{d\theta}{dx}=1/\sqrt{(1-x)^2}$).

(22) Expand $(5-3\tan x)^{-1}$ to 4 terms.

(23) Expand $(1+\theta)^{\frac{1}{m}}$ to 4 terms.

(24) Expand $\left(k-\frac{1}{m}\right)^{k/m}$ to 4 terms.

PART II.

CHIEFLY ABOUT "EPSILON."

CHAPTER VII.

A FIRST MEETING WITH EPSILON: LOGARITHMIC GROWING AND DYING AWAY.

EVERY schoolboy knows what is meant by simple interest; he knows that if £100 produces £3 interest in one year, it is said to be invested at "three per cent.," written, in mathematical symbols, 3%. He knows also that every £100 of a sum of money so invested produces £3 for every year during which it is invested, so that if £P is the sum invested or principal, and n the number of years during which it is invested, the sum, after n years, of the yearly interests (supposing they have been put regularly in a drawer or a stocking just as they were received) will be £$\frac{P}{100}\times n\times 3$. More generally, if the rate of interest is £r for £100 per year, the total interest in n years is £$\frac{P}{100}\times n\times r$.

The yearly growing of the principal can be represented by the straight line AB (see Fig. 2), OA being the original principal and XB the principal plus the interests it has produced. BC is then the total interest produced in n years, and is made up of n equal increments, each of which is $1/n$ of the total increment.

This is what may be called "arithmetical growing."

Obviously there will be a certain number of years for which the total increment will be equal to the

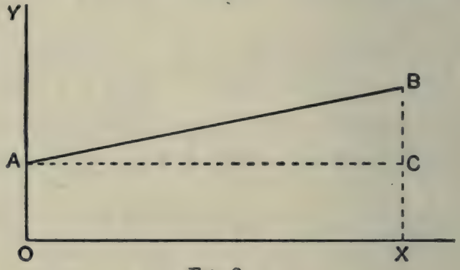


FIG. 2.

original principal. Suppose η is this number of years each yearly interest will be $1/\eta$ of the total interest, tha is £P/η.

Then, after η years, the interests will amount t £$P\eta r/100$, and this is equal to P, so that $\eta r/100=1$ an $\eta=100/r$.

At 4%, for instance, it will take $100/4=25$ years t double any principal. To double it in 24 years woul require a rate of interest $r=100/24=4\frac{1}{6}$ per cent.

Most schoolboys are also acquainted with the meanir of "compound interest." They know it means tha t the end of definite equal periods of time, say at the nd of each year, the interest itself, instead of being ut away in a drawer, is invested at the same rate s the principal which produced it, so that it is as if is principal was growing by continually increasing icrements—but otherwise giving no interest—until at ie end the increased principal is withdrawn from the ivestment. The total amount of increment is the ompound interest.

Let us see what a principal P will become if we ivest it during the time just found to be necessary o double it at simple interest, that is, during $\eta=100/r$ ears.

At the end of the first year, the interest is £$P\times\frac{r}{100}$ nd the principal has become $P_1=P+\frac{Pr}{100}=P\left(1+\frac{r}{100}\right)$, eing the original principal plus the interest. This new rincipal $P_1=P\left(1+\frac{r}{100}\right)$ is re-invested during the second ear, the interest derived from it being of course $P_1\,\frac{r}{100}$ r $P\left(1+\frac{r}{100}\right)\frac{r}{100}$, so that at the end of the second year he principal and the interest together amount to

$$P_2=P\left(1+\frac{r}{100}\right)+P\left(1+\frac{r}{100}\right)\frac{r}{100},$$

hat is, to

$$P\left(1+\frac{r}{100}\right)\left(1+\frac{r}{100}\right) \text{ or } P\left(1+\frac{r}{100}\right)^2,$$

Similarly, at the end of the third year, the principal, together with the compound interest, amount to

$$P_3 = P\left(1+\frac{r}{100}\right)^3.$$

Using the Principle of Mathematical Induction (see p. 55) we see that at the end of n years the principal, swelled by the compound interest, has become

$$P_n = P\left(1+\frac{r}{100}\right)^n.$$

After being invested for η years at r per cent. at compound interest, the principal P becomes

$$P_\eta = P(1+r/100)^\eta.$$

Now, in this particular case

$$\frac{r}{100} = \frac{1}{\eta}, \text{ so that } P_\eta = P\left(1+\frac{1}{\eta}\right)^\eta.$$

If, while we keep the period of investment to th same value of η years, we shorten the time betwee the successive additions to the principal of the interest this principal is producing, these additions will occu more frequently, but the increments will, naturally, b less in amount.

If the interest is added to the principal at the end o every half year, for instance, the interest added the firs time will only be £$r/2$ instead of £r, for every £10 invested, and this will be done 2η times, so that th expression for the final value of the principal become

$$P_\eta = P(1+r/200)^{2\eta} = P(1+1/2\eta)^{2\eta}, \text{ sinec } r/200 = 1/2\eta.$$

If we add the interest to the principal every month, hen the first interest will be $\frac{r}{12}$, and there will be 12η dditions to the principal in η years, so that

$$P_\eta = P(1+r/1200)^{12\eta} = P(1+1/12\eta)^{12\eta}.$$

Similarly, if the operation is done every week, we get

$$P_\eta = P(1+1/52\eta)^{52\eta},$$

or every day, $P_\eta = P(1+1/365\eta)^{365\eta}$,

or every hour, $P_\eta = P(1+1/8{,}760\eta)^{8{,}760\eta}$,

or every minute, $P_\eta = P(1+1/525{,}600\eta)^{525{,}600\eta}$,

or every second, $P_\eta = P(1+1/31{,}536{,}000\eta)^{31{,}536{,}000\eta}$.

We see that both the denominator and the index remain identically the same, and that both increase continually as the number of times the interests are added to the principal during the η years increase indefinitely. If this is done n times a year,

$$P_\eta = P(1+1/n\eta)^{n\eta},$$

and if $n \times \eta = N$ = the total number of additions of the interest to the principal during the η years,

$$P_\eta = P(1+1/N)^N.$$

By the same Principle of Mathematical Induction, we can say that when this number of additions of interest to principal in the η years is *anything* we like, represented by anything we choose, whether N, x, a or *cat*, the principal after η years will be

$$P_\eta = P(1+1/N)^N, \text{ or } P(1+1/x)^x,$$

or $P(1+1/a)^a$, or $P(1+1/\text{cat})^{\text{cat}}$! ....

If we imagine the interest being added *continually* to the principal, as the water of a gradually swelling rivulet adds itself to a lake, instead of being added at short intervals, as if the water was thrown one bucketful of increasing magnitude at a time, then the number of times the addition is performed during the η years is greater than anything one can conceive. Mathematicians express this fact by saying that the number is infinite and represent it by the symbol ∞. We still can write the expression for P_η, it is $P(1+1/\infty)^\infty$, but it has for us no more meaning than $P(1+1/\text{cat})^{\text{cat}}$, above !

Now, if we expand $(1+1/\infty)^\infty$ by the binomial theorem, we get easily an expansion, but it will be meaningless to us ; in fact, the binomial theorem fails to give intelligible results when, in $(a+x)^n$, x is infinite. Yet we can reasonably expect that, since $(1+1/31{,}536{,}000\eta)^{31{,}536{,}000\eta}$ gave some sort of result, $(1+1/\infty)^\infty$ should also give some sort of intelligible result. What can we do ?

Remember what we did when we were confronted with the symbol a^0 ; we sought its value by some other method than the one which gave a^0. Let us try to do the same in this case. We got $(1+1/\infty)^\infty$ from $(1+1/N)^N$ by causing N to grow indefinitely ; but

$$(1+1/N)^N=1+N\times\frac{1}{N}+\frac{N(N-1)}{1\times 2}\,\frac{1}{N^2}+\frac{N(N-1)(N-2)}{1\times 2\times 3}\,\frac{1}{N^3}+\dots,$$

and if we cause N to grow indefinitely in the expressions on both sides of the sign $=$, we must of course get the same result. But if we do this, $(1+1/N)^N$ becomes $(1+1/\infty)^\infty$; then the right-hand expression must give us the value of $(1+1/\infty)^\infty$.

Now, the expansion may be written :

$$1+1+\frac{1-\frac{1}{N}}{1\times 2}+\frac{1-\frac{3}{N}+\frac{2}{N^2}}{1\times 2\times 3}+\dots,$$

and if N grows till it is infinite, when $N=\infty$, since the quotient of unity divided by a very large number is very small, $1/\infty=0$, we get $1/N=0$, $1/N^2=0$, $1/N^3=0$, etc., so that

$$(1+1/\infty)^\infty=1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\dots=2{\cdot}71828182846\dots,$$

so that $P_\eta=\epsilon\times P$. We meet unexpectedly epsilon, the base of Napierian logarithms.

Then, when, at simple interest, the principal P merely doubles itself, at continuous or "true" compound interest, this principal becomes ϵ times greater. We see that just as the ratio of the length of any circumference to its radius is $\pi=3{\cdot}141592\dots$, so the ratio of the true compound increased principal to the original principal, during an interval of time which would double this original principal at simple interest, is $\epsilon=2{\cdot}71828\dots$.

The increase of the principal can also be represented by a graph. Here, the first step or increment is $1/N$ of the original value, so that each ordinate is $1+1/N$ or $(N+1)/N$ of the ordinate before, and as the ordinates grow steadily, each increment is greater than the one before, so that the growth of the principal can be represented by the line AB (see Fig. 3), OA being the original principal, and XB the compound increased principal, so that $CB = 1{\cdot}7183 \times OA$, if the growing follows a true compound interest law.

The characteristic feature of this mode of growing is that the increment at any time is proportional to the actual magnitude, at that time, of the growing

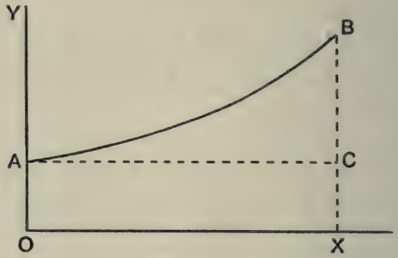


FIG. 3.

thing itself. If this magnitude is A, the increment is A/η, the new magnitude being $A+A/\eta$ or $A(1+1/\eta)$.

Inversely, the " dying away " of a thing may follow a similar law, the decrement being proportional to the actual magnitude of the thing which is diminishing; in this case, the new magnitude is $A-A/\eta$ or $A(1-1/\eta)$. Many physical processes follow a similar law; the loss of temperature of a hot body in any small interval of time is proportional to its excess of temperature above that of the medium in which it is cooling; the loss of electrification of a charged body in a small interval of time is proportional to the quantity of electrification left on it; the loss of light of a beam passing through thin portion of an absorbing medium is proportional o the intensity of the beam entering that portion, and o on.

But whatever made epsilon come into it? Let us ry to find out.

If y_x is the final value of the principal a invested at ompound interest $r/100$ for x years, then

$$y_x = a(1+r/100)^x \text{ after } x \text{ years.}$$

After $x-1$ years $y_{x-1} = a(1+r/100)^{x-1}$.

The ratio of the two values is $(1+r/100)$, and it is the same whichever are the two consecutive values considered.

Let the value of this ratio be p. Then $y_x = ap^x$.

Let also $\log_\epsilon p = C$. Then $p = \epsilon^C$ (see p. 32), and $y_x = a(\epsilon^C)^x$, or $y_x = a\epsilon^{Cx}$.

This is the exponential form of the compound interest law; it is exactly equivalent to the formula given above, $y_x = a(1+r/100)^x$.

This can be easily verified; for instance, £100 at 3 per cent. compound interest becomes in 4 years

$$£100(1{\cdot}03)^4 = £112{\cdot}5509$$

by the first formula; using the other formula we get, since $\log_\epsilon 1{\cdot}03$ is $0{\cdot}0295587$,

$$y_4 = 100 \times 2{\cdot}71828^{0{\cdot}0295587 \times 4}.$$

Using common logarithms for the calculation we get

$$\log_{10} y_4 = 2 + 0{\cdot}118235 \times 0{\cdot}43429 = 2{\cdot}0513483,$$

hence $$y_4 = 112{\cdot}5508.$$

Since it can be so expressed in terms of the base of Napierian logarithms, the compound interest mode of growing is called "logarithmic" growing.

If $p<1$, $\log_\epsilon p=-C$ (see p. 40) and $y=a\epsilon^{-Cx}$.

This is a very important exponential, representing the "logarithmic" dying-away process. ϵ^{-Cx} is the dying-away factor; x is usually a time t. If the constant C be also taken to represent a time, let $C=1/T$, then $y=a\epsilon^{-t/T}$. T is then called the time constant, because if $t=T$, $y=a\epsilon^{-1}=a/\epsilon$, that is, in the time T, y is reduced to $1/\epsilon$ or to 0·3678 of its original value.

The growth of intensity of a continuous electric current after it is suddenly switched on is expressed by such a dying-away expression. Its theoretical value is $I=\frac{E}{R}$, but at first it differs from it by an amount which is rapidly dying away, hence it is given by

$$I=E/R(1-\epsilon^{-t/T}),$$

and T, the time constant, depends on the resistance R and on the self induction L of the circuit; $T=L/R$, so that

$$I=E/R(1-\epsilon^{-Rt/L}).$$

The following worked-out examples on logarithmic growth and decay will help you to work out the exercises which you will find at the end of the chapter.

Example 1. At 3.15 p.m. the temperature of a piece of iron cooling in a room the temperature of which is 20° Cent. is found to be 330° Cent. At 3.25 p.m. it is 86° Cent. Find the time constant, and also at what time the temperature was 100° Cent.

If θ_t is the excess of the temperature of the iron bove that of the room at the time t, and θ_0 the initial xcess, then (see p. 86)

$$\theta_t=\theta_0\epsilon^{-t/T}.$$

Here $$66=310\epsilon^{-10/T}.$$

Solving for T we get

$$\log 66=\log 310-(10/T)\log\epsilon;$$

ence $0{\cdot}6719=4{\cdot}343/T$ and $T=6{\cdot}46$ minutes, or 6 mins. 8 secs.

The equation is therefore, numerically, $\theta_t=\theta_0\epsilon^{-t/6\cdot46}$. Vhen the temperature was 100° Cent., the excess was 0° C., hence $80=310\epsilon^{-t/6\cdot46}$. This time we solve for t ı the same way: $\log 80=\log 310-(t/6{\cdot}46)\log\epsilon$, hence $\cdot4343t/6{\cdot}46=0{\cdot}5883$ and $t=8{\cdot}76$ minutes, or 8^m 46 econds. The temperature was 100° at

$$3^h\ 15^m+8^m\ 46^s=3^h\ 23^m\ 46^s.$$

Example 2. Light is absorbed by fog according to he law $I_l=I_0\epsilon^{-Kl}$, where I_l is the intensity of the ght after passing through a thickness l of fog, K being constant. It is found that the intensity of a source f light is reduced by one half when it is seen through metres of fog. At what distance will the source be just isible to an eye which is able to perceive a light the ntensity of which is one thousandth of the intensity of he source ?

Since, after passing through 5 metres or 500 centi-netres, the light has lost half of its intensity, we have

$$0{\cdot}5I_0=I_0 2{\cdot}7183^{-K\times500},$$

or $0{\cdot}5=2{\cdot}7183^{-500K}$ and $\log 0{\cdot}5=-500K\times\log 2{\cdot}7183.$

$$\bar{1}{\cdot}6990=-500\times0{\cdot}4343K \text{ or } 0{\cdot}301=217{\cdot}15K.$$

$$K=0{\cdot}00139.$$

When light is reduced to one thousandth of its intensity,

$$I_l=0{\cdot}001I_0=I_0 2{\cdot}7183^{-0{\cdot}00139l}$$

and

$$0{\cdot}001=2{\cdot}7183^{-0{\cdot}00139l}.$$

$$\log 0{\cdot}001=-0{\cdot}00139l\times0{\cdot}4343.$$

$$\bar{3}{\cdot}0000=-0{\cdot}4343\times0{\cdot}00139l.$$

$$3=0{\cdot}0006l \text{ and } l=5000 \text{ centimetres.}$$

The light will just be visible at a distance of 50 metres approximately.

Example 3. Light is absorbed by a certain medium according to the law $I_l=kI_0\epsilon^{-Kl}$, where I_0 is the initial intensity of the beam, I_l is the intensity after passing through a thickness l, k and K are constants. If the intensity of a beam of light is reduced by 12% after passing through 10 cms., and by 18% after 20 cms., find the intensity after 1 metre.

We must first find the numerical values of the two constants. We have

$$I_l=0{\cdot}88I_0=kI_0\epsilon^{-10K}.$$

$$I_l=0{\cdot}82I_0=kI_0\epsilon^{-20K}.$$

$$\frac{0{\cdot}88}{0{\cdot}82}=\frac{\epsilon^{-10K}}{\epsilon^{-20K}}=\epsilon^{20K-10K}=\epsilon^{10K}.$$

$$\log 0{\cdot}88-\log 0{\cdot}82=10K\log 2{\cdot}718,$$

hence

$$K=\frac{0{\cdot}00307}{0{\cdot}4343}=0{\cdot}00707.$$

Then

$$0{\cdot}88=k\epsilon^{-0{\cdot}0707}.$$

Solving for k we get

$$\log 0{\cdot}88 = \log k - 0{\cdot}0707 \times \log 2{\cdot}718,$$

$$\cdot 9752 = \log k, \quad k = 0{\cdot}9445.$$

So that the numerical equation is

$$I_l = 0{\cdot}9445 I_0 \epsilon^{-0{\cdot}00707 l}.$$

After 1 metre $I_l = 0{\cdot}9445 I_0 \epsilon^{-0{\cdot}707}.$

$$\frac{I_l}{I_0} = 0{\cdot}9445 \epsilon^{-0{\cdot}707}.$$

$$\log \frac{I_l}{I_0} = \log 0{\cdot}9445 - 0{\cdot}707 \times \log 2{\cdot}718 = \bar{1}{\cdot}6681.$$

$$\frac{I_l}{I_0} = 0{\cdot}4657 \text{ or } I_l = 0{\cdot}466 I_0.$$

The intensity is reduced by 53·4 per cent.

Example 4. In a room at 20° C., a lump of metal cools from 200° C. to 100° C. in 10 minutes. What should be the temperature of the room in order that the same lump of metal should cool twice as quickly through the same range of temperature ?

Here, as before,

$$\theta_t = \theta_0 \epsilon^{-t/T} \text{ or } 80 = 180 \epsilon^{-10/T}.$$

$$\log 80 = \log 180 - \frac{10}{T} \log 2{\cdot}718$$

or $$1{\cdot}9031 = 2{\cdot}2553 - \frac{4{\cdot}343}{T};$$

hence $$T = \frac{4{\cdot}343}{0{\cdot}3522} = 12{\cdot}3.$$

The law of the cooling of this particular lump o[f] metal is therefore $\theta_t = \theta_0 \epsilon^{-t/12\cdot3}$.

If θ_x is the unknown temperature of the room, the[n] since θ_t and θ_0 are the differences of temperature of th[e] lump of metal above that of the room,

$$100 - \theta_x = (200 - \theta_x)\epsilon^{-t/12\cdot3},$$

with $t=5$ minutes.

$$\log(100-\theta_x) = \log(200-\theta_x) - \frac{5}{12\cdot3} \times 0\cdot4343$$

or $$\frac{2\cdot1715}{12\cdot3} = \log\frac{200-\theta_x}{100-\theta_x} = 0\cdot1765;$$

hence $$\frac{200-\theta_x}{100-\theta_x} = 1\cdot5 \text{ very nearly.}$$

It follows that $\theta_x = -100°$ C.

As a check, calculate the time required for the lum[p] to cool when placed in an enclosure at $-100°$ C.

Here the equation becomes $200 = 300\epsilon^{-t/12\cdot3}$.

$$2\cdot301 = 2\cdot477 - \frac{0\cdot4343t}{12\cdot3} \text{ or } 0\cdot352t = 0\cdot176,$$

and $t=5$ minutes.

You can now try the following exercises:

Exercises V. (For Answers, see p. 247.)

1. The temperature of a piece of iron cooling in ai[r] at 0° C. falls from 400° C. to 200° C. in 4 minutes. Ho[w] long will the piece of iron take to further cool from 200° C[.] to 100° C. and 10° C. respectively?

2. How long will it take for a beaker of boiling water cool down to 20° C. in a room the temperature of hich is 16° C., if it cools to 80° C. in 4 minutes ?

3. The quantity of electricity on a body is found be 10 units one hour after charging it, and 2 units minutes later. Find the initial quantity of electricity the leakage follows the law $Q_t = Q_0 \epsilon^{-\mu t}$, where Q_t is e quantity of electricity on the body t minutes after e time at which the quantity had the initial value Q_0, μ ing a constant.

4. In how long will the charge on a body be reduced half its original value if it diminishes by one hundredth the first minute ?

5. Find the resistance R through which a condenser f capacity $K = 3 \times 10^{-6}$ units, charged to an initial otential V_0, is discharging if the potential falls to alf its value in half a minute, and if the fall of potential llows the law $V_t = V_0 \epsilon^{-t/KR}$, t being in seconds.

6. Compare the opacity of two mediums if in one beam of light is reduced in intensity by 50 per cent. n passing through 2 metres of it, while in the other is reduced by 10 per cent. in passing through 40 cms. f it, the law of absorption of the light being $I_l = I_0 \epsilon^{-Kl}$.

7. The pressure p of the atmosphere at an altitude kilometres is given by $p = p_0 \epsilon^{-kh}$, p_0 being the normal ressure at sea level, namely, 76 centimetres, and k eing a constant. Find the average fall of pressure per 00 metres up to a height of 2 kilometres, if, at 1 kilonetre, the pressure is 67 centimetres.

8. The initial strength i_0 of a telephonic curre in a line of length l kilometres falls at the end of th line to a value i_l given by $i_l=i_0\epsilon^{-\beta l}$, where β is constant. If $\beta=0{\cdot}0125$, find the attenuation or dimin tion of intensity at the end of a similar line 10 kilometr in length.

9. The initial strength of a telephonic current reduced by 20 per cent. at the end of a line the length of which is 32 kilometres. Find the length of a simil line for which the current strength is reduced by one hal

10. A beaker of boiling water cools to 50°C. in minutes in a room the temperature of which is $-5°$ At what surrounding temperature would the coolin through the same range take place twice as slowly?

CHAPTER VIII.

A LITTLE MORE ABOUT NAPIERIAN LOGARITHMS.

N a previous chapter we have seen that

$$(1+a)^n = 1 + na + \frac{n(n-1)}{2!}a^2 + \frac{n(n-1)(n-2)}{3!}a^3 + \dots.$$

Now, since this is true for all values of a, it is also rue if a has the value $1/n$, in which case we have

$$\left(1+\frac{1}{n}\right)^n = 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3 + \dots$$

ınd we have seen that when n grows until it is greater han any conceivable quantity, that is, becomes in- initely great, then

$$\left(1+\frac{1}{n}\right)^n = 2{\cdot}71828\ldots = \epsilon.$$

Now

$$\epsilon^x = \left\{\left(1+\frac{1}{n}\right)^n\right\}^x = \left(1+\frac{1}{n}\right)^{nx} = 1 + nx\left(\frac{1}{n}\right) + \frac{nx(nx-1)}{2!}\left(\frac{1}{n}\right)^2 + \frac{nx(nx-1)(nx-2)}{3!}\left(\frac{1}{n}\right)^3 + \dots,$$

or $$\epsilon^x = 1 + x + \frac{x\left(x-\frac{1}{n}\right)}{2!} + \frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{3!} + \ldots \quad (1$$

If we suppose again n to become infinitely great then all the terms such as $\frac{1}{n}, \frac{2}{n} \ldots$, become zero, an we get

$$\epsilon^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots(2$$

But if $1/n < 1$, that is, if $n > 1$, we have seen that th equality

$$\epsilon^x = \left(1+\frac{1}{n}\right)^{nx} = 1 + x + \frac{x\left(x-\frac{1}{n}\right)}{2!} + \frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{3!} + \ldots,$$

is arithmetically true, that is, if we give any valu to x and calculate the value of the left and of the righ hand members respectively, we get the same number N We have therefore $\epsilon^x = N$, or $x = \log_\epsilon N$.

We fall *naturally* upon the system of Napieria logarithms, and for this reason they are also call Natural logarithms. Now, we could give x any val we like, and calculate the number N corresponding it. This, however, would be working the wrong w about; what we want is to find the logarithm of a given number, not the reverse.

Now, the equality (2) remains true for all the valu of x; it is true if instead of x we put anything el instance, cx—the x, of course, being different—and have

$$\epsilon^{cx}=1+cx+\frac{c^2x^2}{2!}+\frac{c^3x^3}{3!}+\dots.$$

Why did we do that? Because we want to bring e Napierian logarithm of any number directly into the pression, and then try to find a value for it; if we ite now $\epsilon^c=a$ then $c=\log_\epsilon a$, and we have

$$\epsilon^{cx}=(\epsilon^c)^x=a^x=1+x\log_\epsilon a + \frac{x^2(\log_\epsilon a)^2}{2!}+\frac{x^3(\log_\epsilon a)^3}{3!}+\dots.$$

This is numerically true, since it is derived from (1), nich we know to be a numerical equality, since in it n is smaller than unity.

To express now $\log_\epsilon a$ as a convergent series, in order to able to calculate it with any approximation we like, ill require the use of a little dodge: since c can be anyting we like, a is necessarily also anything we like. We n therefore put $1+y$ instead of a. We get then

$$(1+y)^x=1+x\log_\epsilon(1+y)+\frac{x^2}{2!}\{\log_\epsilon(1+y)\}^2+\dots \dots(3)$$

it we have also by the binomial theorem:

$$+y)^x=1+xy+\frac{x(x-1)}{2!}y^2+\frac{x(x-1)(x-2)}{3!}y^3+\dots$$

$$=1+xy+\frac{x^2y^2}{2!}-\frac{xy^2}{2!}+\frac{x^3y^3}{3!}-\frac{3x^2y^3}{3!}+\frac{2xy^3}{3!}+\dots$$

$$=1+x\left(y-\frac{y^2}{2!}+\frac{2y^3}{3!}-\dots\right)$$

$$+x^2\left(\frac{y^2}{2!}-\frac{3y^3}{3!}+\dots\right)+\dots.$$

We can, as we have done for the binomial theorer write the coefficient of x so as to put in evidence tl law of formation of the successive terms, as follows :

$$(1+y)^x=1+x\left(y+\frac{-1}{1\times 2}y^2+\frac{(-1)\times(-2)}{1\times 2\times 3}y^3\right.$$
$$\left.+\frac{(-1)\times(-2)\times(-3)}{1\times 2\times 3\times 4}y^4+\ldots\right)+\ldots$$

we get finally

$$(1+y)^x=1+x\left(y-\frac{y^2}{2}+\frac{y^3}{3}-\frac{y^4}{4}+\ldots\right)+\ldots \ldots\ldots($$

Now, the *left* hand expressions in (3) and (4) are iden cally the same, therefore both *right* hand expressio must also be identically equal, and

$$1+x\log_\epsilon(1+y)+\ldots=1+x\left(y-\frac{y^2}{2}+\frac{y^3}{3}-\frac{y^4}{4}+\ldots\right)+\ldots$$

As we have seen before (see p. 76), in such equality, the coefficients of the same power of x a identically equal, so that we have at last

$$\log_\epsilon(1+y)=y-\frac{y^2}{2}+\frac{y^3}{3}-\frac{y^4}{4}+\ldots\ldots\ldots\ldots\ldots$$

This sequence of terms is called the "logarithr series." We arrived at this by a rather long success of steps, but each step was quite easy, and so we to our goal without much effort.

In this particular case, even if $y=1$, the series is s convergent, for

$$\log_\epsilon(1+1)=\log_\epsilon 2=1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\ldots$$
$$=1-0{\cdot}50000+0{\cdot}33333-0{\cdot}25000+\ldots$$

e terms diminishing gradually. We shall be able calculate $\log_\epsilon 2$ as we have calculated the value of itself, but with what labour! Try it, remembering that get four places of decimals correct you should go till you get a term less than 0·00001. The terms minish very slowly, or, to put it in a mathematical rm, the series converges slowly, and we get near the lue we require by zig-zagging, so to speak, each term, hether added or subtracted, carrying us always beyond e mark. It is easy to see that to get down to the rm 0·00001 or 1/100000, we shall have to take 100,000 rms! Nevertheless, it *could* be done; we should find $g_\epsilon 2 \approx 0{\cdot}6931$. If we give now to y the value $\frac{1}{2}$, we get

$$og_\epsilon\left(1+\frac{1}{2}\right)=\log_\epsilon\frac{3}{2}=\log_\epsilon 3-\log_\epsilon 2$$

$$=\frac{1}{2}-\frac{1}{2}\times\frac{1}{2^2}+\frac{1}{3}\times\frac{1}{2^3}-\frac{1}{4}\times\frac{1}{2^4}+\ldots$$

$$=0{\cdot}50000-0{\cdot}12500+0{\cdot}04167-0{\cdot}01562+\ldots.$$

This converges much more rapidly. We get after leven terms

$$g_\epsilon 3-0{\cdot}6931\approx+0{\cdot}5493-0{\cdot}1438 \quad \text{and} \quad \log_\epsilon 3\approx 1{\cdot}0986.$$

Likewise, making $y=\frac{1}{3}$ we get

$$og_\epsilon\left(1+\frac{1}{3}\right)=\log_\epsilon\frac{4}{3}=\log_\epsilon 4-\log_\epsilon 3=\log_\epsilon 4-1{\cdot}0986$$

$$=\frac{1}{3}-\frac{1}{2}\times\frac{1}{3^2}+\frac{1}{3}\times\frac{1}{3^3}-\frac{1}{4}\times\frac{1}{3^4}+\ldots\approx 0{\cdot}2877,$$

nd $\log_\epsilon 4\approx 1{\cdot}3863$,

he series again converging still more rapidly than for $\log_\epsilon 3$, and so on; $y=\frac{1}{4}$ will give $\log_\epsilon 5$, $y=\frac{1}{5}$ wil give $\log_\epsilon 6$, etc., and as the denominator in the value o y gets larger, we want fewer terms to get the logarithn with a given degree of approximation. Howeve mathematicians want to get the values they need mor quickly, by the calculation of fewer terms. This i easily obtained by a little skilful manipulation easy t follow.

We have found $\log_\epsilon(1+y)=y-\frac{y^2}{2}+\frac{y^3}{3}-\frac{y^4}{4}+\ldots$ tru as long as y is not greater than unity, as we have see It will be also true if y is negative, provided that w still have $y<1$. If y is negative, the expression (above becomes

$$\log_\epsilon(1-y)=-y-\frac{y^2}{2}-\frac{y^3}{3}-\frac{y^4}{4}\ldots \ldots\ldots\ldots(\text{(}$$

Subtracting (6) from (5) we have

$$\log_\epsilon(1+y)-\log_\epsilon(1-y)=\log_\epsilon\frac{1+y}{1-y}=y-\frac{y^2}{2}+\frac{y^3}{3}-\frac{y^4}{4}+.$$

$$-\left(-y-\frac{y^2}{2}-\frac{y^3}{3}-\frac{y^4}{4}\ldots\right),$$

or $$\log_\epsilon\frac{1+y}{1-y}=2\left(y+\frac{y^3}{3}+\frac{y^5}{5}+\ldots\right).\ldots\ldots\ldots\ldots($$

We have got rid of half the terms, and precise those, too, which were giving our approach to the fir value the zigzagging feature which made progress slow.

This new expression is true again for all values of provided that $y<1$, since it is derived from (5) and (

Suppose $y=1/(2n+1)$ with $n>1$, then

$$1+y=1+1/(2n+1)=(2n+2)/(2n+1),$$

$$1-y=1-1/(2n+1)=2n/(2n+1),$$

nce $$(1+y)/(1-y)=(2n+2)/2n=(n+1)/n,$$

ıd

$$\log_\epsilon \frac{n+1}{n}=2\left\{\frac{1}{2n+1}+\frac{1}{3(2n+1)^3}+\frac{1}{5(2n+1)^5}+\dots\right\}\dots(8)$$

r replacing in (7) y by $1/(2n+1)$.

If we now make $n=1$ we have

$$\text{g}_\epsilon \frac{2}{1}=2\left\{\frac{1}{3}+\frac{1}{3\times 27}+\frac{1}{5\times 243}+\dots\right\}$$

$$\approx 2(0{\cdot}33333+0{\cdot}01234+0{\cdot}00082+0{\cdot}00007)\approx 0{\cdot}6931,$$

ving $\log_\epsilon 2$ correct to 4 places of decimals with four rms only.

If we make $n=2$ we have likewise

$$\log_\epsilon \tfrac{3}{2}=\log_\epsilon 3-\log_\epsilon 2\approx 0{\cdot}4055 \quad\text{and}\quad \log_\epsilon 3\approx 1{\cdot}0986,$$

ıd so on.

You can therefore calculate a table of Napierian garithms. There is nothing at all mysterious about em, as you see!

Now, if $\epsilon^x=N$, $x=\log_\epsilon N$.

And if $10^y=N$, $y=\log_{10}N$.

But $\epsilon^x=10^y=N$, or $\sqrt[y]{\epsilon^x}=\epsilon^{x/y}=10$,

nce $x/y=\log_\epsilon 10=2{\cdot}3025851\dots$, let us say, $2{\cdot}3026$,

nce $x=2{\cdot}3026\times y$, or $\log_\epsilon N=2{\cdot}3026\times\log_{10}N$,

ıd $\log_{10}N=\log_\epsilon N/2{\cdot}3026=0{\cdot}434294\dots\times\log_\epsilon N$,

y, $0{\cdot}4343\log_\epsilon N$.

It follows that we can readily get the common loga rithm of a number, knowing its Napierian logarithm and *vice versa.* The number 0·4342945 ... is called th *modulus* of common logarithms.

We can, however, calculate common logarithm directly as follows :

We found that

$$\log_\epsilon \frac{n+1}{n} = 2\left(\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \ldots\right).$$

What error do we commit when we neglect all tl terms except the first one ? Evidently

$$2\left(\frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \ldots\right).$$

Now, this is obviously smaller than

$$2\left(\frac{1}{3(2n+1)^2} + \frac{1}{3(2n+1)^3} + \frac{1}{3(2n+1)^4} + \frac{1}{3(2n+1)^5} + \ldots\right.$$

since this last expression contains more terms, and t terms containing the same powers of $2n+1$ have smaller multiplier in the denominator, that is, a larger than the corresponding terms in the fi expression.

Hence, we have

$$\text{error} < \frac{2}{3(2n+1)^2}\left(1 + \frac{1}{2n+1} + \frac{1}{(2n+1)^2} + \frac{1}{(2n+1)^3} + \ldots\right.$$

The expression in the bracket is such that each te is equal to that one immediately before it multipl by a constant factor, here $1/2n+1$; we have seen t ıch a sequence, or series, is called a geometrical proression. It is very easy to get the value of the sum f any number of terms, even if this number is infinitely reat, without calculating the terms themselves, as ollows:

Call Sp the sum of p terms.

$$Sp=1+\frac{1}{2n+1}+\frac{1}{(2n+1)^2}+\dots.$$

We see that the index of the power of $2n+1$ is always qual to the rank of the term, diminished by unity, ne index in the 5th term, for instance, being 4; it ollows that the term of rank p is $\frac{1}{(2n+1)^{p-1}}$.

We have then

$$Sp=1+\frac{1}{2n+1}+\frac{1}{(2n+1)^2}+\dots+\frac{1}{(2n+1)^{p-1}}. \quad (a)$$

Iultiply both sides by $\frac{1}{2n+1}$, we get

$$Sp\times\frac{1}{2n+1}=\frac{1}{2n+1}+\frac{1}{(2n+1)^2}+\dots+\frac{1}{(2n+1)^p}. \quad (b)$$

Subtracting (*b*) from (*a*) we get:

$$Sp-Sp\times\frac{1}{2n+1}=Sp\left(1-\frac{1}{2n+1}\right)=1-\frac{1}{(2n+1)^p},$$

o that

$$Sp=\frac{1}{1-\frac{1}{2n+1}}-\frac{\frac{1}{(2n+1)^p}}{1-\frac{1}{2n+1}}. \quad (c)$$

Now if p becomes greater and greater, $\frac{1}{(2n+1)}$ becomes smaller and smaller, and when p become infinitely great $(2n+1)^p$ becomes infinitely great $\frac{1}{(2n+1)^p}$ becomes zero, and the second term of (c) dis appears, leaving us with

$$Sp=\frac{1}{1-\frac{1}{2n+1}}=\frac{2n+}{2n}.$$

Hence, $\quad \text{error} < \frac{2}{3(2n+1)^2}\times\frac{2n+1}{2n},$

or $\quad \text{error} < 1/3n(2n+1).$

It follows that, if we want to calculate a table common logarithms from 1 to, say, 100,000, it is enoug to calculate them from 10,000 to 100,000, for, as we ha seen, the decimal parts, or *mantissae*, of the logarithn of, say, 3, 71, 508, 8612 are exactly the same as t mantissae of the logarithms of 30,000, 71,000, 50,8(86,120 respectively. We can begin with $n=10{,}000$, the error will be then smaller than $1/(30000\times2000$ as we have just seen, that is, smaller than 0·0000000 and it gets smaller as n increases. It follows th taking only the first term of (8), we shall certai obtain seven places of decimals correctly. We ha then, to that degree of accuracy,

$$\log_e\frac{n+1}{.n}=\frac{2}{2n+1},$$

or

$$\log_{10}\frac{n+1}{n}=\frac{2\times0{\cdot}434294\ldots}{2n+1}.$$

If $n=10^4$, $\log_{10}(n+1)-\log_{10}n=\dfrac{0{\cdot}868588\ldots}{20001}$, and since $\log_{10}n=4$,

$$\log_{10}10001\approx 4+0{\cdot}0000434\approx 4{\cdot}0000434.$$

Now let $n=10001$.

$$\log_{10}\frac{10002}{10001}=\frac{0{\cdot}868588\ldots}{20002},$$

$$\log_{10}10002\approx 4{\cdot}0000434+0{\cdot}0000434\approx 4{\cdot}0000868,$$

and so on, only the difference between two successive logarithms will not always be 0·0000434, it gradually diminishes as n increases.

You know now everything about logarithms, even how to calculate logarithmic tables. It is monotonous work, but there is nothing difficult in it. The actual calculations—divisions, etc.—are performed by calculating machines.

As an exercise, show that

$$\epsilon^{-x}=1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\ldots.$$

$\log_\epsilon 1{\cdot}0001 = {\cdot}0001$

$\log_\epsilon 1{\cdot}001 = {\cdot}001 - \frac{{\cdot}000001}{2} = {\cdot}0010$

$\log_\epsilon 1{\cdot}01 = {\cdot}01 - \frac{{\cdot}0001}{2} + \frac{{\cdot}000001}{3}$

$= {\cdot}009995$

$\log_\epsilon 1{\cdot}1 = {\cdot}1 - \frac{{\cdot}01}{2} + \frac{{\cdot}001}{3} - \frac{{\cdot}0001}{4}$

$= {\cdot}0953$

$\log_{10} 1{\cdot}0001 = {\cdot}00004$

$\log_{10} 1{\cdot}001 = {\cdot}00043$

$\log_{10} 1{\cdot}01 = {\cdot}004343$

$\log_{10} 1{\cdot}1 = {\cdot}0414$

CHAPTER IX.

EPSILON'S HOME: THE LOGARITHMIC SPIRAL.

THE position of a point P (Fig. 4) may be defined by its distance OP from a given fixed point O (called the *Pole*), together with the angle AOP which the line OP makes with a given fixed line, such as OA. The angle AOP is usually represented by the Greek letter theta, θ; the length OP is called the *radius vector*, and is usually represented by the letter r. The position of the point P is then represented by the notation r_θ, meaning a length of length r, making an angle θ with the fixed direction OA, which is agreed to be always horizontal and extending from O to the right. In the Fig. 5 the point Q is given in position by 4_{80°, the length Oa representing one unit of length.

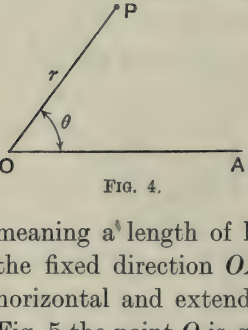


FIG. 4.

To avoid ambiguity, other conventions are necessary. The angles are positive if reckoned from OA in the direction of the arrow; they are negative in the opposite direction. For instance, if the angle AOS is 45°, then S vill be given in position by $2_{-45°}$ or by $2_{+315°}$, since the lirection of the radius vector OS may be reached either by rotating a line initially coincident with OA, and pivoted at O, either through an angle of 315° in the positive direction or through an angle of 45° in the negative direction. Also, for a given angle, when r is positive, its length is taken from O along the arm of the angle, while if r is negative its length is taken in the opposite direction, on the arm of the angle produced backwards. For instance, the position of T is given by $-4_{80°}$; it is also represented by $4_{-260°}$.

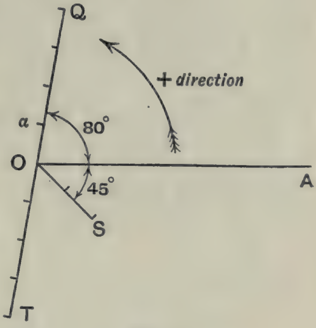


FIG. 5.

Such a way of representing the position of a point is very useful in the study of certain curves, as it enables the curve to be represented by a very simple equation instead of a complicated one. For instance, a circumference of circle of centre O and radius a is represented by the equation $r=a$, simply, since the length r is always the same,

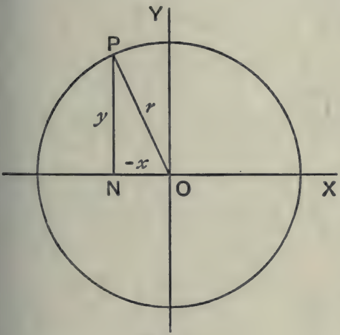


FIG. 6.

namely a, whatever may be the angle θ, which angle therefore does not enter into the equation at all. The same circle, in rectangular or x, y co-ordinates, would be represented by $x^2+y^2=r^2$, since this relation is satisfied for any point P of the circumference, as may be seen in the triangle OPN, Fig. 6. Similarly, suppose that we are dealing with a certain curve such that at any instant, P being a point on the curve, OP is equal in length to the cosine of the angle AOP (Fig. 4). The curve will clearly be represented by $r=\cos\theta$.

You are now acquainted with what mathematicians call "Polar Co-ordinates," a very imposing name for quite a simple thing. (See p. 242, Appendix.)

Among all the curves one can imagine, there are some belonging to a class which has very interesting properties, and which are called *spirals*. These curves start from the pole and describe an endless number of ever widening circumvolutions. Various spirals have different properties. We are concerned here with a spiral which can be drawn by an interesting little apparatus which we shall first describe.

A cylinder O (see Fig. 7) has a compass point and a rectangular slot in which a rod BC can slide smoothly At the end of the rod is a circular frame D fitted with a ring E, which can be turned round so as to allow of the spindle aa of a small sharp wheel F being set in any direction. A small handle G allows the apparatus to be held between the thumb and the two first fingers It will be found that, since the wheel F cannot move sideways owing to its sharp edge, the hand can only move in one direction, namely, in the direction of the plane of the wheel *F*. As the wheel revolves, if the angle ϕ it makes with the direction *AO* be a right angle, it would have no tendency to alter its distance from *O*. If the angle ϕ be less than a right angle, however, the wheel F will tend to get further from *O* as

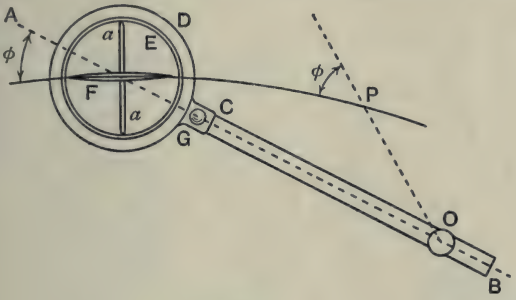


FIG. 7.

it revolves, and it will do so, the rod *CB* sliding in the slot *O*. It is easily seen that, since, for any particular setting of the ring *E*, the plane of the wheel always makes the same angle with the rod *BC*, the trace of the wheel on the paper will always make the same angle with the line joining any point *P* on this trace with the pole *O*. In other words, the curve traced by the wheel, a curve which is evidently a spiral, will make a constant angle with the radius vector. For this reason the curve is called an *equiangular spiral*.

Consider an arc *AR* of equiangular spiral traced by the wheel set in such a way that the constant angle between the curve and the radius vector at any point is 45° (see Fig. 8). Let the arc begin at A, at a distance of 1 inch from the pole O, and let the angle AOR be unity, that is, one radian. Suppose this angle AOR to be divided into a large number, n, of equal angles, AOB, BOC, etc. Then each of these small angles is $1/n$ radian. Since n is large, the angles are very small, and we can therefore consider the small arcs AB, BC, CD, etc., as short straight lines. Drop Aa, Bb, Cc perpendicularly to OB, OC, OD respectively.

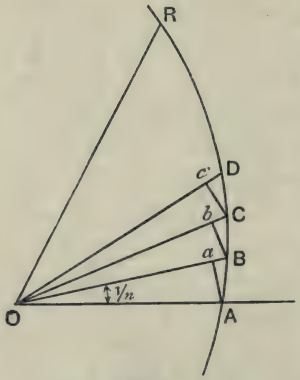


FIG. 8.

The angles aBA, bCB, cDC ... are angles of 45°; hence the small triangles AaB, BbC, CcD ... are isosceles triangles, and $Aa=aB$, $Bb=bC$, $Cc=cD$, etc. The figure does not show the equality of the sides Aa and aB, Bb and bC, etc.; this arises from the fact that, in order to limit the size of the figure, the angles, OBA, OCB, ODC ... etc. ... have actually been made greater than 45°.

Also, since the angles AOB, BOC ... are very small, we may suppose $OA=Oa$, $OB=Ob$, $OC=Oc$, etc., without introducing any appreciable error. Lastly, since length of arc = radius × angle in radians (see p. 44), we have

$$Aa=OA\times 1/n=1\times 1/n=1/n,$$
$$Bb=OB\times 1/n,$$
$$Cc=OC\times 1/n, \text{ etc.}$$

We have therefore :

$$OA = 1,$$

$$OB = Oa + aB = OA + Aa = OA + OA \times 1/n$$
$$= OA(1 + 1/n) = 1 + 1/n,$$

$$OC = OB + bC = OB + Bb = OB + OB \times 1/n$$
$$= OB(1 + 1/n) = (1 + 1/n)^2,$$

$$OD = OC + cD = OC + Cc = OC + OC \times 1/n$$
$$= OC(1 + 1/n) = (1 + 1/n)^3,$$

and so on.

We can make then the following little table :

Radius Vector.	Angle.
1 inch.	0 radian.
$1+1/n$.	$1/n$.
$(1+1/n)^2$.	$2/n$.
$(1+1/n)^3$.	$3/n$.
⋮	⋮
$(1+1/n)^n$.	$n/n=1$ radian.

If the number of angles is indefinitely great, $n = \infty$; but we know that in this case (see p. 84)

$$(1+1/n)^n = \epsilon = 2 \cdot 7183 \ldots .$$

It follows that $(1+1/n) = \sqrt[n]{\epsilon} = \epsilon^{1/n}$,

$$(1+1/n)^2 = (\epsilon^{1/n})^2 = \epsilon^{2/n},$$

$$(1+1/n)^3 = (\epsilon^{1/n})^3 = \epsilon^{3/n} \ldots, \text{ and so on.}$$

So that we have :

Radius Vector r.	Angle θ.
1 inch.	0 radian.
$\epsilon^{1/n}$.	$1/n$.
$\epsilon^{2/n}$.	$2/n$.
⋮	⋮
$\epsilon^{n/n} = \epsilon$.	$n/n=1$ radian.

In every case the length of the radius vector is power of ϵ, the index of which is the correspondin angle in radians. The equation representing the curv is therefore $r=\epsilon^{\theta}$ for this particular spiral. It follow that $\theta=\log_{\epsilon} r$, and the radian measure of any angle is th Napierian logarithm of the length of the correspondin radius vector. For this reaso this type of spiral is also calle the *logarithmic spiral.*

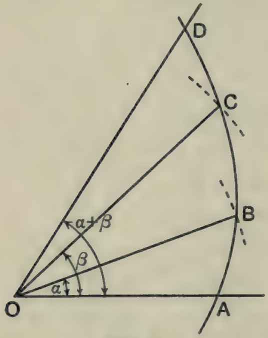


Fig. 9

If we consider a radius vect OB, of angle α (see Fig. 9), an another radius vector OC, angle $\beta=\alpha+1$, then

$$OB=\epsilon^{\alpha}, \quad OC=\epsilon^{\alpha+1},$$

$$OC/OB=\epsilon^{\alpha+1}/\epsilon^{\alpha}=\epsilon.$$

We have then found anoth definition for ϵ. Just as is the value of the ratio of the length of the circu ference of a circle to that of its diameter, ϵ is the val of the ratio of any two radii vectors of the 45° eq angular spiral at an angle of 1 radian to one anoth

If $\alpha=0$, then $\alpha+1=1$ and $OC/OA=\epsilon^{1}/\epsilon^{0}=\epsilon$; t actual value of ϵ can therefore be obtained from t curve by measuring, in inches—since OA is suppos to be 1 inch long—the radius vector corresponding the angle of 1 radian.

We see also that if we want to multiply two numb we may mark off the two numbers, in inches, by me of a compass, at, say, OB and OC (see Fig. 9); add ı the two angles AOB, AOC, which are the logarms of their radii vectors respectively, we get the le AOD, which is the sum of the logarithms, so that length in inches of the corresponding radius vector is the product of the two given numbers.

he equiangular or logarithmic spiral is therefore hing else but a graphical table of logarithms. The e of the system depends on the direction given to spindle of the tracing wheel. For an angle of 45°, system is the Napierian system of logarithms.

'he most general equation of the spiral is $r = ka^{m\theta}$, ere k, a, m are constants. Since a and m are constants, can always find a number n such that $a^m = \epsilon^n$ and equation becomes $r = k\epsilon^{n\theta}$, having only two arbitrary stants, that is, two constants which may take any able independent values. Obviously k gives the e to which the spiral is drawn, and n depends on the le ϕ at which the curve cuts the various radii vectors.

'rom the equation of a curve, all the properties the curve may be investigated mathematically. In present case, the equality $r = k\epsilon^{n\theta}$ implies the fundantal property of this particular spiral, namely, the ıstancy of this angle ϕ.

Now, when we say that a curve makes a certain gle with a line, this is rather a loose way of expressing ngs. It is more accurate to say that the tangent the curve at the point of intersection of the curve l the line make a certain angle with this line. Let therefore examine a little the properties of the ıgent to a curve.

If we consider a chord PP' (see Fig. 10), and if suppose the point P' to approach indefinitely t point P, then the chord PP' gradually approaches t direction PT, which it reaches when P' coincides w P. PT is tangent to the curve AB at P, since it l only one point in common with the curve. This expressed by saying that the tangent is the limit position of the chord PP' wl P' continuously approaches You can easily verify this yourself by drawing any cur taking two points P, P' on putting a pin in at P, a placing a ruler so that it alw touches the pin, drawing a s cession of chords of decreas length. When the length of chord has become very sm you will see that it is aln indistinguishable from a true tangent to the cu drawn at P.

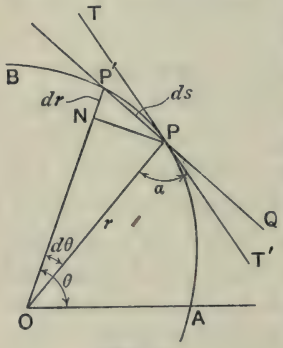


FIG. 10.

The position of the tangent PT is defined by angle α it makes with the radius vector OP. This a is the limit towards which the angle $OP'P$ tends w P' continuously approaches P. This also is easily s

Drop PN perpendicular to OP'. In the right-an triangle $PP'N$ we have:

$$\cos NP'P = NP'/PP', \quad \sin NP'P = NP/PP'.$$

Let $\quad POP' = d\theta, \quad PP' = ds, \quad NP' = dr;$

here, as you know, the letter d placed in front of , s, r, means simply "a little bit of." As a matter of ıct, you see that when P' approaches P, $d\theta$, ds and dr ll three get smaller and smaller.

$NP/PO = \sin PON$, hence $NP = r \sin(d\theta) = r\, d\theta$, for, nce $d\theta$ is very small, $\sin(d\theta) = d\theta$.

When P' practically coincides with P, the angle $NP'P$ ecomes the angle $OPT' = \alpha$. We have then

$$\cos\alpha = dr/ds, \quad \sin\alpha = r\, d\theta/ds, \quad \tan\alpha = \frac{\sin\alpha}{\cos\alpha} = r\, d\theta/dr,$$

r

$$\tan\alpha = r \Big/ \frac{dr}{d\theta}.$$

Let us pause a moment at this stage and apply our resh knowledge to another type of spiral, called the ırchimedian spiral, which has for equation in polar o-ordinates $r = k\theta$, k being a constant. In this curve he radius vector to any point is proportional to the adian measure of the corresponding angle.

Now we have just seen that $\tan\alpha = r \Big/ \frac{dr}{d\theta}$, and in this ase $dr/d\theta = k$, so that $\tan\alpha = r/k$.

It follows that if OA is an arc of Archimedian spiral see Fig. 11), if we draw a circle of radius k, cutting he curve at P, then $r = k$; at P we get for the direction of the tangent $\tan\alpha = r/k = k/k = 1$ and $\alpha = 45°$.

It follows that the tangent PT to the curve at P s the bisector of the right angle OPW.

In the case of the logarithmic spiral, the equation s $r = k\epsilon^{n\theta}$. Do you remember how to get $dr/d\theta$?

Let $\epsilon^{n\theta}=u$, then $du/d\theta=n\epsilon^{n\theta}$. (See *Calculus Made Easy*, p. 150.) $r=ku$ gives $dr/du=k$; then

$$\frac{dr}{du}\times\frac{du}{d\theta}=\frac{dr}{d\theta}=kn\epsilon^{n\theta},$$

hence $\tan\alpha=r\bigg/\dfrac{dr}{d\theta}=r/kn\epsilon^{n\theta}=r/nr=1/n=\text{constant},$

showing that the angle of the tangent to the curve at any point with the radius vector at that point is a

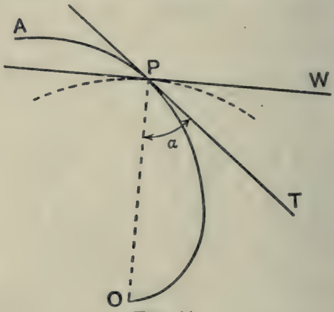


FIG. 11.

constant. This angle is, of course, the angle at which th curve cuts the radius vector, since, just at that point, th curve and its tangent may be considered as coincider along an indefinitely small portion of the curve.

In particular, in the spiral $r=k\epsilon^{\theta}$, $n=1$, $\tan\alpha=$ and the angle α is 45°, whatever the scale, that is, wha ever is the value of k.

Suppose $k=1$; when $\theta=0$, $r=1$. For a comple circumvolution of the spiral $r=\epsilon^{n\theta}$,

$\theta=360°=2\pi$, $r=\epsilon^{n\theta}=\epsilon^{2n\pi}$ and $2n\pi=\log_\epsilon r$,
so that $n=\log_\epsilon r/2\pi$ and $\tan\alpha=1/n=2\pi/\log_\epsilon r$.

In the spiral $r=\epsilon^{\theta}$, after one complete circumvolution, $\theta=2\pi$, $r=\epsilon^{6.2832}=537$ inches, $44\frac{3}{4}$ feet!

If we want r to be 10 inches only, after one complete circumvolution

$$\tan\alpha=2\pi/\log_{\epsilon}10=6\cdot2832/2\cdot3026=2\cdot729\ \ldots$$

and $\alpha=69^{\circ}\ 53'$, while $n=0\cdot3665$, so that $r=\epsilon^{0\cdot3665\theta}$.

This is a very curious spiral, much closer than the 45° logarithmic spiral.

When $r=2$, $2=\epsilon^{0\cdot3665\theta}$, solving for θ as we have learned to do in Chapter III. we find $\theta=1\cdot894$.

Now $\theta/2\pi=1\cdot894/6\cdot2832=0\cdot3010$ of a revolution, and 0·3010 is $\log_{10}2$, the *common* logarithm of 2.

Similarly, if $r=3$ we find $\theta=2\cdot997$, and $\theta/2\pi=0\cdot4771$ of a revolution $=\log_{10}3$, and so on. When $r=10$, we have $\theta=6\cdot2832$ and $\theta/2\pi=1$ revolution $=\log_{10}10$.

That is, for this spiral, the number of revolutions (or the fraction of revolution) is the *common* logarithm of the corresponding radius vector. It follows also that, since $\log 20=\log 10+\log 2$, log 20 corresponds to 1·3010 revolutions, the decimal is the same. We see that a small range of logarithms will really give an unlimited range of values, as we have seen to be the case with common logarithms.

In order to obtain the common logarithmic spiral, however, the only condition needed is $r=\epsilon^{n\theta}$ with $r=10$ when $\theta=1$; then $10=\epsilon^{n}$, $1=n\times\log_{10}\epsilon=0\cdot4343\times n$, and $n=2\cdot3026$, so that the equation of the common logarithmic spiral is $r=\epsilon^{2\cdot3026\theta}$.

Then, also, $\tan\theta=1/2\cdot3026=0\cdot4343$ and $\theta=23\frac{1}{2}$ very nearly. This gives a very wide sweeping spiral, for after one circumvolution we get

$$r = \epsilon^{2 \cdot 3026 \times 6 \cdot 2832} = 1920000 \text{ inches},$$

or 160000 feet, just above 30 miles !

If an arc of circle be described with a radius r, cutting the ϵ and the common spirals at P (see Fig. 12) and θ be measured by means of a protractor divided in radians,

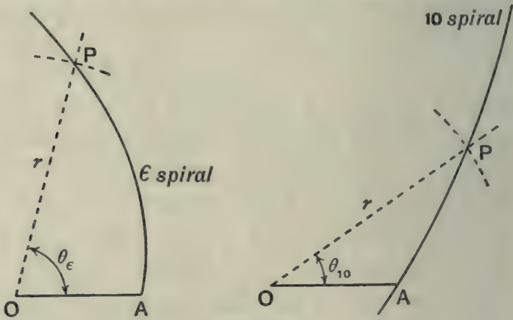


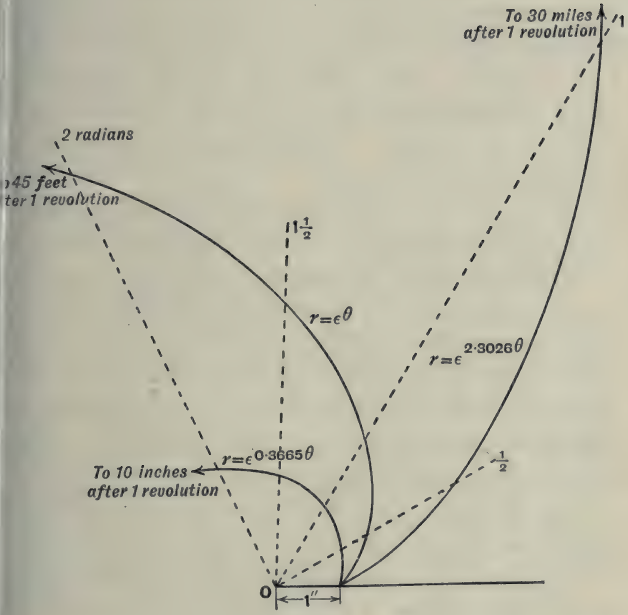
FIG. 12.

θ will be $\log_\epsilon r$ on one spiral and $\log_{10} r$ on the other, as we have seen. In the latter spiral, $r = \epsilon = 2 \cdot 7183$ occurs at an angle of $0 \cdot 4343$ or $1/2 \cdot 3026$ radian, while in the ϵ spiral $r = 10$ occurs at $2 \cdot 3026$ radian. It follows that $2 \cdot 3026 = \log_\epsilon 10$, while $0 \cdot 4343 = \log_{10} \epsilon$, and we see that $\log_{\epsilon\, 10} = 1/\log_{10} \epsilon$.

The three spirals are shown to scale on Plate I.

Note also that since $\log_\epsilon N = 2 \cdot 3026 \times \log_{10} N$, if we make a sector of $2 \cdot 3026$ radian and divide it into ten equal parts subdivided decimally, and if we apply it on the ϵ spiral, we shall, instead of $2 \cdot 3026$, read 1. Every reading, in fact, will be read off as if it was divided

PLATE I.


To 30 miles
after 1 revolution
1
2 radians
45 feet
ter 1 revolution
$1\frac{1}{2}$
$r=\epsilon^{\theta}$
$r=\epsilon^{2\cdot3026\theta}$
$r=\epsilon^{0\cdot3665\theta}$
To 10 inches
after 1 revolution
$\frac{1}{2}$
O
1″

by 2·3026, so that we shall read common logarith directly from the ϵ spiral. Such a sector is the materi ization of the modulus of common logarithms.*

In fact, in every spiral, tan α is the modulus, and c can pass from any equiangular spiral to all the oth by a suitable change in the unit of angle. As an exerci you are advised to plot the spirals $r=\epsilon^{\theta}$, $r=\epsilon^{0\cdot3}$ and $r=\epsilon^{2\cdot3026\theta}$, and verify by actual measurem the mathematical properties of the spirals touc upon in this chapter. It will be both a profitable a an interesting work.

Before we leave the logarithmic spiral, however most curious property calls our attention. If a pi of cardboard is cut so that its outline is a logarith spiral, $r=\epsilon^{\theta}$, and if a small hole is made in the ca board at the place occupied by the pole of the spi then, if the cardboard is made to roll with its sp outline against a straight ruler, the path of the p marked by the point of a pencil inserted through hole, will be found to be a straight line.

You should verify this first by actually cutting piece of cardboard to fit one of the spirals given in Plat and by causing it to roll along a ruler, and marking the position of the pole in every new position of cardboard template.

But the proof is not difficult. Consider the loga mic spiral *ANR* and the straight line *AT* tangen it at *A* (see Fig. 13), A being the point correspond say, to $\theta=0$ and $r=1$—any other point on it would

* E. A. Pochin, *Proc. Phys. Soc.* Vol. XX.

Suppose the spiral rolls on AT, so that N and R come successively in contact with it at N' and R''; then arc $AN = AN'$ and arc $ANR = AR''$, O, O', O'', etc., being the successive positions of the pole.

Now, we have seen above that the tangent to the curve at one point makes a constant angle—here 45°—with the radius vector at that point. It follows that

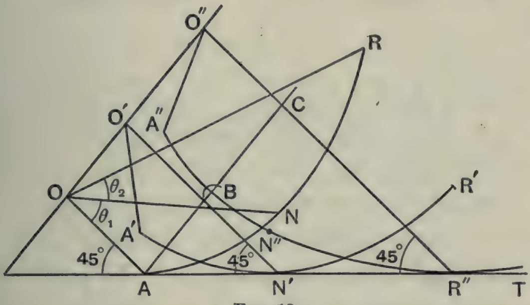


FIG. 13.

OA, $O'N'$, $O''R''$, etc. ... are parallel lines. Draw AB parallel to OO', AC parallel to OO'', etc. (until we have proved that O, O', O'', etc. are on the same straight line, we must assume AB and AC are different lines); then $OA = O'B$, $OA = O''C$, etc., so that

$$OA = O'B = O''C = \dots \text{etc.} \dots = 1.$$

Now, in *Calculus Made Easy* (last edition, pp. 277 and 278) we were shown how to find the length of any arc of the logarithmic spiral $r = \epsilon^{\theta}$, from the point corresponding to $\theta = 0$ (A, in this case) to any other point N, R, etc., corresponding to $\theta = \theta_1$, $\theta = \theta_2$, etc.

We were shown there that the lengths of an arc of the spiral from $\theta=0$ and $\theta=\vartheta_1$ is $\sqrt{2}(\epsilon^{\theta_1}-1)$.

Calculating this for various values of θ, we get the following table :

θ.	r.	Increase of r.	s (measured from A).
0.	$OA=1$.	0.	0.
θ_1.	$O'N'=\epsilon^{\vartheta_1}$.	$BN'=\epsilon^{\vartheta_1}-1$.	$AN'=\sqrt{2}(\epsilon^{\theta_1}-1)$.
θ_2.	$O''R''=\epsilon^{\theta_2}$.	$CR''=\epsilon^{\theta_2}-1$.	$AR''=\sqrt{2}(\epsilon^{\theta_2}-1)$.
⋮	⋮	⋮	⋮
etc.	etc.	etc.	etc.

It follows that the increase of length of the parallel lines OA, $O'N'$, $O''R''$, etc., is proportional to the distances AN', AR'', etc., hence their extremities O, O', O'', etc., must be on same straight line.

We see, as a matter of fact, that $BN'=AN'/\sqrt{2}$, $CR''=AR''/\sqrt{2}$, etc., showing that AN', AR'', etc. are equal in length to diagonals of squares, the sides of which are equal in length to BN', CR'', etc.

Since the angles $AN'B$, $AR''C$, etc., are 45°, the figures ABN', ACR'', etc., are the half squares themselves and the angles ABN', ACR'', etc., are right angles Since OA, $O'N'$, $O''R''$, etc., are parallel, AB, AC, etc. being perpendicular to them must be one same straight line, and as $OA=O'B=O''C$, etc., O, O', O'', etc., are also on one same straight line.

The equiangular spiral is not the only curve with which epsilon is intimately connected. In the next chapters we shall deal with another.

CHAPTER X.

A LITTLE ABOUT THE HYPERBOLA.

N the first chapter of this little book we have seen hat is meant by a "function," and we have seen how he variation of any explicit function with one variable nly can be represented by a curve traced on paper, ith reference to two scales or "axes" usually at right ngles to one another. To every such function correponds a particular curve, and, inversely, to any curve, owever complicated, corresponds a particular equation. n the case of complicated curves, however, the correponding equation may be itself too complicated to be f any use, and while it is easy, though tedious, to plot he curve corresponding to a complicated equation, the everse operation, that is, to find the equation correponding to a given curve, may be impossible, although uch an equation exists, merely because of its extreme omplexity.

Curves which exhibit regular and symmetrical eatures can, besides, usually be obtained by a geometical construction. The position of each point is defined by some geometrical condition, and all the points together constitute the curve. For instance, in the case of the simplest of all the curves, the circle, th condition is that all points are equidistant fror the centre. In the case of the ellipse, the sum of th distances of any point from two fixed points—each calle a focus,—is constant, that is, is the same for every poir of the curve, and so on. We can therefore trace suc curves geometrically, either by finding a succession (points close to one another and then drawing carefull free-hand a smooth curved line through them all, o in the simpler cases, by means of a device which en bodies the mechanical realisation of the particul geometrical construction required, such as a compass t trace a circle, or a loop of thread and two drawing pi to trace an ellipse.

A simple curve may therefore be considered geomet cally, that is, from the point of view of its geometr properties, and also from the point of view of its equatio that is, "analytically."

A circle, of centre O and radius r (see Fig. 6), f instance, has many properties which may be investigat by geometry only, without writing a single algebraic symbol. These properties, however, may also studied by algebra alone, from the equation of the cur without drawing any line or figure whatever; but figur are useful, although not necessary, to illustrate t algebraical analysis and make it clearer to the mir In the case of the circle, for instance, if we adopt O and OY for axes of co-ordinates, whatever may be t position of the point P, it is clear that we shall alwa have the relation $x^2+y^2=r^2$ (see Fig. 6); and this e equation of the circle when the axes of co-ordinates tersect one another at the centre. From this equation ery geometrical property of the circle may be deduced thout having recourse in any way to geometrical nsiderations.

The above equation is an *implicit* function of x and y ee p. 19); it may be written $y=\pm\sqrt{r^2-x^2}$, an explicit function of x, which allows the curve to be plotted

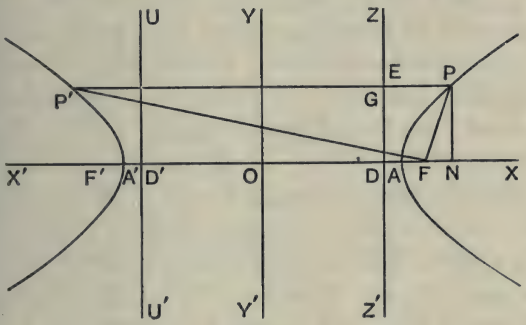


FIG. 14.

by giving various suitable values to the independent variable x.

Among the simpler curves, we are specially concerned with a curve made up of a double branch symmetrical with regard to either the line XX' or the line YY' (see Fig. 14), and called a "hyperbola." Its principal property is that the ratio of the distances of any point P on it to a certain fixed point F—called the *focus*—and to a certain fixed straight line ZZ'—called the *directrix* —has always the same value, which value is greater than unity. In other words, if P is any point on th curve, $PF/PE=e>1$, e being a constant number.

Since there is evidently a point P' such tha $P'F/P'G=e$, e having the same value as before, ther is another branch on the left of ZZ', and this implie the existence of a second focus F' and of a secon directrix UU'.

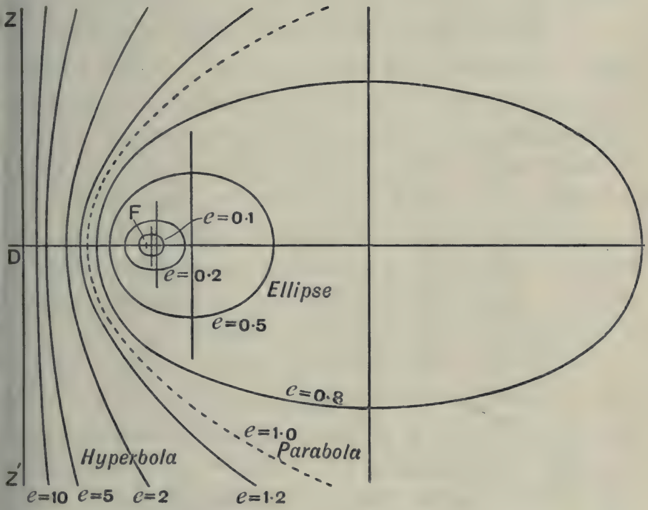
The value of e may be anything. Provided that thi value is greater than unity we get a hyperbola.

If we give to e various values, each particular valu will give a particular curve, but all these curves wi have similar features. In Plate II., for instance, th right-hand branches of hyperbolas for which e has th values 10, 5, 2, 1·2 respectively are shown, the focu and directrix being the same for all. All those curve which are obtained by giving a different value to constant are said to belong to the same "family."

We may remark here that if $e=1$ the left branc ceases to exist, as it is clearly impossible that a poin P' should exist to the left of ZZ' such that $P'F/P'G=$ or $P'F=P'G$. The geometrical properties of the curv are also modified. The curve is, in fact, no more calle a hyperbola; it has become a "parabola," shown dotte on the figure.

If e happens to be smaller than unity a curious thin happens: the branch on the left, which vanished whe e became equal to unity, now reappears on the right with its focus—just as if it had turned right round behin the paper. It meets the branch that had become parabola to form an elongated closed curve—whic

PLATE II.



Z
D
Z'
F
e=0·1
e=0·2
Ellipse
e=0·5
e=0·8
e=1·0
Parabola
Hyperbola
e=10
e=5
e=2
e=1·2

you know is called an "ellipse"—with corresponding new changes of properties. As e becomes smaller and smaller, the two foci approach one another, and the ellipse becomes less and less elongated. Plate II. shows ellipses for which the value of e is 0·8, 0·5, 0·2 and 0·1. Finally, if $e=0$, the curve becomes a circle, with, again, corresponding modifications of geometrical properties.

Here you see that as e becomes smaller the ellipses become smaller also, and when e becomes zero the ellipse is reduced to a mere point at F. This gradual shrinking of the ellipse is due to the fact that we kept the directrix at a constant distance from the point F, as may be easily seen. When $e=0$, in order to have a circle the radius of which is not indefinitely small, the distance DF should be infinitely great, that is, the directrix should be at an infinite distance, which is the same as to say that it would be infinitely remote, or absent; hence a circle has no directrix.

Various curves, all belonging to the parabola family, may be obtained by varying the distance of the focus from the directrix, e, of course, remaining always equal to unity. This is the same as if we varied merely the scale of the figure, however, so that there is really but one parabola. Likewise, for each value of e, by varying the distance of the foci from the directrix and from one another, an infinite number of hyperbolas and ellipses can be obtained, which are only the same curve drawn to a different scale each time. Only when e varies do we obtain really different curves of the same family. There is, likewise, but one circle, as all the circles we may conceive are exactly alike, and differ only in size, that is, in the scale to which they are drawn.

We see, therefore, that the hyperbola family is one of four closely connected types distinguished merely by the value of e, which may be greater than, equal to, or less than unity, or zero; two of these, the parabola and the circle, consist but of a single individual.

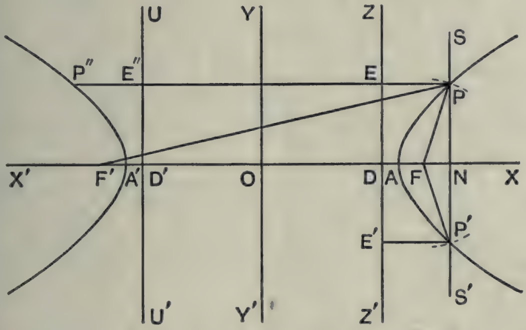


FIG. 15.

Let us now try to apply the geometrical property of the hyperbola to construct points belonging to it, so that, by joining these points by a continuous curved line we may draw the hyperbola corresponding to any relative position of directrix and focus, and to any suitable value of e. We shall only consider the right side branch of the curve, as the left side branch is obtained in exactly the same manner.

If ZZ' be one directrix (see Fig. 15) and F be the corresponding focus, then e being given numerically, if we suppose that the point A is such that $AF/AD=e$, or $AF=e\times AD$, A is evidently a point of the curve. If we take any point N on XX' and draw SS' through N perpendicularly to DF, $DN=EP$ is the distance from the directrix ZZ' of a certain point P of the curve and we simply need to ascertain the position of P along SS' so that the fundamental relation $PF/PE=e$ or $PF=e\times PE$ is satisfied. Since we know the value of e, in order to do this we have only to find the length $e\times ND$, this being the same as $e\times PE=PF$. Once this length is obtained, taking F as centre, we draw two arcs of circle, of radius PF—found as we have explained—intersecting SS' at P and P'. These two points belong to the hyperbola, since they both satisfy the condition defining the curve, namely, $PF/PE=e$ and $P'F/P'E'=e$. Other points may be obtained in a similar manner by selecting another position for the point N.

The curve is very easy to draw, and it will be found that once the position of A is known, three points on on each half of the branch, one situated approximately above or below F, one between A and P, and one beyond P will allow you to draw the curve free-hand with very fair accuracy. The point A is easy to find, because DF is a known given length, and if DA, AF are represented by v, z respectively, then $v+z=DF$ and $z=ev$, from which we get $v+ev=DF$, $v(1+e)=DF$, and $v=DA=DF/(1+e)$.

To get the other branch, find A', so that $FA'=e\times DA'$. This is again quite easy, for if DA', FA' are represented

›y v', z' respectively, then since $FA'-DA'=FD$, $'-v'=FD$ and as $z'=ev'$, $ev'-v'=FD$, so that

$$v'(e-1)=FD \quad \text{and} \quad v'=DA'=FD/(e-1).$$

Bisecting AA' gives O, and

$$OD'=OD, \quad OF'=OF$$

ɡives the position of the directrix UU' and focus F' ›f the left branch respectively. These will vary, of :ourse, for each value of e, with a given position for F ınd D. The construction may then be repeated, or, nore simply, lines such as EP may be produced to neet UU' at E'', and $P''E''$ taken equal to EP; P'' is :hen a point of the left branch of the curve. A and A' ıre each called a *vertex* of the curve, and the length $OA=OA'$ is usually represented by a.

This method of constructing the hyperbola requires, however, scale measurements of lines and arithmetical calculations, and for this reason it is rather cumbersome. There is a much simpler method of constructing the hyperbola, based on another geometrical property which is of more practical importance than the first one we have given, namely, that the difference between the distances of any point on the curve from the two foci is always the same whatever the position of the point, and is equal to the distance AA'; that is, $F'P-FP=AA'$ (see Fig. 15).

This can be easily shown to be the case, for if we drop PN perpendicular to FF', then we have

$$PF=e\times PE=e\times DN=e(ON-OD)$$

and $$PF'=e\times PE''=e\times D'N=e(ON+OD').$$

Hence, subtracting

$$PF' - PF = e(ON + OD' - ON + OD) = 2e \times OD,$$

since $\quad OD = OD',$

and as $2 \times e \times OD$ is a constant length, the first par of the above statement is verified.

Also, since $FA = eDA$ and $FA' = e \times DA'$, we hav

$$FA' - FA = e \times DA' - e \times DA = e(DA' - DA),$$

and this can be written

$$AA' = e\{DA' + DA - DA - DA\}$$

or $\quad AA' = e\{(DA' + DA) - (DA + DA)\} = e(AA' - 2AD$
$\quad = e\{AA' - (AD + A'D')\} = e \times DD' = 2e \times OD.$

As $AA' = 2OA$, it follows that $OA = e \times OD$; but we ha found above $PF' - PF = 2eOD$, hence

$$PF' - PF = 2OA = AA'.$$

We can now quickly obtain as many points the hyperbola as we nee being given the two fc F, F' and the length $2a$, follows:

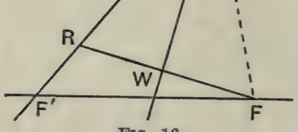


FIG. 16.

Draw any line $F'K$ throu F' (see Fig. 16). Ta $F'R = 2a$. If a is n given, but only the direct ZZ' and the focus F, reme ber we have just seen th $AA' = 2a = 2e \times OD$, hence $a = e \times OD$; now

$$OD = OA - DA = a - DF/(1+e),$$

ınce $DA=DF/(1+e)$, and, since $a=e\times OD$, we have

$$a=e\left(a-\frac{DF}{1+e}\right),$$

nd $a(e-1)=DF\times e/(1+e)$, hence $a=DF\times e/(e^2-1)$, ıence we get a. If now we join RF and bisect it at W ›y a perpendicular to RF meeting $F'K$ at P, P is a ›oint of the curve, since by construction

$$PR=PF,\quad PF'-PF=PF'-PR=F'R=2a.$$

This can also easily be performed by means of a nechanical model, and the curve traced as a continuous

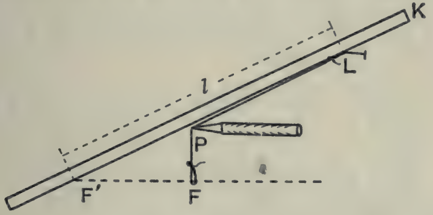


FIG. 17.

line. If we suppose a rod $F'K$ (see Fig 17) so arranged as to turn round a pin F' fixed just on its edge, and fitted with a thread fixed at F and L, if the rod is of such length l that $l-(PL+PF)=2a$, and if the thread is kept taut by a tracing point at P, as shown, then, for any position of the rod $F'K$ we have

$$\begin{aligned}PF'-PF&=PF'+PL-PF-PL\\&=(PF'+PL)-(PF+PL)=l-(PL+PF)=2a,\end{aligned}$$

so that the tracing point P is always on the curve.

Make one with a flat ruler, a piece of string and three pins, and see by yourself how the curves differ when you alter the position of the pin L.

Now that we can draw any hyperbola we choose, let us see how, from the geometrical properties we know, we can derive its equation.

Being given the directrix ZZ' and focus F, we have seen that we can easily obtain the vertices A, A', and the "centre" O by bisecting AA'. The line YY', perpendicular to FF' at O, is evidently an axis of symmetry (see Fig. 15); so is also FF'. Now, a little consideration shows that the equation of a curve will be the simplest when the axes of co-ordinates are axes of symmetry. We shall therefore take YY' and FF' for axes of coordinates.

Remembering that $FA/AD=e$ and $FA'/DA'=e$, we get $FA=eAD$, $FA'=eA'D$, hence (Fig. 15)

$$FA+FA'=e(AD+A'D)=e\times AA';$$

but $FA=F'A'$, hence we may write

$$\begin{aligned}FA+FA'=F'A'+FA'=FA+AA'+F'A'=FF'\\ =2FO=e\times AA'=2e\times AO.\end{aligned}$$

Hence $$OF=ae;$$

but we have seen above that $2OA=2e\times OD$; it follow that $OD=\dfrac{OA}{e}=\dfrac{a}{e}$.

O is our "origin" or intersection of our axes o coordinates. If P is a point on the curve, its coordinate are $ON=x$ and $PN=y$.

Now, $FP=e\times PE$, $\overline{FP}^2=e^2\overline{PE}^2=e^2\overline{ND}^2$,

but $$\overline{FN}^2+\overline{NP}^2=\overline{PF}^2=e^2\overline{ND}^2.$$

Also $FN=ON-OF=x-ae$

$DN=ON-OD=x-a/e.$

Hence $(x-ae)^2+y^2=e^2(x-a/e)^2.$

This, being a relation between x and y and constants a and e, is the equation of the curve. It can be simplified as follows:

Multiplying out, we get

$$x^2-2aex+a^2e^2+y^2=e^2x^2-2aex+a^2$$

or $$y^2+x^2(1-e^2)=a^2(1-e^2),$$

and $$y^2/[a^2(1-e^2)]+x^2/a^2=1.$$

Now, $e>1$, therefore $(1-e^2)$ is negative; let

$$a^2(1-e^2)=-b^2,$$

the equation becomes

$$x^2/a^2-y^2/b^2=1. \quad \ldots\ldots\ldots\ldots(1)$$

In this equation a is half the distance separating the two branches along the line through the foci, and $b=a\sqrt{e^2-1}$.

We can easily find a geometrical definition for b, for, from $x^2/a^2-y^2/b^2=1$, multiplying by a^2b^2, we get

$$b^2x^2-a^2y^2=a^2b^2 \quad \text{or} \quad a^2y^2=b^2(x^2-a^2),$$

hence $y/\pm\sqrt{x^2-a^2}=b/a$ or $y=\pm(b/a)\sqrt{x^2-a^2}$. (2)

This is the equation of the hyperbola when put in the form of an explicit function of x. It can be used for plotting the curve by giving suitable values to x.

Now, on ON as diameter, draw a circle (see Fig. 18).

Draw an arc of centre O and radius $OA=a$, cutting this circle at Q, then $OQ=a$, and, since $ON=x$, and the angle OQN is a right angle,

$$NQ=\pm\sqrt{\overline{ON}^2-\overline{OQ}^2}=\pm\sqrt{x^2-a^2}.$$

With centre N, draw an arc of radius NQ, meeting OF in C, then $CN=QN=\pm\sqrt{x^2-a^2}$; also join CP,

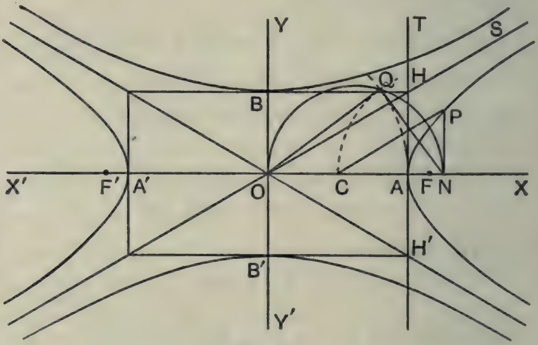


FIG. 18.

and draw OS parallel to CP, meeting at H the tangent AT to the vertex.

The two triangles OAH, CNP have their sides parallel to one another, hence they are equiangular, and we have the proportion

$$NP/CN=AH/OA \quad \text{or} \quad y/\pm\sqrt{x^2-a^2}=AH/a,$$

that is $$y=\pm(AH/a)\sqrt{x^2-a^2},$$

it follows that $AH=b$.

OS has the peculiar property of being gradually and continually approached by the hyperbola with-t ever being touched by it; in other words, the per half of the left branch of the hyperbola gets ntinually nearer to the line OS, as it gets further from without ever touching it, however far the curve may traced. Such a line is called an "asymptote" to e curve.

If we take $OB=AH=b$, then there is a twin sister the original hyperbola, shown on the figure, passing B and having the same asymptotes, the other branch ing symmetrical with respect to XX'. This twin ter is called the "conjugate hyperbola" of the first; see that its b is the a of the first hyperbola, and its is the b of this first hyperbola.

We see that the asymptotes are the diagonals of a ctangle of sides a and b. It follows that, given a and b, drawing a rectangle $2a\times 2b$ and its two diagonals oduced, one can readily draw the two branches of the perbola free-hand with fair accuracy.

A case of particular interest occurs when $a=b$. Then

$$\frac{x^2}{a^2}-\frac{y^2}{a^2}=1 \text{ or } x^2-y^2=a^2 \text{ and } y=\pm\sqrt{x^2-a^2}.$$

ie asymptotes are then at right angles, and the curve then called a "rectangular," or "equilateral," perbola.

Sometimes the two asymptotes are taken for axes of ordinates. Some simple considerations will enable to find its equation in that case.

If the equation of a curve contains a term in y^2, ien, treating it as a quadratic equation and solving for y we get a radical sign with a + and − sign, that is we get two values of y for each value of x. This is the case for the hyperbola referred to the two axes we have used so far.

For instance, suppose that the expression

$$2y^2+3xy-5y+2=0$$

is the equation of a curve. This can be written

$$y^2+\left(\frac{3x-5}{2}\right)y+1=0,$$

and its solution is

$$y=-\frac{3x-5}{4}\pm\sqrt{\left(\frac{3x-5}{4}\right)^2-1}.$$

For each value of x there are two values of y. Similarly, if the equation contains a term in x^2, for each value of y there will be two values of x, as in the case of the hyperbola referred to axes of symmetry.

If the hyperbola is referred to the asymptotes as axes (see Fig. 19), there is only one value of x corresponding to each value of y, and *vice versa*. The new equation of the curve cannot, therefore, contain any terms in x^2 or y^2. It cannot contain any terms in x^3 or y^3, as equations containing such terms are known to give, when plotted, altogether different curves—try a few. Now, if we look at the figure we see that the curve is symmetrical with regard to the point O, that is, when x becomes $-x$, y must become $-y$. It will be readily seen that it follows that there cannot be any terms in x and y in the new equation, or else y would not reverse its sign and y

ep the same value when the sign of x changes. There only one term possible then, besides constants, that

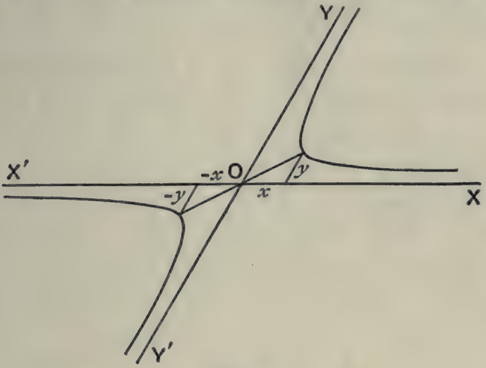


FIG. 19.

, a term containing xy, and the equation of the hyperola referred to its asymptotes is $xy=m$, where m is a

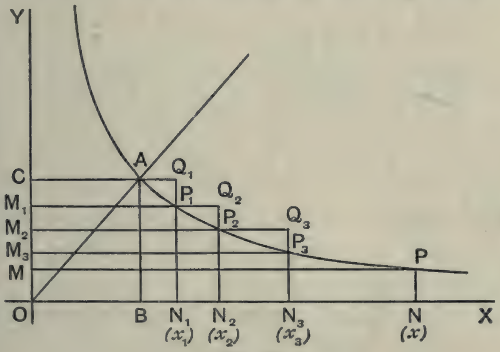


FIG. 20.

onstant. The obvious particular geometrical property of the curve then is that the rectangle of the coordinates of any point P has a constant area, whatever may be the position of P, that is (see Fig. 20),

$$P_1M_1 \times P_1N_1 = P_2M_2 \times P_2N_2 = \dots = AB \times AC = m.$$

The particular case where $m=1$ is specially interesting.

Consider the point P, of abscissa $x>1$; let $x_1=\sqrt[n]{x}$, then $x_1{}^n=x$, and evidently $x_1>1$.

Let the points $P_1, P_2, P_3 \dots P$ have abscissae $x_1, x_2, x_3 \dots x$ such that

$$\begin{aligned} x_2 &= x_1 \times x_1 = x_1{}^2, \\ x_3 &= x_2 \times x_1 = x_1{}^3, \\ &\vdots \\ x &= x_1{}^n. \end{aligned}$$

We can write at the beginning $1=x_1{}^0$, and $x_1=x_1{}^1$, so that we have the complete sequence

$$\begin{aligned} 1 &= x_1{}^0, \\ x_1 &= x_1{}^1, \text{ and since } x_1y_1=1, \ y_1=1/x_1{}^1, \\ x_2 &= x_1{}^2, \text{ and since } x_2y_2=1, \ y_2=1/x_1{}^2, \\ x_3 &= x_1{}^3, \text{ and since } x_3y_3=1, \ y_3=1/x_1{}^3, \\ &\vdots \\ x &= x_1{}^n, \text{ and since } \ xy=1, \ y=1/x_1{}^n. \end{aligned}$$

The first terms are in geometrical progression (see p. 38), having $n-1$ terms, and the constant multiplying factor is x_1.

It follows that we have for the areas of the successive small rectangles $ABN_1Q_1P_1N_1N_2Q_2$, etc.:

Area of ABN_1Q_1

$$=1 \times (x_1-1) = x_1-1,$$

rea of $P_1N_1N_2Q_2$

$$=y_1\times(x_2-x_1)=\frac{1}{x_1}(x_1{}^2-x_1)=\frac{1}{x_1}\times x_1(x_1-1)=x_1-1,$$

rea of $P_2N_2N_3Q_3$

$$=y_2(x_3-x_2)=\frac{1}{x_1{}^2}(x_1{}^3-x_1{}^2)=x_1-1,$$

ıd so on; all these rectangles have the same area. Ve can now form the following table of the total areas ıade up by the rectangles included between the ordinales ıB and Q_1N_1, Q_2N_2, etc., successively, corresponding ɔ the abscissae x_1, x_2, x_3, etc. We have

Abscissa.	Area.
1	0
x_1.	$1(x_1-1)$.
$x_2=x_1{}^2$.	$2(x_1-1)$.
$x_3=x_1{}^3$.	$3(x_1-1)$.
⋮	⋮
$x=x_1{}^n$.	$n(x_1-1)$.

Generally speaking, by the principle of mathematical nduction (see p. 55), when the abscissa is $x_1{}^A$, the total ırea of the rectangles built on it is $A(x_1-1)$.

We see that the abscissae form a geometrical proɔortion of constant multiplying factor x_1, and that the ıreas form an arithmetical progression of constant ıdditive term (x_1-1). We see also that the term of the atter corresponding to 1 in the former is zero.

We have seen in Chapter IV. that when these conditions are satisfied the two progressions form a system of logarithms with a certain base B, which we have to ascertain. The base B will be the term of the geometrical progression—here an abscissa—corresponding to unity in the arithmetical progression—here area unity.

When we wrote $x_1=\sqrt[n]{x}$, we did not specify n; if we suppose it to be very large, then all the rectangles are reduced to very thin strips, which decrease in length very gradually, and all the little triangles resulting from the encroachment of the rectangles beyond the

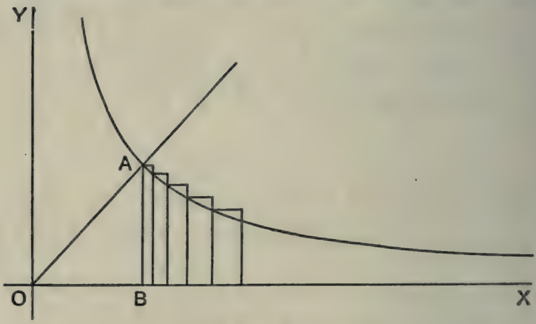


FIG. 21.

hyperbola become very small (see Fig. 21). If n becomes infinitely large, the strips become so narrow that these triangles can be neglected altogether, and the areas tabulated above will become for all practical purposes hyperbolic segments, that is, areas included between the axis of x, the curve itself, the ordinate AB, and any ordinate to the right of AB corresponding to any particular abscissa. For instance, $n(x_1-1)$=area of segment $APNB$ of the rectangular hyperbola, and we have

$$\text{area segment of hyperbola}=\log_B \text{abscissa},$$

B being a certain base which we can find by trial, by easuring, with a planimeter, say, the area of various gments until we found a segment of area unity, the scissa of the right-hand ordinate would then give the lue of B. The only sensible way to do this would be measure the areas corresponding to various abscissae, d to plot the values found. Then the abscissa correponding to area unity on the graph so obtained will ve the value of B.

We can, of course, calculate B exactly; we have en that to an abscissa $x_1{}^A$ corresponds an area (x_1-1). Suppose that this area is the base, that is, $_1{}^A=B$, then $A(x_1-1)=1$, since the areas are numerially the logarithms of the abscissae,

$$\log(x_1{}^A)=\log B=A(x_1-1)=1,$$

hen

$$_1-1=1/A,\quad x_1=1+1/A\quad \text{and}\quad x_1{}^A=B=(1+1/A)^A.$$

ow, we took n infinitely great, and since $x_1=\sqrt[n]{x}$, x_1 infinitely small; also, since $x_1{}^A=B$, $x_1=\sqrt[A]{B}$, so hat A must be infinitely large. It follows that $=(1+1/A)^A$, with A infinitely large. Do you rememer what this is equal to? Epsilon! Epsilon again! see p. 85).

In fact, the areas of the hyperbolic segments and the orresponding abscissae form a system of Napierian ogarithms. We have

$$\left.\begin{array}{l}\text{area of hyperbolic segment}\\ \text{between abscissae 1 and } x\end{array}\right\}=\log_\epsilon x.$$

Now you see why Napierian logarithms are also alled *hyperbolic* logarithms.

Instead of the above method we might use the know-ledge so pleasantly acquired when reading *Calculu Made Easy*, and proceed as follows. We need not tak $m=1$, we can work with the general equation $xy=m$ or $y=m/x$. We know that $\int dx/x=\log_\epsilon x$. The area between $x=x_1$ and $x=x_2$ is (see *Calculus Made Eas* p. 206)

$$A=\int_{x_1}^{x_2} y\,dx=\int_{x_1}^{x_2} m\,dx/x=m\Big[\log_\epsilon x+C\Big]_{x_1}^{x_2}$$

$$=m\{(\log_\epsilon x_2+C)-(\log_\epsilon x_1+C)\}=m\log_\epsilon(x_2/x_1).$$

If $x_1=1$ and $x_2=x$, we get $A=m\log_\epsilon x$.

If $m=1$ then $A=\log_\epsilon x$ as before.

There is no need for the axes of coordinate to be at rig angles to one another; we can start from the general cas

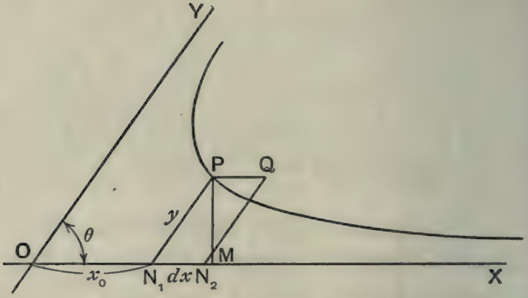


Fig. 22.

If they are inclined at an angle θ, the equation remai the same, but the ordinates are inclined at an angl to the horizontal (see Fig. 22). Each element of ar ıch as PQN_2N_1 is no more a rectangle, but a ırallelogram, and its area A is $N_1N_2 \times PM$, where 'M is the perpendicular distance between PQ and $_2N_1$, or $A = N_1N_2 \times PN_1 \sin\theta$, then

$$dA = y \sin\theta\, dx = m \sin\theta\, dx/x,$$

ence

$$l = \int m \sin\theta\, dx/x = m \sin\theta \int dx/x = m \sin\theta \log_\epsilon x + C.$$

When $x = x_0$, when $A = 0$, then $0 = m \sin\theta \log_\epsilon x_0 + C$ nd $C = -m \sin\theta \log_\epsilon x_0$, hence

$$A = m \sin\theta (\log_\epsilon x - \log_\epsilon x_0) = m \sin\theta \log_\epsilon (x/x_0).$$

If $\theta = 90°$, $\sin\theta = 1$, and $A = m \log_\epsilon (x/x_0)$.

If $m = 1$, $A = \log_\epsilon (x/x_0)$. If P is the apex, $x_0 = 1$, nd we have $A = \log_\epsilon x$ as before.

As an exercise, plot an equilateral hyperbola from its quation $xy = 1$ to a large scale. Measure the areas f various hyperbolic segments by any method you ike, and plot a graph, with areas as ordinates on the orresponding abscissae, and obtain in this way a value or epsilon.

CHAPTER XI.

EPSILON ON THE SLACK ROPE: WHAT THERE IS IN A HANGING CHAIN.

NATURE seems to take a delight in antitheses. He greatest pleasure seemingly is to put in the simples thing the most exquisite mathematical complexity o the most charmingly elaborate delicacy of texture A falling drop of water—what is there more common place? Yet an exhaustive treatment of its feature will tax the power of an able mathematician and fil several volumes. Lower the temperature, and lo! w behold the perfect loveliness of the almost unlimite varieties of the six-branched star patterns we all hav seen in flakes of snow.

What more commonplace, too, than a chain suspende from both ends? Have no misgivings, no attempt wi be made here to study all its many properties; but th little we shall learn about it will show us how wonder fully interesting this simple object is in reality.

We suppose it to be a chain, and not a string, becaus we must surmise a perfect flexibility, always lackin in a string; also, in order to take its natural shape the existence of a certain weight is necessary.

A first glance tells us that the chain hangs in an elegant curve. We have been told that every curve can be mathematically represented by an algebraical expression. Let us see if we can obtain the equation of the curve assumed by a hanging chain: the curve which the French aptly call "chainette"—that is, little chain—and which we pedantically call "catenary,"

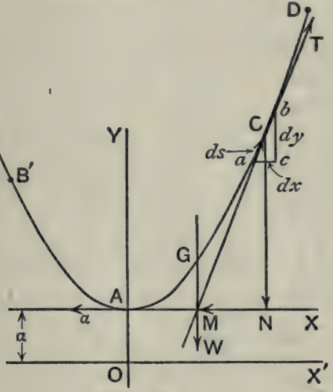


FIG. 23.

from the Greek. The shape of this hanging chain evidently depends on the forces acting upon it. We must therefore investigate those forces.

Consider a portion AC of the chain, A being its lowest point. It is acted upon by three forces:

1. Its weight W (see Fig. 23) acting at the centre of gravity G of the portion of the chain considered, that is AC; since the chain is uniform, $AG = GC$.

2. The tension or pull T exerted by the upper part CD of the chain in resisting the weight of the portion

of the chain situated below C. The direction of this pull is along the curve at C, that is, along the tangent to the curve at C.

3. The horizontal pull a exerted by the portion of the chain situated at the left of A; this pull gradually brings the chain in a horizontal direction at A, it is exerted along the curve at A, that is, horizontally.

The portion of the chain we consider is in the condition mathematicians call "state of equilibrium," that is, it is at rest under the action of the forces acting upon it—forces which balance one another so that there is no tendency for the chain to move in any way.

Now, a force can be represented just in the same manner as, in polar coordinates, we represented the distance of a point from the pole by a line at a certain angle to the horizontal (see p. 106). The direction of the force is represented by an arrow in the direction of the force, the length of the shaft of the arrow representing to a certain scale the magnitude of the force, as seen in the figure for the forces marked T, W and a.

When three forces balance one another in this way, quite elementary books in mechanics show that the lines representing them, if displaced in a suitable manner without altering their direction or length, must necessarily form a triangle, with the directions of the separate forces following one another right round the triangle.

If we produce the lines of action of T and a and displace the line of action of W till it passes through any point of the curve to the right of G, say through the point C, then we form a triangle CMN, and the three sides CN, NM, MC represent therefore the forces W, a and T respectively, to a certain scale. What scale does not actually matter at all; it merely depends on how far we have displaced the line representing the force W. In this triangle we see that

$$\overline{CM}^2 = \overline{MN}^2 + \overline{CN}^2,$$

that is $T^2 = a^2 + W^2$.

To get the equation of the curve we must first select two axes of coordinates, at right angles if possible, as we have done for the hyperbola. Let us take vertical and horizontal lines through A for axes of y and of x respectively. The first one is an "axis of symmetry," and its use will therefore simplify the equation we seek.

We must next find the connection between the geometrical shape of the curve and the forces which compel the chain to take this shape.

Consider a small link ab of the chain. If the length of the portion AC be called s, then the link, being "a little bit of s," will be, as we know, represented by ds. In the small triangle abc, $bc = dy$ and $ca = dx$ will be little bits of the ordinate y and of the abscissa x of the centre c of the link. We can simplify the conditions of the problem by noting that, since the chain is uniform, the weight of any portion is a measure of its length, so that instead of saying a length of 5 inches, of s inches, we may say a length of 5 ounces (or grammes), of s ounces (or grammes). Then we see that, numerically, $s = W$.

The triangles abc, MCN are equiangular, and therefore similar, so that we have

$$dx/ds = MN/MC = a/T = a/(\sqrt{a^2+W^2}) = a/\sqrt{(a^2+s^2)},$$

hence $dx = a\,ds/\sqrt{(a^2+s^2)}$ and $x = a\int ds/\sqrt{(a^2+s^2)}$.

Let us do the integration together: let $\sqrt{(a^2+s^2)} = v-s$, v being a variable (see p. 14), then $v = s+\sqrt{(a^2+s^2)}$.

Squaring, we get

$$a^2+s^2 = v^2-2vs+s^2 \quad \text{or} \quad a^2 = v^2-2vs.$$

To differentiate this, since both v and s are variable quantities, we first suppose s constant and differentiate with respect to v; since the differentiation of a variable gives zero, and since a is a constant, we get

$$d(a^2) = 0 = 2v\,dv - 2s\,dv.$$

We then suppose v constant and differentiate with respect to s; we get

$$d(a^2) = 0 = 0 - 2v\,ds = -2v\,ds.$$

Since the supposed variation of a^2 (variation which is nil really, since a is a constant) is made up of the variation of both v and s, the total variation is made up of both the above variations; it follows that

$$0 = 2v\,dv - 2v\,ds - 2s\,dv$$

or $\quad (v-s)dv = v\,ds \quad$ and $\quad ds = (v-s)dv/v.$

Substitute in the value of x.

$$x = a\int \frac{v-s}{v}\,dv \times \frac{1}{v-s} = a\int \frac{dv}{v}.$$

Surely you remember $\int \frac{dx}{x}$? Whenever the numerator is the differential of the denominator the integral is $\log_\epsilon x + C$. It is because of this that Napierian logarithms are so useful and occur so often; they continually "crop up" in the least expected places.

Here $\quad x = a \log_\epsilon v + C = a \log_\epsilon [s + \surd(a^2 + s^2)] + C,$

where C is, as we know, the integration constant.

What is the value of C? Well, if $x=0$, $s=0$, and then the equality above becomes

$$0 = a \times \log_\epsilon (0 + a) + C,$$

that is
$$C = -a \log_\epsilon a,$$

so that
$$x = a \log_\epsilon [s + \surd(a^2 + s^2)] - a \log_\epsilon a$$

or
$$x = a \log_\epsilon \frac{s + \surd(a^2 + s^2)}{a},$$

and
$$x/a = \log_\epsilon \frac{s + \surd(a^2 + s^2)}{a},$$

that is,

$$[s + \surd(a^2 + s^2)]/a = \epsilon^{x/a} \quad \text{and} \quad s + \surd(a^2 + s^2) = a\epsilon^{x/a},$$

and we find epsilon again appearing on the scenes.

The above seems somewhat arduous because it contains an integration. If we had merely written

$$x = a \int ds/\surd(a^2 + s^2) = a \log_\epsilon \frac{s + \surd(a^2 + s^2)}{a}$$

it would have been much shorter, but you might have felt rather lost!

Now let us get y. We have

$$dy/ds = CN/CM = W/T = W/\surd(a^2+W^2) = s/\surd(a^2+s^2)$$

or $\quad dy = s\,ds/\surd(a^2+s^2) \quad$ and $\quad y = \int s\,ds/\surd(a^2+s^2)$,

and we have another integral to negotiate. Here let $\surd(a^2+s^2)=v$ and $a^2+s^2=u$, then $v=u^{1/2}$ and

$$dv/du = \tfrac{1}{2}u^{-1/2},$$

while $2s\,ds = du$, and $du/ds = 2s$, so that

$$\frac{dv}{du}\times\frac{du}{ds}=\frac{dv}{ds}=\frac{1}{2}u^{-1/2}\times 2s=\frac{s}{\sqrt{a^2+s^2}}.$$

$$dv = s\,ds/\sqrt{a^2+s^2},$$

and $\quad v = \int dv = \int s\,ds/\surd(a^2+s^2) = \surd(a^2+s^2)+C$,

since we have supposed $v=\surd(a^2+s^2)$.

Hence $\quad y = \int s\,ds/\surd(a^2+s^2) = \surd(a^2+s^2)+C$,

C being again the integration constant. To get it[s] value, note that when $y=0$, $s=0$, so that

$$0 = \surd(a^2+0)+C \quad \text{and} \quad C=-a,$$

so that $\quad y+a = \surd(a^2+s^2)$.

Here, again, this seems to be very laborious becaus[e] of the integration; $\int s\,ds/\sqrt{(a^2+s^2)}$ being what mathe[-] maticians call a "standard form," that is, an expres[-] sion the integration of which is so well known that [it] can be written down at once, in reality all we shoul[d] have to do is to write $y=\int s\,ds/\sqrt{(a^2+s^2)}=\sqrt{(a^2+s^2)}-a$, just the same as, to solve the quadratic $x^2+mx+n=0$, we write straight away $x=-\frac{p}{2}\pm\sqrt{\left(\frac{p^2}{4}-q\right)}$ without troubling as to how this last expression is obtained.

We have then got so far :

$$\sqrt{(a^2+s^2)}=y+a \quad \text{and} \quad \sqrt{(a^2+s^2)}+s=a\epsilon^{x/a}.$$

Now $[\sqrt{(a^2+s^2)}+s][\sqrt{(a^2+s^2)}-s]=a^2+s^2-s^2=a^2$

or
$$a\epsilon^{x/a}[\sqrt{(a^2+s^2)}-s]=a^2,$$

and therefore
$$\sqrt{(a^2+s^2)}-s=a\epsilon^{-x/a},$$

but we have seen that

$$\sqrt{(a^2+s^2)}+s=a\epsilon^{+x/a}.$$

Adding, we get $2\sqrt{(a^2+s^2)}=a(\epsilon^{+x/a}+\epsilon^{-x/a})$

or
$$y+a=\frac{a}{2}(\epsilon^{+x/a}+\epsilon^{-x/a}).$$

If we take a new axis of x, OX', at a distance $AO=a$ below AX, then $y+a$ becomes the new y that is, the equation becomes $y=\frac{a}{2}(\epsilon^{+x/a}+\epsilon^{-x/a})$.

This is the equation we sought. It was not so *very* complicated after all, was it ?

As an exercise, take $a=6$, and, taking values of x between 0 and $+4$, plot the values of y.

It is worth noticing that the above reasoning does not require the two points of suspension of the chain to be on the same level. They can be placed anywhere, one high and the other one low, as can be verified experimentally: having hung a light chain on two pins D and B (see Fig. 23), placed in any position, drive a pin through one of the links at B', say, and withdraw the pin at B; no alteration whatever takes place in the shape of the chain.

Remember also that a is the tension or pull at A. Since we have used a unit of weight to express the length AC, and since a is also expressed in terms of a unit of weight—being a pull or force—a represents a length, the length of chain the weight of which is equal to the tension at the vertex, a.

We were able to obtain the value of ϵ from direct measurement of the logarithmic spiral. We have seen also that we can obtain this value from measurements of the rectangular hyperbola. The presence of ϵ in the formula of the catenary suggests that we can also obtain the value of ϵ from measurements on the curve itself. Professor Rollo Appleyard * showed how this could be done in a very elegant manner.

In *Calculus Made Easy* (last edition, p. 272) we have obtained the length s of the catenary whose equation is

$$y=\frac{a}{2}(\epsilon^{x/a}+\epsilon^{-x/a}), \quad \ldots\ldots\ldots\ldots(1$$

and we have found for this length the expression

$$s=\frac{a}{2}(\epsilon^{x/a}-\epsilon^{-x/a}). \quad \ldots\ldots\ldots\ldots(2$$

* See *Proceedings of the Physical Society*, 1914.

At the point F of the curve (see Fig. 24), correspond-g to $x=a$, and $y=y'$, we get, by replacing in (1) ıd (2) x and y by their value

$$y'=\frac{a}{2}\left(\epsilon+\frac{1}{\epsilon}\right), \qquad \text{.........................(3)}$$

ıd

$$s'=\frac{a}{2}\left(\epsilon-\frac{1}{\epsilon}\right), \qquad \text{.........................(4)}$$

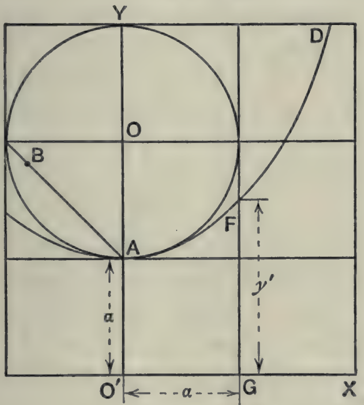


FIG. 24.

' being the length AF of the curve. Adding (3) and (4) ve get $y'+s'=a\epsilon$, and therefore $\epsilon=\dfrac{y'+s'}{a}$.

Also, subtracting (4) from (3) we get

$$y'-s'=\frac{a}{\epsilon} \quad \text{and} \quad \frac{1}{\epsilon}=\frac{y'-s'}{a}.$$

Taking a as unit of length, $a=1$, then $\epsilon=y'+s'$ and $\frac{1}{\epsilon}=y'-s'$, that is, ϵ is represented by the length FG

plus the length AF of the chain. Hence we can get by measurement on the actual chain. This, howeve is not so easy as it appears at first sight. It is eas enough to hang a thin light chain, but once this is don we must find the value of a for the particular curv assumed by the chain. Since a is the horizontal tensio at the vertex A, if one supposes a length of the chai hanging at A and passing over a minute frictionles

FIG. 25.

pulley (see Fig. 25) so place that the tangent to the curve a A is horizontal, then the weigh a of this length of chain wi be equal to the horizontal pu at A, the pulley merely deflec ing the pull due to this weigh from the vertical to the hor zontal direction. A sufficientl small frictionless pulley, howeve could not be obtained, and it was found necessary resort to another device.

If a fine thread is fixed at one end at any point (see Fig. 24), and if, at the other end A, it is ma into a loop through which the chain is passed for portion of its length, and if the end D of the chain moved until the tangent at A is horizontal and t thread AB is at 45° to the horizontal, then the leng of the hanging portion, a, of the chain is equal to t horizontal tension at A.

This follows at once from the equilibrium of forc explained at the beginning of this chapter (see p. 14

here are three forces acting at A: the pull along ıe string AB, the horizontal pull a of the chain AD, ıd the weight of the vertically hanging portion of the ıain, and these three forces balance one another. It llows that the lines representing them form a triangle, ıd as one is horizontal and another is vertical, while ıe third is at 45° to either, it follows that the triangle *ust* be such that the vertical side is of same length s the horizontal side, that is, the weight of the hanging ortion is necessarily equal to a.

The difficulty of deciding when the tangent at A is xactly horizontal is, however, very great, and a small rror in the position of the tangent causes a large error ı y', so that only a rough value of ϵ can be obtained y this means. In order to obtain the correct position, emember that we have found (see *Calculus Made Easy*, ıst edition, p. 262) that, in the case of the catenary, the entre of curvature at the vertex has for coordinates $=0$, $y=2a$, the axis being $O'Y$ and $O'X$ (see Fig. 24). he centre of curvature is therefore at O. We also ound the radius of curvature at the vertex to be $r=a$. Ve have taken a as our unit of length, so that $r=1$. he circle of curvature can be drawn, its radius being he length of chain hanging below A, a length which ve take arbitrarily. The line AB can also be drawn, ınd the end of the thread fastened at any point B on he line AB. The point of suspension D is then noved until a position is found where, while the thread AB is at 45°, the chain in the vicinity of A follows as losely as possible the circle of curvature. A fairly good approximation to the value of ϵ can be obtaine in this way.

We see again that ϵ is not a mere number, but definite length.

Note the physical meaning of the constant a; if i is large, the hanging portion of the chain will be larg the vertex A will be high above the axis $O'X$; the hori

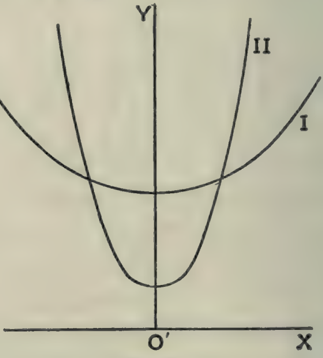
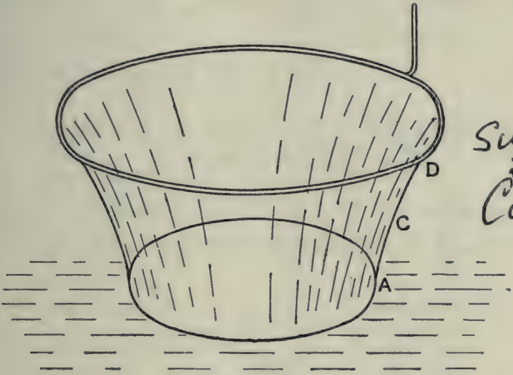


FIG. 26.

zontal tension is great, so that the chain will be widel deflected from the vertical through D, and will affec the form (I) (see Fig. 26). If a is small, the curve wi have its vertex near the axis $O'X$, and as only a sma horizontal force is deflecting the chain, the curve wi affect the shape (II).

After the circle, the catenary is perhaps the curv which has been materialised in man's engineering wor on the greatest scale of all, for the graceful appearanc of a suspension bridge is due to the fact that the cabl o which the bridge itself is suspended are curved so as o form an exact copy of the catenary curve.

The catenary is also naturally obtained by dipping circular wire in soapy water and raising it gently,



Surface is a Catenoid

FIG. 27.

keeping it in a horizontal position. A soap film will then be found to exist between the wire ring and the surface of the water, and the profile ACD of this film is a true catenary (see Fig. 27).

You are advised to plot several catenaries from two points ten units apart, let us say, using various values for a, such as 5, 1 and 0·1.

Two funnels dipped in soap solution

CHAPTER XII.

A CASE OF MATHEMATICAL MIMICRY: THE PARABOLA.

WE have seen (p. 126) that if F is a point (focus) and ZZ' is a straight line (directrix) (see Fig. 28), any point P

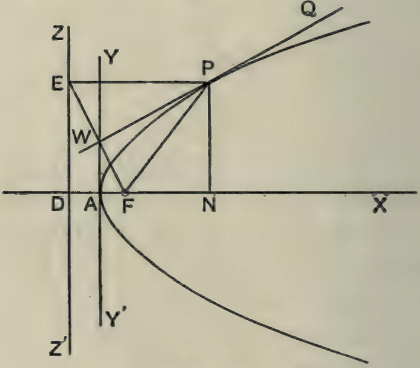


FIG. 28.

such that $PF/PE = e = 1$ (PE being perpendicular t[o] ZZ') belongs to a certain curve which we have called [a] *parabola.* The equation of this curve is very easy t[o] obtain, and it is equally easy to get geometricall[y] various points of the curve for the purpose of drawi[ng] free-hand, as we have done for the hyperbola. Let X be a perpendicular to the directrix through the cus F. A, half-way between D and F, is a point of e curve, since $AF/AD=1$. The distance DA is ually represented by a. Drop PN from P on to DX. hen $FN=DN-2DA=x-2a$. Also $PF=PE=x$, so at, since $\overline{PF}^2=\overline{PN}^2+\overline{NF}^2$,

$$x^2=y^2+(x-2a)^2 \quad \text{and} \quad y^2=4a(x-a).$$

If we draw the tangent YY' at the point A, called the ertex, and take it for the axis of Y, then $x-a$ becomes e new x, and the equation of the curve becomes $^2=4ax$.

By varying a, one can obtain various curves, which, s we have seen, will only be copies of the same curve rawn to a different scale.

If we join EF we see at once a very simple construc-on for the parabola, for, being given the directrix ZZ' nd the focus F, if we take any point E on the former, nd join EF, then by bisecting EF at right angles at W, e have a first line WP on which P is situated, and as is also on the perpendicular EP to ZZ' through E, is at once found, and similarly for other points. It easy to see that the point W is always on the tangent o the curve at the vertex A.

The parabola is specially interesting from a practical oint of view, as the trajectory of a projectile fired from gun and supposed free from the disturbing effect of he atmosphere, is a parabola concave to the ground; lso the parabolic shape is copied in various appliances, such as the so-called parabolic reflectors, throwing ou a parallel beam of light. Here, however, it claims ou interest owing to a remarkable peculiarity.

To put it in evidence, let us go back to the catenary You have plotted one for which, say, $a=5$ (see p. 153 that is, the curve $y=\frac{5}{2}(\epsilon^{x/5}+\epsilon^{-x/5})$. If we take for ax of x a line through the vertex, we know that the equatic is $y+5=\frac{5}{2}(\epsilon^{x/5}+\epsilon^{-x/5})$.

Let us plot this curve from $x=+4$ to $x=-4$. F these two points we have

$$y+5=\tfrac{5}{2}(2{\cdot}2255+0{\cdot}4493)=1{\cdot}687.$$

Let us find the equation of the parabola which pass through these two points, $x=+4$, $y=+1{\cdot}687$, $x=-$ $y=+1{\cdot}687$, and also through the vertex of the catenar The parabola stands now on its vertex, so that i x has become y, and reciprocally. Its equation is ther fore $x^2=4ay$. It will pass through the vertex of t catenary if we take the same axis of y—since bo curves are symmetrical with respect to this axis—a for axis of x the tangent to the vertex of the catenar for we have seen that the above equation correspon to these axes, and that the vertex of the parabola then at their intersection, which is also the vertex of t catenary. The parabola will pass through the tv required points if we give to a such a value that t equation $x^2=4ay$ will be verified when we give x t values ± 4 and y the value $1{\cdot}687$.

We have then

$$(\pm 4)^2=4\times 1{\cdot}687\times a \quad \text{and} \quad a=16/6{\cdot}748=2{\cdot}370.$$

The equation of the parabola is therefore

$$x^2=4\times 2{\cdot}370\times y, \quad \text{or} \quad x^2=9{\cdot}480y.$$

Plot this curve on the same axes as those used for the catenary

$$y+5=\tfrac{5}{2}(\epsilon^{x/5}+\epsilon^{-x/5}).$$

The two curves are so nearly alike that it is hardly possible to show them distinct on a diagram of the size of this page; they would almost exactly coincide. The parabola, then, apes the catenary to an extraordinary extent, so that it is possible to use the one, with the much simpler formula, instead of the other, which is more complicated to calculate. This remarkable mimicry is limited to cases when the two points selected are not far from the vertex, as in the present case. To show this better, the values of the ordinates of the catenary and of the parabola for equal values of x are tabulated below: The last value is given to show how

x	y, Catenary $y+5=\frac{5}{2}(\epsilon^{x/5}+\epsilon^{-x/5})$.	y, Parabola $y=x^2/9{\cdot}480$.	Difference, y parabola $-$ y catenary.
0	0	0	0
1	0·100	0·105	0·005
2	0·405	0·422	0·017
3	0·927	0·949	0·022
4	1·687	1·687	0
10	13·81	10·53	−3·28

the curves gradually separate beyond the two points selected.

The more stretched the chain, that is, the greater the value of a, the less the difference between the two.

It is easy to show the reason for the likeness of the two curves near the vertex; we have seen (see p. 96) that

$$\epsilon^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \text{ and } \epsilon^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots,$$

but the equation of the catenary is

$$y + a = \frac{a}{2}(\epsilon^{x/a} + \epsilon^{-x/a})$$

$$= \frac{a}{2}\left\{1 + \frac{x}{a} + \frac{x^2}{2a^2} + \frac{x^3}{6a^3} + \frac{x^4}{24a^4} + \dots \right.$$

$$\left. + 1 \quad \frac{x}{a} + \frac{x^2}{2a^2} - \frac{x^3}{6a^3} + \frac{x^4}{24a^4} - \dots \right\}$$

$$= a\left(1 + \frac{x^2}{2a^2} + \frac{x^4}{24a^4} - \dots\right).$$

If x/a be small, that is, a large (tight chain) and x small (portion in vicinity of vertex) we can neglect all the higher powers of $\frac{x}{a}$ after the second power, and we have

$$y + a = a + \frac{x^2}{2a} \text{ or } y = \frac{x^2}{2a} \text{ and } x^2 = 2ay,$$

equation of the parabola $x^2 = 4\left(\frac{a}{2}\right)y$.

We obtained the value of ϵ from the catenary, can we then obtain it from the parabola also?

Imagine a parabola cut out of cardboard, make a hole where its focus is, and lay it flat on the table with its vertex against the edge of a straight ruler. Now make the parabola roll along the ruler without sliding on either side of its first position, so that the vertex comes back exactly to the same spot on the edge of the ruler. In each position of the parabola, mark off the position of its focus by pressing the point of a pencil through the hole made at the focus (see Fig. 29). The various points so obtained will lie on a smooth curve; have you any idea as to what this curve is? Let us try to get its equation, it is not too difficult for us,

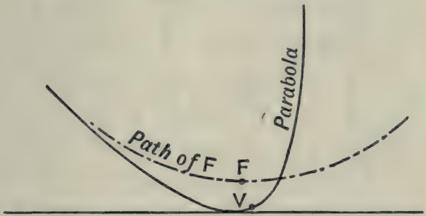


FIG. 29.

although the search is somewhat long. Let us go through it together.

We shall make use of a useful property of the parabola, namely, that if any tangent, such as TT' (see Fig. 32), meets the tangent at the vertex at R, then FR is perpendicular to TT'.* This is not difficult to show, for if we draw any chord PP' (see Fig. 30) and produce it to meet the directrix at K, then $P'K/PK = P'M'/PM = P'F/PF$, since $P'M' = P'F$, $PM = PF$. Since $P'K/PK = P'F/PF$, it follows that, by a well-known Euclid proposition, FK is a bisector of the angle PFQ.

*On Fig. 32 the point M, where TT' is tangent to the curve, happens to be on a perpendicular to AX through F, but the property is true for a tangent at any point.

If now P' is made to approach P so that PP' becomes a tangent to the parabola at P, the angle PFQ becomes

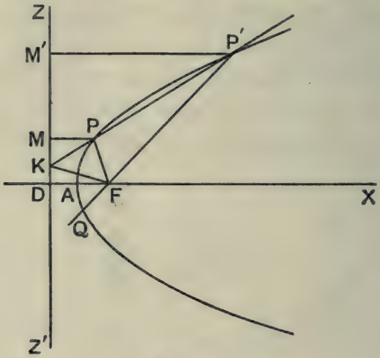


FIG. 30.

two right angles (see Fig. 31). But all the time P' is approaching P, the angle PFK is one half of the angle

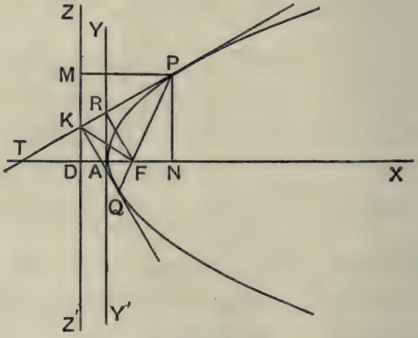


FIG. 31.

PFQ, as we have just seen; this is still true when P' is so near to P as to be *almost* coincident with it, nd it remains true when P' and P are actually oincident. It follows that the angle PFK is a right ngle, and therefore the angle KFQ is a right angle lso.

Now, the triangles MPK, PKF are equal, being ight angled at M, F, with KP common and $MP=PF$, ence angle $MPT=$angle TPF; since MP is parallel o DX, angle $MPT=$angle PTF, hence

angle $PTF=$angle TPF and $FT=FP=MP=DN$.

Hence $TD=FN$, and since $AD=AF$, it follows that $AT=AN$ and that A bisects TN, so that if now we draw the tangent at the vertex, meeting PT at R, R bisects TP, and the triangles TRF, PRF are equal, so that angle $TRF=$angle $PRF=$a right angle.

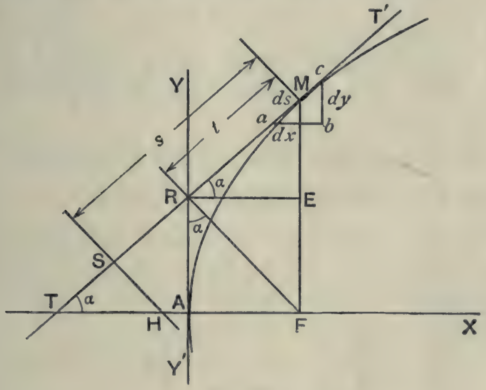


FIG. 32.

Let the perpendicular to AX at F meet the parabola at M (see Fig. 32). Draw TT' tangent at M and let it meet AY at R and AX produced at T. We ar seeking the equation of the curve of which F is on point, when the parabola is rolling along the line TT' When seeking the equation of any curve, it is advisabl to choose for axes of coordinates either axes of sym metry, or else lines playing a specially important par in the generation of the curve. A most important lin is obviously TT', the line on which the parabola i rolling. Now, if s is the length of the parabolic ar AM, and if we take $MS=s$, when the parabola roll on TT' without slipping, it is evident that the point A will touch the line TT' at S. A perpendicular SH to TT at S will evidently be an axis of symmetry of the nev curve. Let us then take as axes of coordinates the tw lines TT' and SH. As F is a point of the new curve then it follows that $FR=y'$ is the y of a point of thi curve, and $SR=x'$ is its x; the coordinates are rect angular, since we have seen above that the angle TR is a right angle.

Let the length RM be represented by t, then

$$SR=s-t=x', \text{ the new } x.$$

To get $s-t$ we shall find expressions for ds and for d respectively (a little bit of s and a little bit of t), an we shall then at once obtain

$$x'=s-t=\int d(s-t)=\int ds-\int dt=\int (ds-dt).$$

To get ds, since $(ds)^2=(dx)^2+(dy)^2$ we shall need d and dy, x and y being the coordinates of M on th parabola.

Let α be the inclination of the tangent TT' to the axis AX; in the small triangle abc $\tan\alpha = dy/dx$. Since $y^2 = 4ax$,

$$dy/dx = 2a/y,$$

so that

$$\tan\alpha = 2a/y,$$

or

$$y = 2a/\tan\alpha = 2a\cot\alpha.$$

Differentiating this we get (see *Calculus Made Easy*, p. 40)

$$dy = \frac{\tan\alpha \times 0 - 2a \times \sec^2\alpha\, d\alpha}{\tan^2\alpha} = -\frac{2a\, d\alpha}{\sin^2\alpha}.$$

Also, we have

$$x = y^2/4a = 4a^2\cot^2\alpha/4a = a\cot^2\alpha = a\cos^2\alpha/\sin^2\alpha.$$

To get dx, differentiate this fraction; we get

$$dx = \frac{\sin^2\alpha \times d(a\cos^2\alpha) - a\cos^2\alpha \times d(\sin^2\alpha)}{\sin^4\alpha},$$

to get $d(a\cos^2\alpha)$ let $z = a\cos^2\alpha$ and $\cos\alpha = v$, then

$$z = av^2, \qquad dz = 2av\, dv, \qquad dv = -\sin\alpha\, d\alpha$$

and

$$dz = -2a\cos\alpha\sin\alpha\, d\alpha.$$

Similarly, $d(\sin^2\alpha) = 2\sin\alpha\cos\alpha\, d\alpha$, so that

$$dx = \frac{-2a\sin^3\alpha\cos\alpha\, d\alpha - 2a\cos^3\alpha\sin\alpha\, d\alpha}{\sin^4\alpha}$$

$$= -\frac{2a(\sin^2\alpha + \cos^2\alpha)\cos\alpha\, d\alpha}{\sin^3\alpha} = -\frac{2a\cot\alpha\, d\alpha}{\sin^2\alpha},$$

since $\sin^2\alpha + \cos^2\alpha = 1$

Now,

$$(ds)^2=(dx)^2+(dy)^2$$

$$=\frac{4a^2\cot^2\alpha\,(d\alpha)^2}{\sin^4\alpha}+\frac{4a^2(d\alpha)^2}{\sin^4\alpha}=\frac{4a^2(1+\cot^2\alpha)(d\alpha)^2}{\sin^4\alpha}$$

$$=\frac{4a^2\operatorname{cosec}^2\alpha\,(d\alpha)^2}{\sin^4\alpha}=\frac{4a^2(d\alpha)^2}{\sin^6\alpha},$$

so that, finally, $ds=-2a\,d\alpha/\sin^3\alpha$. ....(1)

The sign is minus, since, when α increases, s decreases, and *vice versa*.

We have obtained an expression for the length of a small arc of the curve. Let us now seek an expression for the length of "a little bit" of the tangent on which the curve is rolling, that is, let us get dt.

In the right angled triangle REM,

$$RE=RM\cos\alpha,\quad \text{or}\quad x=t\cos\alpha,$$

so that

$$t=x/\cos\alpha=a\cot^2\alpha/\cos\alpha=a\cos\alpha/\sin^2\alpha.$$

Differentiating, and remembering that differentiating $\sin^2\alpha$ gives $2\sin\alpha\cos\alpha\,d\alpha$, we get

$$dt=\frac{\sin^2\alpha\,(-a\sin\alpha\,d\alpha)-a\cos\alpha\times 2\sin\alpha\cos\alpha\,d\alpha}{\sin^4\alpha},$$

or

$$dt=-\frac{a\,d\alpha}{\sin\alpha}-\frac{2a\cos^2\alpha\,d\alpha}{\sin^3\alpha}.$$

Now, we can simplify this by a little artifice: add and subtract, $\frac{2a\,d\alpha}{\sin\alpha}$, that is, $\frac{2a\sin^2\alpha\,d\alpha}{\sin^3\alpha}$, we get

$$dt=\left(\frac{2a\,d\alpha}{\sin\alpha}-\frac{a\,d\alpha}{\sin\alpha}\right)-\left(\frac{2a\cos^2\alpha\,d\alpha}{\sin^3\alpha}+\frac{2a\sin^2\alpha\,d\alpha}{\sin^3\alpha}\right),$$

ıat is, $$dt=\frac{a\,d\alpha}{\sin\alpha}-\frac{2a(\sin^2\alpha+\cos^2\alpha)\,d\alpha}{\sin^3\alpha},$$

ıd $$dt=\frac{a\,d\alpha}{\sin\alpha}-\frac{2a\,d\alpha}{\sin^3\alpha}.\qquad\qquad(2)$$

We can now proceed as outlined above :

$$ds-dt=d(s-t)=-\frac{2a\,d\alpha}{\sin^3\alpha}-\frac{a\,d\alpha}{\sin\alpha}+\frac{2a\,d\alpha}{\sin^3\alpha}=-\frac{a\,d\alpha}{\sin\alpha}.$$

$$s-t=\int d(s-t)=-\int a\,d\alpha/\sin\alpha=-a\int d\alpha/\sin\alpha.$$

To integrate this we must remember what we got ·hen we differentiated $y=\log_\epsilon\tan(x/2)$. (Here is psilon coming on the scene !)

Let $z=\tan\frac{x}{2}$, then $y=\log_\epsilon z$.

$$dy/dz=1/z=1/\tan(x/2),\quad dz/dx=(1/2)\sec^2(x/2)$$

see *Calculus Made Easy*, p. 172, Ex. 6), so that

$$\frac{dy}{dz}\times\frac{dz}{dx}=\frac{dy}{dx}=\frac{1}{2}\times\frac{1}{\tan(x/2)}\times\sec^2(x/2)$$

$$=1/2\sin(x/2)\cos(x/2)=1/\sin 2(x/2)=1/\sin x,$$

o that $dy=dx/\sin x$, and therefore

$$\int dx/\sin x=\int dy=y=\log_\epsilon\tan(x/2)+C;$$

o that here

$$s-t=-a\log_\epsilon\tan(\alpha/2)+C.$$

Find now the value of C.

When $\alpha=90^\circ=\pi/2=1{\cdot}5708$, $t=s=0$,

$$s-t=0=-a\log_\epsilon\tan(\pi/4)+C=-a\log_\epsilon 1+C=0+C,$$

and therefore $C=0$.

Hence $s=t-a\log_\epsilon\tan(\alpha/2)$.

Now $AF=a$. In the right angled triangle AR $AF=FR\times\sin ARF$, and angle ARF=angle $ATR=$ because the sides of the two triangles ARF and AT are perpendicular to one another respectively, so th $FR=a/\sin\alpha$; also $SR=s-t=-a\log_\epsilon\tan(\alpha/2)$. Th coordinates of F on the catenary with respect to th axes TT', SH, are therefore

$$y=a/\sin\alpha,\quad x=-a\log_\epsilon\tan(\alpha/2);$$

the last expression may be written $-x/a=\log_\epsilon\tan(\alpha/2)$ that is, $\tan(\alpha/2)=\epsilon^{-x/a}$.

Now $\cot(\alpha/2)=1/\tan(\alpha/2)=1/\epsilon^{-x/a}=\epsilon^{x/a}$.

We have therefore

$$\epsilon^{x/a}+\epsilon^{-x/a}=\tan(\alpha/2)+\cot(\alpha/2)=\frac{\sin(\alpha/2)}{\cos(\alpha/2)}+\frac{\cos(\alpha/2)}{\sin(\alpha/2)}$$

$$=\frac{2[\sin^2(\alpha/2)+\cos^2(\alpha/2)]}{2\sin(\alpha/2)\cos(\alpha/2)}=\frac{2\times 1}{\sin 2(\alpha/2)},$$

or

$$\epsilon^{x/a}+\epsilon^{-x/a}=2/\sin\alpha=2y/a.$$

It follows that $y=(a/2)(\epsilon^{x/a}+\epsilon^{-x/a})$, which is t equation of the path followed by the focus F duri the rolling of the parabola along TT'. This is a the equation of the catenary. From a parabola can therefore obtain a catenary. The similitude form is not merely a matter of chance, the two cur are cousins, after all!

CHAPTER XIII.

WHERE EPSILON ATTEMPTS TO FORETELL: THE PROBABILITY CURVE AND THE LAW OF ERRORS

> "All Nature is but Art, unknown to thee,
> All Chance, Direction which thou can'st not see,
> All Discord, Harmony not understood."
>
> POPE (*Essay on Man*).

THE future holds very few certainties. It is by no means absolutely certain that to-morrow will come—meaning by it the return of daylight on the portion of the Earth on which we live. Only, its return is so infinitely probable that we are justified in discarding entirely the extremely remote possibility of its failure to return.

On the other hand, it is absolutely certain that time will be going on for ever, even in an absolutely void Universe, with nothing to mark its progress, nothing to give a unit by which this progress may be measured. Equally certain is the fact that space will exist for ever, possibly entirely vacant, devoid of anything that could give the notion of its existence, still less of its magnitude.

Time and space are *abstract things* which can exist by themselves apart from any other consideration—in fact, their absence is inconceivable. The return of daylight is an *event*, and the occurrence of an even postulates the existence of some *concrete thing*, th existence of which at any future time is by no mear certain; hence there is no *certainty* in the definit occurrence of any future event. All that can be sai is that a particular event is *probable*, some events bein more probable than others, according to the case, an for some events, the possibility of the non-occurrenc of which is so exceedingly remote as to be all bu absolutely negligible, the probability is spoken of as certainty, although not strictly so in reality.

There are various degrees of probability, from th so-called certainty to impossibility, which is onl negative certainty. These degrees are usually expresse by the use of words, the only way available whenev means are lacking to express the probability of som event as a mathematical statement which will constitu a definite piece of information. It is "certain" th the daylight will return in a few hours; it is extremel probable that I, a strong healthy man, shall live ti to-morrow to see it; it is very probable that this ol seedy-looking man will do the same. In the case of sick person this is probable, or possible—that is, hard probable—or improbable; in the case of a dead perso as we understand death, it is impossible.

There are two kinds of probabilities. First, there the probability of events that are entirely left to chanc that is, the circumstances determining the occurren of which are absolutely beyond control. Such is, f instance, the probability of throwing double six at dic or of turning up four aces by taking at random four cards out of a pack, or of pulling out a black or a white ball from a bag containing balls of either colour. This kind of probability is specially interesting to the gambler.

Secondly, there is the probability of events that are influenced in some definite way by agents which act consistently so as to eliminate as much as possible all elements of chance, without being able to eradicate them completely. A gunner, for instance, will consistently direct his efforts towards the attainment of a hit exactly at the centre of the target; an astronomer endeavouring to ascertain the mean distance of the moon from the earth, or a physicist engaged in the determination of the specific heat of some substance, will concentrate all their mathematical and experimental skill on the obtainment of a value as nearly correct as possible. Yourself, while trying to extricate the value of epsilon from measurements of an arc of rectangular hyperbola, strove to avoid all causes of discrepancy between your own result and the known value. In other words, the dice are loaded; mere chance is out of the question as a ruling factor. It is not eliminated altogether, however; it remains an important factor in determining the discrepancy between the result sought and the result obtained, between the centre of the target and the spot actually hit, between the values found for the distance of the moon, the specific heat of the substance, or your value of epsilon, and the correct value for these quantities respectively. This kind of probability is specially useful to the scientist.

Of the first kind of probability we shall say little. It is easily expressed numerically, and lends itself to elaborate mathematical treatment, but, as every gambler has found by dire experience, *practical* attempts to verify the mathematical laws which are supposed to govern it generally lead to disappointment even in the simplest cases. For instance, there is evidently one chance in six of throwing one particular number of points in a one-dice throw, but if we throw a dice repeatedly, a great many times, and observe how many times each number of points turns up, we shall probably find that the number of aces, twos, etc., thrown up differs from one sixth of the total number of throws to a greater extent than theoretical considerations would lead us to expect.

As this statement may be criticised as being by no means correct, it seems worth while, in order to avoid misunderstanding, to state in detail the result of one particular experiment made to ascertain to what extent one may expect the mathematical law to be verified; that is, the analysis of the result of twelve hundred throws of a single dice, given in the following table. The throws, performed as uniformly as possible, were divided into 20 sets of 60 throws each, so that, theoretically, each face of the dice should be thrown up ten times in each set. Of course, no one would expect this to occur in any set; but what one would expect is that as the number of throws increased, the result of accumulated sets should approach nearer and nearer to an equal distribution of the throws between the six faces of the ce. This will not generally occur in any *practical* perimeter which is pushed far enough. In the table, e occurrences which are in agreement with the theory e shown in heavier type (their number is surprisingly nall, 17 only, out of 120, in the left half of the table, and only, out of 120, in the right-hand half of the table.

... 60 throws.	No. of times that each face of the dice turned up in each set.						Total number of times that each face of the dice turned up in all the throws.						Theoretical number of times.
	1	2	3	4	5	6	1	2	3	4	5	6	
1	**10**	11	13	11	7	8	**10**	11	13	11	7	8	10
2	14	6	9	12	9	**10**	24	17	22	23	16	18	20
3	9	12	9	11	**10**	9	33	29	31	34	26	27	30
4	8	12	11	9	9	11	41	41	42	43	35	38	40
5	**10**	14	6	5	15	**10**	51	55	48	48	**50**	48	50
6	**10**	**10**	5	12	9	14	61	65	53	**60**	59	62	60
7	6	9	12	11	13	9	67	74	65	71	72	71	70
8	8	7	11	5	17	12	75	81	76	76	89	83	80
9	**10**	7	9	11	11	12	85	88	85	87	100	95	90
10	9	12	6	**10**	15	8	94	**100**	91	97	115	103	100
11	8	8	9	14	8	13	102	108	100	111	123	116	110
12	13	5	11	**10**	7	14	115	113	111	121	130	130	120
13	8	**10**	12	12	8	**10**	123	123	123	133	138	140	130
14	7	7	15	11	**10**	**10**	130	130	138	144	148	150	140
15	11	11	8	7	13	**10**	141	141	146	151	161	160	150
16	12	7	11	8	12	**10**	153	148	157	159	173	170	160
17	12	11	12	**10**	7	8	165	159	169	169	180	178	170
18	8	14	11	11	**10**	6	173	173	**180**	**180**	190	184	180
19	13	1	12	8	17	9	186	174	192	188	207	193	190
20	16	9	9	5	9	12	202	183	201	193	216	205	200

The left half of the table shows the "occurrence" of ach face of the dice, that is, the number of times that ach face of the dice turned up in each set taken individually, while the right half shows this sa number of times for all the sets together, as t experiment went on. The left portion of the ta shows that in the first set the greatest departure any occurrence from the theoretical one, 10, is 3, 30 per cent. ; in the first two sets, taken togeth this greatest departure is 4, and as the theoreti occurrence is 20, this is a discrepancy of 20 per cen while in the first three sets, taken together, the great discrepancy is 4 in 30 or 13·3 per cent., for the first f sets it is 12·5 per cent., for the first five sets it is per cent., and so on. The relative discrepancy decrea gradually as one would expect, as the greater the occ rence, the smaller the discrepancy becomes in co parison, even if it actually increases numerically, it did here, rising from 3 in the first set to 4 in the f two and also in the first three sets together, to 5 in first four and also in the first five sets taken toget What it is intended to convey by the statement t *practical* attempts to verify the theory generally lea disappointment is that this gradual approach t theoretical result does not as a rule continue, even w dealing with a large number of chance-elements (throws in this case). This gradual approach is alre upset when considering the six first sets together (grea departure, 7, or 11·7 per cent., a relative discrepa higher than the one before, which was 10 per cer but one would expect, occasionally, such irregular in a distribution of numbers entirely governed chance, as long as the number of chance-element nall, and the six first sets only represent 360 throws. s we proceed, we would however expect the relative screpancies to diminish in value. But chance always olds in reserve the unexpected, which here appears the 19th set, in which a "two" turned up *once only sixty throws*, while a five turned up seventeen times. his gives for the 1140 throws (a fairly respectable umber of chance-elements) a departure of 17 for the ves, or a relative discrepancy of 8·9 per cent., whereas r the 7 first sets (420 throws) this was 7·1 per cent. nly. In other words, by nearly trebling the number throws, from 420 to 1140, we got further from the eoretical result instead of nearer to it.

It will be objected that 1140 is by no means a great umber. This is a mathematician's argument, and ad the number been 20,000 his objection would have een the same. This is exactly what is meant by the atement that *practical* verifications are disappoint-ng: however far they are pushed, chance will play icks, which will upset all theory, and, in fact, othing but an infinitely great, that is, an *unpractical*, umber of throws, in this case, would allow the variety f these tricks to be exhausted fairly with respect to the x faces of the dice, so as to secure an even distribution; fter a few scores of thousands of throws, several undred throws without a single occurrence of one articular face, or with an abnormal recurrence of the ame face, may perfectly well occur, which will upset verything.

Now, was the deficiency of "twos" and the excess of "fives" due to a defect of the dice? The followin little table, which gives the particulars of groups which the same face of the dice turned up consecutivel will answer the question. From it, it is seen that grou of *three* consecutive "fives" occur four times, a that groups of three consecutive "twos" were obtain three times, and groups of four consecutive "twos were obtained twice; the chance seems therefore

Face of the dice.					
1	2	3	4	5	6
3	3	3	3	3	3
4	4	4	5	3	3
3	3	3	3	3	3
3	3		3	3	3
4	4		3		3
					3
					3

favour the "twos" and not the "fives." Moreov groups of three "sixes" were obtained in not l than seven instances, and a group of five consecuti "fours" was obtained once. From this table one wo expect a deficiency of "threes" and an excess of ac "fours" and "sixes," yet the "threes" were norm the "sixes" only in slight excess, and the "four were deficient in number. The supposition of a "loade dice is therefore not supported by the results genera It is not supported either by an examination of the l hand half of the table showing the distribution of poi in the experiment.

It is, in fact, absolutely hopeless to predict future hance-events from a study of the similar chance-vents which have just taken place.

Not so, however, with the second kind of probability, which is for this reason of much greater interest and cientific utility. The initial numerical nature of the hance-element of the former type of probability is ere generally entirely lacking. We can only replace he looseness inherent to the use of words by the precision of a mathematical expression by gathering statistical observations. In other words, we cannot state *a priori* what the probability is (as one sixth in the case of a single hrow of a dice), but we must observe the number of imes a stated event occurs, and we can then derive rom the observations a numerical statement which vill convey a definite meaning as to the probable occurrence, in the future, of the event under observation. By "probability," then, we mean the value of the ratio :

$$\frac{\text{number of times an event occurs}}{\text{total number of possible occurrences}}.$$

For instance, from the fact that a gunner has hit the centre of the target once for every ten rounds he has fired, we may surmise that the probability of the centre of the target being hit by him in subsequent firing is once out of ten rounds, or one in ten, that is, 1/10, *supposing all the conditions to remain exactly the same.*

This is a precise numerical statement, but it must not be forgotten that, although its verification will in general be found more satisfactory than with the other type of probability, its precision is more apparent tha real; in fact, it is only a *probability*. We have put i italics a sentence that must always be present in one mind: the number expressing the probability is onl of any value if the conditions obtaining while th statistical observations were made continue to exis during the period over which the verification of th probability is pursued.

There is no way of securing this. The best gunn will become affected by fatigue, the rifling of the gu will wear, the wind will vary, the barometric pressu and the temperature will alter, and, with them, t tenuity of the atmospherical resisting medium throug which the shot travels; despite care in manufactu charges are not absolutely uniform, and with a fre batch of ammunition a perceptible difference may pe sist during the rest of the firing, altering permanent the conditions under which it takes place.

Besides, there is always the possibility of the unfor seen. A charge, now and again, *will* be faulty, somethi *will* fail. Clearly, the numerical probability is only guide, and close verification is not to be expected.

Is there any way by which we can make it mo reliable? Yes. By taking a very large number observations under all possible conditions, favoura and otherwise. The figure expressing the probabili of the event will then include every factor which c modify its occurrence, even the unforeseen circumstanc These will have possibly taken place repeatedly if t number of observations is very large, but this very la umber of observations will precisely restrict their ıfluence to its proper magnitude.

A simple example will make this clear. If we take en persons at random, and ask them to put down the orrect time as obtained from their watches, it will be ound most likely that not two watches agree, and that ot one of them is actually quite correct. It is also most kely that some of the watches will be slow and the thers will be fast; it is highly improbable that all hould be either slow or fast. Hence, by taking the verage of the ten times put down we shall get what we nay call the "observed correct time," and this will nost likely be nearer the actual correct time than most of the times put down. But what about the unforeseen? What if one of the persons had just arrived from Cairo ınd had not yet set his watch to Greenwich time, and ıad forgotten to inform us that it is two hours fast? Our "observed correct time" would be seriously in error, being fast by 120/10 minutes, or 12 minutes fast.

Instead of 10, take 100 people; the absent-mindedness of our Cairo friend will only throw out the "observed correct time" by 120/100 minutes, that is, 1 minute and 12 seconds. With 1000 people the effect is reduced to 7·2 seconds.

But with the increased number of people, other unforeseen causes of error will probably have been introduced. What if one watch has stopped altogether, undetected? An ordinary watch, that is, one with a dial divided into 12 hours, cannot possibly be wrong by more than 6 hours, and the greatest possible error arising from the fact would therefore be 720/100=7·2 minut or 7 minutes, 12 seconds with 100 people, and 43· seconds with 1000 people, while with 10,000 people would only amount to 4·32 seconds. If several watch have stopped, it is likely that some will be put dow as fast and others as slow; for instance, if it is actuall 3 p.m. a watch which stopped at 8 a.m. will be believe to be 5 hours fast, while a watch which stopped a 11 a.m. will be thought to be 4 hours slow. In this wa the errors will balance one another to a certain exten as well as being reduced to insignificance by the grea number of observations. By multiplying the observa tions so as to include all possible conditions and unfor seen circumstances a twofold result is therefore achieved errors balance one another and the effect of accident errors is rendered negligible.

There seems, at first sight, to exist a great differen between the case of the watches and the case of th gunner. The "observed correct time" will certain be a very close approximation to the true correct tim that is, the verification will be very good. In the ca of the gunner, the verification may turn out to be ve bad. It is quite possible—although hardly likely—tha although the probability of a hit at the centre of th target is 1/10, five or six shots in succession should happ to be central hits. The uncontrollable element of chan here steps in. However, in the long run, if the conditio of firing remain the same as before, if the estimate based on a sufficiently great number of observatio and if the verification also spreads over a sufficient great number of shots to restore its proper magnitude to the effect of fortuitous circumstances, it will be found that the probability will allow of a fair verification.

In the case of the watches, the numerical result was not a probability of the occurrence of an event at all, it was the value itself of a certain quantity. In the case of the gunner, the fraction 1/10 was the probability of the occurrence of a central hit, not the actual location of the hit itself, hence the difference. The two illustrations are, however, as a matter of fact, exactly similar. Each time noted down is, so to speak, a " shot " aiming at the correct time; for each watch giving exactly the correct time, a central " hit " is secured. If one watch in every ten is exactly correct on the average, the probability is that as we consult a further number of watches, one out of every ten will be correct, yet it is quite possible—although hardly so—that five or six watches examined consecutively should be correct. If the gunner is shooting at a particular mark on a blank target, the mark being invisible to us, by averaging all his shots we shall certainly get a close approximation to the exact position of that mark (see Fig. 33). The above considerations have a great importance in experimental science, for an observer, whether he is attempting to find the moon's distance or the specific heat of a substance, or the value of epsilon from the rectangular hyperbola, is merely trying to secure a central hit at a target—the " bull's eye " is here a mere number—by eliminating, or making allowance for all the factors affecting the accuracy of his "firing," just as the skilled gunner makes allowance for the range, the motion of the target, the "drift" of the projectile, the "jump" of the gun, the wind, the attenuation of the atmosphere in high angle firing, etc. Each result obtained is a "shot" which may usually be recorded on a "target diagram"—as such diagrams are very

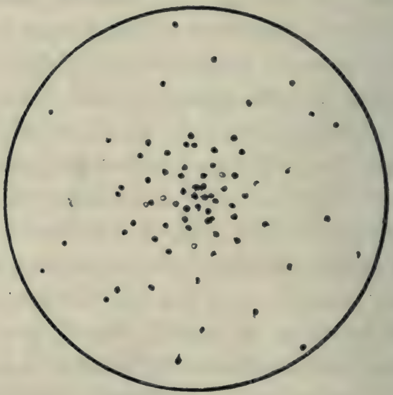


FIG. 33.

aptly termed. The points about which the "shots" cluster represent the average value obtained for the sought quantity; the distance between each "shot" and this point represents the error of the corresponding individual determination.

A particular feature of such observations is that the "bull's eye" is absent, the observer "fires" at a blank target: the correct value of the sought quantity is generally unknown—except in the case of purely experimental observations performed for an educational purpose, like our graphical determinations of epsilon —in fact, the correct value of the sought quantity can only be approximated by taking the arithmetical mean of the results given by all the observations made to ascertain it. This is on the assumption that a sufficiently great number of observations have been made to diminish the effect of exceptionally large errors, and that there is an equal tendency for errors to be in excess or in default, by which it is meant that if one of the values found is too great by a certain amount, there is amongst the other values found one which is too small by practically the same amount, so that, as far as these two values are concerned, their average gives the correct value of the required quantity.

This is no doubt a correct assumption if the errors are governed by chance alone, being what are called "accidental" errors. It is not correct if some cause is making the observed results either consistently too low or consistently too high, introducing what are called "systematic" errors, that is, causing a tendency for the "shot" to deviate always in the same direction, although, owing to the effect of accidental errors, the amount of the deviation is variable. For instance, to return to our illustration of a gunner firing at a target, a defective charge is a cause of accidental error, while the wind, if not allowed for, is a source of systematic errors. The whole skill of an observer is chiefly directed towards the elimination of systematic errors, whether by doing away with their causes or by allowing for their effects. If he is successful, then the arithmetical mean of his results, if fairly numerous observations have been made, will be a very close approximation indeed to the true value he seeks. This true value, let it be remembered, is unknown, and in most cases will remain for ever unknown, being in fact known only from the results of the observations made to ascertain it, observations which are each of them affected by unknown errors which the Calculus of Probabilities enables us to guess more or less successfully, without allowing us really to know if the guess is good or bad. We only know that, given a very large number of observations free from systematic errors, the guess is a good one, but in many cases we probably are never sure there are no unsuspected systematic errors. The only way to approach immunity in this respect is to vary as much as possible the methods of observation and the instruments used; the systematic errors will then assume to a certain extent the character of accidental errors affecting the aggregate of the results obtained, as some will tend to make the results too high, while others will tend to make it too low.

By the error of an individual observation belonging to a group, then, we mean the difference: result of individual observation—arithmetical mean of all the results of the observations belonging to the group. This is called the "absolute error," or, in some cases the "residual" of the individual observation concerned. Now, always supposing that the number of observations or "shots" is very large, it is evident *a priori* that

(1) Errors in one direction (that is, making the result too large, say) will be just as frequent as errors in the opposite direction (making the result too small).

(2) Large errors will not occur as frequently as small ones.

(3) Very large errors in either direction will not occur at all.

If x be the magnitude of an error and y the frequency of its occurrence, that is, its probability, x and y must be connected in such a way that the graph of the function expressing their connection possesses very definite features, illustrating the three facts stated above; that is:

(1) The graph must be symmetrical with respect to the axis of y, since the frequency is the same for positive or negative equal values of x.

(2) The graph must pass through a maximum for $x=0$, since small values of x occur more frequently.

(3) As x increases numerically, y must decrease, and become nil—or at any rate negligibly small—when x becomes large.

And our search for such a function leads us again face to face with epsilon, for such a graph is found to have for equation a function such as $y=\epsilon^{-x^2}$. Why a minus sign? Remember that y must be small when x is large, and this will occur if $y=1/\epsilon^{x^2}$. Why square the index? So as to get the same value of y, whatever may be the sign of the error, that is, of x. The above function gives a very definite curve; we want a function which can adapt itself to all the cases that we may have to consider. We give it more elasticity by introducing two constants. It becomes then $y=k\epsilon^{-a^2x^2}$, where k and a have numerical values depending on the particular conditions governing the distribution of the "shots."

Here it may be asked: "Why use epsilon when any other constant would give the same kind of curve?" The reason lies in the fact that, whether we differentiate or integrate ϵ^x, we still get ϵ^x. To use another constant, such as 2 or 3, instead of 2·718, would not upset the probability apple cart, but it would lead to unwieldy differentials and integrals, and would complicate matters needlessly.

In one respect the mathematical formula fails to represent what occurs in practice. It allows of very great, even of infinitely great values for x, while, in practice, as we have seen, very large errors do not occur. The greatest error of an ordinary clock is 6 hours, it cannot possibly be more; even if fired in the opposite direction to that of the target a shot will not pass an infinite distance away from it; even careless measurements on a clumsily drawn rectangular hyperbola cannot give values less than zero, and will not give values greater than 4 or 5 at the outset. But, with this exception, the formula follows very closely what actually happens in practice, and it is called for this reason the Law of Errors; its graph, called the Probability Curve, gives as ordinate the probability of the occurrence of the corresponding error as abscissa.

We see that k and a can have different values according to the particular way in which errors do occur. If we suppose $a=1$, then $y=k\epsilon^{-x^2}$; if now we give k

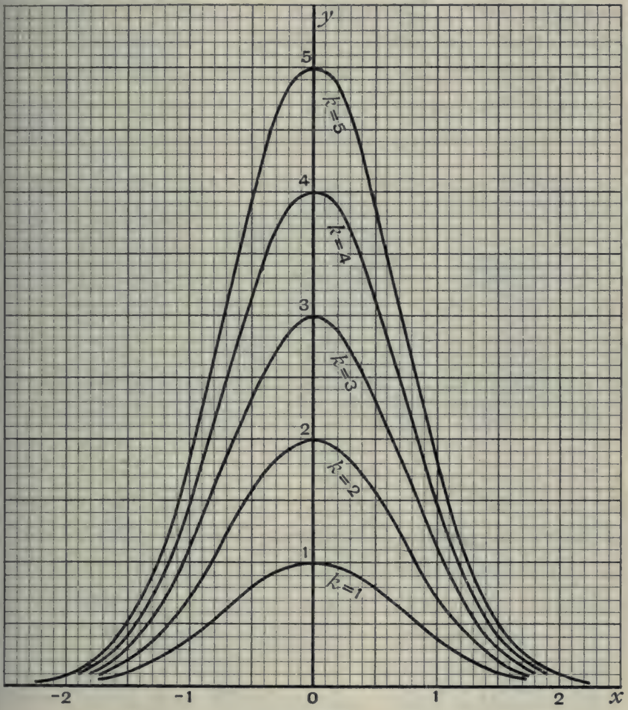


FIG. 34.

values 1, 2, 3, 4, etc., we get, for each value of k, a particular curve as seen in Fig. 34. These various curves are, however, merely copies of the same curve, namely, $y=\epsilon^{-x^2}$, to a different scale of y. This is evident from the position k occupies in the equation; it is simply a multiplier. Now when $x=0$, $y=k\epsilon^0=k$. That is, k is the ordinate at $x=0$, and this is the probable frequency of zero error. The greater the probability of very small errors the more "peaked" will be the curve, the greater will be k, and inversely. By altering in this way the scale to which y is plotted, we introduce in the mathematical equation whatever causes influence

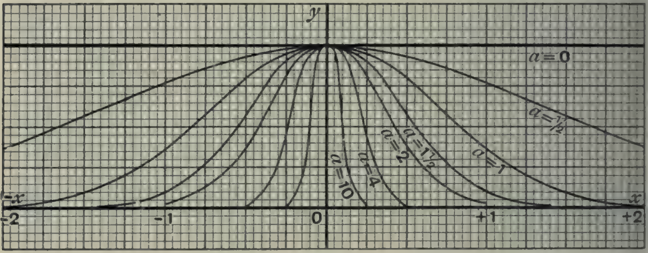


FIG. 35.

the probability of small errors, that is, the accuracy of individual results. Otherwise, k has no effect whatever on the shape of the curve, that is, on the relative distribution of errors; if we double k, we double the number (or probability) of large errors as well as the number (or probability) of small ones.

If we make $k=1$, then $y=\epsilon^{-a^2x^2}$, and if we give to a values 1, 2, 3, 4, etc., for each value of a we again get a particular curve, as shown in Fig. 35. In this case, however, all the curves are different; they all cut the axis of y at the same point, since k is the same for all, but the greater a is the more "peaked" is the urve. The physical fact corresponding to a "peaked" urve is a greater proportion of small errors, that is, greater general accuracy of the group of observations onsidered. For this reason a is called the *accuracy* *iodulus* or *modulus* of *precision*. The actual values of *ı* and k in any particular case depend on the con- istency or otherwise of the observations themselves. 'his will be more evident as we proceed.

Bessel (*Fundamenta Astronomiae*) has examined a eries of 470 astronomical determinations made by 3radley, in order to compare the theoretical frequency ›f the errors with the distribution of the actual departures ›f all the results obtained. He gives: *

Magnitude of error.	Observed number of errors.	Number of errors given by $y=k\epsilon^{-a^2x^2}$.
Between 0″·0 and 0″·1	94	95
,, 0″·1 ,, 0″·2	88	89
,, 0″·2 ,, 0″·3	78	78
,, 0″·3 ,, 0″·4	58	64
,, 0″·4 ,, 0″·5	51	50
,, 0″·5 ,, 0″·6	36	36
,, 0″·6 ,, 0″·7	26	24
,, 0″·7 ,, 0″·8	14	15
,, 0″·8 ,, 0″·9	10	9
,, 0″·9 ,, 1″·0	7	5

There are, besides, eight observed errors greater than 1″·0 against the five given by theory. It will be found that there is generally a tendency for large errors to be somewhat more frequent than theory would lead us to expect.

* See *Taylor's Scientific Memoirs*, vol. ii. 1841.

The close agreement of the observed and calculatec frequencies of errors is illustrated by the diagram o Fig. 36. The curve represents the theoretical frequency while the dots represent the observed frequency plottec at the middle of the successive spaces of 0″·1 representin the successive error magnitudes, that is, at the point

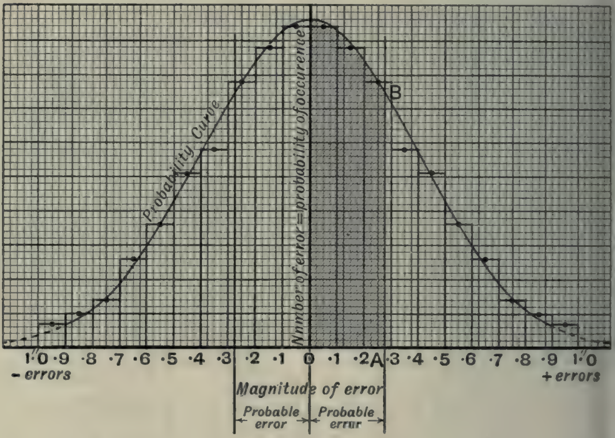


FIG. 36.

corresponding to the mean magnitude of each grou of errors. The agreement is indeed remarkable, an gives confidence in the mathematical deductions whic arise from the law of errors and form the basis of th Calculus of Probabilities.

The arithmetical mean of a number of values obtaine by measuring the same quantity is, however, liable t be affected itself by some error, as it is conceivab hat the sum of all the + errors should not exactly alance the sum of all the − errors, specially if the bservations are not very numerous. From the general eatures of the grouping of the results, however, it is ossible to form an idea as to the probable error of their rithmetical mean, so as to ascertain two values (upper nd lower limits), between which the true value of the uantity measured—*as given by the set of observations nder consideration*—is situated. This possible discrepancy between the arithmetical mean and the true alue is called the "probable error," and is denoted y r. For instance, if the arithmetical mean of the esults of a group of observations is 52·84, and it is ound that the probable error r is 0·02, it means that he true value is somewhere between 52·82 and 52·86, nd this is expressed by stating the result as being 2·84 ± 0·02.

The value of this probable error is such that the umber of errors greater and smaller, respectively, is he same. In other words, if all the errors are arranged n order of magnitude, the error which occupies the osition at the middle of the list is the probable error. Now, the area of the graph representing the law of rrors is evidently proportional to the number of bservations, since the height of each successive strip s proportional to the number of errors of any particular nagnitude, while the width of the strip corresponding o each error magnitude is the same. It follows that the nagnitude of the probable error of the arithmetical nean of a group of observations, as defined above, is the abscissa of the ordinate which divides equally ea of the two equal areas included between the probabili curve, the axis of x and the axis of y respectively (s Fig. 36). In the figure, the shaded area is exactly o quarter of the total area between the graph and t axis of x, and represents one quarter of the number observations. The ordinate AB is merely the pr bability—or frequency of occurrence—of this particul error of value r.

It is not possible to lay too much stress on the fa that this determination is based on a hypothesis whi is only correct when an infinitely large number observations is obtained, by such methods that on chance can affect their results, without the possibil of any systematic errors or of any bias on the part the observer. The first part of this condition cann of course, be actually satisfied; however, the theoreti considerations outlined above are supposed to approximately true in the case of even a limited num of observations, provided these are only affected accidental errors. It is, however, still impossible to how close to the true value is the arithmetical m value derived from the observations, all we can do i ascertain the probability that the error of this ar metical mean value is less than any particular gi value; the true value remains always unknown.

There is another conclusion to be derived from fact that the area of a given portion of the probab curve represents the number of observations corresp ing to the two ordinates between which it is situa nd also the probability that the final error is between he limits indicated by the corresponding abscissae. For the whole area, the two limits are $-\infty$ and $+\infty$. Between these two limits *all* the possible values of he final error are evidently included, so that the probability that the final error is between these limits s a certainty. We can represent certainty by unity, since this corresponds to an occurrence of a hundred per cent. of the possible events. It follows that the area of the curve must be equal to unity—to some scale—in every case, and the constants k and a must be of such value that this is obtained. As a depends on the accuracy and consistency of the observations, k must depend on the value of a. For instance if $a=\frac{1}{2}$ (see Fig. 35), k will be small so as to lower the curve, which otherwise would have too great an area, while if $a=10$, the peak of the curve being very narrow, k must be large so as to give a tall peak, and thereby raise the area to its correct value. In this way, k, that is, the scale of ordinates, adjusts itself automatically, so that the probability curve shall satisfy all the necessary conditions.

CHAPTER XIV.

TAKING A CURVE TO PIECES: EXPONENTIAL ANALYSIS.

HAVE you ever built up a curve from two others? Th
game is as follows: on a sheet of squared paper yo
draw at random two curves such as I. and II. (Fig. 37)

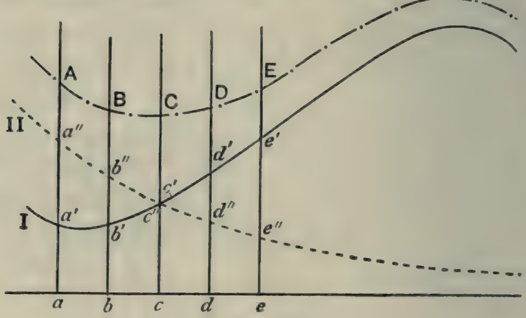


FIG. 37.

Then, at equidistant points *a*, *b*, *c*, *d*, etc., you dra
ordinates, and add the ordinates of one curve to th
corresponding ordinates of the other. For instanc
the length aa' is added to the length aa'' to get a point *A*
the length bb' is added to the length bb'' to get the poi
B, and so on. Finally a curve is drawn through th

points A, B, etc. This curve is built up of the curves I. and II. Of course, the ordinates of more than two curves may be added up in this manner, but we shall concern ourselves chiefly with only two.

Instead of taking two curves at random, one may take curves given by their equation, such as $y_1 = x^2$, $y_2 = x + 2$. Adding the ordinates will give a curve $y = y_1 + y_2 = x^2 + x + 2$. This is a way which can be used for plotting a curve, the equation of which may be split into several terms which are easy to plot separately from tables; for instance, if it were required to plot the graph of $y = \sin\theta + \epsilon^\theta$, the quickest way would be to plot the graph of $y_1 = \sin\theta$ from tables giving sin 0·1, sin 0·2, etc. (since θ is in radians), and on the same piece of squared paper to plot the graph of $y_2 = \epsilon^\theta$ from tables giving $\epsilon^{0\cdot 1}$, $\epsilon^{0\cdot 2}$, etc., then to add the ordinates of both graphs. The resultant graph $y = y_1 + y_2$ would clearly be the graph having for equation $y = \sin\theta + \epsilon^\theta$.

When we consider the inverse problem (that is, given a graph obtained in this way by adding the ordinates of two curves which have been subsequently obliterated to find these two curves again) we discover, as a rule, that this is not possible, except in simple cases, that is, cases in which the curves which have been added are very simple and few in number. It has been found possible to solve the problem of resolving a curve into the curves from which it has been obtained by the addition of ordinates, whatever may be the number of these component curves, when they are all sine curves.

This can be done even when the amplitudes—or greatest ordinates—and the starting points of these sine curves are different, provided their periodic times have some simple relation with one another, that is, provided that the length occupied by one complete portion of one curve—corresponding to 180°—is an exact fraction of the length occupied by a certain number of such complete portions of another curve, so that if both curves were continued indefinitely, both would cross the axis of x (here an angle) together at equidistant intervals. In this case, the operation of resolving a curve into its component sine curves is called "harmonic analysis." It is a process of the greatest scientific value, as it enables the calculator to unravel the complicated result of several causes, each of which acts according to a sine law, and to trace this result to its various simple elemental causes. For instance, the height of the tide at any instant is the result of a great number of causes, each one of which is simple and follows a sine law; and by observing the tides for some time, one can, from the plotted observations deduce the several component curves representing the separate variations of water level which, added together produce the tides, and in this manner deduce the exact features of each particular cause. These, once obtained in any particular case, can be added up again for years to come, and in this manner it is possible to predict the tides a long time in advance.

From some theoretical considerations, it was though by several mathematicians that there should be a solu tion to the problem of splitting a curve into its constituents in the case of a curve built up of several simple exponential curves. Several attempts were made to find the solution of the problem in this case—by Dr. Silvanus Thompson and by the writer among others—but these attempts were either failures or attracted little attention, and remained unknown except to a few, probably because the question was treated in a widely general manner, which caused it to be forbidding to all but skilled mathematicians. For instance, Professor Dale gave, in 1914, a general method of analysis of which the present problem is but a particular case. The solution of this particular case, the "exponential analysis" or the splitting into its constituents of a curve made up of exponential components, has, however, been given quite recently by Mr. J. W. T. Walsh,* in a very simple manner, and we shall explain in detail how this unravelling of the unknown curves can be successfully done.

We should like first to remark that it is commonly thought that there is nothing left to be discovered in elementary mathematics. Nothing is farther from the truth. The writer has occasionally come across original proofs of geometrical propositions discovered—unconsciously, of course—by some of his pupils. It is quite possible for any one to hit upon an untrodden track in the field of elementary mathematics, and to discover curious, important, and sometimes extremely valuable

* See *Proceedings of the Physical Society of London*, vol. xxxii. p. 26.

propositions, either in the domain of arithmetic, algebra, geometry or trigonometry. The problem with which we are dealing is a case in point; here was a problem which ought to have been solvable in a simple manner if only it could be tackled in the proper way—a problem of great importance for its practical applications, an answer to which could apparently have been found by a mere schoolboy, yet which was still begging for solution well into the twentieth century! It is true that the line of approach to the solution did belong to mathematics somewhat beyond the schoolboy's grasp, and that only a student of relatively advanced mathematical attainment would have followed the author's exposition of his method; but this is chiefly because he does not confine himself to a limited number of components at first, and because he leaves in his reasoning gaps which those who are not experts in the handling of mathematical tools would find practically impossible to bridge. Following the author's method step by step in the simple case of two component curves, and entering fully into the mathematical transformations (which mathematicians generally skip, to the discomfiture of their readers, either without any remark at all, or, adding insult to injury, with the casual remark that "one can easily see that" or "it is evident that"), we shall see that the process is very simple—the egg of Columbus itself—once the dodge of smashing the end of the egg on the table is thought of.

Each of the exponential component curves has for equation an expression of the form $y=A\epsilon^{ax}$. Generally peaking, a curve is given, either as such or by several rdinates (in practice the latter are values given by bservations or experiments), this curve being known o have been obtained by the addition of the ordinates f two exponential curves, the equations of which are $y_1=A_1\epsilon^{a_1x}$ and $y_2=A_2\epsilon^{a_2x}$, where A_1, A_2, a_1, a_2 are unknown constants. The equation of the given curve

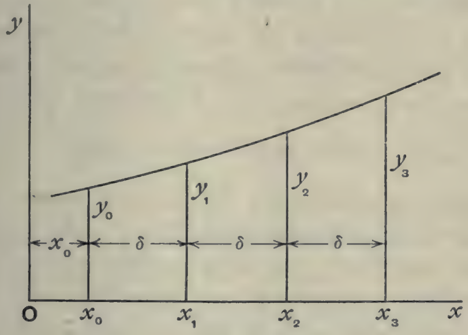


FIG. 38.

is $y=y_1+y_2$ or $y=A_1\epsilon^{a_1x}+A_2\epsilon^{a_2x}$. The problem is to discover the correct values of A_1, A_2, a_1 and a_2.

It must be remembered that only a portion of the curve is given, and that this portion may not include the intersection of the curve with the axis of y, or, in other words, it may not have any point the abscissa of which is zero. It may also be given by ordinates on both sides of the axis of y, none of which, however, correspond to an abscissa $x=0$, the intercept on the axis of y.

Let y_0, y_1, y_2 and y_3 be four equidistant ordinates (see

Fig. 38), the first one of which has x_0 for abscissa, and let δ be the constant difference between the corresponding abscissae, that is, the space between any two consecutive ordinates. Let x be reckoned from the first ordinate y_0, so that the abscissae (expressed in terms of x) corresponding to y_0, y_1, y_2 and y_3 are respectively 0, $x_1=\delta$, $x_2=2\delta$, and $x_3=3\delta$.

The suffixes 0, 1, 2 and 3 indicate the place of each ordinate in the series, or the number of spaces each equal to δ which separate it from the initial ordinate y_0. This number of spaces can be expressed generally by $\dfrac{x-x_0}{\delta}$, where x is reckoned from the origin.

In order to obtain a relation from which we can derive the four unknown quantities A_1, A_2, a_1 and a_2, we must take into consideration the portion of the group of four ordinates y_0, y_1, y_2 and y_3 with respect to the origin, that is, introduce into the equation some quantity which will express whether the ordinates are near or far from the origin. Such a quantity is x_0, and in order to introduce it in the equation, we assume $A_1=k_1\epsilon^{a_1x_0}$ and $A_2=k_2\epsilon^{a_2x_0}$, where k_1 and k_2 are such numbers that the above equations are numerically satisfied. This is the process corresponding to the smashing of the end of the egg on the table; the rest follows as a matter of course.

The equation of the given curve becomes then

$$y=k_1\epsilon^{a_1x_0}\epsilon^{a_1x}+k_2\epsilon^{a_2x_0}\epsilon^{a_2x},$$

that is,
$$y=k_1\epsilon^{a_1(x_0+x)}+k_2\epsilon^{a_2(x_0+x)}.$$

Now, for $y=y_0$, $x=0$, since x is reckoned from the first ordinate y_0, so that

$$y_0=k_1\epsilon^{a_1x_0}+k_2\epsilon^{a_2x_0},$$

that is,

$$y_0=A_1+A_2. \quad\text{.......................(1)}$$

Also, for $y=y_1$, $x=\delta$, so that

$$y_1=k_1\epsilon^{a_1x_0}\epsilon^{a_1\delta}+k_2\epsilon^{a_2x_0}\epsilon^{a_2\delta}.$$

Let $\epsilon^{a_1\delta}=z_1$ and $\epsilon^{a_2\delta}=z_2$, then we have

$$y_1=A_1z_1+A_2z_2. \quad\text{.....................(2)}$$

Similarly, for $y=y_2$, $x=2\delta$, so that

$$y_2=k_1\epsilon^{a_1x_0}\epsilon^{2a_1\delta}+k_2\epsilon^{a_2x_0}\epsilon^{2a_2\delta}$$

$$y_2=A_1(\epsilon^{a_1\delta})^2+A_2(\epsilon^{a_2\delta})^2,$$

or finally

$$y_2=A_1z_1^2+A_2z_2^2. \quad\text{.....................(3)}$$

Finally, for $y=y_3$, $x=3\delta$, so that

$$y_3=k_1\epsilon^{a_1x_0}\epsilon^{3a_1\delta}+k_2\epsilon^{a_2x_0}\epsilon^{3a_2\delta},$$

$$y_3=A_1(\epsilon^{a_1\delta})^3+A_2(\epsilon^{a_2\delta})^3,$$

or, lastly,

$$y_3=A_1z_1^3+A_2z_2^3. \quad\text{.....................(4)}$$

Now, we can imagine an equation with one unknown having two solutions z_1 and z_2; such an equation will be, of course, a quadratic equation of the form

$$z^2+p_1z+p_2=0. \quad\text{.....................(5)}$$

This we shall call the "principal equation."

Since z_1 and z_2 are solutions, if we replace z either by z_1 or by z_2 we shall have a numerical equality, or

$$\left.\begin{aligned} z_1^2+p_1z_1+p_2=0,\\ z_2^2+p_1z_2+p_2=0.\end{aligned}\right\}$$

Nothing is changed when every term of the same equation is multiplied by the same quantity. Multiply all the terms of the first equation by z_1^n and all the terms of the second equation by z_2^n, where n is a whole number which is limited, as we shall see, by the condition that it must be smaller than the number of components we are seeking to find (so that, here, n must be either 0 or 1), we get:

$$\left.\begin{aligned} z_1^2 z_1^n + p_1 z_1 z_1^n + p_2 z_1^n = 0, \\ z_2^2 z_2^n + p_1 z_2 z_2^n + p_2 z_2^n = 0. \end{aligned}\right\}$$

This is the same as

$$\left.\begin{aligned} z_1^{n+2} + p_1 z_1^{n+1} + p_2 z_1^n = 0, \\ z_2^{n+2} + p_1 z_2^{n+1} + p_2 z_2^n = 0. \end{aligned}\right\}$$

Multiply now all the terms of the first equation by A_1 and all the terms of the second equation by A_2; we get:

$$\left.\begin{aligned} A_1 z_1^{n+2} + p_1 A_1 z_1^{n+1} + p_2 A_1 z_1^n = 0, \\ A_2 z_2^{n+2} + p_1 A_2 z_2^{n+1} + p_2 A_2 z_2^n = 0. \end{aligned}\right\}$$

Adding the two members on the left and the two members on the right, we still get an equality:

$$(A_1 z_1^{n+2} + A_2 z_2^{n+2}) + p_1 (A_1 z_1^{n+1} + A_2 z_2^{n+1}) + p_2 (A_1 z_1^n + A_2 z_2^n) = 0. \quad \ldots\ldots(6)$$

The highest power of z_1 we have to deal with in this case is 3, hence $n+2$ cannot be greater than 3, that is, n is either 1 or 0.

If $n=0$ equation (6) becomes, since $z_1^0=1$ and $z_2^0=1$,

$$(A_1 z_1^2 + A_2 z_2^2) + p_1 (A_1 z_1 + A_2 z_2) + p_2 (A_1 + A_2) = 0. \quad (7)$$

Replacing the brackets in equation (7) by their values from equations (3), (2) and (1) respectively, we get :

$$y_2+p_1y_1+p_2y_0=0. \quad \text{.....................(8)}$$

If $n=1$, equation (6) becomes

$$(A_1z_1^3+A_2z_2^3)+p_1(A_1z_1^2+A_2z_2^2) \\ +p_2(A_1z_1+A_2z_2)=0. \quad (9)$$

Replacing the brackets in equation (9) by their values from equations (4), (3) and (2) respectively, we get :

$$y_3+p_1y_2+p_2y_1=0. \quad \text{..................(10)}$$

These equations (8) and (10) we shall call the "preliminary equations," because they are the first ones which are written down and solved when actually dealing with the analysis of an exponential curve into its components. They constitute a system of two equations with two unknowns, p_1 and p_2, since y_0, y_1, y_2 and y_3 are given, hence p_1 and p_2 can be easily calculated.

Replacing p_1 and p_2 by their value in equation (5) and solving that equation, we get z_1 and z_2, the two solutions.

Since $$y_0=A_1+A_2 \quad \text{..........................(11)}$$

and $$y_1=A_1z_1+A_2z_2, \quad \text{..................(12)}$$

knowing z_1 and z_2, since y_0 and y_1 are given, we easily get A_1 and A_2. For this reason we shall call the equations (11) and (12) the "final equations."

Now $z_1=\epsilon^{a_1\delta}$ and z_1, ϵ and δ are known, (δ, remember, is the interval between any two consecutive ordinates among the four equidistant ordinates y_0, y_1, y_2 and y_3) ; it follows that a_1 can easily be calculated.

Similarly, $z_2 = \epsilon^{a_2 \delta}$ gives readily the value of a_2. We have therefore obtained numerically the value of the four constants A_1, A_2, a_1 and a_2, and we can write the numerical equation of the curve. That is all!

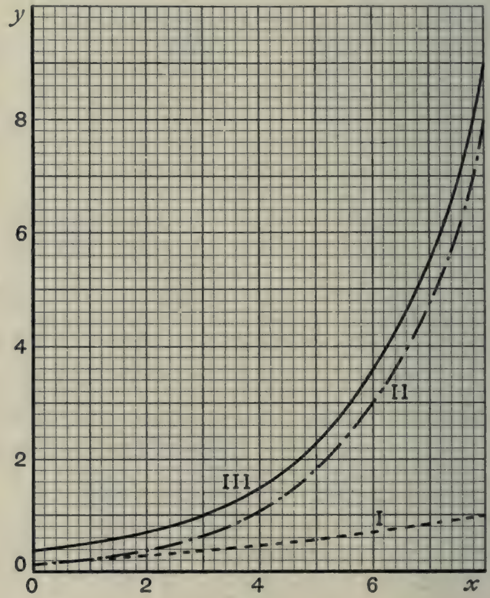


Fig. 39.

Let us apply this method to an example.

Let it be required to analyse the full line curve of Fig. 39 into its two exponential constituents.

We measure four ordinates, say, at $x=0$, $x=2$, $x=4$ and $x=6$, and find $y_0=0{\cdot}35$, $y_1=0{\cdot}71$, $y_2=1{\cdot}56$ and $y_3=3{\cdot}67$.

We get therefore for the preliminary equations :

$$\left.\begin{aligned}1{\cdot}56+0{\cdot}71p_1+0{\cdot}35p_2=0.\\3{\cdot}67+1{\cdot}56p_1+0{\cdot}71p_2=0.\end{aligned}\right\}\quad\begin{aligned}&\text{.................(8')}\\&\text{..............(10')}\end{aligned}$$

The simplest way to solve such equations is as follows :

From (8′) we get $p_2=-\dfrac{1{\cdot}56+0{\cdot}71p_1}{0{\cdot}35}$.

From (10′) we get $p_2=-\dfrac{3{\cdot}67+1{\cdot}56p_1}{0{\cdot}71}$.

Hence $\dfrac{1{\cdot}56+0{\cdot}71p_1}{0{\cdot}35}=\dfrac{3{\cdot}67+1{\cdot}56p_1}{0{\cdot}71}$.

Using four-figure tables and taking the products to three places we get

$$1{\cdot}107+0{\cdot}504p_1=1{\cdot}284+0{\cdot}546p_1,$$

thence $p_1=-4{\cdot}215$.

Replacing in one of the values of p_2, say the first one, we get

$$p_2=-\frac{1{\cdot}56-4{\cdot}215\times0{\cdot}71}{0{\cdot}35}=4{\cdot}093.$$

These two values give the principal equation :

$$z^2-4{\cdot}215z+4{\cdot}093=0, \text{ ....(5')}$$

or $z^2-4{\cdot}215z+4{\cdot}436=-4{\cdot}093+4{\cdot}436$,

where $4{\cdot}436=\left(\dfrac{4{\cdot}215}{2}\right)^2$, so that the left-hand member of the equation is a perfect square ; hence

$$(z-2{\cdot}1075)^2=0{\cdot}343,$$

$$z-2{\cdot}1075=\pm0{\cdot}5857,$$

hence $z_1=2{\cdot}69$ and $z_2=1{\cdot}52$.

Since $y_0=A_1+A_2$ and $y_1=A_1z_1+A_2z_2$, the two final equations are

$$\left.\begin{aligned}0{\cdot}35&=A_1+A_2,\\ 0{\cdot}71&=2{\cdot}69A_1+1{\cdot}52A_2,\end{aligned}\right\}$$

the solutions of which are $A_1=0{\cdot}146$, $A_2=0{\cdot}204$.

We had
$$\begin{aligned}y_0&=A_1+A_2=A_1z_1^0+A_2z_2^0,\\ y_1&=A_1z_1+A_2z_2=A_1z_1^1+A_2z_2^1,\\ y_2&=A_1z_1^2+A_2z_2^2,\\ y_3&=A_1z_1^3+A_2z_2^3;\end{aligned}$$

in each equation the index is equal to the suffix of y, and indicates the corresponding number of spaces, $(x-x_0)/\delta$ (see p. 204), here $(x-0)/2=x/2$, generally, so that, for any ordinate y of abscissa x, we have

$$y=A_1z_1^{x/2}+A_2z_2^{x/2},$$

x being now reckoned *from the origin* throughout. The equation of the given curve is therefore

$$y=0{\cdot}146\times2{\cdot}69^{x/2}+0{\cdot}204\times1{\cdot}52^{x/2}.$$

It must be put in the form $y=A_1\epsilon^{a_1x}+A_2\epsilon^{a_2x}$, that is, we must solve the two equations

$$2{\cdot}69^{x/2}=2{\cdot}718^{a_1x}, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(a)$$

and
$$1{\cdot}52^{x/2}=2{\cdot}718^{a_2x}. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(b)$$

This is easily done by logarithms as follows:

$$(a)\ \tfrac{1}{2}x\log 2{\cdot}69=a_1x\log 2{\cdot}718,\quad a_1=\frac{0{\cdot}2149}{0{\cdot}4343}=0{\cdot}495,$$

$$(b)\ \tfrac{1}{2}x\log 1{\cdot}52=a_2x\log 2.718,\quad a_2=\frac{0{\cdot}0909}{0{\cdot}4343}=0{\cdot}209.$$

The numerical equation of the given curve is therefore

$$y=0{\cdot}146\epsilon^{0\cdot495x}+0{\cdot}204\epsilon^{0\cdot209x}.$$

This can be verified by calculating the values of y for $x=0$, $x=2$, $x=4$ and $x=6$, and comparing with the given ordinates.

The best and quickest way to do these calculations is by tabulating, as follows :

0·495 x.	·4343 × 0·495 x.	+log 0·146	Antilog.	0·209 x.	·4343 × 0·209 x.	+log 0·204.	Antilog.	y calculated.	y given.
			0·146				0·204	0·35	0·35
0·990	0·4300	$\bar{1}$·5944	0·393	0·418	0·1815	$\bar{1}$·4911	0·310	0·703	0·71
1·980	0·8600	0·0244	1·058	0·836	0·3631	$\bar{1}$·6727	0·471	1·529	1·56
2·970	1·2900	0·4544	2·847	1·254	0·5625	$\bar{1}$·8721	0·745	3·592	3·67

(For $x=0$, $\epsilon^{0\cdot495x}=1$ and $\epsilon^{0\cdot209x}=1$, so that the logarithms are not necessary.)

The agreement is satisfactory. The discrepancies are due merely to the fact that the given ordinates were taken to two places of decimals only, and were therefore somewhat inaccurate.

The curve was, as a matter of fact, obtained by plotting the equation

$$y=0{\cdot}15\epsilon^{0\cdot5x}+0{\cdot}2\epsilon^{0\cdot2x}.$$

Here one may ask : " But what would happen if we had given to the two solutions of the equation $z^2+p_1z+p_2=0$, the wrong symbols, that is, called 2·69, z_2, and 1·52, z_1, since nothing indicates *which* is z_1 and *which* is z_2, for it is mere chance which of the two we obtain first in the calculation."

The answer to this is very simple: Nothing would happen! For if z_1 became z_2, and z_2 became z_1, then it would follow that A_1 would become A_2 and A_2 would become A_1, and in the end the two quantities which ought to be together in the equation of the curve would naturally come together, being designated by corresponding symbols, A_1, z_1 in one case, A_2, z_2 in the other. One need not therefore trouble as to which solution is called z_1 or z_2.

As a second example, let us take the same curve, and use the ordinates $y_0=0{\cdot}491$, $y_1=1{\cdot}036$, $y_2=2{\cdot}371$ and $y_3=5{\cdot}778$, corresponding to $x=1$, $x=3$, $x=5$ and $x=7$, respectively.

Following exactly the same method, we get the two preliminary equations:

$$\left.\begin{aligned}2{\cdot}371+1{\cdot}036p_1+0{\cdot}491p_2=0,\\5{\cdot}778+2{\cdot}371p_1+1{\cdot}036p_2=0.\end{aligned}\right\}$$

Solving, we get $p_1=-4{\cdot}189$, $p_2=+4{\cdot}010$.

Hence we have the principal equation:

$$z^2-4{\cdot}189z+4{\cdot}010=0.$$

Solving this equation gives $z_1=2{\cdot}709$ and $z_2=1{\cdot}480$ The two final equations are:

$$\left.\begin{aligned}y_0&=A'+A'',\\y_1&=A'z_1+A''z_2.\end{aligned}\right\}$$

We shall use A' and A'' instead of A_1 and A_2 to remind ourselves of the fact that these values corre spond to values of x reckoned from the point $x=1$, while A_1 and A_2 correspond to values of x referred to the origin $x=0$.

Here, these two equations become numerically

$$\left.\begin{aligned}0{\cdot}491&=A'+A'',\\ 1{\cdot}036&=2{\cdot}709A'+1{\cdot}480A'',\end{aligned}\right\}$$

or

$$\left.\begin{aligned}1{\cdot}330&=2{\cdot}709A'+2{\cdot}709A'',\\ 1{\cdot}036&=2{\cdot}709A'+1{\cdot}480A''.\end{aligned}\right\}$$

From which we get $0{\cdot}294=1{\cdot}229A''$, whence $A''=0{\cdot}239$, and $A'=0{\cdot}491-0{\cdot}239=0{\cdot}252$.

Now, y_0, y_1, y_2 and y_3 correspond to successive numbers of spaces 0, 1, 2, 3, or, generally, $\frac{x-x_0}{\delta}$, here $\frac{x-1}{2}$, so that the general equation in y is really

$$y=A'z_1^{\frac{x-1}{2}}+A''z_2^{\frac{x-1}{2}},$$

or, numerically,

$$y=0{\cdot}252\times 2{\cdot}709^{\frac{x-1}{2}}+0{\cdot}239\times 1{\cdot}480^{\frac{x-1}{2}},$$

where x is reckoned from the point $x=1$.

This equation must be put in the shape

$$y=A_1\epsilon^{a_1x}+A_2\epsilon^{a_2x},$$

where x is reckoned from the origin.

To do this we note that

$$0{\cdot}252\times 2{\cdot}709^{\frac{x-1}{2}}=A_1\epsilon^{a_1x}, \quad (a)$$

and

$$0{\cdot}239\times 1{\cdot}480^{\frac{x-1}{2}}=A_2\epsilon^{a_2x}. \quad (b)$$

Taking logarithms we get

(*a*) $\log 0{\cdot}252+\frac{x-1}{2}\log 2{\cdot}709=\log A_1+a_1 x\log\epsilon,$

or $\bar{1}{\cdot}4014+\frac{x}{2}\times 0{\cdot}4328-\frac{1}{2}\times 0{\cdot}4328$

$$=-0{\cdot}5986+0{\cdot}2164x-0{\cdot}2164$$
$$=-0{\cdot}8150+0{\cdot}2164x$$
$$=\ \bar{1}{\cdot}1850+0{\cdot}2164x=\log A_1+0{\cdot}4343a_1x.$$

Hence,

$$\log A_1=\bar{1}{\cdot}1850,\quad\text{and}\quad A_1=0{\cdot}153,$$

and $0{\cdot}2164=0{\cdot}4343a_1$ or $a_1=\frac{0{\cdot}2164}{0{\cdot}4343}=0{\cdot}498.$

(*b*) Similarly,

$$\log 0{\cdot}239+\frac{x-1}{2}\log 1{\cdot}480=\log A_2+a_2x\log\epsilon,$$

$$\bar{1}{\cdot}3784+\frac{x}{2}\times 0{\cdot}1704-\frac{1}{2}\times 0{\cdot}1704$$
$$=-0{\cdot}6216+0{\cdot}0852x-0{\cdot}0852$$
$$=-0{\cdot}7068+0{\cdot}0852x$$
$$=\ \bar{1}{\cdot}2932+0{\cdot}0852x=\log A_2+0{\cdot}4343a_2x,$$

whence $\log A_2=\bar{1}{\cdot}2932$ and $A_2=0{\cdot}1964.$

(As a check, $A_1+A_2=0{\cdot}3496$ or $0{\cdot}35$ nearly, the ordinate at the origin.)

Also

$$0{\cdot}0852=0{\cdot}4343a_2,\quad\text{or}\quad a_2=\frac{0{\cdot}0852}{0{\cdot}4343}=0{\cdot}1965,$$

the equation of the curve being

$$y=0{\cdot}153\,\epsilon^{0{\cdot}498x}+0{\cdot}1964\,\epsilon^{0{\cdot}1965x}.$$

The equation, as we have seen, is really

$$0{\cdot}15\,\epsilon^{0{\cdot}5x}+0{\cdot}2\,\epsilon^{0{\cdot}2x}.$$

The answer is a satisfactory approximation, considering that the ordinates were given with but three places of decimals, that is, not strictly accurate.

As a third example, let us take the same curve, with the ordinates $y_0=0{\cdot}143$, $y_1=0{\cdot}255$, $y_2=0{\cdot}491$, and $y_3=1{\cdot}036$, corresponding to $x=-3$, $x=-1$, $x=+1$ and $x=+3$ respectively, the value of x_0 being negative, with $x_0=-3$. We get the two preliminary equations:

$$\left.\begin{array}{l} 0{\cdot}491+0{\cdot}255p_1+0{\cdot}143p_2, \\ 1{\cdot}036+0{\cdot}491p_1+0{\cdot}255p_2, \end{array}\right\}$$

which give $p_1=-4{\cdot}422$ and $p_2=4{\cdot}452$.

(It will be found that to get three places of decimals *correct*, one must work the intermediate calculations to five or six places.)

From this we get the principal equation

$$z^2-4{\cdot}422z+4{\cdot}452=0,$$

from which we get $z_1=2{\cdot}872$ and $z_2=1{\cdot}550$.

These values give for the final equations :

$$\left.\begin{array}{l} 0{\cdot}143=A'+A'', \\ 0{\cdot}255=2{\cdot}872A'+1{\cdot}550A'', \end{array}\right\}$$

from which we get $A'=0{\cdot}02523$, $A''=0{\cdot}11778$.

The equation is therefore

$$y=0{\cdot}02523\times 2{\cdot}872^{\frac{x+3}{2}}+0{\cdot}11778\times 1{\cdot}550^{\frac{x+3}{2}},$$

and must be put in the form $y=A_1\epsilon^{a_1x}+A_2\epsilon^{a_2x}$.

We shall work this example with a greater accuracy than heretofore, for a reason which will be soon apparent. Taking logs to five places, we get

(a) $\bar{2}\cdot40192+\frac{x}{2}\times0\cdot45818+\frac{3}{2}\times0\cdot45818$
$$=\log A_1+a_1x\log\epsilon,$$

$$-1\cdot59808+0\cdot22909x+0\cdot68727=\log A_1+0\cdot4343\,a_1x.$$

Hence $\log A_1=\bar{1}\cdot0892$ and $A_1=0\cdot1228$.

Also $0\cdot2291=0\cdot4343\,a_1$ and $a_1=0\cdot528$.

(b) $\bar{1}\cdot07108+\frac{x}{2}\times0\cdot19039+\frac{3}{2}\times0\cdot19039$
$$=\log A_2+a_2x\log\epsilon,$$

$$-0\cdot92892+0\cdot095195x+0\cdot28558=\log A_2+0\cdot4343\,a_2x.$$

Hence $\log A_2=\bar{1}\cdot3567$ and $A_2=0\cdot2273$.

Also $0\cdot095195=0\cdot4343\,a_2$ and $a_2=0\cdot228$.

The equation of the curve is therefore

$$y=0\cdot1228\,\epsilon^{0\cdot528x}+0\cdot2273\,\epsilon^{0\cdot228x}.$$

This is not in good agreement with the known equation of the curve, namely,

$$y=0\cdot15\,\epsilon^{0\cdot5x}+0\cdot2\,\epsilon^{0\cdot2x}.$$

The reason is that the curvature of this portion of the curve is small, and the selected ordinates do not vary so much as in the portions of the curve considered in the previous examples. It follows that a slight inaccuracy in the value of the ordinates, such as occurs when limiting the number of decimals to three, as we have done, introduces a relatively more considerable error than is the case for more curved portions of the curve. In the present case the range of variation of the ordinates is $1{\cdot}036-0{\cdot}143=0{\cdot}893$, and a difference of $0{\cdot}001$ in an ordinate constitutes an error of 1 in 893; while in the second example the range is $5{\cdot}778-0{\cdot}410=5{\cdot}368$, so that a difference of 0·001 constitutes an error of only 1 in 5368. Had we worked with ordinates correct to four places of decimals and calculated with a corresponding accuracy, we would have obtained a closer approximation. (Do it, and satisfy yourself that it is so.)

As a fourth and last example let us consider a curve in which a_1 and a_2 are negative.

Let $y_0=3090{\cdot}15$, $y_1=143{\cdot}56$, $y_2=9{\cdot}40$, $y_3=1{\cdot}10$ be four ordinates corresponding to $x=-10$, $x=-6$, $x=-2$ and $x=+2$; here $x_0=-10$ and $\delta=4$. We have

$$\left.\begin{aligned}1{\cdot}10+9{\cdot}40p_1+143{\cdot}56p_2=0,\\ 9{\cdot}40+143{\cdot}56p_1+3090{\cdot}15p_2=0.\end{aligned}\right\}$$

Whence we get

$$p_1=-0{\cdot}2429 \quad \text{and} \quad p_2=+0{\cdot}008244.$$

We have therefore the principal equation

$$z^2-0{\cdot}2429z+0{\cdot}008244=0,$$

the solutions of which are $z_1=0{\cdot}2022$ and $z_2=0{\cdot}0408$.

So that we have

$$3090{\cdot}15=A'+A'',$$

$$143{\cdot}56=A'z_1+A''z_2=0{\cdot}2022A'+0{\cdot}0408A'',$$

from which $A''=2981{\cdot}8$ and $A'=108{\cdot}3$.

The equation of the curve is therefore

$$y=108{\cdot}3+0{\cdot}2022^{\frac{x+10}{4}}+2981{\cdot}8+0{\cdot}0408^{\frac{x+10}{4}}.$$

From this, we have, taking logarithms

$$(a)\ 2{\cdot}03464+\frac{x}{4}\times\bar{1}{\cdot}30578+\frac{5}{2}\times\bar{1}{\cdot}30578$$
$$=\log A_1+a_1 x\log \epsilon$$

hence $\quad A_1=1{\cdot}99 \quad$ and $\quad a_1=-0{\cdot}3996.$

$$(b)\ 3{\cdot}47449+\frac{x}{4}\times\bar{2}{\cdot}61066+\frac{5}{2}\times\bar{2}{\cdot}61066$$
$$=\log A_2+a_2 x\log \epsilon$$

which gives $\quad A_2=1{\cdot}003 \quad$ and $\quad a_2=-0{\cdot}807.$

The equation of the curve is therefore

$$y=1{\cdot}99\epsilon^{-0{\cdot}3996x}+1{\cdot}003\epsilon^{-0{\cdot}807x}.$$

We have worked this example with a high degree o accuracy, to show that, when the curvature is wel marked, one can get a very close approximation, foı in this case, the equation from which the given dat. was obtained is $y=2\epsilon^{-0{\cdot}4x}+\epsilon^{-0{\cdot}8x}$, practically identica with the equation obtained by calculation.

The previous examples show clearly how to procee whatever may be the given position of the curve whic it is required to analyse in its components.

The calculations are rather tedious, the solution c two equations with two unknowns being necessary t obtain the principal quadratic equation. These calcula tions may be simplified as follows.

When analysing a curve into two components, we have, generally speaking, the three equations :

$$y_2+p_1y_1+p_2y_0=0, \quad \dots\dots\dots\dots(a)$$
$$y_3+p_1y_2+p_2y_1=0, \quad \dots\dots\dots\dots(b)$$
$$z^2+p_1z+p_2=0. \quad \dots\dots\dots\dots(c)$$

We only want p_1 and p_2 in order to find z, otherwise the values of p_1 and p_2 are not required at all. Instead of calculating p_1 and p_2, therefore, we can eliminate these two quantities from the above system of three equations, the result will be an equation containing z and y_0, y_1, y_2 and y_3, the last four quantities being known numerically.

In order to eliminate p_1 between (a) and (b) we multiply (a) by y_2 and (b) by y_1, and we get

$$\left.\begin{aligned} y_2^2+p_1y_1y_2+p_2y_0y_2=0,\\ y_3y_1+p_1y_1y_2+p_2y_1^2=0, \end{aligned}\right\}$$

and, by subtracting :

$$y_3y_1-y_2^2+p_2y_1^2-p_2y_0y_2=0$$

or

$$p_2(y_1^2-y_0y_2)=y_2^2-y_3y_1,$$

so that

$$p_2=\frac{y_2^2-y_3y_1}{y_1^2-y_0y_2}.$$

Similarly, in order to eliminate p_2 between (a) and (b), we multiply (a) by y_1, and (b) by y_0, and we get

$$\left.\begin{aligned} y_2y_1+p_1y_1^2+p_2y_0y_1=0,\\ y_3y_0+p_1y_2y_0+p_2y_1y_0=0, \end{aligned}\right\}$$

and subtracting

$$y_2y_1-y_3y_0+p_1y_1^2-p_1y_2y_0=0$$

or $$p_1(y_1^2 - y_2 y_0) = y_3 y_0 - y_2 y_1,$$

so that $$p_1 = \frac{y_3 y_0 - y_2 y_1}{y_1^2 - y_0 y_2}.$$

Replacing p_1 and p_2 in the third equation (c), we get

$$z^2 + \frac{y_3 y_0 - y_2 y_1}{y_1^2 - y_0 y_2} z + \frac{y_2^2 - y_3 y_1}{y_1^2 - y_0 y_2} = 0,$$

or $$(y_1^2 - y_0 y_2) z^2 + (y_3 y_0 - y_2 y_1) z + (y_2^2 - y_3 y_1) = 0. \qquad (d)$$

This transformation is really what we have done in each case for every particular example worked out in the previous pages, only we worked with particular numerical data instead of generally, as we have just done.

It is really equation (d) which is wanted, and if we could write it down straight away we would be spared the solution of the system of the two preliminary equations. It is not, however, easy to remember as it is, and as a slip in writing this equation would lead naturally to a wrong result, it would not be advisable to adopt this method of shortening the calculation, namely the writing down of equation (d) at once from the numerical values of y_0, y_1, y_2 and y_3, if it were not for the fact that a very simple and easily remembered expression is exactly equivalent to this equation (d).

Consider the following little table :

$$\begin{vmatrix} z^2 & z & 1 \\ y_2 & y_1 & y_0 \\ y_3 & y_2 & y_1 \end{vmatrix}.$$

This is what mathematicians call a "determinant." You need not be frightened by the name ; it is merely a short, convenient and easily remembered way of writing the equation (d).

What has that table to do with equation (d) you will exclaim in wonder: nothing could be more unlike it!

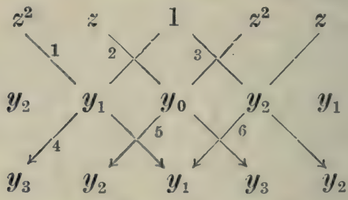
But wait a moment! You must surely notice that both this little table and equation (d) have something in common: all the letters and symbols used in equation (d) are to be found in the table, and reciprocally; in fact, the table is but a short way of representing the algebraical expression constituting the left-hand member of equation (d), so that, given the table, one can readily deduce from it the corresponding expression.

When the table is such a small one, with only nine elements, people who know all about determinants can write down the expression at sight merely from looking at the table. It is easy to learn how to do this, for the letters or digits forming the lines and columns of the table are put precisely in such a definite position that, by picking them up in a definite manner, always the same, the correct expression is obtained, and none other. In the case of a simple determinant of this kind, the expansion or development of the determinant, that is, the writing down of the corresponding algebraical expression, can be done very easily quite mechanically if we proceed as follows:

To the right of the little table above, let us copy the first and the second column, so:

$$\left|\begin{array}{ccc|cc} z^2 & z & 1 & z^2 & z \\ y_2 & y_1 & y_0 & y_2 & y_1 . \\ y_3 & y_2 & y_1 & y_3 & y_2 \end{array}\right.$$

Now let us draw diagonal arrows, like this :



Let us now form the products of the quantities which are on any one arrow, and take this product as being one term, and let us give the sign + to the terms corresponding to the arrows pointing in one direction and the sign − to the terms corresponding to the arrows in the other direction. Which set of terms is + and which is − is not really of any importance, but it is usual to give the sign + to the terms corresponding to the arrows which slope down to the right.

We have, then, for the terms corresponding to the several arrows :

Arrow	No. 1.	$+y_1^2z^2$.
,,	No. 2.	$+y_0y_3z$.
,,	No. 3.	$+y_2^2$.
,,	No. 4.	$-y_1y_3$.
,,	No. 5.	$-y_0y_2z^2$.
,,	No. 6.	$-y_1y_2z$.

Adding algebraically these various terms we have :

$$(y_1^2-y_2y_0)z^2+(y_3y_0-y_2y_1)z+(y_2^2-y_3y_1),$$

which is exactly the left-hand member of the equation (d). To complete the equation it is only necessary to express that this is equal to zero, and we have therefore

$$\begin{vmatrix} z^2 & z & 1 \\ y_2 & y_1 & y_0 \\ y_3 & y_2 & y_1 \end{vmatrix} = 0.$$

Expanding this determinant, as we have just done, gives immediately the principal equation. The arrangement of letters is quite easy to remember without mistakes. It may happen that the signs are everywhere wrong in the principal equation we get in this way. This does not matter in the least, since both sides of the equation (*d*) can be multiplied by -1, which will have for effect to change all the signs. The signs of the principal equation merely depend on the manner in which the preliminary equations have been solved, whether the first equation has been subtracted from the second or the second from the first.

This method, however, can only be used for determinants having nine elements. It will lead to wrong results if applied to determinants having more than three lines and three columns. Other methods, less easy but nevertheless quite simple, must be resorted to when expanding determinants of the latter kind.*

Let us now return to the examples we have worked out, and see how this shorter method works.

* See *Determinants Made Easy*, by the same author, to be published later.

Example 1. Here we have :

$$\begin{vmatrix} z^2 & z & 1 \\ y_2 & y_1 & y_0 \\ y_3 & y_2 & y_1 \end{vmatrix} = 0 \text{ becomes } \begin{vmatrix} z^2 & z & 1 \\ 1{\cdot}56 & 0{\cdot}71 & 0{\cdot}35 \\ 3{\cdot}67 & 1{\cdot}56 & 0{\cdot}71 \end{vmatrix} = 0.$$

Repeating on the right the first two columns on th left, we get :

$$\begin{vmatrix} z^2 & z & 1 \\ 1{\cdot}56 & 0{\cdot}71 & 0{\cdot}35 \\ 3{\cdot}67 & 1{\cdot}56 & 0{\cdot}71 \end{vmatrix} \begin{matrix} z^2 & z \\ 1{\cdot}56 & 0{\cdot}71 \\ 3{\cdot}67 & 1{\cdot}56. \end{matrix}$$

This, as we have seen, leads to the expression :

$$0{\cdot}71 \times 0{\cdot}71 z^2 + 3{\cdot}67 \times 0{\cdot}35 z + 1{\cdot}56 \times 1{\cdot}56$$
$$- 3{\cdot}67 \times 0{\cdot}71 - 1{\cdot}56 \times 0{\cdot}35 z^2 - 1{\cdot}56 \times 0{\cdot}71 z = ($$

Working the products to three places of decimals b four-figure logarithms, we get :

$$0{\cdot}504 z^2 + 1{\cdot}284 z + 2{\cdot}434 - 2{\cdot}606 - 0{\cdot}546 z^2 - 1{\cdot}108 z = 0.$$

That is, $\quad -0{\cdot}042 z^2 + 0{\cdot}176 z - 0{\cdot}172 = 0,$

or multiplying both sides by -1 and dividing by 0·42

$$z^2 - 4{\cdot}215 z + 4{\cdot}095 = 0,$$

which is practically the equation which we obtain before, and we solve it to get the two final equatior as we did above.

Example 2. Proceeding similarly, we write :

$$\begin{vmatrix} z^2 & z & 1 \\ 2{\cdot}371 & 1{\cdot}036 & 0{\cdot}410 \\ 5{\cdot}778 & 2{\cdot}371 & 1{\cdot}036 \end{vmatrix} \begin{matrix} z^2 & z \\ 2{\cdot}371 & 1{\cdot}036 \\ 5{\cdot}778 & 2{\cdot}371, \end{matrix}$$

$$\cdot 036\times 1\cdot 036z^2+0\cdot 41\times 5\cdot 778z+2\cdot 371\times 2\cdot 371$$
$$-1\cdot 036\times 5\cdot 778-0\cdot 41\times 2\cdot 371z^2-2\cdot 371\times 1\cdot 036z=0,$$

$$1\cdot 073z^2+2\cdot 837z+5\cdot 620-5\cdot 985-1\cdot 164z^2-2\cdot 456z=0,$$

at is, $-0\cdot 091z^2+0\cdot 381z-0\cdot 365=0,$

, multiplying both sides by -1 and dividing by 0·091 :

$$z^2-4\cdot 187z+4\cdot 012=0,$$

hich is practically the same equation as the one und by the longer method first shown.

You are advised to try this method and to use it to erify the principal equations obtained in Examples 3 ıd 4 above.

We know now how to perform the analysis of an ponential curve into *two* components. The same ethod exactly applies to three or more components, ıly the short and simple method of developing the eterminant fails in this case, and we are restricted the longer method. The principal equation can be ıdeed expressed as a determinant of sixteen elements nstead of nine), and developed so as to give that quation ; but the method of developing is not so simple s in the case of (9) elements, and we must leave it for nother time.

For three components, $y=A_1\epsilon^{a_1x}+A_2\epsilon^{a_2x}+A_3\epsilon^{a_3x}$, six quidistant ordinates, y_0, y_1, y_2, y_3, y_4 and y_5 are ecessary. (Generally, one needs twice as many ordiates as there are components searched for.)

Proceeding as we have done for two components, w obtain the preliminary equations :

$$y_0 = A_1 + A_2 + A_3, \quad \ldots\ldots\ldots\ldots\ldots\ldots($$

$$y_1 = A_1 z_1 + A_2 z_2 + A_3 z_3, \quad \ldots\ldots\ldots\ldots\ldots($$

$$y_2 = A_1 z_1^2 + A_2 z_2^2 + A_3 z_3^2, \quad \ldots\ldots\ldots\ldots($$

$$y_3 = A_1 z_1^3 + A_2 z_2^3 + A_3 z_3^3, \quad \ldots\ldots\ldots\ldots($$

$$y_4 = A_1 z_1^4 + A_2 z_2^4 + A_3 z_3^4, \quad \ldots\ldots\ldots\ldots($$

$$y_5 = A_1 z_1^5 + A_2 z_2^5 + A_3 z_3^5, \quad \ldots\ldots\ldots\ldots($$

The principal equations must have three solution $z_1 = \epsilon^{a_1\delta}$, $z_2 = \epsilon^{a_2\delta}$ and $z_3 = \epsilon^{a_3\delta}$, and it will be of the form

$$z^3 + p_1 z^2 + p_2 z + p_3 = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots($$

Hence we have

$$\left.\begin{aligned} z_1^3 + p_1 z_1^2 + p_2 z_1 + p_3 &= 0, \\ z_2^3 + p_1 z_2^2 + p_2 z_2 + p_3 &= 0, \\ z_3^3 + p_1 z_3^2 + p_2 z_3 + p_3 &= 0. \end{aligned}\right\}$$

Multiplying both terms of these equations z_1^n, z_2^n, z_3^n respectively, where n is a whole num smaller than 3, so that either $n=0$ or $n=1$ or $n=$ we get :

$$\left.\begin{aligned} z_1^3 z_1^n + p_1 z_1^2 z_1^n + p_2 z_1 z_1^n + p_3 z_1^n &= 0, \\ z_2^3 z_2^n + p_1 z_2^2 z_2^n + p_2 z_2 z_2^n + p_3 z_2^n &= 0, \\ z_3^3 z_3^n + p_1 z_3^2 z_3^n + p_2 z_3 z_3^n + p_3 z_3^n &= 0. \end{aligned}\right\}$$

That is,

$$\left.\begin{aligned} z_1^{n+3} + p_1 z_1^{n+2} + p_2 z_1^{n+1} + p_3 z_1^n &= 0, \\ z_2^{n+3} + p_1 z_2^{n+2} + p_2 z_2^{n+1} + p_3 z_2^n &= 0, \\ z_3^{n+3} + p_1 z_3^{n+2} + p_2 z_3^{n+1} + p_3 z_3^n &= 0. \end{aligned}\right\}$$

Multiplying the terms of these three equations respectively by A_1, A_2, and A_3, we get

$$\left.\begin{aligned}A_1z_1^{n+3}+p_1A_1z_1^{n+2}+p_2A_1z_1^{n+1}+p_3A_1z_1^{n}=0,\\A_2z_2^{n+3}+p_1A_2z_2^{n+2}+p_2A_2z_2^{n+1}+p_3A_2z_2^{n}=0,\\A_3z_3^{n+3}+p_1A_3z_3^{n+2}+p_2A_3z_3^{n+1}+p_3A_3z_3^{n}=0.\end{aligned}\right\}$$

Adding together the three left-hand members and the three right-hand members, we still get an equality :

$$\begin{aligned}(A_1z_1^{n+3}+A_2z_2^{n+3}+A_3z_3^{n+3})&\\+p_1(A_1z_1^{n+2}+A_2z_2^{n+2}+A_3z_3^{n+2})&\\+p_2(A_1z_1^{n+1}+A_2z_2^{n+1}+A_3z_3^{n+1})&\\+p_3(A_1z_1^{n}+A_2z_2^{n}+A_3z_3^{n})=0.&\quad\ldots\ldots\ldots\ldots(8)\end{aligned}$$

When $n=0$, $n=1$ and $n=2$ this equation becomes successively :

$$\begin{aligned}(A_1z_1^3+A_2z_2^3+A_3z_3^3)+p_1(A_1z_1^2+A_2z_2^2+A_3z_3^2)&\\+p_2(A_1z_1+A_2z_2+A_3z_3)+p_3(A_1+A_2+A_3)=0.&\end{aligned}$$

$$\begin{aligned}(A_1z_1^4+A_2z_2^4+A_3z_3^4)+p_1(A_1z_1^3+A_2z_2^3+A_3z_3^3)&\\+p_2(A_1z_1^2+A_2z_2^2+A_3z_3^2)+p_3(A_1z_1+A_2z_2+A_3z_3)=0.&\end{aligned}$$

$$\begin{aligned}(A_1z_1^5+A_2z_2^5+A_3z_3^5)+p_1(A_1z_1^4+A_2z_2^4+A_3z_3^4)&\\+p_2(A_1z_1^3+A_2z_2^3+A_3z_3^3)&\\+p_3(A_1z_1^2+A_2z_2^2+A_3z_3^2)=0.&\end{aligned}$$

Replacing the brackets by their equivalents given by the set of preliminary equations, we have :

$$\left.\begin{aligned}y_3+p_1y_2+p_2y_1+p_3y_0=0,\\y_4+p_1y_3+p_2y_2+p_3y_1=0,\\y_5+p_1y_4+p_2y_3+p_3y_2=0,\end{aligned}\right\}\quad\begin{aligned}\ldots\ldots\ldots\ldots(9)\\\ldots\ldots\ldots\ldots(10)\\\ldots\ldots\ldots\ldots(11)\end{aligned}$$

three equations in which y_0, y_1, y_2, y_3, y_4, and y_5 are known numerically, and p_1, p_2, p_3 are unknown.

They can be solved in the usual manner, and the numer cal values of p_1, p_2 and p_3 ascertained, so that we ge the final equation $z^3+p_1z^2+p_2z+p_3=0$.

This equation being solved gives three solutions, z z_2 and z_3, the values of which, being replaced in an three of the preliminary equations—the three first one are evidently the simplest to use—will give A_1, A and A_3.

From $z_1=\epsilon^{a_1\delta}$ $z_2=\epsilon^{a_2\delta}$ and $z_3=\epsilon^{a_3\delta}$, a_1, a_2 and a_3 ar easily calculated, so that we know fully the three ex ponential components of the curve.

To solve the equation $z^3+p_1z^2+p_2z+p_3=0$, we ma conveniently use a graphical method; for instanc we may put the equation under the form

$$z^3+p_3=-p_1z^2-p_2z,$$

and one may plot

$$y=z^3+p_3 \quad \text{and} \quad y=-p_1z^2-p_2z,$$

and find the points of intersection of the two curve There will be three such points. At each of these thr points the ordinates of both curves are the same. follows that the value of the corresponding absciss that is, the value of z corresponding to this value of y which is common to both curves at one point intersection, is a solution of the equation, since it sat fies simultaneously both equalities

$$y'=z^3+p_3 \quad \text{and} \quad y'=-p_1z^2-p_2z,$$

and therefore satisfies the equation

$$z^3+p_3=-p_1z^2-p_2z \quad \text{or} \quad z^3+p_1z^2+p_2z+p_3=0.$$

These values of z, read off the graph, are not very accurate. To obtain more accurate values, larger scale graphs must be plotted, restricted to the neighbourhood of the points of intersection. These give a closer approximation, which can be made to yield yet a closer result by using it to plot a still more restricted portion of a graph on a still larger scale.

The method is complicated, the plotting of the graphs being somewhat cumbersome. It is possible to simplify the work considerably in transforming the equation $z^3+p_1z^2+p_2z+p_3=0$ into another equation of the form $z^3+q_1z+q_2=0$, that is, not containing any term with z^2. The plotting is reduced then to $y=z^3$, a very simple graph which can be done once for all in ink, and $y=q_1z+q_2$, a straight line which requires but two points to be completely determined in position, and which can be drawn with a light pencil by means of a ruler, and rubbed out afterwards to allow the principal equation of another curve to be solved in a similar manner. The term in z^2 is easily eliminated as follows:

Let $z=z'+k$, then the equation becomes

$$(z'+k)^3+p_1(z'+k)^2+p_2(z'+k)+p_3=0,$$

or

$$z'^3+3z'^2k+3z'k^2+k^3+p_1z'^2+2p_1z'k+p_1k^2$$
$$+p_2z'+p_2k+p_3=0,$$

that is

$$z'^3+(3k+p_1)z'^2+(3k^2+2p_1k+p_2)z'$$
$$+(k^3+p_1k^2+p_2k+p_3)=0.$$

If we chose such a value for k that $3k+p_1=0$, tha is, $k=-p_1/3$, then the coefficient of z'^2 is zero, henc this term vanishes, and the equation becomes

$$z'^3+\left(p_2-\frac{p_1^2}{3}\right)z'+\left(\frac{2p_1^3}{27}-\frac{p_1p_2}{3}+p_3\right)=0,$$

which is of the form $z'^3+q_1z'+q_2=0$.

This equation, solved graphically, as has been ex plained above, gives three values for z'. It must b borne in mind that these values of z' are not the require values of z, since $z=z'+k$, so that one must ad $k=-p_1/3$ to these values of z' to get the sought value of z, that is, $z=z'-p_1/3$.

We shall make the process clear by working fully on example.

Let six ordinates of a compound exponential curv be given :

x:	-2	0	$+2$	$+4$	$+6$	$+8$
y:	$y_0=2{\cdot}415$	$y_1=1{\cdot}350$	$y_2=1{\cdot}155$	$y_3=1{\cdot}755$	$y_4=3{\cdot}768$	$y_5=9{\cdot}2$

and let it be required to resolve this curve into thr components.

Here, $x_0=-2$.

We have the three preliminary equations :

$$\left.\begin{aligned}1{\cdot}755+1{\cdot}155p_1+1{\cdot}350p_2+2{\cdot}415p_3=0.\\3{\cdot}768+1{\cdot}755p_1+1{\cdot}155p_2+1{\cdot}350p_3=0.\\9{\cdot}222+3{\cdot}768p_1+1{\cdot}755p_2+1{\cdot}555p_3=0.\end{aligned}\right\}\begin{aligned}&\ldots\ldots\ldots(\\&\ldots\ldots\ldots(\\&\ldots\ldots\ldots(\end{aligned}$$

Eliminating p_3 between (1) and (2), we get

$$6{\cdot}731+2{\cdot}679p_1+0{\cdot}966p_2=0.$$

Proceeding similarly with (2) and (3), we get

$$8{\cdot}097+3{\cdot}060p_1+1{\cdot}035p_2=0.$$

Solving now

$$\left.\begin{array}{l}6{\cdot}731+2{\cdot}679p_1+0{\cdot}966p_2=0,\\ 8{\cdot}097+3{\cdot}060p_1+1{\cdot}035p_2=0,\end{array}\right\}$$

multiplying the first equation by 1·035 and the second by 0·966, we get

$$\left.\begin{array}{l}6{\cdot}967+2{\cdot}773p_1+1{\cdot}000p_2=0,\\ 7{\cdot}822+2{\cdot}956p_1+1{\cdot}000p_2=0,\end{array}\right\} \quad \text{............(4)}$$

and

$$0{\cdot}855+0{\cdot}183p_1=0 \quad \text{and} \quad p_1=-\frac{0{\cdot}855}{0{\cdot}183}=-4{\cdot}672.$$

Replacing in (4) we get

$$6{\cdot}967-12{\cdot}955+p_2=0 \quad \text{and} \quad p_2=5{\cdot}988.$$

Replacing p_1 and p_2 in (1), we get

$$1{\cdot}755-5{\cdot}396+8{\cdot}084+2{\cdot}415p_3=0.$$

Whence $$p_3=-\frac{4{\cdot}443}{2{\cdot}415}=-1{\cdot}840.$$

(As a check, replacing p_1, p_2 and p_3 by these values in equations (1), (2) and (3) respectively, we get $-0{\cdot}002$, $+0{\cdot}001$ and $+0{\cdot}002$, a sufficiently accurate verification.)

The principal equation is therefore

$$z^3-4{\cdot}672z^2+5{\cdot}988z-1{\cdot}840=0. \quad \text{............(5)}$$

Let $k=-\dfrac{p_1}{3}=+1{\cdot}557$, and let $z=z'+k$;

we get then, as shown above, an equation in z'

without any term in z'^2, and this equation is, as has been shown:

$$z'^3+\left(p_2-\frac{p_1^2}{3}\right)z'+\left(\frac{2p_1^3}{27}-\frac{p_1p_2}{3}+p_3\right)=0.$$

That is, in this case:

$$z'^3-1{\cdot}288z'-0{\cdot}068=0. \quad\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

To solve this graphically we write it in the form $z'^3=1{\cdot}288z'+0{\cdot}068$, and we plot

$$y=z'^3 \quad\text{and}\quad y=1{\cdot}288z'+0{\cdot}068.$$

We get the graphs shown in Fig. 40, from which we get a first approximation

(*a*) $z_1'=+1{\cdot}16$, (*b*) $z_2'=-0{\cdot}04$ and (*c*) $z_3'=-1{\cdot}10$.

For a closer approximation, we plot the same graph between the limits (*a*) $+1{\cdot}1$ and $+1{\cdot}2$ for z_1', (*b*) 0 and $-0{\cdot}1$ for z_2', (*c*) $-1{\cdot}05$ and $-1{\cdot}15$ for z_3'.

This is really quite a simple matter. First, the following table is made (only two points are needed for the straight line, and three points are enough for the curve)

(*a*)			(*b*)		
z'.	$y=z'^3$.	$y=1{\cdot}288z'+0{\cdot}068$.	z'.	$y=z'^3$.	$y=1{\cdot}288z'+0{\cdot}068$
+1·1	+1·331	+1·485	0	0	+0·069
+1·15	+1·521	—	−0·05	−·000125	—
+1·2	+1·728	+1·614	−0·1	−·001	−0·061

(*c*)		
z'.	$y=z'^3$.	$y=1{\cdot}288z'+0{\cdot}068$.
−1·05	−1·158	−1·284
−1·1	−1·331	—
−1·15	−1·521	−1·413

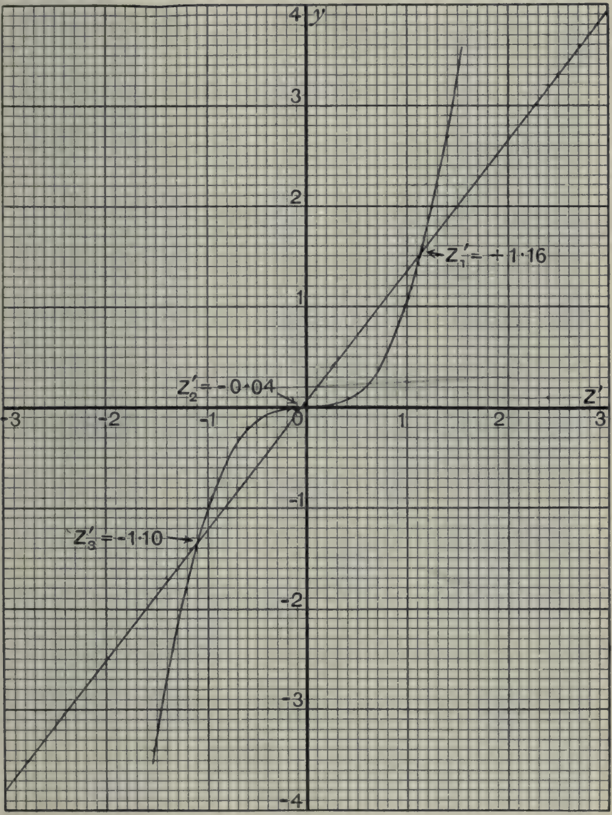


FIG. 40.

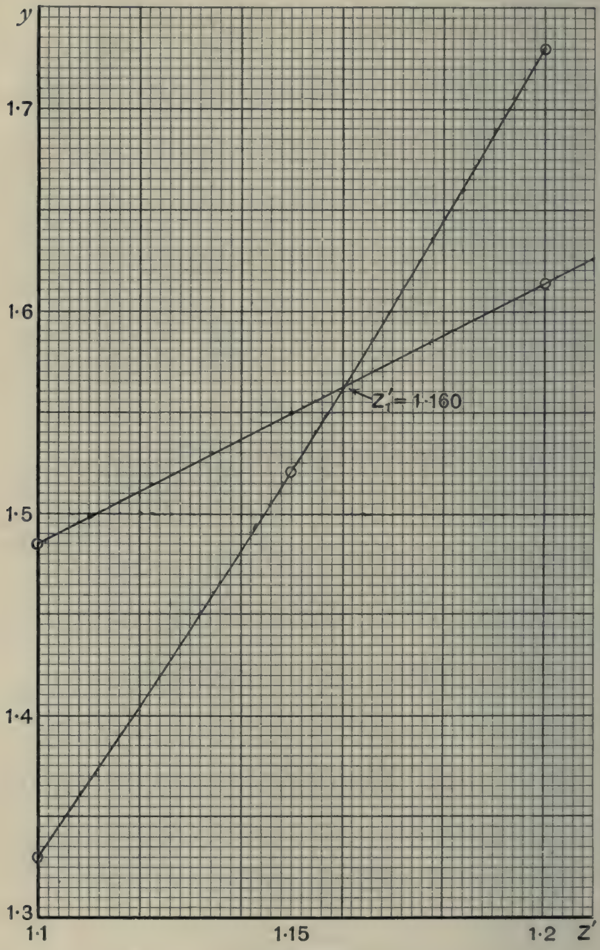


FIG. 41*a*.

The graphs are now plotted (see Figs. 41*a*, 41*b* and 41*c*). The plotted points are shown as the centres of

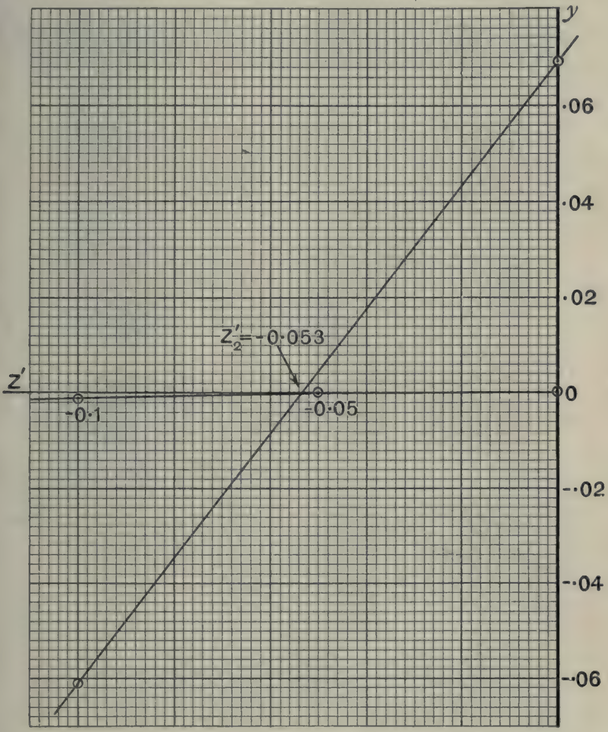


FIG. 41*b*.

small circles. The values obtained from these large scale graphs are respectively

(*a*) $z_1'=1{\cdot}160$, (*b*) $z_2'=-0{\cdot}053$ and (*c*) $z_3'=-1{\cdot}107$.

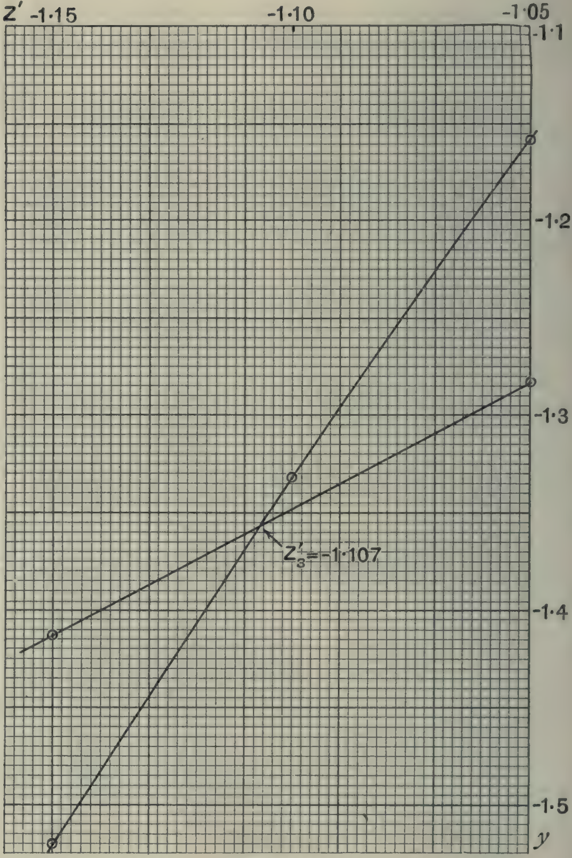


FIG. 41c.

It is advisable to check them by replacing z' in the equation (6) successively by these three values, and verifying that a numerical equality is obtained. Doing this in this case we get $-\cdot 001$, $-0\cdot 002$, $+0\cdot 001$; there is therefore no serious error, bearing in mind that the third place in the values of z is, of course, only approximate, being obtained graphically, and as we raise these values to the cube, a small error is magnified in this check.

The values of z are $z'+k$ or $z'+1\cdot 557$, that is, $z_1=+2\cdot 717$, $z_2=+1\cdot 504$ and $z_3=+0\cdot 450$ respectively.

It is worth while to check these also by replacing in (5), although this check is more laborious. One will then make sure that no mistakes were made in the elimination of the term in z^2. Doing this, we find $-0\cdot 003$, $0\cdot 000$ and $0\cdot 000$, which is quite satisfactory.

We have then

$$\begin{aligned} y_0&=2\cdot 415=A'+A''+A''',\\ y_1&=1\cdot 350=A'z_1+A''z_2+A'''z_3\\ &=2\cdot 717A'+1\cdot 504A''+0\cdot 45A''',\\ y_2&=1\cdot 155=A'z_1^2+A''z_2^2+A'''z_3^2\\ &=7\cdot 382A'+2\cdot 262A''+0\cdot 2025A'''. \end{aligned}$$

Solving for A', A'' and A''', we get

$$\left.\begin{aligned} 6\cdot 562&=2\cdot 717A'+2\cdot 717A''+2\cdot 717A'''\\ 1\cdot 350&=2\cdot 717A'+1\cdot 504A''+0\cdot 450A'''. \end{aligned}\right\}$$

Or $\quad 5\cdot 212= \quad 1\cdot 213A''+2\cdot 267A'''.$

also $\quad 17\cdot 828=7\cdot 382A'+7\cdot 382A''+7\cdot 3821A''',$

$\quad 1\cdot 155=7\cdot 382A'+2\cdot 262A''+0\cdot 2025A'''.$

Or $\quad 16\cdot 673= \quad 5\cdot 120A''+7\cdot 180A'''.$

Now,

$$\left.\begin{aligned}5{\cdot}212&=1{\cdot}213A''+2{\cdot}267A''',\\16{\cdot}673&=5{\cdot}120A''+7{\cdot}180A'''.\end{aligned}\right\}$$

$$A''=\frac{5{\cdot}212-2{\cdot}267A'''}{1{\cdot}213}=\frac{16{\cdot}673-7{\cdot}180A'''}{5{\cdot}120}.$$

$$26{\cdot}685-11{\cdot}607A'''=20{\cdot}224-8{\cdot}709A''',$$

$$6{\cdot}461=2{\cdot}898A''' \quad\text{and}\quad A'''=\frac{6{\cdot}461}{2{\cdot}898}=2{\cdot}229,$$

$$A''=\frac{16{\cdot}673-16{\cdot}011}{5{\cdot}12}=\frac{{\cdot}662}{5{\cdot}12}=0{\cdot}129,$$

$$A'=2{\cdot}415-(A''+A''')=2{\cdot}415-2{\cdot}358=0{\cdot}057.$$

The equation is therefore:

$$y=0{\cdot}057\times2{\cdot}717^{\frac{x+2}{2}}+0{\cdot}129\times1{\cdot}504^{\frac{x+2}{2}}+2{\cdot}229\times0{\cdot}450^{\frac{x+2}{2}}.$$

We can make the same remark as before concerning the allocation of the symbols, z_1, z_2, z_3 to the three solutions of the equation $z^3+p_1z^2+p_2z+p_3=0$. To whichever solution we give any particular symbol does not matter, as, automatically, the symbols A', A'' and A''' will fall to the corresponding values of A, so that the corresponding pairs of values will be correctly coupled together in the equation of the curve.

We proceed now in putting the equation in the general form

$$y=A_1\epsilon^{a_1x}+A_2\epsilon^{a_2x}+A_3\epsilon^{a_3x}$$

as shown in the previous examples.

(*a*) $\log 0{\cdot}057+\frac{x}{2}\log 2{\cdot}717+\log 2{\cdot}717=\log A_1+a_1 x\log \epsilon,$

$$\bar{2}{\cdot}7559+\frac{x}{2}\times 0{\cdot}4341+0{\cdot}4341=\log A_1+0{\cdot}4343\,a_1 x,$$

$$-1{\cdot}2441+0{\cdot}2171x+0{\cdot}4341=\log A_1+0{\cdot}4343\,a_1 x.$$

Hence $-0{\cdot}8100=\log A_1$, or $\bar{1}{\cdot}1900=\log A_1$

and $A_1=0{\cdot}155.$

Also $0{\cdot}2171=0{\cdot}4343\,a_1$, $a_1=\frac{0{\cdot}2171}{0{\cdot}4343}=0{\cdot}50,$

The first component is therefore $0{\cdot}155\epsilon^{0{\cdot}5x}$.

Proceeding similarly, we get :

(*b*) $A_2=0{\cdot}194$, $a_2=0{\cdot}204$, giving for the second component $0{\cdot}194\epsilon^{0{\cdot}204x}$.

(*c*) $A_3=1{\cdot}003$, $a_3=-0{\cdot}399$, say, $-0{\cdot}4$.

the third component being $1{\cdot}003\epsilon^{-0{\cdot}4x}$.

The curve is therefore :

$$y=0{\cdot}155\epsilon^{0{\cdot}5x}+0{\cdot}194\epsilon^{0{\cdot}204x}+1{\cdot}003\epsilon^{-0{\cdot}4x}.$$

As a matter of fact, the given compound curve had been obtained by plotting the equation :

$$y=0{\cdot}15\epsilon^{0{\cdot}5x}+0{\cdot}2\epsilon^{0{\cdot}2x}+\epsilon^{-0{\cdot}4x},$$

so that the result of the analysis is quite satisfactory.

It is so easy to make examples by taking any two or three exponential equations, adding equidistant ordinates and working upon the data so obtained in order to get back to the equations one has started from, that it would seem almost superfluous to give any further exercises. However, we give the following cases, calculated ordinates being given in every case so as to start from a data as accurate as possible.

Exercises VI. (For answers, see p. 247.)

Resolve in two components the exponential curves of which the following ordinates are given:

(1)

x	y
+2	0·42
+3	0·50
+4	0·60
+5	0·71

(2)

x	y
+2	+0·4205
+3	+0·4994
+4	+0·5943
+5	+0·7085

(3)

x	y
−5	0·0787
−3	0·1982
−1	0·4225
+1	0·2896

(4)

t	y
+1	1·589
+2	1·627
+3	1·872
+4	2·244

(5)

x	v
+1	1·589
+1·5	1·573
+2	1·627
+2·5	1·731

(6)

x	y
−10	0·04
−8	0·11
−6	0·29
−4	0·71

(7)

x	y
−10	0·0417
−8	0·1133
−6	0·2882
−4	0·7083

(8)

θ	y
+10	10·589
+20	7·773
+30	5·935
+40	4·705

(9)

x	y
−4	6·33
0	1·50
+4	1·27
+8	1·50

(10)

x	y
−4	0·774
0	0·900
+4	0·999
+8	0·997

(11) Resolve in three components the curve of which he following ordinates are given :

x	y	x	y
−4	6·29	+2	1·11
−2	2·50	+4	1·04
0	1·40	+6	1·03

(12) The decay of activity of a radio-active substance ıas been observed to vary as follows, t being the number of days and B the activity on some arbitrary scale. Analyse the decay of activity into two components. (From *Proc. Phys. Soc.*, vol. xxxii. p. 27.)

t	B
100	278
400	107
700	70
1000	50

APPENDIX

POLAR COORDINATES. (See p. 106.)

THE following exercises on plotting polar coordinate will provide interesting and useful practice, by mean of which one will become familiar with this kind o graphical representation of functions, which is speciall useful in connection with trigonometrical function Some of these give very elegant star, leaf or ros patterns. Plotting the functions in rectangular co ordinates will also be both instructive and interestin

Exercises VII. (See p. 248 for Answers.)

Plot in polar coordinates the following function giving θ values between 0° and 360°.

(1) $r=\sin\theta$. (2) $r=\cos\theta$.

(3) $r=\sin\theta+\cos\theta$. (4) $r=\sin\theta-\cos\theta$.

(5) $r=\cos\theta-\sin\theta$. (6) $r=\sin\frac{\theta}{2}$.

(7) $r=\cos\frac{\theta}{2}$. (8) $r=\frac{\sin\theta}{2}+\cos\frac{\theta}{2}$.

(9) $r=\frac{\sin\theta}{2}-\cos\frac{\theta}{2}$. (10) $r=\sin\frac{\theta}{3}$.

(11) $r=\cos\frac{\theta}{3}$. (12) $r=\sin\frac{\theta}{2}+\frac{\cos\theta}{2}$.

(13) $r=\frac{\cos\theta}{2}-\sin\frac{\theta}{2}$. (14) $r=2+\cos 3\theta$.

(15) Plot on the same pole and on the same scale:

(*a*) $r=\theta$; (*b*) $r=\epsilon^{\theta}$; (*c*) $r=2\epsilon^{\theta/2}$.

(16) $r=4(1-\cos\theta)$. Plot this curve with the same pole and on the same scale as (6) above.

(17) In an aeroplane (monoplane), if P_n is the pressure on the plane when inclined at an angle a to the direction of relative motion of plane and air, and P is the pressure on the plane when the angle a is a right angle, it is found that $P_n=P\frac{2\sin a}{1+\sin^2 a}$. The resistance to advance $R=P_n \sin a=P\frac{2\sin^2 a}{1+\sin^2 a}$. The lifting power at soaring speed L

$$=P_n \cos a=P\frac{2\cos a \sin a}{1+\sin^2 a}.$$

Taking P=unity, plot these three quantities with the same pole and on the same scale, for angles from $a=0°$ to $a=90°$.

ANSWERS

Exercises I. (p. 28.)

(1) $1/a$. (2) $1/x^a$. (3) $2/m^2$. (4) $a/3x$.
(5) $x/2$. (6) $2a^3$. (7) $x^2/81$. (8) $8/a^3$.
(9) $16/x$. (10) $1/8a^x$. (11) $1/3a^2$. (12) $\frac{1}{2}a^{x+3}$.
(13) $\sqrt{a}$. (14) $2a^2$. (15) $2\sqrt[3]{a^2}$. (16) $\dfrac{2}{3\sqrt{x^3}}$.
(17) $9/\sqrt{a}$. (18) $2/\sqrt[3]{x^2}$. (19) $1/2\sqrt{a^x}$. (20) $\frac{1}{2}a^{x+2}$.
(21) $5\sqrt[3]{x}$. (22) $3/\sqrt[5]{m}$. (23) $7/a$. (24) $1/x$.
(25) $x^2/3a^x$. (26) $\sqrt{3}x^3$. (27) $x^3/4a^m$. (28) $2a^5x$.
(29) 1. (30) $9/\sqrt{a}$. (31) $12a^{2x}$. (32) $3^a x^{2am}$.
(33) $1/\sqrt{a}\,\sqrt[3]{x^{10}}$. (34) $6-\dfrac{4}{\sqrt[3]{x^7}}$.
(35) a^{x+3}/x^{3a}. (36) $3^{x-2}a^{x-a}+\dfrac{3^{x-1}}{\sqrt{a^{2a+3}}}$.
(37) $\sqrt[6]{m^{13}}/a^3$. (38) $\frac{1}{3}\sqrt{(k^3/m^5)}$.
(39) $\dfrac{2}{\sqrt{x^{7a}}}$. (40) $\dfrac{1}{\sqrt[4]{a^{3x}}}$.
(41) $\dfrac{6}{\sqrt{a^7}\,\sqrt[3]{x^{11}}}$. (42) $\left(\dfrac{2}{x^{1/3}}\right)\dfrac{a^2-4}{2a}$.

Exercises II. (p. 36.)

(1) 1·544. (2) 2·705. (3) 5·13. (4) 1·807.
(5) $x=6{\cdot}137$, $y=17{\cdot}411$. (6) $k=2{\cdot}735$. (7) $t=3{\cdot}97$
(8) $x=0{\cdot}33$. (9) $x=0{\cdot}753$.

(10) $m=1{\cdot}8$, $n=1{\cdot}2$ (11) $x=1{\cdot}39$

(12) $x=1{\cdot}170$ and $x=-0{\cdot}171$. (13) $y=1{\cdot}44$, $y=1{\cdot}0023$.

(14) $\theta=224^\circ\ 57'$ or $315^\circ\ 3'$. (15) $k=7{\cdot}502$, $x=2{\cdot}283$.

(16) $\log_\epsilon 2=0{\cdot}6931$, $\log_\epsilon 5=1{\cdot}6094$, $\log_\epsilon 10=2{\cdot}3026$.

(17) $\log_\epsilon 3{\cdot}2=1{\cdot}1632$, $\log_\epsilon 0{\cdot}11=-2{\cdot}2073=\bar{3}{\cdot}7927$.

(18) $\log_\epsilon 74{\cdot}3=4{\cdot}3082$, $\log_\epsilon 1{\cdot}808=0{\cdot}5922$,
$\log_\epsilon 10{\cdot}95=2{\cdot}3935$, the answer is 12·27.

(19) $x=1{\cdot}4306$, $\log_5 10=1{\cdot}4306$. (20) $x=0{\cdot}6055$.

(21) $\epsilon=2{\cdot}7182$. (22) 55·35.

(23) $\log_7 3=0{\cdot}5645$, $\log_7 4=0{\cdot}7124$, $\log_7 9=1{\cdot}1291$,
$\log_7 12=1{\cdot}2770$, $\log_7 27=1{\cdot}6937$.

(24) In system of base 1·38. (25) In system of base 2·512.

(26) $\log_{12} 1{\cdot}5=0{\cdot}1631$, $\log_{12}$ answer $=0{\cdot}4893$, answer $=3{\cdot}374$.

(27) $x=+0{\cdot}395$ and $x=-3{\cdot}074$.

(28) $y=1{\cdot}001055$, $x=-0{\cdot}0767$ and $-0{\cdot}2567$.

Exercises III. (p. 49.)

(1) 4·38 inch. (2) 1·244 radians, 40° 58′.

(3) $1^\circ=0{\cdot}01746$, $1'=0{\cdot}000291$, $1''=0{\cdot}00000485$; ·21 inch, ·0035 inch, ·000058 inch.

(4) (*a*) 0·573 inch; (*b*) 2 ft. 10·4 in.; (*c*) 172 feet.

(5) 67° 26′, 1·177 radian. (6) 0·7 radian.

(7) 8·33 inch. (8) 48·2 feet.

(9) (*a*) 3·41; (*b*) 0·83. (10) 0·779.

(11) 84° 0′. (12) 0·783 in., 2·929 in., 3·03 in.

Exercises IV. (p. 77.)

(1) (*a*) $1+14x+84x^2+280x^3+\ldots$.
(*b*) $16x^4+16x^3y+6x^2y^2+xy^3+\ldots$.
(*c*) $a^9x^9+9a^7x^8y+36a^5x^7y^2+84a^3x^6y^3+\ldots$.

(*d*) $1-10y+40y^2-80y^3+\dots$

(*e*) $64-96x+60x^2-20x^3+\dots$

(*f*) $a^7-\frac{21a^6}{b}+\frac{189a^5}{b^2}-\frac{945a^4}{b^3}+\dots$

(*g*) $1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3-\dots$

(*h*) $1+\frac{3}{4}x-\frac{3}{32}x^2+\frac{5}{128}x^3-\dots$

(*k*) $1+\frac{3}{5}x-\frac{3}{25}x^2+\frac{7}{125}x^3-\dots$

(2) $1-6x+27x^2-108x^3+\dots$

(3) $1+5x+15x^2+35x^3+\dots$

(4) $1+3x^2+6x^4+10x^6+\dots$

(5) $\frac{1}{8}-\frac{3x}{16}+\frac{3x^2}{16}-\frac{5x^3}{32}+\dots$

(6) $1+\theta-\frac{1}{2}\theta^2+\frac{1}{2}\theta^3-\dots$

(7) $1-x+\frac{3x^2}{2}-\frac{5x^3}{2}+\dots$

(8) $1+\frac{x}{2}+\frac{3x^2}{8}+\frac{5x^3}{16}+\dots$

(9) $\frac{1}{a^{3/2}}+\frac{3x}{a^{5/2}}+\frac{15x^2}{2a^{7/2}}+\frac{35x^3}{2a^{9/2}}+\dots$

(10) $\epsilon^{5/3}+\frac{5}{6}\epsilon^{2/3}x+\frac{5x^2}{36\epsilon^{1/3}}-\frac{5x^3}{648\epsilon^{4/3}}+\dots$

(11) $\frac{1}{\theta^{1/2}}-\frac{\theta^{x+\frac{1}{2}}}{2}-\frac{\theta^{2x+\frac{3}{2}}}{8}-\frac{\theta^{3x+\frac{5}{2}}}{16}+\dots$

(12) $1+2\sin\theta+3\sin^2\theta+4\sin^3\theta+\dots$

(13) 4·97583.

(14) 6·86828.

(15) 5·00638.

(16) 15·10007.

(17) $1+\theta\cos x+\frac{\theta(\theta-1)}{2}\cos^2 x+\frac{\theta(\theta-1)(\theta-2)}{6}\cos^3 x+\dots$

(18) $1-\tan x\epsilon^2+\frac{\tan x(\tan x-1)}{2}\epsilon^4$

$$-\frac{\tan x(\tan x-1)(\tan x-2)}{6}\epsilon^6+\dots$$

(19) $x^{1/2}-\frac{1}{2x^{1/2}\cos\theta}-\frac{1}{8x^{3/2}\cos^2\theta}-\frac{1}{16x^{5/2}\cos^3\theta}-\dots$

(20) $1+\frac{\epsilon^2}{2}+\frac{\epsilon^3(\epsilon-2)}{8}+\frac{\epsilon^4(\epsilon-2)(\epsilon-4)}{48}+\dots$

(21) $\theta = x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112} + \frac{35x^9}{1152} + \ldots$, $\theta = 0{\cdot}201357$ radian.

(22) $\frac{1}{5} + \frac{3\tan x}{5^2} + \frac{9\tan^2 x}{5^3} + \frac{27\tan^3 x}{5^4} + \ldots$.

(23) $1 + \frac{\theta}{m} + \frac{(1-m)\theta^2}{2m^2} + \frac{(1-m)(1-2m)\theta^3}{6m^3} + \ldots$.

(24) $k^{k/m} - \frac{k}{m^2} k^{\frac{k-m}{m}} + \frac{k(k-m)}{2m^4} k^{\frac{k-2m}{m}}$

$$- \frac{k(k-m)(k-2m)}{6m^6} k^{\frac{k-3m}{m}} + \ldots.$$

Exercises V. (p. 92.)

(1) 4 minutes and 17 minutes 18 seconds respectively.

(2) 14 $^{\text{min.}}$ 42 $^{\text{secs.}}$, 44 $^{\text{min.}}$ 44 $^{\text{secs.}}$

(3) $\mu = 0{\cdot}0805$, $Q_0 = 1252$ units.

(4) $\mu = 0{\cdot}01$ nearly ; $69\frac{1}{2}$ minutes.

(5) 14·43 megohms.

(6) $K_1 = 0{\cdot}00346$, $K_2 = 0{\cdot}00264$, the 1st medium is 1·3 times more opaque.

(7) $k = 0{\cdot}126$; 0·845 centimetre.

(8) 12 per cent.

(9) $\beta = 0{\cdot}00697$, $l = 100$ kilometres very nearly.

(10) The constant T is 15·45 ; 28° Cent.

Exercises VI. (p. 240.)

(NOTE.—The following are the actual equations from which the data given to work from have been calculated, and very close approximations to these should be obtained, as in the worked out examples. In the case of Exercises (1) and (6), in which the data is given to two places only to simplify the calculations, the approximation will not be so good, but on reworking with the same ordinates given to four places (Exercises (2) and (7)) a much closer approximation should be obtained.)

(1) and (2) $y=0{\cdot}1\epsilon^{0\cdot 1x}+0{\cdot}2\epsilon^{0\cdot 2x}$.

(3) $y=\epsilon^{0\cdot 5x}-0{\cdot}5\epsilon^{x}$.
(4) $y=\epsilon^{0\cdot 2t}+\epsilon^{-t}$.

(5) $V\epsilon=^{0\cdot 2x}+\epsilon^{-x}$.
(6) and (7) $y=5\epsilon^{0\cdot 4x}-\epsilon^{0\cdot 3x}$.

(8) $y=10\epsilon^{-0\cdot 05\theta}+5\epsilon^{-0\cdot 01\theta}$.
(9) $y=0{\cdot}5\epsilon^{-0\cdot 6x}+\epsilon^{0\cdot 05x}$.

(10) $y=\epsilon^{0\cdot 05x}-0{\cdot}1\epsilon^{0\cdot 2x}$.
(11) $y=\epsilon^{0\cdot 05x}+0{\cdot}5\epsilon^{-0\cdot 6x}-0{\cdot}1\epsilon^{0\cdot 2x}$.

(12) $B=154{\cdot}3\epsilon^{-0\cdot 00118t}+331{\cdot}4\epsilon^{-0\cdot 00855t}$.

Exercises VII. (p. 242.)

(1) The graph is a circle of diameter = unity, with the pole at its lowest point.

(2) As (1), but with the pole at the left end of a horizontal diameter.

(3) A circle of diameter $=\sqrt{2}$, with the pole at the lower end of a diameter sloping down at 45° to the left.

(4) As (3), the diameter sloping at 45° to the right.

(5) As (4), the pole being at the upper end of the diameter.

(6) A round leaf outline with the point of attachment of the stalk on the right, at the pole. (Length of central rib = unity.)

(7) As (6), but the pole and point of attachment of the stalk are on the left.

(8) and (9). Two dissymmetrical round leaf outlines, greatly overlapping, with both points of attachment of the stalks coinciding at the pole, in the lower portion of the graph, which is symmetrical with respect to a vertical line through the pole.

(10) A nasturtium leaf outline having the point of attachment of the stalk at the pole and a loop between this point and the base of the blade (upper portion).

(11) As (10), but the base of the blade is on the left and the round limb on the right of the graph.

(12) and (13). Two overlapping leaf outlines as (10) or (11), a large one on the right and a smaller one on the left, with a common loop on the right of the pole, between their two bases, the graph being symmetrical with respect to a horizontal line through the pole.

(14) A three bladed ship's propeller with a blade horizontal on the right of the pole, which is at the centre. (Length of blades = 3 units.)

(15) Three spirals.

(16) As (6), but slightly different in shape, as will be seen if both curves are plotted on the same scale and with the same pole, as instructed.

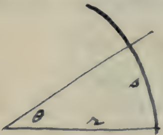
(17) P_n gives a half ellipse on the right of the minor axis which is vertical and of length = unity, the pole being at the lowest point of the ellipse. R gives a half oval on the right of the minor axis of the above ellipse, the point of the oval being at the pole. L gives a smaller and narrower full oval, approximately symmetrical with respect to the radius corresponding to $\theta = 35°$, and the point of which is at the pole.

INDEX

GLASGOW: PRINTED AT THE UNIVERSITY PRESS BY ROBERT MACLEHOSE AND CO. LTD.

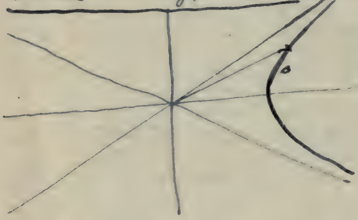
Derivation of the Hyperbolic Funct

Take a circle.



$$\theta = \int_0^s \frac{ds}{r} = \frac{1}{r}\int_0^s ds$$

Take a hyperbola.



Let $\varphi = \int_0^s \frac{ds}{r}$

Find φ

Let hyperbola be

$$x^2 - y^2 = a^2$$

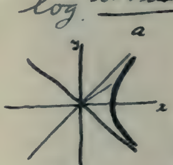
$$y = \sqrt{x^2 - a^2}$$

$$\frac{dy}{dx} = \frac{x}{\sqrt{x^2-a^2}}$$

$$\therefore \left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 = \frac{2x^2-a^2}{x^2-a^2}$$

Also $r = \sqrt{x^2+y^2} = \sqrt{2x^2-a^2}$

$$\therefore \varphi = \int \frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}{\sqrt{2x^2-a^2}} = \int_0^x \frac{dx}{\sqrt{x^2-a^2}} = \log \frac{x+\sqrt{x^2-a^2}}{a}$$



$$\therefore a\epsilon^{\varphi} = x + \sqrt{x^2-a^2}$$

$$\therefore a\epsilon^{-\varphi} = x - \sqrt{x^2-a^2}$$

$$\therefore \frac{x}{a} = \frac{1}{2}\left(\epsilon^{\varphi} + e^{-\varphi}\right).$$ This for a hyperbol x^2-y

Put $\frac{x}{r} = \cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$. This for ⊙ x^2+

On analogy we will call $\frac{e^{\varphi}+e^{-\varphi}}{2}$, co

$$\therefore \frac{x}{a} = \cosh\varphi \qquad \frac{y}{a} = \frac{\sqrt{x^2-a^2}}{a} = \frac{e^{\varphi}-e^{-\varphi}}{2}$$

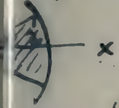
Call this sin

$\frac{y}{x} = \tanh\varphi$ and so on.

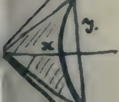
Remember ... $e^{\varphi} = \cosh\varphi + \sin$

…ea of sector of circle $= \frac{\theta}{2\pi}\pi a^2 = \frac{\theta}{2}a^2$ ∴ Shaded are…

$= a^2 \sin^{-1}\frac{y}{a}$



In general $= ab\log(\frac{x}{a}$…



Shaded area $= a^2\log\frac{x+y}{a} = a^2\log\frac{a}{x-y}$

$= a^2\sinh^{-1}\frac{y}{a} = a^2\cosh^{-1}\frac{x}{a}$.

OTHER WORKS ON HIGHER MATHEMATICS.

A TREATISE ON THE DIFFERENTIAL CAL-CULUS. By ISAAC TODHUNTER, F.R.S. Crown 8vo. 12s. 6d. KEY. Crown 8vo. 12s. 6d.

A TREATISE ON THE INTEGRAL CALCULUS AND ITS APPLICATIONS. By ISAAC TODHUNTER, F.R.S. Crown 8vo. 12s. 6d. KEY. Crown 8vo. 12s. 6d.

A FIRST COURSE IN THE DIFFERENTIAL AND INTEGRAL CALCULUS. By Prof. WILLIAM F. OSGOOD, Ph.D. Crown 8vo. 15s. net.

THE DIFFERENTIAL AND INTEGRAL CAL-CULUS. By P. A. LAMBERT. Crown 8vo. 7s. 6d.

DIFFERENTIAL AND INTEGRAL CALCULUS By CLYDE E. LOVE, Ph.D. Extra Crown 8vo. 11s. net.

A BRIEF INTRODUCTION TO THE INFINI-TESIMAL CALCULUS. Designed especially to aid in reading Mathematical Economics and Statistics. By IRVIN FISHER, Ph.D. Globe 8vo. 4s. 6d. net.

THE CALCULUS. By E. W. DAVIS, assisted by W C. BRENKE. Crown 8vo. 14s. net.

ORDINARY DIFFERENTIAL EQUATIONS. By Prof. JAMES M. PAGE, M.A., Ph.D. Crown 8vo. 6s. 6d.

A SHORT COURSE ON DIFFERENTIAL EQUA-TIONS. By Prof. DONALD F. CAMPBELL, Ph.D. Crown 8vo 7s. 6d. net.

LONDON: MACMILLAN & CO., LTD.