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An Extension of the Second Welfare Theorem to
Economies with Nonconvexities and Public Goods

M. Ali Khan
Rajiv Vohra

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College of Commerce and Business Administration

University of Illinois at Urbana-Champaign

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An Extension of the Second Welfare Theorem to
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M. Ali Khan, Professor
Department of Economics

Rajiv Vohra
Brown University

Abstract

In this paper, we report a generalization of the results of Foley
and Guesnerie on the second welfare theorem to economies with
arbitrary nonconvex production sets. The nature of marginal cost
prices in such economies is clarified through the use of the
Clarke tangent cones.

I. Introduction

It has often been argued, beginning with Hotelling [1939],
that even if some firms exhibit increasing returns to scale,
optimality demands the establishment of marginal cost prices. A
rigorous and general proof of such an assertion was first offered
by Guesnerie [1975] but only for economies with certain kinds of
nonconvex production sets. Guesnerie's theorem specifically
rules out production sets with[1] "inward kinks". It has remained
an open question as to whether this restriction on production
sets can be dispensed with. In this paper we show that the
validity of the second welfare theorem does not depend in any way
on the nature of production sets, other than the assumption of
free disposal. We also extend our result to economies with
public goods and in so doing, generalize a corresponding result
of Foley [1970]. This extension is the only result known to us
that provides a normative basis for Lindahl pricing in economies
with increasing returns to scale in production. As such, it is
overdue.

For a proof of his result, Guesnerie modified the, by now
classical, argument of Arrow [1951] and Debreu [1954], whereby
the aggregate endowment is separated from the sum of the "better
than" sets and production sets. By using the fact that the
production sets have no inward kinks, Guesnerie could show that
the resulting hyperplane furnishes marginal cost prices for each
individual set. Unfortunately, we cannot apply this argument in
our generalized set-up[2] and need to consider each set

individually rather than their sum. We thus offer a proof different from that of Arrow-Debreu-Guesnerie. It is worth remarking that the structure of our proof is inspired by the work of Guesnerie on second-best-optimality, see Guesnerie [1979]. However, unlike Guesnerie we do not use a theorem of Laurent [1972]. Our work also suggests that the arguments presented here can be used to generalize Theorem 1 of Guesnerie [1979] to the case where the cones of interior displacement are not necessarily convex.

Section II is devoted to our formalization of marginal cost prices and relates the tangent cones used in this paper to those used by Beato [1982], Brown et al. [1983], Cornet [1982] and Guesnerie [1975,1979]. Section III is devoted to the results and their proofs.

II. Discussion of Tangent Cones and Marginal Rates of Substitution

For concreteness, consider a single input, single output production set as shown in Figure I with the set of technologically efficient points summarized by a "production function" $f(\cdot)$ which associates with every input level a corresponding level of output. The notion that the marginal rate of substitution at a point is given by the derivative of $f(\cdot)$ at that point, provided such a derivative exists, needs no reference. However, the notion that these rates at points of non-differentiability such as (a) and (b) are given by the shaded cones in Figure I is almost as old and can be traced at least to Samuelson's Foundations (1947). Already, there Samuelson argued that necessary conditions for

profit maximization would lead a firm to choose production plans at which the prices would lie in the "cone" of the marginal rates of substitution; such a cone being determined by conventional left and right derivatives.

Subsequent emphasis on the convexity assumption of production sets allowed one to ignore production plans such as (b) and (d) and to formalize the marginal rates of substitution as the cone of "subdifferentials" of a concave function, the existence of which did not even require left and right derivatives. A full development of this point of view can be found in Rockafellar [1970].

In a pioneering paper, Guesnerie [1975] began an investigation of nonconvex production sets. He could not use the definition of marginal rates of substitution in the sense of convex analysis simply because the non-convexity of Y did not allow the use of the separating hyperplane theorems. Instead, Guesnerie followed Dubovickii and Miljurin[3] and considered cones of interior displacement. Heuristically, this simply magnifies and transfers to the origin the "local" shape of the production set at a particular production plan. Thus, the cone of interior displacement at (a) in Figure I is a convex cone generated by the vectors ab and ao but with (a) shifted to the origin. For Guesnerie, the marginal rates of substitution at (a) are the normals to the cone of interior displacement at (a), i.e., the shaded cone at (a). As another example, the cone of interior

displacement at (d) is the half-space under the tangent to $f(\cdot)$ shifted to the origin.[4] In this case, there is a unique normal which coincides with our conventional notion of a marginal rate of substitution in the differentiable case.

The problem with the Dubovickii-Miljurin cone is apparent when we consider the production plan (b). Here the cone of interior displacement is no longer convex as opposed to those at (a), (c) and (d) and it is clear that the normal to such a cone is only the null vector. It is precisely because of this that Guesnerie [1975, 1979] and Beato [1982] rule out production sets with "inward kinks", i.e., points leading to nonconvex cones of interior displacement.

From an economic point of view, it is, of course, clear that the relevant cone of the marginal rates of substitution at (b) is the one shown in Figure 1. It is the set of normals not to the cone of interior displacement but to minus its complement. This simple idea leads us to precisely the Clarke tangent cone which has been recently introduced in the economics literature by Cornet [1982] and used subsequently by Brown et al. [1983]. This cone is always convex and its interior coincides with the cone of interior displacement whenever the set is convex.[5] To bring the circle of ideas back to Samuelson, for the production plan (b) to be a profit-maximizing plan, a necessary condition is that the price vector lies in the normals to the Clarke tangent cone at (b). Indeed, if there is any vector of prices p such that at a production plan y, the necessary conditions for profit maximization are satisfie

at y then p must belong to the set of normals to the Clarke
tangent cone at y (see Clarke [1983] Corollary to Proposition
2.4.3). Thus, if we define the marginal rates of substitution to
be the set of prices which satisfy the necessary conditions for
profit maximization then this set is precisely the set of normals
to the Clarke tangent cone.

We devote the remainder of this section to a formal
presentation of these ideas.

Let R^n denote n-dimensional Euclidean space with R_+^n as
its nonnegative orthant and with $>>$, $>$, $\geqq$ as the ordering on
vectors. For any set $A \subseteq R^n$, Cl A, Bdry A and Int (A) will
be used to denote its closure, boundary and interior respectively.
$B_\varepsilon(y)$ denotes the open ball with center y and radius ε.

Definition 1. The <u>cone of interior displacement</u> for $Y \subseteq R^n$
relative to $y \in R^n$ is the set
$$K(Y,y) = \{x \in R^n | \exists \, \eta > 0, \, \varepsilon > 0, \quad \text{such that}$$
$$\forall \, \lambda \in [0,\eta], \quad \{y\} + \lambda B_\varepsilon(x) \subseteq Y\}.$$

This definition is the one used in Guesnerie [1979] and,
under the assumption of free disposal, identical to the one used
in Guesnerie [1975]. In order to see that this definition
formalizes the fact that the cone of interior displacement
magnifies and transfers to the origin the "local" shape of the
production set at a particular production plan, consider the sets
given by the shaded areas in Figure II and their cones of interior
displacement at the origin. In this case y = 0 in Definition 1
and no transfer to the origin is necessary. In Figure IIa,

$K(Y,0)$ is the interior of the set itself. To see that x is in $K(Y,0)$, we have to find an $\varepsilon > 0$ and a control number $\eta > 0$ such that the ε-ball around x, $B_\varepsilon(x)$, when multiplied by any non-negative number less than or equal to η, remains in the set. This is clearly so for $B_\varepsilon(x)$ shown in Figure IIa for any choice of $\eta < \varepsilon$. In Figure IIb, x is no longer in $K(Y,0)$. Here the control number η is crucial in ruling out x since $\lambda B_\varepsilon(x)$ is in Y for all $\lambda > 1$ but no $\eta > 0$ can be found such that $\lambda B_\varepsilon(x) \subseteq Y$ for all $\lambda \in [0,\eta]$. On the other hand, in Figure IIc, x is in $K(Y,0)$ but $B_\varepsilon(x)$ is not contained in Y. However, $\lambda B_\varepsilon(x) \subseteq Y$ for small enough λ and here too η in controlling for the small enough value of λ, plays a determining role.

The Clarke tangent cone is given in the next definition.

Definition 2. The <u>tangent cone</u> for $Y \subseteq R^n$, relative to $y \in Cl\ Y$ is the set
$$T(Y,y) = \{x \in R^n | \text{ For any sequence } t^k \downarrow 0, y^k \rightarrow y$$
with $y^k \in Cl\ Y, \exists$ a sequence $x^k \rightarrow x$ such that for all large enough $k,\ y^k + t^k x^k \in Cl\ Y\}$.[6]

Note that Clarke defines his tangent cone only for closed sets while we are extending his definition to an arbitrary set by simply considering its closure. In fact, for $y \in Cl\ Y$, by (3.3) of Rockafellar [1980], $T(Cl\ Y,y) = T(Y,y)$. The comparison with $K(Y,y)$ is facilitated if we consider the interior of $T(Y,y)$.

Proposition. For $Y \subseteq R^n$ and $y \in Cl\ Y$, $Int\ T(Y,y)$ is the set

$\{x \in R^n |\ \exists\ \eta > 0,\ \varepsilon > 0,\ \delta > 0,$ such that

$\forall \lambda \in [0,\eta],\quad \{y'\} + \lambda Cl\ B_\varepsilon(x) \subseteq Y$ for all

$y' \in (Cl\ Y \cap Cl\ B_\delta(y))\}.$

Proof. It is clear that, in the set defined in the Proposition, replacing Y by $Cl\ Y$ leaves the set unchanged. Theorem 2 of Rockafellar [1979] can now be used to complete the proof.

The reader should now compare the characterization of $Int\ T(Y,y)$ in the Proposition to that of $K(Y,y)$ in Definition 1. The interior of the Clarke tangent cone simply replaces $\{y\}$ in the definition of $K(Y,y)$ by $\{y'\}$ where y' is in the intersection of a closed δ-ball around y, $Cl\ B_\delta(y)$, with $Cl\ Y$. To see what difference this makes, consider Figure IIa. In this case $Int\ T(Y,0)$ "shrinks" $K(Y,0)$ to the area enclosed by the dotted lines. To see why x does not belong to $Int\ T(Y,0)$, note that one cannot find any ε-ball around x and any δ-ball around 0 and any control number $\eta > 0$ such that $\lambda\ Cl\ B_\varepsilon(x)$ plus the intersection of Y and $Cl\ B_\delta(0)$ is in Y. This can also be seen through Definition 2. Choose $\{y^k\}$ to be a sequence on the lower boundary of Y and one which tends to $y = 0$. There is no way one can find a sequence $\{x^k\}$ tending to x such that $(y^k + t^k x^k)$ is in Y for large enough k and for any sequence $\{t^k\}$ of positive numbers going to zero. This can only be done if x is in the area enclosed by the dotted lines.

As our verbal discussion makes clear, the primary objects of interest are not the tangent cones but their normals, i.e., their polar cones.

Definition 3. The <u>polar cone</u> of a set $A \subseteq R^n$, is the set

$$A^+ = \{p \in R^n \mid pa \leq 0 \quad \forall a \in A\}.$$

Let $N(Y,y) = T(Y,y)^+$. Since $A^+ = (Cl\ A)^+$, $N(Y,y) = (Int\ T(Y,y))^+$.

In the remainder of this section we state and prove properties of $Int\ T(Y,y)$ and its associated normal cone $N(Y,y)$ that we shall need for the proofs of our results. Similar results for cones of interior displacement were used by Guesnerie [1979] in his study on second-best optimality.

Lemma 1. Let $K^1, \ldots, K^m$ be non-empty, open convex cones with vertex 0 such that $\cap_i K^i \neq \{\phi\}$. Then, $(\cap_i K^i)^+ \subseteq \sum_i K^{i+}$.

Lemma 2. Let $Y^i \subseteq R^n$, $i = 1, \ldots, m$ and $y \in \cap_i Cl\ Y^i$. Then,

$$Int\ T(\cap_i Y^i, y) \supseteq \cap_i Int\ T(Y^i, y).$$

Lemma 3. Let $\cap_i Int\ T(Y^i, y) \neq \{\phi\}$ and $\eta \in N(\cap_i Y^i, y)$. Then,

there exist $\eta^i \in N(Y^i, y)$ such that $\eta = \sum_i \eta^i$.

Lemma 4. Let $Y = \Pi_i\ Y^i$ and $y = (y^1 x \ldots x\ y^m) \in Y$, where $y^i \in Y^i \subseteq R^n$, $i = 1, \ldots, m$. Then

$$Int\ T(Y,y) = \Pi_i\ Int\ T(Y^i, y^i)$$

$$N(Y,y) = \Pi_i\ N(Y^i, y^i).$$

Lemma 5. Let $A = \{x \in R^n | f(x) \le 0\}$ where f is a continuous
function. If f is differentiable at x^* and $f(x^*) = 0$,
then $N(A,x^*) = \bigcup_{\lambda \ge 0} \lambda \text{ grad } f(x^*)$.

Lemma 6. Let $Y \subseteq R^n$ and $y \in Cl (Y)$. Then Int $T(Y,y) \ne \{\phi\}$ if
either

(a) Y is convex and has an interior

or

(b) $Y + R^n_{++} \subseteq Y$

or

(c) $Y - R^n_{++} \subseteq Y$.

Before considering the formal proofs of these lemmata, a
heuristic discussion as to their meaning is warranted. This has the
further advantage of familiarizing the reader with the tangent and
normal cones used in this paper. As regards Lemma 1, let K^1 be the
shaded halfspace given by AOB and K^2 the hatched halfspace given
by COD in Figure III. If the frontiers are not included, K^1 and
K^2 are non-empty, open convex cones with vertex 0 and whose
intersection is given by the open cone enclosed by COA. Now the
polar cone of COA is given by FOE and it is easy to see that any
point in FOE can be obtained by the sum of points in OE and OF which
are respectively the polars of K^1 and K^2. This is what Lemma 1
asserts to be the case in general.

For Lemma 2, let Y^1 and Y^2 be as in Figure IV. Then Int
$T(Y^1,y)$ is the halfspace below AB and Int $T(Y^2,y)$ is the area
enclosed by 0C and 0y. Thus Int $T(Y^1,y) \cap$ Int $T(Y^2,y)$ is the
negative orthant. However, $Y^1 \cap Y^2$ is given by the set C0yE and

Int $T(Y^1 \cap Y^2, y)$ is the area enclosed by OH and OA. This contains the negative orthant which is being asserted in general in Lemma 2. It can also be checked that $N(Y^1 \cap Y^2, y)$ is the area enclosed by DOG while $N(Y^1, y)$ is OD and $N(Y^2, y)$ is the area enclosed by GOB. It is now easy to see that any point in DOG can be written as the sum of a point in OD and one within GOB which is being asserted, in general, by Lemma 3.

Lemma 4 states that the interior of the tangent cone of a cartesian product of a finite number of sets at some point is the cartesian product of the interiors of the respective tangent cones at the respective projections of that point, i.e., heuristically, the cartesian product operator and the interior of the tangent cone operator commute. Lemma 5 is an assertion that a normal to a set "enclosed" by a differentiable function is the gradient of the function. Lemma 6 provides conditions typically used in economic theory which guarantee non-emptiness of the tangent cone.

We now present the formal proofs of these lemmata and the reader, if so inclined, may move on to the model and results without any loss of continuity.

Proof of Lemma 1. We first show that $\text{Cl} (\cap_i K^i) \supseteq (\sum_i K^{i+})^+$. Pick $q \in (\sum_i K^{i+})^+$. Then, $qx < 0$ for all $x \in \sum_i K^{i+}$. Since $0 \in K^{i+}$ for all i, $qx^i < 0$ for all $x^i \in K^{i+}$, $i = 1$,

..., m. Hence, $q \in \cap\limits_{i} K^{i++}$. Since K^i is a non-empty

convex cone, by Rockafellar ([1970], p. 121), $K^{i++} = \text{Cl } K^i$.

Thus, $q \in \cap\limits_{i} \text{Cl } K^i$. By Rockafellar ([1970], Theorem 6.5)

$$\text{Cl } (\cap\limits_{i} K^i) = \cap\limits_{i} \text{Cl } K^i \text{ and thus, } \text{Cl } (\cap\limits_{i} K^i) \supseteq (\sum\limits_{i} K^{i+})^{+}.$$

By taking polars and again using the fact that $A^{++} = \text{Cl}$

A if A is a non-empty convex cone, we obtain

$(\cap\limits_{i} K^i)^{+} \subseteq \text{Cl}(\sum\limits_{i} K^{i+})$. It remains only to be shown that

$\sum\limits_{i} K^{i+}$ is closed. By Debreu ([1959], p. 23) $K^{1+} + K^{2+}$ is

closed if K^{1+} and K^{2+} are positively semi-independent.

Thus, to prove that $\sum\limits_{i} K^{i+}$ is closed it suffices to show

that K^{i+} and K^{j+} are positively semi-independent for

$i \neq j$, $i,j = 1, \ldots, m$. Towards this end consider $q^i \in K^{i+}$

and $q^j \in K^{j+}$ such that $q^i + q^j = 0$. We need to show that

$q^i = q^j = 0$. Since K^i and K^j are open and $K^i \cap K^j \neq \{\phi\}$,

there exists $\varepsilon > 0$ and $\bar{x} \neq 0$ such that $B_{\varepsilon}(\bar{x}) \in K^i \cap K^j$.

Now $q^i x < 0$ and $q^j x < 0$ for all $x \in B_{\varepsilon}(\bar{x})$. Since

$q^i = -q^j$, $q^i = q^j = 0$, and the proof is complete.

<u>Proof of Lemma 2.</u> Let $x \in \cap\limits_{i} \text{Int } T(Y^i, y)$. By the Proposition we

can assert that $\exists \; \varepsilon^i > 0, \delta^i > 0, \lambda^i > 0$ such that $y^{i'} + t^i x' \in$

$\text{Cl } Y^i$ for all $y^{i'} \in \text{Cl } Y^i \cap \text{Cl } B_{\delta^i}(y)$,

$t^i \in [0, \lambda^i]$, $x' \in Cl\ B_{\varepsilon^i}(x)$. Let $(\overline{\varepsilon}, \overline{\delta}, \overline{\lambda}) = \underset{i}{Min}'\ (\varepsilon^i, \delta^i, \lambda^i)$.

Then, for all i, $y^{i'} + tx' \in Cl\ Y^i$ for all

$y^{i'} \in Cl\ Y^i \cap Cl\ B_{\overline{\delta}}(y)$, $t \in [0, \overline{\lambda}]$, $x' \in Cl\ B_{\overline{\varepsilon}}(x)$.

This implies that, for all i, $y' + tx' \in Cl\ Y^i$ for all

$y' \in \underset{i}{\cap}\ Cl\ Y^i \cap Cl\ B_{\overline{\delta}}(y)$, $t \in [0, \overline{\lambda}]$, $x' \in Cl\ B_{\overline{\varepsilon}}(x)$, i.e.,

$y' + tx' \in \underset{i}{\cap}\ Cl\ Y^i$. Thus, $x \in Int\ T(\underset{i}{\cap}\ Y^i, y)$.

Proof of Lemma 3. Follows from Lemmata 1 and 2.

Proof of Lemma 4. This is straightforward; the reader may see, for example, the Corollary to Theorem 2.4.5 of Clarke [1983].

Proof of Lemma 5. See Corollary 1 of Theorem 2.4.7 and Definition 2.3.4 of Clarke [1983].

Proof of Lemma 6. If (a) holds we can find $x \in R^n$ such that $x + y \in Int\ (Y)$, and, therefore, $\exists\ \varepsilon > 0$ such that

$Cl\ B_\varepsilon(x) + Cl\ B_\varepsilon(y) \subseteq Y$. This implies that $y' + x' \in Y$

for $y' \in Cl\ Y \cap Cl\ B_\varepsilon(y)$ and $x' \in Cl\ B_\varepsilon(x)$. Since Y is convex, we also have $y' + tx' \in Y$ for all $t \in [0,1]$. Thus, from the Proposition, $x \in Int\ T(Y,y)$.

If (b) holds, consider $\epsilon > 0$ and $x \in R^n_{++}$ such that $Cl\ B_\epsilon(x) \in R^n_{++}$. Then,

$$y' + tx' \in Y \quad \text{for} \quad y' \in Cl\ Y, \quad t > 0$$

and $x' \in Cl\ B_\epsilon(x)$, i.e., $x \in Int\ T(Y,y)$.

If (c) holds, we can show that $-R^n_{++} \subseteq Int\ T(Y,y)$ by the above argument.

III. The Model and Results

We shall index consumers by t, $t = 1, \ldots, T$ each having a consumption set $X^t \subseteq R^n_+$ and a preference relation $\succ_t$. Let the better-than-set for t at x^t be given by $P^t(x^t) = \{y \in X^t | y \succ_t x^t\}$. Firms are indexed by j, $j = 1, \ldots, F$ each having a production set $Y^j \subseteq R^n$. The aggregate endowment is denoted by $w \in R^n_{++}$. An economy is thus denoted by $\mathcal{E} = ((X^t, \succ_t)^T_1, (Y^j)^F_1, w)$ and we shall make the following assumptions on it.

(A.1)   For all t and all $x^t \in X^t$,

$x^t \in Cl\ P^t(x^t)$ and either $P^t(x^t)$ is convex and has an interior or $\{x^t\} + R^n_{++} \subseteq P^t(x^t)$.

(A.2)   For all j, $Y^j - R^n_+ \subseteq Y^j$.

We shall need the following economic concepts.

<u>Definition 4.</u> $((x^{*t}),(y^{*j}))$ is an <u>allocation</u> of $\&$ if for all

$\quad$ t = 1, ..., T, $x^{*t} \in X^t$, for all j = 1, ..., F,

$\quad$ $y^{*j} \in Y^j$ and $\sum_t x^{*t} - \sum_j y^{*j} = w.$

<u>Definition 5.</u> An allocation $((x^{*t}), (y^{*j}))$ is <u>Pareto optimal</u> if

$\quad$ there does not exist any other allocation $((x^t), (y^j))$ such

$\quad$ that $x^t \in P^t(x^{*t})$ for all t.

$\quad$ We can now present

<u>Theorem 1.</u> If $((x^{*t}), (y^{*j}))$ is a Pareto optimal allocation and

$\quad$ (Al) and (A2) are satisfied, then there exists $p^* \in R^n$, $p^* \neq 0$,

$\quad$ such that

$\quad$ (a) $-p^* \in N(P^t(x^{*t}),x^{*t})$ for all t,

$\quad$ (b) $p^* \in N(Y^j,y^{*j})$ for all j.

$\quad$ $N(Y^j,y^{*j})$ is the set of marginal cost prices for j at
y^{*j}. For a convex set the normal cone coincides with the cone of
normals in the sense of convex analysis (see Clarke [1982],
Proposition 2.4.4). Thus, if $P^t(x^{*t})$ is convex, (a) implies
that $\not\exists$ $\bar{x}^t \in P^t(x^{*t})$ with $p^*\bar{x}^t < p^*x^{*t}$ for all t. We leave
it to the reader to compare Theorem 1 to Theorem 6.4 of Debreu
[1959] and Theorem 1 of Guesnerie [1975].

$\quad$ Our proof of Theorem 1 is based on a separating hyperplane
argument involving the interior of the tangent cones of the
production sets and the better-than-sets. In this respect the fact
that the tangent cone is always convex makes it better suited for

our purposes than the cone of interior displacement. It does,

however, involve some additional problems but fortunately these can be

handled without additional assumptions. Firstly, in general, the

interior of the tangent cone is contained in the corresponding cone of

interior displacement[7] and may be empty even if the cone of interior

displacement is not. By Lemma 6, given (A1) and (A2) this does not

occur. Secondly, we cannot use the conventional argument (as in Arrow

[1951] or Debreu [1954]) and separate the tangent space of the

aggregate production set. This is simply because $p \in N(Y,y)$ does

not necessarily imply that there exist $y^i \in Y^i$ such

that $\sum_i y^i = y$ and $p \in N(Y^i, y^i)$ for all i. We circumvent this

problem by applying the separation argument to production sets and

preference sets considered individually.

Before we consider the formalities of the proof of Theorem 1, we

clarify some basic notation. Let $k = n(T+F)$. We shall use the

convention that $v \in R^k$ can be written as $v = ((x^t), (y^j))$ where

x^t and y^j belong to R^n for all t and j. For any v in R^k,

$(v)_{x^t}$ is the projection of the vector v to the coordinates

corresponding to x^t in R^k, i.e., the coordinates $n(t-1)$ to

nt. Similarly, $(v)_{y^j}$ is the projection of v to the coordinates

corresponding to y^j in R^k, i.e., the coordinates $n(T+j-1)$ to

$n(T+j)$. Furthermore, $(v)_{x^t_\ell}$ and $(v)_{y^j_\ell}$ are respectively the $n(t-1)$

$+ \ell$ and $n(T+j-1) + \ell$ coordinates where ℓ runs between 1 and

n. For any set A in R^k, we shall denote an element of its normal cone at a given point by A_α. It is important to be clear that A_α is an element of R^k and that it would have projections as discussed above.

Proof of Theorem 1.

Let $k = n(T+F)$ and any $v \in R^k$ be denoted by $(x^1, \ldots, x^T, y^1, \ldots, y^F)$ where $x^t \in R^n$ and $y^j \in R^n$ for all $t = 1, \ldots, T$ and all $j = 1, \ldots, F$. Let $v^* = ((x^{*t}), (y^{*j}))$ and define the following sets in R^k

$$W_\ell = \{v \in R^k \mid \sum_t x_\ell^t < \sum_j y_\ell^j + w_\ell\} \quad \ell = 1, \ldots, n$$

$$V(x^*) = \Pi_t P^t(x^{*t}) \times \Pi_j Y^j$$

$$M = \bigcap_\ell W_\ell.$$

The basic idea of the proof is to separate the sets Int $T(V(x^*), v^*)$ and Int $T(M, v^*)$. Since these sets lie in R^k, so does the separating hyperplane ρ^*. We then use the construction of M to show that ρ^* is of the form $(p^*, p^* \ldots p^*, -p^*, \ldots -p^*)$ where each p^* is in R^n. The construction of $V(x^*)$ then implies that $-p^*$ is in the normal cone of $P^t(x^{*t})$ at x^{*t} and p^* is in the normal cone of Y^j at y^{*j} for all t and j. We now spell out these ideas in detail.

Clearly, if $((x^{*t}), (y^{*j}))$ is Pareto optimal, (A2) implies that $V(x^*) \cap M = \{\phi\}$. We now show that Int $T(V(x^*), v^*)$ $\cap$ Int $T(M, v^*) = \{\phi\}$. Suppose not. Then, there exists

$v \in$ Int $T(V(x^*),v^*) \cap$ Int $T(M,v^*)$. Notice that by (A1), $v^* \in Cl$ $V(x^*)$. By definition of M, $v^* \in M$ so that the tangent cones are well defined. In accordance with our notational convention, v can be written as $v = ((x^t),(y^j))$. By Lemma 4, $x^t \in$ Int $T(P^t(x^{*t}),x^{*t})$ for all t and $y^j \in$ Int $T(Y^j,y^{*j})$ for all j. By the Proposition, there exist positive real numbers η^t, η^j and η^m such that $x^{*t} + \lambda^t x^t \in P^t(x^{*t})$ $\forall \lambda^t \in [0,\eta^t]$, $y^{*j} + \lambda^j y^j \in Y^j$ $\forall \lambda^j \in [0,\eta^j]$ for all t and j and $v^* + \lambda^m v \in M$ $\forall \lambda^m \in [0,\eta^m]$. Let $\eta =$ Min [Min (η^t), Min (η^j), η^m]. This implies that for $\lambda \in [0,\eta]$, $x^{*t} + \lambda x^t \in P^t(x^{*t})$ $\forall t$, $y^{*j} + \lambda y^j \in Y^j$ for all j and $\sum_t (x^{*t} + \lambda x^t) - \sum_j (y^{*j} + \lambda y^j) < w$. Thus, $v^* + \lambda v \in V(x^*) \cap M$, contradicting $V(x^*) \cap M = \{\phi\}$.

We now verify that Int $T(V(x^*),v^*)$ and Int $T(M,v^*)$ are non-empty. By Lemma 4, Int $T(V(x^*),v^*) = \Pi_t$ Int $T(P^t(x^{*t}),x^{*t})$ $\times \Pi_j$ Int $T(Y^j,y^{*j})$. Given (A1) and (A2), Lemma 6 implies that Int $T(P^t(x^{*t}),x^{*t})$ and Int $T(Y^j,y^{*j})$ are non-empty for all t and j. Thus, Int $T(V(x^*),v^*) \neq \{\phi\}$. It is easy to check that M is convex and has a non-empty interior. We can, therefore, apply Lemma 6 to assert that Int $T(M,v^*) \neq \{\phi\}$.

We can now apply the separating hyperplane theorem to assert that there exists $\rho^* \neq 0$ such that $\rho^* \in N(M,v^*)$ and $-\rho^* \in N(V(x^*),v^*)$. Lemma 3 then implies that there exist vectors $W_{\ell\alpha} \in N(W_\ell,v^*)$ such that

$$(1) \qquad \sum_\ell W_{\ell\alpha} = \rho^*.$$

For any $v \in R^k$ and $\ell = 1, \ldots, n$, let $v_\ell = ((x_\ell^t),(y_\ell^j)) \in R^{T+F}$. Define $g_\ell(v_\ell) = \sum_t x_\ell^t - \sum_j y_\ell^j - w_\ell$ and

$G_\ell = \{v_\ell \in R^{T+F} | g_\ell(v_\ell) < 0\}$. Since $\sum_t x^{*t} - \sum_j y^{*j} = w$, we can appeal

to Lemma 5 to assert that $N(G_\ell, v_\ell^*) = \bigcup_{\lambda > 0} \lambda \text{ grad } g_\ell(v_\ell^*)$. Given the

construction of g_ℓ, this clearly implies that any element of

$N(G_\ell, v_\ell^*)$ must be a $T+F$ dimensional vector of the form $(p_\ell^*, \ldots, p)$

$- p_\ell^*, \ldots, - p_\ell^*)$ for some $p_\ell^* \in R$.

We shall now show that for all t and j,

(i) $(W_{\ell\alpha})_{x_\ell^t} = p_\ell^*$ $\ell = 1, \ldots, n$

(ii) $(W_{\ell\alpha})_{y_\ell^j} = -p_\ell^*$ $\ell = 1, \ldots, n$

(iii) $(W_{\ell\alpha})_{x_q^t} = (W_{\ell\alpha})_{y_q^j} = 0$ $\ell \neq q$.

Notice that W_ℓ has been constructed such that its projection on

coordinates corresponding to v_ℓ is G_ℓ while its projection on the

remaining $(T+F)(n-1)$ coordinates is R. We can, therefore, apply

Lemma 4 to assert that $(W_{\ell\alpha})_{v_\ell} \in N(G_\ell, v_\ell^*)$ and, as argued above, it ca

be written as $(p_\ell^*, \ldots, p_\ell^*, - p_\ell^*, \ldots, - p_\ell^*)$. This establishes (i) an

(ii). Since $N(R, z) = 0$ for any $z \in R$, Lemma 4 also yields (iii).

Letting $p^* = (p_1^*, \ldots, p_n^*)$, we can now assert that $\sum_\ell W_{\ell\alpha}$

$= (p^*, \ldots, p^*, -p^*, \ldots, -p^*)$. By (1) $\rho^* = (p^*, \ldots, p^*, -p^*,$

$\ldots, -p^*)$. Since $\rho^* \neq 0$, $p^* \neq 0$. Now, $-\rho^* \in N(V(x^*), v^*)$ implies,

given Lemma 4 and the definition of $V(x^*)$, that $-p^* \in$

$N(P^t(x^{*t}), x^{*t})$ for all t and $p^* \in N(Y^j, y^{*j})$ for all j.

Our next result extends Theorem 1 to economies with public
goods. Let the first n commodities be private goods and the
next m be public goods in the sense that their consumption is
identical across individuals. An economy with public goods
$\mathcal{E}^G = ((x^t, \succ_t)_1^T, (Y^j)_1^F, w)$ is such that for all t, $X^t =$
(X_π^t, X_g^t) where $X_\pi^t \subseteq R_+^n$, $X_g^t \subseteq R_+^m$ are its projections onto the
space of private and public goods respectively. We assume that
$X_g^t = X_g$ for all t; that $Y^j \subseteq R^{n+m}$ for all j and that
$w = (w_\pi, 0)$, $w_\pi \in R_{++}^n$. Let $x_{\pi \ell}^t$ and $x_{g \ell}^t$ refer to the consumption
of the ℓth private and public good respectively.

<u>Definition 6.</u> $((x^{*t}), (y^{*j}))$ is an <u>allocation</u> of $\mathcal{E}^G$ if for all

$t = 1, \ldots, T$, $x^{*t} \in X^t$, for all $j = 1, \ldots, F$, $y^{*j} \in Y^j$,
$x_g^{*t} = x_g^*$ for all t and $(\sum_t x_\pi^{*t}, x_g^*) - \sum_j y^{*j} = w$.

The definition of a Pareto optimal allocation for $\mathcal{E}^G$ is then
identical to the one given in Definition 5. We shall also need an
assumption on the desirability of public goods.

(A.3)   For all t, if $\bar{x}_g^t < x_g^t$, then $(x_\pi^t, \bar{x}_g^t) \notin P^t(x^t)$.

We can now present our second result.

<u>Theorem 2.</u> If $((x_\pi^{*t}, x_g^{*t}), (y^{*j}))$ is a Pareto optimal allocation

and (A1) - (A3) are satisfied, then there exist $p^* = (p_\pi^*, p_g^*) \in$
R^{n+m}, $p^* \neq 0$ and, for all t, $p_g^{*t} \in R^m$, such that

(a) $\sum_t p_g^{*t} = p_g^*$

(b) $- (p_\pi^*, p_g^{*t}) \in N(P^t(x^{*t}), x^{*t})$ for all t

(c) $p^* \in N(Y^j, y^{*j})$ for all j.

We leave to the reader to compare Theorem 2 to Theorem 1 in Foley [1970].

The proof of Theorem 2 is along the lines of the proof of Theorem 1 with the only difference that additional sets corresponding to public goods have to be introduced in the construction of the set M. An additional notation that has to be kept in mind is that pertaining to the distinction between public and private goods. Thus, for any commodity bundle x^t, in $x^t_{\pi \ell}$ ℓ runs from 1 to n while in $x^t_{g\ell}$, ℓ runs from n+1 to n+m.

Proof of Theorem 2.

Let $k = (n+m)(T+F)$ and define the following sets in R^k.

$$W_{\pi \ell} = \{v \in R^k \mid \sum_t x^t_{\pi \ell} < \sum_j y^j_{\pi \ell} + w_{\pi \ell}\} \quad \ell = 1, \ldots, n$$

$$W^t_{g\ell} = \{v \in R^k \mid x^t_{g\ell} < \sum_j y^j_{g\ell}\} \quad \begin{array}{l} \ell = n+1, \ldots, n+m \\ t = 1, \ldots, T \end{array}$$

$$V(x^*) = \Pi_t P^t(x^{*t}) \times \Pi_j Y^j$$

$$M = \bigcap_\ell W_{\pi \ell} \bigcap_{\ell,t} W^t_{g\ell}.$$

Clearly, given (A3), if $((x^{*t}),(y^{*j}))$ is Pareto optimal $V(x^*) \cap M = \{\phi\}$. This implies, as in the proof of Theorem 1, that Int $T(V(x^*),v^*) \cap$ Int $T(M,v^*) = \{\phi\}$. We can also use the argument of the proof of Theorem 1 to show that Int $T(V(x^*),v^*)$ and Int $T(M,v^*)$ are non-empty.

We can now apply the separating hyperplane theorem to assert that there exists $\rho^* \neq 0$ such that $\rho^* \in N(M, v^*)$ and $-\rho^* \in N(V(x^*), v^*)$. By Lemma 3, there exist vectors $W_{\pi \ell \alpha} \in N(W_{\pi \ell}, v^*)$, $W_{g \ell \alpha}^t \in N(W_{g \ell}^t, v^*)$ such that

$$
(2) \qquad \sum_{\ell} W_{\pi \ell \alpha} + \sum_{t, \ell}{}' W_{g \ell \alpha}^t = \rho^*.
$$

As in the proof of Theorem 1, we can use Lemma 4 to show that $(W_{\pi \ell \alpha})_{x_{\pi \ell}^*} = p_{\pi \ell}^*$ for all t where $p_{\pi \ell}^* \in R$ and $(W_{\pi \ell \alpha})_{y_{\pi \ell}^j} = -p_{\pi \ell}^*$ for all j. Also, $(W_{g \ell \alpha}^t)_{x_{g \ell}^t} = p_{g \ell}^{*t}$ for all t and $(W_{g \ell \alpha}^t)_{y_{g \ell}^j} = -p_{g \ell}^{*t}$ for all j. Thus (2) yields $(\rho^*)_{x^t} = (p_{\pi}^*, p_g^{*t})$ for all t and $(\rho^*)_{y^j} = -(p_{\pi}^*, \sum_t p_g^{*t})$ for all j. Since $\rho^* \neq 0$, $p^* = (p_{\pi}^*, \sum_t p_g^{*t}) \neq 0$. By Lemma 4, $-(p_{\pi}^*, p_g^{*t}) \in N(P^t(x^{*t}), x^{*t})$ for all t and $p^* \in N(Y^j, y^{*j})$ for all j.

Department of Economics, University of Illinois, 1206 South 6th Street, Champaign, IL 61820.

Department of Economics, Brown University, Providence, RI 02912.

FOOTNOTES

1. This is stated precisely below.

2. For a precise reason, see the discussion preceding the proof of Theorem 1 below.

3. See Guesnerie [1975] for references.

4. Similarly the cone of interior displacement at (c) is the half-space under the tangent to $f(\cdot)$ at (c) shifted to the origin.

5. A precise reference is Laurent ([1972], Theorem 1.3.4) and Rockefeller ([1980], Theorem 1).

6. In the definition of $T(Y,y)$, Rockafeller [1979] states "for all k". It is clear that he means "for all large enough k".

7. This follows, for example, from Proposition 3.4 (b) of Cornet [1982] taking account of the fact that $I_0(y)$ and $I_1(y)$ in that paper correspond to Int $T(Y,y)$ and $K(Y,y)$ of this paper.

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FIGURE I

FIGURE IIa

$B_\epsilon(x)$

x

y^k

O

☐ $= Y$

▨ $= K(Y,O)$

FIGURE II b

$B_\epsilon(x)$

x

O

$\square$ = Y

▨ = K(Y,O)

FIGURE II c

$B_\epsilon(x)$

$\square$ = Y

$\boxed{/\!/}$ = K(Y,O)
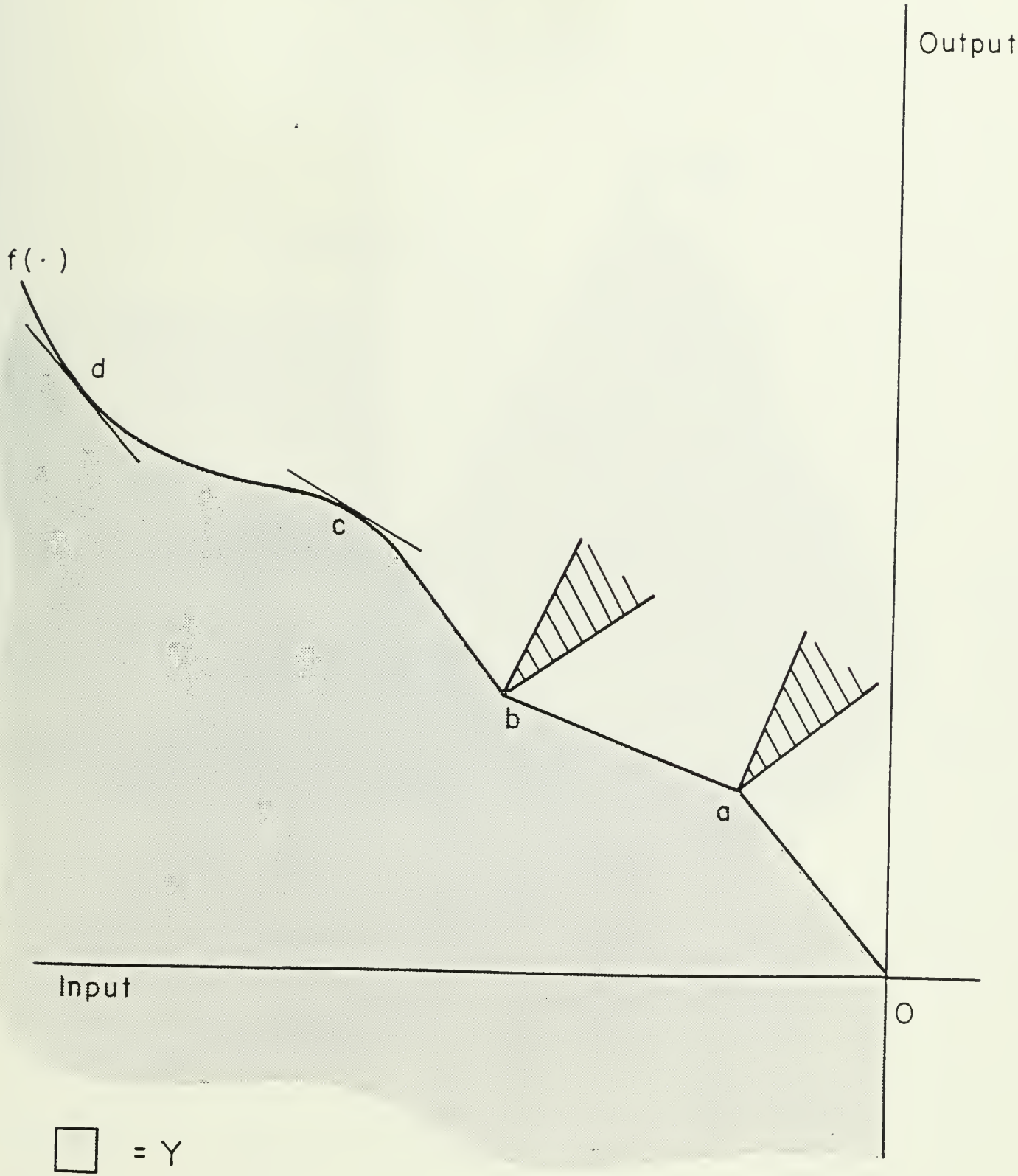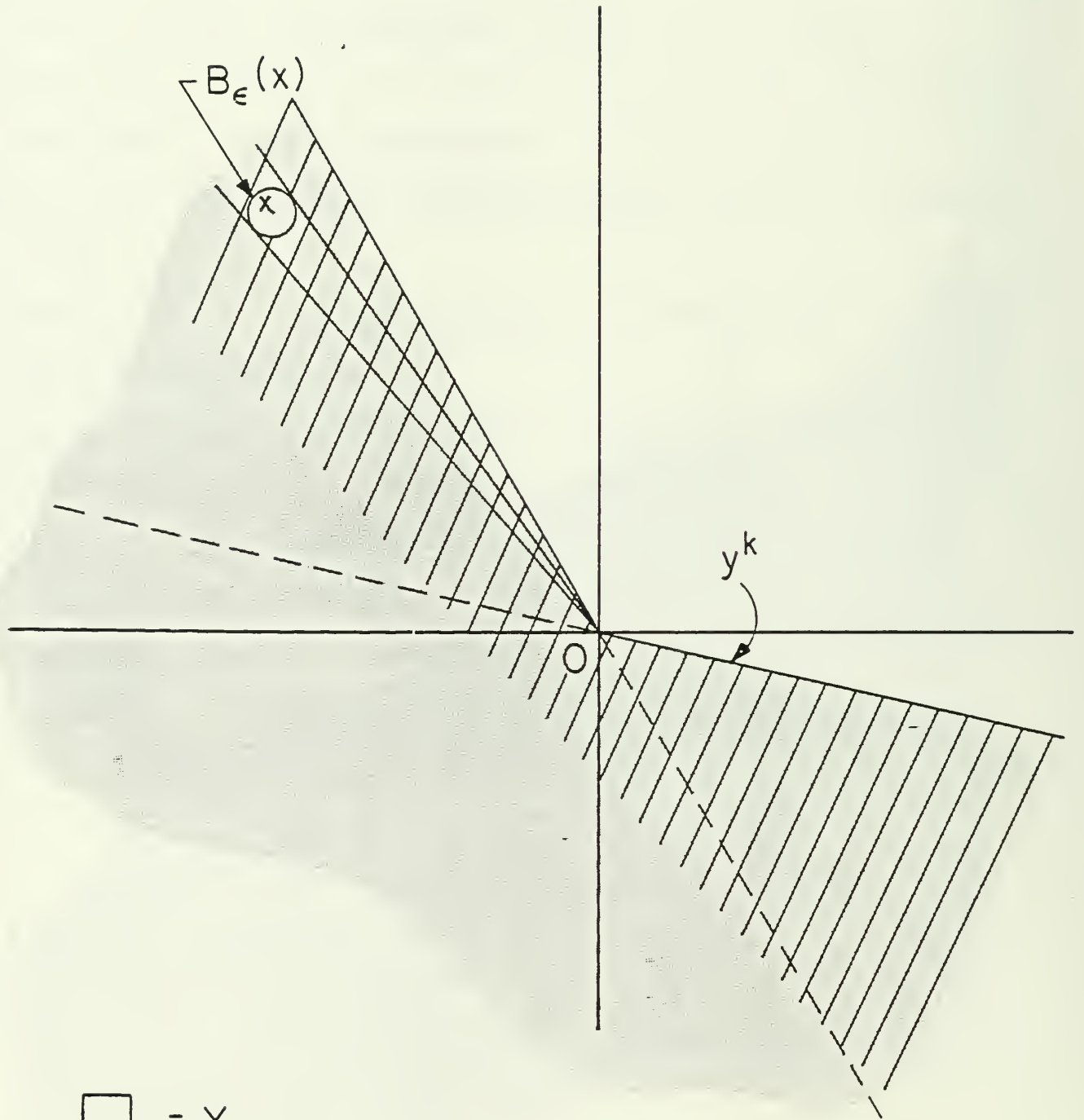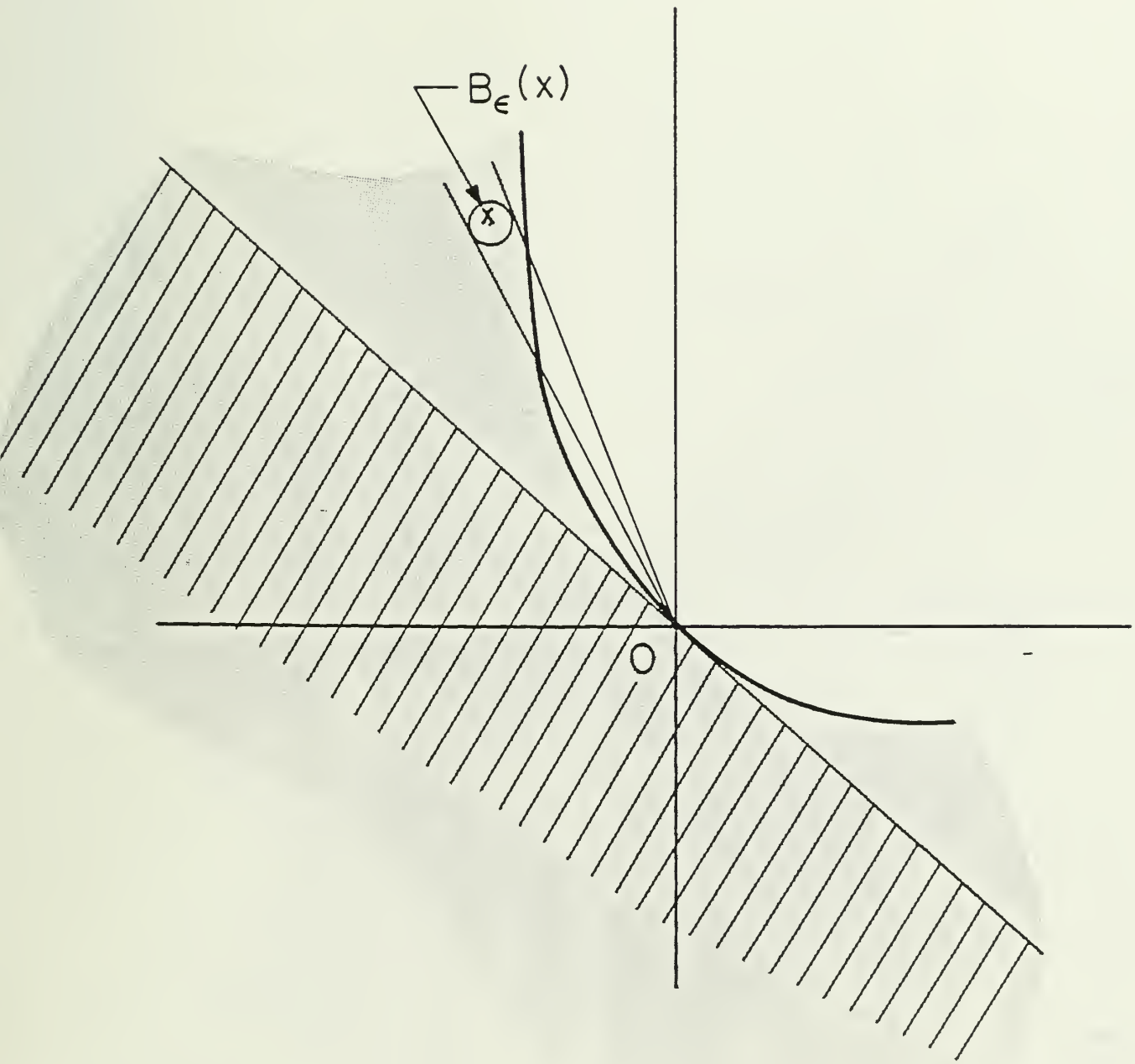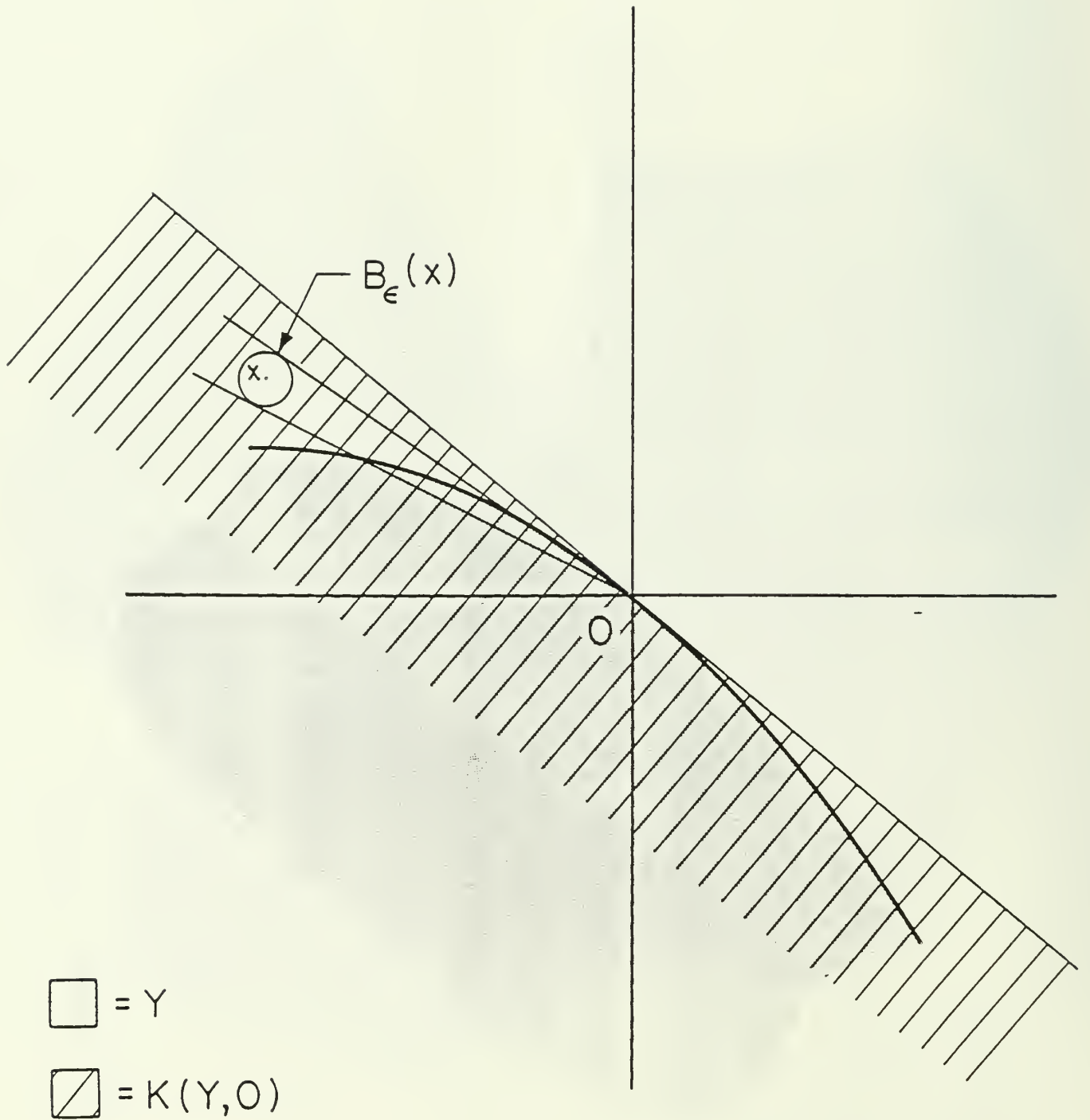
FIGURE III

$\square = K^1$

$\boxed{/} = K^2$

FIGURE IV

= Y¹

= Y²