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W. Hoyle

Roots of Numerical Equations
1826

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EXTRACTION

OF THE

REAL ROOTS

OF

NUMERAL EQUATIONS

OF ALL

DENOMINATIONS.

BY WILLIAM HOYLE.

LONDON:

PRINTED FOR ALL THE BOOKSELLERS,
and
MAY BE HAD OF W. DEAN, MANCHESTER, AND J. WESTALL, ROCHDALE.

1826.

CAJORI

INTRODUCTION.

THE following small treatise being chiefly intended for those who have already made some advance in the science of algebra, it will only be necessary to inform the general reader, that the extraction of the roots of an algebraic equation, and particularly the solution of the irreducible case in cubic equations, has been assiduously sought after during the last four centuries, by the most eminent mathematicians of Europe; amongst whom the following names (with regard to the subject of the present work) stand pre-eminent, viz. Scipio Ferreus, Nicholas Tartalea, Hieronymus Cardan, Lewis Ferrari, Raphael Bombelli, Vieta, Albert Girard, Harriot, Oughtred, Descartes, Sir Isaac Newton, Maclaurin, John and James Bernoulli, Fontaine, Euler, Waring, Simpson, Legendre, and Lagrange.

To the more advanced mathematician, who is already acquainted with the different methods employed by the above mentioned eminent persons, it will be necessary to give a demonstration of the system by which all the real roots (either positive or negative) of a numeral equation, containing only one unknown quantity, may be extracted.

In the following examples a number, as near the real root as possible, is assumed, which is placed in the quotient or root; the said number is then involved to within one power of the given equation, and the different powers multiplied by their proper coefficients, the products are then collected into one sum, due regard being had to the signs + or —, which sum being placed before the numeral (to which the equation is equal), may be supposed to act as a divisor in common arithmetic, the numeral as the dividend, and the root which has been involved as the quotient; multiply the divisor by the quotient and it will be the same as though the root had been involved to the proper powers of the given equation, as may be seen by the following; viz. $(x^4-2x^3+4x^2+5x+1)\times x=x^5-2x^4+4x^3+5x^2+x$; the product of the divisor and quotient is now subtracted from the dividend, the remainder will be the difference between an equation of the same order and coefficients, as the given equation, with the assumed number for its root; to the remainder bring down as many ciphers as there are units in the highest power of the given equation; another figure must be placed in the root, and managed according to the following formula.

The first figure in the root having had its powers extracted, may now be considered as known to be a part of the real root, and therefore may be represented by a, the second figure, which is not yet known to be a part of the real root, by x; the first figure standing to the left may be considered as a tens figure and the second a unit; from whence the present figures in the root may be represented by $a+x$; we must now involve this binomial to the highest power of the given equation; thus, let the given equation be of the fourth order, or $y^4+by^3-cy^2-dy=0$, and the fourth power of $a+x=a^4+4a^3x+6a^2x^2+4ax^3+x^4$, in which we may consider a^4 to have been already extracted, and as the sum will have to be multiplied by x at last, we shall have $4a^3+6a^2x+4ax^2+x^3$, but as a may be considered to be ten times the value of the figure by which it is represented, we have, in the following examples, begun with x^3, and retreated one figure tothe left, it being the same way that is practised in common numeral multiplication, in order to save the repetition of ciphers: now, as y^4 has only 1 for its co-efficient, the sum of $4a^3+6a^2x+4a^2+x^3$ may be now left to represent y^4. Then $\overline{a+x}|^3=a^3+3a^2x+3ax^2+x^3$ where the same allowances for a^3 and a lower power of x will have to be made as before, whence we shall have $3a^2+3ax+x^2$, which (as y^3 is multiplied by $+b$) must be multiplied by b, and then $3a^2b+3ab\,x+bx^2$ will represent $+by^3$.

Then $\overline{a+x}|^2=a^2+2ax+x^2$ where, proceeding as before, we have $-2ac-cx$ to represent $-cy^2$.

Then $a+x$ will be merely $-d$, to represent $-dy$, therefore we shall have

$$\begin{array}{c} 4a^3+6a^2x+4ax^2+x^3 \\ 3a^2b+3abx+bx^2 \\ -2ac-cx \\ -d \end{array}$$

which, added together, will give the second divisor.

But in the addition of numerals it will have to be observed, that x is a decimal, and that therefore as many ciphers must be prefixed to the right of $-d$ as there are units within one in the highest power of the given equation; in the above case it will be $4-1=3$, the number of ciphers; and generally let n represent the number of decimals in the root and m the number of units in the highest power of the given equation, then the number of ciphers to be added will be

$$y^1 = \overline{m-1} \times n$$
$$y^2 = \overline{m-2} \times n$$
$$y^3 = \overline{m-3} \times n, \text{ \&c.}$$

which will be readily seen by adding together the root, square, cube, &c. of a decimal, as $.2$ $\quad .02$

$$.2^2 = .04 \qquad .02^2 = .0004$$
$$.2^3 = .008 \qquad .02^3 = .000008$$
$$.2^4 = .0016 \qquad .02^4 = .00000016$$

where, to make an equal quantity of decimals in each line, it will be necessary to add 3 ciphers to .2 two to .04, 1 to .008, 6 to .02, 4 to .0004, and 2 to .000008.

When the second divisor has been multiplied by the last figure in the root and the product subtracted, to the remainder as many ciphers must be brought down as before, and every thing brought on in the same manner.

The two figures in the root will now be known to be a part of the real root, and therefore will be a, the figure next put in the root will be x.

After the first two or three divisors have been got it will be easily seen what the next quotient figure will be, as the significant figures, or those to the left in the divisors, will not then vary much.

This treatise would not have been published in its present form, if the author could have got it inserted in any of the periodical publications.

The solution of three different forms of the irreducible case in cubic equations was sent to the editor of the Mechanics' Magazine, London, on the 1st of August; its receipt was acknowledged on the 13th of August, but, as it has never yet appeared, what use the editor has made of it is not known. The solution of a cubic equation was sent to the editor of the Kaleidoscope, Liverpool, on the 12th of September, and appeared in it on the 4th of October; but the editor, in a note, November 1st, declined inserting, for the present, any thing more in mathematics.

Oldham, Dec. 1*st*, 1825.

SOLUTION

OF

EQUATIONS,

CONTAINING ONLY ONE UNKNOWN QUANTITY.

Example I.

Given $x^3-15x^2+63x-50=0$.

$1\times1=1$	1		$49)50(1.028=x$
$+63$	-15		49
64	-15		$357604)1000000$
-15			715208
49	1st divisor.		$35425744)284792000$
			283405952
			1386048

$2\times2=$	4	202
$3\times10\times2=$	60	-15
$3\times10\times10=$	300	-3030
	30604	
	$+630000$	
	$+660604$	
	-303000	
	357604 2nd divisor.	

$8\times8=$	64	2048
$3\times102\times8=$	2448	—15
$3\times102\times102=$	31212	———
	———	—30720
	3145744	
	+63000000	
	———	
	+66145744	
	—30720000	
	———	
	35425744	3rd divisor.

SOLUTION 2. $x^3-15x^2+63x=50$

			9)50(6.576=x
			54
$6\times6=36$	6		—
+63	—15	—725)	—4000
—	—		—3625
99	—90		———
—90		—49301)	—375000
—			—345107
9 1st divisor.			———
		—4577004)	—29893000
			—27462024
			———
			—2430976

$5\times5=$	25	125
$3\times6\times5=$	90	—15
$3\times6\times6=$	108	———
	———	—18750
	11725	+18025
	+6300	———
	———	—725 2nd divisor.
	+18025	

7×7= 49	1307
3×65×7= 1365	—15
3×65×65=12675	
	—1960500
1281199	+1911199
+630000	
	—49301 3rd divisor.
+1911199	

6×6= 36	13146
3×657×6= 11826	—15
3×657×657=1294947	
	—197190000
129612996	+192612996
+ 63000000	
	—4577004 4th divisor.
+192612996	

When $x=6.576$.

$x^3=+284.371070976$

$-15x^2=-648.656640$

$+63x=+414.288$

— 50.

+.002430976 the above remainder.

SOLUTION 3. $x^3-15x^2+63x=50.$

7×7=49	7	7)50(7.395=x.
+63	—15	49
+112	—105	189)1000
—105		567
7 1st divisor.		44991)433000
		404919
		5172175)28081000
		25860875
		2220125

3×3=	9	143
3×7×3=	63	—15
3×7×7=	147	
		—2145
	15339	
	+ 6300	
	21639	
	—21450	
	189 2nd divisor.	

9×9=	81	1469
3×73×9=	1971	—15
3×73×73=	15987	
		—22035
	1618491	
	+ 630000	
	2248491	
	—2203500	
	44991 3rd divisor.	

5×5=	25	14785
3×739×5=	11085	—15
3×739×739=	1638363	
		—221775
	163947175	
	+ 63000000	
	226947175	
	—221775000	
	5172175 4th divisor.	

When $x=7.395$.

$$x^3=+404.403154875$$
$$-15x^2=-820.290375$$
$$+63x=+465.885$$

$$+49.997779875$$
$$+\text{ the above remainder }=\ .002220125$$

$$50.0000000000$$

The three values of x $\begin{cases}+1.028\\+6.576\\+7.395\end{cases}$

$+14.999$ $\Big\{$ second term with its sign changed.

Ex. II. Given $x^4-8x^3+14x^2+4x=8$; or to get the negative root $x^4+8x^3+14x^2-4x=8$.

$7\times7\times7=343$ $\qquad$ $7\times7=49$

49
8
392

7	10063)80000(.732	
14	70441	$-.732=x$
9800	29753947)95590000	
3920	89261841	
343		
14063	30855146488)63281590000	
—4000	61710292976	
10063 1st divisor.	1571297024	

3×3×3=	27	3×3=	9	143
4×7×3×3=	252	3×7×3=	63	14
6×7×7×3=	882	3×7×7=	147	——
4×7×7×7=	1372		——	2002
	——		15339	
	1462747		8	
	12271200		——	
	20020000		122712	
	——			
	33753947			
	— 4000000			
	——			
	29753947 2nd divisor.			

2×2×2=	8	2×2=	4	1462
4×73×2×2=	1168	3×73×2=	438	14
6×73×73×2=	63948	3×73×73=	15987	——
4×73×73×73=	1556068		——	20468
	——		1603084	
	1562474488		8	
	12824672000		——	
	20468000000		12824672	
	——			
	34855146488			
	— 4000000000			
	——			
	30855146488 3rd divisor.			

Solution 2. $x^4-8x^3+14x^2+4x=8.$

8	4	2
	—8	14
	—	—
	—32	28
	8	
	28	
	4	
	—	
	+8 1st divisor.	

+8) 8(2.732=x
16
—
—10657)—80000
—74599
——
—16837253)—54010000
—50511759
——
—17057525512)—349824100000
—341150510244
——
—8673589760

$7\times7\times7=\ \ 343$	$7\times7=\ \ 49$
$4\times2\times7\times7=\ 392$	$3\times2\times7=\ 42$
$6\times2\times2\times7=168$	$3\times2\times2=12$
$4\times2\times2\times2=32$	
53063	1669
	—8
	—133520
	+122863
	—10657 2nd divisor.

47	4000
14	65800
	53063
658	+122863

$3\times3\times3=\ \ 27$	$3\times3=\ \ 9$
$4\times27\times3\times3=\ \ 972$	$3\times27\times3=\ \ 243$
$6\times27\times27\times3=\ 13122$	$3\times27\times27=2187$
$4\times27\times27+27=78732$	
80053947	221139
	—8
	—176911200
	+160073947
	3rd divisor —16837253

543	4000000
14	76020000
	80053947
7602	160073947

$$
\begin{array}{rr}
2\times2\times2= & 8 \\
4\times273\times2\times2= & 4368 \\
6\times273\times273\times2= & 894348 \\
4\times273\times273\times273= & 81385668 \\
\hline
 & 81475146488
\end{array}
$$

$$
\begin{array}{rr}
2\times2= & 4 \\
3\times273\times2= & 1638 \\
3\times273\times273= & 223587 \\
\hline
 & 22375084 \\
 & -8 \\
\hline
 & -179000672000 \\
 & +161943146488 \\
\hline
 & -17057525512
\end{array}
$$

4th divisor.

$$
\begin{array}{r}
5462 \\
14 \\
\hline
76468
\end{array}
\qquad
\begin{array}{r}
4000000000 \\
76468000000 \\
81475146488 \\
\hline
161943146488
\end{array}
$$

SOLUTION 3. $x^4-8x^3+14x^2+4x=8$.

$$
\begin{array}{rrr}
343 & 49 & 7 \\
 & -8 & 14 \\
\hline
 & -392 & 9800 \\
 & & 4000 \\
 & & 343 \\
\hline
 & & 14143 \\
 & & -3920 \\
\hline
 & & 10223
\end{array}
$$

$$
\begin{array}{r}
10223)80000(.763=x. \\
71561 \\
\hline
13201896)84390000 \\
79211376 \\
\hline
13171136187)51786240000 \\
39513408561 \\
\hline
12272831439
\end{array}
$$

```
6×6×6=          216      6×6=     36           146
4×7×6×6=      1008      3×7×6= 126            14
6×7×7×6=   1764       3×7×7=147            ----
4×7×7×7=1372                  ------      20440000
          ---------                15996        4000000
          1558696                  —8         1558696
                                  ------     ---------
                               —127968      25998696
                                            12796800
                                           ---------
                              2nd divisor  13201896
```

```
3×3×3=                 27      3×3=           9
4×76×3×3=         2736      3×76×3=      684
6×76×76×3=    103968       3×76×76=17328
4×76×76×76=1755904                  ---------
                ------------              1739649
              1766328187                  —8
                                       ----------
                                     —13917192
```

```
 1523
   14
 -----
21322000000
 4000000000
 1766328187
------------
27088328187
—13917192000
------------
13171136187  3rd divisor.
```

SOLUTION 4. $x^4-8x^3+14x^2+4x=8$.

125	25	5	—1)8(5.236
	—8	14	—5
	—200	70	53288)130000
		4	106576
		125	
		—200	64626747)234240000
			193880241
		1st divisor —1	
			66509117736)403597590000
			399054706416
			4542883584

$2\times2\times2=$	8	$2\times2=$	4	102
$4\times5\times2\times2=$	80	$3\times5\times2=$	30	14
$6\times5\times5\times2=$	300	$3\times5\times5=$	75	142800
$4\times5\times5\times5=$	500		7804	4000
	530808		—8	530808
			—62432	677608
				—624320
				2nd divisor 53288

$3\times3\times3=$	27	$3\times3=$	9
$4\times52\times3\times3=$	1872	$3\times52\times3=$	468
$6\times52\times52\times3=$	48672	$3\times52\times52=$	8112
$4\times52\times52\times52=$	562432		815889
	567317947		—8
			—6527112

1043
14

146020000
4000000
567317947

717337947
—652711200

64626747 3rd divisor.

$6\times 6\times 6=$ 216
$4\times 523\times 6\times 6=$ 75312
$6\times 523\times 523\times 6=$ 9847044
$4\times 523\times 523\times 523=$ 572222668

573208125736

$6\times 6=$ 36
$3\times 523\times 6=$ 9414
$3\times 523\times 523=$ 820587

82152876
—8

—657223008

10466
14

146524000000
4000000000
573208125736

723732125736
—657223008000

4th divisor 66509117736

Four values of x		
	— .732	— 1571297024
	+2.732	— 867358976
	+ .763	+12272831439
	+5.236	+ 4542883584
	+7.999	second term with its sign changed.

Ex. II. Given $x^4-12x^2+12x-3=0$ to find the four roots.

8	2	12		—4) 3(2.858
	—12	8		—8
	—	—24		——
	—24	—		11232)110000
		—4	1st divisor.	89856

34388125)201440000
171940625

36491110112)294993750000
291928880896

3064869104

$8\times8\times8=$ 512	48
$4\times2\times8\times8=$ 512	—12
$6\times2\times2\times8=$192	——
$4\times2\times2\times2=$32	—576
——	
56832	
12000	
—57600	
——	
11232 2nd divisor.	

$5\times5\times5=$ 125	565
$4\times28\times5\times5=$ 2800	—12
$6\times28\times28\times5=$ 23520	——
$4\times28\times28\times28=$87808	—6780
——	
90188125	
12000000	
—67800000	
——	
34388125 3rd divisor.	

$8\times8\times8=$	512	5708
$4\times285\times8\times8=$	72960	-12
$6\times285\times285\times8=$	3898800	
$4\times285\times285\times285=$	92596500	-68496

92987110112
12000000000
-68496000000

36491110112 4th divisor.

Solution 2. $x^4-12x^2+12x-3=0$.

216   6   12000
-12 -7200
216
-72
5016 1st divisor.

5016)3.0000(.606$=x$.
30096
-1594953384)-9600000000
-9569720304
30279696

$6\times6\times6=$	216	1206	12000000000
$4\times60\times6\times6=$	8640	-12	877046616
$6\times60\times60\times6=$	129600		-14472000000
$4\times60\times60\times60=$	864000	-14472	
	877046616	2nd divisor	1594953384

Solution 3. $x^4-12x^2+12x-3=0$.

64   4
-12
-4800
12000
64
7264 1st divisor.

7264)3.0000(.443$=x$.
29056
2217024)9440000
8868096
1748236667)5719040000
5244710001
474329999

4×4×4=	64	84
4×4×4×4=	256	—12
6×4×4×4=	384	
4×4×4×4=	256	1008
	297024	
	—10080000	
	12000000	
	2217024 2nd divisor.	

3×3×3=	27	883
4×44×3×3=	1584	12
6×44×44×3=	34848	
4×44×44×44=	340736	10596
	344236667	
	—10596000000	
	12000000000	
	1748236667 3rd divisor.	

$$x^4-12x^2+12x-3=0,$$

or to get a —root $x^4-12x^2-12x-3=0$.

```
27    3              —21)   3(3.907  { therefore —3.907
    —12                    —63       {  is a root.
    ———                    ———
    —36             72249(660000
    —12                   650241
    +27                   ———————
    ———       1322315847 13)975900000000
    —21 1st divisor.       925621093201
                           ————————————
                            50278906799
```

$9\times9\times9=$	729	69
$4\times3\times9\times9=$	972	—12
$6\times3\times3\times9=$	486	——
$4\times3\times3\times3=$	108	—828
	——	
	167049	
	—82800	
	—12000	
	——	
	+72249 2nd divisor.	

$7\times7\times7=$	343	7807
$4\times390\times7\times7=$	76440	—12
$6\times390\times390\times7=$	6388200	——
$4\times390\times390\times390=$	237276000	—93684
	——	
	237915584743	
	—93684000000	
	—12000000000	
	——	
	132231584743 3d divisor.	

The four roots	2.858	+.003064869104
	.606	—.000030279696
	.443	+.000474329999
	—3.907	—.050278906799
	——	
	0	

Ex. IV. Given $x^5+6x^4-10x^3-112x^2-207x-110=0$.

```
4⁴=256  64      16          —175)  110(4.464=x
         6     —10                 —700
     ——————   —————               ——————
      +384    —160          16676496)81000000
      +256                           66705984
      —160       4                   ————————
      —448    —112    222439918096)1429401600000
      —207    —————                1334639508576
      ————    —448                 —————————————
1st div—175           2310222205747456)9476209142400000
                                        9240888822989824
                                        ————————————————
                                        235320319410176
```

```
   4×4×4×4=       256          4×4×4=      64
  5×4×4×4×4=     1280         4×4×4×4=    256
 10×4×4×4×4=    2560          6×4×4×4=    384
 10×4×4×4×4=   2560           4×4×4×4=256
  5×4×4×4×4=1280                          ——————
              ——————                      297024
              15629056                       +6
                                          ——————
                                        17821440
                                        15629056
                                        ————————
                                        33450496
                                       —16774000
                                        ————————
                             2nd divisor 16676496
```

```
        4×4=  16                 84
      3×4×4= 48                —112
      3×4×4=48                 ————
             ————           —9408000
             5296           —2070000
              —10           —5296000
             ————           ————————
           —52960          —16774000
```

$6\times6\times6\times6=$ 1296
$5\times44\times6\times6\times6=$ 47520
$10\times44\times44\times6\times6=$ 696960
$10\times44\times44\times44\times6=$ 5111040
$5\times44\times44\times44\times44=$ 18740480

192586012496

$6\times6\times6=$ 216
$4\times44\times6\times6=$ 6336
$6\times44\times44\times6=$ 69696
$4\times44\times44\times44=$ 340736

347769176
+6

208661505600
192586012496

401247518096
—178807600000

3rd divisor 222439918096

$6\times6=$ 36
$3\times44\times6=$ 792
$3\times44\times44=$ 5808

588756
—10

—58875600000
—99232000000
—20700000000

—178807600000

886
—112

—99232

$$
\begin{array}{rr}
4\times4\times4\times4= & 256 \\
5\times446\times4\times4\times4= & 142720 \\
10\times446\times446\times4\times4= & 31826560 \\
10\times446\times446\times446\times4= & 3548661440 \\
5\times446\times446\times446\times446= & 197837875280 \\
\hline
 & 198193059832345 6
\end{array}
$$

$$
\begin{array}{rr}
4\times4\times4= & 64 \\
4\times446\times4\times4= & 28544 \\
6\times446\times446\times4= & 4773984 \\
4\times446\times446\times446= & 354866144 \\
\hline
 & 355343827904 \\
 & +6 \\
\hline
 & 2132062967424000 \\
 & 1981930598323456 \\
\hline
 & 4113993565747456 \\
 & -1803771360000000 \\
\hline
\text{4th divisor} & 2310222205747456
\end{array}
$$

$$
\begin{array}{rr}
4\times4= & 16 \\
3\times446\times4= & 5352 \\
3\times446\times446= & 596748 \\
\hline
 & 59728336 \\
 & -10 \\
\hline
 & -597283360000000 \\
 & -999488000000000 \\
 & -207000000000000 \\
\hline
 & -1803771360000000
\end{array}
\qquad
\begin{array}{r}
8924 \\
-112 \\
\hline
999488
\end{array}
$$

$x^5+6x^4-10x^3-112x^2-207x-110=0$, or to find a —root $6x^4-x^5+10x^3-112x^2+207x-110=0$; on trying this, —2 will be found to be a root.

Dividing the equation by $x+2$ we have $x^4+4x^3-18x^2-76x-55=0$; or to find a — root $x^4-4x^3-18x^2+76x-55=0$, whence —1 is evidently a root; and again dividing this equation by $x+1$, we have $x^3+3x^2-21x-55=0$: or to find a — root $3x^2-x^3+21x-55=0$; by inspection —5 is a root, and again dividing, we have $x^2-2x-11=0$, we now have four roots. The original equation will only admit of one more, and by adding the roots together we find it must be —, therefore $x^2+2x-11=0$.

2	44	486	4924
2	20	200	2000
4	64	686	6924

The five roots: +4.464, —2, —1, —5, —2.464; sum 6

```
4)11(2.464   therefore —2.464 is a root.
  8
  —
64)300
   256
   ———
686)4400
    4116
    ————
6924)28400
     27696
     —————
       704
```