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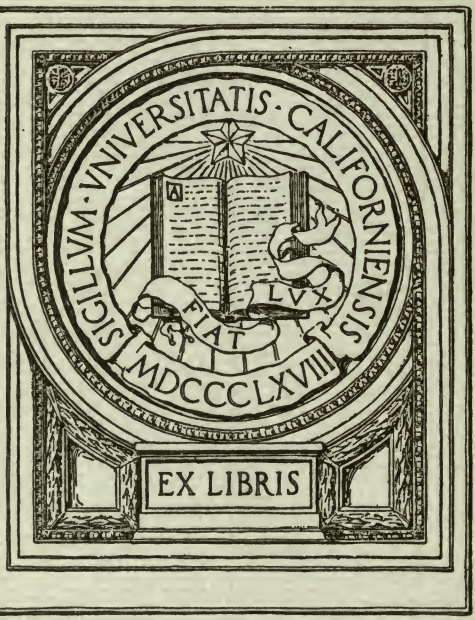

Florence Cajori

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Webster Wells, S. B.

ofessor of Mathematics in the Massachusetts Institute of Technology

FACTORING

BY

WEBSTER WELLS, S. B.

PROFESSOR OF MATHEMATICS IN THE
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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FACTORING

AS PRESENTED IN

WELLS' ESSENTIALS *of* ALGEBRA

I. Advanced processes are not presented too early.
II. Large amount of practice work.

No other algebra has so complete and well-graded development of this important subject, presenting the more difficult principles at those stages of the student's progress when his past work has fully prepared him for their perfect comprehension.

The Chapter on Factoring contains these simple processes :

CASE I. When the terms of the expression have a common monomial factor — thirteen examples.
II. When the expression is the sum of two binomials which have a common binomial factor — twenty examples.
III. When the expression is a perfect trinomial square — twenty-six examples.
IV. When the expression is the difference of two perfect squares — fifty-five examples.
V. When the expression is a trinomial of the form x^2+ax+b — sixty-six examples.
VI. When the expression is the sum or difference of two perfect cubes — twenty examples.
VII. When the expression is the sum or difference of two equal odd powers of two quantities — thirteen examples.

Ninety-three Miscellaneous and Review Examples.

Further practice in the application of these principles is given in the two following chapters — Highest Common Factor and Lowest Common Multiple.

In the discussion of Quadratic Equations, Solution of Equations by Factoring is made a special feature.

Equations of the forms $x^2-5x-24=0$, $2x^2-x=0$, $x^3+4x^2-x-4=0$, and $x^3-1=0$ are discussed and illustrated by thirty examples.

The factoring of trinomials of the form ax^2+bx+c and ax^4+bx^2+c, which involves so large a use of radicals, is reserved until Chapter XXV, where it receives full and lucid treatment.

The treatment of factoring is but one of the many features of superiority in Wells' Essentials of Algebra.

Half Leather, 338 pages. Price $1.10.

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HEATH'S MATHEMATICAL MONOGRAPHS

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CAJORI

PREFACE.

The present work contains a more varied assortment of methods and examples in factoring than are to be found in most high school texts.

Two methods are given for factoring expressions of the forms $x^2 + ax + b$ and $ax^2 + bx + c$; the first by inspection, and the second by the solution of a quadratic equation; the latter is applicable to every case.

In factoring expressions of the form $x^4 + ax^2y^2 + y^4$, examples are given involving radicals, this special case being of importance in higher mathematics.

References made to sections in the author's "Essentials of Algebra" give the authority for the statements in question.

As the examples are different from those in the author's "Academic Algebra" and "Essentials of Algebra," they may be used to supplement the classroom work in those texts.

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FACTORING.

DEFINITIONS.

1. To *Factor* an algebraic expression is to find two or more expressions which, when multiplied together, will produce the given expression.

In the present treatise, we consider only the factoring of *integral* expressions.

2. A *Common Factor* of two or more expressions is an expression which will exactly divide each of them.

FACTORING.

3. It is not always possible to factor a rational and integral polynomial.

There are, however, certain expressions which can always be factored; these will be considered in the present treatise.

4. Case I. *When the terms of the expression have a common factor.*

1. Factor $14\,ab^4 - 35\,a^3b^2$.

Each term contains the monomial factor $7\,ab^2$.

Dividing the expression by $7\,ab^2$, we have $2\,b^2 - 5\,a^2$.

Then, $14\,ab^4 - 35\,a^3b^2 = 7\,ab^2(2\,b^2 - 5\,a^2)$.

2. Factor $(2\,m + 3)x^2 + (2\,m + 3)y^3$.

The terms have the common binomial factor $2\,m + 3$.

Dividing the expression by $2\,m + 3$, we have $x^2 + y^3$.

Then, $(2\,m + 3)x^2 + (2\,m + 3)y^3 = (2\,m + 3)(x^2 + y^3)$.

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3. Factor $(a-b)m+(b-a)n$.

We have $b-a=-(a-b)$. (*Ess. Alg.*, § 41.)

Then, $(a-b)m+(b-a)n=(a-b)m-(a-b)n$

$=(a-b)(m-n)$.

We may also solve Ex. 3 by changing the first term into the form

$$-(b-a)m.$$

Thus, $(a-b)m+(b-a)n=(b-a)n-(b-a)m=(b-a)(n-m)$.

We may thus have more than one form for the factors of an expression.

4. Factor $5\,a(x-y)-3\,a(x+y)$.

$$\begin{aligned}5\,a(x-y)-3\,a(x+y)&=a[5(x-y)-3(x+y)]\\&=a(5\,x-5\,y-3\,x-3\,y)\\&=a(2\,x-8\,y)\\&=2\,a(x-4\,y).\end{aligned}$$

EXERCISES I.

Factor the following:

5. $a^6-5\,a^5-2\,a^4+3\,a^3$.

6. $m^5n^2+m^3n^4-mn^6$.

7. $24\,x^3y-40\,x^2y^2+56\,x^2y$.

8. $(a-2)b^4-(a-2)c^3d^5$.

9. $(3\,x+5)m+(3\,x+5)$.

10. $(m-n)x-(n-m)(y+z)$.

11. $a(a^2-2)+3(2-a^2)$.

12. $(x+y)(m+n)+(x+y)(m-n)$.

13. $a(b+c)-a(b-c)$.

14. $3\,x^2(x-1)-(1-x)$.

15. $(a+m)^2-3(a+m)$.

16. $x^2(5\,y-2\,z)-x^2(2\,y+z)$.

17. $(m-n)^3+2\,m(m-n)^2$.

18. $4\,x(a-b-c)-5\,y(b+c-a)$.

19. $(a-b)(m^2+xz)-(a-b)(m^2-yz)$.

20. $(m-n)^4-2\,m(m-n)^3+m^2(m-n)^2$.

5. The terms of a polynomial may sometimes be so arranged as to show a common polynomial factor; and the expression can then be factored as in Case I.

1. Factor $ab - ay + bx - xy$.

By Sec. 4, $(ab - ay) + (bx - xy) = a(b - y) + x(b - y)$.

The terms now have the common factor $b - y$.

Whence, $ab - ay + bx - xy = (a + x)(b - y)$.

2. Factor $a^3 + 2a^2 - 3a - 6$.

In this case, it is convenient to enclose the last two terms in parentheses preceded by a $-$ sign.

Thus, $a^3 + 2a^2 - 3a - 6 = (a^3 + 2a^2) - (3a + 6)$

$= a^2(a + 2) - 3(a + 2)$

$= (a^2 - 3)(a + 2)$.

EXERCISES II.

Factor the following:

3. $ac + ad + bc + bd$.

4. $xy - 3x + 2y - 6$.

5. $mx + my - nx - ny$.

6. $ab - a - 5b + 5$.

7. $8xy + 12ay + 10bx + 15ab$.

8. $m^4 + 6m^3 - 7m - 42$.

9. $6 - 10a + 27a^2 - 45a^3$.

10. $20ab - 28ad - 5bc + 7cd$.

11. $m^3 - m^2n + mn^2 - n^3$.

12. $a^5b^5 - a^3b^2c^2d^3 - a^2b^3c^3d^2 + c^5d^5$.

13. $63 + 36x^2 + 56x^3 + 32x^5$.

14. $48xy + 18nx - 88my - 33mn$.

15. $mx + my + nx + ny + px + py$.

16. $ax - ay + az - bx + by - bz$.

17. $3am - 6an + 4bm - 8bn + cm - 2cn$.

18. $ax + ay - az - bx - by + bz + cx + cy - cz$.

6. If an expression when raised to the nth power, n being a positive integer, is equal to another expression, the first expression is said to be an *nth Root* of the second.

Thus, if $a^n = b$, a is an nth root of b.

7. The *Radical Sign*, $\sqrt{\ }$, when placed before an expression, indicates some root of the expression.

Thus, $\sqrt{a}$ indicates a *second*, or *square root* of a;

$\sqrt[3]{a}$ indicates a *third*, or *cube root* of a;

$\sqrt[4]{a}$ indicates a *fourth root* of a; and so on.

The *index* of a root is the number written over the radical sign to indicate what root of the expression is taken.

If no index is expressed, the index 2 is understood.

An *even root* is one whose index is an even number; an *odd root* is one whose index is an odd number.

8. A rational and integral expression (*Ess. Alg.*, § 108) is said to be a *perfect square*, a *perfect cube*, or, in general, a *perfect nth power*, when it has, respectively, a rational and integral square, cube, or nth root.

9. Since $(2\,a^2b)^3 = 8\,a^6b^3$, a cube root of $8\,a^6b^3$ is $2\,a^2b$.

Again, since $(m^2)^4 = m^8$, a fourth root of m^8 is m^2.

We also have $(-\,m^2)^4 = m^8$; so that another fourth root of m^8 is $-\,m^2$.

It is evident from this that every positive term which is a perfect nth power, has a *positive* nth root; and in addition, if n is even, a *negative* nth root of the same absolute value.

We shall call the *positive* nth root the *principal* nth root.

It will be understood throughout the remainder of the work, unless the contrary is specified, that when we speak of *the* nth root of a term, we mean the *principal* nth root.

10. *Any Root of a Power.*

Required the value of $\sqrt[n]{a^{mn}}$, where m and n are any positive integers.

We have $(a^m)^n = a^{mn}$.

Then by Sec. 6, $\sqrt[n]{a^{mn}} = a^m$.

11. *Any Root of a Product.*

Let n be a positive integer, and $a, b, c, \cdots$, numbers which are perfect nth powers.

By Sec. 6, $(\sqrt[n]{abc\cdots})^n = abc\cdots$.

Also, $(\sqrt[n]{a} \times \sqrt[n]{b} \times \sqrt[n]{c} \times \cdots)^n = (\sqrt[n]{a})^n \times (\sqrt[n]{b})^n \times (\sqrt[n]{c})^n \times \cdots$

$= abc\cdots$, by Sec. 6.

Then, $\sqrt[n]{abc\cdots} = \sqrt[n]{a} \times \sqrt[n]{b} \times \sqrt[n]{c} \times \cdots$;

for each of these expressions is the nth root of $abc\cdots$; and we assume as an axiom that, if two perfect nth powers are equal, their principal nth roots are equal.

12. *Any Root of a Monomial.*

From Secs. 9, 10, and 11 we have the following rule for finding the principal root of a rational, integral, and positive monomial, which is a perfect power of the same degree as the index of the required root:

Extract the required root of the numerical coefficient, if any, and divide the exponent of each letter by the index of the required root.

Ex. Required the fifth root of $32\,a^{10}b^5c^{15}$.

By the rule, $\sqrt[5]{32\,a^{10}b^5c^{15}} = 2\,a^2bc^3$.

13. A rational and integral trinomial is a perfect square when its first and last terms are perfect squares, and positive, and its second term plus or minus twice the product of their square roots (*Ess. Alg.*, §§ 78, 79).

Thus, $4\,x^2 + 12\,xy^2 + 9\,y^4$ is a perfect square.

To find the square root of a perfect trinomial square, we have the following rule:

Extract the square roots (Sec. 12) *of the first and third terms, and connect the results by the sign of the second term.*

1. Find the square root of $4x^2 + 12xy^2 + 9y^4$.

By the rule, $\sqrt{4x^2 + 12xy^2 + 9y^4} = 2x + 3y^2$.

The expression may also be written in the form

$$(-2x)^2 + 2(-2x)(-3y) + (-3y)^2;$$

which shows that $(-2x) + (-3y)$, or $-2x - 3y$, is also a square root.

But the first form is simpler, and will be used in the examples of the present treatise.

2. Find the square root of $m^2 - 2mn + n^2$.

By the rule, $\sqrt{m^2 - 2mn + n^2} = m - n$.

We may write the expression in the form $n^2 - 2mn + m^2$; in which case, by the rule, the square root is $n - m$.

In the present treatise, when finding the square roots of expressions of the form $m^2 - 2mn + n^2$, we shall always take the square root of the first term as given, minus the square root of the third term as given.

14. Case II. *When the expression is a trinomial perfect square.*

1. Factor $25a^2 + 40ab^3 + 16b^6$.

By Sec. 13, the square root of the expression is $5a + 4b^3$.

Then, $25a^2 + 40ab^3 + 16b^6 = (5a + 4b^3)^2$.

2. Factor $x^2 - 2x(y - z) + (y - z)^2$.

$$x^2 - 2x(y - z) + (y - z)^2 = [x - (y - z)]^2$$
$$= (x - y + z)^2.$$

3. Factor $-m^4 + 4m^2n^2 - 4n^4$.

$$-m^4 + 4m^2n^2 - 4n^4 = -(m^4 - 4m^2n^2 + 4n^4)$$
$$= -(m^2 - 2n^2)^2.$$

EXERCISES III.

Factor the following:

4. $x^2 + 8x + 16$.
5. $9 - 6a + a^2$.
6. $m^2 + 10mn + 25n^2$.
7. $4a^6 - 4a^3bc^2 + b^2c^4$.
8. $x^2y^2 + 14xy + 49$.
9. $36a^2 - 132ab + 121b^2$.
10. $-16a^2 + 24ax - 9x^2$.
11. $81m^2 + 144mn + 64n^2$.
12. $-25x^{10} - 60x^5y^3z^2 - 36y^6z^4$.
13. $64a^2x^2 - 240abxy + 225b^2y^2$.
14. $49m^6 + 168m^3x^4 + 144x^8$.
15. $100a^2b^2 + 180abc^2 + 81c^4$.
16. $144x^4y^2 - 312x^2yz^3 + 169z^6$.
17. $-121a^4m^2 + 220a^2b^2mn - 100b^4n^2$.
18. $169a^8b^2 + 364a^4bc^2d^3 + 196c^4d^6$.
19. $(x+y)^2 + 22(x+y) + 121$.
20. $a^2 - 8a(m-n) + 16(m-n)^2$.
21. $9x^2 - 6x(y+z) + (y+z)^2$.
22. $(m-n)^2 - 2(m-n)n + n^2$.
23. $25(a+b)^2 + 40(a+b)c + 16c^2$.
24. $36(a-x)^2 - 84(a-x)y + 49y^2$.
25. $49m^2 + 42m(m+x) + 9(m+x)^2$.
26. $(a+b)^2 + 4(a+b)(a-b) + 4(a-b)^2$.
27. $9(x+y)^2 - 12(x+y)(x-y) + 4(x-y)^2$.

15. Case III. When the expression is in the form

$$a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.$$

This expression is the square of $a + b + c$ (*Ess. Alg.*, § 187).

1. Factor $9x^2 + y^2 + 4z^2 - 6xy + 12xz - 4yz$.

We must show that this expression is in the form of the square of a trinomial (*Ess. Alg*, § 187), and find its square root.

Now $9x^2$ is the square of $3x$; and hence the first term of the trinomial is $3x$.

We next consider the terms involving the first power of x. $-6xy$ can be written in the form $2(3x)(-y)$, and $12xz$ can be written in the form $2(3x)(2z)$.

This suggests that the second term of the trinomial is $-y$, and the third term $2z$; and since the second, third, and last terms of the given expression are in accordance with this, the expression is a perfect square, and its square root is $3x-y+2z$. Then,

$$9x^2+y^2+4z^2-6xy+12xz-4yz=(3x-y+2z)^2.$$

The square root of the expression is also $-3x+y-2z$; as may be seen by writing it in the form
$(-3x)^2+y^2+(-2z)^2+2(-3x)y+2(-3x)(-2z)+2y(-2z)$.

EXERCISES IV.

Factor the following:

2. $a^2+b^2+c^2-2ab-2ac+2bc$.
3. $x^2+4y^2+9+4xy+6x+12y$.
4. $1+25m^2+36n^2-10m+12n-60mn$.
5. $a^2+81b^2+16+18ab-8a-72b$.
6. $9x^2+y^2+25z^2-6xy-30xz+10yz$.
7. $36m^2+64n^2+x^2+96mn-12mx-16nx$.
8. $16a^4+9b^4+81c^4+24a^2b^2+72a^2c^2+54b^2c^2$.
9. $25x^6+49y^{10}+36z^8-70x^3y^5+60x^3z^4-84y^5z^4$.

16. CASE IV. *When the expression is the difference of two perfect squares.*

We have $a^2-b^2=(a+b)(a-b)$. (*Ess. Alg.*, § 80.)

Then, to obtain the factors we have the following rule:

Extract the square roots of the first square and of the second square; add the results for one factor and subtract the second result from the first for the other.

1. Factor $36\,a^2b^4 - 49\,c^6$.

By the rule, $36\,a^2b^4 - 49\,c^6 = (6\,ab^2 + 7\,c^3)(6\,ab^2 - 7\,c^3)$.

2. Factor $(2\,x - 3\,y)^2 - (x - y)^2$.

By the rule, $(2\,x - 3\,y)^2 - (x - y)^2$

$$= [(2\,x - 3\,y) + (x - y)][(2\,x - 3\,y) - (x - y)]$$
$$= (2\,x - 3\,y + x - y)(2\,x - 3\,y - x + y)$$
$$= (3\,x - 4\,y)(x - 2\,y).$$

A polynomial of more than two terms may sometimes be expressed as the difference of two squares, and factored by the rule of Case IV.

3. Factor $2\,mn + m^2 - 1 + n^2$.

The first, second, and last terms may be grouped together in the order $m^2 + 2\,mn + n^2$; which expression, by Sec. 13, is the square of $m + n$.

Thus,

$$2\,mn + m^2 - 1 + n^2 = (m^2 + 2\,mn + n^2) - 1$$
$$= (m + n)^2 - 1$$
$$= (m + n + 1)(m + n - 1).$$

4. Factor $12\,y + x^2 - 9\,y^2 - 4$.

$$12\,y + x^2 - 9\,y^2 - 4 = x^2 - 9\,y^2 + 12\,y - 4$$
$$= x^2 - (9\,y^2 - 12\,y + 4)$$
$$= x^2 - (3\,y - 2)^2, \text{ by Sec. 13,}$$
$$= [x + (3\,y - 2)][x - (3\,y - 2)]$$
$$= (x + 3\,y - 2)(x - 3\,y + 2).$$

EXERCISES V.

Factor the following:

5. $x^2 - 9$.

6. $16 - a^2$.

7. $81\,m^4 - 25\,n^6$.

8. $64 - 121\,a^6b^2c^4$.

9. $169\,a^8x^2 - 400\,b^4y^6$.

10. $196\,m^4x^{12} - 289\,n^6y^{10}$.

11. $361\,a^8b^{20} - 225$.

12. $625\,a^{22}m^{18} - 324\,b^{14}n^{16}$.

13. $(a-b)^2-c^2$.

14. $(5x+y)^2-x^2$.

15. $a^2-(m+n)^2$.

16. $36m^2-(2m-3)^2$.

17. $(x-y)^2-(m+n)^2$.

18. $(4a+x)^2-(b+3y)^2$.

19. $4(a-b)^2-(c-d)^2$.

20. $(2m+n)^2-(m+2n)^2$.

21. $(6a+x)^2-(a-8x)^2$.

22. $16(2m+7x)^2-49(3n-4y)^2$.

23. $(9x-5y)^2-(6x+7y)^2$.

24. $25(8a-3b)^2-9(4a+5b)^2$.

25. $x^2-2xy+y^2-9$.

26. $a^2+b^2-c^2+2ab$.

27. $4m^2+n^2-p^2-4mn$.

28. $x^2-y^2-2yz-z^2$.

29. $4-a^2+2ab-b^2$.

30. $16m^2-n^2-9p^2-6np$.

31. $m^2-2mn+n^2-x^2+2xy-y^2$.

32. $a^2+2ab+b^2-c^2-2cd-d^2$.

33. $a^2+x^2-b^2-y^2+2ax-2by$.

34. $x^2-y^2+m^2-1-2mx-2y$.

35. $16a^2-8ab+b^2-c^2-10cd-25d^2$.

36. $28xy-36z^2+49y^2+60z-25+4x^2$.

17. CASE V. *When the expression is in the form*

$$x^4+ax^2y^2+y^4.$$

Certain trinomials of the above form may be factored by expressing them as the difference of two perfect squares, and then employing Sec. 16.

1. Factor $a^4+a^2b^2+b^4$.

By Sec. 13, a trinomial is a perfect square if its first and last terms are perfect squares, and positive, and its second term plus or minus twice the product of their square roots.

The given expression may be made a perfect square by adding a^2b^2 to its second term; and this can be done provided we subtract a^2b^2 from the result.

Thus,

$$\begin{aligned} a^4 + a^2b^2 + b^4 &= (a^4 + 2a^2b^2 + b^4) - a^2b^2 \\ &= (a^2 + b^2)^2 - a^2b^2, \text{ by Sec. 13,} \\ &= (a^2 + b^2 + ab)(a^2 + b^2 - ab), \text{ by Sec. 16,} \\ &= (a^2 + ab + b^2)(a^2 - ab + b^2). \end{aligned}$$

2. Factor $9x^4 - 37x^2 + 4$.

The expression will be a perfect square if its second term is $-12x^2$.

Thus,

$$\begin{aligned} 9x^4 - 37x^2 + 4 &= (9x^4 - 12x^2 + 4) - 25x^2 \\ &= (3x^2 - 2)^2 - (5x)^2 \\ &= (3x^2 - 2 + 5x)(3x^2 - 2 - 5x) \\ &= (3x^2 + 5x - 2)(3x^2 - 5x - 2). \end{aligned}$$

The expression may also be factored as follows:

$$\begin{aligned} 9x^4 - 37x^2 + 4 &= (9x^4 + 12x^2 + 4) - 49x^2 \\ &= (3x^2 + 2)^2 - (7x)^2 \\ &= (3x^2 + 7x + 2)(3x^2 - 7x + 2). \end{aligned}$$

EXERCISES VI.

Factor the following:

3. $x^4 + 5x^2 + 9$.

4. $a^4 - 21a^2b^2 + 36b^4$.

5. $4 - 33x^2 + 4x^4$.

6. $25m^4 - 14m^2n^2 + n^4$.

7. $9x^4 + 6x^2y^2 + 49y^4$.

8. $16a^4 - 81a^2 + 16$.

9. $64 - 64m^2 + 25m^4$.

10. $49a^4 - 127a^2x^2 + 81x^4$.

Factor each of the following in two different ways (compare Ex. 2):

11. $x^4 - 17x^2 + 16$.

12. $9 - 148a^2 + 64a^4$.

13. $16m^4 - 104m^2x^2 + 25x^4$.

14. $36a^4 - 97a^2m^2 + 36m^4$.

18. There are many expressions of the form $x^4+ax^2y^2+y^4$ which require radicals for their factoring.

Some of the most important applications of the method in the Integral Calculus, are examples of this nature.

The present section should not be taken up until the pupil is familiar with radicals. (See *Ess. Alg.*, § 287.)

1. Factor $x^4 - 5x^2 + 1$.

$$\begin{aligned} x^4 - 5x^2 + 1 &= (x^4 - 2x^2 + 1) - 3x^2 \\ &= (x^2 - 1)^2 - (x\sqrt{3})^2 \\ &= (x^2 + x\sqrt{3} - 1)(x^2 - x\sqrt{3} - 1). \end{aligned}$$

The expression may also be factored as follows:

$$\begin{aligned} x^4 - 5x^2 + 1 &= (x^4 + 2x^2 + 1) - 7x^2 \\ &= (x^2 + 1)^2 - (x\sqrt{7})^2 \\ &= (x^2 + x\sqrt{7} + 1)(x^2 - x\sqrt{7} + 1). \end{aligned}$$

Certain expressions of the form $x^4 + y^4$ may be factored by the above method.

2. Factor $x^4 + 1$.

$$\begin{aligned} x^4 + 1 &= (x^4 + 2x^2 + 1) - 2x^2 \\ &= (x^2 + 1)^2 - (x\sqrt{2})^2 \\ &= (x^2 + x\sqrt{2} + 1)(x^2 - x\sqrt{2} + 1). \end{aligned}$$

EXERCISES VII.

Factor the following:

3. $x^4 + 7x^2 + 4$.
4. $a^4 + b^4$.
5. $9m^4 - 11m^2 + 1$.
6. $4a^4 + 6a^2 + 9$.
7. $36x^4 - 92x^2 + 49$.
8. $25m^4 + 28m^2n^2 + 16n^4$.

19. Case VI. *When the expression is in the form*

$$x^2 + ax + b.$$

We have $x^2 + (m+n)x + mn = (x+m)(x+n)$, as may be verified by multiplying $x + m$ by $x + n$.

If then a trinomial is in the form $x^2 + ax + b$, and a and b are, respectively, the sum and product of two numbers, the factors are x plus one number, and x plus the other.

The numbers may be found by inspection.

1. Factor $x^2 + 14\,x + 45$.

We find two numbers whose sum is 14, and product 45.

By inspection, we determine that these numbers are 9 and 5.

Then, $x^2 + 14\,x + 45 = (x+9)(x+5)$.

2. Factor $x^2 - 5\,x + 4$.

We find two numbers whose sum is -5, and product 4.

Since the sum is negative, and the product positive, the numbers must both be negative.

By inspection, we determine that the numbers are -4 and -1.

Then, $x^2 - 5\,x + 4 = (x-4)(x-1)$.

3. Factor $x^2 + 6\,x - 16$.

We find two numbers whose sum is 6, and product -16.

Since the sum is positive, and the product negative, the numbers must be of opposite sign, and the positive number must have the greater absolute value.

By inspection, we determine that the numbers are $+8$ and -2. Then, $x^2 + 6\,x - 16 = (x+8)(x-2)$.

4. Factor $x^4 - abx^2 - 42\,a^2b^2$.

We find two numbers whose sum is -1, and product -42.

The numbers must be of opposite sign, and the negative number must have the greater absolute value.

By inspection, we determine that the numbers are -7 and $+6$.

Then, $x^4 - abx^2 - 42\,a^2b^2 = (x^2 - 7\,ab)(x^2 + 6\,ab)$.

5. Factor $1 + 2a - 99a^2$.

We find two numbers whose sum is 2, and product -99.

By inspection, we determine that the numbers are $+11$ and -9.

Then, $1 + 2a - 99a^2 = (1 + 11a)(1 - 9a)$.

If the x^2 term is negative, the entire expression should be enclosed in parentheses preceded by a $-$ sign.

6. Factor $24 + 5x - x^2$.

We have,
$$\begin{aligned} 24 + 5x - x^2 &= -(x^2 - 5x - 24) \\ &= -(x - 8)(x + 3) \\ &= (8 - x)(3 + x). \end{aligned}$$

In case the numbers are large, we may proceed as follows:

Required the numbers whose sum is -26, and product -192.

One number must be $+$, and the other $-$.

Taking in order, beginning with the factors $+1 \times -192$, all possible pairs of factors of -192, one of which is $+$ and the other $-$, we have:

$$\begin{aligned} &+1 \times -192, \\ &+2 \times -96, \\ &+3 \times -64, \\ &+4 \times -48, \\ &+6 \times -32. \end{aligned}$$

Since the sum of $+6$ and -32 is -26, they are the numbers required.

EXERCISES VIII.

Factor the following:

7. $x^2 + 4x + 3$.

8. $x^2 - 7x + 10$.

9. $a^2 + 7a - 18$.

10. $m^2 - 14m - 15$.

11. $y^2 - 16y + 55$.

12. $x^2 + 16x + 39$.

13. $28 + 3c - c^2$.

14. $77 - 4n - n^2$.

15. $a^2 - 14a + 48$.

16. $x^2 + 20x + 51$.

17. $x^2 - 12x - 45$.

18. $n^2 + 14n - 32$.

19. $x^2 - 17x + 52$.

20. $84 + 5x - x^2$.

21. $a^2 + 18a + 56$.
22. $y^2 + 16y - 57$.
23. $x^2 - 10x - 75$.
24. $m^2 + 19m + 90$.
25. $95 - 14n - n^2$.
26. $x^2 - 20x + 96$.
27. $a^2 + 21a + 98$.
28. $x^2 - 7x - 78$.
29. $c^2 - 21c + 104$.
30. $105 - 8m - m^2$.
31. $x^2 - 23x + 76$.
32. $a^2 + a - 110$.
33. $n^2 - 16n - 80$.
34. $a^4 + 18a^2 + 65$.
35. $x^4 + 11x^2 - 12$.
36. $c^6 - 19c^3 + 88$.
37. $x^2y^6 - 13xy^3 - 30$.
38. $a^2b^4 - 23ab^2 + 112$.
39. $n^2x^2 + 25nx + 154$.
40. $126 + 15y^4 - y^8$.
41. $a^4x^4 + 9a^2x^2 - 162$.
42. $m^{10} - 23m^5 + 120$.
43. $(a+b)^2 + 14(a+b) + 24$.
44. $(x-y)^2 - 15(x-y) - 16$.
45. $(m-n)^2 + 21(m-n) - 130$.
46. $(a+x)^2 - 28(a+x) + 192$.
47. $a^2 + 6ax + 5x^2$.
48. $x^2 - 7xy - 8y^2$.
49. $1 + 5a - 14a^2$.
50. $m^2 - 17mn + 66n^2$.
51. $a^2 + 12ab + 27b^2$.
52. $x^2 - 14mx + 40m^2$.
53. $1 - 9x - 36x^2$.
54. $m^2 + 3mn - 54n^2$.
55. $x^2 + 12xy + 20y^2$.
56. $a^2b^2 - 17abc + 60c^2$.
57. $1 - 13n - 68n^2$.
58. $a^2 + 12ax - 85x^2$.
59. $1 + 17mn + 70m^2n^2$.
60. $x^6 - 17x^3yz^2 + 72y^2z^4$.
61. $a^2 + 6ab - 91b^2$.
62. $1 - 3xy - 108x^2y^2$.
63. $a^2 - 32abc + 112b^2c^2$.
64. $x^4y^4 + 7x^2y^2z - 170z^2$.
65. $x^2 - (2m + 3n)x + 6mn$.
66. $x^2 - (a - b)x - ab$.

20. Case VII. *When the expression is in the form*

$$ax^2 + bx + c.$$

If a is a perfect square and b divisible by $\sqrt{a}$, we may factor the expression directly by the method of Sec. 19.

1. Factor $9x^2 - 18x + 5$.

We have, $9x^2 - 18x + 5 = (3x)^2 - 6(3x) + 5$.

We find two numbers whose sum is -6, and product 5. The numbers are -5 and -1.

Then, $9x^2-18x+5=(3x-5)(3x-1)$.

If b is not divisible by $\sqrt{a}$, or if a is not a perfect square, we multiply and divide the expression by a.

2. Factor $6x^2+5x-4$.

Multiplying and dividing the expression by 6, we have

$$6x^2+5x-4=\frac{36x^2+30x-24}{6}=\frac{(6x)^2+5(6x)-24}{6}.$$

The numbers are 8 and -3.

Then, $$6x^2+5x-4=\frac{(6x+8)(6x-3)}{2\times 3}.$$

Dividing the first factor by 2, and the second by 3, we have

$$6x^2+5x-4=(3x+4)(2x-1).$$

In certain cases, the coefficient of x^2 may be made a perfect square by multiplying by a number less than itself.

3. Factor $8x^2+26xy+15y^2$.

Multiplying and dividing by 2, we have

$$\begin{aligned}8x^2+26xy+15y^2&=\frac{16x^2+52xy+30y^2}{2}\\&=\frac{(4x)^2+13y(4x)+30y^2}{2}\\&=\frac{(4x+10y)(4x+3y)}{2}\\&=(2x+5y)(4x+3y).\end{aligned}$$

4. Factor $2+5x-3x^2$.

$$\begin{aligned}2+5x-3x^2&=-(3x^2-5x-2)\\&=\frac{(3x)^2-5(3x)-6}{-3}\\&=\frac{(3x-6)(3x+1)}{-3}\\&=(2-x)(1+3x).\end{aligned}$$

EXERCISES IX.

Factor the following:

5. $2x^2+9x+9$.
6. $3x^2-11x-20$.
7. $4x^2-28x+45$.
8. $6x^2+x-2$.
9. $5x^2-36x+36$.
10. $16x^2+56x+33$.
11. $8n^2+18n-5$.
12. $7+3x-4x^2$.
13. $32-12x-9x^2$.
14. $6x^2+7ax+2a^2$.
15. $25x^2-25mx-6m^2$.
16. $10x^2-39x+14$.
17. $12x_1^2+11x+2$.
18. $20a^2x^2-23ax+6$.
19. $36x^2+12x-35$.
20. $6-x-15x^2$.
21. $5+9x-18x^2$.
22. $72+7x-49x^2$.
23. $24x^2-17nx+3n^2$.
24. $28x^2-x-2$.
25. $21x^2+23xy+6y^2$.
26. $18x^2-27abx-35a^2b^2$.
27. $5-26a^2-24a^4$.
28. $7(a-b)^2-30(a-b)+8$.
29. $12(x+y)^2+5(x+y)+7$.
30. $14(m-n)^2+39a(m-n)+10a^2$.
31. $acx^2-(ad+bc)x+bd$.

21. It is not possible to factor every expression of the form x^2+ax+b by the method of Sec. 19.

Thus, let it be required to factor $x^2+18x+35$.

We have to find two numbers whose sum is 18, and product 35.

The only pairs of positive integral factors of 35 are 7 and 5, and 35 and 1; and in neither case is the sum 18.

22. We will now give a *general* method for factoring any expression of the form ax^2+bx+c.

This method requires a knowledge of quadratic equations; and the present section, and the next, should not be taken up until the pupil is familiar with quadratics. (See *Ess. Alg.*, § 284.)

We have, $$ax^2+bx+c=a\left(x^2+\frac{bx}{a}+\frac{c}{a}\right). \qquad (1)$$

Now let r_1 and r_2 denote the roots of the equation

$$x^2 + \frac{bx}{a} + \frac{c}{a} = 0.$$

Then the equation can be written in the form

$$(x - r_1)(x - r_2) = 0. \quad (\textit{Ess. Alg.}, \S\ 283.)$$

Hence, the *expression* $x^2 + \frac{bx}{a} + \frac{c}{a}$ can be written

$$(x - r_1)(x - r_2).$$

Substituting in (1), we have

$$ax^2 + bx + c = a(x - r_1)(x - r_2).$$

But r_1 and r_2 are the roots of the equation $x^2 + \frac{bx}{a} + \frac{c}{a} = 0$, or $ax^2 + bx + c = 0$; which, we observe, may be obtained by *placing the given expression equal to zero.*

We then have the following rule:

Place the given expression equal to zero, and solve the equation thus formed.

Then, the required factors are the coefficient of x^2 in the given expression, x minus the first root, and x minus the second root.

1. Factor $6x^2 + 7x - 3$.

Solving the equation $6x^2 + 7x - 3 = 0$ (*Ess. Alg.*, § 265), we have

$$x = \frac{-7 \pm \sqrt{49 + 72}}{12} = \frac{-7 \pm 11}{12} = \frac{1}{3} \text{ or } -\frac{3}{2}.$$

Then by the rule, $6x^2 + 7x - 3 = 6(x - \frac{1}{3})(x + \frac{3}{2})$

$$= 3(x - \tfrac{1}{3}) \times (2)(x + \tfrac{3}{2})$$

$$= (3x - 1)(2x + 3).$$

2. Factor $4 + 13x - 12x^2$.

Solving the equation $4 + 13x - 12x^2 = 0$, we have

$$x = \frac{-13 \pm \sqrt{169 + 192}}{-24} = \frac{-13 \pm 19}{-24} = -\frac{1}{4} \text{ or } \frac{4}{3}.$$

Then, $$4 + 13x - 12x^2 = -12(x + \tfrac{1}{4})(x - \tfrac{4}{3})$$
$$= 4(x + \tfrac{1}{4}) \times (-3)(x - \tfrac{4}{3})$$
$$= (1 + 4x)(4 - 3x).$$

3. Factor $2x^2 - 3xy - 2y^2 - 7x + 4y + 6$.

Placing the expression equal to zero, we have

$$2x^2 - 3xy - 2y^2 - 7x + 4y + 6 = 0;$$

or, $$2x^2 - (3y + 7)x = 2y^2 - 4y - 6.$$

Solving this,

$$x = \frac{3y + 7 \pm \sqrt{(3y + 7)^2 + 16y^2 - 32y - 48}}{4}$$

$$= \frac{3y + 7 \pm \sqrt{25y^2 + 10y + 1}}{4} = \frac{3y + 7 \pm (5y + 1)}{4}$$

$$= \frac{8y + 8}{4} \text{ or } \frac{-2y + 6}{4} = 2y + 2 \text{ or } \frac{-y + 3}{2}.$$

Then, $2x^2 - 3xy - 2y^2 - 7x + 4y + 6$

$$= 2[x - (2y + 2)]\left[x - \frac{-y + 3}{2}\right]$$
$$= (x - 2y - 2)(2x + y - 3).$$

EXERCISES X.

Factor the following:

4. $x^2 + 14x + 33$.

5. $x^2 - 13x + 40$.

6. $x^2 - x - 42$.

7. $9 - 8x - x^2$.

The student should now work the examples of Sec. 20 by the method of Sec. 22.

Factor the following:

8. $x^2 - xy - 6y^2 - 6x + 13y + 5$.

9. $x^2 - 3xy - 4y^2 + 6x - 4y + 8$.

10. $x^2 - 6xy + 5y^2 - 2x - 2y - 3$.

11. $2a^2 + 5ab + 2b^2 + 7a + 5b + 3.$

12. $3x^2 + 7xy - 6y^2 - 10xz - 8yz + 8z^2.$

13. $2 - 7y - 7x + 3y^2 + xy - 4x^2.$

23. In factoring an expression of the form $ax^2 + bx + c$, where a is a perfect square, if the method of § 20 can be used, it is usually the shortest.

Another way of factoring an expression of this form is to express it as the difference of two squares, and then apply Sec. 16.

This method is shorter than that of Sec. 22.

As all the examples in the present section involve radicals, the pupil should reserve it until he has studied that subject.

1. Factor $x^2 + 2x - 11$.

The expression $x^2 + 2x$ will become a perfect square by adding to it the square of 1. (*Ess. Alg.*, § 260.)

Then, $x^2 + 2x - 11 = x^2 + 2x + 1 - 1 - 11$

$= (x + 1)^2 - 12$

$= (x+1+\sqrt{12})(x+1-\sqrt{12})$, by Sec. 16,

$= (x + 1 + 2\sqrt{3})(x + 1 - 2\sqrt{3})$.

If the x^2 term is negative, the entire expression should be enclosed in parentheses preceded by a $-$ sign.

2. Factor $4 + 12x - 9x^2$.

We have, $4 + 12x - 9x^2 = -(9x^2 - 12x - 4)$.

The expression $9x^2 - 12x$ will become a perfect square by adding to it the square of $\frac{12}{2\sqrt{9}}$, or 2.

Then, $4 + 12x - 9x^2 = -(9x^2 - 12x + 4 - 4 - 4)$

$= -[(3x - 2)^2 - 8]$

$= (3x-2+\sqrt{8}) \times (-1)(3x-2-\sqrt{8})$

$= (2\sqrt{2} - 2 + 3x)(2\sqrt{2} + 2 - 3x)$.

EXERCISES XI.

Factor the following:

3. $x^2 - 2x - 1$.
4. $4x^2 + 20x + 19$.
5. $16x^2 - 16x + 1$.
6. $2 - 24x - 9x^2$.
7. $36x^2 + 72x + 29$.
8. $11 + 10x - 25x^2$.

24. Case VIII. *When the expression is the cube of a binomial.*

1. Factor $8a^3 - 36a^2b^2 + 54ab^4 - 27b^6$.

We must show that the expression is in the form of the cube of a binomial (see *Ess. Alg.*, § 188), and find its cube root. We can write the expression as follows:

$$(2a)^3 - 3(2a)^2(3b^2) + 3(2a)(3b^2)^2 - (3b^2)^3.$$

This shows that it is a perfect cube, and that its cube root is $2a - 3b^2$.

Then, $8a^3 - 36a^2b^2 + 54ab^4 - 27b^6 = (2a - 3b^2)^3$.

EXERCISES XII.

Factor the following:

2. $x^3 + 3x^2 + 3x + 1$.
3. $8 - 12a + 6a^2 - a^3$.
4. $1 + 9m + 27m^2 + 27m^3$.
5. $64n^3 - 48n^2 + 12n - 1$.
6. $8a^3 + 36a^2b + 54ab^2 + 27b^3$.
7. $125x^3 - 75x^2y + 15xy^2 - y^3$.
8. $a^6 + 18a^4b^3 + 108a^2b^6 + 216b^9$.
9. $125m^3 - 150m^2n + 60mn^2 - 8n^3$.
10. $27a^3b^3 - 108a^2b^2c + 144abc^2 - 64c^3$.
11. $m^6 + 21m^4x^4 + 147m^2x^8 + 343x^{12}$.

25. Case IX. *When the expression is the difference of two perfect cubes.*

The sum or difference of two perfect cubes is divisible by the sum or difference, respectively, of their cube roots.

(*Ess. Alg.*, § 85.)

In either case, the quotient may be obtained by aid of the rules given in *Ess. Alg.*, § 85.

1. Factor $x^6 - 27\,y^9z^3$.

By Sec. 12, the cube root of x^6 is x^2, and of $27\,y^9z^3$ is $3\,y^3z$.

Then one factor is $x^2 - 3\,y^3z$.

Dividing $x^6 - 27\,y^9z^3$ by $x^2 - 3\,y^3z$, the quotient is

$$x^4 + 3\,x^2y^3z + 9\,y^6z^2. \quad (\textit{Ess. Alg.}, \S\ 85.)$$

Then, $x^6 - 27\,y^9z^3 = (x^2 - 3\,y^3z)(x^4 + 3\,x^2y^3z + 9\,y^6z^2)$.

2. Factor $a^6 + b^6$.

One factor is $a^2 + b^2$.

Dividing $a^6 + b^6$ by $a^2 + b^2$, the quotient is $a^4 - a^2b^2 + b^4$.

Then, $a^6 + b^6 = (a^2 + b^2)(a^4 - a^2b^2 + b^4)$.

3. Factor $(x + a)^3 - (x - a)^3$.

$$\begin{aligned}(x + a)^3 - (x - a)^3 &= [(x+a)-(x-a)]\,[(x+a)^2+(x+a)(x-a)+(x-a)^2] \\ &= (x+a-x+a)(x^2+2\,ax+a^2+x^2-a^2+x^2-2\,ax+a^2) \\ &= 2\,a\,(3\,x^2 + a^2).\end{aligned}$$

EXERCISES XIII.

Factor the following:

4. $m^3 + n^3$.
5. $1 - x^3y^3$.
6. $8\,a^3 + 1$.
7. $1 - 27\,n^3$.
8. $a^6 + 1$.
9. $x^6 + y^6z^6$.
10. $64\,m^3 - n^3$.
11. $a^3b^3 + 216\,c^3$.
12. $8\,m^3 + 27\,n^3$.
13. $27\,x^3 - 125\,y^3$.
14. $64 + 125\,a^3b^3$.
15. $343\,a^3 - 64\,m^6$.
16. $125\,x^3 + 512\,y^3z^3$.

17. $216\,a^3m^6 - 343\,n^9$.

18. $729\,a^3b^3 - 8\,c^3d^3$.

19. $(x+y)^3 + (x-y)^3$.

20. $m^3 - (m+n)^3$.

21. $27\,(a-b)^3 + 8\,b^3$.

22. $(2\,a+x)^3 - (a+2\,x)^3$.

23. $(5\,x-2\,y)^3 - (3\,x-4\,y)^3$.

26. Case X. *When the expression is the sum or difference of two equal odd powers of two numbers.*

The sum or difference of two equal odd powers of two numbers is divisible by the sum or difference, respectively, of the numbers. (*Ess. Alg.*, § 87.) The quotient may be obtained by aid of the laws given in *Ess. Alg.*, § 86.

1. Factor $a^5 + 32\,b^5$.

We have, $32\,b^5 = (2\,b)^5$.

Then, one factor is $a + 2\,b$. (*Ess. Alg.*, § 87.)

Dividing $a^5 + 32\,b^5$ by $a + 2\,b$, the quotient is

$a^4 - a^3(2\,b) + a^2(2\,b)^2 - a(2\,b)^3 + (2\,b)^4$. (*Ess. Alg.*, § 86.)

Then, $a^5 + 32\,b^5 = (a+2\,b)(a^4 - 2\,a^3b + 4\,a^2b^2 - 8\,ab^3 + 16\,b^4)$.

EXERCISES XIV.

Factor the following:

2. $x^5 + y^5$.

3. $a^5 - 1$.

4. $1 - m^5n^5$.

5. $a^7 - b^7$.

6. $1 + x^7$.

7. $m^9 + n^9$.

8. $a^9 - 1$.

9. $n^5 + 32$.

10. $32\,a^5 - b^5$.

11. $243\,x^5 + y^5$.

12. $m^{14} + 128\,n^7$.

13. $32\,a^5b^5 - 243\,c^{10}$.

27. By application of the rules already given, an expression may often be resolved into more than two factors.

If the terms of the expression have a common factor, the method of Sec. 4 should be applied first.

1. Factor $2\,ax^3y^2 - 8\,axy^4$.

By Sec. 4, $2\,ax^3y^2 - 8\,axy^4$

$= 2\,axy^2(x^2 - 4\,y^2)$

$= 2\,axy^2(x+2\,y)(x-2\,y)$, by Sec. 16.

2. Factor $a^6 - b^6$.

By Sec. 16, $a^6 - b^6 = (a^3 + b^3)(a^3 - b^3)$.

Whence, by Sec. 25,

$$a^6 - b^6 = (a + b)(a^2 - ab + b^2)(a - b)(a^2 + ab + b^2).$$

3. Factor $x^8 - y^8$.

By Sec. 16, $x^8 - y^8 = (x^4 + y^4)(x^4 - y^4)$

$$= (x^4 + y^4)(x^2 + y^2)(x^2 - y^2)$$
$$= (x^4 + y^4)(x^2 + y^2)(x + y)(x - y).$$

4. Factor $3(m + n)^2 - 2(m^2 - n^2)$.

$$3(m + n)^2 - 2(m^2 - n^2) = 3(m + n)^2 - 2(m + n)(m - n)$$
$$= (m + n)[3(m + n) - 2(m - n)]$$
$$= (m + n)(3m + 3n - 2m + 2n)$$
$$= (m + n)(m + 5n).$$

5. Factor $a(a - 1) - b(b - 1)$.

$$a(a - 1) - b(b - 1) = a^2 - a - b^2 + b$$
$$= a^2 - b^2 - a + b$$
$$= (a + b)(a - b) - (a - b)$$
$$= (a - b)(a + b - 1).$$

EXERCISES XV.

Factor the following:

6. $x^4 - 625$.

7. $a^{12} - 1$.

8. $m^{16} - 1$.

9. $x^6 - 26x^3 - 27$.

10. $(a^2 + 4ab + b^2)^2 - (a^2 + b^2)^2$.

11. $12x^6 - 18x^5 - 6x^4 + 9x^3$.

12. $81m^4 - 256n^8$.

13. $a^{14} - x^{14}$.

14. $x^6 - 16x^3y^3 + 64y^6$.

15. $(16m^2 + n^2)^2 - 64m^2n^2$.

16. $2a^7x - 8a^5x^3 + 2a^3x^5 - 8ax^7$.

17. $9a^2c^2 - 16a^2d^2 - 36b^2c^2 + 64b^2d^2$.

18. $x^{14} - 2x^7 + 1$.

19. $729 - n^6$.

20. $a^2b^3 + a^2y^3 - b^3x^2 - x^2y^3$.

21. $48x^3y - 52x^2y^2 - 140xy^3$.

22. $16\,a^7 - 72\,a^6 + 108\,a^5 - 54\,a^4$.

23. $(m+n)^4 - 2(m+n)^3 + (m+n)^2$.

24. Resolve $a^9 + 512$ into three factors by the method of Sec. 25.

25. $a^2 - m^2 + a + m$.

26. $(x^2 + 4\,x)^2 - 37\,(x^2 + 4\,x) + 160$.

27. $n^{10} - 1024$.

28 $m^3 + m + x^3 + x$.

29. $a^2c^2 - 4\,b^2c^2 - 9\,a^2d^2 + 36\,b^2d^2$.

30. $(m-n)\,(x^2 - y^2) + (x+y)\,(m^2 - n^2)$.

31. $(x-1)^3 + 6\,(x-1)^2 + 9\,(x-1)$.

32. $a^2 - 4\,b^2 - a - 2\,b$.

33. $(m+n)\,(m^2 - x^2) - (m+x)\,(m^2 - n^2)$.

34. $(x^2 + 4\,y^2 - z^2)^2 - 16\,x^2y^2$.

35. $(x^2 - 9\,x)^2 + 4\,(x^2 - 9\,x) - 140$.

36. $a^3b^3 + 27\,a^3y^3 - 8\,b^3x^3 - 216\,x^3y^3$.

37. $(2\,x^2 - 3)^2 - x^2$.

38. $(m^2 + m)^2 + 2\,(m^2 + m)\,(m+1) + (m+1)^2$.

39. $64\,a^3x^3 + 8\,a^3 - 8\,x^3 - 1$.

40. $(4\,a^2 - b^2 - 9)^2 - 36\,b^2$.

41. $(x + 2\,y)^3 - x(x^2 - 4\,y^2)$.

42. $(1 + x^3) + (1+x)^3$.

43. $(a^2 + 6\,a + 8)^2 - 14\,(a^2 + 6\,a + 8) - 15$.

44. $a^4 - 9 + 2\,a\,(a^2 + 3)$.

45. $(x^3 + y^3) - xy\,(x+y)$.

46. $(a^3 - 8\,m^3) - a\,(a - 2\,m)^2$.

47. $9\,a^2\,(3\,a+4)^2 + 6\,a\,(3\,a+4) + 1$.

48. $m^8 - m^5 + 32\,m^3 - 32$.

49. $a\,(a-c) - b\,(b-c)$.

50. $m^2\,(m+p) + n^2\,(n-p)$.

51. $x^9 + 8\,x^6 + x^3 + 8$.

52. $(27\,m^3 - x^3) + (3\,m + x)\,(9\,m^2 - 12\,mx + x^2)$.

53. $(4\,a^2 + 9)^2 - 24\,a\,(4\,a^2 + 9) + 144\,a^2$.

54. $m^9 + m^6 - 64\,m^3 - 64.$

55. $(x^2 + y^2)^3 - x^2y^2(x^2 + y^2).$

56. $a^5 + a^4b + a^3b^2 + a^2b^3 + ab^4 + b^5.$

57. $(8\,n^3 - 27) + (2\,n - 3)(4\,n^2 + 4\,n - 6).$

FACTORING BY SUBSTITUTION.

28. The Factor Theorem.

If any rational integral polynomial, involving x, becomes zero when x is put equal to a, the polynomial has $x - a$ as a factor.

Let the polynomial be

$$A x^n + B x^{n-1} + \cdots + Mx + N.$$

Then, by hypothesis,

$$A a^n + B a^{n-1} + \cdots + Ma + N = 0.$$

Since $Aa^n + Ba^{n-1} + \cdots + Ma + N$ is equal to zero, if we subtract it from the given polynomial, the latter will not be changed; then, the given polynomial is equal to

$$A(x^n - a^n) + B(x^{n-1} - a^{n-1}) + \cdots + M(x - a). \qquad (1)$$

Each expression in parentheses is divisible by $x - a$ (*Ess. Alg.*, § 87); and hence expression (1) has $x - a$ as a factor.

29. 1. Factor $x^3 - 7\,x^2 + 10\,x + 6.$

By Sec. 28, if the expression becomes 0 when x is put equal to a, then $x - a$ is a factor.

The positive and negative integral factors of 6 are 1, 2, 3, 6, -1, -2, -3, and -6.

It is best to try the numbers in their order of absolute magnitude.

If $x = 1$, the expression becomes $1 - 7 + 10 + 6$.
If $x = -1$, the expression becomes $-1 - 7 - 10 + 6$.
If $x = 2$, the expression becomes $8 - 28 - 20 + 6$.
If $x = -2$, the expression becomes $-8 - 28 - 20 + 6$.
If $x = 3$, the expression becomes $27 - 63 + 30 + 6$, or 0.

This shows that $x-3$ is a factor.

Dividing the expression by $x-3$, the quotient is x^2-4x-2.

Then, $x^3-7x^2+10x+6=(x-3)(x^2-4x-2)$.

2. Prove that a is a factor of

$$(a+b+c)(ab+bc+ca)-(a+b)(b+c)(c+a).$$

Putting $a=0$, the expression becomes

$$(b+c)bc-b(b+c)c, \text{ or } 0.$$

Then, by Sec. 28, a is a factor of the expression.

3. Prove that $m+n$ is a factor of

$$m^4-4m^3n+2m^2n^2+5mn^3-2n^4.$$

Putting $m=-n$, we have

$$n^4+4n^4+2n^4-5n^4-2n^4, \text{ or } 0.$$

Then, $m+n$ is a factor.

EXERCISES XVI.

Factor the following:

4. $x^3+4x^2+7x-12$.

5. $x^4-x^3+6x^2+14x+6$.

6. $x^3-x^2-11x-10$.

7. $x^3-9x^2+15x+9$.

8. $x^3-18x+8$.

9. $x^3-5x^3-8x+48$.

10. $x^4+8x^3+13x^2-13x-4$.

11. $3x^4-8x^3+8x^2-14x+12$.

Find, without actual division,

12. Whether $x-3$ is a factor of $x^3-6x^2+13x-12$.

13. Whether $x+2$ is a factor of x^3+7x^2-6.

14. Whether x is a factor of $x(y+z)^2+y(z+x)^2+z(x+y)^2$.

15. Whether a is a factor of $a^3(b-c)^3+b^3(c-a)^3+c^3(a-b)^3$.

16. Whether $x-y$ is a factor of $(x-y)^3+(y-z)^3+(z-x)^3$.

17. Whether $m+n$ is a factor of $m(m+2n)^3-n(2m+n)^3$.

18. Whether $a+b+c$ is a factor of

$$a(b+c)+b(c+a)+c(a+b)+a^2+b^2+c^2.$$

FACTORING OF SYMMETRICAL EXPRESSIONS.

30. Symmetry.

An expression containing two or more letters is said to be *symmetrical* with respect to any two of them when they can be interchanged without altering the value of the expression.

Thus, $a + b + c$ is symmetrical with respect to a and b; for, on interchanging these letters, the expression becomes $b + a + c$, which is equal to $a + b + c$.

An expression containing three or more letters is said to be symmetrical with respect to them when it is symmetrical with respect to any two of them.

Thus, $ab + bc + ca$ is symmetrical with respect to the letters a, b, and c; for if a and b be interchanged, the expression becomes $ba + ac + cb$, which is equal to $ab + bc + ca$.

And in like manner, $ab + bc + ca$ is symmetrical with respect to b and c, and with respect to c and a.

31. Cyclo-Symmetry.

An expression containing n letters, $a, b, c, \cdots, m, n$, is said to be cyclo-symmetrical with respect to them when, if a is substituted for b, b for c, $\cdots$, m for n, and n for a, the value of the expression is not changed.

Thus, the expression $(a - b)(b - c)(c - a)$ is cyclo-symmetrical with respect to a, b, and c; for if a is substituted for b, b for c, and c for a, the expression becomes $(c - a)(a - b)(b - c)$, which is equal to $(a - b)\ (b - c)(c - a)$.

32. Factoring of Symmetrical Expressions.

The method of Sec. 28 is advantageous in the factoring of symmetrical expressions.

1. Factor

$$a(b+c)^2+b(c+a)^2+c(a+b)^2-a^2(b+c)-b^2(c+a)-c^2(a+b).$$

The expression is symmetrical with respect to a, b, and c. Being of the third degree, the only literal factors which

it can have are three of the type a; three of the type $a+b$; or $a+b+c$, and a factor of the second degree.

Putting $a=0$, the expression becomes

$$bc^2+cb^2-b^2c-c^2b, \text{ or } 0.$$

Then, by Sec. 28, a is a factor; and, by symmetry, b and c are factors.

The expression, being of the third degree, can have no other literal factor; but it may have a *numerical* factor.

Let the given expression $= mabc$.

To determine m, let $a=b=c=1$.

Then, $4+4+4-2-2-2=m$, or $m=6$.

Whence, the given expression $= 6\,abc$.

2. Factor $x^2(y+z)+y^2(z+x)+z^2(x+y)+3\,xyz$.

The expression is symmetrical with respect to x, y, and z.

The only literal factors which it can have are three of the type x; three of the type $x+y$; or $x+y+z$, and a factor of the second degree.

It is evident that neither x, y, nor z is a factor.

Putting x equal to $-y$, the expression becomes

$$y^2(y+z)+y^2(z-y)-3\,y^2z,$$

which is not 0.

Then, $x+y$ is not a factor; and, by symmetry, neither $y+z$ nor $z+x$ is a factor.

Putting x equal to $-y-z$, the expression becomes

$$(y+z)^2(y+z)-y^3-z^3-3\,yz(y+z)$$
$$=y^3+3\,y^2z+3\,yz^2+z^3-y^3-z^3-3\,y^2z-3\,yz^2=0.$$

Then, $x+y+z$ is a factor.

The other factor must be of the second degree; and, as it is symmetrical with respect to x, y, and z, it must be of the form

$$m(x^2+y^2+z^2), \text{ or } n(xy+yz+zx).$$

It is evident that the first of these cannot be a factor; for, if it were, there would be terms involving x^3, y^3, and z^3 in the given expression.

Then, the given expression $= n(x+y+z)(xy+yz+zx)$.

To determine n, let $x=1$, $y=1$, and $z=0$.

Then, $1+1=2n$, and $n=1$.

Then, the given expression $=(x+y+z)(xy+yz+zx)$.

3. Factor $ab(a-b)+bc(b-c)+ca(c-a)$.

The expression is cyclo-symmetrical with respect to a, b, and c.

It is evident that neither a, b, nor c is a factor.

The expression becomes 0 when a is put equal to b.

Then, $a-b$ is a factor; and, by symmetry, $b-c$ and $c-a$ are factors.

The expression can have no other literal factor, but may have a numerical one.

Let the given expression $= m(a-b)(b-c)(c-a)$.

To determine m, let $a=2$, $b=1$, and $c=0$.

Then, $2=-2m$, and $m=-1$.

Then, the given expression $=-(a-b)(b-c)(c-a)$.

EXERCISES XVII.

Factor the following:

4. $a^3+a^2b+ab^2+b^3$.

5. $m^3+2m^2n+2mn^2+n^3$.

6. $(ab+bc+ca)(a+b+c)-a^2(b+c)-b^2(c+a)-c^2(a+b)$.

7. $x^2(y+z)+y^2(z+x)+z^2(x+y)+2xyz$.

8. $a(b+c)^2+b(c+a)^2+c(a+b)^2-4abc$.

9. $a^2(b-c)+b^2(c-a)+c^2(a-b)$.

10. $(x+y+z)(xy+yz+zx)-(x+y)(y+z)(z+x)$.

11. $ab(a+b)+bc(b+c)+ca(c+a)+2abc$.

12. $(x+y+z)^3-(x^3+y^3+z^3)$.

13. $(x+y+z)(xy+yz+zx)-xyz$.

14. $(x-y)^3+(y-z)^3+(z-x)^3$.

15. $x^4+x^3y+2x^2y^2+xy^3+y^4$.

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