UC-NRLF
$B 531 964

1005

With the Compliments
of
Little, Brown, & Company

List Price, $1.15 net
Price in Quantities, 92 cents

348

IN MEMORIAM
FLORIAN CAJORI

SIGILLVM · VNIVERSITATIS · CALIFORNIENSIS
FIAT LVX
MDCCCLXVIII

EX LIBRIS

FIRST COURSE IN ALGEBRA

Digitized by the Internet Archive
in 2008 with funding from
Microsoft Corporation

http://www.archive.org/details/firstcourseinalg00wheerich

FIRST COURSE

IN

ALGEBRA

BY

ALBERT HARRY WHEELER

TEACHER OF MATHEMATICS IN THE ENGLISH HIGH SCHOOL AT WORCESTER,
MASSACHUSETTS

WITH EIGHT THOUSAND EXAMPLES

INCLUDING

THREE THOUSAND MENTAL EXERCISES

BOSTON

LITTLE, BROWN, AND COMPANY

1907

Copyright, 1907,
By Little, Brown, and Company.

THE UNIVERSITY PRESS, CAMBRIDGE, U. S. A.

N 48

PREFACE

THIS book may be used by students who have completed a course in arithmetic but who have not previously studied algebra. The statements of all principles and the explanations of all examples have been made as simple and direct as possible. Emphasis has been placed upon the reason for each step taken, whether it be in the proof of a principle or in the explanation of an exercise. Such proofs of principles as may be omitted by the student when taking the subject for the first time have been plainly marked.

The examples are all new and have not been copied from other text-books. There are 4,465 exercises for written solution, 3,301 for mental solution, and 423 explanatory examples, — a total of 8,189 examples.

These examples have been so constructed and graded as to contain a great variety of number-combinations. Thus the student is constantly drilled in arithmetic.

The writer first used mental exercises in the class-room in 1895, and has constantly employed them since that time with the exception of the years 1897–99, which were spent in graduate study at Clark University. The first ten or fifteen minutes of each recitation period are commonly devoted by his pupils to the mental solution of a large number of simple drill problems. In this way all of the pupils have an opportunity to recite several times during every recitation period.

After they gain confidence and a certain amount of skill in solving the mental exercises, it is found that the more difficult problems which are given for written solution are undertaken with readiness.

By the use of the mental exercises the teacher is able to gain a better knowledge of the progress of the pupil than by giving many written examinations, and there is the advantage that mistakes on the part of the pupil can be corrected immediately. Written examinations show the teacher the way the pupil has thought, but mental exercises show the pupil the way to think in the future.

The applied problems are concerned with subjects of modern interest. In particular, problems have been introduced illustrating the applications of certain familiar laws of physics, such as those relating to the lever, falling bodies, expansion of gases, etc. The traditional problems which involve unnatural and absurd situations have been excluded.

The graphs were drawn by the writer, and will be found to be accurate. The explanations of the examples have been given in such a way that all reference to graphs may be omitted if the time devoted to the subject is found to be insufficient for their consideration.

A system of numerical checks is used throughout the book, and the pupil is constantly encouraged to test results obtained rather than to depend upon the authority of the teacher or of a printed list of answers.

In the development of the subject the distinction between natural forms of number and "artificial" or invented forms of number has been constantly kept in view, and by means of the Principle of No Exception the necessity has been shown for the invention of negative number and other forms of "artificial" number.

Simple proofs and illustrations of the principles of equivalence of equations have been given, and the distinction between identical and conditional equalities has been carefully pointed out.

Attention has been given to detail in the classification and arrangement of the subject-matter for the purpose of making it easily available for reference and of simplifying the presentation of the subject.

ALBERT HARRY WHEELER.

WORCESTER, MASS.
April, 1907.

PUBLISHERS' NOTE. — The Brief Edition of this book, which takes the pupil as far as Quadratics, contains 6,327 examples.

TABLE OF CONTENTS

CHAPTER I

FIRST IDEAS

NUMERICAL EXPRESSIONS

CHAPTER II

AN EXTENSION OF THE IDEA OF NUMBER

CHAPTER III

FUNDAMENTAL LAWS OF ALGEBRA

CHAPTER IV

ADDITION AND SUBTRACTION OF POSITIVE AND NEGATIVE NUMBERS

CHAPTER V

Multiplication and Division of Positive and Negative Numbers

I. MULTIPLICATION

II. DIVISION

LITERAL EXPRESSIONS

OPERATIONS WITH INTEGRAL ALGEBRAIC EXPRESSIONS

CHAPTER VI

Addition and Subtraction

CHAPTER VII

Multiplication

CHAPTER VIII

DIVISION

CHAPTER IX

GRAPHICAL REPRESENTATION OF THE VARIATION OF FUNCTIONS OF A SINGLE VARIABLE

CHAPTER X

GENERAL PRINCIPLES GOVERNING THE TRANSFORMATION OF ALGEBRAIC EQUATIONS

CHAPTER XI

EQUATIONS OF THE FIRST DEGREE CONTAINING ONE UNKNOWN

CHAPTER XII

FACTORS OF RATIONAL INTEGRAL EXPRESSIONS

CHAPTER XIII

HIGHEST COMMON FACTOR

CHAPTER XIV

LOWEST COMMON MULTIPLE

CHAPTER XV

RATIONAL FRACTIONS

CHAPTER XVI

FRACTIONAL AND LITERAL EQUATIONS

CHAPTER XVII

SIMULTANEOUS LINEAR EQUATIONS

CHAPTER XVIII

EVOLUTION

CHAPTER XIX

THEORY OF EXPONENTS

CHAPTER XX

IRRATIONAL NUMBERS AND THE ARITHMETIC THEORY OF SURDS

I. IRRATIONAL NUMBERS

II. ARITHMETIC THEORY OF SURDS

CHAPTER XXI

Imaginary and Complex Numbers

I. IMAGINARY NUMBERS

II. COMPLEX NUMBERS

CHAPTER XXII

Equations of the Second Degree Containing One Unknown

CHAPTER XXIII

THEORY OF QUADRATIC EQUATIONS CONTAINING ONE UNKNOWN

CHAPTER XXIV

IRRATIONAL EQUATIONS AND SPECIAL EQUATIONS CONTAINING A SINGLE UNKNOWN

CHAPTER XXV

SYSTEMS OF SIMULTANEOUS EQUATIONS INVOLVING QUADRATIC EQUATIONS

CHAPTER XXVI

Ratio, Proportion, and Variation

I. RATIO

II. PROPORTION

III. VARIATION

CHAPTER XXVII

The Progressions

I. ARITHMETIC PROGRESSION

II. HARMONIC PROGRESSION

III. GEOMETRIC PROGRESSION

CHAPTER XXVIII

THE BINOMIAL THEOREM

FIRST COURSE IN ALGEBRA

CHAPTER I

FIRST IDEAS

1. The student of mathematics who undertakes the study of algebra will find at the outset that the new science is but arithmetic in a different form and under another name.

2. In algebra numbers are represented by letters, and while definite values are not assigned to the different letters employed, they are used with the idea that each letter represents some numerical value, and that different letters commonly represent different values.

3. Letters which stand for numbers whose values remain fixed and unchanged, either throughout the discussion of a particular problem or for all time, are called **constants,** and those whose values may be supposed to change during the discussion of a problem are called **variables.**

Letters which represent numbers whose values are known are called **knowns,** and those whose values may for the moment be unknown are commonly called **unknowns.**

It is customary in elementary algebra to represent constants by the first few letters of the alphabet, a, b, c, and the unknown values, and also variables, by the last few letters, x, y, z.

In many applications of algebra to formulas of physics and other sciences, or even in certain algebraic expressions, any letter or group of letters may be chosen as representing variables or unknowns; in which case the remaining letters, if there be any, may be regarded as constants, or knowns.

4. The **symbols used to denote** the **operations** of addition, subtraction, multiplication, division, and root extraction in arithmetic are used in the same sense in algebra.

5. The symbol for addition, +, is read "plus," and for subtraction, —, is read "minus." These symbols indicate that the numbers before which they are placed are to be added to or subtracted from the numbers which immediately precede them.

Thus, the order of operations is from left to right.

For example, $a + b$, read "a plus b," means that some number represented by b is to be added to some number represented by a. Again, $x - y$, read "x minus y," means that, if x and y represent numbers, the number represented by y is to be subtracted from that represented by x.

6. Multiplication may be indicated in several ways; by the symbol ×; by a dot written higher than a decimal point placed between the multiplicand and multiplier; or, when no possible ambiguity can arise, by simply writing the multiplicand and multiplier consecutively.

Thus, if a and b are taken to represent any two numbers, their product may be written as $a \times b$, $a \cdot b$, or simply as $a\,b$.

In each case the product is read either "a multiplied by b," or "a times b."

The sign for multiplication is usually omitted between two letters, or between a number and a letter.

Thus, $a\,b$ means $a \times b$; $4\,m\,n$ means $4 \times m \times n$.

The sign for multiplication cannot be omitted between numerals in arithmetic because of the positional system of notation employed.

Thus, 72 must be regarded as standing for seventy-two wherever it is found, and not as 7×2, or 14.

To avoid confusion with the decimal point, the dot, when used as a symbol for multiplication, should be written somewhat higher.

Thus, $7 \cdot 2$ may be taken as indicating the product of 7 and 2, while 7.2 will be interpreted as a decimal, 7 *plus* 2 tenths.

7. The symbol for division is ÷. Thus, $a \div b$, read "a divided by b," means that some number represented by a is to be divided by another number represented by b. The number a is regarded as the dividend and the number b as the divisor, as in arithmetic.

As alternative notations for division we have the **fractional notation,** $\frac{a}{b}$, the **ratio notation,** $a : b$, and the **solidus notation,** a/b.

8. In an unbroken chain of additions and subtractions, or in an unbroken chain of multiplications and divisions, the operations are to be carried out successively from left to right.

Thus, from the chain of additions $3 + 4 + 5 + 6$ we obtain 18, by performing the operations successively.

Again, performing the operations of the chain $25 - 8 - 6 - 3$ successively from left to right, we obtain as a result 8.

The result obtained by performing successively the multiplications of the chain $2 \times 4 \times 8$ is 64.

The number resulting from performing the divisions of the chain $8 \div 4 \div 2$, successively from left to right, is 1.

9. However, in a chain of operations containing additions, subtractions, multiplications and divisions together, *the multiplications and divisions must be performed first* in the order as indicated, and then the operations of addition and subtraction in their order.

Thus, to find the number represented by the chain of operations $5 + 12 \times 2 \div 3 - 6 \div 3 + 1$, we may proceed as follows:

Multiplying 12 by 2, and dividing the product 24 by 3, we obtain 8; also, dividing 6 by 3, we have as a quotient 2.

Hence the above chain of operations reduces to the chain $5 + 8 - 2 + 1$.

Performing these last additions and subtractions successively from left to right, we obtain 12 as a result.

Exercise I. 1.

Find the values of the following:

1. $5 + 2 \times 3$.
2. $10 - 4 \times 2$.
3. $12 + 10 \div 2$.
4. $15 - 10 \div 5$.
5. $7 + 6 + 5 \times 4$.
6. $7 + 6 \times 5 + 4$.
7. $7 \times 6 + 5 \times 4$.
8. $8 \times 5 - 6 \times 3$.
9. $20 \div 4 + 28 \div 7$.
10. $8 \div 2 \times 3 \div 4 + 1$.
11. $1 + 2 \times 3 + 4 \times 5$.
12. $5 + 4 \times 3 \div 2 - 1$.
13. $125 \div 25 - 25 \div 5 + 2$.
14. $5 \times 5 - 4 \times 4 - 3 \times 3$.
15. $7 \times 8 \div 14 + 3 \times 2 \div 3$.
16. $8 \div 2 \times 4 - 4 \div 2 \times 6 + 12$.
17. $6 \times 3 \div 2 - 6 \times 2 \div 3 + 7$.
18. $6 \div 2 \times 4 - 4 \div 2 \times 6 + 6 \times 4 \div 2$.
19. $2 \times 3 \times 4 + 3 \times 4 \times 2 + 4 \times 2 \times 3$.
20. $2 \times 4 \times 8 - 8 \div 4 \div 2 + 4 \div 2 \times 8$.
21. $20 \div 4 \times 5 - 20 \div 5 \times 4 - 20 \div 4 \div 5$.

10. Anything which can be multiplied or divided, — that is, which can be increased, separated into parts, or measured, is called **quantity.**

E. g. A line is a quantity because it can be doubled, tripled, halved, etc., and its length can be expressed numerically in terms of another line of definite length, such as a foot or a yard, taken as a unit of measure.

Weight is a quantity, since it can be measured in pounds, ounces, grams, etc.

Time is a species of quantity whose measure can be expressed in terms of seconds, minutes, hours, etc.

Color is not a quantity, for we cannot say that one color is twice as great, or one-half as great as another.

The operations of the mind, such as thought, choice, etc., are not quantities, for they are incapable of measurement, — that is, of direct numerical comparison.

11. Any letter or number upon which an operation is to be performed is called an **operand.**

12. Any combination of numbers, letters, and symbols of operation which may be taken, according to the Principles of Algebra, to represent a number, is called an **algebraic expression**.

13. Parentheses ( ) may be used to denote that an algebraic expression enclosed by them is to be treated as a whole throughout a calculation.

Thus, $3 \times (4 + 5)$ means 3×9, or 27; also $(4 + 1)(7 + 2)$ means 5×9, or 45. (See Chapter III, § 3, also Chapter IV, § 1.)

14. When two numbers are multiplied together the result is called the **product,** and each number is called a **factor** of the product.

Thus, 12 is the product of 4 and 3.

15. By a **continued product** is meant a product composed of three or more factors.

Thus, the continued product of 2, 3 and 4 is 24.

16. An **exponent** is a small number placed at the right of and a little above any number or factor to indicate the number of times that number or factor appears as a factor of a given product.

Thus, 3^2, read "3 to the second power" or "3 square," means 3×3; 2^4, read "2 to the fourth power," means $2 \times 2 \times 2 \times 2$;

a^2b^3, read "a to the second power, times b to the third power," or "a square, times b cube," means $a \times a \times b \times b \times b$.

When no exponent is written, the exponent 1 is understood.

Thus, x means x^1.

17. We shall, at the beginning of the subject, use the terms *sum*, *difference*, *remainder*, *product*, *quotient*, etc., with the same meanings in algebra as in arithmetic.

18. A whole number, or an **integer,** is one or a sum of ones.

Thus, the whole number, or integer, 5 is the sum of five 1's.

19. The **numerical value** of an algebraic expression for specified values of the letters appearing in it, is defined for the present as the number obtained by substituting for the letters given numerical values, and performing the indicated operations.

Thus, if x and y represent 2 and 3 respectively, the numerical value of xy is 6; of $x + y$ is 5; of $x^2 + y^2$ is 13; and that of $5x - 2y$ is 4.

It is to be understood that such values only are to be given to the letters as will allow the operations to be performed. The necessary restrictions on the values of the letters will be explained as they appear in later chapters.

20. An expression is said to be a **function** of some specified letter appearing in it if a change in the value of the letter produces in general a change in the value of the expression.

An expression which is a function of x may be indicated by writing $f(x)$, read "function x."

The expression "function x" suggests to us, *depends for its value upon the value of x.*

E. g. The distance passed over by a person in a given time depends upon the rate at which the person travels, and may be spoken of as being a "function" of the rate.

The time required to build a house depends, among other things, upon the number of workmen employed, and may be said to be a "function" of the number of workmen.

The length of a bar of metal depends upon its temperature, and we may regard the length of a particular bar as being a "function" of the temperature.

21. Although different numbers in arithmetic are represented by different definite number symbols, each number may be represented in an unlimited number of ways by combinations of other numbers.

Thus, the number 24 may be represented by 12×2, 6×4, 8×3, $20 + 4$, $100 \div 5 + 4$, etc.

22. Fixing our attention upon the results of indicated operations, as in the illustration above, we commonly speak of such expressions as 12×2, 6×4, $20 + 4$, etc., as *numbers*, meaning thereby the numbers resulting when we perform these operations.

We say that the number 12×2 *is equal to* the number $20 + 4$, since each represents 24×1.

23. As a symbol of relation we have in algebra, as in arithmetic, the sign of equality =, which may be read "equals," "is equal to," "is replaceable by," etc.

Thus, $8 + 2 = 10$.

24. The statement in symbols that two expressions represent the same number is called an **equation.**

E.g. $5 + 1 = 4 + 2$.

The part at the left of the equality sign is called the **first member,** and the part at the right the **second member,** of the equation.

E.g. In $5 + 1 = 4 + 2$, $5 + 1$ is the first member, and $4 + 2$ is the second member of the equation.

25. *The sign = should never be used except to connect numbers or expressions which are equal, that is, which stand for the same number. It should never be used in place of any form of the verb "to be."*

E.g. We should write, "Ans. is 8," never "Ans. = 8."

26. Two algebraic expressions are **equivalent** when they represent the same numerical value, no matter what particular values may be assigned to the letters appearing in them.

E.g. $3x + x$ is equivalent to $4x$; $5y - 2y$ is equivalent to $3y$.

27. An equality whose members are equivalent expressions is called an **identity.**

In general any values may be assigned at will to the letters appearing in an identity. Such restrictions as may be necessary in certain cases will be explained in a later chapter.

28. An equality which is true for particular values only of certain of the letters appearing in it is called a **conditional equation.**

E. g. The equation $x + 1 = 5$ is a conditional equation, for the first member is equivalent to the second only on condition that x be given the particular value 4.

We shall use the word *equation* to mean conditional equation, that is, an equality *which is not an identity.*

29. In order to distinguish identical from conditional equations we shall use the triple sign of equality, $\equiv$ (read "is the same as," "is identical with," "stands for,") for identical equations, and the double sign of equality $=$ for conditional equations.

To conform to the usage in arithmetic we shall commonly use the double sign of equality $=$ instead of the triple sign $\equiv$ when writing identities in which arithmetic numbers only appear.

Other reasons for this use of the sign will be given in a later chapter.

Thus, instead of $6 + 2 \equiv 8$, we shall write $6 + 2 = 8$.

30. The signs $>$ and $<$, read "is greater than," and "is less than," respectively, are used as symbols of inequality.

E. g. We may denote that 10 is greater than 8 by writing $10 > 8$; that 7 is less than 9 by writing $7 < 9$.

It should be noted that the larger end of each symbol is directed toward the greater quantity.

31. The signs of equality or of inequality, when crossed by lines, are understood as meaning "not equal to," "not greater than," and "not less than."

E. g. $2 \neq 3$ means that 2 is not equal to 3; $6 \not> 9$ means that 6 is not greater than 9; $8 \not< 7$ means that 8 is not less than 7.

EXERCISE I. 2

Find the values represented by the following expressions when the given numerical values are substituted for the letters:

If $a = 4$, $b = 2$, $c = 5$, and $d = 1$, find the value of

1. $a + b$.
2. $a + c$.
3. $c - d$.
4. $b - d$.
5. $6a$.
6. $3bc$.
7. $5cd$.
8. abc.
9. a^2b.

10. $2\,ab^2$.
11. $3\,a^2cd$.
12. $a^2 + ab$.
13. $(a + b)c$.
14. $2\,cd + abc$.
15. $(a + b)(c + d)$.

If $a = 2$, $b = 5$, $c = 1$, $d = 3$, find the numerical values represented by the following expressions:

16. $ab + bc + cd + da$.
17. $ac - bd + ab - cd$.
18. $abc + bcd + cda + abd$.
19. $(a+b)(b+c)+(b+c)(c+d)$.
20. $(a+c)(b+d)+(a+d)(b+c)$.
21. $(a+b)(b-c)+(b+c)(c-d)$.
22. $(a-b)(c-d)+(b-c)(d-a)$.
23. $(a+b)c+(b+c)d+(c+d)a$.

If $a = 6$, $b = 3$, $x = 7$, and $y = 1$, find the values represented by the following expressions:

24. $a^2 + b^2$.
25. $a^2 - b^2$.
26. $a^2 + b + x^2 + y$.
27. $(a + b)^2 + a + b$.
28. $(a^2 + b^2)x + (x^2 + y^2)b$.
29. $a^2 + ab + b^2$.
30. $a^2b + ab^2 - x - y$.
31. $ax^2 - by^2 - x + y$.

Verify the following algebraic identities for particular values of the letters appearing in them, by assigning values to the letters:

Ex. 32. $(a + 1)(a + 2) \equiv a^2 + 3\,a + 2$.

If the members are identical they must represent equal numbers for all values which may be assigned to a.

Accordingly, letting $a = 4$, we obtain, substituting 4 for a,

$$\begin{aligned}(4 + 1)(4 + 2) &\equiv 4^2 + 3 \times 4 + 2\\ 5 \times 6 &\equiv 16 + 12 + 2\\ 30 &\equiv 30.\end{aligned}$$

Accordingly the identity is true for $a = 4$. By substitution it will be found to be true for all other values which may be assigned to a.

33. $(y+5)(y+3) \equiv y(y+8)+15$.
34. $(x + 3)^2 \equiv x(x + 6) + 9$.
35. $(y + 5)^2 \equiv y^2 + 5\,(2\,y + 5)$.
36. $m\,(m+4)+4 \equiv m^2+4\,(m+1)$.

Ex. 37. $(a + b)^2 \equiv a^2 + 2\,ab + b^2$.

Assigning to a and b the values 8 and 2, we obtain by substitution,

$$\begin{aligned}(8 + 2)^2 &\equiv 8^2 + 2 \times 8 \times 2 + 2^2\\ 100 &\equiv 64 + 32 + 4\\ 100 &\equiv 100.\end{aligned}$$

Ex. 38. $(x + a)(x + b) \equiv x^2 + (a + b)x + ab$.

Substituting 4, 3, and 2 for x, a, and b respectively, we obtain

$$(4+3)(4+2) \equiv 4^2 + (3+2)4 + 3 \times 2$$
$$7 \times 6 \equiv 16 + 20 + 6$$
$$42 \equiv 42.$$

39. $(x+y)^2 - 4\,xy \equiv (x-y)^2$.
40. $(a^2+b^2)(x^2+y^2) - (ax+by)^2 \equiv (ay-bx)^2$.
41. $(x^2+y^2)(a^2+b^2) \equiv (ax-by)^2 + (bx+ay)^2$.
42. $(x+y)^3 - x^3 - y^3 \equiv 3\,xy(x+y)$.
43. $a^4 + (a^2+ab+b^2)^2 \equiv (a^2+b^2)\left[a^2+(a+b)^2\right]$.
44. $(x+y)^4 \equiv 2\,(x^2+y^2)(x+y)^2 - (x^2-y^2)^2$.
45. $(a^2+b^2)(c^2+d^2) \equiv (ac+bd)^2 + (ad-bc)^2$.
46. $a^3+b^3 \equiv (a+b)(a^2-ab+b^2)$.
47. $(a-b)^3 + 3\,ab(a-b) \equiv (a+b)^3 - 3\,ab(a+b) - 2\,b^3$.
48. $(x+y)^3 - (x-y)^2(x+y) \equiv 4\,xy(x+y)$.
49. $(a+2)^2 - 4\,(a+1)^2 + 6\,a^2 - 4\,(a-1)^2 + (a-2)^2 \equiv 0$.
50. $(x+y)^3 \equiv x^3 + 3\,x^2y + 3\,xy^2 + y^3$.

32. An **axiom** is the statement of a truth which may be inferred directly from our experience, or from the nature of the things considered.

To be regarded as an axiom, a truth must be such that it is incapable of proof further than its mere statement.

As axioms common to mathematics, we may state the following, which were called by an early writer on mathematics **Common Truths about Things:**

1. *Any number is equal to itself.*

E. g. $4 = 4$.

2. **The Principle of Substitution.** *The numerical value of a mathematical expression is not altered when for any number or expression in it we substitute an equal number or expression.*

That is, the "form" of an expression may be changed without altering its value.

E. g. The value of $4 + 3 + \frac{10}{5}$ remains unaltered if for $\frac{10}{5}$ we substitute its equal value 2; or again, if we substitute 7 for the sum of 4 and 3 and write $7 + 2$.

3. *Numbers equal to the same number are equal.*

E. g. If $2\,x = a$ and $10 = a$, then $2\,x = 10$.

4. *If equal numbers be added to equal numbers the resulting numbers will be equal.*

E. g. If $x = y$, then $x + 3 = y + 3$.

5. *If equal numbers be subtracted from equal numbers the resulting numbers will be equal.*

E. g. If $x + 3 = 10$, then $x = 10 - 3$, or 7.

6. *If equal numbers be multiplied by equal numbers the resulting numbers will be equal.*

E. g. If $\frac{1}{2}x = 5$, then two times $\frac{1}{2}x$ equals two times 5, that is, $x = 10$.

7. *If equal numbers be divided by equal numbers (except zero) the resulting numbers will be equal.*

E. g. If $3x = 12$, then $3x$ divided by 3 equals 12 divided by 3, that is, $x = 4$.

8. *Like roots of equal numbers are equal.*

E. g. If $x^2 = 5^2$, then $x = 5$.

There are two square roots, three cube roots, four fourth roots, etc., of any number, so that when applying this axiom it is necessary to distinguish carefully between these roots. (See Chapter XVIII, Principal Values of Roots.)

33. Substituting the word "identical" for the word "equal" in each of the statements above, we have corresponding axiomatic principles governing identical expressions.

34. If $A \equiv B$ we may immediately write $B \equiv A$, since this is only another way of saying the same thing.

Identities such as those above, which are formed by interchanging the members, are said to be one the **converse** of the other.

Ex. 1. Find the value which must be assigned to a in order that the conditional equation $2a + 3 = 15$ may be true.

Subtracting 3 from each member we obtain

$$2a + 3 - 3 = 15 - 3$$

$$\text{Hence} \qquad 2a = 12.$$

Dividing both members of the last equation by 2, we obtain finally $a = 6$. This value is found to satisfy the original conditional equation.

Exercise I. 3

Find, by applying the axioms, the values which must be assigned to the letters in order that the following conditional equations may be true.

In each case the axiom applied should be stated, and the result obtained should be verified, by substituting for the letter in the given equation the value found.

1. $b + 4 = 10$.
2. $c - 2 = 7$.
3. $d - 3 = 1$.
4. $2m = 10$.
5. $6n = 42$.
6. $\frac{1}{2}x = 8$.
7. $\frac{1}{3}y = 15$.
8. $4a + 1 = 13$.
9. $5b + 2 = 22$.
10. $6c - 7 = 11$.
11. $\frac{1}{2}d + 1 = 17$.
12. $\frac{1}{3}h - 5 = 2$.
13. $\frac{1}{4}y - 1 = 5$.
14. $\frac{2}{3}z = 8$.
15. $\frac{4}{5}a = 12$.
16. $\frac{3}{2}b + 1 = 22$.
17. $\frac{5}{6}c + 3 = 13$.
18. $\frac{3}{4}d - 2 = 4$.
19. $\frac{4}{5}a + 6 = 14$.
20. $\frac{3}{7}b - 1 = 5$.

CHAPTER II

AN EXTENSION OF THE IDEA OF NUMBER

1. In arithmetic we have found it possible to subtract one number from another only when the number subtracted was not greater than the number from which it was taken.

Such combinations of numbers as $6 - 9$, $10 - 11$, etc., are from the point of view of arithmetic wholly destitute of meaning, since there exists no number, that is, no result of counting, which when added to 9 gives 6, or when added to 11 gives the sum 10.

Since such combinations of numbers occur frequently in mathematical work, it becomes necessary for us to give them a meaning if we are to allow them to remain in our calculations. To do this we find it necessary to extend our notion of number. The combination of numbers $6 - 9$, as written, suggests to us a difference, and it will be convenient for us to reckon with it as with every other "real" or "actual" difference, such as $9 - 6$, or $8 - 3$, etc., that is, a difference in which the subtrahend is less than the minuend.

Principle of No Exception

2. Mathematicians are accustomed to apply the names of familiar combinations of numbers and symbols which have recognized meanings to all similar combinations, even when these do not appear at first to admit of meaning, or even to make sense. This principle, that *the old laws of reckoning and the old meanings must be carried over to include all special cases of a given general type, even those which may appear at first to be exceptions*, will appear under many different forms throughout the whole science of mathematics, and will be referred to as the **Principle of No Exception.**

Instead of being an unwarranted stretching of language, as it may appear at first, the Principle of No Exception insists rather on a stretching or broadening of ideas to fit the language used, in order to avoid contradictions which might otherwise arise.

In Arithmetic the primary idea of a fraction is a "part of unity" or a "broken number."

Thus, $\frac{2}{3}$, $\frac{5}{7}$, $\frac{10}{12}$, are fractions in this sense.

In the course of arithmetic work, combinations appear such as $\frac{4}{3}$, $\frac{9}{5}$, $\frac{13}{2}$, etc., which look like fractions, and behave like fractions, but which are not in the original sense "broken numbers." They are not properly fractions, and are accordingly called "improper fractions."

The Principle of No Exception is then applied, and such combinations as $\frac{2}{3}$, $\frac{5}{7}$, $\frac{4}{3}$, $\frac{9}{5}$, etc., are all, without exception, spoken of as fractions, without specifying whether they are proper or improper fractions, so-called.

3. It will now be shown that the application of this idea of No Exception leads us to an extension of our previous notions concerning number, and to the *invention of a new kind of number*, a kind of number which does not appear, as did the primary numbers, as a result of counting, but which nevertheless may be used in our calculations in such a way as always to give sense.

POSITIVE AND NEGATIVE QUANTITIES

4. Certain words, such as

forward — backward,
upward — downward,
north — south,
rising — falling,
profit — loss,
earning — spending,
increasing — diminishing,
positive — negative,

suggest to us a condition of two things such that each tends to destroy the effect produced by the other. One tends to increase whatever the other tends to decrease. The terms are merely relative and imply that, from some point of view, one thing tends to oppose another.

E.g. If travelling east takes us away from some particular place, then from the same point of view, travelling west will take us toward that same place.

In trade, the effect produced by profits offsets the effect produced by losses.

5. Without multiplying illustrations we will remark simply that the terms *positive* and *negative* are used in mathematics in such a way as to imply that there is some opposition such that if, in a calculation, the things denoted as positive should be added, then those called negative should be subtracted. This may be due either to the nature of the things considered, or to the point of view from which we regard them.

6. The opposition between two sets of things is often such that it is of no consequence which is considered as positive. The selection being once made, so long as the things of one set in a calculation are considered as positive quantities, those of the other set must in opposition remain as negative quantities.

7. By the **absolute value** of a quantity expressed in terms of some unit of the same kind, is meant the number of times the unit is contained in the given quantity. This is without regard to the quality of either the quantity or the unit, that is, as to whether both are positive or both negative.

8. If two quantities are such that, when combined or considered as parts of one whole, any given amount of one destroys the effect produced by an amount of the other equal in absolute value to that of the first, these two quantities are called **opposites.**

In mathematical calculations one of two opposite quantities is called **positive** and the remaining one **negative.**

9. If, in any calculation, we choose to regard some quantity as being positive, then all other quantities which tend to increase it must be considered as positive also, and all those which tend to diminish it must be taken as negative.

It is merely a matter of choice which one of two opposite quantities is regarded as positive. On one occasion we may regard motion in one direction, say toward the right, as being positive, and on another we may equally as well choose to regard motion toward the left as being positive. In either case motion in a direction directly opposite to that chosen as positive would be considered as negative motion.

Also, if we choose to call the capital invested in a business positive, then all profits will be positive, since they may be added to and used to increase the capital; all losses and expenses will be negative, for they tend to diminish the capital, since they must be subtracted from it.

10. It is not essential to positive quantities that they be numerically greater than those which are negative. Thus, losses in business, regarded as negative quantities, might greatly exceed gains, which would then be positive quantities.

11. From the nature of things, we may treat positive and negative quantities according to the following Principles:

Principle I. *If a positive and a negative quantity of the same kind are equal in absolute value, either will destroy the effect of the other when both are taken together or combined by addition.*

E. g. Items of income and expense may be regarded as being opposite quantities, and we may call one positive and the other negative; for any item of expense reduces by just an equal amount the effective income.

Principle II. *Positive quantities alone may be added in any order; also negative quantities alone may be added in any order.*

E. g. Since negative quantities are those which are considered as tending to diminish the effect of certain others called positive quantities, the combined effect of several negative quantities will be a negative quantity which is equal to their sum.

There is no contradiction in speaking of adding negative quantities, for *the idea suggested by the terms positive and negative is one of nature or quality, not number or amount.*

If incomes be regarded as positive, expenses must be treated as negative quantities, and we may add all of our expenses and then subtract the sum total from our income to determine our financial condition.

A single negative quantity may "oppose" a positive quantity to produce a decreased "value" indicated by subtraction, while taken with another negative quantity there will be produced an "increased negative effect" which would have to be indicated by addition.

Thus, as before, the total expense results from adding several expenses.

Principle III. *The resultant effect of several combined positive and negative quantities is equal to the numerical difference between the total positive and total negative effects, and has the quality or nature of the greater total.*

E. g. The result of combining expenses of $5 and $10 with items of income of $3, $2, $3, $6, $4, $2, and $1, may be obtained by finding the difference between the total expense, $15, and the total income, $21. This difference would be a balance of $6 in favor of the income. This balance may be taken as a positive quantity.

Principle IV. *The removal or subtraction of a positive quantity has the same effect on an expression in which it occurs as the addition of a negative quantity equal in absolute value to the positive quantity.*

E. g. Consider the items of income \$3, \$2, and \$4 as positive quantities. The effect produced on the total income of neglecting or subtracting one of these items, say the amount of \$4, may also be produced by adding or incurring an expense of \$4, since each results in diminishing the effective income by \$4.

The "not" taking of one thing, say an item of income, amounts in effect to taking an item of opposite character or quality, that is, an "equal" item of expense.

Principle V. *The removal or subtraction of a negative quantity has the same effect on an expression in which it occurs as the introduction of, or addition of, a positive quantity equal in absolute value to the negative quantity.*

E. g. In order to restore a given amount of money to its original value after incurring an expense of \$4, it is necessary to bring about an increase of \$4.

If, instead of spending and then earning equal amounts, we neglect to spend, that is, if we take away or subtract an item of expense, our original capital remains unaltered.

Ex. 1. Classify the changes in temperature from 64° F., to 110° F. and to 32° F., respectively.

Since we have an increase in temperature in changing from 64° F. to 110° F. and a decrease in changing from 64° F. to 32° F., we may regard one of these changes as being positive and the other negative.

Thus, if the increase be taken positive the decrease must be regarded negative.

Exercise II. 1

Classification of Quantities as Being Either Positive or Negative

Classify the following *changes in temperature* as being both positive, both negative, or one positive and the other negative:

1. From 60° F., to 100° F. and to 50° F. respectively.
2. From 68° F., to 90° F. and to 212° F. respectively.
3. From 76° F., to 0° F. and to 32° F. respectively.
4. From 102° F., to 40° F. and to 80° F. respectively.
5. From 0° F., to 17° F. below zero and to 5° F. below zero respectively.
6. From 11° F. below zero, to 21° F. below zero and to 15° F. below zero respectively.

7. From 50° F. above zero, to 10° F. below zero and to 10° F. above zero respectively.

8. From 6° F. above zero, to 20° F. below zero and to 5° F. below zero respectively.

9. From 10° F. below zero, to 18° F. below zero and to 9° F. below zero respectively.

10. From 16° F. below zero, to 14° F. below zero and to 3° F. below zero respectively.

Which of the following cities may be selected as points of reference in order that the distances to the remaining two may be classified as being both positive or both negative?

Ex. 11. Boston, Atlanta, Baltimore.

Boston and Baltimore are both north of Atlanta. Accordingly, the distances of these cities from Atlanta are both measured in the same direction. Accordingly, both may be taken as positive or both negative.

Also, since Baltimore and Atlanta are both south of Boston, the distances from Boston to Baltimore and Atlanta may be taken as both positive or both negative.

12. New York, Philadelphia, Washington, D. C.

13. New Orleans, San Francisco, Montreal.

14. Boston, London, Madrid.

Which of the following cities may be selected as points of reference in order that the distances to the remaining two may be classified as being one positive and the other negative?

15. London, Paris, Rome.

16. St. Petersburg, Calcutta, Pekin.

17. Boston, Buffalo, Chicago.

Regarding a man's *income* as representing a positive quantity, classify the following items as positive or negative quantities wherever possible:

18. (*a*) Money loaned to a friend.
    (*b*) Interest paid on a mortgage.
    (*c*) Interest on money deposited in the bank.
    (*d*) Money drawn out of one bank and deposited in another.
    (*e*) Money paid for house rent.

Regarding *money on hand* as representing a positive value, classify the following items, wherever possible, as positive or negative quantities with reference to the depositor:

19. (*a*) Money deposited in the bank.
    (*b*) Interest received on money deposited in the bank.
    (*c*) Interest paid on a mortgage held by the bank.
    (*d*) Money withdrawn from the bank.

20. Classify the items above with reference to the bank.

With reference to the equator, classify the latitudes of the following places as being positive or negative:

21. Munich, Vienna, Buenos Ayres, Quito, Glasgow, St. Louis, Melbourne, Zanzibar.

Since, starting at the equator, it would be necessary to travel north to reach Munich, Vienna, Glasgow, and St. Louis, and to travel in the opposite direction, that is, south, to reach Buenos Ayres, Quito, Melbourne, and Zanzibar, the distances from the equator to the places first named may be considered as being all positive or all negative. Accordingly, the distances from the equator to the places last named would be regarded as being either all negative or all positive, respectively.

22. Tokio, Jerusalem, Sidney, Stockholm, Honolulu, Rio Janeiro, Cape Town, Tunis.

With reference to the meridian passing through Greenwich, classify the longitudes of the following places as being positive or negative quantities:

23. Shanghai, Minneapolis, Berlin, Naples, Dublin, Ottawa, Havana.

Which of the following dates must be selected for reference in order that the changes in time to the remaining dates may be classified as one positive and the other negative?

24. (*a*) 1492, 1620, 1776.
    (*b*) 1812, 1861, 1863.
    (*c*) 44 B.C., 64 A.D., 753 B.C.
    (*d*) 1815, 1066, 1349.
    (*e*) 490 B.C., 480 B.C., 146 A.D.

THE INVENTION OF NEGATIVE NUMBERS

12. Imagine a series of equal steps or distances to be laid off along a straight line, unlimited in length, taken for convenience in a vertical position, as in Fig. 1. Then, beginning with the lower end of the line and counting upward we will number the points of division, using the primary numerals 1, 2, 3, 4, 5, etc.

If we regard motion upward, or counting upward along the "carrier" line, as being in a positive direction, then motion downward, or counting downward, must be regarded as being negative.

13. Fixing our attention on any particular number we may find a larger number by counting upward, and, except in the case of 1, a smaller number by counting downward. Any number will be relatively positive with regard to another if it be situated above it in position, and relatively negative to it, if it be necessary to count downward from the other to find it.

E. g. Relatively to 8, 10 is positive, while all smaller numbers, as 5, 4, 3, etc., are negative, since we should have to count downward from 8 to reach 5, 4, 3, etc.

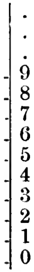


FIG. 1.

14. Observe that it is possible for us to count upward *for any number of spaces*, starting anywhere in the series 1, 2, 3, 4, etc., but it is not possible to count downward for any number of spaces. This is because we must always stop when we reach the lowest point of the line, since there are no numbers below it to count.

Furthermore, we cannot count downward at all if we start at the lowest point, 1, for we have reached the end of our line, and at the same time the "lower" end of our series of integral primary numbers.

15. There is nothing unreasonable in imagining our line to be now extended downward, carrying our series of steps downward indefinitely.

In order to distinguish these newly added downward steps from those above our starting point, which we will take as 0, we may designate them by saying *one below zero*, *two below zero*, *three below zero*, etc.

16. Since starting from 0, it is necessary to count in opposite directions along our "carrier" line to reach numbers having the same number name (as, for example, "four above" zero and "four below"), we may distinguish the two sets of numbers by calling them, relatively to 0, one set positive and the other negative.

17. We will call the numbers "above" zero *positive one*, *positive two*, *positive three*, *etc.* (written $^{+}1$, $^{+}2$, $^{+}3$, etc.), and those "below" zero *negative one*, *negative two*, *negative three*, etc. (written $^{-}1$, $^{-}2$, $^{-}3$, etc.).

In this and the next three chapters we shall indicate the "quality" of a number as being positive or negative by writing before it a small "quality" sign $^{+}$ or $^{-}$. When so used these signs are read *positive* and *negative* respectively, and are called **signs of quality.** They will, by their size and position, be easily distinguished from the larger signs of operation for addition or subtraction, + and −, which are read *plus* and *minus* respectively.

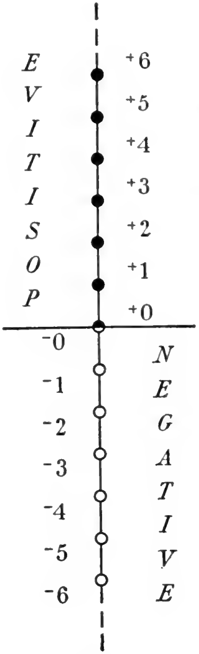


Fig. 2.

18. We have applied the Principle of No Exception to our notion of counting, and have "stretched out," or "extended" our number series to allow of the idea of counting "backward" or "downward" beyond zero.

The important point to be understood is that the positive numbers $^{+}1$, $^{+}2$, $^{+}3$, $^{+}4$, etc., represented on Fig. 2 beside the "black circles" or dots, should be regarded as having arisen as the result of the actual counting of objects.

On the other hand, the negative numbers, $^{-}1$, $^{-}2$, $^{-}3$, $^{-}4$, etc., represented on the figure beside the small rings, or "white circles," were invented simply to serve our convenience, in order that we might represent, or imagine, counting "downward" or "backward" below zero.

These negative numbers may be regarded as being "artificial" numbers, and as being simply the invention of mathematicians to serve as convenient means for simplifying work and interpreting results.

19. Letters were first used to represent negative numbers by Descartes in the first half of the seventeenth century. In a book published in 1545 by an earlier writer, Cardan, they were called "numeri ficti," or imaginary numbers, in contradistinction to "numeri veri," or real numbers.

At the present time the term "imaginary" number is applied to still another form of "invented" number. (See Chapter XXI.)

20. By the **absolute value** of a number or letter is meant its value without regard to its quality as being positive or negative. (See also § 7.)

The absolute value of a number or letter is indicated by writing it between two upright bars, $|\ \ |$.

E.g. $|a|$ means the absolute value of some number a.

We may write $|^{+}a| \equiv |^{-}a|$. This is read: "The absolute value of positive a is the same as the absolute value of negative a."

By the arithmetic value of a number is meant its absolute value.

E.g. The arithmetic value of $^{-}4$ is 4.

21. Just as in arithmetic we regard any whole number as being a repetition of the primary unit 1 a certain number of times (for example $5 \equiv 1 + 1 + 1 + 1 + 1$), so in algebra we regard positive and negative numbers as being repetitions of the **quality units** $^{+}1$ and $^{-}1$.

Of these quality units, $^{+}1$ is taken as the **primary unit.**

A **positive number** is the sum of two or more positive units, and a **negative** number is the sum of two or more negative units.

Positive numbers and negative numbers taken together are called **algebraic numbers.**

E.g. $^{+}5$ denotes five times $^{+}1$, or five positive units. Hence we may write $^{+}5 \equiv + {}^{+}1 + {}^{+}1 + {}^{+}1 + {}^{+}1 + {}^{+}1$.

$^{-}5$ denotes five times $^{-}1$, or five negative units. Hence we may write $^{-}5 \equiv + {}^{-}1 + {}^{-}1 + {}^{-}1 + {}^{-}1 + {}^{-}1$.

22. The **sign of continuation** . . . . ., read *and so on*, is used to indicate that the expression as written may be extended.

Thus, the expression $1 + 2 + 3 + \cdots\cdot$, may be extended, if desired, as for example,

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + \cdots\cdots$$

The sign of continuation may also be used to indicate that certain parts of an expression have been omitted for convenience.

Thus, in the expression $2 + 4 + 6 + \cdots\cdot + 96 + 98 + 100$, the sum of the numbers from 8 to 94 inclusive has been omitted.

23. The symbol ∞ which is read "infinity," is used to represent a number which is greater than any assignable number, however great.

We may symbolize the whole series of algebraic number by $^{-}\infty, \cdots\cdot, {}^{-}3, {}^{-}2, {}^{-}1, {}^{\pm}0, {}^{+}1, {}^{+}2, {}^{+}3, \cdots\cdot, {}^{+}\infty$, the order of ascending magnitude being from the left to the right. The double sign ±, read "positive or negative," is placed before the zero to indicate its exceptional property, namely, that $^{+}0 = {}^{-}0$. We may regard zero as belonging to both the positive and negative parts of the series. It is all that these parts have in common.

CHAPTER III

FUNDAMENTAL LAWS OF ALGEBRA FOR THE ADDITION AND SUBTRACTION OF POSITIVE AND NEGATIVE NUMBERS

1. SINCE in order to obtain correct results we must work with numbers according to certain rules, it is common to speak of them as obeying laws. What we mean, in reality, is that, unless we obey certain rules or laws in performing our calculations, we cannot depend upon the accuracy of our results.

2. It follows from the definitions of the number symbols of arithmetic that since $2 = 1 + 1$, and $3 = 1 + 1 + 1$, that $+ 3 + 2 = + 2 + 3$; and in the same way we may show that $+ 3 + 4 + 7 = + 4 + 3 + 7 = + 7 + 3 + 4$, etc., and in general if a, b, c, etc., represent any whole numbers in arithmetic, $a + b + c \equiv a + c + b \equiv b + c + a \equiv$ etc.

Hence *in a chain of additions the result is independent of the order in which the additions are performed.*

This is called the **Law of Commutation** for addition in arithmetic, and it remains for us to show that it can be applied to expressions which contain both positive and negative numbers, that is, algebraic numbers.

3. As symbols of grouping we have the parentheses ( ), brackets [ ], braces { }, and vinculum $\overline{\qquad}$ which is sometimes written vertically |.

These symbols indicate in each case that the expression included is to be operated with as a whole. (See also Chapter IV.)

E. g. $(3 + 5)$ means that the sum of 3 and 5 is to be used as a whole in an operation, that is, as 8, so that an expression such as $2 \times (3 + 5)$ is to be understood as meaning *two times* 8, *or* 16; while the expression $2 \times 3 + 5$ means *two times* 3, *or* 6, *increased by* 5, — that is, 11, not 16.

Again, $(4 + 2)(8 - 1)$ means $4 + 2$, *that is*, 6, *multiplied by* $8 - 1$, *or* 7, *that is*, 6×7, *or* 42; while $4 + 2 \times 8 - 1$ means 4 *increased by* 2 *times* 8, *or* 16, *and this sum diminished by* 1. Hence the result is 19.

4. It is a matter of experience with us in arithmetic that in a chain of additions *the sum total is not affected by combining two or more parts of the sum.*

This is called the **Law of Association** for addition.

E. g. $3 + 5 + 6$ is equal to $(3 + 5) + 6$, that is, 14 is equal to $8 + 6$. Again, the original expression is equal to $3 + (5 + 6)$, that is, 14 is also equal to $3 + 11$.

If a, b, c, etc., represent any whole numbers in arithmetic, we may state the Law of Association in symbols, by writing

$$\boldsymbol{a + b + c \equiv (a + b) + c \equiv a + (b + c).}$$

5. The foundations of all mathematical knowledge must be laid in definitions.

A **definition** is an explanation of what is meant by any word or phrase, and must be given in terms of things other than those considered. It is essential to a complete definition that it distinguish perfectly the thing defined from everything else.

In mathematics the principal terms may be so defined as to leave not the slightest question respecting their meaning.

6. There are comparatively few mathematical truths or **theorems** which are self-evident. The majority require to be proved by a chain or course of reasoning. The course of reasoning by which the truth of a statement is established is called a **demonstration** or **proof.**

As **symbols of deduction** we have in algebra, as in arithmetic, $\therefore$ meaning *therefore*, and $\because$ meaning *since* or *because.*

STEPS ON A LINE

7. We shall speak of different distances measured on the "carrier" line which "carries" our collection of extended number, $^-3$, $^-2$, $^-1$, $^{\mp}0$, $^+1$, $^+2$, $^+3$, etc., as **steps.** We shall say that a step is *positive* if in taking it we step "up," that is, in the direction of the increasing primary numbers, 1, 2, 3, 4, etc. We shall call it *negative* if we step "down," that is, in the direction $^+6$, $^+5$, $^+4$, $^+3$, $^+2$, $^+1$, or $^-1$, $^-2$, $^-3$, $^-4$, $^-5$, etc. (See Fig. 1.)

8. We shall understand that two steps taken anywhere on the line are equal, when their lengths on the line are equal, and when they are taken in the same direction or sense.

E. g. The step from $^{+}5$ to $^{+}8$ is equal to the one from $^{+}8$ to $^{+}11$; or from $^{+}12$ to $^{+}15$; or from $^{+}20$ to $^{+}23$; etc. It is also equal to the step from $^{-}7$ to $^{-}4$; from $^{-}8$ to $^{-}5$; and these are *all* to be considered as *positive steps.*

As *negative steps,* each having a length of four units, we may select the one from $^{+}10$ "down" to $^{+}6$; from $^{+}4$ to $^{+}0$; from $^{+}2$ to $^{-}2$; from $^{-}0$ to $^{-}4$; from $^{-}5$ to $^{-}9$; etc.

9. If we move along the "carrier" line, taking successively two steps, either both in the same direction, or the first in one direction, then turning around, take the other in the opposite direction, we speak of the process as effecting the **composition** of the two steps.

10. *Two separate steps may, by composition, be combined into a single step.*

E. g. Two separate steps of 3 and 5 units respectively, may by composition be combined into a single step of 8 units in length, and the fact that both are positive or both negative will not affect the result.

11. The single step which may be taken in place of two or more others taken successively, is called the **resultant step.** The resultant step may be defined as the single step which may be taken to produce the same final change of position as that produced by taking several others successively.

If we travel along the line taking different steps successively, observe that *the resultant step is measured from the beginning to the end of the journey. It is sometimes shorter, but never longer than the entire path passed over.*

E. g. If the steps $^{+}7$, $^{-}5$, and $^{+}1$ were taken successively, starting from any point of the line, as say $^{+}3$, we should arrive first at $^{+}10$ on the line; then returning by the step $^{-}5$, stop at $^{+}5$, and the end of our journey would be one unit "higher," or at $^{+}6$. Since the distance from the starting point, $^{+}3$, to the stopping point, $^{+}6$, is three units, taken in an "upward" or positive direction, we say that the resultant of the steps $^{+}7$, $^{-}5$, $^{+}1$, is the single step $^{+}3$.

Similarly, the resultant of $^{+}6$, $^{-}4$, $^{-}5$, taken successively, would be a single negative step of three units, or in symbols, the step $^{-}3$.

12. The following Principles will be seen to apply to steps taken along a line.

Principle I. *The resultant of two steps of the same kind, that is, of two steps taken in the same direction, is a single step of the same kind. Its length is equal to the sum of the lengths of the single steps composing it.*

E. g. The resultant of the separate steps $^+4$, $^+3$, $^+5$, is the single step $^+12$.

Principle II. *The resultant of two steps taken successively in opposite directions is a single step whose length is the difference between the lengths of the separate steps composing it. This is a positive or a negative step, according as the greater one entering into it is positive or negative.*

E. g. The resultant of the two steps $^+12$ and $^-5$ is a step of 12 minus 5, that is, 7 units in length, and since the greater of the two steps entering into it, $^+12$, is positive, we must have as a resultant $^+7$.

Principle III. *The resultant of two or more steps does not depend upon the order in which the steps are taken.*

E. g. The resultant of steps $^+12$, $^-8$, $^+3$ and $^-2$, taken in any order, is the single step $^+5$.

Principle IV. *The resultant of any number of steps is a single step whose length is the difference between the sum of the lengths of the positive steps and the sum of the lengths of the negative steps. This resultant is positive or negative according as the sum total of the positive steps or of the negative steps is the greater.*

E. g. The resultant of the separate steps $^+10$, $^+5$, $^-3$, $^-6$, $^+1$ may be found by taking the resultant of the total positive step $^+(10 + 5 + 1)$ or $^+16$, and the total negative step $^-(3 + 6)$ or $^-9$. The difference between 16 and 9 is 7, and since the greater of the two resultant steps, $^+16$ and $^-9$, is positive, the resultant step is positive, and we have as a resultant $^+7$.

The resultant of the successive steps $^+11$, $^-8$, $^-6$, $^+2$, $^-4$, is found by taking the resultant of the total positive step $^+13$, and the total negative step $^-18$, which is $^-5$.

13. The taking of steps along the number series suggests the operations of arithmetic addition and subtraction.

Taking steps "upward," that is, in the direction of increasing primary numbers, 5, 6, 7, 8, etc., suggests arithmetic addition. Taking steps "downward," that is, in the direction of decreasing primary numbers, 8, 7, 6, 5, etc., suggests arithmetic subtraction.

14. We will now find an **interpretation for combinations of signs of operation and signs of quality** such as $+(^+a)$, $-(^+a)$, $+(^-a)$ and $-(^-a)$.

In order to do this, we will now interpret our signs of operation to mean: plus $+$, *take*, that is, to include with other steps, and minus $-$, *take away*, that is, remove from an expression or neglect in connection with other steps.

These ideas correspond to arithmetic addition, or "taking one number with another," and arithmetic subtraction, which is "taking one number away from another," or neglecting it from a sum.

15. *The effect on a sum total of taking away a positive step is the same as that of performing a negative step of equal numerical value or length.*

E. g. From the step $^+5$ take away the step $^+3$.

We have $$^+5 - {^+3} \equiv {^+(5-3)} \equiv {^+2}.$$

Also, from another point of view,

$$^+5 + {^-3} \equiv {^+2}.$$

We obtain the same result as before, using this time a negative step in an additive sense, instead of using, as in the first place, a positive step in a subtractive sense. (See Fig. 2.)

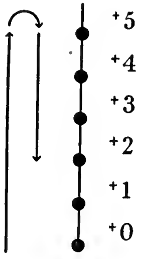


FIG. 2.

16. A negative step produces a decrease of value among positive numbers.

Hence, taking away a decrease amounts to making an increase.

Therefore, $$-(^-a) \equiv +(^+a).$$

Hence, from the illustration above we may draw the following conclusions:

(i.) $+(^+\boldsymbol{a}) \equiv +(^+\boldsymbol{a})$; (iii.) $-(^-\boldsymbol{a}) \equiv +(^+\boldsymbol{a})$;
(ii.) $-(^+\boldsymbol{a}) \equiv +(^-\boldsymbol{a})$; (iv.) $+(^-\boldsymbol{a}) \equiv +(^-\boldsymbol{a})$.

It will be seen that in the identities above, the symbol of operation $+$ occurs in every case on the right of the identity sign; that is, we have transformed our additions and subtractions on the left into additions on the right.

Furthermore, wherever the symbol of subtraction on the left, as in (ii.) and (iii.), has been changed into that of addition on the right, the sign of quality has also been reversed.

17. We may now state the **Law of Signs for Addition and Subtraction.**

Principle: *Additions may be substituted for subtractions, provided that the signs of quality of the numbers subtracted be reversed.*

Ex. 1. From the step ${}^{+}8$ subtract the step ${}^{+}3$.

Indicating the subtraction by writing ${}^{+}8 - {}^{+}3$, we may transform the expression so that instead of a subtraction we shall have an indicated addition. By changing the symbol of operation before 3 from $-$ to $+$, and at the same time reversing the sign of quality and writing ${}^{-}3$, we have

$$ {}^{+}8 - {}^{+}3 \equiv {}^{+}8 + {}^{-}3 \equiv {}^{+}5. $$

Ex. 2. From the step ${}^{+}10$ subtract the step ${}^{-}4$.

Indicating the subtraction as before, we have

$$ {}^{+}10 - {}^{-}4. $$

Substituting an addition for the subtraction, and reversing at the same time the sign of quality, we have

$$ {}^{+}10 - {}^{-}4 \equiv {}^{+}10 + {}^{+}4 \equiv {}^{+}14. $$

Ex. 3. From the step ${}^{-}8$ subtract the step ${}^{+}2$. As before we may write

$$ {}^{-}8 - {}^{+}2 \equiv {}^{-}8 + {}^{-}2 \equiv {}^{-}10. $$

Ex. 4. From the step 9 subtract the step ${}^{-}6$.

We have $$ {}^{-}9 - {}^{-}6 \equiv {}^{-}9 + {}^{+}6 \equiv {}^{-}3. $$

18. If instead of the word "step" we substitute the word "number," or the more general word "quantity," we may regard the principles of this chapter as being principles governing operations with positive and negative numbers or quantities.

19. Whenever reference is made to signs in algebra it is to be understood, unless the contrary is stated, that the + or − signs are meant. Thus, when we speak of the sign of a quantity we shall mean the + or − sign which is prefixed to it or its numerical coefficient, not the × or ÷ signs which may be associated with it.

When we are directed to change the signs of an expression we shall understand that we are to change the + or − signs before every term into − or +, respectively.

CHAPTER IV

ADDITION AND SUBTRACTION OF POSITIVE AND NEGATIVE NUMBERS

Symbols of Grouping.

1. In order to denote that an algebraic expression is to be treated as a whole, in a calculation, it is enclosed within **parentheses** or other symbols of grouping. (See Chapter III, § 3.)

E. g. $(2 + 5 + 7)$ is to be regarded not as three different numbers, 2, 5, and 7, but as the number obtained by adding 5 and 7 to 2, which is 14.

Also, $(5 + 2) \times (3 + 8)$ means 7 multiplied by 11, that is, 77; while $5 + 2 \times 3 + 8$ means 5 increased by 6, increased by 8, that is, 19.

2. Symbols of grouping may be of different kinds; thus, **parentheses** ( ), **braces** { }, **brackets** [ ], etc. The effect in each case is the same, namely, to call our attention to the fact that *whatever is enclosed in them is to be treated or regarded as a whole.*

Occasionally it is convenient to use instead of parentheses a line called a **vinculum,** drawn over an expression.

Thus, $7 + \overline{5 - 2}$ is equivalent to $7 + (5 - 2)$. This notation will be used commonly in connection with fractions and radical or root signs.

E. g. $\frac{2}{3 + 4}$ means 2 divided by $(3 + 4)$, that is, $\frac{2}{7}$.

$\sqrt{11 + 5}$ means the square root of $(11 + 5)$, that is, the square root of 16, which is 4.

3. Expressions containing groups of terms enclosed in parentheses may be treated as follows:

Ex. 1. Consider the expression $25 -(15 + 3)$.

This expression means that the sum of 15 and 3, which is 18, is to be taken from 25 to produce the remainder 7.

To subtract 18 from 25 in a single operation amounts to performing the two separate operations of first subtracting 15 from 25, and then decreasing the remainder by 3. The final result is 7, as before.

Hence $25 - (15 + 3) \equiv 25 - 15 - 3 \equiv 7.$

Accordingly $- (15 + 3) \equiv - 15 - 3.$

Ex. 2. Consider the expression $12 - (9 - 2)$.

This expression means that 2 is to be first subtracted from 9 to produce 7, which is then to be taken from 12, leaving 5 as a final result, that is:

$$12 - (9 - 2) \equiv 5.$$

If we had first subtracted 9 we would have diminished 12 by a number too great by 2.

Hence it would have been necessary to increase this result by 2.

Consequently, we would have obtained the same final result as before by performing the following operations:

$$12 - 9 + 2 \equiv 5.$$

Hence it appears that to subtract $(9 - 2)$, or 7, in one operation, amounts to first subtracting 9 and than adding 2 in two separate operations, that is:

$$- (9 - 2) \equiv - 9 + 2.$$

4. The examples above are illustrations of the General Principle that *we may remove parentheses preceded by the sign of subtraction* —, *provided we at the same time change the signs of operation of the numbers removed, from + to — or from — to +.*

5. In order to prove a theorem true for any algebraic expression, it is only necessary to prove it for an expression containing letters upon whose values no restrictions are placed.

6. We will now proceed to establish the general principles governing the removal or insertion of parentheses preceded by the signs of operation + or — in chains of additions and subtractions for arithmetic numbers. We will then extend these principles to include positive and negative numbers, that is, algebraic numbers.

Representing any arithmetic values by letters a, b, c, at first regarding $a > b > c$, we will deduce the laws for the insertion and removal of parentheses in a chain of additions and subtractions.

$a + (+ b - c)$ means that we are to first diminish b by c, and then to add the result to a. The parentheses about the binomial $(+ b - c)$ indicate that we are to treat the expression included as a whole, and the + sign before the parentheses indicates that we are to use the result of the operation $+ b - c$ to increase a.

Hence, $a + (+ b - c) \equiv a + b - c.$

Consider now, $a - (+ b - c)$.

The inclusion of $+b-c$ in parentheses means that we are first to subtract c from b and then to take the result from a.

If we were first to take b from a, that is, to find the difference $a-b$, we would take away too much from a by the quantity c. Hence, we must increase the remainder by a value equal to c; that is, we must add c to the result.

Therefore, $a-(+b-c)\equiv a-b+c$.

7. Principle I.

(i.) *Parentheses preceded by a + sign may be removed without altering the signs of operation of the separate numbers removed.*

(ii.) *Parentheses preceded by a − sign may be removed, providing the signs of operation preceding all of the numbers removed be changed from + to −, or from − to +.*

8. Since the proof of any identity establishes the truth of its converse, we may state

Principle II.

(i.) *A chain of additions and subtractions may be enclosed within parentheses preceded by the sign of operation for addition + without making any alteration in the signs of operation of the numbers enclosed.* (Associative Law for addition. See Chapter III. § 4.)

(ii.) *A chain of additions and subtractions may be enclosed within parentheses preceded by the sign of operation for subtraction − providing the signs of operation of all of the numbers introduced be changed from + to −, or from − to +.*

9. Since the reasoning above does not depend at all upon the lengths of the chains of additions and subtractions removed or inserted, the principles hold for any chains of additions and subtractions.

E. g. **Inclusion within Parentheses**

Preceded by Sign +	**Preceded by Sign −**
$a-b+c-d-e\equiv a-b+(c-d-e)$	$a-b+c-d-e\equiv a-b+c-(d+e)$
or $\equiv(a-b+c)-d-e.$	or $\equiv a-(b-c+d+e)$
	or $\equiv a-(b-c+d)-e.$

10. When numbers are included within several sets of parentheses, one set within another, the student will find it convenient in certain cases to begin by removing the innermost parentheses first. In other cases it will be better to work with the outer parentheses first.

11. Any change in the form of an expression tending to lessen the number of indicated operations is called a **reduction.**

Ex. 3. Reduce $7-\{4+[2-(8-5)]\}$.

Method I	Method II
Removing inner parentheses first.	Removing outer parentheses first.
$7-\{4+[2-(8-5)]\}\equiv 7-\{4+[2-8+5]\}$	$7-\{4+[2-(8-5)]\}\equiv 7-4-[2-(8-5)]$
$\equiv 7-\{4+2-8+5\}$	$\equiv 7-4-2+(8-5)$
$\equiv 7-4-2+8-5$	$\equiv 7-4-2+8-5$
$\equiv 4.$	$\equiv 4.$

Exercise IV. 1

Find the numerical values of the following expressions:

1. $5+(4-2)+3-(6-3)$.
2. $10-(9-8)-(7-6)$.
3. $(\overline{6-4)+4}-\overline{4-(6-4})$.
4. $12-(2-\overline{4-2})+(2+\overline{4+2})$.
5. $5-(2+\overline{6-3})+7-\overline{8-4}$.
6. $6-\{2+(10-\overline{6+1})\}$.
7. $11-[8-(10-\overline{9-6})]$.
8. $12-\{9-[4+(2-6)]+2\}$.
9. $3+[4-(5-\overline{6-5})+4]-3$.
10. $[15+(9-\overline{6-3})]-[15-(9+\overline{6-3})]$.
11. $20-\{(20-5)-[20-10+(20-15)]\}$.
12. $18-[15-\overline{9-4}-(11-2)-1]$.
13. $11-[6+\overline{8-5}-(12-7)-\overline{8-5}]$.
14. $[21-(7-\overline{8-5})]-[21+(7-\overline{5+8})]$.
15. $24-\{9-(10-6)-[24-\overline{17+5}-(6-4)]\}$.

I. Addition

12. Since positive and negative numbers enter into algebra, we must so extend our idea of addition as to be able to admit of uniting positive and negative numbers in one "sum."

13. As in arithmetic, numbers which enter into a sum are called **summands.**

14. **Subtraction** is the operation by which, when the sum of two expressions is known and one of them is given, the other may be found.

Subtraction may be regarded as the operation which is the inverse of addition, or the process which "undoes" addition.

15. The known sum is called the **minuend,** the given expression to be subtracted the **subtrahend,** and the expression or number to be found is called the **difference** or **remainder.**

The terms *minuend* and *subtrahend* are used in algebra in the same sense as in arithmetic. In arithmetic addition always produces an increase, subtraction a decrease.

16. In algebra, *addition*, *sum*, and *difference* have each a more extended meaning. On account of the introduction of the idea of positive and negative numbers, an "addition" may produce either an increase or a decrease in numerical value; a "subtraction" may produce either a decrease or an increase in numerical value.

17. Addition suggests taking a step forward; subtraction suggests taking a step backward. (See Chapter III. §§ 7, 13.)

E. g. In $5 + 3 = 8$, the second term, $+3$, may be regarded as the step "forward" from 5 to 8, while in $5 - 3 = 2$, -3 may be regarded as the step "backward" from 5 to 2.

18. In the addition of algebraic numbers the two following conditions may arise:

(*a*.) The numbers to be added may both have the same quality, that is be

Both Positive or Both Negative.

Ex. 1. To $^{+}4$ add $^{+}6$.

Here both summands are positive numbers. Consequently 4 positive units, when united by addition with 6 positive units, will produce a total of $(4 + 6)$ positive units, or 10 positive units, which may be expressed as follows:

$$^{+}4 + {}^{+}6 \equiv {}^{+}(4 + 6) \equiv {}^{+}10.$$

Ex. 2. To $^{-}5$ add $^{-}7$.

Here both numbers are negative. By addition, 5 negative units taken with 7 negative units produce a combined result of $(5 + 7)$ negative units; that is, in symbols:

$$^{-}5 + {}^{-}7 \equiv {}^{-}(5 + 7) \equiv {}^{-}12.$$

19. By referring to our scale of extended number it may be seen that to take a step of 4 positive units, and then immediately another step of 6 positive units, amounts to taking in all a single step of 10 positive units. (See Ex. 1.)

But to take successively negative steps of 5 and 7 units respectively amounts to taking a single step of 12 negative units. (See Fig. 1; also Ex. 2, above.)

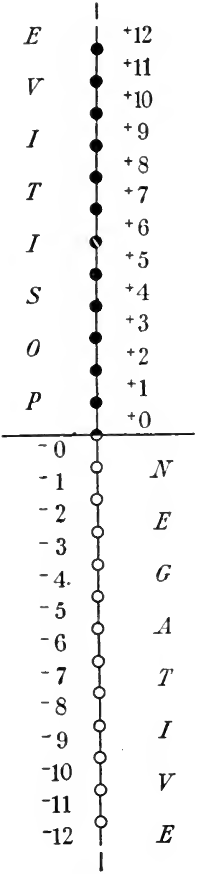


Fig. 1.

20. We may show that the principles applied above may be extended to all positive and negative numbers, by representing by a and b any arithmetic numbers whatsoever, or in symbols as below:

$$\text{(i.)}\quad {}^{+}a + {}^{+}b \equiv {}^{+}(a + b).$$
$$\text{(ii.)}\quad {}^{-}a + {}^{-}b \equiv {}^{-}(a + b).$$

Proof of (i.):

$${}^{+}a + {}^{+}b \equiv {}^{+}(a + b).$$

The positive units represented by ${}^{+}b$ taken together with the positive units represented by ${}^{+}a$ form a combination or group of positive units represented by ${}^{+}(a + b)$.

(ii.) may be proved by similar reasoning.

21. We may now state the following

Principle: *The sum of two algebraic numbers of like sign is an algebraic number of the same sign. It may be found by adding arithmetically the absolute values of the two numbers entering into it.*

(*b.*) The numbers to be added may be of opposite quality, say

One Positive and the Other Negative.

22. Abstract numbers are those which stand alone. They may be thought of as "unnamed numbers," that is, as number names, such as 4, 5, 10, etc., taken by themselves without reference to any particular objects.

23. Concrete numbers are those formed by applying the number names to particular things.

Concrete numbers are "named numbers," as 5 oranges, 4 days, etc.

Positive and negative numbers are "named numbers" and behave like concrete numbers.

As things unlike in kind cannot be "added" or "united" to represent any number of things of either kind alone, so positive and negative numbers, as such, are to be looked upon as being entirely separate and distinct, but with this conditional difference always, that each "offsets" or "destroys" the effect produced by the other upon any particular number or quantity.

24. Hence, positive and negative numbers equal in absolute value, combined by addition, may be neglected in a series of additions and subtractions since together they produce no change in the total result. If either occurred alone, it would produce a change, but both together act in such a way as to "oppose" each other.

E.g. Since $^{+}10 + {}^{-}10 \equiv 0$, $+ {}^{+}15 + {}^{+}10 + {}^{-}10 \equiv {}^{+}15$.
Also, since $- {}^{+}8 - {}^{-}8 \equiv 0$, $+ {}^{+}12 - {}^{+}8 - {}^{-}8 \equiv {}^{+}12$.
and in general $+ {}^{+}a + {}^{-}a \equiv 0$, also $- {}^{+}a - {}^{-}a \equiv 0$.

Ex. 3. To $^{+}9$ add $^{-}2$.

If instead of saying "added to," we use "taken together with," in the addition of algebraic numbers of opposite quality, the student will find that the idea suggested does not involve the difficulty which is frequently encountered because of the idea of an increase which is commonly associated with the word "addition."

When combined together into one whole, the two negative units reduce the number of positive units from 9 to 7.

Hence the result of the combination is 7 positive units, and we may write

$$^{+}9 + {}^{-}2 \equiv {}^{+}(9 - 2) \equiv {}^{+}7.$$

Ex. 4. To $^{-}7$ add $^{+}4$.

The effect of 4 positive units in combination with 7 negative units is to reduce the number of effective negative units by 4. Hence the result of the "addition" or combination is 3 negative units. We may write

$$^{-}7 + {}^{+}4 \equiv {}^{-}(7 - 4) \equiv {}^{-}3.$$

25. It will be noticed that in the examples above the number which is greater in absolute value is the one which determines the quality of the result.

26. It may be observed that the reason why the operation which appears to be subtraction comes under the head of addition is the fact that we are operating with numbers of opposite quality, that is, with positive and negative numbers. The "apparent" subtraction is not strictly addition itself, but belongs rather to the subsequent reduction.

27. The **Commutative Law for Addition,** that is, *the value of a sum does not depend upon the order of adding its parts,* may be shown to hold for both positive and negative numbers.

In symbols $^{+}\boldsymbol{a} + {}^{+}\boldsymbol{b} \equiv + {}^{+}\boldsymbol{b} + {}^{+}\boldsymbol{a}$

Also $^{+}\boldsymbol{a} + {}^{-}\boldsymbol{b} \equiv + {}^{-}\boldsymbol{b} + {}^{+}\boldsymbol{a}.$

E. g. $^{+}6 + {}^{+}3 \equiv {}^{+}3 + {}^{+}6.$ $^{+}7 + {}^{-}5 \equiv {}^{-}5 + {}^{+}7.$

$^{-}7 + {}^{-}4 \equiv {}^{-}4 + {}^{-}7.$ $^{-}2 + {}^{+}8 \equiv {}^{+}8 + {}^{-}2.$

28. From the consideration of the preceding examples we may state the following General Principles:

Principle I.

(i.) *The sum of two or more positive numbers is a positive number whose absolute value is found by taking the sums of the arithmetic values of the positive numbers entering into it.*

In symbols $^{+}\boldsymbol{a} + {}^{+}\boldsymbol{b} \equiv {}^{+}(\boldsymbol{a} + \boldsymbol{b}).$

E. g. $^{+}5 + {}^{+}8 \equiv {}^{+}(5 + 8) \equiv {}^{+}13.$

(ii.) *The sum of two or more negative numbers is a negative number whose absolute value is found by taking the sum of the arithmetic values of the negative numbers entering into it.*

In symbols $^{-}\boldsymbol{a} + {}^{-}\boldsymbol{b} \equiv {}^{-}(\boldsymbol{a} + \boldsymbol{b}).$

E. g. $^{-}7 + {}^{-}3 \equiv {}^{-}(7 + 3) \equiv {}^{-}10.$

29. Principle II. *The algebraic sum of several positive and negative numbers is found by taking the arithmetic difference between the absolute value of the sum total of the positive numbers and the absolute value of the sum total of the negative numbers, and it agrees in quality with the greater sum total.*

In symbols $^{+}\boldsymbol{a} + {}^{-}\boldsymbol{b} \equiv {}^{+}(\boldsymbol{a} - \boldsymbol{b})$, for $a > b$,

Or $\equiv {}^{-}(\boldsymbol{b} - \boldsymbol{a})$, for $a < b$.

Ex. 5. $$\begin{aligned} {}^{+}5 + {}^{+}2 + {}^{-}6 + {}^{-}1 + {}^{-}3 + {}^{+}8 &\equiv {}^{+}(5 + 2 + 8) + {}^{-}(6 + 1 + 3) \\ &\equiv {}^{+}(15) + {}^{-}(10) \\ &\equiv {}^{+}5. \end{aligned}$$

Ex. 6. $$\begin{aligned} {}^{+}3 + {}^{-}6 + {}^{-}12 + {}^{+}4 + {}^{-}3 &\equiv {}^{+}(3 + 4) + {}^{-}(6 + 12 + 3) \\ &\equiv {}^{+}(7) + {}^{-}(21) \\ &\equiv {}^{-}14. \end{aligned}$$

30. By giving concrete meanings to the positive and negative numbers appearing above, for example, letting the positive numbers stand for items of income and the negative numbers for items of expense, it will appear that, taken together, the balance will be $5 in favor of the income in the first example, while in the second example there remains a total unpaid debt of $14.

EXERCISE IV. 2

Simplify the following expressions:

1. ${}^{+}2 + {}^{+}5$.
2. ${}^{+}4 + {}^{+}3$.
3. ${}^{+}6 + {}^{+}8$.
4. ${}^{+}9 + {}^{+}6$.
5. ${}^{-}5 + {}^{-}7$.
6. ${}^{-}8 + {}^{-}10$.
7. ${}^{+}7 + {}^{-}4$.
8. ${}^{+}10 + {}^{-}2$.
9. ${}^{+}11 + {}^{-}9$.
10. ${}^{+}3 + {}^{-}11$.
11. ${}^{+}1 + {}^{-}6$.
12. ${}^{+}5 + {}^{-}13$.
13. ${}^{-}6 + {}^{+}14$.
14. ${}^{+}7 + {}^{-}17$.
15. ${}^{+}12 + {}^{-}12$.
16. ${}^{+}13 + {}^{-}14$.
17. ${}^{+}15 + {}^{+}16$.
18. ${}^{-}17 + {}^{-}8$.
19. ${}^{-}14 + {}^{-}20$.
20. ${}^{-}4 + {}^{-}19$.
21. ${}^{-}16 + {}^{-}3$.
22. ${}^{-}19 + {}^{+}1$.
23. ${}^{-}18 + {}^{+}16$.
24. ${}^{+}19 + {}^{+}19$.
25. ${}^{+}16 + {}^{-}10 + {}^{-}4$.
26. ${}^{+}11 + {}^{+}18 + {}^{-}19$.
27. ${}^{+}1 + {}^{-}3 + {}^{+}20$.
28. ${}^{+}7 + {}^{-}13 + {}^{+}6$.
29. ${}^{+}12 + {}^{-}5 + {}^{-}7$.
30. ${}^{+}13 + {}^{-}14 + {}^{+}15$.
31. ${}^{+}3 + {}^{-}6 + {}^{+}10$.
32. ${}^{+}9 + {}^{-}17 + {}^{-}20$.

II. SUBTRACTION

31. (i.) **Subtraction of Positive Numbers.**

To add two numbers in arithmetic is to take a "positive" step; that is, a step "upward" along the number series in the direction of the increasing values 1, 2, 3, etc.

Hence, to subtract a positive number results in a change of position along the number series in a "downward" direction; that is, in the direction of the decreasing values 8, 7, 6, 5, etc.

A decrease of positive values occurs also whenever we take a negative step, which is one taken in the direction 4, 3, 2, 1. (See Chapter III, § 13.)

32. "Taking away," or subtracting, a positive step amounts to the operation of adding a negative step.

E. g. Using our scale of extended number, we may understand subtracting or "taking away" $^{+}6$ as meaning counting "backward" or "downward" in the direction $^{+}10$, $^{+}9$, $^{+}8$, $^{+}7$, etc., for a distance of 6 units, beginning with some point such as 0, until we finally reach $^{-}6$. By adding a negative step of 6 units we shall arrive at the same point. Hence, we may regard the subtraction of 6 positive units as amounting to the addition of 6 negative units, and write

$$-{}^{+}6 \equiv +{}^{-}6,$$

and in general,

$$-{}^{+}a \equiv +{}^{-}a.$$

A movement "downward" along the number series in the direction $^{+}10$, $^{+}9$, $^{+}8$, $^{+}7$, etc., which takes place among the positive numbers alone, that is, wholly "above" zero, corresponds to an arithmetic subtraction.

33. (ii.) **Addition of Negative Numbers.**

A step "downward," which takes place "below" zero, that is, among the negative numbers alone, amounts to an addition of negative values.

A negative step is taken in a "downward" direction, that is, in the direction of the diminishing values $^{+}4$, $^{+}3$, $^{+}2$, $^{+}1$.

Hence, removing or subtracting a "downward" step amounts to a change in position in an "upward" or positive direction; that is, the subtraction of a negative amounts to the addition of a positive step or number.

That is,

$$-{}^{-}a \equiv +{}^{+}a.$$

E. g.

$$-{}^{-}4 \equiv +{}^{+}4.$$

34. CORRESPONDENCE BETWEEN THE ADDITION AND SUBTRACTION OF POSITIVE AND NEGATIVE NUMBERS

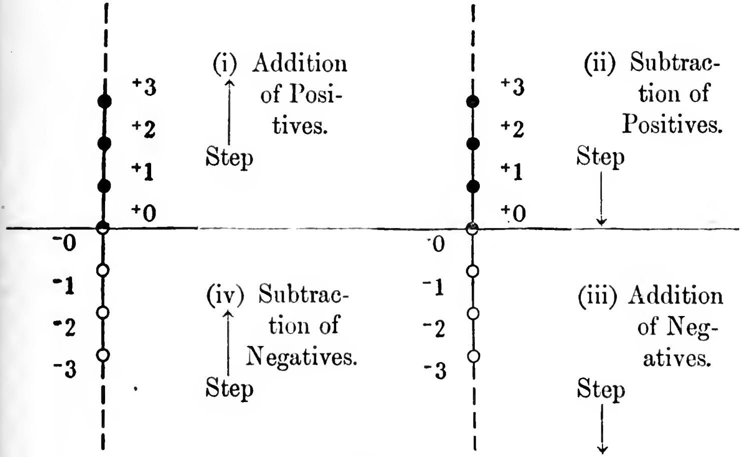


FIG. 2.

Reversing a "downward" step changes it into an "upward" step. Hence, **subtracting a negative amounts to adding a positive;** that is:

$$-\,^{-}a \equiv +\,^{+}a. \quad \text{(See Fig. 2.)}$$

Also, reversing an "upward" step changes it into a "downward" step. Hence, **subtracting a positive amounts to adding a negative,** or:

$$-\,^{+}a \equiv +\,^{-}a. \quad \text{(See Fig. 2.)}$$

35. Hence, for positive and negative numbers we have the following

Principle: *Every operation of subtraction of positive or of negative numbers may be replaced by an equivalent addition; that is, by the addition of a number equal in absolute value, but of reversed quality.*

Hence, *to subtract one number from another, reverse the sign of quality of the subtrahend from + to — or from — to +, and then proceed as in addition.*

In symbols: $^{+}\boldsymbol{a} - {}^{+}\boldsymbol{b} \equiv {}^{+}a + {}^{-}\boldsymbol{b}$.

36. The principles for the removal and insertion of parentheses apply also to positive and negative numbers.

It follows that the Laws of Commutation and Association for successive additions hold also for successive subtractions, or for a chain containing both additions and subtractions.

Ex. 1. From ${}^{+}7$ subtract ${}^{+}2$.

We may express the operation by writing ${}^{+}7 - {}^{+}2$.

Replacing the operation of subtraction by that of addition, reversing at the same time the sign of quality before the 2, we have ${}^{+}7 - {}^{+}2 \equiv {}^{+}7 + {}^{-}2$.

When combined with 7 positive units, 2 negative units reduce the number of positive units by 2, producing as a result of the combination or "addition," 5 positive units.

Hence, the expression above reduces to ${}^{+}(7 - 2) \equiv {}^{+}5$.

We may obtain the same result by another method as follows:

Since 7 positive units amount to 5 positive units increased by 2 positive units, we may write

$$ {}^{+}7 - {}^{+}2 \equiv {}^{+}5 + {}^{+}2 - {}^{+}2. $$

Since adding and subtracting the same number to or from ${}^{+}5$ produces no change in the final result, that is, since $+ {}^{+}2 + {}^{-}2 \equiv 0$, we may write

$$ {}^{+}5 + {}^{+}2 - {}^{+}2 \text{ as } {}^{+}5. $$

Ex. 2. From ${}^{+}3$ subtract ${}^{+}8$.

We may express the subtraction by writing ${}^{+}3 - {}^{+}8$. Replacing the operation of subtraction by that of addition, reversing at the same time the sign of quality of the subtrahend, we have ${}^{+}3 - {}^{+}8 \equiv {}^{+}3 + {}^{-}8$.

Since in combination the 3 positive units reduce the number of negative units from 8 to 5, we may write ${}^{+}3 + {}^{-}8$ as ${}^{-}(8 - 3) \equiv {}^{-}5$.

We may also employ the following method:

Since subtracting 8 positive units in one operation amounts to subtracting successively 3 positive units and 5 positive units in two separate operations, we may write both as

$$ {}^{+}3 - {}^{+}8 \equiv {}^{+}3 - {}^{+}3 - {}^{+}5. $$

Observing that $\quad {}^{+}3 - {}^{+}3 \equiv 0,$

we have as a result ${}^{+}3 - {}^{+}8 \equiv - {}^{+}5 \equiv + {}^{-}5.$

Ex. 3. From ${}^{+}6$ subtract ${}^{-}4$.

We may indicate the operation by writing ${}^{+}6 - {}^{-}4$. Since taking away a negative amounts to adding a positive, we may replace the indicated subtraction by an equivalent addition, and write

$$ {}^{+}6 - {}^{-}4 \equiv {}^{+}6 + {}^{+}4 \equiv {}^{+}(6 + 4) \equiv {}^{+}10. $$

Ex. 4. From ${}^{+}5$ subtract ${}^{-}9$.

Observe that, since the subtraction of a negative amounts to the addition of a positive, we may from the indicated subtraction ${}^{+}5 - {}^{-}9$, obtain

$${}^{+}5 - {}^{-}9 \equiv {}^{+}5 + {}^{+}9 \equiv {}^{+}(5 + 9) \equiv {}^{+}14.$$

Ex. 5. From ${}^{-}12$ subtract ${}^{-}3$.

Writing first as an indicated subtraction, and then transforming into an equivalent addition, we have ${}^{-}12 - {}^{-}3 \equiv {}^{-}12 + {}^{+}3$.

Since in combination with the 12 negative units the 3 positive units diminish the number to 9 negative units, we may write

$${}^{-}12 + {}^{+}3 \text{ as } {}^{-}(12 - 3) \equiv {}^{-}9.$$

From another point of view, 12 negative units may be obtained by combining 9 negative units with 3 negative units.

Hence ${}^{-}12 - {}^{-}3 \equiv {}^{-}9 + {}^{-}3 - {}^{-}3$; and since successively adding and subtracting 3 negative units produce no final change in the value of the original 9 negative units, the second member of the identity reduces to ${}^{-}9$.

Ex. 6. From ${}^{-}4$ subtract ${}^{-}11$.

Replacing the indicated subtraction by an equivalent addition, we have ${}^{-}4 - {}^{-}11 \equiv {}^{-}4 + {}^{+}11$.

In combination with the 11 positive units the 4 negative units produce a decrease in number to 7 positive units. Hence, ${}^{-}4 + {}^{+}11$ may be written as ${}^{+}(11 - 4) \equiv {}^{+}7$.

From another point of view, the subtraction of 11 negative units in one operation amounts to the two separate operations of subtracting 4 and 7 negative units successively. Hence, we may write ${}^{-}4 - {}^{-}11 \equiv {}^{-}4 - {}^{-}4 - {}^{-}7$.

Since the subtraction of 4 negative units from 4 negative units produces zero, we have as a remainder the expression $- {}^{-}7$, which may be transformed into the equivalent expression $+ {}^{+}7$.

Hence ${}^{-}4 - {}^{-}11 \equiv + {}^{+}7$, as above.

Ex. 7. From ${}^{-}1$ subtract ${}^{+}14$.

The subtraction of 14 positive units may be looked upon as amounting to the addition of 14 negative units.

Hence ${}^{-}1 - {}^{+}14 \equiv {}^{-}1 + {}^{-}14$.

In combination by addition, 14 negative units and 1 negative unit amount to 15 negative units.

Hence ${}^{-}1 + {}^{-}14$ may be expressed as ${}^{-}(1 + 14) \equiv {}^{-}15$.

Ex. 8. From ${}^{-}16$ subtract ${}^{+}10$.

Observe that the subtraction of 10 positive units is equivalent to the addition of 10 negative units. We may write ${}^{-}16 - {}^{+}10 \equiv {}^{-}16 + {}^{-}10$.

We have an expression for 16 negative units increased by 10 negative units, resulting in 26 negative units. Hence, $^-16 + {}^-10$ may be written as $^-(16 + 10) \equiv {}^-26$.

37. Inequality. To agree with the ordinary arithmetic notions of inequality, mathematicians have agreed to call one algebraic number, a, greater or less than another, b, according as the reduced value of $a - b$ is positive or negative.

From the definition it appears that any positive number (represented by ^+a) must be considered greater than any negative number (represented by ^-b) since

$$^+a - {}^-b \equiv {}^+a + {}^+b \equiv {}^+(a + b)$$

which is a positive number.

From the same point of view 0 is to be regarded as being greater than any negative number, ^-b, since

$$0 - {}^-b \equiv 0 + {}^+b \equiv {}^+b$$

which is a positive number.

Hence, corresponding to the expression "greater than 0," there follows directly, by application of the Principle of No Exception, also the idea "less than 0."

38. Again, from our previous definition, one negative number, ^-c, is to be regarded as being greater than another negative number, ^-d, according as the reduced value of

$$^-c - {}^-d \equiv {}^-c + {}^+d$$

is positive or negative.

The reduced value of $^-c + {}^+d$ will be negative if c be numerically greater than d, and positive if d be numerically greater than c.

Ex. 9. Compare $^-3$ and $^-4$.

$$^-3 - {}^-4 \equiv {}^-3 + {}^+4 \equiv {}^+1, \text{ a positive number.}$$

Hence $^-3$ is greater than $^-4$.

Ex. 10. Compare $^-5$ and $^-1$.

From the definition we have

$$^-5 - {}^-1 \equiv {}^-5 + {}^+1 \equiv {}^-4, \text{ a negative number.}$$

Therefore $^-5$ is less than $^-1$.

39. Using the symbol ∞, "infinity," to represent any number which is numerically greater than any assignable number, it follows from the reasoning above that the series of extended number may be regarded as being arranged in order of increasing magnitude, from left to right, as follows:

$$^{-}\infty, \cdots\cdot, ^{-}3, ^{-}2, ^{-}1, \pm 0, ^{+}1, ^{+}2, ^{+}3, \cdots\cdot, ^{+}\infty.$$

Observe that 0 may be regarded as belonging to both the positive and negative parts of the series; it is all that they have in common. (See Chapter II. § 23.)

EXERCISE IV. 3

Perform the following indicated subtractions:

1. From $^{+}6$ subtract $^{+}4$.
2. From $^{+}8$ subtract $^{+}1$.
3. From $^{+}10$ subtract $^{+}7$.
4. From $^{+}12$ subtract $^{+}5$.
5. From $^{-}9$ subtract $^{+}3$.
6. From $^{-}4$ subtract $^{+}2$.
7. From $^{-}7$ subtract $^{+}8$.
8. From $^{-}11$ subtract $^{-}12$.
9. From $^{-}5$ subtract $^{-}11$.
10. From $^{-}2$ subtract $^{+}14$.
11. From $^{-}3$ subtract $^{+}19$.
12. From $^{-}14$ subtract $^{+}9$.
13. From $^{-}17$ subtract $^{+}15$.
14. From $^{+}13$ subtract $^{-}20$.
15. From $^{-}18$ subtract $^{+}16$.
16. From $^{-}1$ subtract $^{+}17$.
17. From $^{-}15$ subtract $^{+}18$.
18. From $^{+}19$ subtract $^{-}19$.
19. From $^{-}16$ subtract $^{+}16$.
20. From $^{-}15$ subtract $^{+}14$.
21. From $^{-}9$ subtract $^{+}11$.
22. From $^{-}12$ subtract $^{+}13$.
23. From $^{+}20$ subtract $^{-}17$.
24. From $^{+}18$ subtract $^{+}20$.
25. From $^{-}6$ subtract $^{+}20$.
26. From $^{-}7$ subtract $^{+}18$.
27. From $^{-}8$ subtract $^{+}14$.
28. From $^{-}9$ subtract $^{+}13$.
29. From $^{+}10$ subtract $^{-}12$.
30. From $^{-}11$ subtract $^{+}11$.

Perform the following indicated additions and subtractions:

31. $^{+}2 + ^{+}3 - ^{+}4$.
32. $^{+}1 - ^{+}5 - ^{+}15$.
33. $^{+}5 - ^{+}9 + ^{+}16$.
34. $^{+}5 + ^{-}9 - ^{-}16$.
35. $^{+}5 + ^{+}9 - ^{+}16$.
36. $^{+}3 - ^{+}7 - ^{+}9$.
37. $^{+}11 - ^{+}15 - ^{-}2$.
38. $^{+}17 - ^{-}3 + ^{-}19$.
39. $^{+}10 + ^{-}19 - ^{-}17$.
40. $^{+}20 - ^{-}13 + ^{+}6$.
41. $^{+}5 - ^{-}5 + ^{-}8$.
42. $^{+}13 - ^{-}13 - ^{-}19$.
43. $^{+}1 - ^{+}10 - ^{+}19$.
44. $^{+}14 - ^{-}8 + ^{+}8$.
45. $^{+}16 + ^{-}18 - ^{+}9$.
46. $^{+}2 - ^{-}4 + ^{+}6 - ^{-}8$.
47. $^{+}20 - ^{-}18 + ^{-}9 + ^{+}6$.
48. $^{+}35 - ^{-}15 + ^{-}25$.
49. $^{+}40 - ^{-}20 - ^{+}30$.
50. $^{+}75 + ^{-}50 - ^{+}25$.

Find the values of the following expressions when the given values are substituted for the letters appearing in them.

If $a = {}^{+}1$, $b = {}^{-}5$, $c = {}^{-}3$, $d = {}^{+}4$.

51. $a + b + c + d$.

52. $(a + b) - (c + d)$.

53. $a - (b + c) + d$.

54. $a + (b - c) - d$.

55. $a - [b + (c - d)]$.

56. $a - [b - (c + d)]$.

57. $-[(a + b) - c] - d$.

58. $({}^{+}6 - a) - ({}^{+}6 - b)$.

59. $(c - {}^{+}3) - (d - {}^{-}4)$.

60. $(b - {}^{+}5) - (c - {}^{-}3)$.

CHAPTER V

MULTIPLICATION AND DIVISION OF ALGEBRAIC NUMBERS

I. MULTIPLICATION

1. THE multiplication of abstract numbers is first defined in arithmetic as the taking of one number as many times as there are units in another.

E.g. To multiply 4 by 3, we take as many 4's as there are units in 3. We have $4 \times 3 = 4 + 4 + 4 = 12$.

2. In algebra, as in arithmetic, we call the number multiplied the **multiplicand,** the number which multiplies it the **multiplier,** and the result of the operation the **product.**

Whenever the first of two numbers such as 2×3 is regarded as the multiplier, it is customary to read the product as "2 times 3," while if 3 is regarded as the multiplier and 2 as the multiplicand, we may say "2 multiplied by 3."

3. Our first idea of multiplication is that it is an abbreviated addition.

From this point of view the multiplier must, in the original sense of the word, be the result of counting; that is, it must be a positive whole number.

The multiplicand may be any number previously defined, that is, it may be abstract or concrete, positive or negative, or even zero, but **the multiplier must be an abstract number.**

In a product consisting of two abstract numbers, the one at the right is usually regarded as the multiplier. However, since we speak commonly of 2 books, 3 apples, etc., mentioning the multiplier first, mathematicians find it convenient to arrange a given product containing numbers and letters so that the numerical parts shall occur in the first place.

Repeating a quantity does not alter its nature; hence it follows that **a product must be of the same nature as the multiplicand.** Hence the product will be an abstract or a concrete number, according as the multiplicand is an abstract or a concrete number.

Our first conception of a product can have no meaning when fractional, negative, or zero multipliers are considered, for from the original definition we can multiply by positive whole numbers only. Therefore we apply the Principle of No Exception, and declare that since negative numbers have the forms of differences, multiplications with them may be performed exactly as with real or actual differences.

4. To allow of carrying out the operation of multiplication when the multiplier is a fraction or a negative number, it becomes necessary to give an **extended definition of multiplication:**

To multiply one number by a second number is to do to the first number that which must be done to the positive unit $^{+}1$ *to obtain the second.*

This does not contradict our first definition as given in arithmetic, but includes it in the more general statement.

E.g. $3 = +1 + 1 + 1$.

Hence, doing the same thing to 4 that was done to +1 to obtain 3, we may write as the product of 4 and 3

$$4 \times 3 = +4 + 4 + 4 = 12. \quad \text{(See § 1.)}$$

5. This general definition may be regarded as including the multiplication of fractions. Thus, if we wish to multiply $\frac{2}{5}$ by $\frac{3}{7}$, we must do to $\frac{2}{5}$ that which was done to unity to obtain $\frac{3}{7}$; that is, we must divide $\frac{2}{5}$ into seven equal parts and take three of these equal parts as summands. Each of the equal parts of $\frac{2}{5}$ will be $\frac{2}{5 \cdot 7}$, and by taking three of these parts as summands we shall have

$$\left(\frac{2}{5}\right)\left(\frac{3}{7}\right) = \frac{2}{5 \cdot 7} + \frac{2}{5 \cdot 7} + \frac{2}{5 \cdot 7} = \frac{2 \cdot 3}{5 \cdot 7} = \frac{6}{35}.$$

6. Just as, in arithmetic, numbers result from repetitions of unity, so positive and negative numbers may be regarded as resulting from repetitions of *the unit of positive numbers* $^{+}1$, and *the unit of negative numbers* $^{-}1$.

We may take as **quality units** the numbers $^{+}1$ and $^{-}1$.

7. In the multiplication of positive and negative numbers we may have

I. The Multiplier Positive.

Ex. 1. Multiply $^{+}5$ by $^{+}3$.

By the extended definition of multiplication the product may be obtained by operating upon the multiplicand $^{+}5$ in exactly the same way that we must operate upon the unit of positive numbers $^{+}1$ to obtain the multiplier $^{+}3$.

From the definition of a positive number, $^{+}3 \equiv {}^{+}1 \times 3$.

That is, we obtain the multiplier $^{+}3$ by multiplying the unit of positive numbers $^{+}1$ by 3, retaining its quality as a positive number.

Hence to multiply $^{+}5$ by $^{+}3$ we retain the quality of the multiplicand $^{+}5$ and multiply its absolute value by 3.

That is, $^{+}5 \times {}^{+}3 \equiv {}^{+}(5 \times 3) \equiv {}^{+}15$.

Ex. 2. Multiply $^{-}2$ by $^{+}4$.

Reasoning as before, the product may be obtained by multiplying the absolute value of the multiplicand by 4, retaining its quality as a negative number.

Hence, $^{-}2 \times {}^{+}4 \equiv {}^{-}(2 \times 4) \equiv {}^{-}8$.

8. In the multiplication of positive and negative numbers we may have

II. The Multiplier Negative.

Ex. 3. Multiply $^{+}6$ by $^{-}3$.

By the extended definition of multiplication, the product may be obtained by performing upon the multiplicand $^{+}6$ exactly those operations which must be performed upon the unit of positive numbers $^{+}1$ to produce the multiplier $^{-}3$.

From the definition of a negative number we have

$$^{-}3 \equiv + {}^{-}1 + {}^{-}1 + {}^{-}1, \text{ in terms of negative units.}$$

or

$$^{-}3 \equiv - {}^{+}1 - {}^{+}1 - {}^{+}1, \text{ in terms of positive units.}$$

That is, $^{-}3$ may be obtained from the unit of positive numbers by *reversing the quality of the positive unit and multiplying its absolute value by* 3.

It follows that, to obtain the desired product of $^{+}6$ and $^{-}3$, we may *reverse the quality of the positive unit and multiply its absolute value by* 3.

Hence, $^{+}6 \times {}^{-}3 \equiv {}^{-}(6 \times 3) \equiv {}^{-}18$.

Ex. 4. Multiply $^{-}7$ by $^{-}5$.

Reasoning as above, we may reverse the sign of quality of the multiplicand $^{-}7$ and multiply its absolute value by 5.

Hence $^{-}7 \times {}^{-}5 \equiv {}^{+}(7 \times 5) \equiv {}^{+}35$.

9. It should be observed that, in each of the examples above, *the sign of quality of the product is positive or negative according as the signs of quality of the multiplicand and multiplier are like or unlike.*

10. We will now show that this *Law of Signs* holds for all positive and negative numbers.

Representing by a and b any two arithmetic numbers, that is, integral values,

$$\text{(i.)}\ {}^{+}\boldsymbol{a} \times {}^{+}\boldsymbol{b} \equiv {}^{+}(\boldsymbol{ab}), \qquad \text{(iii.)}\ {}^{-}\boldsymbol{a} \times {}^{+}\boldsymbol{b} \equiv {}^{-}(\boldsymbol{ab}),$$
$$\text{(ii.)}\ {}^{-}\boldsymbol{a} \times {}^{-}\boldsymbol{b} \equiv {}^{+}(\boldsymbol{ab}), \qquad \text{(iv.)}\ {}^{+}\boldsymbol{a} \times {}^{-}\boldsymbol{b} \equiv {}^{-}(\boldsymbol{ab}).$$

Proofs of (i.) and (iii.):

To obtain ${}^{+}b$ from the unit of positive numbers ${}^{+}1$, we retain the quality of the unit and multiply its absolute value by b; it follows from the extended definition of multiplication that to multiply any number by ${}^{+}b$ we retain the quality of the number and multiply its absolute value by b.

$$\text{Hence, (i.)}\ {}^{+}a \times {}^{+}b \equiv {}^{+}(ab).$$
$$\text{(iii.)}\ {}^{-}a \times {}^{+}b \equiv {}^{-}(ab).$$

Proofs of (ii.) and (iv.):

To obtain ${}^{-}b$ from the unit of positive numbers ${}^{+}1$ we may change the sign of quality of the unit and multiply the result by b; accordingly, to multiply any number by ${}^{-}b$, we may reverse the quality of the number and multiply the result by b.

$$\text{Hence (ii.)}\ {}^{-}a \times {}^{-}b \equiv {}^{+}(ab).$$
$$\text{(iv.)}\ {}^{+}a \times {}^{-}b \equiv {}^{-}(ab).$$

The **Law of Signs** for the multiplication of two numbers or quantities may be stated as follows:

The product of two numbers having like quality signs is positive, and the product of two numbers having unlike quality signs is negative.

11. By examining identities (i.) to (iv.), § 10 above, it may be seen that *the signs of quality of both multiplicand and multiplier may be reversed without altering the sign of quality and numerical value of the product.*

E. g. Reversing the signs of multiplicand and multiplier in (i.), we obtain the multiplicand and multiplier in (ii.), but the quality of the product remains unaltered.

Reversing the signs of quality of multiplicand and multiplier in either (iii.) or (iv.) has the effect of an interchange of signs, giving as a result the left member of either (iv.) or (iii.), the quality and numerical value of the product remaining unaltered by the change.

12. Hence we have the **Commutative Law for Signs of Quality in Multiplication,** that is,

$$^{-}a \times {}^{+}b \equiv {}^{+}a \times {}^{-}b \equiv {}^{-}(ab).$$

It follows that, in establishing the Commutative Law for the multiplication of positive and negative numbers, we may establish it for the last form $^{-}(ab)$ only, using the absolute values of a and b.

Exercise V. 1

Simplify the following :

1. $^{+}3 \times {}^{+}2$.
2. $^{+}5 \times {}^{+}4$.
3. $^{+}7 \times {}^{+}3$.
4. $^{+}2 \times {}^{+}9$.
5. $^{+}9 \times {}^{+}5$.
6. $^{+}4 \times {}^{-}7$.
7. $^{+}6 \times {}^{-}11$.
8. $^{+}8 \times {}^{-}12$.
9. $^{+}10 \times {}^{-}13$.
10. $^{+}12 \times {}^{-}8$.
11. $^{-}11 \times {}^{-}10$.
12. $^{-}14 \times {}^{-}6$.
13. $^{-}4 \times {}^{-}15$.
14. $^{-}2 \times {}^{-}17$.
15. $^{-}15 \times {}^{-}3$.
16. $^{-}16 \times {}^{-}5$.
17. $^{+}6 \times {}^{-}20$.
18. $^{-}3 \times {}^{+}18$.
19. $^{-}10 \times {}^{-}19$.
20. $^{+}18 \times {}^{-}10$.
21. $^{+}15 \times {}^{+}1$.
22. $^{-}1 \times {}^{-}14$.
23. $^{+}19 \times {}^{-}4$.
24. $^{+}9 \times {}^{+}12$.
25. $^{-}7 \times {}^{+}9$.
26. $^{-}6 \times {}^{-}8$.
27. $^{-}20 \times {}^{-}15$.
28. $(^{+}2 \times {}^{+}5) + (^{-}5 \times {}^{+}6)$.
29. $(^{+}11 \times {}^{-}11) + (^{+}13 \times {}^{-}2)$.
30. $(^{+}5 \times {}^{-}16) - (^{+}4 \times {}^{+}3)$.
31. $(^{+}1 \times {}^{-}1) - (^{-}20 \times {}^{+}20)$.
32. $(^{-}1 \times {}^{+}3) - (^{+}12 \times {}^{+}12)$.
33. $(^{+}16 \times {}^{-}16) - (^{-}16 \times {}^{+}16)$.
34. $(^{-}8 \times {}^{-}7) - (^{-}7 \times {}^{+}8)$.
35. $(^{+}3 \times {}^{-}4) - (^{-}5 \times {}^{+}6)$.
36. $(^{+}7 \times {}^{+}10) - (^{+}8 \times {}^{+}11)$.
37. $(^{+}10 \times {}^{-}14) + (^{+}15 \times {}^{-}15)$.
38. $(^{+}13 \times {}^{+}20) - (^{+}20 \times {}^{-}14)$.
39. $(^{+}10 \times {}^{-}18) + (^{+}18 \times {}^{+}13)$.

If $a = {}^{+}2$, $b = {}^{-}3$, $x = {}^{-}5$, $y = {}^{+}4$, find the values of the following expressions :

40. $ab + xy$.
41. $ax + by$.
42. $ay - bx$.
43. $ab + a$.
44. $bx + x$.
45. $ab + y$.
46. $x - ab$.
47. $y - bx$.
48. $ab + bx + xy$.
49. $ax + ab + by$.

Find the value of $(a + b) \times c$, when

50. $a = {}^{+}2,\ b = {}^{+}6,\ c = {}^{+}3.$ 52. $a = {}^{+}4,\ b = {}^{-}5,\ c = {}^{-}7.$
51. $a = {}^{-}3,\ b = {}^{+}1,\ c = {}^{+}4.$

Find the value of $(a - b) \times c$ when

53. $a = {}^{-}1,\ b = {}^{+}4,\ c = {}^{-}8.$ 55. $a = {}^{+}10,\ b = {}^{-}10,\ c = {}^{+}10.$
54. $a = {}^{+}6,\ b = {}^{+}9,\ c = {}^{-}1.$

Find the value of $(a + b) \times (c + d)$ when

56. $a = {}^{-}1,\ b = {}^{-}2,\ c = {}^{+}3,\ d = {}^{+}4.$
57. $a = {}^{+}5,\ b = {}^{-}2,\ c = {}^{+}12,\ d = {}^{-}9.$

Find the value of $(a - b) \times (c - d)$ when

58. $a = {}^{+}3,\ b = {}^{+}5,\ c = {}^{-}2,\ d = {}^{-}4.$
59. $a = {}^{+}2,\ b = {}^{+}7,\ c = {}^{-}3,\ d = {}^{+}11.$

13. Whenever both the multiplicand and multiplier are abstract numbers, two fundamental laws hold also in multiplication. These are the Law of Commutation, affecting arrangement, and the Law of Association, affecting grouping. These laws are similar to the laws in addition having the same names. Thus:

(i.) *a times b* $\equiv$ *b times a.* Law of Commutation.
(ii.) *a times (b times c)* $\equiv$ *(a times b) times c.* Law of Association.

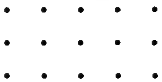
14. The Commutative Law for Multiplication

In a product of two abstract numbers, either number may be taken as the multiplier without affecting the value of the result.

Thus, in symbols, $a \times b \equiv b \times a.$

E. g. $5 \times 3 \equiv 3 \times 5.$

We may show that the law holds for the product of two particular numbers, say 5 and 3, by representing the number 5 by five points in a horizontal row, and constructing three rows, as follows:



Counting the entire set of dots, we may regard it as consisting of three groups of 5 dots each, written 5×3, or again as five groups of 3 dots each, written 3×5. Hence, we may assert that $5 \times 3 = 3 \times 5$.

If, instead of reasoning with particular numbers, we arrange b horizontal rows of a dots each, we shall arrive by similar reasoning at the result, that for all positive integral values of a and b

$$a \times b \equiv b \times a.$$

This is called the Law of Commutation for Multiplication.

It may be shown that the principle applies whenever there are three or more factors.

15. Continued Product.

The product of three or more numbers, a, b, c, and d, written $a \times b \times c \times d$, or $abcd$, is defined to be the number obtained by multiplying a by b, this result by c, and finally the last result by d.

If we represent any three arithmetic whole numbers by a, b, and c, we may indicate the continued product of three positive numbers by writing $(^+a)(^+b)(^+c)$.

(The following proof may be omitted when the chapter is read for the first time.)

By the definition of a continued product we are to understand $(^+a)(^+b)(^+c)$ as meaning that we are to first multiply ^+a by ^+b, and this product by ^+c.

$(^+a)(^+b)(^+c) \equiv [(^+a)(^+b)](^+c)$. By definition of multiplication.

But $(^+a)(^+b) \equiv {}^+(ab)$.

Hence $(^+a)(^+b)(^+c) \equiv [^+(ab)](^+c)$.

Regarding $^+(ab)$ as a single number, and multiplying by ^+c,

$$\equiv {}^+(abc).$$

By assuming some, all, or none of the three factors of the product to be positive numbers, we may extend the principle to include such combinations of positive and negative numbers as the following:

$$\begin{aligned}(^+\boldsymbol{a})(^+\boldsymbol{b})(^+\boldsymbol{c}) &\equiv {}^+(\boldsymbol{ab})(^+\boldsymbol{c}) \equiv {}^+(\boldsymbol{abc}).\\ (^-\boldsymbol{a})(^+\boldsymbol{b})(^+\boldsymbol{c}) &\equiv {}^-(\boldsymbol{ab})(^+\boldsymbol{c}) \equiv {}^-(\boldsymbol{abc}).\\ (^-\boldsymbol{a})(^-\boldsymbol{b})(^+\boldsymbol{c}) &\equiv {}^+(\boldsymbol{ab})(^+\boldsymbol{c}) \equiv {}^+(\boldsymbol{abc}).\\ (^-\boldsymbol{a})(^-\boldsymbol{b})(^-\boldsymbol{c}) &\equiv {}^+(\boldsymbol{ab})(^-\boldsymbol{c}) \equiv {}^-(\boldsymbol{abc}),\end{aligned}$$

and so on for more factors.

16. The essential thing to be observed in the identities above is that a continued product is positive if it contains no negative factors or if it contains an even number of negative factors, and it is negative if it contains an odd number of negative factors.

E. g.

1. $(^+3)(^+2)(^+4)(^+5) \equiv {}^+120.$
2. $(^-4)(^-6)(^+1)(^+7) \equiv {}^+168.$
3. $(^-1)(^-3)(^-8)(^-10) \equiv {}^+240.$
4. $(^-2)(^+3)(^+5)(^+6) \equiv {}^-180.$
5. $(^-5)(^-2)(^-4)(^+9) \equiv {}^-360.$

17. The Associative Law for Multiplication

The value of a product remains unaltered if, in the process of multiplying several numbers, two successive factors are associated or grouped together to form a single product.

E. g. $2 \times 3 \times 4 \times 5 \equiv 2 \times 3 \times (4 \times 5) \equiv (2 \times 3) \times 20 \equiv 6 \times 20 \equiv 120.$

(The following proof may be omitted when the chapter is read for the first time.)

For any three arithmetic whole numbers, a, b, c,

$$abc \equiv a(bc).$$

We have $abc \equiv (ab)c.$ By definition of a product.

Considering the product (ab) as one number, by the Commutative Law for two factors we have:

$$\begin{aligned} &\equiv c(ab), \\ &\equiv (ca)b, && \text{by definition of a product.} \\ &\equiv b(ca), && \text{by Commutative Law.} \\ &\equiv (bc)a, && \text{by definition of a product.} \\ &\equiv a(bc), && \text{by Commutative Law.} \end{aligned}$$

Or, $abc \equiv (ab)c \equiv a(bc).$

18. By repeated applications of the Commutative and Associative Laws for multiplication, it may be shown that both laws hold for three or more factors. That is:

$$\boldsymbol{abc \equiv acb \equiv bac \equiv bca \equiv cab \equiv cba.}$$

Also, $\boldsymbol{abcd \equiv a\ (bcd) \equiv a\ (bc)\ d \equiv b\ (acd) \equiv}$ *etc.*,

and so on for any number of factors.

19. From the above, it appears that *we may arrange the factors of a product in any order, and group them together in any convenient way, without altering the value of the result.*

20. Both the multiplicand and multiplier receive the name of **factor,** since they may be interchanged without altering the value of the product.

E. g. a and b are factors of the product $a \times b$.

Similarly, each number of a continued product, $abcdef \cdots\cdots$, is called a factor of that product.

E. g. 5, a, b, and c are all factors of the continued product $5\ abc$.

21. The Distributive Law for Multiplication

Up to this point we have considered products in which both multiplicand and multiplier consisted of single numbers. In case either or both are sums or differences, we are led to consider the third Fundamental Law of Algebra, namely, the Distributive Law.

In particular, we will show that the product of 3 multiplied by the sum of 4 and 5 is the same as the product of 3 multiplied by 4, increased by the product of 3 multiplied by 5.

Let a series of dots be arranged as below, forming a set of three rows, each containing 9 dots.

The dots may be counted in either of two ways: first, as a single group consisting of three rows containing 9 dots each, that is, 27 dots in all; second, as consisting of one group of three rows containing 4 dots each, and a second group consisting of three rows containing 5 dots each, — the two groups being separated as shown.

Hence we may write

$$3(4+5) = (3 \times 4) + (3 \times 5) = 27.$$

22. The process may be applied to any three whole numbers, a, b, c, and we may assert as a general principle that

The product of an algebraic sum multiplied by a single number may be obtained by multiplying each term of the sum by the given number, and finding the algebraic sum of the results obtained.

Or $$\boldsymbol{a(b+c) \equiv ab + ac.}$$

This is called the Distributive Law for Multiplication, and it may be shown to hold when the multiplier consists of any number of terms, which may be positive or negative, integral or fractional.

23. Zero as a Factor.

It follows directly from the Fundamental Laws that *a product is zero if one of its factors is zero.*

That is $\boldsymbol{a \cdot 0 \equiv 0}$, (Multiplier 0).

$\boldsymbol{0 \cdot a \equiv 0}$. (Multiplicand 0).

(The following proof may be omitted when the chapter is read for the first time.)

We may write $n - n \equiv 0$, as defining zero.

Accordingly, $a \cdot 0 \equiv a(n-n)$ (Multiplier 0).

$\equiv an - an$ By Distributive Law for Multiplication.

$\equiv 0$. By definition of 0.

Also,
$$\begin{aligned} 0 \cdot a &\equiv (n-n)a \quad \text{(Multiplicand 0.)} \\ &\equiv na - na \\ &\equiv 0. \end{aligned}$$

24. Since the proof of an identity establishes at the same time the truth of its converse, it follows that, *if a product is zero, at least one of its factors must be zero,*

that is, if
$$a \cdot b \equiv 0,$$

then either a is 0, or b is 0, or both a and b are 0.

25. The product obtained by using the same factor repeatedly is called a **power** of that factor.

E. g. 3×3 is called the second power of 3, or 3 raised to the second power, since 3 occurs twice as a factor.

Also $2 \times 2 \times 2 \times 2$ is called the fourth power of 2; etc.

26. The number of times a factor appears in a product may be indicated by writing a small number called the **exponent** or the **index of the power** at the right of and immediately above the factor.

E. g. We may write 5^2 instead of 5×5; 4^3 instead of $4 \times 4 \times 4$; etc.

27. The number which is used repeatedly as a factor to obtain a power is called the **base** of the power.

28. The definition of an exponent as given is that it indicates the number of times a factor appears in a product. This definition requires that the exponent should be a positive whole number.

In a later chapter this notion of an exponent will be somewhat extended.

29. In arithmetic a number is defined as being **even or odd** according as it is or is not divisible by 2.

E. g. 2, 4, 6, 10, 16, etc., are even numbers.
3, 5, 7, 11, 17, etc., are odd numbers.

30. A power is defined as being even or odd according as its exponent is even or odd.

E. g. $(^+4)^2$, $(^+6)^4$, $(^-3)^6$, $(^-7)^8$, etc., are even powers,
while $(^+2)^3$, $(^+3)^5$, $(^-1)^7$, $(^-2)^9$, etc., are odd powers.

31. An odd power of a negative base contains an odd number of negative factors, and accordingly, by the rule of signs for continued products, it is of negative quality.

E. g. The power $(^{-}2)^3$ is of negative quality.

For, $$(^{-}2)^3 \equiv (^{-}2)(^{-}2)(^{-}2) \equiv {}^{-}8.$$

32. Whenever we speak of a positive integral power we have reference to the exponent rather than to the value of the base, which may itself be fractional or negative.

E. g. The following are positive integral powers of fractional bases and of negative bases:

$$\left(\frac{3}{4}\right)^2, \quad \left(\frac{1}{2}\right)^3, \quad (-4)^2, \quad \left(-\frac{2}{5}\right)^4$$

33. In operating with powers we are governed by the following Principles:

(i.) *All powers of positive bases are positive.*
(ii.) *Even powers of negative bases are positive.*
(iii.) *Odd powers of negative bases are negative.*

34. Since a product is zero if one or more of its factors is zero, it follows that any positive integral power of zero is zero; that is,

$$\mathbf{0}^m = \mathbf{0}.$$

35. In order to indicate clearly and exactly what number is to be considered as the base, it is often necessary to enclose the number within parentheses, as in the following illustrations:

(i.) **The base a negative number.**

$$(^{-}3)^2 = (^{-}3)(^{-}3) = {}^{+}(3 \times 3) = {}^{+}9.$$

Observe that ${}^{-}a^2$ is not the same as $(^{-}a)^2$.

${}^{-}a^2$ is read "negative a square;" $(^{-}a)^2$ is read "the square of negative a."

We have $${}^{-}a^2 \equiv {}^{-}(a \times a) \equiv {}^{-}a^2,$$
while $$(^{-}a)^2 \equiv (^{-}a)(^{-}a) \equiv {}^{+}a^2.$$

Whenever the symbol before the number or base is regarded as one of operation, as for example, -3^3, we may write

$$-\,3^3 = -\,(3 \times 3 \times 3) = -27.$$

(ii.) **The base not a single number, but either a product or a quotient.**

Ex. 1. $(2 \times 5)^2 = (2 \times 5)(2 \times 5) = (10)(10) = 100.$

The example above should be distinguished from the following:

Ex. 2. $2 \times 5^2 = 2(5 \times 5) = 2(25) = 50.$

Ex. 3.
$$\left(\frac{3}{4}\right)^2 = \frac{3}{4} \times \frac{3}{4} = \frac{9}{16}.$$

If the numerator alone or the denominator alone is to be raised to a power, we may write
$$\frac{3^2}{4} = \frac{9}{4}.$$

Similarly,
$$\frac{3}{4^2} = \frac{3}{16}.$$

(iii.) The base a sum or a difference.

Ex. 4. $(3 + 5)^2 = (3 + 5)(3 + 5) = (8)(8) = 64.$

This should be distinguished from the following:
$$3 + 5^2 = 3 + (5 \times 5) = 3 + 25 = 28,$$
also from
$$3^2 + 5^2 = (3 \times 3) + (5 \times 5) = 9 + 25 = 34.$$

Ex. 5. $(5 - 3)^2 = (2)^2 = 4.$

This should be distinguished from the following:
$$5^2 - 3^2 = 25 - 9 = 16.$$

(iv.) The base a power.

Ex. 6. $(3^2)^3 = (3^2)(3^2)(3^2) = (3 \times 3)(3 \times 3)(3 \times 3) = 9 \times 9 \times 9 = 729.$

The use of exponents above should be distinguished from 3^{2^3}, which may be taken to mean either $(3^2)^3$ (read "the cube of the second power of 3") or $3^{(2^3)}$ (read "3 raised to the power two cubed").

That is, $3^{2^3} = (3^2)^3 = (3^2)(3^2)(3^2) = 9 \cdot 9 \cdot 9 = 729,$

Or $3^{2^3} = 3^{(2^3)} = 3^{2 \times 2 \times 2} = 3^8 = 6561.$

EXERCISE V. 2

Find the values of the following indicated powers:

Arithmetic Numbers

1. 2^2.
2. 2^3.
3. 3^2.
4. 5^2.
5. 2^5.
6. 3^3.
7. 4^3.
8. 8^2.
9. 7^2.
10. 6^3.
11. 2^6.
12. 3^4.
13. 5^3.
14. 12^3.
15. 9^3.
16. 10^4.

Positive and Negative Numbers

17. $(^{+}2)^3$.
18. $(^{+}4)^2$.
19. $(^{+}5)^4$.
20. $(^{+}8)^2$.
21. $(^{+}10)^6$.
22. $(^{-}3)^2$.
23. $(^{-}6)^2$.
24. $(^{-}11)^2$.
25. $(^{-}9)^3$.
26. $(^{-}12)^2$.
27. $(^{-}7)^3$.
28. $(^{-}2)^5$.
29. $(^{-}3)^4$.
30. $(^{-}4)^3$.
31. $(^{-}3)^5$.
32. $(^{+}4)^4$.
33. $(^{-}15)^2$.
34. $(^{-}20)^4$.
35. $(^{+}20)^5$.
36. $(^{-}13)^2$.

Using the letters a, b, c, x, y, z, etc., to represent positive whole numbers, find expressions for the following:

37. $(^{+}2\,a)^3$.
38. $(^{+}3\,b)^2$.
39. $(^{+}4\,c)^4$.
40. $(^{-}2\,x)^3$.
41. $(^{-}3\,y)^2$.
42. $(^{-}4\,z)^4$.
43. $(^{+}5\,ab)^2$.
44. $(^{-}4\,xy)^3$.
45. $(^{-}3\,bc)^4$.
46. $(^{+}2^3)^2$.
47. $(^{+}3^2)^2$.
48. $(^{-}2^2)^2$.

Find the values of the following expressions:

49. $^{+}2^3 + {}^{+}3^2$.
50. $^{+}3^2 + {}^{+}4^2$.
51. $^{+}5^2 - {}^{+}4^2$.
52. $^{+}6^2 - {}^{+}3^2$.
53. $^{-}5^3 + {}^{+}3^4$.
54. $^{-}10^2 + {}^{+}9^2$.
55. $^{+}2^5 - {}^{-}2^5$.
56. $^{+}1^2 + {}^{+}2^2 + {}^{+}3^2 + {}^{+}4^2$.
57. $^{+}1^2 + {}^{+}3^2 + {}^{+}5^2 + {}^{+}7^2$.
58. $^{+}2^2 - {}^{-}3^2 + {}^{+}4^2 - {}^{-}5^2$.
59. $(^{+}4^2 - {}^{+}5^2)^2 - (^{-}6^2 - {}^{-}7^2)^2$.

II. DIVISION

36. The terms **dividend, divisor, quotient, remainder** are used relatively in the same way in algebra as in arithmetic.

Division as an operation is the **inverse of multiplication.**

To divide one number (dividend) by another (divisor), is to find another number (quotient), which when multiplied by the divisor produces the first (dividend).

E. g. To divide 12 by 4 is to find the quotient 3. Multiplying the quotient 3 by the divisor 4 produces the original dividend 12.

37. By the mutual relation of multiplication and division the quotient has the fundamental property that, when multiplied by the divisor, the product is the dividend.

That is, ***Quotient*** × ***Divisor*** ≡ ***Dividend.***

If we represent dividend, divisor, and quotient by D, d and Q respectively, we may indicate the quotient by writing $\frac{D}{d}$, and our definition of division as a process may be symbolized by

$$\frac{D}{d} \times d \equiv D.$$

38. Division, like subtraction, cannot always be performed, but it may always be indicated. It is only in exceptional cases that there can be obtained an integral quotient with no remainder. In this case the dividend is said to be **exactly divisible** by the divisor.

39. The **fractional notation** for a quotient, namely, $\frac{a}{b}$, and the **solidus notation** a/b, are commonly used for division. Primarily, either means that we are to take the bth part of unity a times as a summand. Hence, b times a of the bth parts of unity is equivalent to a times unity; or in symbols,

$$\frac{a}{b} \times b \equiv a.$$

Also, by the definition of division we have

$$(a \div b) \times b \equiv a.$$

Hence $\frac{a}{b}$ has the same meaning as $a \div b$ when a and b are whole numbers.

40. When division can be performed at all, it can lead to but a single result; hence it is called a **determinate process.**

Division by 0 is not an admissible operation.

41. Since multiplication and division are mutually inverse operations, it follows that if any number be successively multiplied by, and then divided by the same number, or be first divided by and then multiplied by the same number, the resulting value will be the same as though no operation had been performed. Or, stated in symbols,

$$(a \div b) \times b \equiv a,$$

and

$$(a \times b) \div b \equiv a.$$

42. It follows, from the definition of division, that *if the product of two factors be divided by either of the factors, the resulting quotient will be the other factor.*

Or,

$$(a \times b) \div a \equiv b,$$

and

$$(a \times b) \div b \equiv a.$$

Since in the product $(a \times b)$ the factors a and b of the dividend are separated by the multiplication sign, it is merely a matter of inspection to obtain the second member of each identity.

43. The Law of Quality Signs for division may be obtained directly from the set of identities in § 10 by applying the definition of division.

(i.) $^{+}(ab) \div {}^{+}b \equiv {}^{+}a$, (iii.) $^{-}(ab) \div {}^{+}b \equiv {}^{-}a$,
(ii.) $^{+}(ab) \div {}^{-}b \equiv {}^{-}a$, (iv.) $^{-}(ab) \div {}^{-}b \equiv {}^{+}a$.

It should be observed that the quotient is positive whenever the signs of quality of the dividend and divisor are like, as in (i.) and (iv.), and the quotient is negative whenever the signs of quality of the dividend and divisor are unlike, as in (ii.) and (iii.).

44. It follows that the quotient obtained by dividing any number by $^{+}1$ is equal to the number itself. It follows, also, that the quotient obtained by dividing any number by $^{-}1$ is a number equal in absolute value to the dividend but opposite in quality.

E. g. $^{+}5 \div {}^{+}1 \equiv {}^{+}5$, $^{-}7 \div {}^{+}1 \equiv {}^{-}7$,
$^{+}6 \div {}^{-}1 \equiv {}^{-}6$, $^{-}8 \div {}^{-}1 \equiv {}^{+}8$.

45. The quotient obtained by dividing 1 by any number is called the **reciprocal** of the number.

E. g. The reciprocal of 5 is $\frac{1}{5}$.

46. Since the product of any number multiplied by its reciprocal is by definition $^{+}1$, it follows that any number and its reciprocal have the same quality.

E. g. The numbers $^{-}3$ and $\frac{1}{^{-}3}$ are reciprocals, and both are negative numbers.

47. *Dividing by any number, except 0, produces the same result as multiplying by the reciprocal of that number.*

Representing any number by A, and any other number different from 0 by d, we may represent the product of A and the reciprocal of d by writing $A \times (1 \div d)$.

If this expression be multiplied by d, the result is A.

Hence, $A \times (1 \div d)$ is equal to the quotient $A \div d$, that is,

$$\boldsymbol{A \times (1 \div d) \equiv A \div d.}$$

E. g. The quotient $12 \div 3$ is equal to the product $12 \times \frac{1}{3}$.

48. As an **extended definition of division,** to correspond to that of multiplication, we have the following:

To divide one number by another is to do to the first that which must be done to the second to obtain the positive unit ${}^{+}1$.

49. In the division of positive and negative numbers, we may have

I. The Divisor Positive

Ex. 1. Divide ${}^{+}24$ by ${}^{+}6$.

By the extended definition of division, the quotient resulting from the division of ${}^{+}24$ by ${}^{+}6$ may be obtained by performing upon the dividend ${}^{+}24$ such operations as must be performed upon the divisor ${}^{+}6$ to obtain the unit of positive numbers ${}^{+}1$.

Since ${}^{+}6 = {}^{+}1 \times 6$, it appears that we may obtain the unit of positive numbers from ${}^{+}6$ by dividing the absolute value of ${}^{+}6$ by 6.

Hence the quotient of ${}^{+}24 \div {}^{+}6$, is a positive number obtained by dividing the absolute value of ${}^{+}24$ by 6; that is:

$${}^{+}24 \div {}^{+}6 \equiv {}^{+}(24 \div 6) \equiv {}^{+}4.$$

Ex. 2. Divide ${}^{-}30$ by ${}^{+}10$.

Reasoning as before, the quotient will be the negative number obtained by dividing the absolute value of the dividend ${}^{-}30$ by 10.

That is, $30 \div {}^{+}10 \equiv {}^{-}(30 \div 10) \equiv {}^{-}3$.

II. The Divisor Negative

Ex. 3. Divide ${}^{+}32$ by ${}^{-}16$.

By the extended definition of division, we may obtain the quotient resulting from the division of ${}^{+}32$ by ${}^{-}16$ by treating the dividend ${}^{+}32$ in the same way as we treat the divisor ${}^{-}16$ to obtain the unit of positive numbers ${}^{+}1$.

By first reversing the quality of ${}^{-}16$ we may, from the positive number thus obtained, ${}^{+}16$, obtain the unit of positive numbers ${}^{+}1$, by dividing the absolute value of the result by 16.

Hence we may obtain the desired quotient by first reversing the quality of the dividend ${}^{+}32$, and dividing the absolute value of the result thus obtained by 16.

That is, ${}^{+}32 \div {}^{-}16 \equiv {}^{-}(32 \div 16) \equiv {}^{-}2$.

Ex. 4. Divide ${}^{-}40$ by ${}^{-}8$.

In order to obtain the unit of positive numbers ${}^{+}1$ from the divisor ${}^{-}8$, we may first reverse the quality of the divisor, obtaining a positive number ${}^{+}8$, and then divide the absolute value of the number thus obtained by 8.

Hence we may perform the same steps with respect to the dividend ${}^{-}40$.

That is, ${}^{-}40 \div {}^{-}8 \equiv {}^{+}(40 \div 8) \equiv {}^{+}5$.

EXERCISE V. 3

Simplify the following:

1. ${}^{+}6 \div {}^{+}2$.
2. ${}^{+}9 \div {}^{+}3$.
3. ${}^{+}10 \div {}^{+}5$.
4. ${}^{+}12 \div {}^{+}6$.
5. ${}^{+}14 \div {}^{+}2$.
6. ${}^{+}16 \div {}^{-}4$.
7. ${}^{+}18 \div {}^{-}9$.
8. ${}^{+}20 \div {}^{-}5$.
9. ${}^{-}8 \div {}^{+}4$.
10. ${}^{-}4 \div {}^{+}2$.
11. ${}^{-}15 \div {}^{-}3$.
12. ${}^{-}12 \div {}^{+}4$.
13. ${}^{-}6 \div {}^{+}6$.
14. ${}^{+}17 \div {}^{-}17$.
15. ${}^{-}19 \div {}^{-}19$.
16. ${}^{+}13 \div {}^{-}13$.
17. ${}^{+}27 \div {}^{-}9$.
18. ${}^{-}33 \div {}^{+}3$
19. ${}^{-}38 \div {}^{-}19$.
20. ${}^{+}40 \div {}^{+}10$.
21. ${}^{-}42 \div {}^{+}14$.
22. ${}^{+}50 \div {}^{-}1$.
23. ${}^{+}56 \div {}^{-}7$.
24. ${}^{-}57 \div {}^{-}19$.
25. ${}^{+}63 \div {}^{-}9$.
26. ${}^{-}64 \div {}^{-}16$.
27. ${}^{+}65 \div {}^{-}13$.
28. ${}^{-}68 \div {}^{+}17$.
29. ${}^{+}70 \div {}^{-}14$.
30. ${}^{-}75 \div {}^{+}5$.
31. $({}^{+}36 \times {}^{-}2) \div {}^{-}8$.
32. $({}^{+}15 \times {}^{-}4) \div {}^{-}12$.
33. $({}^{-}24 \times {}^{-}3) \div {}^{+}9$.
34. $({}^{+}16 \times {}^{+}4) \div {}^{-}8$.
35. $({}^{+}56 \div {}^{-}8) \times {}^{-}7$.
36. $({}^{+}32 \div {}^{-}16) \times {}^{-}5$.
37. $({}^{-}24 \div {}^{-}2) \div {}^{-}2$.
38. $({}^{+}48 \div {}^{-}12) \div {}^{-}4$.

Find the value of $a \div (b + c)$ when

39. $a = {}^{-}27,\ b = {}^{+}5,\ c = {}^{+}4$.
40. $a = {}^{-}39,\ b = {}^{+}15,\ c = {}^{-}2$.

Find the value of $(a + b) \div (c + d)$ when

41. $a = {}^{+}1, b = {}^{+}2, c = {}^{+}4, d = {}^{-}1$.
42. $a = {}^{+}11, b = {}^{-}2, c = {}^{+}6, d = {}^{+}3$.

50. Commutative Law for Division. Since multiplications may be performed in any order, it follows that, in a series of successive divisions also, the operations may be performed in any order; that is,

$$a \div b \div c \equiv a \div c \div b.$$

51. In any chain of operations containing both multiplications and divisions, the quantities may be rearranged in any order, providing the sign of operation, × or ÷, attached to any particular operand, moves with it when it changes from one position to another.

(The following proof may be omitted when the chapter is read for the first time.)

Representing arithmetic whole numbers by a, b, and c, consider $a \times b \div c$.

By the principle of § 41 it follows that, if any number a be divided by any number c, except zero, and this result be then multiplied by c, the result will be the same as if no operation had been performed upon a. That is:

$$a \div c \times c \equiv a.$$

Substituting this expression for a in the given expression, we may write

$$a \times b \div c \equiv (a \div c \times c) \times b \div c.$$

Applying the Commutative Law for multiplication to the two factors c and b, we have:

$$(a \div c \times c) \times b \div c \equiv a \div c \times b \times c \div c.$$

In this chain we may neglect $\times c \div c$ as producing no change in the final result. Therefore $a \times b \div c \equiv a \div c \times b$.

It follows that, in an unbroken chain of multiplications and divisions, the operations may be performed in any order.

E. g. $$2 \div 7 \times 14 = 2 \times 14 \div 7 = 28 \div 7 = 4.$$

52. It follows, from the Law of Commutation for multiplications and divisions occurring together, that *a product of two or more factors may be divided by a number by dividing one of the factors of the product by that number.*

(The following proof may be omitted when the chapter is read for the first time.)

Representing any positive integral numbers by a, b, and c, c not being **0**, we have the following:

$(ab) \div c \equiv a \times b \div c,$	$(ab) \div c \equiv a \times b \div c,$ Notation
$\equiv a \div c \times b,$	$\equiv b \div c \times a,$ Commutative Law
$\equiv (a \div c) \times b,$	$\equiv (b \div c) \times a,$ Notation
	$\equiv a \times (b \div c),$ Commutative Law

From the reasoning above it follows that

$$(ab) \div c \equiv \begin{cases} \text{either } (a \div c) \times b \\ \text{or } \quad a \times (b \div c). \end{cases}$$

Associative Law for Division

PRINCIPLES GOVERNING THE REMOVAL AND INSERTION OF PARENTHESES

53. Parentheses preceded by the sign of multiplication ×.

(The following proof may be omitted when the chapter is read for the first time.)

Representing arithmetic values of whole numbers by a, b, and c, as in similar cases, c not being 0, consider the expression $a \times (b \div c)$ in which parentheses are preceded by a multiplication sign.

Since successive multiplications and divisions by c produce no change of value in the result, we may write

$$a \times (b \div c) \equiv a \times (b \div c) \times c \div c.$$

By the definition of division $(b \div c) \times c \equiv b$.

Hence $$a \times (b \div c) \times c \div c \equiv a \times b \div c.$$

Therefore $$a \times (b \div c) \equiv a \times b \div c.$$

An unbroken chain of multiplications and divisions may be removed from parentheses preceded by the sign of multiplication without altering the signs of multiplication and division preceding the different numbers removed. (Compare with Prin. I. (i.) Chap. IV. § 7.)

In case either the sign of multiplication or the sign of division is required before the first number enclosed within parentheses, and neither sign is written, the sign of multiplication is to be understood.

54. Since the proof of any identity establishes also the truth of its converse, we may state that *an unbroken chain of multiplications and divisions may be enclosed within parentheses, preceded by the symbol of multiplication, without altering the signs of multiplication and division attached to the numbers included.* (Compare with Prin. II. (i.) Chap. IV. § 8.)

55. Parentheses Preceded by the Sign of Division ÷.

(The following proof may be omitted when the chapter is read for the first time.)

Consider the expression $a \div (b \div c)$ in which parentheses are preceded by the sign of division.

It may be seen that the chain of successive operations represented by $\times\, b \div c \times c \div b$, if applied to any number, will not affect its value. Hence, we may write

$$\begin{aligned} a \div (b \div c) &\equiv a \div (b \div c) \times b \div c \times c \div b \\ &\equiv a \div (b \div c) \times (b \div c) \times c \div b. \end{aligned}$$

We may neglect the successive operations represented by $\div\, (b \div c) \times (b \div c)$, as producing no alteration in the value of a.

Hence, $\quad a \div (b \div c) \times (b \div c) \times c \div b \equiv a \times c \div b.$

Applying the Commutative Law for multiplications and divisions, we may write finally

$$a \div (b \div c) \equiv a \div b \times c.$$

An unbroken chain of multiplications and divisions may be removed from parentheses preceded by the sign of division, providing the signs of multiplication and division preceding the different numbers removed be changed from × to ÷, or from ÷ to ×. (Compare with Prin. I. (ii.) Chap. IV. § 7.)

56. Since the proof of any identity establishes also the truth of its converse, it follows that *an unbroken chain of multiplications and divisions may be enclosed within parentheses preceded by the symbol of operation for division, provided the symbols of operation for multiplication and division, attached to all numbers enclosed, be reversed from* × *to* ÷ *or from* ÷ *to* ×. (Compare with Prin. II. (ii.) Chap. IV. § 8.)

E. g. Parentheses preceded by ×

$$\begin{aligned} {}^{+}6 \times ({}^{-}2 \div {}^{-}3) &\equiv {}^{+}6 \times {}^{-}2 \div {}^{-}3 \\ &\equiv {}^{-}12 \div {}^{-}3 \\ &\equiv {}^{+}4. \end{aligned}$$

Parentheses preceded by ÷

$$\begin{aligned} {}^{+}32 \div ({}^{-}8 \div {}^{-}2) &\equiv {}^{+}32 \div {}^{-}8 \times {}^{-}2 \\ &\equiv {}^{-}4 \times {}^{-}2 \\ &\equiv {}^{+}8. \end{aligned}$$

Observe that ${}^{+}32 \div ({}^{-}8 \div {}^{-}2)$ is not equal to ${}^{+}32 \div {}^{-}8 \div {}^{-}2$. For, we have ${}^{+}32 \div {}^{-}8 \div {}^{-}2 \equiv {}^{-}4 \div {}^{-}2 \equiv {}^{+}2$.

57. From the Associative Law for multiplications and divisions we have the following **Law of Signs:**

(i.) $\times (\times\, \boldsymbol{a}) \equiv \times\, \boldsymbol{a}$, (iii.) $\div (\times\, \boldsymbol{a}) \equiv \div\, \boldsymbol{a}$,

(ii.) $\div (\div\, \boldsymbol{a}) \equiv \times\, \boldsymbol{a}$, (iv.) $\times (\div\, \boldsymbol{a}) \equiv \div\, \boldsymbol{a}$.

It appears that *for two like signs either of multiplication or of division, occurring successively as in* (i.) *and* (ii.), *we may substitute the direct sign* × ; *and for two unlike signs occurring successively, as in* (iii.) *and* (iv.), *we may substitute the indirect sign* ÷. (Compare with § 10.)

Exercise V. 4

Simplify the following arithmetic expressions :

1. $4 \times (3 \div 2)$.
2. $1 \div (1 \div 6)$.
3. $1 \div (2 \times 11)$.
4. $5 \div (4 \div 5)$.
5. $27 \div (9 \div 2)$.
6. $21 \div (7 \div 4)$.
7. $12 \div (6 \div 3 \times 2)$.
8. $15 \times (3 \div 5 \times 4)$.
9. $14 \div (7 \div 5 \div 3)$.
10. $5 \div [4 \div 5 \times (3 \div 5)]$.
11. $7 \div [8 \times 3 \div (12 \div 7)]$.
12. $16 \times [9 \div 8 \times (1 \div 3)]$.

Simplify the following algebraic expressions :

13. ${}^{+}10 \times ({}^{+}3 \div {}^{+}2)$.
14. ${}^{+}28 \times ({}^{-}4 \div {}^{-}7)$.
15. ${}^{+}9 \times ({}^{+}8 \div {}^{-}3)$.
16. ${}^{+}3 \div ({}^{+}4 \div {}^{+}5)$.
17. ${}^{+}1 \div ({}^{+}6 \div {}^{-}1)$.
18. ${}^{-}16 \div ({}^{+}8 \div {}^{+}4)$.
19. ${}^{-}27 \div ({}^{-}8 \div {}^{+}3)$.
20. ${}^{-}1 \div ({}^{-}1 \div {}^{-}2)$.

Distributive Law for Division.

58. In division the dividend may be distributed, the signs of the partial quotients following the same law of signs as the partial products in multiplication. Hence, the following

Principle: *The quotient resulting from the division of any algebraic sum by a single algebraic number may be obtained by dividing each term of the dividend by the divisor. The signs prefixed to the partial quotients thus obtained are + or − according as the terms from which they are obtained by division have like or unlike quality signs.*

(The following proof may be omitted when the chapter is read for the first time.)

As in similar cases let a, b, c, and d represent any positive integral values, with the restriction that d shall not be 0. Consider $(a - b + c) \div d$.

By the definition of division, we may write

$$\begin{aligned} +a \div d \times d &\equiv +a, \\ -b \div d \times d &\equiv -b, \\ +c \div d \times d &\equiv +c. \end{aligned}$$

Substituting these values for a, b and c, in the expression $a - b + c$, we may write

$$(a - b + c) \div d \equiv [a \div d \times d - b \div d \times d + c \div d \times d] \div d.$$

Observe that the expression in square brackets may be obtained by multiplying the following expression by d:

$$a \div d - b \div d + c \div d.$$

Hence,
$$\begin{aligned} (a - b + c) \div d &\equiv [(a \div d - b \div d + c \div d)] \times d \div d \\ &\equiv a \div d - b \div d + c \div d. \end{aligned}$$

E. g.
$$\begin{aligned} &[{}^{+}12 \times {}^{-}26 - {}^{+}8 \times {}^{+}13 + {}^{-}11 \times {}^{-}39] \div {}^{+}13 \\ &\equiv [{}^{+}12 \times {}^{-}2 - {}^{+}8 \times {}^{+}1 + {}^{-}11 \times {}^{-}3] \\ &\equiv {}^{-}24 + {}^{-}8 + {}^{+}33 \\ &\equiv {}^{+}1. \end{aligned}$$

59. Although the dividend may be distributed, the divisor cannot be.

E. g. It should be observed that $\dfrac{a+b}{c+d} \equiv \dfrac{a}{c+d} + \dfrac{b}{c+d}$,

but $\dfrac{a+b}{c+d} \not\equiv \dfrac{a+b}{c} + \dfrac{a+b}{d}$.

60. Division of one expression by another may be indicated by writing the dividend as the numerator and the divisor as the denominator of a fraction. When either the dividend or the divisor consists of more than one term, the horizontal dividing line of the fraction, which separates numerator from denominator, serves both as a sign of division and of grouping.

E. g. $$\frac{a+b}{c-d} \equiv (a+b) \div (c-d).$$

61. In a chain of operations involving additions, subtractions, multiplications and divisions, unless the contrary is specified, the multiplications and divisions must be performed first, and then the indicated additions and subtractions.

However, by the use of parentheses we may indicate that additions and subtractions are to be performed before multiplications and divisions.

Ex. 1. $${}^{+}2 \times {}^{+}3 + {}^{-}4 \times {}^{-}5 + {}^{-}8 \div {}^{+}2 \equiv {}^{+}6 + {}^{+}20 + {}^{-}4 \equiv {}^{+}22.$$

Ex. 2. $$\begin{aligned} {}^{+}2 \times ({}^{+}3 + {}^{-}4)({}^{-}5 + {}^{-}8 \div {}^{+}2) &\equiv {}^{+}2 \times ({}^{-}1)({}^{-}5 + {}^{-}4) \\ &\equiv {}^{-}2\,({}^{-}9) \equiv {}^{+}18. \end{aligned}$$

62. From our definition of 0 as resulting from the subtraction of any number from an equal number, it follows that *the quotient obtained by dividing zero by any number different from zero is zero.* For, letting a and x represent any numbers which are different from zero, we have $0 \div x \equiv (a - a) \div x \equiv a \div x - a \div x \equiv 0$.

63. It follows, conversely, that *if a quotient be* 0 *the dividend must be* 0, since from the nature of the case the divisor cannot be 0.

That is, b being any number other than 0,

if $a \div b = 0$, then a must equal 0.

Exercise V. 5

Simplify the following expressions :

1. $^{+}20 \div {}^{-}5 + {}^{-}30 \div {}^{+}6 - {}^{-}40 \div {}^{-}8.$
2. $^{-}24 \times {}^{-}2 - {}^{+}3 \div {}^{-}1 + {}^{-}9 \times {}^{+}5.$
3. $^{+}12 \div {}^{-}3 \times {}^{+}4 - {}^{-}15 \times {}^{-}5 \div {}^{+}3 + {}^{+}1.$
4. $^{-}36 \times {}^{-}2 \div {}^{+}12 - {}^{+}1 \div {}^{+}1 \times {}^{-}2 - {}^{-}3.$
5. $^{+}6 \times {}^{+}5 + {}^{-}7 \times {}^{-}2 + {}^{+}3 \times {}^{-}4 + {}^{-}1 \times {}^{+}9.$
6. $^{-}8 \div {}^{+}2 - {}^{+}15 \times {}^{-}1 - {}^{-}14 \div {}^{-}7 + {}^{-}6 \times {}^{+}2.$
7. $({}^{+}2 + {}^{+}6)({}^{-}3 + {}^{-}4) + {}^{+}50.$
8. $({}^{-}7 - {}^{+}5)({}^{-}1 + {}^{-}10 \div {}^{+}2).$
9. $^{+}3 \div ({}^{-}10 - {}^{-}11)({}^{+}8 - {}^{-}5 \div {}^{-}1).$
10. $({}^{-}2 \times {}^{+}4 - {}^{-}4 \div {}^{+}2) \div ({}^{+}6 \div {}^{-}3 - {}^{-}1 \div {}^{+}1) \times {}^{-}7.$

64.* The Fundamental Laws of ordinary Algebra, as discussed in the preceding pages, have been proved for special cases only, and not in their most general forms.

By the Principle of No Exception they will now be assumed to be extended to include all numbers which are positive or negative, integral or fractional, and will be laid down as being fundamental to the science of Algebra.

Our problem will now be to so define and interpret our symbols and the results of operations with them, that they shall be consistent with the Commutative, Associative, and Distributive Laws, which are three fundamental laws of Algebra.

65.* By means of the Law of Commutation we have obtained the principles governing the insertion and removal of parentheses, and by the aid of these principles we have developed the Law of Distribution.

By applying these principles repeatedly it may be shown that the Distributive Law for Multiplication holds when both multiplicand and multiplier consist of more than one number or term; that is, with the restriction that $a > b$ and $c > d$, we may show that

$$(a - b)(c - d) \equiv ac - bc - ad + bd.$$

By letting a and c each equal zero, we obtain from the above,

$$(-b)(-d) \equiv +bd,$$

from which we obtain the following Law of Quality Signs:

$$(+{}^{-}b)(+{}^{-}d) \equiv +{}^{+}bd.$$

The remaining forms for the Law of Signs for different combinations of signs of operation and of quality (see § 10) may be shown to hold true by a similar course of reasoning.

We thus obtain the Law of Quality Signs as a direct result of the Law of Distribution for Multiplication, and accordingly the Law of Signs may be included among the fundamental laws of operation.

66. We shall, in the following chapters, employ a **single set of signs, + and –**, to denote both the operations of addition and subtraction, and the qualities of the numbers to which they are attached, as being positive or negative.

The symbol of operation + preceding a number or letter which stands first in a chain of additions and subtractions will usually be omitted, whether it denotes the operation of addition or the quality of the number as being positive; but the sign –, whether indicating

---

* This section may be omitted when the chapter is read for the first time.

the operation of subtraction or the negative quality of the number to which it is attached, can never be omitted.

In expressions such as the following

$$(+3) - (-4) + (-5) - (-6)$$

each sign within parentheses is to be interpreted as indicating quality, and each sign outside of the parentheses as indicating an operation.

CHAPTER VI

ADDITION AND SUBTRACTION OF INTEGRAL ALGEBRAIC EXPRESSIONS

Definitions

1. An algebraic expression may for the present be defined to be any collection or combination of letters or of letters and numbers, connected by the signs of operation $+$, $-$, $\times$ and $\div$, which may be used according to the principles and definitions of algebra to represent a number. (Compare with Chap. I. §12.)

2. The parts of an algebraic expression which are separated by the signs plus and minus are called the **terms.**

3. Whenever any term is regarded as being separated into two factors, either factor may be called the co-factor or the **coefficient** of the other.

E. g. In the term $5\,abcd$, 5 is the coefficient of $abcd$, $5\,a$ of bcd, ac of $5\,bd$, etc.

Thus in $4\,xyz$, 4 is the coefficient of xyz; $4\,x$ that of yz; and $4\,xy$ that of z.

When one of the factors of a product is a numeral symbol, it is called the **numerical coefficient** of the product of the other factors.

Unless the contrary is specified, when we speak of the coefficient of a term we mean the numerical coefficient taken together with the sign $+$ or $-$ preceding it. When no numerical coefficient is written, unity is understood.

4. A **power** of a number is a product obtained by using that number two or more times as a factor.

Thus the second power of 2 is 2×2 or 4; the third power is $2 \times 2 \times 2$ or 8; the fifth power is $2 \times 2 \times 2 \times 2 \times 2$ or 32.

5. For the present we shall define an **exponent** as an integral number written at the right of and a little above a number or expression to show how many times the number or expression is to be taken as a factor.

Thus a^2, read "a square" or "a to the second power," means $a \cdot a$; a^3, read "a cube" or "a to the third power," means $a \cdot a \cdot a$; a^6 means $a \cdot a \cdot a \cdot a \cdot a \cdot a$; etc.

x^n may be interpreted as meaning a number x taken as a factor n times, and not until definite values are assigned to x and n will the expression be regarded as having a definite numerical value. (See also Chap. XIX §§ 1–4.)

The same notation may be applied to expressions in which two or more numbers or letters appear.

E. g.

$$(ab)^4 \text{ means } (ab)(ab)(ab)(ab);$$

$$(x+y)^4 \text{ means } (x+y)(x+y)(x+y)(x+y);$$

$$(5-z)^3 \text{ means } (5-z)(5-z)(5-z).$$

When no exponent is written, the exponent 1 is understood.

Thus, 2 is called the first power of 2; 5 the first power of 5; a the first power of a; the exponent 1 being understood in each case.

6. The expression "no exponent" must not be confused with the "exponent zero." We shall show later that any number with the exponent zero may be regarded as standing for unity or 1.

7. One power is *higher* or *lower* than another according as its exponent is greater than or less than that of the other.

E. g. The fourth power of a number is "higher" than the second power; the sixth is higher than the fifth; etc.

8. A power is *even* or *odd* according as its exponent is even or odd.

9. Any letter or number which is raised to a power, is called a **base.**

10. Each literal factor of a term is called a **dimension** of the term.

It is customary to write the literal factors of a term in alphabetical order, unless for some particular reason a different arrangement is required.

11. The number obtained by adding the exponents of the literal factors of a term is called the **degree** of the term as a whole.

E. g. x^3, x^2y, abc, are of the third degree ;
$abcde$, x^5, x^4y, m^3n^2, are of the fifth degree; etc.

12. Similar or like terms are terms which contain the same letters, affected by the same exponents. These terms may, however, differ in their numerical coefficients.

Thus, $5\ xy$ and $-10\ xy$ are like terms; so also are $7\ a^2b^3c$ and $9\ a^2b^3c$.

13. An algebraic expression is called a **monomial, binomial, or trinomial**, according as it consists of one, two, or three terms. Algebraic expressions of two or more terms are commonly spoken of as **polynomials** or as **multinomials.**

14. A **monomial** is **integral** if the letters which it contains enter by multiplication only, and none enter by division, that is, if none appear in the denominator of any fraction.

E. g. abc, mn^2, $2\ xy$, $15\ xy^2z^3$, are all monomials.
$a+b$, $2\ x-3\ y$, $\frac{a}{b}+1$ are binomials.
$x+y+z$, ax^2+bx+c, $2\ m+3\ x-11$, are trinomials.
$a^4+a^3+a^2+a+1$ is a polynomial.

All of these expressions, with the exception of the monomials, are polynomials or multinomials.

15. A **polynomial** is said to be **integral** with respect to a specified letter when this letter does not appear in the denominator of any fraction; that is, when it does not enter any term through the process of division. In the opposite case it is said to be **fractional.**

E. g. The expression $x^3+\frac{a^2-b^2}{b}x^2-\frac{(a-b)^2}{(a+b)^2}x+\frac{a^2+2\ ab+b^2}{a}$ is integral with respect to x, but fractional with respect to a and b.

In order that a polynomial be integral with respect to a given letter it is necessary that all of the terms which appear in it be integral with respect to that letter.

E. g. $x^3+5\ x^2-x+1$ is an integral polynomial of the third degree with reference to x, since the highest power of x which appears is the third.

We may regard ax^3+bx^2+cx+d as a polynomial of the third degree with reference to x alone, or of the fourth degree if no particular letter is specified. This is because the term ax^3 is of the third degree with reference to x alone, but of the fourth degree with reference to a and x taken together.

16. An expression in which all of the terms are of the same degree, reckoned with reference to all of the letters, is **homogeneous.**

E. g. $x^4 + x^3y + x^2y^2 + xy^3 + y^4$ and $abc + 5\,a^3 + b^2c$ are homogeneous, and of the fourth and third degrees respectively.

17. One of the equal factors of a number is called a **root** of the number. According as there are two, three, or four equal factors, etc., each is called a **square root, cube root, fourth root, etc.**

Thus, 2 is one of the square roots of 4; 3 of 9; 7 of 49; etc. 2 is a cube root of 8; 3 of 27; 5 of 125; etc.

18. A root of a number is indicated commonly by means of a root or **radical sign** $\sqrt{\ }$.

A small number, called the **index of the root,** placed thus, $\sqrt[2]{\ }$, $\sqrt[3]{\ }$, $\sqrt[4]{\ }$, etc., is used to indicate the order of the root, that is, whether it be a second, third, or fourth root, etc. If no index be expressed, the index 2 is understood.

Thus, $\sqrt[3]{27}$ is 3, $\sqrt[4]{16}$ is 2, $\sqrt[5]{32}$ is 2, and $\sqrt{16} \equiv \sqrt[2]{16}$ is 4.

19. An expression is **rational** with respect to specified letters when it does not contain indicated roots of these letters. In the contrary case, it is said to be **irrational.**

According to this definition, a rational expression may contain indicated roots of the numerical parts.

E. g. $2\,x^2yz, \dfrac{m+n}{3\,a}, \dfrac{\sqrt{5}\,abc}{9\,xy}, \dfrac{+3\,a+\sqrt{4}\,b}{2}$, are rational with respect to the letters, for no letter appears under a radical sign. On the other hand, $\sqrt{x^2-y^2}$ and $\dfrac{2\,a-\sqrt{a^2+b^2}}{\sqrt{a-b}}$ are irrational with respect to the letters.

20. A **polynomial** is **rational, irrational, integral, or fractional** according as its terms are rational, irrational, integral, or fractional.

21. A **polynomial** is said to be **arranged** with reference to a specified letter when the exponent of that letter in successive terms increases or decreases in numerical value.

E. g. The expression $4 + 2x + 6x^2 + 5x^3 + x^4$ is arranged according to increasing powers of x; in the following form it is arranged according to decreasing powers of x; $x^4 + 5x^3 + 6x^2 + 2x + 4$.

The polynomial $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$ is arranged according to descending powers, 5, 4, 3, 2, 1, of a, and according to the ascending powers, 1, 2, 3, 4, 5, of b.

ADDITION OF MONOMIALS

22. The algebraic sum of two numbers or quantities may always be indicated by writing them one after the other, separated by the sign of addition, each term being preceded by the sign of quality of its numerical coefficient.

This collecting of several numbers or quantities into one algebraic expression is what in algebra is called **addition.** The resulting expression is called the **sum.**

23. It is commonly understood that, after the parts of a sum are written consecutively as parts of one algebraic expression, like or similar terms are to be united.

This "reduction" is not addition, but amounts simply to changing the form of an expression so that it shall have as few terms as possible.

(a) Addition of Dissimilar Terms.

24. *If several quantities or terms to be added are unlike, they cannot be united to form any particular amount or number of either. The sum may be indicated by writing the terms one after the other with the proper signs.*

E. g. The sum of a and b may be indicated by writing $a + b$, but not until particular values are given to a and b can we find a single *number* representing their sum.

If a represents 2 and b represents 5, then $a + b$ represents 7.

If a represents $+ 3$ and b represents $- 3$, then $a + b$ represents 0.

Ex. 1. Express the sum of $3x$ and $-2y$.

Since these terms are unlike, we cannot combine them into a single term, but we may express the sum by writing the terms separated by a positive sign, considering the second term as a negative number; that is

$$3x + (-2y) \equiv 3x - 2y.$$

(b) Addition of Similar Terms.

25. *Like or similar terms may be united by addition into a single term.*

Ex. 2. Find the sum of $5\,ab$, $3\,ab$, and ab.

Since we have no knowledge of the value represented by the product of the letters a and b, we may add the terms with respect to the product ab, as a concrete number.

Indicating the sum by writing $5\,ab + 3\,ab + ab$, we find that we have $(5 + 3 + 1)\,ab$, that is, $9\,ab$.

Hence we may write $5\,ab + 3\,ab + ab \equiv 9\,ab$.

Ex. 3. Find the sum of $4\,mn^2$, $-2\,mn^2$, and $-5\,mn^2$.

Observe that the terms are similar with respect to the literal parts, mn^2. Hence we may indicate the sum by writing as a coefficient to this common literal part the algebraic sum of the numerical coefficients considered as positive and negative numbers; that is, we may write

$$4\,mn^2 - 2\,mn^2 - 5\,mn^2 \equiv (4 - 2 - 5)\,mn^2 \equiv -3\,mn^2.$$

26. The addition of terms in algebra is performed according to the following **Principles:**

(i.) *The sum of two monomials may be indicated by writing them with their signs of quality one after the other, separated by the sign +.*

(ii.) *To add like terms, find the algebraic sum of their coefficients considered as positive or negative numbers, and prefix this as a coefficient to their common parts.*

Mental Exercise VI. 1

Perform the following indicated additions:

1. $\begin{array}{r} a \\ \underline{a} \end{array}$ 2. $\begin{array}{r} -b \\ \underline{-b} \end{array}$ 3. $\begin{array}{r} 2\,c \\ \underline{3\,c} \end{array}$ 4. $\begin{array}{r} 5\,d \\ \underline{d} \end{array}$

5. $\begin{array}{r} -4\,g \\ \underline{-\ \ g} \end{array}$ 6. $\begin{array}{r} 6\,h \\ \underline{4\,h} \end{array}$ 7. $\begin{array}{r} 5\,x \\ \underline{-2\,x} \end{array}$ 8. $\begin{array}{r} -7\,x \\ \underline{11\,x} \end{array}$

9. $\begin{array}{r} -8\,x \\ \underline{8\,x} \end{array}$ 10. $\begin{array}{r} -12\,y \\ \underline{-13\,y} \end{array}$ 11. $\begin{array}{r} 9\,z \\ \underline{-19\,z} \end{array}$ 12. $\begin{array}{r} 13\,w \\ \underline{-20\,w} \end{array}$

13. $\begin{array}{r} 8\,ab \\ \underline{2\,ab} \end{array}$ 14. $\begin{array}{r} 6\,bc \\ \underline{9\,bc} \end{array}$ 15. $\begin{array}{r} -\ \ 3\,ad \\ \underline{-11\,ad} \end{array}$ 16. $\begin{array}{r} 16\,xy \\ \underline{-9\,xy} \end{array}$

17. $\begin{array}{r} 4\,c^2x \\ \underline{5\,c^2x} \end{array}$ 18. $\begin{array}{r} 9\,by^2 \\ \underline{8\,by^2} \end{array}$ 19. $\begin{array}{r} -17\,c^2d^2 \\ \underline{21\,c^2d^2} \end{array}$ 20. $\begin{array}{r} -26\,x^3y^2 \\ \underline{31\,x^3y^2} \end{array}$

21.	ax	22.	$5by$	23.	$6c^2$	24.	$7dw$
	$2ax$		$4by$		$2c^2$		$-5dw$
	$3ax$		$2by$		$5c^2$		$3dw$
25.	$4xw$	26.	$11y^3$	27.	$12z^2w$	28.	$-20w^2$
	$-2xw$		$-3y^3$		$4z^2w$		$10w^2$
	$-xw$		$3y^3$		$-4z^2w$		$-5w^2$
29.	$-3abc$	30.	$-13bxy$	31.	$14cm^2$	32.	$-17dh^2$
	$-5abc$		$-7bxy$		$-16cm^2$		$-13dh^2$
	$-7abc$		$-9bxy$		$18cm^2$		$29dh^2$
33.	$22a^2bc$	34.	$28ab^2c$	35.	$31xyz^2$	36.	$-43x^2y^2$
	a^2bc		$-ab^2c$		$-11xyz^2$		$-27x^2y^2$
	$-3a^2bc$		$-5ab^2c$		$-xyz^2$		x^2y^2
37.	$7a^2b$	38.	$9bc$	39.	$-5x^2y$	40.	$7xyz$
	$5a^2b$		$-10bc$		$9x^2y$		$-8xyz$
	$8a^2b$		$12bc$		$-13x^2y$		$-9xyz$
	$6a^2b$		$-14bc$		$18x^2y$		$10xyz$
41.	$32ab$	42.	$13xy$	43.	$18x^2yz^2$	44.	$14x^3y^3z^3$
	$7ab$		$15xy$		$12x^2yz^2$		$14x^3y^3z^3$
	$11ab$		$17xy$		$16x^2yz^2$		$22x^3y^3z^3$
	$2ab$		$19xy$		$14x^2yz^2$		$25x^3y^3z^3$
45.	$13abx$	46.	$-23b^2c^3d$	47.	$-25a^3mw$	48.	$8mnx^2$
	$-15abx$		$37b^2c^3d$		$-35a^3mw$		$17mnx^2$
	$27abx$		$-6b^2c^3d$		$-45a^3mw$		$17mnx^2$
	$12abx$		$-b^2c^3d$		$95a^3mw$		$-33mnx^2$

49.	$\frac{1}{2}a$	50.	b	51.	$\frac{2}{5}c$	52.	$\frac{2}{3}d$	53.	$\frac{6}{7}h$	54.	$\frac{3}{4}k$
	$3a$		$\frac{1}{3}b$		$-\frac{1}{5}c$		$\frac{3}{4}d$		$\frac{1}{2}h$		$-\frac{4}{3}k$
55.	$\frac{5}{9}m$	56.	$\frac{3}{10}n$	57.	$.2a$	58.	$.04b$	59.	$5.67x$	60.	$2.001y$
	$\frac{3}{5}m$		$\frac{3}{4}n$		$.3a$		$.05b$		$-2.31x$		$-.102y$

61. $0.7c + 0.02c$.
62. $8.1d + 0.05d$
63. $1.1e + .11e$.
64. $10.1x - 1.01x$.
65. $.68y + 0.25y$.
66. $1.001z + 9.099z$.
67. $100.1w - 1.001w$.
68. $11.01a - 10.11a$.

Add the following, with reference to the similar parts. *In case two unlike terms contain the same letter or factor, we may perform the addition with reference to this letter as a summand, regarding the remaining factors as coefficients.*

Ex. 69. Find the sum of xz and yz.

Regarding x and y as coefficients of z, the sum may be expressed by writing the sum of x and y as a coefficient of z, as follows:

$xz + yz \equiv (x + y)z$, or in vertical arrangement, $$\begin{array}{r} xz \\ \underline{yz} \\ (x+y)z. \end{array}$$

70. $\begin{array}{l} ax \\ \underline{bx} \end{array}$	71. $\begin{array}{l} cy \\ \underline{dy} \end{array}$	72. $\begin{array}{l} mz \\ \underline{2\,z} \end{array}$	73. $\begin{array}{l} 3\,w \\ \underline{yw} \end{array}$
74. $\begin{array}{r} ad \\ \underline{d} \end{array}$	75. $\begin{array}{l} 6\,h \\ \underline{bh} \end{array}$	76 $\begin{array}{r} cx \\ \underline{-\,dx} \end{array}$	77. $\begin{array}{r} 3\,k \\ \underline{-\,hk} \end{array}$
78. $\begin{array}{r} 2\,ab \\ \underline{3\,b} \end{array}$	79. $\begin{array}{l} a^2d \\ \underline{b^2d} \end{array}$	80. $\begin{array}{r} x^2w \\ \underline{-\,y^2w} \end{array}$	81. $\begin{array}{l} x \\ \underline{xy} \end{array}$
82. $\begin{array}{l} bc \\ \underline{b} \end{array}$	83. $\begin{array}{l} ab \\ \underline{bc} \end{array}$	84. $\begin{array}{l} xy \\ \underline{yz} \end{array}$	85. $\begin{array}{l} 6\,cd \\ \underline{5\,bd} \end{array}$
86. $\begin{array}{l} 2\,ax \\ \underline{3\,bx} \end{array}$	87. $\begin{array}{r} 5\,xy \\ \underline{-\,2\,xz} \end{array}$	88. $\begin{array}{r} -\,9\,ab \\ \underline{-\,4\,ac} \end{array}$	89. $\begin{array}{r} 12\,abc \\ \underline{-\,5\,abd} \end{array}$

Expressions which contain a common binomial or a common polynomial factor may be added with reference to this common factor.

We may regard the factors which are not common as being coefficients with reference to the common factors and, finding their sum as positive and negative numbers, prefix this as a coefficient to the common factor.

Ex. 90. Find the sum of $a(x + y)$ and $b(x + y)$.

Regarding a and b as coefficients of the common factors of the two given terms, we may write their sum as a coefficient to the common part.

$$a(x + y) + b(x + y) \equiv (a + b)\,(x + y).$$

Ex. 91. Find the sum of $3(x^2 - y^2)$, $4\,a(x^2 - y^2)$, and $-\,2\,b(x^2 - y^2)$.

Adding the coefficients 3, $4\,a$, and $-\,2\,b$ with respect to the common factor $x^2 - y^2$, we have

$$3(x^2 - y^2) + 4\,a(x^2 - y^2) - 2\,b(x^2 - y^2) \equiv (3 + 4\,a - 2\,b)\,(x^2 - y^2).$$

92. $\begin{array}{r} x(c-d) \\ \underline{y(c-d)} \end{array}$

93. $\begin{array}{r} a(b+c) \\ \underline{(b+c)} \end{array}$

94. $\begin{array}{r} x(y-z) \\ \underline{-(y-z)} \end{array}$

95. $\begin{array}{r} 2a(m+n) \\ \underline{b(m+n)} \end{array}$

96. $\begin{array}{r} 2ab(g-k) \\ \underline{3cd(g-k)} \end{array}$

97. $\begin{array}{r} a^2(x+y) \\ \underline{-b^2(x+y)} \end{array}$

98. $\begin{array}{r} a(x^2-y^2) \\ \underline{b(x^2-y^2)} \end{array}$

99. $\begin{array}{r} -(c^2+1) \\ \underline{2d(c^2+1)} \end{array}$

SUBTRACTION OF MONOMIALS

27. The algebraic difference of two numbers or quantities may always be indicated by writing the subtrahend after the minuend, separating the two by the sign of operation for subtraction —, each being preceded by its proper quality sign.

From the principles affecting operations with positive and negative numbers it appears that *instead of a subtraction of a number we may substitute the addition of a number equal to it in absolute value but of opposite quality.*

That is, *to subtract a given number or term we change its sign of quality and then proceed as in addition.*

As in the case of addition, this collecting of several numbers or quantities into one algebraic expression is what in algebra is called **subtraction.** The resulting expression is called the **difference.**

Ex. 1. From $5x$ subtract $2x$.

Indicating the process by the *horizontal arrangement*, we have

$$5x - 2x \equiv (5-2)x \equiv 3x.$$

When employing the *vertical arrangement*, $\begin{array}{c} 5x \\ \underline{2x} \end{array}$, instead of actually altering the sign of quality of the subtrahend $2x$, we should make the change mentally when performing the operation, and write the result immediately underneath as in arithmetic.

Ex. 2. From $-3y$ subtract $7y$.

Using the horizontal arrangement, we have

$$-3y - (+7y) \equiv -3y - 7y \equiv (-3-7)y \equiv -10y.$$

Using the vertical arrangement, we have $\begin{array}{r} -3y \\ \underline{+7y} \\ -10y \end{array}$.

28. Caution. The student should be careful, when employing the vertical arrangement for subtraction, not to actually change the sign of the subtrahend on paper. Such a change of sign is confusing when the work is reviewed.

29. It is customary, whenever possible, to so arrange an expression as to have the first term preceded by a positive rather than a negative sign.

E. g. We should consider $-b+a$ to be arranged in better order if written $+a-b$.

Mental Exercise. VI. 2

Perform the following indicated subtractions:

1. $\begin{array}{r}6a\\ \underline{4a}\end{array}$
2. $\begin{array}{r}9b\\ \underline{6b}\end{array}$
3. $\begin{array}{r}11c\\ \underline{3c}\end{array}$
4. $\begin{array}{r}-12d\\ \underline{-4d}\end{array}$
5. $\begin{array}{r}-10e\\ \underline{-7e}\end{array}$
6. $\begin{array}{r}-13m\\ \underline{5m}\end{array}$
7. $\begin{array}{r}14n\\ \underline{-8n}\end{array}$
8. $\begin{array}{r}-16q\\ \underline{-12q}\end{array}$
9. $\begin{array}{r}5r\\ \underline{7r}\end{array}$
10. $\begin{array}{r}8s\\ \underline{10s}\end{array}$
11. $\begin{array}{r}-3x\\ \underline{-8x}\end{array}$
12. $\begin{array}{r}-4y\\ \underline{6y}\end{array}$
13. $\begin{array}{r}-15z\\ \underline{z}\end{array}$
14. $\begin{array}{r}-17w\\ \underline{17w}\end{array}$
15. $\begin{array}{r}-18h\\ \underline{-18h}\end{array}$
16. $\begin{array}{r}2a^2\\ \underline{a^2}\end{array}$
17. $\begin{array}{r}5b^3\\ \underline{7b^3}\end{array}$
18. $\begin{array}{r}4d^3\\ \underline{-11d^3}\end{array}$
19. $\begin{array}{r}-19x^2\\ \underline{-23x^2}\end{array}$
20. $\begin{array}{r}21y^4\\ \underline{-19y^4}\end{array}$
21. $\begin{array}{r}6ab\\ \underline{27ab}\end{array}$
22. $\begin{array}{r}31ac\\ \underline{-26ac}\end{array}$
23. $\begin{array}{r}-29bd\\ \underline{41bd}\end{array}$
24. $\begin{array}{r}33mw\\ \underline{-27mw}\end{array}$
25. $\begin{array}{r}24a^2b\\ \underline{-14a^2b}\end{array}$
26. $\begin{array}{r}-37cd^2\\ \underline{-18cd^2}\end{array}$
27. $\begin{array}{r}42m^2n^2\\ \underline{-31m^2n^2}\end{array}$
28. $\begin{array}{r}-58x^3y^2\\ \underline{46x^3y^2}\end{array}$
29. $\begin{array}{r}16abc\\ \underline{-73abc}\end{array}$
30. $\begin{array}{r}-25cdy\\ \underline{71cdy}\end{array}$
31. $\begin{array}{r}56bgm\\ \underline{-17bgm}\end{array}$
32. $\begin{array}{r}-43a^2bc\\ \underline{28a^2bc}\end{array}$
33. $\begin{array}{r}62ab^2c\\ \underline{-47ab^2c}\end{array}$
34. $\begin{array}{r}-77abc^2\\ \underline{-38abc^2}\end{array}$
35. $\begin{array}{r}76a^2b^2c\\ \underline{-67a^2b^2c}\end{array}$
36. $\begin{array}{r}-39ab^2c^2\\ \underline{93ab^2c^2}\end{array}$
37. $\begin{array}{r}45a^2bc^2\\ \underline{54a^2bc^2}\end{array}$
38. $\begin{array}{r}-56x^3y^2z\\ \underline{67x^3y^2z}\end{array}$
39. $\begin{array}{r}-101a^2xy\\ \underline{-75a^2xy}\end{array}$
40. $\begin{array}{r}123m^3nw\\ \underline{95m^3nw}\end{array}$
41. $\begin{array}{r}158syw\\ \underline{85syw}\end{array}$
42. $\begin{array}{r}-193gnq\\ \underline{89gnq}\end{array}$
43. $\begin{array}{r}-150chw\\ \underline{91chw}\end{array}$
44. $\begin{array}{r}-239bk^2\\ \underline{167bk^2}\end{array}$
45. $\begin{array}{r}2a\\ \underline{\frac{1}{3}a}\end{array}$
46. $\begin{array}{r}5b\\ \underline{\frac{1}{6}b}\end{array}$
47. $\begin{array}{r}\frac{1}{2}c\\ \underline{\frac{1}{4}c}\end{array}$
48. $\begin{array}{r}\frac{3}{5}d\\ \underline{\frac{1}{2}d}\end{array}$

49. $\frac{1}{3}h$ $\frac{1}{7}h$	53. $.5x$ $.2x$	57. $1.6a$ $.1a$	61. $.36g$ $3.64g$
50. $\frac{2}{3}k$ $-\frac{1}{4}k$	54. $.7y$ $.3y$	58. $2.28b$ $.02b$	62. $.345x$ $-.655x$
51. $\frac{4}{5}m$ $\frac{5}{4}m$	55. $.08z$ $.04z$	59. $.33c$ $.03c$	63. $-.583y$ $-5.417y$
52. $-\frac{4}{7}n$ $\frac{5}{8}n$	56. $.11w$ $.03w$	60. $2.51d$ $.25d$	64. $1.001z$ $-.011z$

In case two unlike terms contain the same letter or factor, we may perform the subtraction with reference to this letter or factor, regarding the remaining factors as coefficients.

Ex. 65. From ax subtract bx.

Since a and b are the coefficients of x, we may express the difference by writing the algebraic difference $(a - b)$ as a coefficient of the common letter x, or $(a - b)x$.

Perform the following subtractions with reference to similar parts:

66. ac bc	71. bw $-nw$	76. $-bnq$ $-hnq$	81. $-7xy^2$ y^2
67. dy xy	72. bm^2 hm^2	77. $5axyz$ $2bxyz$	82. ab bc
68. aw mw	73. $-gy^3$ $-my^3$	78. $2ab$ b	83. xy xz
69. $-ct$ $-dt$	74. axy bxy	79. $-3cd$ $-d$	84. x^2y^2 $-y^2z^2$
70. $-ay$ xy	75. m^2xw $-n^2xw$	80. $5x^2y$ $-y$	85. xyz yzw

The difference between two compound expressions which contain the same polynomial factor may be found with reference to this polynomial factor.

Ex. 86. From $a(x+y)$ subtract $-b(x+y)$.

Regarding a and $-b$ as coefficients of the binomial factor $(x+y)$ we may find the difference by prefixing the algebraic difference of a and $-b$, as positive and negative numbers, as a coefficient to the common factor $(x+y)$ as follows:

$$\begin{array}{r} a(x+y) \\ -b(x+y) \\ \hline (a+b)(x+y) \end{array}$$

87. $\begin{array}{r} 5(a+b) \\ \underline{2(a+b)} \end{array}$

88. $\begin{array}{r} 8(c-a) \\ \underline{-3(c-a)} \end{array}$

89. $\begin{array}{r} -6(m-n) \\ \underline{9(m-n)} \end{array}$

90. $\begin{array}{r} -18(g+h) \\ \underline{-18(g+h)} \end{array}$

91. $\begin{array}{r} x(a+b) \\ \underline{y(a+b)} \end{array}$

92. $\begin{array}{r} c(m-n) \\ \underline{d(m-n)} \end{array}$

93. $\begin{array}{r} h(m-x) \\ \underline{-c(m-x)} \end{array}$

94. $\begin{array}{r} a(c+1) \\ \underline{-b(c+1)} \end{array}$

95. $\begin{array}{r} a(b+1) \\ \underline{(b+1)} \end{array}$

96. $\begin{array}{r} m(n-1) \\ \underline{-(n-1)} \end{array}$

97. $\begin{array}{r} (x-1) \\ \underline{x(x-1)} \end{array}$

98. $\begin{array}{r} 2(c+g) \\ \underline{3b(c+g)} \end{array}$

99. $\begin{array}{r} xy(z-w) \\ \underline{z(z-w)} \end{array}$

100. $\begin{array}{r} ab(a-bc) \\ \underline{c(a-bc)} \end{array}$

101. $\begin{array}{r} x(xy-z) \\ \underline{yz(xy-z)} \end{array}$

102. $\begin{array}{r} ax(bx-cy) \\ \underline{-by(bx-cy)} \end{array}$

30. A polynomial is said to be **reduced** when its like or similar terms have all been combined, as far as possible; that is, when it contains no similar terms.

Ex. 1. Reduce $3x+2y+5z-2x-3y+7z+4y$ to simplest form.

We have $3x+2y+5z-2x-3y+7z+4y \equiv x+3y+12z$.

Exercise VI. 3

Reduce each of the following polynomials to simplest form:

1. $2a-4b+6c-a+5b-2c$.
2. $5x+7y-2z-4x-6y+z$.
3. $11a-2d+3b+4d-b+a$.
4. $12a-b+d+3c+2b-2c-a+3b$.

5. $13x - 13y + 14z - 14w - 2z + 2y - 2w + 4z$.
6. $6x + y - 10z - 3y - 2x + 9z + 4x - 5y + 8z$.
7. $3a + 5b - c + 2 - 2a - 4b + 6c + 7 - a$.
8. $5k - 4m - 8n + 9 - 3k + 4m + 8n - 9 - 2k$.
9. $4a^2 - 2ab + 10c^2 + 3ab + a^2 + 2b^2 - 3c^2 - ab + 6b^2$.
10. $9x^2 - xy + 3 - 5x^2 + 8y^2 + xy + 5 - 4x^2 - 8y^2$.

31. The Check of Arbitrary Values. Since, unless the contrary is expressly stated, the result of any operation with letters is obtained without restricting the values of the letters, we may assume that they have such values as we choose to assign to them. This simple check of arbitrary values will be found in most cases to be sufficient to indicate errors in a calculation if there be any.

Whenever numerical checks are used, it should be understood that the substitutions are to be made in the example as originally given, and also in the final result. All intermediate steps leading from the first indicated operations to the final result should be neglected. If the original indicated operations, when performed with numerical values, produce the same result as is found by substituting numerical values in the final result — that is, **if our work "balances" — we have a check** (except in certain very special cases) upon the accuracy of all of the intermediate steps which have been neglected in the checking process.

If, in checking examples, the value 1 be assigned to any particular letter, errors among exponents may not be detected, since all integral powers of unity are 1.

32. A check upon an operation is another operation which is such as to verify the result first obtained.

E.g. Such a check is the employment of addition to verify an example in subtraction, or the multiplication of a divisor by the quotient to obtain the dividend in an example in division.

ADDITION OF POLYNOMIALS

33. To add a polynomial, it is sufficient to add its terms successively.

When adding two polynomials it will be found convenient *first to arrange the terms according to the powers of some letter of reference,* *and then to write the polynomials, one under the other, in such a way that similar terms shall be in the same vertical column.*

Then add the columns separately and connect their sums by the resulting signs.

Ex. 1. Add $3xy^2 - 3x^2y + x^3 - y^3$, $x^3 + 2y^3 + 2x^2y$, and $-x^2y + 4xy^3 - y^3$.

Arrange the polynomials according to descending powers of x, and write them so that similar terms shall appear in vertical columns, as below. Adding the first column, we have $2x^3$; adding the second column, $-2x^2y$; adding the third column, $7xy^2$; in the fourth column the sum is 0.

Hence the resulting sum is $2x^3 - 2x^2y + 7xy^2$.

Check. Let $x = 2$, $y = 3$.

$$\begin{array}{lllll}
x^3 & -3x^2y & +3xy^2 & -y^3 & \ldots\ldots\ -1 \\
x^3 & +2x^2y & & +2y^3 & \ldots\ldots\ \ \ 86 \\
 & -x^2y & +4xy^2 & -y^3 & \ldots\ldots\ \ \ \underline{33} \\
\hline
 & & & & \qquad\qquad 118 \\
2x^3 & -2x^2y & +7xy^2 & & \ldots\ldots\ 118 \\
 & & & & \qquad\qquad\ \ 0
\end{array}$$

We may check the result by substituting the values 2 and 3 for x and y respectively in the given expressions, obtaining the values -1, 86 and 33, as shown above.

The algebraic sum of these values, 118, should "balance" with the result obtained by substituting the same values, $x = 2$ and $y = 3$, in the result of the algebraic addition, $2x^3 - 2x^2y + 7xy^2$. This value is found to be 118, and according to this test the work is correct.

34.* Detached Coefficients. In performing an example in addition, when writing several similar terms in a column, it is not strictly necessary to write the literal factors every time, provided that it is understood that they are the same as those of the term at the head of the column.

E. g. Instead of

$$\begin{array}{r}
+5a^3b^2c \\
-3a^3b^2c \\
+2a^3b^2c \\
\hline
4a^3b^2c
\end{array}
\quad \text{write} \quad
\begin{array}{r|l}
+5 & a^3b^2c \\
-3 & \\
+2 & \\
\hline
4 & a^3b^2c
\end{array}$$

---

* This section may be omitted when the chapter is read for the first time.

Ex. 2. Add the following, using Detached Coefficients:

$3x^2+2xy+5xy^2-y^3$, $3xy-2xy^2+x^3-2y^3$, and $3x^3+2x^2-xy+3xy^2$.

Check. Let $x=2$, $y=4$.

$$\begin{array}{r|r|r|r|r|lr}
 & 3 & x^2+2 & xy+5 & xy^2- & y^3\ \ldots\ldots & +124 \\
 & x^3 & +3 & -2 & -2 & \ \ \ \ \ldots\ldots & -160 \\
3 & +2 & -1 & +3 & & \ \ \ \ \ldots\ldots & +120 \\
\hline
 & & & & & & 84 \\
\multicolumn{6}{l}{4\ x^3+5\ x^2+4\ xy+6\ xy^2-3\ y^3\ \ldots\ldots} & \underline{84} \\
 & & & & & & 0
\end{array}$$

Exercise VI. 4

Perform the following indicated additions; the answers to the first twenty-eight examples may be obtained mentally:

1. $\begin{array}{r} 3a+2b \\ a+5b \\ \hline \end{array}$

2. $\begin{array}{r} 6a-9b \\ 8a-3b \\ \hline \end{array}$

3. $\begin{array}{r} 7b-11c \\ 4b-19c \\ \hline \end{array}$

4. $\begin{array}{r} 6d+\ \ g \\ 6d-12g \\ \hline \end{array}$

5. $\begin{array}{r} 5a+2b \\ 3a+\ \ b \\ \hline \end{array}$

6. $\begin{array}{r} 6c-3d \\ 4c-2d \\ \hline \end{array}$

7. $\begin{array}{r} 8m-11n \\ 4m+14n \\ \hline \end{array}$

8. $\begin{array}{r} 16x+21y \\ 32x-23y \\ \hline \end{array}$

9. $\begin{array}{r} 7a+b \\ 2a+b \\ \hline \end{array}$

10. $\begin{array}{r} 9b-3c \\ 9b+8c \\ \hline \end{array}$

11. $\begin{array}{r} 5x-7y \\ -5x+7y \\ \hline \end{array}$

12. $\begin{array}{r} 3a-6b \\ 3a-6b \\ \hline \end{array}$

13. $\begin{array}{r} -10h-15y \\ -3h+\ \ 4y \\ \hline \end{array}$

14. $\begin{array}{r} -16m-20q \\ -17m-11q \\ \hline \end{array}$

15. $\begin{array}{r} 15ab-\ \ \ \ cd \\ -23ab+29cd \\ \hline \end{array}$

16. $\begin{array}{r} 9a-14bc \\ -26a+33bc \\ \hline \end{array}$

17. $\begin{array}{r} 2a+3b-4c \\ a-5b+7c \\ \hline \end{array}$

18. $\begin{array}{r} x-3y+5z \\ -2x+4y-6z \\ \hline \end{array}$

19. $\begin{array}{r} 8a+6b-4c \\ -3a-5b+7c \\ \hline \end{array}$

20. $\begin{array}{r} 2a+\ \ 5b-\ \ 7c \\ 3a-11b+14c \\ \hline \end{array}$

21. $\begin{array}{r} 4b-12c-24d \\ 6b+12c-\ \ 7d \\ \hline \end{array}$

22. $\begin{array}{r} 9x-8y+3z \\ 3x-8y+9z \\ \hline \end{array}$

23. $\begin{array}{r} 10a-7b+12c \\ 3a-7b-10c \\ \hline \end{array}$

24. $\begin{array}{r} 3x-4y+5z \\ 5x+4y-3z \\ \hline \end{array}$

25. $$\begin{array}{r} 4a^2+9b^2-16 \\ 5a^2+3b^2-\ \ 1 \\ \hline \end{array}$$

26. $$\begin{array}{r} 10a-\quad\ m+15w \\ -a+19m-28w \\ \hline \end{array}$$

27. $$\begin{array}{r} 13a^2-17b^2+\quad\ c^2 \\ 41a^2-21b^2-33c^2 \\ \hline \end{array}$$

28. $$\begin{array}{r} 11x^2+13xy+21y^2 \\ 11x^2-31xy-12y^2 \\ \hline \end{array}$$

Find the sums of the following groups of expressions:

29. $a+2b+c$; $3a+b+c$; $a+b+4c$.
30. $2x+y+3z$; $x+4y+z$; $5x+6y+8z$.
31. $5b-c+2d$; $3b-2c+4d$; $b-3c+d$.
32. $6x-9y-2z$; $7x-10y-3z$; $4x-5y-8z$.
33. $7a-2b+6x$; $5a+8b-4x$; $a-9b-3x$.
34. $5b+9d-4w$; $8b-7d-3w$; $6b-2d-w$.
35. $g-4h+8k$; $7g-h-2k$; $3g+5h-6k$.
36. $m-3n+7y$; $5m-8n+2y$; $9m-4n+6y$.
37. $2a-5b-3c$; $-9a-b-4c$; $8a+6b+7c$.
38. $6x+3y-2z$; $-4x-7y-8z$; $-x+5y+9z$.
39. $3a-2b+4c-10$; $3b-2c+8$; $-2a-c+1$.
40. $2y^3-z^2+10$; z^2-w^2-9; y^3+w^2.
41. $4m^3+x^2-7$; $-6m^3+4x$; $7m^3-x^2-3x+7$.
42. $c^3-4c^2d-2cd^2+2d^3$; $3c^2d+2cd^2$; c^2d-d^3.
43. $9p^2q-6pq^2-7p^2q^3-q^4$; $-8p^2q+6pq^2+7p^2q^3$.
44. $4ab-2ac+10abc$; $-6ab-9abc+2bc$; $3ab+4ac-abc-3bc$.
45. $7x^3+8x-11$; $-4x^2+12$; $-5x^3-6x-2$; $-x^3+5x^2-x+2$.

SUBTRACTION OF POLYNOMIALS

35. *To subtract a polynomial we may subtract successively the terms of which it is composed. Hence we may reverse the sign of quality of each of its terms and then proceed as in addition.*

Ex. 1. Subtract $4x^2+9xy-7y^2$ from $6x^2+xy-3y^2$.

We may indicate the operation by writing

$$6x^2+xy-3y^2-(4x^2+9xy-7y^2).$$

Removing the terms from the parentheses preceded by the minus sign, and reversing the signs of quality of the terms removed, we have

$$6x^2+xy-3y^2-(4x^2+9xy-7y^2)\equiv 6x^2+xy-3y^2-4x^2-9xy+7y^2$$

Combining like terms, $\equiv 2x^2-8xy+4y^2.$

Whenever, as in the example above, the operation of subtraction is indicated by enclosing the subtrahend in parentheses preceded by a minus sign, we may remove the parentheses and actually change the signs of the numbers removed.

If, however, the subtrahend is simply written underneath the minuend and we are directed to subtract one expression from the other, it is better not to change the signs on paper, but to make the change mentally while performing the calculation. (Compare with § 28.)

The following arrangement is usually more convenient:

$$\begin{array}{lr} & \text{Check. Let } x = 4,\ y = 3. \\ 6x^2 + xy - 3y^2 \ldots\ldots & 92 \\ 4x^2 + 9xy - 7y^2 \ldots\ldots & \underline{110} \\ \hline & -18 \\ 2x^2 - 8xy + 4y^2 \ldots\ldots & \underline{\underline{-18}} \\ & 0 \end{array}$$

EXERCISE VI. 5

Perform the following indicated subtractions, reversing the signs mentally; the answers to the first sixteen examples may be obtained mentally:

1. $\begin{array}{r} 2a + b \\ \underline{a - b} \end{array}$

2. $\begin{array}{r} 3a + 4b \\ \underline{2a - 5b} \end{array}$

3. $\begin{array}{r} 4x - 7y \\ \underline{5x - 6y} \end{array}$

4. $\begin{array}{r} 7b + c \\ \underline{7b - 2c} \end{array}$

5. $\begin{array}{r} 6r - s \\ \underline{3r - s} \end{array}$

6. $\begin{array}{r} 2a + x \\ \underline{-3a - 2x} \end{array}$

7. $\begin{array}{r} -2m - 3 \\ \underline{-3m - 4} \end{array}$

8. $\begin{array}{r} 2 + a \\ \underline{3 - 2a} \end{array}$

9. $\begin{array}{r} -10x - y \\ \underline{-11x + y} \end{array}$

10. $\begin{array}{r} -ac - b \\ \underline{2ac + 2b} \end{array}$

11. $\begin{array}{r} 4x - 5y \\ \underline{-3x - 6y} \end{array}$

12. $\begin{array}{r} xy - zw \\ \underline{-xy + zw} \end{array}$

13. $\begin{array}{r} -ax + by \\ \underline{2ax - 2by} \end{array}$

14. $\begin{array}{r} 3a + b + c \\ \underline{2a + 2b - c} \end{array}$

15. $\begin{array}{r} -4x - 5y + 6z \\ \underline{5x + 6y + 7z} \end{array}$

16. $\begin{array}{r} 8a - 7b + 6c \\ \underline{-6a + 7b - 8c} \end{array}$

17. From $6a - 10b - 7$ subtract $4a - 5b - 2$.
18. From $5a + 8b - 9$ subtract $2a + 3b - 4$.
19. Subtract $a^2 + 2ab + b^2$ from $2a^2 - 2ab + 2b^2$.
20. From $3x^2 + 4xy + 8$ subtract $4x^2 - 2xy - 3$.
21. Subtract $x^2 + 3xy + y^2$ from $4x^2 + 10xy + 13y^2$.
22. Subtract $x^3 + x^2y + xy^2 + y^3$ from $x^3 + y^3$.
23. From $a^3 + 3a^2b + 3ab^2 + b^3$ subtract $a^3 + b^3$.
24. Subtract $m^3 - 3m^2n + 3mn^2 - n^3$ from $m^3 - n^3$.
25. From $a^3 + a^2 + a$ subtract $-2a^3 - 3a^2 + a$.
26. From $ab + bc + ca$ subtract $-ab + bc - ca$.
27. Subtract $2a^2 - ab - 5b^2$ from $3a^2 + ab - 4b^2$.
28. Subtract $6x^2 - 7xy - 8y^2$ from $7x^2 - 8xy - 7y^2$.
29. Subtract $5x^2y^2 - 10xy^3 + 16y^4$ from $5x^2y^2 + 10xy^3 + 16y^4$.
30. Subtract $a^3 + 3a^2b + 3ab^2 + b^3$ from $a^3 - 3a^2b + 3ab^2 - b^3$.
31. From $x^4 + 5x^3y + 6x^2y^2 + 3xy^3 + y^4$ subtract $5x^3y + 6x^2y^2 + 3xy^3$.
32. Subtract $c^3 - c^2d + d^3$ from $3c^3 - c^2d + 4cd^2 + 5d^3$.
33. From $5g^5 + 6g^3h^2 + 3g^2h + 7h^5$ subtract $3g^5 + 6g^3h^2 - 5h^5$.
34. Subtract $4a^4 + 7a^3b + 6a^2b^2 + b^4$ from $4a^4 + 10a^3b - 15a^2b^2 + 9b^4$.
35. From $12a^3 + 19a^2b + 17ab^2 + 21b^3$ subtract $5a^3 + 17a^2b - ab^2 - 20b^3$.
36. From $13d^5 - 23d^4r + 11d^3r^2 + 14r^5$ subtract $13d^5 - 24d^4r + 10d^3r^2 + 14r^5$.

When we are required to subtract the sum of several polynomials from the sum of several others, we may treat the problem as one of addition by actually changing the signs of all those expressions which are to be subtracted and then proceeding with the resulting expressions as an example in addition.

37. From the sum of $a + 2b + 3c$ and $3a - b + 4c$, subtract $2a - b - 5c$.
38. From the sum of $5x - y + 4z$ and $4x - 2y - 3z$, subtract $3x + 3y - 5z$.
39. Subtract $3a + 2b + 2c - 3d$ from the sum of $a - b + c - d$ and $2a - b + 2c - d$.

40. Subtract $7a - 3b - c - d$ from the sum of $6a - b + 4c + d$ and $3a + 4b - 2c + d$.

41. From the sum of $a^2 + ab + b^2$ and $3a^2 - 2ab - 4b^2$, subtract $2a^2 - 3ab - 3b^2$.

42. Subtract $ab + 3bc - 4cd$ from the sum of $6ab - 4bc + cd$ and $-4ab + 2bc - 3cd$.

43. From the sum of $ab + bc$, $bc - cd$ and $da - cd$, subtract $ab + bc - cd + da$.

44. From the sum of $3ab + bc$, $-4bc + cd$ and $2da - 2cd$, subtract $2ab - 2bc + cd + da$.

CHAPTER VII

MULTIPLICATION OF INTEGRAL ALGEBRAIC EXPRESSIONS

(For Definitions, See Chapter V)

Principles Relating to Powers

1. We have already defined a^n to mean the product of n factors each equal to a. Accordingly, we have as a

(i.) **Definition Formula:** $a^n \equiv \overbrace{a \times a \times a \times \cdot \cdot \cdot \cdot \cdot \times a}^{n \text{ factors}}$,
n being understood to be a positive whole number, and a being different from zero. (See Chapter VI. § 5).

E.g. $2^5 = 2 \times 2 \times 2 \times 2 \times 2$; $a^3 \equiv a \times a \times a$.

Product of Powers of the Same Base

As a direct consequence of the definition formula above, we have for equal bases the following

(ii.) **Distribution Formula** $a^m \times a^n \equiv a^{m+n}$.

If a be any number other than zero, we may indicate the product obtained by multiplying a^m by a^n by writing

$$a^m \times a^n \equiv \overbrace{(aaa \cdot \cdot \cdot \cdot \cdot a)}^{m \text{ factors}}\overbrace{(aaa \cdot \cdot \cdot \cdot \cdot a)}^{n \text{ factors}} \equiv a^{m+n}.$$

The principle expressed by the distribution formula may be stated as follows:

Law of Indices. *The product obtained by multiplying a power of a given base by another power of the same base is a power of the same base, the exponent of which is found by adding the exponents of the factors.*

E.g. $b^3 \times b^2 \equiv b^5$.

In order to find the power of a power of a base we may employ the (iii.) **Association Formula** $(a^m)^n \equiv \overbrace{a^m \times a^m \times a^m \times \cdots\cdots \times a^m}^{n\text{ factors}}$

$$\equiv a^{\overbrace{m+m+m+\cdots\cdots+m}^{n\text{ terms}}}$$

$$\equiv a^{mn}.$$

That is, the power of a power of a base is a power of the base, the exponent of which is found by multiplying the exponent of the given power of the base by the exponent of the required power.

E. g. $(c^3)^2 \equiv c^6.$

Powers of Products of Different Bases

(iv.) **Distribution Formula** $(\boldsymbol{ab})^m \equiv \boldsymbol{a}^m\boldsymbol{b}^m.$

From the definition of a power, we have

$$(ab)^m \equiv \overbrace{(ab) \times (ab) \times (ab) \times \cdots\cdots \times (ab)}^{m\text{ factors}}.$$

$$\equiv \overbrace{a \times a \times a \times \cdots\cdots \times a}^{m\text{ factors}} \times \overbrace{b \times b \times b \times \cdots\cdots \times b}^{m\text{ factors}}.$$

$$\equiv a^m b^m.$$

The n^{th} power of a product of several factors may be written as the product of the n^{th} powers of these factors, and conversely.

Ex. 1. $(2\,a)^3 \equiv 2^3a^3$
$\equiv 8\,a^3.$

Ex. 2. $(5\,b^4)^2 \equiv 5^2(b^4)^2$
$\equiv 25\,b^8.$

Ex. 3. $(-\,c)^3 \equiv (-\,1)^3c^3$
$\equiv -c^3.$

Ex. 4. $(-\,5\,x)^2 \equiv (-\,5)^2x^2$
$\equiv 25\,x^2.$

It may be shown that the laws above hold for more than two exponents or for more than two factors. That is,

Corresponding to (ii.), above, we have

$$\boldsymbol{a}^m \times \boldsymbol{a}^n \times \boldsymbol{a}^p \times \cdots\cdots \times \boldsymbol{a}^w \equiv \boldsymbol{a}^{m+n+p+\cdots\cdots+w}.$$

Corresponding to (iii.)

$$(((\boldsymbol{a}^m)^n)^p)^{q\cdots\cdots} \equiv \boldsymbol{a}^{mnpq\cdots\cdots}.$$

Corresponding to (iv.)

$$(\boldsymbol{abcd}\cdots\cdots\boldsymbol{x})^m \equiv \boldsymbol{a}^m \times \boldsymbol{b}^m \times \boldsymbol{c}^m \times \boldsymbol{d}^m \times \cdots\cdots \times \boldsymbol{x}^m.$$

MENTAL EXERCISE VII. 1

Express each of the following products of different powers of the same base as a single power of the given base :

1. $2^4 \times 2^3$.
2. $4^2 \times 4$.
3. $3^5 \times 3$.
4. $5^4 \times 5$.
5. $2^5 \times 2^2$.
6. $a^2 \times a^4$.
7. $b^3 \times b^7$.
8. $c^4 \times c^6$.
9. $d^5 \times d^2$.
10. $e^7 \times e^4$.
11. $g^8 \times g^3$.
12. $h^6 \times h^9$.
13. $k^5 \times k^8$.
14. $m^7 \times m^{10}$.
15. $n^9 \times n^8$.
16. $x^8 \times x^7$.
17. $a^2 \times a^3 \times a^4$.
18. $b^3 \times b^5 \times b^6$.
19. $c^5 \times c^4 \times c^7$.
20. $d^8 \times d^{10} \times d^4$.
21. $h^6 \times h^7 \times h$.
22. $m^2 \times m^8 \times m^{10}$.
23. $n^6 \times n^8 \times n^3$.
24. $x^8 \times x^9 \times x^{10}$.
25. $a^{2m} \times a^{3m}$.
26. $b^{4n} \times b^{2n}$.
27. $c^{5r} \times c^r$.
28. $d^{n-2} \times d^2$.
29. $h^{n+5} \times h^{n-5}$.
30. $x^{a+b+1} \times x^{a-b-1}$.
31. $x^a \times x^b \times x^c$.
32. $y^{2n} \times y^{3n} \times y^{4n}$.
33. $z^{2r} \times z^{3s} \times z^{5t}$.

Express each of the following powers of powers as a single power of the given base :

34. $(2^2)^3$.
35. $(3^8)^2$.
36. $(5^2)^2$.
37. $(6^2)^2$.
38. $(10^2)^3$.
39. $(7^2)^2$.
40. $(a^2)^2$.
41. $(b^2)^3$.
42. $(c^8)^2$.
43. $(d^2)^4$.
44. $(e^3)^4$.
45. $(g^2)^5$.
46. $(g^4)^5$.
47. $(h^2)^6$.
48. $(k^6)^2$.
49. $(m^5)^5$.
50. $(n^6)^7$.
51. $(p^8)^9$.
52. $(q^9)^7$.
53. $(r^8)^{10}$.
54. $(s^{12})^{12}$.
55. $(a^2)^m$.
56. $(b^n)^2$.
57. $(c^n)^3$.
58. $(d^4)^n$.
59. $(h^m)^m$.
60. $(k^n)^n$.
61. $(m^{2x})^x$.
62. $(n^{3y})^{2y}$.
63. $(x^{4a})^{4b}$.

Express the following powers of products as products of powers :

64. $(3a)^2$.
65. $(4b)^3$.
66. $(6c)^2$.
67. $-(2d)^4$.
68. $(3m)^4$.
69. $(-7x)^2$.
70. $(-4y)^3$.
71. $(-2z)^5$.
72. $-(2ab)^5$.
73. $(3bc)^4$.
74. $(2xy)^6$.
75. $(-3yz)^3$.
76. $-(5yw)^3$.
77. $-(-3ac)^3$.
78. $(a^2b)^3$.
79. $(bc^2)^4$.
80. $(c^2d^2)^5$.
81. $(m^nn)^6$.
82. $(x^2y^3)^7$.
83. $(ab^2c)^2$.
84. $-(a^2bc)^2$.
85. $(ab^2c^3)^4$.
86. $(a^5b^4c^3)^2$.
87. $(-2x^3y^4z^6)^2$.
88. $-(7x^5y^7z^2)^2$.
89. $-(-4x^4)^4$.
90. $(ab)^c$.
91. $(bc)^d$.
92. $(a^2b)^n$.
93. $(cd^3)^r$.
94. $-(xy^2)^n$.
95. $(a^mb)^m$.
96. $(x^ay^b)^c$.
97. $-(a^xb^y)^c$.
98. $(a^{2x}b^y)^{2z}$.
99. $(2x^{3y})^{4z}$.

Express the following products of powers as powers of products:

100. $2^3 \times 3^3$.
101. $5^2 \times 2^2$.
102. $3^2 \times 7^2$.
103. $4^2 \times 3^2$.
104. $a^2 \times b^2$.
105. $b^3 \times c^3$.
106. $m^4 \times n^4$.
107. $a^2 \times b^2 \times c^2$.
108. $x^3 \times y^3 \times z^3$.
109. $8\,a^3b^3$.
110. $25\,x^2y^2z^2$.
111. $32\,a^5b^5x^5$.
112. $64\,a^6x^6z^6$.
113. $(-a)^4b^4$.
114. $(-x)^3y^3z^3$.

2. Product of Two or More Monomials. The product obtained by multiplying one monomial by another is the monomial obtained by multiplying together all the factors of the two in any order.

Hence, ***to multiply one monomial by another,*** *determine first the sign of the quality of the product. Then, for a numerical coefficient, write the product of the numeral factors of the monomials, followed by the product of the different letters, each letter having for an exponent the sum of the exponents of this letter in the two monomials.*

Proceed similarly for products of three or more monomials.

Ex. 1. Multiply $3\,x^2y^3$ by $2\,xy^4z$.

Both terms are understood to be positive since no signs are written before them. We have as a numerical coefficient the product of 3 and 2; x occurs to the $2 + 1$ or 3rd power; y to the $3 + 4$ or 7th power; z to the 1st power.

Hence, $$3\,x^2y^3 \times 2\,xy^4z \equiv 6\,x^3y^7z.$$

Ex. 2. Find the product of $5\,ab^2c$ and $-2\,abc^2$.

Since the terms are of opposite quality, the product is a negative number.

Hence, $$(5\,ab^2c) \times (-2\,abc^2) \equiv -10\,a^2b^3c^3.$$

Ex. 3. Find the continued product of $3\,a^2b$, $2\,bc$ and $-5\,ac$.

Since two of the given numerical coefficients are positive numbers and one is a negative number, we may write for a numerical coefficient the product of 3, 2, and 5, or 30, prefixing the $-$ sign to indicate its quality. The literal factor a occurs $2 + 1$ or 3 times; b, $1 + 1$ or 2 times; c, $1 + 1$ or 2 times.

Hence, $$(3\,a^2b)(2\,bc)(-5\,ac) \equiv -30\,a^3b^2c^2.$$

The student should check the examples above numerically.

MENTAL EXERCISE VII. 2

Perform the following indicated multiplications :

1. $\begin{array}{r} a \\ \underline{2} \end{array}$
2. $\begin{array}{r} 3 \\ \underline{b} \end{array}$
3. $\begin{array}{r} -c \\ \underline{4} \end{array}$
4. $\begin{array}{r} d \\ \underline{-5} \end{array}$
5. $\begin{array}{r} 2n \\ \underline{3} \end{array}$
6. $\begin{array}{r} 4x \\ \underline{6} \end{array}$
7. $\begin{array}{r} -7y \\ \underline{5} \end{array}$
8. $\begin{array}{r} 9z \\ \underline{-6} \end{array}$
9. $\begin{array}{r} -11w \\ \underline{-8} \end{array}$
10. $\begin{array}{r} c \\ \underline{d^2} \end{array}$
11. $\begin{array}{r} -h \\ \underline{-x^2} \end{array}$
12. $\begin{array}{r} ab \\ \underline{-c} \end{array}$
13. $\begin{array}{r} -xz \\ \underline{y} \end{array}$
14. $\begin{array}{r} -bc \\ \underline{-a} \end{array}$
15. $\begin{array}{r} ab \\ \underline{cd} \end{array}$
16. $\begin{array}{r} ay \\ \underline{bx} \end{array}$
17. $\begin{array}{r} mw \\ \underline{nx} \end{array}$
18. $\begin{array}{r} a^2b \\ \underline{c^2} \end{array}$
19. $\begin{array}{r} xy^2 \\ \underline{z} \end{array}$
20. $\begin{array}{r} b^3y \\ \underline{-n^2} \end{array}$
21. $\begin{array}{r} 2a^2 \\ \underline{a} \end{array}$
22. $\begin{array}{r} 3b^4 \\ \underline{b^2} \end{array}$
23. $\begin{array}{r} 5c^5 \\ \underline{-c^6} \end{array}$
24. $\begin{array}{r} -7d^5 \\ \underline{-d^7} \end{array}$
25. $\begin{array}{r} 3h^2 \\ \underline{2h} \end{array}$
26. $\begin{array}{r} 5m^3 \\ \underline{6m} \end{array}$
27. $\begin{array}{r} 8c^4 \\ \underline{3c^2} \end{array}$
28. $\begin{array}{r} 9d^3 \\ \underline{8d^4} \end{array}$
29. $\begin{array}{r} 10g^5 \\ \underline{-4g^3} \end{array}$
30. $\begin{array}{r} 11h^4 \\ \underline{-7h^6} \end{array}$
31. $\begin{array}{r} 13k^5 \\ \underline{3k^2} \end{array}$
32. $\begin{array}{r} 7ab \\ \underline{4a} \end{array}$
33. $\begin{array}{r} 14bc \\ \underline{5bc} \end{array}$
34. $\begin{array}{r} 4ab \\ \underline{11bc} \end{array}$
35. $\begin{array}{r} -6xy \\ \underline{8xz} \end{array}$
36. $\begin{array}{r} 12mn \\ \underline{12mx} \end{array}$
37. $\begin{array}{r} 15a^2b \\ \underline{-4bc} \end{array}$
38. $\begin{array}{r} 11bc^2 \\ \underline{9bd} \end{array}$
39. $\begin{array}{r} 16m^2n \\ \underline{-6mn^2} \end{array}$
40. $\begin{array}{r} 17a^3b \\ \underline{11ab^2} \end{array}$
41. $\begin{array}{r} -7x^2yz \\ \underline{13yz^2} \end{array}$
42. $\begin{array}{r} 19a^2bc \\ \underline{-5abc^2} \end{array}$
43. $\begin{array}{r} 18x^3y^2z \\ \underline{6xy^2z^3} \end{array}$
44. $\begin{array}{r} 12a^2b^4c^6 \\ \underline{11a^5b^3c} \end{array}$
45. $\begin{array}{r} a^mb^n \\ \underline{ab} \end{array}$
46. $\begin{array}{r} x^ay^b \\ \underline{xy} \end{array}$
47. $\begin{array}{r} x^ny^r \\ \underline{-xy} \end{array}$
48. $\begin{array}{r} -a^{2n}x^r \\ \underline{-ay} \end{array}$
49. $\begin{array}{r} a^{3n}b^{5x} \\ \underline{x^{2n}b^x} \end{array}$
50. $\begin{array}{r} 32x^my^{2m}z^{8m} \\ \underline{-3xyz} \end{array}$
51. $\begin{array}{r} 14a^{2m+1}b^{n-1} \\ \underline{-5a^3b^2} \end{array}$
52. $\begin{array}{r} a^{n-1}b^nc^{n+1} \\ \underline{a^{n+1}b^nc^{n-1}} \end{array}$

53. $ab \times bc \times ca$.

54. $xy \times xz \times xw$.

55. $a^2b \times b^2c \times c^2a$.

56. $abc \times ab \times a$.

57. $a^2b^3 \times b^2c^4 \times c^2d.^5$

58. $- a^3x \times b^3y \times c^3x^2y^2$.

59. $a^2bc \times ab^3c \times abc^4$.

60. $x^2yz^3 \times xy^4z^2 \times y^3z^5w^6$.

61. $(8\,a^2b)(2\,ac^4)(6\,b^3c)$.

62. $-(9\,m^3x)(3\,m^2x)(2\,mnx)$.

63. $(3\,a^3b^3)(4\,b^2c^3)(-5\,c^2a^3)$.

64. $(7\,x^2z^3)(-3\,x^5y^4)(5\,y^3z^2)$.

65. $-(6\,a^3bc)(2\,a^2c^2)(3\,b^4c^5)$.

66. $(-8\,x^6y^2z)(-3\,xy^7)(4\,x^3z^3)$.

67. $(-2\,a^3bc)(-4\,ab^2d)(-3\,bc^2d^3)$.

68. $(-5\,x^4yz^5)(2\,x^3z^2w)(-6\,yzw^2)$.

69. $(-7\,a^2b^2x^3)(2\,a^3b^2y^2)(-6\,b^2x^3y^2)$.

70. $(-9\,a^2b^3c^4)(-5\,a^3b^4d^5)(-2\,b^4c^5d^6)$.

3. Two terms are said to be of the **same type** when they may be derived, one from the other, by interchanging the letters of one or more pairs of letters.

E. g. The terms x^2yz, y^2xz and z^2xy are all of one type. The second expression may be obtained from the first by interchanging the letters x and y, and the third may be obtained from the first by interchanging z and x.

Two expressions are said to have the **same form** or to be of the **same type,** with reference to certain specified letters, if to every term in either there corresponds one and only one term of the same type in the other.

E. g. The expressions $a^2 + 2\,ab + b^2$ and $m^2 + 2\,mn + n^2$ are of the same type.

4. When any one term of a particular type is given, the literal parts of all others of the same type for a given set of letters may be written at once.

E. g. For three letters, x, y, and z, the terms of the type x^2 are x^2, y^2, and z^2. The terms of the type xy are xy, yz, and zx. The terms of the type x^2y are x^2y, x^2z, y^2x, y^2z, z^2x, and z^2y.

5. It is customary when writing integral expressions to arrange consecutively those terms which are of the same type, and to group those of the same degree. When so arranged an expression is said to be in **standard form.**

E. g. The following expression is in standard form:

$$x^3 + y^3 + z^3 + x^2y + x^2z + y^2x + y^2z + z^2x + z^2y + xyz.$$

6. A **variable term** of a polynomial is a term in which one or more letters appear. This name is appropriate because the numerical value of a term changes or varies as different numerical values are given to the letter or letters appearing in it.

If a term appearing in a polynomial does not contain any letter, but instead consists of a numeral, it is called a **constant term,** because its value remains unaltered when numerical values are assigned to the letters appearing in the remaining terms.

7. A polynomial in which the terms are arranged according to the powers of some letter is said to be **complete** when no powers of this letter are missing, from the highest one contained down to and including the constant term.

E. g. The polynomials $x^3 + x^2 + x + 1$ and $3\,ax^4 + 2\,bx^3 - cx^2 + dx - 2$ are both complete.

A polynomial is said to be **incomplete** with reference to the powers of a specified letter if, when its terms are arranged according to the powers of this letter, one or more powers are missing.

E. g. In $x^3 - 1$ the powers missing are x^2 and x; in $x^4 + x^2 + 1$ the powers missing are x^3 and x; in $x^3 + x^2 + x$ the constant term is missing.

Multiplication of a Polynomial by a Monomial

8. As a direct application of the Distributive Law for Multiplication in its first form, we have

$$x(a + b - c) \equiv xa + xb - xc. \qquad (1)$$

Since the value of a product remains unchanged when we alter the order of the factors (Commutative Law), we have

$$\boldsymbol{(a + b - c)x \equiv ax + bx - cx.} \qquad (2)$$

From the expression above, it appears that *a polynomial may be multiplied by a monomial by multiplying each term of the polynomial by the monomial (observing the law of signs), and taking the algebraic sum of the partial products thus obtained.*

Ex. 1. Multiply $3\,x^2 + x - 2$ by $3\,ab$.

We may obtain the product by multiplying the successive terms $3\,x^2$, x, and -2 by $3\,ab$ and finding the algebraic sum of the partial products thus obtained.

$$(3\,x^2 + x - 2) \times 3\,ab \equiv 9\,abx^2 + 3\,abx - 6\,ab.$$

Instead of employing the horizontal arrangement of the work, as above, we may use the vertical arrangement corresponding to the one commonly used in arithmetic, as shown below :

		Check. Let $a = 1$, $b = 2$, $x = 3$.
Multiplicand	$3x^2 + x - 2$	28
Multiplier	$3ab$	6
		168
Product	$9abx^2 + 3abx - 6ab$	168
		0

9. It should be observed that, since in algebra we do not use the positional system of notation employed in arithmetic, we may write the multiplier in any convenient position beneath the multiplicand.

The partial products may then be obtained by beginning either with the first term or the last term of the multiplier and multiplying the terms of the multiplicand successively by it, either from left to right or from right to left.

MENTAL EXERCISE VII. 3

Perform the following indicated multiplications :

1. $\begin{array}{l} a + b \\ 2 \\ \hline \end{array}$

2. $\begin{array}{l} b - c \\ 3 \\ \hline \end{array}$

3. $\begin{array}{l} x + y \\ z \\ \hline \end{array}$

4. $\begin{array}{l} y - z \\ w \\ \hline \end{array}$

5. $\begin{array}{l} 2m + 3n \\ 4 \\ \hline \end{array}$

6. $\begin{array}{l} 6h - 8k \\ 5 \\ \hline \end{array}$

7. $\begin{array}{r} 2x + 3y \\ -4 \\ \hline \end{array}$

8. $\begin{array}{r} 7m - n \\ -8 \\ \hline \end{array}$

9. $\begin{array}{l} 4a + b \\ 3c \\ \hline \end{array}$

10. $\begin{array}{l} d - 5h \\ 4x \\ \hline \end{array}$

11. $\begin{array}{r} 6m - 5n \\ -2y \\ \hline \end{array}$

12. $\begin{array}{l} ab + cd \\ xy \\ \hline \end{array}$

13. $\begin{array}{l} ab + ac \\ bc \\ \hline \end{array}$

14. $\begin{array}{l} xy - xz \\ xy \\ \hline \end{array}$

15. $\begin{array}{l} ab - bc \\ ca \\ \hline \end{array}$

16. $\begin{array}{l} a^2 - b^2 \\ ab \\ \hline \end{array}$

17. $\begin{array}{r} x^2 - y^2 \\ -xy \\ \hline \end{array}$

18. $\begin{array}{l} 2mn + 3bc \\ 4ax \\ \hline \end{array}$

19. $\begin{array}{l} 7ax + 5by \\ 3mn \\ \hline \end{array}$

20. $\begin{array}{r} 8cd + 9bh \\ -2kr \\ \hline \end{array}$

21. $\begin{array}{l} 4ax^2 + 3by^2 \\ 2cd \\ \hline \end{array}$

22. $\begin{array}{l} 3a^2b + 5ab^2 \\ 4ab \\ \hline \end{array}$

23. $\begin{array}{r} 6xy^2 - 7y^2z \\ -2xyz \\ \hline \end{array}$

24. $\begin{array}{r} 4ax^2 + 3by^2 \\ -3abh \\ \hline \end{array}$

25. $6xyz^2 + 5x^2yz$
$\underline{4xyz}$

26. $9a^2b^2c + 12ab^2c^3$
$\underline{3abc}$

27. $11x^4y^2z^5 - 6x^3y^6$
$\underline{7x^7z^8}$

28. $a^3 + a^2 + a + 1$
$\underline{2a}$

29. $b^3 - b^2 + b - 1$
$\underline{4b^2}$

30. $a^2 - 2ab + b^2$
$\underline{3ab}$

31. $m^2 - 2mn + n^2$
$\underline{-2mn}$

32. $ab + bc + ca$
$\underline{abc}$

33. $abc + abd + bcd$
$\underline{abcd}$

34. $(a + b + c)abc.$
35. $(x^2 + y^2 + z^2 + xyz)xyz.$
36. $(1 + a + a^2 + a^3)a^4.$
37. $(x^4 - x^3 + x^2 - x + 1)x^2.$
38. $(a^3 + a^2b + ab^2 + b^3)a^4.$
39. $(x^3 - x^2y + xy^2 - y^3)xy.$
40. $(a^3 + b^2 + c)ab^2c^3.$
41. $(xyz - yzw - zwx)xyzw.$
42. $(x^4 + x^3y + x^2z + xw)xyzw.$
43. $(2a^2y - 3b^2z - 4c^2w)5abc.$
44. $(1 - a^4b^2 + a^3b^3c - a^2b^4cd)abcd.$
45. $(4a^2x - 3b^2y + 2x^2y^2)5cxy.$
46. $(8a^2b^3 + 13b^2c^3 + 6c^2d^3)(-4abcd).$
47. $(a^n + a^{n+1} + a^{n+2} + a^{n+3})a.$
48. $(b^{4n} + b^{3n} + b^{2n} + b^n)b.$
49. $(c^{n-3} + c^{n-2} + c^{n-1} + c^n)c.$
50. $(d^{n-1} + d^n + d^{n+1} + d^{n+2})d^n.$
51. $(n^{2r-3} + n^{2r-1} + n^{2r+1} + n^{2r+3})n^2.$
52. $(m^{4r-3} - m^{3r-4} + m^{2r-1} - m^{r-2})m^5.$
53. $(a^{n-2} + a^{n-1}b^n + a^nb^{n+1} + b^{n+2})ab.$
54. $(x^{n-5} - x^{n-3}y^{n-1} + x^{n-1}y^{n-3} - y^{n-5})x^4y^4.$
55. $(a^{2x+2} + a^{2x+1}b^{2x} - a^{2x}b^{2x+1} - b^{2x+2})a^xb^x.$
56. $(m^{a+1} - m^{a+2}n^2 + m^{a+3}n^3 - n^4)m^{a-1}n^{a+1}.$

Multiplication of One Polynomial by Another

10. We have, by the Distributive Law for multiplication,

$$\boldsymbol{(a + b - c)(x - y) \equiv ax + bx - cx - ay - by + cy.}$$

Hence, *to multiply one polynomial by another, multiply each term* *of the multiplicand by each term of the multiplier, and write the algebraic sum of the resulting partial products.*

Ex. 1. Multiply $2x^2+3x-7$ by $3x-5$.

Two arrangements of the work are shown below:

$$\begin{array}{lclr}
\text{Form I} & & \text{Form II} & \\
 & & & \text{Check. Let } x=2. \\
2x^2+3x-7 \;\ldots\ldots & \text{Multiplicand} & \ldots\ldots\; 2x^2+3x-7 \;\ldots & 7 \\
\underline{3x-5} \;\ldots\ldots\ldots & \text{Multiplier} & \ldots\ldots\; \underline{3x-5} \;\ldots & \underline{1} \\
6x^3+9x^2-21x & \text{Partial} & -10x^2-15x+35 & 7 \\
\underline{\quad -10x^2-15x+35} & \text{Products} & \underline{6x^3+9x^2-21x} & \\
6x^3-x^2-36x+35 \;\ldots & \text{Product} & \ldots\; 6x^3-x^2-36x+35 \;\ldots & \underline{\underline{7}} \\
 & & & 0
\end{array}$$

In Form I the first and second rows of partial products are obtained by multiplying the terms of the multiplicand successively by $3x$ and by -5 respectively.

In Form II the arrangement corresponds to the one adopted in arithmetic multiplication.

Ex. 2. Multiply $3x^3+x^2-4x+5$ by x^2-x-3.

For convenience in performing the work, it is usually best to arrange the given polynomials according to increasing or decreasing powers of some letter.

Arranging both multiplicand and multiplier according to descending powers of x, we have:

$$\begin{array}{lllr}
 & & & \text{Check. Let } x=2. \\
\text{Multiplicand} \ldots\ldots & & 3x^3+x^2-4x+5 \;\ldots\ldots & 25 \\
\text{Multiplier} \ldots\ldots & & \underline{x^2-x-3} \;\ldots\ldots & \underline{-1} \\
\text{Partial} & \left\{\begin{array}{l}\text{using } x^2 \\ \text{using } -x \\ \text{using } -3\end{array}\right. & \begin{array}{l}3x^5+x^4-4x^3+5x^2 \\ \quad -3x^4-x^3+4x^2-5x \\ \underline{\qquad\quad -9x^3-3x^2+12x-15}\end{array} & -25 \\
\text{Products} & & & \\
\text{Reduced Product} \ldots & & 3x^5-2x^4-14x^3+6x^2+7x-15 & \underline{\underline{-25}} \\
 & & & 0
\end{array}$$

11. From the Distributive Law for the multiplication of polynomials, it appears that the product of an algebraic sum of m terms and an algebraic sum of n terms will contain $m \times n$ partial products if all be written directly without the collection of like terms or the suppression of terms which mutually destroy each other. This principle relating to the number of terms in the reduced product will sometimes serve as a check upon accuracy.

E. g. When $2x + 3y + z + w$ is multiplied by $a + b - c$ there will be twelve separate partial products in the result, for the first polynomial contains four and the second contains three separate terms.

12.* **Detached Coefficients.** When two expressions are both arranged according to either descending or ascending powers of some letter, the work of multiplication may be shortened by writing only the numerical coefficients of the different terms, writing 0 in place of each missing term.

Ex. 3. Multiply $3x^2 + x - 4$ by $2x^2 - 3x + 5$.

Using Literal Factors. Using Detached Coefficients. Check. $x = 1$

$$\begin{array}{rrrrr}
3x^2 & +\ x & -\ 4 & & \\
2x^2 & -3x & +\ 5 & & \\
\hline
6x^4 & +2x^3 & -\ 8x^2 & & \\
 & -9x^3 & -\ 3x^2 & +12x & \\
 & & +15x^2 & +\ 5x & -20 \\
\hline
6x^4 & -7x^3 & +\ 4x^2 & +17x & -20.
\end{array}
\qquad
\begin{array}{rrrrrr}
3 & +1 & -\ 4 & & & \ldots\ldots\ldots\ldots 0 \\
2 & -3 & +\ 5 & & & \ldots\ldots\ldots\ldots \underline{4} \\
 & & & & & \overline{0} \\
6 & +2 & -\ 8 & & & \\
 & -9 & -\ 3 & +12 & & \\
 & & +15 & +\ 5 & -20 & \\
\hline
6 & -7 & +\ 4 & +17 & -20 & (1)\ .\ .\ \underline{\underline{0}} \\
\multicolumn{5}{r}{6x^4 - 7x^3 + 4x^2 + 17x - 20.} & (2)\ .\ .\ \overline{0}
\end{array}$$

The highest power of x in the product is x^4, and since the terms in the multiplicand and multiplier are arranged according to decreasing powers of x, and no powers are missing, the remaining powers in the product will follow in successive terms.

Hence from (1), we may write as the product required

$$6x^4 - 7x^3 + 4x^2 + 17x - 20. \qquad (2)$$

By the use of detached coefficients the labor of writing all the literal factors and their exponents has been saved.

Ex. 4. Multiply $5x^3 - 6x + 3$ by $2x^2 - 4$.

Supplying the "missing terms," it may be seen that the multiplicand and multiplier are equivalent to $5x^3 + 0x^2 - 6x + 3$ and $2x^2 + 0x - 4$ respectively.

The process, using detached coefficients, is shown below:

$$\begin{array}{rrrrrr}
5 & +0 & -\ 6 & +3 & & \\
2 & +0 & -\ 4 & & & \\
\hline
10 & +0 & -12 & +6 & & \\
 & -20 & -0 & +24 & -12 & \\
\hline
10 & +0 & -32 & +6 & +24 & -12
\end{array}$$

---

* This section may be omitted when the chapter is read for the first time.

The term containing the highest power of x in the product is obtained by multiplying $5x^3$ by $2x^2$. Accordingly beginning with x^5 we may write the successive terms of the reduced product as follows:

$$10x^5 + 0x^4 - 32x^3 + 6x^2 + 24x - 12,$$

that is, $10x^5 - 32x^3 + 6x^2 + 24x - 12$.

The detached coefficients may be used for a numerical check as in Ex. 3.

13. A polynomial is said to be **homogeneous** if the sums of the exponents of all of the letters appearing in the different terms are the same.

E. g. The polynomial $a^4 - a^3b + a^2b^2 - ab^3 + b^4$ is homogeneous and of the fourth degree with reference to a and b, since the sum of the exponents in each of the terms is 4.

The polynomial $5x^6 + 2x^4y^2 - x^3y^3 + 3xy^5 - y^6$ is homogeneous and of the sixth degree with reference to x and y.

Homogeneity as a Check upon Accuracy

14.* *The product of two homogeneous expressions is a homogeneous expression.*

For if the homogeneous multiplicand is of the m^{th} degree, every term will, by definition, be of the m^{th} degree. Also, if each term of the homogeneous multiplier be of the n^{th} degree, then since each term of the product arises from multiplying a term of the multiplicand by a term of the multiplier, each of the partial products must be of the $(m + n)^{\text{th}}$ degree; that is, the product must be a homogeneous expression of the $(m + n)^{\text{th}}$ degree.

Although the proof is given for two factors only, it may be extended to include any number of factors.

This property of homogeneous expressions may be used as a check upon accuracy, since if the product obtained by the multiplication of one homogeneous expression by another is not homogeneous, then we know at once that there must be some mistake in the work.

E. g. The product obtained by multiplying $a^3 + a^2b + ab^2 + b^3$ by $a^2 - ab + b^2$ is a homogeneous expression of the fifth degree, and may contain terms such as a^5, a^4b, a^3b^2, a^2b^3, ab^4, and b^5. It cannot contain terms such as a^4, a^3b, a^2b^2, ab^5, etc.

---

* This section may be omitted when the chapter is read for the first time.

EXERCISE VII. 4

Perform the following indicated multiplications, checking all results numerically :

1. $\begin{array}{l} a+b \\ \underline{x+y} \end{array}$

2. $\begin{array}{l} b+c \\ \underline{b+d} \end{array}$

3. $\begin{array}{l} c-d \\ \underline{d-e} \end{array}$

4. $\begin{array}{l} f+g \\ \underline{f-g} \end{array}$

5. $\begin{array}{l} k^2+k \\ \underline{k\ \ +l} \end{array}$

6. $\begin{array}{l} r^2-r^3 \\ \underline{r^4+r^5} \end{array}$

7. $\begin{array}{l} 2m+3n \\ \underline{4m+5n} \end{array}$

8. $\begin{array}{l} 6t-10n \\ \underline{7t+\ \ 3n} \end{array}$

9. $\begin{array}{l} a^2+2ab+b^2 \\ \underline{a\ \ +\ \ \ \ b\ \ \ \ \ } \end{array}$

10. $(a^2 - ab + b^2)(a + b)$.
11. $(2x^2 + 5xy + 7y^2)(2x - 3y)$.
12. $(2c^2 + 3cd + 4d^2)(6c - 5d)$.
13. $(w^4 + w^2 + 1)(w^4 - w^2 + 1)$.
14. $(a^2 + 2a - 3)(a^2 + a - 6)$.
15. $(5r^2 + 4r - 2)(3r^2 - 2r - 5)$.
16. $(a^2 - 3ab - b^2)(a^2 + 3ab + b^2)$.
17. $(g^3 + g^2y^2 + y^3)(g^3 - g^2y^2 + y^3)$.
18. $(k^2 - ky + y^2)(2k^2 - 3ky + y^2)$.
19. $(h^3 + h^2y + hy^2 + y^3)(h - y)$.
20. $(d^3 - 4d^2 + 3d + 1)(d^2 - 2d + 5)$.
21. $(y^3 - 5y^2 + 1)(2y^3 + 5y + 1)$.
22. $(s^6 - 3s^4 + 2s^2 - 1)(s^3 - s + 1)$.
23. $(6k^5 + 4k^3 + 2k + 1)(k^4 - k^2 - 1)$.
24. $(a^2 - ab + b^2 + a + b + 1)(a + b - 1)$.
25. $(1 - 2m + 2m^2 - m^3)(1 + 2m + 2m^2 + m^3)$.

Multiply

26. $x^2 + y^2 + z^2 - xy - zx - zy$ by $x + y + z$.
27. $x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5$ by $x - y$.
28. $x^8 - x^6y^3 + x^4y^6 - x^2y^9 + y^{12}$ by $x^2 + y^3$.
29. $a^2b - b^2c^3 - c^3d^2 + d^2a$ by $a^2bc^2 - ab^2c$.
30. $5a^3b^3 - 4b^3c^3 + 5c^3a^3$ by $4ab - 5bc + 4ca$.
31. $a^2bc - ab^2c + abc^2$ by $ab^2c^2 - a^2bc^2 + a^2b^2c$.
32. $ab^2c^3 - ab^3c^2 + a^2bc^3$ by $a^2b^3c - a^3bc^2 + a^3b^2c$.

No change in the process is necessary when some or all of the coefficients are fractional.

Multiply

33. $\frac{1}{2}a^2+\frac{1}{3}a+\frac{1}{4}$ by $\frac{1}{3}a+\frac{1}{4}$.
34. $\frac{1}{4}x^2-\frac{1}{3}x+\frac{1}{2}$ by $\frac{1}{3}x-\frac{1}{2}$.
35. $\frac{1}{2}a^2-\frac{1}{3}ab+\frac{1}{4}b^2$ by $\frac{1}{2}a+\frac{1}{3}b$.
36. $\frac{2}{3}m^2+mn+\frac{3}{2}n^2$ by $\frac{3}{2}m+\frac{2}{3}n$.
37. $\frac{1}{2}a^3-\frac{5}{6}a^2+\frac{3}{2}a-1$ by $6a-\frac{1}{15}$.
38. $\frac{1}{2}-\frac{2}{3}a+\frac{3}{4}a^2-\frac{4}{5}a^3$ by $2-5a$.

Ex. 39. Find the product of $4a^{n+1}-3a^n+7a^{n-1}$ and $5a^{n+1}-2a^n$.

The process of multiplication is not affected in any way because the letter n appears as an exponent. We shall consider for the present that n represents a positive whole number. The separate partial products are obtained by adding the exponents of the like factors entering into them.

E.g. The first term of the multiplicand may be multiplied by the first term of the multiplier as follows:

$$4a^{n+1}\times 5a^{n+1}\equiv 20a^{(n+1)+(n+1)}\equiv 20a^{2n+2}.$$

The remaining partial products may be found in a similar way.

Multiply

40. $2a^{n+3}-5a^{n+2}+7a^{n+1}$ by $3a^n-4a$.
41. $4a^{n+2}+3a^{n+1}+2a^n+1$ by $a-1$.
42. $5x^{4n}+4x^{3n}+3x^{2n}+2x^n$ by $2x-1$.
43. $a^{n-3}b^3-a^{n-2}b^2+a^{n-1}b$ by $a^3b^n-a^nb^3$.
44. $3a^{m+1}-7a^m+4a^{m-1}$ by $2a^m-5a^{m-1}$.
45. $x^{2n}+x^n+1$ by $x^{2n}-x^n-1$.
46. $x^{m+1}y^{n-1}+x^{n-1}y^{m+1}$ by $x^{n-1}y^{m+1}-x^{m+1}y^{n-1}$.
47. $x^{2n+1}-x^{2n}+x^{2n-1}$ by $x^{2n-1}+x^{2n-2}+x^{2n-3}$.
48. $x^{n-a}+x^{n-b}+x^{n-c}$ by $x^{n+a}+x^{n+b}+x^{n+c}$.
49. $a^{m+n}+a^m+1$ by $a^{m-n}-a^m+1$.

15. Removal of Parentheses. Parentheses may be removed by applying the Distributive Law for Multiplication.

Ex. 1. Simplify $6a-5\{a-4[3+2(a-1)]\}$.

We have,
$$\begin{aligned}6a-5\{a-4[3+2(a-1)]\}&\equiv 6a-5\{a-4[3+2a-2]\}\\&\equiv 6a-5\{a-4[1+2a]\}\\&\equiv 6a-5\{a-4-8a\}\\&\equiv 6a-5\{\ \ -4-7a\}\\&\equiv 6a+20+35a\\&\equiv 41a+20.\end{aligned}$$

EXERCISE VII. 5

Simplify each of the following expressions :

1. $1 + 2\{1 + 3[1 + 4(1 + 5x)]\}$.
2. $x - \{(y - z) - [x + y - z - 2(x - y + z)]\}$.
3. $2 + 2\{2 - 2[2 + 2(2 - 2x)]\}$
4. $(x + 1) - 2\{(x + 2) + 3[(x + 3) - 4(x + 4)]\}$.
5. $a - \{b + c[d - e(f + g)]\}$.
6. $5\{4[3(2 + a)]\} - 5\{-4[-3(2 - a)]\}$.
7. $7\{b - 4[b - 4(b + y)]\} - 6\{b - 4[b - 2(b - y)]\}$.
8. $a\{1 + b[1 + c(1 + d)]\} - d\{1 + c[1 + b(1 + a)]\}$.
9. $a\{b - c[a - b(a + b + c) - (b + a)] - c - b\}$.
10. $b^2 - b\{b + c[a(b - c) + b(c - a) + c(a - b)]\}$.

STANDARD IDENTITIES

16. Special Products. Just as in Arithmetic we find it necessary to commit to memory the Multiplication Table, so in Algebra certain products occur so frequently that it is important to memorize them.

17. A polynomial expression is said to be an **expansion** of a second polynomial expression if it is obtained by raising the second expression to some power.

E. g. The expansion of $(a + b)^2$ is $a^2 + 2ab + b^2$.

18. Square of a Binomial Sum.

Theorem I. $(a + b)^2 \equiv a^2 + 2ab + b^2$.

The square of a binomial sum is equal to the square of the first term, increased by twice the product of the two, plus the square of the second.

Ex. 1. $(x + 4)^2 \equiv x^2 + 8x + 16$. Check. Let $x = 3$. $49 = 49$.

Ex. 2. $(m + n)^2 \equiv m^2 + 2mn + n^2$. Check. Let $m = n = 4$. $64 = 64$.

Ex. 3. $(2x + 3y)^2 \equiv (2x)^2 + 2 \cdot 2x \cdot 3y + (3y)^2$
$\equiv 4x^2 + 12xy + 9y^2$. Check. Let $x = 3$, $y = 2$. $144 = 144$.

MENTAL EXERCISE VII. 6

Expand each of the following binomial sums :

1. $(a + 3)^2$.
2. $(b + 4)^2$.
3. $(c + 6)^2$.
4. $(c + 7)^2$.
5. $(5 + d)^2$.
6. $(8 + h)^2$.
7. $(k + 9)^2$.
8. $(m + 10)^2$.
9. $(11 + n)^2$.
10. $(x + 12)^2$.
11. $(13 + y)^2$.
12. $(z + 14)^2$.
13. $(15 + w)^2$.
14. $(2a + 1)^2$.
15. $(3b + 1)^2$.
16. $(4c + 1)^2$.
17. $(5d + 1)^2$.
18. $(4e + 3)^2$.
19. $(5h + 4)^2$.
20. $(3k + 5)^2$.
21. $(6g + 7)^2$.
22. $(5 + 8h)^2$.
23. $(9 + 6k)^2$.
24. $(7 + 9m)^2$.
25. $(10x + 3)^2$.
26. $(4a + 5b)^2$.
27. $(3c + 7d)^2$.
28. $(6h + 10k)^2$
29. $(5m + 13n)^2$.
30. $(2r + 17y)^2$.
31. $(16s + 3w)^2$.
32. $(12x + 5y)^2$.
33. $(7t + 20z)^2$.
34. $(18k + 10n)^2$.
35. $(19a + 2c)^2$.
36. $(3b + 18d)^2$.
37. $(20c + 5g)^2$.
38. $(10b + 16k)^2$.
39. $(14d + 5n)^2$.
40. $(12h + 11r)^2$.
41. $(21s + 2v)^2$.
42. $(8ab + 5)^2$.
43. $(9xy + 10)^2$.
44. $(11a + 4bc)^2$.
45. $(7x + 4yz)^2$.
46. $(6ab + 5cd)^2$.
47. $(10xw + 9yz)^2$.
48. $(15ac + 5bd)^2$.

19. Square of a Binomial Difference.

Theorem II. $(a - b)^2 \equiv a^2 - 2ab + b^2$.

The square of a binomial difference is equal to the square of the first term, diminished by twice the product of the two, plus the square of the second.

Check. Let $x = 6$.

Ex. 1. $(x - 4)^2 \equiv x^2 - 8x + 16$. $4 = 4$.

Check Let $z = 2$.

Ex. 2. $(3 - z)^2 \equiv 9 - 6z + z^2$. $1 = 1$.

Check.

Ex. 3. $(3x - 5y)^2 \equiv (3x)^2 - 2 \cdot 3x \cdot 5y + (5y)^2$ Let $x = 4$, $y = 2$.

$\equiv 9x^2 - 30xy + 25y^2$. $4 = 4$.

MENTAL EXERCISE VII. 7

Expand each of the following binomial differences :

1. $(a - 3)^2$.
2. $(b - 7)^2$.
3. $(c - 8)^2$.
4. $(4 - d)^2$.
5. $(5 - e)^2$.
6. $(6 - f)^2$.

7. $(g - 9)^2$.
8. $(10 - h)^2$.
9. $(h - 11)^2$.
10. $(12 - m)^2$.
11. $(n - 14)^2$.
12. $(16 - r)^2$.
13. $(s - 17)^2$.
14. $(18 - t)^2$.
15. $(x - 19)^2$.
16. $(2a - 1)^2$.
17. $(7b - 1)^2$.
18. $(9c - 1)^2$.
19. $(1 - 4d)^2$.
20. $(1 - 5e)^2$.
21. $(8g - 1)^2$.
22. $(3k - 5)^2$.
23. $(4h - 7)^2$.
24. $(6v - 5)^2$.
25. $(7w - 9)^2$.
26. $(8 - 6s)^2$.
27. $(11 - 4t)^2$.
28. $(13r - 3)^2$.
29. $(2s - 15)^2$.
30. $(4q - 8)^2$.
31. $(9a - 3b)^2$.
32. $(6c - 7d)^2$.
33. $(5h - 12r)^2$.
34. $(9p - 7q)^2$.
35. $(8x - 7z)^2$.
36. $(14t - 4y)^2$.
37. $(10q - 8z)^2$.
38. $(6r - 11y)^2$.
39. $(15m - 5s)^2$.
40. $(12n - 20y)^2$.
41. $(17g - 5r)^2$.
42. $(2ab - 18)^2$.
43. $(5 - 4cg)^2$.
44. $(5ab - 10cd)^2$.
45. $(9xw - 4yz)^2$.
46. $(3mx - 4ny)^2$.
47. $(2ay - 9bx)^2$.
48. $(7cd - 6hk)^2$.

20. Multiplication of the Sum of Two Terms by their Difference.

Theorem III. $(a + b)(a - b) \equiv a^2 - b^2$.

The product obtained by multiplying the sum of two terms by their difference is equal to the difference of the squares of these terms.

Ex. 1. $(x + 5)(x - 5) \equiv x^2 - 25$.
Ex. 2. $(9 + m)(9 - m) \equiv 81 - m^2$.
Ex. 3. $(5x + 6y)(5x - 6y) \equiv 25x^2 - 36y^2$.

Let the student check each of the examples above.

MENTAL EXERCISE VII. 8

Obtain the following products :

1. $(x + y)(x - y)$.
2. $(c + k)(c - k)$.
3. $(r + w)(r - w)$.
4. $(m + q)(m - q)$.
5. $(a + z)(a - z)$.
6. $(x + 3)(x - 3)$.
7. $(y + 4)(y - 4)$.
8. $(5 + z)(5 - z)$.
9. $(6 + w)(6 - w)$.
10. $(k + 9)(k - 9)$.
11. $(11 + g)(11 - g)$.
12. $(h + 13)(h - 13)$.
13. $(14 + k)(14 - k)$.
14. $(m + 17)(m - 17)$.
15. $(16 + n)(16 - n)$.
16. $(2a + 1)(2a - 1)$.
17. $(3b + 1)(3b - 1)$.
18. $(c + 2d)(c - 2d)$.

19. $(c + 7d)(c - 7d)$.
20. $(a + 5b)(a - 5b)$.
21. $(a + 8c)(a - 8c)$.
22. $(4a + 3b)(4a - 3b)$.
23. $(5c + 7q)(5c - 7q)$.
24. $(8k + 3w)(8k - 3w)$.
25. $(9n + 11z)(9n - 11z)$.
26. $(12m + 8r)(12m - 8r)$.
27. $(7g + 15t)(7g - 15t)$.
28. $(13n + 9s)(13n - 9s)$.
29. $(15q + 12z)(15q - 12z)$.
30. $(17d + 6v)(17d - 6v)$.
31. $(18n + 3s)(18n - 3s)$.
32. $(10p + 16t)(10p - 16t)$.
33. $(9q + 20r)(9q - 20r)$.
34. $(19a + 5h)(19a - 5h)$.
35. $(6c + 12g)(6c - 12g)$.
36. $(4s + 14z)(4s - 14z)$.
37. $(3ab + 5)(3ab - 5)$.
38. $(2cd + 9)(2cd - 9)$.
39. $(7 + 6my)(7 - 6my)$.
40. $(8 + 9nz)(8 - 9nz)$.
41. $(4ab + 5c)(4ab - 5c)$.
42. $(10x + 7yz)(10x - 7yz)$.
43. $(11bk + 9w)(11bk - 9w)$.
44. $(8gx + 10hy)(8gx - 10hy)$.
45. $(12am + 13bn)(12am - 13bn)$.
46. $(7abc + 8d)(7abc - 8d)$.
47. $(14x + 11yzw)(14x - 11yzw)$.
48. $(15ab + 10cd)(15ab - 10cd)$.
49. $(16bc + 15mn)(16bc - 15mn)$.
50. $(17mn + 18pq)(17mn - 18pq)$.

21. Square of a Polynomial. Consider the square of a polynomial consisting of three terms :

$$\begin{aligned}(a + b + c)^2 &\equiv (a + b + c)(a + b + c)\\ &\equiv (a + b + c)\,a\\ &\quad + (a + b + c)\,b\\ &\quad + (a + b + c)\,c.\end{aligned}$$

From the arrangement of the work above it appears that each partial product obtained by multiplying any one of the terms in parentheses by the factor outside must be of the second degree. The only possible terms which can arise in this way will be those which are the squares of the given letters, such as a^2, b^2, and c^2 and those which are the products of all possible pairs of the letters, such as ab, ac, and bc.

By examining the identity above it may be seen that a^2 will occur but once, that is, as a result of the multiplication in the first line; b^2 will occur but once, that is, as a result of the multiplication in the second line; c^2 once only, and that from the multiplication in the third line.

Furthermore, it appears that any product such as ab, of two different letters, will occur twice and twice only.

Thus, ab will arise from the multiplication in the first line and also from that in the second line; ac from that in the first and third lines; bc from that in the second and third lines.

The reasoning employed above may be extended to include a chain of additions and subtractions containing any number of terms. Hence we have

Theorem IV. $(a + b + c)^2 \equiv a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.$

The square of any polynomial is equal to the sum of the squares of the different terms, increased by twice the product of each term and every term which follows it.

22. The square of a polynomial may also be obtained by suitably grouping the terms and then applying directly the theorem for the square of a binomial.

E.g. Find the square of $a + b + c$.

Regarding $a + b$ in the polynomial as a single term, we may write

$$a + b + c \equiv (a + b) + c.$$

Hence,
$$\begin{aligned}[(a + b) + c]^2 &\equiv (a + b)^2 + 2(a + b)c + c^2\\ &\equiv a^2 + 2ab + b^2 + 2ac + 2bc + c^2\\ &\equiv a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.\end{aligned}$$

Ex. 1. Find the square of $(a + 3b + 5c + 7d)$.

$$(a + 3b + 5c + 7d)^2 \equiv a^2 + (3b)^2 + (5c)^2 + (7d)^2$$

Multiplying $2a$ by following terms . $+ 2a(3b) + 2a(5c) + 2a(7d)$

Multiplying $6b$ by following terms . . . . . $+ 2(3b)(5c) + 2(3b)(7d)$

Multiplying $10c$ by $7d$ . . . . $+ 2(5c)(7d)$.

$$\begin{aligned}&\equiv a^2 + 9b^2 + 25c^2 + 49d^2\\ &\quad + 6ab + 10ac + 14ad\\ &\quad + 30bc + 42bd\\ &\quad + 70cd.\end{aligned}$$

Check. $a = b = c = d = 2$.
$1024 = 1024$.

Ex. 2. $(5x - 2y - 4z)^2 \equiv 25x^2 + 4y^2 + 16z^2 - 20xy - 40xz + 16yz.$

Check. $x = 4, y = 3, z = 1.$ $100 = 100.$

MENTAL EXERCISE VII. 9

Expand each of the following polynomials:

1. $(a + 2b + 3c)^2$.
2. $(4c + d + 5e)^2$
3. $(6g + 3n + w)^2$.
4. $(4x + 7y + 2z)^2$.
5. $(5a - b + 2c)^2$.
6. $(4x + 3y - z)^2$.

7. $(6a + 2b - 3c)^2$.
8. $(9a + 2b + 1)^2$.
9. $(7x - 3y + 2)^2$.
10. $(a + b + c + 1)^2$.
11. $(x + y - z - 1)^2$.
12. $(a - b + c - 2)^2$.
13. $(3x - y + 2z - w)^2$.
14. $(a - 2b + c - 5d)^2$.
15. $(7 - a - b - c)^2$.
16. $(2a + 3b + 4c + 5d)^2$.
17. $(a - b - c - d - e)^2$.
18. $(a - 2b + c - 3d + e)^2$.
19. $(4x + 2y - z + 3w - 5)^2$.
20. $(x - 2y + 3z - 2w + 1)^2$.

23. Product of Two Binomials in which the First Terms are Equal and the Second Terms are Unequal.

Theorem V. $(x + a)(x + b) \equiv x^2 + (a + b)x + ab$.

The product of two binomials having equal first terms, but unequal second terms, is equal to the square of the common first term, increased by the product of the algebraic sum of the second terms and the first term, plus the product of the second terms.

Ex. 1. Multiply $(x + 2)$ by $(x + 3)$.

The two binomials have equal first terms, x, but unequal second terms, 2 and 3. Hence, we may write:

$(x + 2)(x + 3) \equiv x^2 + (2 + 3)x + 2 \cdot 3$ Check. $x = 2$.
$\equiv x^2 + 5x + 6.$ $20 = 20$.

Ex. 2. $(2x + 5)(2x + 6) \equiv (2x)^2 + (5 + 6)2x + 5 \cdot 6$ Check. $x = 2$.
$\equiv 4x^2 + 22x + 30.$ $90 = 90$.

Ex. 3. $(x + 7)(x - 2) \equiv x^2 + (7 - 2)x + 7(-2)$ Check. $x = 3$.
$\equiv x^2 + 5x - 14.$ $10 = 10$.

Ex. 4. $(3a + b)(3a - 2b) \equiv (3a)^2 + (b - 2b)3a + b(-2b)$ Check. $a = 3$,
$b = 4$.
$\equiv 9a^2 - 3ab - 2b^2.$ $13 = 13$.

Mental Exercise VII. 10

Obtain the following products:

1. $(a + 2)(a + 1)$.
2. $(x + 3)(x + 2)$.
3. $(x + 4)(x + 1)$.
4. $(x - 4)(x - 5)$.
5. $(a + 5)(a + 2)$.
6. $(x + 5)(x + 3)$.
7. $(x + 6)(x + 4)$.
8. $(x + 6)(x + 8)$.
9. $(x + 7)(x + 2)$.
10. $(x + 7)(x - 5)$.
11. $(x + 7)(x + 6)$.
12. $(x + 8)(x - 3)$.

13. $(x + 8)(x + 7)$.
14. $(x + 9)(x - 2)$.
15. $(m + 9)(m - 6)$.
16. $(x - 10)(x - 2)$.
17. $(x - 3)(x + 10)$.
18. $(x - 5)(x + 11)$.
19. $(x - 2)(x - 12)$.
20. $(x - 10)(x + 8)$.
21. $(x + 11)(x - 4)$.
22. $(x + 4)(x + 13)$.
23. $(a + 9)(a + 10)$.
24. $(a + 11)(a + 7)$.
25. $(m - 11)(m + 10)$
26. $(x - 12)(x + 1)$.
27. $(a - 13)(a - 2)$.
28. $(a + 3)(a + 14)$.
29. $(a + 12)(a - 1)$.
30. $(c + 13)(c - 3)$.
31. $(a - 3)(a - 16)$.
32. $(b + 15)(b + 4)$.
33. $(a + 10)(a + 16)$.
34. $(a + 5)(a + 18)$.
35. $(a + 19)(a - 2)$.
36. $(c - 3)(c - 17)$.
37. $(y + 14)(y - 8)$.
38. $(m + 17)(m - 2)$.
39. $(a - 10)(a - 19)$.
40. $(h + 18)(h - 10)$.
41. $(a - 20)(a - 15)$.
42. $(x - 20)(x - 5)$.
43. $(m - 8)(m + 20)$.
44. $(x - 16)(x + 20)$
45. $(a + 15)(a + 14)$.
46. $(b - 22)(b - 3)$.
47. $(a + 25)(a + 8)$.
48. $(c + 20)(c + 30)$.
49. $(2a + 3)(2a + 5)$.
50. $(4b + 5)(4b + 1)$.
51. $(6c + 1)(6c - 3)$.
52. $(7d - 2)(7d - 4)$.
53. $(5g + 8)(5g - 6)$.
54. $(8h - 11)(8h - 1)$.
55. $(3k - 1)(3k - 19)$.
56. $(9m - 2)(9m - 4)$.
57. $(3n + 5)(3n - 8)$.
58. $(11x - 9)(11x + 2)$.
59. $(12y + 7)(12y - 5)$.
60. $(10z - 3)(10z + 4)$.

24. The Product of Two Binomials of the forms $ax + b$ and $cx + d$, the first terms of which contain a common factor x, may be found by applying the Law of Distribution for Multiplication.

Theorem VI. $(ax + b)(cx + d) \equiv acx^2 + (ad + bc)x + bd.$

That is, *the product of two binomials of the types* $ax + b$ *and* $cx + d$ *is equal to the product of the first terms of the binomials, increased by the sum of the "cross products," plus the product of the second terms.*

Such products as adx and bcx, shown above, are commonly called **cross products,** since when the given expressions are arranged as multiplicand and multiplier in the vertical form for multiplication, in order to obtain these products we must cross over from one column to another as suggested below:

$$\begin{array}{c} ax + b \\ \times \\ \underline{cx + d} \\ bcx + adx \text{ or } (ad + bc)\,x. \end{array}$$

The remaining terms of the product, acx^2 and bd, are obtained by multiplying together terms which are in the same columns.

Ex. 1. Multiply $(2a + 5)$ by $(3a + 7)$. Check. $a = 2$

$(2a + 5)(3a + 7) \equiv 6a^2 + 29a + 35.$ $117 = 117.$

25. Products of expressions of the types $ax + by$ and $cx + dy$ may also be found by reference to the Theorem above.

Check. $x = y = 2.$

Ex. 2. $(3x + 8y)(5x - 2y) \equiv 15x^2 + 34xy - 16y^2.$ $132 = 132.$

MENTAL EXERCISE VII. 11

Obtain each of the following indicated products:

1. $(3a + 2)(a + 1)$.
2. $(2b + 1)(3b + 4)$.
3. $(4c + 1)(2c + 3)$.
4. $(5d + 7)(d + 1)$.
5. $(g + 2)(6g + 5)$.
6. $(7h + 3)(2h + 1)$.
7. $(8k + 5)(k - 1)$.
8. $(9m - 2)(m + 1)$.
9. $(6n - 1)(4n + 3)$.
10. $(5r - 4)(2r - 3)$.
11. $(7s - 9)(s - 4)$.
12. $(11x - 8)(3x + 2)$.
13. $(10y - 7)(5y - 4)$.
14. $(12z + 5)(7z - 3)$.
15. $(13w + 6)(5w - 2)$.
16. $(8s - 15)(3s + 5)$.
17. $(9t + 11)(4t - 5)$.
18. $(15v + 14)(5v - 4)$.
19. $(6k + 17)(2k - 5)$.
20. $(10a + 19)(2a + 1)$.
21. $(16b - 9)(5b + 3)$.
22. $(14d + 11)(4d + 3)$.
23. $(20h + 17)(5h - 4)$.
24. $(13a - 18)(2a - 3)$.
25. $(7q + 12)(4q - 7)$.
26. $(6g + 7)(11g - 10)$.
27. $(12c - 13)(5c + 7)$.
28. $(8w + 9)(7w - 8)$.
29. $(19k + 7)(3k - 1)$.
30. $(17c - 4)(3c - 10)$.
31. $(13b + 9)(5b - 4)$.
32. $(18h + 7)(8h - 3)$.
33. $(15g - 11)(6g + 5)$.
34. $(16m + 9)(5m + 3)$.
35. $(14n + 13)(7n - 6)$.
36. $(5r + 3s)(4r + 7s)$.
37. $(9p - 4q)(5p - 2q)$.
38. $(12b + 7d)(8b - 5d)$.
39. $(15c - 13g)(4c + 3g)$.
40. $(14h + 17q)(3h - 4q)$.
41. $(16m - 9w)(8m + 5w)$.
42. $(13r + 9z)(7r - 4z)$.

26. The product of two polynomials consisting of the same terms arranged in the same order, but in which the signs of one or more pairs of corresponding terms differ, may often be found by so grouping the terms as to make the polynomials appear as the sum and difference of the same combinations of terms.

Ex. 1. Multiply $x + y + z$ by $x + y - z$.

The terms of the polynomials may be grouped as follows:

$$x + y + z \equiv (x + y) + z \text{ and } x + y - z \equiv (x + y) - z.$$

Hence, we may write $[(x + y) + z][(x + y) - z] \equiv x^2 + 2\,xy + y^2 - z^2.$

Ex. 2. $(x + y + 3)(x - y + 3) \equiv (x + 3)^2 - y^2$

$$\equiv x^2 + 6\,x + 9 - y^2.$$

The student should check the examples above numerically.

Mental Exercise VII. 12

Obtain each of the following products:

1. $(m + n + q)(m + n - q)$.
2. $(a + c + 4)(a + c - 4)$.
3. $(b + d + 2)(b + d - 2)$.
4. $(a + b + 1)(a - b + 1)$.
5. $(2 + c + d)(2 - c + d)$.
6. $(a + b + c)(a - b - c)$.
7. $(x + y + z)(x - y - z)$.
8. $(m + n + 1)(m - n - 1)$.
9. $(b + c + 3)(b - c - 3)$.
10. $(a + b - c)(a - b + c)$.
11. $(x + y - z)(x - y - z)$.
12. $(m + n - 5)(m - n + 5)$.

Ex. 13. $(a + b + 3)(a + b + 2) \equiv (a + b)^2 + 5\,(a + b) + 6$

$$\equiv a^2 + 2\,ab + b^2 + 5\,a + 5\,b + 6.$$

Ex. 14. $(x - y + 7)(x - y - 4) \equiv (x - y)^2 + 3\,(x - y) - 28$

$$\equiv x^2 - 2\,xy + y^2 + 3\,x - 3\,y - 28.$$

15. $(m + n + 2)(m + n + 1)$.
16. $(x + y + 4)(x + y + 5)$.
17. $(a + b + 6)(a + b + 3)$.
18. $(b + c + 7)(b + c + 2)$.
19. $(x - y + 4)(x - y + 1)$.
20. $(y - z + 5)(y - z + 3)$.
21. $(a - b + 6)(a - b - 2)$.
22. $(b - c + 9)(b - c - 5)$.
23. $(c - x - 3)(c - x - 4)$.
24. $(d - y - 7)(d - y - 2)$.
25. $(m - q - 10)(m - q - 6)$.
26. $(m - w - 11)(m - w - 7)$.

Ex. 27. $(a + b + c + d)(a + b - c - d) \equiv (a + b)^2 - (c + d)^2$

$$\equiv a^2 + 2\,ab + b^2 - c^2 - 2\,cd - d^2.$$

28. $(x+y+z+w)(x+y-z-w)$.
29. $(a+b+c+1)(a+b-c-1)$.
30. $(x+y+z+3)(x+y-z-3)$.
31. $(a-b+c-d)(a-b-c+d)$.
32. $(x-y+z-w)(x-y-z+w)$.
33. $(a-b+c-2)(a-b-c+2)$.

Powers of a Binomial.

27. The Binomial Theorem. The student should obtain the following identities by actual multiplication: Check. Let $a = b = \mathbf{1}$.

$$\begin{aligned}
(a+b)^1 &\equiv a + b \dots\dots\dots\dots\dots\dots & 2^1 &= 2.\\
(a+b)^2 &\equiv a^2 + 2\,ab + b^2 \dots\dots\dots\dots & 2^2 &= 4.\\
(a+b)^3 &\equiv a^3 + 3\,a^2b + 3\,ab^2 + b^3 \dots\dots & 2^3 &= 8.\\
(a+b)^4 &\equiv a^4 + 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 + b^4 \dots & 2^4 &= 16.
\end{aligned}$$

. . . .

It will be well for the student to extend this set of identities as far as the tenth or twelfth powers of the binomial $(a + b)$.

By inspection of the identities above, we shall discover certain laws of coefficients and exponents which will be found to hold true for any positive integral powers of any binomial. (See Chap. XXVIII. §§ 2, 3).

CHAPTER VIII

DIVISION OF INTEGRAL ALGEBRAIC EXPRESSIONS

(For Definitions, see Chapter V.)

1. WHEN an expression A can be produced by multiplying together two others, B and C, then B and C are called factors of A. The expression A is said to be exactly divisible by B, and also by C.

In multiplication we are given two factors to find their product, while in division we assume a given expression to be the product of two factors, and, having one of them, our problem is to find the other.

2. Since the operation of division is the inverse of that of multiplication, it follows that *if we multiply the quotient by the divisor we shall always obtain the dividend.*

3. To divide one power of a base by another power of the same base, we may apply the following

Law of Exponents: *The quotient obtained by dividing any power of a given base by a lower power of the same base is a power of that base. Its exponent is found by subtracting the exponent of the divisor from the exponent of the dividend.*

$$\text{That is, }\left\{\begin{array}{lll}\text{(i.)} & a^m \div a^n \equiv a^{m-n}, & \text{if } m > n.\\ \text{(ii.)} & a^m \div a^n \equiv 1 \div a^{n-m}, & \text{if } m < n.\end{array}\right.$$

If m and n are positive whole numbers, we may from the definition of a power write the following:

(i.) $m > n$.

$$\begin{aligned} a^m \div a^n &\equiv (\overbrace{a\times a\times a\times\cdots\cdots\times a}^{m\text{ factors}})\div(\overbrace{a\times a\times a\times\cdots\times a}^{n\text{ factors}})\\ &\equiv (\overbrace{a\times a\times a\times\cdots\times a}^{(m-n)\text{ factors}})\times(\overbrace{a\times a\times a\times\cdots\times a}^{n\text{ factors}})\div(\overbrace{a\times a\times a\times\cdots\times a}^{n\text{ factors}})\\ &\equiv (\overbrace{a\times a\times a\times\cdots\times a}^{(m-n)\text{ factors}})\\ &\equiv a^{m-n}, \text{ by notation.}\end{aligned}$$

(ii.) $m < n$.

$$a^m \div a^n \equiv \overbrace{(a\times a\times a\times \cdots \times a)}^{m\text{ factors}} \div \overbrace{(a\times a\times a\times a\times \cdots\cdots \times a)}^{n\text{ factors}}$$

$$\equiv \overbrace{(a\times a\times a\times \cdots \times a)}^{m\text{ factors}} \div \overbrace{(a\times a\times a\times \cdots \times a)}^{m\text{ factors}} \times \overbrace{(a\times a\times a\times \cdots \times a)}^{(n-m)\text{ factors}}$$

$$\equiv 1 \div \overbrace{(a\times a\times a\times a\times \cdots\cdots \times a)}^{(n-m)\text{ factors}}$$

$$\equiv 1 \div a^{n-m}, \text{ by notation.}$$

4. According as the exponent of one power is greater than or less than the exponent of another power, the first power is said to be **higher** or **lower** than the second.

E. g. a^5 is a higher power than a^3, or a^2; but it is a lower power than a^8 or a^9.

5. It will be shown in a later chapter that the Law of Exponents may be applied whenever the exponents are positive or negative, integral or fractional numbers. (See Chapter XIX. § 4.)

One power divided by another power of the same base.

6. The quotient resulting from the division of one power of a base by another power of the same base may be found by applying the Law of Exponents.

Ex. 1. $a^6 \div a^4 \equiv a^{6-4} \equiv a^2.$

Ex. 2. $b^9 \div b^5 \equiv b^{9-5} \equiv b^4.$

Ex. 3. $(-x)^8 \div (-x)^6 \equiv (-x)^{8-6} \equiv (-x)^2.$

Since all even powers of negative bases are positive numbers, we have $(-x)^2 \equiv x^2$.

Let the student check the above results numerically.

MENTAL EXERCISE VIII. 1

Express the following quotients of different powers of the same bases as single powers of positive bases :

1. $3^5 \div 3^2$.
2. $2^7 \div 2^4$.
3. $3^6 \div 3^5$.
4. $6^3 \div 6$.
5. $5^7 \div 5^4$.
6. $4^9 \div 4^6$.
7. $7^{10} \div 7^9$.
8. $2^{11} \div 2^5$.
9. $3^{12} \div 3^2$.
10. $4^{10} \div 4^8$.
11. $6^6 \div 6^4$.
12. $(-3)^8 \div (-3)^5$.
13. $(-6)^9 \div (-6)^7$.
14. $(-2)^8 \div (-2)$.
15. $(-2)^{10} \div (-2)^5$.
16. $a^8 \div a^6$.
17. $b^{13} \div b^{10}$.
18. $c^{12} \div c^7$.
19. $h^{11} \div h^3$.
20. $g^{12} \div g^{10}$.
21. $m^{14} \div m^{13}$.

22. $k^{16} \div k^{12}$.
23. $(-n)^{17} \div (-n)^{7}$.
24. $(-r)^{15} \div (-r)^{9}$.
25. $(-x)^{18} \div (-x)^{11}$.
26. $(-y)^{13} \div (-y)^{11}$.
27. $(-z)^{19} \div (-z)^{5}$.
28. $a^{m} \div a$.
29. $b^{n} \div b^{2}$.
30. $c^{2r} \div c$.
31. $d^{3s} \div d^{4}$.
32. $m^{a} \div m^{b}$.
33. $a^{m} \div a^{b}$.
34. $c^{r} \div c^{x}$.
35. $-d^{x} \div d^{5}$.
36. $(-g)^{k} \div (-g)^{6}$.
37. $(-h)^{y} \div (-h)^{z}$.
38. $a^{m+1} \div a$.
39. $b^{n+2} \div b$.
40. $c^{2n} \div c^{n}$.
41. $d^{4r} \div d^{2r}$.
42. $g^{5x} \div g^{2x}$.
43. $a^{2b+1} \div a$.
44. $b^{8c+2} \div b$.
45. $c^{4d+5} \div c^{2}$.
46. $d^{3m+6} \div d^{3}$.
47. $h^{a+b} \div h^{a}$.
48. $k^{m+n} \div k^{n}$.
49. $a^{2m+5} \div a^{m}$.
50. $b^{4n+7} \div b^{2n}$.
51. $c^{5r-2} \div c^{8r}$.
52. $d^{8x+2} \div d^{2x}$.
53. $a^{8r+2} \div a^{r+1}$.
54. $b^{4s+6} \div b^{2s+4}$.
55. $c^{5n+6} \div c^{4n+5}$.
56. $d^{x} \div d^{x-1}$.
57. $g^{y} \div g^{y-2}$.
58. $h^{r} \div h^{r-8}$.
59. $k^{r+1} \div k^{r-1}$.
60. $m^{2n} \div m^{n-1}$.
61. $n^{8r+2} \div n^{2r-2}$.
62. $q^{4x+3} \div q^{4x-5}$.
63. $x^{2m+n} \div x^{2n-1}$.
64. $y^{8n+r} \div y^{r-2}$.
65. $z^{4x+y} \div z^{4x-y}$.
66. $a^{3b+5c} \div a^{b+2c}$.
67. $b^{4x+7y} \div b^{8x+5y}$.
68. $c^{6m+5n} \div c^{4m+2n}$.
69. $d^{2d+8} \div d^{d+8}$.
70. $x^{a+b+c} \div x^{a}$.
71. $y^{m+n+8} \div y^{n}$.
72. $z^{r+s+5} \div z^{5}$.
73. $a^{m+n+r} \div a^{m+n}$.
74. $b^{x+y+z} \div b^{x+z}$.
75. $c^{r+s+4} \div c^{s+4}$.
76. $a^{x+y+7} \div a^{x+y+1}$.
77. $x^{a+b+c} \div x^{a+5}$.
78. $y^{m+n+8} \div y^{n+6}$.
79. $z^{a+b+4} \div z^{a+2}$.
80. $a^{2m+n+1} \div a^{m+n+1}$.
81. $b^{8r+2s+4} \div b^{8r+s+8}$.
82. $x^{4a+8b+2c} \div x^{3a+2b+c}$.
83. $y^{5a+6b+7c} \div y^{4a+8b+2c}$.

Division of One Monomial by Another

7. It may happen that the factors of a given divisor may be found by inspection among the factors of the dividend.

In this case the quotient is by definition that part of the dividend remaining after striking out the factors of the dividend which are equal to those of the given divisor.

Ex. 1. Divide xyz by y.

We may write $xyz \div y \equiv xzy \div y$. By the Commutative Law for Multiplication.

$\equiv xz$. Since $\times\, y \div y$ may be neglected.

Ex. 2. Divide $20\,x^3$ by $5\,x$.

20 is 4×5, x^3 is $x^2 \times x$.

Hence, $20\,x^3 \div 5\,x \equiv (4\,x^2 \times 5\,x) \div 5\,x \equiv 4\,x^2$.

Ex. 3. If we are required to divide mn by x, then, since x does not "appear" as a factor of the dividend mn, we may indicate the quotient by writing $mn \div x \equiv \frac{mn}{x}$.

Ex. 4. Divide $12\,ab^2c^3$ by $4\,ab^2d$.

$$12\,ab^2c^3 \div 4\,ab^2d \equiv (12 \div 4) \times (a \div a) \times (b^2 \div b^2) \times (c^3 \div d)$$
$$\equiv 3\,c^3 \div d \equiv \frac{3\,c^3}{d}.$$

Ex. 5. $18\,x^2yz \div 7\,x^2yw \equiv (18 \div 7) \times (x^2 \div x^2) \times (y \div y) \times (z \div w)$

$$\equiv 18 \div 7 \times z \div w$$
$$\equiv 18\,z \div 7\,w \equiv \frac{18\,z}{7\,w}.$$

Let the student check these examples numerically.

8. The division of one integral monomial by another may be indicated by writing the divisor beneath the dividend, separating the two by a horizontal stroke of division.

It follows that, since the operation of division introduces neither the symbol of addition +, nor the symbol of subtraction —, that the quotient obtained by dividing one monomial by another is always a monomial.

The quotient obtained by dividing one monomial by another, is the quotient of their numerical coefficients (considered as positive or negative numbers), multiplied by the quotient of their literal factors.

MENTAL EXERCISE VIII. 2

Perform the following indicated divisions :

Divisor)$\underline{\text{Dividend}}$
Quotient.

1. $2\,b\,)\,\underline{6\,b^3}.$
2. $3\,c\,)\,\underline{12\,c^4}.$
3. $5\,d\,)\,\underline{25\,d^6}.$
4. $4\,h\,)\,\underline{24\,h^2}.$
5. $6\,m\,)\,\underline{-\,30\,m^6}.$
6. $-\,7\,n\,)\,\underline{28\,n^3}.$
7. $-\,8\,x\,)\,\underline{-\,16\,x^5}$
8. $2\,a^2\,)\,\underline{8\,a^3}.$
9. $3\,b^2\,)\,\underline{15\,b^4}.$
10. $4\,c^3\,)\,\underline{20\,c^6}.$
11. $-\,9\,d^4\,)\,\underline{-\,18\,d^{10}}.$
12. $7\,h^5\,)\,\underline{14\,h^5}.$
13. $6\,ab\,)\,\underline{-\,12\,a^2b^3}.$
14. $4\,cd\,)\,\underline{8\,c^2d}.$
15. $-\,3\,gx\,)\,\underline{21\,gx^6}.$
16. $-11\,a^2m\,)\,\underline{-22\,a^3m^4}.$
17. $12\,b^2k^7\,)\,\underline{36\,b^7k^9}.$
18. $9\,xyz\,)\,\underline{-\,27\,x^2y^3z^4}.$

19. $15a^2b^3c\,)\,\underline{30a^3b^3c^2}.$

20. $-13ac^3y^5\,)\,\underline{52a^2c^4y^6}.$

21. $14a^2b^3c^2\,)\,\underline{56a^3b^4c^5d}.$

22. $16x^5y^2z\,)\,\underline{-96x^6y^2z^2w^3}.$

23. $-15a^2b^4y^3\,)\,\underline{75a^3b^4x^2y^4}.$

24. $-9b^6c^4z^5\,)\,\underline{-72b^7c^4y^6z^5}.$

25. $42a^8 \div 6a^2.$

26. $39b^7 \div 13b^4.$

27. $72a^4b^6 \div 12a^3b^2.$

28. $56b^3d^7 \div 8b^2d^6.$

29. $44x^4y^5 \div (-11x^3y^5).$

30. $75m^2w^6 \div (-15m^2w^5).$

31. $(-90n^6z^7) \div (-10n^3z^3).$

32. $(-96g^6h^7) \div (-16g^4h^4).$

33. $(-72h^4k^3) \div 18hk^3.$

34. $80a^3b^3c^3 \div 16a^2b^2c^2.$

35. $63x^4y^2z^3 \div 7x^2y^2z^2.$

36. $(-60a^3b^2d) \div (-15abd).$

37. $-45m^4n^3w^2 \div 9m^2n^3w^2.$

38. $50g^5h^4k^5 \div 25g^5k^5.$

39. $51x^6y^6z^6 \div (-17x^4y^4z^4).$

40. $(-57a^4b^5y^6) \div (-19a^3b^4y^5).$

41. $a^nb^2 \div ab.$

42. $c^3d^r. \div cd.$

43. $m^xn^y \div mn.$

44. $x^{2a}y^3 \div x^ay.$

45. $a^{3m}x^{2n} \div a^mx^n.$

46. $b^{4r}y^{2x} \div b^{2r}y^x.$

47. $c^{x+1}w^{y+1} \div cw.$

48. $b^{n+2}x^{n+1} \div b^2x^n.$

49. $d^{r+4}q^{r+2} \div d^{r+1}q^2.$

50. $a^{2m+3}b^{3n+2} \div a^2b^2.$

51. $c^{3x+5}d^{2x+4} \div c^{x+2}d^{2x+3}.$

52. $x^{4a+2b}y^{3a+4c} \div x^{2a+b}y^{2a+3c}.$

Division of a Polynomial by a Monomial.

9. The quotient resulting from the division of a polynomial by a monomial may be obtained as a direct application of the Distributive Law for division. That is, since

$$(a + b - c) \div d \equiv a \div d + b \div d - c \div d,$$

it follows that we may *divide each term of the polynomial dividend by the monomial divisor and write the algebraic sum of the resulting partial quotients.*

Ex. 1. Divide $15a^4b^2 - 10a^2b^3 + 5ab^5$ by $5b^2$.

$$(15a^4b^2 - 10a^2b^3 + 5ab^5) \div 5b^2 \equiv 15a^4b^2 \div 5b^2 - 10a^2b^3 \div 5b^2 + 5ab^5 \div 5b$$

$$\equiv 3a^4 - 2a^2b + ab^3.$$

Check. Let $a = 3$, $b = 2$.

$$(4860 - 720 + 480) \div 20 = 243 - 36 + 2$$

$$231 = 231.$$

Mental Exercise VIII. 3

Perform the following indicated divisions:

1. $2a\overline{)\,6ab + 8ac}.$

2. $3b\overline{)\,12bc + 18bx}.$

3. $6d\overline{)\,30dxy + 42d}.$

4. $5m\overline{)\,5m - 10mn}.$

5. $7n\overline{)\,14n^2 + 21nx}.$

6. $4c\overline{)\,8cy - 28cw}.$

7. $8x\overline{)\,24xy + 16x^2}.$

8. $9y\overline{)\,27y^2z + 45y^4w}.$

9. $6w\overline{)\,30x^2w^2 + 48y^3w^4}.$

10. $3a^2\overline{)\,6a^3b + 12a^2c}.$

11. $\dfrac{8ab^4 + 14b^5c^2}{2b^3}.$

12. $\dfrac{4c^5 - 8c^4d^2}{4c^4}.$

13. $\dfrac{4abc + 10abd}{2ab}.$

14. $\dfrac{15x^2y + 18xy^2}{3xy}.$

15. $\dfrac{32m^3n^2 - 40m^2n^3}{4mn}.$

16. $\dfrac{18a^2b^2 - 12a^3bc}{6a^2b}.$

17. $\dfrac{14xy^3 - 35x^2y^2z}{7xy^2}.$

18. $\dfrac{16m^3x^2y - 44m^2x^3z}{4m^2x^2}.$

19. $\dfrac{10a^3b^5c - 15a^2bc^2 + 5a^2bc}{5a^2bc}.$

20. $\dfrac{9a^2bc^4 + 12a^4b^2c + 24ab^4c^2}{3abc}.$

21. $\dfrac{21a^3b^4c^2 - 7a^2b^3c^4 - 28a^4b^2c^3}{7a^2b^2c^2}.$

22. $\dfrac{30x^3y^5z^4 + 12x^5y^4z^3 - 18x^4y^3z^5}{-\ 6x^3y^3z^3}.$

23. $\dfrac{25m^6n^3z^5 - 30m^3n^5z^6 - 45m^5n^6z^3}{-5m^3n^3z^3}.$

24. $\dfrac{28r^5s^2w^6 + 32r^6s^3w^7 - 44r^3s^4w^8}{4r^3s^2w^5}.$

25. $\dfrac{40a^3bd^5 + 24a^2b^2d^3 + 32a^4b^3d^4}{8a^2bd^3}.$

26. $\dfrac{27m^4n^5z^6 - 18m^5n^6z^7 - 54m^6n^7z^8}{-9m^3n^4z^6}.$

27. $\dfrac{33\,x^7y^7z^7 - 44\,x^5y^6z^5 - 22\,x^6y^4z^6}{11\,x^5y^4z^5}.$

28. $\dfrac{49\,g^4h^7k^5 + 42\,g^6h^5k^7 + 35\,g^5h^6k^4}{-7\,g^4h^5k^4}.$

29. $\dfrac{-24\,a^6c^5h^7 - 12\,a^7c^6h^5 - 48\,a^5c^7h^6}{-12\,a^5c^5h^5}.$

30. $\dfrac{-26\,b^3h^5n^7 - 13\,b^2h^4n^6 - 39\,b^4h^6n^8}{-13\,b^2h^4n^6}.$

31. $(\frac{4}{5}\,a^3bc + \frac{2}{3}\,ab^3c + \frac{4}{15}\,abc^3) \div 2\,abc.$

32. $(\frac{3}{4}\,a^2b^3c^2 + \frac{3}{5}\,a^3b^2c^3 + \frac{3}{7}\,a^2b^3c^3) \div 3\,a^2b^2c^2.$

33. $(\frac{1}{2}\,xy^2z^2 - 2\,x^2y^3z - \frac{1}{4}\,x^3yz^2) \div \frac{1}{2}\,xyz.$

34. $(\frac{5}{7}\,m^5n^3w^4 - \frac{10}{7}\,m^3n^4w^5 - \frac{15}{9}\,m^4n^5w^3) \div 5\,m^3n^3w^3.$

35. $(\frac{2}{3}\,a^4bd^3 - \frac{5}{6}\,a^3b^4d - \frac{1}{2}\,a^2b^2d^2) \div (-\frac{1}{3}\,a^2bd).$

36. $[3(a+b)^3 + 9(a+b)^4 + 6(a+b)^5] \div 3(a+b)^2.$

37. $[5(x+y)^6 - 15(x+y)^5 - 30(x+y)^4] \div 5(x+y)^3.$

38. $[4(b-d)^5 + 20(b-d)^6 + 16(b-d)^7] \div 4(b-d)^5.$

39. $[6(g^2-h^2) + 18(g-h)^2 + 12(g-h)^3] \div 6(g-h).$

40. $(a^{m+6} + a^{m+5} + a^{m+4}) \div a.$

41. $(b^{n+7} + b^{n+5} + b^{n+3}) \div b^2.$

42. $(c^{2n+3} + c^{2n+2} + c^{2n+1}) \div c^{2n}.$

43. $(d^{a+2} - d^{a+4} + d^{a+6}) \div d^{a+1}.$

44. $(x^{3n+4} - x^{3n+3} - x^{3n+2}) \div x^{2n+2}.$

45. $(y^{a} + y^{2a} + y^{3a} + y^{4a}) \div y^{a}.$

46. $(z^{5n} + z^{4n} + z^{3n} + z^{2n}) \div z^{2}.$

47. $(a^{4x+y} + a^{3x+2y} + a^{2x+3y} + a^{x+4y}) \div a^{x+y}.$

48. $(b^{m+4n} - b^{2m+3n} + b^{3m+2n} - b^{4m+n}) \div b^{m+n}.$

49. $(a^{2m+1}b^3 + a^{2m+2}b^5 + a^{2m+3}b^7) \div a^{2m}b^2.$

Division of One Polynomial by Another.

10. An expressed quotient is called a **fraction;** the dividend is called the **numerator** and the divisor the **denominator.**

11. The process by which the division of one polynomial by another is performed, may be made to depend upon the following

Fundamental Principle: *The quotient obtained by dividing an integral function of x by an integral function of x which is of degree not higher than that of the dividend can be transformed into the sum of an integral function of x (the integral quotient) and a fraction the* *numerator of which is the remainder resulting from the division, and the denominator of which is the given divisor.*

The degree of the integral quotient obtained by the above Division Transformation is equal to the excess of the degree of the dividend over that of the divisor.

The degree of the remainder which is used as the numerator of the fractional part of the transformed function is accordingly less than the degree of the divisor which is used as a denominator.

(The following proof may be omitted when the chapter is read for the first time.)

Let the dividend D and the divisor d be integral functions of some letter x, the degree of the divisor being not higher than that of the dividend with reference to x.

Letting Q stand for the integral part of the quotient and R for the remainder, we have

$$\frac{D}{d} \equiv Q + \frac{R}{d}.$$

We may construct the following identity:

$$\begin{aligned} D \div d &\equiv D \div d + Q - Q. && \text{Since } Q - Q = 0. \\ &\equiv Q + (D \div d - Q). && \text{Commutative and Associative Laws.} \\ &\equiv Q + (D \div d - Q) \times d \div d. && \text{Since} \times d \div d = 1. \\ &\equiv Q + (D \div d \times d - Q \times d) \div d. && \text{Distributive Law.} \\ &\equiv Q + (D - Qd) \div d. && \text{Since} \div d \times d = 1. \\ &\equiv Q + R \div d. && \text{Since } D - Qd \text{ is by definition the same as } R. \end{aligned}$$

Division of one Polynomial by Another.

DEVELOPMENT OF THE PROCESS

12. In order to clearly understand the **process of division,** it is well to obtain the product of two given integral polynomials, and then, after having examined carefully the manner in which it is built up, to reverse certain of the steps and processes. Then starting with the **reduced product as a dividend,** and one of the factors, say the **multiplicand, as a divisor,** we may find the other, **the multiplier,** which we shall now call **the quotient.**

13. For convenience, we shall select two polynomials in which no powers are missing, and shall arrange them according to descending powers of the same letter, say a.

Multiply $a^3 + 3a^2b + 3ab^2 + b^3$ by $a^2 + 2ab + b^2$.

Form I

Multiplicand		$a^3 + 3a^2b + 3ab^2 \quad + b^3$	Divisor.
Multiplier		$a^2 + 2ab \; + b^2$	Quotient.
Partial Products.	1st row	$a^5 + 3a^4b + 3a^3b^2 + a^2b^3$	
	2nd row	$+ 2a^4b + 6a^3b^2 + 6a^2b^3 + 2ab^4$	
	3rd row	$+ a^3b^2 + 3a^2b^3 + 3ab^4 + b^5$	
Reduced Product .		$a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$	Dividend.

14. Observe that *the number of terms in each horizontal row of partial products corresponds to the number of terms in the multiplicand*, and there are as many rows as there are separate terms in the multiplier. The degree of the first term of each row with reference to the letter of arrangement is higher than that of any following term.

15. If now we interchange the given polynomials and use the first as a multiplier, and the second as a multiplicand, we shall obtain the same reduced product and the same partial products as before; but their orders of arrangement will be different, as in Form II.

Form II

Check for both forms.
Let $a = 3$, $b = 2$.

Multiplicand . .		$a^2 + 2ab \; + b^2$	25
Multiplier . . .		$a^3 + 3a^2b + 3ab^2 \quad + b^3$	125
Partial Products	1st row	$a^5 + 2a^4b + a^3b^2$	3125
	2nd row	$+ 3a^4b + 6a^3b^2 + 3a^2b^3$	
	3rd row	$+ 3a^3b^2 + 6a^2b^3 + 3ab^4$	
	4th row	$+ a^2b^3 + 2ab^4 + b^5$	
Reduced Product		$a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$. . .	3125
			0

16. Each horizontal row of partial products in Form II corresponds to an oblique or diagonal row containing the same terms in Form I, and each row in Form I has a corresponding oblique or diagonal row in Form II.

E. g. The first horizontal row in Form II, $a^5 + 2a^4b + a^3b^2$, appears as the first diagonal row $a^5 + 2a^4b + a^3b^2$ in Form I (§ 13). Also, the terms in the first horizontal row of Form I, $a^5 + 3a^4b + 3a^3b^2 + a^2b^3$, are found as the first terms of the different rows of Form II; that is, they occur in the first diagonal row. The same is true for the other rows of partial products.

17. The arrangement of the partial products for a given example depends simply on which of the two given polynomials is chosen as multiplicand, and which as multiplier.

18. In either Form I or Form II, the terms of any particular row are obtained by multiplying successively the terms of the multiplicand by one of the terms of the multiplier.

E. g. *To obtain the first row of partial products use* as a multiplier *the first term* of the polynomial chosen as multiplier; *for the second row use the second term* of the polynomial multiplier; *for the third row the third term*, etc.

19. The first term of each row is obtained by multiplying the first term of the multiplicand by the term of the multiplier corresponding to the number of the row.

E. g.
In Form I $\begin{cases} a^5 \text{ is obtained by multiplying } a^3 \text{ by } b^2; & (1) \\ 2a^4b \quad " \quad " \quad " \quad a^3 \text{ by } 2ab; & (2) \\ a^3b^2 \quad " \quad " \quad " \quad a^3 \text{ by } b^2. & (3) \end{cases}$

Similar results may be seen to be true by examining Form II. (§ 15).

20. It follows that, if we know the first term of any specified row, we may, by dividing it by the first term of the multiplicand, obtain the term of the polynomial multiplier corresponding in number to the number of the row.

E. g. In Form I (§ 13) we have:

From the first row of partial products . . . . . . $a^5 \div a^3 \equiv a^2$, (1)
which is the first term of the polynomial multiplier.

From the second row of partial products . . . $2a^4b \div a^3 \equiv 2ab$, (2)
which is the second term of the polynomial multiplier.

From the third row of partial products . . . $a^3b^2 \div a^3 \equiv b^2$, (3)
which is the third term of the polynomial multiplier.

21. From the arrangement of the partial products in columns, the first partial product, a^5, is the first term a^5 of the reduced product in either Form I or Form II. (§§ 13, 15).

22. If now, returning to Form I, we *divide* a^5 *by* a^3, we obtain as a quotient the first term a^2 of the polynomial multiplier.

That is, $a^5 \div a^3 \equiv a^2$. (See (1) § 20.)

23. *If we multiply the terms of the multiplicand successively by this multiplier*, a^2, we obtain the first horizontal row of partial products as in Form I (§ 13). By subtracting the terms of this first row from the reduced product, we have as below:

Step (i.)	Reduced product,	$a^5 + 5\,a^4b + 10\,a^3b^2 + 10\,a^2b^3 + 5\,ab^4 + b^5$
	First row of partial products,	$a^5 + 3\,a^4b + 3\,a^3b^2 + a^2b^3$
	First partial remainder,	$2\,a^4b + 7\,a^3b^2 + 9\,a^2b^3 + 5\,ab^4 + b^5.$

24. The result of the subtraction, $2\,a^4b + 7\,a^3b^2 + 9\,a^2b^3 + 5\,ab^4 + b^5$, we shall call the **first partial remainder.** From the nature of the case the first term, $2\,a^4b$, is the same as the first term of the second row of partial products in Form I (§ 13). It is also the sum of all of the partial products in Form I, except those in the first row which were subtracted.

25. *Dividing the first term* $2\,a^4b$ *of the first partial remainder by* a^3, we obtain $2\,a^4b \div a^3 \equiv 2\,ab$. (See (2) § 20.)

The term $2\,ab$ thus obtained is equal to the second term of the original multiplier. Referring to Form I (§ 13) it may be seen that this step in the process consists in dividing the term of highest degree with reference to the letter of arrangement (that is, the first term of the second row of partial products in Form I) by the first term of the multiplicand above.

26. By multiplying the multiplicand, as in Form I (§ 13), by $2\,ab$ as a multiplier, we obtain the second row of partial products $2\,a^4b + 6\,a^3b^2 + 6\,a^2b^3 + 2\,ab^4$.

27. *Subtracting these terms from the first partial remainder* obtained above in § 23 (i.), we have

Step (ii.)	First partial remainder,	$2\,a^4b + 7\,a^3b^2 + 9\,a^2b^3 + 5\,ab^4 + b^5,$
	Second row of partial products,	$2\,a^4b + 6\,a^3b^2 + 6\,a^2b^3 + 2\,ab^4,$
	Second partial remainder,	$a^3b^2 + 3\,a^2b^3 + 3\,ab^4 + b^5.$

28. The first term a^3b^2 of the *second partial remainder* is the same as the first term of the third row of partial products in Form I (§ 13).

29. As we proceed in our work any particular remainder will, from the nature of the case, be the sum of the rows of partial products below the last one subtracted. (See Form I, § 13.)

30. Dividing, as before, the first term a^3b^2 by the first term of the multiplicand a^3, we have as in (3) § 20,

$$a^3b^2 \div a^3 \equiv b^2,$$

which is the third term of the original polynomial multiplier.

31. As before, we use this as a multiplier with the entire multiplicand, and obtain the third row of partial products in Form I (§ 13), $a^3b^2 + 3a^2b^3 + 3ab^4 + b^5$.

32. *Subtracting these terms from the second partial remainder,* we find as below, that the third partial remainder vanishes, that is, it is zero :

Step (iii.)	Second partial remainder,	$a^3b^2 + 3a^2b^3 + 3ab^4 + b^5$
	Third row of partial products,	$\underline{a^3b^2 + 3a^2b^3 + 3ab^4 + b^5}$
		0

33. The process stops here, as we should naturally expect, since we have recovered all of the terms of the original polynomial multiplier a^2, $+ 2ab$ and b^2. (See Form I, § 13.)

34. Summary. By this process we have succeeded in finding the original polynomial multiplier, when the reduced product and the original multiplicand were given. The process would have been exactly the same if we had started with the reduced product and the original multiplier, $a^2 + 2ab + b^2$, to find the original multiplicand $a^3 + 3a^2b + 3ab^2 + b^3$. In that case, however, there would have been this exception, that the successive rows of partial products would have corresponded to the rows of partial products in Form II (§ 15), where the polynomial to be found is used as a multiplier.

35. We have thus developed a method for finding the original multiplier when the multiplicand and the reduced product are given. The method holds good also for finding the original multiplicand when the multiplier and the reduced product are given.

36. It will be seen from the discussion above that the reduced product takes the place, in our problem of division, of the dividend, the original multiplicand of the divisor, and the original multiplier of the quotient.

CORRESPONDENCE BETWEEN EXAMPLES

IN

DIVISION MULTIPLICATION

DIVIDEND | DIVISOR . . . . . MULTIPLICAND
. | QUOTIENT . . . . MULTIPLIER
. . . . REDUCED PRODUCT

37. The rows of partial products in the process of multiplication correspond to those appearing in the process of division.

38. The Division Transformation. Whenever division is possible, we may carry out the different steps of the process as follows:

First arrange the terms *of both dividend and divisor according to ascending or descending powers of some letter.*

Place the divisor, for convenience, at the right of the dividend. Since the different terms of the quotient are to be used as multipliers during the process, it will be convenient to write the quotient, term by term, immediately below the divisor.

Divide *the first term of the dividend by the first term of the divisor, and write the result as the first term of the quotient.*

Multiply *the whole divisor by this first term of the quotient, and write the resulting partial products under the dividend.*

Subtract *from the dividend the polynomial composed of the partial products, and bring down the result as the first partial remainder.*

Divide *the first term of the first partial remainder by the first term of the divisor as before, and write the result as the second term of the quotient.*

Multiply *the whole divisor by the second term of the quotient, subtract the resulting product from the first partial remainder, and write the result as a second partial remainder.*

Repeat the operations above *either until the remainder is 0, or until as many terms of the quotient are found as are desired. In the latter case, add algebraically a fraction having for numerator the remainder at this stage, and for denominator the divisor.*

ARRANGEMENT OF THE WORK

39. A convenient arrangement of the different steps of the process is shown below:

	DIVIDEND	DIVISOR
Multiplying divi-	$a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$	$a^3+3a^2b+3ab^2+b^3$
sor by a^2 . .	$a^5+3a^4b+3a^3b^2+a^2b^3$	$a^2+2ab+b^2$
		QUOTIENT
1st partial remainder .	$2a^4b+7a^3b^2+9a^2b^3+5ab^4+b^5$	
Mult'g divisor by $2ab$.	$2a^4b+6a^3b^2+6a^2b^3+2ab^4$	Check. Let
2nd partial remainder . . .	$a^3b^2+3a^2b^3+3ab^4+b^5$	$a=3, \quad b=2.$
Multiplying divisor by b^2 . .	$a^3b^2+3a^2b^3+3ab^4+b^5$	$3125\underline{)125}$
		Quot't sh'd be 25
		Quotient is . 25
		0

We have carried out the process above on the assumption that the degree of the dividend, with reference to the letter of arrangement, was at least as high as that of the divisor, — that is, that division was possible. It is the exception rather than the rule that we find an integral quotient when dividing one integral polynomial by another. We therefore apply the Principle of No Exception, and assert that division may be performed, if indeed it can be begun at all, by the steps of the process above.

40. In the division transformation two cases may arise:

First, it may be possible, or second, it may be impossible, to find as a quotient an integral function of x.

In the first case the division, if carried out, is said to be **exact** and there is no remainder. It may be shown that when division is exact, the form of the quotient will be the same whether the division be carried out with both dividend and divisor arranged according to descending or ascending powers of some specified letter.

Ex. 1. Divide $x^2+10x+21$ by $x+3$.

The process is shown below, at the left with the dividend and divisor arranged according to descending powers, and at the right with both arranged according to ascending powers.

Descending Powers.

$$\begin{array}{r|l} x^2+10x+21 & x+3 \\ \cline{2-2} x^2+3x & x+7 \\ \cline{1-1} 7x+21 & \\ \underline{7x+21} & \\ \hline \end{array}$$

Ascending Powers.

$$\begin{array}{r|l} 21+10x+x^2 & 3+x \\ \cline{2-2} 21+7x & 7+x \\ \cline{1-1} 3x+x^2 & \\ \underline{3x+x^2} & \\ \hline \end{array}$$

The student should check the example numerically.

41. The form of the quotient obtained when both dividend and divisor are arranged according to ascending powers of some letter of arrangement, is not the same as that of the quotient obtained when the terms are arranged according to descending powers.

Ex. 2. Divide $8x^2 + 17x + 14$ by $x + 2$.

The process is shown below, at the left with both dividend and divisor arranged according to descending powers of x, and at the right with both arranged according to ascending powers of x. Let the student check each result numerically.

Descending Powers.

$$\begin{array}{r|l}
8x^2+17x+14 & x+2 \\ \cline{2-2}
\underline{8x^2+16x} & 8x+1+\dfrac{12}{x+2} \\
x+14 & \\
\underline{x+2} & \\
12 &
\end{array}$$

Ascending Powers.

$$\begin{array}{r|l}
14+17x+8x^2 & 2+x \\ \cline{2-2}
\underline{14+7x} & 7+5x+\dfrac{3x^2}{2+x} \\
10x+8x^2 & \\
\underline{10x+5x^2} & \\
3x^2 &
\end{array}$$

42. When the operation of division is performed with both dividend and divisor arranged according to descending powers of some letter, it will happen either that division will be exact, or that we shall arrive sooner or later at a remainder which is of lower degree with reference to the letter of arrangement than the divisor. Here, for the present, the operation of division terminates.

If, however, division be carried out with both dividend and divisor arranged according to ascending powers of some letter, it will happen that when division is not exact the degrees of the remainders will be successively higher and higher, and an unlimited number of terms may be obtained in the quotient.

Ex. 3. Divide $3x + 4x^2$ by $1 + x + x^2$.

$$\begin{array}{r|l}
3x+4x^2 & 1+x+x^2 \\ \cline{2-2}
\underline{3x+3x^2+3x^3} & 3x+x^2-4x^3+\cdots\cdots \\
x^2-3x^3 & \\
\underline{x^2+x^3+x^4} & \\
-4x^3-x^4 & \\
\underline{-4x^3-4x^4-4x^5} & \\
3x^4+4x^5 & \\
\cdots\cdots \text{ etc.} &
\end{array}$$

The quotient obtained in the example above may be checked at any stage of the process by adding to the quotient at that stage a fraction whose numerator is the corresponding remainder and whose denominator is the divisor, and then making numerical substitutions.

43. *We may continue the operation of division as long as the first term of the arranged partial remainder is divisible by the first term of the divisor.*

By bringing down the terms of the dividend in the successive partial remainders only as they are actually needed, we may save labor when carrying out the process.

44. Detached Coefficients in Division. Detached coefficients may be used in division as well as in addition and multiplication.

To illustrate the use of detached coefficients in division, the following example is performed first in the ordinary way and then again by using detached coefficients.

Ex. 4.

$$\begin{array}{r|l}
6x^6 - 17x^5 + 31x^4 - 44x^3 + 55x^2 - 40x + 14 & 2x^4 - 3x^3 + 5x^2 - 6x + 7 \\ \cline{2-2}
6x^6 - 9x^5 + 15x^4 - 18x^3 + 21x^2 \phantom{{}- 40x + 14} & 3x^2 - 4x + 2 \\ \cline{1-1}
-8x^5 + 16x^4 - 26x^3 + 34x^2 - 40x \phantom{{}+ 14} & \qquad\quad \text{Check.} \quad \text{Let } x = 2. \\
-8x^5 + 12x^4 - 20x^3 + 24x^2 - 28x \phantom{{}+ 14} & \qquad\qquad\qquad 138\underline{)23} \\ \cline{1-1}
4x^1 - 6x^3 + 10x^2 - 12x + 14 & \text{Quotient should be } 6 \\
4x^4 - 6x^3 + 10x^2 - 12x + 14 & \text{Quotient is } \qquad \underline{\underline{6}} \\ \cline{1-1} \cline{1-1}
 & \qquad\qquad\qquad\quad 0
\end{array}$$

Using detached coefficients:

$$\begin{array}{rrrrrrr|rrrrr}
6 & -17 & +31 & -44 & +55 & -40 & +14 & 2 & -3 & +5 & -6 & +7 \\ \cline{8-12}
6 & -9 & +15 & -18 & +21 & & & 3 & -4 & +2 & & \\ \cline{1-7}
 & -8 & +16 & -26 & +34 & -40 & & 3x^2 & -4x & +2 & & \\
 & -8 & +12 & -20 & +24 & -28 & & & & & & \\ \cline{2-7}
 & & 4 & -6 & +10 & -12 & +14 & & & & & \\
 & & 4 & -6 & +10 & -12 & +14 & & & & & \\ \cline{3-7} \cline{3-7}
\end{array}$$

45. Numerical Checks in Division. Since zero cannot be used as a divisor, it follows that care must be taken when employing numerical checks in division to avoid giving to the letters such values as would cause the divisor to become zero.

E. g. When checking the quotient obtained by dividing $x^2 - 7x + 12$ by $x - 3$, we must avoid giving the value 3 to x, for in this case the divisor $x - 3$ would represent the value zero.

EXERCISE VIII. 4

When performing the following divisions, check all results numerically. Divide:

1. $a^2 + 6a + 8$ by $a + 2$.
2. $6b^2 + 5b - 6$ by $3b - 2$.
3. $5c^2 + c - 6$ by $5c + 6$.
4. $21d^2 + 38d + 16$ by $7d + 8$.
5. $1 + 2c + 2c^2 + c^3$ by $1 + c$.
6. $6a^2 + 8a + 28$ by $3a + 7$.
7. $6a^2 + 13ab + 6b^2$ by $2a + 3b$.
8. $8x^2 - 22xy + 15y^2$ by $2x - 3y$.
9. $a^6 + b^6$ by $a^2 + b^2$.
10. $m^3 + 2n^3 + 3m^2n + 4mn^2$ by $m + n$.
11. $k^3 + k^2m - km^2 - m^3$ by $k - m$.
12. $a^4 - 2a^3b + 2ab^3 - b^4$ by $a^2 - b^2$.
13. $16x^4 - 1$ by $2x - 1$.
14. $32m^5 + 1$ by $2m + 1$.
15. $3a^6 - 24$ by $a^2 - 2$.
16. $6d^3 - d^2 - 12d + 4$ by $3d^2 + 4d - 1$.
17. $z^4 + z^2w^2 + w^4$ by $z^2 + zw + w^2$.
18. $q^4 - q^3 - q^2 + q$ by $q^2 + q + 1$.
19. $h^4 - 6h^3z + 9h^2z^2 - 4z^4$ by $h^2 + 3hz - 2z^2$.
20. $c^4 - c^3 - 8c^2 + 10c - 10$ by $c^2 + 2c - 2$.
21. $15a^4 - a + 8a^2 - 1 - 19a^3$ by $5a^2 - 3a - 1$.
22. $6y^4 - 13xy^3 + 13x^2y^2 - 13x^3y - 5x^4$ by $2y^2 - 3xy - x^2$.
23. $c^5 - 6c^4 + 16c^3 - 25c^2 + 13c + 5$ by $c^3 - 4c^2 + 3c + 1$.
24. $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$ by $x^2 - 2xy + y^2$.
25. $s^5 + 3s^4 - 20s^3 - 60s^2 + 64s + 192$ by $s^3 + 9s^2 + 26s + 24$.
26. $3x^5 - 8x^4 - 5x^3 + 26x^2 - 28x + 24$ by $x^3 - 2x^2 - 4x + 8$.
27. $a^5 + 10a^3b^2 + b^5 + 10a^2b^3 + 5a^4b + 5ab^4$ by $a^2 + 2ab + b^2$.
28. $3x^5 + 7x^4y - 11x^3y^2 - 11x^2y^3 + 6xy^4 - 18y^5$ by $x + 3y$.
29. $1 + 2a^2 - 7a^4 - 16a^6$ by $1 + 2a + 3a^2 + 4a^3$.
30. $v^6 - 2v^3 + 1$ by $v^2 - 2v - 1$.
31. $3x^6 + 43x^2 - 6x^3 - 30x + 80 - 32x^4 + 20x^5$ by $x + 8$.
32. $15a^6 + 16a^5 + 8a^4 - 9a^3 - 7a^2 + 19a - 42$ by $5a^2 + 2a - 7$.
33. $c^7 - 6c^6d^3 + 14c^5d^6 - 12c^4d^9$ by $c^3 - 2c^2d^3$.
34. $x^3 + y^3 + z^3 - 3xyz$ by $x + y + z$.

46. The ordinary process for "long division" may also be employed when the coefficients are fractional.

Ex. 1. Divide $\frac{1}{12}x^3 + \frac{1}{72}x^2y + \frac{1}{4}y^3$ by $\frac{1}{3}x + \frac{1}{2}y$.

$$\begin{array}{rrr|l}
\frac{1}{12}x^3 + \frac{1}{72}x^2y & + \frac{1}{4}y^3 & & \frac{1}{3}x + \frac{1}{2}y \\
\cline{4-4}
\underline{\frac{1}{12}x^3 + \frac{1}{8}x^2y} & & & \frac{1}{4}x^2 - \frac{1}{3}xy + \frac{1}{2}y^2. \\
-\frac{1}{9}x^2y & + \frac{1}{4}y^3 & & \\
-\frac{1}{9}x^2y & - \frac{1}{6}xy^2 & & \\
\cline{1-3}
 & \frac{1}{6}xy^2 & + \frac{1}{4}y^3 & \\
 & \frac{1}{6}xy^2 & + \frac{1}{4}y^3. & \\
\cline{2-3}
\end{array}$$

Let the student check this example numerically.

EXERCISE VIII. 5

Divide:

1. $\frac{1}{6}x^3 - \frac{5}{36}x^2y + \frac{1}{6}xy^2 + \frac{2}{9}y^3$ by $\frac{1}{2}x + \frac{1}{3}y$.
2. $\frac{2}{3}b^{15} - \frac{503}{160}b^{12} + \frac{803}{240}b^9 - \frac{803}{360}b^6 + \frac{503}{300}b^3 - \frac{3}{2}$ by $\frac{8}{15}b^3 - \frac{15}{8}$.
3. $\frac{1}{6}k^8 - \frac{11}{60}k^7 + \frac{223}{1260}k^6 - \frac{29}{420}k^5 + \frac{1}{42}k^4$ by $\frac{1}{3}k^2 - \frac{1}{5}k + \frac{1}{7}$.
4. $a^5 - \frac{10}{3}a^4 + 10a^3 - 30a^2 + 90a - 27$ by $81a - 27$.
5. $\frac{1}{8}t^2 - \frac{5}{4}t^3 + \frac{5}{2}t^4 - 5t^5 + 8t^6$ by $\frac{1}{16}t - \frac{1}{2}t^2$.
6. $m^7 - \frac{7}{6}m^6 + \frac{9}{35}m^5 + \frac{4}{3}m^4 - \frac{26}{35}m^3 - \frac{12}{7}m^2 + \frac{61}{30}m - 1$ by $\frac{1}{7}m^2 - \frac{1}{6}m + \frac{1}{5}$.
7. $a^5 - \frac{67}{90}a^4 + \frac{83}{210}a^3 + \frac{111}{140}a^2 - \frac{123}{350}a + \frac{2}{5}$ by $\frac{2}{3}a^2 - \frac{1}{5}a + \frac{2}{7}$.
8. $\frac{3}{5}x^7 - \frac{43}{24}x^6 + \frac{81}{20}x^5 - \frac{443}{90}x^4 + \frac{59}{12}x^3 - 4x^2$ by $\frac{2}{5}x^2 - \frac{3}{4}x + \frac{4}{3}$.
9. $a^{m+1} + a^mb + ab^m + b^{m+1}$ by $a^m + b^m$.
10. $2a^{3m} - 6a^{2m}b^m + 6a^mb^{2m} - 2b^{3m}$ by $2a^m - 2b^m$.
11. $a^{4n} + a^{2n}b^{2n} + b^{4n}$ by $a^{2n} + a^nb^n + b^{2n}$.

47. The Remainder Theorem. *When a rational integral expression containing one unknown, x, arranged according to descending powers of x is divided by $x - a$, the remainder may be obtained by substituting a for x in the original expression.*

Let an expression or function of x, arranged according to descending powers of x, be represented by $f(x)$. When $f(x)$ is divided by $x - a$, denote the quotient by Q and the remainder by R.

Then, from the identical relation of division, we have

$$f(x) \equiv (x - a)\,Q + R.$$

Since no restriction has been placed upon the value of x we may assign to it any value we please, such as a. Representing the result of substituting a for x wherever x appears in the identity above, we may write

$$f(a) \equiv (a - a)\,Q + R.$$

Since $(a-a)$ is 0, the product of $(a-a)$ multiplied by the quotient Q vanishes, and we have $f(a) \equiv R$.

It follows that *we may obtain the remainder after division by $x-a$, by writing a in place of x in the original function.*

48. The remainder obtained by dividing an expression such as $Ax^4+Bx^3+Cx^2+Dx+E$ by a binomial divisor $x-a$, may be obtained by replacing x in the original expression by a. We may show this by actually carrying out the process of division as below:

$$\begin{array}{l|l}
Ax^4+Bx^3+Cx^2+Dx+E & x-a \\ \cline{2-2}
\underline{Ax^4-Aax^3} & \boldsymbol{Ax^3+(Aa+B)x^2+(Aa^2+Ba+C)x} \\
\quad +(\boldsymbol{Aa+B})\boldsymbol{x}^3+Cx^2 & \qquad\qquad +(\boldsymbol{Aa^3+Ba^2+Ca+D}) \\
\quad \underline{+(Aa+B)x^3-(Aa+B)ax^2} & \\
\qquad\qquad +(\boldsymbol{Aa^2+Ba+C})\boldsymbol{x}^2+Dx & \\
\qquad\qquad \underline{+(Aa^2+Ba+C)x^2-(Aa^2+Ba+C)ax} & \\
\qquad\qquad\qquad\qquad +(\boldsymbol{Aa^3+Ba^2+Ca+D})\boldsymbol{x}+E & \\
\qquad\qquad\qquad\qquad \underline{+(Aa^3+Ba^2+Ca+D)x-(Aa^4+Ba^3+Ca^2+Da)} & \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad +\boldsymbol{Aa^4+Ba^3+Ca^2+Da+E.} & \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{Remainder} &
\end{array}$$

It may be seen that wherever a appears in the remainder $Aa^4+Ba^3+Ca^2+Da+E$, x is found in the given dividend $Ax^4+Bx^3+Cx^2+Dx+E$.

49. Synthetic Division. If we write only the coefficients of the dividend and place the second term of the binomial divisor with its sign changed, that is, $-(-a) \equiv +a$, at the right as below,

$$+A \quad +B \quad +C \quad +D \quad +E\,\underline{)\,a}$$

we may obtain the coefficients shown in heavy-face type in § 48 by the following process, known as Synthetic Division:

$$\begin{array}{llllll}
+A & +B & +C & +D & +E & \underline{)+a} \\
 & \underline{+Aa} & \underline{+(Aa^2+Ba)} & \underline{+(Aa^3+Ba^2+Ca)} & \underline{+(Aa^4+Ba^3+Ca^2+Da)} & \\
+A & +(\boldsymbol{Aa+B}) & +(\boldsymbol{Aa^2+Ba+C}) & +(\boldsymbol{Aa^3+Ba^2+Ca+D}), & +(\boldsymbol{Aa^4+Ba^3+Ca^2+Da+E}) & \\
 & & & & \text{Remainder.} &
\end{array}$$

50. Write under the long line the first coefficient $+A$ of the original expression, then multiply it by a, and write the product Aa diagonally above the line in the second place, that is, under B. Then, adding Aa and B, we obtain the heavy-face coefficient $\boldsymbol{Aa+B}$, which is written immediately underneath.

Continue this process, that is, multiply this last expression $\boldsymbol{Aa+B}$ by a; write the result, Aa^2+Ba, above the line under C in the third place. Then we have, adding Aa^2+Ba to C, the second heavy-face coefficient $\boldsymbol{Aa^2+Ba+C}$.

Continuing this process, we obtain as the last sum the remainder $\boldsymbol{Aa^4+Ba^3+Ca^2+Da+E}$.

Ex. 1. Divide $4x^5+3x^4-2x^3+x^2-x+5$ by $x-2$.

Writing the coefficients only, with $+2$ in the divisor's place, we may proceed as follows:

Coefficients of Dividend. **Modified Synthetic Divisor.**

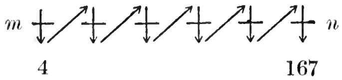
$$\begin{array}{ll} & 4+\ \ 3-\ \ 2+\ \ 1-\ \ 1+\ \ 5\ \underline{)+2} \\ & \quad +\underline{8}+\underline{22}+\underline{40}+\underline{82}\ +\underline{162} \\ m & \underbrace{4+11+20+41+81,}_{\text{Coefficients of Quotient.}}\ +167\quad n \\ & \qquad\qquad\qquad\qquad\qquad\ \text{Remainder.} \end{array}$$

First bring down the 4 underneath the line. Multiplying the 4 by 2 we obtain 8; adding 3, 11; multiplying by 2, 22; combining with -2, $+20$; multiplying by 2, $+40$; adding 1, 41; multiplying by 2, 82; combining with -1, 81; multiplying by 2, 162, which when combined with 5 gives the remainder sought, 167.

Using as coefficients the numbers below the broken line, mn, with the exception of the last, we may construct the quotient by writing in the literal factors. Since the first term of the dividend $4x^5$, divided by the first term of the divisor x, produces the quotient $4x^4$, we begin in the first term with the highest power x^4.

The result thus obtained is $4x^4+11x^3+20x^2+41x+81+\dfrac{167}{x-2}$.

51. It will be noticed that in carrying out this process we work back and forth across the horizontal line mn in the directions indicated by the arrows in the accompanying figure. We begin at 4 and arrive finally at the remainder 167.



52. In case any powers of the polynomial dividend are lacking, their places must be indicated, when this process is applied, by writing in terms with zero coefficients, as in the following example:

Ex. 2. Divide $5x^4 + 3x^2 - 2$ by $x - 3$.

This expression is equivalent to

$$5x^4 + 0x^3 + 3x^2 + 0x - 2.$$

The work may be arranged as follows:

$$\begin{array}{rrrrrl} 5 & +\ 0 & +\ 3 & +\ \ 0 & -\ \ 2 & \underline{)+3} \\ \underline{} & +\ \underline{15} & +\ \underline{45} & +\ \underline{144} & +\ \underline{432} & \\ 5 & +\ 15 & +\ 48 & +\ 144, & +\ 430 & \end{array}$$

Remainder.

The highest power of x in the quotient is x^3, since $5x^4$ divided by x is $5x^3$.

We have as a result $5x^3 + 15x^2 + 48x + 144 + \dfrac{430}{x-3}$.

53. In order to divide an expression arranged according to descending powers of x by $x + a$, that is, by $x - (-a)$, the same process is employed, except that $-a$ is used in the same way as $+a$ was used when the divisor was $x - a$.

Ex. 3. Find the remainder when $4x^4 - 3x^3 - 5x^2 - x + 10$ is divided by $x + 2$.

$$\begin{array}{rrrrrl} 4 & -\ 3 & -\ 5 & -\ \ 1 & +\ 10 & \underline{)-2} \\ \underline{} & -\ \underline{8} & +\ \underline{22} & -\ \underline{34} & +\ \underline{70} & \\ 4 & -\ 11 & +\ 17 & -\ 35, & +\ 80 & \end{array}$$

Remainder.

54. If when a function of x, $f(x)$, is divided by $x - a$ the division is exact, that is, if there be no remainder, the identity $f(x) \equiv (x - a)Q + R$ reduces to $f(x) \equiv (x - a)Q$. By substituting a for x this last identity reduces to $f(a) \equiv 0$.

Hence we have the following

Factor Theorem: If, *when a is substituted for x in an expression arranged according to descending powers of x, the expression becomes* 0, *then $x - a$ will be an exact divisor of the expression.*

EXERCISE VIII. 6

Find the quotient and remainder when

1. $x^2 + 5x + 7$ is divided by $x + 3$.
2. $y^2 + 7y + 11$ is divided by $y + 4$.
3. $z^2 - 17z + 39$ is divided by $z + 3$.
4. $w^2 - 48w + 97$ is divided by $w + 10$.
5. $a^3 + 11a^2 - 19a + 117$ is divided by $a + 4$.

6. $b^3 - 19b^2 + 18b - 17$ is divided by $b - 5$.
7. $c^3 - 14c^2 + 28c - 42$ is divided by $c - 6$.
8. $d^4 - 5d^3 + 6d^2 - 10d + 1$ is divided by $d + 1$.
9. $k^4 + 2k^3 + 3k^2 + 4k + 5$ is divided by $k - 5$.
10. $3x^5 - 5x^4 + 6x^3 - 2x^2 + x - 1$ is divided by $x - 2$.

Employing the Remainder Theorem, find the numerical value of the following expressions when x is given the value indicated:

11. $x^3 + 3x^2 + 4x + 3$, when $x = 1$.
12. $2x^3 + x^2 + 7x + 4$, when $x = 1$.
13. $x^3 + 2x^2 + 3x + 5$, when $x = 2$.
14. $x^3 + 2x^2 + 5x + 7$, when $x = 2$.
15. $x^3 - 8x^2 + 17x - 10$, when $x = 1$.
16. $x^3 + 5x^2 + 3x + 1$, when $x = 2$.
17. $x^3 + 9x^2 - 7x + 5$, when $x = 1$.
18. $x^3 - 7x^2 + 13x - 6$, when $x = 2$.
19. $3x^3 + 2x^2 + x + 5$, when $x = 2$.
20. $2x^3 - 5x^2 + 4x + 7$, when $x = 3$.
21. $3x^3 + x^2 - 7x + 9$, when $x = 1$.
22. $4x^3 - 9x^2 - x + 6$, when $x = 1$.
23. $2x^3 + x^2 + 3x + 2$, when $x = 2$.
24. $4x^3 - 3x^2 + 2x + 5$, when $x = 3$.
25. $3x^3 + 7x^2 - 4x + 3$, when $x = -3$.
26. $4x^3 - 5x^2 + 3x - 4$, when $x = -2$.
27. $2x^3 - 9x^2 + 4x + 5$, when $x = 5$.
28. $3x^3 + 11x^2 - 7x + 5$, when $x = -1$.
29. $2x^3 + 11x^2 - 3x + 10$, when $x = -5$.
30. $6x^3 - 3x^2 + 7x + 4$, when $x = -2$.
31. $11x^3 - 9x^2 + 7x + 1$, when $x = -1$.
32. $x^4 + 2x^3 + 3x^2 + x + 1$, when $x = 1$.
33. $x^4 - x^3 - 7x^2 + x - 6$, when $x = 1$.
34. $x^4 - 3x^3 + x^2 + 2x + 5$, when $x = 3$.
35. $x^3 + 2x^2 + 7$, when $x = 2$.
36. $x^5 + 1$, when $x = 3$.
37. $x^3 + 4x^2 + 12$, when $x = 4$.
38. $x^3 + 15x + 3$, when $x = 5$.
39. $2x^3 + 7x + 20$, when $x = 2$.

40. $2x^3 + 5x^2 - 33x + 12$, when $x = 3$.
41. $3x^3 - 6x^2 + 5x + 21$, when $x = 2$.
42. $2x^4 + 5x^3 + 2x + 1$, when $x = 1$.
43. $3x^4 + 5x^2 + 4x + 1$ when $x = -1$.

Applications of the Remainder Theorem.

55. (i.) *The binomial difference* $x^m - y^m$ *is always divisible without remainder by the difference* $x - y$.

For, substituting y for x, we have $y^m - y^m \equiv 0$.

Hence by the Remainder Theorem the division is exact.

(ii.) *The binomial sum* $x^m + y^m$ *is never divisible without remainder by the difference* $x - y$.

For, substituting y for x, we have $y^m + y^m \equiv 2y^m \not\equiv 0$.

Hence by the Remainder Theorem the division is not exact.

(iii.) *The binomial difference* $x^m - y^m$ *is or is not divisible without remainder by the sum* $x + y$ *according as* m *is even or odd.*

For, substituting $-y$ for x, we have $(-y)^m - y^m$ (1).

Examining this "remainder" for both even and odd values of m, we draw the following conclusions:

If m be even, then, since all even powers of negative numbers are positive numbers, $(-y)^m - y^m$ becomes $y^m - y^m \equiv 0$.

By the Remainder Theorem the division in this case is exact.

If m be odd, then, since all odd powers of negative numbers are negative numbers, $(-y)^m - y^m$ becomes $-y^m - y^m \equiv -2y^m \not\equiv 0$.

By the Remainder Theorem the division in this case is not exact.

(iv.) *The binomial sum* $x^m + y^m$ *is or is not divisible without remainder by the sum* $x + y$ *according as* m *is odd or even.*

For, replacing x by $(-y)$, we have $(-y)^m + y^m$.

Examining this "remainder" for both odd and even values of m, we find that:

If m be odd, $(-y)^m + y^m$ becomes $-y^m + y^m \equiv 0$.

Hence by the Remainder Theorem the division in this case is exact.

If m be even, $(-y)^m + y^m$ becomes $y^m + y^m \equiv 2y^m \not\equiv 0$.

Hence by the Remainder Theorem the division in this case is not exact.

56. By (i.) $x^m - y^m$ is **always** divisible by $x - y$ whether m be odd or even, and by (iii.) it is also divisible by $x + y$ when m is even. Hence it follows that when m is even, $x^m - y^m$ is divisible by **either** $x - y$ or $x + y$.

57. Since, when m is even, the binomial difference $x^m - y^m$ is divisible by both the sum $x + y$ and the difference $x - y$, it follows that it is divisible by the product $(x + y)(x - y) \equiv x^2 - y^2$.

58. The **general principles** established above may be stated as follows:

I. *The* ***sum*** *of the same* ***odd powers*** *of two numbers is exactly divisible by the* ***sum of the numbers.***

E. g. $$\frac{a^5 + b^5}{a + b} \equiv a^4 - a^3b + a^2b^2 - ab^3 + b^4.$$

II. *The* ***difference*** *of the same* ***odd powers*** *of two numbers is exactly divisible by the* ***difference of the numbers.***

E. g. $$\frac{a^7 - b^7}{a - b} \equiv a^6 + a^5b + a^4b^2 + a^3b^3 + a^2b^4 + ab^5 + b^6.$$

III. *The* ***difference*** *of the same* ***even powers*** *of two numbers is exactly divisible by* ***either*** *the sum or the difference of the numbers.*

E. g. (1) $$\frac{a^4 - b^4}{a + b} \equiv a^3 - a^2b + ab^2 - b^3.$$

(2) $$\frac{a^4 - b^4}{a - b} \equiv a^3 + a^2b + ab^2 + b^3.$$

IV. *The* ***sum*** *of the same* ***even powers*** *of two numbers is* ***not exactly divisible*** *by either the sum or the difference of the two numbers.*

E. g. $$\left.\begin{array}{l}\dfrac{a^2 + b^2}{a + b}\\[2ex] \dfrac{a^2 + b^2}{a - b}\end{array}\right\} \text{The division in each case is not exact.}$$

Law of Polynomial Quotients.

59. It may be shown by actual division that the polynomial quotients obtained by dividing $a^n \pm b^n$ by $a \pm b$ have the following forms:

If n is an odd positive integer, we have

$$\text{(i.)}\ \frac{a^n + b^n}{a + b} \equiv a^{n-1} - a^{n-2}b + a^{n-3}b^2 - a^{n-4}b^3 + \cdots\cdots - ab^{n-2} + b^{n-1}.$$

The signs of the terms of the quotient are alternately + and −.

If n is an odd positive integer, we have

$$\text{(ii.)}\ \frac{a^n - b^n}{a - b} \equiv a^{n-1} + a^{n-2}b + a^{n-3}b^2 + a^{n-4}b^3 + \cdots\cdots + ab^{n-2} + b^{n-1}.$$

The signs of the terms of the quotient are all positive.

If n is an even positive integer, we have

$$\text{(iii.)}\ \frac{a^n - b^n}{a \pm b} \equiv a^{n-1} \mp a^{n-2}b + a^{n-3}b^2 \mp a^{n-4}b^3 + \cdots\cdots + ab^{n-1} \mp b^{n-1}.$$

In the expression above, the double sign ±, read "plus or minus," is used with the double sign ∓, read "minus or plus," to indicate that according as the upper or lower sign is used in the divisor, the corresponding upper or lower sign must be used in the quotient.

Hence, using the upper signs, the signs of the quotient are alternately positive and negative when the divisor is a sum. Using the lower signs, the signs of the quotient are all positive when the divisor is a difference.

MENTAL EXERCISE VIII. 7

Obtain each of the following quotients mentally, stating in each case the general principle applied: (See § 58.)

1. $\frac{x^3 + y^3}{x + y}$.
2. $\frac{a^7 + b^7}{a + b}$.
3. $\frac{c^3 - d^3}{c - d}$.
4. $\frac{m^9 - n^9}{m - n}$.
5. $\frac{a^6 - b^6}{a + b}$.
6. $\frac{a^6 - b^6}{a - b}$.
7. $\frac{a^{11} + b^{11}}{a + b}$.
8. $\frac{x^{10} - y^{10}}{x - y}$.
9. $\frac{a^3 + 8}{a + 2}$.
10. $\frac{x^3 + 1}{x + 1}$.
11. $\frac{y^3 - 27}{y - 3}$.
12. $\frac{1 + x^5}{1 + x}$.
13. $\frac{1 - x^9}{1 - x}$.
14. $\frac{a^{12} - b^{12}}{a - b}$.
15. $\frac{a^8 - b^8}{a + b}$.

CHAPTER IX

GRAPHICAL REPRESENTATION OF THE VARIATION OF FUNCTIONS OF A SINGLE VARIABLE

1. Any expression which depends for its value upon the value assigned to some specified variable contained in it is called a **function** of this specified variable.

E.g. The expression $x + 1$ does not represent any particular number until some definite value is assigned to x. Hence we say that $x + 1$ is a function of x.

Similarly the expression $x^2 + 2x + 3$ is a function of x. If x be 2, the expression stands for the number 11; if x be 5, the expression represents 38.

2. An expression containing several variables may be regarded as being a function of any one of them, or of a combination of two or more taken together.

E.g. $x^2 + xy + y^2$ may be regarded as being a function of either x or y separately, or of x and y together.

3. From our experience with algebraic expressions we have found that definite numbers were commonly obtained when particular values were assigned to the letters appearing in them.

E.g. For $x = 2$, the following functions of one letter each represent 7: $x^2 + 3, x + 5, 3x + 1, 4x - 1$ and $\frac{6x + 23}{5}$.

4. We found also that a given expression commonly assumed different values when different values were substituted for some particular letter or letters appearing in it.

The expression $x^2 - 10x + 21$, regarded as a function of x, will represent different values as x is given successively the values 0, 1, 2, 3, 4, 5, ..., 10.

We will tabulate the resulting values of the function, writing the results as in the accompanying table.

When x has the values 0, 1, 2, 3, · · · · · , the resulting values of the expression are 21, 12, 5, 0, · · · · · respectively.

x	Function $x^2 - 10x + 21$
⋮	⋮
0	21
1	12
2	5
3	0
4	-3
5	-4
6	-3
7	0
8	5
9	12
10	21
⋮	⋮

5. Writing the expression $x^2 - 10x + 21$ in the form $(x-3)(x-7)$, it appears that, for values of x equal to 3 or 7, the expression becomes 0. It cannot become 0 for any other values, since neither of the factors $x - 3$ nor $x - 7$ can become 0 for other values.

For any value of x greater than 7, the expression will represent a positive number, since each factor $x - 3$ and $x - 7$ will, in this case, be a positive number.

6. Also, for all negative values of x, the expression will represent a positive number, since for negative values of x, $x - 3$ and $x - 7$ are both negative, and their product is accordingly a positive number.

We may thus, by tabulating results, obtain an idea of the variation in the value of the expression under examination as the letters are given different values.

7. We will now explain a graphic method for representing the variation, or change in numerical value, of a given function as the letters appearing in it are given different values.

We shall obtain a diagram or picture which will represent to us the variation above and below zero in the numerical value of a given function in much the same way as a profile map of some section of a country gives us at a glance a different and far better idea of the relative elevations of places than can be obtained from a table of estimated distances above or below the sea level.

Specification of Points in a Plane

8. We will draw two perpendicular straight lines, $X'OX$ and $Y'OY$, as **axes of reference** separating a plane into four parts or regions. By common consent among mathematicians these parts into which the plane is separated are called **quadrants.** The parts containing the Roman numerals I, II, III, IV in the accompanying diagram are called the *first*, *second*, *third*, and *fourth* quadrants respectively.

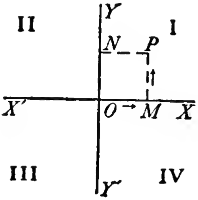


Fig. 1.

9. Letting P represent any point in the plane, we will suppose that lines such as MP and NP are drawn parallel respectively to the lines of reference or axes $Y'OY$ and $X'OX$.

10. Starting at the point of intersection O of the axes of reference, we may imagine taking a first step OM equal to NP along the axis $X'OX$ from O to M; then turning about at M and moving away from the axis in a direction MP parallel to the other axis of reference $Y'OY$, we shall arrive at the point P by taking a second step from M to P.

By taking steps of different lengths, we shall arrive at different points in the plane.

11. We shall call the lines of reference $X'OX$ and $Y'OY$ the ***x*-axis and the *y*-axis** respectively, and shall call their intersection O, from which the first step of any pair is always taken, the **origin.**

12. We shall speak of steps as being x-steps or y-steps, according as they are taken parallel to the x-axis or the y-axis.

13. If we agree to call steps taken along the x-axis toward the right, as indicated by the arrow, positive steps, then those toward the left will be negative. If y-steps be positive when taken upward, as indicated by the arrow, then those downward will be negative.

14. Starting from the origin O, and moving toward the right or left, then either up or down, we may, by taking the proper positive or negative x-steps and y-steps, reach points lying in any of the four quadrants.

15. If, starting from O, we first take a positive x-step and then take a y-step, we shall pass into either the first or the fourth quadrants, according as the y-step is positive or negative.

If our first x-step be negative, then the y-step which follows it will take us into either the second or the third quadrants.

Hence the signs of the steps which may be taken to reach a point determine the quadrant in which it lies.

16. With reference to any particular point, P, the corresponding x-step is called the **abscissa,** from the Latin meaning "to cut off," and the y-step is called the **ordinate,** from the Latin meaning "to set in order or arrange."

17. Since the abscissa and ordinate taken together enable us to locate any particular point, they are together called the **coördinates** of any particular point, and the axes parallel to which they are drawn are called the **coördinate axes.**

18. Whenever, in the system of graphic representation which we are presenting, the position of a point is described by means of its coördinates, the x-step or abscissa is understood to be the one named first, unless the contrary is stated.

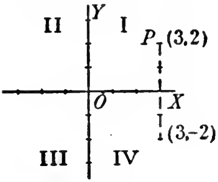


FIG. 2.

19. By the notation (3, 2) we shall understand that the abscissa of the point represented is 3 and its ordinate 2. After assuming some convenient unit of length, we may locate the point by first measuring off a distance of three units from the origin O along the x-axis toward the right. Then, turning about in a direction parallel to the y-axis, we shall find the point, P, situated at a distance of two units measured in a positive direction, that is, upward. (See Fig. 2.)

20. The point (3, − 2) is situated in the fourth quadrant at a distance of three units to the right of the y-axis, and two units below the axis of X, as in Fig. 2.

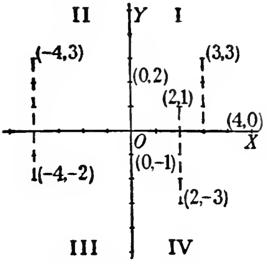


FIG. 3.

21. Such points as those represented by (− 4, + 3), (− 4, − 2), (0, 2), (0, − 1), (2, 1), (2, − 3), (3, 3), (4, 0), etc., will be readily located by stepping off the indicated distances, first toward right or left from the origin, then up or down, according as the signs of the given coördinates are + or −. (See Fig. 3.)

22. The scale units in terms of which the x-steps and y-steps are expressed are understood to be the same unless the contrary is stated.

It may happen for special purposes that it is convenient to choose a scale unit for the y-steps different from that chosen for the x-steps.

23. The operation of marking any point on a diagram when the coördinates of the point are given, is called **plotting the point.**

EXERCISE IX. [1]

Plot the points whose coördinates are

1. $(2, 4)$.
2. $(3, 6)$.
3. $(1, 7)$.
4. $(5, 8)$.
5. $(1, -5)$.
6. $(2, -4)$.
7. $(3, -1)$.
8. $(-3, 11)$.
9. $(4, -4)$.
10. $(-2, 4)$.
11. $(0, -4)$.
12. $(-1, -1)$.
13. $(3, 0)$.
14. $(0, 2)$.
15. $(0, 0)$.
16. $(-6, -6)$.

24. If we regard the sets of values in the table calculated from the function $x^2 - 10x + 21$ (see §§ 4, 5) as representing the x-coördinates and y-coördinates of different points, we may obtain as many points in a plane as we please, such as (1, 12), (2, 5), (3, 0), etc. We shall assume the numbers in the column under x as abscissas, and those in the column under $x^2 - 10x + 21$ as ordinates, as in Fig. 4.

25. By assigning fractional values to x we may obtain corresponding values of the function.

E. g. If x be given values between 0 and 1, say .1, .2, .3, .4, etc., we may calculate as corresponding values of y, 20.01, 19.04, 18.09, 17.16, etc.

The points corresponding to these values taken as coördinates will be found to lie between the points (0,21) and (1,12).

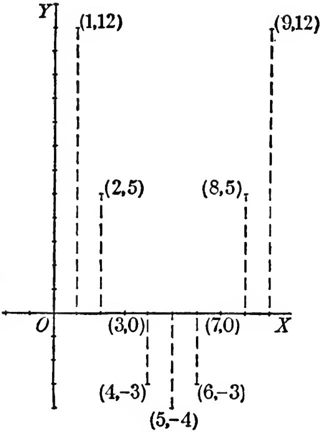


FIG. 4.

26. Continuity. If we imagine that the value of x changes continuously from 0 to 1, then from 1 to 2, and on to 3, 4, 5, etc., passing through all intermediate values without at any stage making a sudden "jump" from one value to another, then the point P, determined by x and y (Fig. 2) will trace out a continuous line, — that is, one having no breaks or discontinuities in it.

27. A continuous line passing through all of the points which may be plotted for a given function is called the **graph of the function.**

28. The graph of a given function is, from its construction, a "point picture" of an algebraic expression.

A portion of the graph of the function $x^2 - 10x + 21$ will be found by drawing a continuous line through the points plotted in Fig. 4. The "accuracy" of the graph, that is, the "likeness of the picture," will depend largely upon the lengths of the scale units adopted in its construction. (See Fig. 5.)

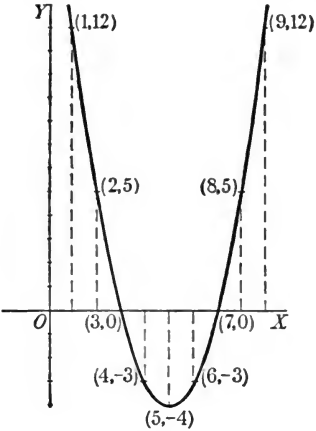


Fig. 5.

29. The equality $y = x^2 - 10x + 21$, or in general, $y = f(x)$, is called **the equation of the graph or curve.**

To every pair of values satisfying the equality $y = f(x)$ there corresponds a point on the graph; and conversely, the coördinates of all points on the graph, and no other points, satisfy the equation.

30. Whenever we construct a graph by drawing a continuous line through different points which are separately plotted, we assume in the operation that the graph of the function is continuous between any two points through which the continuous line passes.

We need not, at present, be at all concerned with the subject of "breaks" or discontinuities in the graph, since it may be proved that *the graph of every rational integral function of one variable has no discontinuities.* It may also be shown, for finite values of the variable, that *a rational fractional function has no discontinuities except for such values of the variable as make the denominator zero.*

31. As an illustration of a break or discontinuity in a graph or curve see the accompanying figure where we suppose that the point which is tracing the curve, on arriving at P, jumps immediately to Q through a finite distance PQ, and then moves on along the branch of the curve QQ'.

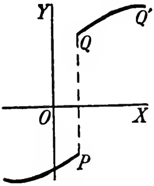


FIG. 6.

TYPICAL GRAPHS

32. Consider a function of the form ax^2, in which a is positive.

By assigning some numerical value to a throughout the discussion, we may, by giving different values successively to x, calculate the corresponding values of the function ax^2. If we represent the value of the function by y, we may write $y = ax^2$. If, for the present purpose, we let $a = 2$, we may obtain the graph of $y = 2x^2$ by plotting the points whose abscissas and ordinates are given by x and $2x^2$ respectively.

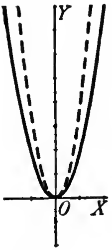


FIG. 7.

33. Since the number represented by a is positive and x^2 is an even power, ax^2 will be positive for all values of x. It follows that there can be no negative ordinates, and the graph can enter neither the third nor the fourth quadrants. A portion of the graph of the function $2x^2$ which has the form ax^2 is exhibited in Fig. 7 in full lines, and we have shown in dotted lines on the same diagram a portion of the graph of another function, $4x^2$, whose algebraic form is the same as that of the function $2x^2$.

It will be seen that the graphs have the same general "shape" except that one comes to a "sharper" point than the other.

34. In case the number represented by a is negative, that is, if in particular a represents -2, the ordinates corresponding to the function $-2x^2$ will be negative for both positive or negative values of x. Hence, no part of the graph of $-2x^2$ can enter either the first or the second quadrants. The graphs of the two functions $2x^2$ and $-2x^2$ having the form ax^2, but in one of which the number represented by a is positive and in the other negative, will have

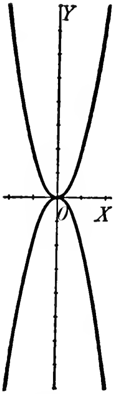


FIG. 8.

the same shape and size, but will be situated symmetrically with respect to the axis of X as in Fig. 8.

35. If we imagine the x-axis to be a plane mirror perpendicular to the y-axis, either graph may be regarded as being the reflection of the other in the x-axis as a mirror.

By interchanging the letters x and y, we may, from the expression of equality $y = ax^2$, obtain $x = ay^2$. Hence, when plotting this last function, $x = ay^2$, we may simply interchange the values represented by x and y, and make the same measurements as before for the function $y = ax^2$.

36. Interchanging these values amounts to revolving the entire system toward the right about the origin as a center, keeping the axes and graph in the same relative positions, through an angle such that the y-axis shall swing around into a position originally occupied by the x-axis. Hence, if in Fig. 9 the dotted line represents a portion of the graph of $2x^2$ in its original position, and we imagine the system to be revolved about the origin O until OY comes into the position originally occupied by OX, the graph may be supposed to turn about and appear in a position indicated by the full line. It may then be taken as the graph of the function $2y^2$.

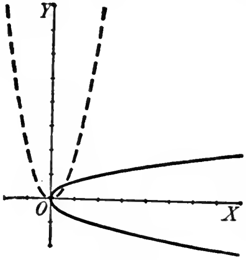


Fig. 9.

37. Consider now the function $ax^2 + b$. To obtain the graph of this function we have simply to add b to each of the ordinates calculated for the graph of ax^2. Hence, the x-coördinates will be the same for both of the functions ax^2 and $ax^2 + b$, but the y-ordinates of the function $ax^2 + b$ will be greater by b than those of the function ax^2. It follows that the two graphs have the same size and shape, but because of the greater length of the y-ordinates the graph of the function $ax^2 + b$ is situated "higher up" in the plane than the graph of the function ax^2. That is, it will be farther away from the origin in the y-direction, as in Fig. 10. In this figure we have given a and b the values 2 and 3 respect-

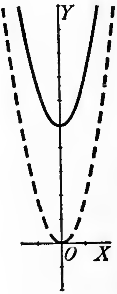


Fig. 10

ively, and have shown a portion of the graph of the function $2x^2 + 3$ in full lines and a portion of the graph of the function $2x^2$ in dotted lines.

38. In Fig. 11 we have shown portions of the graphs of (i.) $2x^2$, (ii.) $2x^2 + 3$, and (iii.) $x^2 + 2x - 3$.

It may be noted that, so far as the portions of the graphs shown are concerned, they have in all cases the same general shape.

39. We will now show representative graphs of certain common mathematical curves, leaving the student to plot the functions in the next exercise, comparing the forms of the graphs obtained with those shown in Figures 12 to 16.

We cannot always safely assign a name to a curve simply because a certain portion of it resembles very closely the shape of a curve whose name is known, but we may at least assert that the portions of the curves obtained appear to have the same general form.

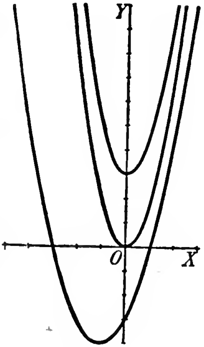


FIG. 11.

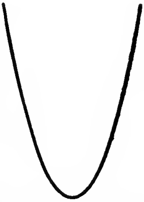


FIG. 12. *Parabola*

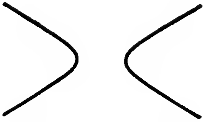


FIG. 13. *Hyperbola*

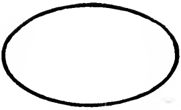


FIG. 14. *Ellipse*

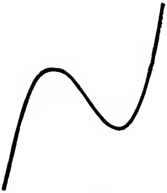


FIG. 15. *Cubic Curve*

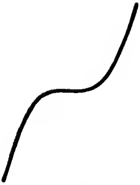


FIG. 16. *Cubic Curve*

40. The student, after having constructed a few of the simple typical graphs, will understand the term "algebraic form" in a new light. It will be seen that there is a striking similarity in shape among the "point pictures" or graphs of algebraic expressions which have the same algebraic form.

41. Although the number of different graphs which may be constructed by plotting functions is as endless as the number of expressions themselves which may be written, comparatively few of these forms will be encountered in elementary work, and these can readily be separated into a certain small number of typical forms which may be easily recognized.

We present a few of the more common forms in order that the student may become somewhat acquainted with certain of the graphs which are of practical and historic interest.

Exercise IX. 2

Obtain portions of the graphs of the following equations :

1. $y = x.$
2. $y = -x.$
3. $y = 2x.$
4. $y = -4x.$
5. $x = 2.$
6. $y = 3.$
7. $y = 0.$
8. $x = 0.$
9. $y = x + 1.$
10. $y = -x + 1.$
11. $y = 3x + 4.$
12. $y = -2x + 5.$
13. $y = 4x^2.$
14. $y = x^2.$
15. $y = x^2 + 2x + 1.$
16. $y = x^2 + 7x + 12.$
17. $y = x^2 - 7x + 9.$
18. $y = x^2 - 6x + 9.$
19. $y = -4x^2 + 20x - 23.$
20. $y = -2x^2 + x.$
21. $y = \frac{1}{x}\cdot$
22. $y = \frac{1}{x^2}\cdot$
23. $y = \frac{1}{x+1}\cdot$
24. $y = \frac{1}{x-1}\cdot$
25. $y = \frac{1}{x+2}\cdot$
26. $y = \frac{x^2}{x+1}\cdot$
27. $xy = 24.$
28. $xy = -1.$
29. $y = x^3 - 9x^2 + 11x + 21.$
30. $y = x^3 - 4x.$
31. $y = x^3 - 8x^2 + 23x + 2.$
32. $y = x^3 + x^2 + x + 1.$
33. $y = x^3 + 3x^2 + 2x + 1.$
34. $y = x^3 - 3x^2 + 2x - 1.$
35. $y = -x^3 + 3x^2 - 2x + 1.$
36. $y = x^3 + 1.$
37. $y = x^3 + 3x^2 + 6x + 11.$
38. $y = x^3 - 3x^2 + 6x - 11.$
39. $y = x^3 + 4x^2 + 2x + 5.$
40. $y = x^3 - 4x^2 + 2x - 5.$
41. $y = x^4 - 10x^2 + 12.$

INVERSE USE OF GRAPHS

42. When constructing graphs we have assumed that, starting from the origin O, every point in the plane may be reached by taking definite x-steps and y-steps. It may be seen that, when the location of a particular point is given relatively to the axes of reference, we may by reversing the process use a scale and obtain by actual measurement approximate values for the x-coördinate and the y-coördinate of the point.

The accuracy of the numerical results thus obtained will depend upon the accuracy with which the measurements are made.

43. The graph of a function is obtained by assigning values to the variable appearing in it, and by locating points whose coördinates are the values assigned to the variable and the corresponding values calculated for the function.

If the graph of a function be given, we may measure the coördinates of different points situated on it and estimate approximately the values which must be given to the variable in order that the function shall have certain specified values.

44. We may find an approximate value for $\sqrt{7}$ by using the graph of $x = \sqrt{y}$ as follows:

Since like powers of equal expressions are equal, from $x = \sqrt{y}$ it follows that $x^2 = y$.

A portion of the graph of $x^2 = y$ may be obtained by using as abscissas different values assigned to x, and for ordinates the corresponding values calculated for the function x^2.

When to x is given successively the values 0, 1, 2, 3, 4, etc., the corresponding values of the function x^2 are 0, 1, 4, 9, 16, etc., as shown in the accompanying table.

By using these pairs of values, (0,0), (1,1), (2,4), (3,9), (4,16), etc., as abscissas and ordinates, points upon the graph may be located. (See Fig. 17.)

Abscissa x	Ordinate $x^2 = y$
⋮	⋮
0	0
1	1
2	4
3	9
4	16
⋮	⋮

The accuracy of the figure obtained by drawing a continuous line through the points thus located will depend largely upon the lengths of the scale units employed when setting off the abscissas and ordinates.

To find $\sqrt{7}$ our problem is simply to find the abscissa of a point on the graph corresponding to the ordinate whose length is 7. To do this we may start at the origin and move upward along the y-axis through a distance equal to seven of the scale units used in constructing the graph. Then, tracing along the horizontal line passing through this point to the graph, we shall find the point whose ordinate has the required length, 7.

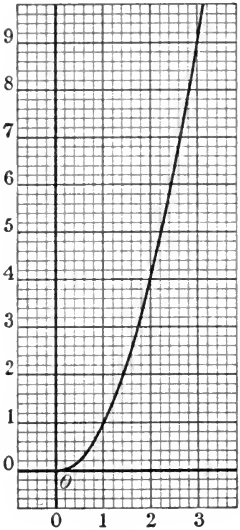


Fig. 17.

The $\sqrt{7}$ which is represented by the length of the abscissa x, corresponding to the point thus located on the graph, may be readily estimated by measuring the distance passed over in tracing across from the axis of Y to the graph. Thus, from Fig. 17, in which each small square represents .2, the value of $\sqrt{7}$ is found to be 2.6 +, nearly.

45. Approximate values for the square roots of other numbers, such as 2, 3, 5, 6, 8, etc., may be found in a similar way by tracing across the figure from the axis of Y to the graph along horizontal lines situated at distances of 2, 3, 5, 6, 8, etc., units respectively, above the x-axis.

We may thus obtain as approximations for the square roots of 2, 3, 5, 6, 8, etc., the numbers 1.4 +, 1.7 +, 2.2 +, 2.5 +, 2.8 +, etc.

CHAPTER X

GENERAL PRINCIPLES GOVERNING TRANSFORMATIONS OF ALGEBRAIC EQUATIONS

IDENTICAL EQUATIONS

1. AN **equality or equation** is the assertion in symbols that two different expressions represent the same number.

2. Two expressions are said to be **identical** if they are exactly alike, or if either can be reduced to the form of the other by applying the laws of reckoning and the definitions of the symbols and functions considered.

E. g. The expressions $a + b$ and $a + b$ are identical since they are exactly alike.

The expressions $2a + 3a$ and $5a$ are identical since $2a + 3a$ may by addition be reduced to $5a$.

Since the expression $(a + b)c$ may by multiplication be transformed into the expression $ac + bc$, it follows that $(a + b)c$ is identical with $ac + bc$.

3. Two numerical expressions which represent the same number are said to be identical.

E. g. Since $7 + 5$ and 4×3 each represent 12, it follows that $7 + 5$ and 4×3 are identical.

4. The triple sign of equality, $\equiv$, commonly called the **identity sign,** — which is read, "is identical with," "may be transformed into" or "becomes," — is written between two expressions to denote that they are identical.

5. In an identity the expression at the left of the identity sign is called the **first member,** and the expression at the right of the sign the **second member.**

E. g. In the identity $2a + 3a \equiv 5a$, the expression $2a + 3a$ is the first member and $5a$ is the second member.

6. Since either member of an algebraic identity may be reduced to the form of the other, it follows directly that, *if both members are finite expressions, they must represent equal numerical values for all values of the letters which appear in them.*

E. g. Since $(a+b)^2$ can be expressed in the form $a^2+2ab+b^2$, it follows that $(a+b)^2 \equiv a^2+2ab+b^2$ for all values which may be assigned to a and b.

7. In dealing with identical equations we are governed by the following

General Principle: *Either member of an algebraic identity may be reduced directly to the form of the other, or both members may be reduced to a common third form, by applying the principles and definitions of algebra.*

8. In applying this principle it will in certain cases be found convenient to transform one member of an algebraic identity directly to the form of the other.

E. g. In the identity $(x+3)(x+2) \equiv x^2+5x+6$, the first member may be reduced directly to the form of the second by performing the indicated multiplication, and in a later chapter a method will be shown for reversing the process and reducing the second member to the form of the first.

In some cases the two members of an algebraic identity may be of such forms that one does not readily reduce to the form of the other. Hence, in such cases, it is convenient to transform both members to a common third form.

E. g. In the identity $(a+b)^2-2ab \equiv \frac{a^4-b^4}{a^2-b^2}$, we shall by performing the indicated operations in the members separately, obtain in each case the expression a^2+b^2.

Accordingly, by reducing the members of the given identity to the common third form, a^2+b^2, we shall derive the identity $a^2+b^2 \equiv a^2+b^2$.

9. ***An equality may be proved to be an identity,*** *either by showing that one member may be reduced directly to the form of the other, or by showing that both members may be reduced to a common third form for all values which may be given to the letters which appear in them.*

Mental Exercise X. 1

Show that each of the following equalities is an identity :

1. $2a + 3a \equiv 5a$.
2. $4b + 6b \equiv 10b$.
3. $8c - 2c \equiv 6c$.
4. $11d \equiv 6d + 5d$.
5. $13e \equiv 20e - 7e$.
6. $6m + 9m \equiv 4m + 11m$.
7. $7n + 10n \equiv 20n - 3n$.
8. $12x - 4x \equiv 17x - 9x$.
9. $2(3a + 4b) \equiv 6a + 8b$.
10. $3(5b + 7c) \equiv 15b + 21c$.
11. $4(6c - 5) \equiv 24c - 20$.
12. $13a + 7a \equiv (5 \times 4)a$.
13. $16b + 3b \equiv 38b \div 2$.
14. $19a^2 + 5a^2 \equiv 8a \times 3a$.
15. $23b^2 + 5b^2 \equiv 7b \times 4b$.
16. $31c^2 - 9c^2 \equiv 11c \times 2c$.
17. $(x + 2)(x - 2) \equiv x^2 - 4$.
18. $(y + 3)(y - 3) \equiv y^2 - 9$.
19. $(z + 8)(z - 8) \equiv z^2 - 64$.
20. $(1 + m)(1 - m) \equiv 1 - m^2$.
21. $(a + 3)^2 \equiv a^2 + 6a + 9$.
22. $(6 + b)^2 \equiv 36 + 12b + b^2$.
23. $(9 - c)^2 \equiv 81 - 18c + c^2$.
24. $a(a + 4) + 4 \equiv a^2 + 4(a + 1)$.
25. $b^2 + 16(b + 4) \equiv b(b + 16) + 64$.
26. $x(x - 20) + 100 \equiv x^2 - 20(x - 5)$.
27. $x(x - 6) + 3(2x - 3) \equiv x^2 - 9$.
28. $x(x - 12) + 12(x - 3) \equiv x^2 - 36$.
29. $x(x + 18) - 9(2x + 9) \equiv x^2 - 81$.
30. $x(x - 22) + 11(2x - 11) \equiv x^2 - 121$.
31. $x(x + 24) - 24(x + 6) \equiv x^2 - 144$.
32. $28(x + 7) - x(x + 28) \equiv 196 - x^2$.
33. $(a + 1)^2 - 4a \equiv (a - 1)^2$.
34. $(b + 2)^2 - 8b \equiv (b - 2)^2$.
35. $(a + 3)^2 - 12a \equiv (a - 3)^2$.
36. $(c + 5)^2 - 20c \equiv (c - 5)^2$.
37. $(d - 4)^2 + 16d \equiv (d + 4)^2$.
38. $(m - 6)^2 + 24m \equiv (m + 6)^2$.
39. $(7 - n)^2 + 28n \equiv (7 + n)^2$.
40. $(2a + b)^2 - 8ab \equiv (2a - b)^2$.
41. $(3b - 4c)^2 + 48bc \equiv (3b + 4c)^2$.
42. $(a + 1)^2 - (a - 1)^2 \equiv 4a$.
43. $(b + 2)^2 - (b - 2)^2 \equiv 8b$.
44. $(c + 5)^2 - (c - 5)^2 \equiv 20c$.
45. $(7 - x)^2 - (7 + x)^2 \equiv -28x$.
46. $(9 - y)^2 - (9 + y)^2 \equiv -36y$.
47. $(a - 5b)^2 - (a + 5b)^2 \equiv -20ab$.
48. $(a + b)^2 + (a - b)^2 \equiv 2a^2 + 2b^2$.

49. $(b + c)^2 + (b - c)^2 \equiv 2\,b^2 + 2\,c^2$.
50. $(m - n)^2 + (m + n)^2 \equiv 2\,(m^2 + n^2)$.
51. $(1 - x)^2 + (1 + x)^2 \equiv 2 + 2\,x^2$.
52. $x(x + y) + y^2 \equiv x^2 + y(x + y)$.
53. $a(a - b) + b^2 \equiv a^2 - b(a - b)$.
54. $x^2 + y(y - x) \equiv x(x - y) + y^2$.
55. $ab + c(a + b) \equiv b(a + c) + ac$.
56. $a(b + c) + bc \equiv c(a + b) + ab$.
57. $xy + z(x - y) \equiv xz + y(x - z)$.
58. $a(b + c) - c(a + b) \equiv b(a - c)$.
59. $x(y - z) + z(x + y) \equiv y(x + z)$.
60. $x(z + w) + y(z + w) \equiv z(x + y) + w(x + y)$.
61. $x^3 + (x^2 + xy + y^2)y \equiv x(x^2 + xy + y^2) + y^3$.
62. $x(y - z) + y(z - x) + z(x - y) \equiv 0$.

CONDITIONAL EQUATIONS

10. The number which an expression such as $x + 3$ may represent depends entirely upon the particular value which may be given to x.

E. g.
If x be 1, then $x + 3$ represents 4.
If x be 6, then $x + 3$ represents 9.
If x be 0, then $x + 3$ represents 3.
If x be -10, then $x + 3$ represents -7.
etc. etc.

It should be understood that, unless some condition is imposed which restricts x to some particular value, the expression $x + 3$ taken by itself may represent any number whatever.

If an expression such as $x + 3$ be assumed to represent some particular number, such as 5, it may be seen that a restriction is placed upon the value of x by this assumed condition. We shall find that this condition is satisfied providing x is restricted to the value 2.

11. Equalities which are true only on condition that specified letters or sets of letters appearing in them be given definite values or sets of values, are called **conditional equations.**

12. In a conditional equation, the expression at the left of the sign of equality is called the **first member** of the equation and the expression at the right of the sign the **second member.**

13. The terms of the algebraic expressions which form the members of a conditional equation are called the **terms** of the equation.

14. In order to distinguish conditional equations (which are true for *particular values only* of the letters appearing in them) from identities (which are true for *all values* of the letters), we shall use the *double sign of equality*, $=$, when writing conditional equations, and the *triple sign of equality*, $\equiv$, when writing identical equations in which one or more letters appear.

When writing *numerical identities*, that is, identities in which numbers alone appear, we shall use the double sign of equality, $=$.

All definite arrangements of number symbols used in arithmetic to represent numbers (except such as contain zero as a divisor) represent definite numerical values. Hence, when two such arrangements of symbols are written as members of a numerical equality, no condition affecting the value of either can exist. That is, a numerical equality is always an identity.

The use of the sign $=$ for numerical identities in algebra conforms with the use of this sign in arithmetic.

15. Although two different algebraic expressions may represent unequal numbers when numerical values are given to the letters appearing in them, it may happen that they represent equal numbers on condition that some particular value or set of values is assigned to certain specified letters appearing in them.

E. g. It may be seen that the two expressions $2x + 5$ and $x + 8$ represent unequal numbers when x is given the values 1, 2, or 5.

If x be 1, then $2x + 5$ represents 7, and $x + 8$ represents 9.

If x be 2, then $2x + 5$ represents 9, and $x + 8$ represents 10.

If x be 5, then $2x + 5$ represents 15, and $x + 8$ represents 13.

On condition that x be given the particular value 3, the expressions $2x + 5$ and $x + 8$ each represent 11, and accordingly we may construct the conditional equation $2x + 5 = x + 8$.

The equality $4x - 7 = 2x + 3$ is a conditional equation in which the members represent the number 13 on condition that x be given the particular value 5.

16. A conditional equation is said to be **integral, fractional, rational, or irrational,** with respect to certain specified letters appearing in it, according as the algebraic terms appearing in its members are integral, fractional, rational, or irrational with respect to these letters.

E. g. The conditional equation $x^3 + 7x^2 = 4$ is integral and rational with respect to x, since x does not appear in the denominator of a fraction in any term, nor under a radical sign.

The conditional equation $\frac{5}{x+1} = x - 3$ is fractional with reference to x, since x appears in the denominator of the fraction in the first member. It is also rational with reference to x, since x does not appear under a radical sign.

The conditional equation $\sqrt{x+2} + x = 10$ is irrational with reference to x, since x appears under the radical sign in the first term.

17. The **degree** of a conditional equation which is integral and rational with respect to one or more specified letters is equal to the degree of the term of highest degree with reference to these letters.

E. g. The conditional equation $x^3 + 2x^2 - 5x + 1 = 0$ is of the third degree with reference to x.

The conditional equation $x^2y + 2x - y^2 = 9$ is of the third degree with reference to x and y together.

It should be observed that the definition given for the degree of a conditional equation requires that the equation be neither fractional nor irrational with reference to the letters in terms of which its degree is reckoned.

Equations which are either fractional or irrational with reference to certain letters appearing in them are not spoken of as having degree.

E. g. The conditional equations $\frac{5}{x-1} = x - 3$ and $\sqrt{x+2} + x = 10$ cannot be considered as having degree.

18. Since, when a conditional equation is constructed, we may not know the value or values which must be assigned to a specified letter or set of letters appearing in it in order that the expressed equality may be true, it is consistent to speak of these letters as **unknowns.**

E. g. In the conditional equation $3x = 12$, the unknown x must have the value 4 on condition that $3x$ shall represent 12.

The members of the conditional equation $5x + 2 = 6x - 1$ represent the same number, 17, only on condition that the unknown x is given the value 3.

In the conditional equation $x^2 - 5x + 6 = 0$ the unknown x may have either of the values 2 or 3.

19. The particular values which must be given to the unknown letters, in order that both members of a conditional equation may represent the same number, are called the **solutions, or roots,** of the equation.

20. **To solve** a conditional equation is to find its root, or its roots if it has more than one, or to show that it has no root.

21. A number or quantity is said to **satisfy** a given conditional equation if, when it is substituted for the unknown in the equation, both members may be reduced to the same form, and hence may take the same value.

E. g. The number 5 satisfies the conditional equation $4x + 1 = 6x - 9$, since when x is replaced by 5 we obtain $4 \times 5 + 1 = 6 \times 5 - 9$, or $21 = 21$.

22. Two conditional equations are said to be **equivalent,** with respect to a specified unknown, x, when they have the same solutions with respect to x.

From this definition it follows that every solution of either equation must be a solution of the other also; that is, neither equation can have any solution which the other has not.

E. g. The conditional equation $3x - 7 = 5 - x$ is equivalent to the conditional equation $4x = 12$. Both equations are satisfied when x is given the value 3, and neither equation is satisfied for any other value of x.

23. If the solution of a given conditional equation cannot be obtained immediately by inspection, it is often possible to derive from it an equivalent conditional equation in which the members are of such forms that the solution may be readily obtained.

E. g. It may be seen that in the conditional equation $x + 1 = 5$ the unknown x must have the value 4; while the fact that the unknown may have either of the values $1\frac{1}{5}$ or $4\frac{1}{2}$ would not appear immediately on inspection of the conditional equation $\frac{x-2}{x-3} = \frac{10x}{27}$.

GENERAL PRINCIPLES GOVERNING TRANSFORMATIONS OF CONDITIONAL EQUATIONS

24. Throughout all of the discussions which follow in this chapter we shall assume that none of the functions considered become infinite for any values which may be given to the letters which appear in them.

Whenever, in this and the following chapters, the word *equation* is used, it will be understood that a *conditional equation is meant*, unless the contrary is expressly stated.

25. Principle I. SUBSTITUTION.

If for any expression in an equation we substitute an identical expression, the original and the derived equations will be equivalent.

E. g. Performing the indicated operations we may reduce the first member of the conditional equation, $7x + 8 - 2(3x + 4) = 10x - 5(2x - 1)$, to x, and the second member to 5. That is, we may replace the first and second members of the given conditional equation by the identical expressions x and 5, and obtain immediately the equivalent equation $x = 5$.

26. Principle II. ADDITION AND SUBTRACTION.

If identical expressions be added to or subtracted from both members of an equation, the original and derived equations will be equivalent.

This principle follows from the Fundamental Laws of Algebra.

(The following proof may be omitted when the chapter is read for the first time.)

For, if A and B represent expressions which contain one or several unknowns, and C is either a constant or any function of the unknowns, it follows that from $A = B$, we may obtain either of the equivalent equations

$$A + C = B + C \text{ or } A - C = B - C.$$

If, when certain values are given to the unknowns, A and B take the same numerical value, that is, if $A \equiv B$, then for the same values of the unknowns $A + C$ will take the same value as $B + C$, and $A - C$ will take the same value as $B - C$.

Hence, all of the solutions of $A = B$ are solutions of both

$$A + C = B + C \text{ and } A - C = B - C.$$

Conversely: Every solution of either $A + C = B + C$ or of $A - C = B - C$ is also a solution of $A = B$.

For, if when certain values are given to the unknowns, $A + C$ and $B + C$ have equal values, and also $A - C$ and $B - C$ have equal values, — that is, if $A + C \equiv B + C$ and $A - C \equiv B - C$, — then for the same values of the unknowns we shall have $A + C - C \equiv B + C - C$ and also

$$A - C + C \equiv B - C + C; \text{ that is, in either case } A \equiv B.$$

Hence, every solution of either of the derived equations $A + C = B + C$ or of $A - C = B - C$ is also a solution of the original equation $A = B$.

Hence, the two equations are equivalent, since *no additional solution is gained by the transformation.*

27. From the principle above we have the following

APPLICATIONS:

(i.) **Transposition of Terms.** *Any term may be stricken out from either member of an equation provided that a term equal in absolute value, but opposite in sign, be written in the other member of the equation.*

This operation, which has the effect of carrying a term over from one side to the other of the equality sign in an equation and at the same time changing its sign from + to − or from − to +, is called **transposition.**

E. g. From the conditional equation $4x - 5 = 3x + 2$, we may, by transposing the terms -5 and $3x$, obtain the equivalent equation

$$4x - 3x = 2 + 5.$$

Combining the terms in the members separately, we obtain the equivalent equation $x = 7$.

It should be observed that, instead of transposing -5 from the first member to the second member of the given equation $4x - 5 = 3x + 2$, we may add 5 to both members and obtain $4x - 5 + 5 = 3x + 2 + 5$.

By combining the terms in the first member of this last equation, we obtain the equivalent equation $4x = 3x + 2 + 5$.

We may cause the term $3x$ to disappear from the second member by subtracting $3x$ from each member of the equation. Hence, from the derived equation $4x = 3x + 2 + 5$, we may obtain the equivalent equation $4x - 3x = 3x + 2 + 5 - 3x$, which is equivalent to $4x - 3x = 2 + 5$.

This equation, as before, is equivalent to the conditional equation $x = 7$.

(ii.) *Identical terms may be stricken out from both members of an equation.*

For, if either of two identical terms be transposed from one member of the equation to the other, then both terms will appear with opposite signs in the same member. When combined, these terms will produce zero, and accordingly will disappear from the derived equation.

E.g. If $x + a = 5 + a$, we may immediately strike out a from both members, obtaining as an equivalent equation $x = 5$.

(iii.) *From any equation we may derive an equivalent equation by reversing the sign of every term in each member from + to — or from — to +.*

This operation, of reversing the signs of all of the terms, has the effect of transposing every term in the equation from each member to the other, and then interchanging the members of the resulting equation.

E.g. Reversing the signs of all of the terms of the conditional equation $-10x + 1 = -9x - 2$, we obtain the equivalent equation $+10x - 1 = +9x + 2$. From this equation we obtain, by transposing and combining terms, the equivalent equation $x = 3$.

An integral equation is said to be in **standard form** if the second member is zero; if the first member is reduced to simplest form and arranged according to descending powers of the unknown; and if the coefficient of the highest power of the unknown is positive.

E. g. Each of the following equations is in standard form:

$$x^2 - 8x + 12 = 0,$$
$$2x^3 + 3x^2 - 7x + 1 = 0.$$

(iv.) It follows directly from the principle above that *any conditional equation may be reduced to standard form by transposing the terms from the second to the first member, after which the second member of the equivalent derived equation will be zero.*

This operation, of transposing to the first member every term appearing in the second member of a given equation, has the effect of subtracting the original second member of the equation from each member of the given equation, producing an equivalent equation whose second member is zero.

E.g. From the conditional equation $x^2 = 7x - 12$ we obtain the equivalent equation $x^2 - 7x + 12 = 0$, in standard form.

By principles to be shown later, this equation will be found to have the two solutions $x = 3$ and $x = 4$.

MENTAL EXERCISE X. 2

From each of the following conditional equations derive an equivalent equation by transposing to the first member the terms containing x, and to the second member all other terms; then simplify the members separately:

1. $x - 5 = 0$.
2. $x + 7 = 0$.
3. $x - a = 0$.
4. $x + b = 0$.
5. $0 = 9 - x$.
6. $0 = 12 - x$.
7. $0 = - x + 14$.
8. $0 = - x - 17$.
9. $x - 2 = 1$.
10. $x - 8 = 2$.
11. $x - 9 = 13$.
12. $x + 3 = 7$.
13. $x + 7 = 8$.
14. $6 + x = 11$.
15. $8 + x = 14$.
16. $x + 9 = 2$.
17. $x + 11 = 12$.
18. $x + 5 = 5$.
19. $x - 4 = 4$.
20. $x + 7 = - 7$.
21. $3 = 4 - x$.
22. $4 = 9 - x$.
23. $5 = 11 - x$.
24. $6 = 4 - x$.
25. $12 = 5 - x$.
26. $8 = - 3 - x$.
27. $- 15 = 11 - x$.
28. $- 5 = - x - 2$.
29. $- 12 = - x - 18$.
30. $x - 1 - a = 0$.
31. $x + 2 - b = 0$.
32. $x - 4 + c = 0$.
33. $x + 9 + d = 0$.
34. $2x = 13 + x$.
35. $4x = 3x + 5$.
36. $6x = 8 + 5x$.
37. $9x = 8x - 15$.
38. $23x = 22x - 24$.
39. $3x - 25 = 2x$.
40. $12x - 7 = 11x$.
41. $15x + 4 = 14x$.
42. $- 3x - 10 = - 4x$.
43. $17 + 30x = 29x$.
44. $14 - 8x = - 9x$.
45. $2x + 5 = x + 7$.
46. $3x - 1 = 2x + 6$.
47. $4x - 7 = 3x - 2$.
48. $5x + 8 = 4x + 5$.
49. $7x + 3 = 6x - 5$.
50. $10x + 11 = 9x + 7$.
51. $13x - 6 = 12x - 5$.
52. $6x + 17 = 23 + 5x$.
53. $15x - 1 = 11 + 14x$.
54. $13 + 12x = 11x + 21$.
55. $4 - 15x = 11 - 16x$.
56. $16 - 17x = - 18x - 19$.
57. $5 - 19x = - 6 - 20x$.
58. $- 19x - 33 = - 20x - 31$.
59. $- 28x - 37 = - 40 - 29x$.
60. $- 31x - 11 = 29 - 32x$.
61. $21x + \frac{1}{3} = 1 + 20x$.
62. $26x + \frac{1}{5} = 25x + 1$.

63. $27x + \frac{1}{3} = 2 + 26x.$
64. $23x - \frac{1}{5} = 22x + 1.$
65. $34x + \frac{5}{9} = 33x + 1.$
66. $\frac{3}{2}x + 1 = \frac{1}{2}x + 6.$
67. $\frac{7}{4}x + 2 = \frac{3}{4}x + 9.$
68. $\frac{8}{3}x - 7 = 9 + \frac{5}{3}x.$
69. $\frac{4}{7}x - 13 = 4 - \frac{3}{7}x.$
70. $\frac{3}{8}x - 10 = 3 - \frac{5}{8}x.$
71. $\frac{5}{9}x - \frac{1}{7} = \frac{6}{7} - \frac{4}{9}x.$
72. $\frac{4}{11}x - \frac{1}{3} = \frac{5}{3} - \frac{7}{11}x.$
73. $\frac{1}{2}x - \frac{15}{11} = \frac{18}{11} - \frac{1}{2}x.$
74. $\frac{7}{12}x + 18 = -\frac{5}{12}x - 18.$
75. $\frac{13}{9}x - 23 = 23 + \frac{4}{9}x.$
76. $35x + 1 = a + 34x.$
77. $37x - 2 = 36x + b.$
78. $40x + c = 39x + 5.$
79. $42x - d = 41x - 6.$
80. $m - 48x = 4 - 49x.$
81. $50x + \frac{1}{2} = 49x + a.$
82. $54x + b = \frac{1}{3} + 53x.$
83. $61x + b = a + 60x.$
84. $63x - c = 62x + b.$

From each of the following conditional equations derive an equivalent equation by omitting the identical terms from both members:

85. $x + a = a + c.$
86. $x + b = a + b.$
87. $x - d = b - d.$
88. $m + d = m + x.$
89. $a - b = x - b.$
90. $ax + x = ax + 2.$
91. $bx + x = 3 + bx.$
92. $mx - 1 = x + mx.$
93. $x + a = a.$
94. $x - cx = -d - cx.$

From each of the following conditional equations derive an equivalent equation in the standard form $A = 0$:

95. $x^2 - x = 12.$
96. $x^2 - 5 = 4x.$
97. $x^2 + 6x = 16.$
98. $x^2 = 7x - 30.$
99. $x^2 = 22 - 9x.$
100. $x^2 = 10x + 39.$
101. $2x^2 = 3 - 5x.$
102. $3x^2 - 5 = 14x.$
103. $23x = 6 - 4x^2.$
104. $10x^2 = 13x + 3.$
105. $6x^2 - 4x = 5x^2 + 21.$
106. $2x^2 + 4x = 3x + 3.$
107. $8x^2 + x = 7x - 1.$
108. $7x^2 + 7x = 10 - 5x^2.$
109. $3x^2 + 9x + 6 = 4x + 8.$
110. $4x^2 - 13 = 3x^2 + 5 - 3x.$
111. $x^2 - 12x + 4 = 10x - 7x^2 - 11.$
112. $9x^2 + 6x = 6x + 36 + 8x^2.$
113. $5x - 9 = 3 - 3x^2 + 5x.$
114. $x^3 + 5x^2 + 9x + 7 = x^3 + 4x^2 - 11.$
115. $x^3 - 3x^2 + x + 6 = x^3 - 4x^2 + 11x - 18.$
116. $x^4 + 7x^2 - 4x - 5 = x^4 + 6x^2 - 5x + 7.$

28. Principle III. MULTIPLICATION

If both members of an integral equation be multiplied by the same number or expression, the original and the derived equations will be equivalent, provided that the multiplier used is neither zero nor infinitely great, and that it does not contain the unknown letter or letters.

(The following proof may be omitted when the chapter is read for the first time.)

Let a conditional equation be represented by $A = B$ (1), in which either A or B, or both, are functions of some unknown letter, x.

If C is a number which is neither zero nor infinitely great, or if C represents an expression which does not contain the unknown letter x, then the given equation $A = B$ is equivalent to the derived equation $AC = BC$ (2).

To show this we will write the derived equation (2) in the standard form $AC - BC = 0$ (3).

Since C is assumed to be neither zero nor infinitely great, we may divide the expression $AC - BC$ by C and write equation (3) in the form $C(A - B) = 0$ (4).

Every value of x which satisfies the original equation $A = B$ reduces the factor $A - B$ to zero.

Since C is assumed to have a value which is not infinitely great, it follows that any value which, when substituted for x, reduces the factor $A - B$ to zero, reduces the product $C(A - B)$ to zero. Accordingly such a value of x satisfies the equation $C(A - B) = 0$.

It follows that every solution of (1) is also a solution of (3), and hence of $AC = BC$ (2).

That is, no solution of (1) *is lost by the transformation.*

To show that equation (1) is equivalent to equation (2) it remains for us to show that no solutions have been gained in passing from equation (1) to equation (2).

Every value of x which satisfies $C(A - B) = 0$ (4) reduces the product $C(A - B)$ to zero.

In order that the product of two factors shall become zero it is necessary and sufficient that one of these factors shall become zero, the other factor not becoming infinitely great.

Accordingly, since the value of C is assumed to be neither zero nor infinitely great, it is necessary that the remaining factor $A - B$ should become zero.

Every value which, when substituted for x, reduces $A - B$ to zero, must satisfy the equation $A - B = 0$. It follows that every solution of

$C(A - B) = 0$ is a solution also of $A - B = 0$, and hence is a solution of $A = B$ (1).

That is, no solution is gained by the transformation.

Since, by the reasoning above, solutions are neither gained nor lost in passing from the given equation $A = B$ to the derived equation $AC = BC$, these equations are equivalent.

E. g. From the equation $x + 1 = \frac{2}{3}x + \frac{7}{3}$, we may, by multiplying both members by the constant multiplier 3, derive the equivalent equation

$$3x + 3 = 2x + 7.$$

Transposing and combining terms, we obtain from this last equation $x = 4$.

Since the equation $x = 4$ has the single solution 4, it follows that the given equation to which it is equivalent must have the solution $x = 4$, and no other.

29. Caution. To be certain that solutions have not been gained in solving conditional equations, it is necessary that we know that the multipliers with which we affect the forms of given equations are different from zero.

30. The solutions which may be gained when the members of a conditional equation are transformed by multiplication may be determined by the following

Principle Relating to Extra Roots: *The strange or extra solutions which may be introduced into the members of a derived equation by multiplying the members of a given equation by an integral expression containing the unknown number, are the values of the unknown which, when substituted, reduce the multiplier to zero.*

(The following proof may be omitted when the chapter is read for the first time.)

If C were a function of x it might happen that, when particular values were given to x, C might become zero and $A - B$ become different from zero.

Such values of x would satisfy the equation $C(A - B) = 0$ without reducing the factor $A - B$ to zero, and hence without satisfying the given equation $A = B$ (1) § 28.

Thus, if C were a function of x, solutions of the derived equation $AC = BC$ might exist which were not solutions also of the original equation $A = B$. In such a case the given and derived equations would not be equivalent.

E. g. If both members of the equation $3x - 16 = x - 2$ (1) were multiplied by the expression $x - 3$, containing the unknown, we should obtain the equation $(3x - 16)(x - 3) = (x - 2)(x - 3)$, (2).

The derived equation (2) would have, not only the solution $x = 7$ of the original equation (1), but also the solution $x = 3$ which does not satisfy the original equation (1). This extra solution $x = 3$ would have been introduced by means of the multiplier $x - 3$ which becomes zero for $x = 3$.

By using the multiplier $x - 3$ we should thus gain a solution in passing from the given equation (1) to the derived equation (2).

31. Principle IV. DIVISION.

If both members of an equation be divided by the same number, the original and the derived equations will be equivalent, provided that the divisor used is neither zero nor infinitely great, and that it does not contain the unknown letter or letters.

(The following proof may be omitted when the chapter is read for the first time.)

Instead of division by a number D we may substitute multiplication by its reciprocal $\frac{1}{D}$.

Hence, since D can become neither zero nor infinitely great, it follows that $1/D$ can become neither infinitely great nor zero.

Accordingly, the principle under consideration is proved by the course of reasoning employed for the proof of Principle III § 28, provided that $1/D$ is represented in that proof by C.

E. g. From the equation $2x - 4 = 10$, we shall, by dividing both members by the constant 2, derive the equivalent equation $x - 2 = 5$. Transposing and combining terms, we have $x = 7$.

Since the given and derived equations are equivalent, it follows that 7, which is the single solution of the last equation, must be the single solution of the given equation.

32. Caution. Beginners often make the error of dividing both members of an equation by an expression containing the unknown, and thus lose as solutions of the given equation such values as would, when substituted for the unknown, reduce to zero the divisor thus used and rejected.

33. Principle Relating to Loss of Roots: *If identical expressions containing the unknown be removed by division from both members of an equation, the given and the derived equations will not be equivalent.*

The solutions of the given equation which are not solutions of the derived equation also are those values of the unknown which, if substituted, would reduce to zero the expression removed by division.

(The following proof may be omitted when the chapter is read for the first time.)

It follows from the reasoning employed in the proof of the principle of § 30 that the solutions of the equation $AC = BC$ are the same as those of the two equations $A = B$ and $C = 0$, provided that A, B, and C are functions of the unknown.

Hence, it follows that if C, which is common to both members of the equation $AC = BC$, be removed by division and rejected, the solutions of the equation formed by placing this divisor equal to zero, that is $C = 0$, are thus lost.

E. g. If we divide both members of the conditional equation

$$(x - 1)(x - 4) = 2(x - 1)$$

by the divisor $x - 1$ containing the unknown x, we shall obtain as a derived equation $x - 4 = 2$. This derived equation, $x - 4 = 2$, is satisfied by the single value $x = 6$. We have, by the process, lost a root, since the original equation is satisfied by the value $x = 1$ in addition to the value $x = 6$.

It should be observed that the expression $x - 1$, which was removed from both members of the given equation by division, becomes zero when x is given the value 1, which is the root which was lost when passing from the given to the derived equation.

The conditional equation $x^2 = 6x$ is satisfied by the values $x = 0$ and $x = 6$, and by no others. If, however, we derive the equation $x = 6$ by removing by division and rejecting x from both members of $x^2 = 6x$, we shall lose one of the solutions of the original equation, namely $x = 0$.

34. From the principles of §§ 28, 31 we have the following

APPLICATIONS :

(i.) *If the terms of an equation are integral with reference to some specified letter, x, and the coefficients of x are fractional, we may derive an equivalent equation in which the coefficients of x are integral, by multiplying all of the terms of the given equation by the least number which contains all of the denominators of the fractional coefficients exactly as divisors.*

E. g. If we multiply the terms of both members of the conditional equation $6x - 106 = \frac{3}{4}x + \frac{1}{10}x + 100$ by 20, which is the least number which can be divided without remainder by all of the denominators of the fractional coefficients, we shall obtain the equivalent equation, $120x - 2120 = 15x + 2x + 2000$, in which the coefficients are integral.

Transposing and collecting terms in the derived equation, we obtain $103x = 4120$, the solution of which is found to be $x = 40$.

(ii.) *From an integral equation may be derived an equivalent equation in which the coefficient of any specified term shall have any desired value.*

It is often desirable so to transform the terms of an equation that the coefficient of the highest power of the unknown shall be unity.

E. g. By dividing each term of the conditional equation $3x^2 + 4x = 7$ by 3, we may derive the equivalent equation $x^2 + \frac{4}{3}x = \frac{7}{3}$, in which the coefficient of the highest power of x is unity.

35. When by means of any step we derive an equation which is equivalent to another, this step is said to be **reversible,** since, either equation being equivalent to the other, we may derive either equation from the other, and hence take the step forward or backward without gaining or losing solutions.

36. Observe that we may derive an equivalent equation by adding to or subtracting from both members of a given equation the same expression containing the unknown. But in case we multiply or divide both members by an expression containing the unknown, the resulting equation may or may not be equivalent to the one from which it is derived, because in certain cases we may gain or lose solutions.

Mental Exercise X. 3

From each of the following conditional equations derive an equivalent equation in which the coefficient of x is unity:

1. $2x = 8$.
2. $3x = 15$.
3. $4x = 12$.
4. $5x = 30$.
5. $6x = 42$.
6. $7x = 56$.
7. $9x = 81$.
8. $14 = 2x$.
9. $18 = 3x$.
10. $25 = 5x$.
11. $33 = 11x$.
12. $42 = 14x$.
13. $8x = -24$.
14. $9x = -54$.
15. $11x = -77$.
16. $-16 = 8x$.
17. $-7 = 7x$.
18. $-22 = 2x$.
19. $3x = 1$.
20. $4x = 1$.
21. $5x = -1$.
22. $1 = 6x$.
23. $1 = 8x$.
24. $-1 = 9x$.
25. $2x = 0$.
26. $0 = 3x$.
27. $5x = 2$.
28. $7x = 3$.
29. $8x = 5$.
30. $9x = 2$.
31. $3 = 11x$.
32. $4 = 13x$.
33. $-5 = 14x$.
34. $3x = 4$.
35. $2x = 5$.
36. $4x = 7$.
37. $5x = 13$.
38. $6x = 19$.
39. $7x = 29$.

40. $8x = 43$.
41. $9x = -28$.
42. $12x = -25$.
43. $24 = 7x$.
44. $32 = 9x$.
45. $43 = 11x$.
46. $-49 = 13x$.
47. $-52 = 12x$.
48. $6x = 3$.
49. $8x = 2$.
50. $18x = 6$.
51. $42x = 7$.
52. $65x = 5$.
53. $34x = 2$.
54. $50x = -10$.
55. $3x = a$.
56. $4x = b$.
57. $5x = c$.
58. $6x = d$.
59. $7x = 2m$.
60. $8x = 3n$.
61. $9x = 5k$.
62. $6a = 11x$.
63. $12b = 17x$.
64. $10x = 8a$.
65. $12x = 14b$.
66. $16x = 18c$.
67. $21d = 14x$.
68. $33k = 6x$.
69. $3x = \frac{1}{2}$.
70. $4x = \frac{1}{3}$.
71. $5x = \frac{1}{7}$.
72. $6x = \frac{1}{4}$.
73. $8x = \frac{1}{8}$.
74. $9x = -\frac{1}{9}$.
75. $2x = \frac{4}{5}$.
76. $3x = \frac{12}{7}$.
77. $4x = \frac{20}{21}$.
78. $5x = \frac{25}{6}$.
79. $\frac{1}{3}x = 2$.
80. $\frac{1}{2}x = 3$.
81. $\frac{1}{4}x = 5$.
82. $\frac{1}{6}x = 4$.
83. $\frac{1}{7}x = 9$.
84. $\frac{1}{11}x = 1$.
85. $\frac{1}{12}x = 12$.
86. $\frac{1}{5}x = -10$.
87. $\frac{1}{10}x = -3$.
88. $\frac{1}{13}x = -13$.
89. $\frac{1}{2}x = 0$.
90. $6 = \frac{1}{7}x$.
91. $12 = \frac{1}{5}x$.
92. $-14 = \frac{1}{3}x$.
93. $\frac{1}{2}x = \frac{1}{3}$.
94. $\frac{1}{3}x = \frac{1}{5}$.
95. $\frac{1}{4}x = \frac{1}{7}$.
96. $\frac{1}{6}x = \frac{1}{9}$.
97. $\frac{1}{8}x = \frac{1}{14}$.
98. $\frac{1}{10}x = \frac{1}{10}$.
99. $\frac{1}{21}x = \frac{1}{28}$.
100. $\frac{1}{25}x = \frac{1}{30}$.
01. $\frac{1}{5}x = \frac{1}{4}$.
102. $\frac{1}{7}x = \frac{1}{2}$.
103. $\frac{1}{9}x = \frac{1}{5}$.
104. $\frac{1}{10}x = \frac{1}{3}$.
105. $\frac{1}{4}x = \frac{1}{2}$.
106. $\frac{1}{12}x = \frac{1}{3}$.
107. $\frac{1}{32}x = \frac{1}{4}$.
108. $\frac{1}{48}x = \frac{1}{8}$.
109. $\frac{1}{3} = \frac{1}{27}x$.
110. $\frac{1}{5} = \frac{1}{55}x$.
111. $\frac{1}{6} = \frac{1}{6}x$.
112. $\frac{4}{3}x = 1$.
113. $\frac{5}{2}x = 1$.
114. $\frac{6}{5}x = 1$.
115. $\frac{9}{5}x = 1$.
116. $\frac{2}{7}x = 1$.
117. $\frac{3}{10}x = 1$.
118. $\frac{2}{3}x = 0$.
119. $1 = \frac{12}{7}x$.
120. $1 = \frac{3}{8}x$.
121. $1 = \frac{5}{11}x$.
122. $\frac{9}{4}x = 2$.
123. $\frac{13}{3}x = 4$.
124. $\frac{5}{2}x = 3$.
125. $\frac{4}{3}x = 5$.
126. $\frac{6}{5}x = 4$.
127. $\frac{9}{10}x = 8$.
128. $\frac{7}{8}x = 9$.
129. $\frac{5}{6}x = 6$.
130. $\frac{7}{5}x = 7$.
131. $\frac{8}{9}x = 6$.
132. $\frac{4}{9}x = \frac{3}{7}$.
133. $\frac{3}{5}x = \frac{4}{7}$.
134. $\frac{3}{4}x = \frac{8}{11}$.
135. $\frac{5}{4}x = \frac{4}{5}$.
136. $\frac{5}{2}x = \frac{8}{3}$.
137. $\frac{8}{3}x = \frac{2}{3}$.
138. $\frac{5}{6}x = \frac{5}{7}$.
139. $\frac{7}{8} = \frac{10}{9}x$.
140. $\frac{10}{13} = \frac{3}{2}x$.
141. $\frac{7}{10} = \frac{2}{3}x$.
142. $\frac{11}{13} = \frac{3}{4}x$.
143. $\frac{1}{2}x = \frac{1}{3}a$.
144. $\frac{1}{3}x = \frac{1}{4}b$.
145. $\frac{1}{5}x = \frac{1}{8}c$.
146. $\frac{1}{8}x = \frac{1}{11}d$.
147. $\frac{1}{9}x = \frac{1}{4}a$.
148. $\frac{1}{15}x = \frac{1}{7}b$.
149. $\frac{1}{6}x = \frac{1}{3}m$.
150. $\frac{1}{20}x = \frac{1}{5}a$

151. $\frac{1}{34}x = \frac{1}{17}b.$
152. $\frac{1}{4}x = \frac{1}{6}a.$
153. $\frac{1}{18}x = \frac{1}{27}b.$
154. $\frac{1}{28}x = \frac{1}{70}c.$
155. $\frac{1}{12}x = \frac{1}{8}d.$
156. $\frac{1}{16}x = \frac{1}{14}h.$
157. $\frac{1}{24}x = \frac{1}{18}k.$
158. $\frac{1}{5}a = \frac{1}{2}x.$
159. $\frac{1}{17}b = \frac{1}{3}x.$
160. $\frac{1}{19}c = \frac{1}{8}x.$
161. $\frac{1}{3}d = \frac{1}{23}x.$
162. $\frac{1}{4}g = \frac{1}{19}x.$
163. $\frac{1}{34}h = \frac{1}{16}x.$
164. $\frac{1}{42}k = \frac{1}{24}x.$
165. $\frac{1}{52}m = \frac{1}{32}x.$
166. $\frac{1}{18}a = \frac{1}{21}x.$
167. $\frac{1}{27}a = \frac{1}{45}x.$
168. $\frac{1}{36}b = \frac{1}{60}x.$
169. $\frac{1}{2}x = \frac{3}{7}a.$
170. $\frac{1}{3}x = \frac{4}{3}b.$
171. $\frac{1}{4}x = \frac{2}{5}c.$
172. $\frac{1}{6}x = \frac{3}{7}d.$
173. $\frac{4}{9}m = \frac{1}{5}x.$
174. $\frac{6}{5}m = \frac{1}{6}x.$
175. $\frac{2}{3}x = \frac{1}{7}a.$
176. $\frac{3}{4}x = \frac{1}{5}b.$
177. $\frac{4}{7}x = \frac{1}{4}c.$
178. $\frac{3}{2}x = \frac{4}{5}a.$
179. $\frac{2}{7}x = \frac{3}{11}b.$
180. $\frac{3}{8}x = \frac{5}{7}c.$
181. $\frac{2}{9}x = \frac{3}{10}d.$
182. $\frac{7}{3}x = \frac{8}{5}m.$
183. $\frac{4}{11}a = \frac{5}{9}x.$
184. $\frac{8}{3}b = \frac{6}{5}x.$
185. $\frac{4}{3}c = \frac{10}{9}x.$
186. $\frac{9}{5}d = \frac{5}{9}x.$

CHAPTER XI

EQUATIONS OF THE FIRST DEGREE CONTAINING ONE UNKNOWN

1. **A simple or linear equation** containing a single unknown is an equation which is of the first degree with reference to the unknown appearing in it.

2. **Solution of a Linear Equation.** By applying the Principles of Chapter X., any integral rational equation containing one unknown, that is any linear equation, may be transformed into an equivalent equation of the standard form

$$ax + b = 0, \tag{1}$$

in which a and b are either simple or compound expressions, both free from the unknown. The term b, which is free from the unknown, may be zero, but the coefficient a cannot be zero.

Equation (1) may be written in the equivalent form

$$ax = -b. \tag{2}$$

Since $a \neq 0$, we may divide both members by a, and obtain

$$x = \frac{-b}{a}. \tag{3}$$

It follows that any linear equation in the form $ax + b = 0$, containing one unknown, has one solution and one only, $x = \frac{-b}{a}$.

Ex. 1. Solve the equation $15x - 11 = 5x + 9$. (1)

Transposing the terms $5x$ and -11 we obtain

$$15x - 5x = 9 + 11, \tag{2}$$

which is equivalent to equation (1) by Chapter X. § 27 (i.) Principle II.

Combining like terms, we obtain

$$10x = 20, \tag{3}$$

which is equivalent to equation (2) by Chapter X, § 25 Principle I.

Dividing both members by the coefficient of x, we obtain the solution

$$x = 2.$$

Since throughout the entire process all of the equations obtained are equivalent to one another, the solution of the last, $x = 2$, is a solution of, and the only solution of, the first.

We may verify the solution by substituting this value, $x = 2$, in the original equation as follows:

$$15 \cdot 2 - 11 = 5 \cdot 2 + 9$$
$$19 = 19.$$

Ex. 2. Solve the equation $12 - 7x = x - 18$. (1)

Transposing the terms so that all terms containing x shall appear in the first member, and all terms free from x in the second, we have the equivalent equation

$$-7x - x = -18 - 12. \qquad (2)$$

Combining terms, we derive the equivalent equation

$$-8x = -30. \qquad (3)$$

Dividing both members by the coefficient, -8, of x, we obtain as a solution of this last equation

$$x = \frac{-30}{-8} = \frac{15}{4}.$$

Since, in obtaining the successive equations we have applied the principles for deriving equivalent equations, the result $\frac{15}{4}$, which is the only solution of the last equation, must be a solution, and the only solution of the given equation.

Substituting this value in the original equation, we obtain

$$12 - 7\left(\tfrac{15}{4}\right) = \tfrac{15}{4} - 18$$
$$-14\tfrac{1}{4} = -14\tfrac{1}{4}.$$

Ex. 3. Solve the equation $\frac{1}{5}(x - 5) + \frac{2}{3}(x - 3) = x - \frac{7}{5}$. (1)

Multiplying both members of the equation by 15, which is the least number which contains the denominators of the different fractions exactly as divisors, we obtain the equivalent equation

$$3(x - 5) + 5 \cdot 2(x - 3) = 15x - 3 \cdot 7. \qquad (2)$$

Performing the indicated operations, we obtain

$$3x - 15 + 10x - 30 = 15x - 21,$$

and hence $$13x - 45 = 15x - 21. \qquad (3)$$

By Principle I, Chapter X, § 25, this equation is equivalent to equation (2) above.

Transposing terms, we derive from (3) the equivalent equation

$$13x - 15x = -21 + 45. \tag{4}$$

From (4) we obtain,

$$-2x = +24. \tag{5}$$

Dividing both members by the coefficient, −2, of x, we obtain the solution

$$x = \frac{24}{-2} = -12.$$

Since all of the steps used in the process of deriving these equations successively are reversible, the solution of the last equation must be a solution of, and the only solution of, the original equation.

Verifying the accuracy of the result by substituting in the first equation, we have

$$\begin{aligned} \tfrac{1}{5}(-12-5) + \tfrac{2}{3}(-12-3) &= -12 - \tfrac{7}{5} \\ -13\tfrac{2}{5} &= -13\tfrac{2}{5}. \end{aligned}$$

3. It is possible that roots may be either gained or lost during the process of solution of a conditional equation. (See Chapter X. §§ 30, 33).

The substitution in the original equation of any roots which may satisfy any one of the derived equations does not establish thereby the equivalence of the equations.

By such a substitution we simply determine whether or not the values substituted are solutions of the original equation.

The equivalence of the given and derived equations must be determined by examining the *process* of derivation.

Ex. 4. Solve the equation $(x+2)(x+3)+4=(x+2)^2$. (1)

Performing the indicated multiplications, we obtain

$$x^2 + 5x + 6 + 4 = x^2 + 4x + 4. \tag{2}$$

Equal terms, such as x^2 or 4, which occur in both members, may be stricken out, since if transposed they would by combination produce zero.

Hence we have

$$\begin{aligned} 5x - 4x &= -6 \qquad (3) \\ x &= -6. \end{aligned}$$

No root is gained or lost by any of the operations performed on the members of these equations. Hence, the root of the last equation is a solution of, and the only solution of, the original equation.

Verifying the result by substitution, we obtain

$$\begin{aligned} (-6+2)(-6+3)+4 &= (-6+2)^2 \\ 16 &= 16. \end{aligned}$$

4. The ultimate test of the correctness of every solution is that, when the value found is substituted for the letter representing it, the equation is satisfied. No matter how we may obtain the value of an unknown letter, even if it be by mere guessing or by inspection, if it stands this test of substitution, it is a solution.

5. The process of solving a conditional equation consists of obtaining and substituting for a given equation another equation which has all of the solutions of the first, and, if possible, no more solutions, and which is of such form that the relations expressed between the letter whose value is to be found and the remaining quantities in the equation is less complicated. This process is continued until, if possible, an equation is finally obtained which may be solved by inspection.

6. Suggestions Concerning the Solution of Simple Equations Containing One Unknown Number

(i.) Remove fractional coefficients, if there be any, by multiplying both members of the equation by the least number which contains the denominators of the different fractions exactly as divisors.

(ii.) Perform all such indicated operations as are necessary to separate the terms of the equation into two distinct groups, — one group consisting of all of the terms containing the unknown number (and no other terms), and a second group consisting of all terms which do not contain the unknown number.

The terms which contain the unknown number are commonly transposed to the first member of the derived equation, and all terms which are free from the unknown are transposed to the second member of the equation.

(iii.) All numerical or all monomial or polynomial factors *not containing the unknown numbers*, which are common to all of the terms of both members of the equation, should be removed by division and rejected as soon as discovered.

(iv.) Combine into one term all of the terms containing the unknown number, and into another term the remaining terms which are free from the unknown.

(v.) Divide both members of the equation by the coefficient of the unknown number.

(vi.) The expression found for the unknown number should be reduced to simplest form.

Exercise XI. 1 (Mental and Written Examples)

Solve the following equations, verifying all results by substitution. The first sixty-four examples may be solved mentally:

1. $4x + 5 = 3x + 7$.
2. $6x + 11 = 5x + 17$.
3. $10x - 7 = 9x + 8$.
4. $7x - 10 - 6x = 0$.
5. $5x + 11 - 4x = 0$.
6. $11x - 6 - 9x = 0$.
7. $3x + 1 = x - 1$.
8. $9x + 1 = 3x + 5$.
9. $13x + 4 = 11x + 10$.
10. $6x - 11 = 2x + 9$.
11. $13x - 18 = x + 6$.
12. $8x - 13 = 3x - 53$.
13. $22x + 15 = 19x - 12$.
14. $15x + 37 = 3x + 13$.
15. $12x + 1 = 28 + 3x$.
16. $19 + 17x = 59 - 3x$.
17. $15x - 13 = 29 + 8x$.
18. $21 + 22x = 8x - 35$.
19. $7x + 2 = 4x + 7$.
20. $17 - 18x = 87 - 25x$.
21. $15 + 11x = 79 - 5x$.
22. $2x + 23 = 5x + 2$.
23. $4x - 23 = 1 - 4x$.
24. $2x - 37 = 7x + 3$.
25. $3x + 5 = 9x + 59$.
26. $5x + 1 + 2x = 15$.
27. $6x - 2 + 3x = 25$.
28. $7x + 10 - 4x = 40$.
29. $9x - 2 - 4x = -57$.
30. $12x = 8 + 30 - x$.
31. $5x - 4 + 6x - 7 = 0$.
32. $13x - 50 - 2x = x$.
33. $19x - 5 + 2x = 39$.
34. $18x - 33 - 7x = 3x - 1$.
35. $4x + 5 + 6x = 7 + 8x + 4$.
36. $5x + 7 + 3x = x + 10 + 6x$.
37. $8x + 3 + 6x = 4x + 11 + 9x$.
38. $9x - 7 - 5x = 11x + 5 - 8x$.
39. $9x - 7 - 4x = 10x + 5 - 7x$.
40. $15 - 7x + 6 = 12 - 2x + 19$.
41. $5x + 12 - 8x = 19 - 13x + 2$.
42. $22x - 9 - 6x = 5x - 6 + x$.
43. $7x - 5 + 2x = 3x + 8 + x$.
44. $4x + 9 - 7x = 8x - 11 + 3x$.
45. $11 + 7x - 18 - 3x = 9 + x + 5$.
46. $12 + 11x + 3 = 5x - 2 - 4x$.
47. $19x + 9 - 12x + 6 = 2x + 35 + 3x$.
48. $25 + 12x - 23 + 14x = 25x + 12 - 23x + 2$.
49. $8 - 4x - 2 + 9x = 7 + 2x - 19 - 6x$.
50. $7x + 9 - 3x + 5 = 4x - 11 + 2x + 45$.
51. $3x + 13 + 5x - 7 = x + 7 + 2x + 2$.
52. $14x - 1 + 3x + 5 = 7x + 2 - 4x - 8$.
53. $5x - (2x + 3) = 12$.
54. $3x - (13 - x) = 61$.
55. $12 - (4x + 7) = 13$.
56. $17x + 5(2 - 3x) = 18$.

57. $9x - 2(1 + 4x) = 3.$

58. $17 - 3(x + 11) = -7x.$

59. $5(x - 4) = 4(x - 3).$

60. $3(x - 6) - 2(4 - x) = 0.$

61. $7(3 - x) - 4(7 - 2x) = 0.$

62. $6(x + 5) - 12 = 3(3x - 1) + 4x.$

63. $22 - 5(3 - 2x) = x - 4(x + 8).$

64. $8(x - 7) - 6(x - 5) = 5(x - 4) - 4(x - 3).$

65. $(x + 1)^2 = x^2 + 5.$

66. $(x - 3)^2 = x^2 - 21.$

67. $(x + 4)^2 = x(x + 3).$

68. $(3x + 1)^2 - 2x = 9x^2 + 13.$

69. $(x + 1)(x + 2) = x^2 + 11.$

70. $(x + 3)(x + 5) = x^2 + 31.$

71. $(x + 1)(x + 5) = (x + 2)(x + 3).$

72. $(x - 10)(x - 7) = (x - 9)(x - 6).$

73. $(x + 2)(x + 4) = (x + 3)(x + 1) + 1.$

74. $(x - 4)(x + 1) - (x - 5)(x - 2) = 0.$

75. $(x - 6)(x - 1) - (x + 7)(x + 3) = 0.$

76. $(2x + 1)(3x + 1) = (6x - 1)(x + 2).$

77. $(16x - 5)(3x + 4) = (12x - 1)(4x + 3).$

78. $(2x + 5)(5x - 4) - 5x = (10x - 3)(x + 1) + 8.$

79. $\frac{2}{3}x = 8 - \frac{1}{3}x.$

80. $\frac{3}{5}x = 1 - \frac{2}{5}x.$

81. $\frac{3}{2}x = 8 + \frac{1}{2}x.$

82. $\frac{2}{3}x = 4 + \frac{1}{3}x.$

83. $\frac{5}{2}x - 7 = \frac{3}{2}x.$

84. $\frac{5}{6}x - 1 = 1 - \frac{1}{6}x.$

85. $\frac{1}{3}x + \frac{1}{4}x = 2.$

86. $\frac{1}{2}x - \frac{1}{5}x = 4.$

87. $\frac{1}{4}x - \frac{1}{7}x = 12.$

88. $\frac{1}{2}x + \frac{1}{3}x = \frac{1}{4}.$

89. $\frac{2}{3}x + \frac{2}{5}x = \frac{5}{6}.$

90. $\frac{4}{3}x - 5 = \frac{3}{4}x - 4.$

91. $\frac{1}{2}x - \frac{1}{3}x + \frac{1}{4}x = \frac{1}{5}.$

92. $\frac{1}{3}x + \frac{1}{4}x + \frac{1}{5}x = x - 13.$

93. $\frac{1}{2}(x + 3) = 4.$

94. $\frac{1}{3}(x - 4) = 1.$

95. $\frac{4}{5}(x + 1) - 2 = 0.$

96. $\frac{1}{2}(x + 4) - \frac{1}{6}x = 8.$

97. $\frac{1}{3}(x + 7) = \frac{1}{15}(x + 6).$

98. $\frac{1}{7}(5x - 1) - 8 = \frac{1}{3}(4x - 2).$

99. $\frac{1}{8}(1 - x) - \frac{1}{10}(2 - x) = \frac{1}{12}(3 + x).$

100. $\frac{1}{3}(x - 15) - \frac{1}{18}(9x - 2) = \frac{1}{4}x + \frac{1}{9}.$

101. $\frac{1}{5}(x + \frac{1}{2}) - \frac{1}{2}(x - \frac{1}{5}) = 10.$

102. $\frac{3}{4}(4x - \frac{1}{3}) + \frac{4}{3}(3x - \frac{1}{4}) = \frac{1}{12}.$

103. $\frac{5}{6}(12x - \frac{6}{5}) - \frac{6}{7}(14x + \frac{7}{6}) = 84.$

104. $\frac{1}{4}x - 1\frac{1}{4} + x = \frac{1}{3}(6x - 9) - \frac{2}{3}x.$

Equations in which Decimal Fractions appear among the Coefficients

Ex. 105. $.5x = .015$.

By multiplying both members of the equation by 1000 we shall obtain an equivalent equation in which the decimal coefficients are replaced by integral coefficients.

That is, $500x = 15$

Therefore, $x = \frac{3}{100}$

Or, $x = .03$, in decimal notation.

Check. $.5 \times (.03) = .015$

$.015 = .015$.

106. $.01x = 200$.

107. $.7x = .07$.

108. $.5x = 1$.

109. $.3x = 3$.

110. $.2x = 4$.

111. $.25x = 1.25$.

112. $.2x = 48 - .04x$.

113. $x - .1 = 1 - .1x$.

114. $.2x + 3 - .04x = 3.8$.

115. $.2x + .04 = .25x - .26$.

116. $.093 - .1x = .02x - .13x + .01$.

117. $x - 10 + .1x = 1100 - .01x$.

118. $3 - .2x + 30 = .02x - 300 + .002x$.

PROBLEMS

7. A **problem** is a question proposed for solution.

8. A problem is said to be **determinate** if it has a limited or finite number of solutions.

In the contrary case, it is said to be **indeterminate**.

9. To solve a given problem is to find the values of certain unknown quantities whose relations with one another and with certain known quantities are given.

The relations between the known and unknown quantities are called the **conditions of the problem.**

In solving a problem which admits of algebraic solution, the first step to be taken is to discover the relations between the unknown and the known quantities, as given in the statement of the problem.

10. The beginner will find it helpful, whenever relations are given between general numbers, to consider an analogous arithmetic problem, choosing definite numerical values in place of the given general numbers.

E. g. By how much does a exceed 15?

Consider a similar example in numbers. By how much does 21 exceed 15?

The excess of 21 over 15 is the difference between 21 and 15, that is, $21 - 15$.

By the same reasoning it appears that a must exceed 15 by the difference between a and 15, that is, by $a - 15$.

ALGEBRAIC EXPRESSION

Mental Exercise XI. 2

1. By how much does x exceed y?
2. By how much does a exceed b?
3. By how much does x exceed 25?
4. By how much does 30 exceed y?
5. By how much does a exceed 1?
6. What number must be added to x to obtain a?
7. By what number must x be diminished to equal z?
8. What number is less than 10 by a?
9. What number is greater than 15 by b?
10. If a represents an integer how may the next greater integer be represented? The next less?
11. If b represents an odd integer how may the next greater odd integer be represented? The next less?
12. If $2c$ represents an even integer how may the next greater even integer be represented?
13. Find an expression for three consecutive integers of which a is the least.
14. Find an expression for three consecutive integers of which b is the greatest.
15. Find an expression for three consecutive integers of which c is the one between the other two.
16. If a number represented by x is separated into two parts, one of which is 5, what is the other part?
17. If a number represented by a is separated into two parts one of which is b, what is the other part?
18. Find an expression for the greater of two numbers if the less is l and the difference is d.
19. A man sold a horse for \$$h$ and gained \$$g$ on the cost. Find an expression for the cost of the horse.
20. A boy is 15 years old now. Find an expression for his age x years ago. How old will he be in y years?
21. If a boy is y years old now how old will he be 5 years from now? How old was he three years ago?

22. A man has $\$a$ and spends $\$x$. Find an expression for the sum remaining.

23. In uniform motion, Distance = Rate × Time.

How far can a person walk in a hours at the rate of b miles per hour?

24. How long will a train require to move uniformly a distance of a miles at the rate of x miles per hour?

25. If a man's expenses are $\$x$ per week how much will they be for a year?

26. How much will a man whose wages are $\$a$ per day earn in b days? In cd days?

27. Express $\$a$ in terms of cents.

28. Express b cents in terms of dollars.

29. Express d dimes in terms of dollars.

30. Express $\$a$ and b dimes in terms of cents.

31. Express x half dollars and y quarters in terms of cents.

32. Express y yards in terms of feet.

33. Express f feet in terms of inches.

34. Express h inches in terms of yards.

35. Express m inches in terms of feet.

36. Express x feet plus y inches in terms of inches.

37. Express a yards plus b feet plus c inches in terms of inches.

38. Find an expression for a gallons in terms of quarts. In terms of pints.

39. Find an expression for x pounds in terms of ounces and of t tons in terms of pounds.

40. Find an expression for a gallons plus b quarts in terms of quarts.

41. Find an expression for x pounds plus y ounces in terms of ounces.

42. Find an expression for a acres in terms of square rods.

43. Find an expression for x acres plus y square rods in terms of square rods.

44. Express h hours in terms of minutes.

45. Express a minutes in terms of hours. In terms of seconds.

46. Express a days plus b hours in terms of hours.

47. Express m hours plus n minutes in terms of seconds.

48. What is the cost of m books at n cents each?

49. If b books cost $\$d$ find an expression for the cost of one book.

50. If one book cost c cents how many cents do b books cost? How many dollars?

51. If the interest on \$1 for one year is r cents what will be the interest on $\$d$ for one year at the same rate?

52. Find an expression for one per cent of a; five per cent of b; seventy five per cent of c; thirty-seven and one-half per cent of d.

53. Find an expression for a per cent of b.

54. Find an expression for x per cent of x.

55. Find an expression in square feet for the area of a rectangular room which is a feet long and b feet wide. In square yards.

56. What is the area in square feet of a square room each of whose sides is x feet in length? In square yards?

57. Find an expression in square yards for the area of a rectangular room which is x yards long and y yards wide. In square feet.

EXERCISE XI. 3

Translate the following statements into algebraic language and express the given conditions by means of conditional equations, and solve. The solutions of the equations should in every case be examined to see if they satisfy the conditions of the given problems:

1. Find two numbers whose sum is 46, which are such that the greater number shall exceed the less by 12.

In this problem the values of the two numbers required are unknown, but by the statement we know that if x represents the less number, $x + 12$ must represent the greater.

Since the sum of the numbers is 46, we have

$$(\text{first number}) + (\text{second number}) = 46.$$

Hence, $$x \quad + \quad (x + 12) \quad = 46.$$

Solving, we find that $x = 17$.

Accordingly the less number is 17 and the greater number which is represented by $x + 12$ must be 29.

These numbers satisfy the conditions of the problem.

2. Find a number which, when multiplied by 7, exceeds 36 by as much as the number itself falls short of 36.

Let x represent the required number.

Translating the conditions of the given problem into algebraic language, we obtain the conditional equation

$$7x - 36 = 36 - x,$$

whose solution is found to be $x = 9$, which is the required number.

It will be found that 9 satisfies the conditions of the given problem.

Suggestions for stating and solving simple problems.

First, decide what unknown number is to be found, and represent it by some letter, say x.

It should be observed that the letters appearing in conditional equations can represent abstract numbers only.

E. g. The letter x cannot represent a given sum of money or a given distance, but it may stand for the *number* of dollars in a given sum, or the *number* of units of length, say feet, yards, or miles, in terms of which a given distance is expressed.

It is essential that we should be consistent in expressing the known and unknown numbers of the problem in terms of the same unit.

E. g. If the unknown letter represents the number of miles in a given distance, then the remaining numbers in the conditional equation must be expressed in terms of miles.

Second, examine the statement of the problem to discover two independent conditions which may lead to two different algebraic expressions for the same number.

Third, obtain *two* algebraic expressions for this number.

Fourth, use these expressions as members of a conditional equation, and solve.

Fifth, examine the solutions of the conditional equation, and determine whether or not they satisfy the stated conditions of the problem.

3. Find two numbers whose sum is 28 and whose difference is 6.

4. Find three consecutive numbers whose sum is 42.

Suggestion. Let x represent the first number, $x + 1$ the second number, and $x + 2$ the third number.

5. Find three consecutive numbers whose sum is 96.

6. Find four consecutive numbers whose sum is 206.

7. Find three consecutive even numbers whose sum is 54.

8. Find two consecutive numbers such that seven times the first shall exceed four times the second by 29.

9. Find a set of six consecutive numbers such that the sum of the first and last shall be 95.

10. What number is it whose double is 25 more than its third part?

11. Find two numbers differing by 12 whose sum is twice their difference.

Let x represent the greater number.

Then $x - 12$ represents the less number.

We have $$x + (x - 12) = 2 \cdot 12$$

From which $$x = 18.$$

Accordingly the greater number is 18, and the less number, represented by $x - 12$, is 6.

These numbers will be found to satisfy the conditions of the given problem.

12. Find two numbers whose sum is 36 and whose difference is 10.

13. Separate 63 into two parts such that one part shall be greater than the remaining part by 5.

14. Separate 147 into two parts such that the greater part shall exceed the less by 7 more than $\frac{1}{3}$ of the less part.

15. Separate 72 into two parts such that three-fourths of the less part shall exceed three-eighths of the greater by 9.

16. Find a number such that if one be added to three-fourths of the number the result will be four more than one-half of the number.

17. One-half of a certain number exceeds the sum of its fifth and sixth parts by 4. Find the number.

18. The sum of the third and fourth parts of a certain number exceeds the sum of the fifth and sixth parts by 13. Find the number.

19. If $5x + 4$ stands for 49, for what number will $x - 2$ stand?

On condition that $5x + 4$ shall represent 49, it is necessary that x shall satisfy the conditional equation $5x + 4 = 49$.

The solution of this equation is found to be $x = 9$.

Accordingly, the expression $x - 2$ must represent 7.

20. The sum of the two digits of a number is 11. If 27 be added to the number the digits will be interchanged. What is the number?

Let x represent the digit in units' place.

It follows that $11 - x$ represents the digit in tens' place.

The given number may be represented by

$$10\,(\text{digit in tens' place}) + (\text{digit in units' place}),$$

that is, by $$10(11 - x) + x.$$

If the digits are interchanged, the resulting number will be represented by

$$10x + (11 - x).$$

By the conditions of the problem, we have

$$10(11 - x) + x + 27 = 10x + (11 - x).$$

Solving, we find that $x = 7$.

Hence the digit in units' place is 7, and the digit in tens' place, which is represented by $11 - x$, must be 4.

Accordingly, the required number is $4 \times 10 + 7$, that is, 47.

It may be seen that 47 satisfies the conditions of the problem, for the sum of the digits is 11, and if 27 be added to 47 the resulting number is 74, which may also be obtained by interchanging the digits of 47.

21. The sum of the two digits of a number is 12. If 36 be added to the number the digits will be interchanged. What is the number?

22. The sum of the three digits of a number is 9, and the digit in hundreds' place is three times that in units' place. If 180 be added to the number, the digits in hundreds' and tens' places will be interchanged. What is the number?

23. Find such a number that, when used to diminish each of the two indicated factors of the two unequal products $45 \cdot 75$ and $51 \cdot 66$, the resulting products will be equal.

24. A's age exceeds B's by 14 years. Eight years ago A was three times as old as B. Find the present age of each.

Let b represent the number of years in B's present age.

Then, by the conditions of the problem, the number of years in A's present age is represented by $b + 14$.

The numbers of years in A's and B's ages eight years ago would accordingly be represented by $(b + 14) - 8$ and $b - 8$, respectively.

Since at that time A's age was three times that of B's, we may form the conditional equation

$$(b + 14) - 8 = 3(b - 8).$$

Solving, we obtain $b = 15.$

Accordingly, B's present age is 15 years, and A's present age, which is represented by $b + 14$, must be 29 years.

These numbers are found to satisfy both the statement of the problem and the algebraic equation.

25. The ages of A and B are such that five years ago A's age was four times that of B's, while five years hence his age will be twice that of B's. Find the present age of each.

26. The sum of the ages of A and B is 36 years, and six years hence A's age will be three times that of B's. Find their present ages.

27. In a company of 22 persons a resolution is carried by a majority of 12, all voting. How many voted for the measure?

28. In an informal ballot a resolution was adopted by a majority of six votes, but in a formal vote one-third of those who had before voted for it voted against it, and the resolution was lost by a majority of four votes. How many voted each way in the formal ballot?

29. A grocer estimated that his supply of sugar would last eight weeks. He sold on an average 50 pounds a day more than he expected. It lasted him six weeks. How much did he have?

30. Determine how an amount of $135 must be divided among three persons in such a way that the share of the first shall be three times that of the second, and the share of the second twice that of the third.

Let x represent the number of dollars in the share of the third. Then the number of dollars in the shares of the second and first will be represented by $2x$ and $6x$ respectively.

By the conditions of the problem, we may construct the conditional equation $x + 2x + 6x = 135$, whose solution is found to be $x = 15$.

Accordingly, the share of the third is $15; the share of the second represented by $2x$, is $30; and the share of the first, represented by $6x$, is $90.

These amounts are found to satisfy the conditions of the given problem.

31. A sum of $7924 was bequeathed to three persons with the stipulation that the first was to receive twice as much as the second and one-half as much as the third. Determine the amounts.

32. A man wishes to divide the sum of $99 into five parts in such a way that the first part shall exceed the second by $3, be less than the third part by $10, greater than the fourth part by $9 and less than the fifth part by $16. Find the parts.

33. A paymaster, wishing to use $25,662 on pay-day, requested the paying teller to make up the amount in the following way: A certain number of $100 bills, three times as many fifties, four times as many twenties as fifties, twice as many tens as fifties, three times as many fives as tens, as many twos as tens, as many ones as twos.

How many bills of each denomination were given?

34. At two stations, A and B, on a line of railway, the prices of coal are $3.50 per ton and $4 per ton respectively. If the distance between A and B be 150 miles and coal can be shipped for one-half a cent per ton per mile, find the place on the railway between A and B at which it will be indifferent to a customer whether he buys coal from A or from B.

35. A farmer estimated that his supply of feed for his 50 cows would last only 12 weeks. How many cows must he sell in order that the supply may last 20 weeks?

36. A contractor undertakes to complete a certain amount of work in a given time. By the terms of the contract he is to receive $12 for each day's work during the given time, and is to forfeit $5 for each day taken beyond that time. If the total amount received was $167 for 21 days' work, find the time for the original contract.

37. It was estimated that a certain amount of earth could be excavated by a steam shovel alone in 12 days, or by a gang of laborers alone in 28 days. After being used a certain number of days the shovel was disabled, and the work was then completed by the men, who worked 2 days less than the time during which the shovel had been used. During how many days was the shovel used?

38. Sixty laborers were engaged to remove an embankment. Some of

them were engaged at the rate of $1.10 a day, and the others at the rate of $1.60 a day. The memorandum having been lost, it was required to find how many worked at each rate, if the total amount paid was $80.

39. The help of a certain factory, numbering 316, consists of men and boys. If the weekly pay of each man is $12 and that of each boy $4 find the number of each, if the weekly pay roll amounts to $2688.

40. It is observed that a square room requires one and one-ninth square yards less of carpeting than a rectangular room whose length is one yard longer, and width two feet less, than the side of the square room. Find the area of each of the rooms.

41. A man has $4200 in four banks. He has twice as much in the second bank as in the first, as much in the third as in the first and second together, and twice as much in the fourth as in the first and third together. How much money has he in the fourth bank?

42. A man invests $\frac{1}{5}$ of his capital at 4 per cent, $\frac{1}{4}$ at 3 per cent, and the remainder at $3\frac{1}{2}$ per cent, and thus secures an annual income of $1390. What is his capital?

43. A man invests $\frac{5}{6}$ of his capital at 3 per cent and the rest at $3\frac{1}{2}$ per cent, and thus receives an annual income of $55.50. What is his capital?

44. A man desires to invest his capital of $10,000, partly at 5 per cent and partly at 3 per cent. How must he invest the amounts so that his yearly income shall be at the rate of $4\frac{1}{4}$ per cent?

45. How may $3600 be invested, partly in 5 per cent stock whose market value is eighty cents on a dollar, and the remainder in 6 per cent stock selling for one dollar and twenty cents on a dollar, in order that the income from the two sources may be the same?

46. How long will it take an investment of $5730 to amount to $6589.50 at 3 per cent simple interest?

47. At what time between three and four o'clock are the hands of a watch together?

Let x represent the number of minute spaces passed over by the minute-hand, reckoned from the position occupied by the hand at 3 o'clock, to the point where it overtakes the hour-hand. Then, during the same time, the hour-hand will pass over $x/12$ minute spaces.

Since at 3 o'clock the hour-hand is 15 minute spaces ahead of the minute-hand, it follows that the distance x passed over by the minute-hand must be greater by 15 minute spaces than the distance $x/12$ passed over by the hour-hand.

Accordingly, we have $x - x/12 = 15,$
from which we obtain $x = 16\frac{4}{11}$, which is the number of minutes after 3 o'clock when the hands of the watch are together.

48. At what time between 9 and 10 o'clock are the hands of a watch opposite each other?

49. At what time between 7 and 8 o'clock will the hands of a clock be at right angles to each other?

50. At what time between 5 and 6 o'clock is the minute hand of a watch two minutes in advance of the hour hand?

51. Two trains start at the same time from different stations 400 miles apart. One travels at the rate of 48 miles an hour and the other at the rate of 32 miles an hour. How far does the faster train travel before meeting the slower one?

52. Two hours after a train left a certain station a second train was dispatched, and it overtook the first train in four hours. To accomplish this it was necessary for the second train to run 15 miles an hour faster than the first. How many miles per hour did the trains run?

53. Where must a side track be placed on a single-track railway in order that an express train, travelling at the rate of 46 miles an hour, may not be delayed by an accommodation train travelling toward it at the rate of 29 miles an hour, the two trains starting at the same time from two places 60 miles apart?

54. If an outward trip of an excursion train is made at the rate of 20 miles an hour and the return trip at 16 miles an hour, the whole time being 9 hours, what is the distance?

55. Two ships start from a given port at the same time, one going north at the rate of 11 miles per hour and the other going south at the rate of 7 miles per hour. How long after starting will they at these rates be exactly 108 miles apart?

56. Two ferry boats, whose rates are 15 miles and 12 miles an hour respectively, start simultaneously from opposite shores of a river, three-fourths of a mile wide. Where will they meet?

Problems in Science

57. A certain grade of sulphuric acid is known to be 94 per cent pure. How much distilled water must be added to a gallon of this sulphuric acid in order that the mixture may be 75 per cent pure?

Represent by x the number of gallons of distilled water to be added. Then the number of gallons of the mixture will be represented by $1 + x$.

Since 75 per cent of this diluted solution is pure acid, we have as an expression for the number of gallons of acid $.75(1 + x)$.

By adding water, the amount of acid, which was given as 94 per cent of the original gallon, has been neither increased nor decreased.

Hence we may use these two expressions for the same amount of acid as the members of a conditional equation, as follows:

$$.75(1 + x) = .94,$$

from which we obtain $x = .253\frac{1}{3}$.

Accordingly it is necessary to add $.253\frac{1}{3}$ gallons of distilled water.

It may be seen that this amount of water satisfies the condition of the problem as stated, for by the addition of $.253\frac{1}{3}$ gallons of water the mixture is made to consist of $1.253\frac{1}{3}$ gallons, and 75 per cent of this mixture, that is 94 per cent of one gallon, is pure acid.

58. How much water must be added to a pint of alcohol 85 per cent pure, in order to make the mixture three-fourths water?

59. How much 12 per cent solution of a certain chemical must be added to a gallon of 4 per cent solution to raise it to a 6 per cent solution?

60. How much water must be added to 6 quarts of acid which is 10 per cent of full strength to make the mixture $8\frac{1}{2}$ per cent of full strength?

61. How much water must be added to a gallon of three per cent solution of a certain chemical to reduce it to a one per cent solution?

62. How many ounces of pure silver must be melted with 300 ounces of silver 600 fine in order to make a bar of metal 800 fine?

By 600 fine is meant the number of parts in 1000 which are pure metal.

63. How many ounces each of two bars of silver which are 800 fine and 725 fine respectively must be melted together to make a bar of 60 ounces which shall be 775 fine?

64. How many pounds of pure copper must be melted with 500 pounds of gold 67/72 pure in order to make the composition 9/10 pure gold?

65. It has been said that the crown of Hiero of Syracuse, which was part gold and part silver, weighed 20 pounds in the air and $18\frac{3}{4}$ pounds when weighed while immersed in water. Find how much gold and how much silver it contained, knowing that $19\frac{1}{4}$ pounds of gold and $10\frac{1}{2}$ pounds of silver each lose one pound when weighed immersed in water.

66. The pendulum of a clock swings 364 times in 5 minutes while that of a second clock swings 233 times in 4 minutes. After how long will the second have swung 582 times more than the first?

CHAPTER XII

FACTORS OF RATIONAL INTEGRAL EXPRESSIONS

1. FROM the point of view of multiplication, the numbers or expressions which are multiplied together to form a product are called **factors** of the product.

E. g. Since $2 \times 5 = 10$, 2 and 5 are factors of 10.

Since $2(m + n) \equiv 2m + 2n$, 2 and $m + n$ are factors of $2m + 2n$.

Similarly $a + b$ and $a - b$ are factors of $a^2 - b^2$.

Also, $x + 5$ and $x + 5$ are factors of $x^2 + 10x + 25$.

2. Our present problem of factoring is that of finding two or more numbers or expressions which may be multiplied together to produce a given expression as a product.

3. Since for the present it will seldom be necessary to use factors of given expressions, except such as are altogether integral and rational, we shall in this chapter consider only integral rational expressions, and shall restrict the word "factor" to mean only such a factor as is entirely integral and rational.

4. An integral rational algebraic expression is said to be **prime** if it has no exact integral rational divisors except itself and unity (positive or negative).

E. g. The following expressions are algebraically prime:

$$a,\ b + c,\ x^2 + y^2 + 1.$$

5. An integral rational algebraic expression which is not prime is said to be **composite.**

E. g. The following expressions are composite:

$$a^2,\ b^2 - c^2,\ mx^2 + my^2 + m.$$

6. When completely factored, all of the prime, integral, rational divisors of a given polynomial expression are made to appear, and a chain of additions and subtractions is transformed into one of multiplications.

E. g. The number represented by the chain of additions $10a^2 + 70a + 120$ may be expressed also by the following chain of multiplications:

$$2 \cdot 5\ (a+3)(a+4). \qquad \text{Check. } a = 2.$$

That is, $10a^2 + 70a + 120 \equiv 2 \cdot 5\ (a+3)(a+4).$ $\qquad 300 = 300.$

Hence it appears that the prime, integral, rational factors or divisors of $10a^2 + 70a + 120$ are 2, 5, $a + 3$, and $a + 4$.

7. The separate factors of an expression are never of degree higher than that of the given expression.

8. The whole number of algebraic expressions which may be written is unlimited, and of these some do not admit of being expressed as the product of two or more integral rational factors.

Hence, in many cases it is impossible to tell beforehand whether or not integral rational factors of a given expression exist, or to determine their degrees even if they do.

9. Since factoring is commonly used as a means for abbreviating algebraic work, it is necessary for the beginner to become thoroughly familiar with certain methods which may be employed in elementary work.

10. In examining an expression for factors, it is natural to determine first what factors, if any, are common to all of the terms.

Expressions in which all of the Terms contain a Common Monomial Factor

11. *If some letter is common to all of the terms of an expression it follows that every term, and accordingly the whole expression, is exactly divisible by that letter.*

This follows from the converse of the distributive law of multiplication,

$$xa + xb + xc + \cdot\cdot\cdot\cdot\cdot \equiv x(a + b + c + \cdot\cdot\cdot\cdot\cdot).$$

12. All expressions should be examined first for common factors, for it often happens, after removing a common factor, that the form of the resulting expression is such that some other process may be then applied to simplify further the factors which may thus be obtained.

Ex. 1. Factor $20a^3 + 15a^2 + 10a$.

Observe that the monomial factor $5a$ is found in every term, and that this is all that is common to all of the terms.

Using $5a$ as a divisor, the quotient obtained is $4a^2 + 3a + 2$.

Hence $20a^3 + 15a^2 + 10a \equiv 5a(4a^2 + 3a + 2)$. Check. Let $a = 2$.

$$160 + 60 + 20 = 10\ (16 + 6 + 2)$$
$$240 = 240.$$

It appears that the factors of the given expression are 5, a, and the expression $4a^2 + 3a + 2$.

Ex. 2. Factor $44a^4b^3 - 22a^3b^4$.

Since $22a^3b^3$ is all that is common to both terms, we have

Check. Let $a = 2$, $b = 1$.

$$44a^4b^3 - 22a^3b^4 \equiv 22a^3b^3(2a - b).$$

$$44 \cdot 16 - 22 \cdot 8 = 22 \cdot 8\ (4 - 1)$$
$$528 = 528.$$

The different prime factors of the given expression are 2, 11, a, and b, and the binomial $(2a - b)$.

EXERCISE XII. 1

Obtain factors of the following expressions, verifying all results by numerical checks:

1. $ab + ac$.
2. $bcd + bce$.
3. $3x + 3y + 3z$.
4. $6xy + 12zy + 30wy$.
5. $14x + 14$.
6. $4x^3 + 4x^2$.
7. $5a^2 - 6ab$.
8. $9x^2 - 3x$.
9. $8a^2b - 8ab^2$.
10. $x^5 + x^3 + x^2$.
11. $5a^3b^2 + 3a^2b - 4a^2b^2$.
12. $7x^4 - 14x^3 + 21x^2$.
13. $9a^2c + 18ac^2 - 9ac$.
14. $3a^2bc + 3ab^2c + 3abc^2$.
15. $m^3n^2 - 3m^2n^2 - m^2n^3$.
16. $a^3b^2 - b^2c - b^2d^2 + b^2$.
17. $5a^2cd + 10ac^2e - 5acf$.
18. $14a^3b^2 + 28a^2b^2 - 7a^2b^3$.
19. $20a^2b^2c^3 + 25a^2b^3c^2 - 30a^3b^2c^2$.
20. $38x^3y^4 + 57x^4y^3 - 19x^3$.
21. $34a^3b^3 - 51a^3bc + 85a^2bc^2$.
22. $32x^2y^3z^2 - 48xy^4z^2 + 64x^2y^4z$.
23. $77m^5nr - 99m^5n + 88mrs^2$.
24. $13a^3yx^2 + 15ag^2x^2 - 2a^2g^2x - 9agx$.
25. $ab^2c^2d^2 + a^2bc^2d^2 + a^2b^2cd^2 + a^2b^2c^2d$.
26. $x^3y^3z^2 + x^2y^3z^3 + x^3y^2z^3$.
27. $a^2b^2c^3d^4 + ab^3c^3d^4 + ab^2c^4d^4$.
28. $3x^3y^2z - 6xy^3z^2 + 3x^2yz^3$.
29. $ax^my^m - bx^my^m + cx^my^m - dx^my^m$.
30. $a^{m+1}b^{m+1}c^m - a^mb^{m+1}c^{m+1} + a^{m+1}b^mc^{m+1}$.

13. If all of the terms of a given expression do not contain a common factor, it may be possible to so arrange them in groups that there shall be

Groups of Terms Having a Common Factor

The process for factoring such expressions may be obtained by applying the converse of the law of distribution for multiplication. Consider the identity

$$\begin{aligned}(a+b)(x+y+z) &\equiv (a+b)x+(a+b)y+(a+b)z\\ &\equiv ax+bx+ay+by+az+bz.\end{aligned}$$

By reversing the steps in this process the last expression, which consists of a chain of additions, may be transformed into the original expression, which is the product of two factors.

14. The complete distributed product of the sum of two terms, a and b, multiplied by the sum of three other terms, x, y, and z, different from the first, will contain 2×3 or 6 terms, ax, bx, ay, by, az and bz.

Consequently, when factoring $ax+bx+ay+by+az+bz$, we may reverse the process, and when seeking groups of terms which have common factors, we may look for either three groups of two terms each or two groups of three terms each.

Ex. 1. Factor $(a^2+2)(m-n)-(3a^2-b)(m-n)$.

Since the factor $m-n$ occurs in both terms we may add with respect to $m-n$ as a summand, and obtain

$$\begin{aligned}(a^2+2)(m-n)-(3a^2-b)(m-n) &\equiv (a^2+2-3a^2+b)(m-n)\\ &\equiv (2-2a^2+b)(m-n).\end{aligned}$$

Check. Let $a=2$, $b=3$, $m=5$, and $n=4$.

$$\begin{aligned}(4+2)(5-4)-(12-3)(5-4) &= (2-8+3)(5-4)\\ -3 &= -3.\end{aligned}$$

Ex. 2. Factor $ax-5ay+2bx-10by$.

There is no monomial factor common to all of the terms, but there are groups of terms which have common monomial factors.

Since the expression contains four terms, we may expect to find that it can be separated into a product of two factors, each of which contains two terms.

The factor a is found to be common to the two terms ax and $-5ay$, and the factor $2b$ is common to the two terms $2bx$ and $-10by$.

Separating the monomial factors, a and $2b$, by division, from the terms containing them, we obtain

$$ax - 5ay + 2bx - 10by \equiv a(x - 5y) + 2b(x - 5y).$$

The binomial factor $x - 5y$ is common to the terms $a(x - 5y)$ and $2b(x - 5y)$.

Hence, using $x - 5y$ as a divisor, the expression $a(x - 5y) + 2b(x - 5y)$ may be written in the form $(a + 2b)(x - 5y)$.

The work may be arranged as follows:

$$\begin{aligned} ax - 5ay + 2bx - 10by &\equiv a(x - 5y) + 2b(x - 5y) \\ &\equiv (a + 2b)(x - 5y). \end{aligned}$$

Check.

Let $a = 2$, $b = 3$, $x = 6$, $y = 1$.

$12 - 10 + 36 - 30 = 8 \cdot 1$

$8 = 8.$

The same result can be obtained by grouping the terms ax and $2bx$, and also $-5ay$ and $-10by$.

The student should carry out the work with this grouping.

15. It should be remembered that *an expression is not factored unless it is written as a single product, not as the sum of several products separated by + or − signs.*

Thus, in the example above, the given expression is not factored when written in the form $a(x - 5y) + 2b(x - 5y)$, though certain groups of its terms are factored.

Ex. 3. Factor $a^3 + a^2 + a + 1$.

We have
$$\begin{aligned} a^3 + a^2 + a + 1 &\equiv a^2(a + 1) + (a + 1) \\ &\equiv (a^2 + 1)(a + 1). \end{aligned}$$

Check. $a = 2$.

$8 + 4 + 2 + 1 = 5 \cdot 3$

$15 = 15.$

The student should group the terms a^3 and a, and also a^2 and 1, and obtain the same result.

16. It often happens that an expression contains groups of factors which differ only in sign.

Ex. 4. Factor $x^2(a - 1) - y^2(1 - a)$.

Since $1 - a \equiv -(a - 1)$, we may transform the expression so that both groups of terms shall contain the common factor $a - 1$; we do this by writing

$$\begin{aligned} x^2(a - 1) - y^2(1 - a) &\equiv x^2(a - 1) + y^2(a - 1) \\ &\equiv (x^2 + y^2)(a - 1). \end{aligned}$$

Check.

Let $x = 2$, $a = 3$, $y = 4$.

$4 \cdot 2 - 16(-2) = 20 \cdot 2$

$40 = 40.$

Exercise XII. 2

Obtain factors of the following expressions, checking all results numerically:

1. $ac + ad + bc + bd$.
2. $xy + xw - yz - zw$.
3. $bs - st - bc + ct$.
4. $m^3 - 2m^2 - 3m + 6$.
5. $a^3 - 7a^2 - 3a + 21$.
6. $2m + 3mn + 3n + 2m^2$.
7. $x^2 - 4x + xy - 4y$.
8. $a^3 - a^2 + a - 1$.
9. $y^3 + 4y^2 + 2y + 8$.
10. $m^2 - m - a + am$.
11. $1 + x + y + xy$.
12. $a^2 + abm - 4ab - 4mb^2$.
13. $6x^2 + 3xy - 2ax - ay$.
14. $3a^3 - 15a + 10b - 2a^2b$.
15. $ax + ay + bx + by + cx + cy$.
16. $by - b - cy + c - dy + d$.
17. $mx - my + mz - nx + ny - nz$.
18. $2mn - 2ny - mx + xy + 2n^2 - nx$.
19. $21b - 5a + 3ab - 2ac - 14c - 35$.
20. $xz + x - 5yz - 5y - 6z - 6$.
21. $ax + ay - bx - by + bz - az$.
22. $ax - bx + x - ay + by - y$.
23. $cx - dx + cy - dy + cz - dz$.
24. $am - bn + an - bm + cm + cn$.
25. $abx + bcx + cax + aby + bcy + cay$.
26. $8ax + 15by + 12bx + 10ay$.
27. $a^2c + b^2e + a^2d + b^2d + a^2e + b^2c$.
28. $ax + by + ay + bz + az + bx + cx + cy + cz$.
29. $af - bg + ch + ag - bh + cf + ah - bf + cg$.
30. $ar + bs - ct + as + bt - cr + at + br - cs$.

17. Any identity obtained by the process of multiplication may, by being "read backward," be used as a model form to give a factorization.

Type I. Trinomial Squares

18. From Theorems I and II, Chapter VII. §§ 18, 19, we obtain

$$a^2 \pm 2ab + b^2 \equiv (a \pm b)^2.$$

The type expression $a^2 \pm 2ab + b^2$ consists of two squares, a^2 and b^2, having positive signs; the remaining term $\pm 2ab$ is double the product of the numbers a and b, whose squares are represented by a^2 and b^2.

The sign of the middle term $\pm\ 2ab$ is positive or negative according as the signs of a and b are the same or different.

19. The student should become familiar with expressions of the type $a^2 \pm 2ab + b^2$, and should be able, when any two of the three terms a^2, $\pm\ 2ab$, or b^2 are given, to supply the third term which is necessary to make the expression a trinomial square.

Ex. 1. Supply the term which is necessary to make $a^2 + (\quad) + 25$ a trinomial square.

Assuming that a^2 and 25 are the first and third terms respectively of a trinomial square of the form $a^2 \pm 2ab + b^2$, it appears that the missing middle term $+(\quad)$, represented by $\pm\ 2ab$, may be obtained by finding $\pm$ double the product of the square roots, a and 5, of the first and third terms a^2 and 25 respectively; that is, the missing term is $\pm\ 2 \cdot a \cdot 5 \equiv \pm\ 10a$.

Since the missing middle term is assumed to be positive, the required trinomial square is $a^2 + 10a + 25$.

Mental Exercise XII. 3

Find the terms which if supplied will make the following expressions trinomial squares:

1. $a^2 + (\quad) + 9$.
2. $c^2 + (\quad) + 64$.
3. $d^2 + (\quad) + 100$.
4. $x^2 - (\quad) + 16$.
5. $y^2 - (\quad) + 49$.
6. $z^2 - (\quad) + 81$.
7. $d^2 + (\quad) + 121$.
8. $e^2 - (\quad) + 144$.
9. $4a^2 + (\quad) + 1$.
10. $9b^2 + (\quad) + 1$.
11. $16c^2 + (\quad) + 1$.
12. $36d^2 - (\quad) + 1$.
13. $25g^2 - (\quad) + 1$.
14. $9x^2 + (\quad) + 4$.
15. $16y^2 + (\quad) + 25$.
16. $49z^2 - (\quad) + 4$.
17. $9w^2 - (\quad) + 9$.
18. $16a^2 - (\quad) + 16$.
19. $b^2 + (\quad) + c^2$.
20. $a^2 - (\quad) + x^2$.
21. $4d^2 - (\quad) + x^2$.
22. $9h^2 - (\quad) + k^2$.
23. $x^2 + (\quad) + 4y^2$.
24. $y^2 - (\quad) + 64z^2$.
25. $z^2 - (\quad) + 81w^2$.
26. $1 + (\quad) + x^2$.
27. $1 - (\quad) + y^2$.
28. $1 - (\quad) + 169a^2$.
29. $1 - (\quad) + 196b^2$.
30. $4a^2 + (\quad) + 9n^2$.
31. $1 + (\quad) + 121c^2$.
32. $1 - (\quad) + 225d^2$.

33. $1 + (\quad) + 256\,k^2.$

34. $16\,b^2 + (\quad) + 25\,c^2.$

35. $9\,d^2 - (\quad) + 16\,k^2.$

36. $25\,x^2 + (\quad) + 64\,y^2.$

37. $49\,x^2 + (\quad) + 16\,y^2.$

38. $36\,a^2 - (\quad) + 4\,b^2.$

39. $64\,b^2 - (\quad) + 9\,c^2.$

40. $49\,x^2 + (\quad) + 49\,y^2.$

41. $81\,a^2 + (\quad) + 25\,b^2.$

42. $121\,a^2 - (\quad) + 16\,b^2.$

43. $9\,d^2 + (\quad) + 100\,m^2.$

44. $4\,y^2 + (\quad) + 169\,z^2.$

45. $81\,c^2 - (\quad) + 121\,w^2.$

46. $100\,h^2 - (\quad) + 64\,q^2.$

47. $225\,r^2 - (\quad) + 36\,v^2.$

48. $25\,s^2 - (\quad) + 256\,t^2.$

49. $196\,n^2 + (\quad) + 25\,r^2.$

Ex. 50. $a^2 + 2\,ab + (\quad).$

In order that $2\,ab$ shall be the middle term of a trinomial square of which one of the terms is a^2, the term supplied must be the square of the quotient obtained by dividing $2\,ab$ by two times the square root, a, of the term a^2.

From $2\,ab \div 2\,a$ we obtain b, which is accordingly the term whose square b^2 is required.

Hence the desired trinomial square is $a^2 + 2\,ab + b^2$.

51. $c^2 + 2\,cd + (\quad).$

52. $a^2 + 2\,ac + (\quad).$

53. $b^2 + 2\,bm + (\quad).$

54. $c^2 - 2\,ck + (\quad).$

55. $d^2 + 4\,d + (\quad).$

56. $x^2 + 6\,x + (\quad).$

57. $y^2 + 10\,y + (\quad).$

58. $y^2 - 14\,y + (\quad).$

59. $z^2 + 18\,z + (\quad).$

60. $a^2 - 24\,a + (\quad).$

61. $4\,a^2 + 12\,a + (\quad).$

62. $9\,b^2 + 18\,b + (\quad).$

63. $25\,c^2 + 40\,c + (\quad).$

64. $49\,d^2 - 28\,d + (\quad).$

65. $(\quad) + 2\,m + 1.$

66. $(\quad) + 2\,x + 1.$

67. $(\quad) + 4\,y + 1.$

68. $(\quad) + 10\,z + 1.$

69. $(\quad) + 16\,w + 1.$

70. $(\quad) + 4\,a + a^2.$

71. $(\quad) + 6\,b + b^2.$

72. $(\quad) + 12\,c + c^2.$

73. $(\quad) - 14\,d + d^2.$

74. $(\quad) + 2\,ak + k^2.$

75. $(\quad) + 2\,mn + n^2$

76. $(\quad) - 2\,ay + y^2.$

77. $(\quad) - 4\,az + z^2.$

78. $(\quad) - 10\,by + y^2.$

79. $(\quad) + 12\,b + 4\,b^2.$

80. $(\quad) + 20\,c + 4\,c^2.$

81. $(\quad) - 42\,ab + 9\,b^2.$

82. $(\quad) + 96\,ay + 64\,y^2.$

83. $(\quad) - 90\,bn + 81\,n^2.$

84. $(\quad) - 66\,cx + 9\,x^2.$

85. $(\quad) + 120\,aw + 25\,w^2.$

86. $(\quad) - 130\,hy + 25\,y^2.$

20. Before attempting to factor a trinomial in which two of the terms are squares, the student should examine the middle term to find whether or not it satisfies the conditions for a trinomial square.

Ex. 1. Factor $x^2 + 10x + 25$.

Two of the terms, x^2 and 25 are positive squares, and to be a trinomial square the remaining or "middle term" should be double the product of the square roots of these terms, x and 5, that is $2(x)(5) \equiv 10x$.

Since the middle term is $10x$, we may write

$$x^2 + 10x + 25 \equiv (x + 5)^2.$$

Check. Let $x = 2$.
$4 + 20 + 25 = 7^2$
$49 = 49.$

Ex. 2. Factor $49x^2 - 28xy + 4y^2$.

We find that the terms $49x^2$ and $4y^2$ are the squares of $7x$ and of $2y$ respectively. The remaining term $-28xy$ is double the product of these terms $7x$ and $-2y$.

Hence we have

$$49x^2 - 28xy + 4y^2 \equiv (7x - 2y)^2.$$

Check. Let $x = 2$, $y = 3$.
$196 - 168 + 36 = 8^2$
$64 = 64.$

Ex. 3. Factor $72x^3y^3 - 81x^6 - 16y^6$.

The expression should be placed in "minus parentheses," because the squares $81x^6$ and $16y^6$ should appear as positive numbers to satisfy the type expression $a^2 \pm 2ab + b^2$.

Accordingly,
$$\begin{aligned} 72x^3y^3 - 81x^6 - 16y^6 &\equiv -[81x^6 - 72x^3y^3 + 16y^6] \\ &\equiv -[(9x^3)^2 - 72x^3y^3 + (4y^3)^2] \\ &\equiv -(9x^3 - 4y^3)^2. \end{aligned}$$

Check. $x = 2$, $y = 1$.
$576 - 5184 - 16 = -(72 - 4)^2$
$-4624 = -4624.$

EXERCISE XII. 4

Factor the following expressions, verifying all results by numerical checks :

1. $x^2 + 2x + 1$.
2. $m^2 - 2m + 1$.
3. $z^2 - 2zw + w^2$.
4. $a^2 + 2ak + k^2$.
5. $r^2 - 2rv + v^2$.
6. $q^2 - 2qx + x^2$.
7. $a^2 + 4a + 4$.
8. $b^2 + 8b + 16$.
9. $c^2 + 12c + 36$.
10. $d^2 - 18d + 81$.
11. $25 + 10a + a^2$.
12. $49 - 14y + y^2$.
13. $64 - 16z + z^2$.
14. $100 + 20g + g^2$.

15. $4a^2 + 4a + 1$.
16. $9b^2 + 6b + 1$.
17. $36c^2 + 12c + 1$.
18. $81n^2 - 18n + 1$.
19. $16a^2 + 24ab + 9b^2$.
20. $9c^2 + 30cd + 25d^2$.
21. $25k^2 + 70kw + 49w^2$.
22. $121s^2 + 44st + 4t^2$.
23. $49h^2 - 140hy + 100y^2$.
24. $16g^2 + 72gm + 81m^2$.
25. $121k^2 - 242ks + 121s^2$.
26. $36m^2 + 108mn + 81n^2$.
27. $81d^2 - 126de + 49e^2$.
28. $64h^2 + 320hw + 400w^2$.
29. $225r^2 - 120rs + 16s^2$.
30. $169t^2 - 260tw + 100w^2$.
31. $100a^2 - 280ab + 196b^2$.
32. $256c^2 + 96cd + 9d^2$.
33. $64a^2b^2 + 80abc + 25c^2$.
34. $16x^2 - 56xyz + 49y^2z^2$.
35. $4c^2 - 36cde + 81d^2e^2$.
36. $25a^2b^2 + 40abcd + 16c^2d^2$.
37. $36a^2x^2 - 84abxy + 49b^2y^2$.
38. $9b^2y^2 + 48bcyz + 64c^2z^2$.
39. $49a^2y^2 + 42abxy + 9b^2x^2$.
40. $4a^2z^2 - 20acmz + 25c^2m^2$.
41. $25a^2b^2c^2 - 60abcd + 36d^2$.
42. $81x^2 + 180xyzw + 100y^2z^2w^2$.
43. $(x + y)^2 + 2(x + y) + 1$.
44. $(a - b)^2 - 2(a - b) + 1$.
45. $(m + n)^2 + 6(m + n) + 9$.
46. $(x - y)^2 + 10(x - y) + 25$.
47. $16 + 8(a + b) + (a + b)^2$.
48. $36 - 12(a - x) + (a - x)^2$.
49. $(a + b)^2 + 2(a + b)(x + y) + (x + y)^2$.
50. $(a - b)^2 - 2(a - b)(x - y) + (x - y)^2$.
51. $(b + c)^2 - 8(b + c)(x + y) + 16(x + y)^2$.
52. $9(a + b)^2 + 12(a + b)(c + d) + 4(c + d)^2$.
53. $49(d + k)^2 - 70(d + k)(m + w) + 25(m + w)^2$.

21. If an expression has the form of a trinomial square, $a^2 \pm 2ab + b^2$, we shall call the "middle term," represented by $\pm 2ab$, the **finder term.**

The finder term, $\pm 2ab$, of an integral rational trinomial square $a^2 \pm 2ab + b^2$, contains as factors the terms a and $\pm b$ of the binomial $a \pm b$ of which the trinomial is the square.

We may often, by using some particular term as a finder term, select from a set of four or more terms a group of three terms, which taken together form a trinomial square.

Ex. 1. Factor $x^2 + y^2 - 2x + 2xy + 1 - 2y$.

Since there are three separate squares, x^2, y^2 and 1, and three terms, $2xy$, $-2x$ and $-2y$, each having a coefficient 2, we are led to suspect the existence of one or more trinomial squares among the terms of the given expression.

We may group x^2, y^2, and $2xy$, since the product of the square roots x and y of x^2 and y^2 is doubled to form the "finder term" $2xy$.

We may write,

$$x^2+y^2-2x+2xy+1-2y \equiv x^2+2xy+y^2-2x-2y+1$$
$$\equiv (x+y)^2-2x-2y+1.$$

Since $(x+y)^2$ and 1 are squares, we may look for *twice the product of their square roots*, $(x+y)$ and 1, that is, for $2(x+y)$.

This product is obtained by writing $-2x-2y$ in the form $-2(x+y)$.

Accordingly $(x+y)^2-2x-2y+1$ may be written as $(x+y)^2-2(x+y)+1$, which is the square of $x+y-1$.

Check.

Hence, $x^2+y^2-2x+2xy+1-2y \equiv (x+y-1)^2$.

Let $x=2$, $y=3$.

$$4+9-4+12+1-6=(2+3-1)^2.$$
$$16=16.$$

We may obtain the same result by grouping either x^2, $-2x$ and 1, or y^2, $-2y$ and 1.

$$\text{That is, } x^2+y^2-2x+2xy+1-2y \equiv \begin{cases} \text{either } (x-1)^2+2\,(?)+(?)^2 \\ \text{or } \quad (y-1)^2+2\,(?)+(?)^2 \end{cases}$$
$$\equiv (x+y-1)^2.$$

The student should carry out the solutions as suggested above.

Ex. 2. Factor $m^2+2mn+n^2+2mz+2nz+z^2$.

The terms may be grouped in any one of the ways (1), (2), or (3) as suggested below:

$$\left\{\begin{array}{l}(1)\ m^2+2mn+n^2\\(2)\ m^2+2mz+z^2\\(3)\ n^2+2nz+z^2\end{array}\right. \begin{array}{l}\text{with the cor-}\\\text{responding}\\\text{groups}\end{array} \left\{\begin{array}{l}(1)+2(m+n)z\\(2)+2(m+z)n\\(3)+2(n+z)m\end{array}\right. \begin{array}{l}\text{leaving}\\\text{the}\\\text{squares}\end{array} \left\{\begin{array}{l}(1)+z^2.\\(2)+n^2.\\(3)+m^2.\end{array}\right.$$

Of these three possible arrangements, we will use the first,

$$(1)\ m^2+2mn+n^2+2(m+n)z+z^2 \equiv (m+n)^2+2(m+n)z+z^2$$
$$\equiv [(m+n)+z]^2$$
$$\equiv [m+n+z]^2.$$

Check.

Let $m=2$, $n=3$, $z=1$.

$$4+12+9+4+6+1=(2+3+1)^2$$
$$36=36.$$

The student should show that the adoption of either of the arrangements of the terms, (2) or (3), leads to the same result.

EXERCISE XII. 5

Obtain factors of the following expressions, checking all results numerically:

1. $a^2+2ab+b^2+2a+2b+1$.
2. $m^2+2mn+n^2+10m+10n+25$.

3. $a^2 + b^2 + x^2 + 2\,ab + 2\,ax + 2\,bx.$
4. $x^2 + 6\,xy + 9\,y^2 + 2\,x + 6\,y + 1.$
5. $c^2 + d^2 + e^2 - 2\,cd - 2\,ce + 2\,de.$
6. $c^2 + d^2 + 4 - 2\,cd + 4\,c - 4\,d.$
7. $x^2 + y^2 + z^2 + 2\,zx - 2\,zy - 2\,xy.$
8. $4\,a^2 + b^2 + 1 + 4\,ab + 4\,a + 2\,b.$
9. $9\,a^2 + 4\,b^2 + c^2 - 12\,ab - 6\,ac + 4\,bc.$
10. $a(a + 2\,k) + w(w + 2\,a) + k(k + 2\,w).$
11. $b(b + 2\,c) + c(c + 2\,d) + d(d + 2\,b).$
12. $a(a - 2\,b) + b(b + 2\,c) + c(c - 2\,a).$
13. $(x^2 + 2\,yz) + (y^2 + 2\,xy) + (z^2 + 2\,xz).$
14. $a^2 + b^2 + c^2 - 2(ab - ac + bc).$
15. $4x^2 + y^2 + 9\,z^2 + 2(2\,xy - 6\,xz - 3\,yz).$

Type II. The Difference of Two Squares
$x^2 - y^2$

22. From the converse of the identity in Chap. VII. § 20, we have

$$x^2 - y^2 \equiv (x + y)(x - y).$$

Or, *the difference of the squares of two numbers may be written as the product of the sum and difference of the numbers.*

Ex. 1. Factor $64\,x^2 - 9$.

Since $64\,x^2$ is the square of $8\,x$, and 9 is the square of 3, we may write as factors of $64\,x^2 - 9$ the product of the sum $8\,x + 3$ and the difference $8\,x - 3$ of these same numbers.

Check. Let $x = 2$.

That is, $$64\,x^2 - 9 \equiv (8\,x + 3)(8\,x - 3).$$

$256 - 9 = 19 \cdot 13$

$247 = 247.$

Ex. 2. Factor $100\,a^4b^2c^6 - 25\,d^2$.

The terms of the binomial quotient $4\,a^4b^2c^6 - d^2$, obtained by dividing the terms of the given expression by the common numerical factor 25, are squares.

Hence, the binomial difference, $4\,a^4b^2c^6 - d^2$, may be expressed as the product of the sum $2\,a^2bc^3 + d$, multiplied by the difference $2\,a^2bc^3 - d$.

Hence, we have $100\,a^4b^2c^6 - 25\,d^2 \equiv 25\,[4\,a^4b^2c^6 - d^2]$

$\equiv 25\,(2\,a^2bc^3 + d)(2\,a^2bc^3 - d).$

Check. Let $a = 3$, $b = 2$, $c = 1$, $d = 4$.

$32000 = 32000.$

Mental Exercise XII. 6

Factor each of the following expressions :

1. $a^2 - 4$.
2. $b^2 - 16$.
3. $c^2 - 25$.
4. $d^2 - 49$.
5. $h^2 - 81$.
6. $25 - x^2$.
7. $36 - y^2$.
8. $64 - z^2$.
9. $81 a^2 - 1$.
10. $49 b^2 - 1$.
11. $100 a^2 - b^2$.
12. $4 c^2 - 25 d^2$.
13. $9 r^2 - 16 w^2$.
14. $16 q^2 - 49 t^2$.
15. $25 g^2 - 64 m^2$.
16. $1 - 121 n^2$.
17. $1 - 169 k^2$.
18. $144 h^2 - 25 t^2$.
19. $9 h^2 - 100 q^2$.
20. $64 a^2b^2 - 25$.
21. $49 x^2y^2 - 64$.
22. $81 - 36 m^2w^2$.
23. $36 - 81 c^2t^2$.
24. $25 p^2 - 144 x^2w^2$.
25. $100 a^2b^2 - 81 c^2d^2$.
26. $196 x^2y^2 - 100 z^2w^2$.
27. $169 a^2c^2 - 225 b^2d^2$.
28. $256 a^2d^2 - 144 b^2c^2$.
29. $289 a^2x^2 - 49 b^2y^2$.
30. $324 a^2n^2 - 64 c^2w^2$.
31. $121 x^4 - 1$.
32. $64 x^6 - 9$.
33. $25 y^8 - 16$.
34. $36 z^{10} - 49$.
35. $64 k^{12} - 100$.
36. $81 - 121 a^{14}$.
37. $144 b^2 - 121 c^4$.
38. $16 a^2 - 81 b^6$.
39. $49 x^8 - 36 y^{12}$.
40. $9 a^{2n} - 25$.
41. $x^{2m} - 64 y^{2r}$.
42. $49 a^{2s} - 100 b^{2t}$.
43. $180 a^{2n} - 121 n^2$.
44. $64 a^{2r} - 49 r^2$.
45. $x^{2x} - 1$.
46. $4 a^2b^4c^6 - 1$.
47. $9 x^8y^4z^2 - 1$.
48. $16 a^{10}b^6c^4 - d^2$.
49. $25 a^{10}b^8c^6 - d^4e^2$.
50. $m^2a^{2n} - 1$.
51. $r^2s^{2r} - t^2$.
52. $a^{2b} - b^{2a}$.
53. $4 x^{2x} - 1$.
54. $9 a^{2a} - b^2$.
55. $16 x^{2y} - y^2$.
56. $4 a^{4y} - 9$.

23. The difference of the squares of either binomials or polynomials may be factored by applying the method under consideration.

Ex. 1. Factor $(a+b)^2-(c+d)^2$.

We have

$$\begin{aligned}(a+b)^2-(c+d)^2 &\equiv [(a+b)+(c+d)][(a+b)-(c+d)]\\ &\equiv [a+b+c+d][a+b-c-d].\end{aligned}$$

Check. Let $a=b=3$, $c=d=2$.

$$36-16=10\times 2.$$
$$20=20.$$

Ex. 2. Factor $(x-y)^2-(m-n)^2$.

We have

$$\begin{aligned}(x-y)^2-(m-n)^2 &\equiv [(x-y)+(m-n)][(x-y)-(m-n)]\\ &\equiv [x-y+m-n][x-y-m+n].\end{aligned}$$

Check. Let $x=4$, $y=2$, $m=3$, $n=2$.

$$2^2-1^2=3\cdot 1$$
$$3=3.$$

EXERCISE XII. 7

Obtain factors of the following expressions, checking all results numerically:

1. $(a+b)^2-1$.
2. $(a+b)^2-4c^2$.
3. $(x-y+z)^2-1$.
4. $(a+b-c)^2-9d^2$.
5. $(a+b)^2-(x-y+z)^2$.
6. $(a-b)^2-(x+y-z)^2$.
7. $(x-y-z)^2-(a+b)^2$.
8. $(a+b-c)^2-9(x-y)^2$.
9. $9(x+y)^2-16(z+w)^2$.
10. $9(a-b)^2-49(c-d)^2$.
11. $4x^2-49(a+b+c)^2$.
12. $64(g+h)^2-121(k+r)^2$.
13. $36(g-x)^2-169(h-y)^2$.
14. $225(a+b)^2-225(c+d)^2$.
15. $49(a^2+b)^2-144(a+b^2)^2$.
16. $16(a^2+b^2)^2-121(a+b)^2$.
17. $64(a+b+c)^2-169d^2$.
18. $121(a-b+c)^2-225d^2$.
19. $25(a-b+c)^2-81(x-y-z)^2$.
20. $100x^2y^2z^2-196a^2(b+c)^2$.
21. $121a^2(b+c)^2-81$.
22. $169b^2(c+d)^2-196e^2$.
23. $196x^2(y-z)^2-225w^2$.
24. $36c^2(d-e)^2-121x^2(y+z)^2$.
25. $144x^2(a+b+c)^2-225d^2$.
26. $64x^2(m-n)^2-225k^2z^2$.
27. $81a^2(m+n)^2-256b^2(x+y-z)^2$.

24. Many polynomial expressions may be so transformed, by suitably grouping the terms, as to appear as the difference of two squares.

Ex. 1. Factor $a^2+2ab+b^2-c^2+2cd-d^2$.

The terms may be so grouped that the expression will appear as the difference of two trinomial squares, as follows :

$$\begin{aligned} a^2 + 2\,ab + b^2 - c^2 + 2\,cd - d^2 &\equiv (a^2 + 2\,ab + b^2) - (c^2 - 2\,cd + d^2) \\ &\equiv (a+b)^2 - (c-d)^2 \\ &\equiv [\,(a+b) + (c-d)][(a+b) - (c-d)] \\ &\equiv [a + b + c - d][a + b - c + d]. \end{aligned}$$

Check. Let $a = 4$, $b = 3$, $c = 2$, $d = 1$.

$$16 + 24 + 9 - 4 + 4 - 1 = 8 \cdot 6$$

$$48 = 48.$$

Exercise XII. 8

Factor the following expressions, checking all results numerically :

1. $x^2 + 2\,xy + y^2 - z^2$.
2. $m^2 + 2\,mn + n^2 - w^2$.
3. $a^2 - 2\,ab + b^2 - y^2$.
4. $c^2 - 2\,cd + d^2 - k^2$.
5. $a^2 + 2\,ab + b^2 - 4$.
6. $c^2 - 2\,cd + d^2 - 9$.
7. $a^2 + 2\,ab + b^2 - 1$.
8. $g^2 - 2\,gk + k^2 - 4\,m^2$.
9. $h^2 - 2\,hx + x^2 - 16\,y^2$.
10. $a^2 + 6\,ab + 9\,b^2 - c^2$.
11. $x^2 - 10\,xy + 25\,y^2 - z^2$.
12. $m^2 - 12\,mn + 36\,n^2 - t^2$.
13. $1 - a^2 - 2\,ab - b^2$.
14. $1 - x^2 - 2\,xy - y^2$.
15. $4 - c^2 - 2\,cd - d^2$.
16. $9 - m^2 + 2\,mn - n^2$.
17. $25\,a^2 - 10\,ab + b^2 - s^2$.
18. $c^2 - 18\,c + 81 - d^2$.
19. $h^2 - 20\,h + 100 - k^2$.
20. $16\,a^2 - 8\,ab + b^2 - 16$.
21. $49\,d^2 - 14\,dr + r^2 - 49$.
22. $81\,p^2 - 36\,px + 4\,x^2 - 1$.
23. $1 - 9\,a^2 - 30\,ab - 25\,b^2$.
24. $49\,a^2 - 28\,ab + 4\,b^2 - 9\,c^2$.
25. $36\,d^2 + 60\,dz + 25\,z^2 - 121\,w^2$.
26. $64\,c^2 - 48\,cn + 9\,n^2 - 16\,v^2$.
27. $a^2b^2 + 28\,ab + 196 - 169\,c^2$.
28. $c^2g^2 - 24\,cg + 144 - 121\,k^2$.
29. $a^2 + 2\,ab + b^2 - c^2 - 2\,cd - d^2$.
30. $c^2 - 2\,ch + h^2 - m^2 - 2\,mr - r^2$.
31. $k^2 + 2\,kn + n^2 - r^2 + 2\,rs - s^2$.
32. $h^2 - 2\,ht + t^2 - n^2 + 2\,ny - y^2$.
33. $9\,a^2 + 6\,ab + b^2 - c^2 - 4\,cd - 4\,d^2$.
34. $x^2 - 10\,xy + 25\,y^2 - 36\,z^2 - 12\,zw - w^2$.
35. $a^2 - 14\,ab + 49\,b^2 - x^2 - 16\,xy - 64\,y^2$.
36. $100\,b^2 - 20\,bd + d^2 - y^2 + 18\,yz - 81\,z^2$.
37. $48\,a^2 - 3\,b^2$.
38. $7\,x^2 - 63\,y^2$.
39. $45 - 125\,a^2b^2$.
40. $28\,x^2y^2 - 7\,(c + d)^2$.
41. $ax^3y - axy^3$.
42. $x^2 - y^2 + xz + yz$.
43. $x^2 - y^2 + x - y$.
44. $a^3b - ab^3 + a^2b + ab^2$.

25. An expression of the form $x^4 \pm hx^2y^2 + y^4$, where h has such a value that the trinomial is not a square, may be transformed into one by adding a square represented by $k^2x^2y^2$, and combining the term thus introduced with $\pm hx^2y^2$.

By subtracting $k^2x^2y^2$, the value of the transformed expression will become equal to that of the given expression, but will appear as the difference of two squares, in which form it may be factored.

26. The value of the term $k^2x^2y^2$, used in making the transformation, is obtained by subtracting the given term $\pm hx^2y^2$ of the expression $x^4 \pm hx^2y^2 + y^4$, which is not a trinomial square, from the required middle term of the trinomial square of which x^4 and y^4 are the first and third terms respectively.

Ex. 1. Factor $x^4 + 9x^2 + 25$.

If x^4 and 25 are the first and third terms of a trinomial square, the middle term must be $\pm 10x^2$.

Subtracting the given middle term, $9x^2$, of the expression $x^4 + 9x^2 + 25$ which is not a trinomial square, from the required middle term $\pm 10x^2$ of the required trinomial square $x^4 + 10x^2 + 25$, we obtain x^2 and also $-19x^2$.

Using the square x^2, we may transform the given expression and obtain its factors as follows:

$$\begin{aligned} x^4 + 9x^2 + 25 &\equiv x^4 + 10x^2 + 25 - x^2 \\ &\equiv (x^2+5)^2 - x^2 \\ &\equiv (x^2+5+x)(x^2+5-x). \end{aligned}$$

Check. Let $x = 1$.
$1 + 9 + 25 = 7 \cdot 5$
$35 = 35$.

If, by adding and subtracting $-19x^2$, the given trinomial had been so transformed as to have a negative middle term, $-10x^2$, the transformed expression would have appeared as a sum $(x^2+5)^2 + 19x^2$, and in this form would have had no "real" factors.

Ex. 2. Factor $9x^4 - 12x^2y^2 + 16y^4$.

A trinomial square of which $9x^4$ and $16y^4$ are the first and third terms must have for a middle term $\pm 2(3x^2)(4y^2) \equiv \pm 24x^2y^2$.

Subtracting the given middle term $-12x^2y^2$ from the required middle term $\pm 24x^2y^2$, we obtain $36x^2y^2$, and also $-12x^2y^2$.

Using the square, $36x^2y^2$, we may transform the given expression and obtain its factors as follows:

$$\begin{aligned} 9x^4 - 12x^2y^2 + 16y^4 &\equiv 9x^4 + 24x^2y^2 + 16y^4 - 36x^2y^2 \\ &\equiv (3x^2+4y^2)^2 - (6xy)^2 \\ &\equiv [3x^2+4y^2+6xy][3x^2+4y^2-6xy]. \end{aligned}$$

Check.
$x = y = 1$.
$13 = 13$.

The student should explain the reason for not using $-24x^2y^2$ as a "middle term" in the transformed trinomial above.

EXERCISE XII. 9

Factor the following expressions, checking all results numerically:

1. $x^4 + x^2y^2 + y^4$.
2. $x^8 + x^4y^4 + y^8$.
3. $x^4 + x^2 + 1$.
4. $a^4 - 7a^2 + 1$.
5. $x^4 + 3x^2 + 36$.
6. $a^4 - 14a^2 + 1$.
7. $a^4 + 9a^2 + 25$.
8. $4x^4 - 13x^2 + 1$.
9. $25a^4 - 11a^2 + 1$.
10. $x^4 - 13x^2y^2 + 4y^4$.
11. $a^4 + 24a^2b^2 + 196b^4$.
12. $4a^4 + 16a^2 + 25$.
13. $4x^4 - 8x^2y^2 + y^4$.
14. $9x^4 + 3x^2y^2 + 4y^4$.
15. $9k^4 - 21k^2 + 4$.
16. $100a^4 - 49a^2 + 4$.
17. $36c^4 - 40c^2 + 9$.
18. $36m^4 + 116m^2n^2 + 121n^4$.
19. $25 + 9m^4 + 26m^2$.
20. $25a^4 + 101a^2b^2 + 121b^4$.
21. $49x^4 + 64y^4 + 87x^2y^2$.
22. $25m^4 + 36n^4 + 35m^2n^2$.
23. $25a^8 + 81b^8 + 41a^4b^4$.
24. $x^{12} - 34x^6y^6 + 81y^{12}$.

TYPE III. Trinomials of the Type $x^2 + sx + p$

27. From the distributive law for multiplication, we have

$$(x + a)(x + b) \equiv x^2 + (a + b)x + ab.$$

By comparing this result with the type expression, it appears that the coefficient, $a + b$, of x corresponds to the coefficient s in the type expression, and the constant term ab corresponds to the term p.

Hence, reading the identity backward, the necessary and sufficient conditions that the trinomial $x^2 + sx + p$ may be factored, are that the first term should be a positive square with the coefficient 1, and that the coefficient, s, of x should be the sum of two numbers whose product is the constant term p.

28. *To factor trinomials of the type $x^2 + sx + p$, examine first the product term p, and separating it into pairs of factors, select that pair whose sum is the coefficient, s, of x.*

Then write two binomial factors, each having x for a first term, and for a second term one of the factors of the pair whose sum is s.

Ex. 1. Factor $x^2 + 7x + 12$.

This expression is of the type $x^2 + sx + p$, because the first term, x^2, is positive, with the coefficient 1; the second term, $7x$, contains x to the first power; and the third term, 12, is free from x.

We find that the only possible pairs of integral factors of 12, whose product is 12, are the numbers in vertical columns:

$$\left\{\begin{array}{lll} 12, & 6, & 4. \\ 1, & 2, & \underline{3.} \\ & & 7. \end{array}\right.$$

The sum of the factors 4 and 3 is 7, which is the coefficient of x in the term $7x$. Check. $x = 2$. $4 + 14 + 12 = 6 \cdot 5$

Hence, we may write, $x^2 + 7x + 12 \equiv (x + 4)(x + 3)$. $30 = 30$.

29. Whenever the double signs ± and ∓ (read "plus or minus" and "minus or plus" respectively) are used in algebra before numbers which are *not separated by an equality sign*, it is agreed that, whenever we take the upper or lower sign of either double sign, we must take the corresponding sign of the other double sign. (See also Chap. XXII, § 12.)

We must never take the upper sign of one double sign and the lower sign of the other in the same calculation.

$$\text{E.g. } 5 \pm 3 \mp 2 \text{ means} \begin{cases} \text{either } 5 + 3 - 2 = 6, \\ \text{or } 5 - 3 + 2 = 4. \end{cases}$$

$$\text{But it means} \begin{cases} \text{neither } 5 + 3 + 2 = 10, \\ \text{nor } 5 - 3 - 2 = 0. \end{cases}$$

Ex. 2. Factor $x^2 + 8x - 20$.

Comparing with the type expression $x^2 + sx + p$, it appears that $x^2 + 8x - 20$ satisfies the conditions for form, and that of any pair of factors of -20 whose product is -20, one factor must be positive and the other negative.

As possible pairs of integral factors of -20, we have:

$$\left\{\begin{array}{lll} \pm 20, & \pm 10, & \pm 5. \\ \mp 1, & \underline{\mp 2,} & \mp 4. \\ & \pm 8 & \end{array}\right.$$

The coefficient, 8, of x, of the term $8x$, is the sum of the factors 10 and -2, which are found in the second column. Check. $x = 3$.

Hence we may write, $x^2 + 8x - 20 \equiv (x + 10)(x - 2)$. $9 + 24 - 20 = 13 \cdot 1$

$13 = 13$.

Ex. 3. Factor $x^2 - 9x + 14$.

The expression $x^2 - 9x + 14$ is of the type $x^2 + sx + p$, on condition that -9 is represented by s and 14 by p.

Since the product term, 14, is positive, the signs of the factors of the pair whose product is 14 and whose sum is -9, must be like, and to produce the sum -9, these numbers must both be negative.

Hence, the possible pairs of factors of 14, whose product is 14, are

$$\begin{cases} -14, & -7. \\ -\ \ 1, & \underline{-2.} \\ & -9. \end{cases}$$

The sum of the factors -7 and -2 is -9, which is the coefficient of x in the term $-9x$.

Hence we may write, $x^2-9x+14 \equiv (x-7)(x-2)$.

Check. $x=3$.
$9-27+14=-4\cdot 1$
$-4=-4$.

Ex. 4. Factor $(a+b)^2+5(a+b)+6$.

This expression is of the type x^2+sx+p, on condition that the binomial $a+b$ in the given expression is represented by x in the type expression.

Hence, we have,

$$\begin{aligned}(a+b)^2+5(a+b)+6 &\equiv [(a+b)+3][(a+b)+2] \\ &\equiv (a+b+3)(a+b+2).\end{aligned}$$

Check.
Let $a=b=2$.
$42=42$.

30. If the sign of the coefficient of the highest power of the letter with respect to which a given expression is to be factored is negative, we may, by placing the given expression in minus parentheses, so transform it as to obtain an expression in which the coefficient of the highest power is positive.

In this form it may be treated by the methods for factoring expressions of the form x^2+sx+p.

Ex. 5. Factor $12+4x-x^2$.

We have
$$\begin{aligned}12+4x-x^2 &\equiv -[x^2-4x-12] \\ &\equiv -(x-6)(x+2) \\ &\equiv (6-x)(2+x).\end{aligned}$$

Check. Let $x=2$.
$12+8-4=4\cdot 4$
$16=16$.

Exercise XII. 10

Obtain factors of the following expressions, checking all results numerically:

1. a^2+3a+2.
2. $x^2+9x+14$.
3. $a^2+7a+10$.
4. $x^2+8x+15$.
5. $x^2+10x+24$.
6. $x^2+11x+18$.
7. $x^2-12x+20$.
8. $a^2-15a+26$.
9. $x^2+7x-18$.
10. $a^2+12a-45$.

11. $x^2 + 5x - 24$.
12. $m^2 + 9m + 8$.
13. $a^2 + 8a - 9$.
14. $x^2 - 9x + 20$.
15. $x^2 + 19x + 18$.
16. $m^2 - m - 110$.
17. $c^2 + 10c - 39$.
18. $x^2 - 11x - 12$.
19. $a^2 - 23a + 60$.
20. $a^2 - 18a + 32$.
21. $m^2 + 3m - 54$.
22. $m^2 + 15m - 34$.
23. $a^2 + 17a + 42$.
24. $a^2 + 18a + 77$.
25. $a^2 + 23a + 90$.
26. $x^2 - 25x + 100$.
27. $c^2 + 50c + 600$.
28. $a^2 + 17a - 38$.
29. $c^2 - 28c + 75$.
30. $a^2 - 26a + 88$.
31. $c^2 + 44c - 45$.
32. $c^2 + 22c + 112$.
33. $x^2 + 14x + 13$.
34. $x^2 + 16x + 55$.
35. $y^2 + 19y + 60$.
36. $z^2 + 32z + 112$.
37. $d^2 - 13d - 68$.
38. $s^2 + 11s - 180$.
39. $w^2 + 35w + 300$.
40. $w^2 + 5w - 300$.
41. $c^2 - 15c - 250$.
42. $a^2b^2 + 9ab + 20$.
43. $m^2n^2 - 4mn - 140$.
44. $b^2c^2 + 26bc + 165$.
45. $c^2y^2 + 36cy + 320$.
46. $a^2k^2 + 39ak + 380$.
47. $a^2m^2 - 3am - 340$.
48. $(x + y)^2 + 7(x + y) + 10$.
49. $(x + y)^2 + 11(x + y) + 28$.
50. $(a - b)^2 + 13(a - b) + 42$.
51. $(x - y)^2 + (x - y) - 72$.
52. $(a - b)^2 - 7(a - b) - 18$.

31. Certain expressions of the form $x^{2m} + sx^m + p$, containing only two powers of a particular letter, and one of the powers, x^{2m}, being the square of the other, x^m, may be written in the form $(x^m)^2 + s(x^m) + p$, and factored.

Ex. 1. Factor $x^{10} + 6x^5 + 8$.

Since x^{10} is the square of x^5, we may write,

$$\begin{aligned} x^{10} + 6x^5 + 8 &\equiv (x^5)^2 + 6(x^5) + 8 \\ &\equiv (x^5 + 4)(x^5 + 2). \end{aligned}$$

Check. Let $x = 1$.

$$1 + 6 + 8 = 5 \cdot 3$$
$$15 = 15.$$

Ex. 2. Factor $x^6 + 6x^3 - 27$.

Observe that x^6 is the square of x^3, and that the factors 9 and -3 of -27 produce by addition the coefficient, $+6$, of x^2.

Hence we may write,

$$x^6 + 6x^3 - 27 \equiv (x^3 + 9)(x^3 - 3).$$

Check. Let $x = 2$.

$$64 + 48 - 27 = 17 \cdot 5$$
$$85 = 85.$$

Ex. 3. Factor $25 m^2 + 15m + 2$.

Observe that $25 m^2 \equiv (5 m)^2$, and that we may write

$$25 m^2 + 15 m + 2 \equiv (5 m)^2 + 3(5 m) + 2.$$

The expression $(5 m^2) + 3 (5 m) + 2$ is of the type $x^2 + sx + p$; $5 m$ corresponding to x, 3 to s, and 2 to the constant term p.

Hence, since the sum of the factors 2 and 1 of the constant term 2, is 3, we may write

$$\begin{aligned} 25 m^2 + 15 m + 2 &\equiv (5 m)^2 + 3(5 m) + 2 \\ &\equiv (5 m + 2)(5 m + 1). \end{aligned}$$

Check. Let $m = 2$.
$100 + 30 + 2 = 12 \cdot 11,$
$132 = 132.$

32. Expressions of the form $x^2 + sxy + py^2$, which are homogeneous and of the second degree with reference to the letters x and y, may be factored by the methods employed for factoring $x^2 + sx + p$, by separating the term py^2 into two factors, each containing y, the sum of which is the coefficient, sy, of x in the term sxy.

Ex. 4. Factor $x^2 + 14 xy + 33 y^2$.

The term $33 y^2$ is the product of the factors $11 y$ and $3 y$, the sum of which is the coefficient, $14 y$, of x in the term $14 xy$.

Hence we have $x^2 + 14 xy + 33 y^2 \equiv (x + 11 y)(x + 3 y)$.

Check.
$x = 2, y = 3.$
$385 = 385.$

Exercise XII. 11

Obtain factors of the following expressions, checking all results numerically :

1. $x^4 + 15 x^2 + 56$.
2. $a^4 + 9 a^2 + 18$.
3. $a^6 + 19 a^3 + 90$.
4. $b^8 + 5 b^4 + 6$.
5. $x^{10} + 22 x^5 + 40$.
6. $x^4 + 2 x^2 - 15$.
7. $x^6 - 18 x^3 + 72$.
8. $x^{2n} + 11 x^n + 30$.
9. $x^{2m} - 14 x^m + 48$.
10. $x^{2a} - 2 x^a - 35$.
11. $4 a^2 + 16 a + 7$.
12. $4 x^2 + 20 x + 9$.
13. $4 y^2 - 28 y + 13$.
14. $9 b^2 - 18 b + 8$.
15. $25 c^2 - 35 c + 12$.
16. $49 d^2 + 14 d - 3$.
17. $16 h^2 + 40 h + 9$.
18. $81 k^2 - 45 k - 14$.
19. $36 x^2 + 24 x - 5$.
20. $64 y^2 - 80 y - 11$.
21. $a^2 + 15 ab + 26 b^2$.
22. $x^2 - 12 xy + 27 y^2$.

23. $b^2 - 19bc + 34c^2$.
24. $x^2 - 21xy + 38y^2$.
25. $c^2 - 12cd - 28d^2$.
26. $m^2 + 10mn - 39n^2$.
27. $x^2 + 21xy - 46y^2$.
28. $a^2 - 16ac - 57c^2$.
29. $x^2 + 10xy - 56y^2$.
30. $h^2 - 13hk - 68k^2$.

TYPE IV. Trinomials of the Type $ax^2 + bx + c$

33. The expression $ax^2 + bx + c$ is called the *general expression of the second degree* with reference to a single variable, x, because by assigning numerical values to a, b, and c, the expression $ax^2 + bx + c$ may be made to represent any expression of the second degree with reference to a single variable, x.

We will show two methods for factoring expressions of the type

$$ax^2 + bx + c.$$

34. First Method. A method for obtaining the factors of an expression of the type $ax^2 + bx + c$ may be obtained as follows:

If the expression $ax^2 + bx + c$ be assumed to be the product of two binomial factors of the forms $mx + n$ and $m'x + n'$, the product $(mx + n)(m'x + n')$, which may be expressed as

$$mm'x^2 + (mn' + m'n)x + nn' \text{ must represent } ax^2 + bx + c.$$

It should be observed that the term $(mn' + m'n)x$ is the sum of the two factors $mn'x$ and $m'nx$ of the product $mm'nn'x^2$, obtained by multiplying the term $mm'x^2$ by the term nn' which is free from x.

It follows that a given expression of the type $ax^2 + bx + c$ can be factored by inspection, provided that the product acx^2 can be separated into two factors of which the sum is bx.

To factor a trinomial expression of the type $ax^2 + bx + c$, *multiply the term* ax^2 *by the term* c, *and find two factors of the product* ax^2c *(each containing* x*) of which the sum is the middle term* bx.

The required factors of the trinomial expression may then be obtained by grouping the terms of the polynomial expression thus obtained.

Ex. 1. Factor $4a^2 + 15a + 14$.

The product obtained by multiplying the term $4a^2$ by 14 is $56a^2$; and $56a^2$ is the product of two terms, $8a$ and $7a$, of which the sum is the middle term, $15a$.

Hence we have,

$$\begin{aligned} 4a^2+15a+14 &\equiv 4a^2+8a+7a+14 && \text{Check. } a=2.\\ &\equiv 4a(a+2)+7(a+2) && 16+30+14=15\cdot 4\\ &\equiv (4a+7)(a+2). && 60=60.\end{aligned}$$

Ex. 2. Factor $6b^2-13b-8$.

Multiplying $6b^2$ by -8, we obtain $-48b^2$ which is the product of the factors $3b$ and $-16b$ of which the sum is the middle term, $-13b$.

Hence, we have,

$$\begin{aligned} 6b^2-13b-8 &\equiv 6b^2+3b-16b-8 && \text{Check. } b=2.\\ &\equiv 3b(2b+1)-8(2b+1) && 24-26-8=-2\cdot 5\\ &\equiv (3b-8)(2b+1). && -10=-10.\end{aligned}$$

35. Second or Trial Method. Certain expressions of the type ax^2+bx+c may be factored by inspection as follows:

Separate the term ax^2 into all possible pairs of factors, each factor containing x; also separate c into all possible pairs of factors.

For the first terms of the required binomial factors of ax^2+bx+c, use the factors of one of the pairs of factors whose product is ax^2, and for the second terms of these binomial factors use the factors of one of the pairs of factors whose product is c.

The different pairs of factors selected for first and second terms of the required binomial factors must be so chosen that, when the binomials are arranged as multiplicand and multiplier, the sum of the cross products shall be the middle term bx of the expression ax^2+bx+c.

36. For convenience, when applying this method, we shall suppose that a given expression of the type ax^2+bx+c has been so transformed that the number represented by a is positive. The binomial factors may then be so chosen that their first terms will be positive. Hence, when selecting the terms for these binomial factors, it is necessary to consider only the signs of the second terms.

37. If the third term, c, of an expression of the type ax^2+bx+c be positive, the signs of any pair of factors of c must be like. Hence the signs of the second terms of the required binomial factors are the same, and must be like the sign of the middle term of the given trinomial which is the sum of the cross products containing the factors of c.

If the third term, c, be negative, the signs of any pair of its factors must be unlike. Accordingly, the signs of the second terms of the required binomial factors are unlike.

The signs of these second terms must be so taken that, when the required binomial factors are used as multiplicand and multiplier, the sign of the greater cross product is like the sign of the middle term of the given trinomial.

Ex. 3. Factor $6x^2 + 13x + 5$.

This expression is of the type $ax^2 + bx + c$, 6 corresponding to a, 13 to b, and 5 to c. The "trial" pairs of factors of $6x^2$ and 5 may be arranged as follows:

For first terms of the binomial factors,	For second terms of the binomial factors,
we have $\left\{\begin{matrix} 6x, & 3x. \\ x, & 2x. \end{matrix}\right.$	we have $\left\{\begin{matrix} 5. \\ 1. \end{matrix}\right.$

The first terms of the required binomial factors must be terms in the same column, — that is, either $6x$ and x, or $3x$ and $2x$, respectively, — and the second terms of the binomial factors must be 5 and 1.

We may now arrange the different pairs of factors, $6x$ and x, $3x$ and $2x$, as first terms of "trial" binomial factors, and 5 and 1 as second terms, as follows:

	Selecting one pair of columns	Interchanging terms of second column
First Binomial.	$6x + 5$	$6x + 1$
Second Binomial.	$x + 1$	$x + 5$
	$6x^2 + 11x + 5.$	$6x^2 + 31x + 5.$

	Selecting a second pair of columns	Interchanging terms of second column
First Binomial.	$3x + 5$	$3x + 1$
Second Binomial.	$2x + 1$	$2x + 5$
	$6x^2 + 13x + 5.$	$6x^2 + 17x + 5.$

Examining the expressions resulting from the multiplications of the different pairs of binomials above, we find that in the first expression of the second row we have obtained the product $6x^2 + 13x + 5$, in which the middle term $13x$ is the same as the middle term of the given expression.

Hence, we may write

$$6x^2 + 13x + 5 \equiv (3x + 5)(2x + 1).$$

Check. Let $x = 2$.

$24 + 26 + 5 = 11 \cdot 5$

$55 = 55.$

Ex. 4. Factor $12x^2 + 8x - 15$.

This expression is of the type $ax^2 + bx + c$, 12 corresponding to a, 8 to b, and -15 to c.

The sign of the constant term -15 is minus, and accordingly the signs of the second terms of the binomial factors must be unlike, one positive and the other negative.

The "trial" pairs of factors of $12x^2$ and of -15 may be arranged as below, the double signs $\pm$ and $\mp$ being used to indicate that the factors of -15 are of opposite sign, one positive and the other negative:

For first terms of the binomial factors,	and	For second terms of the binomial factors,
we have $\begin{cases} 12x,\ 6x,\ 4x. \\ \quad x,\ 2x,\ 3x. \end{cases}$		we have $\begin{cases} \pm 15,\ \pm 5. \\ \mp\ \ 1,\ \mp 3. \end{cases}$

As in example 3, we may construct different "trial" binomial factors by selecting combinations of first terms and second terms from among the different columns above.

Since in all cases the products arising from the multiplications of the first terms of the "trial" binomials as arranged below are $12x^2$, and the products of the second terms are -15, we have, for convenience, shown only the cross-product terms.

With a given 1st column.	Selecting a 2nd column.	Interchanging terms of 2nd column.	New 2nd column.	Interchanging terms of 2nd column.
1st Binomial	$12x \pm 15$	$12x \mp 1$	$12x \pm 5$	$12x \mp 3$
2nd Binomial	$x \mp 1$,	$x \pm 15$,	$x \mp 3$,	$x \pm 5$.
Cross Product	$\pm 3x$	$\pm 179x$	$\mp 31x$	$\pm 57x$
With a second 1st column.				
1st Binomial	$6x \pm 15$	$6x \pm 1$	$6x \pm 5$	$6x \mp 3$
2nd Binomial	$2x \mp 1$,	$2x \pm 15$,	$2x \mp 3$,	$2x \pm 5$.
Cross Product	$\pm 24x$	$\pm 88x$	$\mp 8x$	$\pm 24x$
With a third 1st column.				
1st Binomial	$4x \pm 15$	$4x \mp 1$	$4x \pm 5$	$4x \mp 3$
2nd Binomial	$3x \mp 1$,	$3x \pm 15$,	$3x \mp 3$,	$3x \pm 5$.
Cross Product	$\pm 41x$	$\pm 57x$	$\pm 3x$	$\pm 11x$

The middle term, $8x$, of the given expression $12x^2 + 8x - 15$ is the cross-product, $8x$, which arises from the multiplication of the trial binomials

$$\begin{array}{c} 6x - 5 \\ \underline{2x + 3} \\ +8x. \end{array}$$

Hence we have,

$$12x^2 + 8x - 15 \equiv (6x - 5)(2x + 3).$$

Check. Let $x = 2$.
$48 + 16 - 15 = 7 \cdot 7$
$49 = 49.$

38. It will not usually be necessary to write all possible combinations of trial binomial factors as in Ex. 4, above, since by inspection we may perform the cross multiplications mentally, and unsuitable combinations of terms may be rejected at once.

When constructing the different binomial factors, we should be guided by the following

Principle: *If no monomial factor is common to all of the terms of a given expression, there can be no monomial factor common to the terms of any polynomial factor of the given expression.*

E. g. Since no monomial factor is common to all of the terms of the expression $12x^2 + 8x - 15$ (see Ex. 4 § 37), it follows that such "trial" binomial factors as $12x \pm 15$, $12x \mp 3$, $6x \pm 15$, $6x \mp 3$, $3x \pm 15$ and $3x \mp 3$ must be rejected.

Ex. 5. Factor $8x^2 - 10x + 3$.

Observe that, since the sign of the constant term 3 is +, and of the middle term $-10x$ is −, the signs of the second terms of the binomial factors must be like, and both −.

The factors of $8x^2$ and of 3 may be arranged for first terms and for second terms of "trial" binomial factors as follows:

	First terms	Second terms
First Binomial	$\cancel{8x}$, $\cancel{x}$, $4x$, $\cancel{2x}$,	-3, $-\cancel{1}$.
Second Binomial	$\cancel{x}$, $\cancel{8x}$, $2x$, $\cancel{4x}$,	-1, $-\cancel{3}$.

Cancelling terms which we find cannot be used, we have finally,

$$8x^2 - 10x + 3 \equiv (4x - 3)(2x - 1).$$

Check. Let $x = 2$.
$32 - 20 + 3 = 5 \cdot 3$
$15 = 15.$

39. Expressions of the form $ax^2 + bxy + cy^2$, which are homogeneous and of the second degree with reference to two letters,

x and y, may be factored by the methods employed for factoring expressions of the type $ax^2 + bx + c$.

When factoring such expressions it is necessary that each of the factors of the term ax^2 selected as the first terms of the required binomial factors should contain x, and that each of the factors of the term cy^2, selected as second terms, should contain y.

E. g. The expression $6m^2 + 25mn + 11n^2$ may be expressed as the product of the factors $3m + 11n$ and $2m + n$.

EXERCISE XII. 12

Factor the following expressions, checking all results by making numerical substitutions:

1. $3x^2 + 4x + 1$.
2. $2x^2 + 5x + 2$.
3. $5x^2 + 26x + 5$.
4. $4x^2 + 11x + 7$.
5. $3x^2 + 8x + 4$.
6. $2x^2 + 7x + 3$.
7. $7x^2 + 33x + 20$.
8. $9x^2 + 34x + 21$.
9. $10x^2 + 19x + 9$.
10. $6x^2 + 41x + 30$.
11. $7x^2 + 13x - 2$.
12. $8a^2 - 5a - 3$.
13. $3a^2 - 5a - 22$.
14. $6a^2 - 19a + 15$.
15. $4a^2 + 8a - 45$.
16. $13x^2 + 9x - 4$.
17. $2a^2 - 15a - 8$.
18. $11a^2 - 21a + 10$.
19. $12b^2 - b - 13$.
20. $13a^2 - 27a + 2$.
21. $14m^2 + 73m + 15$.
22. $14a^2 - 33a + 18$.
23. $54a^2 - 15a - 50$.
24. $36m^2 + 27m + 2$.
25. $30m^2 - 47m - 5$.
26. $8b^2 - 14b - 39$.
27. $7x^2 - 10x + 3$.
28. $30c^2 - 31c + 8$.
29. $56x^2 + 65x - 9$.
30. $40y^2 + 14y - 33$.
31. $7x^2 + 13 + 20x$.
32. $28x^2 + 17 - 48x$.
33. $48x^2 - 23x - 13$.
34. $18x^2 + 37x + 19$.
35. $20a^2 - 17a - 3$.
36. $8a^2b^2 + 103ab - 13$.
37. $17a^2x^2 - 69ax + 4$.
38. $20m^2 + 19mn + 3n^2$.
39. $7b^2 + 41bc - 6c^2$.
40. $40x^2 - 83xy + 42y^2$.
41. $9x^2 - 4xy - 13y^2$.
42. $9x^2 + 55xy - 56y^2$.
43. $6a^2b^2 + 7abc - 5c^2$.
44. $3(4x^2 - 21) - x$.
45. $13m^2 - (m + 14)$.
46. $15 + w(19w - 34)$.
47. $h(22h - 19) - 15$.
48. $b(1 + 15b) - 16$.
49. $2(3y^2 - 28z^2) + 41yz$.
50. $7c(29d + 14c) + 14d^2$.

Type V. Binomial Sums and Differences of Like Powers

$a^n \pm b^n$

40. From §§ 55 – 58, Chap. VIII., we have the following principles:

(i.) *The **sum** of the same **odd powers** of two numbers is equal to the **sum** of the two given numbers multiplied **by a polynomial factor.***

(ii.) *The **difference** of the same **odd powers** of two numbers is equal to the **difference** of the two given numbers, multiplied **by a polynomial factor.***

(iii.) *The **difference** of the same **even powers** of two numbers is equal to the **sum** of the two given numbers, multiplied **by a polynomial factor,** or is equal to the **difference** of the two given numbers, multiplied **by a polynomial factor.***

The polynomial factors of binomial sums or differences of like powers are formed according to the Law of Quotients. (See Chap. VIII. § 59.)

Ex. 1. Factor $m^5 + 32$.

To apply the principles of § 40, it is necessary that the terms of the binomial be expressed as like powers.

Since 32 may be expressed as 2^5, it follows that $m^5 + 32$ may be expressed as $m^5 + 2^5$.

The binomial $m^5 + 2^5$ is the sum of the same odd powers, of m and 2 and hence by Principle (i.) the binomial sum $m + 2$ must be one of the required factors.

Hence we have,

$$m^5 + 2^5 \equiv [m + 2][m^4 - m^3 \cdot 2 + m^2 \cdot 2^2 - m \cdot 2^3 + 2^4]$$

That is, $m^5 + 32 \equiv [m + 2][m^4 - 2m^3 + 4m^2 - 8m + 16]$. Check. $m = 2$.
$64 = 64$.

Ex. 2. Factor $125x^3 - y^3$.

We may express $125x^3 - y^3$ as $(5x)^3 - y^3$, which is the difference of the same odd powers of $5x$ and y.

It follows from (ii.) that $5x - y$ is one of the required factors of this difference.

Hence we have,

$$(5x)^3 - y^3 \equiv [5x - y][(5x)^2 + (5x)y + y^2]$$ Check. $x = 3, y = 2$.
$3375 - 8 = 13 \cdot 259$

That is, $125x^3 - y^3 \equiv [5x - y][25x^2 + 5xy + y^2]$. $3367 = 3367$.

Ex. 3. Factor $a^6 - 27\,b^3$.

The binomial $a^6 - 27\,b^3$ may be expressed as $(a^2)^3 - (3\,b)^3$ which is the difference of the same odd powers of a^2 and $3\,b$.

By Principle (ii.), one of the factors of $(a^2)^3 - (3\,b)^3$ is the binomial difference $a^2 - 3\,b$.

Hence we have,

$$(a^2)^3 - (3\,b)^3 \equiv [a^2 - 3\,b][(a^2)^2 + a^2\,(3\,b) + (3\,b)^2]$$

That is, $a^6 - 27\,b^3 \equiv [a^2 - 3\,b][a^4 + 3\,a^2b + 9\,b^2]$.

Check.
$a = 3,\ b = 2.$
$513 = 513.$

Ex. 4. Factor $a^6 + b^6$.

The binomial $a^6 + b^6$, which is the sum of the same even powers of a and b, may be expressed as $(a^2)^3 + (b^2)^3$, which is the sum of the same odd powers of a^2 and b^2.

Considered as the sum of the same odd powers of a^2 and b^2, it follows from Principle (i.) that one of the factors of $(a^2)^3 + (b^2)^3$ must be $a^2 + b^2$.

Hence we have,

$$(a^2)^3 + (b^2)^3 \equiv [a^2 + b^2][(a^2)^2 - (a^2)(b^2) + (b^2)^2]$$

That is, $a^6 + b^6 \equiv [a^2 + b^2][a^4 - a^2b^2 + b^4]$.

Check.
$a = 2,\ b = 1.$
$64 + 1 = 5\,(16 - 4 + 1)$
$65 = 65.$

EXERCISE XII. 13

Factor the following expressions, checking all results numerically:

1. $x^3 + 1$.
2. $x^3 + 8$.
3. $27\,a^3 + 1$.
4. $x^6 - 1$.
5. $a^5 + 1$.
6. $32\,a^5 + 1$.
7. $c^3 - 64\,d^3$.
8. $x^6 - 64\,y^6$.
9. $81\,a^4 - b^4$.
10. $125 + 27\,a^3$.
11. $x^7 - y^7$.
12. $a^{12} + b^{12}$.
13. $16\,a^4 - 1$.
14. $1000\,x^3 - y^3$.
15. $343\,a^3 - b^3$.
16. $m^7 + 128$.
17. $a^6 + 729\,b^6$.
18. $32\,a^5 - 243\,b^5$.
19. $8\,x^3 + 27\,y^3$.
20. $32\,a^{10} + 1$.
21. $1728 - 27\,y^3$.
22. $125\,x^6 - 8\,y^3$.
23. $16\,a^4 - 625\,b^4$.
24. $81\,x^4 - 256\,y^4$.
25. $8\,c^3 + 343\,d^3$.
26. $216\,h^3 + 729\,y^3$.
27. $d^3 + 512\,x^3$.
28. $64\,a^6 + 1$.
29. $128\,a^7 - 1$.
30. $1024\,a^5 + b^5$.
31. $3125 + c^5$.
32. $1296 - a^4$.

33. $2401 - x^4$.	37. $x^5 - 243\,y^5$.
34. $x^3 - 216\,y^3$.	38. $1000\,m^3 - z^6$.
35. $64\,a^6 + b^3$.	39. $1728\,a^3 - b^{12}$.
36. $y^9 - 512\,z^3$.	40. $3125\,c^5 - 32\,d^5$.

Polynomials of at Least the Third Degree with Reference to Some Letter, x

41. Consider the following product of binomials:

$$(x-a)(x-b)(x-c) \equiv x^3 - (a+b+c)\,x^2 + (ab+ac+bc)\,x - abc.$$

It will be seen that the first term, x^3, is the product of the first terms of the binomial factors, and that the last or constant term, $-abc$, is the product of the second terms, $-a$, $-b$ and $-c$, of the binomial factors.

It may be seen that if two or more binomial factors of the form $x - a$ are multiplied together to form a product, the term of highest degree with reference to x will be the product of the first terms of the binomial factors, and that the constant term, which is the term free from x, will be the product of the second terms of all of the binomial factors.

Hence, *if binomial factors of the form $x - a$ exist for any given integral polynomial whose highest power with reference to x has the coefficient unity, we may discover their second terms among the factors of the term that is free from the letter of arrangement.*

We may accordingly construct different "trial" binomial factors, by writing x minus each of these factors in turn. Then we may apply the Remainder Theorem to discover which of these "trial" factors, if any, are factors of the given expression.

Ex. 1. Factor $x^3 - 6\,x^2 + 11\,x - 6$.

Since the expression is of the third degree with respect to x, we shall look for three binomial factors of the first degree with respect to x, having the form $x - a$.

Since the term free from x, -6, is negative, the signs of an odd number of second terms of the binomial factors must be minus. Neglecting the sign, the integral factors of 6 are 1, 2, 3, and 6. We may assume as "trial" binomial factors, $x - 1$, $x - 2$, $x - 3$, and $x - 6$.

It may happen that, of the final set of three binomial factors selected, two binomials may be sums instead of differences.

Applying the Remainder Theorem, we find by trial that $x - 6$ is not a factor of the given expression.

It will be convenient to use the method of synthetic division as below:

$$\begin{array}{rrrrl} 1 & -6 & +11 & -6 & \underline{)+6} \\ & +6 & +0 & +66 & \\ \hline 1 & +0 & +11, & +60 & \end{array}$$

The remainder is 60. Hence the division is not exact.

By trial, we find that the remaining factors are contained exactly as divisors in the given expression. Hence, we may write

$$x^3 - 6x^2 + 11x - 6 \equiv (x-1)(x-2)(x-3).$$

Check. Let $x = 4$.

$$64 - 96 + 44 - 6 = 3 \cdot 2 \cdot 1$$

$$6 = 6.$$

42. After having found one factor of a given expression, it will often be possible to save work by dividing the expression by this factor, and then examining the quotient for the remaining factors of the given expression.

Ex. 2. Factor $x^3 + 6x^2 - x - 30$.

The second terms of such binomial factors of the form $x - a$ as may exist will be found among the integral factors of -30. Since the expression is of the third degree with reference to x, we may expect to find three binomial factors.

The integral factors of 30 are 1, 2, 3, 5, 6, 10, 15, and 30. Since 30 is negative, at least an odd number of the factors of 30, selected for second terms of the required binomial factors, must have minus signs. We may assume as "trial" binomial factors $x - 1$, $x - 2$, $x - 3$, $x - 5$, $x - 6$, $x - 10$, $x - 15$, and $x - 30$.

Applying the Remainder Theorem, we find that $x - 2$ is a factor of the given expression. Since $x - 2$ is a factor of the given expression, the quotient obtained by dividing the expression by $x - 2$ must be the product of the remaining factors.

When dividing the expression $x^3 + 6x^2 - x - 30$ by $x - 2$ it will be found convenient to apply the method of synthetic division, as shown below:

$$\begin{array}{rrrrl} 1 & +6 & -1 & -30 & \underline{)2} \\ & +2 & +16 & +30 & \\ \hline 1 & +8 & +15, & 0 & \end{array}$$

By using the coefficients 1, 8 and 15 and supplying the proper powers of x we may immediately construct the desired quotient $x^2 + 8x + 15$, Hence we may write

$$x^3 + 6x^2 - x - 30 \equiv (x - 2)(x^2 + 8x + 15).$$

The trinomial factor $x^2 + 8x + 15$ is the product of the binomial factors $x + 3$ and $x + 5$.

Hence we have finally,

$$x^3 + 6x^2 - x - 30 \equiv (x - 2)(x + 3)(x + 5).$$

Check. Let $x = 3$.

$$27 + 54 - 3 - 30 = 1 \cdot 6 \cdot 8$$
$$48 = 48.$$

EXERCISE XII. 14

Obtain factors of the following expressions, checking all results numerically :

1. $x^3 + 6x^2 + 11x + 6.$
2. $x^3 + 8x^2 + 17x + 10.$
3. $x^3 + 8x^2 + 19x + 12.$
4. $x^3 + 11x^2 + 34x + 24.$
5. $x^3 + 10x^2 + 31x + 30.$
6. $x^3 + 8x^2 + 5x - 14.$
7. $x^3 + 3x^2 - 13x - 15.$
8. $x^3 + x^2 - 22x - 40.$
9. $x^3 - 2x^2 - 29x - 42.$
10. $x^3 - 27x - 54.$
11. $x^3 + 4x^2 - 28x + 32.$
12. $x^3 - 14x^2 + 51x - 54.$
13. $x^3 - 18x^2 + 87x - 70.$
14. $x^3 - 10x^2 - 17x + 66.$
15. $x^3 - 17x^2 + 55x - 39.$
16. $x^3 - 3x^2 - 34x - 48.$
17. $x^3 + 13x^2 + 54x + 72.$
18. $x^3 - 13x^2 + 55x - 75.$
19. $x^3 - 3x^2 - 24x + 80.$
20. $x^3 + 66x^2 + 129x + 64.$

Determine by the Remainder Theorem whether or not $x + 2$ is a factor of each of the following expressions :

21. $x^3 + 4x^2 + 2x + 4.$
22. $x^3 - x^2 - 5x + 2.$
23. $x^3 - 2x^2 + 4x - 6.$
24. $x^3 + 2x^2 - 29x - 30.$
25. $x^3 + 3x^2 + 3x + 2.$
26. $x^3 + x^2 - 10x + 8.$
27. $x^3 - 10x^2 + 27x - 18.$
28. $x^3 + 10x^2 + 29x + 20.$
29. $x(x^2 - 9) - 2(3x^2 - 7).$
30. $x(x^2 - 10) - 3(x^2 - 8).$
31. $x^3 + 6x^2 - x - 18.$
32. $x^3 - 25x + x^2 - 46.$

Suggestions Relating to Methods

43. Although no rules can be given which will apply in all cases, the student will find that the following general directions will serve to systematize the work when factoring a given expression :

(1) *Remove all monomial factors which are common to all of the terms of a given expression.*

(2) *Examine the resulting expression and determine whether or*

not it can be simplified by applying one or more of the methods shown in this chapter.

Ex. 1. Factor $11x^2 + 22xy + 11y^2$.

Dividing all of the terms by the common factor 11, we obtain the quotient $x^2 + 2xy + y^2$, which is the square of $x + y$.

Hence we have

$$11x^2 + 22xy + 11y^2 \equiv 11(x^2 + 2xy + y^2)$$
$$\equiv 11(x + y)^2.$$

Check. Let $x = 2$, $y = 3$.

$$44 + 132 + 99 = 11 \cdot 25$$
$$275 = 275.$$

(3) *If a given expression is arranged according to descending or ascending powers of some particular letter, factors will frequently be suggested by the form of the expression.*

Ex. 2. Factor $x^2(y - a) + y^2(a - x) + a^2(x - y)$.

Performing the multiplications, and rearranging the terms according to descending powers of x, we obtain the expression

$$x^2(y - a) - x(y^2 - a^2) + (ay^2 - a^2y),$$

in which the binomial factor $y - a$ is common to the different groups of terms.

Hence we have

$$x^2(y - a) + y^2(a - x) + a^2(x - y) \equiv x^2(y - a) - x(y^2 - a^2) + (ay^2 - a^2y)$$
$$\equiv x^2(y - a) - x(y + a)(y - a) + ay(y - a)$$
$$\equiv [x^2 - x(y + a) + ay][y - a]$$
$$\equiv (x - y)(x - a)(y - a).$$

Check.

Let $a = 1$, $x = 3$, $y = 2$.

$$9 + 4(-2) + 1 = 2$$
$$2 = 2.$$

(4) *Occasionally different methods of factoring may be applied to different groups of terms of a given expression. It may happen that these groups, after being separately factored, are found to contain one or more common factors which in turn may suggest a method to be applied to complete the factorization desired.*

Ex. 3. Factor $x^3 + y^3 + ax + ay + bx + by$.

In the given expression, the binomial sum of the same odd powers of x and y, $x^3 + y^3$, may be expressed as $(x + y)(x^2 - xy + y^2)$, and the remaining terms $ax + ay + bx + by$ may be grouped and expressed as the product $(a + b)(x + y)$.

Hence we have

$$x^3 + y^3 + ax + ay + bx + by \equiv (x^3 + y^3) + [(ax + ay) + (bx + by)]$$
$$\equiv (x + y)(x^2 - xy + y^2) + (a + b)(x + y)$$
$$\equiv (x^2 - xy + y^2 + a + b)(x + y).$$

Check.

Let $x = 3$, $y = 2$, $a = 4$, $b = 5$.

$$80 = 16 \cdot 5$$
$$80 = 80.$$

(5) *The Remainder Theorem may be applied when a given expression is of the third or higher degree with reference to some letter of arrangement.*

44. When for any reason a given expression appears to have more than one distinct set of prime factors, we shall find, upon closer examination, that factors which apparently differ, are identical. This is because some of the factors which were supposed at first to be prime still admit of further reduction, or else differ only in sign.

Ex. 4. Factor $x^4 - y^4$.

Using different methods, we obtain

Method I. (See § 22) $x^4 - y^4 \equiv (x^2 + y^2)(x + y)(x - y)$.

Method II. (See § 40) $x^4 - y^4 \equiv \begin{cases} \text{either } (x + y)(x^3 - x^2y + xy^2 - y^3), \\ \text{or } \quad (x - y)(x^3 + x^2y + xy^2 + y^3). \end{cases}$

Check. Let $x = 3$ and $y = 2$.

Method I. $81 - 16 = 13 \cdot 5 \cdot 1$

$65 = 65.$

Method II. $81 - 16 = \begin{cases} \text{either } 5 \cdot 13, \\ \text{or } \quad 1 \cdot 65. \end{cases}$

$65 = 65.$

Neither of the polynomial factors obtained by Method II is prime.

Considering the first polynomial factor, $x^3 - x^2y + xy^2 - y^3$, we find that

$$\begin{aligned} x^3 - x^2y + xy^2 - y^3 &\equiv (x^3 - y^3) - (x^2y - xy^2) \\ &\equiv (x - y)(x^2 + xy + y^2) - xy(x - y) \\ &\equiv [(x^2 + xy + y^2) - xy][x - y] \\ &\equiv (x^2 + y^2)(x - y). \end{aligned}$$

Accordingly, the first set of factors obtained by Method II reduces to

$$(x + y)(x^2 + y^2)(x - y).$$

The factors in this set are identical with those obtained by Method I.

In a similar way we may show that the second set of factors,

$$(x - y)(x^3 + x^2y + xy^2 + y^3),$$

obtained by Method II, may also be reduced to the set of factors

$$(x - y)(x^2 + y^2)(x + y).$$

Ex. 5. Factor $7ax - 3a^2 - 2x^2$.

We may place the expression in minus parentheses and arrange the terms according to descending powers of a, and factor as follows:

$$7\,ax - 3\,a^2 - 2\,x^2 \equiv -\,(3\,a^2 - 7\,ax + 2\,x^2) \qquad \text{Check. Let } a = 3,\ x = 2.$$
$$\equiv -\,(3\,a - x)(a - 2\,x). \qquad 7 = 7.$$

If, however, the expression be placed in minus parentheses and the terms be arranged according to descending powers of x, we have

$$7\,ax - 3\,a^2 - 2\,x^2 \equiv -\,(2\,x^2 - 7\,ax + 3\,a^2) \qquad \text{Check. Let } x = 2,\ a = 3.$$
$$\equiv -\,(2\,x - a)(x - 3\,a). \qquad 7 = 7.$$

We shall find that the factors first obtained, $-\,(3\,a - x)(a - 2\,x)$, differ from the factors $-\,(2\,x - a)(x - 3\,a)$ only in the signs of the terms of the binomials.

We may show that the factors of the second set are identical with those of the first set, as follows:

$$\begin{aligned} -\,(2\,x - a)(x - 3\,a) &\equiv -\,(2\,x - a)(x - 3\,a)(-\,1)(-\,1) \\ &\equiv -\,(2\,x - a)(-\,1)(x - 3\,a)(-\,1) \\ &\equiv -\,(-\,2\,x + a)(-\,x + 3\,a) \\ &\equiv -\,(a - 2\,x)(3\,a - x). \end{aligned}$$

45. Certain expressions may be factored by applying any one of several different methods.

Ex. 6. Factor $16\,a^4 - 41\,a^2c^2 + 25\,c^4$.

First Method. On examination, we find that two of the terms $16\,a^4$ and $25\,c^4$ are squares. A trinomial square having these same two terms would have as a middle term twice the product of the square roots of these terms, — that is, $40\,a^2c^2$. The sign of this term will be plus or minus according as the trinomial is the square of a sum or the square of a difference.

Solution I. Assuming first that the middle term is $+\,40\,a^2c^2$, we shall obtain the factors as follows:

The difference between $40\,a^2c^2$ and $-\,41\,a^2c^2$ is $81\,a^2c^2$.

Accordingly we have,

$$\begin{aligned} 16\,a^4 - 41\,a^2c^2 + 25\,c^4 &\equiv 16\,a^4 - 41\,a^2c^2 + 25\,c^4 + 81\,a^2c^2 - 81\,a^2c^2 \\ &\equiv 16\,a^4 + 40\,a^2c^2 + 25\,c^4 - 81\,a^2c^2 \\ &\equiv (4\,a^2 + 5\,c^2)^2 - (9\,ac)^2 \\ &\equiv [4\,a^2 + 5\,c^2 + 9\,ac][4\,a^2 + 5\,c^2 - 9\,ac] \\ &\equiv (4\,a + 5\,c)(a + c)(4\,a - 5\,c)(a - c). \end{aligned}$$

Solution II.

Assuming secondly that the middle term is $-\,40\,a^2c^2$, we obtain the same result, as follows:

$$\begin{aligned} 16\,a^4 - 41\,a^2c^2 + 25\,c^4 &\equiv 16\,a^4 - 41\,a^2c^2 + 25\,c^4 + a^2c^2 - a^2c^2 \\ &\equiv 16\,a^4 - 40\,a^2c^2 + 25\,c^4 - a^2c^2 \\ &\equiv (4\,a^2 - 5\,c^2)^2 - (ac)^2 \\ &\equiv [4\,a^2 - 5\,c^2 + ac]\,[4\,a^2 - 5\,c^2 - ac] \\ &\equiv (4\,a + 5\,c)(a - c)(4\,a - 5\,c)(a + c). \end{aligned}$$

Second Method. We may apply the methods of §§ 33–39 to the expression as given, and thus obtain the same factors, as follows:

$$\begin{aligned} 16a^4 - 41a^2c^2 + 25c^4 &\equiv (16a^2 - 25c^2)(a^2 - c^2) \\ &\equiv (4a + 5c)(4a - 5c)(a + c)(a - c). \end{aligned}$$

Check. Let $a = 3$, $c = 2$.

$$1296 - 1476 + 400 = 22 \cdot 2 \cdot 5 \cdot 1$$
$$220 = 220.$$

46. Any homogeneous function of two letters may be factored, provided that it is possible to factor the non-homogeneous expression resulting from giving the value unity to one of the letters.

Ex. 7. Factor $x^3 - 9x^2y + 26xy^2 - 24y^3$.

By assigning the value 1 to y, the expression $x^3 - 9x^2y + 26xy^2 - 24y^3$, which is homogeneous with reference to x and y, reduces to the non-homogeneous expression $x^3 - 9x^2 + 26x - 24$.

The expression $x^3 - 9x^2 + 26x - 24$ may be factored as follows:

$$x^3 - 9x^2 + 26x - 24 \equiv (x - 2)(x - 3)(x - 4).$$

From this identity we may obtain the factors of the given expression by introducing such powers of y as are necessary to make the members homogeneous expressions with reference to x and y.

Hence we have,

Check

$$x^3 - 9x^2y + 26xy^2 - 24y^3 \equiv (x - 2y)(x - 3y)(x - 4y).$$

Let $x = 3$, $y = 2$.

$$27 - 162 + 312 - 192 = (-1)(-3)(-5)$$
$$-15 = -15.$$

EXERCISE XII. 15 **Miscellaneous**

Obtain factors of the following, checking all results numerically:

1. $a^3 + a^2 + a$.
2. $xy + 4x - 3y - 12$.
3. $a^2 + 22a + 121$.
4. $x^2 - 144$.
5. $x^2 + 3x + 2$.
6. $2x^2 + 11x + 12$.
7. $3a^2 - 12$.
8. $3k^2 + 33k + 72$.
9. $km + 2lm - sk - 2ls$.
10. $x^2 + 5x - 6$.
11. $49h^2 - 36y^2$.
12. $x^{2m} - y^{2m}$.
13. $x^2 - 22x + 105$.
14. $x^3 + x^2 - x - 1$.
15. $x^2 + cx + 2dx + 2cd$.
16. $(a-b)^2 - 11(a-b) - 12$.
17. $a^3 - 343$.
18. $2a^3 + 54$.
19. $7a^7 - 7a$.
20. $5m + 6m^2 + 1$.

21. $m^2 + m - 2$.
22. $2a^3 + 28a^2 + 66a$.
23. $3a^2 + 30a + 27$.
24. $13x^2 + 25x - 2$.
25. $2x^2 + 12x + 18$.
26. $2a^2 + 3a + 1$.
27. $6a^2 - 3a - 3$.
28. $169c^2 - 9d^2$.
29. $56 - 15x + x^2$.
30. $15x^2 + 2xy - 24y^2$.
31. $75a^2 - 3b^2$.
32. $5x^2 + 20x + 20$.
33. $15a^2 + 41a + 14$.
34. $a + a^6$.
35. $x^6 + 19x^3 + 88$.
36. $a^2b^2 + 30ab + 104$.
37. $a^2x^2 + 3abx + 2b^2$.
38. $14 - 21m - 14m^2$.
39. $128m^4 - 18n^2$.
40. $c^4 - 5c^2d^2 + 4d^4$.
41. $27c^2 + 18c + 3$.
42. $x^2y^2z^2 - 4xyz - 12$.
43. $8a^2b^2c^2 - 18c^4$.
44. $x^4 - 21x^2 + 80$.
45. $25a^{10} - 26a^5 + 1$.
46. $144x^2 - 625y^2$.
47. $16a^4 - 41a^2c^2 + 25c^4$.
48. $64a^5 + 2$.
49. $8a^9 + 729$.
50. $b^2 + 288 - 34b$.
51. $x^6 + 25x^3 + 24$.
52. $50 - 20x + 2x^2$.
53. $147a^2 - 75$.
54. $x^8 - 38x^4 + 105$.
55. $8 - 9x^4 + x^8$.
56. $64 - a^6$.
57. $6a^2 + 150 - 60a$.
58. $3x^2 + 36xy + 108y^2$.
59. $4x^2 - 28xy + 49y^2$.
60. $80a^2 - 20a^2b^2$.
61. $64r^2 + 80rs + 25s^2$.
62. $x^3 - 7x^2 + 14x - 8$.
63. $(a + b)^3 - 1$.
64. $9a^2 + 24ab + 16b^2$.
65. $(a + b)^2 + 8(a + b) + 15$.
66. $(a + b)^2 + 5(a + b) + 6$.
67. $3c^2 - 14cy + 8y^2$.
68. $a^{2m} + 2a^mb^m + b^{2m}$.
69. $30a^2 - 154a + 20$.
70. $x^{2m} - y^2$.
71. $(c + d)^2 + 12(c + d) + 20$.
72. $a^2 - 2ab + b^2 - 11a + 11b - 12$.
73. $4x^4 - 13x^2y^2 + 9y^4$.
74. $a^6 - 50a^3 + 49$.
75. $x^3 + 9x^2 + 26x + 24$.
76. $a^3 - 9a^2b + 23ab^2 - 15b^3$.
77. $(a + b)^2 - 2(a + b) + 1$.
78. $d^4 + 11(2d^2 + 11)$.
79. $18h^2 - 31hk + 6k^2$.
80. $16a^2 - (2b + 3c)^2$.
81. $a^3 - 13a + 12$.
82. $12a^2 - 12b(2a - b)$.
83. $(c + d)^2 + 10(c + d) + 21$.
84. $2a(a + 6) + 18$.
85. $x^2y^2z^2 - 8xyzw - 20w^2$.
86. $4m(m + 3) + 9$.
87. $2a(4a - 19) + 35$.
88. $(2x + 3y)^2 - (3a + b)^2$.
89. $x^2 - y^2 - 2yz - z^2$.
90. $49m^4 - 65m^2n^2 + 16n^4$.
91. $a^2 - 7(2a - 7)$.
92. $a^7 + 8a$.
93. $y^{2m} - 68y^m - 140$.
94. $4x(x + y) + y^2$.

95. $3r^2 + 4(2r + 1)$.

96. $x^3 - 3xy^2 + 2y^3$.

97. $a^2 + 49b^2 - 1 + 14ab$.

98. $c^2 + d^2 + 2cd - a^2 - b^2 - 2ab$.

99. $c^2 - 2cd + d^2 - 2(c - d)e + e^2$.

100. $x^2 + (c + 2d)x + 2cd$.

101. $a^4 - a^3 - 7a^2 + a + 6$.

102. $m^2 + n^2 + 2mn - a^2 - b^2 - 2ab$.

103. $1 + 2cd - c^2 - d^2$.

104. $h^2 + 2hm + m^2 + 2hy + 2my + y^2$.

105. $16x^4 - 81y^4$.

106. $9 + 49x^2 - 58x^4$.

107. $x^7 - 9x$.

108. $a^2b^3 - 16b^3c^2$.

109. $x^3 + y^3 + x + y$.

110. $x^4 - y^4 - (x + y)(x - y)$.

111. $a^2b + 3ab^2 - 3a^3 - b^3$.

112. $a^2 + b^2 - c^2 - m^2 - 2ab + 2mc$.

113. $4x^{2n} - 32x^ny^n + 64y^{2n}$.

114. $x^2 + 2xy + y^2 + 8xz + 8yz + 15z^2$.

115. $4a^2 - 25b^2 + 2a + 5b$.

116. $x^2y^2z^2 - 8axyz - 20a^2$.

117. $a^4 + 9a^2b^3 + 18b^6$.

118. $y^{2m} - 168y^m - 340$.

119. $a^3b^2 - c^3b^2 - a^3d^2 + c^3d^2$.

120. $a^{12} + 4b^{12} - 5a^6b^6$.

121. $ab(x^2 + 1) + x(a^2 + b^2)$.

122. $1 + b - 56b^2$.

123. $1 - 17x^2 + 16x^4$.

124. $4 - 52x^7 + 169x^{14}$.

125. $abc^3 + 3abc^2 - abc - 3ab$.

126. $4(a^2 + c^2)(a^2 - c^2) + 3b^3(4a^2 - 4c^2)$.

127. $a^3 - b^3 - a(a^2 - b^2) + b(a - b)^2$.

128. $9a^6 + 71a^3 - 8$.

129. $(x + y)(x + y + 7) + 10$.

130. $a^4 - b^2(11a^2 - b^2)$.

131. $na^{2n} - 19na^n + 34n$.

132. $a^{10} + 18a^5b^5 - 144b^{10}$.

133. $bcx^2 + (ac + bd)x + ad$.

134. $x^{2m} + (a + b)x^m + ab$.

135. $25a^8 + 81b^8 + 41a^4b^4$.

136. $a^2(a+1) + b^2(b+1) + 2ab$.

137. $x^2(x+1) - 2xy - y^2(y-1)$.

138. $a^2(a+3b) + b^2(b+3a)$.

139. $x^2 + 2ax + a^2 - x - a$.

140. $8a^8 - 8$.

141. $16a^{16} - 16$.

142. $12a^{12} + 12$.

143. $6a^2 + 5ax + x^2$.

144. $14a^2 - 109ab - 24b^2$.

145. $7x^2 + 47xy + 30y^2$.

146. $a^2 - d^2 - b(2a - b) + c(2d - c)$.

147. $c^2 + d^2 - e^2 - f^2 + 2(ef - cd)$.

148. $x^4 - 2x^3 + 2x - 1$.

149. $a^2x + b^2y + b^2x + a^2y + 2(abx + aby)$.

150. $(a+b)^2 + (b+d)^2 - (c+d)^2 - (c+a)^2$.

Application of the Principles of Factoring to the Solution of Equations

47. To solve an equation containing one unknown is to find such a value, or values, for the unknown as will, when substituted for the unknown, make the two members of the equation identical.

Ex. 1. Solve $x^2 + 15 = 8x$. (1)

By transposing the term $8x$ to the first member we have,

$$x^2 - 8x + 15 = 0.$$

Factoring, $$(x-5)(x-3) = 0. \quad (2)$$

By §§ 27, 25, Chap. X., the derived equation (2) is equivalent to the original equation (1).

If either of the factors $x - 5$ or $x - 3$ becomes zero for any particular value of x, the other remaining finite, the product of the two factors will become zero. Hence for such a value of x the first member of the equation will take the same value as the second, that is, it will become zero.

If x be given either of the values 5 or 3, one of the factors will become zero, and the other a finite number.

Accordingly these values are solutions of equation (2).

It may be seen that, by placing the factor $x - 5$ of the first member of equation (2) equal to zero and solving the equation thus formed, we shall obtain $x = 5$, which is one of the solutions of equation (2).

It appears that *the values for the unknown may be found by solving the separate equations formed by writing equal to zero each of the factors of the first member of the equation obtained by transposing all of the terms of the given equation to the first member.*

Accordingly, from (2) we may write

$$(x-5)=0, \quad \text{also} \quad (x-3)=0,$$

Hence, $\quad x=5, \qquad x=3.$

By substitution, these values are found to satisfy the original equation.

Ex. 2. Solve $x^3+x^2=12\,x$. (1)

Transposing and factoring, we obtain the equivalent equation

$$x(x+4)(x-3)=0. \tag{2}$$

Since this equation is satisfied by any value of x which makes any one of the factors of the first member zero (the other factors remaining finite), we may place each of the factors of (1) equal to zero, and solve the resulting equations.

$$x=0, \qquad x+4=0, \qquad x-3=0.$$

Therefore, $\quad x=0, \qquad x=-4, \quad$ and $\quad x=3.$

These values are all solutions of the given equation.

Substituting 0 for x in (1), $0=0.$	Sub. -4 for x in (1), $(-4)^3+(-4)^2=12\,(-4)$ $-48=-48.$	Sub. 3 for x in (1), $3^3+3^2=12\cdot 3$ $36=36.$

48. These examples illustrate the following

Principle of Equivalence: *If the terms of an integral equation be all transposed to one member, and if this member be factored and the separate factors be placed equal to zero, the set of equations thus obtained will be equivalent to the original equation.*

That is, in particular, the equation

$$(x-a)(x-b)(x-c)=0, \tag{1}$$

is equivalent to the following set of equations:

$$x-a=0 \qquad x-b=0, \qquad \text{and } x-c=0. \tag{2}$$

Solving these equations separately we obtain the following solutions of the given equation:

$$x=a, \qquad x=b, \qquad \text{and} \qquad x=c.$$

The equivalence may be established as follows:

It may be seen that when x is given any value which reduces one of the factors of the first member of equation (1) to zero, the same value will also reduce the first member of one of the equations of set (2) to zero. Accordingly such a value of x satisfies equation (1), and also one of the equations of set (2).

Since the factors of the first member of equation (1) are the first members of the equations of set (2), it follows that every solution of equation (1) must also be a solution of one of the equations of set (2).

Furthermore, any value which, when substituted for x, satisfies one of the equations of set (2), must reduce the first member of one of the equations of set (2) to zero. Accordingly such a value of x will reduce one of the factors of the first member of equation (1) to zero, and hence will satisfy equation (1).

Hence solutions are neither gained nor lost in passing from the single equation (1) to the set of separate equations (2); that is, equation (1) is equivalent to the set of equations (2).

49. *If, after having transposed all of the terms of an equation to one member, it is possible to separate the resulting member into factors, we may completely solve the original equation, provided that these factors are of such forms that we are able to solve the equations formed by equating them separately to zero.*

50. According as the unknown quantity appears in an integral rational equation to the first, second, third, or fourth powers, the equation is said to be *linear*, *quadratic*, *cubic*, or *biquadratic*.

Ex. 3. Solve the quadratic equation $2x^2 = x + 6$.

Transposing the terms to the first member, we have,

$$2x^2 - x - 6 = 0.$$

Factoring, $$(2x + 3)(x - 2) = 0.$$

This single quadratic equation is equivalent to the following set of two linear equations:

$$2x + 3 = 0, \quad \text{and} \quad x - 2 = 0.$$

Solving these equations separately, we obtain the following values which are the required solutions of the original equation:

$$x = -\tfrac{3}{2}, \quad \text{and} \quad x = 2.$$

Verifying these solutions by substituting in the original equation, we have:

Substituting $-\frac{3}{2}$ for x,

$$2\left(-\tfrac{3}{2}\right)^2 = \left(-\tfrac{3}{2}\right) + 6$$

$$\tfrac{9}{2} = \tfrac{9}{2}.$$

Substituting 2 for x,

$$2\,(2)^2 = 2 + 6$$

$$8 = 8.$$

EXERCISE XII. 16

Solve the following equations, regarding the letters appearing in them as unknowns, and verify all solutions by substituting for the letters in the original equations the particular values found:

1. $x^2 - 4x + 3 = 0$.
2. $y^2 - 5y + 4 = 0$.
3. $a^2 - 6a - 7 = 0$.
4. $b^2 - 7b + 10 = 0$.
5. $c^2 - 8c + 12 = 0$.
6. $g^2 - 9g + 18 = 0$.
7. $h^2 + 5h + 6 = 0$.
8. $k^2 + 6k + 8 = 0$.
9. $m^2 + 4m - 5 = 0$.
10. $n^2 + 7n - 8 = 0$.
11. $z^2 + 11z + 30 = 0$.
12. $w^2 + 11w + 24 = 0$.
13. $c^2 + 11c + 10 = 0$.
14. $g^2 - 10g + 25 = 0$.
15. $r^2 + 14r + 49 = 0$.
16. $s^2 - 18s + 81 = 0$.
17. $x^2 + 12x + 36 = 0$.
18. $y^2 - 16y + 64 = 0$.
19. $z^2 - 2z = 0$.
20. $w^2 - 3w = 0$.
21. $c^2 + 4c = 0$.
22. $d^2 = 6d$.
23. $m^2 = 7m$.
24. $5n = n^2$.
25. $8p = p^2$.
26. $q^2 = -9q$.
27. $h^2 = -10h$.
28. $k^2 - 4 = 0$.
29. $a^2 - 9 = 0$.
30. $b^2 - 16 = 0$.
31. $v^2 = 25$.
32. $m^2 = 36$.
33. $n^2 = 49$.
34. $d^2 = d$.
35. $k^2 - k = 6$.
36. $h^2 - h = 12$.
37. $r^2 + r = 20$.
38. $s^2 - s = 42$.
39. $x^2 + x = 56$.
40. $t^2 - t = 72$.
41. $y^2 + 2y = 15$.
42. $z^2 + 3z = 28$.
43. $w^2 + 4w = 45$.
44. $a^2 + 6a = 16$.
45. $b^2 - 5b = 50$.
46. $c^2 - 7c = 18$.
47. $d^2 - 8d = 48$.
48. $h^2 + 12 = 7h$.
49. $k^2 + 40 = 13k$.
50. $m^2 + 32 = 12m$.
51. $n^2 + 13 = 14n$.
52. $r^2 - 14 = 5r$.

53. $s^2 - 21 = 4s$.
54. $t^2 - 26 = 11t$.
55. $v^2 - 35 = 2v$.
56. $b^2 - 38 = 17b$.
57. $k^2 - 48 = 13k$.
58. $h^2 - 33 = -8h$.
59. $m^2 - 34 = -15m$.
60. $r^2 - 27 = -6r$.
61. $2z^2 - 7z + 3 = 0$.
62. $3g^2 - 7g + 2 = 0$.
63. $13n^2 - 14n + 1 = 0$.
64. $5d^2 - 21d + 4 = 0$.
65. $7w^2 - 15w + 2 = 0$.
66. $11x^2 - 34x + 3 = 0$.
67. $5y^2 - 7y + 2 = 0$.
68. $7z^2 - 10z + 3 = 0$.
69. $2w^2 - 5w + 3 = 0$.
70. $3a^2 + 4a + 1 = 0$.
71. $5b^2 + 6b + 1 = 0$.
72. $7c^2 + 22c + 3 = 0$.
73. $11d^2 + 23d + 2 = 0$.
74. $5g^2 + 7g + 2 = 0$.
75. $6k^2 + 5k + 1 = 0$.
76. $10m^2 + 7m + 1 = 0$.
77. $15n^2 + 8n + 1 = 0$.
78. $20r^2 + 12r + 1 = 0$.
79. $20s^2 + 9s + 1 = 0$.
80. $21t^2 + 10t + 1 = 0$.
81. $18x^2 - 11x + 1 = 0$.
82. $26y^2 - 15y + 1 = 0$.
83. $30z^2 - 13z + 1 = 0$.
84. $w^3 = 64w$.
85. $a^3 = 81a$.
86. $b^3 = 100b$.
87. $c^3 = c^2$.
88. $c^3 - 6c^2 - 7c = 0$.
89. $d^3 - 7d^2 - 8d = 0$.
90. $h^3 + 3h^2 - 10h = 0$.
91. $k^3 + 4k^2 - 12k = 0$.
92. $m^3 + 6m^2 - 7m = 0$.
93. $n^3 - 8n^2 - 9n = 0$.
94. $p^3 + 10p^2 - 11p = 0$.
95. $q^3 + 4q^2 - 21q = 0$.
96. $r^4 - 11r^2 + 28 = 0$.
97. $x^4 - 12x^2 + 35 = 0$.
98. $y^4 - 13y^2 + 36 = 0$.
99. $z^4 + 9z^2 + 14 = 0$.
100. $w^4 + 10w^2 + 16 = 0$.

CHAPTER XIII

HIGHEST COMMON FACTOR

1. IN this chapter we shall undertake to find whether or not a rational integral expression can be found by which two or more given expressions which are rational and integral with reference to certain letters, *a*, *b*, *c*, *x*, *y*, *z*, etc., can be divided.

2. Two given integral expressions are said to be **prime to each other** if there exists no expression which is integral with reference to the letters involved by which the given expressions may be divided without remainder.

E. g. $x^2 + 3x + 5$ and $x^2 + 2x + 3$ are prime to each other because no monomial or polynomial divisor can be found by which both of these expressions can be divided without remainder.

3. **A common factor** of two or more integral algebraic expressions is an integral expression by which each of them can be divided without remainder.

4. **The highest common factor** (H. C. F.) of two or more integral algebraic expressions is the product consisting of the entire group of factors, numerical and literal, by which each of the given expressions can be divided without remainder.

5. From this definition it appears that, since the highest common factor is the entire common factor, it must be of the highest possible degree with reference to any particular letter.

When the given expressions are monomials, the highest common factor must contain the numerical greatest common divisor (G. C. D.) of the numerical coefficients.

When the given expressions are polynomials, the highest common factor may be the product of a monomial and a polynomial factor. In that case the monomial factor is the highest common factor of all of the terms of the given expressions, if they have common factors; the polynomial factor is the polynomial expression of highest degree with reference to some particular letter by which both of the given expressions can be divided without remainder.

6. When the given expressions are monomials, the degree of the highest common factor is usually reckoned by taking into account all of the letters entering into it.

HIGHEST COMMON FACTOR

BY FACTORING

Monomial Expressions

Ex. 1. Find the highest common factor of $a^5x^2y^3z$ and $a^4x^3y^4w$.

A divisor may be constructed containing the letters a, x, and y, which are common to the given expressions.

Observe that a is found in each of the expressions to at least the fourth power, x to at least the second power, and y to the third power.

As there can be no common factor of higher degree with reference to any of the letters, the entire common factor is $a^4x^2y^3$.

The degree of this highest common factor may be reckoned in terms of any particular letter, but it is usual in such cases to reckon it in terms of all of the letters.

Accordingly the highest common factor $a^4x^2y^3$ is considered as being of the ninth degree.

Ex. 2. Find the H. C. F. of $a^6b^2c^3$, $a^3b^4c^2$, and a^7b^3c.

The highest common factor is a^3b^2c, for each of the letters a, b, and c is found in every one of the expressions, and a^3, b^2, and c are the highest powers of a, b, and c by which all of the given expressions can be divided exactly.

Ex. 3. Find the H. C. F. of $8a^5b^2c$ and $12a^3b^4d$.

Observe that the greatest common divisor of the numerical coefficients 8 and 12 is 4. The literal parts have the highest common factor a^3b^2.

Accordingly the highest common factor sought must be the product of the greatest common divisor, 4, and a^3b^2; that is, $4a^3b^2$.

7. To find the H. C. F. of two or more monomial expressions:

Construct a term containing every letter and prime numerical factor which is common to all of the given expressions, taking each to the lowest power which is found in any one of them.

8. It should be observed that no mention is made in the definition of the highest common factor of that factor's numerical value.

The numerical value of the highest common factor of two given expressions is not equal to the arithmetic greatest common divisor of the numerical values of the given expressions when numerical values are given to the letters.

Hence, numerical checks cannot be used for highest common factors.

9. There is no fundamental connection between the ideas of greatest common divisor in arithmetic and highest common factor in algebra. The word "highest," used in the definition of the algebraic highest common factor, refers simply to the degree of the divisor, either with reference to a particular letter or with reference to all of the letters in a term. (See § 5.)

The word "greatest" is used in the definition of the greatest common divisor of two or more numbers in arithmetic, because the greatest common divisor is the *greatest* divisor by which two or more given expressions can be divided without remainder.

10. It does not follow that one expression represents a greater number than another because it contains higher powers of one or more letters.

E. g. The value of a^2 is numerically less than that of a when a is positive and less than unity. The value of a^2 is equal to that of a when a is unity, and is numerically greater than a for all values of a which are neither unity nor numerically less than unity.

For, when $a = \frac{1}{2}$, $a^2 = \frac{1}{4}$; that is, $a^2 < a$.
$a = 1$, $a^2 = 1$; that is, $a^2 = a$.
$a = 3$, $a^2 = 9$; that is, $a^2 > a$.
$a = -2$, $a^2 = 4$; that is, $a^2 > a$.

EXERCISE XIII. 1

Find the highest common factor of the expressions in each of the following groups:

1. ax^2, ay^2, az^2.
2. a^2bc, ab^2c, abc^2.
3. x^4y^2z, x^3y^3zw, y^4z^2w.
4. a^5b^4c, b^5c^4a, c^5a^4b.
5. ab^2cm, a^2cmy, ab^3cz.
6. ab^2c, a^3bd, a^2b^3e.
7. $x^2y^5z^7$, $x^9y^4z^6$, $x^3y^2z^5$.
8. $10\,a^2$ and $25\,a^3$.
9. $12\,b^2c$ and $16\,bc^2$.
10. $18\,abc$ and $24\,bcd$.

11. $56\,a^4b$ and $14\,a^4b^2$.
12. $42\,a^2y^3$ and $72\,x^3y^2$.
13. $77\,a^2b$ and $33\,ab^2c^2d$.
14. $100\,a^2b^2c^2$ and $55\,a^4b^3c$.
15. $6\,abd^2e$, $18\,a^2b^2de$, $30\,a^3b^3d^3e^3$.
16. $3\,a^3bd$, $27\,ab^2d^3e$, $9\,a^2bde^2$.
17. $2\,abx$, $10\,a^2yx$, $8\,ax^2z$.
18. $4\,xyz^3$, $12\,x^2y^2z$, $16\,x^2y^3z^2$.
19. $20\,x^2y^2zw$, $5\,xyz^2$, $15\,xyzw$.
20. $35\,a^2bcd$, $7\,abcd$, $49\,a^2b^2cd$.
21. $5\,a^4bc^3$, $10\,ab^3c$, $35\,a^2b$.
22. $4\,x^2y^3$, $6\,x^2y^3z^2$, $10\,x^2y^2z$, $12\,x^2y^2z$.
23. $40\,a^3bc$, $24\,ab^3$, $16\,abz$, $88\,a^2by$.
24. $14\,x^2y^2$, $21\,x^3y$, $42\,xy^2z$, $35\,xyz^2$.
25. $3\,ax^2y^2z$, $12\,xy^2m$, $18\,x^3y^2z^2$, $3\,xy^2z$.
26. $50\,x^2y^2zw$, $125\,x^2y^2z^2w$, $100\,x^2y^2z^2w$, $75\,x^2y^2z$.
27. $18\,x^2yz^2w$, $27\,xy^2zw^2$, $81\,x^2yzw^2$, $45\,xy^2z^2w$.

11. When two given expressions are polynomials, the degree of the highest common factor is usually reckoned with respect to some particular letter.

Highest Common Factor of Polynomial Expressions

12. If the given expressions can be readily factored, we may obtain the highest common factor by inspection as follows:

Obtain the prime factors of each of the given expressions. Write the product containing each of the prime factors common to all of the given expressions, taking each factor to the lowest power which is found in any one of them.

Ex. 1. Find the H. C. F. of $a^2 + 2\,ab + b^2$, $a^2 - b^2$, and $3\,ab + 3\,b^2$.

The work may be arranged as follows:

$$a^2 + 2\,ab + b^2 \equiv (a + b)^2$$
$$a^2 - b^2 \equiv (a + b)(a - b)$$
$$3\,ab + 3\,b^2 \equiv 3\,b(a + b).$$

The only factor which is common to all of the expressions is the first power of $a + b$.

Hence the highest common factor is $a + b$.

Ex. 2. Find the H. C. F. of $x^2 + 11\,x + 30$, $x^2 + 3\,x - 10$, and $4\,x + 20$.

We have
$$x^2 + 11\,x + 30 \equiv (x + 5)(x + 6)$$
$$x^2 + 3\,x - 10 \equiv (x + 5)(x - 2)$$
$$4\,x + 20 \equiv 4\,(x + 5)$$

Hence the H. C. F. is $x + 5$.

Ex. 3. Find the H. C. F. of $7a^3b^2(a+b)^3(a-b)^2(x+y)$ and $6a^2b^3(a+b)^2(a-b)^4(x-y)$.

The factors a^2, b^2, $(a+b)^2$, and $(a-b)^2$, and no others, are common to both expressions.

Hence the H. C. F. is their product, $a^2b^2(a+b)^2(a-b)^2$.

EXERCISE XIII. 2

Find the highest common factor of the expressions in each of the following groups:

1. $(a+b)^2$, $(a+b)^3$.
2. $(x+y)^2$, x^2-y^2.
3. $5a(m-n)$, $15a^2(m^2-n^2)$.
4. $(a-2)^2$, a^2-2a.
5. x^2-16, $x^2-9x+20$.
6. x^2+5x+6, x^2+6x+9.
7. a^2-9a, $a^2-11a+18$.
8. $a^2+2ab+b^2$, $(a+b)^3$.
9. a^2-b^2, $a-b$, $a^2-2ab+b^2$.
10. $2x-4y$, x^2-4y^2, $x^2-4xy+4y^2$.
11. $3ab+3b$, $2ax+2x$, $8a^2z+8az$.
12. $a+b$, $a^2+2ab+b^2$, a^3+b^3.
13. $x^2+8x+15$, x^2-x-12, x^2+6x+9.
14. $xz+xw-yz-yw$ and x^2-y^2.
15. x^4-y^4, x^3+y^3, $x+y$.
16. $4a^2+20a+25$, $2a+5$, $8a^3+125$.
17. $3a^2+7ab-20b^2$, a^2-16b^2, $a+4b$.
18. $2ac+3ad-2bc-3bd$ and $4c^2-9d^2$.
19. $a^2+3a-10$, a^2+6a+5, $a^2+2a-15$.
20. $2x^2+x-6$, x^2+3x+2, x^2-4.
21. $2x^2-xy-y^2$, x^2-y^2, $x^2-2xy+y^2$.
22. $56a^3-126a$, $2a-3$, $4a^2-12a+9$.
23. $81a^2-72ab+16b^2$, $9a-4b$, $81a^2-16b^2$.
24. $xz+xw+yz+yw$, $x^2+2xy+y^2$, $ax+bx+ay+by$.
25. $(a-b)^4$, $a^4-2a^2b^2+b^4$, $a^3-a^2b-ab^2+b^3$.

13. The term "greatest common divisor" is not appropriate when applied to algebraic expressions.

When numerical values are assigned to the letters appearing in two algebraic expressions and also to the letters appearing in their highest common factor, it may happen that the value represented by the highest common factor is not the greatest common divisor of the values represented by the given expressions.

E.g. The expressions $x^2 + 1$ and $x + 1$ have no highest common factor and are algebraically prime. Yet, if 3 be substituted for x, $x^2 + 1$ becomes 10, and $x + 1$ becomes 4. The greatest common divisor of 10 and 4 is 2.

The highest common factor of $x^2 + 7x + 12$ and $x^2 + 10x + 21$ is $x + 3$. If x is given some particular value, such as 11, then the expression $x^2 + 7x + 12$ becomes 210 and the expression $x^2 + 10x + 21$ becomes 252, while the highest common factor, $x + 3$, of the algebraic expressions, becomes 14. The greatest common divisor of the numerical values 210 and 252, however, is 42, not 14, which was obtained by substituting 11 for x in the algebraic highest common factor $x + 3$.

If, however, 4 be substituted for x, the values of these expressions and their highest common factor are 56, 77, and 7, respectively. The value, 7, obtained from the highest common factor, is in this case the greatest common divisor of the numerical values 56 and 77 which are represented by the algebraic expressions.

HIGHEST COMMON FACTOR OF POLYNOMIALS

14. When two integral functions of some common letter, x, cannot be readily factored by inspection, the process for finding the highest common factor, or of showing that the functions are prime to each other, may be made to depend upon the following principles:

Principle I. *If an integral expresssion be divisible without remainder by another integral expression which is of the same or of lower degree with reference to some common letter of arrangement, the expression used as divisor is the highest common factor of the two expressions.*

For, if the divisor be contained without remainder in the dividend, it is a factor of the dividend, and hence, by definition, must be the highest common factor.

Principle II. *The highest common factor (if there be one) of two integral polynomial expressions is also the highest common factor of the divisor and the integral remainder obtained by dividing one expression by the other.*

(The following proof may be omitted when the chapter is read for the first time.)

Let D and d represent any two integral expressions arranged according to descending powers of some common letter, x, the degree of the divisor

d being not higher than that of the dividend D with reference to the letter of arrangement.

Let both the quotient Q and the remainder R, obtained by dividing the dividend D by the divisor d, be integral.

(1) *If D and d have a highest common factor, denoted by h, then d and R will have a highest common factor which is the same expression, h.*

If all of the terms of either of the given expressions D or d have common numerical or monomial factors, these should first be removed by division. If the common factors thus removed have a highest common factor this should be set aside as a factor of the required highest common factor of the two given expressions D and d.

Accordingly we shall assume in the following proof that the given expressions D and d have neither numerical nor monomial factors common to all of their terms.

Representing by h the highest common factor of the dividend D, and the divisor d, we may write

$$\left.\begin{aligned} D &\equiv mh, \\ d &\equiv nh. \end{aligned}\right\} \qquad (1)$$

The factors m and n of the right members of identities (1) must be prime to each other, otherwise h would not be the highest common factor of D and d.

Since the dividend D is equal to the divisor d multiplied by the quotient Q, plus the remainder R, we may write

$$D \equiv dQ + R. \qquad (2)$$

Substituting in (2) the values for D and d, from (1), we obtain

$$mh \equiv nhQ + R.$$

Hence $$mh - nhQ \equiv R.$$

Or, $$h(m - nQ) \equiv R. \qquad (3)$$

Since both members of the last identity are integral expressions, and h is a factor of the first member, it must be a factor of the second member also. That is, h must be a factor of R.

Since h is the highest common factor of D and d, it follows that h must be a factor of d.

We have shown by the reasoning above that h is a factor of the remainder R also.

Hence d and R must have a common factor which is at least h.

We will show that d and R have no factor in common other than h.

An expression as a whole is exactly divisible by another expression if its terms are separately divisible by the other expression.

The highest common factor of d and R must be a divisor of the expression $dQ + R$, for it is contained in each of the terms dQ and R.

It follows from the identity $D \equiv dQ + R$ (2) that every factor of the second member $dQ + R$ must also be a factor of the first member D. Such a factor which is common to both members of the identity $D \equiv dQ + R$ must be a factor of both d and D.

Hence the highest common factor of d and R must be a factor also of D, that is, *the highest common factor of d and R cannot exceed h* which is the highest common factor of d and D.

It follows from the reasoning above that *the highest common factor of two given expressions is preserved in the integral remainder (if there be one) after division.*

If this remainder be used as a new divisor and the divisor first used be taken for a new dividend, the principle will hold as before, and the highest common factor will be carried over again, and will be found in the second remainder (if there be one) after division.

(2) *If, however, D and d be prime to each other, then d and R will also be prime to each other.*

If the dividend D and the divisor d have no factors in common, that is, if they are prime to each other, it follows that d and R can have no factors in common.

This is because all of the factors which are common to d and R are contained in each of them, and from the identity $D \equiv dQ + R$ (2), it follows that such factors must be contained also in D.

Hence it appears that if d and R have any common factor it will contradict our assumption that D and d are prime to each other.

From the reasoning above it appears that *if D and d have no highest common factor the divisor d and the remainder R will have no highest common factor.*

15. Development of the Process. From the principle above it appears that, instead of examining two given expressions to determine whether or not they have a highest common factor, we may divide one expression by the other and examine for a highest common factor the divisor and the remainder resulting from the division.

16. The "division" may be carried out according to descending powers of some letter of arrangement until a remainder is obtained which is either a constant or an expression of lower degree than the divisor with reference to the common letter of arrangement according to which the division is being performed.

If the remainder is an expression containing the letter according to which the division is being performed, it may be taken as a new divisor, and the divisor previously used may be taken as a new dividend.

It follows that by repeating this process we must sooner or later reach a stage at which:

Either the remainder-divisor is contained exactly in the corresponding remainder-dividend,

Or, the last remainder is a constant free from the letter of arrangement. In this last case the process of "division" must stop, that is, become "inexact" at this point.

17. In case the last remainder-divisor is contained in the corresponding remainder-dividend, it must, by Principle I, § 14, be the highest common factor of "itself" and the corresponding dividend. Hence, it must be the H. C. F. of all previous pairs of corresponding remainder-divisors and remainder-dividends, and consequently must be the H. C. F. of the original divisor and the original dividend.

18. If, however, the last remainder is different from zero, and is a constant free from the letter of arrangement according to which the "division" is being performed, then the last remainder-divisor and remainder-dividend have no highest common factor, and accordingly the preceding pairs of remainder-divisors and remainder-dividends have none. Therefore the original functions have no highest common factor; that is, they must be prime to each other.

19.* Denoting the original expressions by D and d, and the successive integral quotients by $Q_1, Q_2, Q_3, \cdots \cdot$, and the successive remainders by $R_1, R_2, R_3, \cdots \cdot$, we may indicate the process as follows:

Indicated Process	**Pairs of expressions which all have the same H. C. F.**
$d\,\underline{)\,D}$	$d,\ D$
$R_1\,\underline{)\,d}$	$R_1,\ d.$
$R_2\,\underline{)\,R_1}$	$R_2,\ R_1.$
$R_3\,\underline{)\,R_2}$	$R_3,\ R_2.$
$R_4\,\underline{)\,R_3}$	$R_4,\ R_3.$
etc.	etc.

---

* This section may be omitted when the chapter is read for the first time.

$$\begin{array}{rl} \text{In this process,}\ D & \equiv d\ Q_1 + R_1 \\ d & \equiv R_1 Q_2 + R_2 \\ R_1 & \equiv R_2 Q_3 + R_3 \\ R_2 & \equiv R_3 Q_4 + R_4 \\ & \ldots\ldots\ldots\ldots \end{array}$$

20. The process above, for finding the highest common factor of two integral functions, consists in substituting for the original pair of functions a second pair, for these a third pair, and so on, all pairs of functions having the same highest common factor.

Ex. 1. Find the H. C. F. of $x^3 - x^2 + 2$ and $x^3 - 2x^2 + 3$.

In order to make the arrangement of the process correspond closely with that shown in the next section, § 21, we shall, throughout the work, write the divisor at the left and the "quotient" at the right of the dividend.

The first stage of the work is carried out below:

$$\begin{array}{rrl} \text{Divisor} & \text{Dividend} & \text{Quotient} \\ x^3 - x^2 + 2\ ) & x^3 - 2x^2 + 3 &)\ \underline{1} \\ & \underline{x^3 - x^2 + 2} & \\ \text{First Remainder}\ \ldots & -x^2 + 1 & \end{array}$$

First Stage

The remainder from the division, $-x^2 + 1$, is of lower degree with reference to x than the divisor $x^3 - x^2 + 2$.

By Principle II. § 14, we know that the H. C. F., if there be one, of the divisor and dividend must be contained as a factor of the remainder, $-x^2 + 1$, and must be the H. C. F. of this remainder $-x^2 + 1$ and the divisor $x^3 - x^2 + 2$.

For the second stage of the work we will use this first remainder as a *new divisor* and the first divisor, $x^3 - x^2 + 2$, as a new or *second dividend*, as follows:

$$\begin{array}{rrl} & \text{Original Divisor} & \text{Quotient} \\ \text{First Remainder-Divisor}\ \ -x^2 + 1) & x^3 - x^2 + 2 &)\ \underline{-x + 1} \\ & \underline{x^3 - x\phantom{{}+2}} & \\ & -x^2 + x + 2 & \\ & \underline{-x^2 \phantom{{}+x} + 1} & \\ \text{Second Remainder}\ \ldots\ldots & +x + 1 & \end{array}$$

Second Stage

This second remainder, $x + 1$, must contain the H. C. F. of the original expressions. The work of the third stage of the process may be carried out by taking this second remainder, $x + 1$, as a *new divisor* and the corresponding divisor, $-x^2 + 1$, as a new or *third dividend*, as follows:

	Second Remainder-Divisor	First Remainder-Dividend	Quotient
H. C. F. sought	$x+1$	$)-x^2+1$	$)\underline{-x+1}$
		$\underline{-x^2-x}$	
		$+x+1$	
Third Stage		$\underline{\underline{+x+1}}$	
		0	

At this stage of the work, the divisor $x+1$ is contained without remainder in the corresponding dividend $-x^2+1$.

Hence, by Principle I, § 14, it must be the H. C. F. of "itself" and the corresponding dividend $-x^2+1$, and by Principle II, § 14, it must be the H. C. F. of all previous pairs of corresponding divisors and dividends, and hence, finally, of the original expressions.

21. The different steps of the process may be arranged in the following compact **oblique form:**

	$x^3-x^2+2\ )\ x^3-2x^2+3\ )\underline{\ 1\ }$	**First Stage**
	$\underline{x^3-\quad x^2+2}$	
First Remainder-divisor	$-\quad x^2+1\ )\ x^3-x^2+2\ )\underline{-x+1}$	
	$\underline{x^3-x}$	
	$-x^2+x+2$	**Second Stage**
	$\underline{-x^2\qquad+1}$	
Second Remainder-divisor	**H.C.F. sought** . . . . $x+1\ )-x^2+1\ )\underline{-x+1}$	
	$\underline{x^2-x}$	
	$+x+1$	**Third Stage**
	$\underline{\underline{+x+1}}$	
	0	

22. Whenever during the process of "division" a "quotient" is obtained which is not integral, we may apply the following **Principle:** *At any stage of the process of finding the highest common factor, any remainder-dividend or corresponding remainder-divisor may be multiplied by or divided by any number or expression which is not already a factor of the other.*

23. **Caution.** If at any stage of the process any common factor is removed by division from both the dividend and the corresponding divisor, this common factor must be set aside to be used as a multiplier of the polynomial highest common factor resulting from the "division" process.

24. In practice it is often convenient to remove numerical factors from divisors by division and to introduce numerical factors into dividends by multiplication.

By thus transforming the terms of the divisors and dividends it is possible to avoid fractional "quotients" at any stage of the process.

Ex. 2. Find the H. C. F. of $3x^4 + 2x^3 + 4x^2 + x + 2$ and $2x^4y + 5x^3y + 5x^2y + 3xy$.

We will first remove the common monomial factor xy from the terms of the second expression by division, as follows:

$$2x^4y + 5x^3y + 5x^2y + 3xy \equiv xy(2x^3 + 5x^2 + 5x + 3).$$

Neither of the factors removed (x nor y) is a common factor of all of the terms of the first expression $3x^4 + 2x^3 + 4x^2 + x + 2$. Hence, neither x nor y can be contained as a factor of the highest common factor of the given expressions.

Accordingly the factors x and y which were removed by division from the second expression may be neglected.

The expressions $2x^3 + 5x^2 + 5x + 3$ and $3x^4 + 2x^3 + 4x^2 + x + 2$ may now be used as divisor and dividend respectively to find the desired highest common factor.

The fractional "quotient," $\frac{3}{2}x$, which would be obtained by dividing the first term $3x^4$ of the dividend by the first term $2x^3$ of the divisor, may be avoided by applying the principle of § 22, that is, by multiplying all of the terms of the dividend $3x^4 + 2x^3 + 4x^2 + x + 2$ by 2.

Since the factor, 2, thus introduced into the dividend by multiplication, is not also a factor of all of the terms of the divisor, the value of the highest common factor sought will not be affected.

The steps of the process are shown below :

$$
\begin{array}{lrl}
\textbf{Modified Divisor.} & \textbf{Original Dividend.} & \textbf{Place for Quotients.}\\
2x^3+5x^2+5x+3\,) & 3x^4+\ 2x^3+\ 4x^2+\ \ x+\ 2 & \\
 & \underline{2\ \text{(To avoid fractional coefficients).}} & \\
 & 6x^4+\ 4x^3+\ 8x^2+\ 2x+\ 4\,) & \underline{3x}\\
 & \underline{6x^4+15x^3+15x^2+\ 9x\qquad} & \\
 & -11x^3-\ 7x^2-\ 7x+\ 4 & \textbf{First Stage.}\\
 & \underline{2\ \text{(To avoid fractional coef.).}} & \\
 & -22x^3-14x^2-14x+\ 8\,) & \underline{-11}\\
 & \underline{-22x^3-55x^2-55x-33} & \\
\textbf{Numerical factor 41 removed, since it cannot belong to the H. C. F. sought.} & 41\,)\,\underline{+41x^2+41x+41} & \\
 & x^2+\ \ x+\ 1 &
\end{array}
$$

The remainder $x^2 + x + 1$ is of lower degree with reference to x than the corresponding modified divisor $2x^3 + 5x^2 + 5x + 3$.

The process may now be continued by using this remainder $x^2 + x + 1$ as a new divisor, and the first divisor, $2x^3 + 5x^2 + 5x + 3$, as a new dividend, as follows :

$$
\begin{array}{llrl}
 & \textbf{Remainder-divisor} & \textbf{Modified divisor used as dividend} & \textbf{Place for quotients} \\
\textbf{H. C. F. sought} & x^2 + x + 1\ ) & 2x^3 + 5x^2 + 5x + 3 &)\ \underline{2x} \\
 & & \underline{2x^3 + 2x^2 + 2x} & \\
 & & 3\ \underline{)\ 3x^2 + 3x + 3} & \textbf{Second Stage} \\
\textbf{Numerical factor removed} \ \ldots\ldots & & x^2 + \ \ x + 1 &)\ \underline{1} \\
 & & \underline{\underline{x^2 + \ \ x + 1}} & \\
 & & 0 &
\end{array}
$$

The numbers written in the place for quotients cannot be considered quotients in the ordinary sense, owing to the introduction and rejection of factors during the work.

The division becomes exact when $x^2 + x + 1$ is used as a divisor. Hence, by Principles I and II, § 14, it must be the H. C. F. of the given expressions.

25. It will be seen in the work above that, each time a remainder is taken as a new divisor, the first term is contained an integral number of times in the first term of the corresponding dividend. It will be found that this very seldom happens in practice.

26. It should be observed that the successive "divisions" performed when carrying out the process for finding the highest common factor are not commonly "divisions" in the ordinary sense, since at different stages of the work we may introduce or remove factors from either the dividend or the divisor.

27. For this reason the expressions written in the places of quotients are not "quotients" in the ordinary sense, and since in the result we are not at all concerned with the "quotients," we may neglect writing them altogether.

28. It will be seen that in the "oblique" arrangement of the work used in the examples of §§ 21, 24, it is necessary to copy again each divisor when it is used as a new dividend. Furthermore, when arranged in compact form, the work tends to extend downward in an oblique direction, toward the right.

These objections may be overcome by adopting a vertical arrangement for the work.

The contrast between the oblique and the vertical arrangements may be shown by carrying out the work of the example of § 24 in vertical arrangement.

The given expressions are separated by vertical lines, as shown below, and the "quotients" are placed in the side columns nearest their dividends.

The divisors will be found sometimes on the left and sometimes on the right of the corresponding dividends.

Place for Quotients	Dividend and Divisor		Place for Quotients
	$2x^3+5x^2+5x+3$	$3x^4+2x^3+4x^2+x+2$	
$2x$	$2x^3+2x^2+2x$	2 (To avoid frac. coef.)	
	$3\,)\,3x^2+3x+3$	$6x^4+4x^3+8x^2+2x+4$	$3x$
1	x^2+x+1	$6x^4+15x^3+15x^2+9x$	
	x^2+x+1	$-11x^3-7x^2-7x+4$	**First Stage**
	0	2 (To avoid frac. coef.)	
		$-22x^3-14x^2-14x+8$	-11
		$-22x^3-55x^2-55x-33$	
		$41\,)\,41x^2+41x+41$	
		x^2+x+1 **H. C. F. sought.**	**Second Stage**

29. In finding the H. C. F. by the long "division" process, the simple numerical or monomial factors must first be removed by division from the terms of the expressions.

The H. C. F. (if there be any) of these factors thus removed must be set aside to be used as a multiplier of the polynomial highest common factor resulting from the "division" process.

Ex. 3. Find the H. C. F. of $2a^5b^3-4a^4b^4+4a^3b^5-2a^2b^6$ and $a^5b-2a^4b^2+a^3b^3$.

	$a^3b \,)\, \underline{a^5b + 2a^4b^2 + a^3b^3}$	$2a^2b^3 \,)\, \underline{2a^5b^3 - 4a^4b^4 + 4a^3b^5 - 2a^2b^6}$	
	$a^2 - 2ab + b^2$	$a^3 - 2a^2b + 2ab^2 - b^3$	a
a	$\underline{a^2 - ab}$	$\underline{a^3 - 2a^2b + ab^2}$	
$-b$	$-ab + b^2$	$b^2 \,)\, \underline{ab^2 - b^3}$	
	$\underline{\underline{-ab + b^2}}$	**H.C.F. of modified expressions.** $a - b$	
	0		

The H. C. F. of a^3b and $2a^2b^3$, which were removed by division at the beginning of the work, is a^2b. Hence the H. C. F. sought is the product of a^2b and the polynomial highest common factor $(a - b)$. That is, the highest common factor is $a^2b(a - b)$.

30. The process for finding the highest common factor has the peculiarity of not only furnishing the highest common factor (if it exists), but also of indicating when there is none.

31. To find the H.C.F. of two integral functions (if it exists) we may proceed as follows:

After having first removed all common monomial factors, treat the modified expressions as dividend and divisor.

If the degrees of the expressions are the same with reference to the common letter of arrangement either expression may be used as divisor, but if the degrees are not the same then the expression of lower degree must be used as divisor.

Continue the process of "division" until the degree of the remainder with reference to the letter of arrangement is at least as low as the degree of the divisor.

Using this remainder as a new divisor, and the first divisor as a new dividend, the process may be repeated until finally either the "division" becomes exact, — in which case the last divisor used, multiplied by such common factors as may have been removed at the beginning of the work, is the H.C.F. sought, — or until a remainder is obtained which is free from the letter of arrangement, in which case no H.C.F. exists.

During the process the separate dividends and divisors may be divided by or multiplied by such numerical or literal factors as are necessary to avoid fractional coefficients in the "quotients" in the course of the "division."

Exercise XIII. 3

Find the H. C. F. of each of the following groups of expressions :

1. $x^3 + 2x^2 - 8x - 16,\ x^3 + 3x^2 - 8x - 24$.
2. $z^2 - 5z + 4,\ z^3 - 5z^2 + 4$.
3. $a^3 - a^2 - 5a - 3,\ a^3 - 4a^2 - 11a - 6$.
4. $b^3 + 2b^2 - 13b + 10,\ b^3 + b^2 - 10b + 8$.
5. $4c^3 - 3c^2 - 24c - 9,\ 8c^3 - 2c^2 - 53c - 39$.
6. $2d^2 - 5d + 2,\ 12d^3 - 8d^2 - 3d + 2$.
7. $g^3 + 2g^2 + 2g + 1,\ g^3 - 2g - 1$.
8. $h^4 - 2h^2 + 1,\ h^4 - 4h^3 + 6h^2 - 4h + 1$.
9. $2s^3 + 3s^2t - t^3,\ 4s^3 + st^2 - t^3$.
10. $w^4 - 2w^3 + w,\ 2w^4 - 2w^3 - 2w - 2$.
11. $2x^2 - 5x + 2$ and $2x^3 - 3x^2 - 8x + 12$.
12. $4a^3 - a^2b - ab^2 - 5b^3$ and $7a^3 + 4a^2b + 4ab^2 - 3b^3$.
13. $3y^4 - 4y^3 + 2y^2 + y - 2$ and $3y^4 + 8y^3 - 5y^2 - 6y$.
14. $b^3 + 3b^2 + 4b + 2$ and $2b^3 + b^2 + 1$.
15. $2c^4 + 9c^3 + 14c + 3$ and $3c^4 + 15c^3 + 5c^2 + 10c + 2$.
16. $3d^3 - 3d^2 - 33d + 9$ and $6d^4 + 6d^3 - 42d^2 - 18d$.
17. $3h^3 - 9h^2 + 21h - 63$ and $2h^4 + 19h^2 + 35$.
18. $6a^3 - 6a^2b + 2ab^2 - 2b^3$ and $4a^2 - 5ab + b^2$.
19. $m^4 + 7m^3 + 7m^2 - 15m$ and $2m^3 - 4m^2 - 26m + 220$.
20. $4n^5 + 14n^4 + 20n^3 + 70n^2$ and $8n^7 + 28n^6 - 8n^5 - 12n^4 + 56n^3$.
21. $2w^4 - 2w^3 + 4w^2 + 2w + 6$ and $3w^4 + 6w^3 - 3w - 6$.
22. $x^4 + x^3 - 9x^2 - 3x + 18$ and $x^5 + 6x^2 - 49x + 42$.
23. $8z^4 - 6z^3 + 3z^2 - 3z + 1$ and $18z^3 - 3z^2 - 15z + 6$.
24. $64g^5 - 3g^2h^3 + 5gh^4$ and $20g^4 - 3g^3h + h^4$.
25. $2s^4 - 7s^3 + 9s^2 - 8s - 5$ and $6s^4 - 11s^3 - 16s^2 + 15s$.

The H. C. F. of three or more expressions may be obtained by first finding the H. C. F. of any two of them, then the H. C. F. of this result and a third expression, and so on.

This is because any factor which is common to three or more expressions must be a factor of the H. C. F. of any two of them.

26. $m^2 + m - 6,\ m^3 - 2m^2 - m + 2$ and $m^3 + 3m^2 - 6m - 8$.
27. $b^3 + 7b^2 + 5b - 1,\ 3b^3 + 5b^2 + b - 1$ and $3b^3 - b^2 - 3b + 1$.
28. $a^3 - 6a^2 + 11a - 6,\ a^3 - 9a^2 + 26a - 24$ and $a^3 - 8a^2 + 19a - 12$.

29. $c^3 - 9c^2 + 26c - 24$, $c^3 - 10c^2 + 31c - 30$ and $c^3 - 11c^2 + 38c - 40$.

30. $d^3 - 1$, $d^3 - d^2 - d - 2$ and $d^3 - 2d^2 - 2d - 3$.

31. $c^5 - c^3 - c^2 - 2c - 1$, $c^5 + c^3 + c^2 + 1$ and $c^5 + 2c^4 + c^3 + c^2 - 1$.

32. $h^6 - 1$, $h^6 - h^5 - h^3 - h - 1$ and $h^6 - h^5 + h^4 - h^3 + h^2 - h$.

33. $5x^4 + 7x^3 - 7x^2 - 6x + 4$, $5x^4 + 12x^3 - 15x^2 - 14x + 12$ and $5x^4 - 3x^3 + 4x^2 + 8x - 8$.

34. $4s^4 + 4s^3 - 13s^2 - 15s$, $4s^3 - 8s^2 - 7s + 15$ and $4s^4 - 2s^3 - 4s^2 - 3s - 15$.

CHAPTER XIV

LOWEST COMMON MULTIPLE

1. A **common multiple** of two or more integral algebraic expressions is an integral expression which may be divided by each of them without remainder.

2. The **lowest common multiple** or L. C. M. of two or more integral algebraic expressions is the integral expression of lowest degree which may be divided by each of them without remainder.

The lowest common multiple of two numbers or expressions which are prime to each other must accordingly be their product.

When dealing with monomials, the degree of the lowest common multiple may be reckoned in terms of several letters, while if we are considering integral polynomial functions of some single letter, say x, the degree of the lowest common multiple is determined by the powers of the common letter of arrangement, x.

Lowest Common Multiple of Monomials and Polynomials which can be readily factored

3. In order to be exactly divisible by each of the given expressions, the lowest common multiple of two or more given expressions must contain every prime factor of each of them. Each prime factor appearing in it must be raised to a power equal to the highest power of this factor which is found in any one of the given expressions.

*Hence, **to find the lowest common multiple of two or more expressions,** construct an expression consisting of the product of all of the different prime factors (numerical and literal) found in the given expressions, each prime factor being raised to the highest power which is found in any one of them.*

Ex. 1. Find the L. C. M. of $32a^2b$, $6a^3b^2c$, and $12ab^3c^2d$.

We may exhibit the prime factors of the numerical coefficients, together with the literal factors, by writing the expressions as follows:

$$32\,a^2b \equiv 2^5 \cdot a^2b$$
$$6\,a^3b^2c \equiv 2 \cdot 3 \cdot a^3b^2c$$
$$12\,ab^3c^2d \equiv 2^2 \cdot 3 \cdot ab^3c^2d$$

The different prime factors are found to be 2, 3, a, b, c, and d, and the L. C. M. must be constructed by writing the product of these prime factors, each factor being raised to the highest power which is found in any one of the given expressions.

Hence the L. C. M. is $2^5 \cdot 3 \cdot a^3b^3c^2d \equiv 96\,a^3b^3c^2d$.

Ex. 2. Find the L. C. M. of $6\,ab^2(a+b)^2$ and $4\,a^2b(a^2-b^2)$.

$$6\,ab^2(a+b)^2 \equiv 2 \cdot 3\,ab^2(a+b)^2$$
$$4\,a^2b(a^2-b^2) \equiv 2^2 \cdot a^2b(a+b)(a-b)$$

Hence the L. C. M. is

$$2^2 \cdot 3 \cdot a^2b^2(a+b)^2(a-b) \equiv 12\,a^2b^2(a+b)^2(a-b)$$
$$\equiv 12\,a^5b^2 + 12\,a^4b^3 - 12\,a^3b^4 - 12\,a^2b^5.$$

Ex. 3. Find the L. C. M. of $x^2+7x+12$, x^2-16, and x^2+6x+9.

$$x^2+7\,x+12 \equiv (x+3)(x+4)$$
$$x^2-16 \equiv (x+4)(x-4)$$
$$x^2+6\,x+9 \equiv (x+3)(x+3)$$

Hence the L. C. M. is $(x+3)^2(x+4)(x-4) \equiv x^4+6\,x^3-7\,x^2-96\,x-144$.

Exercise XIV. 1

Find the lowest common multiple of the expressions in each of the following groups :

1. a, b, c.
2. xy, yz, zw.
3. a^2b, b^2c, c^2a.
4. b^3cd, bc^3d, bcd^3.
5. g^2h, g^2x, g^2w.
6. $4\,r$, $6\,s$, $9\,t$.
7. $8\,mn$, $10\,m^2n$, $15\,mn^2$.
8. $5\,k^2zy$, $6\,m^2xy$, $10\,n^2wy$.
9. ab^2, abc^2, $abcd^2$.
10. $14\,sx^4w$, $28\,ty^4w$, $2\,x^4y^4$.
11. $6\,g^2h^2$, $16\,g^3k^2$, $9\,g^4s^2$, $18\,g^5t^2$.
12. $3\,c^2d^2e^2$, $4\,d^2e^3f^2$, $5\,x^3y^4z^3$, $6\,y^3z^4w^3$.
13. $x^2+5\,x+6$, $x^2+6\,x+8$.
14. $y^2+2\,y-15$, $y^2-4\,y+3$.
15. $z^2-15\,z+54$, $z^2-18\,z+81$.
16. $ab-5\,b$, a^2-25, $a^2-10\,a+25$.
17. m^2-n^2, m^3-n^3, $m^2+2\,mn+n^2$.
18. $r+s$, r^2+s^2, r^3+s^3.

19. $w^4 - 1,\ w^2 - 1,\ w - 1.$
20. $h + 1,\ h + 2,\ h^2 + 5h + 6.$
21. $x^2 - 1,\ y^2 - 1,\ x^2 - y^2.$
22. $2a^2 + a - 3,\ 3a^2 + a - 4,\ 4a^2 + a - 5.$
23. $5c^2 + 26c + 5,\ 5c^2 + 31c + 6,\ 5c^2 + 36c + 7.$
24. $cx + dx + cy + dy,\ x(x + 2y) + y^2,\ c^2 - d^2.$

Lowest Common Multiple by means of Highest Common Factor

4. If two integral polynomial functions of a single letter cannot be readily factored by inspection, we may find their lowest common multiple by making use of their highest common factor.

For, representing any two integral expressions, which are not prime to each other, by A and B, and denoting their highest common factor by h, we may write

$$\left.\begin{aligned} A &\equiv ah, \\ B &\equiv bh. \end{aligned}\right\} \qquad (1)$$

Since h represents the highest common factor of A and B, it contains every factor which is common to A and B, and hence a and b must be prime to each other.

The lowest common multiple of A and B, represented by L, must be the lowest common multiple of the right members ah and bh of the identities (1).

Hence we have $\qquad L \equiv abh.$

The value of the right member abh remains unaltered if it be successively multiplied by and divided by h.

Hence
$$\begin{aligned} L &\equiv abh \times h \div h \\ &\equiv \frac{(ah)(bh)}{h}. \end{aligned}$$

That is,
$$L \equiv \frac{AB}{h}. \qquad (2)$$

Or, the lowest common multiple of two integral expressions may be found by dividing their product by their highest common factor.

5. *The lowest common multiple of two expressions may be found by dividing either one of them by their highest common factor, and multiplying the quotient by the other expression.*

For, from $L \equiv \frac{AB}{h}$, (See § 4)

we have $L \equiv \frac{(ah)B}{h}$,

or $L \equiv aB$.

Also from $L \equiv \frac{AB}{h}$,

we have $L \equiv \frac{A(bh)}{h}$,

or $L \equiv Ab$.

6. *The product obtained by multiplying the lowest common multiple of two expressions by their highest common factor is equal to the product of the two given expressions.*

For, from the identity $L \equiv \frac{AB}{h}$,

we obtain, multiplying both numbers by h,

$$Lh \equiv AB.$$

Ex. 1. Find the L. C. M. of $x^3 + 5x^2 + x - 10$ and $x^3 + x^2 - x + 2$.

The H. C. F. is found to be $x + 2$. The quotient obtained by dividing the first expression $x^3 + 5x^2 + x - 10$ by $x + 2$ is $x^2 + 3x - 5$. Hence, the L. C. M. must be the product obtained by multiplying this quotient by the other expression, that is,

$$(x^3 + x^2 - x + 2)(x^2 + 3x - 5) \equiv x^5 + 4x^4 - 3x^3 - 6x^2 + 11x - 10.$$

EXERCISE XIV. 2

Find the lowest common multiple of the expressions in each of the following groups:

1. $a^4 + 5a^3 + 5a^2 - 5a - 6$ and $a^3 + 6a^2 + 11a + 6$.
2. $b^3 + 3b^2 - b - 3$ and $b^3 + 4b^2 + b - 6$.
3. $c^3 + c^2 - 8c - 6$ and $2c^3 - 5c^2 - 2c + 2$.
4. $6x^3y - 7ax^2y - 20a^2xy$ and $6x^2 + 2ax - 8a^2$.
5. $2d^4 - 2d^3 - 2d - 2$ and $d^4 - 2d^3 + d$.
6. $y^3 - 6y^2 + 11y - 6$ and $y^3 - 8y^2 + 19y - 12$.
7. $14s^3 - 4s^2 - 10$ and $14s^3 + 24s^2 + 20s + 10$.
8. $2r^3 + 12r^2 + 22r + 12$ and $r^4 + r^3 - 4r^2 - 4r$.
9. $6z^3 - 8z^2 - 17z - 6$ and $12z^3 - z^2 - 21z - 10$.
10. $w^3 + 3w^2 + 4w + 2$ and $2w^3 + w^2 + 1$.
11. $h^3 - h^2 + h + 3$ and $h^4 + h^3 - 3h^2 - h + 2$.
12. $6k^3 + 15k^2 - 6k + 9$ and $9k^3 + 6k^2 - 51k + 36$.

To find the lowest common multiple of three or more integral polynomial algebraic expressions first find the lowest common multiple,

L_1, of any two of them, then the lowest common multiple, L_2, of this result and a third expression, and so on, until all of the given expressions have been used.

The lowest common multiple last obtained will be the expression of lowest degree which may be divided without remainder by each of the given expressions.

13. $a^3 + a^2 - 10a + 8$; $a^2 - 3a + 2$ and $a^3 - 4a^2 + 5a - 2$.

14. $m^3 + 2am^2 + 4a^2m + 8a^3$; $m^3 - 2am^2 + 4a^2m - 8a^3$ and $m^3 + 4a^2m + am^2 + 4a^3$.

15. $2n^2+3n-20$; $6n^3-25n^2+21n+10$ and $2n^3-5n^2+6n-15$.

16. $x^2 - 3bx + 2b^2$; $x^2 - 5bx + 4b^2$ and $3x^2 - 19bx + 28b^2$.

MENTAL EXERCISE XIV. 3

Review

Simplify each of the following :

1. $(x + y)(x^2 - xy + y^2)$.
2. $(a - b)(a^2 + ab + b^2)$.
3. $(b^5 + c^5) \div (b + c)$.
4. $(c^6 - d^6) \div (c^2 - d^2)$.

Distinguish between

5. $a + bc$ and $(a + b)c$.
6. $x + yz + w$, $(x + y)z + w$ and $(x + y)(z + w)$.

Perform the following multiplications :

7. $(a + b + x)(a + b - x)$.
8. $(a + b + 7)(a + b - 7)$.
9. $(x - y + 1)(x + y - 1)$.
10. $(a+b+m+n)(a+b-m-n)$.
11. Find the continued product of $x + y$, $x^2 - y^2$, $x^2 + y^2$ and $x^4 + y^4$.
12. Show that $x^4 - 2x^3 + x^2$ is the square of a binomial.

Are the following expressions conditionally or identically equal ?

13. $3x$ and $2x + x$.
14. $3x$ and $2 + 1$.
15. $(a + b)^2$ and $a^2 + 2ab + b^2$.
16. $a + b$ and $b + a$.
17. $a + b$ and $b + d$.
18. $\frac{x}{2}$ and 1.

Are the following expressions identical ?

19. $a^3 + b^3$ and $(a + b)^3$.
20. $(a+b)^2(a-b)^2$ and $(a^2-b^2)^2$.
21. $a^2 - b^2$ and $(a - b)^2$.
22. $(x - 1)^2$ and $x^2 - 1$.

Of the following equations, select those which are equivalent to the equation $2x - 3y = 5$:

23. $4x - 6y = 10$; $6x - 9y = 15$; $8x + 12y = 30$; $10x - 15y = 25$.

Replace each of the following single equations by a set of separate equations which, taken together, are equivalent to it:

24. $(x + 5)(x - 3) = 0$.
25. $(y - 2)(y + 7) = 0$.
26. $(z - 4)(z - 8) = 0$.
27. $(w + 1)(w + 9) = 0$.
28. $(a + 6)(a + 10)(a + 12) = 0$.
29. $(b - 2)(b + 5)(b - 15) = 0$.

Construct single equations which are equivalent to the following pairs of separate equations:

30. $x - 2 = 0$ and $x - 3 = 0$.
31. $y - 4 = 0$ and $y - 6 = 0$.
32. $z + 5 = 0$ and $z - 1 = 0$.
33. $w + 7 = 0$ and $w + 8 = 0$.
34. $x = 1$ and $x = 2$.
35. $y = 5$ and $y = 7$.

Solve the following conditional equations for x, y, z and w:

36. $7x - 7 = 0$.
37. $2y - 8 = 0$.
38. $3z - 12 = 0$.
39. $6 - 2w = 0$.

Show that the following identities are true:

40. $2(a + b)3(a - b) \equiv 6(a^2 - b^2)$.
41. $a(a^2 + b^2) - a^2(a - b) \equiv b(a^2 + b^2) + b^2(a - b)$.

CHAPTER XV

RATIONAL FRACTIONS

1. In order to obtain as a quotient a number previously defined, that is, a positive or a negative number, the dividend must be a multiple of the divisor.

E. g. If the divisor be 7, the dividend must be some multiple of 7, that is,

$$7,\ 14,\ 21, \text{ etc., or } -7,\ -14,\ -21, \text{ etc.}$$

We may obtain by actual division

$$21 \div 7 = 3, \quad 35 \div 7 = 5, \quad -14 \div 7 = -2, \text{ etc.},$$

the quotients in all cases being numbers previously defined, that is, either positive or negative whole numbers.

2. From this point of view a combination of symbols has no meaning when it consists of a dividend which is not a multiple of the divisor.

E. g. In this sense, $12 \div 7$, $3 \div 7$, $-2 \div 7$, have no meaning, since in such cases the division can never be performed exactly.

It will be noted that expressions such as those above have the forms of quotients, and by the Principle of No Exception our idea of number must be extended to include such *quotient forms* as numbers. We shall admit such expressions to our calculations and reckon with them as with ordinary quotients.

3. Negative numbers were invented because of the impossibility of subtraction in all cases, and broken numbers or fractions because of the impossibility of division in all cases. Combining these two extended ideas of number, we are led to the idea of negative fractional numbers.

Thus, the idea of "fractured or broken numbers" arises from division in a way similar to that in which the idea of negative numbers arose in connection with subtraction.

4. A fraction is expressed by writing the dividend above the divisor and separating the two by a horizontal line or "stroke of division." Thus, the fractional notation $\frac{a}{b}$ and the solidus notation a/b each express the same as $a \div b$.

5. In the quotient symbol, or fraction, the divisor or part written below the horizontal line is called the **denominator** of the fraction, since it *names* or *denominates* the number of parts into which unity is supposed to be divided. The number above the line indicates how many of these parts are to be taken. Hence this number is called the **numerator,** since it *enumerates* or *counts* the parts.

6. When either the numerator or the denominator of a fraction is a polynomial, the horizontal stroke of division separating them serves both as a sign of division and as a sign of grouping.

E. g.
$$x + \frac{y+z}{3} \equiv x + (y+z) \div 3.$$
$$\frac{a+b}{c+d} \equiv (a+b) \div (c+d).$$

7. Algebraic fractions have the same properties and are governed by the same rules of calculation as arithmetic fractions.

E. g. As $1/4$ is to be regarded as being one of the parts obtained by separating or dividing unity into four equal parts, so $1/b$ is to be taken as representing one of the b equal parts into which unity may be divided, according to our extended idea of number. Furthermore, as $3/4$ is understood as meaning three of four equal parts of unity, a/b is to be understood as meaning a of the b equal parts of unity.

8. For a broken number or fraction, we may write as our

$$\textbf{Definition Formula, } \frac{a}{b} \times b \equiv a.$$

From this it appears that *the product obtained by multiplying the quotient symbol by the divisor is identically equal to the dividend.*

9. The **terms of a fraction** are the numbers or expressions separated by the line or symbol of division.

10. Since zero can never be used as a divisor, the ordinary laws of reckoning cannot be applied to fractions whose denominators are zero.

11. A **rational algebraic fraction** is the quotient obtained by dividing one rational integral function by another.

12. A fraction is said to be **proper** if the degree of the numerator is lower than that of the denominator when the degrees of both are reckoned in terms of some *common* letter of reference.

E. g. $\frac{2}{x+5}$, $\frac{x}{x^2+1}$ are proper fractions.

13. A fraction is said to be **improper** if the degree of the numerator is equal to or greater than that of the denominator, reckoned in terms of some *common* letter of reference.

E. g. $\frac{a}{a+2}$, $\frac{b^2}{b+1}$ are improper fractions,

but $\frac{a}{a^2+2}$, $\frac{b^2}{b^4+1}$ are proper fractions.

14. We say that a given value has the **form of a fraction** when it can be expressed as a quotient.

E. g. Each of the following fractions represents the value 3:

6/2, 30/10, 12/4, 3/1.

Again, a^5b/a^3, a^3b^3/ab^2, $7\,a^4b^4/7\,a^2b^3$, all represent a^2b.

15. To **reduce** a fraction is to change its form without altering its value.

16. When a quotient can be so transformed as to become integral, the dividend is said to be **exactly divisible** by the divisor.

17. When a quotient cannot be so transformed as to become integral it is said to be *fractional*, or for emphasis, *essentially fractional.*

18. A fraction considered as a whole, that is, as a quotient, must be regarded as possessing quality, that is, as being either positive or negative.

The quality of a fraction as a whole may be indicated by writing either + or — as a quality sign directly before the horizontal stroke of division, separating the numerator from the denominator.

E. g. The expression $+\frac{a+b}{c}$ is to be understood as meaning that the fraction as a whole is positive, while the expression $-\frac{x}{y-z}$ is to be understood as meaning that the quotient resulting from dividing x by $y-z$ is negative.

19. In a fraction, the numerator as a whole, and also the denominator as a whole, may each be regarded as possessing quality. Hence, as each may have a sign independent of the sign of the fraction as a whole, we are led to consider **three signs** in connection with any fraction, as follows:

$$+\frac{+a}{+b};\quad +\frac{-a}{+b};\quad +\frac{+a}{-b};\quad +\frac{-a}{-b};$$
$$-\frac{+a}{+b};\quad -\frac{-a}{+b};\quad -\frac{+a}{-b};\quad -\frac{-a}{-b}.$$

20. Since a fraction is an indicated quotient, the fundamental laws of signs for multiplication and division may be applied directly. (See Chap. V. §§ 9, 43.)

Hence we have the following **Principles relating to the signs of a fraction:**

(i.) *The signs of both numerator and denominator as wholes may be changed from + to – or from – to + without altering the value of the quotient, and hence, without affecting the sign before the whole fraction.*

E. g. $+\frac{-a}{-b}\equiv+\frac{+a}{+b};\ -\frac{+a}{-b}\equiv-\frac{-a}{+b};\ -\frac{-a}{-b}\equiv-\frac{+a}{+b};\ -\frac{+a}{-b}\equiv-\frac{-a}{+b}.$

(The following proof may be omitted when the chapter is read for the first time.)

The changes of signs in the first illustration above may be explained as follows:

$$\begin{aligned}+\frac{-a}{-b}&\equiv+[-a]\div[-b]\\&\equiv+[-a]\div[-b]\times(-1)\div(-1)\\&\equiv+[(-a)\times(-1)]\div[(-b)\times(-1)]\\&\equiv+[+a]\div[+b]\\&\equiv+\frac{+a}{+b}.\end{aligned}$$

The changes of signs in the remaining illustrations may be explained in a similar way.

Ex. 1. Express $\frac{-x-y}{n-m}$ as an equivalent fraction containing the least possible number of negative signs.

Reversing the signs of both numerator and denominator as wholes, we have
$$\frac{-x-y}{n-m}\equiv\frac{-(-x-y)}{-(n-m)}\equiv\frac{x+y}{m-n}.$$

Since if the sign of either the whole numerator or of the whole denominator be changed, the sign of the fraction as a quotient will be changed, it follows that the sign of the fraction will be restored by reversing at the same time the sign before the fraction as a whole. Hence,

(ii.) *The sign before a fraction may be changed, provided that the sign of either the whole numerator or of the whole denominator be also changed.*

E. g. $-\frac{-a}{+b} \equiv +\frac{+a}{+b}$; $-\frac{+a}{-b} \equiv +\frac{+a}{+b}$; $-\frac{-a}{+b} \equiv +\frac{+a}{+b}$; $-\frac{-a}{-b} \equiv +\frac{-a}{+b}$.

The changes of signs in these illustrations may be explained by using a method of reasoning similar to that employed for the illustration of Principle (i.)

Ex. 2. Transform $-\frac{1}{b-a}$ into an equivalent positive fraction.

An equivalent positive fraction may be obtained by reversing the quality of the fraction as a whole, and also the quality of the denominator as a whole.

Accordingly, we have $-\frac{1}{b-a} \equiv \frac{1}{-(b-a)} \equiv \frac{1}{a-b}$.

Exercise XV. 1

Express each of the following negative fractions as an equivalent positive fraction :

1. $-\frac{1}{-a}$.
2. $-\frac{2}{-b}$.
3. $-\frac{-c}{5}$.
4. $-\frac{1-a}{2}$.
5. $-\frac{3}{b-5}$.
6. $-\frac{y-x}{z}$.
7. $-\frac{m}{b-c}$.
8. $-\frac{-5}{x-4}$.
9. $-\frac{c-a}{b+c}$.
10. $-\frac{x+y}{z-y}$.
11. $-\frac{-a}{-b+c}$.
12. $-\frac{1}{b-a+c}$.
13. $-\frac{1}{c-b-a}$.
14. $-\frac{x-y+z}{z-x-y}$.
15. $-\frac{-b+a}{-x-y-z}$.
16. $-\frac{1}{(y-x)(x+y)}$.
17. $-\frac{a}{(a-b)(c-b)}$.
18. $-\frac{-a}{(b-a)(-b-c)}$.

Show that each of the following identities is true :

19. $\frac{1}{b-a} \equiv -\frac{1}{a-b}.$

20. $\frac{-1}{c-b} \equiv \frac{1}{b-c}.$

21. $\frac{y-x}{3} \equiv -\frac{x-y}{3}.$

22. $\frac{y-x}{w-z} \equiv \frac{x-y}{z-w}.$

23. $\frac{-a-b}{b-a} \equiv \frac{a+b}{a-b}.$

24. $\frac{x}{z-y} \equiv -\frac{x}{y-z}.$

25. $\frac{a-2}{3} \equiv -\frac{2-a}{3}.$

26. $\frac{1}{b-a-c} \equiv -\frac{1}{a-b+c}.$

27. $\frac{1}{c-b-a} \equiv -\frac{1}{a+b-c}.$

28. $-\frac{b-c-a}{b} \equiv \frac{a-b+c}{b}.$

29. $\frac{b-c-a}{c-a-b} \equiv \frac{a-b+c}{a+b-c}.$

30. $\frac{c}{a(c-b)} \equiv -\frac{c}{a(b-c)}.$

31. $\frac{1}{(a-b)(c-b)} \equiv -\frac{1}{(a-b)(b-c)}.$

32. $\frac{1}{(b-a)(c-b)} \equiv \frac{1}{(a-b)(b-c)}.$

33. $\frac{1}{(y-x)^2} \equiv \frac{1}{(x-y)^2}.$

34. $-\frac{1}{(y-x)^3} \equiv \frac{1}{(x-y)^3}.$

35. $\frac{-1}{(1-x)^5} \equiv \frac{1}{(x-1)^5}.$

36. $\frac{1}{(b^2-a^2)^3} \equiv -\frac{1}{(a^2-b^2)^3}.$

37. $\frac{1}{(y-x)(z-y)(z-x)} \equiv \frac{1}{(x-y)(y-z)(z-x)}.$

38. $\frac{1}{(a-b)(c-b)(c-a)} \equiv -\frac{1}{(a-b)(b-c)(c-a)}.$

39. $\frac{a-c}{(a-b)(c-b)} \equiv \frac{c-a}{(a-b)(b-c)}.$

40. $\frac{(y-x)(y-z)}{(w-z)(w-x)} \equiv \frac{(x-y)(y-z)}{(z-w)(w-x)}.$

41. $-\frac{c-a-b}{(b-c-a)(a-b-c)} \equiv \frac{a+b-c}{(a-b+c)(b+c-a)}.$

21. An **expression** is said to be **fractional** if it contains one or more fractional terms; if it contains both fractional and integral expressions, it is said to be **mixed.**

An expression is said to be **entirely fractional** if it contains fractions only.

E. g. $\frac{a}{b}+\frac{c}{d}-\frac{x+y}{z}$ is entirely fractional;

while $m^2-n^2+\frac{r}{s}+\frac{x}{y}$ is a mixed expression.

22. Two fractions are said to be **equal or equivalent** when the terms of either may be obtained from the terms of the other by multiplying or dividing both numerator and denominator by the same number or expression, zero excepted.

E. g. $\frac{4}{8}$ is equal to $\frac{1}{2}$, since the numerator and denominator of the second fraction can be obtained from the numerator and denominator of the first fraction by dividing each by 4.

Also, $\frac{a^2b}{ab^2}\equiv\frac{a}{b}$, since by dividing the numerator and denominator of the first fraction by the same expression, ab, we obtain the numerator and denominator of the second fraction.

Reduction to Lowest Terms.

23. A fraction whose terms are wholly rational and integral is said to be in **lowest terms** or simplified, when its numerator and denominator have no common integral factors.

(The following proof may be omitted when the chapter is read for the first time.)

Consider the fraction $\frac{ah}{bh}$, a, b, and h being positive integers. Then, if a and b be prime to each other, and also to h, h will be the H. C. F. of the numerator ah and the denominator bh.

By the Laws of Association and Commutation for multiplication and division, and the definition of a quotient, we have

$$\frac{ah}{bh}\equiv(ah)\div(bh)$$

$$\equiv a\times h\div b\div h$$

$$\equiv a\div b\times h\div h$$

Or $$\frac{ah}{bh}\equiv\frac{a}{b}.$$

Read backward and forward, this identity establishes the following

Principle: *If both numerator and denominator of a fraction be multiplied by or divided by the same number, except zero, the value of the fraction will remain unaltered.*

24. From this principle it follows that *a fraction may be reduced to lowest terms if both numerator and denominator be divided by their highest common factor.*

Ex. 1. Reduce $18\,a^3b^2c\,/\,45\,a^5bc$ to lowest terms.

The H. C. F. of numerator and denominator is found to be $9\,a^3bc$.

Hence,
$$\frac{18\,a^3b^2c}{45\,a^5bc} \equiv \frac{(9\,a^3bc)(2\,b)}{(9\,a^3bc)(5\,a^2)} \equiv \frac{2\,b}{5\,a^2}.$$

Check. $a = 2,\ b = 3,\ c = 4.$

$\frac{5184}{17280} = \frac{6}{20}$

$\frac{3}{10} = \frac{3}{10}.$

25. This operation of removing common factors from both numerator and denominator of a fraction by division is called **cancellation.**

The student should *never strike out equal terms* which are found in both numerator and denominator, *unless they are factors* of both the whole numerator and the whole denominator.

E.g. In $\dfrac{5\,x^2 + 21}{5\,x^2 - 14}$ we must not strike out the first terms, $5\,x^2$, nor attempt to remove 7 from 21 and 14, for neither $5\,x^2$ nor 7 is a factor of both the entire numerator and the entire denominator.

Ex. 2. Reduce $12\,x^2y^3\,/\,(24\,x^2y - 12\,xy^2)$ to lowest terms.

The terms of the denominator contain $12\,xy$ as a common factor. Hence, we may write
$$\frac{12\,x^2y^3}{24\,x^2y - 12\,xy^2} \equiv \frac{12\,x^2y^3}{12\,xy\,(2\,x - y)}.$$

Dividing both numerator and denominator by the common factor, $12\,xy$,
$$\equiv \frac{xy^2}{2\,x - y}.$$

Check. $x = 2,\ y = 3.$
$18 = 18.$

Ex. 3. Reduce $(6\,x^3y - 15\,x^2y^2)\,/\,(10\,x^2y^2 - 25\,xy^3)$ to lowest terms.
$$\frac{6\,x^3y - 15\,x^2y^2}{10\,x^2y^2 - 25\,xy^3} \equiv \frac{3\,x^2y\,(2\,x - 5\,y)}{5\,xy^2\,(2\,x - 5\,y)} \equiv \frac{3\,x}{5\,y}.$$

Check. $x = 3,\ y = 1.$
$\frac{9}{5} = \frac{9}{5}.$

It is sometimes necessary to find the H. C. F. of numerator and denominator by the division process.

Ex. 4. Reduce $(2x^3 + x^2 - 16x + 15)/(6x^3 - 11x^2 + 7x - 6)$ to lowest terms.

By the division process, the H. C. F. of numerator and denominator is found to be $(2x - 3)$. Hence we may write

$$\frac{2x^3 + x^2 - 16x + 15}{6x^3 - 11x^2 + 7x - 6} \equiv \frac{(2x-3)(x^2 + 2x - 5)}{(2x-3)(3x^2 - x + 2)}$$

$$\equiv \frac{x^2 + 2x - 5}{3x^2 - x + 2}.$$

Check. $x = 2.$
$\frac{1}{4} = \frac{1}{4}.$

EXERCISE XV. 2

Simplify the following fractions, checking all results by substituting such numerical values as do not reduce the denominators to zero:

1. $\frac{xy}{xz}$.
2. $\frac{ab}{a^2}$.
3. $\frac{7}{21b^2}$.
4. $\frac{20z}{5w}$.
5. $\frac{28a^2}{14a^3}$.
6. $\frac{x^2y}{xy^2}$.
7. $\frac{25h^2}{5h^3}$.
8. $\frac{8bc}{24c^2}$.
9. $\frac{18cd}{9d^2}$.
10. $\frac{a^2bc}{ab^2c}$.
11. $\frac{x^2yz}{xyz^2}$.
12. $\frac{a^4b^3c^2}{a^2b^3c^4}$.
13. $\frac{4ac^2x}{4cx^2}$.
14. $\frac{12xz^3}{4xyz^2}$.
15. $\frac{11c^2xy}{33x^2yz^2}$.
16. $\frac{19ab^2cd^3}{57abc^2d^2}$.
17. $\frac{a^m}{a^{m+1}}$.
18. $\frac{b^{m-1}}{b^m}$.
19. $\frac{c^{n-1}}{c^{n+1}}$.
20. $\frac{x^{n+2}}{x^{n+5}}$.
21. $\frac{x^{r-5}}{x^{r-3}}$.
22. $\frac{a^{2n}}{a^{2n+1}}$.
23. $\frac{x^{n-r}}{x^{n-r+1}}$.
24. $\frac{a^nb^{n+1}}{a^{n+1}b^n}$.
25. $\frac{x^{n+2}y^{n-1}}{x^{n-3}y^{n+5}}$.
26. $\frac{ab}{ac + ad}$.
27. $\frac{xy}{yz - yw}$.
28. $\frac{m}{m^2 + m}$.
29. $\frac{2a^2}{2a + 2}$.
30. $\frac{am + an}{ax + ay}$.

31. $\frac{ab - bc}{ab + bc}$.

32. $\frac{4a + 8b}{12c + 16d}$.

33. $\frac{5x - 15y}{10x + 20z}$.

34. $\frac{6c + 12d}{3m - 9n}$.

35. $\frac{a + b}{(a + b)^2}$.

36. $\frac{x + 1}{x^2 - 1}$.

37. $\frac{2x + 1}{4x^2 - 1}$.

38. $\frac{(c + d)^2}{c^2 - d^2}$.

39. $\frac{m^2 - n^2}{(m - n)^2}$.

40. $\frac{d^2 - 1}{d^3 + 1}$.

41. $\frac{x^2 - 1}{x^3 - 1}$.

42. $\frac{a^2 - b^2}{a^3 + b^3}$.

43. $\frac{x - 1}{x^2 - 2x + 1}$.

44. $\frac{y - 4}{y^2 - 8y + 16}$.

45. $\frac{a + 1}{a^2 + 3a + 2}$.

46. $\frac{x - 5}{x^2 - 7x + 10}$.

47. $\frac{a^3 - 3a^2}{a^2 - 6a + 9}$.

48. $\frac{m^4 + 2m^3}{m^2 + 4m + 4}$.

49. $\frac{c^2 - 2cd + d^2}{c^3 - d^3}$.

50. $\frac{x^2 - xy + y^2}{x^3 + y^3}$.

51. $\frac{a^2 + ab + b^2}{a^3 - b^3}$.

52. $\frac{a^2 + 2a + 1}{a^2 + 3a + 2}$.

53. $\frac{b^2 - 5b + 6}{b^2 - 4b + 4}$.

54. $\frac{c^2 + 6c + 9}{c^2 + c - 6}$.

55. $\frac{x^2 - 3x - 10}{x^2 - 6x + 5}$.

56. $\frac{y^2 - 8y + 12}{y^2 - 9y + 18}$.

57. $\frac{1 + 5x + 6x^2}{1 + 6x + 8x^2}$.

58. $\frac{2a^2 + 5a + 3}{5a^2 + 12a + 7}$.

59. $\frac{a^2 + 3a + 2}{2a^2 + 5a + 2}$.

60. $\frac{k^2 - 7k + 12}{k^2 - 5k + 6}$.

61. $\frac{(a + b)^2 - c^2}{a^2 - (b + c)^2}$.

62. $\frac{x^2 - (y - z)^2}{(x + y)^2 - z^2}$.

63. $\frac{a^2 - (b + c)^2}{(a + c)^2 - b^2}$.

64. $\frac{(x - y)^2 - z^2}{x^2 - (y + z)^2}$.

65. $\frac{x^3 + x^2 + 3x - 5}{x^2 - 4x + 3}$.

66. $\frac{x^3 + 3x^2 + 3x + 2}{x^3 - 2x^2 - 2x - 3}$.

26. If the degree of the numerator be equal to or higher than that of the denominator, the form of a fractional expression may be such as to admit of transformation into an equivalent integral expression, or into a mixed expression.

Ex. 1.
$$\frac{a^2-b^2}{a\ -b} \equiv \frac{(a+b)(a-b)}{(a-b)} \qquad \text{Check.}\quad a=3,\ b=2.$$
$$\equiv a+b. \qquad\qquad 5=5.$$

Ex. 2. Reduce $(a^3+2\,a^2b+3\,b^2)/a^2$ to lowest terms.

By the Distributive Law for division, we may find the result by dividing each of the terms of the dividend by the divisor, and write

$$\frac{a^3+2\,a^2b+3\,b^2}{a^2} \equiv \frac{a^3}{a^2}+\frac{2\,a^2b}{a^2}+\frac{3\,b^2}{a^2} \qquad \text{Check.}\quad a=2,\ b=3.$$
$$\equiv a+2\,b+\frac{3\,b^2}{a^2}. \qquad\qquad \tfrac{59}{4}=\tfrac{59}{4}.$$

Ex. 3. Reduce $\dfrac{x^3+2\,x^2-x+2}{x^2+2\,x+3}$ to lowest terms.

Using the denominator as divisor, and carrying out the process for long division, we obtain x as an integral quotient and $-4\,x+2$ as a remainder.

That is,
$$\frac{x^3+2\,x^2-x+2}{x^2+2\,x+3} \equiv x+\frac{-4\,x+2}{x^2+2\,x+3} \qquad \text{Check.}\quad x=2.$$
$$\equiv x-\frac{4\,x-2}{x^2+2\,x+3}. \qquad\qquad \tfrac{16}{11}=\tfrac{16}{11}.$$

EXERCISE XV. 3

Reduce the following improper fractions to integral or mixed expressions, checking all results numerically :

1. $\dfrac{a^2-9}{a\ +3}$.
2. $\dfrac{b^3-1}{b\ -1}$.
3. $\dfrac{x^2+1}{x}$.
4. $\dfrac{4\,d^4+2\,d^2+1}{2\,d^2}$.
5. $\dfrac{h^3-3}{h\ -3}$.
6. $\dfrac{s^4-1}{s\ +1}$.
7. $\dfrac{r^3+1}{r\ -1}$.
8. $\dfrac{x^2-4\,x-13}{x+4}$.
9. $\dfrac{n^2-5\,n-12}{n-2}$.
10. $\dfrac{c^2+9\,c+20}{c+4}$.
11. $\dfrac{3\,a^3}{a-b}$.
12. $\dfrac{z^4-w^4}{z\ -w}$.
13. $\dfrac{s^4+t^4}{s\ +t}$.
14. $\dfrac{a^4}{a+x}$.
15. $\dfrac{d^3+3\,d^2e+3\,de^2+e^3}{d^2+2\,de\ +e^2}$.

Reduction of Fractions to Equivalent Fractions having a Common Denominator

27. Two or more fractions are said to have a **common denominator** when their denominators do not differ.

E. g. $\frac{x}{y}$ and $\frac{z}{y}$, or $\frac{a}{a^2-b^2}$ and $\frac{c+d}{a^2-b^2}$.

28. The **lowest common denominator** (or L. C. D.) of two or more fractions is the lowest common multiple (or L. C. M.) of their denominators, used as a new denominator.

29. Any number of separate rational fractions may be transformed into equivalent fractions which have equal denominators.

(The following proof may be omitted when the chapter is read for the first time.)

Let a/b and c/d represent two fractions, and, for the present, let their terms a, b, c, and d be rational and integral. Let the L. C. M. of the denominators b and d of the fractions be L, and let $bm \equiv L$, and $dn \equiv L$.

In these identities the letters m and n represent integers since the lowest common multiple of two integers is an integer, and m and n must be prime to each other, for if they were not L would not be the *lowest* common multiple.

Multiplying both numerator and denominator of the first fraction a/b by the factor m, necessary to produce the L. C. D. of the denominators b and d,

we have
$$\frac{a}{b} \equiv \frac{am}{bm} \equiv \frac{am}{L}.$$

Similarly,
$$\frac{c}{d} \equiv \frac{cn}{dn} \equiv \frac{cn}{L}.$$

The two original fractions, a/b and c/d, are now expressed as the equivalent fractions, am/L and cn/L respectively, which have equal denominators, L.

From the proof above we have the following:

To reduce two or more fractions to equivalent fractions having the lowest common denominator, first find the lowest common multiple of their denominators. Then multiply both the denominator and numerator of each fraction by the number or factor which, when taken with the denominator of the fraction, will produce as a product the required lowest common multiple of all of the denominators.

Ex. 1. Reduce $2/3a$, $4/5a^2$, $6/15a^3$ to equivalent fractions having the L. C. D.

The L. C. D. is $15a^3$. Hence we have

$$\frac{2}{3a} \equiv \frac{2 \cdot 5a^2}{3a \cdot 5a^2} \equiv \frac{10a^2}{15a^3}.$$

$$\frac{4}{5a^2} \equiv \frac{4 \cdot 3a}{5a^2 \cdot 3a} \equiv \frac{12a}{15a^3}.$$

$$\frac{6}{15a^3} \equiv \qquad\qquad \frac{6}{15a^3}.$$

Let the student check the example above.

Ex. 2. Reduce the fractions $(a-b)/(a+b)$, $(a+b)/(a-b)$ to equivalent fractions having the L. C. D.

The L. C. D. of the two fractions is found to be $(a+b)(a-b)$. Therefore:

$$\frac{a-b}{a+b} \equiv \frac{(a-b)(a-b)}{(a+b)(a-b)} \equiv \frac{(a-b)^2}{a^2-b^2}. \quad \text{Check. } a=3, b=2. \quad \tfrac{1}{5}=\tfrac{1}{5}.$$

$$\text{Similarly, } \frac{a+b}{a-b} \equiv \frac{(a+b)(a+b)}{(a-b)(a+b)} \equiv \frac{(a+b)^2}{a^2-b^2}. \quad \text{Check. } a=3, b=2. \quad 5=5.$$

The student will find it a good plan, when transforming a fraction to an equivalent fraction having a different denominator, first to copy the fraction in its original form and then to make the necessary alterations by inserting the proper factors in both denominator and numerator. This is better than to attempt to copy the fraction and make the transformation at the same time.

E. g. Thus, in the example above, we may write as a first step,

$$\frac{a+b}{a-b} \equiv \frac{(a+b)\qquad\qquad}{(a-b)\qquad\qquad}.$$

As a second step we will insert the factor $(a+b)$ in both denominator and numerator, as below:

$$\frac{a+b}{a-b} \equiv \frac{(a+b)(a+b)}{(a-b)(a+b)} \equiv \frac{(a+b)^2}{a^2-b^2}.$$

EXERCISE XV. 4

Reduce the fractions in each of the following groups to equivalent fractions having the L. C. D.:

1. $\frac{a}{2}, \frac{a}{3}.$

2. $\frac{b}{5}, \frac{b}{7}.$

3. $\frac{c}{8}, \frac{c}{12}.$

4. $\frac{d}{5}, \frac{d}{20}.$

5. $\frac{1}{a}, \frac{1}{b}.$

6. $\frac{2}{x^2}, \frac{2}{y^2}.$

7. $\frac{5}{x^2}, \frac{6}{x}.$

8. $\frac{2}{a^2}, \frac{7}{a}.$

9. $\frac{3}{x}, \frac{4}{y}.$

10. $\frac{6}{z}, \frac{8}{w}.$

11. $\frac{5}{2a}, \frac{6}{7a}.$

12. $\frac{8}{ab}, \frac{10}{bc}.$

13. $\frac{a}{b}, \frac{b}{c}.$

14. $\frac{x}{y}, \frac{y}{x}.$

15. $\frac{a}{bc}, \frac{b}{cd}.$

16. $\frac{x^2}{yz}, \frac{y^2}{zw}.$

17. $\frac{1}{3ab}, \frac{2}{4bc}.$

18. $\frac{a}{5bc}, \frac{a^2}{10cd}.$

19. $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}.$

20. $\frac{x}{2y}, \frac{x}{4y}, \frac{x}{6y}.$

21. $\frac{b}{a^3}, \frac{b^2}{a^2}, \frac{b^3}{a}.$

22. $\frac{1}{2x}, \frac{1}{3y}, \frac{1}{4z}.$

23. $\frac{x}{y}, \frac{y}{z}, \frac{z}{x}.$

24. $\frac{a}{bc}, \frac{b}{ca}, \frac{c}{ab}.$

25. $\frac{2}{a}, \frac{1}{2a}, \frac{2}{a^2}.$

26. $\frac{x}{y+z}, \frac{y}{z+x}.$

27. $\frac{1}{x+1}, \frac{1}{x^2-1}.$

28. $\frac{1}{a-b}, \frac{1}{a^2-b^2}.$

29. $\frac{x+y}{x-y}, \frac{x-y}{x+y}.$

30. $\frac{1}{(x-y)(y-z)}, \frac{1}{(y-z)(z-x)}.$

31. $\frac{2}{x^2-9}, \frac{3}{x+3}, \frac{4}{x-3}.$

32. $\frac{1}{a-b}, \frac{1}{b-c}, \frac{1}{c-a}.$

30.* *If all of the terms of two equal fractions, x/y and n/d, be positive whole numbers, then if n/d be in lowest terms, it follows that $x = pn$ and $y = pd$, where p represents a positive whole number.*

$$\text{For, if} \qquad \frac{x}{y} = \frac{n}{d} \tag{1}$$

$$\text{then} \qquad x = \frac{ny}{d}. \tag{2}$$

Or, the quotient obtained by dividing ny by d is a positive whole number x, and since n is prime to d, y must be some multiple of d, say $y = pd$, where p represents a positive integer.

Hence, (2) becomes $x = \dfrac{npd}{d}$, or $x = pn$.

31.* *In particular, if x/y is in lowest terms, x and y can have no common factor p; hence, putting $p = 1$, we have*

$$x = n, \qquad\qquad y = d.$$

Addition and Subtraction of Fractions

32. Applying the Distributive Law for Division (See Chapter V. §§ 58,59), a, b, and d being taken as positive integers as in other proofs, we may write

$$\frac{a}{d} + \frac{b}{d} \equiv a \div d + b \div d \quad \text{and} \quad \frac{a}{d} - \frac{b}{d} \equiv a \div d - b \div d$$

$$\equiv (a + b) \div d \qquad\qquad \equiv (a - b) \div d$$

$$\equiv \frac{a + b}{d}. \qquad\qquad \equiv \frac{a - b}{d}.$$

Hence, the ***sum*** *of two fractions having a common denominator is a fraction having for numerator the sum of the numerators of the given fractions, and for denominator their common denominator.*

Also, the ***difference*** *of two fractions having a common denominator is a fraction having for numerator the difference of the numerators of the given fractions, and for denominator their common denominator.*

---

* This section may be omitted when the chapter is read for the first time.

Ex. 1. Find the sum of a/bc and b/cd.

The lowest common denominator is bcd.

Hence,
$$\frac{a}{bc}+\frac{b}{cd}\equiv\frac{ad}{bcd}+\frac{b^2}{bcd}$$

Writing the sum of the numerators over the lowest common denominator, $\equiv\dfrac{ad+b^2}{bcd}$.

Check. $a=2$, $b=3$, $c=4$, $d=5$. $\frac{19}{60}=\frac{19}{60}$.

Ex. 2.
$$\frac{2a+b}{3b}+\frac{a-2b}{4b}\equiv\frac{(2a+b)4+(a-2b)3}{12b}$$

Performing the multiplications, $\equiv\dfrac{8a+4b+3a-6b}{12b}$

Combining like terms, $\equiv\dfrac{11a-2b}{12b}$.

Check. $a=3$, $b=2$. $\frac{29}{24}=\frac{29}{24}$.

Instead of finding a common denominator at once for all of the fractions of a given set, it is sometimes desirable to combine the fractions by groups, and after reduction of the groups separately, to combine the results thus obtained.

Ex. 3. Simplify $\dfrac{3}{x+2}+\dfrac{5}{x-1}-\dfrac{5}{x+1}-\dfrac{3}{x-2}$.

The given fractions may be rearranged as follows:

$$\begin{aligned}\frac{3}{x+2}-\frac{3}{x-2}+\frac{5}{x-1}-\frac{5}{x+1}&\equiv\frac{3(x-2)}{(x+2)(x-2)}-\frac{3(x+2)}{(x-2)(x+2)}\\&\quad+\frac{5(x+1)}{(x-1)(x+1)}-\frac{5(x-1)}{(x+1)(x-1)}\\&\equiv\frac{3(x-2)-3(x+2)}{(x+2)(x-2)}+\frac{5(x+1)-5(x-1)}{(x+1)(x-1)}\\&\equiv\frac{3x-6-3x-6}{x^2-4}+\frac{5x+5-5x+5}{x^2-1}\\&\equiv\frac{-12}{x^2-4}+\frac{10}{x^2-1}\\&\equiv\frac{-12(x^2-1)+10(x^2-4)}{(x^2-4)(x^2-1)}\\&\equiv\frac{-2x^2-28}{(x^2-4)(x^2-1)}\\&\equiv-\frac{2x^2+28}{(x^2-4)(x^2-1)}.\end{aligned}$$

Check. $x=3$. $-\frac{23}{20}=-\frac{23}{20}$.

Ex. 4. Simplify $\dfrac{2a-3b}{a+b}-\dfrac{3a-4b}{2a-b}$.

The L. C. D. of the fractions is $(a+b)(2a-b)$.

Hence,

$$\frac{2a-3b}{a+b}-\frac{3a-4b}{2a-b}\equiv\frac{(2a-3b)(2a-b)}{(a+b)(2a-b)}-\frac{(3a-4b)(a+b)}{(2a-b)(a+b)}$$

$$\equiv\frac{(2a-3b)(2a-b)-(3a-4b)(a+b)}{(a+b)(2a-b)}$$

$$\equiv\frac{(4a^2-8ab+3b^2)-(3a^2-ab-4b^2)}{(a+b)(2a-b)}$$

$$\equiv\frac{4a^2-8ab+3b^2-3a^2+ab+4b^2}{(a+b)(2a-b)}$$

$$\equiv\frac{a^2-7ab+7b^2}{(a+b)(2a-b)}.$$

Check. $a=4,\ b=2.$
$-\frac{1}{3}=-\frac{1}{3}.$

EXERCISE XV. 5

Simplify the following expressions, reducing all results to lowest terms, and check numerically :

1. $\frac{a}{b}+\frac{c}{b}.$
2. $\frac{a}{b}+\frac{b}{c}.$
3. $\frac{ab}{c}+\frac{bc}{a}.$
4. $\frac{a}{bc}+\frac{b}{ca}.$
5. $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}.$
6. $\frac{a+b}{c}+\frac{b+c}{a}.$
7. $\frac{a-b}{a}-\frac{b-c}{b}.$
8. $\frac{a-b}{a}-\frac{b-c}{b}-\frac{c-a}{c}.$
9. $\frac{a}{xy}-\frac{b}{yz}.$
10. $\frac{x}{yz}+\frac{y}{zx}+\frac{z}{xy}.$
11. $\frac{a}{a-b}-\frac{b}{a+b}-\frac{b^2}{a^2-b^2}.$
12. $\frac{1}{a-b}-\frac{1}{a+b}.$
13. $\frac{1}{a+b}+\frac{1}{a+c}.$
14. $\frac{a}{a+b}+\frac{b}{a-b}.$
15. $\frac{a+b}{2}+\frac{a-b}{3}.$
16. $\frac{x-y}{4}+\frac{x+y}{5}.$
17. $\frac{a+b}{a}+\frac{b+c}{b}.$
18. $\frac{x-y}{5}-\frac{x-y}{3}.$

19. $\frac{a+2}{2}+\frac{a-2}{2}$.

20. $\frac{2}{a^2+1}-\frac{2}{a^2-1}$.

21. $\frac{4}{x-4}-\frac{5}{x+5}$.

22. $\frac{x}{x+2}-\frac{x}{x+3}$.

23. $\frac{2m}{m^2-1}-\frac{1}{m+1}$.

24. $\frac{x+y}{6}+\frac{x-y}{3}-\frac{x+y}{9}$.

25. $\frac{b+y}{b-y}-\frac{b}{b+y}+\frac{y^2}{y^2-b^2}$.

26. $\frac{1}{a^2+ab}-\frac{1}{b^2+ab}$.

27. $\frac{1}{3x-x^2}-\frac{1}{x^2-9}$.

28. $\frac{a+b}{a-b}+\frac{a-b}{a+b}$.

29. $\frac{x-1}{x+1}+\frac{x+1}{x-1}$.

30. $\frac{c+d}{c-d}-\frac{c-d}{c+d}$.

31. $\frac{x+1}{x-1}+\frac{x^2+1}{x^2-1}$.

32. $\frac{a^2+a+1}{a+1}+\frac{a^2-a+1}{a-1}$.

33. $\frac{a-1}{a^2-a+1}-\frac{1+a}{a^2+a+1}$.

34. $\frac{5}{x+y}+\frac{8}{x-y}-\frac{3y}{x^2-y^2}$.

35. $\frac{3}{x(x+3)}+\frac{6}{x(x-3)}-\frac{9}{x^2-9}$.

36. $\frac{5}{x+5}+\frac{5}{x-5}-\frac{10x}{x^2-25}$.

37. $\frac{b}{a+b}-\frac{b^2}{(a+b)^2}-\frac{a^2b}{(a+b)^3}$.

38. $\frac{3}{(a+x)^2}+\frac{3}{(a-x)^2}+\frac{9}{x^2-a^2}$.

39. $\frac{c^2}{(c-a)(c-b)}-\frac{a^2}{(a-c)(b-a)}-\frac{b^2}{(b-c)(a-b)}$.

40. $\frac{x}{(x-y)(x-z)}+\frac{y}{(y-z)(y-x)}+\frac{z}{(z-x)(z-y)}$.

41. $\frac{b+1}{(b-c)(b-a)}+\frac{c+1}{(c-b)(c-a)}-\frac{a+1}{(a-b)(c-a)}$.

42. $\frac{1}{(b-c)(b-a)}+\frac{1}{(b-c)(c-a)}+\frac{1}{(b-a)(c-a)}$.

43. $\frac{4}{(a-1)(a-3)}+\frac{2}{(a-2)(3-a)}-\frac{2}{(2-a)(1-a)}$.

44. $\frac{3}{a+4}-\frac{4}{a-5}-\frac{3}{a-4}+\frac{4}{a+5}.$

45. $\frac{4}{x-2}+\frac{5}{x-1}-\frac{4}{x+2}-\frac{5}{x+1}.$

Reduction of Integral or Mixed Expressions to Fractional Forms

33. Any integer or an integral expression may be written in the form of a fraction having any required denominator.

This is because successive multiplications and divisions by the same number or expression produce no change in the value of a given number or expression.

In symbols, $$\boldsymbol{a \equiv a \times b \div b \equiv \frac{ab}{b}}.$$

Hence a given number or expression may be expressed as a fraction having a required denominator provided that the numerator of the desired fraction is the product obtained by multiplying the given number or expression by the required denominator.

Ex. 1. Reduce $1+\frac{2b}{a}+\frac{b^2}{a^2}$ to an improper fraction.

Expressing 1 and $\frac{2b}{a}$ as fractions having for denominators the L. C. D. of a and a^2, which is a^2, we have

$$1+\frac{2b}{a}+\frac{b^2}{a^2}\equiv\frac{a^2}{a^2}+\frac{2ab}{a^2}+\frac{b^2}{a^2}$$
$$\equiv\frac{a^2+2ab+b^2}{a^2}.$$

Check. Let $a=b=2$.

$4=4$.

EXERCISE XV. 6

Write the following expressions as fractions with the denominators indicated :

1. 3 with denominator x.
2. 5 “ “ $10a$.
3. 7 “ “ $x+y$.
4. ab “ “ a.
5. xy with denominator xy.
6. $m+n$ “ “ $m+n$.
7. $x+y$ “ “ $x-y$.
8. a^2+ab+b^2 “ “ $a-b$.

Reduce the following mixed expressions to the forms of improper fractions, checking results numerically :

9. $x + \frac{y}{z}$.

10. $a - \frac{b}{c}$.

11. $a + 2 + \frac{1}{a}$.

12. $x - 2 + \frac{1}{x}$.

13. $a + b + \frac{b^2}{a}$.

14. $m - n + \frac{n^2}{m}$.

15. $x^2 - \frac{9y^2}{4z^2}$.

16. $m^2n - \frac{mn^2}{m - n}$.

17. $x - 4 + \frac{6x + 3}{5x}$.

18. $x + y + \frac{y}{x + y}$.

19. $a + b + \frac{1}{a + b}$.

20. $\frac{n^2}{m - n} + m + n$.

21. $\frac{6}{a + 3} + a - 2$.

22. $\frac{6}{x + 4} + x - 3$.

23. $\frac{9}{b + 2} + b + 4$.

24. $a - b - \frac{a^2 + b^2}{a + b}$.

25. $a - 4 - \frac{2a + 3}{a - 4}$.

26. $r^2 + r + 1 + \frac{1}{r - 1}$.

34. The following principles are fundamental to the processes of multiplication and division involving fractions.

Principle I. *To multiply a fraction by a whole number, we may either multiply the numerator alone by the given multiplier, or divide the denominator alone by the given multiplier.*

That is,
$$\frac{a}{b} \times c \equiv \frac{a \times c}{b},$$

or,
$$\frac{a}{b} \times c \equiv \frac{a}{b \div c}.$$

(The following proof may be omitted when the chapter is read for the first time.)

For convenience of proof we shall assume that the numerator and denominator of the fraction a/b and the multiplier c represent positive integers.

In the proof below, which depends upon the Fundamental Laws for Multiplication and Division and also upon the definition of a fraction, the required product is obtained at the left by multiplying the numerator alone by the given multiplier, and at the right by dividing the denominator alone by the given multiplier.

We have
$$\begin{aligned}\frac{a}{b}\times c &\equiv (a \div b)\times c\\ &\equiv a \times c \div b\\ &\equiv (a \times c)\div b\\ &\equiv \frac{a \times c}{b}.\end{aligned}$$

and also
$$\begin{aligned}\frac{a}{b}\times c &\equiv (a \div b)\times c\\ &\equiv a \div b \times c\\ &\equiv a \div (b \div c)\\ &\equiv \frac{a}{b \div c}.\end{aligned}$$

E. g. Multiply 5/24 by 3.

The product may be obtained in either of the two following ways :

Multiplying the numerator alone by the multiplier, we have

$$\tfrac{5}{24}\times 3 = \tfrac{15}{24} = \tfrac{5}{8}.$$

Dividing the denominator alone by the multiplier, we have

$$\tfrac{5}{24}\times 3 = \frac{5}{24 \div 3} = \tfrac{5}{8}.$$

Principle II. *To divide a fraction by a whole number we may either divide the numerator alone by the given divisor, or multiply the denominator alone by the given divisor.*

That is,
$$\frac{a}{b}\div c \equiv \frac{a \div c}{b},$$

or,
$$\frac{a}{b}\div c \equiv \frac{a}{b \times c}.$$

(The following proof may be omitted when the chapter is read for the first time.)

In the following proof, where a, b, and c represent positive integers, the quotient is obtained at the left by dividing the numerator alone by the divisor, and at the right by multiplying the denominator alone by the divisor.

We have
$$\begin{aligned}\frac{a}{b}\div c &\equiv (a \div b)\div c\\ &\equiv a \div b \div c\\ &\equiv a \div c \div b\\ &\equiv (a \div c)\div b\\ &\equiv \frac{a \div c}{b}.\end{aligned}$$

and also
$$\begin{aligned}\frac{a}{b}\div c &\equiv (a \div b)\div c\\ &\equiv a \div b \div c\\ &\equiv a \div (b \times c)\\ &\equiv \frac{a}{b \times c}.\end{aligned}$$

E. g. Divide 15/17 by 5.

Dividing the numerator alone by the divisor,

we have $\tfrac{15}{17}\div 5 = \dfrac{15 \div 5}{17} = \tfrac{3}{17}.$

Multiplying the denominator alone by the divisor,

we have $\tfrac{15}{17}\div 5 = \dfrac{15}{17 \times 5} = \tfrac{15}{85} = \tfrac{3}{17}.$

Multiplication of an Integer by a Fractional Multiplier

35. To multiply one number by another is, by the extended definition of multiplication, to perform upon the multiplicand exactly those operations which must be performed upon the unit of positive numbers to produce the multiplier.

E. g. Let it be required to multiply a by c/d.

To obtain the multiplier from $^+1$, we may first divide $^+1$ into d equal parts, each equal to $^+1/d$, and then take c of these parts as summands, and thus obtain the multiplier c/d.

Hence to multiply a by c/d, we may first divide the multiplicand, a, into d equal parts, each part being represented by a/d, and then take c of these parts as summands. The result thus obtained is the desired product $\frac{a \times c}{d}$.

That is,

$$a \times \frac{c}{d} \equiv \frac{ac}{d}.$$

Hence, ***to multiply a whole number by a fraction,*** *multiply the whole number by the numerator of the fraction and divide the result by the denominator.*

E. g. Multiply 12 by 2/3.

We have

$$12 \times \tfrac{2}{3} = \frac{12 \times 2}{3} = \tfrac{24}{3} = 8.$$

Multiplication of One Fraction by Another

36. Let it be required to multiply a/b by c/d, regarding a, b, c, and d as positive whole numbers in order to simplify proofs.

Applying Principles I. and II., § 34, and applying a course of reasoning similar to that used above, it appears that we may perform the operation by first separating a/b into d equal parts, each of these parts being represented by $a/(b \times d)$.

By Principle II., § 34, taking c of these parts as summands, we obtain as a final result

$$a/bd + a/bd + \cdot\cdot\cdot\cdot\cdot \text{ to } c \text{ summands, that is, } ac/bd.$$

Hence, we may write

$$\frac{a}{b} \times \frac{c}{d} \equiv \frac{a \times c}{b \times d}.$$

Accordingly, ***the product of two fractions*** *is a fraction whose numerator is the product of the given numerators, and whose denominator is the product of the given denominators.*

37. It may be shown that fractions obey the Commutative Law for Multiplication.

Hence, to find the product of two fractions, either fraction may be used as multiplier.

Ex. 1.

$$\frac{5\,a^2b}{3\,cd^2} \times \frac{2\,abx}{7\,c^2dy} \equiv \frac{(5\,a^2b)(2\,abx)}{(3\,cd^2)(7\,c^2dy)} \equiv \frac{10\,a^3b^2x}{21\,c^3d^3y}.$$

Check. $a = b = 2$, $c = d = 3$, $x = y = 1$. $\frac{320}{15309} = \frac{320}{15309}$.

Ex. 2.

$$\frac{5\,(a+b)^2}{c^2-d^2} \times \frac{(c-d)^2}{10\,(a^2-b^2)} \equiv \frac{5\,(a+b)^2(c-d)^2}{10\,(c+d)(c-d)(a+b)(a-b)}$$

Cancelling factors common to both numerator and denominator $\equiv \frac{(a+b)(c-d)}{2(a-b)(c+d)}$.

Check. $a = 5$, $b = 2$, $c = 4$, $d = 3$. $\frac{1}{6} = \frac{1}{6}$.

38. Any factors which occur in the numerator of one fraction and in the denominator of another may be cancelled at the outset, instead of being carried over into the terms of the required product and then stricken out.

E.g. In Example 2 of § 37, 5 in the numerator of the first fraction may be cancelled with 5 which is a factor of the denominator of the second; $(a + b)$ in the numerator of the first fraction with $(a + b)$ in the denominator of the second; and $(c - d)$ in the denominator of the first fraction with $(c - d)$ in the numerator of the second fraction.

MENTAL EXERCISE XV. 7

Express each of the following products as a single fraction in lowest terms:

1. $\frac{a}{c} \times d.$
2. $\frac{x}{y^2} \times y.$
3. $\frac{2\,a}{b} \times 3\,c.$
4. $\frac{9}{10\,x} \times 2\,x^2.$
5. $\frac{ab}{c} \times c^2d.$
6. $x \times \frac{y}{2}.$
7. $2 \times \frac{a}{b}.$
8. $4 \times \frac{3}{m}.$
9. $a^2 \times \frac{bc}{a}.$
10. $16\,x^2 \times \frac{-y}{8\,x}.$
11. $18\,a^2x \times \left(-\frac{1}{6\,a}\right).$
12. $4\,b \times \left(-\frac{1}{20\,ab^2}\right).$
13. $\frac{a}{2} \times \frac{2}{3}.$
14. $\frac{b}{3} \times \frac{4}{5}.$
15. $\frac{5}{6} \times \frac{c}{10}.$

16. $\frac{8}{11} \times \frac{2z}{d}$.

17. $\frac{a}{d} \times \frac{b}{c}$.

18. $\frac{x^2}{y} \times \frac{y}{z}$.

19. $\frac{a}{b^2} \times \frac{b}{c}$.

20. $\frac{b^2}{c} \times \frac{c}{b^3}$.

21. $\frac{m^3}{n^2} \times \frac{n^3}{m^4}$.

22. $\frac{ab}{x} \times \frac{xy}{bc}$.

23. $\frac{ax^2}{y} \times \frac{b}{xy^2}$.

24. $\frac{2a}{3b} \times \frac{4c}{5d}$.

25. $\frac{1}{xy} \times \frac{y^2}{z}$.

26. $\frac{1}{ab} \times \frac{1}{ac}$.

27. $\frac{3}{x^2y} \times \frac{3}{yz^2}$.

28. $\frac{x}{y} \times \left(-\frac{a}{b}\right)$.

29. $\frac{m^2}{16} \times \left(-\frac{8}{m}\right)$.

30. $-\frac{12}{ab^3} \times \frac{abc}{3}$.

31. $\left(-\frac{x^2}{2y}\right)\left(\frac{4y^4}{x^3}\right)$.

32. $\left(\frac{abx}{cy}\right)\left(-\frac{acy}{bx}\right)$.

33. $\left(-\frac{a}{bcy}\right)\left(-\frac{by}{acz}\right)$.

34. $\frac{a}{2} \times \frac{b}{3} \times \frac{c}{4}$.

35. $\frac{a}{b} \times \frac{b}{c} \times \frac{c}{a}$.

36. $\frac{x}{5} \times \frac{10}{y} \times \frac{z}{2}$.

37. $\frac{2a}{3} \times \frac{b}{4c} \times \frac{6d}{e}$.

38. $\frac{a}{bc} \times \frac{ac}{b} \times \frac{ab}{c}$.

EXERCISE XV. 8

Express the following products as single fractions in lowest terms, checking all results numerically :

1. $\frac{6a^2b}{5xy^2} \times \frac{25x^2y}{18ab}$.

2. $\frac{5a^2b^2}{7xy^2} \times \frac{14xy^3}{15a^3b}$.

3. $\frac{12abc}{7yzw} \times \frac{9xyz}{28bcd}$.

4. $\frac{13akt}{6rs^2m} \times \frac{6r^2sm}{11k^2tx}$.

5. $\frac{9a^3x^2}{4bcd} \times \frac{2b^2}{3acx} \times \frac{2cd}{3abx}$.

6. $\frac{5xz^3}{21y^2} \times \frac{3yz^2}{8x^2} \times \frac{7z^4}{30x^2y^2}$.

7. $\frac{a+b}{2} \times \frac{6}{a^2-b^2}$.

8. $\frac{x^2-y^2}{10} \times \frac{5}{x-y}$.

9. $\frac{x}{x+y} \times \frac{y}{x-y}$.

10. $\frac{a+b}{c+d} \times \frac{a-b}{c-d}$.

11. $\frac{x^3 + 1}{14} \times \frac{2}{x + 1}.$

12. $\frac{51}{x^3 - 1} \times \frac{x - 1}{17}.$

13. $\frac{a + 1}{x^2 - 1} \times \frac{x - 1}{a + 1}.$

14. $\frac{a^2 + 2a + 1}{3xy} \times \frac{6xy}{a^2 - 1}.$

15. $\frac{x + 3y}{2a + 1} \times \frac{4a^2 - 1}{x^2 - 9y^2}.$

16. $\frac{4a - b}{m - 5n} \times \frac{m^2 - 25n^2}{16a^2 - b^2}.$

17. $\frac{x^2 + 7x + 12}{abc} \times \frac{bcd}{x + 3}.$

18. $\frac{(a + 2)^2}{a^2 + 8a + 15} \times \frac{a + 5}{a + 2}.$

19. $\frac{a^2 + 3a}{3bc} \times \frac{3abc}{a + 3}.$

20. $\frac{a^2 + ab + b^2}{x + y} \times \frac{(x + y)^3}{a^3 - b^3}.$

21. $\frac{a^2 + a}{x^2 - xy + y^2} \times \frac{x^3 + y^3}{ab + b}.$

22. $\frac{x + z}{x - y} \times \frac{x^2 - y^2}{x^2 + xz}.$

23. $\frac{(a + b)^2}{a^2 - b^2} \times \frac{a - b}{a + b}.$

24. $\frac{m^2 - n^2}{x + y} \times \frac{x(x + y)}{m - n}.$

25. $\frac{x^2 + xy}{abcx} \times \frac{3a^2bdy}{3x + 3y}.$

26. $\frac{x + 5}{a - 3} \times \frac{5a - 15}{10 + 2x}.$

27. $\frac{2ax + a^2}{b^2 + ab} \times \frac{a^2 - b^2}{a^2 - 4x^2}.$

28. $\frac{m^2 - 2mn + n^2}{m^2 + mn} \times \frac{m^2}{m^2 - n^2}.$

29. $\frac{x^2 + x - 6}{x^2 - x - 6} \times \frac{x^2 + x - 12}{x^2 - x - 12}.$

30. $\frac{(a - b)b}{a^2 - 2ab + b^2} \times \frac{xy(a^2 - b^2)}{a^2 + 2ab + b^2}.$

31. $\frac{(a - b)b}{a^2 + b^2 + 2ab} \times \frac{a(b + a)}{a^2 + b^2 - 2ab} \times \frac{xz(a^2 - b^2)}{a^2b^2y}.$

32. $\frac{a^2 - 4a + 3}{a^2 - 6a + 8} \times \frac{a^2 - 4a + 4}{2a^4 + a^2} \times \frac{a^2 - 2a - 8}{a^2 - 5a + 6}.$

33. $\frac{a^2 + b^2 + 2ab - c^2}{a^2 - 2bc - b^2 - c^2} \times \frac{a^2 - 2ac - b^2 + c^2}{b^2 - 2bc - a^2 + c^2}.$

Power of a Fraction

39. A fraction may be raised to a power by applying the definition of a power and the principle for the multiplication of fractions.

E. g. $\left(\frac{a}{b}\right)^m \equiv \frac{a^m}{b^m}$, *m being considered, for the present, a positive integer.*

Ex. 1.
$$\left(\frac{3\,a^3b^2c}{2\,d^3}\right)^3 \equiv \frac{3^3\,(a^3)^3(b^2)^3c^3}{2^3\,(d^3)^3} \equiv \frac{27\,a^9b^6c^3}{8\,d^9}.$$

Check.
$a = b = c = d = 2.$
$1728 = 1728.$

Since the converse of any identity is true, it follows from

$$\left(\frac{a}{b}\right)^m \equiv \frac{a^m}{b^m}, \text{ that } \frac{a^m}{b^m} \equiv \left(\frac{a}{b}\right)^m.$$

Ex. 2.
$$\frac{(x^2+5\,x+6)^2}{(x+3)^2} \equiv \left(\frac{x^2+5\,x+6}{x+3}\right)^2 \equiv \left[\frac{(x+3)(x+2)}{(x+3)}\right]^2 \equiv (x+2)^2.$$

Check. Let $x = 2$.
$16 = 16.$

Ex. 3. Express $\frac{a^2-10\,a+25}{a^2+\ \ 8\,a+16}$ as the power of a fraction.

$$\frac{a^2-10\,a+25}{a^2+\ \ 8\,a+16} \equiv \frac{(a-5)^2}{(a+4)^2} \equiv \left(\frac{a-5}{a+4}\right)^2.$$

Check. Let $a = 6$.
$\frac{1}{100} = \frac{1}{100}.$

Mental Exercise XV. 9

Express the following powers of quotients as quotients of powers:

1. $\left(\frac{2}{3}\right)^2$.
2. $\left(\frac{3}{5}\right)^2$.
3. $\left(-\frac{2}{3}\right)^4$.
4. $\left(-\frac{4}{5}\right)^3$.
5. $-\left(\frac{1}{6}\right)^2$.
6. $\left(\frac{7}{8}\right)^2$.
7. $\left(-\frac{5}{3}\right)^3$.
8. $-\left(\frac{4}{5}\right)^4$.
9. $-\left(\frac{1}{2}\right)^5$.
10. $\left(\frac{3}{2}\right)^6$.
11. $\left(-\frac{5}{6}\right)^3$.
12. $-\left(-\frac{3}{4}\right)^3$.
13. $\left(\frac{a}{b}\right)^2$.
14. $\left(\frac{x}{3}\right)^2$.
15. $\left(\frac{2}{y}\right)^3$.
16. $\left(-\frac{c}{5}\right)^2$.
17. $\left(-\frac{3}{w}\right)^3$.
18. $\left(\frac{2\,a}{3\,b}\right)^2$.

19. $\left(-\frac{4x}{5y}\right)^3$.

20. $-\left(\frac{2}{m}\right)^5$.

21. $\left(-\frac{3}{2a}\right)^4$.

22. $\left(\frac{1}{ab}\right)^6$.

23. $\left(\frac{1}{a^2}\right)^4$.

24. $\left(\frac{ab}{x^3}\right)^5$.

Express the following quotients of powers as powers of fractions :

25. $\frac{9}{25}$.

26. $\frac{16}{49}$.

27. $\frac{25}{81}$.

28. $-\frac{8}{125}$.

29. $\frac{16}{81}$.

30. $-\frac{1}{32}$.

31. $-\frac{27}{64}$.

32. $-\frac{125}{64}$.

33. $\frac{144}{169}$.

34. $\frac{81}{625}$.

35. $\frac{289}{400}$.

36. $\frac{196}{225}$.

37. $\frac{121}{256}$.

38. $\frac{49}{324}$.

39. $-\frac{729}{1000}$.

40. $\frac{4}{a^2}$.

41. $\frac{b^2}{9}$.

42. $\frac{16x^2}{25}$.

43. $\frac{36}{m^2n^2}$.

44. $\frac{49x^6}{a^2}$.

45. $\frac{m^9}{8n^3}$.

46. $\frac{81x^4y^2}{16a^2b^6}$.

47. $\frac{8a^3b^3}{27x^6y^9}$.

48. $\frac{a^2+2ab+b^2}{(x+y)^2}$.

49. $\frac{(m-n)^2}{c^2+2cd+d^2}$.

50. $\frac{(a+b)^2}{(a^2-b^2)^2}$.

51. $\frac{(x-y)^2}{(x^2-y^2)^2}$.

40. *If n is prime to d, then the fraction n/d will be in lowest terms and every positive integral power of the fraction n/d will be a fraction in lowest terms.*

Consider $\left(\frac{n}{d}\right)^p \equiv \frac{n^p}{d^p}$, *in which n, p, and d are positive integers.*

Since n and d have no factors in common, the powers n^p and d^p cannot have any factors in common, and hence n^p/d^p must be in lowest terms.

Division of a Whole Number by a Fraction

41. To divide one number by another is, by the extended definition of division, to perform upon the dividend exactly those operations which must be performed upon the divisor to produce positive unity ($^+1$).

(The following proof may be omitted when the chapter is read for the first time.)

Let it be required to divide a by c/d, a, c, and d being taken for convenience as positive whole numbers.

To obtain $^+1$ from the divisor c/d we may reverse the steps which would have to be taken to obtain c/d from $^+1$, that is, we may, by applying Principle II , § 34, divide the fraction c/d by c, obtaining as a quotient $1/d$ as follows:

$$\frac{c}{d} \div c \equiv \frac{c \div c}{d} \equiv \frac{1}{d}.$$

Multiplying this result by d, we have by Principle I., § 34,

$$\frac{\mathbf{1}}{\boldsymbol{d}} \times \boldsymbol{d} \equiv \mathbf{1}.$$

Hence, to divide a by c/d we may perform upon it successively the operations above· that is, first dividing a by c, we have

$$a \div c \equiv \frac{a}{c},$$

and then multiplying this result by d,

we have $$\frac{a}{c} \times d \equiv \frac{ad}{c}.$$

Hence, $$\boldsymbol{a} \div \frac{\boldsymbol{c}}{\boldsymbol{d}} \equiv \frac{\boldsymbol{ad}}{\boldsymbol{c}}.$$

That is, ***to divide a whole number by a fraction,*** *we may multiply the whole number by the denominator of the fraction and divide the result by the numerator.* (Compare with § 35.)

Division of One Fraction by Another

(The following proof may be omitted when the chapter is read for the first time.)

42. Let the terms of two given fractions be represented by the positive integers a, b, c and d.

Representing the fractions by a/b and c/d we may write

$\frac{a}{b} \div \frac{c}{d} \equiv (a \div b) \div (c \div d)$	By the definition of a fraction.
$\equiv a \div b \div c \times d$	Removing parentheses preceded by the sign of divison.
$\equiv a \div c \div b \times d$	By the Commutative Law for Division.
$\equiv (a \div c) \div (b \div d)$	By the Associative Law for Division.
$\equiv \frac{a \div c}{b \div d}.$	By the Notation for a fraction.

That is, ***to divide one fraction by a second fraction,*** *divide the numerator of the first by the numerator of the second for a new numerator, and the denominator of the first by the denominator of the second for a new denominator.* (Compare with § 36.)

Ex. 1. $\frac{35}{48} \div \frac{7}{8} = \frac{35 \div 7}{48 \div 8} = \frac{5}{6}.$

Ex. 2. $\frac{10\,a^2b^3}{21\,x^2y} \div \frac{2\,ab}{3\,x} \equiv \frac{10\,a^2b^3 \div 2\,ab}{21\,x^2y \div 3\,x} \equiv \frac{5\,ab^2}{7\,xy}.$ Check. $a = b = 2,$ $x = y = 3.$ $\frac{40}{63} = \frac{40}{63}.$

This process is useful only in those cases in which the numerator and denominator of the divisor are factors of the numerator and denominator of the dividend as shown above.

43. The **reciprocal** of a fraction is unity divided by the fraction.

That is, the reciprocal of $\frac{a}{b}$ is $1 \div \frac{a}{b}\cdot$

This result may be reduced to the form $\frac{b}{a}\cdot$

The fraction b/a is commonly referred to as being the reciprocal of the fraction a/b.

It should be observed that the reduced value, b/a, of the reciprocal, $1 \div a/b$, of the fraction, a/b, may be obtained by interchanging the terms a and b of the given fraction a/b.

The product obtained by multiplying the reciprocal of a fraction by the fraction is unity.

That is, $$\left(1 \div \frac{a}{b}\right) \times \frac{a}{b} \equiv 1\cdot$$

44. It will appear upon examination of the expression

$$a \div \frac{c}{d} \equiv a \times \frac{d}{c}$$

that the operation of dividing a by $\frac{c}{d}$ has the same effect as that of multiplying a by the reciprocal of the divisor, that is by $\frac{d}{c}$ (see § 41).

Hence, for the operation of division by a fraction may be substituted that of multiplication by the reciprocal of the fraction.

Division of One Fraction by Another

45. Let it be required to divide a/b by c/d; a, b, c, and d representing positive integers.

Applying Principles I. and II., § 34, and employing a course of reasoning similar to that used above, we obtain the result as follows:

We may either separate the fraction a/b into c equal parts, and then take d of these parts as summands, obtaining ad/bc, or we may substitute for the operation of division that of multiplication, using the reciprocal of the divisor as a multiplier, and immediately write

$$\frac{a}{b} \div \frac{c}{d} \equiv \frac{a}{b} \times \frac{d}{c} \equiv \frac{ad}{bc}.$$

Hence it follows that *the quotient obtained by dividing one fraction by another is equal to the product of the dividend and the reciprocal of the divisor.*

Ex. 3. $$\frac{3a^2b}{4cd^2} \div \frac{2c^2x}{5by} \equiv \frac{3a^2b}{4cd^2} \times \frac{5by}{2c^2x}$$
$$\equiv \frac{(3a^2b)(5by)}{(4cd^2)(2c^2x)}$$
$$\equiv \frac{15a^2b^2y}{8c^3d^2x}.$$

Check. $a = b = 2$, $c = d = 3$, $x = y = 1$. $\frac{10}{81} = \frac{10}{81}$.

Ex. 4. Simplify $\dfrac{2a^2 + 7ab + 6b^2}{ab - b^2} \times \dfrac{a - b}{4a^2 - 9b^2} \div \dfrac{a + 2b}{b}$.

Factoring and substituting $\times \dfrac{b}{a + 2b}$ for $\div \dfrac{a + 2b}{b}$, we may write

$$\frac{2a^2 + 7ab + 6b^2}{ab - b^2} \times \frac{a - b}{4a^2 - 9b^2} \div \frac{a + 2b}{b} \equiv$$
$$\frac{(2a + 3b)(a + 2b)}{b(a - b)} \times \frac{a - b}{(2a + 3b)(2a - 3b)} \times \frac{b}{a + 2b}$$
$$\equiv \frac{1}{2a - 3b}.$$

Check. $a = 4$, $b = 2$. $\frac{1}{2} = \frac{1}{2}$.

Since the factors are removed from the numerator by division, not by subtraction, it follows that when we strike out the last factor, whichever that may be, the quotient 1 remains.

46. Mixed numbers appearing in either dividend or divisor should be reduced to fractional form before the division is carried out.

Ex. 5. Divide $\dfrac{a}{a+b}+1+\dfrac{b}{a-b}$ by $\dfrac{a^2}{a^2-b^2}$.

Expressing the dividend as an improper fraction and multiplying by the reciprocal of the divisor, we have

$$\left(\frac{a}{a+b}+1+\frac{b}{a-b}\right)\div\frac{a^2}{a^2-b^2}\equiv\frac{a^2-ab+a^2-b^2+ab+b^2}{a^2-b^2}\times\frac{a^2-b^2}{a^2}$$

$$\equiv\frac{2\,a^2}{a^2}$$

$$\equiv 2.$$

Check. $a=3,\ b=2.$
$2=2.$

Mental Exercise XV. 10

Express each of the following quotients in simplest form :

1. $\dfrac{a^2}{b}\div a.$
2. $\dfrac{5\,m}{n}\div m.$
3. $\dfrac{12\,x}{y}\div 2\,x.$
4. $\dfrac{15\,c}{y}\div(-3\,c).$
5. $\dfrac{2}{3}\div x.$
6. $\dfrac{4}{5}\div(-y).$
7. $\dfrac{1}{2}\div z.$
8. $\dfrac{9}{10}\div 3\,a.$
9. $\dfrac{7}{8}\div 2\,b.$
10. $\dfrac{2\,a}{5}\div 3.$
11. $\dfrac{4\,b}{7}\div(-5).$
12. $\dfrac{2\,a}{x}\div 2\,y.$
13. $\dfrac{3\,b}{2\,z}\div 6.$
14. $x\div\left(-\dfrac{2}{3}\right).$
15. $y\div\dfrac{1}{2}.$
16. $z\div\dfrac{5}{6}.$
17. $2\,w\div\left(-\dfrac{4}{5}\right).$
18. $3\,a\div\dfrac{3}{4}.$
19. $\dfrac{a}{b}\div\dfrac{2}{3}.$
20. $\dfrac{x}{y}\div\dfrac{4}{5}.$
21. $\dfrac{7}{8}\div\dfrac{m}{n}.$
22. $\dfrac{9}{10}\div\dfrac{d}{c}.$
23. $\dfrac{1}{2}\div\dfrac{x}{y}.$
24. $\dfrac{1}{3}\div\dfrac{a}{b}.$
25. $\dfrac{m}{n}\div\dfrac{1}{4}.$
26. $\dfrac{z}{w}\div\left(-\dfrac{1}{5}\right).$
27. $\dfrac{a}{b}\div\dfrac{d}{c}.$
28. $\dfrac{x}{y}\div\dfrac{y}{x}.$
29. $\dfrac{m^2}{n^2}\div\dfrac{m}{n}.$
30. $\dfrac{a^2}{b^2}\div\dfrac{b}{a}.$
31. $\dfrac{ab}{c}\div\dfrac{b}{c}.$
32. $\dfrac{x}{yz}\div\dfrac{yz}{x}.$
33. $\dfrac{abc}{xy}\div\dfrac{ab}{xyz}.$

EXERCISE XV. 11

Express the following quotients as single fractions in lowest terms, checking all results numerically :

1. $\frac{7a}{3b} \div \frac{9b}{14a}.$

2. $\frac{2a^2}{5b} \div \frac{8a^4}{15b^4}.$

3. $\frac{9ab^2}{7x^2y} \div \frac{12a^2b}{35xy^2}.$

4. $\frac{9x}{10y} \div \frac{11z}{12w}.$

5. $\frac{6dy}{5cy} \div \frac{4bz}{3aw}.$

6. $\frac{12ad}{23bc} \div \frac{32bc}{21ad}.$

7. $\frac{3a^2b}{4cd^2} \div \frac{4ab^2}{3c^2d}.$

8. $\frac{1}{1-x} \div \frac{1}{1+x}.$

9. $\frac{x-1}{x} \div \frac{y}{x+1}.$

10. $\frac{a+1}{5} \div \frac{a^2-1}{10a}.$

11. $\frac{a+1}{2} \div \frac{3}{a+1}.$

12. $\frac{x-2}{3} \div \frac{5}{x+2}.$

13. $\frac{m+n}{8} \div \frac{6}{m+n}.$

14. $\frac{a+2}{5} \div \frac{1}{a+3}.$

15. $\frac{b-6}{10} \div \frac{3}{b+1}.$

16. $\frac{c+4}{5} \div \frac{4}{c+5}.$

17. $\frac{a+b}{x+y} \div \frac{x-y}{a-b}.$

18. $\frac{10a^2}{(a+b)^2} \div \frac{5a}{a+b}.$

19. $\frac{9x^2}{x^2-9} \div \frac{3x}{x-3}.$

20. $\frac{18x}{16x^2-1} \div \frac{6x}{4x-1}.$

21. $\frac{2}{2x+3y} \div \frac{4}{4x^2-9y^2}.$

22. $\frac{c^2-9}{c^2-c-2} \div \frac{c-3}{c^2+c-6}.$

23. $\frac{m^2-m-2}{m^2-m-6} \div \frac{m^2-2m}{2m+m^2}.$

24. $\frac{a-3}{a^2-2a+4} \div \frac{a^2-9}{a^3+8}.$

25. $\frac{(x+y)^2}{x+3y} \div \frac{(x^2-y^2)^2}{x^3+27y^3}.$

26. $\frac{33}{a^3+b^3} \div \frac{11}{a^2-ab+b^2}.$

27. $\frac{x^3-y^3}{14} \div \frac{x^2+xy+y^2}{7}.$

28. $\frac{a^3+1}{x^2-4y^2} \div \frac{a^2-a+1}{x-2y}.$

29. $\frac{(a+b)^2-c^2}{abc} \div \frac{a+b+c}{bc}$. 32. $\left(1+\frac{x}{y}\right) \div \left(1-\frac{y^2}{x^2}\right)$.

30. $\frac{(x-y)^2-z^2}{xy^2z} \div \frac{x-y-z}{xyz}$. 33. $\left(4-\frac{3}{a+1}\right) \div \left(6+\frac{5}{a^2-1}\right)$.

31. $\left(\frac{x}{a}+\frac{y}{b}\right) \div \left(\frac{a}{x}+\frac{b}{y}\right)$. 34. $\left(1-\frac{y}{x}\right) \div \left(x^2-\frac{y^3}{x}\right)$.

35. $\left(\frac{b}{c}+\frac{c}{a}-\frac{a}{b}\right) \div \left(\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right)$.

36. $\frac{3a^2+a-24}{6ax-20x} \times \frac{6a^2}{a^2-9} \times \frac{a-3}{3a-8} \div \frac{3a^2+9a}{3a^2-a-30}$.

37. $\frac{(a+b)^2-y^2}{a^2-ay+ab} \times \frac{a}{(a+y)^2-b^2} \div \frac{ab-by-b^2}{(a-b)^2-y^2}$

38. $\frac{m^2+m-6}{m^2-m-30} \times \frac{m^2+8m+15}{m^2-4m+4} \div \left(\frac{m^2+4m+3}{m^2-4m-12} \times \frac{m+3}{m-2}\right)$.

Complex Fractions

47. Since in algebraic expressions we cannot restrict ourselves to using positive integers only, we shall find it necessary to admit to our calculations fractions whose numerators and denominators, either or both, contain terms which in themselves may be positive or negative, integral or fractional.

By the Principle of No Exception we shall so extend our ideas concerning fractions that whatever may be the nature of the numbers entering into the terms of any particular fraction, the operands must in all cases be governed by the laws demonstrated for fractions whose terms consist of positive whole numbers only.

48. A **complex fraction** is a fraction containing one or more fractions among the terms of its numerator and denominator.

E. g. The following expressions are complex fractions:

$$\frac{a}{\frac{b}{c}}, \quad \frac{\frac{a+b}{c}}{d}, \quad \frac{\frac{1}{a}-\frac{1}{b}}{\frac{c+d}{ab}}.$$

49. When simplifying a complex fraction it is often convenient to obtain the reduced value of the numerator alone and of the denominator alone before attempting to perform the indicated division.

Ex. 1. Simplify $$\frac{\frac{1}{a}-\frac{1}{b}}{\frac{1}{a^2}-\frac{1}{b^2}}.$$

We have, $$\frac{\frac{1}{a}-\frac{1}{b}}{\frac{1}{a^2}-\frac{1}{b^2}} \equiv \frac{\frac{b-a}{ab}}{\frac{b^2-a^2}{a^2b^2}} \equiv \frac{b-a}{ab} \div \frac{b^2-a^2}{a^2b^2}$$

$$\equiv \frac{b-a}{ab} \times \frac{a^2b^2}{b^2-a^2} \equiv \frac{ab}{a+b}.$$

Check. $a=3, b=2.$ $\frac{6}{5}=\frac{6}{5}.$

50. It is sometimes desirable, when simplifying complex fractions, to multiply both the numerator and denominator by the L. C. M. of the denominators of the fractional terms.

E. g. In the example above we might have multiplied the complex numerator and also the complex denominator by the L. C. M. of their denominators, a^2b^2, and have written

$$\frac{\frac{b-a}{ab}}{\frac{b^2-a^2}{a^2b^2}} \equiv \frac{\frac{b-a}{ab} \times a^2b^2}{\frac{b^2-a^2}{a^2b^2} \times a^2b^2} \equiv \frac{(b-a)ab}{b^2-a^2} \equiv \frac{ab}{a+b}.$$

It will be observed that in this particular instance we have not used the reciprocal of the divisor as a multiplier, but have adopted another method.

Ex. 2. $$\frac{\frac{a+b}{a-b}-\frac{a-b}{a+b}}{\frac{a+b}{a-b}+\frac{a-b}{a+b}} \equiv \frac{\left(\frac{a+b}{a-b}-\frac{a-b}{a+b}\right)(a+b)(a-b)}{\left(\frac{a+b}{a-b}+\frac{a-b}{a+b}\right)(a+b)(a-b)}$$

$$\equiv \frac{(a+b)^2-(a-b)^2}{(a+b)^2+(a-b)^2}$$

$$\equiv \frac{2ab}{a^2+b^2}.$$

Check. $a=3, b=2.$ $\frac{12}{13}=\frac{12}{13}.$

EXERCISE XV. 12

Simplify the following complex fractions, checking all results numerically:

1. $\dfrac{1}{x+\dfrac{y}{z}}.$

2. $\dfrac{\dfrac{1}{c}}{a+\dfrac{b}{c}}.$

3. $\dfrac{\dfrac{1}{x}+\dfrac{1}{y}}{\dfrac{y}{x}-\dfrac{x}{y}}.$

4. $\dfrac{1+\dfrac{1}{x}}{1-\dfrac{1}{y}}.$

5. $\dfrac{1+\dfrac{m}{n}}{1-\dfrac{n^2}{m^2}}.$

6. $\dfrac{\dfrac{1}{x}-\dfrac{1}{y}}{\dfrac{1}{x^2}-\dfrac{1}{y^2}}.$

7. $\dfrac{\dfrac{x}{a}+\dfrac{b}{y}}{\dfrac{x}{a}-\dfrac{y}{z}}.$

8. $\dfrac{1-\dfrac{1}{x+1}}{1+\dfrac{1}{x-1}}.$

9. $\dfrac{\dfrac{x}{y}-\dfrac{y}{z}-\dfrac{z}{x}}{\dfrac{1}{y}-\dfrac{1}{z}-\dfrac{1}{x}}.$

10. $\dfrac{a-2-\dfrac{14}{a+3}}{a-1-\dfrac{21}{a+3}}.$

11. $\dfrac{\dfrac{b+c}{c}+\dfrac{c}{b+c}}{\dfrac{1}{b}+\dfrac{1}{c}}.$

12. $\dfrac{a+b+\dfrac{b^2}{a}}{a+b+\dfrac{a^2}{b}}.$

13. $\dfrac{\dfrac{1}{1-x}+\dfrac{1}{1+a}}{\dfrac{1}{1-x}-\dfrac{1}{1+a}}.$

14. $\dfrac{1-\dfrac{2b-2c}{a+b-c}}{1+\dfrac{2c}{a-b-c}}.$

15. $\dfrac{\dfrac{a+b}{c-d}+\dfrac{a-b}{c+d}}{\dfrac{a+b}{c+d}+\dfrac{a-b}{c-d}}.$

16. $\dfrac{c-3-\dfrac{1}{c-3}}{c-2-\dfrac{3}{c-4}}\times\dfrac{c-4-\dfrac{1}{c-4}}{c-5+\dfrac{3}{c-1}}.$

17. $\dfrac{\dfrac{c^2+y^2}{y}-c}{\dfrac{1}{y}-\dfrac{1}{c}}\div\dfrac{c^3+y^3}{c^2-y^2}.$

18. $\dfrac{\dfrac{1+2k}{1-2k}-\dfrac{1-2k}{1+2k}}{\dfrac{1-2k}{1+2k}+\dfrac{1+2k}{1-2k}}.$

19. $\dfrac{\dfrac{1}{xy}+\dfrac{1}{xz}+\dfrac{1}{yz}}{\dfrac{x^2-(y+z)^2}{xy}}.$

20. $\dfrac{\dfrac{a}{b}-\dfrac{b}{a}}{\dfrac{a}{b}+\dfrac{b}{a}} \div \dfrac{\dfrac{a}{b}-\dfrac{b}{a}}{\dfrac{1}{a}+\dfrac{1}{b}}.$

22. $\dfrac{3}{w+1} - \dfrac{2w-1}{w^2+\dfrac{w}{2}-\dfrac{1}{2}}.$

21. $\dfrac{\dfrac{x}{y^2}+\dfrac{y}{x^2}}{\dfrac{1}{x^2}-\dfrac{1}{xy}+\dfrac{1}{y^2}}.$

23. $\dfrac{y+\dfrac{1}{z}}{y+\cfrac{1}{z+\cfrac{1}{x}}} - \dfrac{1}{z(yzx+y+x)}.$

24. $\dfrac{\dfrac{x}{y}-\dfrac{y}{x}}{\dfrac{x}{y}+1+\dfrac{y}{x}} + \dfrac{1+\dfrac{y}{x}-\dfrac{y^2}{x^2}}{\dfrac{x}{y}-\dfrac{y^2}{x^2}}.$

Continued Fractions

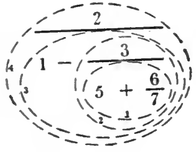
51. Rational functions of the form

$$\cfrac{a}{b+\cfrac{c}{d+\cfrac{e}{f+\cfrac{g}{h}}}}$$

are commonly called **continued fractions.**

52. Certain simple forms of continued fractions may be simplified by beginning at the last mixed expression, $f+\frac{g}{h}$, and after reducing this, by proceeding upward, reducing successively the next higher complex and mixed fractional expressions, until finally all have been used.

Ex. 1. Simplify $\dfrac{2}{1-\cfrac{3}{5+\cfrac{6}{7}}}.$



In the accompanying figure we have indicated the successive steps of the process, 1, 2, 3, and 4, by enclosing the different mixed and complex fractional expressions in different "spaces" bounded by curved lines.

The fraction may be simplified as follows:

$$\boxed{5+\frac{6}{7}} = \frac{41}{6}, \qquad \boxed{1-\frac{18}{41}} = \frac{23}{41},$$

$$\boxed{\frac{3}{\frac{41}{6}}} = \frac{18}{41}, \qquad \boxed{\frac{2}{\frac{23}{41}}} = \frac{82}{23}.$$

By reducing the improper fraction $\frac{82}{23}$ we obtain the mixed number $3\frac{13}{23}$, which is the value of the given continued fraction.

Ex. 2. Simplify the following continued fraction $\dfrac{a}{c-\dfrac{b}{b-\dfrac{c}{a}}}$.

Simplify the lowest mixed expression and divide b by the result. Subtract the result of this operation from c. Use the reciprocal of the remainder as a multiplier of a, to obtain the reduced value $\dfrac{a^2b-ac}{abc-ab-c^2}$ of the given continued fraction.

Check. $a=4$, $b=3$, $c=2$.
$5=5$.

Exercise XV. 13

Simplify each of the following :

1. $\dfrac{1}{a+\dfrac{1}{a+\dfrac{1}{a}}}$.

2. $\dfrac{a}{x+\dfrac{b}{y+\dfrac{c}{z}}}$.

3. $\dfrac{a}{b-\dfrac{b}{a-\dfrac{a}{b}}}$.

4. $\dfrac{a}{b-\dfrac{c}{d-\dfrac{e}{f}}}$.

5. $1-\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{x}}}$.

6. $\dfrac{x}{1-\dfrac{x}{2-\dfrac{x}{3-\dfrac{x}{4}}}}$.

7. $\dfrac{a-2}{a-2-\dfrac{a}{a-\dfrac{a-1}{a-2}}}$.

8. $\dfrac{1}{m-\dfrac{m^2-1}{m+\dfrac{1}{m-1}}}$.

53. Fractions whose terms contain binomial sums and differences may be simplified as follows :

Ex. 1. Simplify $\frac{b+c}{(a-b)(a-c)}+\frac{a+c}{(b-c)(b-a)}+\frac{a+b}{(c-a)(c-b)}$.

Among the six binomial factors of the denominators there are but three which are essentially different, such pairs as $a-b$ and $b-a$ differing only in sign.

We may accordingly choose for the form of the L. C. D. the expression $(a-b)(b-c)(c-a)$. Hence we may write:

$$\frac{b+c}{(a-b)(a-c)}+\frac{a+c}{(b-c)(b-a)}+\frac{a+b}{(c-a)(c-b)}$$
$$\equiv\frac{-(b+c)}{(a-b)(c-a)}+\frac{-(a+c)}{(b-c)(a-b)}+\frac{-(a+b)}{(c-a)(b-c)}$$

(The – signs are written in the numerators to compensate for the change of sign resulting from the alteration of the signs of the factors in the corresponding denominators.)

$$\equiv\frac{-(b+c)(b-c)-(c+a)(c-a)-(a+b)(a-b)}{(a-b)(b-c)(c-a)}$$
$$\equiv\frac{-b^2+c^2-c^2+a^2-a^2+b^2}{(a-b)(b-c)(c-a)}.$$

(It will be seen that the mutually destructive terms in the numerator disappear through addition and subtraction, for example, $a^2-a^2\equiv 0$.

Hence, when striking out the last pair, whichever that may be, we must write 0 as a resulting numerator.)

$$\equiv\frac{0}{(a-b)(b-c)(c-a)}.$$

$\equiv 0$, providing $a\neq b\neq c$. Check. $a=4$, $b=3$, $c=2$.
$0=0$.

This is because the value of a fraction is 0 if the numerator be 0 and the denominator different from 0.

54. Complex Fractions Involving **Decimal Fractions.**

Although no new principles are introduced by the appearance of decimal fractions, much labor may be saved by means of certain special devices.

E. g. By expressing all of the decimal fractions and whole numbers in a complex fraction as decimal fractions of the same order, we may disregard the decimal points altogether, since this amounts to multiplying both numerator and denominator of the complex fraction by the number representing the order of the decimal fractions.

Ex. 2. $$\frac{3.06\,a + .0204}{.255} \equiv \frac{30600\,a + 204}{2550}$$

$$\equiv \frac{300\,a + 2}{25}, \text{ in algebraic form,}$$

Check. $a = 2.$

$24.08 = 24.08.$

$$\equiv 12\,a + .08, \text{ in decimal notation.}$$

EXERCISE XV. 14 Miscellaneous

Simplify the following fractional expressions, checking all results numerically:

1. $1 - a + a^2 - \dfrac{a^3}{1 + a}.$

2. $x^2 + xy + y^2 + \dfrac{y^3}{x - y}.$

3. $\dfrac{a^2b - b^3}{a} \times \dfrac{3\,a}{2\,ab - 2\,b^2}.$

4. $\dfrac{m^2 - a^2}{az} \times \dfrac{m^2 + a^2}{a - m}.$

5. $\dfrac{x^4 - y^4}{x + y} \times \dfrac{x^2}{xy - y^2}.$

6. $\dfrac{y + y^2}{3\,a^2} \div \dfrac{2\,ay + 2\,ay^2}{7}.$

7. $\dfrac{m^2 - n^2}{m^2 - 9\,x^2} \times \dfrac{3\,mx + m^2}{m^2 + mn}.$

8. $\dfrac{d^2 - 4}{d^2 - 1} \times \dfrac{d^2 - 1}{2\,d} \times \dfrac{d - 2}{2 + d}.$

9. $\left(\dfrac{1}{a^2} - \dfrac{1}{b^2}\right)\left(\dfrac{ab}{a + b}\right).$

10. $\left(x^2 - \dfrac{y^3}{x}\right)\left(\dfrac{x}{x - y}\right).$

11. $\left(\dfrac{a^2}{x} - \dfrac{x^2}{a}\right)\left(\dfrac{a}{x^2} + \dfrac{x}{a^2}\right).$

12. $\dfrac{4}{(a - 1)(a - 3)} + \dfrac{2}{(a - 2)(3 - a)} - \dfrac{2}{(2 - a)(1 - a)}.$

13. $\dfrac{2}{a} - \dfrac{a + b - c}{ab} + \dfrac{2}{b} - \dfrac{a + c - b}{ac} + \dfrac{2}{c} - \dfrac{b + c - a}{bc}.$

14. $\dfrac{k^2 - 2\,k + 1}{k^2 - 5\,k + 6} \times \dfrac{k^2 - 4\,k + 4}{k^2 - 4\,k + 3} \times \dfrac{k^2 - 6\,k + 9}{k^2 - 3\,k + 2}.$

15. $\dfrac{3\,a^2 + a - 24}{6\,ax - 20\,x} \times \dfrac{6\,a^2}{a^2 - 9} \times \dfrac{a - 3}{3\,a - 8} \times \dfrac{3\,a^2 + 9\,a}{3\,a^2 - a - 30}.$

16. $\left(\dfrac{a^2}{x} - 8\,x + \dfrac{12\,x^3}{a^2}\right) \div \left(a - \dfrac{2\,x^2}{a}\right).$

17. $\left(x - a - \dfrac{x^2 + a^2}{x + a}\right)\left(x + a - \dfrac{x^2 + a^2}{x - a}\right).$

18. $\left(\frac{12}{m-3}+1-\frac{4}{m-1}\right)\left(\frac{4}{m+1}+1-\frac{12}{m+3}\right)$.

19. $\left(\frac{a}{2+a}-\frac{2-a}{a}\right)\div\left(\frac{a}{a+2}+\frac{2-a}{a}\right)$.

20. $\frac{m^2+mn}{m^2+n^2}\times\left(\frac{m}{m-n}-\frac{n}{m+n}\right)$.

21. $\left(2+\frac{m}{m-3}\right)\times\frac{9-m^2}{4-m^2}\times\frac{m+2}{m^2+m-6}-\frac{2}{m+2}$.

22. $\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\times\frac{abcd}{ac+ab+bc}$.

23. $\frac{a^2bc}{(a-b)(a-c)}+\frac{b^2ca}{(b-c)(b-a)}+\frac{c^2ab}{(c-a)(c-b)}$.

24. $\frac{r+1}{(r-s)(r-t)}+\frac{s+1}{(s-t)(s-r)}+\frac{t+1}{(t-r)(t-s)}$.

25. $\frac{c^2}{(c^2-d^2)(c^2-e^2)}+\frac{d^2}{(d^2-e^2)(d^2-c^2)}+\frac{e^2}{(d^2-c^2)(e^2-d^2)}$.

26. $\frac{d+b-c}{(d-b)(d-c)}+\frac{c+d-b}{(c-d)(c-b)}+\frac{b+c-d}{(b-c)(b-d)}$.

Indeterminate Forms

Sections 55–72 may be omitted when the chapter is read for the first time.

55. It frequently happens that when particular values are assigned to the letters appearing in the terms of a fraction, either the denominator or the numerator or both become zero.

E. g. $\frac{x+1}{x-1}$ for $x=1$ becomes $\frac{2}{0}$;

for $x=-1$ becomes $\frac{0}{-2}$;

$\frac{x^2-4}{x+2}$ for $x=2$ becomes $\frac{0}{4}$;

for $x=-2$ becomes $\frac{0}{0}$;

$\frac{x^3-x^2}{x^2-x}$ for $x=0$ becomes $\frac{0}{0}$;

for $x=1$ becomes $\frac{0}{0}$.

For all other finite values of the letters the expressions above assume perfectly definite values.

56. According to definitions previously given, expressions in which zero appears as a divisor have been excluded from calculations as being meaningless as numbers.

In order that such "number forms" may without exception be admitted to our calculations, we proceed to extend our idea of the value of an expression.

Since $\text{Quotient} \times \text{Divisor} \equiv \text{Dividend},$

we may regard $\frac{0}{0}$ as meaning $\text{Quotient} \times 0 \equiv 0.$

Hence, since the product of any number and zero is zero, we may interpret $\frac{0}{0}$ as representing any number.

57. If a variable be supposed to change in value in such a way as to *become and remain* as nearly equal as we please to some definite fixed value or constant, the variable is said to approach the constant as its **limit.**

The symbol $\doteq$ is read "approaches as a limit."

E.g. The expression $\lim\limits_{x \doteq a}(x+b) = a+b$ is read, "the limit of $(x+b)$ as x approaches a is equal to $a+b$."

The fraction $\dfrac{x^2-25}{x-5}$ assumes the form $\dfrac{0}{0}$ when $x=5$.

We may write $\dfrac{x^2-25}{x-5} \equiv \dfrac{(x+5)(x-5)}{x-5} \equiv (x+5)\left(\dfrac{x-5}{x-5}\right)$.

If we suppose that x approaches 5 as a limit, then so long as x has a value different from 5, $\dfrac{x-5}{x-5}$ will have the value unity (since the numerator and denominator are equal), while $x+5$ will differ from 10 by exactly that value by which x differs from 5.

Accordingly we can make the value of $(x+5)\left(\dfrac{x-5}{x-5}\right)$ as nearly equal to 10 as we please by giving to x a value sufficiently near to 5.

Accordingly, $\lim\limits_{x \doteq 5}\left(\dfrac{x^2-25}{x-5}\right) \equiv \lim\limits_{x \doteq 5}\left[(x+5)\left(\dfrac{x-5}{x-5}\right)\right] = 10.$

58. We shall define the **value of an expression** for any particular value of its variable to be the *limit* (if there be one) approached by the expression as its variable approaches the particular value as a limit.

Although this general definition may be used in all cases, we shall employ it only when, by using the definition given in Chap. I., § 19, we fail to obtain a definite number.

59. ***To find the value of a fraction which assumes the indeterminate form*** **0/0** *for some particular value of the variable appearing in it, find the limit approached by the fraction when the variable approaches the given particular value as its limit.*

60. It should be observed that a rational fraction assumes the form 0/0 because some factor common to both numerator and denominator becomes zero for some particular value of the letter appearing in it.

61. Since the symbol 0/0 does not represent the same value all of the time, but assumes different values according to circumstances, we interpret 0/0 as representing an **indeterminate value.**

Ex. 1. Find the limiting value of the fraction $(x^2 - 6x + 9)/(x - 3)$ as $x \doteq 3$.

We may write $\dfrac{x^2 - 6x + 9}{x - 3} \equiv (x - 3)\left(\dfrac{x - 3}{x - 3}\right)$.

For all values of x different from 3, $x - 3 \neq 0$.

Hence the value obtained by multiplying $x - 3$ by $\dfrac{x - 3}{x - 3}$, which is unity when x is different from 3, is the value of the factor $x - 3$.

As x approaches 3 as a limit the expression $x - 3$ approaches zero as a limit.

Accordingly the value of the given fraction is taken as zero when $x \doteq 3$.

Another method for finding the limiting value of an indeterminate fraction is shown in the following example:

Ex. 2. Find the value of $(x^2 - 49)/(x + 7)$ when $x \doteq -7$.

We may indicate that x differs from -7 by writing $x = -7 + h$, letting h represent a value which may be made as small as we please.

Accordingly, substituting $h - 7$ for x, $\dfrac{x^2 - 49}{x + 7}$ becomes

$$\frac{(h - 7)^2 - 49}{(h - 7) + 7} \equiv \frac{h^2 - 14h + 49 - 49}{h - 7 + 7},$$

$$\equiv \frac{h^2 - 14h}{h},$$

for all values of $h \neq 0$, $\qquad \equiv h - 14.$

Hence $(x^2 - 49)/(x + 7)$ differs from -14 by h, which may become and remain as small as we please.

Accordingly the given fraction approaches -14 as a limit as x approaches -7.

62. Consider the fraction a/x in which the numerator a is regarded as having some fixed or constant value, different from zero, while the value of the denominator is subject to change.

As the denominator is given successively smaller and smaller values (1/10, 1/100, 1/1000, etc.), the numerator retaining some constant value, it may be seen that as the value of the denominator decreases, the value of the fraction increases.

E.g. $$\frac{a}{\frac{1}{10}} \equiv 10\,a, \quad \frac{a}{\frac{1}{100}} \equiv 100\,a, \quad \frac{a}{\frac{1}{1000}} \equiv 1000\,a, \text{ etc.}$$

By giving to the denominator a value small enough, the value of the fraction a/x can be made greater than any assignable number.

63. The symbol ∞, read "infinity", is commonly used to denote all numbers or values which are greater than any assignable arithmetic number or value.

The expression $x = \infty$, read "x increases in value without limit" or "x is infinite", is to be understood as meaning that x has no definite fixed value, but that it *may assume different values which are very great* and are beyond the range of computation or imagination.

64. The symbol for an infinite number, ∞, should never be treated as representing a definite value, and it is not subject to the Laws of Algebra.

Corresponding to the ideas of positive and negative numbers, we have $^{+}\infty$, read "positive infinity," and $^{-}\infty$, read "negative infinity."

65. It is impossible to separate unity, or in fact any number, into such small parts that one of these parts shall have no value at all, that is, be zero. Hence it may be seen that, although the successive denominators (1/10, 1/100, 1/1000, etc.) of the complex fractions in § 62 become smaller and smaller in value, we can never, by diminishing the denominators in this way, obtain zero as a denominator.

66. A variable whose value may become indefinitely small without ever becoming zero is called an **infinitesimal.**

67. The symbol $\circ$ (horizontal zero), — read "diminishes indefinitely in value without ever becoming zero," or "is indefinitely or infinitely small," or "is nearly zero," — has been used by certain writers to denote an infinitesimal variable.

E. g. $x = \circ$ means "is nearly zero," that is, has a very small value.
$x = 0$ means "is exactly zero," that is, has no value.

68. Such numbers as are neither infinite nor infinitesimal in value are called **finite numbers.**

69. Interpretation of $\frac{a}{0}$. It may be seen from the preceding paragraphs that if the numerator of a fraction remains constant in value, the value of the fraction as a whole increases when the value of the denominator decreases; that is, for a given numerator, the smaller the denominator the greater the value of the fraction.

Hence, as the denominator becomes infinitesimal the fraction becomes infinite. Hence we may write $\frac{a}{\circ} = \infty$.

Accordingly, although strictly speaking $\frac{a}{0}$ has no meaning, we shall define it to have the same meaning as $\frac{a}{\circ}$; that is, we shall interpret $\frac{a}{0} = \infty$.

70. Interpretation of $\frac{a}{\infty}$. It may be seen that if the numerator of a fraction remains fixed in value, the value of the fraction as a whole becomes smaller indefinitely, as the value of the denominator increases indefinitely.

Accordingly we may write $\frac{a}{\infty} = \circ$.

It should be observed that $\frac{a}{\infty}$ is defined to mean *nearly zero*, that is $\circ$, not *exactly zero*, which is 0.

71. By the Principle of No Exception we shall define $\frac{0}{0}$ to mean $\frac{\circ}{\circ}$, (which may have any finite value), $\frac{a}{0}$, to mean $\frac{a}{\circ}$, that is ∞, (which represents any infinite value), and $\frac{a}{\infty}$ to mean $\circ$ (an infinitesimal value).

The expressions $\frac{0}{0}, \frac{a}{0}, \frac{a}{\infty}$ are commonly called **indeterminate forms.**

72. The symbol $]_s$, written at the right of a fraction or other expression containing a single variable, is to be understood to denote that the value represented by the subscript, s, is to be substituted for the variable.

E.g. The expression $\left.\frac{x+1}{x+2}\right]_3$ means $\frac{3+1}{3+2} = \frac{4}{5}$.

EXERCISE XV. 15

Find the limiting values of the following expressions for the values specified :

1. $\left.\frac{x^2-1}{x-1}\right]_1$.

2. $\left.\frac{a^2-25}{a+5}\right]_{-5}$.

3. $\left.\frac{x^3+1}{x+1}\right]_{-1}$.

4. $\left.\frac{b^5-1}{b-1}\right]_1$.

5. $\left.\frac{x^2-x-2}{x-2}\right]_2$.

6. $\left.\frac{x^3-4x^2+x+6}{x^2-5x+6}\right]_3$.

7. $\left.\frac{x^3-3x^2+3x-1}{x-1}\right]_1$.

8. $\left.\frac{x^3-6x^2+12x-8}{x^2-4x+4}\right]_2$.

Factors of Fractional Expressions

73. Expressions which are fractional with respect to specified variables may be factored by the methods which are employed for factoring integral expressions.

Ex. 1. $$a^2 + \frac{a}{c} + \frac{a}{b} + \frac{1}{bc} \equiv a\left(a + \frac{1}{c}\right) + \frac{1}{b}\left(a + \frac{1}{c}\right)$$
$$\equiv \left(a + \frac{1}{b}\right)\left(a + \frac{1}{c}\right).$$

Check.
$a = 1, b = 2, c = 3.$
$2 = 2.$

Ex. 2. $$\frac{9a^2}{16} - \frac{b^2}{c^2} \equiv \left(\frac{3a}{4} + \frac{b}{c}\right)\left(\frac{3a}{4} - \frac{b}{c}\right).$$

Check.
Let $a = 3, b = 2, c = 1.$
$\frac{17}{16} = \frac{17}{16}.$

EXERCISE XV. 16

Factor the following expressions, checking all results numerically:

1. $\frac{1}{a^2} + \frac{2}{ab} + \frac{1}{b^2}.$

2. $a^2 + \frac{2a}{b} + \frac{1}{b^2}.$

3. $x^2y^2 - \frac{1}{z^2}.$

4. $\frac{m^2}{n^2} - 2 + \frac{n^2}{m^2}.$

5. $\frac{a^2}{b^2} + \frac{b^2}{a^2} + 1.$

6. $\frac{x^2}{y^2} + \frac{5x}{y} + 6.$

7. $\frac{6a^2}{b^2} + \frac{13a}{b} + 5.$

8. $a^3 - \frac{1}{a^3} + a - \frac{1}{a}.$

9. $a^3 + a + \frac{1}{a} + \frac{1}{a^3}.$

10. $a^2 + a + 2 + \frac{1}{a} + \frac{1}{a^2}.$

11. $\frac{1}{x^3} - \frac{y^3}{8}.$

12. $\left(\frac{a^2+4}{a}\right)^2 - 7\left(\frac{a^2+4}{a}\right) + 12.$

MENTAL EXERCISE XV. 17 Review

Obtain the following products:

1. $(x-1)(1-x).$
2. $(2-b)(b-2).$
3. $(4+d)(d-4).$
4. $(c+5)(-c-5).$

Simplify each of the following:

5. $1 \div a/b.$
6. $a \div 1/b.$
7. $1 \div 1/xy.$
8. $xy \div x/y.$

Distinguish between

9. $\frac{1}{x} + \frac{1}{y}$ and $\frac{1}{x+y}.$

10. $\frac{1}{a} - \frac{1}{b}$ and $\frac{1}{a-b}$

11. $\frac{a+1}{2}$ and $\frac{a}{2} + 1.$

Simplify each of the following:

12. $\frac{a}{b+c} + \frac{a}{b-c}.$

13. $\frac{x}{y-z} - \frac{x}{y+z}.$

Supply the terms which make the following expressions trinomial squares:

14. $a^2+8a+(\ \ ).$
15. $b^2-10b+(\ \ ).$
16. $c^4+2c^2+(\ \ ).$
17. $d^6+6d^3+(\ \ ).$
18. $m^4+2m^3+(\ \ ).$
19. $n^6+4n^4+(\ \ ).$
20. $9h^4+6h^3+(\ \ ).$
21. $16x^8-24x^7+(\ \ ).$
22. $25y^{10}-30y^7+(\ \ ).$

Are the following expressions conditionally or identically equal?

23. $6 + 2$ and $5 + 3$.

24. $6x + 2x$ and $5 + 3$.

25. $6x + 2$ and $5x + 3$.

26. $6x + 2x$ and $5x + 3x$.

27. To which of the equations, $5x + 2y = 7$, $3x - 4y = 8$, $4x - 3y = 5$, are the following equations equivalent?

$$10x + 4y = 14; \quad 15x + 6y = 21; \quad 6x - 8y = 16;$$
$$9x - 12y = 24; \quad 16x - 12y = 20.$$

Are the following expressions identical?

28. $\frac{bh}{2}$, $\frac{1}{2}bh$, $\frac{b}{2}h$, and $b\frac{h}{2}$.

29. $\frac{x+y}{2}$, $\frac{x}{2} + \frac{y}{2}$ and $\frac{1}{2}(x+y)$.

Show that the following identities are true:

30. $\frac{1}{(b-a)(c-b)} \equiv \frac{1}{(a-b)(b-c)}$.

31. $-\frac{1}{(d-c)^3} \equiv \frac{1}{(c-d)^3}$.

32. $\frac{1}{(b-a)^2} \equiv \frac{1}{(a-b)^2}$.

33. $\frac{(a-x)^2}{a^2} \equiv \left(1 - \frac{x}{a}\right)^2$.

CHAPTER XVI

FRACTIONAL AND LITERAL EQUATIONS

Equations which are Algebraically Rational and Fractional with Reference to a Single Unknown Having Numerical Coefficients

1. An equation is said to be **fractional** with respect to any specified letter if that letter appears in the denominator of a fraction in either member of the equation.

E. g. The equation $\frac{2x}{3x+2} + \frac{5}{x-1} = 7$ is fractional with respect to x, while $\frac{3x}{a+b} - \frac{4x}{2a} = \frac{b}{a-b}$ is integral with respect to x but fractional with respect to a and b.

2. The only equations which are spoken of as having **degree** are those which are entirely rational and integral with respect to the unknowns appearing in them.

Hence the term *degree* does not apply to equations which are irrational or fractional with reference to a specified unknown.

3. If a given fractional equation cannot be solved immediately by inspection, its solution may be made to depend upon the solution of an integral equation derived from it by multiplying both members by the lowest common denominator of all of the fractions appearing in it.

This process of deriving an integral equation from a fractional equation is spoken of as **clearing the fractional equation of fractions.**

4. The equivalence of the given fractional and the derived integral equations may be determined by the following

Principle: *If both members of an equation whose terms are rational and fractional with reference to a single unknown, x, be multiplied by such an integral function of x as is necessary to clear* *the members of fractions, then the derived integral equation will be equivalent to the given fractional equation.*

(The following proof may be omitted when the chapter is read for the first time.)

Let the terms of a given equation which is rational and fractional with reference to a specified unknown, x, be all transposed to the first member, and then added algebraically.

Let the resulting fractional first member, *reduced to lowest terms*, be represented by $\frac{N}{D}$, in which N and D represent expressions which are rational and integral with reference to the unknown x.

Then, by the principles of equivalence, the given fractional equation will be equivalent to the derived fractional equation $\frac{N}{D} = 0$. (1).

Since $\frac{N}{D}$ is in lowest terms, it follows that N and D have no factor in common.

If for any value of x, such as $x = a$, both N and D should become zero, it would follow from the Factor Theorem that N and D would have the factor $x - a$ in common, and accordingly the fraction $\frac{N}{D}$ would not be in lowest terms.

Accordingly, when for any particular value of x the numerator N of the fraction $\frac{N}{D}$, which is in lowest terms, becomes zero, the denominator D must be different from zero.

The necessary and sufficient condition that the fraction $\frac{N}{D}$ in lowest terms shall become zero is that the numerator N shall become zero, the denominator D remaining finite and different from zero.

Hence any solution of the fractional equation $\frac{N}{D} = 0$ (2) must be a solution also of the integral equation $N = 0$ (3), obtained by multiplying both members of the fractional equation (2) by the multiplier D, which is necessary to clear equation (2) of fractions.

Hence no solutions of the fractional equation (2) are lost by multiplying both members by the multiplier D.

Since any value of x which reduces N to zero cannot reduce D to zero also (because N/D is in lowest terms), it follows that any solution of the integral equation $N = 0$ (3) must be a solution also of the fractional equation $\frac{N}{D} = 0$ (2).

Hence no solutions are gained by multiplying the fractional equation (2) by the multiplier D.

Accordingly, the given fractional equation and the derived integral equation $N = 0$ (3) are equivalent.

5. Although an integral equation which is equivalent to a given rational fractional equation may be derived by transposing all of the terms of the fractional equation to the first member, uniting these terms into a single fraction, reducing this fraction to lowest terms, and then clearing the equation of fractions, it is not always convenient to carry out the steps of the process in this order.

6. We shall consider in this chapter rational fractional equations containing a single unknown, the solutions of which may be made to depend upon the solutions of linear equations.

Other fractional equations will be discussed in a later chapter.

Ex. 1. Solve the fractional equation $\frac{11}{x+1} = \frac{9}{2x-11}$. (1)

We may derive an integral equation by multiplying both members of (1) by the product $(x+1)(2x-11)$ of the denominators and obtain

$$\frac{11(x+1)(2x-11)}{x+1} = \frac{9(x+1)(2x-11)}{2x-11}.$$

Or, $$11(2x-11) = 9(x+1). \quad (2)$$

From the integral equation (2) we may obtain the equivalent integral equation $$22x - 121 = 9x + 9, \quad (3)$$

the single solution of which is found to be $x = 10$.

Since neither of the solutions, $x = -1$ or $x = 11/2$, of the equation formed by placing the multiplier $(x+1)(2x-11)$ equal to zero, is a solution of the derived integral equation (3), it may be seen that the single solution $x = 10$ of the derived integral equation (3) must be the solution of the original fractional equation (1), and there can be no other solution.

The solution may be verified by substituting 10 for x in (1), when we shall obtain the identity $1 = 1$.

We may obtain the graph of the equation $\frac{11}{x+1} = \frac{9}{2x-11}$ as follows:

Transpose $9/(2x-11)$ to the first member and then write the first member of the equation thus formed equal to y, as follows:

$$\frac{11}{x+1} - \frac{9}{2x-11} = y. \quad (4)$$

Different pairs of corresponding values of x and y, which may be taken as abscissas and ordinates respectively of points on the graph, may be obtained as follows:

By assigning different values to x and substituting these values for x in (4) we may calculate corresponding values for y.

It will be found convenient to transform the first member of equation (4) by addition, and to obtain the values of y by substituting values of x in the transformed equation (5).

$$\frac{13(x-10)}{(x+1)(2x-11)} = y. \tag{5}$$

After having computed pairs of values for x and y, points on the graph may be located. (See Fig. 1.)

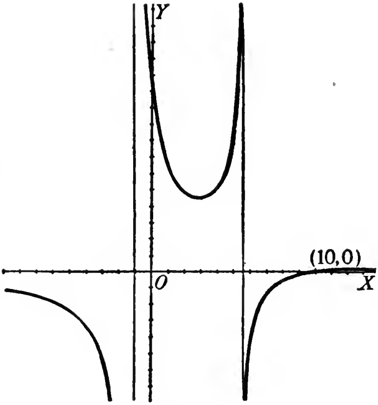


Fig. 1.

7. If a fractional term of an equation is preceded by a negative sign, then when deriving an integral equation (by multiplying every term of the equation by the lowest common multiple of the denominators of the different terms) it is necessary to change the sign of every term of the numerator from + to −, or from − to +.

Ex. 2. Solve $\dfrac{x-2}{x+2} - \dfrac{x+2}{x-2} - \dfrac{x\ -1}{x^2-4} = 0.$

Clearing of fractions by multiplying every term by $x^2 - 4$, which is the lowest common denominator of the fractions, and changing the signs of the terms in the numerator $x - 1$ of the last fraction, we have

$$(x-2)^2 - (x+2)^2 - x + 1 = 0$$
$$x^2 - 4x + 4 - x^2 - 4x - 4 - x + 1 = 0$$
$$-9x = -1$$
$$x = \tfrac{1}{9}.$$

By substituting 1/9 for x in the given equation, the solution $x = 1/9$ may be verified as follows:

$$\frac{-17}{19} + \frac{19}{17} - \frac{72}{323} = 0$$
$$-289 + 361 - 72 = 0$$
$$0 = 0.$$

8. In some cases, before clearing of fractions, it is well to unite some of the fractions appearing in an equation.

Ex. 3. Solve $$\frac{1}{x-7}-\frac{1}{x-3}=\frac{1}{x-5}-\frac{1}{x-1}. \qquad (1)$$

Uniting fractional terms in each member separately,

$$\frac{(x-3)-(x-7)}{(x-7)(x-3)}=\frac{(x-1)-(x-5)}{(x-5)(x-1)}. \qquad (2)$$

Simplifying the numerators, we have

$$\frac{4}{(x-7)(x-3)}=\frac{4}{(x-5)(x-1)}. \qquad (3)$$

Instead of obtaining an integral equation by dividing both members by 4 and clearing of fractions, we may proceed as follows:

Since the members of the equation are equal fractions having equal numerators, we may equate the denominators at once, and write

$$(x-7)(x-3)=(x-5)(x-1).$$

From this equation we obtain

$$x^2-10x+21=x^2-6x+5.$$

Collecting terms, $-4x=-16.$

Finally, we obtain $x=4.$

Since the root 4 could not have been introduced when (3) was being cleared of fractions, it must be the single solution of the original equation (1).

The solution may be verified by substituting 4 for x in the original equation.

General Directions for Solving Fractional Equations

9. Although special devices may be employed to obtain the solutions of fractional equations, the following general directions will often enable the student to avoid unnecessary work, and also to avoid introducing into the derived integral equations roots which do not satisfy the given fractional equations.

1. Before clearing of fractions, all fractions should be reduced to lowest terms.

2. Fractions having a common denominator should be combined.

3. Wherever possible, the denominators of fractions in lowest terms should be factored and the factored forms retained until an integral equation is derived, for the forms of these factors may suggest a simple grouping of the terms of the equation.

4. When clearing of fractions, use the lowest common multiple of the denominators of the fractions in lowest terms as a multiplier.

EXERCISE XVI. 1

Solve the following fractional equations for the letters appearing in them; the first one hundred and twenty equations may be solved mentally:

1. $\frac{2}{x} = 1.$
2. $\frac{3}{y} = 1.$
3. $1 = \frac{6}{z}.$
4. $1 = \frac{8}{w}.$
5. $\frac{5}{m} = -1.$
6. $\frac{-4}{n} = 1.$
7. $\frac{1}{x} = 2.$
8. $\frac{1}{y} = 3.$
9. $4 = \frac{1}{z}.$
10. $6 = \frac{1}{w}.$
11. $\frac{1}{2x} = 1.$
12. $\frac{1}{3y} = 1.$
13. $\frac{1}{9z} = -1.$
14. $\frac{2}{5a} = 1.$
15. $\frac{3}{7b} = 1.$
16. $1 = \frac{6}{7c}.$
17. $1 = \frac{-5}{11d}.$
18. $\frac{1}{2x} = 3.$
19. $\frac{1}{3y} = 5.$
20. $-8 = \frac{7}{3w}.$
21. $5 = \frac{1}{5k}.$
22. $2 = \frac{3}{4x}.$
23. $6 = \frac{5}{2y}.$
24. $10 = \frac{13}{3z}.$
25. $\frac{5}{x} - 1 = 0.$
26. $\frac{12}{y} - 1 = 0.$
27. $\frac{17}{z} - 1 = 0.$
28. $1 - \frac{15}{w} = 0.$
29. $1 - \frac{13}{a} = 0.$
30. $\frac{1}{x} - 6 = 0.$
31. $\frac{1}{y} - 5 = 0.$
32. $\frac{1}{z} - 8 = 0.$
33. $14 - \frac{1}{a} = 0.$
34. $17 - \frac{1}{b} = 0.$
35. $\frac{2}{c} - 3 = 0.$
36. $\frac{4}{d} + 5 = 0.$
37. $\frac{9}{h} + 13 = 0.$
38. $\frac{1}{x} - 2 = 3.$
39. $\frac{1}{y} - 4 = 7.$
40. $\frac{1}{z} - 1 = 1.$
41. $\frac{1}{a} + 3 = 6.$
42. $\frac{1}{b} + 7 = 5.$

43. $\frac{1}{c} + 8 = 2.$

44. $5 + \frac{1}{w} = 8.$

45. $9 + \frac{1}{a} = 7.$

46. $10 + \frac{1}{b} = 6.$

47. $4 - \frac{1}{x} = 3.$

48. $5 - \frac{1}{y} = 2.$

49. $6 - \frac{1}{z} = 10.$

50. $7 - \frac{1}{w} = 12.$

51. $\frac{3}{x + 2} = 1.$

52. $\frac{5}{y + 3} = 1.$

53. $\frac{2}{z - 3} = 1.$

54. $\frac{4}{w - 6} = 1.$

55. $\frac{5}{1 - a} = 1.$

56. $\frac{6}{3 - b} = 1.$

57. $5 = \frac{1}{5 + c}.$

58. $1 = \frac{7}{d - 7}.$

59. $1 = \frac{8}{8 - g}.$

60. $\frac{2}{2x + 1} = 1.$

61. $\frac{3}{3y + 1} = 1.$

62. $\frac{4}{x + 4} = 1.$

63. $\frac{3}{h + 4} = 2.$

64. $\frac{4}{y + 2} = 5.$

65. $\frac{1}{x + 2} = 2.$

66. $\frac{1}{z - 3} = 4.$

67. $\frac{2}{m - 5} = 7.$

68. $\frac{1}{n} = \frac{1}{3}.$

69. $\frac{1}{y} = \frac{1}{5}.$

70. $\frac{1}{z} = -\frac{1}{7}.$

71. $\frac{1}{9} = \frac{1}{w}.$

72. $\frac{1}{x + 2} = \frac{1}{3}.$

73. $\frac{1}{y - 4} = \frac{1}{4}.$

74. $\frac{1}{z + 6} = \frac{1}{5}.$

75. $\frac{1}{2a + 1} = \frac{1}{a + 3}.$

76. $\frac{1}{3b - 2} = \frac{1}{2b + 2}.$

77. $\frac{2}{5c - 1} = \frac{2}{3c + 5}.$

78. $\frac{1}{x} = \frac{2}{3}.$

79. $\frac{1}{y} = \frac{6}{5}.$

80. $\frac{1}{z} = -\frac{8}{9}.$

81. $\frac{4}{7} = \frac{1}{x}.$

82. $\frac{3}{8} = \frac{2}{y}.$

83. $\frac{1}{5} = \frac{5}{z}.$

84. $\frac{1}{5} = \frac{3}{2w}.$

85. $\frac{2}{7a} = \frac{5}{9}.$

86. $\frac{3}{7b} = -\frac{7}{3}.$

87. $\frac{4}{5} = -\frac{7}{6c}.$

88. $\frac{2}{x} + \frac{3}{x} = 5.$

89. $\frac{4}{y} + \frac{6}{y} = 9.$

90. $\frac{8}{z} - \frac{3}{z} = 4.$

91. $\frac{9}{a} - \frac{5}{a} = 2.$

92. $\frac{7}{b} + 1 = 10 - \frac{2}{b}.$

93. $\frac{4}{c} + 3 = \frac{3}{c} + 4.$

94. $\frac{2x+3}{x+4} = 1.$

95. $\frac{5x+1}{2x+7} = 1.$

96. $\frac{3m-8}{m} = 1.$

97. $\frac{4n-7}{n} = 3.$

98. $\frac{5g-9}{2g} = 2.$

99. $\frac{7h}{3h+4} = 2.$

100. $\frac{13k}{4k-5} = 3.$

101. $\frac{w+7}{w+1} = 3.$

102. $\frac{a+1}{a-2} = 4.$

103. $\frac{b-1}{b+3} = 5.$

104. $\frac{1}{2x} + \frac{1}{x} = 3.$

105. $\frac{1}{d} + \frac{2}{3d} = 2.$

106. $\frac{1}{3x} + \frac{1}{2x} = 1.$

107. $\frac{3}{2y} - \frac{5}{4y} = 2.$

108. $\frac{3}{5z} + \frac{1}{2z} = 3.$

109. $\frac{2}{x+5} = \frac{1}{x+2}.$

110. $\frac{3}{x+4} = \frac{2}{x+2}.$

111. $\frac{2}{x-3} = \frac{1}{x-2}.$

112. $\frac{7}{x+3} = \frac{6}{x-2}.$

113. $\frac{5}{x-1} = \frac{4}{x+1}.$

114. $\frac{3}{2x+1} = \frac{1}{x-1}.$

115. $\frac{7}{3x-1} = \frac{2}{x-1}.$

116. $\frac{3}{5x-1} = \frac{1}{2x-1}.$

117. $\frac{5}{3x+2} = \frac{7}{5x-2}.$

118. $\frac{6}{3x+4} = \frac{3}{2x-1}.$

119. $\frac{4}{7x+3} = \frac{3}{6x+2}.$

120. $\frac{6}{2x-3} = \frac{11}{4x-5}.$

121. $\frac{x+1}{x-2} = \frac{x-3}{x+5}.$

122. $\frac{4x-3}{2x-1} = \frac{4x-7}{2x-5}.$

123. $\frac{6x-2}{3x+4} = \frac{2x+1}{x+3}.$

124. $\frac{1}{(x+1)(x+4)} = \frac{1}{(x+2)(x+5)}.$

125. $\frac{1}{(x+14)(x-7)} = \frac{1}{(x-13)(x-6)}.$

126. $3 - \frac{12}{x+1} = 7 - \frac{4x+30}{x+3}.$

127. $\frac{1}{x-2}+\frac{2}{x-3}=\frac{3}{x-4}$.

128. $\frac{7}{x-7}-\frac{3}{x-3}=\frac{4}{x-4}$.

129. $\frac{1}{x-2}-\frac{1}{x-1}=\frac{1}{x-4}-\frac{1}{x-3}$.

130. $\frac{1}{x-9}-\frac{1}{x-11}=\frac{1}{x-15}-\frac{1}{x-17}$.

Equations in which Numbers other than the Unknown are Represented by Letters

10. Equations in which coefficients and known numbers are represented by letters are called **literal** equations.

In the preceding chapters principles were developed which may be applied to obtain the solutions of literal equations containing ing one unknown number.

11. A literal equation is said to be solved with reference to a specified letter when the value of this letter is expressed in terms of the remaining letters appearing in the equation. We shall commonly refer to the value thus obtained as the **expressed value** of the unknown.

It should be understood that no numerical value is obtained for the unknown by this process.

Numerical values for the specified unknown letters can be obtained only when definite values are assigned to the letters which are regarded as representing known values.

Literal Equations which are Integral with Reference to the Unknown

Ex. 1. Regarding x as the unknown, solve $x+d=c$.

Transposing, we have, $x=c-d$.

This expressed value is found by substitution to satisfy the given equation.

Ex. 2. Regarding x as the unknown, solve $ax+n=m$.

We have, $ax=m-n$.

Therefore $x=\frac{m-n}{a}$.

This expressed value will be found to satisfy the given equation.

Mental Exercise XVI. 2

Regarding x, y, z, and w as unknowns, solve the following literal equations which are integral with reference to x, y, z, and w:

1. $x - b = a$.
2. $x + d = c$.
3. $n = m - z$.
4. $k + h = w$.
5. $x + 1 = c$.
6. $y + 4 = d$.
7. $z - 6 = k$.
8. $w - g = 7$.
9. $ax = b$.
10. $by = c$.
11. $az = - h$.
12. $a = bw$.
13. $- c = dx$.
14. $ax - b = 1$.
15. $ax + c = 1$.
16. $by - 2 = c$.
17. $3 + mz = n$.
18. $ax + b = c$.
19. $by - c = a$.
20. $cz - b = - a$.
21. $ax + b = ac$.
22. $a = bx + c$.
23. $r = sx - t$.
24. $m = n - qy$.
25. $ax = b + c$.
26. $2\,abx = a + b$.
27. $a^2x = a + 1$.
28. $(m + n)x = 2\,mn$.
29. $(a + 5)\,y = 5\,a$.
30. $(3 - m)\,z = 3\,m$.
31. $ax = a^2$.
32. $b^2x = b$.
33. $abx = bc$.
34. $ab^2cx = a^2bc^2$.
35. $- abx = a - b$.
36. $(a + b)x = c + d$.
37. $(c - 2)\,y = c + 3$.
38. $(a + b)y = a^2 - b^2$.
39. $(m - n)\,z = m^2 - n^2$.
40. $(a^2 - 4)x = a - 2$.
41. $(b^2 - 1)\,y = b + 1$.
42. $\frac{x}{a} = b$.
43. $\frac{y}{b} = c$.
44. $\frac{y}{r + s} = rs$.
45. $\frac{z}{m - n} = mn$.
46. $\frac{x}{a + 3} = 3\,a$.
47. $\frac{x}{a + 1} = a^2$.
48. $\frac{x}{c} = \frac{1}{b}$.
49. $\frac{x}{a} = \frac{b}{c}$.
50. $\frac{x}{c} = \frac{c}{d}$.
51. $\frac{x}{2} = \frac{2}{a}$.
52. $\frac{y}{b} = \frac{b}{5}$.

53. $\frac{my}{n} = 1.$

54. $\frac{dw}{h} = -1.$

55. $\frac{bx}{c} = \frac{1}{d}.$

56. $\frac{ax}{b} = \frac{b}{a}.$

57. $x - m = n - x.$

58. $a - x = x - a.$

59. $b - x = x - c.$

60. $dy - 1 = 1 - dy.$

61. $ax - b = b - ax.$

62. $hz - 3 = 3 - hz.$

63. $a - bx = bx - a.$

64. $b - cx = cx - b.$

65. $1 - ax = ax - 1.$

66. $2 - cy = cy - 2.$

67. $ax + bx = a + b.$

68. $h + mx = hx + m.$

69. $bx - cx = b - c.$

70. $ky - k = ry - r.$

71. $by - b = ny - n.$

72. $mx + n^2 = nx + m^2.$

73. $ax = bx + 1.$

74. $dy = 1 - ny.$

75. $2\,ay = 3\,by + 4.$

76. $abx + bcx = ca.$

77. $ax - bx - 1 = cx.$

78. $ay + by = d - cy.$

79. $a(y - b) = c.$

80. $a(x + 1) = b.$

81. $c(by + 1) = d.$

82. $b(b - y) = a.$

83. $a(1 - bx) = b.$

84. $\frac{x}{a} - 1 = b.$

85. $\frac{y}{d} + 1 = c.$

86. $\frac{x}{a + b} - b = a.$

87. $\frac{ax}{b} + \frac{bx}{a} = 2.$

12. Literal Equations as Formulas. Instead of stating a mathematical law or principle in words, it is often more convenient to express it by means of a literal equation in which the letters used are understood as standing for particular quantities.

When so used, a literal equation is called a **formula.**

In applications of mathematics to other sciences, it is a common practice to state principles by means of formulas.

E. g. The identity $(a \pm b)^2 \equiv a^2 \pm 2\,ab + b^2$ is a formula for finding the square of a binomial sum or difference.

The literal equation $i = p \times r \times t$ may be used as a formula for computing simple interest, provided that the letters i, p, r, and t are understood as representing interest, principle, rate, and time respectively.

13. When solving numerical equations, certain terms are often so combined as to cause particular numerical constants to disappear from the calculation; but, when solving literal equations or formulas, it often happens that none of the given letters disappear, but may be traced throughout the entire work.

14. The solution of a particular literal equation may be used to obtain the solutions of an indefinite number of numerical equations of corresponding type, that is, of equations in which numerical constants appear in place of the literal constants of the given literal equation.

The particular literal equation thus solved may be used as a formula for obtaining the solutions of the numerical equations which it may be taken as representing.

E. g. Let it be required to solve several equations such as the following:

$$\begin{aligned} 6x + 4 &= 5x + 7, &(1)\\ 8x + 3 &= 2x + 11, &(2)\\ 2x + 9 &= x + 14, &(3)\\ 3x - 10 &= 4x + 18. &(4) \end{aligned}$$

All of the equations are of the type $ax + b = cx + d$, in which a, b, c, and d represent in the first equation 6, 4, 5, and 7 respectively; in the second equation 8, 3, 2, and 11 respectively; etc.

We may obtain the solution of the literal equation as follows:

From $\quad ax + b = cx + d,$

we obtain $\quad ax - cx = d - b.$

Hence, $\quad x(a - c) = d - b,$

and finally, $\quad x = \dfrac{d - b}{a - c}.$

From the process of derivation the successive equations are equivalent, and hence solutions have neither been gained nor lost.

This solution may be verified by direct substitution.

The solution $x = (d - b)/(a - c)$, which is a literal equation, may be used as a formula for obtaining numerical values for x in the given equations by substituting for a, b, c, and d the values which they represent.

For equation (1), we have, $x = \dfrac{7 - 4}{6 - 5} = 3.$

For equation (2), we have, $x = \dfrac{11 - 3}{8 - 2} = \dfrac{4}{3}.$

For equation (3), we have, $x = \frac{14 - 9}{2 - 1} = 5.$

For equation (4), we have, $x = \frac{18 + 10}{3 - 4} = -28.$

Ex. 1. Solve $a(x - a) - 2ab = b(b - x).$ (1)

Removing parentheses by performing the indicated multiplications, we obtain by Principle I, Chap. X. § 25, the equivalent equation

$$ax - a^2 - 2ab = b^2 - bx. \qquad (2)$$

Transposing to the first member the terms containing x, and to the second member those free from x, we obtain by Chap. X. § 27, the equivalent equation

$$ax + bx = b^2 + a^2 + 2ab. \qquad (3)$$

Factoring the members separately, we obtain by Principle I, Chap. X. § 25, the equivalent equation

$$(a + b)x = (a + b)^2. \qquad (4)$$

Therefore $$x = \frac{(a + b)^2}{a + b} \equiv a + b,$$

which by the process of derivation must satisfy the original equation and be its only solution.

We may verify this result by substituting in equation (1) as follows:

$$\begin{aligned} a(a + b - a) - 2ab &= b[b - (a + b)] \\ a(b) - 2ab &= b(b - a - b) \\ ab - 2ab &= b(-a) \\ -ab &= -ab. \end{aligned}$$

Numerical Checks for the Solutions of Literal Equations

15. Whenever numerical values are assigned to the letters representing known quantities in a literal equation, a numerical equation is obtained. The solutions of this numerical equation are equal to the numerical values found by substituting the same numerical values for the known letters in the expressed value for the unknown which is the solution of the given literal equation.

It follows that the values assigned to the known letters and the value calculated for the unknown letter will, if substituted for the known and unknown letters respectively in the given literal equation, reduce it to a numerical identity.

Regarding x as the unknown, we obtain from the literal equation $a(x-a)-2ab=b(b-x)$, the expressed value $x=a+b$. (See Ex. 1. § 14.)

By assigning particular numerical values to a and b we may, from the expressed value $x=a+b$, calculate a numerical value for x.

Thus, if $a=1$, and $b=2$, it follows from $x=a+b$ that $x=3$.

Substituting 1 for a, 2 for b and 3 for x in the given literal equation, $a(x-a)-2ab=b(b-x)$, we obtain the numerical identity, $1(3-1)-2\cdot 1\cdot 2=2(2-3)$, which reduces to $-2=-2$.

Accordingly, we have by this method verified for particular values of the letters the solution $x=a+b$ of the given literal equation.

From the nature of the method it may be seen that if the check holds in a particular case it holds in all cases.

Hence we have verified the solution of the given literal equation.

Exercise XVI. 3

Solve the following literal equations for x, verifying all results either by substituting the literal solutions directly, or by making proper numerical substitutions :

1. $a(x+b)=b(x+a)$.
2. $ax+bc=d(b+x)$.
3. $(x+a)(x+b)=x(x+c)$.
4. $m=a+(n-1)x$.
5. $(m-n)x-m^2=(m+n)x$.
6. $a(1+x)+b(1+x)=x(a+b+1)$.
7. $a(x-1)+(a-1)x=a+x$.
8. $a(a-2x)+b(b-2x)+2ab=0$.
9. $3(3x-b)+2b=b(bx-3)+6$.
10. $hk(x^2-1)=(h+kx)(k+hx)$.
11. $(x+a)(x-b)=(x+a-b)^2$.
12. $(b-c)(x-b)=(b-a)x$.
13. $(x-a)(b-c)=(a-c)(x-b)$.
14. $(a+b)^2+(a-x)(b-x)=(x+a)(x+b)$.
15. $(a-x)(x+b)-c(a+c)=(c-x)(x+c)+ab$.
16. $\dfrac{x}{a+b}+b=a$.

17. $\frac{x}{c+d} - d = c.$

18. $\frac{3x}{a} + \frac{2x}{b} = 3.$

19. $\frac{cx}{d} + \frac{dx}{c} = 1.$

20. $\frac{x}{a} + b = \frac{x}{b} + a.$

21. $\frac{x}{g} - h = \frac{x}{h} - g.$

22. $\frac{x}{m} + \frac{a-x}{n} = b.$

23. $\frac{x}{a} - \frac{m-x}{b} = n.$

24. $\frac{x}{b} = x - b + \frac{1}{b}.$

25. $bx + b = \frac{x}{b} + \frac{1}{b}.$

26. $\frac{x-a}{b} = \frac{x-b}{a}.$

27. $\frac{d+x}{k} = \frac{x}{d+k}.$

28. $\frac{x-a}{b} + \frac{x-b}{a} = \frac{a^2+b^2}{ab}.$

29. $\frac{b^2-ax}{b} = b - \frac{a^2-bx}{a}.$

30. $\frac{x}{b} - 2 - \frac{b}{a} + \frac{x}{a} - \frac{a}{b} = 0.$

31. $\frac{cx}{c-d} = c - x.$

32. $bx - ac = \frac{c^2x}{c-b}.$

33. $\frac{a+x}{a+b} + \frac{ax}{b} = \frac{b}{a}.$

34. $\frac{ax}{bc} + \frac{bx}{ac} + \frac{cx}{ab} = f.$

Literal Equations which are Fractional with Reference to the Unknown

16. When solving literal equations which are fractional with reference to certain specified letters it is often convenient to obtain the solutions by deriving integral equations in which the unknown letters are found in the second member instead of in the first.

Ex. 1. Solve for x, $$\frac{m}{x} = n.$$

Clearing of fractions, we have
$$m = nx.$$

Hence, $$\frac{m}{n} = x.$$

Ex. 2. Solve for x, $$\frac{ab - bc}{x - b} = c.$$

Clearing of fractions we obtain
$$ab - bc = cx - bc.$$

From which, $$ab = cx.$$

Hence, $$\frac{ab}{c} = x.$$

Mental Exercise XVI. 4

Solve the following equations for x, y, z, and w:

1. $\frac{a}{x} = 1.$

2. $\frac{b}{y} = 1.$

3. $1 = \frac{d}{z}.$

4. $1 = \frac{m}{w}.$

5. $\frac{1}{x} = m.$

6. $\frac{2}{y} = n.$

7. $\frac{4}{z} = k.$

8. $\frac{7}{w} = q.$

9. $\frac{c}{x} = 2.$

10. $\frac{k}{y} = 3.$

11. $4 = \frac{r}{z}.$

12. $8 = \frac{t}{w}.$

13. $\frac{b}{x} = c.$

14. $\frac{bc}{x} = a.$

15. $\frac{d}{y} = fg.$

16. $\frac{a}{x} = \frac{2}{3}.$

17. $\frac{b}{y} = \frac{5}{9}.$

18. $\frac{m}{x} = -1.$

19. $\frac{n}{y} = -2.$

20. $9 = -\frac{b}{y}.$

21. $\frac{1}{ax} = 1.$

22. $\frac{2}{by} = 1.$

23. $\frac{3}{cy} = -1.$

24. $4 = \frac{3}{mz}.$

25. $\frac{mn}{x} = q.$

26. $\frac{r}{sx} = t.$

27. $\frac{a+b}{x} = a - b.$

28. $\frac{c-d}{y} = c + d.$

29. $m + n = \frac{m-n}{z}.$

30. $k - 1 = \frac{k+1}{w}.$

31. $\frac{1}{x} = \frac{a}{b}.$

32. $\frac{d}{n} = \frac{1}{x}.$

33. $\frac{g}{n} = \frac{1}{y}.$

34. $\frac{1}{x} = -\frac{c}{d}.$

35. $\frac{3}{x} = \frac{4}{a}.$

36. $\frac{5}{y} = \frac{b}{6}.$

37. $\frac{m}{2} = \frac{3}{z}.$

38. $\frac{a}{x} = \frac{b}{a}.$

39. $\frac{5}{h} = \frac{h}{z}.$

40. $\frac{b}{c} = \frac{c}{y}.$

41. $\frac{a}{2} = \frac{2}{ax}.$

42. $\frac{b}{3} = \frac{3}{by}.$

43. $\frac{6}{cz} = \frac{c}{6}$.

44. $\frac{8}{dw} = \frac{d}{8}$.

45. $\frac{a}{b} = \frac{b}{ax}$.

46. $\frac{m}{ny} = \frac{n}{m}$.

47. $\frac{2a}{bx} = \frac{c}{2a}$.

48. $\frac{c}{w-1} = 1$.

49. $\frac{b}{y-1} = 1$.

50. $\frac{1}{x-a} = 1$.

51. $\frac{1}{x-c} = \frac{1}{b}$.

52. $\frac{1}{y-b} = \frac{1}{c}$.

53. $\frac{1}{z+d} = \frac{1}{m}$.

54. $\frac{1}{w-g} = \frac{1}{g}$.

55. $\frac{1}{a} = \frac{1}{x-a}$.

56. $\frac{1}{b} = \frac{1}{y-b}$.

57. $\frac{1}{z+a} = \frac{1}{3a}$.

58. $\frac{1}{w+2b} = \frac{1}{5b}$.

59. $\frac{1}{x-a} = \frac{1}{a-b}$.

60. $\frac{1}{x+c} = \frac{1}{c+d}$.

61. $\frac{1}{z-g} = \frac{1}{g-h}$.

62. $\frac{1}{2x-m} = \frac{1}{m-2}$.

63. $\frac{1}{3y-t} = \frac{1}{t-3}$.

64. $\frac{1}{x-a} = \frac{1}{b+c}$.

65. $\frac{1}{x+b} = \frac{1}{a+c}$.

66. $\frac{1}{x+c} = \frac{1}{a+b}$.

67. $\frac{1}{ax+b} = \frac{1}{b+c}$.

68. $\frac{a}{x-b} = 1$.

69. $\frac{c}{y+d} = 1$.

70. $\frac{2k}{z+h} = 1$.

71. $\frac{4m}{w+m} = 1$.

72. $\frac{a}{x-a} = 2$.

73. $\frac{b}{y-b} = 3$.

74. $\frac{h}{z-h} = 4$.

75. $\frac{1}{x-b} = 2$.

76. $\frac{1}{y-c} = 3$.

77. $\frac{1}{z+d} = 4$.

78. $\frac{1}{x-2} = a$.

79. $\frac{1}{y-3} = b$.

80. $\frac{1}{z+5} = h$.

81. $\frac{1}{w+7} = k$.

82. $5 = \frac{c}{x-2}$.

83. $\frac{b}{x-4} = 3$.

84. $\frac{a}{x-1} = 2$.

85. $\frac{a}{x-b} = c$.

86. $\frac{m}{y-m} = n$.

87. $\frac{a}{bx+c} = d$.

Ex. 1. Solve $\frac{ax-b}{ax}=\frac{bx-a}{bx}-ab.$ (1)

Clearing of fractions by multiplying each of the terms by the lowest common denominator, *abx*, of the fractional terms, we obtain

$$abx-b^2=abx-a^2-a^2b^2x. \quad (2)$$

Hence, $$a^2b^2x=b^2-a^2. \quad (3)$$

Therefore, $$x=\frac{b^2-a^2}{a^2b^2}.$$

We may apply the method of § 15 and verify this solution as follows:

If we let $a=2$ and $b=3$, it follows that $x=\frac{9-4}{4\cdot 9}=\frac{5}{36}.$

Substituting 2, 3, and 5/36 for a, b, and x respectively, in the given literal equation, we obtain the following numerical equality which is found to be a numerical identity:

$$\frac{2\left(\frac{5}{36}\right)-3}{2\left(\frac{5}{36}\right)}=\frac{3\left(\frac{5}{36}\right)-2}{3\left(\frac{5}{36}\right)}-2\cdot 3$$

$$-\frac{49}{5}=-\frac{49}{5}.$$

Exercise XVI. 5

Regarding x as the unknown, solve the following literal equations:

1. $g-\frac{h}{x}=\frac{g}{x}-h.$

2. $\frac{n}{x}=m(n-t)+\frac{t}{x}.$

3. $a=\frac{b}{c+dx}.$

4. $b=\frac{c-x}{ax}.$

5. $n=\frac{e-a}{x}+1.$

6. $m=\frac{a+3(x-m)}{x}.$

7. $\frac{dh}{dx+hx}=1.$

8. $r=\frac{x-a}{x-s}.$

9. $\frac{x+b}{x-b}=\frac{3}{4}.$

10. $\frac{x-a}{x-b}=\frac{a^2}{b^2}.$

11. $\frac{b}{a}=\frac{x-a^2}{x-b^2}.$

12. $\frac{c-dx}{cx-d}=\frac{c}{d}.$

13. $\frac{bx^2}{c-ax}+b+\frac{bx}{a}=0.$

14. $\frac{x}{x-a}=\frac{a+b}{b}.$

15. $\dfrac{g+h}{g-x}=\dfrac{g}{g-h}$.

16. $\dfrac{x+p}{x-q}=\dfrac{p-q}{p+q}$.

17. $\dfrac{e-x}{x-d}=\dfrac{d-x}{x-e}$.

18. $\dfrac{m-x}{n-x}=\dfrac{m-1}{n-1}$.

19. $\dfrac{x-b}{x-4}=\dfrac{x-c}{x-5}$.

20. $\dfrac{a-x}{a+x}=\dfrac{x}{a-x}$.

21. $\dfrac{x-a}{3}-\dfrac{3}{x+a}=\dfrac{x+a}{3}$.

22. $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}+\dfrac{1}{d}=\dfrac{1}{x}$.

23. $\dfrac{x-a}{2bc}+\dfrac{x-b}{2ac}+\dfrac{x-c}{2ab}=\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}$.

24. $\dfrac{a}{b-c}+\dfrac{b-c}{x}=\dfrac{a}{b+c}+\dfrac{b+c}{x}$.

25. $\dfrac{x^2+b^2}{4x^2-b^2}+\dfrac{1}{4}=\dfrac{x}{2x+b}$.

EXERCISE XVI. 6

Problems Solved by Fractional Equations

Solve the following problems, examining the solutions to see if they satisfy the conditions of the problems as stated:

1. What must be the value of a in order that $(3a+10)/(14a-4)$ shall have the value $1/2$?

2. Find two numbers, whose sum is 73, which are such that the quotient obtained by dividing the greater by the less is 3 and the remainder 13.

If x represents the greater number, $73-x$ will represent the less number.

By the conditions of the problem, we have the conditional equation

$$\frac{x}{73-x}=3+\frac{13}{73-x},$$

from which x is found to be 58, which is the greater number.

Accordingly the less number, which is represented by $73-x$, is 15.

These numbers are found to satisfy the conditions of the problem.

3. Find two numbers, whose sum is 36, which are such that the quotient obtained by dividing the less by the greater is $2/7$.

4. The sum of two numbers is 28, and the quotient obtained by dividing the less by the greater is $3/4$. Find the numbers.

5. The sum of two numbers is 59, and if the greater be divided by the less, the quotient is 3 and the remainder is 3. Find the numbers.

6. The sum of two numbers is 116, and if the greater be divided by the less the quotient is 8 and the remainder 8. Find the numbers.

7. The difference between two numbers is 60; if the greater is divided by the less the quotient is 7 and the remainder 6. Find the numbers.

8. What number must be added to the numerator and also to the denominator of the fraction 41 / 57 in order that the resulting fraction shall equal 11 / 15?

9. What number must be added to the numerator and subtracted from the denominator of the fraction 7 / 12 in order that the result shall be equal to the reciprocal of the given fraction?

10. When four is subtracted from the numerator of a fraction of which the numerator is three less than its denominator, the value of the fraction becomes one-eighth. What is the original fraction?

11. The reduced value of a certain fraction is 3/7 and its denominator exceeds its numerator by 20. Find the fraction.

12. The value of a fraction is 1/12. If its numerator is increased by 5 and its denominator by 4, the resulting fraction will be equal to 1/5. Find the fraction.

13. Separate 580 into two parts such that when the greater part is divided by the less the quotient is 12 and the remainder is 21.

14. The reciprocal of a number is equal to four times the reciprocal of the sum of the number and 18. Find the number.

15. The figure in units' place of a number expressed by two figures exceeds the figure in tens' place by 4. If the number, increased by 11, is divided by the sum of the figures in units' and tens' places, the quotient is 5. What is the number?

16. The figure in tens' place of a number of two figures exceeds the figure in units' place by 5, and if the number increased by 5 is divided by the sum of its figures the quotient is 8. Find the number.

17. A can do a piece of work in 16 days, and B in 12 days. How many days will be required if both work together?

If x represents the number of days which will be required when A and B work together, then $1/x$ will represent the fractional part of the work which will be performed in one day. According to the statement of the problem, A in one day can perform 1/16 of the work, while B can perform 1/12.

By the conditions of the problem, we have

$$\frac{1}{16} + \frac{1}{12} = \frac{1}{x}.$$

Hence $$x = 6\tfrac{6}{7}.$$

Accordingly $6\frac{6}{7}$ days will be required when both work together.

This result will be found to satisfy the conditions of the problem as stated.

18. A and B together can paint a house in 12 days, A and C together in 16 days, and A alone in 20 days. In what time can B and C together paint it? In what time can A, B, and C working together paint it?

19. A can do a piece of work in 12 days, B in 15 days, and A, B, and C together in 5 days. In how many days can C do the work?

20. A can do a piece of work in 10 days, B in 8 days, and C in 5 days. How many days will be required if all work together?

21. A can do a certain piece of work in $2\frac{1}{3}$ days, B in $3\frac{1}{4}$ days, and C in $4\frac{1}{5}$ days. If A, B, and C work together, how long will it take them to do the work?

22. A sum of $1200 was to be divided equally among a certain number of persons. If there had been four more persons, each would have received 2/3 as much. How many persons were there?

23. A cask may be emptied by any one of three taps. It can be emptied by the first alone in 25 minutes, by the second alone in 30 minutes, and by the third alone in 40 minutes. What time would be required to empty the cask by using all three together?

24. A vat in a paper mill can be filled by one pipe in one and one-third hours, by a second in two and one-half hours, and by a third in four hours. What time will be required to fill it when all are running together?

25. A train runs 164 miles in a given time. If it were to run 3 miles an hour faster it would go 12 miles farther in the same time. Find the train's rate of speed in miles per hour.

26. It is observed that a steamer can run 60 miles with the current in the same time that it can run 36 miles against the current. Find the rate of the current in miles per hour, knowing that the steamer can run 12 miles an hour in still water.

27. A can row five miles and B four miles an hour in still water. A is 12 miles farther up stream than B, and they row toward each other until they meet 4 miles above B's starting-place. Find the rate of the current in miles per hour.

GENERAL PROBLEMS

17. Since a particular letter may represent more than one value, it is customary to speak of numbers represented by letters as **literal or general numbers.**

18. A **general problem** is a problem in which the numbers whose values are supposed to be known are represented by letters; that is, in which the known numbers are general numbers.

EXERCISE XVI. 7

Problems Involving Literal Equations

Find the general solution of each of the following problems:

1. Separate the number a into two parts such that m times the first part shall exceed n times the second part by b.

If x stands for one of the required numbers, $a - x$ will represent the other. By the conditions of the given problem we have

$$mx = n \times (a - x) + b,$$

from which we obtain $$x = \frac{an + b}{m + n}.$$

Hence, one of the parts into which a is required to be separated is represented by $$\frac{an + b}{m + n}.$$

The other part, represented by $a - x$, may be found as follows:

$$a - x = a - \frac{an + b}{m + n} \equiv \frac{am - b}{m + n}.$$

By giving particular values to the letters appearing in the statement of any general problem, it is possible to obtain as many separate special problems of a given type as may be desired. The solution of the general problem will in every case be the solution of all of the special problems of the given type.

E. g. The following is a special problem which is of the same type as Ex. 1 which is a general problem:

Separate the number 19 into two parts such that 10 times the first part shall exceed 3 times the second part by 8.

The solutions of this special problem may either be obtained directly by solving a conditional equation, or by substituting the values $a = 19$, $b = 8$, $m = 10$ and $n = 3$ for the letters appearing in the expressions $\frac{an + b}{m + n}$ and $\frac{am - b}{m + n}$, which are found by solving the general problem.

The solutions of the special problem are found to be 5 and 14.

2. Separate a into two parts such that m times the first part shall equal n times the second part.

3. The sum of two numbers is s and their difference is d. What are the numbers?

4. Separate a into three parts such that the first part shall be m times the second part, and the second part n times the third part.

5. Separate d into two parts such that when one part is divided by the other the quotient shall be q and the remainder r.

6. What number must be added to each term of the fraction a/b in order that the resulting fraction shall be equal to c/d.

7. A can do a piece of work in a hours, and B can do the same piece of work in b hours. How many hours will be required if both work together?

8. One pipe can fill a tank in a hours and a second pipe can fill it in b hours. If a third pipe can empty it in c hours, how many hours will be required to fill the tank when the three pipes are open?

Discuss the problem for $a + b \gtreqless c$.

9. A tank can be filled from three taps. By using the first alone it is filled in a minutes, by using the second alone, in b minutes, and by using the third alone, in c minutes. In how many minutes would it be filled if the taps were all open at the same time?

10. In how many years will P dollars amount to A dollars at r per cent simple interest per year?

11. What principal at r per cent interest per year will amount to A dollars in t years?

12. An alloy of two metals is composed of a parts of one to b parts of the other. How many pounds of each are required to make c pounds of the alloy?

13. Two trains, A and B, d miles apart, start at the same time and travel toward each other at the rates of a miles per hour and b miles per hour respectively. How far will each have travelled when they meet?

14. In a certain time a train ran a miles. If it had run b miles an hour faster it would have gone c miles farther in the same time. Find the rate of the train in miles per hour.

15. If A and B can travel at the rates of a and b miles an hour respectively, how far must A travel to overtake B, if both move in the same direction, and B be given a start of s miles?

16. A naphtha launch can run a miles an hour in still water. If it can run b miles against the current in the same time that it can run c miles with the current, what is the rate of the current in miles per hour?

17. A crew can row a certain distance up a stream in a hours and can row back again in b hours. If the rate of the crew in still water is s miles an hour, find the velocity of the stream in miles per hour.

18. A dealer mixes a pounds of tea worth x cents a pound with b pounds of tea worth y cents a pound and with c pounds of tea worth z cents a pound. Find the value, v, of the mixture in cents per pound.

19. Pieces of money of one denomination are of such value that a pieces are equal in value to one dollar, and pieces of money of another denomination are of such value that b pieces are equal in value to one dollar. Find how many pieces of each denomination must be taken on condition that c pieces of money shall be equal in value to one dollar.

THE INTERPRETATION OF THE SOLUTIONS OF PROBLEMS

19. It often happens that there are restrictions on the nature of the unknown numbers of a given problem which cannot be translated into algebraic language, and hence cannot be expressed by means of algebraic conditional equations.

If, for example, the unknown number of a problem represents a number of men, it is implied that the number sought is *integral*, yet this implied condition cannot be translated into algebraic language and expressed in a conditional equation.

20. It appears that, when solving the conditional equations arising from the translation into algebraic language of the conditions of a stated problem, all that we know at the outset is that the solutions of the problem, if indeed any exist, must be found among the algebraic solutions of the conditional equations.

If it happens that none of these solutions are consistent with the stated conditions of the problem, we conclude that the concrete problem as stated has no solution.

Ex. 1. At an entertainment 75 cents was charged for each reserved seat ticket and 35 cents for each admission ticket. The ticket seller showed by his account that 500 tickets were sold, for which he received $236. How many people bought reserved seat tickets?

Let x stand for the number of people who bought reserved seat tickets at $0.75 each. Then $500 - x$ will stand for the number of people who bought admission tickets at $0.35 each.

By the conditions of the problem we have

$$.75\,x + .35\,(500 - x) = 236.$$

Solving, we obtain $x = 152\frac{1}{2}$.

The result $x = 152\frac{1}{2}$ satisfies the conditional equation but not the implied conditions of the problem, since it is impossible to give a sensible interpretation to a fractional number as representing the number of people.

It appears, then, that the conditions of the problem as stated are inconsistent when applied to people.

If the amount received had been given as $235 instead of $236 the number of people would have been found to be equal to 150; that is, we would have obtained an answer which could have been given a sensible interpretation.

21. In connection with the interpretation of solutions, the following problem has become classical:

The Problem of the Couriers. Two couriers, A and B, are travelling in the same direction along the same road at the uniform rates of m miles an hour and n miles an hour respectively. At a specified time, say at noon, B is d miles in advance of A. Will they ever be together, and if so, when?

Let x represent the number of hours after 12 o'clock when they will be together.

During that time A will travel mx miles, and B will travel nx miles.

Since at noon B is d miles in advance of A, we may form the conditional equation $mx = nx + d$ (1), of which the solution is found to be

$$x = \frac{d}{m - n} \cdot \quad (2)$$

We will now examine this expressed value of x and determine what restrictions, if any, must be placed upon the numbers represented by d, m, and n in order that the value of x shall be consistent with the stated conditions of the problem.

1. If A is to overtake B after 12 o'clock, the value of x must be *positive.* For this it is sufficient that d, m, and n all represent positive numbers, and that $m > n$. The assumption that $m > n$ implies that A is travelling faster than B and accordingly will overtake him.

2. If m, n, and d were all positive and $m < n$, the value of x would be *negative,* and accordingly we should interpret the negative quality of x as indicating that A and B *had been* together $d/(n - m)$ hours before noon.

3. If $m = n$, then $m - n = 0$, and since we cannot divide any number by zero it appears that when $m = n$, we cannot obtain the solution (2) from the given equation (1).

If, however, m instead of being equal to n differs from n by a very small amount, the difference $m - n$ will be different from zero, and accordingly the smaller the value of $m - n$, the larger for a given value of d will the value of the fraction $d/(m - n)$ become.

This is commonly expressed by saying that as m becomes nearly equal to n, x becomes *infinite.*

This should be interpreted as another way of saying that as A's rate becomes more and more nearly equal to B's, the time required for A to overtake B will become correspondingly greater and greater. Finally if A's rate is equal to B's, A will never overtake B.

Accordingly, *an infinite solution may be interpreted as meaning that it is impossible to find the solution under the assumed conditions.*

4. If we assume that $m = n$ and also that $d = 0$, the equation (1) is satisfied by any value which we may assign to x and the general solution assumes the *indeterminate* form $x = 0/0$.

This may be interpreted as meaning that, since the distance d between A and B is zero, and their rates of travelling, m and n, are equal, that if at any time they are together they will always be together.

22. The student will commonly find no difficulty in giving sensible interpretations to the solutions of particular problems whether these solutions be positive or negative, fractional, zero, indeterminate, or infinite.

Whenever it is impossible to give a sensible interpretation to any particular solution which may be obtained, it will be a good exercise for the student to examine the data of a given problem and if possible to ascertain the cause of the inconsistency.

Problems in Physics

23. The Horizontal Lever. A straight horizontal lever at rest, supported at some point F called the *fulcrum*, and acted upon at distances of a units and b units from F by two parallel vertical forces having the same directions, which are represented numerically by A and B, will remain at rest provided that the numbers represented by a, b, A, and B satisfy the conditional equation $Aa = Bb$.

24. The product obtained by multiplying the number which represents the force A by the number which represents the distance a of the point of application of the force from the point of support F, is called the *moment of the force* A with respect to the point of support F.

25. Since the forces A and B tend to produce rotation of the horizontal bar in opposite directions about the fulcrum F as a point of support, it may be seen that the condition of equilibrium is that the moment Aa shall be equal to the moment Bb.

See Fig. 2, in which vertical forces A and B have the same direction and act upon the horizontal lever at points situated at distances a and b on opposite sides of the fulcrum F. In Fig. 3 the forces A and B have opposite directions and act on the horizontal lever at points which are situated on the same side of the fulcrum F.

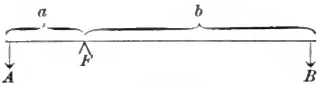


FIG. 2.

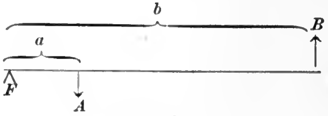


FIG. 3.

26. A horizontal bar in equilibrium will remain at rest if vertical forces, represented by A, B, C, D, E, etc., acting at distances a, b, c, d, e, etc., from the fulcrum F, satisfy a conditional equation such as $Aa + Cc = Bb + Dd + Ee$.

27. It is a principle that a mass may be treated in calculations as if it were concentrated at a certain point called the *center of gravity* of the mass.

EXERCISE XVI. 8

Solve each of the following problems :

1. How heavy a stone can a man, by exerting a force of 160 pounds, lift with a crowbar 6 feet in length, if the fulcrum be one foot from the stone (neglecting the weight of the crowbar) ?

2. A wheelbarrow is loaded with 50 bricks, each weighing 6 pounds. What lifting force must be applied at the handles to raise the load (neglecting the weight of the wheelbarrow), provided that the center of gravity of the load is 2 feet from the center of the wheel and the hands are placed at a distance of 4 feet from the center of the wheel ?

3. A beam 20 feet in length and weighing 50 pounds is supported at a point 4 feet from one end. What force must be applied at the end farthest from the point of support to keep the beam in equilibrium ? What force must be applied at the end nearer the point of support ?

4. A board 15 feet in length and weighing 21 pounds is supported at a point 2 feet from the center. If the board is kept in equilibrium by a stone placed on it at a point 3 feet from the fulcrum, find the weight of the stone.

5. A horizontal bar 18 inches in length is in equilibrium when forces of 4 pounds and 2 pounds respectively are acting downward at its ends. Find the position of the point of support.

6. A basket weighing 100 pounds is suspended at a point two feet from the end of a stick which is 8 feet in length and which weighs three pounds. If the stick is being carried by two boys, one at each end, how many pounds does each boy lift?

7. Two boys, one at each end of a stick 12 feet in length which weighs 5 pounds, raise a certain weight which is suspended from the stick. How heavy is the weight and at what point does it hang, if one boy lifts 35 pounds and the other lifts 30 pounds?

Since the boys lift 35 and 30 pounds respectively, and the weight of the stick is 5 pounds, it follows that the weight carried must be 60 pounds.

If the stick be assumed to be uniform, it may be seen that one boy will carry $32\frac{1}{2}$ pounds and the other boy $27\frac{1}{2}$ pounds of the weight.

We will represent by x the number of feet from the center of gravity of the stick to the point at which the weight is suspended on the side of the center of gravity nearer the boy exerting the greater force.

It may be seen that, with respect to the center of gravity, regarded as a fixed point, the weight of 60 pounds which is carried and the force of $27\frac{1}{2}$ pounds exerted at one end of the stick both tend to produce rotation of the stick about its center of gravity in one direction, while the force of $32\frac{1}{2}$ pounds exerted at the other end of the stick tends to produce rotation of the stick about its center of gravity in the opposite direction.

Since the stick is in equilibrium, the sums of the moments of these forces must be equal.

Hence we have the following conditional equation:

$$60x + (27\tfrac{1}{2}) \times 6 = (32\tfrac{1}{2}) \times 6.$$

Solving, we obtain $x = \frac{1}{2}$.

Hence, the weight is suspended from the stick at a point which is 6 inches from the center of gravity on the side nearer the boy exerting the greater force.

This value will be found to satisfy the conditions of the given problem.

8. A safety valve having an area of 4 square inches is held down by a lever which is hinged at one end.

The lever is 10 inches long and the point of application of the valve is 2 inches from the hinged end of the lever. If a weight of 12 pounds is placed on the free end, find the pressure per square inch on the valve which will lift the safety valve, disregarding the weight of the lever.

9. A dog-cart carrying a load of 576 pounds is found, when on a level road, to exert a pressure of only 8 pounds on the horse's back. If the dis-

tance from the point of support on the horse's back to the axle be 6 feet, find the distance of the center of gravity of the load from the axle.

10. A board of uniform thickness, weighing 30 pounds, is balanced when supported at a point 4 feet from one end and when a weight of 70 pounds is placed one foot from this end. Find the length of the board.

11. A beam of uniform thickness, 20 feet in length, is supported at a point 8 feet from one end. If the beam is balanced when a weight of 80 pounds is placed on the end nearer the fulcrum and a weight of 30 pounds is placed on the end farther from the fulcrum, what is the weight of the beam?

EXERCISE XVI. 9 **Review**

Simplify each of the following:

1. $(a^2 + a + 1)\left(\frac{1}{a^2} - \frac{1}{a} + 1\right)$. 3. $(a^n - b^n)(a^n + b^n)(a^{2n} + b^{2n})$.

2. $\left(\frac{a^2}{b^2} + \frac{a}{b} + 1\right)\left(\frac{a^2}{b^2} - \frac{a}{b} + 1\right)$. 4. $\left(\frac{a}{b} + \frac{c}{d}\right)(ad - bc)$.

5. $(x + y)\left(\frac{1}{x} + \frac{1}{y}\right)$.

6. Find the remainder when $6x^4 - 7x^3 + 5x^2 - 2x + 3$ is divided by $x - 5$.

7. Factor $10(a^2 + 1) - 29a$.

8. Factor $x^2 + 2xy + y^2 + x + y$.

9. Factor $(a - b)^3 - 8$.

10. Find the L. C. M. of $a^4 + a^2 + 1$ and $a^4 - a^2 + 1$.

Simplify each of the following:

11. $(a^3 + \frac{1}{27}) \div (a + \frac{1}{3})$. 13. $\frac{a^2 + b^2}{a - b} + \frac{a^3 - b^3}{a^2 + b^2}$.

12. $(a^3 - \frac{1}{125}) \div (a^2 + \frac{1}{5}a + \frac{1}{25})$. 14. $\frac{(5 + x)^2}{(5 - x)^2} - \frac{5 + x}{5 - x}$.

15. $$\frac{a - \frac{1}{b}}{c} + \frac{b - \frac{1}{c}}{a} + \frac{c - \frac{1}{a}}{b}.$$

16. Show that $(x^2 + 3x + 2)(x^2 + 7x + 12) \equiv (x^2 + 4x + 3)(x^2 + 6x + 8)$.

17. Find the value of $\frac{a - b + 1}{a + b - 1}$, when $a = \frac{m + 1}{mn + 1}$ and $b = \frac{mn + m}{mn + 1}$.

18. Divide the product of $a + b - c$, $b + c - a$ and $c + a - b$ by $a^2 - b^2 - c^2 - 2bc$.

CHAPTER XVII

SIMULTANEOUS LINEAR EQUATIONS

GENERAL PRINCIPLES OF EQUIVALENCE

1. Two or more conditional equations are said to be **simultaneous** with reference to two or more unknowns appearing in them when each unknown letter is assumed to represent the same number wherever it appears in all of the equations.

2. A set or group of simultaneous equations is called a **system** of simultaneous equations.

E. g. The equations $3x + 2y = 14$ (1) and $x + 5y = 9$ (2) are simultaneous on condition that x represents the same number in (1) as in (2), and that y has the same value in one equation as it has in the other.

3. A solution of a conditional equation containing two or more unknowns is any set of values which, when substituted for the unknowns, reduces the conditional equation to an identity.

E. g. The sets of two values, $x = 2$, $y = 4$; $x = 0$, $y = 7$; $x = 6$, $y = -2$, etc.; are solutions of the conditional equation $3x + 2y = 14$ containing two unknowns.

4. A solution of a system of simultaneous conditional equations is any set of values of the unknowns which satisfies all of the equations of the system.

E. g. The single set of two values $x = 4$, $y = 1$ is the single solution of the system of two simultaneous conditional equations $3x + 2y = 14$ (1), and $x + 5y = 9$ (2).

5. The word "solution" may be used to denote either the process of solving an equation or system of equations, or the value or values obtained by the process.

6. If the number of solutions, — that is, the number of different sets of values which satisfy all of the equations of a given system of simultaneous conditional equations, — is limited or finite, the system is said to be **determinate.**

If, however, the number of different sets of values which satisfy all of the equations of a system be unlimited or infinite, the system is said to be **indeterminate.**

7. Two conditions restricting the values of two or more unknown numbers are said to be **consistent** if both conditions can be satisfied by the same values of the unknowns.

In the contrary case, the conditions are said to be **inconsistent.**

E. g. If we are required to find two numbers whose sum is 10 and difference 8, the conditions restricting the values of the unknown numbers are consistent, since we can find two numbers, 9 and 1, which satisfy them.

Two conditions requiring that the sum of two unknown numbers shall be 10 and also 8 are inconsistent, since it is impossible to find two such numbers.

8. The **graph of a conditional equation** of the first degree containing two unknowns is a straight line.

This may be shown directly by applying certain simple properties of plane triangles.

(The following proof is offered for such students as are acquainted with a few of the simple principles of geometry, and may be omitted when the chapter is read for the first time.)

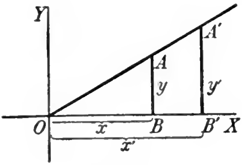


Fig. 1.

Let A and A' represent any two points on the graph of a given equation $y = ax$, located by means of the coördinates (x, y) and (x', y'), so taken that corresponding values of x and y satisfy the given equation. (See Fig. 1.)

Draw straight lines from the origin O to A and A'.

Since the values x, y, and x', y', are assumed to satisfy the equation $y = ax$, we have $y = ax$ and $y' = ax'$.

Hence $$\frac{y}{x} = a, \text{ and } \frac{y'}{x'} = a.$$

Therefore $$\frac{y}{x} = \frac{y'}{x'}.$$

The ordinates y and y' are taken parallel to the axis of Y, and accordingly the triangles OBA and $OB'A'$ are similar, since they have an angle of one equal to an angle of the other, and the included sides proportional.

The corresponding angles AOB and $A'OB'$ are consequently equal, and the lines OA and OA' coincide.

It follows that either of the points A or A' lies on the straight line drawn from the origin to the other point, and since A and A' represent any two points on the graph, they represent all points on the graph, which must accordingly be a straight line passing through the origin.

It may be observed that the inclination of the line with the x-axis depends wholly upon the value of a, since for a given value of x the length of y is equal to the product ax. The line will slope upward or downward toward the right according as a is positive or negative.

The graph of the equation $y = ax + b$ may be obtained by adding b to each of the ordinates calculated for the graph of the equation $y = ax$. The x-coördinates will be the same for the graphs of both of the equations, but the y-ordinates of the graph of the equation $y = ax + b$ will be greater by b than those of the graph of the equation $y = ax$.

It may be seen that the figure $AA'A''A'''$ is a parallelogram by construction, and accordingly the straight line $A'''A''$ is parallel to the straight line AA'. (See Fig. 2.)

Accordingly, the graph of the equation $y = ax + b$ is a straight line which is parallel to the graph of the equation $y = ax$. (Compare with Chapter IX. § 37.)

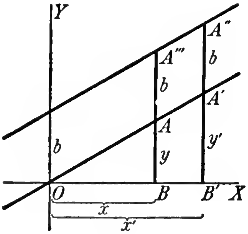


FIG. 2.

9. Since the graph of every equation of the first degree containing two unknowns, such as x and y, is a straight line, an equation of the first degree with reference to the unknowns appearing in it is commonly called a **simple or linear equation,** that is, the equation of a line.

10. ***To obtain graphically the solution of a system of two linear equations containing two unknowns,*** *we plot the graphs representing the equations and, locating their intersection, if there be one, measure the x-coördinate and y-coördinate corresponding to this point and estimate the corresponding numerical values, attaching the proper quality signs determined by the quadrant in which the point lies.*

11. Since any two straight lines lying in the same plane, which are not coincident, must either intersect or be parallel, it follows that pairs of equations of the first degree containing two unknowns may be separated into three classes: one class consisting of such pairs of equations as are represented graphically by intersecting straight lines (*independent equations*); a second class consisting of those pairs of equations which are represented by lines which do not meet, that is, which are parallel (*inconsistent equations*); and a third class consisting of those pairs of equations which may be reduced to exactly the same form, that is, which represent coincident lines (*equivalent equations*).

12. Two or more conditional equations which express different consistent conditions restricting the values of the same unknowns are called **independent equations.**

Of two equations which are independent, neither can be transformed into the other.

E. g. The two conditional equations $2x + y = 12$ and $3x + 7y = 29$ are independent, since neither can by any transformation be made to take the form of the other.

13. The point of intersection of the graphs of two linear equations containing two unknowns may be located by means of the two coördinates which are equal to the values found by solving algebraically the two equations of which they are the graphs.

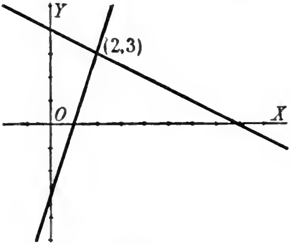


FIG. 3.

E. g. Consider the two independent linear equations $x + 2y = 8$ and $3x - y = 3$.

Any set of values satisfying either equation may be taken as coördinates locating a definite point on the graph.

Hence the values $x = 2$, $y = 3$, which are the common solution of the two given equations, are equal to the coördinates $x = 2$ and $y = 3$ of the point common to the two lines in Fig. 3.

14. Two or more conditional equations which express consistent conditions existing between the unknowns appearing in them are called **consistent equations.**

E. g. The two conditional equations $x + y = 12$ and $x - y = 6$ are consistent since both are satisfied by the values $x = 9$ and $y = 3$.

15. Consider the conditional equations $x + y = 7$, $x - y = 1$, $3x + 2y = 18$, and $x - 4y = -8$.

Each of these equations is satisfied by the values $x = 4$ and $y = 3$, and the equations are all independent and consistent. Any two of them will serve to determine the values of x and y.

Referring to the accompanying figure, in which portions of the graphs of these equations are plotted, it appears that these equations represent separate straight lines, all passing through a common point A whose coördinates, $x = 4$ and $y = 3$, are equal to the common solutions of the equations.

From the illustration it appears that, since any particular point such as A is located definitely by means of any two straight lines passing through it, and not more than two lines are necessary to locate the point, so *two independent conditional equations of the first degree containing two unknowns determine the values of two unknown numbers, and more than two equations are unnecessary.*

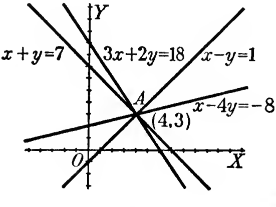


FIG. 4.

16. Two conditional equations which express inconsistent conditions restricting the values of the unknowns appearing in them are called **inconsistent equations.**

E. g. The conditional equations $x + y = 7$ and $x + y = 5$ are inconsistent.

17. Two or more inconsistent equations can have no solution in common.

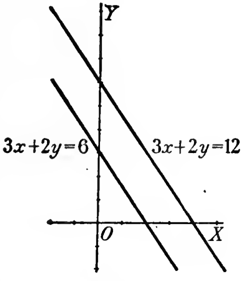


FIG. 5.

E. g. Consider the two conditional equations $3x + 2y = 6$ and $3x + 2y = 12$.

Since it is impossible that $3x + 2y$ should equal 6 and also 12 at the same time, these must be classed as inconsistent equations.

On attempting to solve the equations as simultaneous equations we shall find that they have no common solution.

If their graphs are plotted with reference to the same axis of reference we shall find that they appear to be parallel straight lines. (See Fig. 5.)

18. Since the point of intersection, if there be one, of the graphs of two linear equations containing two unknowns is located by means of coördinates equal to the values which form the common solution of the given equations, it follows that if the equations have no common solution their graphs, which are straight lines, can have no point in common, and accordingly must be parallel straight lines.

19. Two conditional equations containing two or more unknowns are said to be **equivalent** when every solution of either equation is at the same time a solution of the other equation; that is, when any set of values satisfying either equation satisfies the other equation also. (See also Chap. X. § 22.)

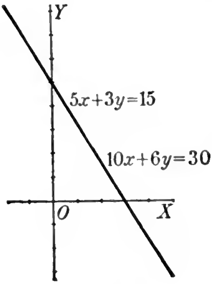


FIG. 6.

E. g. Since two conditional equations are equivalent when each equation is satisfied by all of the solutions of the other, it follows that the graphs of two equivalent equations such as $5x+3y=15$ and $10x+6y=30$ must contain the same points, and neither graph can contain any point which the other does not.

Hence the graphs must be coincident lines. (See Fig. 6.)

20. It may be observed that independent conditional equations express different consistent relations between the unknowns, while equivalent equations express the same relations between the unknowns.

E. g. Any one of the following equations is equivalent to any other, since of any pair of equations either equation may be transformed into the other:

$$3x+2y=7,\ 6x+4y=14,\ 2x+5y=3y-x+7,\ \frac{3x}{7}+\frac{2y}{7}=1.$$

EXERCISE XVII. 1

Of the following equations select those which are equivalent to the equation $2x+3y=10$:

1. $4x+6y=20.$
2. $6x+12y=30.$
3. $3x+3y=10-x.$
4. $x+\frac{3y}{2}=5.$
5. $2(x+y+1)=12-y.$

Of the following equations select those which are equivalent to the equation $4x - 2y = 3$:

6. $12x + 8y = 9.$

7. $8x - 4y = 6.$

8. $2x - y = \frac{3}{2}.$

9. $\frac{x}{2} - \frac{y}{4} = \frac{3}{8}.$

10. $20x - 10y = 16.$

Among the following sets of equations,

(i.) which have equations which are independent and consistent? (ii.) equivalent? (iii.) inconsistent?

11. $2x + 3y = 10,$
$4x + 6y = 20.$

12. $5x - 7y = 9,$
$2x + y = 7.$

13. $3x + 8y = 12,$
$6x + 16y = 22.$

14. $x + y - 2 = 0,$
$x - y = 1.$

15. $3x - y = 12,$
$x - \frac{y}{3} = 4.$

16. $12x - 9y = 18,$
$8x - 6y = 14.$

17. $3x - 2y = 0,$
$2x - 3y = 0.$

18. $4x + 5y = 6,$
$5x + 6y = 7.$

19. $10x + 8y = 3,$
$5x - 4y = 6.$

20. $2x + y = 3,$
$x + 3y = 2.$

21. $2x - 3 = 4y,$
$2y - 3 = 4x.$

22. $5 + x = 6y,$
$y + 5x = 6.$

21. It can be shown that the necessary and sufficient condition that a system of simultaneous linear equations shall have a definite number of solutions is that there shall be the same number of independent and consistent equations as there are unknowns whose values are to be found.

22. Two systems of equations are **equivalent** when every solution of either system is also a solution of the other.

E.g. The equations in groups I. and II. below form equivalent systems, for the equations in either group are satisfied by the solution $x = 1$ and $y = 3$; we shall show later that they are satisfied by no other solution.

$$\left.\begin{aligned} 2x + 5y &= 17, \\ x + y &= 4. \end{aligned}\right\} \text{System I.} \qquad \left.\begin{aligned} 2x + 5y &= 17, \\ 2x + 2y &= 8. \end{aligned}\right\} \text{System II.}$$

ELIMINATION

23. Any process by means of which the members of two or more equations may be combined to produce a derived equation in which fewer unknowns appear than in the equations whose members have been combined to produce it, is called a **process of elimination.**

24. Any **unknown** which is found in two or more given equations but which does not appear in an equation derived from them is said to have been **eliminated.**

25. The general principles governing the derivation of equivalent equations containing one unknown, proved in Chapter X., hold true also for equations containing any number of unknowns.

26. The processes of elimination employed in the solution of a system of simultaneous linear equations may be made to depend upon the following

GENERAL PRINCIPLES

Principle (i.) *If, for any equation of a system of simultaneous equations an equivalent equation be substituted, the system composed of this derived equation, taken together with the remaining original equations, will be equivalent to the original system of equations.*

(The following proof may be omitted when the chapter is read for the first time.)

Let a system of simultaneous equations containing two unknowns, say x and y, be represented by

$$\left.\begin{matrix} A = C, \\ B = D. \end{matrix}\right\} \text{I. Given System.}$$

Let the equation $B' = D'$ be derived from the equation $B = D$, and let $B' = D'$ be equivalent to $B = D$.

Then the system composed of the other given equation, $A = C$, and the derived equation, $B' = D'$, will be equivalent to the original System I., for by the definition of equivalent equations, the equivalent equations $B = D$ and $B' = D'$ have the same solutions.

$$\left.\begin{matrix} A = C, \\ \\ B' = D'. \end{matrix}\right\} \text{II. Equivalent Derived System.}$$

Hence, any set of values which satisfies either of the equations $B = D$ or $B' = D'$ and also the equation $A = C$ must satisfy both the given and the derived systems; that is, the systems of equations I. and II. are equivalent.

The reasoning may be extended to include systems of three or more equations containing three or more unknowns.

27. In elementary algebra, methods of elimination by substitution, by comparison, and by addition or subtraction are commonly employed.

I. Elimination by Substitution

28. The method of elimination by substitution may be made to depend upon the following

Principle (ii.): *If in any equation belonging to a system of simultaneous equations the value of one of the unknowns be expressed in terms of the remaining unknown numbers and known numbers appearing in the same equation, and this expressed value thus obtained be substituted for the same unknown wherever it appears in the remaining equation or equations of the system, then the derived system will be equivalent to the given system.*

(The following proof may be omitted when the chapter is read for the first time.)

We will represent the linear equations composing a given system of two simultaneous equations, in which two unknowns, x and y, appear, by

$$\left.\begin{array}{ll} A = C, & (1) \\ B = D. & (2) \end{array}\right\} \text{I. Given System.}$$

Representing by E the expressed value of one of the unknowns, say y, in terms of the remaining unknown x, and of the known numbers appearing in one of the equations, — say equation (1), — we may derive the equivalent equation

$$y = E. \quad (3)$$

Substituting this expressed value, E, for y wherever y is found in the remaining equation, — say equation (2) of System I., — we may represent the derived equation by

$$B' = D'. \quad (4)$$

We are to show that System II., composed of equations (3) and (4), is equivalent to the given system of equations (1) and (2), that is, to System I.

$$\left.\begin{array}{ll} y = E, & (3) \\ B' = D'. & (4) \end{array}\right\} \text{II.} \begin{array}{c} \text{Equivalent} \\ \text{Derived} \\ \text{System.} \end{array}$$

Since $y = E$ is equivalent to equation (1), the system composed of this equation and the remaining original equation (2), that is, System III., must be equivalent to the original System I.

$$\left.\begin{array}{ll} y = E, & (3) \\ B = D. & (2) \end{array}\right\} \text{III.} \quad \begin{array}{c}\text{Equivalent} \\ \text{Derived} \\ \text{System.}\end{array}$$

Any solution of System III. satisfies equations (2) and (3), that is, makes each of them an identity. Hence we have

$$y \equiv E, \text{ and } B \equiv D.$$

Any solution therefore which satisfies (3) and (2) must also satisfy (2) after E has been substituted for y, that is, must satisfy the equation $B' = D'$. (4)

It follows that any solution of System III. is also a solution of System II.

Furthermore, any solution of System II. makes $y \equiv E$, and also $B' \equiv D'$.

Hence, any solution of System II. must also satisfy $B' = D'$ after y has been substituted for E, that is, must satisfy the equation $B = D$ (2).

Also, since the equation $y = E$ is equivalent to the equation $A = C$, any solution of System II. must satisfy also System I.

Hence Systems I. and II. are equivalent.

The reasoning may be extended to include a system of three or more equations containing three or more unknowns.

Systems of Linear Equations containing two Unknowns

29. The method of elimination by substitution may be used to advantage whenever one of the given equations contains a single unknown.

Ex. 1. Solve the following system of equations :

$$\left.\begin{array}{rl} x - 6 = 0, & (1) \\ 3x + 5y = 23. & (2) \end{array}\right\} \text{I. Given System.}$$

From equation (1) we obtain directly

$$\left.\begin{array}{l} x = 6. \quad (3) \quad \dots\dots \\ \text{Substituting this value for } x \text{ in equation (2) we have} \\ 3 \cdot 6 + 5y = 23 \\ y = 1. \quad (4) \quad \dots\dots \end{array}\right\} \text{II.} \quad \begin{array}{c}\text{Equivalent} \\ \text{Derived} \\ \text{System.}\end{array}$$

Hence the solution of System II., and consequently of System I. is

$$\left.\begin{array}{l} x = 6, \\ y = 1. \end{array}\right\}$$

In Fig. 7 portions of the graphs of the equations $x - 6 = 0$ and $3x + 5y = 23$ are shown, and the intersection of these graphs is a point, of which the numerical values of the coördinates are equal to the values of x and y, found by solving the given equations.

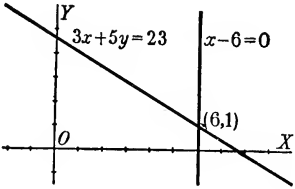


Fig. 7.

The student may establish the equivalence of the equations in Systems I. and II.

We may verify the solution by substituting the values 6 and 1 for x and y respectively, in the original equations (1) and (2).

Substituting in (1),	Substituting in (2),
$6 - 6 = 0$	$18 + 5 = 23$
$0 = 0.$	$23 = 23.$

Ex. 2. Solve the system of simultaneous equations

$$\left.\begin{aligned} 2x + y &= 15, \quad (1) \\ 5x - 2y &= 6. \quad (2) \end{aligned}\right\} \text{I. Given System.}$$

We are to find a value for x and also one for y which will satisfy both of the given equations.

Although we do not at first know the values of x and y, we proceed upon the assumption that each letter has the same value in one equation that it has in the other.

From equation (1) we may obtain the expressed value of y in terms of x and the numbers entering into the equation.

That is, $$y = 15 - 2x. \qquad (3)$$

Substituting $15 - 2x$ for y in equation (2), we obtain the following derived equation in which y does not appear :

$$5x - 2(15 - 2x) = 6. \qquad (4)$$

The derived System II., composed of equations (3) and (4), is equivalent to the original system.

$$\left.\begin{aligned} y &= 15 - 2x, \quad (3) \\ 5x - 2(15 - 2x) &= 6. \quad (4) \end{aligned}\right\} \text{II. Equivalent Derived System.}$$

From equation (4) we obtain, $x = 4$.

Substituting this value in equation (3), we obtain $y = 7$.

Hence, the solution of System II., and consequently of System I., is

$$\left.\begin{aligned} x &= 4, \\ y &= 7. \end{aligned}\right\}$$

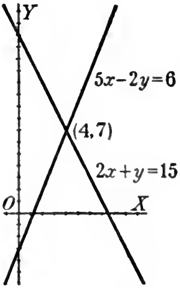


FIG. 8.

In Fig. 8 portions of the graphs of the equations $2x + y = 15$ and $5x - 2y = 6$ are shown, and the intersection of these graphs is a point, of which the numerical values of the coördinates are equal to the values of x and y, found by solving the given equations.

We may verify the solution by substituting the values 4 and 7, for x and y respectively, in the original equations (1) and (2).

Substituting in (1),

$$2 \cdot 4 + 7 = 15$$
$$15 = 15.$$

Substituting in (2),

$$5 \cdot 4 - 2 \cdot 7 = 6$$
$$6 = 6.$$

The equivalence of the equations employed in the process of solution may be established as follows:

Equation (3) is equivalent to equation (1) by Chap. X. § 27 (i.). The system composed of equations (4) and (3) is equivalent to the system composed of equations (2) and (1) by § 26.

It follows that the solution of System II. must be the solution of System I.

30. The **general method of solution** may be stated as follows:

Obtain from one of the equations the expressed value of one of the unknowns in terms of the other; substitute this expressed value for the same unknown wherever it appears in the remaining equation of the system, and solve the resulting equation.

The value of the remaining unknown may be found by substituting the value of the unknown just found, either in one of the original equations or in the expressed value of the other unknown.

31. An objection to this method is that, unless the coefficient of the unknown to be eliminated is unity in the equation from which the expressed value of this unknown is obtained, it may happen that fractions are introduced into the derived equation.

32. The preceding examples illustrate the principle that the solution of a system of simultaneous equations which consists of as many equations as there are unknown numbers is finally made to depend upon the solution of a single equation containing a single unknown.

EXERCISE XVII. 2

Solve each of the following systems of equations by the method of substitution, verifying all results numerically :

1. $x + 9y = 19,$
   $8x - y = 79.$

2. $x + 11y = 0,$
   $x + y = 10.$

3. $3x - 2y = 4,$
   $5x + y = 11.$

4. $x - y = 8,$
   $5x + 6y = 51.$

5. $2x = 12,$
   $3x + 5y = 53.$

6. $3x - y = 6,$
   $x + 9y = 86.$

7. $2x + y = 21,$
   $2y + x = 12.$

8. $3x - 12y = 0,$
   $x - 2y - 14 = 0.$

9. $4x = 3y,$
   $7x = 5y + 1.$

10. $7x = 4y,$
    $10x = 3y + 19.$

11. $12x - 17y - 2 = 0,$
    $x - y - 1 = 0.$

12. $4x + 2y - 12 = 0,$
    $x + y - 4 = 0.$

13. $x = 3y - 2,$
    $y = 3x + 2.$

14. $x = y,$
    $8x + 3y = 11.$

15. $x - y = 0,$
    $2x + 3y = 5.$

16. $25x - 6y = 3,$
    $5x + 27y = 10.$

17. $5x - y - 5 = 0,$
    $7x + 3y - 24 = 0.$

18. $x + y = 2a,$
    $(a - b)x = (a + b)y.$

19. $cx - by = 0,$
    $bx - cy = a.$

20. $ax + y = b,$
    $x + cy = d.$

33. The method of elimination by substitution is sometimes employed in a special form called the method of

Elimination by Comparison

An unknown is eliminated by the method of comparison by obtaining from each of two given equations the expressed value of this unknown, and then constructing an equation the members of which are the two expressed values thus obtained.

Ex. 1. Solve the system of equations

$$\left.\begin{aligned} 3x+4y&=40, \quad (1)\\ 9x-5y&=\ \ 1. \quad (2)\end{aligned}\right\}\text{ I. Given System.}$$

From equation (1) we obtain the expressed value of x,

$$x=\frac{40-4y}{3}. \quad (3) \quad \ldots\ldots\ldots$$

From equation (2) we obtain the expressed value of x,

$$x=\frac{1+5y}{9}. \quad (4) \quad \ldots\ldots\ldots$$

II. Equivalent Derived System.

Since these expressed values of x represent the same number, they may be used as members of an equation; or, from another point of view, we may substitute for x in one of the equations, say (4), the expressed value of x from the other equation, say (3), and obtain

$$\frac{40-4y}{3}=\frac{1+5y}{9}. \qquad (5)$$

Hence, $y=7 \ \ldots\ldots$

From either equation (3) or equation (4) we may obtain x by substituting 7 for y.

From equation (4), $x=\dfrac{1+5\cdot 7}{9}$

Hence, $x=4 \ \ldots\ldots$

III. Equivalent Derived System.

The values $\left.\begin{aligned}x&=4,\\ y&=7,\end{aligned}\right\}$ are the solution of the given equations (1) and (2), and by Principles (i.) and (ii.), the systems of equations I., II. and III. are equivalent.

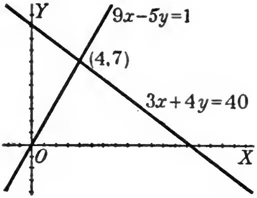


Fig. 9.

Hence, the solution of the given System I. is the single solution of System III.

In Fig. 9, portions of the graphs of the given equations $3x+4y=40$ and $9x-5y=1$ are shown, and the numerical values of the coördinates $x=4$ and $y=7$ of the point of intersection of the graphs are equal to the numerical values of x and y found by solving the algebraic equations.

The solution may be verified by substituting the values 4 and 7 for x and y respectively in the given equations (1) and (2).

Substituting in (1),
$$3 \cdot 4 + 4 \cdot 7 = 40$$
$$40 = 40.$$

Substituting in (2),
$$9 \cdot 4 - 5 \cdot 7 = 1$$
$$1 = 1.$$

34. The preceding example illustrates the following rule for elimination by comparison :

From each of two given equations find the expressed value of one of the unknowns and form an equation the members of which are these expressed values.

Solve the equation thus obtained for the single unknown appearing in it. The value of the remaining unknown may be found by substituting the value of the first unknown, thus obtained, for the first unknown wherever it appears, either in one of the original equations or in the expressed value of the remaining unknown.

Exercise XVII. 3

Solve the following systems of equations, eliminating the unknown numbers by the method of comparison :

1. $x = 3y - 4,$
   $x = 4y - 7.$

2. $5x - 2y = 11,$
   $2x - 3y = 0.$

3. $3x + 8y = 19,$
   $7x - 2y = 1.$

4. $8x = 6y,$
   $10x = 27y - 4.$

5. $3x + 7y = 42,$
   $5x + 6y = 53.$

6. $2x - 5y = - 23,$
   $3x - 4y = - 3.$

7. $y = 2x + 1,$
   $y = 3x - 5.$

8. $3x - 4y - 19 = 0,$
   $7x + 2y - 50 = 0.$

9. $5x + 6y = 7,$
   $8x + 9y = 10.$

10. $11x - 9y = 7,$
    $9x - 10y = 11.$

11. $bx + ay = 2ab,$
    $ax + by = a^2 + b^2.$

12. $x + ay + a^2 = 0,$
    $x + by + b^2 = 0.$

II. Elimination by Addition or Subtraction

35. The method of elimination by addition or subtraction may be made to depend upon the following

Principle (iii.) *If, for any equation of a system of simultaneous equations, an equation be substituted which is derived from two or* *more of the given equations of the system by either adding or subtracting the corresponding members of these equations, the resulting system will be equivalent to the one given.*

(The following proof may be omitted when the chapter is read for the first time.)

If d and n represent any finite numbers, d being different from zero, the System I. is equivalent to the derived System II.

$$\left.\begin{aligned} A &= 0, \quad (1) \\ B &= 0. \quad (2) \end{aligned}\right\} \text{I. Given System} \qquad \left.\begin{aligned} dA + nB &= 0, \quad (3) \\ B &= 0. \quad (2) \end{aligned}\right\} \text{II.} \begin{array}{l}\text{Equivalent} \\ \\ \text{Derived System.}\end{array}$$

Every solution of System I., that is, every set of values making both A and B zero, must make both dA and nB zero, and accordingly must satisfy System II.

Furthermore, any set of values which satisfies System II., making B and $dA + nB$ zero, must make dA zero; and since d is different from zero, the remaining factor A of the term dA must be zero.

Hence A and B must both become zero for any particular set of values which satisfies System II.

Accordingly, such a set of values must satisfy System I., that is, Systems I. and II. must be equivalent.

36. The elimination of unknowns by the method of addition or subtraction will, in the majority of cases, be found to be more convenient than the method of elimination by substitution. This is because fractions are not introduced into the derived equations during the process.

37. The process of elimination by addition or subtraction is effected by so transforming the given equations that the unknown number to be eliminated appears in two equations with coefficients which differ, if at all, only in sign. Then, either by adding or subtracting the corresponding members of the transformed equations, the terms containing this unknown may be made to disappear, and the resulting derived equation will be free from this unknown number. After solving this last equation for the single unknown appearing in it, the value of the other unknown may be found, either by substitution or by repeating the process above.

Ex. 1. Solve the system of linear equations

$$\left.\begin{aligned} 3x + 4y &= 35, \quad (1) \\ 6x - 5y &= 44. \quad (2) \end{aligned}\right\} \text{I. Given System.}$$

To eliminate x it is necessary to obtain equations equivalent to equations (1) and (2) in which the coefficients of x differ, if at all, only in sign.

Multiplying both members of equation (1) by 2, we obtain the equivalent equation $6x + 8y = 70$, (3), which, if taken with equation (2), forms a System II. which is equivalent to the given System I.

$$\left.\begin{array}{rl} 6x+8y=70, & (3)\\ \underline{6x-5y=44.} & (2)\end{array}\right\}\text{ II.}\qquad\begin{array}{c}\text{Equivalent}\\ \text{Derived}\\ \text{System.}\end{array}$$
$$13y=26.\quad(4)$$

By subtracting the members of equation (2) from the corresponding members of equation (3), the terms containing x disappear and the derived equation (4) contains y only.

Solving (4), we obtain, $y = 2$. (5)

By Principle (iii.) § 35, equation (5) taken together with either of the original equations, say (2), forms a System III. which is equivalent to the given system.

$$\left.\begin{array}{rl} y=2, & (5)\\ 6x-5y=44. & (2)\end{array}\right\}\text{ III.}\qquad\begin{array}{c}\text{Equivalent}\\ \text{Derived}\\ \text{System.}\end{array}$$

By substituting the value 2 for y in one of the given equations, say (2), we shall obtain an equation in which x is the only unknown number. Solving this equation we shall obtain the value $x = 9$.

Instead of obtaining x by substituting the value found for y we may transform the given equations in such a way that their members may be combined to eliminate y, as follows:

$$\left.\begin{array}{rl} \text{Multiplying the members of (1) by 5,} & \\ 15x+20y=175. & (6)\ \ldots\\ \text{Multiplying the members of (2) by 4,} & \\ \underline{24x-20y=176.} & (7)\ \ldots\end{array}\right\}\text{ IV.}\qquad\begin{array}{c}\text{Equivalent}\\ \text{Derived}\\ \text{System.}\end{array}$$
$$\begin{array}{lrl}\text{By addition,} & 39x & =351\\ \text{Hence,} & x & =9.\ (8)\end{array}$$

We may verify the solution $\left.\begin{array}{l}x=9,\\ y=2,\end{array}\right\}$ by substituting these values in the given equations, obtaining the identities $35 = 35$ and $44 = 44$.

By Principle III, Chap. X. § 28, equation (1) in System I. is equivalent to equation (3) in System II.

By Principle (iii.) § 35, equation (5), obtained from equations (3) and (2) of System II., taken together with equation (2), forms System III. which is equivalent to the original System I. Hence the solution of

System III. must be the solution of the given system of equations (1) and (2).

By a similar course of reasoning, we may show that System IV. is equivalent to System I.

Hence, the solution of IV., which is the same as that of III., is the solution of the original equations.

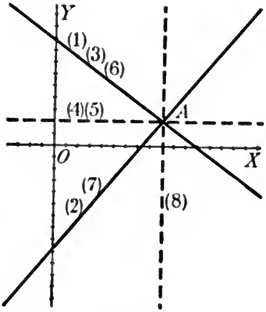


Fig. 10.

The results of the algebraic work may be illustrated graphically by the accompanying figure.

The graphs are numbered to correspond to the numbers of the different equations. It may be seen that the point marked A is the intersection of the graphs (1) and (2), representing the given equations (1) and (2). It is also located by the graphs (3) and (2), which represent the equations of System II.; by the graphs (5) and (2), which represent the equations of System III.; by the graphs (6) and (7), which represent the equations of System IV.; and finally by the graphs (8) and (5), which represent the solution, $x = 9$, $y = 2$, of the given equations.

38. The preceding example illustrates the following rule for elimination by addition or subtraction:

First, reduce both equations to the form $ax + by = c$. *Multiply or divide both members of the equations, if necessary, by such constants as are required to make the absolute values of the coefficients of one of the unknowns equal in both equations.*

Combine the corresponding members of the derived equations by addition or subtraction, according as the signs of the coefficients of the unknown number to be eliminated are unlike or like.

Solve the resulting equation for the single unknown appearing in it.

To find the remaining unknown, substitute the value of the unknown just found for the same unknown wherever it appears in one of the original equations, and solve the derived equation for the single unknown appearing in it; or repeat the process of elimination by addition or subtraction for the remaining unknown.

EXERCISE XVII. 4

Solve the following sets of simultaneous equations, eliminating the unknowns by the process of addition or subtraction, verifying all results :

1. $9x + 7y = 25,$
$9x - 7y = 11.$

2. $3x + 4y = 47,$
$3x + 2y = 31.$

3. $6x + 5y = 23,$
$4x - 5y = 7.$

4. $2x + 5y = 4,$
$4x - 10y = 48.$

5. $7x + y = 16,$
$3x - y = 4.$

6. $x + 2y = 14,$
$2x + 3y = 23.$

7. $5x - 8y = -39,$
$8x - 5y = 0.$

8. $2x - 7y = 2,$
$10x - 21y = -18.$

9. $5x + 3y - 90 = 0,$
$2x - 7y - 94 = 0.$

10. $3x + 4y + 5 = 0.$
$5x + 4y + 3 = 0.$

11. $8x - 9y + 1 = 0,$
$16x + 27y - 17 = 0.$

12. $3(x + y) = 57,$
$5(x - y) = 15.$

13. $ax + by = (a - b)^2,$
$ax - by = a^2 - b^2.$

14. $x - y = m - n,$
$mx - ny = 2(m^2 - n^2).$

15. $d^2x - e^2y = f^2,$
$d^3x - c^3y = f^3.$

16. $rx + sy = 2rs,$
$sx + ry = r^2 + s^2.$

17. $k^2x + m^2y = 0,$
$kx + my = k + m.$

18. $x + ay + a^2 = 0,$
$x + by + b^2 = 0.$

19. $(p + q)x - (p - q)y = 3,$
$(p - q)x + (p + q)y = 3.$

20. $0.8x + 0.1y = 0.19,$
$0.6x + 0.9y = 0.39.$

21. $0.5x + 0.4y = 0.13,$
$0.7x + 0.3y = 0.13.$

22. $0.3x + 0.2y = 9.5,$
$0.2x + 0.3y = 10.5.$

23. $x - \frac{7y}{3} = -21,$
$x + \frac{y}{5} = 17.$

24. $\frac{x}{3} + 3y = 15,$
$\frac{y}{4} + 4x = 37.$

25. $x + \frac{7y}{3} = \frac{27}{2},$
$\frac{17y}{5} + x = \frac{15}{2}.$

26. $$\begin{aligned}\frac{x+3}{2}+5y&=9,\\ \frac{y+9}{10}-\frac{x-2}{3}&=0.\end{aligned}$$

27. $$\begin{aligned}\frac{x+2}{3}-\frac{y+1}{4}&=0,\\ \frac{3x-1}{4}-\frac{2y+3}{5}&=0.\end{aligned}$$

28. $$\begin{aligned}\frac{x+y+9}{x-y+9}&=3,\\ \frac{x-y-9}{x+y-9}&=5.\end{aligned}$$

General Solution of a System of Two Consistent, Independent, Linear Equations Containing Two Unknown Numbers

39. A system of two consistent, independent, linear equations, containing two unknowns, has one and only one solution.

This may be shown by solving the following set of simultaneous equations :

$$\left.\begin{aligned}a_1x+b_1y&=c_1, \ (1)\\ a_2x+b_2y&=c_2, \ (2)\end{aligned}\right\}\text{I. Given System.}$$

In these equations a_1, b_1, c_1, etc., are read "a sub-one," "b sub-one," "c sub-one," etc.

The subscripts $_1$ and $_2$ are used for convenience to indicate that the letters to which they are attached are found in either the first or the second equation, respectively.

By applying the principles governing the derivation of equivalent equations, it may be seen that equations (1) and (2) may be taken as representing any system of two linear equations containing two unknowns, x and y; a_1, b_1, c_1, a_2, b_2, and c_2 being known numbers the values of which do not depend upon the values of x and y.

We may apply the method of elimination by subtraction as follows :

Eliminating y

$$\begin{aligned}a_1b_2x+b_1b_2y&=c_1b_2\\ a_2b_1x+b_2b_1y&=c_2b_1\\ \hline (a_1b_2-a_2b_1)x&=c_1b_2-c_2b_1 \ (3)\\ x&=\frac{b_2c_1-b_1c_2}{a_1b_2-a_2b_1} \ (5)\end{aligned}$$

Eliminating x

$$\begin{aligned}a_1a_2x+b_1a_2y&=c_1a_2\\ a_2a_1x+b_2a_1y&=c_2a_1\\ \hline (b_1a_2-b_2a_1)y&=c_1a_2-c_2a_1 \ (4)\\ y&=\frac{a_1c_2-a_2c_1}{a_1b_2-a_2b_1} \ (6)\end{aligned}$$

By Principle (i.), § 26, and Principle (iii.), § 35, the equations (3) and (4) taken together, or either of the equations (3) or (4), taken together with one of the given equations (1) or (2) constitute a system equivalent to the given system.

The single solution of equations (3) and (4), that is, equations (5) and (6) taken together, is accordingly the solution of, and the only solution of, the given system.

It may be seen that the given equations (1) and (2) are equivalent, provided that

either $(a_1b_2 - a_2b_1) = 0$ and $(c_1b_2 - c_2b_1) = 0$, from (3),

or $(a_1b_2 - a_2b_1) = 0$ and $(c_1a_2 - c_2a_1) = 0$, from (4).

It may also be seen that the given equations are inconsistent, provided that $(a_1b_2 - a_2b_1) = 0$ and $(c_1b_2 - c_2b_1) \neq 0$, from (3),

or $(a_1b_2 - a_2b_1) = 0$ and $(c_1a_2 - c_2a_1) \neq 0$, from (4).

It follows that the condition that the given equations shall be independent and consistent is that $a_1b_2 - a_2b_1 \neq 0$.

Systems of Linear Equations Containing Three Or More Unknowns

40. It may be shown that a system of three or more consistent, independent, linear equations has in general a single definite solution, provided that the number of equations is equal to the number of unknowns appearing in them.

The solution may be obtained by applying the methods already shown for systems containing two unknowns.

41. To obtain the solution of a system of three consistent, independent, linear equations containing three unknowns, all of the unknowns appearing in each of the given equations, we may proceed as follows:

Using any two of the given equations, eliminate one of the unknowns; then, using one of these same equations with the remaining equation of the system, eliminate the same unknown as before.

Two derived equations will thus be obtained which, taken together with one of the original equations, will form a system equivalent to the given system.

Solve these two derived equations, for the two unknowns appearing in them, by the methods previously shown; substitute the values of the *unknowns thus found for these unknowns in one of the original equations to obtain the value of the third unknown.*

42. Whenever a system consists of four equations containing four unknowns, by elimination we may obtain: first, a system of three equations containing three unknowns; then from this, a system of two equations containing two unknowns; and finally, a single equation containing a single unknown.

The value of the single unknown obtained by solving this last equation may be substituted in one of the equations containing two unknowns to obtain the value of a second unknown; substituting the values of these two unknowns in one of the equations containing three unknowns, we may find the value of a third unknown; the value of the fourth unknown may be obtained by substituting the values of the three unknowns for these unknowns in one of the equations containing four unknowns.

Graphical Record of the Process of Solution

43. The following device will be found to be helpful in planning and keeping record of the different steps taken when solving a system of simultaneous equations.

Since, by transposing all of the terms of a given equation to the first member, we may derive an equivalent equation in which the second member is zero, it is seen that the first member of the equation thus transformed is a function of the unknowns appearing in it. It follows that any equation containing a single unknown, x, may be represented by the notation $f(x) = 0$, read "function of x equal to zero"; an equation containing two unknowns, x and y, by the notation $f(x, y) = 0$, read "function of x and y equal to zero"; and an equation containing three unknowns, x, y, and z, by the notation $f(x, y, z) = 0$, read "function of x, y, and z equal to zero."

Different equations may be denoted by subscripts.

Thus, $f_1(x, y, z) = 0$, read "function one of x, y, and z equal to zero," $f_2(x, y, z) = 0$, read "function two of x, y, and z equal to zero," and $f_3(x, y, z) = 0$, read "function three of x, y, and z equal to zero," represent three different equations which may be referred to as equations (1), (2), and (3), each containing x, y, and z.

The letters written within each parenthesis must in every case be the same as the different unknown letters appearing in the equation numbered to correspond to the subscript of the symbol.

By this notation our attention is directed simply to the fact that certain equations contain particular unknowns, and nothing is indicated as to the forms of the equations in which the unknowns are found.

If, using equations (1) and (2), one of the unknowns, say y, is eliminated, the derived equation will contain the two unknowns x and z. This derived equation may be represented by the symbol $f_4(x, z) = 0$. The derivation of equation (4) from equations (1) and (2) may be suggested as follows:

$$\begin{tikzcd}
f_1(x, y, z) = 0 \arrow[dr, "a"] & \\
f_2(x, y, z) = 0 \arrow[r, "b"] & f_4(x, z) = 0.
\end{tikzcd}$$

If, during the process of elimination, the members of equations (1) and (2) are multiplied by a and b respectively, this may be indicated by placing a and b on the leading lines, as shown above.

A second equation (5) containing the same unknowns, x and z, which may be represented by the symbol $f_5(x, z) = 0$, may be derived by using either equations (1) and (3) or equations (2) and (3).

If equations (1) and (3) are used, the derivation may be suggested as follows:

$$\begin{tikzcd}
f_1(x, y, z) = 0 \arrow[dr, "c"] & \\
f_3(x, y, z) = 0 \arrow[r, "d"] & f_5(x, z) = 0.
\end{tikzcd}$$

If equations (4) and (5) are used to eliminate z to obtain a single equation (6) in which x alone appears, we may suggest the derivation as follows:

$$\begin{tikzcd}
f_4(x, z) = 0 \arrow[dr, "e"] & \\
f_5(x, z) = 0 \arrow[r, "f"] & f_6(x) = 0.
\end{tikzcd}$$

The forms of the equations may be such that it is unnecessary to use multipliers represented by a, b, c, d, e, and f, which are shown on the leading lines.

It will be seen that the solution of the given system of three equations, (1), (2), and (3), containing three unknowns, is made to depend upon the solution of a system of two equations, (4) and (5), in which two unknowns, x and z, are found. The solution of this system in turn depends upon the solution of a single equation (6) in which a single unknown, x, appears.

44. It will be found that this device will enable the student to plan the solution of a system of equations intelligently, and to carry out the work systematically, with the advantage that at any stage of the process the record will show the reason for each step taken, and serve also as a guide for such additional steps as are necessary to complete the solution.

45. To illustrate the use of this graphical record, we will solve a system of three simultaneous, independent, linear equations containing three unknowns.

Ex. 1. Solve the following system of simultaneous equations:

$$\left.\begin{array}{r} 4x+3y+9z=53, \ (1) \\ 11x-2y+8z=75, \ (2) \\ 6x+\ \ y+5z=47. \ (3) \end{array}\right\} \text{Given System.}$$

An examination of the equations shows that it will be convenient to select y as the unknown to be eliminated first.

The unknown, y, may be eliminated by using the first and third equations and also the second and third equations.

Two equations which may be called equations (4) and (5) will thus be obtained, in each of which the two unknowns x and z will be found.

After the equations (4) and (5) are obtained the remaining steps of the process of solution may be determined.

We begin the record of the process of solution by writing the symbols representing the set of given equations; that is, the equations numbered 1, 2, and 3.

The elimination of y from equations (1) and (3), and also from equations (2) and (3), to produce the derived equations numbered (4) and (5), may be suggested as follows:

$$\begin{tikzcd}
f_1(x, y, z)=0 \arrow[dr] & \\
f_2(x, y, z)=0 \arrow[dr] & f_4(x, z)=0 \\
f_3(x, y, z)=0 \arrow[ur, "3"] \arrow[r, "2"] & f_5(x, z)=0
\end{tikzcd}$$

The multipliers, 3 and 2, which are used to multiply the members of equation (3) in preparation for the elimination of y to obtain equations (4) and (5) respectively, are written on the leading lines, as shown above.

Equations (4) and (5) may be derived as follows:

(1) Unaltered. $4x+3y+9z=53$

(3) Modified.
Members
mult. by 3. $\underline{18x+3y+15z=141}$

By subtraction, $14x \quad +6z=88,$

Hence, $7x \quad +3z=44.$ (4)

(2) Unaltered. $11x-2y+8z=75$

(3) Modified.
Members
mult. by 2. $\underline{12x+2y+10z=94}$

By addition, $23x \quad +18z=169.$ (5)

The solution of the given system of three equations containing three unknowns, x, y, and z, is thus made to depend upon the solution of a derived system consisting of two equations containing two unknowns, x and z.

$$\left.\begin{array}{rl} 7x+3z=44, & (4) \\ 23x+18z=169. & (5) \end{array}\right\}$$

It should be understood that this derived system is not equivalent to the given system consisting of three equations.

However, the derived system consisting of equations (4) and (5), taken together with one of the given equations, will form a system of equations which is equivalent to the given system of three equations.

Using equations (4) and (5), we may eliminate z by multiplying the members of equation (4) by 6 and then combining the members of the resulting equation by addition with the members of equation (5). A sixth equation, containing a single unknown, x, will thus be obtained, and this may be solved for x.

The graphical record of the process of solution may now be completed by indicating the elimination of z by using equations (4) and (5) to obtain equation (6), as follows:

$$\begin{tikzcd}
f_1(x, y, z)=0 \arrow[dr] & & \\
f_2(x, y, z)=0 \arrow[dr] & f_4(x, z)=0 \arrow[dr, "6"] & \\
f_3(x, y, z)=0 \arrow[ur, "3"] \arrow[r, "2"] & f_5(x, z)=0 \arrow[r] & f_6(x)=0
\end{tikzcd}$$

The process of the derivation of equation (6) may be carried out as follows:

(4) Modified.
Mem. mult. by 6. $42x+18z=264$

(5) Unaltered. $\underline{23x+18z=169}$

By subtraction, $19x \quad =95.$

Hence, $x \quad =5.$ (6)

By substituting the value of x, which is 5, for x in either of the equations (4) or (5) and solving the resulting equation, we shall obtain the value of z, as follows:

Using (4) we have, $7 \cdot 5+3z=44.$ Hence $z=3.$

To obtain the value of y we may select an equation containing y, say equation (3), and substitute for x and z the values 5 and 3 respectively, and solve the resulting equation as follows:

$$6 \cdot 5 + y + 5 \cdot 3 = 47.$$

Hence, $$y = 2.$$

The solution of the given system of simultaneous equations is the set of values

$$\left.\begin{aligned} x &= 5, \\ y &= 2, \\ z &= 3. \end{aligned}\right\}$$

From the process of derivation, it will be seen that roots have neither been gained nor lost during the transformation. Hence, these values constitute the only solution of the given set of simultaneous equations.

Substituting in the original equations for x, y, and z the values 5, 2, and 3 respectively, we obtain the following numerical identities: $53 = 53$, (1); $75 = 75$, (2); and $47 = 47$, (3).

Ex. 2. Solve the following system of four independent, simultaneous, linear equations containing four unknowns, x, y, z, and w.

$$\left.\begin{array}{rrrrll} 3x + & y - & 2z + & w = & 1, & (1) \\ 2x - & 4y & & + 5w = & 5, & (2) \\ & 3y + & 2z - & 4w = & 16, & (3) \\ 4x - & y + & 3z & = & 35. & (4) \end{array}\right\} \text{Given System.}$$

The accompanying graphical outline will serve to indicate the different steps of the process.

$$\begin{tikzcd}
f_1(x, y, z, w) = 0 \arrow[ddr] & & & \\
f_2(x, y, \quad w) = 0 \arrow[r] & f_2(x, y, w) = 0 \arrow[dr] \arrow[ddr] & & \\
f_3(\quad y, z, w) = 0 \arrow[r] \arrow[dr, "3"] & f_5(x, y, w) = 0 \arrow[r, "11"'] & f_7(x. w) = 0 \arrow[dr] & \\
f_4(x, y, z, \quad) = 0 \arrow[r, "2"] & f_6(x, y, w) = 0 \arrow[r] & f_8(x, w) = 0 \arrow[r] & f_9(w) = 0.
\end{tikzcd}$$

Equations (5) and (6) may be derived as follows:

(1) Unaltered.	$3x + y - 2z + w = 1$	(3) Modified. Mem. mult. by 3.	$9y + 6z - 12w = 48$
(3) Unaltered.	$3y + 2z - 4w = 16$	(4) Modified. Mem. mult. by 2.	$8x - 2y + 6z = 70$
(5) By addit'n,	$3x + 4y - 3w = 17.$	(6) By subtraction.	$8x - 11y + 12w = 22$

It will be seen that, when z is eliminated by using equations (1), (3), and (4), the derived equations (5) and (6) contain the unknowns x, y, and w.

Hence, equation (2) may be carried over unaltered as the remaining equation necessary to complete the set of three equations containing three unknowns, x, y, and w.

The solution of the given system of equations is thus made to depend upon the solution of the three following equations:

$$\left.\begin{array}{rl} 3x + 4y - 3w = 17, & (5) \\ 2x - 4y + 5w = 5, & (2) \\ 8x - 11y + 12w = 22. & (6) \end{array}\right\}$$

Equations (7) and (8) may be derived as follows:

(5) Unaltered $3x+4y-3w=17$

(2) Unaltered $2x-4y+5w=5$

By addition, $5x \quad +2w=22$. (7)

(2) Modified Mem. mult. by 11. $22x-44y+55w=55$

(6) Modified Mem. mult. by 4. $32x-44y+48w=88$

(8) By subtraction, $10x \quad -7w=33$.

The solution of the given system of three equations has now been made to depend upon the solution of the following system of two equations:

$$\left.\begin{array}{rl} 5x + 2w = 22, & (7) \\ 10x - 7w = 33. & (8) \end{array}\right\}$$

Equation (9) may be derived as follows:

(7) Modified Mem. mult. by 2. $10x + 4w = 44$

(8) Unaltered, $10x - 7w = 33$

By subtraction $11w = 11$

$w = 1$. (9)

By substituting the value 1 for w in equation (7) or equation (8) we may obtain the value of x.

Using (7), we have $5x + 2 \cdot 1 = 22$. Hence $x = 4$.

By substituting the values 4 and 1 for x and w respectively in equations (5), (2), or (6), we may find the value of y.

Using (5), we have $3 \cdot 4 + 4y - 3 \cdot 1 = 17$. Hence $y = 2$.

The value of z may be found from equations (1), (3), or (4), by substituting the values 4, 2, and 1 for x, y, and w respectively.

Using (3), and substituting the values for y and w, we have

$$3 \cdot 2 + 2z - 4 \cdot 1 = 16. \quad \text{Hence } z = 7.$$

The solution of the given set of simultaneous equations is thus found to be the following set of values:

$$\left.\begin{aligned} x &= 4, \\ y &= 2, \\ z &= 7, \\ w &= 1. \end{aligned}\right\}$$

From the nature of the derivation of the successive equations it may be seen that solutions have neither been gained nor lost. Hence the solution found is the only solution of the given system of simultaneous equations.

Substituting these values for x, y, z, and w, respectively, in the given equations, we obtain the following numerical identities:

$$1 = 1,\ (1);\ 5 = 5,\ (2);\ 16 = 16,\ (3);\ \text{and } 35 = 35,\ (4).$$

EXERCISE XVII. 5

Solve the following systems of simultaneous equations, verifying all solutions:

1. $$\begin{aligned} 3x + 2y &= 13, \\ 3y + 2z &= 8, \\ 3z + 2x &= 9. \end{aligned}$$

2. $$\begin{aligned} x - y &= 3, \\ y - z &= 1, \\ z + x &= 6. \end{aligned}$$

3. $$\begin{aligned} x - \frac{y}{3} &= 4, \\ y - \frac{z}{5} &= 6. \\ z - \frac{x}{7} &= 8. \end{aligned}$$

4. $$\begin{aligned} 2x + 5y - 3z &= 23, \\ 3x + 2y + 7z &= 41, \\ 5x - 4y + 6z &= 35. \end{aligned}$$

5. $$\begin{aligned} 5x + 4y + 3z &= 35, \\ 4x + 3y + 2z &= 25, \\ 3x + 2y - z &= 15. \end{aligned}$$

6. $$\begin{aligned} x + y + z &= 19, \\ y &= 2x - 3, \\ z &= y - 10. \end{aligned}$$

7. $$\begin{aligned} 3x + 4y + 5z &= -68, \\ 2x + y &= -2, \\ 4y - z &= -14. \end{aligned}$$

8. $$\begin{aligned} x - y - 2z &= -4, \\ x - 2y - z &= -4, \\ 2x - y - z &= 0. \end{aligned}$$

9. $$\begin{aligned} 6x - y &= 3z - 36, \\ y - 3z &= 3x - 39, \\ z - 2x &= 3y + 2. \end{aligned}$$

10. $$\begin{aligned} 2x - 3y &= 4z - 21, \\ 2y - 3z &= 4x - 14, \\ 2z - 3x &= 4y - 10. \end{aligned}$$

11. $$\begin{aligned} 2x - 3y + 2z &= \tfrac{7}{36}, \\ 3x - 2y + 3z &= \tfrac{1}{2}, \\ 2x + 3y - 2z &= \tfrac{17}{36}. \end{aligned}$$

12. $2x + 3y + 4z + 5 = 0,$
$2x + 3y - 4z + 6 = 0,$
$2x - 3y + 4z + 7 = 0.$

13. $10x - 8y + z = 40,$
$x + y + 2z = 5,$
$3x - z + 5 = 0.$

14. $0.3x + 0.5y = 0.8,$
$0.4x + 0.7z = 1.8,$
$0.1y + 0.1z = 0.3.$

15. $0.1x + 0.3y = 1.9,$
$0.2x + 0.4z = 3.2,$
$0.5y + 0.1z = 3.1.$

16. $x + y = c,$
$y + z = b,$
$z + x = a.$

17. $x + y = a + b,$
$y + z = b + c,$
$z + x = c + a.$

18. $x + 3y = a,$
$y + 3z = b,$
$z + 3x = c.$

19. $x + y - z = a,$
$x - y + z = b,$
$z + y - x = c.$

20. $bx + ay + cz = a,$
$cx + by + az = b,$
$ax + cy + bz = c.$

21. $2x - 3y = 6,$
$4y - 5z = 7,$
$2z - 3u = 8,$
$4u - 5x = 9.$

22. $2x - y = 8,$
$3y - z = 13,$
$4z - w = 16,$
$5w - x = 13.$

23. $3x + 4y - 2z = 20,$
$2x - 7y + 5u = -9,$
$8x + 2z + 3u = 27,$
$2y - 3z + 4u = 17.$

24. $7x + y - 4z = w,$
$x + w - y = 0,$
$2z - w + 3y = 15,$
$3y - 7x - 2z = 3w - 8.$

25. $x + y + z = 3,$
$y + z + u = 4,$
$z + u + x = 5,$
$u + x + y = 6.$

26. $x + y + z + u = 12,$
$x + y + z + v = 14,$
$x + y + u + v = 16,$
$x + z + u + v = 18,$
$y + z + u + v = 20.$

Systems of Fractional Equations Solved like Equations of the First Degree

46. Certain systems of fractional equations which are linear with reference to the reciprocals of the unknowns may be solved without clearing of fractions before eliminating the unknowns.

Ex. 1. Solve the system of fractional equations

$$\left.\begin{aligned}\frac{15}{x}+\frac{18}{y}&=9, \quad (1)\\ \frac{20}{x}-\frac{6}{y}&=2. \quad (2)\end{aligned}\right\}\text{I. Given System.}$$

Instead of clearing of fractions before eliminating either of the unknowns, in which case we should introduce into the equations terms containing both x and y, we will eliminate the reciprocals of the unknowns, $\frac{1}{x}$ and $\frac{1}{y}$, and then from the derived equations obtain values for x and y.

Multiplying members of equation (2) by 3. $\quad \frac{60}{x}-\frac{18}{y}=6, \quad (3)$

(1) Unaltered. $\quad \frac{15}{x}+\frac{18}{y}=9. \quad (1)$

By addition, $\quad \frac{75}{x}=15.$

Hence, $\quad x=\frac{75}{15}=5.$

The value of y may be obtained by substituting 5 for x in one of the given equations and solving the resulting equation for y, or by repeating the process of elimination as follows:

Multiplying members of equation (1) by 4. $\quad \frac{60}{x}+\frac{72}{y}=36. \quad (4)$

Multiplying members of equation (2) by 3. $\quad \frac{60}{x}-\frac{18}{y}=6. \quad (5)$

By subtraction, $\quad \frac{90}{y}=30.$

Hence, $\quad y=\frac{90}{30}=3.$

Thus, it appears that the following set of values is the solution of the given system of simultaneous equations:

$$\left.\begin{aligned}x&=5,\\ y&=3.\end{aligned}\right\}$$

Substituting these values in the original equations, we obtain the numerical identities $9=9$, (1), and $2=2$, (2).

From the process of derivation it may be seen that solutions have been neither gained nor lost. Hence the set of values found is the only solution of the given system of equations.

47. When solving systems of fractional equations it should be kept in mind that during the process of clearing the fractional equations of fractions no solutions will be lost, but it may happen that extra solutions will be introduced. (Compare with Chapter X. § 30, also Chapter XVI. § 4.)

EXERCISE XVII. 6

Solve the following systems of equations which are fractional with reference to the unknowns, verifying all results obtained :

1. $\frac{1}{x}-\frac{1}{y}=m,$
$\frac{1}{x}+\frac{1}{y}=n.$

2. $\frac{9}{x}-\frac{10}{y}=1,$
$\frac{12}{x}+\frac{15}{y}=7.$

3. $\frac{1}{x}-\frac{2}{y}+\frac{3}{4}=0,$
$\frac{2}{x}+\frac{3}{y}-\frac{4}{5}=0.$

4. $\frac{4}{x}+\ \frac{7}{y}=\frac{5}{6},$
$\frac{6}{x}-\frac{14}{y}=-\frac{1}{2}.$

5. $\frac{10}{x}+\frac{9}{y}=8,$
$\frac{8}{x}+\frac{3}{y}=5.$

6. $\frac{3}{x}+\frac{4}{y}=\frac{7}{24},$
$\frac{8}{x}-\frac{6}{y}=\frac{1}{12}.$

7. $\frac{1}{2x}+\frac{1}{3y}=8,$
$\frac{1}{4x}+\frac{1}{9y}=\frac{10}{3}.$

8. $\frac{1}{5x}-\frac{1}{12y}=\frac{9}{2},$
$\frac{1}{15x}+\frac{1}{6y}=5.$

9. $\frac{1}{2x}+\frac{2}{3y}=15,$
$\frac{4}{5x}+\frac{5}{6y}=20.$

10. $\frac{8}{ax}+\frac{3}{by}=\frac{1}{c},$
$\frac{4}{ax}+\frac{9}{by}=\frac{1}{2d}.$

11. $\frac{a}{x}+\frac{b^2}{y}=c,$
$\frac{a^2}{x}-\frac{b}{y}=c^2.$

12. $11x-\frac{4}{y}=3,$
$10x-\frac{3}{y}=4.$

13. $\frac{a}{x} + \frac{b}{y} = 1,$

$\frac{b}{x} - \frac{a}{y} = 3.$

14. $\frac{a}{x} + \frac{b}{y} = \frac{c}{d},$

$\frac{a}{x} - \frac{b}{y} = \frac{d}{c}.$

15. $\frac{3}{x} + \frac{1}{3y} - 3 = 0,$

$\frac{1}{2x} - \frac{2}{y} + \frac{1}{2} = 0.$

16. $\frac{3}{x} + \frac{7}{y} = \frac{43}{20},$

$2y - 6x = -\frac{7xy}{10}.$

17. $\frac{a}{bx} + \frac{b}{ay} = a + b,$

$\frac{b}{x} + \frac{a}{y} = a^2 + b^2.$

18. $5x - 5y = 4xy,$

$6y + 6x = 5xy.$

HINT. Divide both members of each of the equations by xy.

19. $\frac{4}{x+1} + \frac{5}{y-2} = 9,$

$\frac{5}{x+1} - \frac{3}{y-2} = 2.$

20. $\frac{2}{x+2} + \frac{3}{y+3} - 4 = 0,$

$\frac{3}{x+2} - \frac{2}{y+3} + 4 = 0.$

21. $\frac{1}{x} + \frac{1}{y} = 3,$

$\frac{1}{x} + \frac{1}{z} = 4,$

$\frac{1}{y} + \frac{1}{z} = 5.$

HINT. From the sums of the corresponding members of all of the equations, subtract the corresponding members of each of the equations in turn, and solve.

22. $\frac{1}{x} - \frac{2}{y} + 3 = 0,$

$\frac{1}{y} - \frac{3}{z} + 4 = 0,$

$\frac{1}{z} - \frac{4}{x} + 5 = 0.$

23. $\frac{1}{x} + \frac{1}{y} = \frac{1}{a},$

$\frac{1}{y} + \frac{1}{z} = \frac{1}{b},$

$\frac{1}{z} + \frac{1}{x} = \frac{1}{c}.$

24. $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{2},$

$\frac{1}{x} - \frac{1}{y} + \frac{1}{z} = \frac{1}{3},$

$\frac{1}{x} + \frac{1}{y} - \frac{1}{z} = \frac{1}{4}.$

25. $\frac{2}{x}+\frac{3}{y}+\frac{1}{z}=4,$

$\frac{3}{x}+\frac{1}{y}+\frac{2}{z}=5,$

$\frac{1}{x}+\frac{2}{y}-\frac{3}{z}=6.$

26. $\frac{1}{x}+\frac{2}{y}+\frac{3}{z}=20,$

$\frac{2}{x}+\frac{3}{y}+\frac{1}{z}=17,$

$\frac{3}{x}+\frac{1}{y}+\frac{2}{z}=17.$

27. $\frac{xy}{x+y}=a,$

$\frac{yz}{y+z}=b,$

$\frac{zx}{z+x}=c.$

HINT. Write the first equation in the form $\frac{1}{y}+\frac{1}{x}=\frac{1}{a}\cdot$

Write the others in similar forms.

PROBLEMS INVOLVING SIMULTANEOUS EQUATIONS

48. Whenever the unknown numbers of a problem are obtained by solving algebraic conditional equations, it is necessary that the number of independent consistent equations be equal to the number of unknowns whose values are to be found.

49. It is often a matter of choice whether a particular problem shall be solved by using a single equation containing one unknown, or a system of two or more independent equations containing two or more unknowns.

EXERCISE XVII. 7

1. Separate 101 into two such parts that 2/5 of the greater shall exceed 2/3 of the less by 2.

Let x stand for the greater of the two numbers into which 101 is separated, and let y represent the less number.

By the conditions of the problem we obtain the conditional equations

$$x+y=101,$$

$$\frac{2x}{5}-\frac{2y}{3}=2.$$

The solution of these equations is found to be $x=65$, which is the greater number, and $y=36$, which is the less number.

These numbers are found to satisfy the conditions of the given problem.

2. Find a fraction such that, if 7 be added to both numerator and denomi-

nator, its value becomes 4/5, and if 2 be subtracted from both numerator and denominator, its value becomes 1/2.

Let x represent the numerator and y the denominator of the fraction. Then by the conditions of the given problem we have

$$\frac{x+7}{y+7}=\frac{4}{5},$$

and

$$\frac{x-2}{y-2}=\frac{1}{2}.$$

The solution of these two equations is found to be $x=5$ (which is the numerator of the fraction), and $y=8$ (which is the denominator).

Accordingly the required fraction is 5/8.

This fraction will be found to satisfy the conditions of the problem as stated.

3. The difference between two numbers is 5 and their sum is 29. Find the numbers.

4. Two numbers are to each other in the ratio of 7 to 9, and if 50 be subtracted from each of the numbers the remainders will be to each other as 1 is to 2. Find the numbers.

5. Separate 109 into two parts such that 3/8 of the greater part shall exceed 4/9 of the less by 4.

6. If three times the greater of two numbers be divided by the less, the quotient is 3 and the remainder is 15; and if four times the less be divided by the greater, the quotient is 3 and the remainder is 14. What are the numbers?

7. What is that fraction which equals 1/5 when 1 is added to the numerator, and equals 1/6 when 1 is added to the denominator?

8. Find a fraction which is equal to 1/5 when its numerator and denominator are each diminished by 2, and is equal to 1/3 when its terms are increased by 3.

9. If 1 is added to the numerator of a certain fraction its value becomes 5/7, and if 1 is added to the denominator the value becomes 3/5. What is the fraction?

10. Find a fraction such that if 1 be added to both numerator and denominator the value becomes 1/2, while if 1 be subtracted from both numerator and denominator the value becomes 7/16.

11. If 3 be added to both numerator and denominator of a certain fraction its value becomes 7/9; if 3 be subtracted from both numerator and denominator its value becomes 1/3. Find the fraction.

12. If the numerator of a certain fraction be multiplied by 2 and its denominator be increased by 5, the value of the fraction becomes 1/2; if

the denominator be multiplied by 2 and the numerator be increased by 18, the value of the fraction becomes unity. Find the numerator and denominator of the fraction.

13. The numerator and denominator of a certain proper fraction each consists of the same two figures whose sum is 9, written in different orders. If the value of the fraction be 3/8, find the numerator and denominator.

14. The numerator and denominator of a certain improper fraction each consists of the same two figures whose sum is 6, written in different orders. If the value of the fraction be 7/4, find the numerator and denominator.

15. Separate 100 into three parts such that if the second part be divided by the first the quotient is 3 and the remainder 2; and if the third be divided by the second the quotient is 3 and the remainder is 1.

16. A number expressed by two figures is equal to 7 times the sum of its figures. If 27 be subtracted from the number, the figures in tens' and units' places are interchanged. Find the number.

17. Separate the two numbers 75 and 70 into two parts each, such that the sum of one part of the first and one part of the second shall equal 100, and the difference of the remaining parts shall equal 25.

18. Separate the two numbers 60 and 50 into two parts each, such that the sum of one part of the first and one part of the second shall equal 75, and the difference of the remaining parts shall equal 5.

19. Separate a into two parts such that $1/m$th of the greater part shall exceed $1/n$th of the less by b.

20. A number is composed of two figures whose sum is 12. If the figures in tens' and units' places are interchanged the number is increased by 18. Find the number.

21. A farmer bought 100 acres of land for $3304. If part of it cost him $50 an acre and the remainder $18 an acre, find the number of acres bought at each price.

22. If five pounds of sugar and ten pounds of coffee together cost $3.80, and at the same price ten pounds of sugar and five pounds of coffee cost $2.35, what is the price of each per pound?

23. A man invested $5000, a part at 5 per cent and the remainder at 4 per cent interest. If the annual income from both investments was $235, what were the separate amounts invested?

24. There are two pumps drawing water from a tank. When the first works three hours and the second five hours, 1350 cubic feet of water are withdrawn. When the first works four hours and the second three hours, 1250 cubic feet of water are withdrawn. How many cubic feet of water can each pump discharge in one hour?

25. A plumber and his helper together receive $4.80. The plumber works 5 hours and the helper 6 hours. At another time the plumber works 8 hours and the helper $9\frac{1}{2}$ hours, and they receive \$7.65. What are the wages of each per hour?

26 Three men and three boys can do in 4 days a certain amount of work which can be done in 6 days by one man and 5 boys. How long would it require one man alone, or one boy alone, to do the work?

27. A certain piece of work can be completed by 3 men and 6 boys in 2 days. At another time it is observed that an equal amount of work is performed in 3 days by 1 man and 8 boys. Find the length of time required for one man alone or for one boy alone to do the given amount of work.

28. Two persons, A and B, can complete a certain amount of work in l days; they work together m days, when A stops; B finishes it in n days. Find the time each would require to do it alone.

29. A steamer makes a trip of 70 miles up a river and down again in 24 hours, allowing 5 hours for taking on a cargo. It is observed that it requires the same time to go $2\frac{1}{2}$ miles up the river as 7 miles down the river. Find the number of hours required for the up trip and for the down trip respectively.

30. A train ran a certain distance at a uniform rate. If the rate had been increased by 4 miles an hour the journey would have required 16 minutes less, but if the rate had been diminished by 4 miles an hour the journey would have required 20 minutes more. Find the length of the journey and the rate of the train in miles per hour.

31. A steamer runs a miles up a river and back again in t hours. It is observed that it requires the same time to go b miles with the stream as it does to go c miles against it. Find expressions for the number of hours required for the up and down trips respectively, and also for the velocity of the stream in miles per hour.

32. Three trains start for a certain city, the second h hours after the first and the third k hours after the first. The second and third run at the rates of a and b miles an hour respectively. If all three arrive together, find an expression for the distance and for the rate of the first train in miles per hour.

33. A marksman fires at a target 600 yards distant. He hears the bullet strike 4 seconds after he fires. An observer, standing 525 yards from the target and 300 yards from the marksman, hears the bullet strike 3 seconds after he hears the report of the rifle. Find the velocity of the sound in yards per second and also the velocity of the bullet in yards per second, supposing each to be uniform.

34. Having given two alloys of the following composition: A, composed

of 4 parts (by weight) of gold and 3 of silver; B, 2 parts of gold and 7 of silver; how many ounces of each must be taken to obtain 6 ounces of an alloy containing equal amounts (by weight) of gold and silver?

35. Two alloys, A and B, contain: A, 2 parts (by weight) of tin and 9 parts of copper; B, 7 parts of tin and 3 parts of copper. To obtain 1000 pounds of alloy containing (by weight) 5 parts of tin and 16 parts of copper, how many pounds of each must be taken and melted together?

36. A bar of metal contains 20.625 per cent pure silver, and a second bar 12.25 per cent. How many ounces of each bar must be used if, when the parts taken are melted together, a new bar weighing 50 ounces is obtained, of which 15 per cent is pure silver?

37. A and B run two quarter-mile races. In the first race A gives B a start of 2 seconds and beats him by 20 yards. In the second race A gives B a start of 6 yards and beats him by 4 seconds. Find the rates of A and B in yards per second.

38. A and B run a race of 500 yards. In the first trial A gives B a start of 7 yards and wins by 10 seconds. In the second trial A gives B a start of 56 yards and wins by 2 seconds. Find the rates of A and B in yards per second.

39. In a race of one hundred yards A beats B by $\frac{1}{5}$ of a second. In the second trial A gives B a start of 3 yards, and B wins by $1\frac{1}{13}$ yards. Find the time required for A and B each to run 100 yards.

40. A and B run a race of 440 yards. In the first trial A gives B a start of 65 yards and wins by 20 seconds. In the second trial A gives B a start of 34 seconds and B wins by 8 yards. Find the rates of A and B in yards per second.

Solve the following problems, employing equations containing three or more unknowns:

41. If 270 is added to a certain number of three figures the figures in tens' and hundreds' places are interchanged. When 198 is subtracted from the number the figures in hundreds' and units' places are interchanged. The figure in hundreds' place is twice that in units' place. Find the number.

42. A number is expressed by three figures whose sum is 10. The sum of the figures in hundreds' and units' places is less by 4 than the figure in tens' place, and if the figures in units' and tens' places are interchanged the resulting number is less by 54 than the original number. Find the original number.

43. Find three numbers such that the sum of the reciprocals of the first and second is 1/2; of the second and third, 1/3; and of the third and first, 1/4.

44. Separate 400 into 4 parts such that if the first part be increased by 9, the second diminished by 9, the third multiplied by 9, and the fourth divided by 9, the results will all be equal.

45. A and B together can do a certain piece of work in $7\frac{1}{5}$ days, A and C in 6 days. All three work together for 2 days, when A stops and B and C finish the work in $2\frac{4}{7}$ days. How long would it require each man alone to do the work?

46. A and B can do a piece of work in r days, A and C can do the same work in s days, and B and C can do it in t days. Find in how many days each can do the work alone.

47. In a mile race A can beat B by 60 yards and can beat C by 230 yards. By how much can B beat C?

Represent the rates of A, B, and C in yards per second by a, b, and c respectively.

The time required for A to run one mile or 1760 yards is $1760/a$ seconds. Since A beats B by 60 yards, B in the same time runs 1700 yards. The time required by B is $1700/b$ seconds.

Accordingly we have the conditional equation

$$\frac{1760}{a} = \frac{1700}{b}. \quad (1)$$

Similarly, since A can beat C by 230 yards, we have the conditional equation

$$\frac{1760}{a} = \frac{1530}{c}. \quad (2)$$

From (1) and (2) we obtain $\frac{1700}{b} = \frac{1530}{c}$, (3), from which we find that C's rate and B's rate must satisfy the conditional equation $c = \frac{9}{10} b$. (4)

If B and C run a mile race the time required for B is $1760/b$ seconds.

Let x represent the number of yards by which B beats C. Then the time required for C to run $1760 - x$ yards is $(1760 - x)/c$ seconds.

Accordingly, we have the conditional equation

$$\frac{1760}{b} = \frac{1760 - x}{c}. \quad (5)$$

Substituting for c in (5) the expression $\frac{9}{10} b$ from (4), and solving the resulting equation for x, we obtain $x = 176$, which is equal to the number of yards by which B beats C.

This value is found to satisfy the conditions of the given problem.

48. In a race of 500 yards A can beat B by 20 yards, and C by 30 yards. By how many yards can B beat C?

49. Having given 3 bars of metal, the first containing (by weight) 6 parts of gold, 2 parts of silver, and 1 part of lead; the second, 3 parts of gold, 4

parts of silver, and 2 parts of lead; the third, 1 part of gold, 3 parts of silver, and 5 parts of lead; find how many ounces of each must be taken to obtain 12 ounces of an alloy containing equal amounts (by weight) of gold, silver, and lead.

50. Of three bars of metal, the first contains 13 parts (by weight) of silver, 5 parts of copper, and 2 parts of tin; the second, 35 parts of silver, 4 parts of copper, and 1 part of tin; the third, 8 parts of silver, 7 parts of copper, and 5 parts of tin. How many ounces of each bar must be used if when the parts taken are melted together a bar is obtained which weighs 10 ounces, of which 5 ounces are silver, 3 ounces are copper, and 2 ounces are tin?

CHAPTER XVIII

EVOLUTION

1. A POWER has been defined as the product of two or more equal factors. (See Chap. V. § 25.)

With respect to the power, each of the equal factors is called a **root.**

According as there are two, three, four, or n equal factors, each is called a **square root, cube root, fourth root, or nth root.**

E. g. Since $3^2 = 3 \times 3 = 9$, 3 is a square root of 9.

Again, since $(-3)^2 = (-3)(-3) = +9$, -3 is also a square root of 9.

Also, since $2^3 = 2 \times 2 \times 2 = 8$, 2 is a cube root of 8.

We shall see later that there are in all three expressions whose cubes are 8, hence there are three different cube roots of 8.

2. The radical or root sign $\sqrt{\ }$ written before a number is a sign of operation which is commonly used to denote that a root is to be taken. This symbol is an abnormal form of the initial letter r, from the Latin *radix* meaning root.

3. The number or expression whose root is required is called the **radicand.**

E. g. The expression $\sqrt{9}$ means that the square root of 9 is to be taken. The number 9 is called the radicand.

4. A number called the **index** of the root is written before and directly above the radical sign to indicate which root is required.

E. g. The symbols $\sqrt[2]{\ }$, $\sqrt[3]{\ }$, $\sqrt[4]{\ }$, $\sqrt[n]{\ }$, denote that the second, third, fourth, and nth roots, respectively, of the numbers or expressions before which they are placed are to be taken.

In case no index is written, the index 2 is understood, so that $\sqrt{\ }$ indicates that the square root is required.

5. The radical sign affects only the *factor* before which it is placed. Accordingly if the radicand consists of more than one factor or more than one term, it must either be enclosed in parentheses, or an overline or vinculum joined to the radical sign must be used to denote that the root of the expression underneath is to be taken as a whole.

E. g. Observe that the expression $\sqrt{}4 \times 9$ means that the square root of 4 is to be taken and the result is to be multiplied by 9. Hence, since the square root of 4 is either $+2$ or -2, it follows that the expression $\sqrt{}4 \times 9$ represents either $+18$ or -18.

The expression is considered to be arranged in better form if written as $9\sqrt{}4$.

If the square root of the product of 4 and 9 is required, we may write either $\sqrt{}(4 \times 9)$ or $\sqrt{4 \times 9}$. Since the square root of the product 4×9, which is 36, is either $+6$ or -6, either of these expressions may be taken as meaning $+6$ and also -6.

The expression $\sqrt{36 + 64}$ means that the square root of the sum of 36 and 64, which is 100, is to be taken.

Hence $\sqrt{36 + 64}$ means either $+10$ or -10.

6. According as their indices are equal or unequal, two roots are said to be **like or unlike** without regard to the equality or inequality of the radicands.

E. g. The expressions $\sqrt[3]{x}$ and $\sqrt[3]{y}$ are like roots,

while $\sqrt{m}$ and $\sqrt[3]{m}$ are unlike roots.

7. A root is said to be **even or odd** according as its index is even or odd.

E. g. The expressions $\sqrt{a}$, $\sqrt[4]{b}$, $\sqrt[6]{7}$, $\sqrt[8]{10}$ denote even roots,

while $\sqrt[3]{c}$, $\sqrt[5]{8}$, $\sqrt[7]{33}$, $\sqrt[9]{10}$ denote odd roots.

8. A number or expression which can be expressed either as an integer or as a fraction of which the terms are finite integers, without using an indicated root, is said to be **rational or commensurable.**

In the contrary case it is said to be **irrational or incommensurable.**

E. g. Since $\sqrt{16}$ can be expressed as either $+ 4$ or $- 4$ it follows that $\sqrt{16}$ is a rational number; while $\sqrt{2}$, which cannot be expressed either as a whole number or as a fraction whose terms are finite integers, is an irrational or incommensurable number.

9. A number or expression is said to be an ***n*th power** if its nth root is a rational number or expression.

E. g. The number 36 is a square because either of its square roots, $+ 6$ or $- 6$, is a rational number.

The number 27 is a cube because one of its cube roots is a rational number 3.

The number 17 is not an nth power, because a rational number cannot be found which is its nth root.

10. As a **formal definition of a root** (letting n represent a positive whole number) we have

$$(\sqrt[n]{a})^n \equiv a.$$

Or $\sqrt[n]{a}$ is one of the n equal factors of a.

GENERAL PRINCIPLES GOVERNING ROOT EXTRACTION

Number of Roots.

(i) *A positive number has at least two even roots which are equal in absolute value but opposite in sign.*

We may use $(\pm 3)^2 = 9$ as a convenient abbreviation for the two identities $(+ 3)^2 = 9$ and $(- 3)^2 = 9$.

Since $(+ 3)^2 = 9$ and $(- 3)^2 = 9$ it follows that $\sqrt{9}$ is $+ 3$ and also $- 3$. It is convenient to show that $\sqrt{9}$ has two values by writing $\sqrt{9} = \pm 3$. It should be understood that $\sqrt{9} = \pm 3$ is not an identity, but is simply a convenient abbreviation for two identities $\sqrt{9} = + 3$ and $\sqrt{9} = - 3$.

We may write $\sqrt{a^2} \equiv \pm a$ as a convenient abbreviation for the two identities $\sqrt{a^2} \equiv + a$ and $\sqrt{a^2} \equiv - a$.

Similarly $\sqrt[2n]{b^{2n}} \equiv \pm b$ is to be understood as meaning that $\sqrt[2n]{b^{2n}} \equiv + b$ and $\sqrt[2n]{b^{2n}} \equiv - b$.

(ii.) *There exists at least one odd root of any positive or negative number which has the same sign as the number itself.*

For since $(+c)^{2n+1} \equiv +c^{2n+1}$, we have $\sqrt[2n+1]{c^{2n+1}} \equiv +c$. (1).
Also since $(-c)^{2n+1} \equiv -c^{2n+1}$, we have $\sqrt[2n+1]{-c^{2n+1}} \equiv -c$. (2).
Changing the signs of both members of (1), we obtain

$$-\sqrt[2n+1]{c^{2n+1}} \equiv -c. \qquad (3).$$

It follows from (2) and (3) that $\sqrt[2n+1]{-c^{2n+1}} \equiv -\sqrt[2n+1]{c^{2n+1}}$.

From the reasoning above it appears that:

(iii.) *For the operation of finding an odd root of a negative number may be substituted that of finding a like odd root of a positive number having the same absolute value, provided that a negative sign is prefixed to the result.*

E. g. $\sqrt[3]{-27} = -\sqrt[3]{27} = -3$.

11. The **principal root** of a positive number is its single positive root.

E. g. The principal square root of 4 is 2; of 9 is 3; of 16 is 4; etc.

12. The **principal odd root** of a negative number is its single negative root.

E. g. The principal cube root of -27 is -3; the principal fifth root of -32 is -2; etc.

13. The radical sign will be used to denote the principal root only, unless the contrary is expressly stated.

That is, $\sqrt[n]{a^n} \equiv |a|$.

In order that the signs of operation + and − may be applied to rational and irrational numbers without exception, we shall understand that whenever a term of an algebraic expression is affected by a radical sign the principal root of the radicand is to be taken; and this root is to be combined with the other terms of the expression by addition or subtraction, according as the radical sign is preceded by a + or a − sign.

It should be observed, however, that the root of a monomial which is not a term of an algebraic expression may, according to circumstances, be positive or negative.

Thus, $\sqrt{25} = \pm 5$; $\sqrt{4} = \pm 2$.

It should be observed that although $\sqrt{25} = \pm 5$ and $\sqrt{4} = \pm 2$, the expression $\sqrt{25} + \sqrt{4} = 5 + 2 = 7$; also, the expression $\sqrt{25} - \sqrt{4} = 5 - 2 = 3$.

The expression $\sqrt{25} + \sqrt{4}$ does not mean $\pm 5 \pm 2$, which is either $+ 7$ or $- 7$.

Whenever an even root is required, and we have means for knowing that a given radicand is an even power of a negative number, it is necessary that a negative number be taken as the root.

Thus, if in a calculation a for some reason is regarded as being negative, then if a is raised to the second power the result a^2 must be considered as the square of a negative number, not as the square of a positive number.

Accordingly, in such a case we would have $\sqrt{a^2} \equiv - | a |$.

In particular, $\sqrt{(-1)(-1)} = \sqrt{(-1)^2} = -1$.

14. Since $(+5)^2 = +25$ and $(-5)^2 = +25$, it may be seen that -25 cannot be obtained by multiplying together two like factors which are either both positive or both negative. Accordingly it is impossible to express $\sqrt{-25}$ either as a positive or a negative number.

Representing any even number by $2n$, it may be seen that $(\pm a)^{2n} \equiv + a^{2n}$. It is therefore impossible to express $\sqrt[2n]{-a^{2n}}$ either as a positive or a negative number.

In Chap. XXI we shall deal with indicated even roots of negative numbers which are commonly called **imaginary numbers.**

To distinguish them from these so-called imaginary numbers, all other numbers, such as those with which we have previously dealt, are called **real numbers.**

The following principles and proofs apply to real numbers only.

15. The operation of finding any required power of a given number or expression is called **involution,** and that of finding any required root of a given number or expression is called **evolution.**

16. It may be shown, if we extend somewhat our idea of number, that there are two square roots, three cube roots, four fourth roots, etc., and n nth roots of a given number or expression.

Roots of Monomials

Principles Governing Operations with Radical Symbols

17. As fundamental to root extraction we have the following

Principle: *Like principal roots of equal numbers or expressions are equal.*

18. In the statements of the following principles the principal root is the root meant in each case, and for convenience of proof we shall restrict the radicand a to represent a positive whole number the value of which cannot become infinitely great.

19. Root of a Power.

(i.) *The exponent of the rth root of a given base a^{rn} is found by dividing the exponent rx of the power by the index r of the indicated root.*

That is, $\sqrt[r]{a^{nr}} \equiv a^n.$

By (iii.) Chap. VII. § 1, $a^{nr} \equiv (a^n)^r.$

Hence $\sqrt[r]{a^{nr}} \equiv \sqrt[r]{(a^n)^r}.$

Or $\sqrt[r]{a^{nr}} \equiv a^n.$

E. g. $\sqrt[2]{a^6} \equiv a^3.$

20. Root of a Product.

(ii.) *The rth root of a product of two or more factors is equal to the product of the rth roots of the given factors.*

That is, $\sqrt[r]{(abcd \cdots)} \equiv \sqrt[r]{a}\sqrt[r]{b}\sqrt[r]{c}\sqrt[r]{d} \cdots.$

(The following proof may be omitted when the chapter is read for the first time.)

We will establish this principle for a radicand consisting of two factors, and by similar reasoning the result may be extended to include three or more factors.

Let p represent the positive value of $\sqrt[r]{a}\sqrt[r]{b}$.

That is, $p \equiv \sqrt[r]{a}\sqrt[r]{b}.$

Raising both members to the rth power, we have

$$p^r \equiv [\sqrt[r]{a}\sqrt[r]{b}]^r$$

$$\equiv (\sqrt[r]{a})^r(\sqrt[r]{b})^r$$

Or, $p^r \equiv ab.$

Accordingly p is one of the rth roots of the product ab, and since it is real and positive, it must be the principal root.

Hence, $\sqrt[r]{ab} \equiv \sqrt[r]{a}\sqrt[r]{b}.$

Similarly, $\sqrt[r]{abcd \cdots\cdots} \equiv \sqrt[r]{a}\sqrt[r]{b}\sqrt[r]{c}\sqrt[r]{d} \cdots\cdots.$

Ex. 1. $\sqrt[2]{49 \times 81} = \sqrt[2]{49} \times \sqrt[2]{81} = 7 \times 9 = 63.$

Ex. 2. $\sqrt[3]{-8a^3b^6} \equiv \sqrt[3]{-8}\sqrt[3]{a^3}\sqrt[3]{b^6} \equiv -2ab^2.$

Mental Exercise XVIII. 1

Find the indicated root of each of the following expressions :

1. $\sqrt[3]{b^6}.$
2. $\sqrt[3]{-x^3}.$
3. $\sqrt{a^4}.$
4. $\sqrt{a^2b^4}.$
5. $\sqrt{c^8d^{10}}.$
6. $\sqrt[3]{h^9k^3}.$
7. $\sqrt{a^4b^8c^6}.$
8. $\sqrt[3]{-x^3y^6z^9}.$
9. $\sqrt[3]{x^9y^3z^{12}}.$
10. $\sqrt[5]{c^{10}x^5z^5}.$
11. $\sqrt[4]{x^4y^8z^{16}}.$
12. $\sqrt[7]{x^{14}y^7z^{21}}.$
13. $\sqrt{4a^{14}}.$
14. $\sqrt{25b^{26}}.$
15. $\sqrt[3]{-8x^3}.$
16. $\sqrt[3]{27a^6b^9}.$
17. $\sqrt[3]{-27b^{27}}.$
18. $\sqrt[4]{16a^4b^{16}}.$
19. $\sqrt[3]{64a^9b^{12}c^{15}}.$
20. $\sqrt[3]{-343a^9b^{12}c^{15}}.$
21. $\sqrt{441a^{10}b^6x^4}.$
22. $\sqrt[4]{1296b^4c^{16}d^8}.$
23. $\sqrt[5]{-32a^{10}d^{15}}.$
24. $\sqrt[5]{1024a^{10}x^{15}z^{20}}.$
25. $\sqrt[6]{64a^6b^{24}}.$
26. $\sqrt[m]{a^{4m}b^{2m}c^m}.$
27. $\sqrt[n]{b^{6n}c^{4n}d^n}.$

21. Root of a Root.

(iii.) *The nth root of the rth root of any radicand is equal to the nrth root of the given radicand.*

That is, $\sqrt[n]{\sqrt[r]{a}} \equiv \sqrt[nr]{a}.$

(The following proof may be omitted when the chapter is read for the first time.)

If we consider principal roots only, $\sqrt[r]{a}$ must have a positive value.

The nth root of the positive value of $\sqrt[r]{a}$ must itself have a positive value.

If this nth root be represented by p, we have $p \equiv \sqrt[n]{\sqrt[r]{a}}.$

Raising both members of the equation to the nth power, we have $p^n \equiv \sqrt[r]{a}.$

Raising both members to the rth power, we have $(p^n)^r \equiv a.$

Hence, $p^{nr} \equiv a.$ (See (iii.) Chap. VII. § 1.)

Accordingly p is one of the nrth roots of a, and since it is positive, it must be the principal root.

Hence, $$p \equiv \sqrt[nr]{a}.$$

Consequently $$\sqrt[n]{\sqrt[r]{a}} \equiv \sqrt[nr]{a}.$$

Ex. 1. $\sqrt{\sqrt[3]{64}} = \sqrt[6]{64}$
$= 2.$

Ex. 2. $\sqrt[3]{\sqrt{64\,a^6b^{12}}} \equiv \sqrt[6]{2^6a^6b^{12}}$
$\equiv 2\,ab^2.$

22. Power of a Root.

(iv.) *The pth power of the rth root of any radicand is equal to the rth root of the pth power of the given radicand.*

That is, $(\sqrt[r]{a})^p \equiv \sqrt[r]{a^p}$.

(The following proof may be omitted when the chapter is read for the first time.)

Since a is positive, we have

$$a \equiv \sqrt[p]{a^p}.$$

Accordingly, $$\sqrt[r]{a} \equiv \sqrt[r]{\sqrt[p]{a^p}}.$$
$$\equiv \sqrt[rp]{a^p}.$$

We have $$\sqrt[rp]{a^p} \equiv \sqrt[p]{\sqrt[r]{a^p}}.$$

Hence, $$\sqrt[r]{a} \equiv \sqrt[p]{\sqrt[r]{a^p}}.$$

Raising both members to the pth power,

$$(\sqrt[r]{a})^p \equiv (\sqrt[p]{\sqrt[r]{a^p}})^p$$

Or, $$(\sqrt[r]{a^p}) \equiv \sqrt[r]{a^p}.$$

Ex. 3. $(\sqrt[2]{4})^3 = \sqrt[2]{4^3} = \sqrt[2]{64} = 8.$

EXERCISE XVIII. 2

Write the following roots of roots as rational expressions :

1. $\sqrt{\sqrt[3]{a^6b^{18}}}$.
2. $\sqrt[4]{\sqrt[2]{c^8d^{16}}}$.
3. $\sqrt[2]{\sqrt[3]{64\,x^{12}}}$.
4. $\sqrt[3]{\sqrt[2]{8^2y^{24}}}$.
5. $\sqrt{\sqrt{16\,m^4n^8}}$.
6. $\sqrt[3]{\sqrt[5]{3^{15}x^{15n}y^{30}}}$.
7. $\sqrt[3]{\sqrt[3]{3^{9n}a^9b^{36}}}$.
8. $\sqrt{\sqrt[n]{a^{2n}b^{4n}}}$.
9. $\sqrt[n]{\sqrt[m]{x^{mn}y^{3mn}}}$.
10. $(\sqrt{ab})^4$.
11. $(\sqrt[3]{x^2yz})^6$.
12. $(\sqrt{2\,xy})^8$.
13. $(\sqrt{2\,ax})^{10}$.
14. $(\sqrt[3]{5\,a^2bc})^{12}$.
15. $(\sqrt[3]{3\,x^my^n})^9$.
16. $(\sqrt[n]{a^2b^3c^{n-1}})^{2n}$.
17. $(\sqrt[m]{a^nb})^{mn}$.
18. $(\sqrt[3]{x^2y^nz^{2n}})^{3n+3}$.

23. Root of a Quotient.

(v.) *The rth root of a quotient is equal to the quotient obtained by dividing the rth root of the dividend by the rth root of the divisor.*

That is, $$\sqrt[r]{\frac{a}{b}} \equiv \frac{\sqrt[r]{a}}{\sqrt[r]{b}}.$$

(The following proof may be omitted when the chapter is read for the first time.)

If a and b be both positive, b being different from zero, and we represent the value of the positive fraction a/b by f and of the whole number b by w, we have, applying

(i) above $$\sqrt[r]{fw} \equiv \sqrt[r]{f}\,\sqrt[r]{w}.$$

Hence $$\sqrt[r]{\frac{a}{b} \times b} \equiv \left(\sqrt[r]{\frac{a}{b}}\right)\sqrt[r]{b}.$$

Therefore, $$\sqrt[r]{a} \equiv \left(\sqrt[r]{\frac{a}{b}}\right)\sqrt[r]{b}.$$

Dividing both members by $\sqrt[r]{b}$, we have

$$\frac{\sqrt[r]{a}}{\sqrt[r]{b}} \equiv \frac{\left(\sqrt[r]{\frac{a}{b}}\right)\sqrt[r]{b}}{\sqrt[r]{b}}.$$

Or $$\frac{\sqrt[r]{a}}{\sqrt[r]{b}} \equiv \sqrt[r]{\frac{a}{b}}.$$

Ex. 1. $$\sqrt{\frac{25}{36}} = \frac{\sqrt{25}}{\sqrt{36}} = \frac{5}{6}.$$

Mental Exercise XVIII. 3

Simplify each of the following roots of quotients:

1. $\sqrt{\frac{x^2}{9}}$.

2. $\sqrt{\frac{25}{y^2}}$.

3. $\sqrt{\frac{a^2b^2}{c^2}}$.

4. $\sqrt{\frac{36\,x^4}{49\,y^8}}$.

5. $\sqrt[3]{\frac{8}{c^3}}$.

6. $\sqrt[3]{\frac{27}{d^6}}$.

7. $\sqrt[3]{\frac{64\,x^9}{27\,y^6}}$.

8. $\sqrt{\frac{16\,a^2b^4x^8}{9\,d^4e^{16}}}$.

9. $\sqrt{\frac{81\,b^{10}c^{16}}{64\,d^4x^8}}$.

10. $\sqrt[3]{\frac{8\,a^3b^6}{27\,c^9}}$.

11. $\sqrt[3]{\frac{x^6y^{12}z^{24}}{8\,w^3}}$.

12. $\sqrt[3]{\frac{-64\,a^{15}x^{21}}{b^3y^{30}}}$.

13. $\sqrt[4]{\dfrac{81\,c^{12}d^{24}}{16\,a^4x^{28}}}$.

14. $\sqrt[4]{\dfrac{81\,b^{12}c^{16}}{625\,d^8x^4}}$.

15. $\sqrt[5]{\dfrac{a^{15}x^{25}z^{30}}{b^5y^{10}}}$.

16. $\sqrt[5]{-\dfrac{32\,c^{10}d^5e^{20}}{x^5y^{15}}}$.

17. $\sqrt[n]{\dfrac{a^nb^{2n}}{c^{3n}}}$.

18. $\sqrt[m]{\dfrac{x^{mn}y^m}{z^{2m}w^{3m}}}$.

SQUARE ROOTS OF POLYNOMIALS

24. We have shown in Chap. XII. §§ 18, 19, a method for obtaining by inspection the equal factors, and hence the square root, of a trinomial which is a square.

We will now present another method.

Representing any trinomial square by $a^2 + 2\,ab + b^2$, we may obtain the square root as follows:

Arranged according to descending powers of a, we may write

$$a^2 + 2\,ab + b^2 \equiv a^2 + (2\,a + b)b. \qquad (1)$$

We may obtain the first term in the required square root by taking the square root, a, of the first term, a^2.

Subtracting the square of a from the given trinomial, we obtain as a *first remainder* $2\,ab + b^2$, which may be written in the form $(2\,a + b)b$.

The second term of the required square root may be found by dividing the first term $2\,ab$ of the first remainder $2\,ab + b^2$, arranged according to descending powers of a, by a *trial divisor* $2\,a$, which is formed by multiplying by 2 the part of the root already found.

From (1) it appears that by increasing $2\,a$ by b we may obtain a *complete divisor*, $2\,a + b$, which when multiplied by b will produce the terms $2\,ab + b^2$ of the first remainder.

Subtracting the product of $2\,a + b$ and b, that is, $2\,ab + b^2$, from the first remainder, $2\,ab + b^2$, we obtain zero as a *second remainder*.

The steps of the process are shown below:

	Given Expression	Square Root
First term of root, $\sqrt{a^2} \equiv a$.	$a^2 + 2\,ab + b^2$	$a + b$
$a^2 \equiv$	a^2	
Trial divisor, $2\,a$.	$+ 2\,ab + b^2$	
Second term of root, $2\,ab \div 2\,a \equiv b$.		
Complete divisor, $2\,a + b$.		
$(2\,a + b)b \equiv$	$2\,ab + b^2$	

Ex. 1. Find the square root of $9x^4 - 42x^2y^3 + 49y^6$.

	Given Expression	Square Root
First term of root, $\sqrt{9x^4} \equiv 3x^2$.	$9x^4 - 42x^2y^3 + 49y^6$	$3x^2 - 7y^3$
$(3x^2)^2 \equiv$	$9x^4$	
Trial divisor, $2(3x^2) \equiv 6x^2$.	$-42x^2y^3 + 49y^6$	
Second term of root, $-42x^2y^3 \div 6x^2 \equiv -7y^3$.		
Complete divisor, $6x^2 - 7y^3$.		
$(6x^2 - 7y^3)(-7y^3) \equiv$	$-42x^2y^3 + 49y^6$	

25. We will now show that the steps of the process may be repeated to obtain the square root of any polynomial square which contains more than three terms.

The square of a polynomial may be written as follows:

For two terms,

$$(a+b)^2 \equiv \left.\begin{array}{l} a^2 + 2ab \\ \quad + b^2 \end{array}\right\} (1)$$

For three terms,

$$(a+b+c)^2 \equiv \left.\begin{array}{l} a^2 + 2ab + 2ac \\ \quad + b^2 + 2bc \\ \quad\quad + c^2 \end{array}\right\} (2)$$

For four terms,

$$(a+b+c+d)^2 \equiv \left.\begin{array}{l} a^2 + 2ab + 2ac + 2ad \\ \quad + b^2 + 2bc + 2bd \\ \quad\quad + c^2 + 2cd \\ \quad\quad\quad + d^2 \end{array}\right\} (3)$$

Factoring the groups of terms which appear in the vertical columns, the identities (1), (2), and (3) become respectively:

$$(a+b)^2 \equiv a^2 + (2a+b)b. \qquad (1)$$

$$(a+b+c)^2 \equiv a^2 + (2a+b)b + (2a+2b+c)c. \qquad (2)$$

$$(a+b+c+d)^2 \equiv a^2 + (2a+b)b + (2a+2b+c)c + (2a+2b+2c+d)d. \qquad (3)$$

It may be seen that, with each new letter added on the left, a new group of terms in parentheses is added on the right. This new group consists of the product of twice the sum of all previous letters plus the last letter, multiplied by the last letter.

In (1), (2), and (3) the expressions in parentheses are in each case the complete divisors used in the extraction of the square root of a polynomial at the successive stages of the process.

Ex. 2. Find the square root of

$$a^2 + 2\,ab + b^2 + 2\,ac + 2\,bc + c^2 + 2\,ad + 2\,bd + 2\,cd + d^2.$$

$$\begin{array}{r|l|l}
 & \textbf{Given Expression} & \textbf{Square Root} \\
\sqrt{a^2} \equiv a. & a^2+2\,ab+b^2+2\,ac+2\,bc+c^2+2\,ad+2\,bd+2\,cd+d^2 & \underline{a+b+c+d} \\
a^2 \equiv & a^2 & \\
\hline
2 \times a \equiv 2\,a. & 2\,ab & \\
2\,ab \div 2\,a \equiv b. & & \\
2\,a + b. & & \\
(2\,a + b)b \equiv & 2\,ab+b^2 & \\
\hline
2\,(a+b) \equiv 2\,a+2\,b. & 2\,ac & \\
2\,ac \div 2\,a \equiv c. & & \\
2\,a + 2\,b + c. & & \\
(2\,a + 2\,b + c)c \equiv & 2\,ac+2\,bc+c^2 & \\
\hline
2(a+b+c) \equiv 2\,a+2\,b+2\,c. & 2\,ad & \\
2\,ad \div 2\,a \equiv d. & & \\
2\,a+2\,b+2\,c+d. & & \\
(2\,a+2\,b+2\,c+d)d \equiv & 2\,ad+2\,bd+2\,cd+d^2 & \\
\hline
\end{array}$$

26. The method employed in §§ 24 and 25 for extracting the square root of a polynomial may be stated in the following form as a rule.

Rule for finding the principal square root of a polynomial square.

Write the given polynomial according to descending or ascending powers of some letter of arrangement.

Extract the square root of the first term and write the result as the ***first term*** *of the required square root.*

Subtract the square of the first term of the root from the given polynomial, and arrange the ***first remainder*** *according to the powers of the letter of arrangement, and in the same order as before.*

Divide the first term of the remainder by twice the first term of the root, write the quotient as the second term of the root, and add it also to the ***trial divisor*** *to form the* ***complete divisor.***

Subtract from the first remainder the product of the complete divisor multiplied by the term of the root last found, and arrange the remainder, if there be one, as a ***second remainder.***

Repeat the process, using as a trial divisor at each stage of the work twice the part of the root already found.

Ex. 3. Find the square root of $29a^2-40a^5+16a^6-46a^3+4+49a^4-12a$. Arranged according to descending powers of a we have

$$\begin{array}{r|l|l}
 & \textbf{Given Expression} & \textbf{Square Root} \\
\sqrt{16a^6}\equiv 4a^3. & 16a^6-40a^5+49a^4-46a^3+29a^2-12a+4 & \underline{4a^3-5a^2+3a-2} \\
(4a^3)^2\equiv & 16a^6 & \\
\hline
2\times 4a^3\equiv \quad 8a^3. & -40a^5 & \\
-40a^5\div 8a^3\equiv -5a^2. & & \\
8a^3-5a^2. & & \\
(8a^3-5a^2)(-5a^2)\equiv & -40a^5+25a^4 & \\
\hline
2(4a^3-5a^2)\equiv 8a^3-10a^2. & +24a^4 & \\
24a^4\div 8a^3\equiv 3a. & & \\
8a^3-10a^2+3a. & & \\
(8a^3-10a^2+3a)3a\equiv & +24a^4-30a^3+\ 9a^2 & \\
\hline
2(4a^3-5a^2+3a)\equiv 8a^3-10a^2+6a. & -16a^3+20a^2 & \\
-16a^3\div 8a^3\equiv -2. & & \\
8a^3-10a^2+6a-2. & & \\
(8a^3-10a^2+6a-2)(-2)\equiv & -16a^3+20a^2-12a+4 & \\
\hline
\end{array}$$

In practice the student should obtain the successive terms of the root by performing the divisions, $-40a^5\div 9a^3\equiv -5a^2$, etc., mentally.

EXERCISE XVIII. 4

Find the square roots of the following expressions :

1. $a^4+2a^3-a^2-2a+1$.
2. $16x^4-24x^3+25x^2-12x+4$.
3. $81x^4+54x^3+81x^2+24x+16$.
4. $9x^6+24x^5+22x^4+38x^3+41x^2+10x+25$.
5. $25b^6-30b^4-20b^3+9b^2+12b+4$.
6. $36a^2+48ab+12ac+16b^2+8bc+c^2$.
7. $a^4+\dfrac{11a^2b^2}{12}+\dfrac{b^4}{9}-a^3b-\dfrac{ab^3}{3}$.
8. $x^4+3+\dfrac{1}{x^4}-2x^2-\dfrac{2}{x^2}$.
9. $4-4y+13y^2+16y^6+17y^4-22y^3-24y^5$.
10. $4x^{4n}+12x^{3n}+29x^{2n}+30x^n+25$.

CUBE ROOTS OF POLYNOMIALS

27. A process for finding the cube root of a polynomial which is a cube may be developed as follows :

We know that $(a+b)^3\equiv a^3+3a^2b+3ab^2+b^3$.

Hence, $\sqrt[3]{a^3+3a^2b+3ab^2+b^3} \equiv \sqrt[3]{a^3+(3a^2+3ab+b^2)b} \equiv a+b.$ (1)

It may be seen that the first term a of the required cube root is the cube root of the first term of the given expression arranged according to descending powers of a.

Subtracting the cube of the first term a of the root from the given expression, we obtain as a *first remainder* $3a^2b + 3ab^2 + b^3$.

Dividing the first term of this first remainder, arranged according to descending powers of a, by a trial divisor, $3a^2$, which is obtained by taking three times the square of the first term of the root already found, we obtain the second term, b, of the root.

To obtain a *complete divisor* we may, as indicated in (1), add to the trial divisor, $3a^2$, the term $3ab$ which is three times the product obtained by multiplying the first term a of the root by the second term b, and add also the square of the second term b of the root, that is, b^2.

If the *complete divisor* thus obtained, $3a^2 + 3ab + b^2$, be multiplied by b and the product be subtracted from the first remainder, $3a^2b + 3ab^2 + b^3$, we have zero as a *second remainder*, and accordingly the process stops here and the required cube root is $a + b$.

The different steps of the process are shown below:

	Given Expression	Cube Root
First term of root, $\sqrt[3]{a^3} \equiv a.$	$a^3 + 3a^2b + 3ab^2 + b^3$	$a + b.$
$a^3 \equiv$	a^3	
Trial divisor, $3 \times a^2 \equiv 3a^2.$	$3a^2b + 3ab^2 + b^3$	
Second term of root, $3a^2b \div 3a^2 \equiv b.$		
Complete divisor, $3a^2 + 3ab + b^2.$		
$(3a^2 + 3ab + b^2)b \equiv$	$3a^2b + 3ab^2 + b^3$	

Ex. 1. Find the cube root of $27x^6 - 54x^4y^3 + 36x^2y^6 - 8y^9$.

The process may be carried out as follows:

	Given Expression	Cube Root
First term of root, $\sqrt[3]{27x^6} \equiv 3x^2.$	$27x^6 - 54x^4y^3 + 36x^2y^6 - 8y^9$	$3x^2 - 2y^3$
$(3x^2)^3 \equiv$	$27x^6$	
Trial divisor, $3(3x^2)^2 \equiv 27x^4.$	$-54x^4y^3 + 36x^2y^6 - 8y^9$	
Second term of root, $-54x^4y^3 \div 27x^4 \equiv -2y^3.$		
Complete div., $3(3x^2)^2 + 3(3x^2)(-2y^3) + (-2y^3)^2.$		
$(27y^4 - 18x^2y^3 + 4y^6)(-2y^3) \equiv$	$-54x^4y^3 + 36x^2y^6 - 8y^9$	

28. It may be shown that, by properly grouping the terms, the cube root of any polynomial cube may be obtained by repeating the steps of the process for finding the cube root of $a^3 + 3a^2b + 3ab^2 + b^3$.

The cube of a polynomial of three terms may be arranged as follows:

$$\left.(a+b+c)^3 \equiv (a+b)^3 + 3(a+b)^2c + 3(a+b)c^2 + c^3,\right\}(1)$$

$$\left.\begin{array}{cclll} \equiv & a^3 & +3a^2b & +3(a+b)^2c \\ & & +3ab^2 & +3(a+b)c^2 \\ & & +\quad b^3 & +\qquad c^3, \end{array}\right\}(2)$$

$$\left.\begin{array}{ccl|ll|l} \equiv & a^3 & +3a^2 & b & +3(a+b)^2 & c. \\ & & +3ab & & +3(a+b)c & \\ & & +\quad b^2 & & +\qquad c^2 & \end{array}\right\}(3)$$

Similarly for four terms:

$$\left.\begin{array}{ccl|ll|ll|l} (a+b+c+d)^3 \equiv & a^3 & +3a^2 & b & +3(a+b)^2 & c & +3(a+b+c)^2 & d. \\ & & +3ab & & +3(a+b)c & & +3(a+b+c)d & \\ & & +\quad b^2 & & +\qquad c^2 & & +\qquad d^2 & \end{array}\right\}(4)$$

In (3) and (4) the expressions at the left of the vertical bars are the complete divisors used in the extraction of the cube root at the successive stages of the process.

29. From §§ 27 and 28 it appears that we can find the cube root of a polynomial cube of any number of terms as follows:

Rule for finding the principal cube root of a polynomial cube.

Arrange the terms of the given expression and all successive remainders according to descending or ascending powers of some letter.

For the ***first term*** *of the required root write the cube root of the first term of the arranged expression, and subtract its cube from the given expression to obtain the* ***first remainder.***

To obtain the next term of the root, divide the first term of the arranged first remainder by a trial divisor which is three times the square of the part of the root already found.

Construct a ***complete divisor*** *by adding to the trial divisor three times the product of the part of the root previously found and the term of the root last obtained, and add also the square of the term of the root last obtained.*

Subtract from the first remainder the product obtained by multiplying the complete divisor by the term of the root last obtained.

Repeat these steps, with successive remainders, until zero is obtained as a remainder.

Ex. 2. Find the cube root of $8x^6-36x^5+66x^4-63x^3+33x^2-9x+1$.

Arranged according to descending powers of x, we have:

	Given Expression	Cube Root
First term of root, $\sqrt[3]{8x^6}\equiv 2x^2$.	$8x^6-36x^5+66x^4-63x^3+33x^2-9x+1$	$2x^2-3x+1$
$(2x^2)^3\equiv$	$8x^6$	
Trial divisor, $3(2x^2)^2\equiv 12x^4$.	$-36x^5$	
Second term of root, $-36x^5\div 12x^4\equiv -3x$.		
Complete div., $3(2x^2)^2+3(2x^2)(-3x)+(-3x)^2$.		
$(12x^4-18x^3+9x^2)(-3x)\equiv$	$-36x^5+54x^4-27x^3$	
Trial divisor, $3(2x^2-3x)^2\equiv 12x^4-36x^3+27x^2$.	$+12x^4-36x^3$	
Third term of root, $12x^4\div 12x^4\equiv 1$.		
Complete div., $3(2x^2-3x)^2+3(2x^2-3x)\,1+(1)^2$.		
$(12x^4-36x^3+33x^2-9x+1)\,1\equiv$	$+12x^4-36x^3+33x^2-9x+1$	

EXERCISE XVIII. 5

Find the cube roots of the following expressions:

1. $27x^3+54x^2+36x+8$.
2. $8x^3+84x^2y+294xy^2+343y^3$.
3. $512a^3-1344a^2b+1176ab^2-343b^3$.
4. $8a^6-36a^5b+66a^4b^2-63a^3b^3+33a^2b^4-9ab^5+b^6$.
5. $125a^9-225a^7+150a^6+135a^5-180a^4+33a^3+54a^2-36a+8$.
6. $343a^6+441a^5b+777a^4b^2+531a^3b^3+444a^2b^4+144ab^5+64b^6$.
7. $729x^6+972x^5y+918x^4y^2+496x^3y^3+204x^2y^4+48xy^4+8y^6$.
8. $a^{12}-3a^{10}-3a^9+6a^7+8a^6+3a^5-3a^4-7a^3-6a^2-3a-1$.
9. $\frac{a^6}{27}+\frac{a^5}{6}+\frac{a^4}{3}+\frac{3a^3}{8}+\frac{a^2}{4}+\frac{3a}{32}+\frac{1}{64}$.
10. $\frac{x^3}{8}+\frac{3x^2}{4}+\frac{3x}{2}+\frac{5}{2}+\frac{6}{x}+\frac{6}{x^2}+\frac{6}{x^3}+\frac{12}{x^4}+\frac{8}{x^6}$.

SQUARE ROOTS OF ARITHMETIC NUMBERS

30. Any arithmetic number may be written in the form of a polynomial whose different terms contain different powers of 10, and hence the method for extracting the square root of a polynomial may be applied to arithmetic numbers.

E. g. $$625 = 6 \cdot 100 + 2 \cdot 10 + 5$$
$$= 6 \cdot 10^2 + 2 \cdot 10 + 5$$
Or, $$= 4 \cdot 10^2 + 20 \cdot 10 + 25.$$

In this last form, in which the coefficients of the first and last terms are squares, the square root of 625, considered as a polynomial, may be found by the algebraic process.

	Given Expression.	Square Root.
First term of root, $\sqrt{4 \cdot 10^2} = 2 \cdot 10.$	$4 \cdot 10^2 + 20 \cdot 10 + 25$	$2 \cdot 10 + 5$
$(2 \cdot 10)^2 =$	$4 \cdot 10^2$	
Trial divisor, $2(2 \cdot 10) = 4 \cdot 10$	$+ 20 \cdot 10 + 25$	
Second term of root, $20 \cdot 10 \div 4 \cdot 10 = 5.$		
Complete divisor, $4 \cdot 10 + 5$		
$(4 \cdot 10 + 5)5 =$	$20 \cdot 10 + 25$	

The square root of 625 is thus found to be $2 \cdot 10 + 5$, that is 25.

31. Consider the following relations between powers and roots:

Since	$1^2 = \mathbf{1},$	we have	$\sqrt{1}$	$= \mathbf{1}.$
		$\therefore$	$\sqrt{\left(\text{any number between 1 and 100}\right)}$	$= \left(\text{a number between 1 and 10}\right)$
	$10^2 = \mathbf{100},$		$\sqrt{100}$	$= \mathbf{10}.$
		$\therefore$	$\sqrt{\left(\text{any number between 100 and 10000}\right)}$	$= \left(\text{a number between 10 and 100}\right)$
	$100^2 = \mathbf{10000},$		$\sqrt{10000}$	$= \mathbf{100}.$
		$\therefore$	$\sqrt{\left(\text{any number between 10000 and 1000000}\right)}$	$= \left(\text{a number between 100 and 1000}\right)$
	$1000^2 = \mathbf{1000000},$		$\sqrt{1000000}$	$= \mathbf{1000}.$
	etc.			etc.

From the relations above it may be seen that the square of an integral number having a specified number of figures may be expressed as an integral number having either twice as many figures as the given number, or as one having a number of figures one less than twice as many.

Accordingly, *the number of figures in the integral part of the square root of an integral number may be found by separating the figures of the given integral number into groups of two figures each, beginning at units' place.*

The number of figures in the square root will be equal to the num-

ber of groups thus obtained, provided that any single figure which remains on the left is counted as a complete group.

E. g. The number of figures in the square root of $\widehat{9}\widehat{7}\,\widehat{41}\,\widehat{69}$ is three; in the square root of $\widehat{5}\ \widehat{48}\,\widehat{02}\,\widehat{81}$ is four.

Ex. 1. Find the square root of 3249.

Since, beginning with the units' figure 9, we can separate the figures into two groups of two figures each, $\widehat{32}\,\widehat{49}$, it appears that the integral part of the required square root must be a number of two figures.

Any given number expressed in the common system of arithmetic notation having two or more figures may be regarded as being composed of a certain number, t, of tens, increased by some number, u, of units. Hence we may represent any number by $t\cdot 10+u$.

Accordingly, if $t\cdot 10+u$ represents the number which is the required square root of the given number, 3249, we may represent the square of the square root, that is $\widehat{32}\,\widehat{48}$, by

$$\begin{aligned}(t\cdot 10+u)^2 &\equiv (t\cdot 10)^2+2\,(t\cdot 10)\,u+u^2\\ &\equiv t^2\,(100)\ +2\,tu\cdot 10+u^2\end{aligned}$$

Since, depending upon the value of t, $t^2(100)$ represents a number having *three* or *four* figures, it appears that for this example t must have such a value that its square t^2 shall not be greater than 32.

The square next less than 32 is 25, hence we *assume* that t^2 represents 25, or that $t=5$.

			Given Number	Square Root
$\sqrt{(t\cdot 10)^2}$;	$\sqrt{32\cdot 10^2}$	$=(5+\ )\cdot 10$	$\widehat{32}\ \widehat{49}$	$5\cdot 10+7$
$(t\cdot 10)^2$;	$(5\cdot 10)^2=$		25 00	
$2\,(t\cdot 10)$;	$2\,(5\cdot 10)$	$=100.$	7 49	
$749\div 100=7+$				
u;	$7=$	7		
$2\,(t\cdot 10)+u$;		107		
$[2\,(t\cdot 10)+u]\,u$;		$107\times 7=$	7 49	

Observe that the first remainder 749 contains the two terms, $2\,(t\cdot 10)\,u$ and u^2, *combined* into one arithmetic sum. Accordingly we cannot, as in the algebraic process, obtain at this step the exact value of the next term of the root represented by u, by dividing 749 by 100.

It should be understood that, in the arithmetic process above, the quotient 7 + *suggests* but does not definitely determine the value of the root figure sought. We may *assume* 7 as the root figure desired, and determine the accuracy of our assumption by the next remainder.

Since in this example the second remainder is zero, we find that 7 is the second figure of the required square root, 5 · 10 + 7, which is 57.

If in the first step of this example the square less than 32 had been assumed to be 16 instead of 25, the first root figure would have been 4 representing 40, instead of 5 representing 50.

In this case the integral part of the quotient resulting from the division of the corresponding remainder 1649 by the trial divisor 80 would have been a number of *two figures*, 20 +, instead of a number equal to or less than 9.

This would have indicated that the square assumed, 16, was too small.

32. Since there are twice as many figures at the right of the decimal point in the square of a decimal fraction as there are in a given decimal fraction, it follows that *the number of figures at the right of the decimal point in the square root of a given decimal fraction may be found by separating the figures of the given decimal fraction into groups of two figures each from left to right, beginning at the decimal point.*

33. When finding the square root of an arithmetic number it is often convenient to refer to the formula

$$\sqrt{t^2 + 2\,tu + u^2} \equiv \sqrt{t^2 + (2\,t + u)u} \equiv t + u.$$

At any stage of the process, t represents the part of the root already found considered as representing tens, and u represents the next figure of the root, which accordingly may be considered as representing units with reference to the part of the root already found.

Thus, it may be seen that we may use $2\,t$ and $2\,t + u$ to suggest the different trial and complete divisors respectively. It should be observed that in each case the number represented by t is to be multiplied by 10.

Ex. 2. Find the square root of 3642.1225.

	Given Number	Square Root
$\sqrt{t^2}$; $\sqrt{36} = 6.$	$\overset{\frown}{36}\overset{\frown}{42}.\overset{\frown}{12}\overset{\frown}{25}$	60.35
t^2; $6^2 =$	36	
$2t$; $2(60) = 120.$ **42 ÷ 120 is not an integer.** **Hence root figure is 0.**	42	
$2t$; $2(600) = 1200.$ $4212 \div 1200 = 3 +$ u; $3 = 3.$ $2t + u$; $1203.$	42 12	
$(2t + u)u$; $1203 \times 3 =$	36 09	
$2t$; $2(6030) = 12060.$ $60325 \div 12060 = 5 +$ u; $5 = 5$ $2t + u$; $12065.$	6 03 25	
$(2t + u)u$; $12065 \times 5 =$	6 03 25	

34. The examples of §§ 31, 33 illustrate the following rule :

Separate the figures of the given number into groups of two figures each, beginning at units' place.

Find the greatest square which is not greater than the number represented by the figures in the first group at the left, and write its square root as the ***first figure of the required root.***

Subtract the square of the root figure thus found from the first group at the left, and to the remainder annex the next group of figures for a ***"first remainder".***

Divide this "remainder" by a ***trial divisor*** *which is obtained by taking twice the part of the root already found, considered as representing tens, and write the integral part of the quotient as the next root figure.*

Add the root figure last found to the trial divisor to form a ***complete divisor,*** *and multiply this complete divisor by the figure of the root last found, subtracting the product from the "remainder" last found to obtain a* ***new "remainder".***

Repeat these steps with successive groups of figures either until all of the groups have been used or until as many figures have been obtained for the root as are desired.

Formula $\sqrt{t^2 + 2tu + u^2} \equiv \sqrt{t^2 + (2t + u)u} \equiv t + u.$

35. Whenever, during the process of extracting the square root, the product of the complete divisor multiplied by the figure of the root last found is greater than the remainder last found, it is necessary to choose a lower root figure and construct a new complete divisor.

36. The number of figures at the right of the decimal point in the square root of a decimal fraction is equal to the number of groups of two figures each at the right of the decimal point of the given number.

37. Since the position of the decimal point in the square root of a decimal fraction may be determined by means of the number of groups of two figures each at the left of the decimal point in the given number whose root is to be found, it follows that we may disregard the decimal point altogether when constructing the different trial and complete divisors.

38. The square root of a fraction may be obtained either by finding the square roots of the numerator and denominator separately, or by first reducing it to a decimal fraction and then extracting the square root.

E. g. The number 3642.1225 of example (2) § 33 might have been written as $\frac{36421225}{10000}$.

Hence, $$\sqrt{3642.1225} = \sqrt{\frac{36421225}{10000}} = \frac{6035}{100} = 60.35.$$

Exercise XVIII. 6

Find the square roots of the following numbers :

1. 2209.
2. 6241.
3. 26244.
4. 64009.
5. 643204.
6. 6625.96.
7. 64.1601.
8. 4.008004.
9. 4096.256004.
10. 4.141225.
11. .00009409.
12. 1.00020001.
13. 49.61511844.
14. .3603841024.

To four decimal places :

15. 1.00001.
16. 10000.00001.
17. 59.

CUBE ROOTS OF ARITHMETIC NUMBERS

39. Consider the following relations between powers and roots :

Since

$$\begin{array}{lll}
\mathbf{1^3 = 1,} & \text{we have } \sqrt[3]{1} & = \mathbf{1.} \\
 & \therefore \quad \sqrt[3]{\left(\begin{matrix}\text{any number between} \\ 1 \text{ and } 1000\end{matrix}\right)} & = \left(\begin{matrix}\text{a number between} \\ 1 \text{ and } 10\end{matrix}\right) \\
\mathbf{10^3 = 1000,} & \sqrt[3]{1000} & = \mathbf{10.} \\
 & \therefore \quad \sqrt[3]{\left(\begin{matrix}\text{any number between} \\ 1000 \text{ and } 1000000\end{matrix}\right)} & = \left(\begin{matrix}\text{a number between} \\ 10 \text{ and } 100\end{matrix}\right) \\
\mathbf{100^3 = 1000000,} & \sqrt[3]{1000000} & = \mathbf{100.} \\
 & \therefore \quad \sqrt[3]{\left(\begin{matrix}\text{any number between} \\ 1000000 \text{ and } 1000000000\end{matrix}\right)} & = \left(\begin{matrix}\text{a number between} \\ 100 \text{ and } 1000\end{matrix}\right) \\
\mathbf{1000^3 = 1000000000,} & \sqrt[3]{1000000000} & = \mathbf{1000.} \\
\text{etc.} & & \text{etc.}
\end{array}$$

From the relations above it may be seen that *the number of figures in the integral part of the cube root of an integral number may be found by separating the figures of the given integral number into groups of three figures each, beginning at units' place.*

E.g. The number of figures in the cube root of $\overset{\frown}{638}\ \overset{\frown}{277}\ \overset{\frown}{381}$ is three ; in the cube root of $\overset{\frown}{1}\ \overset{\frown}{728}$ is two.

40. If t represents the number of tens and u the number of units in terms of which an arithmetic number may be regarded as being expressed, a process for finding the cube root of an arithmetic number may be developed by referring to the identity

$$\sqrt[3]{t^3 + 3\,t^2u + 3\,tu^2 + u^3} \equiv \sqrt[3]{t^3 + (3\,t^2 + 3\,tu + u^2)u} \equiv t + u.$$

Ex. 1. Find the cube root of 79507.

Since, beginning with the units' figure 7 to separate the figures into groups of *three* figures each we obtain two groups, $\overset{\frown}{79}\ \overset{\frown}{507}$, we find that the integral part of the required cube root must be a number of two figures.

If the required cube root is expressed as a number consisting of t tens plus u units, then its cube, that is, 79507, must be represented by

$$\begin{aligned}(t \cdot 10 + u)^3 &\equiv (t \cdot 10)^3 + 3\,(t \cdot 10)^2u + 3\,(t \cdot 10)u^2 + u^3 \\ &\equiv t^3 \cdot 1000 + 3\,t^2u \cdot 100 + 3\,tu^2 \cdot 10 + u^3 \\ &\equiv t^3 \cdot 1000 + [3\,t^2 \cdot 100 + 3\,tu \cdot 10 + u^2]\,u.\end{aligned}$$

From the term $t^3 \cdot 1000$ it appears that t^3 must be a cube of which the value is not greater than the number represented by the figures in the first group at the left which is 79.

The integral cube next less than 79 is 64, hence we *assume* that t^3 represents 64, or that t is 4.

		Given Number	Square Root
$\sqrt[3]{t^3\cdot 1000}$;	$\sqrt[3]{79\cdot 1000} = (4 +)\cdot 10$	$\overset{\frown}{79}\ \overset{\frown}{507}$	$4\cdot 10 + 3$
$(t\cdot 10)^3$;	$(4\cdot 10)^3 =$	64 000	
$3\,t^2\cdot 100$;	$3\cdot 4^2\cdot 100 = 4800$	15 507	
$15507 \div 4800 = 3 +$			
$3\,tu\cdot 10$;	$3\cdot 4\cdot 3\cdot 10 = 360$		
u^2;	$3^2 = 9$		
$3\,t^2\cdot 100 + 3\,tu\cdot 10 + u^2$;	5169		
$(3\,t^2\cdot 100 + 3\,tu\cdot 10 + u^2)\,u$;	$5169\times 3 =$	15 507	

It should be understood that the quotient 3 + obtained by dividing the remainder 15507 by the trial divisor 4800 *suggests* but does not definitely determine the value of the next root figure, 3.

Whenever the root figure thus found leads us to construct a complete divisor which when multiplied by this root figure produces a product which is greater than the remainder from which it is to be subtracted, it is necessary to try the next less integer and construct a new complete divisor.

41. Since there are three times as many figures at the right of the decimal point in the cube of a given decimal fraction as there are in the given decimal fraction, it follows that *the number of figures at the right of the decimal point in the cube root of a given decimal fraction may be found by separating the figures of the given decimal fraction into groups of three figures each from left to right, beginning at the decimal point.*

42. When constructing the trial and complete divisors during the process of finding the cube root of an arithmetic number, it will be convenient to refer to the formula

$$\sqrt[3]{t^3 + 3\,t^2u + 3\,tu^2 + u^3} \equiv \sqrt[3]{t^3 + (3\,t^2 + 3\,tu + u^2)u} \equiv t + u,$$

in which, at any stage of the process, t represents the part of the root already found, considered as representing tens.

Thus, the expressions $3\,t^2$ and $3\,t^2 + 3\,tu + u^2$ may be used to suggest the different trial and complete divisors respectively.

The number represented by t should in each case be multiplied by 10.

Ex. 2. Find the cube root of 12.326391.

		Given Number	Cube Root
$\sqrt[3]{t^3}$;	$\sqrt[3]{12} = 2 +$	$\overset{\frown}{12}.\ \overset{\frown}{326}\ \overset{\frown}{391}$	2.31
t^3;	$2^3 =$	8	
$3t^2$;	$3(2 \cdot 10)^2 = 1200.$	4 326	
$4326 \div 1200 = 3 +$			
$3tu$;	$3(2 \cdot 10)3 = 180$		
u^2;	$3^2 = 9$		
$3t^2 + 3tu + u^2$;	1389		
$(3t^2 + 3tu + u^2)u$;	$1389 \times 3 =$	4 167	
$3t^2$;	$3(23 \cdot 10)^2 = 158700.$	159 391	
$159391 \div 159700 = 1 +$			
$3tu$;	$3(23 \cdot 10)1 = 690$		
u^2;	$1^2 = 1$		
$3t^2 + 3tu + u^2$;	159391		
$(3t^2 + 3tu + u^2)u$;	$159391 \times 1 =$	159 391	

Ex. 3. Find the cube root of 204336469.

When carrying out the work we find that after multiplying the first complete divisor 8764 by the figure of the root last found 8, and subtracting the product 70112 from the first "remainder" 79336, a new "remainder" 9224 is obtained which is greater than the complete divisor 8764. However the complete divisor constructed by using 9 as a root figure instead of 8 is not contained in the corresponding "remainder" 9 times.

The student should carry out the process.

43. The examples of §§ 40, 42 illustrate the following rule:

Separate the figures of the given number into groups of three figures each, beginning at units' place.

Find the greatest cube which is not greater than the number represented by the figures in the first group at the left, and write its cube root as the ***first figure of the required root.***

Subtract the cube of the root figure thus found from the first group at the left, and to the remainder annex the next group of figures to obtain a ***"first remainder."***

Divide this "remainder" by a ***trial divisor*** *which is obtained by taking three times the square of the part of the root already found, considered as representing tens, and write the integral part of the quotient as the next root figure.*

Construct a ***complete divisor*** *by adding to the trial divisor three times the product of the part of the root previously found, considered as representing tens, multiplied by the root figure last found, and add also the square of the root figure last found.*

Multiply this complete divisor by the root figure last found and subtract the product from the remainder last found to obtain a ***new "remainder"****.*

Repeat these steps with successive groups of figures either until all of the groups of figures have been used or until as many figures of the root have been obtained as are desired.

Formula $\sqrt[3]{t^3+3t^2u+3tu^2+u^3} \equiv \sqrt[3]{t^3+(3t^2+3tu+u^2)u} \equiv t+u.$

44. Whenever the product obtained by multiplying the complete divisor by the figure of the root last found is greater than the remainder from which it is to be subtracted, it is necessary to choose a less number for the last root figure and construct a new complete divisor.

EXERCISE XVIII. 7

Find the cube root of each of the following numbers :

1. 32768.
2. 42875.
3. 68921.
4. 21952.
5. 140608.
6. 1061208.
7. 28652616.
8. 94818816.
9. 569722789.
10. 967361669.
11. 448399762.264.
12. 1.003003001.

To four decimal places :

13. 10.
14. 34903.588968101.
15. 4.
16. $\frac{4}{7}$.
17. $\frac{3}{8}$.

CHAPTER XIX

THEORY OF EXPONENTS

EXTENSION OF THE MEANING OF EXPONENT

1. IN a previous chapter we have defined an exponent as a positive whole number, and it has been proved that, when m and n are positive integers, operations with numbers affected by exponents are governed by the following **Index Laws:**

I. **Distribution Formulas** *Bases Equal*

$$\begin{cases} a^m \times a^n \equiv a^{m+n}. & \\ a^m \div a^n \equiv a^{m-n}. & m > n. \\ a^m \div a^n \equiv 1 \div a^{n-m}. & n>m. \end{cases}$$

II. **Association Formula** $(a^m)^n \equiv a^{m\cdot n} \equiv (a^n)^m.$

III. **Distribution Formulas** *Exponents Equal*

$$\begin{cases} (a \times b)^m \equiv a^m b^m. \\ (a \div b)^m \equiv a^m \div b^m. \end{cases}$$

2. These laws were established for positive integral exponents only.

It may be shown by repeated applications of the formulas above that these fundamental laws may be extended to apply to three or more factors. We have the following general formulas :

(i.) $a^m \times a^n \times a^p \times a^q \times \cdots \equiv a^{m+n+p+q+\cdots\cdots}.$

(ii.) $((((a^m)^n)^p)^q)^r \cdots\cdots \equiv a^{mnpqr\cdots\cdots}.$

(iii.) $(abcd \cdots\cdots)^m \equiv a^m \times b^m \times c^m \times d^m \times \cdots\cdots.$

3. According to definitions already given, no meaning can be attached to such an expression as $a^{\frac{1}{3}}$, for it is absurd to speak of taking a as a factor one-third of a time.

Similarly, since we cannot use b as a factor "minus two times," we have excluded from our calculations such expressions as b^{-2}, x^{-n}, etc.

4. When applying the Fundamental Index Laws we shall find that in certain cases exponents are obtained which are not positive whole numbers. Hence, in order that these Fundamental Index Laws may hold without exception in all cases, it is necessary to remove the restriction that an exponent must be a positive whole number. We must accordingly investigate the meanings which must be attached to numbers other than integers when used as exponents.

5. It is essential that all exponents, without exception, should obey the same fundamental laws; hence we shall impose the restriction that, no matter what may be the nature of the number n, *a^n must have such a meaning that in all cases it obeys all of the Fundamental Laws of Algebra*, and in particular the

Fundamental Index Law $a^m \times a^n \equiv a^{m+n}$.

It will be found that as a consequence the remaining laws of indices will be obeyed. (See § 1.)

Interpretation of Zero and Unity as Exponents

6. Whenever in the operation of the index law $a^m \div a^n \equiv a^{m-n}$, the exponents m and n are equal, we obtain zero as an exponent.

E. g.

$$a^5 \div a^5 \equiv a^{5-5} \equiv a^0. \qquad b^p \div b^p \equiv b^{p-p} \equiv b^0.$$

7. Since the fundamental law $a^m \times a^n \equiv a^{m+n}$ is to hold, whatever may be the nature of the exponents involved, we have, taking m equal to zero,

$$a^0 \times a^n \equiv a^{0+n} \equiv a^n.$$

Hence, dividing both members by a^n, we have

$$a^0 \times a^n \div a^n \equiv a^n \div a^n$$

Therefore, $\mathbf{a^0 \equiv 1}$.

Accordingly, when a has any finite value whatever different from zero, a^0 is defined as meaning 1.

E. g. $3^0 = 1$, $5^0 = 1$, $x^0 = 1$, $(a + b)^0 = 1$, etc.

8. If $m - n = 1$, we obtain unity as an exponent by applying the index law, $a^m \div a^n \equiv a^{m-n}$.

E. g. $a^7 \div a^6 \equiv a^1$.

9. Since, in the illustration above, the quotient is the base, it may be seen that the use of unity as an exponent does not contradict our previous definitions. Hence we define the first power of a base to be the base.

That is, $$a^1 \equiv a.$$

A Negative Integer as an Exponent

10. Negative integral exponents arise when we attempt to divide a given power of a base by a higher power of the same base.

E. g. $$3^4 \div 3^6 = 3^{4-6} = 3^{-2}.$$

11. Since negative exponents are to be subject to the Fundamental Index Law $a^m \times a^n \equiv a^{m+n}$, we have, if $n = -m$,

$$a^m \times a^{-m} \equiv a^0 \equiv 1.$$

Dividing both members by a^m, $a \neq 0$,

$$a^{-m} \equiv \frac{1}{a^m}.$$

That is, *we define any base, a, with a negative exponent, $-m$, as being equal to the reciprocal of the base with a positive exponent of which the absolute value is equal to that of the given exponent.*

12. From the identity $a^{-m} \equiv \frac{1}{a^m}$ it appears that a negative exponent, $-m$, indicates that the reciprocal $1/a$ of the base a is to be used as a factor m times.

Ex. 1. $2^{-4} = \frac{1}{2^4} = \frac{1}{16}$. Ex. 2. $5^{-2} = \frac{1}{5^2} = \frac{1}{25}$.

Ex. 3. $$\frac{a^{-3}}{a^4} \equiv \frac{1}{a^3 \cdot a^4} \equiv \frac{1}{a^7}.$$

Ex. 4. $$(\tfrac{4}{5})^{-2} = \frac{1}{(\frac{4}{5})^2} = (\tfrac{5}{4})^2 = \tfrac{25}{16}.$$

13. A negative integral power of zero is defined to be an infinite number, for $$0^{-p} \equiv \frac{1}{0^p} \equiv \frac{1}{0} \equiv \infty.$$

14. Since $$a^{-n} \equiv \frac{1}{a^n}, \qquad (1)$$

it follows that
$$\frac{1}{a^{-n}} \equiv \frac{1}{\dfrac{1}{a^n}}$$

That is,
$$\frac{1}{a^{-n}} \equiv a^n. \tag{2}$$

Hence, it follows from (1) and (2) that *a factor may be transferred from the numerator to the denominator of a fraction, or from the denominator to the numerator, provided that the quality of its exponent is changed from + to − or from − to +.*

Write each of the following expressions, using positive exponents only :

Ex. 5. $\dfrac{a^{-2}b}{c} \equiv \dfrac{b}{a^2c}$.

Ex. 6. $\dfrac{2x^{-3}y}{3z^2w^{-4}} \equiv \dfrac{2yw^4}{3x^3z^2}$.

Ex. 7. $5a^{-1}b^2c^{-4}d^3 \equiv \dfrac{5b^2d^3}{ac^4}$.

MENTAL EXERCISE XIX. 1

Find the numerical value of each of the following expressions :

1. 2^{-1}.
2. 3^{-2}.
3. 5^{-2}.
4. 2^{-4}.
5. 3^{-3}.
6. -8^{-2}.
7. 3^{-4}.
8. 2^{-5}.
9. -2^{-2}.
10. -3^{-3}.
11. $-(-2)^{-2}$.
12. $-(-2)^{-3}$.
13. $-(-2)^{-5}$.
14. $\dfrac{1}{2^{-1}}$.
15. $\dfrac{1}{3^{-2}}$.
16. $\dfrac{1}{5^{-4}}$.
17. $\dfrac{1}{(-2)^{-3}}$.
18. $-\dfrac{1}{(-3)^{-2}}$.
19. $\dfrac{1}{-(-4)^{-3}}$.
20. $\dfrac{3}{2^{-1}}$.
21. $\dfrac{5}{2^{-3}}$.
22. $\dfrac{4^{-1}}{3}$.
23. $\dfrac{5^{-2}}{2}$.
24. $-\dfrac{2^{-3}}{4}$.
25. $(\frac{1}{2})^{-1}$.
26. $(\frac{1}{4})^{-3}$.
27. $(\frac{2}{3})^{-1}$.
28. $(\frac{3}{4})^{-3}$.
29. $(\frac{4}{5})^{-2}$.
30. $-(-\frac{3}{5})^{-4}$.
31. $.2^{-1}$.
32. $.5^{-1}$.
33. $.1^{-1}$.
34. $.1^{-2}$.
35. $.01^{-1}$.
36. $.2^{-2}$.
37. 2^0.
38. 5^0.
39. $\dfrac{7^0}{3}$.
40. $\dfrac{4}{3^0}$.
41. $\dfrac{8^0}{5^0}$.

42. 3×8^0.
43. $3 + 8^0$.
44. $5^0 + 9^0$.
45. $10^0 - 6^0$.
46. $2^0 - 8^0$.
47. $(\frac{2}{3})^0$.
48. $\frac{1}{4^0}$.
49. $\frac{2^0}{3^0 + 4^0}$.
50. $(a + b)^0$.
51. $x^0 - (x - y)^0$.

Write each of the following expressions with positive exponents :

52. x^{-2}.
53. y^{-3}.
54. $-ab^{-1}$.
55. $c^{-2}d$.
56. $a^{-3}b^2$.
57. $-x^4y^{-5}$.
58. $m^{-2}n^{-2}$.
59. $-x^{-3}y^{-4}$.
60. $dx^{-4}y$.
61. $3a^{-1}$.
62. $2^{-1}b$.
63. $3^{-2}x^{-2}$.
64. $5x^{-2}y$.
65. $-7mx^{-3}$.
66. $9a^{-4}b^{-5}$.
67. $4x^{-1}yz^{-2}$.
68. $6^{-2}a^3b^{-2}$.
69. $2a^{-3}b^{-2}c^{-1}$.
70. $5^{-2}a^{-3}b^{-4}$.
71. $\frac{1}{a^{-2}}$.
72. $\frac{2}{h^{-3}}$.
73. $\frac{4x}{c^{-1}}$.
74. $\frac{6a}{b^{-1}c^{-1}}$.
75. $\frac{a^{-1}b}{c}$
76. $\frac{dx^{-2}}{y}$.
77. $-\frac{m^{-1}x}{n^{-2}}$.
78. $\frac{ky^{-2}}{z^{-3}}$.
79. $\frac{b^{-2}}{cd^{-2}}$.
80. $\frac{a^{-1}b^{-2}}{c}$.
81. $\frac{-xy}{z^{-3}}$.
82. $-\frac{x^{-2}}{y^{-2}}$.
83. $\frac{xy^{-3}}{z^{-1}w}$.
84. $\frac{a^{-2}b}{cd^{-3}}$.
85. $\frac{xy^{-4}}{zw^{-2}}$.
86. $\frac{b^{-1}cd^{-1}}{3}$.
87. $\frac{a^{-2}b^{-3}}{c^{-3}d^{-4}}$.
88. $\frac{2x^{-3}y^4}{z^{-5}w^6}$.
89. $\frac{4m^{-2}b^{-2}}{5na^{-4}}$.
90. $\frac{2}{(a + b)^{-2}}$.
91. $\frac{(x + y)^{-2}}{(a + b)^3}$.
92. $\frac{(m - n)^2}{(x - y)^{-2}}$.
93. $\frac{3(a + b)^{-4}}{4(x - y)^{-2}}$.
94. $\frac{6(c + d)^{-1}}{7(a - b)^{-2}}$.
95. $-(x - y)^{-1}$.
96. $a^{-1} + b^{-1}$.
97. $x^{-2} + x^{-1}$.
98. $m^{-3} - n^{-3}$.
99. $x^2 + 2 + x^{-2}$.
100. $a^{-2}+2a^{-1}b^{-1}+b^{-2}$.

In each of the following expressions transfer the factors from the denominator to the numerator:

101. $\frac{x}{y^2}$.

102. $\frac{2}{z}$.

103. $\frac{a^2}{b^3}$.

104. $\frac{c}{d^{-2}}$.

105. $\frac{a^{-2}}{b}$.

106. $\frac{1}{x^{-2}}$.

107. $-\frac{1}{y^2}$.

108. $-\frac{3}{a^3}$.

109. $\frac{3xy}{z}$.

110. $-\frac{1}{a^{-2}}$.

111. $\frac{5}{a^{-2}b^3}$.

112. $\frac{6}{x^3y^{-4}}$.

113. $\frac{8}{a^{-3}b^{-2}}$.

114. $\frac{a^2b^2}{c^2}$.

115. $\frac{xyz}{w}$.

116. $\frac{ace}{b^{-1}d^{-1}}$.

117. $\frac{1}{a^{-2}b^{-3}c}$.

118. $-\frac{1}{x^2y^2z^2}$.

119. $\frac{1}{x+y}$.

120. $\frac{(a+b)(x-y)}{(a-b)(x+y)}$.

121. $\frac{11(a+b)^{-3}}{3^{-2}(x-y)^{-2}}$.

A Positive Fraction as an Exponent

15. Positive fractions as exponents arise from an attempt to find the rth root of a^p when r and p are integers and p is not an exact multiple of r.

$$\sqrt[r]{a^p} \equiv a^{\frac{p}{r}}.$$

$$\sqrt[3]{5^2} = 5^{\frac{2}{3}}.$$

Since it is absurd to speak of taking 5 as a factor two-thirds of a time, it becomes necessary to find a meaning for an exponent which is a fraction.

16. We may obtain an interpretation for a positive fractional exponent n/d in which the numerator n and the denominator d represent any finite positive integers and d is different from zero.

We may assume that the denominator d is positive and that the numerator n has the sign of the fraction which may be either positive or negative.

If a be real and positive, then since fractional exponents are to obey the Fundamental Index Law $a^m \times a^n \equiv a^{m+n}$, we must have

$$\left(a^{\frac{n}{d}}\right)^d \equiv \overbrace{a^{\frac{n}{d}} \times a^{\frac{n}{d}} \times a^{\frac{n}{d}} \times \cdots\cdots \times a^{\frac{n}{d}}}^{d \text{ factors in all}}$$

$$\equiv a^{\overbrace{\frac{n}{d}+\frac{n}{d}+\frac{n}{d}+\cdots\cdots+\frac{n}{d}}^{d \text{ terms in all}}}$$

$$\equiv a^{\frac{n}{d}\times d}$$

$$\equiv a^n$$

Hence, $\left(a^{\frac{n}{d}}\right)^d \equiv a^n.$

That is, restricting the roots to principal values, the dth power of $a^{\frac{n}{d}}$ is equal to the nth power of a.

Or, $a^{\frac{n}{d}}$ is the principal dth root of a^n.

This may be expressed by means of the radical notation $\sqrt[d]{a^n}$.

Hence, $a^{\frac{n}{d}} \equiv \sqrt[d]{a^n}.$

In order that $a^{\frac{n}{d}}$ shall, without exception, represent the same value as $\sqrt[d]{a^n}$, it is necessary that the restrictions relating to roots expressed by the radical notation hold for roots expressed by means of the fractional notation $a^{\frac{n}{d}}$. (See Chapter XVIII. § 13.)

In particular, we shall understand when $a^{\frac{n}{n}}$ is a term of an algebraic expression that $a^{\frac{n}{n}} \equiv |a|$ unless the contrary is expressly stated.

If the signs of operation, + and —, are to be applied without exception to rational numbers and to irrational numbers which are indicated by the notation of fractional exponents, it is necessary that $a^{\frac{n}{d}}$ should be understood as representing the principal value of the root unless it is known that a is the nth power of a number which is not positive. In this case the root taken will be a number which is not positive.

E. g. If x for some reason is regarded as representing the square of a negative number, then $\sqrt{x}$, represented by $x^{\frac{1}{2}}$, must be a negative number.

In particular, $[(-3)^2]^{\frac{1}{2}} = -3.$

We shall consider that a root of a monomial expressed by the notation of fractional exponents which is not a term of an algebraic expression may, according to circumstances, be either positive or negative.

That is, $$49^{\frac{1}{2}} = \pm 7.$$

17. It should be observed that the denominator of a fractional exponent is the index of the root which must be taken, and that the numerator is the power to which the base must be raised.

In particular, if $n = 1$, $a^{\frac{1}{d}} \equiv \sqrt[d]{a}$.

18. Interpreting $a^{\frac{n}{d}}$ as meaning the principal dth root of a^n, we have by Chap. XVIII. § 22 (iv.),

$$a^{\frac{n}{d}} \equiv \sqrt[d]{a^n} \equiv (\sqrt[d]{a})^n.$$

Ex. 1. $$8^{\frac{2}{3}} = \sqrt[3]{8^2} = 4.$$

It is often better to obtain first the indicated root of the base and then the required power of this result, rather than first to raise the base to a power and then obtain the indicated root of this result.

Ex. 2. $$216^{\frac{2}{3}} = (\sqrt[3]{216})^2 = 6^2 = 36.$$

19. To agree with the interpretation for a negative exponent, we shall define $a^{-\frac{n}{d}}$ to be the reciprocal of $a^{\frac{n}{d}}$.

That is, $$a^{-\frac{n}{d}} \equiv \frac{1}{a^{\frac{n}{d}}}.$$

20. It should be understood that the expressions "fractional power," "negative power," etc., refer to the exponent of the power and not to the value of the power itself.

E. g. The one-third power of 8 is the whole number 2.

That is, $$8^{\frac{1}{3}} = 2.$$

Also the "minus second power" of 3 is the fraction 1/9.

That is, $$3^{-2} = \tfrac{1}{9}.$$

21. *Both terms of a fractional exponent may be multiplied by or divided by the same number (except zero) without altering the value of a given expression.*

That is, $$a^{\frac{n}{d}} \equiv a^{\frac{pn}{pd}}.$$

Let x represent the value of $a^{\frac{n}{d}}$, the base a being positive.

Then $$x \equiv a^{\frac{n}{d}}.$$

Raising both members to the dth power, we have,

$$x^d \equiv a^n.$$

Raising both members to the pth power, we have

$$x^{pd} \equiv a^{pn}.$$

Extracting the pdth root of both members,

$$x \equiv a^{\frac{pn}{pd}}.$$

Hence, $$a^{\frac{n}{d}} \equiv a^{\frac{pn}{pd}}.$$

22. It should be understood that because the fractional exponent n/d is equal in value to the fractional exponent pn/pd, it does not follow as a consequence that the dth root of the nth power of a given base is equal to the pdth root of the pnth power of the same base.

The demonstration of § 21 depends upon the principles that *like powers of equal numbers are equal*, and *like roots of equal numbers are also equal.* It does not depend upon the principle that both terms of a fraction may be multiplied by or divided by the same number (except zero) without altering the value of the fraction.

That is, it does not follow that $a^{\frac{n}{d}} \equiv a^{\frac{pn}{pd}}$ because $n/d \equiv pn/pd$.

The laws governing operations with and upon fractions which are factors of the terms of an expression must be shown to apply to fractions which are used as exponents before they can be applied to transform fractional exponents.

23. It may be seen that if the restriction be removed that principal roots only be taken, the principle of § 21 does not always hold.

E. g. $9^{\frac{4}{2}}$ has a value which is different from 9^2 if any other than the principal values of roots be taken.

For $9^{\frac{4}{2}} \equiv (9^4)^{\frac{1}{2}} = (6561)^{\frac{1}{2}} = \pm 81$, while $9^2 = +81$.

If only principal roots be taken it may be seen that $9^{\frac{4}{2}}$ has the single value $+81$, which is equal to the value of 9^2.

To find the value of $9^{\frac{4}{2}}$ we would commonly proceed as follows:

$$9^{\frac{4}{2}} = 9^2 = 81.$$

From the illustrations above it may be seen that, for principal values only of roots, $9^{\frac{4}{2}} = 9^2$.

That is, if a fractional exponent is not in lowest terms, by taking only the principal values of roots we shall obtain the same result as by reducing the fractional exponent to lowest terms and then proceeding with the transformed expression.

24. Whenever a fractional exponent, n/d, is negative, we shall understand that d is positive and n is negative.

E. g. The expression $9^{-\frac{1}{2}}$ will be understood as meaning

$$9^{-\frac{1}{2}} = 9^{\frac{-1}{2}} = (9^{-1})^{\frac{1}{2}} = (\tfrac{1}{9})^{\frac{1}{2}} = \pm \tfrac{1}{3}\cdot$$

Mental Exercise XIX. 2

Find the value of each of the following expressions :

1. $9^{\frac{1}{2}}$.
2. $8^{\frac{1}{3}}$.
3. $16^{\frac{1}{4}}$.
4. $4^{\frac{3}{2}}$.
5. $8^{\frac{2}{3}}$.
6. $25^{\frac{3}{2}}$.
7. $49^{-\frac{1}{2}}$.
8. $64^{-\frac{1}{3}}$.
9. $81^{\frac{3}{4}}$.
10. $100^{-\frac{5}{2}}$.
11. $125^{-\frac{2}{3}}$.
12. $144^{-\frac{1}{2}}$.
13. $64^{-\frac{5}{6}}$.
14. $121^{\frac{3}{2}}$.
15. $169^{-\frac{3}{2}}$.
16. $125^{-\frac{4}{3}}$.
17. $(\frac{4}{9})^{\frac{1}{2}}$.
18. $(\frac{25}{36})^{-\frac{1}{2}}$.
19. $(\frac{81}{16})^{\frac{1}{4}}$.
20. $(\frac{8}{125})^{-\frac{1}{3}}$.
21. $(\frac{27}{64})^{\frac{2}{3}}$.
22. $.25^{-\frac{1}{2}}$.
23. $.343^{-\frac{2}{3}}$.
24. $.008^{-\frac{4}{3}}$.

Express by the radical notation the following relations which are expressed by the notation of fractional exponents :

25. $a^{\frac{1}{2}}$.
26. $b^{\frac{1}{3}}$.
27. $c^{\frac{3}{4}}$.
28. $d^{\frac{1}{4}}$.
29. $x^{-\frac{1}{2}}$.
30. $y^{-\frac{1}{3}}$.
31. $e^{\frac{2}{5}}$.
32. $x^{\frac{3}{2}}$.
33. $y^{\frac{5}{3}}$.
34. $z^{\frac{9}{7}}$.
35. $a^{\frac{m}{2}}$.
36. $b^{\frac{3}{n}}$.
37. $c^{\frac{m}{n}}$.
38. $a^{\frac{b}{a}}$.
39. $b^{\frac{c}{a}}$.
40. $c^{\frac{a}{d}}$.
41. $d^{\frac{5}{a}}$.
42. $z^{\frac{6}{5r}}$.
43. $y^{-\frac{5}{2n}}$.
44. $z^{\frac{2n}{9}}$.
45. $w^{\frac{3m}{2n}}$.
46. $3^{\frac{m}{s}}$.
47. $2^{\frac{2}{r}}$.
48. $x^{\frac{n}{n+1}}$.

49. $y^{-\frac{r}{r+1}}$.
50. $z^{\frac{m-1}{m+1}}$.
51. $a^{\frac{1}{2}}b^{\frac{1}{2}}$.
52. $m^{\frac{1}{2}}n^{\frac{1}{3}}$.
53. $b^{\frac{3}{4}}c^{\frac{4}{5}}$.
54. $x^{\frac{1}{m}}y^{\frac{1}{n}}$.
55. $a^{\frac{2}{n}}x^{\frac{n}{2}}$.
56. $a^{\frac{1}{b}}b^{\frac{1}{a}}$.
57. $-c^{\frac{1}{r}}d^{-\frac{1}{s}}$.
58. $x^{\frac{y}{z}}y^{\frac{z}{x}}$.
59. $b^{-\frac{2}{3}}y^{-\frac{3}{2}}$.
60. $-a^{-\frac{1}{b}}b^{-\frac{1}{c}}$.
61. $x^{\frac{x}{y}}y^{\frac{y}{z}}z^{\frac{z}{x}}$.
62. $a^{\frac{1}{2}}b^{-\frac{1}{3}}c^{-\frac{1}{4}}$.
63. $a^{\frac{b}{c}}b^{\frac{c}{a}}c^{\frac{a}{b}}$.
64. $3x^{\frac{1}{2}}$.
65. $5y^{\frac{1}{3}}$.
66. $4z^{-\frac{1}{4}}$.
67. $7w^{\frac{2}{5}}$.
68. $-8a^{\frac{1}{n}}$.
69. $2c^{\frac{c}{2}}$.
70. $10m^{\frac{1}{10}}$.
71. $12a^{\frac{1}{3}}b^{\frac{3}{4}}$.
72. $-6a^2b^{\frac{1}{2}}$.
73. $7c^{\frac{1}{3}}d^2$.
74. $11ab^{\frac{2}{9}}$.
75. $ab^{\frac{a}{b}}$.
76. $(ab^2)^{\frac{1}{3}}$.
77. $(cd^3)^{\frac{1}{5}}$.
78. $(x^2y^4)^{\frac{1}{7}}$.
79. $(z^2w^4)^{\frac{1}{5}}$.
80. $2^{\frac{n+2}{2}}$.
81. $1 \div a^{\frac{1}{2}}$.
82. $2 \div b^{\frac{1}{3}}$.
83. $3 \div 4c^{\frac{1}{5}}$.
84. $-6 \div 11d^{\frac{1}{7}}$.
85. $x \div y^{\frac{1}{4}}$.
86. $x^{\frac{1}{y}} \div z$.
87. $(a+b)^{\frac{1}{2}}$.
88. $(c-d)^{\frac{1}{3}}$.
89. $(x+y)^{\frac{3}{4}}$.
90. $(z-w)^{\frac{5}{6}}$.
91. $(m-n)^{-\frac{1}{mn}}$.
92. $(a^2-b^2)^{\frac{1}{2}}$.
93. $(a^3+b^3)^{\frac{1}{3}}$.
94. $(c^2-4)^{\frac{1}{4}}$.
95. $(b+c)^{\frac{d}{a}}$.
96. $(a+b)^{\frac{1}{a+b}}$.

Express by the notation of positive fractional exponents the roots which in the following expressions are indicated by the radical notation :

97. $\sqrt{a}$.
98. $\sqrt[3]{b}$.
99. $\sqrt[4]{c}$.
100. $\sqrt[5]{x}$.
101. $\sqrt[5]{d^2}$.
102. $-\sqrt[4]{x^3}$.
103. $\sqrt{-x^3}$.
104. $\sqrt[n]{y}$.
105. $\sqrt[n]{n}$.
106. $\sqrt[n]{z^{n-1}}$.
107. $3\sqrt{x}$.
108. $2\sqrt[3]{y}$.
109. $4\sqrt[3]{z^2}$.
110. $-5\sqrt[5]{y^2}$.
111. $7a\sqrt[5]{bc}$.
112. $6\sqrt[7]{x^5y^3}$.
113. $\sqrt{1 \div a}$.
114. $-\sqrt{1 \div c}$.
115 $\sqrt[3]{1 \div b^2}$.
116. $-\sqrt[3]{1 \div d^2}$.
117. $\sqrt[3]{2 \div c^n}$.
118. $\sqrt[m]{1 \div m}$.
119. $\sqrt{a^{-1}}$.
120. $-\sqrt{x^{-1}}$.
121. $\sqrt[3]{b^{-2}}$.
122. $\sqrt[5]{c^{-3}}$.
123. $\sqrt[n]{x^{-m}}$.
124. $\sqrt[3]{1 \div x^{-2}}$.
125. $5\sqrt[3]{ab^2c}$.
126. $m\sqrt[4]{m^2n^3w}$.
127. $\sqrt{\frac{a}{b}}$.
128. $\sqrt{\frac{c^2}{d}}$.
129. $\left(\sqrt{\frac{a}{b}}\right)^b$.
130. $-\sqrt[5]{ab^3c^4}$.
131. $\sqrt[n]{a^{n-1}b}$.
132. $2\sqrt[a]{a^2}$.
133. $-a\sqrt[b]{b^a}$.
134. $\sqrt{\sqrt{m}}$.
135. $\sqrt[3]{\sqrt{n}}$.
136. $\sqrt[3]{a\sqrt{b}}$.
137. $-a\sqrt{b\sqrt{c}}$.
138. $x^3\sqrt[3]{y^2\sqrt{z}}$.
139. $\sqrt[3]{-x^2y\sqrt[3]{z^2}}$.
140. $2\sqrt[3]{3\sqrt[4]{5^n}}$.

Products of Powers and Quotients of Powers.

25. The laws relating to products of powers and quotients of powers of the same base may be applied without exception when exponents are zero, negative numbers, or fractions.

This is because exponents which are zero, negative numbers, or fractions, were so defined as to obey the Fundamental Index Law $a^m \times a^n \equiv a^{m+n}$, and accordingly the derived law $a^m \div a^n \equiv a^{m-n}$.

Ex. 1. Simplify $9^{\frac{3}{2}} \times 27^{\frac{2}{3}}$.

We have
$$\begin{aligned} 9^{\frac{3}{2}} \times 27^{\frac{2}{3}} &= (9^{\frac{1}{2}})^3 \times (27^{\frac{1}{3}})^2 \\ &= 3^3 \times 3^2 \\ &= 3^5 \\ &= 243. \end{aligned}$$

Ex. 2. Simplify $a^5 \times a^{-8}$.

We have
$$\begin{aligned} a^5 \times a^{-8} &\equiv a^{5-8} \\ &\equiv a^{-3} \\ &\equiv \frac{1}{a^3}. \end{aligned}$$

Ex. 3. Simplify $b^{-\frac{1}{2}} \times b^{\frac{2}{3}}$.

We have
$$\begin{aligned} b^{-\frac{1}{2}} \times b^{\frac{2}{3}} &\equiv b^{-\frac{1}{2}+\frac{2}{3}} \\ &\equiv b^{\frac{1}{6}}. \end{aligned}$$

Ex. 4. Simplify $8c^{\frac{4}{5}} \div 2c^{\frac{1}{2}}$.

We have
$$\begin{aligned} 8c^{\frac{4}{5}} \div 2c^{\frac{1}{2}} &\equiv 4c^{\frac{4}{5}-\frac{1}{2}} \\ &\equiv 4c^{\frac{3}{10}}. \end{aligned}$$

Exercise XIX. 3

Simplify the following products and quotients, applying the

$$\text{Index Laws}: \begin{cases} a^m \times a^n \equiv a^{m+n}. \\ a^m \div a^n \equiv a^{m-n}. \end{cases}$$

1. $4^{\frac{1}{2}} \times 8^{\frac{4}{3}}$.
2. $25^{\frac{3}{2}} \times 125^{\frac{2}{3}}$.
3. $64^{\frac{5}{6}} \times 16^{\frac{5}{4}}$.
4. $4^{\frac{1}{2}} \times 2^{-2}$.
5. $3 \times 9^{-\frac{1}{2}}$.
6. $25^{-\frac{1}{2}} \times 5^{-2}$.
7. $32^{\frac{2}{5}} \div 4^2$.
8. $8^{-\frac{1}{3}} \times 2^{-1}$.
9. a^5a^{-3}.
10. $b^{-6}b^{-2}$.
11. c^8c^{-8}.
12. $z \cdot z^{-\frac{1}{3}}$.

13. $(5\,d^3)(4d^0)$.

14. $x^4x^{-7}x^{-5}$.

15. $\frac{4}{5}\,y^{\frac{1}{2}} \times \frac{5}{4}\,y^{\frac{1}{3}}$.

16. $8w^{-\frac{3}{4}} \times 2\,w^{-\frac{1}{4}}$.

17. $m^{\frac{1}{2}}m^{\frac{1}{3}}m^{\frac{1}{4}}$.

18. $n^an^{-b}n^c$.

19. $(x^{a-b})(x^{b-c})(x^{c-a})$.

20. $(y^az^b)(y^{-b}z^{-a})$.

21. $x^9 \div x^{-2}$.

22. $y^7 \div y^{-10}$.

23. $z^{-4} \div z^{-6}$.

24. $r^{\frac{1}{2}} \div r^{\frac{1}{3}}$.

25. $4\,s^{\frac{1}{6}} \div 2\,s^{\frac{2}{3}}$.

26. $d^{-\frac{2}{5}} \div d^{-\frac{7}{5}}$.

27. $g^{n+2} \div 3\,g^2$.

28. $\frac{2}{3}\,h^{p-q} \div h^{q-p}$.

29. $-\,12a^{\frac{5}{3}}b^2 \div 3\,ab^{\frac{3}{2}}$.

30. $a^{\frac{2}{3}}b^{\frac{3}{2}} \div a^{\frac{3}{2}}b^{\frac{2}{3}}$.

31. $16\,n^4y^{\frac{1}{3}} \div 8\,n^{\frac{1}{4}}y^3$.

32. $a^{\frac{2}{3}}b^{\frac{3}{4}}c^{\frac{4}{5}} \div a^{\frac{5}{4}}b^{\frac{4}{3}}c^{\frac{3}{2}}$.

33. $\dfrac{-\,a^{-4}}{-\,a^{-5}}$.

34. $\dfrac{(-\,a)^3}{(-\,a)^2}$.

35. $\dfrac{3\,b^{-8}}{(-\,3\,b)^3}$.

36. $\dfrac{15\,a^{-4}}{-\,3\,a^{-9}}$.

37. $\dfrac{-\,20\,c^{-7}n^5z^{-3}}{5\,dn^4z^{-4}}$.

38. $\dfrac{-\,18\,m^{-4}x^3}{-\,2\,m^{-4}y^{-3}}$.

39. $\dfrac{x^{\frac{a}{2}}y^{\frac{b}{3}}}{x^{\frac{2}{a}}y^{\frac{3}{b}}}$.

40. $x^{\frac{m}{n}}x^{\frac{n}{m}}x^{-\frac{1}{mn}}$.

41. $x^{\frac{m+n}{2}}x^{\frac{m-n}{2}}$.

42. $(hy^az^b)\,(-\,ky^{-\frac{9}{a}}z^{-\frac{4}{b}})$.

43. $(b^{\frac{1}{x-y}}c^{\frac{x}{y}})(b^{\frac{1}{x+y}}c^{-\frac{y}{x}})$.

26. Since exponents which are zero, negative numbers, or fractions, have been so defined as to obey the Fundamental Index Law $a^m \times a^n \equiv a^{m+n}$, and accordingly the derived law $a^m \div a^n \equiv a^{m-n}$, it remains for us to show that with the meanings thus obtained these exponents obey all of the laws relating to positive integral exponents.

Unless the contrary is explicitly stated we shall understand the word "exponent" to have any of the meanings previously defined.

27. From the Index Law $a^m \times a^n \equiv a^{m+n}$, I, we may derive the law $a^m \div a^n \equiv a^{m-n}$, and from the law $(ab)^n \equiv a^n\,b^n$ we may derive the law $\left(\dfrac{a}{b}\right)^n \equiv \dfrac{a^n}{b^n}$.

Hence it is necessary and sufficient for us to show that the index laws $(a^m)^n \equiv a^{mn}$, II, and $(ab)^n \equiv a^n b^n$, III, apply for all commensurable exponents.

Powers of Powers.

28.* Proof that the index law $(a^m)^n \equiv a^{mn}$ applies for all commensurable exponents.

(i.) If n be a positive integer, then for all values of m, we have,

$$\begin{aligned}(a^m)^n &\equiv \overbrace{a^m \times a^m \times a^m \times \cdots\cdots \times a^m}^{n \text{ factors}}\\ &\equiv a^{mn}.\end{aligned}$$

(ii.) Let n be a positive fraction, p/r, in which p and r are positive integers.

Then $\quad (a^m)^n \equiv (a^m)^{\frac{p}{r}}$

Raising $(a^m)^{\frac{p}{r}}$ to the rth power, we have

$$\begin{aligned}[(a^m)^{\frac{p}{r}}]^r &\equiv (a^m)^{\frac{p}{r}\times r}\\ &\equiv (a^m)^p.\end{aligned}$$

Since p is a positive integer it follows that

$$(a^m)^p \equiv a^{mp}.$$

Writing the principal rth roots of both members, it follows that

$$(a^m)^{\frac{p}{r}} \equiv a^{\frac{mp}{r}}.$$

Hence, substituting for p/r its value n, we have

$$(a^m)^n \equiv a^{mn}.$$

(iii.) Let n be any negative number denoted by $-q$.

Then

$$\begin{aligned}(a^m)^n &\equiv (a^m)^{-q}\\ &\equiv \frac{1}{(a^m)^q}\\ &\equiv a^{-mq}.\end{aligned}$$

Hence, substituting for $-q$ its value n, we have,

$$(a^m)^n \equiv a^{mn}.$$

(iv.) Let n be zero.

Then $\quad (a^m)^n \equiv a^{mn}$ is satisfied if $a \neq 0$ and $n = 0$.

For, $\quad (a^m)^0 \equiv a^{m0}$, means $1 \equiv 1$.

---

* This section may be omitted when the chapter is read for the first time.

It follows from the reasoning of (i.), (ii.), (iii.) and (iv.) that for all commensurable values of the exponents we have

$$(a^m)^n \equiv a^{mn} \equiv (a^n)^m.$$

29. Substituting $1/r$ for m, and p for n, it may be seen that from $(a^m)^n \equiv (a^n)^m$ it follows that $(a^{\frac{1}{r}})^p \equiv (a^p)^{\frac{1}{r}}$.

That is, *the pth power of the rth root of any base, a, which is different from zero, is equal to the rth root of the pth power of the base.*

30. It should be observed that if the restriction regarding principal roots is removed, the relation $(a^{\frac{1}{r}})^p \equiv (a^p)^{\frac{1}{r}}$ does not always hold.

E. g. $(9^{\frac{1}{2}})^2 = 9$, but $(9^2)^{\frac{1}{2}} = \pm 9$.

Hence, $(9^{\frac{1}{2}})^2 \neq (9^2)^{\frac{1}{2}}$ except for principal values of the roots.

Ex. 1. Simplify $(16^{\frac{1}{4}})^{-8}$.

We have
$$(16^{\frac{1}{4}})^{-8} = (2)^{-8} = \frac{1}{2^8} = \frac{1}{8}.$$

Mental Exercise XIX. 4

Find simplified expressions for the following indicated powers of powers, applying the Index Law $(a^m)^n \equiv a^{mn}$:

1. $(c^2)^{-8}$.
2. $(d^{-4})^2$.
3. $(a^{-2})^{-4}$.
4. $(b^{-5})^{-8}$.
5. $(x^{-3})^0$.
6. $(y^0)^{-2}$.
7. $(z^0)^0$.
8. $(w^{\frac{1}{2}})^0$.
9. $(g^{10})^{\frac{1}{2}}$.
10. $(h^{\frac{1}{3}})^{12}$.
11. $(k^{-6})^{\frac{1}{3}}$.
12. $-(m^8)^{\frac{1}{4}}$.
13. $(n^{-\frac{2}{3}})^{\frac{1}{2}}$.
14. $-(t^{\frac{3}{4}})^{\frac{1}{3}}$.
15. $(x^{\frac{4}{5}})^{\frac{3}{2}}$.
16. $(y^{-\frac{1}{2}})^{-4}$.
17. $(z^{-\frac{2}{5}})^{-5}$.
18. $-(w^{\frac{1}{2}})^{\frac{1}{3}}$.
19. $-(m^{-1})^{-1}$.
20. $-(m^{-2a})^3$.
21. $(h^{2r})^2$.
22. $(k^{3n})^2$.
23. $(d^{-5x})^2$.
24. $(n^{4r})^{-5}$.
25. $-(q^{2n})^{-4}$.
26. $(x^{3a})^{-6}$.
27. $(a^2)^b$.
28. $-(b^3)^c$.
29. $(c^4)^{-d}$.
30. $(d^5)^{-x}$.

31. $(h^{-2})^y$.

32. $(k^{-3})^{-z}$.

33. $-(m^2)^{3a}$.

34. $(n^3)^{2b}$.

35. $(x^{-1})^{-3x}$.

36. $(a^b)^c$.

37. $-(b^x)^y$.

38. $(c^m)^{-y}$.

39. $(d^{-r})^x$.

40. $(h^{-m})^{-n}$.

41. $(k^a)^{\frac{1}{b}}$.

42. $(m^{\frac{1}{n}})^m$.

43. $(n^{\frac{1}{x}})^{\frac{1}{y}}$.

44. $(x^a)^{-\frac{1}{n}}$.

45. $(y^{-m})^{\frac{1}{r}}$.

46. $(z^{-\frac{1}{a}})^{\frac{1}{b}}$.

47. $(w^{-\frac{1}{m}})^{\frac{1}{n}}$.

48. $-(a^{-\frac{b}{2}})^4$.

49. $(b^{3x})^{-\frac{1}{6}}$.

50. $(a^b)^b$.

51. $(b^n)^n$.

52. $(c^{-m})^m$.

53. $(d^r)^{-r}$.

54. $(x^{-a})^{-a}$.

55. $(y^{2x})^x$.

56. $-(z^a)^{3a}$.

57. $(w^{4n})^{-n}$.

58. $(a^{2x})^{2x}$.

59. $(b^{-3x})^{3x}$.

60. $(c^{-4n})^{-4n}$.

61. $((x^a)^b)^c$.

62. $((y^m)^{-n})^{-r}$.

63. $((z^{\frac{1}{2}})^{\frac{3}{a}})^{\frac{b}{4}}$.

64. $((a^m)^{-\frac{1}{n}})^{-n}$

65. $(a^{m+1})^2$.

66. $(b^{n-2})^3$.

67. $(c^{r+3})^4$.

68. $(d^2)^{n+1}$.

69. $(h^3)^{r-1}$.

70. $(k^4)^{z+2}$.

71. $(m^{r+1})^2$.

72. $(n^{a-1})^3$.

73. $(x^{a+b})^{\frac{1}{2}}$.

74. $(y^{c-d})^{\frac{1}{3}}$.

75. $(z^{2n+1})^3$.

76. $(w^{3r-2})^4$.

77. $(m^{5a-1})^{-1}$.

78. $(a^{4x+2})^{\frac{1}{2}}$.

79. $(b^{6x-9})^{\frac{1}{3}}$.

80. $(d^{\frac{1}{5}})^{5n-10}$.

81. $(h^{\frac{1}{3}})^{6r+3}$.

82. $(k^{a+2})^{a-2}$.

83. $(a^{m+n})^{m-n}$.

84. $(x^{a+1})^{a+2}$.

85. $(y^{b+3})^{b+5}$.

86. $(z^{m-2})^{m+3}$.

87. $(w^{n+5})^{n-2}$.

88. $(b^{c+d})^{\frac{1}{a+b}}$.

89. $(c^{a^2-b^2})^{\frac{1}{a-b}}$.

90. $(d^{\frac{1}{x+y}})^{(x^2-y^2)^2}$.

Powers of Products and Powers of Quotients.

31.* Proof that for all commensurable values of the exponents we may apply the Index Law $(ab)^n \equiv a^n b^n$.

(i.) Let n be a positive fraction.

The law has been shown to apply for all positive integral values of the exponents.

---

* This section may be omitted when the chapter is read for the first time.

Substituting p/r for n we may write

$$(ab)^n \equiv (ab)^{\frac{p}{r}}.$$

Writing the rth power of $(ab)^{\frac{p}{r}}$, we have

$$[(ab)^{\frac{p}{r}}]^r \equiv (ab)^p.$$
$$\equiv a^pb^p, \text{ since } p \text{ is a positive integer.}$$

Furthermore, $$(a^{\frac{p}{r}}b^{\frac{p}{r}})^r \equiv (a^{\frac{p}{r}})^r(b^{\frac{p}{r}})^r \equiv a^pb^p.$$

Hence, $$[(ab)^{\frac{p}{r}}]^r \equiv (a^{\frac{p}{r}}b^{\frac{p}{r}})^r.$$

Writing the principal rth roots, we have

$$(ab)^{\frac{p}{r}} \equiv a^{\frac{p}{r}}b^{\frac{p}{r}}.$$

Or, $(ab)^n \equiv a^nb^n$, when n is a positive fraction.

(ii.) Let n have any negative value represented by $-q$.

Then
$$(ab)^n \equiv (ab)^{-q}$$
$$\equiv \frac{1}{(ab)^q}$$
$$\equiv \frac{1}{a^qb^q}$$
$$\equiv a^{-q}b^{-q}.$$

Or, $$(ab)^n \equiv a^nb^n.$$

(iii.) Let n have the value zero.

Then $(ab)^0 \equiv a^0b^0$, because $(ab)^0$ means 1, and a^0b^0 means 1×1 for bases a and b, which are different from zero.

It follows from the reasoning of (i.), (ii.), and (iii.) that for all commensurable values of the exponent we have

$$(ab)^n \equiv a^nb^n.$$

Ex. 1. Simplify $(a^{\frac{1}{2}}b^{-\frac{1}{3}})^{12}$.

We have
$$(a^{\frac{1}{2}}b^{-\frac{1}{3}})^{12} \equiv a^6b^{-4}$$
$$\equiv \frac{a^6}{b^4}.$$

Ex. 2. Simplify $\left(\frac{x^{-2}}{y^3}\right)^{-4}$.

We have
$$\left(\frac{x^{-2}}{y^3}\right)^{-4} \equiv \frac{x^8}{y^{-12}}$$
$$\equiv x^8y^{12}.$$

MENTAL EXERCISE XIX. 5

Simplify each of the following expressions, applying either the index law $(ab)^n \equiv a^n b^n$, or the index law $(a \div b)^n \equiv a^n \div b^n$:

1. $(a^{\frac{1}{2}}b^3)^4$.
2. $(b^5c^{-2})^3$.
3. $(c^{-1}d^4)^{-2}$.
4. $(d^{-2}h^{-3})^5$.
5. $(m^5n^{-4})^{-1}$.
6. $(x^{-2}y^{-6})^{-2}$.
7. $-(a^{\frac{1}{2}}x^2)^4$.
8. $-(b^3y^{\frac{1}{3}})^6$.
9. $-(c^{-\frac{1}{3}}z^4)^{\frac{1}{2}}$.
10. $(d^5w^{-\frac{1}{5}})^{-10}$.
11. $a^{-\frac{1}{3}}n^{-\frac{1}{4}})^{\frac{1}{5}}$.
12. $(b^{\frac{1}{6}}x^{-\frac{1}{4}})^{-8}$.
13. $(a^{\frac{1}{3}}b^{\frac{1}{2}})^6$.
14. $(b^{\frac{1}{4}}c^{\frac{1}{6}})^{12}$.
15. $(c^{-2}d^{-\frac{1}{4}})^{-\frac{1}{2}}$.
16. $(k^{-\frac{2}{3}}z^{\frac{3}{2}})^{-\frac{1}{6}}$.
17. $(a^{-2} \div b^3)^2$.
18. $(b^5 \div c^4)^{-1}$.
19. $(c^{-2} \div d^3)^{-2}$.
20. $(a^{-8} \div x^4)^{-3}$.
21. $(b^{-1} \div c^{-2})^3$.
22. $(c^{-2} \div y^{-1})^{-4}$.
23. $(a^{-1} \div b^{-1})^{-1}$.
24. $(a^{\frac{1}{4}} \div b^{\frac{1}{6}})^2$.
25. $-(b^{\frac{1}{3}} \div x^{\frac{1}{2}})^6$.
26. $(c^{\frac{1}{4}} \div y^{\frac{1}{2}})^{-4}$.
27. $(d^{-\frac{1}{3}} \div q)^{-8}$.
28. $(h \div m^{-2})^{-\frac{1}{2}}$.
29. $(a^{2b} \div x^{-4})^{\frac{1}{2}}$.
30. $(b^{-3c} \div d^{\frac{1}{3}})^{\frac{1}{3}}$.
31. $(2a)^{-2}$.
32. $(4b^{-1})^3$.
33. $(2c^{-2})^3$.
34. $(3d^3)^{-2}$.
35. $(4x)^{\frac{1}{2}}$.
36. $-(9y^2)^{\frac{1}{2}}$.
37. $(16z^2)^{-\frac{1}{2}}$.
38. $(25z^{-4})^{-\frac{1}{2}}$.
39. $(36^{-1}ab^{-2})^{-\frac{1}{2}}$.
40. $(9^{-3}m^{-4}n^{-6})^{-\frac{1}{2}}$.
41. $(a^2b^{-3}c^4)^{-2}$.
42. $(b^{\frac{1}{2}}c^3d^{-\frac{1}{3}})^{-6}$.
43. $(a^{-2}b^{\frac{1}{2}}c^{-4})^{-2}$.
44. $(b^6c^{-4}y^{-\frac{1}{2}})^{-\frac{1}{2}}$.
45. $(d^{-6}x^{-3}y^{-1})^{-\frac{1}{3}}$.
46. $(2a^{-8}b^{-1}c)^{-2}$.
47. $(3^{-1}x^3y^{-2}z)^{-2}$.
48. $(4^{-\frac{1}{3}}a^3b^{-2}c^{-8})^{-6}$.

32. It is customary to regard a fractional power $a^{\frac{p}{q}}$ as being higher or lower than another fractional power $a^{\frac{r}{s}}$, according as $\frac{p}{q} - \frac{r}{s}$ is a positive or a negative number.

E. g. We shall consider that $x^{\frac{2}{3}}$ is a higher power than $x^{\frac{1}{2}}$ since $\frac{2}{3} - \frac{1}{2} = \frac{1}{6}$, which is a positive number.

But $x^{\frac{2}{3}}$ is a lower power than x since $\frac{2}{3} - 1 = -\frac{1}{3}$, which is a negative number.

EXERCISE XIX. 6

The following examples illustrate the application of the principles of earlier chapters to expressions whose terms involve negative and fractional exponents.

Multiply:

1. $a^{\frac{3}{4}} + b^{\frac{5}{6}}$ by $a^{\frac{1}{2}} + b^{\frac{2}{3}}$.
2. $x^{\frac{1}{2}} - y^{\frac{1}{3}}$ by $x^{\frac{1}{3}} - y^{\frac{1}{2}}$.
3. $x^{\frac{1}{3}} - y^{\frac{3}{2}}$ by $x^{\frac{2}{3}} - y^{-\frac{1}{2}}$.
4. $x^{-\frac{1}{3}} + y^{-\frac{1}{2}}$ by $x^{\frac{4}{3}} - y^{\frac{2}{3}}$.
5. $a^{-\frac{1}{6}} - b^{\frac{4}{5}}$ by $a^{-\frac{5}{6}} - b^{\frac{1}{5}}$.
6. $b^{-\frac{1}{4}} + c^{\frac{3}{2}}$ by $b^{\frac{5}{4}} + c^{-\frac{1}{2}}$.
7. $a^{\frac{4}{5}} + x^{\frac{1}{3}}$ by $a^{\frac{1}{5}} + x^{\frac{2}{3}}$.
8. $x^{-3} - 3x^{-2} + 1$ by $x^{-4} + 2x^{-1}$.
9. $2x^{\frac{3}{4}} - 3x^{\frac{1}{4}} - 4 + x^{-\frac{1}{4}}$ by $3x^{\frac{5}{4}} + x - 2x^{\frac{3}{4}}$.
10. $a^{-3}b^{-2} + 2a^{-1}b^{2} - 3a^{2}b^{-4}$ by $a^{2}b^{-3} + 3a^{4}b$.

Divide:

11. $x^{-5} - 3x^{-3} + 2x^{-2}$ by $x^{-2} - 2x^{-1} + 1$.
12. $a^{\frac{5}{4}} - a^{\frac{1}{2}}b^{\frac{5}{6}} - a^{\frac{3}{4}}b^{\frac{2}{3}} + b^{\frac{3}{2}}$ by $a^{\frac{1}{2}} - b^{\frac{2}{3}}$.
13. $x - y$ by $x^{\frac{1}{2}} - y^{\frac{1}{2}}$.
14. $x^{-6} - 3x^{-4}y^{-2} + 3x^{-2}y^{-4} - y^{-6}$ by $x^{-2} - y^{-2}$.
15. $a + b$ by $a^{\frac{1}{3}} + b^{\frac{1}{3}}$.
16. $x^{-1}y - 5xy^{-1} + 4x^{3}y^{-3}$ by $x^{2}y^{-1} + x^{3}y^{-2} - 2x^{4}y^{-3}$.
17. $a - b$ by $a^{\frac{1}{4}} - b^{\frac{1}{4}}$.
18. $2a^{-\frac{8}{3}}b^{3} - 4a^{-\frac{4}{3}}b^{\frac{3}{2}} + 2$ by $2a^{-2}b^{\frac{9}{4}} - 4a^{-\frac{4}{3}}b^{\frac{3}{2}} + 2a^{-\frac{2}{3}}b^{\frac{3}{4}}$.

Exercise XIX. 7. **Miscellaneous**

Simplify each of the following expressions:

1. $4^{\frac{1}{2}} \times 16^{-\frac{3}{4}} \times 64^{\frac{1}{3}}$.
2. $\dfrac{16^{\frac{1}{2}} \times 125^{-\frac{1}{3}}}{25^{\frac{1}{2}} \times 32^{\frac{2}{5}}}$.
3. $9^{0} + 9^{\frac{1}{2}} - 9^{-\frac{1}{2}} + 9^{\frac{3}{2}}$.
4. $32^{0} + 32^{\frac{1}{5}} + 32^{\frac{2}{5}} + 32^{\frac{3}{5}} + 32^{\frac{4}{5}} + 32$.
5. $2(a^{-6})^{-4} - (a^{-12})^{-2} - (a^{-8})^{-3}$.
6. $\dfrac{2^{n+4} - 2 \cdot 2^{n}}{2 \cdot 2^{n+3}}$.
7. $\dfrac{4(2^{n-1})^{n} \cdot 2^{-1}}{(2^{n+1})^{n-1} \cdot 2^{-n}}$.

8. $2^{\frac{2}{3}} \cdot 6^{\frac{1}{3}} - 9^{-\frac{2}{3}} \cdot 3^{\frac{5}{3}} - \dfrac{576^{\frac{1}{6}}}{2}$.

9. $\left(\dfrac{1}{a+b}\right)^{-2}$.

10. $(x \div x^{\frac{1}{a}})^{a}$.

11. $\left(\dfrac{1}{1 \div a^{\frac{5}{4}}}\right)^{\frac{4}{5}}$.

12. $\left(\dfrac{a^0 + b^0}{2}\right)^2$.

13. $\dfrac{a^{-1} - b^{-1}}{a^{-1}b^{-1}}$.

14. $[(x^a)^{a-\frac{1}{a}}]^{\frac{1}{a-1}}$.

15. $(x^{\frac{a}{a-1}})^{a^2-1} \div \dfrac{(x^{\frac{n}{2}})^2}{(x^2)^{\frac{1}{2}}}$

16. $\left(\dfrac{a^{\frac{1}{x}}}{b^{-\frac{1}{y}}}\right)^{xy} \div \left(\dfrac{a^{\frac{1}{x-y}}}{b^{\frac{1}{x+y}}}\right)^{x^2-y^2}$.

17. $\dfrac{3^n \times (3^{n-1})^n}{3^{n+1} \times 3^{n-1} \times 9^{-n}}$.

18. $(((a^{\frac{2m}{n}})^{-\frac{b}{c}})^{\frac{n}{m}})^{\frac{c}{b}} \div (((a^{-\frac{n}{m}})^{\frac{c}{b}})^{-\frac{m}{n}})^{\frac{b}{c}}$.

19. $\dfrac{1 - a^{-2}b^2}{a^{-1} - a^{-2}b} \times \dfrac{a^{-1}b^{-1}}{a^{-1} + b^{-1}}$.

20. $\left(\dfrac{a^{-1} - b^{-1}}{a^{-1}b - ab^{-1}}\right)\left(\dfrac{b-a}{ab^{-1}}\right) \div \left(\dfrac{a^{-1}b^{-1}}{1 + ab^{-1}}\right)$.

21. $\left(\dfrac{x^a}{y^b z^c}\right)^{\frac{1}{a+c}} \div \left(\dfrac{x^{a-1}}{z^{c+1}y^{b+1}}\right)^{\frac{1}{abc}}$.

22. $[n^{\frac{a+b}{b-c}}]^{\frac{1}{c-a}} \times [n^{\frac{b+c}{c-a}}]^{\frac{1}{a-b}} \times [n^{\frac{c+a}{a-b}}]^{\frac{1}{b-c}}$.

CHAPTER XX

IRRATIONAL NUMBERS AND THE ARITHMETIC THEORY OF SURDS

I. Irrational Numbers

1. Two numbers are said to be **commensurable,** that is, to have a common measure, if both can be divided without remainder by the same integral or fractional number.

E. g. The numbers 9 and 12 are commensurable because both contain 3 exactly as a divisor.

Any two whole numbers, any two fractions whose numerators and denominators consist of a limited or finite number of figures, or any whole number and any fraction, are commensurable numbers.

E. g. The two fractions $\frac{11}{19}$ and $\frac{3}{2}$ are commensurable, since they can be expressed as integral multiples of a fraction having as a denominator the common denominator of the two fractions; that is, $\frac{11}{19}$ can be expressed as 22 times $\frac{1}{38}$, and $\frac{3}{2}$ as 57 times $\frac{1}{38}$.

2. Two numbers which are not commensurable with reference to each other are said to be **incommensurable.**

3. Since any whole number or any fraction is commensurable with respect to unity, whole numbers and fractions are commonly called commensurable numbers. (Compare with Chap. XVIII. § 8.)

4. Since any whole number can be expressed as a fraction it follows that *any number is commensurable if it can be expressed as the quotient of one whole number divided by another whole number.*

E. g. The number 5 is commensurable because 5 can be expressed as a fraction, such as $\frac{10}{2}$.

The square root of 5 is an incommensurable number because it cannot be expressed exactly by any fraction.

5. In order that the rth root of a commensurable number c may be expressed as a commensurable number n, it is necessary and sufficient that c be the rth power of some commensurable number n.

For if $$\sqrt[r]{c} = n, \text{ then } c = n^r.$$

E. g. Since 9 is the square of the commensurable number 3, it follows that $\sqrt{9}$ is a commensurable number. The number 10 is not the square of a commensurable number. Accordingly, the square root of 10 is an incommensurable number.

6. The classification of numbers as being either commensurable or incommensurable may be made to depend upon the following **Principles Relating to Fractions:**

(i.) *The rth power of an integer is an integer.*

For, since the operation of division does not enter into the process of involution, a fraction cannot result from using an integer any number of times as a factor.

(ii.) *The rth power of a fraction in lowest terms is a fraction in lowest terms.* (See Chap. XV. § 40.)

It follows from the reasoning of (i.) and (ii.) that:

(a) *The rth root of a positive integral number which is not the rth power of another integer cannot be expressed either as an integer or a fraction, that is, as a commensurable number.*

E. g. Since 2 is not the square of any whole number, it follows that $\sqrt{2}$ cannot be expressed either as a whole number or as a fraction.

(b) *The rth root of a positive fraction in lowest terms of which the numerator or denominator, or both, are not rth powers of positive integers, cannot be expressed either as an integer or a fraction, that is, as a commensurable number.*

E. g. Since 2 and 5 are prime to each other, and neither is the square of a whole number, it follows that $\sqrt{\frac{2}{5}}$ cannot be expressed as a commensurable number.

7. It is not possible to express the value of the rth root of a number which is not an rth power exactly as an integer or as a fraction in lowest terms. Still it may be shown that it is possible to find as **approximate values** two fractions, one greater than and the other less than the true value of the indicated root, which shall differ from each other, and consequently differ from the true value of the required root, by as small a value as we please.

(The following illustration may be omitted when the chapter is read for the first time.)

8. Different methods may be applied to obtain approximate values for a particular incommensurable number, such as $\sqrt{2}$.

Since 2 is not the square of any integer, its square root cannot be expressed exactly either as an integer or as a fraction. (See (i.) and (ii.), § 6.)

Since 2 is greater than 1^2 and less than 2^2, it follows that $\sqrt{2}$ must be greater than 1 and less than 2.

That is, $$1^2 = 1 < 2 < 4 = 2^2.$$

Hence, $\sqrt{2}$ lies between 1 and 2.

Or, $$1 < \sqrt{2} < 2.$$

The following are the only fractions having 10 for a common denominator, the values of which lie between 1 and 2:

$$\tfrac{11}{10},\ \tfrac{12}{10},\ \tfrac{13}{10},\ \tfrac{14}{10},\ \tfrac{15}{10},\ \tfrac{16}{10},\ \tfrac{17}{10},\ \tfrac{18}{10},\ \tfrac{19}{10}.$$

Writing the squares of these fractions, we obtain,

$$\tfrac{121}{100},\ \tfrac{144}{100},\ \tfrac{169}{100},\ \tfrac{196}{100},\ \tfrac{225}{100},\ \tfrac{256}{100},\ \tfrac{289}{100},\ \tfrac{324}{100},\ \tfrac{361}{100}.$$

We find that $(\tfrac{14}{10})^2 = \tfrac{196}{100} < 2 < \tfrac{225}{100} = (\tfrac{15}{10})^2$.

Hence, since 2 lies between $(1.4)^2$ and $(1.5)^2$, it follows that

$$1.4 < \sqrt{2} < 1.5.$$

The following are the only fractions having 100 for a common denominator, the values of which lie between $\tfrac{14}{10}$ and $\tfrac{15}{10}$:

$$\tfrac{141}{100},\ \tfrac{142}{100},\ \tfrac{143}{100},\ \tfrac{144}{100},\ \tfrac{145}{100},\ \tfrac{146}{100},\ \tfrac{147}{100},\ \tfrac{148}{100},\ \tfrac{149}{100}.$$

We find that the square of $\tfrac{141}{100}$ is less than, and the square of $\tfrac{142}{100}$ is greater than 2.

That is, $(\tfrac{141}{100})^2 = \tfrac{19881}{10000} < 2 < \tfrac{20164}{10000} = (\tfrac{142}{100})^2$.

Hence, since 2 lies between $(1.41)^2$ and $(1.42)^2$, it follows that

$$1.41 < \sqrt{2} < 1.42.$$

Continuing the process indefinitely, it is possible to find two fractions, one greater than and the other less than the true value of $\sqrt{2}$, which differ from each other, and accordingly from the true value of $\sqrt{2}$ by a value less than any assignable number, however small.

Consider the following relations :

$$\begin{array}{llll}
1^2 & = 1 & < 2 < 4 & = 2^2 \\
1.4^2 & = 1.96 & < 2 < 2.25 & = 1.5^2 \\
1.41^2 & = 1.9881 & < 2 < 2.0164 & = 1.42^2 \\
1.414^2 & = 1.999396 & < 2 < 2.002225 & = 1.415^2 \\
1.4142^2 & = 1.99996164 & < 2 < 2.00024449 & = 1.4143^2 \\
1.41421^2 & = 1.9999899241 & < 2 < 2.0000182084 & = 1.41422^2 \\
\text{etc.} & & & \text{etc.}
\end{array}$$

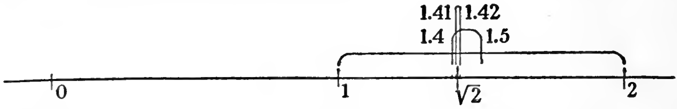
Since the true value of $\sqrt{2}$ lies between any two corresponding numbers of which the squares are indicated in the first and last columns above, it follows that the difference between the true value of $\sqrt{2}$ and either of the numbers must be less than the difference between the two numbers themselves.

Any number of which the square is indicated in either column may be taken as an approximation to the true value of $\sqrt{2}$.

The numbers of which the squares are indicated in the first column are all less than, and those of which the squares are inindicated in the last column are all greater than, the true value of $\sqrt{2}$.

Hence, by extending the process indefinitely, as close an approximation to $\sqrt{2}$ as may be required can be obtained.

9. If we represent the different numbers, 1, 1.4, 1.41, etc., and 2, 1.5, 1.42, etc., which are approximate values of $\sqrt{2}$, by distances from a fixed point 0 measured along a straight line, we may suggest as in the accompanying figure that the successive pairs of values, 1 and 2, 1.4 and 1.5, 1.41 and 1.42, etc., approach nearer and nearer to the true value of $\sqrt{2}$.



10. By the reasoning employed in §§ 8, 9, any indicated root of an incommensurable number can be obtained.

It should be understood that, although it may not be possible to express the true value of a specified root exactly in terms of either an integer or a fraction, yet this value is definite.

11. In order that there may be no exception to operations involving indicated roots, we shall assume that, even when the radicand c is not the rth power of a commensurable number, the following identity is true:

$$(\sqrt[r]{c})^r \equiv c.$$

E. g. $(\sqrt{3})^2 = 3.$ $(\sqrt{\tfrac{5}{11}})^2 = \tfrac{5}{11}\cdot$

12. Two irrational numbers are said to be equal or unequal according as their approximate values are equal or unequal.

13. By means of the Principles of Variables and Limits it may be shown that the Fundamental Laws of Algebra apply for incommensurable as well as for commensurable numbers.

II. ARITHMETIC THEORY OF SURDS

14. A **rational number** has been defined as a number which can be expressed as the quotient obtained by dividing one whole number by another. (See Chapter XVIII, § 8.)

E. g. The number 4 which can be expressed as a fraction such as $\frac{8}{2}$ is a rational number.

Any fraction such as $\frac{2}{3}$ is a rational number.

A number which is not rational is said to be **irrational.**

E. g. Since $\sqrt{2}$ cannot be expressed exactly as a fraction, it follows that $\sqrt{2}$ is an irrational number.

15. An indicated root of a number or expression is frequently spoken of as a **radical.**

E. g. $\sqrt{2}$, $\sqrt{16}$, $\sqrt[3]{5}$, $\sqrt{x+y}$.

16. A **radical expression** is an expression containing one or more radicals.

E. g. $3\sqrt{5}$, $\sqrt{b}$, $\sqrt{x}+1$, $\sqrt{a-\sqrt{b}}$.

17. A **surd** is an irrational or incommensurable root of a rational or commensurable number.

E. g. $\sqrt{2}$, $\sqrt[3]{4}$, $\sqrt{\tfrac{5}{11}}$ are surd numbers, for in each case the radicand is a rational number which cannot be obtained by raising another rational number to a power the exponent of which is equal to the index of the indicated root.

Observe that $\sqrt{9}$ is not a surd, since $\sqrt{9}=3$.

Expressions such as $\sqrt{\sqrt{5}-\sqrt{2}}$, $\sqrt{\sqrt{3}+1}$, are not surds in the sense of the definition, since the radicands $\sqrt{5}-\sqrt{2}$ and $\sqrt{3}+1$ are incommensurable numbers.

18. Although, from the point of view of algebra, $\sqrt{n}$ is an irrational function, yet it may or may not be arithmetically irrational according to the particular values assigned to n.

E. g. If $n=4, 9, 16, \cdots\cdots$, $\sqrt{n}$ is not a surd;
while if $n=2, 3, 5, 7, \cdots\cdots$, $\sqrt{n}$ is a surd.

19. According as the index of the indicated root is 2, 3, 4, 5, $\cdots\cdots$, n, surds are classified as being of the second order or *quadratic*, as $\sqrt{3}$; of the third order or *cubic*, as $\sqrt[3]{5}$; of the fourth order or *biquadratic*, as $\sqrt[4]{10}$; of the fifth order or *quintic*, as $\sqrt[5]{2}$; of the nth order or *n-tic*, as $\sqrt[n]{x}$; etc.

20. A single surd number or any rational multiple of a single surd number is called a **simple monomial surd number.**

E. g. $\sqrt{5}, 3\sqrt{2}, 2\sqrt{x}$.

21. The sum of two simple monomial surd numbers, or of a simple monomial surd number and a rational number, is called a **simple binomial surd number.**

E. g. $\sqrt{a}+\sqrt{b}, \quad \sqrt[3]{2}+1$.

The expression $\sqrt{x+y}$ is not a simple binomial surd number, but is a simple monomial surd number of which the radicand is a binomial.

22. The principles established in the chapter on Evolution apply for irrational as well as for rational roots.

We shall, in the present chapter, consider such radicands only as are positive, and shall use only the principal values of the indicated roots.

REDUCTION OF SURDS TO SIMPLEST FORM

23. A surd is said to be in **simplest form** when the expression under the radical sign is integral, and is not a power of a rational expression the exponent of which is equal to the index or which contains as a factor any factor of the index of the indicated root; and when there appears under the radical sign no factor raised to a power of which the exponent is equal to or greater than the index of the indicated root.

E. g. The following surds are all in simplest form:

$$\sqrt{5},\quad \sqrt{a},\quad \sqrt[3]{a^2},\quad \sqrt[4]{(a+b)^3}.$$

The following surds are not in simplest form:

$$\sqrt{\tfrac{2}{3}},\quad \sqrt[6]{8},\quad \sqrt[8]{x^6},\quad \sqrt{18}.$$

For, in $\sqrt{\frac{2}{3}}$ the radicand $\frac{2}{3}$ is not integral.

In $\sqrt[6]{8}$ the radicand 8 is a power, $8 = 2^3$, of which the exponent 3 is a factor of the index 6 of the radical. We have $\sqrt[6]{8} = \sqrt{2}$.

The radicand x^6 of $\sqrt[8]{x^6}$ may be expressed as a power, $(x^3)^2$, of which the exponent 2 is a factor of the index 8 of the radical, and we have $\sqrt[8]{x^6} \equiv \sqrt[4]{x^3}$.

The radicand of the surd $\sqrt{18}$ contains as a factor 9, the square root of which can be extracted.

We have $\sqrt{18} = \sqrt{9 \cdot 2} = 3\sqrt{2}$. (See Chap. XVIII. § 20.)

I. The Radicand an Integer.

24. When the radicand is an integer, the reduction of a surd to simplest form depends upon the following principles:

$$\sqrt[r]{a^{rn}} \equiv a^n,$$

$$\sqrt[r]{ab} \equiv \sqrt[r]{a}\sqrt[r]{b}. \quad \text{(See Chap. XVIII. § 20.)}$$

Hence the following:

Separate the expression under the radical sign into two groups of factors, one group containing all of the factors the exponents of which are equal to or multiples of the index of the required root, and the second group containing all factors of lower degree.

Extract the indicated root of the first group of factors and multiply the result by the coefficient of the given surd, if there be one, and write the product as the coefficient of the indicated root of the remaining second group of factors.

Ex. 1. Reduce $\sqrt{a^2b^5c}$ to simplest form.

We may separate the radicand into the two groups of factors, a^2b^4 and bc. The first group consists of all of the powers of highest degree of which the exponents are exactly divisible by the index 2 of the indicated root, and the second group contains all of the remaining factors of degree lower than the second.

Hence,
$$\begin{aligned}\sqrt{a^2b^5c} &\equiv \sqrt{(a^2b^4)(bc)}\\ &\equiv \sqrt{a^2b^4}\,\sqrt{bc}\\ &\equiv ab^2\sqrt{bc}.\end{aligned}$$

In practice we may omit writing the step in the second line above. It is given here simply to show that the expression in the third line results from the expression in the first line by applying the law

$$\sqrt[r]{a^rb} \equiv \sqrt[r]{a^r}\,\sqrt[r]{b}.$$

Ex. 2. Simplify $5\sqrt{12}$.

We have, $5\sqrt{12} = 5\sqrt{4 \cdot 3} = 5\sqrt{4}\sqrt{3} = 10\sqrt{3}$.

It is sometimes convenient to write the prime factors of the expression under the radical sign before attempting to express it as a product of two factors or groups of factors.

Ex. 3.
$$\begin{aligned}\sqrt[3]{4320} &= \sqrt[3]{2^5 \cdot 3^3 \cdot 5}\\ &= \sqrt[3]{(2^3 \cdot 3^3)(2^2 \cdot 5)}\\ &= \sqrt[3]{(2^3 \cdot 3^3)}\,\sqrt[3]{2^2 \cdot 5}\\ &= 2 \cdot 3\sqrt[3]{2^2 \cdot 5}\\ &= 6\,\sqrt[3]{20}.\end{aligned}$$

MENTAL EXERCISE XX. 1

Reduce each of the following surds to simplest form :

1. $\sqrt{8}$.
2. $\sqrt{12}$.
3. $\sqrt{20}$.
4. $\sqrt{28}$.
5. $\sqrt{40}$.
6. $\sqrt{52}$.
7. $\sqrt{60}$.
8. $3\sqrt{24}$.
9. $5\sqrt{44}$.
10. $6\sqrt{56}$.
11. $\sqrt{18}$.
12. $\sqrt{45}$.
13. $\sqrt{54}$.
14. $\sqrt{90}$.
15. $\sqrt{99}$.
16. $2\sqrt{27}$.
17. $3\sqrt{63}$.
18. $\sqrt{32}$.
19. $\sqrt{48}$.
20. $\sqrt{80}$.
21. $2\sqrt{96}$.
22. $3\sqrt{160}$.
23. $\sqrt{75}$.
24. $\sqrt{125}$.
25. $3\sqrt{50}$.
26. $\sqrt{175}$.
27. $4\sqrt{250}$.
28. $\sqrt{72}$.
29. $\sqrt{108}$.
30. $2\sqrt{180}$.
31. $3\sqrt{98}$.
32. $5\sqrt{128}$.
33. $6\sqrt{162}$.
34. $7\sqrt{200}$.
35. $\sqrt[3]{16}$.
36. $\sqrt[3]{24}$.
37. $\sqrt[3]{32}$.
38. $6\sqrt[3]{40}$.
39. $3\sqrt[3]{48}$.
40. $5\sqrt[3]{56}$.
41. $4\sqrt[3]{72}$.
42. $7\sqrt[3]{80}$.
43. $8\sqrt[3]{88}$.
44. $\sqrt[3]{54}$.

45. $4\sqrt[3]{81}$.
46. $2\sqrt[3]{108}$.
47. $\sqrt[4]{32}$.
48. $5\sqrt[4]{80}$.
49. $3\sqrt[4]{96}$.
50 $\sqrt[5]{64}$.
51. $\sqrt[5]{96}$.
52. $3\sqrt{242}$.
53. $4\sqrt{288}$.
54. $10\sqrt{1000}$.

55. $3\sqrt[3]{250}$.
56. $5\sqrt[3]{600}$.
57. $9\sqrt[3]{128}$.
58. $6\sqrt{500}$.
59. $12\sqrt{700}$.
60. $5\sqrt{243}$.
61. $4\sqrt{343}$.
62. $2\sqrt{512}$.
63. $3\sqrt{450}$.
64. $3\sqrt[3]{375}$.

65. $\sqrt{a^3}$.
66. $\sqrt{b^5}$.
67. $\sqrt{c^7}$.
68. $a\sqrt{b^3}$.
69. $b\sqrt{c^5}$.
70. $c\sqrt{c^8}$.
71. $d\sqrt{d^5}$.
72. $\sqrt{x^2y^3}$.
73. $\sqrt{ab^3}$.
74. $\sqrt{c^3d^3}$.

75. $x\sqrt{y^2z^9}$.
76. $m\sqrt{m^2n}$.
77. $\sqrt{h^5x^4}$.
78. $\sqrt{b^7y^6}$.
79. $\sqrt{m^8n^7}$.
80. $\sqrt{a^2b^3c^4}$.
81. $\sqrt{x^4y^5z^6}$.
82. $\sqrt{a^{10}b^8c^5}$.
83. $x\sqrt{x^4yz^3}$.
84. $\sqrt{4\,a^5b}$.

85. $\sqrt{9\,c^3d^2}$.
86. $3\sqrt{16\,h^4k^3}$.
87. $5\,c\sqrt{8\,ab^2}$.
88. $3\,b\sqrt{12\,c^3d^4}$.
89. $6\,c\sqrt{24\,a^2b^3c}$.
90. $6\,x\sqrt{28\,xy^2z^8}$.
91. $3\sqrt[3]{8\,a^4b}$.
92. $5\sqrt[3]{40\,a^3b^3c^3}$.
93. $4\,a\sqrt{27\,a^4b^3c^2}$.
94. $2\,y\sqrt{56\,x^4y^5z^4}$.
95. $3\,c\sqrt[3]{48\,a^5b^3c^5}$.
96. $5\,s\sqrt[3]{81\,r^2s^3w^4}$.
97. $2\sqrt[4]{32\,g^6h^4k^5}$.
98. $3\,b\sqrt[4]{64\,a^4bc^5}$.
99. $4\,q\sqrt[4]{243\,m^4n^4q^5}$.

100. $\sqrt{(a+b)^2c}$.
101. $\sqrt{(x-y)^2z^2w}$.
102. $\sqrt{(m+n)^2mn}$.
103. $\sqrt{(b-c)^2b^3c^3}$.
104. $\sqrt{ac^2+bc^2}$.
105. $\sqrt{my^2-ny^2}$.
106. $\sqrt[m]{a^mb}$.
107. $\sqrt[n]{b^{n+1}}$.
108. $\sqrt[r-1]{c^r}$.
109. $\sqrt[n+1]{a^{n+2}}$.
110. $\sqrt[n-2]{b^{n-1}}$.
111. $\sqrt[n]{c^{2n}d}$.
112. $\sqrt[2n]{x^{2n}y^{4n}z}$.
113. $\sqrt[n-r]{a^{n-r+1}}$, $n > r$.
114. $\sqrt[r]{a^{r+1}b^{r+2}c^{r+3}}$, $r > 3$.

II. The Radicand a Fraction.

25. To reduce a surd to simplest form when a fraction appears under the radical sign:

Multiply both terms of the fractional radicand by the number or expression of lowest degree that will make the denominator a power of which the exponent is equal to or a multiple of the index of the indicated root.

Express the transformed radicand as the product of two groups of factors, one of which contains the denominator of the transformed radicand and also all factors of the numerator which have exponents which are equal to or a multiple of the index of the indicated root; the second group contains all factors of the numerator of which the exponents are not equal to or multiples of the index of the indicated root.

Extract the indicated root of the first group of factors and multiply the result by the coefficient of the given surd, if there be one, and write the product, reduced to simplest form, as the coefficient of the indicated root of the remaining second group of factors.

Ex. 1. Simplify $\sqrt{\frac{2}{3}}$.

We may obtain a fraction the denominator of which is the square of a rational number by multiplying the numerator and denominator of 2 / 3 by 3.

Hence we have, $\sqrt{\frac{2}{3}} = \sqrt{\frac{6}{9}} = \frac{\sqrt{6}}{\sqrt{9}} = \frac{\sqrt{6}}{3}$.

Ex. 2. Simplify $\sqrt{\frac{9a^2}{8b}}$.

The radicand $9a^2/8b$ may be transformed into an equivalent fraction of which the denominator is the square of a rational expression by multiplying both numerator and denominator by $2b$.

Hence we have, $\sqrt{\frac{9a^2}{8b}} \equiv \sqrt{\frac{9a^2 \cdot 2b}{16b^2}} \equiv \frac{\sqrt{9a^2}\sqrt{2b}}{\sqrt{16b^2}} \equiv \frac{3a}{4b}\sqrt{2b}$.

EXERCISE XX. 2

Reduce each of the following surds to simplest form:

1. $\sqrt{\frac{1}{2}}$.
2. $\sqrt{\frac{1}{3}}$.
3. $\sqrt{\frac{1}{5}}$.
4. $5\sqrt{\frac{1}{6}}$.
5. $4\sqrt{\frac{1}{7}}$.
6. $11\sqrt{\frac{1}{11}}$.
7. $\sqrt{\frac{1}{8}}$.
8. $\sqrt{\frac{1}{12}}$.
9. $\sqrt{\frac{1}{18}}$.
10. $30\sqrt{\frac{1}{20}}$.
11. $28\sqrt{\frac{1}{32}}$.
12. $4\sqrt{\frac{5}{8}}$.
13. $\frac{1}{2}\sqrt{\frac{1}{3}}$.
14. $\frac{1}{4}\sqrt{\frac{1}{6}}$.
15. $\frac{1}{8}\sqrt{\frac{1}{8}}$.
16. $\frac{1}{9}\sqrt{\frac{1}{27}}$.
17. $\sqrt[3]{\frac{1}{2}}$.
18. $\sqrt[3]{\frac{1}{3}}$.
19. $\sqrt[3]{\frac{1}{5}}$.
20. $\sqrt[3]{\frac{1}{6}}$.
21. $\sqrt[3]{\frac{1}{4}}$.
22. $3\sqrt[3]{\frac{1}{9}}$.
23. $5\sqrt[3]{\frac{1}{25}}$.
24. $4\sqrt[3]{\frac{1}{16}}$.
25. $\frac{1}{9}\sqrt[3]{\frac{1}{9}}$.
26. $\frac{1}{10}\sqrt[3]{\frac{1}{10}}$.
27. $12\sqrt[3]{\frac{1}{36}}$.
28. $\frac{1}{2}\sqrt[4]{\frac{1}{2}}$.

29. $9\sqrt[4]{\frac{1}{3}}$.

30. $4\sqrt[4]{\frac{1}{8}}$.

31. $8\sqrt[5]{\frac{1}{4}}$.

32. $\sqrt[5]{\frac{1}{16}}$.

33. $\sqrt[6]{\frac{7}{32}}$.

34. $\sqrt{\frac{4}{7}}$.

35. $\sqrt{\frac{9}{10}}$.

36. $\sqrt{\frac{16}{3}}$.

37. $\sqrt{\frac{25}{32}}$.

38. $\frac{1}{6}\sqrt{\frac{36}{5}}$.

39. $\sqrt[3]{\frac{8}{9}}$.

40. $\sqrt[3]{\frac{27}{16}}$.

41. $\sqrt[3]{\frac{125}{36}}$.

42. $\frac{1}{3}\sqrt[3]{\frac{27}{4}}$.

43. $\sqrt[3]{-\frac{5}{4}}$.

44. $\sqrt[5]{-\frac{32}{81}}$.

45. $-7\sqrt[3]{-\frac{1}{49}}$.

46. $\frac{1}{2}\sqrt[4]{\frac{5}{27}}$.

47. $\sqrt{\frac{1}{a}}$.

48. $a\sqrt{\frac{1}{x}}$.

49. $\sqrt{\frac{1}{c^3}}$.

50. $\sqrt{\frac{1}{d^5}}$.

51. $\sqrt{\frac{1}{h^7}}$.

52. $\sqrt[3]{\frac{1}{m}}$.

53. $b\sqrt[3]{\frac{1}{n}}$.

54. $\sqrt[3]{\frac{1}{x^2}}$.

55. $\frac{1}{x}\sqrt[3]{\frac{1}{y^7}}$.

56. $\frac{1}{z}\sqrt[3]{\frac{1}{z^4}}$.

57. $a\sqrt[4]{\frac{1}{a}}$.

58. $\frac{1}{b}\sqrt[4]{\frac{1}{b}}$.

59. $\sqrt[4]{\frac{1}{c^3}}$.

60. $\sqrt[5]{\frac{1}{d}}$.

61. $\sqrt[5]{\frac{1}{d^2}}$.

62. $\sqrt[6]{\frac{1}{h}}$.

63. $\sqrt{\frac{4a}{b}}$.

64. $\sqrt{\frac{c}{9d}}$.

65. $\sqrt{\frac{16}{25x}}$.

66. $\sqrt{\frac{a^2}{x}}$.

67. $\sqrt{\frac{b^2}{y}}$.

68. $\sqrt{\frac{c^3}{z}}$.

69. $\sqrt{\frac{d^5}{w}}$.

70. $\sqrt{\frac{a}{bc}}$.

71. $\sqrt{\frac{x^2}{yz}}$.

72. $\sqrt[3]{\frac{ab}{cd}}$.

73. $\sqrt{\frac{m}{n^2x}}$.

74. $\sqrt{\frac{a^2}{bc^2}}$.

75. $\sqrt{\frac{b^2}{c^2d}}$.

76. $\sqrt{\frac{2}{3x^2y}}$.

77. $\sqrt[3]{\frac{a}{b^2c}}$.

78. $\sqrt{\frac{1}{9a^2b}}$.

79. $\sqrt{\frac{c^3}{d^2x}}$.

80. $\sqrt{\frac{1}{a^2b}}$.

81. $cm\sqrt{\frac{1}{cm}}$.

82. $ab\sqrt{\frac{b}{a}}$.

83. $\frac{a}{b}\sqrt{\frac{c}{d}}$.

84. $\frac{x}{y}\sqrt{\frac{1}{xy}}$.

85. $\frac{a^2}{b}\sqrt{\frac{c}{a^3}}$.

86. $\frac{b}{a}\sqrt[3]{\frac{a}{b}}$.

87. $\frac{x^2}{y^2}\sqrt[3]{\frac{y}{x}}$.

88. $\sqrt[4]{\frac{1}{27a^3b^2}}$.

89. $\sqrt[4]{\frac{5}{8x^3y^4}}$.

90. $\sqrt{\frac{1}{a+b}}$.

91. $ab\sqrt{\frac{1}{a-b}}$.

92. $\frac{1}{x-y}\sqrt{\frac{1}{x+y}}$.

93. $(a^2-b^2)\sqrt{\frac{1}{a+b}}$.

94. $(x+y)\sqrt{\frac{x-y}{x+y}}$.

95. $\frac{x+y}{x-y}\sqrt{\frac{x+y}{x-y}}$.

96. $(a^2-b^2)\sqrt{\frac{a+b}{a-b}}$.

97. $(a+b)\sqrt{\frac{1}{a^2-b^2}}$.

98. $\frac{1}{y-1}\sqrt{\frac{y-1}{y+1}}$.

99. $\sqrt{\frac{a+1}{a+2}}$.

100. $\sqrt{\frac{x-3}{x-1}}$.

101. $\sqrt[n]{\frac{1}{a}}$.

102. $\sqrt[x]{\frac{1}{b^2}}$, $x > 2$.

103. $\sqrt[n+1]{\frac{1}{a^n}}$

104. $\sqrt[n]{\frac{1}{a^{n-1}}}$.

105. $\sqrt[n]{\frac{1}{x^{n-2}}}$, $n > 2$.

106. $\sqrt[m]{\frac{1}{a^n}}$, $m > n$.

107. $\sqrt[2n]{\frac{1}{x^n}}$.

108. $\sqrt[3n]{\frac{1}{a^{2n}}}$.

109. $\sqrt[a+2]{\frac{1}{x^a}}$.

110. $\sqrt{\frac{1}{x^{2n+1}}}$.

111. $\sqrt[2n]{\frac{1}{a^{n+2}}}$.

112. $\sqrt[3]{\frac{1}{z^{3n-1}}}$

113. $\sqrt[3]{\frac{1}{w^{3n+1}}}$.

114. $\sqrt[3]{\frac{1}{y^{3n+2}}}$.

115. $n\sqrt[n]{\frac{1}{n^{n-1}}}$.

116. $\sqrt[n-2]{\frac{1}{x^{n-3}}}$.

117. $x\sqrt[n]{\frac{y^n}{x^{n-1}}}$.

118. $\frac{a}{b}\sqrt[n]{\frac{b^{n+1}}{a^{n-1}}}$.

119. $\frac{1}{x}\sqrt[a]{\frac{1}{x^{a-b}}}$

120. $\sqrt[8]{\frac{1}{x^n}}$, $n < 8$

121. $\sqrt[3]{\frac{1}{x^{3a+6b+2c}}}$.

III. Reduction of a surd to an equivalent surd of lower order, when the index of the root and the exponent of the radicand have a factor in common.

26. The reduction depends upon the following

Principle: *The value of a surd remains unaltered if the index of the root and the exponent of the radicand be both multiplied by or divided by the same number.*

$$\sqrt[rn]{a^{pn}} \equiv \sqrt[r]{\sqrt[n]{a^{pn}}} \equiv \sqrt[r]{a^p}.\quad \text{(See Chap. XVII. §§ 19, 21.)}$$

Ex. 1. Reduce $\sqrt[6]{8}$ to an equivalent surd of lower order.

Expressing the radicand 8 as a power, 2^3, we have,

$$\sqrt[6]{8} = \sqrt[6]{2^3} = \sqrt{2}.$$

Ex. 2. Reduce $\sqrt[4]{25\,a^2b^6}$ to simplest form.

$$\sqrt[4]{25\,a^2b^6} \equiv \sqrt[4]{(5\,ab^3)^2} \equiv \sqrt[2]{5\,ab^3} \equiv b\sqrt{5\,ab}.$$

MENTAL EXERCISE XX. 3

Simplify each of the following:

1. $\sqrt[4]{25}$.
2. $\sqrt[4]{36}$.
3. $\sqrt[6]{4}$.
4. $\sqrt[6]{9}$.
5. $\sqrt[9]{27}$.
6. $\sqrt[8]{16}$.
7. $\sqrt[6]{a^3b^3}$.
8. $\sqrt[4]{a^2b^2}$.
9. $\sqrt[4]{49\,a^4}$.
10. $\sqrt[4]{x^2y^4}$.
11. $\sqrt[6]{125\,a^3}$.
12. $\sqrt[8]{a^2b^4c^6}$.
13. $\sqrt[10]{x^5y^5}$.
14. $\sqrt[10]{100\,a^2}$.
15. $\sqrt[14]{a^7b^{14}}$.
16. $\sqrt[10]{a^6b^8}$.
17. $\sqrt[8]{x^6y^4z^2}$.
18. $\sqrt[8]{16\,a^4x^4}$.
19. $\sqrt[12]{a^3b^6c^9}$.
20. $\sqrt[12]{x^4y^6z^8}$.
21. $\sqrt[12]{m^6n^6}$.
22. $\sqrt[2n]{a^2}$.
23. $\sqrt[3n]{b^3}$.
24. $\sqrt[4n]{c^2}$.
25. $\sqrt[3a]{x^3}$.
26. $\sqrt[5n]{y^n}$.
27. $\sqrt[ax]{y^x}$.
28. $\sqrt[nr]{z^n}$.
29. $\sqrt[an]{x^{ab}y^{ac}}$.
30. $\sqrt[2n]{x^{2r}y^{2s}}$.
31. $\sqrt[4]{\frac{1}{4}}$.
32. $\sqrt[4]{\frac{1}{9}}$.
33. $\sqrt[4]{\frac{1}{25}}$.
34. $\sqrt[4]{\frac{1}{a^2}}$.
35. $\sqrt[6]{\frac{1}{c^3}}$.
36. $\sqrt[8]{\frac{1}{d^4}}$.
37. $\sqrt[2n]{\frac{1}{a^n}}$.
38. $\sqrt[3n]{\frac{1}{b^n}}$.
39. $\sqrt[4n]{\frac{1}{c^{3n}}}$.
40. $\sqrt[6n]{\frac{1}{d^{4n}}}$.

ADDITION AND SUBTRACTION OF SURDS

27. Two surds are said to be **similar or like** if they can be expressed as rational multiples of the same monomial surd.

E. g. $\sqrt{8}$ and $\sqrt{2}$ are similar surds, for $\sqrt{8}$ may be expressed as a multiple of $\sqrt{2}$, as follows: $\sqrt{8} = 2\sqrt{2}$.

$\sqrt{27}$ and $\sqrt{48}$ are similar surds, since each may be expressed as a multiple of $\sqrt{3}$, as follows:

$$\sqrt{27} = 3\sqrt{3}, \text{ and } \sqrt{48} = 4\sqrt{3}.$$

$\sqrt{75}$ and $\sqrt{\frac{1}{3}}$ are similar surds, since $\sqrt{75} = 5\sqrt{3}$, and $\sqrt{\frac{1}{3}} = \frac{1}{3}\sqrt{3}$.

Two surds are said to be **dissimilar or unlike** if they cannot be expressed as rational multiples of the same monomial surd.

E. g. $\sqrt{2}$ and $\sqrt{3}$ are dissimilar surds.

28. *To add or subtract like surds, first reduce them to simplest form and then find the algebraic sum or difference of the coefficients as a coefficient of the common surd factor.*

Ex. 1. Find the sum of $\sqrt{12}$, $\sqrt{3}$ and $18\sqrt{1/3}$.

Reducing the surds to simplest form, we have,

$$\sqrt{12}+\sqrt{3}+18\sqrt{\frac{1}{3}}=2\sqrt{3}+\sqrt{3}+6\sqrt{3}$$
$$=9\sqrt{3}.$$

29. Unlike surds cannot be expressed as a single surd by addition or subtraction.

Ex. 2. Simplify $2\sqrt{48}+5\sqrt{20}-3\sqrt{1/3}+\sqrt{45}$.

$$2\sqrt{48}+5\sqrt{20}-3\sqrt{\frac{1}{3}}+\sqrt{45}=8\sqrt{3}+10\sqrt{5}-\sqrt{3}+3\sqrt{5}$$
$$=7\sqrt{3}+13\sqrt{5}.$$

Exercise XX. 4

Simplify each of the following :

1. $\sqrt{3}+4\sqrt{3}$.
2. $2\sqrt{5}+3\sqrt{5}$.
3. $4\sqrt{13}-7\sqrt{13}$.
4. $\sqrt{3}+\sqrt{12}$.
5. $\sqrt{2}+\sqrt{18}$.
6. $2\sqrt{6}+\sqrt{24}$.
7. $4\sqrt{3}+2\sqrt{48}$.
8. $\sqrt{45}-2\sqrt{5}$.
9. $2\sqrt{72}-5\sqrt{2}$.
10. $\sqrt{50}+\sqrt{8}$.
11. $3\sqrt{80}-2\sqrt{20}$.
12. $5\sqrt{72}-2\sqrt{32}$.
13. $3\sqrt[3]{16}+\sqrt[3]{54}$.
14. $5\sqrt[3]{108}-\sqrt[3]{32}$.
15. $5\sqrt[4]{2}-\sqrt[4]{32}$.
16. $\sqrt[4]{2}+\sqrt[4]{512}$.
17. $\sqrt{2}+\sqrt{\frac{1}{2}}$.
18. $\sqrt{3}-\sqrt{\frac{1}{3}}$.
19. $\sqrt{\frac{2}{3}}-\sqrt{6}$.
20. $\sqrt{\frac{3}{5}}+\sqrt{\frac{5}{3}}$.
21. $\sqrt{\frac{3}{2}}-\sqrt{\frac{2}{3}}$.
22. $\sqrt{10}+\sqrt{40}+\sqrt{90}$.
23. $\sqrt{2}+\sqrt{50}-\sqrt{72}$.
24. $\sqrt{6}-\sqrt{24}+\sqrt{54}$.
25. $2\sqrt{2}+\sqrt{18}+\sqrt{32}$.
26. $4\sqrt{3}+2\sqrt{12}+\sqrt{75}$.
27. $2\sqrt{11}+5\sqrt{44}-3\sqrt{99}$.
28. $\sqrt{\frac{1}{3}}+\sqrt{3}+\sqrt{\frac{4}{3}}$.
29. $\sqrt{\frac{2}{3}}+\sqrt{\frac{3}{2}}+\frac{1}{6}\sqrt{6}$.
30. $2\sqrt{x}+3\sqrt{x}$.
31. $a\sqrt{2}+b\sqrt{2}$.
32. $c\sqrt{w}-d\sqrt{w}$.

33. $a\sqrt{m} + \sqrt{m}$.

34. $\sqrt{4x} + \sqrt{9x}$.

35. $\sqrt{16y} - \sqrt{36y}$.

36. $2\sqrt{64z} - \sqrt{49z}$.

37. $\sqrt{a} + \sqrt{a^3}$.

38. $\sqrt[3]{a} + \sqrt[3]{a^4}$.

39. $\sqrt[3]{c^4} - c\sqrt[3]{c}$.

40. $3\sqrt{81m} + 2\sqrt{25m}$.

41. $16\sqrt{9a} - 9\sqrt{16a}$.

42. $\sqrt[3]{8x} + \sqrt[3]{x^4}$.

43. $a\sqrt{a^3} + \sqrt{a^5}$.

44. $m\sqrt[4]{m^5} + \sqrt[4]{m^9}$.

45. $\sqrt{d} + \sqrt{dx^2}$.

46. $a\sqrt{b^2c} + \sqrt{c}$.

47. $\sqrt[3]{27y^4} - \sqrt[3]{125y}$.

48. $\sqrt{a^2bc} + \sqrt{b^3c}$.

49. $b\sqrt{a^3bc^3} + ac\sqrt{ab^3c}$.

50. $\sqrt{\frac{a^2}{2}} + \sqrt{2}$.

51. $5\sqrt{\frac{c^2}{5}} - c\sqrt{5}$.

52. $\sqrt{(a+b)^3} + \sqrt{a+b}$.

53. $\sqrt{x-y} - \sqrt{(x-y)^3}$.

54. $\sqrt{(a^2-b^2)(a+b)} + \sqrt{a-b}$.

55. $\sqrt{2} + \sqrt{2a^2} + \sqrt{8b^2}$.

56. $\sqrt{a} - 2\sqrt{a^3} + \sqrt{a^5}$.

57. $\sqrt{\frac{1}{a}} + \sqrt{\frac{1}{a^3}} + \sqrt{\frac{1}{a^5}}$.

58. $\frac{1}{2}\sqrt{ab} - \sqrt{\frac{a}{b}} + \sqrt{\frac{1}{ab}}$.

59. $\sqrt{\frac{x}{yz}} + \sqrt{\frac{y}{zx}} + \sqrt{\frac{z}{xy}}$.

60. $a\sqrt{xy} + \sqrt{b^2xy} - \sqrt{c^2xy}$.

61. $a\sqrt{a^3b} + \sqrt{4a^3b^3} + \sqrt{ab^5}$.

62. $\sqrt[3]{m^2n} + \sqrt[3]{m^5n} + \sqrt[3]{m^2n^4}$.

63. $\sqrt[3]{x^2y} + \sqrt[3]{\frac{y}{x}} - \sqrt[3]{\frac{x^2}{y^2}}$.

64. $2\sqrt{ab} + 3\sqrt{a^3b} + 4\sqrt{ab^3} + 5\sqrt{a^3b^3}$.

65. $3\sqrt{7} - \sqrt{28} + \sqrt{40} + \sqrt{90}$.

66. $2\sqrt{6} - \sqrt{54} + \sqrt{11} - \sqrt{44}$.

67. $\sqrt{a} + \sqrt{b} + \sqrt{a^3} + \sqrt{b^3}$.

68. $\sqrt{a} - \sqrt{ab^2} + \sqrt{x} + \sqrt{xy^2}$.

Reduction to Equivalent Forms

30. Any finite rational number can be expressed in the form of a surd of any desired order by applying the law $a \equiv \sqrt[r]{a^r}$.

Raise the given number to a power of which the exponent is equal to the order of the required radical and, using this result as a radicand, express the required root.

Ex. 1. Express 3 as a surd of the fourth order.

We have $3 = \sqrt[4]{3^4} = \sqrt[4]{81}$.

31. *The coefficient of a surd may be placed under the radical sign and made to appear as a factor of the radicand by raising it to a power the exponent of which is equal to the index of the indicated root.*

This is an application of the principle $a \equiv \sqrt[r]{a^r}$.

We have $a\sqrt[r]{b} \equiv \sqrt[r]{a^r}\sqrt[r]{b} \equiv \sqrt[r]{a^r b}$.

32. A surd is said to be **entire** if its coefficient is unity.

E.g. $\sqrt{2}$, $\sqrt{ab}$, $\sqrt[3]{xy}$, are entire surds.

33. A surd is said to be **mixed** if its coefficient is a number different from unity.

E.g. $3\sqrt{5}$, $a\sqrt[3]{b}$, $(m+n)\sqrt{m-n}$, are mixed surds.

Ex. 2. Express the mixed surds $2\sqrt{3}$, $3\sqrt{2}$, and $\frac{1}{4}\sqrt{8}$ as entire surds.

We have

$$2\sqrt{3} = \sqrt{4 \cdot 3} = \sqrt{12},$$
$$3\sqrt{2} = \sqrt{9 \cdot 2} = \sqrt{18},$$
$$\tfrac{1}{4}\sqrt{8} = \sqrt{\tfrac{1}{16} \cdot 8} = \sqrt{\tfrac{1}{2}}.$$

MENTAL EXERCISE XX. 5

Reduce to the forms of surds of the orders indicated :

1. 2, 2nd order.
2. 3, 4th order.
3. $\frac{1}{2}$, 4th order.
4. $\frac{3}{4}$, 3rd order.
5. 6, 3rd order.
6. 2, 5th order.
7. 12, 3rd order.
8. 4, 4th order.
9. $\frac{1}{3}$, 5th order.
10. a, 6th order.
11. $\frac{x}{y}$, 5th order.
12. a^m, nth order.

Transform each of the following mixed surds into an entire surd :

13. $2\sqrt{3}$.
14. $3\sqrt{5}$.
15. $4\sqrt{2}$.
16. $2\sqrt[3]{2}$.
17. $3\sqrt[3]{3}$.
18. $5\sqrt[3]{2}$.
19. $4\sqrt[3]{3}$.
20. $6\sqrt[3]{10}$.

21. $\frac{1}{2}\sqrt{2}$.
22. $\frac{1}{3}\sqrt{6}$.
23. $\frac{1}{5}\sqrt{5}$.
24. $\frac{1}{6}\sqrt{7}$.
25. $3\sqrt{\frac{1}{3}}$.
26. $5\sqrt{\frac{1}{10}}$.
27. $\frac{1}{10}\sqrt{5}$.
28. $\frac{1}{2}\sqrt{\frac{1}{3}}$.
29. $\frac{1}{4}\sqrt{\frac{1}{5}}$.
30. $\frac{3}{4}\sqrt{\frac{3}{4}}$.
31. $\frac{2}{3}\sqrt{\frac{3}{2}}$.
32. $\frac{1}{3}\sqrt[3]{\frac{1}{3}}$.
33. $\frac{2}{5}\sqrt[3]{\frac{2}{5}}$.
34. $a\sqrt{b}$.
35. $b\sqrt{c}$.
36. $c\sqrt[3]{d}$.
37. $m\sqrt[4]{n}$.
38. $x\sqrt{x}$.
39. $y\sqrt[3]{y}$.
40. $ab\sqrt{c}$.
41. $xy\sqrt{yz}$.
42. $x^2\sqrt{y}$.
43. $\frac{1}{y}\sqrt{x}$.
44. $\frac{1}{a}\sqrt{a}$.
45. $b\sqrt{\frac{1}{b}}$.
46. $xy\sqrt{\frac{x}{y}}$.
47. $\frac{a}{b}\sqrt{ab}$.
48. $\frac{1}{b}\sqrt{\frac{1}{b}}$.
49. $\frac{a}{2}\sqrt{\frac{2}{a}}$.
50. $\frac{3}{b}\sqrt{\frac{b}{3}}$.
51. $\frac{x^2}{y^2}\sqrt{\frac{y}{x}}$.
52. $\frac{m}{n}\sqrt[3]{\frac{m}{n}}$.
53. $\frac{x}{2}\sqrt[3]{\frac{4}{x^2}}$.
54. $a\sqrt[n]{b}$.
55. $b\sqrt[n]{c}$.
56. $c\sqrt[n]{c}$.
57. $d\sqrt[n]{d^2}$.
58. $x^2\sqrt[n]{x}$.
59. $x\sqrt[b]{x^a}$.
60. $a^n\sqrt[r]{a}$.

Change of Order.

34. Surds of different orders can be transformed into equivalent surds of the same order by applying the principle $\sqrt[r]{a^p} \equiv \sqrt[rn]{a^{pn}}$. (See § 26).

Using radical symbols, we may proceed as follows:

As a common index for all of the transformed surds, write the lowest common multiple of all of the indices of the given indicated roots. Then raise each radicand to a power the exponent of which is equal to the number by which the root index must be multiplied to produce the lowest common multiple of the indices.

Ex. 1. Transform $\sqrt{3}$, $\sqrt[3]{2}$ and $\sqrt[6]{5}$ into equivalent surds of the same order.

The lowest common multiple of the indices 2, 3, and 6 is 6.

Hence,
$$\sqrt[2]{3} = \sqrt[6]{3^3} = \sqrt[6]{27},$$
$$\sqrt[3]{2} = \sqrt[6]{2^2} = \sqrt[6]{4},$$
$$\sqrt[6]{5} = \sqrt[6]{5}.$$

35. When transforming surds of different orders into equivalent surds of the same order it is often convenient to use the notation of fractional exponents, and to proceed as follows:

Express each of the indicated roots by using the notation of fractional exponents. Reduce the fractional exponents thus obtained to equivalent fractional exponents having a lowest common denominator. Express the results thus obtained in the radical notation, observing that the numerator of the fractional exponent denotes the power to which the radicand is to be raised and that the denominator is the index of the required root.

Ex. 2. Which is the greater, $\sqrt[6]{5}$ or $\sqrt[8]{7}$?

We have $\sqrt[6]{5} = 5^{\frac{1}{6}} = 5^{\frac{4}{24}} = \sqrt[24]{5^4} = \sqrt[24]{625}$;

also, $\sqrt[8]{7} = 7^{\frac{1}{8}} = 7^{\frac{3}{24}} = \sqrt[24]{7^3} = \sqrt[24]{343}$.

Since $625 > 343$, it follows that $\sqrt[24]{625} > \sqrt[24]{343}$.

Hence, $\sqrt[6]{5} > \sqrt[8]{7}$.

EXERCISE XX. 6

Express as equivalent surds of the same order:

1. $\sqrt{2}$ and $\sqrt[3]{5}$.
2. $\sqrt{3}$ and $\sqrt[3]{4}$.
3. $\sqrt{5}$ and $\sqrt[3]{10}$.
4. $\sqrt{6}$ and $\sqrt[3]{14}$.
5. $\sqrt{7}$ and $\sqrt[4]{50}$.
6. $\sqrt[3]{2}$ and $\sqrt[4]{3}$.
7. $\sqrt[3]{3}$ and $\sqrt[4]{5}$.
8. $\sqrt[5]{2}$ and $\sqrt[6]{3}$.
9. $\sqrt{3}$ and $\sqrt[5]{15}$.
10. $\sqrt[3]{9}$ and $\sqrt[6]{80}$.
11. $\sqrt{3}$ and $\sqrt[8]{75}$.
12. $\sqrt{2}$ and $\sqrt[7]{12}$.
13. $\sqrt{a}$ and $\sqrt[3]{a^2}$.
14. $\sqrt[3]{b^2}$ and $\sqrt[4]{b^3}$.
15. $\sqrt{x}$ and $\sqrt[5]{x^4}$.
16. $\sqrt[n]{y}$ and $\sqrt[2n]{y}$.
17. $\sqrt[2n]{z}$ and $\sqrt[3n]{z}$.
18. $\sqrt[n]{w}$ and $\sqrt[n+1]{w}$.

Which is the greater,

19. $3\sqrt{5}$ or $4\sqrt{3}$?
20. $4\sqrt{6}$ or $5\sqrt{3}$?
21. $6\sqrt{2}$ or $2\sqrt{6}$?
22. $5\sqrt{7}$ or $8\sqrt{3}$?
23. $\sqrt[3]{36}$ or $2\sqrt[3]{5}$?
24. $3\sqrt[3]{3}$ or $2\sqrt[3]{10}$?
25. $\sqrt[4]{3}$ or $\sqrt{2}$?
26. $\sqrt{5}$ or $\sqrt[4]{20}$?
27. $\sqrt{6}$ or $2\sqrt[4]{2}$?
28. $\sqrt[3]{7}$ or $\sqrt[6]{50}$?
29. $\sqrt[3]{14}$ or $2\sqrt[6]{3}$?
30. $\sqrt{3}$ or $\sqrt[6]{29}$?
31. $\sqrt{6}$ or $\sqrt[6]{200}$?
32. 2 or $\sqrt[3]{7}$?
33. 3 or $\sqrt[3]{30}$?
34. 4 or $2\sqrt[3]{3}$?
35. 5 or $3\sqrt[4]{7}$?
36. 3 or $2\sqrt[5]{10}$?
37. $\sqrt{a}$ or $\sqrt[3]{a^2}$, for $a > 1$?
38. $\sqrt[3]{b^2}$ or $\sqrt[5]{b^3}$, for $b > 1$.

MULTIPLICATION OF SURDS

36. The product of two monomial surds of equal orders may be found by applying the principle $\sqrt[r]{a}\sqrt[r]{b} \equiv \sqrt[r]{ab}$. (See § 24.)

If the given surds are of different orders, they must first be transformed into equivalent surds of the same order.

Multiply the coefficients together for a new coefficient, and the expressions under the radical signs for a new radicand, and reduce the result to simplest form.

Ex. 1. $5\sqrt{6} \times 3\sqrt{2} = 5 \cdot 3\sqrt{6 \cdot 2} = 15\sqrt{12} = 30\sqrt{3}$.

Ex. 2. $5\sqrt{5} \times \sqrt[3]{2} = \sqrt[6]{5^3} \times \sqrt[6]{2^2} = \sqrt[6]{5^3 \times 2^2} = \sqrt[6]{500}$.

It is often convenient to obtain the prime factors of the radicands before multiplying.

Ex. 3. $\sqrt{35} \times \sqrt{91} = \sqrt{5 \cdot 7} \times \sqrt{13 \cdot 7} = \sqrt{7^2 \cdot 5 \cdot 13} = 7\sqrt{65}$.

EXERCISE XX. 7

Write each of the following products in simplest form :

1. $\sqrt{3} \times \sqrt{5}$.
2. $\sqrt{2} \times \sqrt{6}$.
3. $\sqrt{5a} \times \sqrt{10a}$.
4. $\sqrt{3x} \times \sqrt{15}$.
5. $\sqrt{6b} \times \sqrt{12bc}$.
6. $\sqrt[3]{18x} \times \sqrt[3]{3x^2}$.
7. $\sqrt[4]{4x^3y} \times \sqrt[4]{8xy^3}$.
8. $\sqrt[6]{8m^2} \times \sqrt[6]{16m^3}$.
9. $5\sqrt{15} \times \sqrt{5w}$.
10. $2\sqrt{14c} \times 3\sqrt{7d}$.
11. $6\sqrt{12} \times 4\sqrt{8a}$.
12. $8\sqrt{18} \times 9\sqrt{20}$.
13. $\sqrt[3]{45} \times \sqrt[3]{18}$.
14. $\sqrt{2a} \times \sqrt[4]{6ax}$.
15. $\sqrt[2]{14} \times \sqrt[3]{21}$.
16. $\sqrt[3]{abc} \times \sqrt[3]{a^2b^2c}$.
17. $\sqrt[3]{xy^2z} \times \sqrt[3]{xy^2z^2}$.
18. $\sqrt[3]{2a^2bc^2} \times \sqrt[3]{4a^2b^2c}$.
19. $\sqrt[4]{ab^2c^3} \times \sqrt[4]{a^3bc}$.
20. $\sqrt[4]{ab^2c^3} \times \sqrt[4]{b^3c^2d}$.
21. $\sqrt[5]{xy^2z^3w^4} \times \sqrt[5]{y^3w}$.
22. $\sqrt{\frac{1}{2}} \times \sqrt{\frac{1}{3}}$.
23. $\sqrt{\frac{5}{6}} \times \sqrt{\frac{2}{5}}$.
24. $\sqrt{\frac{7}{11}} \times \sqrt{\frac{11}{7}}$.
25. $4\sqrt{2\frac{1}{8}} \times \sqrt{3\frac{5}{2}}$.
26. $\sqrt{\frac{a}{c}} \times \sqrt{\frac{b}{c}}$.
27. $\sqrt{\frac{x}{y}} \times \sqrt{\frac{x}{z}}$.
28. $\sqrt{\frac{a}{b}} \times \sqrt{\frac{c}{d}}$.
29. $\sqrt{\frac{1}{ab}} \times \sqrt{\frac{a}{b}}$.
30. $\sqrt{\frac{c}{x}} \times \sqrt{\frac{x}{c}}$.

31. $\sqrt{\frac{a}{bc}} \times \sqrt{\frac{c}{ab}}$.

32. $\sqrt[3]{\frac{xy}{z}} \times \sqrt[3]{\frac{y}{zx}}$.

33. $\sqrt[n]{a} \times \sqrt[n]{a^{n-1}b}$.

34. $\sqrt[n]{b^2} \times \sqrt[n]{b^{n-1}}$.

35. $\sqrt[n]{c^{n-4}} \times \sqrt[n]{c^5d}$.

36. $\sqrt[n]{x^{n-1}} \times \sqrt[n]{x^{n+1}}$.

37. $n\sqrt[n]{n^{n-1}} \times \sqrt[n]{n^2}$.

38. $\sqrt[3]{36} \times \sqrt[3]{30}$.

39. $\sqrt[3]{33} \times \sqrt{55}$.

40. $\sqrt{\frac{3}{5}} \times \sqrt[3]{\frac{5}{6}}$.

41. $4\sqrt[4]{\frac{1}{6}} \times 6\sqrt[6]{\frac{1}{4}}$.

42. $3\sqrt[3]{3a^3} \times 9\sqrt[9]{9a^9}$.

MULTIPLICATION OF POLYNOMIALS INVOLVING SURDS

37. The product of two polynomials the terms of which contain surds may be obtained as follows:

Multiply each term of the multiplicand by each term of the multiplier.

Ex. 1.
$$\begin{aligned}(\sqrt{2}-5\sqrt{3}+\sqrt{6}-7)\times 3\sqrt{6} &= 3\sqrt{12}-15\sqrt{18}+3\sqrt{36}-21\sqrt{6}\\ &= 6\sqrt{3}-45\sqrt{2}+18-21\sqrt{6}\\ &= 18-45\sqrt{2}+6\sqrt{3}-21\sqrt{6}.\end{aligned}$$

Ex. 2. Multiply $5\sqrt{5}+2\sqrt{2}$ by $4\sqrt{5}-3\sqrt{2}$.

$$\begin{array}{rrr}5\sqrt{5}+ & 2\sqrt{2} & \\ 4\sqrt{5}- & 3\sqrt{2} & \\ \hline 100+ & 8\sqrt{10} & \\ & -15\sqrt{10} & -12 \\ \hline 100- & 7\sqrt{10} & -12 = 88-7\sqrt{10}.\end{array}$$

38. Two binomial quadratic surds which differ only in the sign of one of the surd terms are called **conjugate surds.**

E. g. $\sqrt{a}+\sqrt{b}$ and $\sqrt{a}-\sqrt{b}$; $5+4\sqrt{3}$ and $5-4\sqrt{3}$; $\sqrt{6}+\sqrt{7}$ and $-\sqrt{6}+\sqrt{7}$.

39. *The product of two conjugate surds is a rational number or expression.*

Representing two conjugate surds by $\sqrt{x}+\sqrt{y}$ and $\sqrt{x}-\sqrt{y}$, we have, $(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y}) \equiv (\sqrt{x})^2-(\sqrt{y})^2 \equiv x-y$.

Ex. 3. $(2\sqrt{7}+\sqrt{11})(2\sqrt{7}-\sqrt{11}) = 28-11 = 17$.

EXERCISE XX. 8

Simplify each of the following expressions:

1. $(\sqrt{11} - \sqrt{13} + \sqrt{7})\sqrt{3}$.
2. $(\sqrt{2} + \sqrt{7} - \sqrt{3})\sqrt{5}$.
3. $(\sqrt{7} - \sqrt{11} + \sqrt{13})\sqrt{6}$.
4. $(\sqrt{5} - \sqrt{10} - \sqrt{15})\sqrt{7}$.
5. $(\sqrt{2} - \sqrt{5} + \sqrt{10})\sqrt{10}$.
6. $(\sqrt{2} + \sqrt{7} + 2\sqrt{14})\sqrt{14}$.
7. $(\sqrt{3} - \sqrt{5} + \sqrt{10})\sqrt{15}$.
8. $(\sqrt{10} + 2\sqrt{3} - \sqrt{5})\sqrt{5}$.
9. $(3\sqrt{2} - 2\sqrt{3} + \sqrt{12})\sqrt{6}$.
10. $(\sqrt{5} - \sqrt{21} + 4\sqrt{27})2\sqrt{3}$.
11. $(2\sqrt{5} - 4\sqrt{10} - \sqrt{30})3\sqrt{5}$.
12. $(\sqrt{a} + \sqrt{b} + \sqrt{c})\sqrt{abc}$.
13. $(\sqrt{xy} + \sqrt{yz} + \sqrt{zx})\sqrt{xyz}$.
14. $(\sqrt[3]{2} + \sqrt[3]{6} - \sqrt[3]{10})\sqrt[3]{4}$.
15. $(\sqrt[3]{9} - 2\sqrt[3]{3} - \sqrt[3]{18})\sqrt[3]{3}$.
16. $(\sqrt[3]{4} + \sqrt[3]{9} + \sqrt[3]{36})\sqrt[3]{6}$.
17. $(\sqrt[3]{100} - \sqrt[3]{25} + \sqrt[3]{4})\sqrt[3]{10}$.
18. $(\sqrt[3]{a^2bc} + \sqrt[3]{ab^2c} + \sqrt[3]{abc^2})\sqrt[3]{abc}$.
19. $(\sqrt[3]{a^2c} - \sqrt[3]{b^2d} + \sqrt[3]{c^2d})\sqrt[3]{abcd}$.
20. $(\sqrt[n]{a^{n-1}} + \sqrt[n]{b^{n-1}} + \sqrt[n]{c^{n-1}})\sqrt[n]{abc}$.
21. $(\sqrt{3} + 4)(\sqrt{2} + 3)$.
22. $(2 + \sqrt{3})(3 - \sqrt{2})$.
23. $(\sqrt{6} - 5)(\sqrt{3} + 5)$.
24. $(2\sqrt{7} + 7\sqrt{2})(3\sqrt{7} + 8\sqrt{2})$.
25. $(10\sqrt{6} - 6\sqrt{10})(5\sqrt{3} + 3\sqrt{5})$.
26. $(\sqrt[3]{2} + \sqrt[3]{3})(\sqrt[3]{4} + \sqrt[3]{5})$.
27. $(1 + \sqrt{2} - \sqrt{3})(\sqrt{2} - \sqrt{6})$.
28. $\sqrt{3 + \sqrt{5}} \times \sqrt{3 - \sqrt{5}}$.
29. $\sqrt{6 + 3\sqrt{3}} \times \sqrt{6 - 3\sqrt{3}}$.
30. $(\sqrt{2} - \sqrt{3} + \sqrt{5})(\sqrt{2} + \sqrt{3} - \sqrt{5})$.
31. $(\sqrt{\tfrac{1}{6}} - \sqrt{\tfrac{1}{7}} + \sqrt{\tfrac{1}{8}})(\sqrt{6} + \sqrt{7} - \sqrt{8})$.
32. $(\sqrt{\tfrac{5}{6}} - \sqrt{\tfrac{6}{7}} - \sqrt{\tfrac{7}{5}})(\sqrt{\tfrac{6}{5}} - \sqrt{\tfrac{7}{6}} - \sqrt{\tfrac{5}{7}}$.
33. $(\sqrt{2} - \sqrt{3})(\sqrt{3} - \sqrt{5})(\sqrt{5} - \sqrt{7})$.
34. $(a\sqrt{b} - \sqrt{ab} + b\sqrt{a})(\sqrt{a} - \sqrt{b})$.
35. $(\sqrt{ab} + \sqrt{bc} + \sqrt{ca})(\sqrt{a} + \sqrt{b} + \sqrt{c})$.
36. $(\sqrt{a} + \sqrt{b} + \sqrt{c})(\sqrt{a} + \sqrt{b} - \sqrt{c})(\sqrt{a} - \sqrt{b} + \sqrt{c})$
$(\sqrt{a} - \sqrt{b} - \sqrt{c})$.

INVOLUTION OF MONOMIAL SURDS

40. From the principle $(\sqrt[r]{a})^p \equiv \sqrt[r]{a^p}$, it follows that:

An entire monomial surd may be raised to a power by raising the radicand to the indicated power.

Ex. 1. $(\sqrt[3]{5})^4 = \sqrt[3]{5^4} = 5\sqrt[3]{5}.$

Ex. 2. $(2\sqrt{6a})^3 \equiv 2^3\sqrt{6^3a^3} \equiv 48a\sqrt{6a}.$

Ex. 3. $(\sqrt[6]{5a})^2 \equiv (5a)^{\frac{2}{6}} \equiv (5a)^{\frac{1}{3}} \equiv \sqrt[3]{5a}.$

MENTAL EXERCISE XX. 9

Reduce the following indicated powers to simplest form:

1. $(\sqrt[3]{2})^2$.
2. $(\sqrt[3]{5})^2$.
3. $(\sqrt[4]{3})^3$.
4. $(\sqrt{2})^3$.
5. $(\sqrt{5})^3$.
6. $(\sqrt[5]{8})^2$.
7. $(2\sqrt{3})^3$.
8. $(3\sqrt{5})^3$.
9. $(4\sqrt[3]{4})^2$.
10. $(2\sqrt[6]{2})^5$.
11. $(-2\sqrt{7})^3$.
12. $(\sqrt[3]{-4})^4$.
13. $(\sqrt{a})^3$.
14. $(\sqrt{b^3})^3$.
15. $(\sqrt{x})^5$.
16. $(\sqrt{y^5})^5$.
17. $(\sqrt[3]{c})^2$.
18. $(\sqrt[3]{d})^4$.
19. $(\sqrt[3]{h})^5$.
20. $(\sqrt[3]{-a})^2$.
21. $(\sqrt[3]{-b^2})^2$.
22. $(-\sqrt[3]{-x})^4$.
23. $(a\sqrt{x})^3$.
24. $(-b\sqrt{y})^3$.
25. $(c^2\sqrt{z})^4$.
26. $(c\sqrt[3]{m})^2$.
27. $(d\sqrt[3]{n})^4$.
28. $(a\sqrt[3]{b^2c})^2$.
29. $(b\sqrt[3]{cd^2})^2$.
30. $(x^3\sqrt[3]{y^2z})^2$.
31. $(2x\sqrt{x})^3$.
32. $(3y\sqrt[3]{y^2})^2$.
33. $(5a^2\sqrt[3]{5a^2})^2$.
34. $(\sqrt[n]{a})^2$, $n > 2$.
35. $(\sqrt[n]{b})^3$, $n > 3$.
36. $(x\sqrt[n]{y})^4$, $n > 4$.

DIVISION OF SURDS

41. The quotient obtained by dividing one entire monomial surd by another may be found by applying the principle

$$\frac{\sqrt[r]{a}}{\sqrt[r]{b}} \equiv \sqrt[r]{\frac{a}{b}}.$$

The process for finding the quotient of one mixed surd divided by another may be made to depend upon the principle above.

Since two surds of different orders can be transformed into equivalent surds of the same order, it follows that it is necessary to state the process only for mixed surds of the same order.

The index of the indicated root of the radicand of the quotient obtained by dividing one monomial surd by another of the same order is equal to the common index of the indicated roots of the radicands of the dividend and divisor.

For the coefficient of the radical part of the quotient divide the coefficient of the dividend by the coefficient of the divisor, and for the radicand of the quotient divide the radicand of the dividend by the radicand of the divisor.

The result should be reduced to simplest form.

Ex. 1. $8\sqrt{14} \div 4\sqrt{2} = (8 \div 4)\sqrt{14 \div 2} = 2\sqrt{7}.$

Ex. 2. Divide $21\sqrt{2}$ by $7\sqrt[3]{6}$.

We have, $\dfrac{21\sqrt{2}}{7\sqrt[3]{6}} = \dfrac{3\sqrt[6]{2^3}}{\sqrt[6]{(2\cdot 3)^2}} = 3\sqrt[6]{\dfrac{2^3}{2^2\cdot 3^2}} = 3\sqrt[6]{\dfrac{2\cdot 3^4}{3^2\cdot 3^4}} = \sqrt[6]{162}.$

Exercise XX. 10

Simplify each of the following quotients :

1. $\sqrt{6} \div \sqrt{2}$.
2. $\sqrt{10} \div \sqrt{5}$.
3. $\sqrt{14} \div \sqrt{2}$.
4. $\sqrt{21} \div \sqrt{3}$.
5. $\sqrt{15} \div \sqrt{5}$.
6. $3\sqrt{22} \div \sqrt{11}$.
7. $5\sqrt{26} \div \sqrt{13}$.
8. $8\sqrt{30} \div 2\sqrt{2}$.
9. $\sqrt{39} \div 2\sqrt{13}$.
10. $\sqrt{51} \div 3\sqrt{17}$.
11. $\sqrt[3]{4} \div \sqrt[3]{2}$.
12. $\sqrt[3]{9} \div \sqrt[3]{3}$.
13. $\sqrt[3]{12} \div \sqrt[3]{4}$.
14. $5\sqrt[3]{18} \div \sqrt[3]{3}$.
15. $\sqrt[4]{27} \div \sqrt[4]{3}$.
16. $\sqrt[4]{50} \div \sqrt[4]{2}$.
17. $\sqrt[6]{48} \div \sqrt[6]{6}$.
18. $\sqrt{7} \div \sqrt{2}$.
19. $\sqrt{3} \div \sqrt{5}$.
20. $\sqrt{10} \div \sqrt{3}$.
21. $\sqrt{13} \div \sqrt{7}$.
22. $\sqrt{10} \div \sqrt{6}$.
23. $\sqrt{15} \div \sqrt{10}$.
24. $\sqrt{21} \div \sqrt{14}$.
25. $\sqrt[3]{2} \div \sqrt[3]{4}$.
26. $\sqrt[3]{10} \div \sqrt[3]{12}$.
27. $\sqrt[3]{3} \div \sqrt[3]{2}$.
28. $\sqrt[3]{5} \div \sqrt[3]{4}$.
29. $5\sqrt{2} \div \sqrt{10}$.
30. $3\sqrt{5} \div \sqrt{15}$.
31. $6\sqrt{7} \div \sqrt{42}$.
32. $10\sqrt{3} \div \sqrt{6}$.
33. $14\sqrt{2} \div \sqrt{14}$.
34. $\sqrt{7} \div 3\sqrt{21}$.
35. $\sqrt{11} \div 2\sqrt{22}$.
36. $\sqrt{3} \div 4\sqrt{15}$.
37. $\sqrt{2} \div 5\sqrt{6}$.
38. $\sqrt[3]{a^2} \div \sqrt{a}$.
39. $\sqrt[3]{b^2c} \div \sqrt[3]{c}$.
40. $\sqrt[3]{a^2bc} \div \sqrt[3]{ab}$.
41. $\sqrt[3]{xy^2z} \div \sqrt[3]{xyz}$.
42. $\sqrt[4]{a^3} \div \sqrt[4]{a}$.

43. $\sqrt[4]{x^2y^3} \div \sqrt[4]{xy}$.
44. $\sqrt[4]{ab^2c^3} \div \sqrt[4]{ab^2c}$.
45. $\sqrt{a} \div \sqrt{b}$.
46. $\sqrt{c} \div \sqrt{x}$.
47. $\sqrt{d} \div \sqrt{y}$.
48. $\sqrt[3]{a} \div \sqrt[3]{b^2}$.
49. $\sqrt[3]{b^2} \div \sqrt[3]{c}$.
50. $a\sqrt{y} \div \sqrt{a}$.
51. $b\sqrt{x} \div \sqrt{b}$.
52. $a\sqrt{b} \div b\sqrt{a}$.
53. $\sqrt{xyz} \div xy\sqrt{z}$.
54. $ab\sqrt{c} \div a\sqrt{bc}$.
55. $\sqrt{xyz} \div \sqrt{yzw}$.
56. $\sqrt[3]{xy^2} \div \sqrt[3]{yz^2}$.
57. $\sqrt{5a} \div \sqrt{2b}$.
58. $\sqrt{2c} \div \sqrt{3d}$.
59. $\sqrt{7x} \div \sqrt{14y}$.
60. $\sqrt{10ab} \div \sqrt{15bc}$.
61. $6\sqrt{3xy} \div 3\sqrt{6xz}$.
62. $\sqrt[n]{a^2} \div \sqrt[n]{a}$, $n > 2$.
63. $\sqrt[n]{a^{n-1}} \div \sqrt[n]{ab^{n-1}}$, $n > 1$.
64. $(\sqrt{30} + \sqrt{42}) \div \sqrt{6}$.
65. $(12\sqrt{35} - \sqrt{45}) \div 3\sqrt{5}$.
66. $(8\sqrt[3]{51} - 5\sqrt[3]{33}) \div (-2\sqrt[3]{3})$.
67. $(\sqrt{15} - \sqrt{6} + \sqrt{2} - \sqrt{3}) \div \sqrt{3}$.
68. $(\sqrt{2} + \sqrt{8} - \sqrt{21}) \div \sqrt{2}$.
69. $(8\sqrt{7} - 6\sqrt{5} + 4\sqrt{3}) \div 4\sqrt{2}$.
70. $(10\sqrt{15} - 5\sqrt{3} + 3\sqrt{5}) \div 30\sqrt{15}$.
71. $(2\sqrt{\tfrac{1}{2}} - 8\sqrt[3]{\tfrac{1}{4}} - 4) \div \sqrt[3]{4}$.

RATIONALIZATION

42. To **rationalize** a surd expression is to free it from indicated roots.

If the product of two irrational factors is a rational number, either factor is called the **rationalizing factor** of the other.

E.g. The rationalizing factor of $\sqrt[3]{3}$ is $\sqrt[3]{9}$, since $\sqrt[3]{3} \times \sqrt[3]{9} = \sqrt[3]{27} = 3$, which is a rational number.

43. From the identity $\sqrt[r]{a^p} \times \sqrt[r]{a^{r-p}} \equiv \sqrt[r]{a^r} \equiv a$, it may be seen that when p is less than r the rationalizing factor of a simple entire monomial surd represented by $\sqrt[r]{a^p}$ is $\sqrt[r]{a^{r-p}}$.

Ex. 1. Rationalize the denominator of $\dfrac{3}{\sqrt{2}}$.

By multiplying both terms of the fraction $3/\sqrt{2}$ by $\sqrt{2}$ the value of the fraction remains unaltered and the denominator is made rational.

We have
$$\frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{\sqrt{2}\sqrt{2}} = \frac{3\sqrt{2}}{2}.$$

Ex. 2. Rationalize the denominator of $\dfrac{6\sqrt[3]{7}}{\sqrt[3]{2}}$.

The rationalizing factor of $\sqrt[3]{2}$ is $\sqrt[3]{2^2} = \sqrt[3]{4}$.

Hence $$\frac{6\sqrt[3]{7}}{\sqrt[3]{2}} = \frac{6\sqrt[3]{7}\sqrt[3]{4}}{\sqrt[3]{2}\sqrt[3]{4}} = \frac{6\sqrt[3]{28}}{2} = 3\sqrt[3]{28}.$$

44. From the identity $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) \equiv a - b$, it appears that *a binomial quadratic surd may be rationalized by multiplying by the conjugate surd.*

Ex. 3. Reduce $(3 + \sqrt{5})/(3 - \sqrt{5})$ to an equivalent fraction whose denominator is rational.

The rationalizing factor of the denominator $3 - \sqrt{5}$ is the conjugate surd $3 + \sqrt{5}$.

Hence,

$$\frac{3 + \sqrt{5}}{3 - \sqrt{5}} = \frac{(3 + \sqrt{5})(3 + \sqrt{5})}{(3 - \sqrt{5})(3 + \sqrt{5})} = \frac{9 + 6\sqrt{5} + 5}{9 - 5} = \frac{14 + 6\sqrt{5}}{4} = \frac{7 + 3\sqrt{5}}{2}.$$

Ex. 4. Divide $\sqrt{2} + \sqrt{3} + \sqrt{6}$ by $\sqrt{3} + 1$.

Expressing the quotient as a fraction, and rationalizing the divisor, we have,

$$\frac{\sqrt{2}+\sqrt{3}+\sqrt{6}}{\sqrt{3}+1} = \frac{(\sqrt{2}+\sqrt{3}+\sqrt{6})(\sqrt{3}-1)}{(\sqrt{3}+1)(\sqrt{3}-1)} = \frac{3+2\sqrt{2}-\sqrt{3}}{2} = \frac{3}{2}+\sqrt{2}-\frac{1}{2}\sqrt{3}.$$

45. From the identity,

$$(\sqrt{a}+\sqrt{b}+\sqrt{c})(\sqrt{a}-\sqrt{b}+\sqrt{c})(\sqrt{a}+\sqrt{b}-\sqrt{c})(\sqrt{a}-\sqrt{b}-\sqrt{c}) \equiv a^2+b^2+c^2-2ab-2ac-2bc,$$

it appears that the rationalizing factor of any one of the factors of the first member is the product of the remaining three factors.

E. g. The trinomial surd $(\sqrt{2} - \sqrt{3} + \sqrt{5})$ may be made rational by multiplying by the product

$$(\sqrt{2} + \sqrt{3} + \sqrt{5})(\sqrt{2} + \sqrt{3} - \sqrt{5})(\sqrt{2} - \sqrt{3} - \sqrt{5}).$$

When rationalizing a trinomial surd it will often be found convenient to group the terms of the trinomial and regard the expression as a binomial, of which one of the terms is a sum or a difference. After multiplying by the conjugate binomial factor, the terms of the resulting expression may be combined and the process of rationalization may be again applied.

To rationalize either the denominator or the numerator of a fraction, multiply both numerator and denominator by the rationalizing factor of the term to be rationalized.

Ex. 5. Rationalize the denominator of $\sqrt{3}/(\sqrt{10} - \sqrt{6} + \sqrt{3})$.

We have,

$$\frac{\sqrt{3}}{\sqrt{10} - \sqrt{6} + \sqrt{3}} = \frac{\sqrt{3}(\sqrt{10} - \sqrt{6} - \sqrt{3})}{(\sqrt{10} - \sqrt{6} + \sqrt{3})(\sqrt{10} - \sqrt{6} - \sqrt{3})}$$

$$= \frac{\sqrt{30} - 3\sqrt{2} - 3}{13 - 4\sqrt{15}}$$

$$= \frac{(\sqrt{30} - 3\sqrt{2} - 3)(13 + 4\sqrt{15})}{(13 - 4\sqrt{15})(13 + 4\sqrt{15})}$$

$$= \frac{\sqrt{30} - 12\sqrt{15} + 21\sqrt{2} - 39}{-71}.$$

$$= \tfrac{39}{71} - \tfrac{21}{71}\sqrt{2} + \tfrac{12}{71}\sqrt{15} - \tfrac{1}{71}\sqrt{30}.$$

46. The object of rationalizing the denominator of a given fraction is to avoid the use of a divisor consisting of a non-terminating decimal.

E. g. To find the value of $\frac{\sqrt{2}}{\sqrt{3}}$, correct to four places of decimals, if the denominator is not rationalized, it is necessary to divide 1.41421 + by 1.73205 +, as follows:

$$\frac{\sqrt{2}}{\sqrt{3}} = \frac{1.41421 +}{1.73205 +} = .8164 + .$$

While, by rationalizing the denominator of the fraction, we may obtain the required value by dividing 2.4492 + by 3, as follows:

$$\frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{6}}{3} = \frac{2.4492 +}{3} = .8164 + .$$

Exercise XX. 11

Rationalize the denominators of each of the following:

1. $\frac{1}{\sqrt{6}}$.
2. $\frac{20}{\sqrt{5}}$.
3. $\frac{14}{\sqrt{7}}$.
4. $\frac{33}{\sqrt{11}}$.
5. $\frac{a}{\sqrt{b}}$.
6. $\frac{5}{\sqrt{10}}$.
7. $\frac{1}{\sqrt[3]{5}}$.
8. $\frac{4}{\sqrt[3]{4}}$.

9. $\dfrac{c}{\sqrt[3]{d^2}}$.

10. $\dfrac{3}{\sqrt{21}}$.

11. $\dfrac{12}{\sqrt{28}}$.

12. $\dfrac{18}{\sqrt{27}}$.

13. $\dfrac{15}{2\sqrt{5}}$.

14. $\dfrac{22}{3\sqrt{11}}$.

15. $\dfrac{7}{5\sqrt{14}}$.

16. $\dfrac{3}{2\sqrt[3]{9}}$.

17. $\dfrac{6}{7\sqrt[5]{16}}$.

18. $\dfrac{8}{3\sqrt[5]{8}}$.

19. $\dfrac{8}{9\sqrt[7]{64}}$.

20. $\dfrac{\sqrt{13}}{\sqrt{2}}$.

21. $\dfrac{\sqrt{17}}{\sqrt{34}}$.

22. $\dfrac{\sqrt{10}}{\sqrt{12}}$.

23. $\dfrac{2\sqrt{3}}{3\sqrt{2}}$.

24. $\dfrac{\sqrt{5}}{2\sqrt{6}}$.

25. $\dfrac{5\sqrt{19}}{\sqrt{20}}$.

26. $\dfrac{2}{\sqrt{3}+1}$.

27. $\dfrac{7}{\sqrt{11}-2}$.

28. $\dfrac{10}{\sqrt{3}+2}$.

29. $\dfrac{11\sqrt{6}}{\sqrt{5}+4}$.

30. $\dfrac{4\sqrt{13}}{\sqrt{3}-1}$.

31. $\dfrac{\sqrt{x}}{\sqrt{x}-y}$.

32. $\dfrac{5}{\sqrt{x+2}}$.

33. $\dfrac{5}{\sqrt{x}+2}$.

34. $\dfrac{4\sqrt{3}+6}{\sqrt{2}}$.

35. $\dfrac{a}{\sqrt{b-c}}$.

36. $\dfrac{a}{\sqrt{b}-\sqrt{c}}$.

37. $\dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$.

38. $\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$.

39. $\dfrac{\sqrt{7}+\sqrt{10}}{\sqrt{7}+\sqrt{2}}$.

40. $\dfrac{\sqrt{8}-3}{4+\sqrt{2}}$.

41. $\dfrac{a-\sqrt{x}}{a+\sqrt{x}}$.

42. $\dfrac{\sqrt{a}-\sqrt{b}}{\sqrt{x}+\sqrt{y}}$.

43. $\dfrac{\sqrt{5}+\sqrt{6}}{\sqrt{7}+\sqrt{8}}$.

44. $\dfrac{2\sqrt{3}-3\sqrt{2}}{3\sqrt{6}-2\sqrt{2}}$.

45. $\dfrac{2\sqrt{3}-4\sqrt{5}}{9\sqrt{8}-7\sqrt{6}}$.

46. $\dfrac{a\sqrt{b}+b\sqrt{a}}{\sqrt{b}+\sqrt{a}}$.

47. $\dfrac{x\sqrt{y}+y\sqrt{x}}{y\sqrt{x}-x\sqrt{y}}$.

48. $\dfrac{\sqrt{a^2+5}+3}{\sqrt{a^2+5}-3}$.

49. $\dfrac{b^2}{a+\sqrt{a^2-b^2}}$.

50. $\dfrac{6}{\sqrt{x+6}-\sqrt{x-6}}$.

51. $\dfrac{\sqrt{a-3}+\sqrt{a}}{\sqrt{a-3}-\sqrt{a}}$.

52. $\dfrac{2\sqrt{a+b}+3\sqrt{a-b}}{3\sqrt{a+b}-2\sqrt{a-b}}$.

53. $\dfrac{\sqrt{a+3}+\sqrt{a-3}}{\sqrt{a+3}-\sqrt{a-3}}$.

54. $\dfrac{\sqrt{x+1}+\sqrt{x+2}}{\sqrt{x-1}-\sqrt{x-2}}$.

55. $\dfrac{\sqrt{x^2-1}-\sqrt{x^2+1}}{\sqrt{x^2-1}+\sqrt{x^2+1}}$.

56. $\dfrac{2}{1+\sqrt{2}+\sqrt{3}}$.

57. $\dfrac{\sqrt{5}-\sqrt{7}}{2+\sqrt{5}+\sqrt{6}}$.

58. $\dfrac{\sqrt{10}+\sqrt{2}-\sqrt{5}}{\sqrt{10}-\sqrt{2}+\sqrt{5}}$.

59. $\dfrac{\sqrt{3}+\sqrt{5}+\sqrt{2}}{2\sqrt{15}+6}$.

60. $\dfrac{x+a+\sqrt{x^2-a^2}}{x+a-\sqrt{x^2-a^2}}$.

47.* By means of the identities in Chapter VIII. § 59 a rationalizing factor can be found for any binomial surd $\sqrt[p]{x} \pm \sqrt[q]{y}$.

(i.) $\sqrt[p]{x}-\sqrt[q]{y}$ can be rationalized.

By letting $\sqrt[p]{x}=a$, and $\sqrt[q]{y}=b$, and representing the lowest common multiple of p and q by n, it may be seen that a^n and b^n are both rational.

For all values of n we have,

$$(a-b)(a^{n-1}+a^{n-2}b+\cdots\cdots+ab^{n-2}+b^{n-1}) \equiv a^n-b^n.$$

Hence, the rationalizing factor of $\sqrt[p]{x}-\sqrt[q]{y}$ may be written by referring to the polynomial factor above.

Ex. 1. Find the rationalizing factor of $\sqrt[3]{5}-\sqrt{2}$.

The lowest common multiple of the exponents 3 and 2 is 6. Accordingly, it is necessary to multiply the binomial by such a factor as will raise both terms to the sixth power.

The polynomial rationalizing factor may be found as follows:

Representing $\sqrt[3]{5}=5^{\frac{1}{3}}$ by a and $\sqrt{2}=2^{\frac{1}{2}}$ by b, it follows that the binomial $\sqrt[3]{5}-\sqrt{2}$ may be represented by the binomial $a-b$.

If $a-b$ be multiplied by $a^5+a^4b+a^3b^2+a^2b^3+ab^4+b^5$ the product will be a^6-b^6.

Hence the rationalizing factor of the given binomial may be constructed by substituting $5^{\frac{1}{3}}$ for a and $2^{\frac{1}{2}}$ for b in the polynomial

$$a^5+a^4b+a^3b^2+a^2b^3+ab^4+b^5.$$

---

* This section may be omitted when the chapter is read for the first time.

Hence the rationalizing factor of $\sqrt[3]{5} - \sqrt{2}$ is

$$5^{\frac{5}{3}} + 5^{\frac{4}{3}}2^{\frac{1}{2}} + 5^{\frac{3}{3}}2^{\frac{2}{2}} + 5^{\frac{2}{3}}\cdot 2^{\frac{3}{2}} + 5^{\frac{1}{3}}\cdot 2^{\frac{4}{2}} + 2^{\frac{5}{2}}.$$

This factor reduces to $5\sqrt[3]{25} + 5\sqrt[3]{5}\sqrt{2} + 10 + 2\sqrt[3]{25}\sqrt{2} + 4\sqrt[3]{5} + 4\sqrt{2}$.

(ii.) If n be even, $\sqrt[p]{x} + \sqrt[q]{y}$ can be rationalized, since

$$(a + b)(a^{n-1} - a^{n-2}b + \cdot\cdot\cdot\cdot\cdot + ab^{n-2} - b^{n-1}) \equiv a^n - b^n.$$

(iii.) If n be odd, $\sqrt[p]{x} + \sqrt[q]{y}$ can be rationalized, since

$$(a + b)(a^{n-1} - a^{n-2}b + \cdot\cdot\cdot\cdot\cdot - ab^{n-2} + b^{n-1}) \equiv a^n + b^n.$$

Exercise XX. 12

(This exercise may be omitted when the chapter is read for the first time.)

Find the rationalizing factor of each of the following binomial surds:

1. $1 + \sqrt[3]{2}$.
2. $2 + \sqrt[3]{3}$.
3. $5 - \sqrt[3]{4}$.
4. $\sqrt[4]{5} + 1$.
5. $\sqrt{3} + \sqrt[3]{2}$.
6. $\sqrt{5} + \sqrt[3]{6}$.
7. $\sqrt[3]{7} - \sqrt{10}$.
8. $\sqrt{5} - \sqrt[4]{6}$.
9. $\sqrt[3]{6} + \sqrt[5]{9}$.
10. $\sqrt[5]{2} - \sqrt[4]{3}$.

Factors Involving Surds.

48. Extending the idea of "factor" to include expressions in which surd numbers appear among the coefficients, we may, by applying the principles of Chapter XII., transform certain expressions so that they shall appear as products of *factors involving surds.*

We will now consider the problem of factoring the general expression of the second degree containing one unknown, x:

$$ax^2 + bx + c,\ a \neq 0.$$

We may write, $ax^2 + bx + c \equiv \quad ax^2 + \dfrac{abx}{a} + \dfrac{ac}{a}$

$$\equiv a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right].$$

In the trinomial square $a^2 \pm 2ab + b^2$ the third term b^2 is the square of the quotient obtained by dividing the "middle term" or "finder term" $\pm 2ab$ by twice the square root of the first term a^2. (See Chapter XII. § 21.)

That is, $$b^2 \equiv \left(\frac{\pm 2ab}{2a}\right)^2.$$

The process of obtaining a trinomial square, $a^2 \pm 2ab + b^2$, by adding the term b^2 to a binomial such as $a^2 \pm 2ab$, is called **completing the square** with reference to $a^2 \pm 2ab$. (See also Chapter XXII. §§ 18–20.)

We may complete the square with reference to the binomial $x^2 + \frac{b}{a}x$ which appears in the expression $a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right]$ as follows:

Dividing the "finder term" $\frac{b}{a}x$ by twice the square root of the first term x^2 we obtain $\frac{b}{2a}$ which is the term whose square must be added to complete the square with reference to $x^2 + \frac{b}{a}x$.

Hence,

$$ax^2 + bx + c \equiv a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right]$$

$$\equiv a\left[\left(x + \frac{b}{2a}\right)^2 - \left(\sqrt{\frac{b^2}{4a^2} - \frac{c}{a}}\right)^2\right]$$

$$\equiv a\left[\left(x + \frac{b}{2a}\right)^2 - \left(\sqrt{\frac{b^2 - 4ac}{4a^2}}\right)^2\right]$$

$$\equiv a\left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right).$$

Ex. 1. Factor $x^2 - 10$. Check. Let $x = 1$.

$$x^2 - 10 \equiv x^2 - (\sqrt{10})^2 \qquad -9 = -9.$$

(See Chapter XII. § 22.) $\equiv [x + \sqrt{10}][x - \sqrt{10}]$.

Ex. 2. Factor $x^2 + 6x + 4$. Check. Let $x = 1$.
$11 = 11$.

Using $6x$ as a "finder term," we may complete the square with reference to $x^2 + 6x$ as follows:

We have $$\frac{6x}{2x} \equiv 3.$$

Hence, $$x^2 + 6x + 4 \equiv x^2 + 6x + 3^2 - 9 + 4$$

(See Chap. XII. § 18.) $\equiv (x+3)^2 - (\sqrt{5})^2$

(See Chap. XII. § 22.) $\equiv [x+3+\sqrt{5}][x+3-\sqrt{5}].$

Ex. 3. Factor $3x^2 + 8x - 5$. Check. Let $x = 1$.

The following method may be employed : $6 = 6$.

$$3x^2 + 8x - 5 \equiv \tfrac{1}{3}[9x^2 + 24x - 15].$$

The square may be completed with reference to $9x^2 + 24x$ by using $24x$ as a "finder term," as follows:

We have, $$\frac{24x}{2(3x)} \equiv 4.$$

Hence, $$\begin{aligned} 3x^2 + 8x - 5 &\equiv \tfrac{1}{3}[9x^2 + 24x + 4^2 - 16 - 15] \\ &\equiv \tfrac{1}{3}[(3x+4)^2 - (\sqrt{31})^2] \\ &\equiv \tfrac{1}{3}[3x+4+\sqrt{31}][3x+4-\sqrt{31}]. \end{aligned}$$

Observe that in each of the examples above the factors obtained are of the *first degree* with reference to the letters appearing in them.

Exercise XX. 13

Obtain factors containing surds for each of the following :

1. $x^2 - 3$.
2. $x^2 - 5$.
3. $x^2 - 8$.
4. $x^2 - 50$.
5. $9y^2 - 13$.
6. $16z^2 - 5$.
7. $z^2 + 8z + 1$.
8. $m^2 + 6m + 2$.
9. $a^2 + 12a - 3$.
10. $b^2 - 10b + 20$.
11. $r^2 + 2r - 1$.
12. $m^2 - 10m - 17$.
13. $x^2 - 14x - 18$.
14. $9d^2 - 12d + 1$.
15. $a^2 + 16a + 19$.
16. $b^2 - 18b + 57$.
17. $c^2 - 10c - 100$.
18. $x^2 - x - 1$.
19. $x^2 - x + 1$.
20. $x^2 - 3x - 5$.
21. $a^2 - \frac{2a}{5} - \frac{97}{75}$.
22. $b^2 - \frac{4b}{3} - \frac{49}{72}$.

Evolution of Surds

49. A root of a monomial surd may be found by applying the principles $$\begin{cases} \sqrt[r]{\sqrt[n]{a}} \equiv \sqrt[rn]{a}. \\ \sqrt[r]{\sqrt[n]{a}} \equiv \sqrt[n]{\sqrt[r]{a}}. \end{cases}$$ (See Chap. XVIII. § 21.)

Ex. 1. $$\sqrt[2]{\sqrt[3]{7}} = \sqrt[6]{7}.$$

Ex. 2. $$\sqrt[3]{\sqrt[5]{8a^3}} \equiv \sqrt[5]{\sqrt[3]{8a^3}} \equiv \sqrt[5]{2a}.$$

EXERCISE XX. 14

Simplify each of the following :

1. $\sqrt[3]{\sqrt[3]{a}}$.
2. $\sqrt{\sqrt[3]{25}}$.
3. $\sqrt[3]{\sqrt[4]{x^3y^6}}$.
4. $\sqrt[3]{\sqrt{6a^3}}$.
5. $\sqrt[4]{\sqrt[5]{a^6}}$.
6. $\sqrt[2]{\sqrt[4]{8}}$.
7. $\sqrt{\sqrt{\sqrt{a^{14}}}}$.
8. $\sqrt[3]{a^2\sqrt{a^3}}$.
9. $\sqrt[3]{8\sqrt{8x^6}}$.
10. $-\sqrt{-\sqrt[3]{a^6b^{12}c^{18}}}$.
11. $\sqrt[n]{\sqrt{x^n}}$.
12. $\sqrt{2\sqrt{2}}$.
13. $2\sqrt{2\sqrt{2\sqrt{2}}}$.
14. $\sqrt[a]{\sqrt[b]{\sqrt[c]{x^{ab}}}}$.
15. $3\sqrt[3]{3\sqrt[3]{3}}$.
16. $\sqrt{2x^2\sqrt{2y^4\sqrt{2z^8}}}$.

PROPERTIES OF QUADRATIC SURDS

50.* In the statements and proofs of the following principles, the radicands are restricted to positive commensurable values.

(i.) *The product or the quotient of two similar quadratic surds is rational.*

For if a, b, and c be rational numbers,

$$a\sqrt{c} \times b\sqrt{c} \equiv ab\sqrt{c^2} \equiv abc.$$

$$\frac{a\sqrt{c}}{b\sqrt{c}} \equiv \frac{a}{b}.$$

(ii.) *The product or the quotient of two dissimilar quadratic surds is a quadratic surd.*

For, in simplest form, every quadratic surd has as a radicand one or more prime factors raised to the first power only.

Two dissimilar surds cannot have all of these factors alike, and accordingly their product must, after it is simplified, have at least one of these factors to the first degree as a radicand.

From this it follows that

(iii.) *The sum or the difference of two dissimilar quadratic surds cannot be equal either to a rational number or to a single surd.*

---

* This section may be omitted when the chapter is read for the first time.

$$\sqrt{a} \pm \sqrt{b} \neq c, \quad \text{when} \quad a \neq b, \tag{1}$$

and $$\sqrt{a} \pm \sqrt{b} \neq \sqrt{d}, \qquad a \neq b. \tag{2}$$

For if $\sqrt{a} \pm \sqrt{b} = c$, we should have

$$a \pm 2\sqrt{a}\,\sqrt{b} + b = c^2,$$

or $$\sqrt{a}\,\sqrt{b} = \frac{c^2 - a - b}{\pm 2}.$$

Hence, we should have the product of two dissimilar quadratic surds equal to a rational number, which is impossible by (ii.) above.

It follows that $\sqrt{a} \pm \sqrt{b}$ cannot be equal to c, when $a \neq b$.

A similar method of proof holds for (2).

(iv.) *The square root of a rational number cannot be expressed as the sum of another quadratic surd and a rational number.*

That is, if $\sqrt{a}$ and $\sqrt{b}$ are quadratic surds, and c is any rational number, it follows that $\sqrt{a} \neq \sqrt{b} + c$.

For, if $$\sqrt{a} = \sqrt{b} + c \tag{1}$$

we have, squaring, $$a = b + 2c\sqrt{b} + c^2.$$

Therefore, $$\sqrt{b} = \frac{a - b - c^2}{2c}. \tag{2}$$

That is, if (1) be true, we have in (2) a surd number $\sqrt{b}$ equal to a rational expression, which is impossible.

Accordingly, $\sqrt{a}$ cannot be equal to $\sqrt{b} + c$.

(v.) *In any equation containing quadratic surds and rational numbers, the surd numbers in one member are equal to the surd numbers in the other member, and the rational numbers in one member are equal to the rational numbers in the other member.*

That is, if $\sqrt{x} + y = \sqrt{a} + b$, (1)
it follows that $x = a$, and $y = b$, where a, b, x, and y are all commensurable numbers and $\sqrt{a}$ and $\sqrt{x}$ are surds.

For, if $y \neq b$, let $y = b \pm n$, where $n \neq 0$. (2)

Substituting $b \pm n$ for y in (1), we obtain,

$$\sqrt{x} + b \pm n = \sqrt{a} + b.$$

Or, $$\sqrt{x} = \sqrt{a} \mp n,$$

which is impossible by Principle (iv.) above.

Hence we cannot assume that y is different from b, as in (2) above.

It follows that $y = b$, and hence $\sqrt{x} = \sqrt{a}$, or $x = a$.

(vi.) If
$$\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}, \qquad (1)$$
then
$$\sqrt{a - \sqrt{b}} = \sqrt{x} - \sqrt{y}, \qquad (2)$$
provided that a, b, x, and y are commensurable, and that $a > \sqrt{b}$.

For, from (1), $$a + \sqrt{b} = x + 2\sqrt{xy} + y.$$

Hence, by Principle (v.)
$$a = x + y, \text{ and } \sqrt{b} = 2\sqrt{xy}.$$

Hence $$a - \sqrt{b} = x + y - 2\sqrt{xy}.$$

Or $$\sqrt{a - \sqrt{b}} = \sqrt{x} - \sqrt{y}.$$

51.* Square Root of a Binomial Quadratic Surd may be obtained by applying the principles of § 50.

Ex 1. Find the square root of $14 + 2\sqrt{33}$.

Let $$\sqrt{14 + 2\sqrt{33}} = \sqrt{x} + \sqrt{y}. \qquad (1)$$

Then by (vi.), $$\sqrt{14 - 2\sqrt{33}} = \sqrt{x} - \sqrt{y}. \qquad (2)$$

Multiplying the members of (1) by the corresponding members of (2) we have,
$$\sqrt{196 - 132} = x - y.$$

Or $$x - y = 8. \qquad (3)$$

From (1), squaring, $$14 + 2\sqrt{33} = x + 2\sqrt{xy} + y. \qquad (4)$$

By (v.), $$x + y = 14. \qquad (5)$$

Solving (3) and (5), $$x = 11, \quad y = 3.$$

Hence, from (1), $$\sqrt{14 + 2\sqrt{33}} = \sqrt{11} + \sqrt{3}.$$

52.* Solution by Inspection.

From the identity, $(\sqrt{a} \pm \sqrt{b})^2 \equiv a + b \pm 2\sqrt{ab}$,

it follows that, $$\sqrt{a} \pm \sqrt{b} \equiv \sqrt{a + b \pm 2\sqrt{ab}}.$$

It should be observed that the expression $a + b \pm 2\sqrt{ab}$ consists of a surd term $\pm 2\sqrt{ab}$ and a rational binomial $a + b$.

---

* This section may be omitted when the chapter is read for the first time.

The radicand of the surd term $\pm 2\sqrt{ab}$ is the product of two factors a and b of which the sum is the rational binomial $a + b$.

Hence, to find by inspection the square root of a binomial surd which is a square, we may proceed as follows:

Transform the given binomial quadratic surd so that the coefficient of the surd term shall be 2; *then find by inspection two factors of the radicand of the surd term of which the sum is the rational term of the transformed binomial surd.*

The square root required is the sum or the difference of the square roots of the numbers thus obtained, according as the given binomial surd is a sum or a difference.

Ex. 2. Find by inspection the square root of $53 - 10\sqrt{6}$.

We have $$\sqrt{53 - 10\sqrt{6}} = \sqrt{53 - 2\sqrt{150}}.$$

The two factors of 150, of which the sum is 53, are 50 and 3.

Hence $$\sqrt{53 - 10\sqrt{6}} = \sqrt{50} - \sqrt{3} = 5\sqrt{2} - \sqrt{3}.$$

53.* It may be shown that, if x and y are positive rational numbers, $\sqrt{x \pm \sqrt{y}}$ can be expressed as a simple binomial surd, provided that $x^2 - y$ is the square of a rational number.

EXERCISE XX. 15

(This exercise may be omitted when the chapter is read for the first time.)

Find the square root of each of the following binomial quadratic surds:

1. $8 - 2\sqrt{15}$.
2. $7 - 2\sqrt{12}$.
3. $17 + 2\sqrt{70}$.
4. $5 + \sqrt{24}$.
5. $13 - \sqrt{168}$.
6. $19 - \sqrt{192}$.
7. $64 + 6\sqrt{7}$.
8. $18 - 8\sqrt{5}$.
9. $32 + 10\sqrt{7}$.
10. $18 - 3\sqrt{20}$.
11. $28 + 7\sqrt{12}$.
12. $51 + 7\sqrt{8}$.
13. $88 - 9\sqrt{28}$.
14. $6 + 3\sqrt{3}$.
15. $3 + \sqrt{5}$.

---

* This section may be omitted when the chapter is read for the first time.

CHAPTER XXI

IMAGINARY AND COMPLEX NUMBERS

I. IMAGINARY NUMBERS

1. AN even root of a negative number cannot be expressed either as a positive or as a negative number. (See Chap. XVIII. §§ 7, 14.)

By the Law of Signs in multiplication the product of an even number of positive or of negative numbers is positive; hence a negative number cannot result as the product of an even number of positive or of negative factors alone.

E. g. Since $(\pm 5)^2 = 25$, $\sqrt{-25}$ cannot be expressed either as a positive or as a negative number.

2. In order that an indicated even root of a negative number may be admitted to our calculations, we shall assume that the identity $(\sqrt[r]{a})^r \equiv a$ holds without exception for negative as well as for positive values of the radicand a. (See Chap. XVIII. § 10.)

E. g. $\sqrt{-1}$ is defined to be such a number that its square shall equal -1, that is, $(\sqrt{-1})^2 = -1$.

3. An even root of a negative number is called an **imaginary number.**

E. g. $\sqrt{-1}$, $\sqrt{-2}$, $\sqrt{-4}$, $\sqrt{-x}$, and $\sqrt[2n]{-a}$ are all imaginary numbers.

4. The initial letter i of the word *imaginary* is commonly used to represent $\sqrt{-1}$, which is taken as the **unit of imaginary numbers.** Hence $i^2 = -1$. (See § 2.)

5. Imaginary numbers have no existence in an arithmetic sense, and hence, when first introduced into mathematical science, were called "imaginary" numbers before their meaning and use in connection with other "kinds" of number were understood.

6. To distinguish them from imaginary numbers, all other numbers previously defined, — such as rational or irrational numbers, whether they be positive or negative, integral or fractional, — are called **real numbers.**

Certain other names have been suggested for "imaginary" numbers and "real" numbers, but we shall employ these commonly accepted terms.

7. In order to be able to operate with imaginary numbers by the same rules as with real numbers, we must assume that all positive or negative multiples, or fractional parts of the unit of imaginaries, $\sqrt{-1}$ or i, are numbers, and that they obey all of the Laws of Algebra.

E. g. Just as

$$4 = 1 + 1 + 1 + 1, \text{ so } 4\sqrt{-1} = \sqrt{-1} + \sqrt{-1} + \sqrt{-1} + \sqrt{-1};$$

and as

$$\tfrac{3}{4} = \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{4}, \qquad \text{so } \tfrac{3}{4}\sqrt{-1} = \tfrac{1}{4}\sqrt{-1} + \tfrac{1}{4}\sqrt{-1} + \tfrac{1}{4}\sqrt{-1}.$$

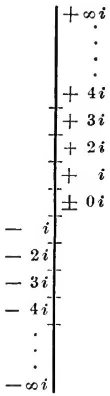
8. Multiplication by *i* may be defined by assuming that the Commutative Law holds for imaginary numbers:

that is, $$a \times \sqrt{-1} \equiv \sqrt{-1} \times a.$$

Or $$ai \equiv ia.$$

E. g. $3 \times \sqrt{-1} = \sqrt{-1} \times 3 = \sqrt{-1} + \sqrt{-1} + \sqrt{-1}.$

By multiplying each of the numbers of the extended series of whole numbers by i, we may form the series of purely imaginary whole numbers.



9. Powers of *i*. It should be observed that

$$\sqrt{-1}\ \sqrt{-1} = \sqrt{(-1)^2} = -1,$$

and that $$\sqrt{-1}\ \sqrt{-1} = \sqrt{(-1)^2} \neq \sqrt{+1^2} = +1.$$

This is because, whenever the radicand is known to be the square of a negative number, such as $(-1)^2$, the square root must be a negative number, -1. (See Chapter XVIII. § 13.)

We may obtain the following powers by multiplication:

$$\begin{aligned}
\sqrt{-1} &= \sqrt{-1},\\
(\sqrt{-1})^2 &= \quad -1,\\
(\sqrt{-1})^3 &= (\sqrt{-1})^2(\sqrt{-1}) = (-1)\sqrt{-1} = -\sqrt{-1},\\
(\sqrt{-1})^4 &= (\sqrt{-1})^3(\sqrt{-1}) = (-\sqrt{-1})\sqrt{-1}\\
&\qquad\qquad = -(\sqrt{-1})^2 = -(-1) = +1,\\
(\sqrt{-1})^5 &= (\sqrt{-1})^4\sqrt{-1} = \sqrt{-1}.
\end{aligned}$$

The values of the first four powers of $\sqrt{-1}$ are all different, and the value of the fifth power is the same as that of the first. Consequently, since each power is obtained by multiplying the power next preceding it by $\sqrt{-1}$, it follows that the results obtained above must recur in groups of four different values.

That is,

$$\begin{aligned}
i &\equiv i^5 \equiv i^9 \equiv i^{13} \equiv i^{17} \equiv \cdots\cdots \equiv i^{4n+1} \equiv +\sqrt{-1},\\
i^2 &\equiv i^6 \equiv i^{10} \equiv i^{14} \equiv i^{18} \equiv \cdots\cdots \equiv i^{4n+2} \equiv -1,\\
i^3 &\equiv i^7 \equiv i^{11} \equiv i^{15} \equiv i^{19} \equiv \cdots\cdots \equiv i^{4n+3} \equiv -\sqrt{-1},\\
i^4 &\equiv i^8 \equiv i^{12} \equiv i^{16} \equiv i^{20} \equiv \cdots\cdot \equiv i^{4n} \quad \equiv +1,
\end{aligned}$$

n being zero or any positive integer.

From the reasoning above, it follows that:

(a) *Any even power of i is equal to one of the real numbers* -1 *or* $+1$, *and any odd power of i is equal to one of the imaginary numbers* $+\sqrt{-1}$ *or* $-\sqrt{-1}$.

(b) *According as the remainder obtained by dividing the exponent m of a given power of i by 4 is* 1, 2, 3, *or* 0, *the value of i^m is i, i^2, i^3 or i^4; that is, i, -1, $-i$, or $+1$, respectively.*

E. g. $\quad i^{27} \equiv i^{4\cdot 6+3} \equiv i^3 \equiv -i;\ i^{34} \equiv i^{4\cdot 8+2} \equiv i^2 \equiv -1.$

10. As a result of assuming that the Commutative Law holds, we have the Distributive Law $(x \pm y)\,i \equiv xi \pm yi$; and also the Associative Law $\quad xiyi \equiv i^2xy \equiv -xy.$

11. Division by *i*. To conform with the definition of division, $\frac{ni}{i}$ must be such a number that, when multiplied by i, the product is ni.

By our definition, $n \times i \equiv ni.$

Hence $n \times i \div i \equiv ni \div i.$

Or $n \equiv \frac{ni}{i}.$

12. The square root of any negative number is an imaginary number and may be expressed in the form $\sqrt{-a} \equiv i\sqrt{a}$.

For, since $(\sqrt{a}\sqrt{-1})^2 \equiv (\sqrt{a})^2(\sqrt{-1})^2 \equiv -a$

and $(\sqrt{-a})^2 \equiv -a,$

it follows that $(\sqrt{-a})^2 \equiv (\sqrt{a}\sqrt{-1})^2.$

Accordingly, for principal values of the roots,

$$\sqrt{-a} \equiv \sqrt{-1}\sqrt{a} \equiv i\sqrt{a}.$$

Of the two values, $+\sqrt{-a}$ and $-\sqrt{-a}$, of the square root of any given negative number $-a$, the first one, $+\sqrt{-a}$, is selected as denoting the *principal value* of the square root of the given negative number.

Ex. 1. $\sqrt{-25} = \sqrt{25(-1)} = \sqrt{25}\sqrt{-1} = 5\sqrt{-1} = 5i.$

Ex. 2. $3\sqrt{-36} = 3\sqrt{36(-1)} = 3\sqrt{36}\sqrt{-1} = 3 \cdot 6\sqrt{-1} = 18i.$

MENTAL EXERCISE XXI. 1

Reduce each of the following powers of i to one of the numbers $1, -1, i$ or $-i$:

1. i^{21}.
2. i^{23}.
3. i^{34}.
4. i^{36}.
5. i^{39}.
6. i^{43}.
7. i^{50}.
8. $-i^{59}$.
9. $-i^{60}$.
10. i^{73}.
11. i^{82}.
12. i^{101}.
13. $(\sqrt{-1})^9$.
14. $(\sqrt{-1})^{12}$.
15. $(\sqrt{-1})^{14}$.
16. $(\sqrt{-1})^{15}$.
17. $(\sqrt{-1})^{25}$.
18. $(\sqrt{-1})^{32}$.
19. $(\sqrt{-1})^{27}$.
20. $(\sqrt{-1})^{53}$.
21. $-(\sqrt{-1})^{62}$.
22. $(\sqrt{-1})^{84}$.
23. $(\sqrt{-1})^{99}$.
24. $(\sqrt{-1})^{102}$.

Express each of the following imaginary numbers as a multiple of the unit of imaginary numbers $\sqrt{-1}$:

25. $\sqrt{-9}$.
26. $\sqrt{-16}$.
27. $\sqrt{-25}$.
28. $2\sqrt{-36}$.
29. $-3\sqrt{-49}$.
30. $5\sqrt{-64}$.
31. $-7\sqrt{-81}$.
32. $9\sqrt{-100}$.
33. $4\sqrt{-121}$.
34. $\sqrt{-a^2}$.
35 $\sqrt{-b^2}$.
36. $2\sqrt{-c^2}$.
37. $-3\sqrt{-d^2}$.
38. $x\sqrt{-y^2}$.
39. $-a\sqrt{-z^2}$.
40. $\sqrt{-4a^2}$.

41. $\sqrt{-49b^2}$.
42. $x\sqrt{-x^2}$.
43. $y\sqrt{-y^2}$.
44. $-c\sqrt{-c^2}$.
45. $\sqrt{-a^4}$.
46. $\sqrt{-b^6}$.
47. $\sqrt{-c^8}$.
48. $\sqrt{-d^{10}}$.
49. $\sqrt{-h^{12}}$.
50. $\sqrt{-4a^4}$.
51. $\sqrt{-9b^4}$.
52. $\sqrt{-16c^6}$.
53. $\sqrt{-25d^8}$.
54. $2\sqrt{-36x^4}$.
55. $3\sqrt{-49y^4}$.
56. $5\sqrt{-64z^8}$.
57. $(\sqrt{-a})^{16}$.
58. $(\sqrt{-b})^{19}$.
59. $(\sqrt{-c})^{29}$.
60. $(\sqrt{-x})^{34}$.

Addition and Subtraction of Imaginary Numbers

13. Since imaginary numbers may be written so as to appear as arithmetic multiples of the units i and $-i$, they may be combined by addition or subtraction by the same principles as real numbers.

In all operations involving imaginary numbers, it will be convenient to write the terms of given expressions as multiples of the unit of imaginaries, $\sqrt{-1} \equiv i$.

It should be observed that $\sqrt{-a} \equiv \sqrt{-1}\sqrt{a} \equiv i\sqrt{a}$.

Ex. 1. $\sqrt{-25}+\sqrt{-9}-\sqrt{-1}=5\sqrt{-1}+3\sqrt{-1}-\sqrt{-1}=7\sqrt{-1}.$

Or $\qquad =5i \quad +3i \quad -i \quad =7i.$

Ex. 2. $\sqrt{-16a^2}-\sqrt{-49b^2}\equiv 4ai-7bi\equiv(4a-7b)i.$

Ex. 3. $i^9+i^{11}\equiv i+(-i)\equiv 0.$

Ex. 4. $i^8+i^{20}\equiv 1+1\equiv 2.$

EXERCISE XXI. 2

Simplify each of the following:

1. $2\sqrt{-1}+3\sqrt{-1}$.
2. $\sqrt{-4}+\sqrt{-1}$.
3. $\sqrt{-16}+\sqrt{-9}$.
4. $\sqrt{-49}-\sqrt{-25}$.
5. $\sqrt{-64}+\sqrt{-100}$.
6. $\sqrt{-121}-\sqrt{-144}$.
7. $2\sqrt{-81}+3\sqrt{-1}$.
8. $4\sqrt{-9}-6\sqrt{-1}$.
9. $\sqrt{-169}-2\sqrt{-36}$.
10. $\sqrt{-196}-4\sqrt{-9}$.
11. $a\sqrt{-100}-2\sqrt{-49a^2}$.
12. $8\sqrt{-64x^2}-10x\sqrt{-25}$.
13. $\sqrt{-121}-2\sqrt{-49}+3\sqrt{-4}$.
14. $13\sqrt{-y^2}-\sqrt{-169y^2}+6\sqrt{-16y^2}$.
15. $\sqrt{-144a^2}-\sqrt{-64a^4}-\sqrt{-16a^6}$.
16. $9\sqrt{-4a^4}-4a^6\sqrt{-9a^{-8}}-a^8\sqrt{-36a^{-12}}$.
17. $\sqrt{-x^2}-\sqrt{-y^2}-\sqrt{-z^2}$.

18. $\sqrt{-2} + \sqrt{-18} - \sqrt{-50}$.

19. $\sqrt{-12} + 5\sqrt{-75} - 10\sqrt{-108}$.

20. $3\sqrt{-20} - 2\sqrt{-45} + 5\sqrt{-125}$.

21. $i^6 + i^7 + i^8 + i^9$.

22. $i^{12} - 11\,i^{10}$.

23. $3\,i^3 - 4\,i^4 + 5\,i^5$.

24. $6 - i^6$.

25. $7\,i - i^7$.

26. $i^{31} - i^{19}$.

27. $i^{17} - i^{71}$.

28. $13\,i^{31} - 31\,i^{13}$.

29. $i^2 + i^4 + i^8$.

30. $i^{16} - i^{14} - i^{10}$.

31. $8\,i^8 - 4\,i^4 + 2\,i^2$.

32. $i^3 + i^4 + i^5 + i^6$.

33. $i^{20} - i^{18} + i^{16} - i^{14}$.

34. $2\,i^2 - 3\,i^4 + 4\,i^8 - 5\,i^{10}$.

Multiplication of Imaginary Numbers

14. In case either the multiplier or multiplicand, or both, are imaginary numbers, the following principles apply:

$$\begin{cases} \text{(i.)} & \sqrt{a}\sqrt{-b} \equiv \sqrt{a}\,i\sqrt{b} \equiv i\sqrt{ab} \equiv \sqrt{-ab}. \\ \text{(ii.)} & \sqrt{-a}\sqrt{-b} \equiv i\sqrt{a}\,i\sqrt{b} \equiv i^2\sqrt{ab} \equiv -\sqrt{ab}. \end{cases}$$

Ex. 1. $\sqrt{-4} \times \sqrt{49} = 2i \times 7 = 14i$.

Ex. 2. $\sqrt{-2} \times \sqrt{-5} = i\sqrt{2} \times i\sqrt{5} = i^2\sqrt{10} = -\sqrt{10}$.

Exercise XXI. 3

Simplify each of the following:

1. $2\,i \times i$.

2. $3\,i \times i$.

3. $i \times 6\,i$.

4. $4\,i \times 5\,i$.

5. $8\,i \times 3\,i$.

6. $-7\,i \times i$.

7. $4\,i \times (-3\,i)$.

8. $-9\,i \times (-i)$.

9. $-10\,i \times (-2\,i)$.

10. $\sqrt{3} \times \sqrt{-2}$.

11. $\sqrt{5} \times \sqrt{-4}$.

12. $\sqrt{-2} \times \sqrt{-1}$.

13. $\sqrt{-1} \times \sqrt{-3}$.

14. $\sqrt{-2} \times \sqrt{-3}$.

15. $\sqrt{-5} \times \sqrt{-6}$.

16. $-\sqrt{-1} \times \sqrt{-2}$.

17. $-\sqrt{-1} \times (-\sqrt{-3})$.

18. $-2\sqrt{-3} \times (-3\sqrt{-2})$.

19. $2\sqrt{-4} \times 3\sqrt{-9}$.

20. $-\sqrt{-16} \times \sqrt{-1}$.

21. $\sqrt{-2} \times \sqrt{-6}$.

22. $\sqrt{-5} \times \sqrt{-10}$.

23. $\sqrt{-6} \times \sqrt{-3}$.

24. $\sqrt{-14} \times \sqrt{-7}$.

25. $\sqrt{-8} \times \sqrt{-3}$.

26. $ia \times i$.

27. $ib \times i$.

28. $ix \times iy$.
29. $-ia \times ib$.
30. $-ic \times (-iw)$.
31. $2ix \times im$.
32. $3ia \times iy$.
33. $ic \times 2id$.
34. $ix \times ix$.
35. $3ig \times 3ik$.
36. $-6ik \times 3iy$.
37. $\sqrt{-c} \times \sqrt{-1}$.
38. $ix^2 \times ix$.
39. $iab \times iac$.
40. $iabc \times ia$.
41. $iabc \times ibcd$.
42. $\sqrt{-d} \times \sqrt{-1}$.
43. $-\sqrt{-h} \times \sqrt{-1}$.
44. $\sqrt{-x} \times \sqrt{-y}$.
45. $\sqrt{-m} \times \sqrt{-n}$.
46. $\sqrt{-a^2} \times \sqrt{-1}$.
47. $\sqrt{-2a} \times \sqrt{-y}$.
48. $\sqrt{-3b} \times \sqrt{-c}$.
49. $2\sqrt{-c} \times \sqrt{-3c}$.
50. $5\sqrt{-5d} \times \sqrt{-2h}$.
51. $\sqrt{-3a} \times \sqrt{-3b}$.
52. $\sqrt{-5c} \times \sqrt{-6c}$.
53. $\sqrt{-10a} \times \sqrt{-2a}$.
54. $\sqrt{-3b} \times \sqrt{-6b}$.
55. $\sqrt{-21d} \times \sqrt{-7d}$.
56. $\sqrt{-xy} \times \sqrt{-x}$.
57. $\sqrt{-xyz} \times \sqrt{-xz}$.
58. $x\sqrt{-yz} \times xy\sqrt{-z}$.
59. $x\sqrt{-x^2} \times x^3\sqrt{-x^4}$.
60. $3i \times 2i \times i$.
61. $5i \times 3i \times i$.
62. $6i \times 4i \times i$.
63. $-7i \times 5i \times i$.
64. $-8i \times 3i \times 2i$.
65. $ia \times ib \times ic$.
66. $ix \times iy \times iz$.
67. $2ix \times 4ix \times 6ix$.
68. $ia^3 \times ia^2 \times ia$.
69. $\sqrt{-3} \times \sqrt{-2} \times \sqrt{-5}$.
70. $\sqrt{-9} \times \sqrt{-4} \times \sqrt{-1}$.
71. $\sqrt{-25} \times \sqrt{-49} \times \sqrt{-16}$.
72. $\sqrt{-64} \times \sqrt{-36} \times \sqrt{-9}$.
73. $\sqrt{-12} \times \sqrt{-3} \times \sqrt{-1}$.
74. $\sqrt{-14} \times \sqrt{-7} \times \sqrt{-2}$.
75. $\sqrt{-15} \times \sqrt{-5} \times \sqrt{-3}$.
76. $\sqrt{-a} \times \sqrt{-b} \times \sqrt{-c}$.
77. $\sqrt{-2c} \times \sqrt{-3d} \times \sqrt{-k}$.
78. $\sqrt{-3x} \times \sqrt{-4y} \times \sqrt{-5z}$.
79. $\sqrt{-3abc} \times \sqrt{-2ab} \times \sqrt{-a}$.
80. $\sqrt{-x^2} \times \sqrt{-y^2} \times \sqrt{-z^2}$.
81. $(-\sqrt{-a^2})(-\sqrt{-b^2})(-\sqrt{-c^2})$.
82. $ia \times ib \times ic \times id$.
83. $ib \times id \times ix \times iy$.
84. $2ia \times 3ih \times im \times i$.

85. $iabc \times iab \times ia \times i$.

86. $5\,ib \times 4\,ib \times ic \times i$.

87. $\sqrt{-2} \times \sqrt{-3} \times \sqrt{-4} \times \sqrt{-1}$.

88. $\sqrt{-5} \times \sqrt{-6} \times \sqrt{-2} \times \sqrt{-1}$.

89. $\sqrt{-6} \times \sqrt{-2} \times \sqrt{-3} \times \sqrt{-1}$.

90. $\sqrt{-10} \times \sqrt{-5} \times \sqrt{-2} \times \sqrt{-1}$.

91. $i^3 \times i$.

92. $i^4 \times i^2$.

93. $i^5 \times i^4$.

94. $2\,i^6 \times 4\,i^3$.

95. $-7\,i^7 \times 3\,i^3$.

96. $i^4 \times i^3 \times i^2$.

97. $i^5 \times i^6 \times i^8$.

98. $-i^3 \times i^5 \times i^7$.

99. $3\,i^4 \times 4\,i^5 \times i^2$.

100. $5\,i^3 \times 4\,i^4 \times 3\,i^5$.

101. $-7\,i^8 \times 5\,i^6 \times 3\,i^4$.

102. $i^2 \times i^4 \times i^6 \times i^8$.

103. $2\,i^5 \times 3\,i^4 \times 4\,i^3 \times 5\,i^2$.

Division of Imaginary Numbers

15. In case either the dividend or divisor, or both, are imaginary numbers, the following principles apply:

$$\left\{\begin{array}{ll}
\text{(i.)} & \dfrac{\sqrt{-a}}{\sqrt{b}} \equiv \dfrac{i\sqrt{a}}{\sqrt{b}} \equiv i\sqrt{\dfrac{a}{b}} \equiv \sqrt{-\dfrac{a}{b}}. \\[2ex]
\text{(ii.)} & \dfrac{\sqrt{a}}{\sqrt{-b}} \equiv \dfrac{\sqrt{a}}{i\sqrt{b}} \equiv \dfrac{i}{i^2}\sqrt{\dfrac{a}{b}} \equiv -\sqrt{-\dfrac{a}{b}}. \\[2ex]
\text{(iii.)} & \dfrac{\sqrt{-a}}{\sqrt{-b}} \equiv \dfrac{i\sqrt{a}}{i\sqrt{b}} \equiv \sqrt{\dfrac{a}{b}}.
\end{array}\right.$$

Ex. 1. $$\frac{\sqrt{-12}}{\sqrt{6}} \equiv \frac{i\sqrt{12}}{\sqrt{6}} \equiv i\sqrt{2}.$$

Ex. 2. $$\frac{\sqrt{-2}}{\sqrt{-3}} \equiv \frac{i\sqrt{2}}{i\sqrt{3}} \equiv \sqrt{\frac{2}{3}} \equiv \frac{1}{3}\sqrt{6}.$$

Ex. 3. $$\frac{1}{-i} \equiv -\frac{i}{i^2} \equiv -\frac{i}{-1} \equiv +i.$$

16. In the last example, the operation of "realizing" the imaginary denominator, or making it "real," suggests the operation of "rationalizing" an irrational denominator.

Exercise XXI. 4

Simplify each of the following:

1. $\sqrt{-22} \div \sqrt{11}$.
2. $\sqrt{-39} \div \sqrt{13}$.
3. $\sqrt{-30} \div \sqrt{3}$.
4. $\sqrt{-6} \div \sqrt{-2}$.
5. $\sqrt{-14} \div \sqrt{-7}$.
6. $\sqrt{-15} \div \sqrt{-3}$.
7. $-\sqrt{-26} \div \sqrt{-13}$.
8. $\sqrt{-28} \div (-\sqrt{-7})$.
9. $\sqrt{-20} \div \sqrt{-5}$.
10. $12\,i \div 3\,i$.
11. $18\,i \div 6\,i$.
12. $24\,i \div 8\,i$.
13. $35\,i \div 5\,i$.
14. $\sqrt{-7} \div \sqrt{-2}$.
15. $\sqrt{-6} \div \sqrt{-5}$.
16. $\sqrt{-1} \div \sqrt{-10}$.
17. $\sqrt{-11} \div \sqrt{-3}$.
18. $2\sqrt{-5} \div \sqrt{-4}$.
19. $3\sqrt{-6} \div \sqrt{-9}$.
20. $7\sqrt{-14} \div \sqrt{-2}$.
21. $9\sqrt{-16} \div 16\sqrt{-9}$.
22. $49\sqrt{-25} \div 25\sqrt{-49}$.
23. $\sqrt{-a} \div \sqrt{-1}$.
24. $-\sqrt{-b} \div \sqrt{-1}$.
25. $\sqrt{-c} \div (-\sqrt{-1})$.
26. $-\sqrt{-d} \div (-\sqrt{-1})$.
27. $\sqrt{-abc} \div \sqrt{-a}$.
28. $-\sqrt{-x} \div \sqrt{-y}$.
29. $\sqrt{-a^2} \div \sqrt{-b^2}$.
30. $-\sqrt{-x^2} \div \sqrt{-y^2}$.
31. $\sqrt{-a^2} \div \sqrt{-a}$.
32. $\sqrt{-b} \div \sqrt{-b^2}$.
33. $\sqrt{-abc} \div \sqrt{-bcd}$.
34. $x\sqrt{-y} \div y\sqrt{-x}$.
35. $a\sqrt{-a} \div b\sqrt{-b}$.
36. $i\sqrt{7} \div i\sqrt{10}$.
37. $i\sqrt{2} \div i\sqrt{5}$.
38. $-i\sqrt{11} \div i\sqrt{3}$.
39. $\frac{2}{\sqrt{-3}}$.
40. $\frac{4}{\sqrt{-5}}$.
41. $\frac{6}{\sqrt{-2}}$.
42. $\frac{12}{\sqrt{-6}}$.
43. $\frac{11}{\sqrt{-11}}$.
44. $\frac{-9}{\sqrt{-3}}$.
45. $\frac{1}{i}$.
46. $\frac{2}{i^2}$.
47. $\frac{3}{i^3}$.
48. $\frac{4}{i^4}$.
49. $\frac{5}{i^5}$.
50. $\frac{8}{i^8}$.
51. $\frac{9}{i^9}$.
52. $\frac{11}{i^{12}}$.
53. $\frac{13}{i^{10}}$.

54. $-\frac{15}{i^{14}}$. 55. $\frac{6}{i^7}$. 56. $-\frac{1}{i^6}$.

17. Both negative and fractional numbers were included in our extended number system by bringing in the idea of measuring distances or counting in opposite directions.

From the point of view of the primary numbers 1, 2, 3, 4, etc., negative numbers and fractional numbers both have an existence as imaginary as "imaginary numbers."

It remains for us to show that imaginary numbers may be given a graphical interpretation.

18. By means of the principle of geometry that *in a right triangle the square on the hypotenuse is equivalent to the sum of the squares on the remaining two sides*, we may represent graphically any surd number.

E. g. If in a right triangle the sides including the right angle are each one unit in length, the hypotenuse has a length represented by $\sqrt{2}$. (See Fig. 1.)

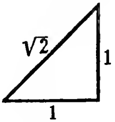


Fig. 1

Using the length thus found for $\sqrt{2}$, we may find a length representing $\sqrt{3}$. (See Fig. 2.)

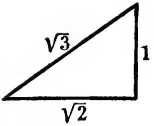


Fig. 2.

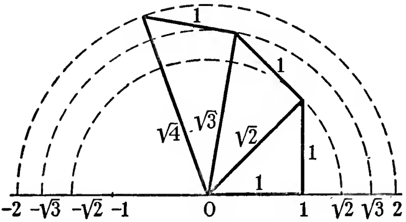


Fig. 3.

By setting off lengths thus found along a straight line, as in Fig. 3, definite points can be located on the line, representing $+\sqrt{2}$, $+\sqrt{3}$, $+\sqrt{4} = 2$, $-\sqrt{2}$, $-\sqrt{3}$, $-\sqrt{4} = -2$, etc.

By separating the line from 0 to 1 into equal parts, we may locate points representing positive fractions, such as $+\frac{1}{3}$, $+\frac{1}{2}$, $+\frac{3}{2}$, etc.

In a similar way points representing negative fractions, such as $-\frac{1}{2}$, $-\frac{6}{7}$, etc., may be located.

Graphical Representation of Imaginary Numbers

19. Consider the following products consisting of a positive arithmetic number a multiplied by -1 and by $(-1)^2$.

Multiplying a by -1, *once*, $a \times (-1) \equiv -a$.

Multiplying a by -1, *twice*, $a \times (-1)(-1) \equiv a \times (+1) \equiv +a$.

It appears that multiplying a by -1 *once* reverses the quality of a, while multiplying by -1 *twice successively* reverses and then restores the quality of a.

Accordingly, if $+a$ represents a length OA_1, measured along our "carrier line" in the *positive* direction, since multiplying by -1 reverses the quality of $+a$, to represent $-a$ we must measure an equal length OA_3 in the opposite or *negative* direction from the same starting point or *origin*, O.

We may think of the line a as having made a "half revolution" about the *origin* O from the position OA_1 to the position OA_3.

Multiplying $-a$ by -1 will produce a second "half revolution;" hence multiplying $+a$ by -1 *twice successively* may be thought of as producing a "complete revolution" of a about the origin O.

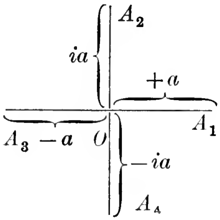
20. Observe that by multiplying $+a$ by $\sqrt{-1} \equiv i$ *once*, *twice*, *three times*, and *four times* successively, we obtain the numbers $+ia$, $-a$, $-ia$ and $+a$, respectively, as follows:

$$\begin{aligned} ai &\equiv +ia, \\ aii &\equiv -a, \\ aiii &\equiv -ia, \\ aiiii &\equiv +a. \end{aligned}$$

It appears that the quality of a may be reversed either by multiplying by i *twice* or by multiplying by -1 *once*; and the quality of a remains unaltered when a is multiplied by i *four times* or by -1 *twice* successively.

If a represents a given distance measured along a fixed line in one direction, it may be seen that the quality of a may be reversed by turning a about one of its end points through a *half revolution*; and the quality of a remains unaltered when a is turned about one of its end points through a *complete revolution*.

Hence, since performing the same algebraic operation *two times* (multiplying a by $\sqrt{-1}$ two times) has the same effect on a as turning a about one of its end points through a *half revolution*, it may be seen that it is consistent to interpret multiplying a by i *once* as causing a *quarter revolution* of the line a about one of its end points.



If we let OA_1 represent the original position of the line a, then, having made a *quarter revolution*, OA_2 will represent ia; at a *half revolution*, OA_3 will represent $-a$; at a *three-quarters revolution*, OA_4 will represent $-ia$; while OA_1 will represent the position of the line a after having made a *complete revolution* about the point O.

It is customary to suppose the line OA_1 to swing *upward* and around the point O in the direction opposite to the movement of the hands of a clock, that is, *counter-clockwise.*

It follows that the positive imaginary number $+ia$ may be represented graphically by measuring a length "upward" along a line *at right angles to a in its original position*, OA_1. The negative imaginary number $-ia$ will accordingly be represented by measuring *downward* along the same perpendicular line a distance equal to a.

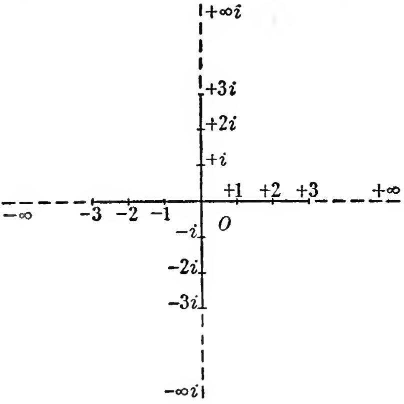


FIG. 4.

The line A_3OA_1 is called the **axis of real numbers** to distinguish it from the **axis of imaginary numbers,** A_4OA_2, which is drawn at right angles to the line A_3OA_1.

We may represent the two series of numbers, "real and imaginary," as in the accompanying figure. It should be observed that the series of real numbers and the series of imaginary numbers have in common the value zero, and no other value. This value zero is represented by the point O, which is the intersection of the axis of real numbers and the axis of imaginary numbers.

21. One imaginary number, ia, is said to be equal to another, ib, if $a = b$. We cannot say that a *real* number is equal to, greater than, or less than an *imaginary* number.

22. Negative, fractional, surd, and imaginary "numbers" are all "extended" or "invented" numbers with reference to arithmetic numbers, that is, to "positive whole numbers" which alone are the result of counting. Of these "artificial" or "invented" numbers, each may have a graphical interpretation, and accordingly from this point of view one "form" is no more "imaginary" than another.

II. Complex Numbers

23. The algebraic sum of a real number and an imaginary number is called a **complex number.**

E. g. $2 + \sqrt{-3}$, $5 - 8i$, $a \pm ib$.

If a and b represent real numbers, the general expression for a complex number is $a + ib$.

In particular, if $a = 0$, $a + ib$ becomes ib, which is an *imaginary* number.

If $b = 0$, $a + ib$ becomes a, which is a *real* number.

24. Two complex numbers, which differ only in the signs of the terms containing the imaginary unit i, are called **conjugate complex numbers.**

E. g. $a + ib$, and $a - ib$ are conjugate complex numbers.

25. Two complex numbers, $x + iy$ and $a + ib$, are said to be **equal** if their real terms, x and a, are equal, and their imaginary terms, iy and ib, are equal.

26. *In order that a complex number, $x + iy$, shall be equal to zero, it is necessary and sufficient that the real part shall equal zero and the imaginary part shall equal zero.*

For if $x + iy = 0 = 0 + i0$, then by § 25, $x = 0$, and also $y = 0$.

27. If either x or y is infinite, the complex number $x + iy$ is said to be infinite.

28. The Four Fundamental Operations Involving Complex Numbers are defined by assuming that the fundamental laws of algebra, as proved for real numbers, apply also for complex numbers.

29. Addition and Subtraction of Complex Numbers. *The sum of two complex numbers is in general a complex number obtained by adding the real parts and the imaginary parts separately.*

For, since complex numbers are assumed to obey the Fundamental Laws of Algebra, it may be seen that

$$(a + ib) \pm (x + iy) \equiv (a \pm x) + i(b \pm y).$$

The principle applies for three or more complex numbers.

Ex. 1. $(3 + 5\sqrt{-1}) + (6 - 2\sqrt{-1}) = (3 + 6) + (5 - 2)\sqrt{-1}$
$= 9 + 3i.$

30. *The sum of two conjugate complex numbers is a real number.*

For, $(a + ib) + (a - ib) \equiv (a + a) + i(b - b) \equiv 2a.$

31. Multiplication of Complex Numbers is defined by assuming that the Distributive Law for Multiplication (Chapter V. § 21) applies to complex numbers.

$$\begin{aligned}(a + ib)(x + iy) &\equiv (a + ib)x + (a + ib)iy\\ &\equiv ax + ibx + aiy + ibiy\\ &\equiv (ax - by) + i(bx + ay).\end{aligned}$$

32. It may be shown, by applying the Associative Law (Chapter III. § 4), *that in general the product of two or more complex numbers can be expressed as a complex number.*

Ex. 2. $(4 + 7i)(2 - 5i) \equiv 8 + 14i - 20i - 35i^2 \equiv 43 - 6i.$

Ex. 3. $(a + ib)^2 \equiv a^2 + 2aib + i^2b^2 \equiv a^2 - b^2 + 2iab.$

33. *The product of two conjugate complex numbers is a number which is real and positive.*

For, $(a + ib)(a - ib) \equiv a^2 - i^2b^2 \equiv a^2 + b^2.$

Ex. 4. $(-5+2\sqrt{-3})(-5-2\sqrt{-3})=(-5)^2-2^2(\sqrt{-3})^2=25+12=37.$

34. Division of Complex Numbers. *The quotient obtained by dividing one complex number by another can be expressed as a complex number.*

For,
$$\frac{a+ib}{x+iy} \equiv \frac{(a+ib)(x-iy)}{(x+iy)(x-iy)}$$
$$\equiv \frac{(ax+by)+i(bx-ay)}{x^2+y^2}$$
$$\equiv \frac{ax+by}{x^2+y^2} + i\,\frac{bx-ay}{x^2+y^2}.$$

35. From the reasoning above it appears that *a fraction the denominator of which is a complex number can be expressed as a complex number.*

Ex. 5.
$$\frac{2-3i}{4+5i} = \frac{(2-3i)(4-5i)}{(4+5i)(4-5i)}$$
$$= \frac{-7-22i}{41}$$
$$= -\tfrac{7}{41} - \tfrac{22}{41}i.$$

Ex. 6.
$$\frac{7-2\sqrt{-6}}{\sqrt{10}-3\sqrt{-5}} = \frac{(7-2i\sqrt{6})(\sqrt{10}+3i\sqrt{5})}{(\sqrt{10}-3i\sqrt{5})(\sqrt{10}+3i\sqrt{5})}$$
$$= \frac{7\sqrt{10}+6\sqrt{30}+i(21\sqrt{5}-4\sqrt{15})}{55}.$$

EXERCISE XXI. 5

Simplify each of the following expressions:

1. $(2+3i)+(5+2i)$.
2. $(5+8i)+(2+4i)$.
3. $(7-10i)+(2+9i)$.
4. $(4-11i)-(6+3i)$.
5. $(1-12i)-(12-i)$.
6. $(9-5i)-(5+9i)$.
7. $(3+4\sqrt{-1})+(7+2\sqrt{-1})$.
8. $(8-7\sqrt{-1})+(8-10\sqrt{-1})$.
9. $(4-12\sqrt{-1})-(9-11\sqrt{-1})$.
10. $(6-2\sqrt{-1})-(5+4\sqrt{-1})$.
11. $(2+\sqrt{-9})+(4+\sqrt{-16})$.
12. $(3+\sqrt{-25})+(6+\sqrt{-4})$.
13. $(5-\sqrt{-36})-(6+\sqrt{-49})$.
14. $(25-\sqrt{-36})-(36-\sqrt{-25})$.
15. $(8-\sqrt{-8})-(18-\sqrt{-18})$.
16. $(5+2\sqrt{-3})+(7-3\sqrt{-12})$.
17. $(11+2\sqrt{-20})-(4-3\sqrt{-180})$.
18. $(6-5\sqrt{-28})-(12+4\sqrt{-63})$.

19. $(a + 3i) + (2 + bi)$.
20. $(x - 4i) - (y + 6i)$.
21. $(m - 8i) - (n - 10i)$.
22. $(2a + 5bi) + (6a - bi)$.
23. $(5 + 3i)(5 - 3i)$.
24. $(-6 + 7i)(-6 - 7i)$.
25. $(7 + 4i)(7 - 4i)$.
26. $(8 + \sqrt{-2})(8 - \sqrt{-2})$.
27. $(7 + 3\sqrt{-5})(7 - 3\sqrt{-5})$.
28. $(3 + 2i)(4 + 5i)$.
29. $(8 - 6i)(2 + 5i)$.
30. $(2 - i)(1 - 2i)$.
31. $(\sqrt{2} + i)(\sqrt{3} + i)$.
32. $(\sqrt{2} + \sqrt{-1})(\sqrt{2} - \sqrt{-1})$.
33. $(\sqrt{3} + \sqrt{-5})(\sqrt{3} - \sqrt{-5})$.
34. $(3\sqrt{5} + i\sqrt{3})(3\sqrt{5} - i\sqrt{3})$.
35. $(6\sqrt{7} + 8\sqrt{-9})(6\sqrt{7} - 8\sqrt{-9})$.
36. $(2 + 3i)(\sqrt{5} + i\sqrt{6})$.
37. $(1 + i)^2$.
38. $(3 - i)^2$.
39. $(1 + i)^3$.
40. $(3 + 4i)^2$.
41. $1 \div (2 + 3i)$.
42. $5 \div (2 - \sqrt{-3})$.
43. $10 \div (\sqrt{5} - \sqrt{-2})$.
44. $51 \div (1 - 5\sqrt{-2})$.
45. $\dfrac{2 + 3i}{2 - 3i}$.
46. $\dfrac{9 + \sqrt{-9}}{8 + \sqrt{-8}}$.
47. $\dfrac{1 + \sqrt{-1}}{1 - \sqrt{-1}}$.
48. $(b + a\sqrt{-a})(a + b\sqrt{-b})$.
49. $(y\sqrt{y} + x\sqrt{-x})(x\sqrt{x} - y\sqrt{-y})$.

50. $(-1+\sqrt{-1})(-1-\sqrt{-1})$.
51. $(a\sqrt{-a}-b\sqrt{-b})^2$.
52. $(1-i)^2(1+i)$.
53. $(1-i)(2-i)(3-i)$.
54. $(1+i)(1-3i^3)(1+5i^5)$.

Complex Factors of Rational Integral Expressions.

36. The method of Chap. XX. § 48, may be applied to obtain factors of expressions of the form ax^2+bx+c which are the products of complex factors.

The following identity was obtained in Chapter XX. § 48:

$$ax^2+bx+c \equiv a\left[x+\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}\right]\left[x+\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}\right].$$

If the expression b^2-4ac be negative, $\sqrt{b^2-4ac}$ will be imaginary, and accordingly the factors of ax^2+bx+c, represented by the expressions in square brackets, will be complex.

Ex. 1. Factor x^2+3x+4.

We may complete the square with reference to x^2+3x by using $3x$ as a finder term as follows:

We have $$\frac{3x}{2x}\equiv\frac{3}{2}.$$

Hence,
$$\begin{aligned}x^2+3x+4 &\equiv x^2+3x+(\tfrac{3}{2})^2-\tfrac{9}{4}+4,\\ &\equiv (x+\tfrac{3}{2})^2+\tfrac{7}{4},\\ &\equiv (x+\tfrac{3}{2})^2-(\sqrt{-\tfrac{7}{4}})^2,\\ &\equiv (x+\tfrac{3}{2}+\tfrac{\sqrt{-7}}{2})(x+\tfrac{3}{2}-\tfrac{\sqrt{-7}}{2}).\end{aligned}$$

Square Root of a Complex Number.

37.* Corresponding to the principle employed when finding the square root of a simple binomial surd (see Chap. XX. § 50 (vi.)), we have the following

Principle: *If the square root of a complex number can be expressed as a complex number, then the square root of the conjugate complex number can also be expressed as a complex number.*

That is, if $$\sqrt{a+ib}=\sqrt{x}+i\sqrt{y}, \tag{1}$$

---

* This section may be omitted when the chapter is read for the first time.

it follows that $\sqrt{a - ib} = \sqrt{x} - i\sqrt{y}$. (2)

If a, b, x, and y are real numbers, we obtain, by squaring both members of (1), $a + ib = x - y + 2\,i\sqrt{xy}$.

Hence, by § 25, $a = x - y$, and also $b = 2\sqrt{xy}$.

Accordingly we may construct the expression

$$a - ib = x - y - 2\,i\sqrt{xy}$$
$$= (\sqrt{x} - i\sqrt{y})^2.$$

Hence, $\sqrt{a - ib} = \sqrt{x} - i\sqrt{y}$.

38.* It follows that *the square root of a complex number can be expressed as a complex number.*

For, from § 37, if $\sqrt{a + ib} = \sqrt{x} + i\sqrt{y}$, (1)

it follows that $\sqrt{a - ib} = \sqrt{x} - i\sqrt{y}$, (2)

in which a, b, x, and y are real numbers.

Hence, from (1) and (2) we obtain by multiplication,

$$\sqrt{(a + ib)(a - ib)} = (\sqrt{x} + i\sqrt{y})(\sqrt{x} - i\sqrt{y}),$$

or, $\sqrt{a^2 + b^2} = x + y$. (3)

Squaring both members of equation (1),

$$a + ib = x - y + 2\,i\sqrt{xy}. \quad (4)$$

Hence, by § 25, $a = x - y$. (5)

Solving equations (3) and (5) for x and y, we have

$$x = \frac{\sqrt{a^2 + b^2} + a}{2} \; (6) \text{ and } y = \frac{\sqrt{a^2 + b^2} - a}{2}. \quad (7)$$

Accordingly, from (1),

$$\sqrt{a + ib} = \sqrt{\frac{\sqrt{a^2 + b^2} + a}{2}} + i\sqrt{\frac{\sqrt{a^2 + b^2} - a}{2}}. \quad (8)$$

Since a and b are real numbers, it follows that $a^2 + b^2$ is positive; hence $\sqrt{a^2 + b^2}$ is a real number, and accordingly the right member of (8) is a complex number.

Ex. 2. Express $\sqrt{-i}$ as a complex number.

We may write $\sqrt{-i} \equiv \sqrt{0 - i}$.

---

* This section may be omitted when the chapter is read for the first time.

Comparing $\sqrt{0-i}$ with the form $\sqrt{a+bi}$, it appears that $a=0$ and $b=-1$; hence, carrying out the process shown above, or substituting immediately in (8), we obtain,

$$\sqrt{-i} \equiv \sqrt{0-i}$$

$$\sqrt{-i} \equiv \sqrt{\frac{\sqrt{0+(-1)^2}+0}{2}} + i\sqrt{\frac{\sqrt{0+(-1)^2}-0}{2}}$$

$$\equiv \sqrt{\frac{-1}{2}} + i\sqrt{\frac{-1}{2}}$$

$$\equiv i\sqrt{\frac{1}{2}} + i^2\sqrt{\frac{1}{2}}$$

$$\equiv i\sqrt{\frac{1}{2}} - \sqrt{\frac{1}{2}}$$

$$\equiv (i-1)\sqrt{\frac{1}{2}}$$

$$\equiv \frac{(i-1)\sqrt{2}}{2}.$$

Exercise XXI. 6

Find complex factors for each of the following:

1. a^2+b^2.
2. $25x^2+1$.
3. $x^2+6x+10$.
4. x^2-4x-8.
5. $2x^2+3x+4$.
6. $9x^2-8x-7$.

(The following examples may be omitted when the chapter is read for the first time.)

Express as complex numbers the square roots of the following complex numbers:

7. $-1+2\sqrt{-2}$.
8. $3-4i$.
9. $41-14\sqrt{-8}$.
10. $-36-9\sqrt{-48}$.
11. $-1-6\sqrt{-10}$.
12. $-1-2i\sqrt{2}$.

Graphical Representation of Complex Numbers.

39. Applying the method of §§ 19, 20, for the representation of an imaginary number, we may represent a complex number $a+ib$ graphically by means of a point, B. This point is located by first measuring a distance OA equal to a from the origin O along the axis of *real numbers*, and then from the point A thus reached measuring a length AB, equal to b, in a direction parallel to the axis of *imaginary numbers*, that is, at right angles to the axis of real numbers. (See Fig. 5.)

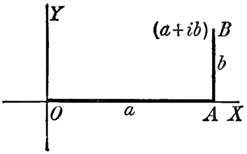


FIG. 5.

The point B may be called the **graph of the complex number** $a + ib$. It may be seen that the graphs of all complex numbers which have equal real parts a, lie on the same straight line, AB, parallel to the axis of imaginary numbers, OY. It may also be seen that the graphs of all complex numbers which have the same imaginary part, ib, lie on the same straight line, passing through the point B, parallel to the axis of real numbers, OX. (See Fig. 6.)

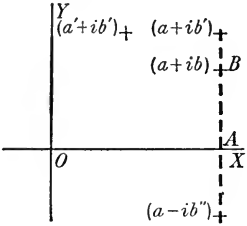


FIG. 6.

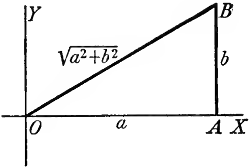


FIG. 7.

40.* The length of the line OB may be shown by principles of elementary geometry to be equal to the *positive value* of $\sqrt{a^2 + b^2}$. The positive value of $\sqrt{a^2 + b^2}$ is called the **modulus** of the complex number $a + ib$. (See Fig. 7.)

E. g. The modulus of either of the conjugate complex numbers, $4 + 3i$ or $4 - 3i$, is $+\sqrt{4^2 + 3^2} = +5$.

41.* The modulus of a complex number may be taken as representing the *absolute* or *arithmetic* value of the given complex number.

42.* One complex number, $a + ib$, is said to be numerically equal to, greater than, or less than another complex number, $x + iy$, according as the modulus $\sqrt{a^2 + b^2}$ of the first is equal to, greater than, or less than the modulus $\sqrt{x^2 + y^2}$ of the second.

---

* This section may be omitted when the chapter is read for the first time.

43.* It may be shown that the points representing all complex numbers having equal moduli lie on the circumference of the same circle, while all those representing complex numbers of greater or less *absolute value* lie without or within the circle according as their moduli are greater than, or less than, the modulus of the given complex number.

E. g. The complex numbers $4+3i, 4-3i, -4+3i, -4-3i, 3+4i, 3-4i, -3+4i, -3-4i$, etc., all have the same modulus, $\sqrt{4^2+3^2} = +5$. Each of these complex numbers may be represented by a definite point situated on a circle, of which the center is the origin and the radius is 5. This circle passes also through the points representing the real numbers $+5$ and -5 and the imaginary numbers $+5i$ and $-5i$. (See Fig. 8.)

All numbers of greater or less *absolute value* may be represented by points situated outside of, or inside of this circle, respectively.

E. g. The point representing the complex number $6+2i$, of which the modulus is $\sqrt{40} = 6+$, lies without the circle, while the point representing the complex number $4+2i$, of which the modulus is $\sqrt{20} = 4+$, lies within the circle. (See Fig. 8.)

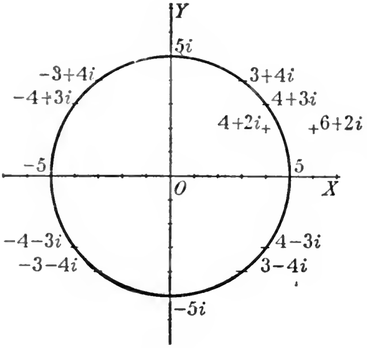


FIG. 8.

44. By applying the Principle of No Exception, we have extended our idea of number from the primary arithmetic whole number, to negative, fractional, irrational, imaginary, and finally to complex number.

We have shown in §§ 29–35 that the fundamental operations of addition, subtraction, multiplication, and division, when applied to complex numbers, result in general in complex numbers.

It may be shown by principles and methods beyond the limits of elementary algebra that, whenever the direct operations of addition, multiplication, and involution, or the indirect operations of subtraction, division, and root extraction, are applied to any of the kinds of

---

* This section may be omitted when the chapter is read for the first time.

number, including complex number, the results in each case lead to no new kind of number.

The number represented by $(a + ib)^{x+iy}$ is a complex number.

Accordingly, the complex number may be regarded as *the most general kind of number*, and with its inclusion the number system of algebra is complete.

Mental Exercise XXI. 7 **Review**

1. Show that $x^6 + 6x^5 + 9x^4$ is the square of a binomial.

2. Find the continued product of $x^2 - y^2$, $x^2 + y^2$, $x^4 + y^4$ and $x^8 + y^8$.

Obtain each of the following quotients :

3. $(a - b) \div (b - a)$.
4. $(c - 4) \div (4 - c)$.
5. $(5 - d) \div (d - 5)$.
6. $(x^2 - y^2) \div (y - x)$.

Square each of the following :

7. a.
8. b^2.
9. c^3.
10. $\frac{1}{2}$.
11. $\frac{1}{3}$.
12. $-\frac{1}{4}$.
13. $\frac{2}{5}$.
14. $.2$.
15. $.3$.
16. $.5$.
17. $.06$.
18. $a^{\frac{1}{2}}$.
19. $b^{\frac{2}{3}}$.
20. c^n.
21. $d^{\frac{1}{n}}$.
22. $.01$.
23. $c^{-\frac{1}{2}}$.
24. $d^{-\frac{2}{3}}$.
25. -3.
26. $-x^3$.

Find the value of

27. $\dfrac{2^2}{3}$.
28. $\left(\dfrac{2}{3}\right)^2$.
29. $\dfrac{2}{3^2}$.

Express as a single power of 5 :

30. 25^3; 125^2; 625^4.

Find the value of

31. $3^2 - 2^3$.
32. $2^3 + 2^{-3}$.

Express as a power of a base:

33. $(2^x)^x$.
34. $(y^y)^y$.
35. $(z^2)^{2z}$.

36. Find the values of $(\frac{1}{2})^{-2}$ and -2^{-2}.

Express the following with positive exponents :

37. $a^{-1} \div b^{-\frac{1}{2}}$.
38. $x^{-1} \div y^{-2}$.
39. $(1/m)^{-1} \div (1/n)^{-1}$.

Show that the following identities are true:

40. $\dfrac{y-x}{z-y} \equiv \dfrac{x-y}{y-z}$.

41. $\dfrac{(a-b)^2}{a^2} \equiv 1 - 2\left(\dfrac{b}{a}\right) + \left(\dfrac{b}{a}\right)^2$.

42. $(x-y)(z-y)(x-z) \equiv (x-y)(y-z)(z-x)$.

Simplify each of the following:

43. $\sqrt[6]{8\,a^3b^6}$.

44. $\sqrt[8]{16\,a^6y^4}$.

45. $\sqrt[9]{27\,m^3n^6}$.

46. Regarding x as the unknown, solve $\dfrac{a}{x} + \dfrac{b}{x} + \dfrac{c}{x} = d$.

Distinguish between

47. $\frac{3}{2}a$ and $a^{\frac{3}{2}}$.

48. x^{-1} and $-x$.

49. $(\frac{2}{3}) - 1$, $(\frac{2}{3})^{-1}$, and $(\frac{2}{3})(-1)$.

50. $a^{-1} + b^{-1}$ and $\dfrac{1}{a+b}$.

51. $-1 \div a$ and $1 \div a^{-1}$.

52. $\dfrac{0}{a}$ and $\dfrac{1}{a^0}$.

53. $0 \cdot 5$, 5^0, 0^5, $\dfrac{5}{0}$ and $\dfrac{0}{5}$.

54. $(-a)^{-2}$ and $-a^{-2}$.

55. $(-b)^{-3}$ and $-b^{-3}$.

Which of the following complex numbers have equal moduli?

56. $3 + 4i$, $4 - 3i$, $5 + 12i$, $-3 + 4i$, $-12 - 5i$.

Simplify each of the following:

57. $\dfrac{(b-a)^2}{a^2 - b^2}$.

58. $\dfrac{\frac{1}{2} + 1}{\frac{1}{2}}$.

59. $\dfrac{\frac{1}{3} + 3}{\frac{1}{3}}$.

60. $\dfrac{\frac{1}{2} + \frac{1}{3}}{\frac{1}{6}}$.

61. $\dfrac{\frac{1}{4} + \frac{1}{5}}{20}$.

62. $\dfrac{\dfrac{a}{b} - a}{b}$.

63. $\dfrac{\dfrac{x}{y} - y}{y}$.

64. $\dfrac{\dfrac{z}{w} - w}{z}$.

65. $(a^{-2} + a^2)(a^2 - a^{-2})$.

66. $\left(\dfrac{a^{\frac{1}{3}}}{3} + 3\right)\left(\dfrac{a^{\frac{1}{3}}}{3} - 3\right)$.

67. $(a^{2x} - b^{\frac{y}{2}})^2$.

CHAPTER XXII

EQUATIONS OF THE SECOND DEGREE CONTAINING ONE UNKNOWN

1. If the members of an equation containing the second power of one unknown quantity, x, are rational and integral with respect to x, and if they contain no powers of x other than the second and the first, the equation is said to be of the second degree with reference to x, or **quadratic**.

E. g.
$$x^2 + 5x + 6 = 0,$$
$$x^2 + 3x = 7 - 2x^2 + 5x.$$

2. From every quadratic equation containing one unknown quantity, x, may be obtained, by applying the principles of Chapter X, an equivalent equation of the standard form $ax^2 + bx + c = 0$.

In this equation, which will be referred to as the **Standard Quadratic Equation,** a, b, and c represent known quantities, *a being positive and different from zero.* The first term, ax^2, represents the sum of all of the terms containing x^2; the second term, bx, is the sum of all of the terms containing x to the first power, and the third or known term, c, stands for the sum of all of the terms that are free from the unknown, x.

E. g. From $4x - 2x^2 + 8 = 3 - 2x - 5x^2$ may be derived the equivalent quadratic equation in standard form, $3x^2 + 6x + 5 = 0$.

In the equation $3x^2 + 6x + 5 = 0$, the numbers 3, 6, and 5 take the place of a, b, and c respectively in the standard quadratic equation $ax^2 + bx + c = 0$.

3. In the standard quadratic equation $ax^2 + bx + c = 0$, either of the letters b or c may represent zero, causing the corresponding terms to disappear from the equation, but *a cannot be zero,* for in that case there would be no term containing x^2; hence the equation would not be quadratic.

In all that follows we shall therefore assume that $a \neq 0$, and that a, b, and c are all real quantities.

4. If, when reduced to the standard form $ax^2 + bx + c = 0$, a quadratic equation contains all of the terms represented by ax^2, bx and c, it is said to be **complete.**

If either of the terms represented by bx or c be missing, the equation is said to be **incomplete.**

E. g. $7x^2 + 3x - 5 = 0$ and $x^2 - 3x + 10 = 0$ are complete quadratic equations, while $3x^2 - 4 = 0$ and $9x^2 = 2x$ are incomplete quadratic equations.

Graphs of Quadratic Equations.

5. The graph of a quadratic equation having the form of the standard quadratic equation $ax^2 + bx + c = 0$ may be obtained as follows:

Let y represent the value of the expression $ax^2 + bx + c$ when a particular number is substituted for x. The value assigned to x and the corresponding value calculated for y may be taken as the x-coördinate and the y-coördinatè respectively of a point on the graph of the given quadratic equation $ax^2 + bx + c = y$. By assigning different values to x and calculating corresponding values for y, the coördinates of as many points on the graph of the given quadratic equation as may be required may be obtained.

E. g. We may obtain a portion of the graph of the quadratic equation $x^2 + 4x = 5$ as follows:

Transposing the terms of the given equation to the first member and representing the value of this first member by y, we have $x^2 + 4x - 5 = y$.

Substituting different values for x in the equation $x^2 + 4x - 5 = y$ we may calculate corresponding values for y.

$$\begin{array}{llrlr}
\text{If} & x = 0, & \text{we have} & 0 + 0 - 5 = y. & \text{Hence,}\quad -5 = y. \\
 & x = 1, & & 1 + 4 - 5 = y. & 0 = y. \\
 & x = 2, & & 2^2 + 4\cdot2 - 5 = y. & 7 = y. \\
 & x = 3, & & 3^2 + 4\cdot3 - 5 = y. & 16 = y. \\
 & \text{etc.} & & \text{etc.} & \text{etc.}
\end{array}$$

The values shown in the accompanying table may be readily calculated.

$x^2 + 4x - 5 = y$	
x	y
⋮	⋮
6	55
5	40
4	27
3	16
2	7
1	0
0	-5
-1	-8
-2	-9
-3	-8
-4	-5
-5	0
-6	7
⋮	⋮

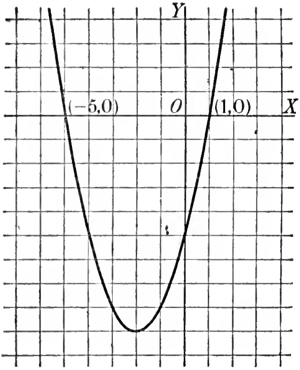


FIG. 1. $x^2 + 4x - 5 = y$.

The general shape of the graph may be determined as follows:

By factoring the first member of $x^2 + 4x - 5 = y$,

we obtain
$$(x + 5)(x - 1) = y.$$

For all values of x greater than $+1$ or less than -5, the two factors $(x + 5)$ and $(x - 1)$ have like signs, and consequently the corresponding values of y are positive, and the points on the graph of which these values of x and y are coördinates must lie *above* the axis of X.

For all values of x lying between $+1$ and -5, the factors $(x + 5)$ and $(x - 1)$ have opposite signs, one positive and the other negative. It follows that the product $(x + 5)(x - 1)$, represented by y, is negative. Hence the values of y, obtained by substituting values lying between $+1$ and -5 for x, must be negative, and the points on the graph of which these values of x and y are coördinates must be found *below* the axis of X.

The graph crosses the axis of X in the two points $(-5, 0)$ and $(+1, 0)$, and in no others, as shown in Fig. 1.

The values -5 and $+1$, which locate the points of "crossing," are the values which reduce the expression $x^2 + 4x - 5$ to zero, and hence are the roots of the equation $x^2 + 4x - 5 = 0$.

6. We have illustrated in this example the principle that *every quadratic equation containing one unknown has two roots, and cannot have more than two roots.*

7. **Approximate values for the real roots** of a quadratic equation may be obtained by first constructing its graph, and then measuring the distances along the axis of X from the origin to the intersections (if there be any) of the graph and this axis, and estimating the numerical values of x in terms of the scale unit according to which the graph is constructed.

8. If, instead of crossing the axis of X, the graph lies wholly upon one side of it, and simply touches it in one point, it is convenient to say that there are two values of x which satisfy the equation, but that these values are equal.

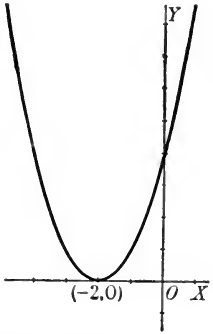


FIG. 2. $x^2 + 4x + 4 = y$.

E. g. The solutions of $x^2 + 4x + 4 = 0$, which is equivalent to $(x + 2)(x + 2) = 0$, are $x = -2$, and $x = -2$.

By referring to Fig. 2, it will be seen that the graph of $x^2 + 4x + 4 = y$ touches the axis of X at the point $(-2, 0)$, and does not cross it.

9. If the graph neither crosses nor touches the axis of X, there are no "real" points of intersection with the axis, and in such cases it will be found that there are no "real" solutions to the equation, but there are two "imaginary" values which may be found by solving the equation.

E. g. By methods which will be shown later, the solutions of $x^2 + 4x + 5 = 0$ are found to be

$$x = -2 + \sqrt{-1} \text{ and } x = -2 - \sqrt{-1}.$$

These solutions are complex numbers, and it will be seen, by referring to Fig. 3, that the corresponding graph of $x^2 + 4x + 5 = y$ neither crosses nor touches the axis of X; that is, the points of crossing are "imaginary."

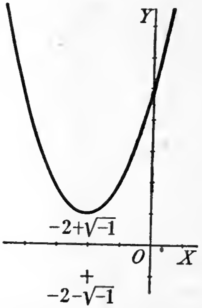


FIG. 3. $x^2+4x+5=y$.

I. INCOMPLETE QUADRATIC EQUATIONS

10. Any incomplete quadratic equation in which the first power of the unknown is missing, may be reduced to the form

$$ax^2 + c = 0. \tag{1}$$

We have assumed a to be different from zero (see § 3); hence, dividing both members by a and transposing, we may write

$$x^2 = \frac{-c}{a}. \tag{2}$$

In this form, the **general solution** of the equation may be obtained by either of the following methods.

11. First Method. We may employ the principles of **factoring** by first expressing $\frac{-c}{a}$ as the square of its square root,

that is, $$\frac{-c}{a} \equiv \left(\sqrt{\frac{-c}{a}}\right)^2.$$

Hence, we have $$x^2 = \left(\sqrt{\frac{-c}{a}}\right)^2. \tag{3}$$

Transposing, $$x^2 - \left(\sqrt{\frac{-c}{a}}\right)^2 = 0. \tag{4}$$

Accordingly, $$\left(x + \sqrt{\frac{-c}{a}}\right)\left(x - \sqrt{\frac{-c}{a}}\right) = 0. \tag{5}$$

The single equation in this form is equivalent to the two separate equations formed by equating to zero each of the factors. (Compare with Chap. XII. § 48.)

Hence we may write

$$x + \sqrt{\frac{-c}{a}} = 0, \qquad \text{and} \qquad x - \sqrt{\frac{-c}{a}} = 0. \tag{6}$$

The solutions of these equations are found to be

$$x = -\sqrt{\frac{-c}{a}}, \text{ and } \qquad x = +\sqrt{\frac{-c}{a}}. \tag{7}$$

It should be observed that the two roots are equal in absolute value, but opposite in sign.

If c and a have unlike signs, both roots will be real, but if c and a have like signs, both roots will be imaginary.

12. Second Method. By extracting the square roots of both members of $x^2 = \frac{-c}{a}$ (see (2) §10), we may obtain

$$x = \pm\sqrt{\frac{-c}{a}},$$

which is a convenient abbreviation for the separate equations,

$$x = +\sqrt{\frac{-c}{a}}, \text{ and } x = -\sqrt{\frac{-c}{a}}. \quad \text{(See (7) § 11.)}$$

Observe that, while it is not incorrect to write the double sign ± before both members of the equation after extracting the square roots, it is unnecessary.

This is because the equation $x = \pm\sqrt{\frac{-c}{a}}$ might have been written $\pm x = \pm\sqrt{\frac{-c}{a}}$, which is a convenient abbreviation for the following set of four equations:

$$+x = +\sqrt{\frac{-c}{a}}, \quad (1) \qquad +x = -\sqrt{\frac{-c}{a}}, \quad (2)$$

$$-x = +\sqrt{\frac{-c}{a}}. \quad (3) \qquad -x = -\sqrt{\frac{-c}{a}}. \quad (4)$$

By changing the signs of the terms in both members of (3) and (4), we obtain (2) and (1) respectively.

Hence, when extracting the square roots of both members of an equation, it is sufficient to write the double sign ± before the square root of one member only.

13. Observe that, since there can be two, and only two square roots of a given quantity, an incomplete quadratic equation of the type $ax^2 + c = 0$ can have two, and only two roots.

Ex. 1. Solve the incomplete quadratic equation,

$$5x^2 - 50 = 2x^2 + 25.$$

Transposing, collecting terms, and dividing the terms of the resulting members by the coefficient of x^2, we obtain the equivalent equation

$$x^2 = 25.$$

Extracting the square roots of both members,

$$x = \pm 5.$$

Accordingly the solutions are

$x = +5$, and $x = -5$. (See Fig. 4.)

By substitution, these values are found to be solutions of the original equation.

Substituting $+5$,

$$5(+5)^2 - 50 = 2 \cdot 5^2 + 25$$
$$75 = 75.$$

Substituting -5,

$$5(-5)^2 - 50 = 2(-5)^2 + 25$$
$$75 = 75.$$

Ex. 2. Solve $3x^2 = (3 + x)(3 - x)$.

We may derive, $3x^2 = 9 - x^2$

Hence, $x^2 = \frac{9}{4}$.

Accordingly, $x = \pm \frac{3}{2}$.

The given equation is found to be satisfied by both of these values.

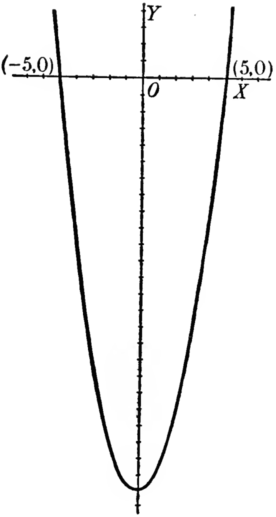


Fig. 4. $x^2 - 25 = y$.

Substituting $+\frac{3}{2}$,

$$3(\tfrac{3}{2})^2 = (3 + \tfrac{3}{2})(3 - \tfrac{3}{2})$$
$$\tfrac{27}{4} = \tfrac{27}{4}.$$

Substituting $-\frac{3}{2}$,

$$3(-\tfrac{3}{2})^2 = (3 - \tfrac{3}{2})(3 + \tfrac{3}{2})$$
$$\tfrac{27}{4} = \tfrac{27}{4}.$$

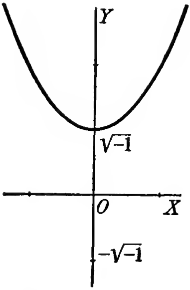


Fig. 5. $x^2 + 1 = y$.

Ex. 3. Solve $x^2 + 1 = 0$.

Transposing, $x^2 = -1$.

Extracting the square roots, $x = \pm\sqrt{-1}$.

The solutions, $x = +\sqrt{-1}$ and $x = -\sqrt{-1}$ (see Fig. 5), are both found to satisfy the original equation.

Substituting $+\sqrt{-1}$,

$$(\sqrt{-1})^2 + 1 = 0$$
$$-1 + 1 = 0$$
$$0 = 0.$$

Substituting $-\sqrt{-1}$,

$$(-\sqrt{-1})^2 + 1 = 0$$
$$-1 + 1 = 0$$
$$0 = 0.$$

14. Any incomplete quadratic equation containing no term free from x may be reduced to the form

$$ax^2 + bx = 0. \qquad (1)$$

In this form its solution may be obtained by factoring.

Factoring (1), we have $x(ax + b) = 0$, (2)

which is equivalent to the set of two separate equations,

$$x = 0, \qquad \text{and} \qquad ax + b = 0.$$

The solutions of these equations are found to be

$$x = 0, \qquad \text{and} \qquad x = \frac{-b}{a}. \qquad (3)$$

Ex. 4. Solve $3x^2 = 6x$.

Transposing and factoring, $3x(x - 2) = 0$.

Hence, $x = 0$, and $x - 2 = 0$.

Hence the solutions are

$$x = 0, \qquad \text{and} \qquad x = 2.$$

(See Fig. 6.)

These values are found by substitution to satisfy the given equation.

Substituting 0, $0 = 0$.

Substituting 2, $3 \cdot 2^2 = 6 \cdot 2$

$12 = 12$.

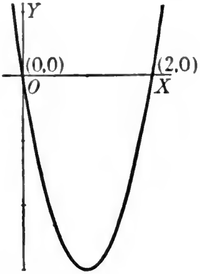


FIG. 6. $3x^2 - 6x = y$.

15. By the Principles of Equivalence, roots have neither been gained nor lost in making the different transformations in the preceding examples; hence the solutions shown are the only ones which satisfy the given equations.

EXERCISE XXII. 1

Solve the following Incomplete Quadratic Equations, verifying results:

1. $6x^2 - 54 = 0$.
2. $15x^2 = 64 - x^2$.
3. $3x^2 - 16 = x^2 + 16$.
4. $5x^2 - 8 = 2x^2 + 19$.
5. $9x^2 - 8 = 12x^2 - 11$.
6. $3x^2 + 11 = 7x^2 - 5$.
7. $5x^2 - 7 = 7x^2 - 9$.
8. $(x + 5)^2 = 10x + 34$.
9. $(x + 3)^2 - 6(x + 3) = 7$.
10. $ax^2 - b = a - bx^2$.
11. $11x^2 + 7 = 5x^2 + 43$.
12. $6x^2 - 5 = x^2 + 45$.
13. $\frac{2x^2 - 7}{13} = 5$.
14. $\frac{3x^2 + 13}{16} = 10$.
15. $\frac{x^2 + 5}{2} = \frac{x^2 + 2}{3}$.
16. $\frac{3x^2 - 2}{5} + \frac{x^2 + 4}{2} = 4$.

17. $2(x-1)(2x+1)+(8+x)(10-x)=4x^2+29$.
18. $2(x-3)(x+4)=(x+2)(x+5)-x^2-5x$.
19. $(x+2)(x^2+4)=(x+1)^3+x-2$.
20. $bx^2=(a-b)^2(a+b)-ax^2$.

II. COMPLETE QUADRATIC EQUATIONS

16. After having transformed a given quadratic equation into an equivalent quadratic equation in standard form, $ax^2+bx+c=0$, the solution may be obtained by different methods. These methods, however, are all based upon "factoring."

17. If the factors of the expression represented by ax^2+bx+c can be readily obtained by inspection, the method shown in Chap. XII. § 47, may be applied to the equation $ax^2+bx+c=0$, and we can obtain its

Solution by Factoring

Ex. 1. Solve $x^2+4x-5=0$.

Factoring, $(x+5)(x-1)=0$.

This single equation is equivalent to the set of two linear equations obtained by writing the factors separately equal to zero.

That is, $x+5=0$, and $x-1=0$.

From which $x=-5$, and $x=+1$.

(Compare with § 5; see also Fig. 1.)

Ex. 2. Solve $6x^2+3=11x$.

In standard form, $6x^2-11x+3=0$.

Factoring, $(3x-1)(2x-3)=0$.

This equation is equivalent to the set of two linear equations,

$$3x-1=0, \quad \text{and} \quad 2x-3=0.$$

The solutions are

$$x=\tfrac{1}{3}, \quad \text{and} \quad x=\tfrac{3}{2}.$$

(See Fig. 7.)

Verifying these results by substituting in the given equation, we have,

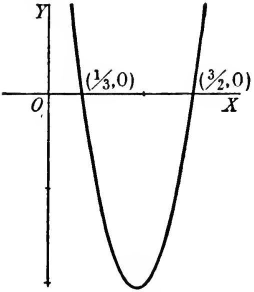


FIG. 7. $6x^2-11x+3=y$.

Substituting $\frac{1}{3}$,

$$6(\tfrac{1}{3})^2+3=11(\tfrac{1}{3})$$
$$\tfrac{11}{3}=\tfrac{11}{3}.$$

Substituting $\frac{3}{2}$,

$$6(\tfrac{3}{2})^2+3=11(\tfrac{3}{2})$$
$$\tfrac{33}{2}=\tfrac{33}{2}.$$

Exercise XXII. 2

Solve the following Complete Quadratic Equations by the Method of Factoring, verifying results :

1. $x^2 - x = 20.$
2. $x^2 + 20 = 9x.$
3. $x^2 - 4x = 45.$
4. $x^2 + 3x = 40.$
5. $15x = x^2 + 36.$
6. $110 = x^2 - x.$
7. $x(x - 10) = 11.$
8. $x(x - 12) = - 27.$
9. $x(x + 15) = 54.$
10. $x(x - 14) = 51.$
11. $2x^2 + 5x + 3 = 0.$
12. $3x^2 - 7x + 2 = 0.$
13. $3x^2 + x - 14 = 0.$
14. $5x^2 - 3x - 2 = 0.$
15. $6x^2 + 7x = 5.$
16. $12x^2 + 10 = 23x.$
17. $8x^2 = 9(3x - 1).$
18. $(x + 3)^2 + (x - 4)^2 = 65.$
19. $(x + 2)^2 - 17 = 2(x - 1)^2.$
20. $(x-3)^2 - 3(x-5)^2 - 4 = 0.$
21. $x^2 + ax - x = a.$
22. $x^2 + a = a^2 + x.$
23. $x^2 + x(b - a) = ab.$
24. $c^3x + x^2 = c^2x + c^5.$

18. If the factors of the expression represented by $ax^2 + bx + c$, which is the first member of the standard quadratic equation $ax^2 + bx + c = 0$, cannot be readily obtained by inspection, we may obtain the

Solution by Completing the Square

19. The expression "completing the square" may be given a geometric significance.

By attaching a strip a units in width to each of two adjacent sides of a square, of which each side is x units in length, a geometric figure may be constructed which is made up of parts represented by x^2, ax, and ax, that is, $x^2 + 2ax$, as shown by the heavy lines in Fig. 8.

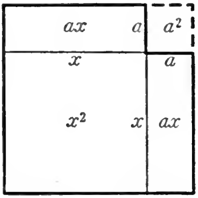


Fig. 8.

The figure may be made a "complete" square, each side of which is $(x + a)$ units in length, by adding the part represented by a^2.

By adding a^2 to $x^2 + 2ax$ we obtain the algebraic trinomial $x^2 + 2ax + a^2$, which is the square of the binomial $x + a$.

The algebraic operation of obtaining the trinomial square

$x^2 + 2ax + a^2$, by adding the square a^2 to the binomial $x^2 + 2ax$, is commonly called "completing the square" with reference to the binomial $x^2 + 2ax$.

Ex. 1. Complete the square with respect to the binomial $x^2 + 6x$.

The term, the square of which must be added to the binomial $x^2 + 6x$ to obtain a trinomial square of the form $x^2 + 6x + (\ )^2$, may be found by dividing $6x$ by twice the square root of the term x^2.

That is,
$$\frac{6x}{2x} \equiv 3.$$

Adding the square of 3 to the binomial $x^2 + 6x$, we obtain the required trinomial square $x^2 + 6x + 3^2 \equiv x^2 + 6x + 9$.

Ex. 2. Complete the square with respect to the binomial $25a^2 + 40ab$.

We have
$$\frac{40ab}{2 \cdot 5a} \equiv 4b.$$

Accordingly the required trinomial square is

$$25a^2 + 40ab + (4b)^2 \equiv 25a^2 + 40ab + 16b^2.$$

Mental Exercise XXII. 3

Complete the square with respect to each of the following binomials :

1. $x^2 + 2x$.
2. $y^2 + 4y$.
3. $z^2 + 8z$.
4. $a^2 + 12a$.
5. $b^2 - 14b$.
6. $c^2 - 16c$.
7. $d^2 + 18d$.
8. $g^2 - 20g$.
9. $81h^2 - 18h$.
10. $121c^2 - 22c$.
11. $b^2 + 2bc$.
12. $x^2 + 22xy$.
13. $a^2 - 24ab$.
14. $4x^2 - 12xy$.
15. $9h^2 + 60hk$.
16. $4b^2 + 28by$.
17. $25z^2 - 60zw$.
18. $64a^2 + 48ab$.
19. $9a^2 + 66ab$.
20. $121c^2 - 44cd$.
21. $x^2y^2 - 2xy$.
22. $c^2d^2 + 2cd$.
23. $25g^2k^2 - 40gk$.
24. $81c^2d^2 - 36cd$.
25. $49b^2c^2 + 70bc$.
26. $144c^2x^2 + 120cx$.
27. $16b^4 + 8b^2$.
28. $9c^4 - 12c^2d^2$.
29. $4h^4k^4 + 36h^2k^2$.
30. $36c^6 - 12c^3d^3$.
31. $25h^8 + 30h^4k^4$.
32. $49d^6 + 84d^3h^2$.

33. $81x^{2m} + 36x^{m}$.

34. $100x^{2m}y^{2m} - 60x^{m}y^{m}$.

35. $c^{6x}y^{4x} + 2c^{3x}y^{2x}$.

36. $121x^{4n}y^{2n} + 44x^{2n}y^{n}$.

37. $x^2 + 3x$.

38. $z^2 - 7z$.

39. $y^2 + 5y$.

40. $a^2 + a$.

41. $4a^2 + 5a$.

42. $9b^2 + 8b$.

43. $16c^2 - 10c$.

44. $25d^2 - 5d$.

45. $36h^2 - 13h$.

46. $9m^2 + m$.

47. $16n^2 - n$.

48. $a^2 + \frac{a}{2}$.

49. $x^2 + \frac{x}{3}$.

50. $y^2 + \frac{y}{4}$.

51. $z^2 - \frac{z}{5}$.

52. $w^2 + \frac{2w}{3}$.

53. $a^2 + \frac{4a}{5}$.

54. $b^2 + \frac{3b}{2}$.

55. $c^2 - \frac{5c}{3}$.

56. $d^2 - \frac{7d}{5}$.

57. $4x^2 + \frac{x}{3}$.

58. $9y^2 + \frac{y}{2}$.

59. $16z^2 + \frac{z}{5}$.

60. $4w^2 - \frac{2w}{5}$.

61. $9a^2 + \frac{6a}{7}$.

62. $25b^2 - \frac{20b}{3}$.

20. To solve a complete quadratic equation in the standard form $ax^2 + bx + c = 0$, by the method of "completing the square," our problem is to obtain an equivalent equation, one member of which contains all of the terms in which x appears, and which has the form of a trinomial square.

The values for x may then be found by the methods used in the previous sections of this chapter.

Ex. 1. Solve $x^2 + 6x = 7$.

To have the form of a trinomial square, the first member must contain two terms which are squares and positive, and another term which is twice the product of the square roots of these two.

The first term, x^2, is the square of x. Hence, using the remaining term, $6x$, as a "finder term" (see Chap. XII, § 21), we may obtain the square root of the missing "square-term" by dividing the "finder term" by twice the square root of the first term.

That is,
$$\frac{6x}{2x} \equiv 3.$$

By adding the square of 3 to $x^2 + 6x$, we obtain the complete trinomial square $x^2 + 6x + 3^2$.

We may obtain an equation which is equivalent to the one given, having this expression as a first member, by adding to both members of the given equation 3^2 or 9.

That is, $$x^2 + 6x + 3^2 = 7 + 9.$$

Or, $$(x + 3)^2 = 4^2.$$

Extracting the square roots of both members, we obtain

$$x + 3 = \pm 4,$$

which is a convenient abbreviation for the set of two equations,

$$x + 3 = +4, \quad \text{and} \quad x + 3 = -4.$$

The solutions are

$$x = +1, \quad \text{and} \quad x = -7.$$

These values are found by substitution to be solutions of the original equation.

This equation may also be readily solved by factoring. The student should obtain the graph.

Ex. 2. Solve $$x^2 - 8x + 11 = 0.$$

The first term, x^2, is a square and is positive; hence, using the second term $-8x$ as a "finder term," we may obtain the "missing term" which is required to complete the square, with reference to the binomial $x^2 - 8x$.

That is, $\frac{-8x}{2x} \equiv -4.$

Adding the square of -4 to both members, and transposing $+11$ to the second member, we obtain the equivaent equation,

$$x^2 - 8x + 16 = -11 + 16.$$

Hence, $(x - 4)^2 = 5.$

Extracting the square roots,

$$x - 4 = \pm\sqrt{5}.$$

It follows that $x - 4 = +\sqrt{5}$, and $x - 4 = -\sqrt{5}$.

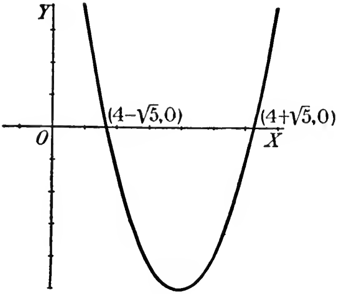


FIG. 9. $x^2 - 8x + 11 = y$.

Hence,

$$x = 4 + \sqrt{5}, \quad \text{and} \quad x = 4 - \sqrt{5}. \quad \text{(See Fig. 9.)}$$

These solutions may be verified by substituting in the given equation.

Substituting $4 + \sqrt{5}$,

$$(4 + \sqrt{5})^2 - 8(4 + \sqrt{5}) + 11 = 0$$
$$21 + 8\sqrt{5} - 21 - 8\sqrt{5} = 0$$
$$0 = 0.$$

Substituting $4 - \sqrt{5}$,

$$(4 - \sqrt{5})^2 - 8(4 - \sqrt{5}) + 11 = 0$$
$$21 - 8\sqrt{5} - 21 + 8\sqrt{5} = 0$$
$$0 = 0.$$

Although the values $x = 4 + \sqrt{5}$ and $x = 4 - \sqrt{5}$ are mathematically *exact*, it is often convenient to use *approximate* results that are correct to some specified number of decimal places.

Finding the value of $\sqrt{5}$ correct to four places of decimals, *approximations* to the true values of x obtained above are

$x = 4 + 2.2360 +$ and $x = 4 - 2.2360 +$,

that is, $x = 6.2360 +$, and $x = 1.7640 +$.

Ex. 3. Solve $8x^2 - 7x - 1 = 0$.

Observe that the equation as written contains no term which is "entirely" a square.

By either dividing or multiplying both members by such a number as will transform the term $8x^2$ into a square, an equivalent equation may be obtained which can be solved by the method of completing the square.

Dividing $8x^2$ by either 8 or 2, we obtain the squares x^2 or $4x^2$, respectively, or by multiplying by either 2 or 8, we obtain the squares $16x^2$ or $64x^2$, respectively.

We may obtain an equation containing the term x^2, equivalent to the given equation, by dividing the separate terms by 8.

Accordingly, transposing the known term to the second member, we obtain

$$x^2 - \frac{7x}{8} = \frac{1}{8}.$$

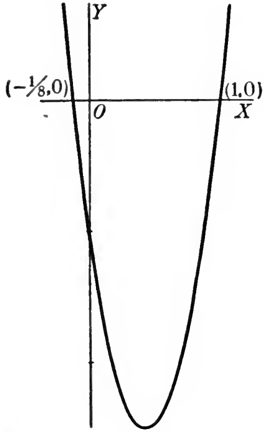


FIG. 10. $8x^2 - 7x - 1 = y$.

To find the term required to "complete the square" we may use $-\frac{7x}{8}$ as a "finder term," as follows:

$$-\frac{7x}{8} \div 2x \equiv -\frac{7x}{8 \cdot 2x} \equiv -\tfrac{7}{16}.$$

Completing the square with reference to $x^2 - \frac{7x}{8}$, by adding to both members $(-\frac{7}{16})^2 = +\frac{49}{256}$, we obtain the equivalent equation

$$x^2 - \frac{7x}{8} + (-\tfrac{7}{16})^2 = \tfrac{1}{8} + \tfrac{49}{256}.$$

Or, $$(x - \tfrac{7}{16})^2 = \tfrac{81}{256}.$$

Extracting the square roots of both members, we obtain

$$x - \tfrac{7}{16} = \pm \tfrac{9}{16}.$$

Hence, $x - \frac{7}{16} = +\frac{9}{16}$, and $x - \frac{7}{16} = -\frac{9}{16}$.

Accordingly, $x = 1$, and $x = -\frac{1}{8}$.

(See Fig. 10.)

These values are found, by substitution, to satisfy the given equation.

Ex. 4. Solve $x^2 + x + 1 = 0$.

Using x as a "finder term," we have $\frac{x}{2x} \equiv \frac{1}{2}$.

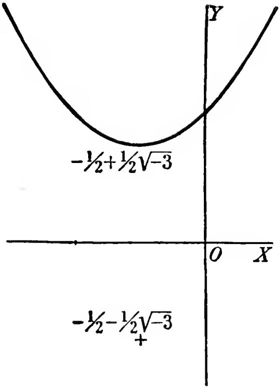


FIG. 11. $x^2 + x + 1 = y$.

Adding the square of $\frac{1}{2}$ to both members of the equation, and transposing the known term to the right member, we obtain,

$$x^2 + x + (\tfrac{1}{2})^2 = -1 + \tfrac{1}{4},$$

which may be written in the form

$$(x + \tfrac{1}{2})^2 = -\tfrac{3}{4}.$$

Extracting the square roots of both members, we obtain,

$$x + \tfrac{1}{2} = \pm\sqrt{-\tfrac{3}{4}}.$$

Hence,

$$x + \tfrac{1}{2} = +\sqrt{-\tfrac{3}{4}}, \text{ and } x + \tfrac{1}{2} = -\sqrt{-\tfrac{3}{4}}.$$

Hence the required solutions are

$$x = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{-3}, \text{ and } x = -\tfrac{1}{2} - \tfrac{1}{2}\sqrt{-3}.$$

(See Fig. 11.)

These values will be found, by substitution, to be solutions of the original equation.

Substituting $-\frac{1}{2} + \frac{1}{2}\sqrt{-3}$,

$$(-\tfrac{1}{2} + \tfrac{1}{2}\sqrt{-3})^2 + (-\tfrac{1}{2} + \tfrac{1}{2}\sqrt{-3}) + 1 = 0$$

$$\tfrac{1}{4} - \tfrac{1}{2}\sqrt{-3} - \tfrac{3}{4} - \tfrac{1}{2} + \tfrac{1}{2}\sqrt{-3} + 1 = 0$$

$$0 = 0.$$

Similarly, substituting $-\frac{1}{2} - \frac{1}{2}\sqrt{-3}$, we obtain $0 = 0$.

EXERCISE XXII. 4

Complete the square, and solve each of the following equations, verifying all results obtained. Whenever surds appear in the *exact* solutions, *approximate* values correct to four places of decimals should be obtained.

1. $x^2 + 2x = 15$.
2. $x^2 + 4x = 60$.
3. $x^2 - 6x = 16$.
4. $x^2 + 8x = 65$.
5. $x^2 - 2x = 8$.
6. $x^2 + 10x = 11$.
7. $x^2 - 6x = -8$.
8. $x^2 - 12 = 4x$.
9. $x^2 + 1 = 2x$.
10. $x^2 - 51 = 14x$.
11. $x^2 - 242 = 11x$.
12. $x(x + 6) = -9$.
13. $x(x - 12) = 64$.
14. $63 - 2x = x^2$.

15. $x^2 + 7x + 5 = 0.$
16. $x^2 + 6x + 10 = 0.$
17. $7(x - 3) = x^2.$
18. $(x - 2)(x - 3) = 4.$
19. $2x^2 + 2x - 3 = 0.$
20. $3x^2 + x = 4.$
21. $5x^2 - 6x + 7 = 0.$
22. $6x^2 + 10x = 9.$
23. $12x^2 - 13x = 32.$
24. $9x^2 + 6x - 4 = 0.$
25. $6x^2 + x = 6.$
26. $5x^2 - x = \frac{1}{5}.$
27. $x^2 + \frac{3x}{2} = 17.$
28. $x^2 + \frac{16x}{3} = 4.$

The methods employed in the solution of numerical equations may be applied also to literal equations.

Ex. 29. Solve $ax^2 - (a - b)x - b = 0.$

We may obtain an equivalent equation containing the square a^2x^2 by multiplying every term of the given equation by a.

Hence, we have, $\quad a^2x^2 - a(a - b)x - ab = 0.$

Using $-a(a - b)x$ as a "finder term," we may obtain the term required to complete the square, as follows:

$$\frac{-a(a-b)x}{2ax} \equiv \frac{-(a-b)}{2}.$$

Completing the square with reference to $a^2x^2 - a(a - b)x$, and transposing the known term $-ab$ to the second member, we obtain,

$$a^2x^2 - a(a-b)x + \left(-\frac{a-b}{2}\right)^2 = +ab + \left(-\frac{a-b}{2}\right)^2.$$

Or, $$a^2x^2 - a(a-b)x + \frac{(a-b)^2}{4} = +ab + \frac{(a-b)^2}{4}.$$

Writing the first member as the square of a binomial and combining the terms in the second member, we obtain

$$\left(ax - \frac{a-b}{2}\right)^2 = \frac{4ab + (a-b)^2}{4}.$$

Or, $$\left(ax - \frac{a-b}{2}\right)^2 = \frac{(a+b)^2}{4}.$$

Hence, $$ax - \frac{a-b}{2} = \pm\frac{a+b}{2}.$$

This is a convenient abbreviation for the set of two separate equations,

$$ax - \frac{a-b}{2} = +\frac{a+b}{2}, \quad \text{and} \quad ax - \frac{a-b}{2} = -\frac{a+b}{2}.$$

We obtain the solutions as follows :

$$ax = +\frac{a-b}{2} + \frac{a+b}{2} \qquad ax = +\frac{a-b}{2} - \frac{a+b}{2}$$

$$ax = \frac{a-b+a+b}{2} \qquad ax = \frac{a-b-a-b}{2}$$

$$ax = \frac{2a}{2} \qquad ax = \frac{-2b}{2}$$

$$x = 1. \qquad x = -\frac{b}{a}.$$

These values will be found, by substitution, to satisfy the given equation.

30. $x^2 - 2dx = c^2 - d^2$.
31. $x^2 - 4cx = 12c^2$.
32. $x^2 - 3a^2 = 2ax$.
33. $x^2 + 2ax + b^2 = 0$.
34. $a^2x^2 - 2ax = b - a^2$.
35. $11mx = 6x^2 - 7m^2$.
36. $4(3x^2 - 5d^2) = dx$.
37. $2(6x^2 + 5h^2) + 23hx = 0$.
38. $x^2 - (a+b)x + 5ab = 2(a^2 + b^2)$.

21. Hindu Method. In the method employed for solving the preceding equations it will be observed that, to complete the square, it was necessary to add the square of a fraction whenever the quotient obtained by dividing the "finder term" by twice the square root of the term containing x^2 was fractional.

Quadratic equations may be solved by a method employed by the Hindus which allows of the completion of the square without introducing fractions during the process.

The solution of the following equation illustrates the Hindu Method.

Ex. 1. Solve $3x^2 - 13x + 11 = 0$.

The first term may be made a square, $36x^2$, by multiplying by four times the coefficient of x^2, that is, by $4 \times 3 = 12$.

Accordingly, multiplying all of the terms of the given equation by 12, we obtain the equivalent equation

$$36x^2 - 156x + 132 = 0.$$

We can find the number, the square of which must be added to complete the square with reference to the binomial $36x^2 - 156x$, as follows:

$$\frac{-156x}{2 \cdot 6x} \equiv -13.$$

It will be observed that -13 is the coefficient of x in the given equation.

Adding the square of -13 to both members of $36x^2 - 156x + 132 = 0$ and transposing 132 to the second member, we obtain

$$36x^2 - 156x + 169 = -132 + 169.$$

Hence, $$(6x - 13)^2 = 37.$$

Therefore, $$6x - 13 = \pm\sqrt{37}.$$

Hence, $6x - 13 = +\sqrt{37}$, and $6x - 13 = -\sqrt{37}$.

The *exact* solutions of these equations are

$$x = \frac{13 + \sqrt{37}}{6} \quad \text{and} \quad x = \frac{13 - \sqrt{37}}{6}.$$

By extracting $\sqrt{37}$, correct to four places of decimals, we obtain the following *approximate* values:

$$x = \frac{13 + 6.0827 +}{6} \quad \text{and} \quad x = \frac{13 - 6.0827 +}{6}$$

Or, $x = 3.1804 +$, and $x = 1.1528 +$.

The student should verify the exact results by substituting in the given equation.

22. It should be observed that, *when the Hindu Method is applied to the solution of a quadratic equation in the form* $ax^2 + bx + c = 0$, *the separate terms are all multiplied by four times the coefficient* a *of* x^2, *and the square is completed with reference to* $ax^2 + bx$ *by adding the square of the coefficient of* x, *represented by* b.

EXERCISE XXII. 5

Solve each of the following equations by the Hindu Method, verifying all *exact* rational results. Whenever surds appear in the *exact* solutions, *approximate* values correct to four places of decimals should be obtained.

1. $3x^2 + 10x + 8 = 0.$
2. $8x^2 + 26x + 15 = 0.$
3. $5x^2 - 16x + 121 = 0.$
4. $3x^2 + 50 = 25x.$
5. $15x^2 + 16x = 15.$
6. $10x^2 - 9 = 43x.$
7. $25x^2 + 20x = 33.$
8. $x - 1 = 20x^2.$
9. $(2x - 7)^2 = 6x.$
10. $(x + 1)(2x + 3) = x^2 - 11.$
11. $3(x+1)(x-1) = 2(27 - 5x).$
12. $(2x + 1)(x + 2) = (x - 1)^2.$
13. $\frac{x^2}{20} - \frac{1}{4} = \frac{x}{5}.$
14. $\frac{x^2}{3} - \frac{x}{7} = \frac{20}{21}.$
15. $\frac{x^2}{11} + \frac{1}{11} = \frac{x}{5}.$
16. $\frac{x(2x - 5)}{30} + \frac{1}{6} = \frac{x}{5}.$

General Solution of $ax^2 + bx + c = 0$

23. The general solution of the standard quadratic equation

$$ax^2 + bx + c = 0, \qquad (1)$$

in which a is assumed to represent a positive number different from zero, may be obtained as follows:

From (1) we may obtain the equivalent equation

$$x^2 + \frac{bx}{a} = \frac{-c}{a}.$$

Using $\frac{bx}{a}$ as a "finder term," it appears from $\frac{bx}{2x \cdot a} \equiv \frac{b}{2a}$, (2)

that the square may be completed with reference to $x^2 + \frac{bx}{a}$ by adding the square of $\frac{b}{2a}$.

Accordingly, $$x^2 + \frac{bx}{a} + \left(\frac{b}{2a}\right)^2 = \frac{-c}{a} + \frac{b^2}{4a^2}. \qquad (3)$$

Hence, $$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}. \qquad (4)$$

Extracting the square roots of both members,

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}. \qquad (5)$$

Instead of writing the two separate linear equations which, taken together, form a system of which (5) is a convenient abbreviation, we may proceed as follows:

Transposing $\frac{b}{2a}$ to the right member and simplifying the radical, we obtain,

$$x = -\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac}.$$

Therefore, $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \qquad (6)$$

The separate expressions for the value of x may be obtained by using either the + or the − sign before the radical.

Accordingly, from (6) we obtain the following set of two equations in which the value of x is expressed in terms of the known numbers a, b, and c:

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \qquad (7)$$

In obtaining these solutions we have employed only those methods of derivation which lead to equivalent equations. Hence, the "expressed" values found are the solutions of the original equation, and there are no others.

By using the double sign ± before the radical, as in (6), we can verify the two results at the same time.

Substituting the expressed values (6) for x in equation (1), we have,

$$a\left(\frac{-b \pm \sqrt{b^2-4ac}}{2a}\right)^2 + b\left(\frac{-b \pm \sqrt{b^2-4ac}}{2a}\right) + c = 0$$

$$\frac{b^2 \mp 2b\sqrt{b^2-4ac} + b^2 - 4ac}{4a} + \frac{-2b^2 \pm 2b\sqrt{b^2-4ac}}{4a} + \frac{4ac}{4a} = 0$$

$$b^2 \mp 2b\sqrt{b^2-4ac} + b^2 - 4ac - 2b^2 \pm 2b\sqrt{b^2-4ac} + 4ac = 0$$

$$0 = 0.$$

24. After a given quadratic equation has been reduced to the standard form, $ax^2 + bx + c = 0$, the solutions may be obtained immediately by substituting in the general solution of the standard equation, $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$, the values corresponding to a, b, and c, respectively, in the given equation reduced to standard form.

Ex. 1. Solve $x^2 + 6x - 216 = 0.$

Referring to the standard quadratic equation, $ax^2 + bx + c = 0$, we find that 1 corresponds to a, 6 to b, and -216 to c.

Hence, substituting these values for a, b, and c, respectively, in the general formula

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a},$$

we have

$$x = \frac{-6 \pm \sqrt{6^2 - 4(-216)}}{2}$$

$$= \frac{-6 \pm \sqrt{36 + 864}}{2}$$

$$= \frac{-6 \pm 30}{2}.$$

That is, $x = 12$, and $x = -18$.

These values are found by substitution to satisfy the original equation. Hence, they are its roots.

Ex. 2. Solve $6x^2 - 5x - 375 = 0.$

Referring to the standard quadratic equation, $ax^2 + bx + c = 0$, it appears that 6 corresponds to a, -5 to b, and -375 to c.

Hence, substituting these values in the general formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

we have
$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(6)(-375)}}{2 \cdot 6}$$

$$= \frac{+5 \pm \sqrt{25 + 9000}}{12}$$

$$= \frac{+5 \pm 95}{12}.$$

That is, $x = 8\frac{1}{3}$, and $x = -7\frac{1}{2}$.

Both of these values satisfy the original equation, and hence they are its roots.

Ex. 3. Solve $x^2 + 11 = 8x$.

Observe that, when reduced to the standard form $x^2 - 8x + 11 = 0$, the numbers 1, -8 and 11 correspond to a, b and c respectively in the standard quadratic equation $ax^2 + bx + c = 0$.

Substituting these values in the general formula,

we obtain
$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4 \cdot 11}}{2}$$

$$= \frac{+8 \pm \sqrt{64 - 44}}{2}$$

$$= \frac{8 \pm \sqrt{20}}{2}$$

$$= \frac{8 \pm 2\sqrt{5}}{2}$$

That is, $x = 4 \pm \sqrt{5}$.

(Compare the solutions above with those shown in Ex. 2, § 20, and also see Fig. 9.)

Ex. 4. Solve $x^2 + 4x + 5 = 0$. (See § 9; also Fig. 3.)

Referring to the standard quadratic equation, we find that a, b, and c represent the values 1, 4, and 5 respectively in the given equation.

Hence, substituting these values for a, b, and c in the general solution,

we obtain
$$x = \frac{-4 \pm \sqrt{16 - 4 \cdot 5}}{2}$$

$$= \frac{-4 \pm \sqrt{-4}}{2}$$

$$= \frac{-4 \pm 2\sqrt{-1}}{2}$$

That is, $x = -2 \pm \sqrt{-1}.$

Let the student check these solutions, using the double sign before $\sqrt{-1}$.

Ex. 5. Solve $ax^2 - (a - b)x - b = 0.$

It will be observed that the coefficient a of x^2 in the given equation corresponds to the coefficient a of x^2 in the standard quadratic equation $ax^2 \div bx + c = 0$; that $-(a - b)$ which is the coefficient of x, corresponds to b in the standard equation; and that the known term $-b$ in the given equation corresponds to the known term $+c$ in the standard equation.

Hence, substituting these values in the general formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

we have,

$$x = \frac{-[-(a-b)] \pm \sqrt{[-(a-b)]^2 - 4a(-b)}}{2a}$$

$$= \frac{+(a-b) \pm \sqrt{a^2 - 2ab + b^2 + 4ab}}{2a}$$

$$= \frac{+(a-b) \pm (a+b)}{2a}.$$

Hence, $x = \frac{+a-b+a+b}{2a}$ and $x = \frac{+a-b-a-b}{2a}$

$x = \frac{2a}{2a}$ $\qquad$ $x = \frac{-2b}{2a}$

$x = 1.$ $\qquad$ $x = \frac{-b}{a}.$

(Compare this method of solution with that in Ex. 29, Exercise XXII. 4.)

EXERCISE XXII. 6

Solve the following equations by the formula, verifying such results as are neither irrational nor imaginary. Whenever surds appear in the *exact* solutions, *approximate* solutions correct to four places of decimals should be obtained.

1. $x^2 - 6x + 5 = 0.$
2. $x^2 + 9x = -20.$
3. $x^2 - 12 = x.$
4. $x^2 - 45 = 4x.$
5. $x^2 + 11x = 12.$
6. $x^2 + 8x = 1.$
7. $x^2 - 12x + 16 = 0.$
8. $x^2 - 14x + 9 = 0.$
9. $x^2 + 16 = 6x.$
10. $x^2 - 6x + 1 = 0.$
11. $x^2 + 16 = 4x.$
12. $4x^2 - 3x = 85.$

13. $2x^2 = 5x + 117.$

14. $x^2 + 25 = 18x.$

15. $6x^2 + 7x + 8 = 0.$

16. $7x^2 - 6x + 5 = 0.$

17. $x(x + 12) = -27.$

18. $15x(x - 1) = 2(x - 2).$

19. $3x(x - 1) = 2(1 + 2x).$

20. $(3x - 2)^2 = 7x.$

21. $(x - 2)^2 + 5(x - 3)^2 = 0.$

22. $2x^2 - 1.1x = 4.2.$

23. $\dfrac{x + 12}{12} = x^2 - \dfrac{x}{2}.$

24. $\dfrac{x^2}{21} + \dfrac{1}{7} = \dfrac{x}{3}.$

25. $\dfrac{x^2}{2} - \dfrac{x}{3} + \dfrac{1}{4} = 0.$

26. $x^2 - 2mx + n^2 = 0.$

27. $x^2 + 4ab = 2ax + 2bx.$

28. $a(x^2 - 1) = (a^2 - 1)x.$

29. $acx^2 + bd = adx + bcx.$

30. $x^2 - \dfrac{a^2 + b^2}{ab}x + 1 = 0.$

31. $cx + \dfrac{ac}{a + b} = (a + b)x^2.$

32. $x^2 + d^2 + 2c(x - d) = 2dx.$

33. $abcx^2 - (a^2b^2 + c^2)x + abc = 0.$

Rational Fractional Equations, Containing One Unknown

25. We shall now consider equations, which are rational and fractional with reference to a specified unknown, the solutions of which may be made to depend upon the solutions of quadratic equations.

26. If a fractional equation cannot be solved by inspection, then its solution may be made to depend upon the solution of a rational integral equation. This integral equation may be derived from the given fractional equation by multiplying the terms of both members by the lowest common multiple of the denominators of all of the fractions which appear in the different terms of the equation.

27. The derived integral equation will not always be equivalent to the given fractional equation, for in exceptional cases it may happen that extra roots which do not satisfy the given fractional equation are introduced into the derived equation.

28. Such extra roots as may be introduced during the process of clearing a given fractional equation of fractions may be determined by an examination of the equation, as may be seen by considering the following

Principle Relating to Extra Roots: *If an integral equation* *is derived from an equation which is rational and fractional with reference to a specified unknown, x, by multiplying both members of the fractional equation by a function of x, it may have solutions which do not satisfy the given fractional equation.*

(The following proof may be omitted when the chapter is read for the first time.)

Let all the terms of a given equation which is rational and fractional with reference to a specified unknown, x, be transposed to the first member and then added algebraically.

Let $\frac{N}{D}$ represent the resulting rational fractional expression, N and D being rational integral polynomials with reference to the unknown, x. Then since the second member of the derived fractional equation is zero, it follows that the given fractional equation will be equivalent to the derived fractional equation $\frac{N}{D} = 0$ (1).

We have seen in Chapter XVI, § 4, that if $\frac{N}{D}$ be reduced to lowest terms, the fractional equation $\frac{N}{D} = 0$ (1) is equivalent to the integral equation $N = 0$ (2). The latter is derived by multiplying both members of $\frac{N}{D} = 0$ by the multiplier D, which is *necessary* in order to clear (1) of fractions.

If, however, $\frac{N}{D}$ is not in lowest terms with reference to x, then the values of x which reduce the factor common to N and D to zero will reduce both the numerator and denominator of the fraction $\frac{N}{D}$ to zero, and for such values of x the true value of $\frac{N}{D}$ *may* be different from zero.

Accordingly, such values of x *may not* satisfy the fractional equation $\frac{N}{D} = 0$.

Hence, such solutions of the derived integral equation $N = 0$ *as are solutions also of the equation constructed by placing the multiplier* D *equal to zero, may not be solutions also of the fractional equation* $\frac{N}{D} = 0$, *and accordingly must be rejected.*

Hence, if when deriving the integral equation $N = 0$, a multiplier D is used which contains factors which are not necessary to clear the fractional equation $\frac{N}{D} = 0$ of fractions, extra solutions may be introduced by the process.

Ex. 1. Solve $\frac{1}{x-5}+\frac{2}{x-3}=1.$

The fractions are in lowest terms and an integral equation may be derived by multiplying the separate terms by $(x-5)(x-3)$ which is the lowest common multiple of the denominators of the fractions in the given equation.

Accordingly, $x-3+2(x-5)=(x-5)(x-3).$

Reducing to standard form, we obtain

$$x^2-11x+28=0,$$

the solutions of which are found to be

$$x=4, \qquad \text{and} \qquad x=7.$$

It should be observed that neither of these values is a solution of the equation $(x-5)(x-3)=0$ which is obtained by equating the multiplier, $(x-5)(x-3)$, to zero. Hence, neither of these values can have been introduced as an extra root during the process of solution.

The values 4 and 7 are found by substitution to be the solutions of the original equation.

Substituting 4,	Substituting 7,
$\frac{1}{-1}+\frac{2}{1}=1.$	$\frac{1}{2}+\frac{2}{4}=1.$
$1=1.$	$1=1.$

Ex. 2. Solve $$\frac{3x-1}{x^2-1}-3=\frac{1}{x-1}. \qquad (1)$$

Observe that the fractions are in lowest terms. Using as a multiplier x^2-1, which is the lowest common multiple of the denominators, we may derive the integral equation,

$$3x-1-3(x^2-1)=x+1. \qquad (2)$$

From this we obtain the equivalent equation in standard form,

$$3x^2-2x-1=0. \qquad (3)$$

Factoring, $$(3x+1)(x-1)=0. \qquad (4)$$

This last equation is equivalent to the set of two linear equations formed by writing the factors of the first member separately equal to zero.

$$3x+1=0, \qquad \text{and} \qquad x-1=0. \qquad (5)$$

The solutions of these equations are

$$x=-\tfrac{1}{3}, \qquad \text{and} \qquad x=1.$$

It should be observed that the value $x=-\frac{1}{3}$ is not a solution of the equation obtained by equating the multiplier x^2-1 to zero. Hence it cannot have been introduced during the process of solution.

By substitution $x = -\frac{1}{3}$ is found to be a solution of the given equation. (See Fig. 12.)

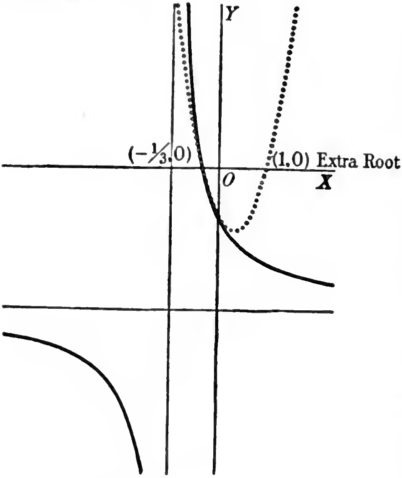


Fig. 12. { Full line $\frac{2}{x+1} - 3 = y$. Dotted line $3x^2 - 2x - 1 = y$.

The remaining value, $x = 1$, is a solution of the multiplier equation, $x^2 - 1 = 0$, and accordingly may have been introduced as an extra root during the process of solution.

By substitution it is found that this value does not satisfy the given equation, and accordingly it must be rejected as being an extra root. (See Fig. 12.)

Hence, $x = -\frac{1}{3}$ is the single solution of the given equation.

If, instead of deriving an integral equation from the fractional equation in the form as given, we first write equation (1) in the form

$$\frac{3x-1}{x^2-1} - \frac{1}{x-1} - 3 = 0,$$

we may obtain

$$\frac{3x-1}{x^2-1} - \frac{x+1}{(x-1)(x+1)} - 3 = 0.$$

Combining the fractions, $$\frac{2(x-1)}{(x+1)(x-1)} - 3 = 0,$$

which reduces to $$\frac{2}{x+1} - 3 = 0.$$

Hence, $$2 - 3x - 3 = 0.$$

The single solution of this last equation is $x = -\frac{1}{3}$.

It will be observed that, when deriving the integral equation in the first solution, we used a multiplier containing more factors than were necessary to clear of fractions, and accordingly introduced an extra solution into the derived equation.

29. It is often advisable, before deriving an integral equation, to combine the fractions in a given fractional equation, and to reduce to lowest terms all of the fractions then appearing.

Ex. 3. Solve $$\frac{1}{x-3} + \frac{1}{x-7} = \frac{1}{x+2} + \frac{1}{x-2}.$$

Instead of deriving an integral equation at once by multiplying the terms by the lowest common multiple of the denominators, we shall find it to our advantage to transpose the fractional terms in such a way as to obtain differences between two pairs of fractions, as follows:

$$\frac{1}{x-3}-\frac{1}{x+2}+\frac{1}{x-7}-\frac{1}{x-2}=0.$$

Combining the first and second fractions, and also the third and fourth fractions, we obtain

$$\frac{5}{(x-3)(x+2)}+\frac{5}{(x-7)(x-2)}=0.$$

Multiplying the terms of this equation by the lowest common multiple of the denominators, $(x-3)(x+2)(x-7)(x-2)$, and dividing both members by 5, we obtain

$$(x-7)(x-2)+(x-3)(x+2)=0,$$

which reduces to $x^2-5x+4=0.$

The solutions of this equation are

$$x=4, \text{ and } x=1.$$

Neither of these values is a solution of the equation

$$(x-3)(x+2)(x-7)(x-2)=0$$

obtained by equating to zero the multiplier used in deriving the integral equation $x^2-5x+4=0$. Hence neither value can be an extra root introduced during the process of solution, and accordingly these values must be solutions of the given equation.

Both values are found by substitution to satisfy the given equation.

Ex. 4. Solve $\frac{x}{x-2}-\frac{x-6}{(x-4)(x-2)}=\frac{x}{x-3}-\frac{2(x-6)}{(x-5)(x-3)}$.

Clearing of fractions by multiplying all of the terms by the lowest common multiple of the denominators, $(x-2)(x-4)(x-3)(x-5)$, and combining terms, we obtain the integral equation $x^2-5x-6=0$, the solutions of which are found to be $x=3$, and $x=2$.

Since these values are solutions of the equation

$$(x-2)(x-4)(x-3)(x-5)=0,$$

formed by placing equal to zero the multiplier employed in clearing the given equation of fractions, they must be rejected as being extra roots introduced during the process of solution.

By substituting these values, it will be found that neither satisfies the original equation.

Accordingly the fractional equation, as given, has no finite solution.

Exercise XXII. 7

Solve the following equations, rejecting extra solutions, and verifying all others :

1. $\frac{x}{2}+\frac{2}{x}=\frac{x}{8}+\frac{8}{x}.$
2. $\frac{7}{6x^2}-\frac{6}{7x^2}=\frac{13}{42}.$
3. $\frac{2}{x^2}+\frac{3}{5x^2}=\frac{13}{5}.$
4. $\frac{x-6}{8}=\frac{8}{x+6}.$
5. $\frac{x+3}{4}=\frac{4}{x-3}.$
6. $\frac{x-6}{x-4}=x.$
7. $2-\frac{2x+1}{3}=\frac{1}{2x-1}.$
8. $\frac{40}{x-5}+\frac{27}{x}=13.$
9. $\frac{4x-3}{1-x}=3+\frac{x^2}{1-x}.$
10. $x+\frac{3x-8}{x-3}+\frac{2x-7}{x-3}=9.$
11. $x-\frac{5x-7}{x-4}=\frac{2x-21}{x-4}-1.$
12. $\frac{1}{x}+\frac{1}{13}=\frac{1}{x+13}.$
13. $\frac{x}{x-1}=\frac{2}{3}+\frac{x+1}{x}.$
14. $x-\frac{7x-30}{x-5}=\frac{3x-20}{x-5}-2.$
15. $\frac{4}{x-6}-\frac{24}{x(x-6)}=1.$
16. $x-\frac{7x+23}{x+2}=\frac{8x+7}{x+2}-11.$
17. $\frac{x+5}{x-5}+\frac{x-5}{x+5}=\frac{37}{6}.$
18. $\frac{x+4}{x-4}+\frac{x-4}{x+4}=\frac{82}{9}.$
19. $\frac{4x-6}{x^2-9}-\frac{1}{x-3}-1=0.$
20. $\frac{6}{x-1}-1=\frac{12}{x^2-1}.$
21. $\frac{6+x}{1+x}+\frac{x-5}{2x}=\frac{15}{8}.$
22. $\frac{2x}{x-2}+\frac{3x-14}{x(x-2)}=3.$
23. $\frac{4}{x-2}+\frac{x-18}{x^2-4}=1.$
24. $\frac{1}{x-1}+\frac{2}{x+2}=\frac{1}{x+1}.$
25. $\frac{5x-6}{x-2}+\frac{x-18}{x^2-4}=6.$
26. $\frac{9}{x+2}+\frac{7}{x}=\frac{20}{x+3}.$
27. $\frac{1}{3}+\frac{1}{3+x}+\frac{1}{4+x}=0.$
28. $\frac{1}{x-1}+\frac{2}{x-2}=\frac{6}{x-3}.$

29. $\frac{1}{x-4}-\frac{3}{(x-1)(x-4)}=\frac{8}{x+6}.$

30. $\frac{1}{x-5}-\frac{11}{(x+6)(x-5)}=\frac{1}{3(x-6)}.$

31. $\frac{2x-7}{x-1}+1=\frac{2x+3}{x}-\frac{17}{20}.$

32. $\frac{4(x+5)}{x-3}=\frac{7x-10}{7}.$

33. $\frac{2x+3}{x-5}=\frac{7x-6}{x+7}.$

34. $\frac{2}{x+3}+\frac{12}{x^2-9}=1.$

35. $\frac{5}{x^2-1}-\frac{2}{x+1}=\frac{1}{8}.$

36. $\frac{7}{x-5}+\frac{17x-155}{x^2-25}=2.$

37. $\frac{x}{x-1}+\frac{x-3}{x^2-1}=\frac{5}{4}.$

38. $\frac{5}{x-6}+\frac{4}{5}=\frac{17x-42}{x^2-36}.$

39. $\frac{1}{x-7}-\frac{14}{x^2-49}=\frac{1}{17}.$

40. $\frac{x}{x-3}-\frac{9(x-1)}{x^2-9}=\frac{2}{5}.$

41. $\frac{1}{1-x}-\frac{1}{1-2x}=\frac{1}{1-3x}.$

42. $\frac{10}{x}-\frac{9}{x+1}=\frac{8}{x+2}.$

43. $\frac{4}{4-x}-\frac{3}{3-x}=\frac{1}{1-4x}.$

44. $\frac{5}{5-x}-\frac{3}{3-x}=\frac{16}{x-2}.$

45. $\frac{x-1}{x-3}-\frac{5}{2}=\frac{5-x}{x^2-5x+6}.$

46. $\frac{x-5}{x-7}+\frac{x+4}{2x+5}=\frac{25}{7}.$

47. $\frac{1}{x-2}-\frac{x+11}{(2x+9)(x-2)}=\frac{1}{3(2x-5)}.$

48. $\frac{1}{3x-2}-\frac{x+1}{(4x-1)(3x-2)}=\frac{3}{5(2x+1)}.$

49. $\frac{x}{2(x+2)}+\frac{3x+2}{x^2-4}=\frac{9}{10}.$

50. $\frac{2}{x-2}+\frac{x-4}{(x-1)(x-2)}=\frac{5}{x+1}.$

51. $\frac{x-4}{x-9}-\frac{3x-12}{x+4}=\frac{5}{x-9}.$

52. $\frac{2}{x+5}-\frac{x-3}{(x+1)(x+5)}=\frac{5}{x+9}.$

53. $\frac{1}{x+4}-\frac{7}{(x-3)(x+4)}=\frac{2}{x+2}.$

54. $\frac{2}{x+5}-\frac{x+3}{(x+4)(x+5)}=\frac{1}{4(x-8)}.$

55. $\frac{2}{x+2}-\frac{x-8}{x^2-x-6}=\frac{2}{x+1}.$

56. $\frac{1}{x-5}+\frac{1}{x^2-11x+30}=\frac{5}{x+10}.$

57. $\frac{1}{x+6}-\frac{1}{2(x-3)}=\frac{1}{x^2+13x+42}.$

58. $\frac{1}{2x-1}-\frac{3}{(4x+1)(2x-1)}=\frac{1}{8x-1}.$

59. $\frac{1}{x-4}+\frac{2}{3x-7}=\frac{5}{(2x-3)(x-4)}.$

60. $\frac{5}{5x-1}-\frac{26}{(x+5)(5x-1)}=\frac{1}{3x-5}.$

61. $\frac{3}{6x-1}+\frac{26}{(5x-9)(6x-1)}=\frac{3}{x+3}.$

62. $\frac{7}{3x-1}+\frac{2x-10}{(x+1)(3x-1)}=\frac{4}{3x+1}.$

63. $\frac{2}{3x-2}+\frac{7x-16}{(4x+3)(3x-2)}=\frac{2}{6x-1}$.

64. $\frac{15}{5x-2}+\frac{4(5x-11)}{(x+2)(5x-2)}=\frac{15}{3x+2}$.

65. $\frac{1}{2x-5}-\frac{2(5x+2)}{(12x-1)(2x-5)}=\frac{1}{4x+5}$.

66. $\frac{1}{2x+3}-\frac{4(x+1)}{(6x+7)(2x+3)}=\frac{3}{6x+11}$.

67. $\frac{x}{x-6}-\frac{5x-6}{(x-2)(x-6)}=\frac{x^2-7}{(x-3)(x-1)}-\frac{3}{x-1}$.

68. $\frac{x-5}{x-7}-\frac{x+17}{(x+5)(x-7)}=\frac{x^2-5x-15}{(x-8)(x+1)}-\frac{1}{x+1}$.

69. $\frac{x}{x-1}+\frac{2}{(x-3)(x-1)}=\frac{x}{x+1}+\frac{2x-1}{(x-2)(x+1)}$.

70. $\frac{x}{x-4}-\frac{13x+24}{(x+11)(x-4)}=\frac{x}{x-2}-\frac{7x+8}{(x+9)(x-2)}$.

71. $\frac{x}{x-3}-\frac{2(x-9)}{(x-7)(x-3)}=\frac{x}{x-2}-\frac{x-8}{(x-5)(x-2)}$.

72. $\frac{x}{x-6}-\frac{3(5x+8)}{(x+13)(x-6)}=\frac{x}{x+4}-\frac{5x+4}{(x+8)(x+4)}$.

73. $\frac{x}{x+3}+\frac{12(x+2)}{(x-1)(x+3)}=\frac{x}{x-2}+\frac{7x-20}{(x+1)(x-2)}$.

74. $\frac{x}{x-4}+\frac{4(x-5)}{(x-3)(x-4)}=\frac{x}{x-2}+\frac{6(x-3)}{(x+1)(x-2)}$.

75. $\frac{x}{x+2}+\frac{2(3x+8)}{(x+4)(x+2)}=\frac{x}{x+3}+\frac{7x+15}{(x+1)(x+3)}$.

EXERCISE XXII. 8. **Miscellaneous**

Solve the following equations, verifying such solutions as are neither irrational nor imaginary :

1. $x^2-15x+54=0$.
2. $2x^2-7x=15$.
3. $3(x^2-1)=8x$.
4. $(x+4)^2=9x^2$.
5. $55(x^2-x)=11x$.
6. $(x+2)(x+3)=20$.

7. $(x+1)(x-4)=50.$

8. $\dfrac{2}{x-1}-10x=9.$

9. $6=\dfrac{x}{2x^2-1}.$

10. $x+\dfrac{1}{2}=2+\dfrac{1}{x}.$

11. $x+\dfrac{1}{5}=5+\dfrac{1}{x}.$

12. $10(x+2)(x-2)=41x.$

13. $x^2+x=d^2+d.$

14. $x+\dfrac{1}{b}=b+\dfrac{1}{x}.$

15. $x+\dfrac{mn}{x}=m+n.$

16. $x^2+n^2=m^2+2nx.$

17. $x(2x-a)+x(x-a)=bx.$

18. $\dfrac{x}{2}+\dfrac{2}{x}=\dfrac{3}{2}+\dfrac{2}{3}.$

19. $\dfrac{x}{a}+\dfrac{a}{x}=\dfrac{b}{a}+\dfrac{a}{b}.$

20. $\dfrac{x}{x-3}+\dfrac{8}{x+2}=3.$

21. $\dfrac{3}{x-5}-5=-\dfrac{2x}{x-3}.$

22. $\dfrac{1}{x}-\dfrac{1}{2}=\dfrac{1}{x-2}.$

23. $\dfrac{7x}{8}+\dfrac{8}{7x}=\dfrac{9}{10}.$

24. $\dfrac{5x}{7}+\dfrac{1}{x}=\dfrac{7}{5}.$

25. $\dfrac{x}{5}-\dfrac{5}{6}=\dfrac{6}{5}-\dfrac{5}{x}.$

26. $\dfrac{3x}{2}+\dfrac{2}{x}=\dfrac{2x}{3}.$

27. $\dfrac{1}{x}+x=3+\dfrac{3}{x}.$

28. $\dfrac{2x}{3}-4x^2=\dfrac{1}{2}.$

29. $\dfrac{15x}{x-1}=11(x-1).$

30. $\dfrac{1}{x+1}-\dfrac{1}{x+2}-\dfrac{1}{x+3}=0.$

31. $\dfrac{x+16}{8}-\dfrac{64-x}{x-4}=\dfrac{x}{4}-13.$

32. $\dfrac{x}{x+1}+\dfrac{x+1}{x}=\dfrac{13}{6}.$

33. $\dfrac{1}{x}-\dfrac{1}{6}=\dfrac{1}{x-6}.$

34. $\dfrac{1}{x}+\dfrac{1}{8-x}=\dfrac{1}{8}.$

35. $\dfrac{x}{x+1}+\dfrac{x+1}{x}=\dfrac{113}{56}.$

36. $\dfrac{x+2}{x+3}+\dfrac{x+3}{x+2}=\dfrac{41}{20}.$

37. $\dfrac{1}{x-2}+1=\dfrac{6-x}{x^2-4}+\dfrac{1}{x+2}.$

38. $(2x+1)(2x-1)+(x+2)(x-2)=x(x+3).$

39. $(x+2)(x-3)+(x+9)(x+3)=3x+15.$

40. $\frac{1}{x}-\frac{1}{d}=\frac{1}{x-d}.$

41. $\frac{a}{x-b}+\frac{b}{x-a}=2.$

42. $\frac{a}{x}+\frac{x}{a}=\frac{33a^2-x^2}{ax}.$

43. $\frac{x}{a^2}+\frac{a^2}{x}=\frac{x}{b^2}+\frac{b^2}{x}.$

44. $x^2+8x=16g+2gx.$

45. $x^2+2bx=a^2+2ab.$

46. $(x+b)^2+6(x+b)+9=0.$

47. $x^2+4ab=2x(b+a).$

48. $2x(7x-a)=(a+x)(a-x).$

49. $\frac{m^2}{x^2}=\frac{m+1}{x+1}.$

50. $\frac{x-4m+n}{n-m}=\frac{9n+x}{x}.$

51. $\frac{(x+1)(x-2)}{2}-\frac{(x-1)(x+2)}{3}=4.$

52. $(a-x)^2+(b-x)^2=(b-a)^2.$

53. $(x-g)^2+(x-h)^2=g^2+h^2.$

54. $(c-a)x^2+(a-b)x+b-c=0.$

55. $\frac{1}{a+b+x}=\frac{1}{a}+\frac{1}{b}+\frac{1}{x}.$

56. $\frac{x+c}{x-d}+\frac{x-c}{x+d}=1+\frac{c^2+2d^2}{x^2-d^2}.$

57. $\frac{1}{x}+\frac{1}{x+4}=\frac{1}{x+1}+\frac{7}{7x+3}$

58. $\frac{x}{b+x}+\frac{x}{c+x}=\frac{a}{a+b}+\frac{a}{a+c}.$

PROBLEMS SOLVED BY MEANS OF QUADRATIC EQUATIONS

30. In the solution of problems by means of conditional equations, it is often convenient to represent the unknown quantities by such "initial letters" as may suggest the quantities considered.

E.g. In the case of a body moving at a uniform rate during a specified time, the distance passed over may be found as follows:

$$\textit{distance} = \textit{rate} \times \textit{time}.$$

In particular, if a train moves uniformly at the rate of forty miles an hour for two hours, the distance travelled will be eighty miles.

If we represent the number of units in terms of which distance, rate, and time are expressed by the initial letters d, r, and t, respectively, we may express the general relation between distance, rate, and time for uniform motion by the formula $d = r \times t$.

For simple interest we have the relation

$$\textit{interest} = \textit{principal} \times \textit{rate} \times \textit{time}.$$

Representing the numbers, in terms of which interest, principal, rate, and time are expressed, by the initial letters i, p, r, and t, respectively, we have the formula $i = p \times r \times t$.

31. When using a particular formula, the letters which are to be regarded as representing unknowns must be determined by the nature of the problem to the solution of which it is applied.

Solution of Formulas for Specified Letters

32. The Greek letter π, read "pi," is used in mathematical calculations to represent a certain incommensurable constant the approximate value of which may for many practical purposes be taken equal to 22/7. In particular, the ratio of the circumference to the diameter of a circle is equal to π.

EXERCISE XXII. 9

In the following formulas find the expressed values of the letters specified in terms of those remaining:

1. Solve for R, $s = \pi R^2$.

2. Solve for E, $w = \dfrac{E^2t}{R}$.

3. Solve for t, $S = \frac{1}{2}at^2$.

4. Solve for s, $F = \dfrac{ms^2}{r}$.

5. Solve for r, $f = \frac{mm_1}{r^2}$.

6. Solve for a, $s = \frac{a^2}{4}\sqrt{3}$.

7. Solve for n, $d = \frac{2(ln - s)}{n(n-1)}$.
Find the value of n, if $d = 2$, $l = 19$, and $s = 96$.

8. Solve for R, $V = \frac{1}{3}\pi R^2 h$.
Find the value of R, if $V = 132$, and $h = 14$.

9. Solve for R, $T = 2\pi R(H + R)$.
Find the value of R, if $T = 352$, and $H = 10$.

10. Solve for t, $ab = t^2 + pq$.
Find the value of t, if $a = 16$, $b = 14$, $p = 7$, and $q = 8$.

11. Solve for c, $a^2 = b^2 + c^2 + 2cp$.
Find the value of c, if $a = 18$, $b = 11$, and $p = \frac{11}{15}$.

12. Solve for m, $2a^2 + 2b^2 = c^2 + 4m^2$.
Find the value of m, if $a = 12$, $b = 16$, and $c = 20$.

13. Solve for n, $s = \frac{n}{2}[2a + (n-1)d]$.
Find the value of n, if $s = 272$, $a = 6$, and $d = 8$.

14. Solve for a, $s = \frac{l+a}{2} + \frac{l^2 - a^2}{2d}$.
Find the value of a, if $s = 70$, $l = 16$, and $d = 2$.

15. Solve for l, $d = \frac{l^2 - a^2}{2s - l - a}$.
Find the value of l, if $d = 15$, $a = 18$, and $s = 1206$.

The Solution of Problems

33. Whenever the translation into algebraic language of stated relations between the unknown quantities of a problem leads to one or more conditional equations of the second degree, we may expect to find two or more solutions.

It may happen, however, that one or both of the solutions of the algebraic equations must be rejected as not fulfilling the conditions expressed by the given problem.

Solutions consisting of negative numbers or fractions cannot, in certain cases, be given a sensible interpretation.

E. g. If the number of people in a given assembly be in question, negative or fractional answers must necessarily be rejected.

In the case of a body in motion, fractions have a significance as indicating definite distances, while negative answers may be interpreted as meaning a reversal in the direction of the motion.

34. Positive results will in general be found to satisfy all the conditions of a given problem.

A negative result will, in general, satisfy the conditions of such problems as are concerned with abstract numbers.

Whenever the unknown quantities referred to in a problem are of such nature as to admit of being taken in opposite senses, that is, of being regarded as positive or negative, then in such cases it is usually possible to give a sensible interpretation to a negative result.

Ex. 1. Find two numbers of which the sum is 30, and of which the product is 221.

If x stands for one of the required numbers, then by one of the conditions of the problem, $30 - x$ will stand for the other.

We may express the remaining condition, that the product is 221, by means of the conditional equation

$$x(30 - x) = 221.$$

Solving, $x = 17$, and $x = 13$.

Hence one of the numbers is 17 or 13, and substituting either of these values for x in the expression for the other number, $30 - x$, we obtain 13 and 17 respectively.

These values satisfy both the algebraic equation and also the conditions of the problem as stated, and hence are its solutions.

35. Imaginary results must always be interpreted as indicating inconsistent relations among the conditions of a given problem as stated.

Ex. 2. Separate 50 into two parts the product of which is 630.

We may represent the required numbers by x and $50 - x$ respectively.

Expressing the condition that the product of these numbers is 630, we may construct the conditional equation $x(50 - x) = 630$, the solutions of which are both found to be imaginary.

These imaginary results must be interpreted as indicating that *two such numbers cannot exist.*

36. When expressing the relations of a given problem by means of one or more conditional equations, all that we know at the outset regarding the number of the solutions is that they must be found among the abstract numbers which constitute the algebraic solutions of the equations; if none of the numbers thus found can be given a reasonable interpretation, then the given concrete problem as stated has no solutions.

EXERCISE XXII. 10

Solve the following problems, employing equations containing one unknown quantity:

1. Separate 42 into two parts such that one part is the square of the other.

2. The sum of two numbers is 45 and their product is 476. Find the numbers.

3. Find two consecutive integers the product of which is 702.

4. Find two consecutive even integers the product of which is 528.

5. Find two consecutive odd integers the product of which is 1023.

6. Find two consecutive integers, the sum of the squares of which is 481.

7. Find two consecutive even integers, the sum of the squares of which is 2180.

8. Find a number which equals 4 more than the square of its fourth part.

9. Find three consecutive even integers such that the product of the first and third shall equal three times the second.

10. Find three consecutive even integers such that the square of the greatest shall be equal to the sum of the squares of the other two.

11. Find three consecutive integers, the sum of the products of which, by pairs, is 674.

12. Find four consecutive even integers such that the product of the first and third shall equal the sum of the second and fourth.

13. Find four consecutive odd integers such that twice the product of the second and fourth is equal to eleven times the sum of the first and third.

14. Find three consecutive integers the sum of which is one-third the product of the first two.

15. Find two consecutive integers such that the sum of their reciprocals is $\frac{13}{42}$.

16. Separate a number represented by a into two parts such that one part shall be the square of the other.

17. Find a number such that, if 87 be subtracted from it, the remainder equals the quotient obtained by dividing 270 by the number.

18. Find a positive fraction such that two times its square is 3 more than the fraction.

19. The denominator of a certain fraction exceeds its numerator by 3, and the reciprocal of the fraction exceeds the fraction by 39/40. Find the fraction.

20. The numerator of a certain improper fraction exceeds the denominator by 5, and the fraction exceeds its reciprocal by 45/14. Find the fraction.

21. Find two numbers in the ratio 2 : 7, the sum of the squares of which is 212.

22. Find a number such that the sum of its third part and its square is 1100.

23. Find a number such that one-half its square shall exceed the square of one-half the number by one-half the number.

24. If the seventh part and the eighth part of a certain number are multiplied together, and the product is divided by 3, the quotient is $298\frac{2}{3}$. Find the number.

25. It is found that when a number which is the product of three consecutive integral numbers is divided separately by each of these three factors, the sum of the quotients thus obtained is 191. What are the numbers?

26. I have thought of a number. I multiply it by $2\frac{1}{5}$, then add 4 to the product; I then multiply the result by three times the number thought of, and finally divide by 5 and subtract from the quotient five times the number originally thought of, obtaining thus 20. What was the original number?

27. It is necessary to construct a coal bin to hold 6 tons of coal. Allowing 40 cubic feet of space per ton of coal, what must be the dimensions if, the depth being 6 feet, the length is equal to the sum of the width and the depth?

28. A crew can row 8 miles, down a stream and back again, in 3 hours and 40 minutes. If the rate of the stream is two and one-half miles an hour, find the rate of the crew in still water in miles per hour.

29. A man bought two farms for $3600 each. The larger contained 15 acres more than the smaller, but $8 more per acre was paid for the smaller than for the larger. How many acres did each contain?

30. If $3000 amounts to $3213.675 when put at compound interest for two years, interest being compounded annually, what is the rate per cent per year?

31. If $4250 amounts to $4508.825 when placed at compound interest for 2 years, interest being compounded annually, find the rate per cent per year.

32. Telegraph poles are placed at equal intervals along a certain railway. In order that there should be two less poles per mile it would be necessary to increase the space between every two consecutive poles by 24 feet. Find the number of poles to a mile.

33. A man bought a certain number of shares of a railway stock for $6600. The next day they declined in value $12 a share, when he could have bought five shares more for the same amount. Find the price paid per share.

34. A broker purchased a certain number of shares of stock for $2560. After reserving 10 shares, those remaining were sold for $2450 at an advance of $3 a share on the cost price. How many shares did he buy?

35. It is desired to carpet a floor in the form of a rectangle 15 feet long by 12 feet wide, with a carpet having a plain color border of uniform width. Allowing $1.44 per square yard for the center and $0.45 per square yard for the border, determine the width of the border in order that the entire expense may be $18.68.

36. Two steamers ply between two ports, a distance of 475 miles. One goes half a mile an hour faster than the other, and requires two and one-half hours less for the voyage. Find the rates of the steamers in miles per hour.

Let x represent the rate of the slower steamer in miles per hour.

Then the rate of the faster boat in miles per hour will be represented by $x + 1/2$.

From the general formula expressing the relation between distance, rate, and time for uniform motion, we have, by the conditions of the problem, $475/x$ and $475/(x + \frac{1}{2})$ as representing the times required for the slower and faster boats respectively to make the entire trip.

By the conditions of the problem, the time required by the faster boat is $2\frac{1}{2}$ hours less than that taken by the slower. Hence we obtain the conditional equation $475/x = 475/(x + \frac{1}{2}) + \frac{5}{2}$.

From this equation we obtain the integral equation

$$2x^2 + x - 190 = 0,$$

the solutions of which are found to be

$$x = \tfrac{19}{2}, \quad \text{and} \quad x = -10.$$

Since, from the nature of the problem, the forward motion only of the boats is considered, we shall use the positive value, $x = 9\frac{1}{2}$, and reject the negative value $x = -10$.

Accordingly, $x + 1/2$, which is the rate of the faster boat, is 10 miles an hour.

The values $9\frac{1}{2}$ and 10 will be found to satisfy the condition of the given problem.

37. An engineer, on a trip of 108 miles, found it necessary at the thirty-sixth milestone to decrease his speed to a rate 9 miles an hour less, with the result that he was on the road 24 minutes longer than would have been the case had no alteration in speed been made. Find the rate in miles per hour before the speed was changed and also the time required to make the entire trip.

Let the speed, in miles per hour before the change was made, be represented by x.

By the conditions of the problem the time, $36/x$, required to cover the first 36 miles, taken together with that for the remaining 72 miles at the decreased speed, $x - 9$, that is, $72/(x - 9)$, is equal to the time which would have been required to run the entire distance of 108 miles at x miles per hour, — that is, $108/x$, increased by $24/60$ hours.

Hence we have the conditional equation

$$\frac{36}{x} + \frac{72}{x-9} = \frac{108}{x} + \frac{24}{60},$$

which reduces to $$x^2 - 9x - 1620 = 0.$$

Of the two solutions of this equation, $x = 45$ and $x = -36$, the negative value cannot be admitted, since only the forward motion of the train is in question.

Hence, as a solution of the problem, we find that at first the train was going at the rate of 45 miles per hour.

The time in hours required to cover the entire distance is expressed by

$$36/x + 72/(x - 9).$$

Substituting 45 for x in this expression we obtain as the number of hours required for the entire trip, $\frac{36}{45} + \frac{72}{36}$, which reduces to $2\frac{4}{5}$.

These values, 45 and $2\frac{4}{5}$, will be found to satisfy the conditions of the given problem.

38. In answering a false alarm, a fire engine travelled a distance of 3/5 of a mile at the rate of 5 miles an hour faster than when returning. If it returned immediately on reaching the "alarm box" and was gone from the station $16\frac{1}{2}$ minutes in all, what was its rate at first in miles per hour?

39. A and B run a half-mile race. A, who is faster than B by 1/2 a yard a second, allows B a start of 1/4 of a minute, and beats him by 5 yards. Find their respective rates in yards per second.

40. A warship which is approaching a port is discovered when it is 12 miles away. A flotilla of torpedo boats, the maximum speed of which is known to exceed that of the warship by 7 miles an hour, is sent out $5\frac{3}{5}$ minutes later to meet the warship. They intercept it when it has covered half the distance to the port. Find the rate of the warship in miles per hour.

41. After having gone 40 miles of his trip at a uniform rate, an engineer found that his train was behind time. He immediately increased the speed of the engine to a rate 4 miles an hour more, and completed the trip of 62 miles, arriving at the terminus 3 minutes earlier than would have been the case if no change in rate had been made. Find the rate of the train in miles per hour.

42. A man travels 24 miles by an accommodation train and returns by an express which runs 10 miles an hour faster. Find the rates of the two trains in miles per hour, provided that the time occupied for the two trips was one hour and twenty-four minutes.

43. It is found that two steam fire engines can, by working together, pump all of the water out of a partly filled cellar in $22\frac{1}{2}$ minutes. The more powerful one alone would have been able to perform the work in 24 minutes less than the other one alone. Find the time required by each one working alone.

44. After travelling 8 miles in an automobile, a man found that, on account of an accident to the machine, it was necessary to walk back. If the rate of the automobile exceeded the man's rate when walking by 17 miles an hour, and he was 2 hours, 16 minutes longer in returning than going, find the rate of the machine in miles per hour.

Problems in Physics

37. If a moving body passes over equal distances in successive equal intervals of time, the motion of the body is said to be *uniform.* If the distances passed over in successive equal intervals of time are not equal, the motion is said to be *variable.*

38. When the motion of a body is uniform its *velocity* is defined to be the number of units of distance passed over by the body in one unit of time. When the motion of a body is not uniform the velocity at any instant is defined to be the number of units of distance which would be passed over in the next unit of time if the motion of the body were to become uniform at that instant.

39. If the velocity of a moving body increases during successive intervals of time, the motion is said to be *accelerated.*

Acceleration is defined to be the rate at which velocity changes.

Since velocity is distance per unit of time, it follows that acceleration is distance per unit of time per unit of time.

Acceleration is said to be positive if the velocity increases in successive intervals of time, and negative if the velocity decreases.

E. g. A body which is falling freely from any point above the surface of the earth moves toward the earth with uniformly accelerated motion. A body which is thrown upward moves away from the surface of the earth with a uniformly retarded motion.

40. For uniform motion, the velocity, v, expressed in feet per second, of a body which in t seconds passes over a total distance, S, expressed in feet, may be found from the formula

$$v = \frac{S}{t}.$$

41. Falling Bodies. If a body in a state of rest starts to fall from any point above the surface of the earth, and is acted upon by the force of gravity alone, the total distance S, expressed in feet, passed over in t seconds, is found by the following formula. In this formula g is a numerical constant of which the approximate value may be taken as 32 in the following examples.

$$S = \tfrac{1}{2}gt^2.$$

It should be understood that the results obtained by using the formulas in the following examples are only approximate, because the value 32 substituted for g is approximate and because no allowance is made for the retarding influence due to the resistance of the air.

42. The velocity, v, expressed in feet per second, of a falling body at the end of t seconds, may be found by the formula

$$v = gt.$$

EXERCISE XXII. 11

Solve the following problems relating to moving bodies:

1. What velocity, expressed in feet per second, will a body acquire by falling 5 seconds?

2. In what time will a falling body acquire a velocity of 224 feet per second?

3. What is the height of a tower, if a stone dropped from it requires 3 seconds to reach the ground?

4. A stone dropped from the top of a cliff is observed to reach the bottom in 5 seconds. Find the height of the cliff.

5. A balloon is moving horizontally at a height of one mile above the ground. How long will it take a bag of ballast to reach the ground?

By substituting for t in the formula $S = \frac{1}{2} gt^2$ the value $\frac{v}{g}$ obtained from $v = gt$, we obtain

$$v^2 = 2\,gS.$$

This formula may be used to find the velocity, v, in feet per second, acquired by a body falling a distance of S feet.

6. What velocity, expressed in feet per second, will a body acquire by falling a distance of 576 feet?

7. What velocity, expressed in feet per second, would a body acquire in falling a distance of 500 feet?

The velocity of a body which is thrown vertically upward with a velocity of v_1 feet per second will be retarded by an amount equal to g feet per second per second. The time of the ascent is found by dividing the initial velocity, v_1, expressed in feet per second, by the constant g.

That is, $$t = \frac{v_1}{g}.$$

It may be shown that, if there were no retarding influence due to the resistance of the air, the times required by the body in ascending and descending would be equal, and that it would return to its starting point with a velocity equal to that with which it was thrown upward. Hence it follows that the height to which a body will rise when thrown vertically upward with an initial velocity of v feet per second is given by S in the formula $S = \frac{v^2}{2\,g}$.

8. A stone thrown vertically upward strikes the ground after an interval of 10 seconds. With what velocity was it thrown and to what height did it rise?

9. How high is a tree if it requires three seconds for a stone which is thrown over it to reach the ground?

10. With what velocity, expressed in feet per second, must an arrow be shot vertically upward to reach the top of a tower which is 169 feet high?

If a falling body is given an initial velocity downward of v_1 feet per second, the total space S, expressed in feet per second, passed over by the body in t seconds, may be found by the formula

$$S = v_1 t + \tfrac{1}{2} gt^2.$$

11. Find the distance passed over in three seconds by a body which is thrown vertically downward with a velocity of 24 feet per second.

12. Find the time, expressed in seconds, required for a body which is thrown vertically downward with an initial velocity of 5 feet per second to move a distance of 475 feet.

13. A balloon is two miles from the ground and is descending at the rate of eight feet per second when a sand bag is dropped. Find the number of seconds required for the sand bag to reach the ground.

If a body be thrown vertically upward, with an initial velocity of v feet per second, the velocity of the body will be retarded by an amount equal to g feet per second per second.

Accordingly, since the acceleration due to gravity acts in such a way as to diminish the upward motion of the body, it follows that the initial velocity and the acceleration due to gravity must be considered as positive and negative numbers.

Hence, if a body is projected vertically upward with an initial velocity of v_1 feet per second, the velocity v_t at the end of t seconds may be found from the formula

$$v_t = v_1 - gt.$$

The height, expressed in feet, to which the body will rise, is represented by S in the following formula:

$$S = v_1 t - \tfrac{1}{2} gt^2.$$

14. A balloon is 500 feet from the ground and ascending at the rate of 12 feet per second when a sand bag is dropped. How many seconds will be required for the sand bag to reach the ground?

15. If a body is projected upward with an initial velocity of 160 feet per second, what is the height to which it will rise?

16. A rifle bullet is shot vertically upward with an initial velocity of 400 feet per second. Find the height to which it will rise.

17. A bullet is fired vertically upward with a velocity of 100 feet per second. Find the time required for it to reach a point 156 feet above the ground, and also the velocity with which it passes this point.

CHAPTER XXIII

THEORY OF QUADRATIC EQUATIONS CONTAINING ONE UNKNOWN

1. FROM every quadratic equation containing one unknown, an equivalent quadratic equation in the standard form $ax^2 + bx + c = 0$ can be derived by making suitable transformations. The equation $ax^2 + bx + c = 0$ has been shown to have two solutions. Hence, it follows that *every quadratic equation containing one unknown has two roots.* (See Chapter XXII. § 23.)

Denoting the two solutions of the standard quadratic equation $ax^2 + bx + c = 0$ by x_1 and x_2, we may write $x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$, and $x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$.

In all of the discussions which follow, it will be assumed that a, b, and c have real rational values.

2. It follows from the Remainder Theorem (Chapter VIII.) that if, when any value represented by r is substituted for x the value of the quadratic expression $ax^2 + bx + c$ becomes zero, then $x - r$ is a factor of $ax^2 + bx + c$.

Since $ax^2 + bx + c$ is of the second degree with reference to x, it cannot be the product of more than two factors which are each of the first degree with reference to x.

The roots of the quadratic equation $ax^2 + bx + c = 0$ are the roots of the equations obtained by equating the factors of the first member $ax^2 + bx + c$ to zero.

Hence, it follows that *the quadratic equation* $ax^2 + bx + c = 0$ *cannot have more than two roots.* (See Chapter XXII. § 6.)

Nature of the Roots

3. In the general solution $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ of the standard quadratic equation $ax^2 + bx + c = 0$, the expression $b^2 - 4ac$ is called the **discriminant** of the quadratic equation. This is because it affords means for discovering the nature of the roots of a given equation, — that is, for determining whether the roots are positive or negative, equal or unequal, rational or irrational, real or imaginary.

4. When values represented by a, b, and c, in the standard quadratic equation $ax^2 + bx + c = 0$, are selected from a given equation and substituted in the discriminant $b^2 - 4ac$, it may be seen that the resulting value must be zero, a positive number or a negative number.

By referring to the general solution

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

of the quadratic equation $ax^2 + bx + c = 0$, it may be seen that:

(i.) *If $b^2 - 4ac$ be zero, the roots are real, rational, and equal.*

That is, $x_1 = -\frac{b}{2a}$, and $x_2 = -\frac{b}{2a}$. (See § 1.)

(ii.) *If $b^2 - 4ac$ be positive, the roots are real and unequal, and rational or irrational, according as the value represented by $b^2 - 4ac$ is, or is not, the square of a rational number.*

(iii.) *If $b^2 - 4ac$ be negative, the expression $\sqrt{b^2 - 4ac}$ represents an imaginary quantity, and the roots are conjugate complex numbers.*

It follows from the general solution that irrational and also complex roots enter in pairs.

Determine, without solving, the nature of the roots of each of the following equations:

Ex. 1. Examine $x^2 + 4x + 4 = 0$.

Substituting in the discriminant $b^2 - 4ac$ the values from the given equation represented by a, b, and c in the standard quadratic equation $ax^2 + bx + c = 0$, we obtain

$$4^2 - 4 \cdot 4 = 0.$$

Since the value represented by the discriminant is zero, it follows that the roots of the equation are real, rational, and equal. (See (i.), § 4, and also Chap. XXII. § 8, Fig. 2.)

Ex. 2. Examine $x^2 + 4x - 5 = 0$.

Substituting in the discriminant $b^2 - 4ac$ the values from the given equation represented by a, b, and c in the standard quadratic equation, we obtain

$$4^2 - 4 \cdot 1(-5) = 36.$$

Hence, by (ii.), § 4, the roots are real and unequal. Also, since the value of the discriminant is the square of a rational number, — that is, 36 is the square of the rational number 6, — the roots are rational and unequal. (See Chap. XXII. § 5, Fig. 1.)

Ex. 3. Examine $x^2 - 8x + 11 = 0$.

Substituting in the discriminant $b^2 - 4ac$ the values from the given equation represented by a, b, and c in the standard quadratic equation, we have

$$(-8)^2 - 4 \cdot 1 \cdot 11 = 20.$$

Since the value represented by the discriminant is a positive number, 20, it follows from (ii.), § 4, that the roots are real and unequal.

Also, since 20 is not the square of a rational number, that is, $\sqrt{20}$ is irrational, the roots are conjugate irrational numbers.

Ex. 4. Examine $x^2 + 4x + 5 = 0$.

The value represented by $b^2 - 4ac$ is negative, that is,

$$4^2 - 4 \cdot 1 \cdot 5 = -4.$$

Accordingly, by (iii.), § 4, the roots are imaginary; that is, they are conjugate complex numbers. (See Chap. XXII, § 9, Fig. 3.)

Ex. 5. Examine $mx^2 + (m + n)x + n = 0$, $m \neq n$.

Comparing with the standard quadratic equation $ax^2 + bx + c = 0$, we find that m is represented by a, $(m + n)$ by b, and n by c.

Hence, substituting these values in the discriminant $b^2 - 4ac$, we obtain

$$(m + n)^2 - 4mn \equiv (m - n)^2.$$

Since the square of any real number is positive, it follows from (ii.), § 4, that the roots are real, rational, and unequal, for real values of m and n.

Exercise XXIII. 1

Determine, without solving the equations, the nature of the roots of the following:

1. $x^2 + 10x + 25 = 0$.
2. $9x^2 - 24x + 16 = 0$.
3. $2x^2 + 3x - 27 = 0$.
4. $6x^2 - x - 15 = 0$.

5. $3x^2 - 5x + 1 = 0.$
6. $x^2 + 6x - 247 = 0.$
7. $3x = x^2 + 4.$
8. $x^2 + 3x = 180.$
9. $7x^2 - 8x + 2 = 0.$
10. $4x^2 - 10x + 3 = 0.$
11. $x^2 - x + 1 = 0.$
12. $x^2 + 4x - 77 = 0.$
13. $3x^2 + 2x + 2 = 0.$
14. $2 + 9x - 8x^2 = 0.$
15. $4x^2 - 5x + 3 = 0.$
16. $x^2 - x - 342 = 0.$
17. $6x^2 + 5x + 4 = 0.$
18. $x^2 - x - 756 = 0.$

19. Prove that the roots of $x^2 - 2ax + a^2 = b^2 + c^2$ are real.

20. Prove that the roots of $2ax^2 + (2a + 3b)x + 3b = 0$ are real for all real values of a and b.

21. Prove that $3cx^2 - (2c + 3d)x + 2d = 0$ has rational roots.

22. Show that the roots of $5x^2 + 4ax + a^2 = 0$ are imaginary.

23. Show that the roots of $(a + b)^2x^2 - 2(a^2 - b^2)x + (a - b)^2 = 0$ are neither irrational nor imaginary.

Determine the value which k must have in order that the following equations shall have equal roots:

Ex. 24. $$x^2 + (k - 3)x + k = 0.$$

The condition for equal roots is that the discriminant $(k - 3)^2 - 4k$ shall be zero. (See (i.), § 4.)

Accordingly, placing the discriminant equal to zero, we obtain the conditional equation,

$$(k - 3)^2 - 4k = 0,$$

the solutions of which are $k = 9$, and $k = 1$. (See (i.) § 4.)

It will be found, if 9 is substituted for k, that the given equation will reduce to $x^2 + 6x + 9 = 0$, which has equal roots.

Also, substituting 1 for k, the given equation reduces to $x^2 - 2x + 1 = 0$, which also has equal roots.

25. $x^2 = 2k(x - 4) + 15.$
26. $x^2 + 2(k + 2)x + 9k = 0.$
27. $x^2 - 2kx + 6x + 4k = 0.$
28. $x^2 + k(2x - 8) = 15.$
29. $x^2 = (k - 1)x - 2(k - 1).$
30. $(2k + 1)x^2 + 3kx + k = 0.$
31. $x^2 + 3kx + 4k + 1 = 0.$
32. $12kx^2 - 2x + 3k = 0.$
33. $x^2 - (2k - 3)x + 2k = 0.$
34. $(8k + 5)x^2 - 12kx + 1 = 0.$
35. $(k+2)x^2+3k+4=5(k-1)x.$
36. $x^2+(6k+7)x+6k+22=0.$
37. $(k+6)x^2-3(k-2)x=1-k.$
38. $(k-11)x^2+3k+4=2(k-1)x.$

Relations Between Roots and Coefficients

5. Representing the two roots of the standard quadratic equation $ax^2 + bx + c = 0$ by

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \text{ and } x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a},$$

as in § 1, above, it follows immediately by addition that

$$x_1 + x_2 = \frac{[-b + \sqrt{b^2 - 4ac}] + [-b - \sqrt{b^2 - 4ac}]}{2a},$$

or $$x_1 + x_2 = \frac{-b}{a}. \tag{1}$$

Writing the product of the roots, we have

$$\begin{aligned} x_1x_2 &= \left[\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right]\left[\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right] \\ &= \frac{(-b)^2 - (\sqrt{b^2 - 4ac})^2}{4a^2} \\ &= \frac{+4ac}{4a^2}. \end{aligned}$$

Hence

$$x_1x_2 = \frac{c}{a}. \tag{2}$$

If the terms of a quadratic equation in standard form,

$$ax^2 + bx + c = 0,$$

be all divided by a, which is the coefficient of x^2, the coefficient of x^2 will be unity in the derived equivalent equation

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0. \tag{3}$$

Referring to (1) and (2), it may be seen that in (3):

(i.) *The sum of the roots* x_1 *and* x_2 *is equal to the coefficient of* x *with its sign changed* $-\frac{b}{a}$.

(ii.) *The product of the roots is equal to the term free from* x, that is, $\frac{c}{a}$.

Principles (i.) and (ii.) may be used as checks upon the solution of a quadratic equation.

E. g. The roots of $2x^2 + 7x + 3 = 0$ are -3 and $-1/2$.

Dividing the terms by 2, which is the coefficient of x^2, we obtain the equivalent equation, $$x^2 + \frac{7}{2}x + \frac{3}{2} = 0.$$

It may be seen that the sum of the roots -3 and $-1/2$ is the coefficient of x with its sign changed, that is, $-\frac{7}{2}$, and the product of the roots is $\frac{3}{2}$.

6. From $x_1x_2 = \frac{c}{a}$, (2), obtained in § 5, it appears, if the roots x_1 and x_2 are real, that according as the numbers represented in the standard equation by a and c have like or unlike signs, the quotient $\frac{c}{a}$ will be positive or negative, and the roots of the given equation will be both positive or both negative.

E. g. Consider $$2x^2 - 7x + 3 = 0.$$
Using the discriminant we find that the roots cannot be imaginary, since $(-7)^2 - 4 \cdot 2 \cdot 3 = 25$, which is a positive number.

The quotient $\frac{3}{2}$, represented by $\frac{c}{a}$, is positive. Hence the roots cannot differ in sign, and must be either both positive or both negative.

It may also be seen that the roots of the equation $2x^2 + 5x - 3 = 0$ are also both real, but they have opposite signs because $\frac{-3}{2}$ is negative.

7. From $x_1 + x_2 = \frac{-b}{a}$, (1), § 5, it appears that the sum of the roots is represented by $\frac{-b}{a}$. Hence *the roots either both agree in sign with the quotient* $\frac{b}{a}$ *with its sign changed, or if they differ in sign, the sign of the greater root must agree with the reversed sign of the quotient,* $\frac{b}{a}$. (See (i.) § 5.)

From this it follows that, if the number represented by a in the standard quadratic equation $ax^2 + bx + c = 0$ is positive, the root which is numerically the greater is opposite in sign to the sign of the number represented by b.

E. g. The roots of $2x^2 - 5x - 3 = 0$ are both real, but they have opposite signs, since $\frac{-3}{2}$ is negative.

The greater root must be positive because its sign must agree with the reversed sign of the quotient $-\frac{5}{2}$.

For convenience of reference, the illustrations used above are given in tabular form as follows:

Equation.	$\frac{c}{a}$	The roots are	$\frac{b}{a}$	Signs of roots are	Greater root
$2x^2 + 7x + 3 = 0$	$\frac{3}{2}$ is +	$-3, \frac{-1}{2}$	$\frac{7}{2}$ is +	Both −	−
$2x^2 - 7x + 3 = 0$	$\frac{3}{2}$ is +	$+3, \frac{+1}{2}$	$\frac{-7}{2}$ is −	Both +	+
$2x^2 + 5x - 3 = 0$	$\frac{-3}{2}$ is −	$-3, \frac{+1}{2}$	$\frac{5}{2}$ is +	Different	−
$2x^2 - 5x - 3 = 0$	$\frac{-3}{2}$ is −	$+3, \frac{-1}{2}$	$\frac{-5}{2}$ is −	Different	+

Formation of an Equation having Specified Roots

8. A quadratic equation having specified roots may be constructed by applying Principles (i.) and (ii.) § 5.

We may substitute $x_1 + x_2$ for $\frac{-b}{a}$ and x_1x_2 for $\frac{c}{a}$ in the quadratic equation $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$ and obtain

$$x^2 + [-(x_1 + x_2)]x + [x_1x_2] = 0.$$

A quadratic equation having specified roots may be obtained by constructing a quadratic equation of which the coefficient of x^2 is unity, the coefficient of x is the sum of the roots with sign changed, and the term free from x is the product of the specified roots.

Whenever fractions appear in any of the terms of an equation thus constructed an equivalent integral equation may be obtained by multiplying the terms by the lowest common multiple of all the denominators of the fractions.

Ex. 1. Construct the quadratic equation the roots of which are 5 and 7.

The sum of the roots is $5 + 7 = 12$, and the product is $5 \cdot 7 = 35$.

Hence, changing the sign of the sum 12, we may write as the required equation, $x^2 - 12x + 35 = 0$.

Ex. 2. Construct the equation the roots of which are $+3$ and -8.

The sum of the roots is $+3-8=-5$, and the product is $3(-8)=-24$.

Changing the sign of the sum, -5, the equation required is

$$x^2+5x-24=0.$$

Ex. 3. Construct the equation the roots of which are $\frac{7}{8}$ and $-\frac{2}{3}$.

The sum of the roots is $\frac{7}{8}-\frac{2}{3}=\frac{5}{24}$.

The product of the roots is $(\frac{7}{8})(-\frac{2}{3})=-\frac{7}{12}$.

Changing the sign of the sum, we may construct the equation

$$x^2-\tfrac{5}{24}x-\tfrac{7}{12}=0.$$

Or, $$24x^2-5x-14=0.$$

Ex. 4. Construct the equation the roots of which are $\frac{m}{n}$ and -1.

The sum of the roots is $\frac{m}{n}-1\equiv\frac{m-n}{n}$.

The product of the roots is $\left(\frac{m}{n}\right)\left(-1\right)\equiv-\frac{m}{n}$.

Using the sum, with its sign changed, as the coefficient of x, we have,

$$x^2-\frac{m-n}{n}x-\frac{m}{n}=0.$$

Or, $$nx^2-(m-n)x-m=0.$$

Ex. 5. Construct the equation the roots of which are the conjugate irrational numbers $2+\sqrt{5}$ and $2-\sqrt{5}$.

The sum of these values is $(2+\sqrt{5})+(2-\sqrt{5})=4$.

The product is $(2+\sqrt{5})(2-\sqrt{5})=4-5=-1$.

Reversing the sign of the sum, we may write the equation

$$x^2-4x-1=0.$$

Ex. 6. Construct the equation the roots of which are the complex numbers $\frac{-1+\sqrt{-2}}{3}$ and $\frac{-1-\sqrt{-2}}{3}$.

The sum of these values is $\frac{-1+\sqrt{-2}}{3}+\frac{-1-\sqrt{-2}}{3}=\frac{-2}{3}$.

The product is,

$$\left(\frac{-1+\sqrt{-2}}{3}\right)\left(\frac{-1-\sqrt{-2}}{3}\right)=\frac{(-1)^2-(\sqrt{-2})^2}{9}=\frac{1}{3}.$$

Using the sum with its sign changed, we have for the equation

$$x^2+\tfrac{2}{3}x+\tfrac{1}{3}=0.$$

Or, $$3x^2+2x+1=0.$$

9. *The roots of a quadratic equation in the standard form $ax^2 + bx + c = 0$ are the roots of the two linear equations formed by equating to zero the two linear factors the product of which is the first member of the equation.* (See Chap. XII. § 48.)

It follows that, by reversing the process, *we may construct a quadratic equation having specified roots by equating to zero the product of the two linear factors which, when equated to zero and considered as equations, have as roots the given values.*

Ex 7. Construct the equation the roots of which are 5 and 7.

We may indicate that these are roots x_1 and x_2 of an equation by writing

$$x_1 = 5 \quad \text{and} \quad x_2 = 7.$$

Accordingly, $x_1 - 5 = 0$, and $x_2 - 7 = 0$.

These two equations taken together are equivalent to the single quadratic equation

$$(x_1 - 5)(x_2 - 7) = 0.$$

Performing the indicated multiplication and neglecting subscripts, we obtain $x^2 - 12x + 35 = 0$. (Compare with Ex. 1, § 8.)

10. It will be found that, whenever the roots are irrational or imaginary, the method of § 8 is to be preferred.

EXERCISE XXIII. 2

Construct integral equations the roots of which are

1. 3 and 5.
2. 2 and 9.
3. 6 and 8.
4. 4 and 7.
5. 1 and 2.
6. -5 and -8.
7. -7 and -10.
8. -3 and $+5$.
9. $+2$ and -15.
10. 2 and -2.
11. 0 and 4.
12. $\frac{2}{7}$ and $\frac{3}{7}$.
13. $\frac{1}{2}$ and $\frac{1}{5}$.
14. $\frac{2}{3}$ and $\frac{7}{8}$.
15. $-\frac{5}{6}$ and $\frac{3}{10}$.
16. $-\frac{4}{3}$ and 2.
17. $\frac{8}{9}$ and 3.
18. $5/a$ and $-b/6$.
19. m/n and n/m.
20. $b/2c$ and $c/2b$.
21. $\dfrac{a-b}{a+b}$ and $+1$.
22. $\dfrac{a}{b+c}$ and $\dfrac{b-c}{a}$.
23. $2\sqrt{5}$ and $-2\sqrt{5}$.
24. $\sqrt{3}$ and $-\sqrt{3}$.
25. $1+\sqrt{6}$ and $1-\sqrt{6}$.

26. $2+\sqrt{-7}$ and $2-\sqrt{-7}$.
28. $c+\sqrt{c^2-1}$ and $c-\sqrt{c^2-1}$.

27. $a+\sqrt{b}$ and $a-\sqrt{b}$.
29. $\frac{3+4\sqrt{-5}}{6}$ and $\frac{3-4\sqrt{-5}}{6}$.

11.* One Root Known. When one root of a given quadratic equation is known, by applying principle (i.), § 5, the other root may be found immediately without solving the equation.

Ex. 1. Knowing that one root of $x^2+5x-24=0$ is 3, find the other.

Since the coefficient, 5, of x with its sign changed is the sum of the roots, we may obtain the required root, -8, by subtracting the given root 3 from -5, that is, $-5-3=-8$.

The root may also be obtained by dividing the known term -24 by 3. (Compare with Ex. 2 § 8.)

Ex. 2. Knowing that one root of the equation $x^2-4x-1=0$ is $2-\sqrt{5}$, we can find the remaining root by subtracting $2-\sqrt{5}$ from the coefficient of x with its sign changed.

We have, $+4-(2-\sqrt{5})=2+\sqrt{5}$. (Compare with Ex. 5, § 8.)

Since in an equation in which the coefficients are rational, irrational or complex roots enter in conjugate pairs, if indeed they enter at all, it follows that we may obtain the required root immediately by writing the binomial $2+\sqrt{5}$, which is the conjugate of the one given, $2-\sqrt{5}$.

Exercise XXIII. 3

(This exercise may be omitted when the chapter is read for the first time.)

Find, without solving, the remaining root of each of the following equations, when one of the roots is given :

1. $x^2-3x-130=0$, one root being 13.
2. $x^2+2x-323=0$, one root being -19.
3. $x^2-21x-130=0$, one root being -5.
4. $4x^2+28x-15=0$, one root being $\frac{1}{2}$.
5. $18x^2-117x+37=0$, one root being $\frac{1}{3}$.
6. $25x^2-85x-18=0$, one root being $-\frac{1}{5}$.
7. $3x^2-5x-308=0$, one root being 11.
8. $98x^2-7x-6=0$, one root being $\frac{2}{7}$.
9. $x^2-(a+b)x+2ab-2b^2=0$, one root being $a-b$.

*** This section may be omitted when the chapter is read for the first time.**

10. $2ax^2 + (4 - ab)x = 2b$, one root being $b/2$.
11. $cdx^2 - (c + d)x + 1 = 0$, one root being $1/c$.
12. $6a^2x^2 + 5ax + 1 = 0$, one root being $-1/2a$.
13. $4x^2 - 4ax = b^2 - a^2$, one root being $(a + b)/2$.
14. $ab^2cx^2 - b(c - a)x - 1 = 0$, one root being $1/ab$.
15. $4abx^2 + 2(a^2 + b^2)x + ab = 0$, one root being $-a/2b$.
16. $acx^2 + b(a^2 + c^2)x + ab^2c = 0$, one root being $-ab/c$.

Mental Exercise. XXIII. 4. **Review**

Cube each of the following:

1. x.
2. y^2.
3. z^3.
4. $-a$.
5. $-b^2$.
6. $-\frac{c}{2}$.
7. $\frac{d}{-3}$.
8. $-\frac{2}{4}$.
9. $x^{\frac{1}{2}}$.
10. $y^{\frac{1}{3}}$.
11. $z^{\frac{2}{3}}$.
12. $\frac{3m^{\frac{3}{2}}}{2}$.
13. a^{-1}.
14. $b^{-\frac{2}{3}}$.
15. $c^{-\frac{3}{4}}$.
16. $d^{-\frac{5}{3}}$.

Express each of the following numbers as a power of 2:

17. $(4^3)^2$.
18. $(2^3)^4$.
19. $(16^{\frac{1}{2}})^5$.
20. $(32^{\frac{2}{5}})^3$.

Solve each of the following equations:

21. $\frac{x^2}{9} = 1$.
22. $25y^2 = 1$.
23. $\frac{9z^2}{16} = 1$.
24. $\frac{w^2}{36} = 4$.

Factor each of the following expressions:

25. $a^2 - 10ab + 25b^2 - 36$.
26. $x^2 + 18xy + 45y^2$.
27. $9m^4 + 23m^2n^2 + 16n^4$.
28. $15x^2 + 13xy + 2y^2$.

Solve each of the following equations:

29. $x^{\frac{1}{2}} = 3$.
30. $y^{\frac{1}{3}} = 2$.
31. $z^{\frac{1}{2}} = \frac{1}{2}$.
32. $w^{\frac{1}{5}} = -2$.
33. $x^{\frac{1}{3}} = -1$.
34. $\sqrt{\frac{x}{2}} = 3$.
35. $\frac{\sqrt{z}}{2} = 3$.
36. $x^{-1} = -1^{-1}$
37. $\sqrt[n]{x} = n$.

Express the following as positive fractions:

38. $-\frac{1}{1 - a}$.
39. $-\frac{b - 2}{3}$
40. $-\frac{y - x}{x + z}$.

Express the following without negative exponents:

41. $(-a)^{-2}$. 42. $(-b)^{-3}$. 43. $(-c^2)^{-5}$.

Simplify each of the following expressions:

44. $-2a \cdot a^{-2}$. 46. $a(a^a)^a$. 48. $(2a \div b^2)^{-2}$.

45. $\frac{ab^c}{cb^a}$. 47. $\frac{b^{-b}}{b}$. 49. $-(x^{-1} \div y^{-1})^{-1}$.

50. $(m^{\frac{1}{n}}n^{\frac{1}{m}})^{mn}$.

Distinguish between

51. $a^{-\frac{1}{b}}$ and $\frac{1}{a^b}$. 52. $-a^{\frac{1}{x}}$ and $a^{-\frac{1}{x}}$.

53. Express $\left(\frac{a}{b}\right)^{\frac{b}{a}}$ as a quotient of powers.

Find the value of

54. $.2^{-1}$. 55. $.04^{-1}$. 56. $.5^{-2}$. 57. $(.02)^{-2}$.

Rationalize the denominators of each of the following:

58. $\frac{1}{\sqrt{2}}$. 60. $\frac{1}{\sqrt[3]{a^2}}$. 62. $\frac{1}{\sqrt[3]{5b}}$.

59. $\frac{1}{\sqrt[3]{4}}$. 61. $\frac{2}{\sqrt{2a}}$. 63. $\frac{2}{\sqrt{2c}}$.

What roots may possibly be introduced by squaring both members of each of the following equations?

64. $x = 5$. 66. $z - 1 = 6$. 68. $y - 3 = 7$.

65. $y + 2 = 3$. 67. $x - 2 = 0$. 69. $z + 8 = 3$.

Distinguish between

70. $(-\sqrt{2})^2$ and $(\sqrt{-2})^2$. 71. $-\sqrt[3]{8}$ and $\sqrt[3]{-8}$.

72. $\sqrt{6} - \sqrt{6}$ and $\sqrt{6} + \sqrt{-6}$.

Simplify each of the following expressions:

73. $(\sqrt{5} + \sqrt{-5})^2$. 74. $(\sqrt{-7} - \sqrt{7})^2$. 75. $(2a^{\frac{1}{2}} + \frac{1}{2})^2$.

Solve each of the following equations:

76. $0 = 2\sqrt{7}\sqrt{x - 7}$. 77. $\sqrt{x - a} = 0$.

78. $(x - 3)(9 \cdot 7 \cdot \frac{1}{2} + 32 \cdot 16 - 5) = 0$.

Show that the following identities are true:

79. $-i \equiv \frac{1}{i}\cdot$ 80. $-i \equiv i^{-1}$. 81. $-i^{-2} \equiv 1$.

82. $(a-b)(c-b)(d-c)(d-a) \equiv (a-b)(b-c)(c-d)(d-a)$.

83. Rationalize $\frac{1}{\sqrt{2}}\cdot$ 84. Realize $\frac{1}{\sqrt{-2}}\cdot$

85. Find the values of $(a^0 + b^0)^2$ and $(a^2 + b^2)^0$.

Simplify each of the following:

86. $-(a^{-1}b^{-2}c^{-3})^{-4}$. 87. $-(x^{-1} \div y^{-1})^{-2}$. 88. $(n\sqrt[n]{n})^n$.

CHAPTER XXIV

IRRATIONAL EQUATIONS AND SPECIAL EQUATIONS CONTAINING A SINGLE UNKNOWN

Irrational Equations

1. An **irrational equation** is an equation in which one or more terms are irrational with reference to the unknown.

E.g.
$$\sqrt{x^2 - 3} = x - 1.$$
$$\sqrt{x + 5} + \sqrt{x + 12} = 7.$$
$$\sqrt{x + 4} - 2\sqrt{x + 13} + \sqrt{x + 22} = 0.$$

2. It is understood that, when the terms of an equation are affected by radical signs, principal values only of the roots are to be taken, unless we have means for knowing that other values should be taken. (See Chap. XVIII. § 13, also Chap. XXIV. Ex. 3, § 6, and Ex. 5 and Ex. 6, § 10.)

3. To be consistent it is necessary that a specified letter represent the same numerical value wherever it appears in the terms of a given conditional equation.

E.g. In the equation $x^2 - 5x + 6 = 0$, x may represent either 3 in every term or 2 in every term, but never 3 in one term and 2 in another term.

Thus the equation $x^2 - 5x + 6 = 0$ represents either of the following numerical identities:

$$3^2 - 5 \cdot 3 + 6 = 0 \quad \text{or} \quad 2^2 - 5 \cdot 2 + 6 = 0.$$

It is also necessary that a specified symbol of operation, such as $\sqrt{\ }$, represent the same root wherever it may appear among the different terms of an equation.

It may be seen that if $\sqrt{\ }$ is to be considered as representing the positive value of the root in one term, to be consistent we must understand it as representing the positive value of the root in every other term of the equation.

Thus, $\sqrt{}$ cannot represent a positive value of the root in one term of an equation and a negative value in another term.

4. When solving irrational equations it is necessary to consider the following

Principle Relating to Extra Roots: *Whenever both members of an equation are raised to the same positive integral power no solutions of the original equation are lost in the process; but solutions may be gained which satisfy the derived equation, yet do not satisfy the original equation.*

Let the members of a given equation be represented by

$$A = B. \tag{1}$$

Raising both members to the nth power, n being a positive integer, we obtain $A^n = B^n$. (2)

Transposing, $A^n - B^n = 0$. (3)

For all positive integral values of n, $A^n - B^n$ is exactly divisible by $A - B$, and the quotient Q resulting from the division is of degree $n - 1$.

Accordingly, factoring $A^n - B^n$, we may from (3) obtain

$$(A - B)Q = 0. \tag{4}$$

This last equation (4) is equivalent to the set of two equations

$$A - B = 0, \quad (5) \quad \text{and} \quad Q = 0. \quad (6)$$

Equation (5) is equivalent to the original equation $A = B$ (1), and (6) is an additional equation introduced by the process of raising both members of (1) to the nth power.

Since equation (2) and its equivalent equation (4) are satisfied by any solution either of equation (5) or of equation (6), it follows that any solution of the additional equation $Q = 0$ (6), which is not at the same time a solution of equation (5) must have been introduced by the process of raising the members of (1) to the nth power.

E. g. If both members of $x + 2 = 5$ be raised to the second power, we shall obtain the equation $(x + 2)^2 = 25$, which has two solutions,

$$x = 3 \quad \text{and} \quad x = -7.$$

The first of these values, 3, is the single solution of the given equation, while the second value, -7, does not satisfy the given equation, but was introduced by the process of squaring.

5. *Irrational equations are usually prepared for solution as follows:*

First the terms are so transposed that one member of the equation consists of one of the irrational terms appearing in the equation, all other terms being transposed to the remaining member.

Both members of the equation are then raised to the lowest power necessary to rationalize the term which has been separated from the others.

If irrational terms still appear after the operation, the process is repeated until an equation is obtained which is entirely rational with respect to the unknown quantity appearing in it.

All solutions of the rational equation thus obtained which do not satisfy the given irrational equation must be rejected as being extra solutions introduced during the process of rationalization.

Ex. 1. Solve $$14 - x = \sqrt{100 - x^2}. \quad (1)$$

Squaring both members, $$196 - 28x + x^2 = 100 - x^2. \quad (2)$$

The equivalent equation $$x^2 - 14x + 48 = 0 \quad (3)$$

has the two solutions, $x = 8$ and $x = 6$.

We find by substitution that both values satisfy the given equation. Hence in this case no solutions have been introduced by the process of squaring the members of the given equation.

If the given equation (1) be written in the form

$$14 - x - \sqrt{100 - x^2} = 0, \quad (4)$$

it may be seen that a rational equation can be derived from it by using the rationalizing factor $14 - x + \sqrt{100 - x^2}$.

Any root which may possibly be introduced into the derived equation by the use of this factor must be a root of the equation obtained by placing this factor equal to zero, — that is, of the equation

$$14 - x + \sqrt{100 - x^2} = 0,$$

or, $$14 - x = -\sqrt{100 - x^2}. \quad (5)$$

Either of the solutions of equation (3), $x = 8$ or $x = 6$, if substituted in (5), makes the first member positive and the second member negative, provided that we take only the principal value of the root.

Hence, with the restriction that the principal value only of the root is to be taken, these values could not have been introduced during the process of rationalization.

6. It should be observed that we may write an equality which expresses a relation between two numbers or expressions which is inconsistent with the laws governing relations between numbers.

Such an equality is sometimes spoken of as an "impossible equation."

E. g. Thus, since it is impossible that a positive number be equal to a negative number, equation (5) of § 5 is an illustration of an "impossible equation."

Rational, fractional equations having no finite solutions may be written.

E. g. $\frac{1}{x-2} = \frac{1}{x-3}$ expresses a condition which cannot be satisfied by any finite value of x, that is, it is a so-called "impossible equation."

It may be seen that by clearing of fractions we obtain the inconsistent relation $x - 2 = x - 3$, or $-2 = -3$, which is impossible.

Ex. 2. Solve $x - 5 + \sqrt{x-5} = 0.$ (1)

Transposing, $\sqrt{x-5} = 5 - x.$ (2)

Squaring both members, $x - 5 = 25 - 10x + x^2.$ (3)

Equation (3) is equivalent to $x^2 - 11x + 30 = 0,$ (4)

of which the solutions are $x = 6$, and $x = 5$.

Both values are roots of the rational equations (3) and (4), but if we restrict the roots to principal values only, equation (1) is satisfied by the value $x = 5$, but not by the value $x = 6$.

It appears that $x = 6$ is an extra root introduced by squaring the members of (2).

It should be observed that if, instead of first transposing the terms of equation (1) and then squaring, we had obtained a rational equation by multiplying both members of (1) by the rationalizing factor $x - 5 - \sqrt{x-5}$, we would have obtained the same rational equation (4) as before, and consequently the same solutions, $x = 6$, and $x = 5$.

We find that both of these values satisfy the "multiplier equation," $x - 5 - \sqrt{x-5} = 0$, formed by equating the rationalizing factor to zero. Hence, either of the values $x = 6$ or $x = 5$ might have been introduced during the process of rationalization.

It is, therefore, necessary to substitute in the original equation to find whether or not either or both of these values can be accepted as solutions of the given equation (1).

Ex. 3. Solve $x + \sqrt{x} - 2 = 0$.

It should be observed that this equation is quadratic with reference to $\sqrt{x}$. Hence we may factor with reference to $\sqrt{x}$, and obtain,

$$(\sqrt{x} + 2)(\sqrt{x} - 1) = 0.$$

Hence, $\sqrt{x} + 2 = 0$, and $\sqrt{x} - 1 = 0$.
Or, $\sqrt{x} = -2$, and $\sqrt{x} = 1$.
Hence, $x = 4$, and $x = 1$.

It should be observed that the value 4 is the square of the negative number -2. Hence, when substituting 4 for x in the given equation, it is necessary that $\sqrt{x}$ be considered equal to -2.

With this understanding, it will be found that the value 4 satisfies the given equation.

For, we have $4 + (-2) - 2 = 0$. That is, $0 = 0$.

The remaining value $x = 1$ will be found by substitution to satisfy the given equation.

For, we have $1 + 1 - 2 = 0$. That is, $0 = 0$.

7. If A, B, and C be any functions of x, it may be shown that the equations

$$\sqrt{A} + \sqrt{B} + \sqrt{C} = 0, \qquad (1)$$
$$\sqrt{A} + \sqrt{B} - \sqrt{C} = 0, \qquad (2)$$
$$\sqrt{A} - \sqrt{B} + \sqrt{C} = 0, \qquad (3)$$
$$\sqrt{A} - \sqrt{B} - \sqrt{C} = 0, \qquad (4)$$

when rationalized, all lead to the same derived equation

$$A^2 + B^2 + C^2 - 2\,AB - 2\,BC - 2\,CA = 0. \qquad (5)$$

Accordingly, equation (5) is equivalent to the set of four equations (1), (2), (3), and (4).

It follows that, when solving an irrational equation having the form of any one of the irrational equations (1), (2), (3), or (4), we shall obtain the solutions, not only of the given equation, but also of the equations represented by the remaining three equations of the set, obtained by changing among themselves in all possible ways the signs of the radicals of the given equation.

8. We cannot speak of the degree of an equation which is irrational with reference to the unknown, and we have no means for knowing before solving, as we have in the case of integral equations, the number of roots of a given irrational equation.

9. Often no solution of the derived rational equation will satisfy the given irrational equation, if the roots are restricted to principal values only.

Ex. 4. Solve $1 - \sqrt{x} + \sqrt{2x+1} = 0.$ (1)

Transposing, $1 + \sqrt{2x+1} = \sqrt{x}.$ (2)

Squaring both members, $1 + 2\sqrt{2x+1} + 2x + 1 = x.$ (3)

Collecting terms and transposing, $2\sqrt{2x+1} = -2 - x.$ (4)

Again squaring both members, $4(2x+1) = 4 + 4x + x^2.$ (5)

The equivalent equation $x^2 - 4x = 0,$ (6)

has as solutions $x = 0,$ and $x = 4.$

We find by substitution that neither of these values satisfies the given equation (1). Hence equation (1) *has no root.*

It may be seen that, if the signs of the terms of the given equation are changed among themselves in all possible ways, we shall obtain the set of four irrational equations which all lead, when rationalized, to the rational equation (6). (See § 7, (1), (2), (3), (4), (5).)

Substituting the values $x = 0$ and $x = 4$ in these equations, we find that

(i.) Neither value satisfies the given equation

$$1 - \sqrt{x} + \sqrt{2x+1} = 0, \quad (1)$$

(ii.) One value $x = 0$, satisfies the additional equation

$$1 - \sqrt{x} - \sqrt{2x+1} = 0, \quad (7)$$

(iii.) Neither value satisfies the additional equation

$$1 + \sqrt{x} + \sqrt{2x+1} = 0, \quad (8)$$

(iv.) Both values satisfy the additional equation

$$-1 - \sqrt{x} + \sqrt{2x+1} = 0. \quad (9)$$

10. Certain irrational equations may be so written as to appear in quadratic form with respect to some expression appearing in them, and the principles for the solution of quadratic equations may then be applied to obtain their solutions, if indeed any solutions exist.

Ex. 5. Solve $x^2 + 5x - 2\sqrt{x^2 + 5x + 3} - 12 = 0.$ (1)

Observe that we may duplicate the expression $x^2 + 5x + 3$, which appears under the radical sign, by adding 3 to the expression $x^2 + 5x$ which appears outside.

Accordingly, adding 3 and also subtracting 3 from the first member of (1), we obtain the equivalent equation

$$x^2 + 5x + 3 - 2\sqrt{x^2 + 5x + 3} - 15 = 0. \quad (2)$$

Factoring with reference to $\sqrt{x^2 + 5x + 3}$, we obtain

$$[\sqrt{x^2 + 5x + 3} - 5][\sqrt{x^2 + 5x + 3} + 3] = 0. \quad (3)$$

The roots of equation (3) include all of the roots of the following equations:

$$\sqrt{x^2 + 5x + 3} - 5 = 0, \quad (4) \qquad \text{and} \qquad \sqrt{x^2 + 5x + 3} + 3 = 0. \quad (5)$$

The roots of equation (4) are found to be $x = \dfrac{-5 \pm \sqrt{113}}{2}$. These values will be found by substitution to satisfy the given equation.

From equation (5) we obtain

$$\sqrt{x^2 + 5x + 3} = -3. \quad (6)$$

Hence, $x^2 + 5x + 3 = 9$. (7)

Or, $x^2 + 5x - 6 = 0$.

Factoring, $(x + 6)(x - 1) = 0$.

Hence, $x = -6$, and $x = 1$.

Since the right member, 9, of equation (7) was obtained by squaring a negative number in the second member of equation (6), it follows that when the values -6 and $+1$ are substituted for x in the expression $x^2 + 5x + 3$ appearing under the radical sign in equation (6), the expression will represent the square of a negative number. Accordingly, when finding the value of $\sqrt{x^2 + 5x + 3}$, after having substituted -6 and $+1$ for x, it is necessary to consider that the result is a negative number.

With this understanding, it may be seen that the values -6 and $+1$ satisfy the given equation.

For, substituting -6, we have, $36 - 30 - 2(-3) - 12 = 0$. Hence, $0 = 0$.

Substituting -1, we have, $1 + 5 - 2(-3) - 12 = 0$. Hence, $0 = 0$.

Ex. 6. Solve $2x^2 - 2x - \sqrt{x^2 - x + 4} + 2 = 0$. (1)

Equation (1) may be written in the equivalent form

$$2x^2 - 2x + 8 - \sqrt{x^2 - x + 4} + 2 - 8 = 0. \quad (2)$$

Or, $$2(x^2 - x + 4) - \sqrt{x^2 - x + 4} - 6 = 0. \quad (3)$$

Equation (3) is quadratic with respect to the expression $\sqrt{x^2 - x + 4}$. Hence, factoring with reference to $\sqrt{x^2 - x + 4}$, we have,

$$(2\sqrt{x^2 - x + 4} + 3)(\sqrt{x^2 - x + 4} - 2) = 0. \quad (4)$$

Placing these factors equal to zero, we obtain the equations

$$2\sqrt{x^2 - x + 4} + 3 = 0, \text{(5) and } \sqrt{x^2 - x + 4} - 2 = 0. \qquad (6)$$

From equation (5) we obtain $\sqrt{x^2 - x + 4} = -\frac{3}{2}$, the solutions of which are found to be $x = \dfrac{1 \pm \sqrt{-6}}{2}$.

It should be observed that when these values are substituted for x, $\sqrt{x^2 - x + 4}$ must be a negative number, $-\frac{3}{2}$.

Accordingly, with this understanding, these imaginary values will be found to satisfy the given equation.

Equation (6) is found to have the solutions $x = 0$ and $x = 1$, both of which satisfy the given equation.

EXERCISE XXIV. 1

Solve the following irrational equations, verifying integral or fractional results and rejecting "extra roots":

1. $\sqrt{5 + x^2} - 3 = 0$.
2. $\sqrt{x^2 + 15} - 7 = 0$.
3. $\sqrt{3x + 4} + \sqrt{2x + 9} = 1$.
4. $\sqrt{16 + x} + \sqrt{1 - x} = 5$.
5. $\sqrt{3x + 10} = \sqrt{5x - 1} - 1$.
6. $\sqrt{x + 6} = \sqrt{3x - 8} - 2$.
7. $\sqrt{25 - x} + \sqrt{16 + x} = 9$.
8. $\sqrt{x + 13} + \sqrt{2x - 45} = 7$.
9. $\sqrt{x + 5} + \sqrt{2x + 8} = 7$.
10. $\sqrt{2x + 1} + \sqrt{x + 5} = 6$.
11. $x\sqrt{x^2 + 20} + x\sqrt{x^2 + 10} = 5$.
12. (a) $2 + \sqrt{3x - 2} - \sqrt{8x} = 0$.
    (b) $2 - \sqrt{3x - 2} - \sqrt{8x} = 0$.
    (c) $2 - \sqrt{3x - 2} + \sqrt{8x} = 0$.
    (d) $\sqrt{3x - 2} - 2 - \sqrt{8x} = 0$.
13. (a) $\sqrt{x + 2} + \sqrt{x - 13} + \sqrt{x - 5} = 0$.
    (b) $\sqrt{x + 2} - \sqrt{x - 13} - \sqrt{x - 5} = 0$.
    (c) $\sqrt{x + 2} - \sqrt{x - 13} + \sqrt{x - 5} = 0$.
    (d) $\sqrt{x + 2} + \sqrt{x - 13} - \sqrt{x - 5} = 0$.
14. $\sqrt{x + 12} + \sqrt{x - 12} - \sqrt{x + 23} = 0$.
15. $\sqrt{5 - x} + \sqrt{2 + x} = \sqrt{14}$.
16. $\sqrt{5 - x} + \sqrt{8 - x} = \sqrt{13 - 2x}$.
17. $\sqrt{2x + 9} - \sqrt{x - 4} = \sqrt{x + 1}$.
18. $\sqrt{5 + 2x} = \sqrt{3 + x} + \sqrt{2 + x}$.

19. $\sqrt{4-x}=\sqrt{6+x}+\sqrt{9+x}$.

20. $\sqrt{2x+9}+\sqrt{x-4}=\sqrt{x+41}$.

21. $\sqrt{3x+1}+\sqrt{x-1}=\sqrt{2x-6}$.

22. $\sqrt{3x+4}-\sqrt{2x+6}=\sqrt{x-14}$.

23. $\sqrt{(x-4)(x-3)}+\sqrt{(x-2)(x-1)}=\sqrt{2}$.

24. $\sqrt{(4+x)(x+1)}+\sqrt{(4-x)(x-1)}=4\sqrt{x}$.

25. $\sqrt{a+x}-\sqrt{a-x}=\sqrt{a}$.

26. $\sqrt{a^2+x}+\sqrt{b^2-x}=a+b$.

27. $\sqrt{a-x}+\sqrt{b-x}=\sqrt{a+b-2x}$.

28. $\sqrt[3]{x-a}-\sqrt[3]{x-b}=\sqrt[3]{b-a}$.

29. $\dfrac{3}{\sqrt{x-4}}+\sqrt{x-4}=4$.

30. $\dfrac{3x+\sqrt{3x-3}}{3x-\sqrt{3x-3}}=1$.

31. $\dfrac{1}{x+\sqrt{x^2-1}}-\dfrac{1}{x-\sqrt{x^2-1}}=2$.

32. $\dfrac{\sqrt{2x-3}}{\sqrt{3x-2}}-\dfrac{\sqrt{3x-2}}{\sqrt{2x-3}}=-\dfrac{7}{12}$.

33. $2\sqrt{2x}+\sqrt{2x+9}=\dfrac{65}{\sqrt{2x+9}}$.

34. $\sqrt{x+2}+\dfrac{1}{\sqrt{x+2}}=x+3$.

35. $\dfrac{x+1}{\sqrt{x}}=\dfrac{b+1}{\sqrt{b}}$.

36. $\dfrac{a-x}{\sqrt{a-x}}+\dfrac{x-b}{\sqrt{x-b}}=\sqrt{a-b}$.

37. $\dfrac{1}{k-\sqrt{k^2-x^2}}-\dfrac{1}{k+\sqrt{k^2-x^2}}=\dfrac{k}{x^2}$.

38. $\sqrt{\dfrac{m}{x}}-\sqrt{\dfrac{n}{x}}=\sqrt{\dfrac{x}{n}}-\sqrt{\dfrac{x}{m}}$.

39. $\dfrac{\sqrt{a^2+x^2}+\sqrt{a^2-x^2}}{\sqrt{a^2+x^2}-\sqrt{a^2-x^2}}=\dfrac{\sqrt{a}+\sqrt{c}}{\sqrt{a}-\sqrt{c}}.$

40. $\sqrt{x^2-x+1}-\sqrt{x^2+x+1}=c.$

41. $x^2-3x-\sqrt{x^2-3x}-2=0.$

42. $\sqrt{10-x^2-x}=8-x^2-x.$

43. $2x^2+x+\sqrt{2x^2+x}-42=0.$

44. $2x^2+6x+\sqrt{x^2+3x}=10.$

45. $2x^2+x-3\sqrt{2x^2+x+4}=6.$

46. $2x^2-10x+12-2\sqrt{x^2-5x+8}=0.$

47. $3x^2-4x+\sqrt{3x^2-4x-6}=18.$

11. At least one solution of certain equations which have the special form $x^{\frac{p}{r}}=a$ may be obtained by the following process:

From the equation $x^{\frac{p}{r}}=a,$

we obtain, $(x^{\frac{p}{r}})^{\frac{r}{p}}=a^{\frac{r}{p}}.$

Therefore $x=a^{\frac{r}{p}}.$

12. It should be observed that more solutions exist than are commonly obtained by the process above.

Ex. 1. Find one or more solutions of $x^{\frac{1}{2}}=3.$

From $x^{\frac{1}{2}}=3,$

we have, $(x^{\frac{1}{2}})^2=3^2.$

That is, $x=9.$

In this case the solution obtained is the only one which exists for the given equation.

Ex. 2. Find one or more solutions of $y^{\frac{3}{2}}=8.$

From $y^{\frac{3}{2}}=8,$

we have $(y^{\frac{3}{2}})^{\frac{2}{3}}=8^{\frac{2}{3}}.$

That is, $y=\sqrt[3]{64}=4.$

The solution obtained, $y=4$, is in this case one of the three possible solutions of the given equation.

The complete solution of the equation may be obtained as follows:

From $y^{\frac{3}{2}} = 8,$

we obtain $y^3 = 64.$

Hence $y^3 - 64 = 0.$

Factoring $(y - 4)(y^2 + 4y + 16) = 0.$

Solving the equations obtained by placing these factors separately equal to zero, we have

from $y - 4 = 0,$ and from $y^2 + 4y + 16 = 0,$

$y = 4,$ and $y = -2 \pm 2\sqrt{-3}.$

Accordingly, the three solutions of the given equation are the *real number* 4 and the *conjugate complex numbers* $-2 + 2\sqrt{-3}$ and $-2 - 2\sqrt{-3}$.

These solutions will be found, by substitution, to satisfy the given equation.

MENTAL EXERCISE XXIV. 2

Obtain one or more solutions of each of the following equations, regarding x, y, z, and w as unknowns and all other letters as representing known numbers:

1. $x^{\frac{1}{2}} = 2.$
2. $x^{\frac{1}{3}} = 3.$
3. $x^{\frac{1}{4}} = 2.$
4. $x^{\frac{1}{3}} = -3.$
5. $y^{\frac{1}{3}} = -6.$
6. $-y^{\frac{1}{3}} = 4.$
7. $-y^{\frac{1}{3}} = 5.$
8. $\sqrt{z} = 5.$
9. $\sqrt{z} = 7.$
10. $\sqrt[3]{z} = 4.$
11. $\sqrt[3]{w} = 3.$
12. $\sqrt[3]{w} = -4.$
13. $\sqrt[4]{x} = 3.$
14. $\sqrt[4]{x} = 2.$
15. $\sqrt[5]{x} = 1.$
16. $\sqrt[5]{y} = -2.$
17. $\sqrt[3]{z} = \frac{1}{2}.$
18. $x^{\frac{1}{2}} = b^3.$
19. $x^{\frac{1}{4}} = c^4.$
20. $y^{\frac{1}{3}} = ab^2.$
21. $y^{\frac{1}{4}} = h^2m^3.$
22. $z^{\frac{1}{2}} = ab^2.$
23. $w^{\frac{1}{3}} = -bc^3.$
24. $w^{\frac{2}{3}} = 4.$
25. $x^{\frac{3}{4}} = 27.$
26. $x^{\frac{3}{4}} = 8.$
27. $y^{\frac{3}{2}} = 27.$
28. $y^{\frac{2}{3}} = 8.$
29. $z^{\frac{2}{3}} = 16.$
30. $z^{\frac{2}{3}} = 9.$
31. $\sqrt[3]{x^2} = 36.$
32. $\sqrt[5]{y^2} = 16.$
33. $x^{\frac{3}{2}} = 64.$
34. $x^{\frac{5}{6}} = 32.$
35. $x^{\frac{5}{3}} = -1.$
36. $x^{\frac{3}{2}} = 8.$
37. $x^{\frac{1}{3}} = a^3.$
38. $x^{\frac{3}{2}} = a^3.$
39. $x^{\frac{2}{3}} = a^6.$
40. $y^{\frac{2}{5}} = b^4.$
41. $z^{\frac{2}{3}} = a^{-2}.$
42. $y^{\frac{3}{2}} = m^{-9}.$
43. $x^{\frac{3}{2}} = \frac{a^3}{8}.$
44. $w^{\frac{2}{3}} = \frac{a^6}{9}.$
45. $\sqrt{2x} = 4.$

46. $\sqrt{3y} = 6.$

47. $\sqrt{5z} = 10.$

48. $\sqrt{2w} = 8.$

49. $\sqrt[3]{4x} = -2.$

50. $\sqrt[3]{2x} = 2.$

51. $\sqrt[3]{4y} = 4.$

52. $\sqrt{6z} = 2.$

53. $\sqrt{10z} = 5.$

54. $\sqrt{3w} = 4.$

55. $\sqrt{5w} = 6.$

56. $x^{\frac{1}{2}} = \frac{1}{3}.$

57. $y^{\frac{1}{3}} = \frac{1}{2}.$

58. $z^{\frac{1}{2}} = \frac{1}{2}.$

59. $x^{\frac{1}{2}} = \frac{1}{a}.$

60. $x^{\frac{1}{2}} = \frac{a}{b}.$

61. $\sqrt{w} = \frac{4}{5}.$

62. $\sqrt{w} = \frac{1}{5}.$

63. $\sqrt{x} = \frac{3}{2}.$

64. $\sqrt[3]{x} = \frac{3}{4}.$

65. $\sqrt[3]{x} = -\frac{1}{2}.$

66. $2x^{\frac{1}{2}} = 3.$

67. $3x^{\frac{1}{2}} = 4.$

68. $2x^{\frac{1}{3}} = 5.$

69. $3x^{\frac{1}{4}} = 2.$

70. $2x^{\frac{1}{4}} = 5.$

71. $\frac{1}{3}x^{\frac{1}{2}} = \frac{1}{4}.$

72. $3\sqrt{y} = 2.$

73. $3\sqrt{z} = 4.$

74. $4\sqrt{w} = 1.$

75. $5\sqrt{x} = 6.$

76. $2\sqrt{3y} = 5.$

77. $3\sqrt{2z} = 2.$

78. $4\sqrt{5w} = 5.$

79. $\sqrt{ax} = b.$

80. $\sqrt{cx} = d.$

81. $\sqrt{by} = 1.$

82. $a\sqrt{w} = b.$

83. $c\sqrt{z} = d.$

84. $d\sqrt{w} = 1.$

85. $a\sqrt{x} = 2.$

86. $\sqrt{\frac{y}{m}} = n.$

87. $\sqrt{\frac{x}{a}} = b.$

88. $\sqrt{z} = \frac{a}{b}.$

89. $\sqrt{aw} = \frac{1}{k}.$

90. $\frac{\sqrt{ax}}{a} = 1.$

91. $\sqrt[n]{y} = 2.$

92. $\sqrt[n]{y} = 3.$

93. $\sqrt[n]{2z} = 1.$

94. $\sqrt[n]{2w} = 3.$

95. $\sqrt[n]{3x} = 3.$

96. $\sqrt[n-1]{y} = 2.$

97. $x^{\frac{1}{2}} = a + b.$

98. $y^{\frac{1}{2}} = a - b.$

99. $z^{\frac{1}{3}} = c + d.$

100. $w^{\frac{1}{3}} = m - n.$

101. $\sqrt{\frac{x}{2}} = 3.$

102. $\frac{\sqrt{x}}{2} = 3.$

103. $\sqrt{\frac{y}{3}} = 5.$

104. $\frac{\sqrt{y}}{3} = 5.$

105. $2\sqrt{\frac{z}{3}} = 7.$

106. $\frac{1}{4}\sqrt{\frac{x}{3}} = \frac{1}{3}.$

107. $\frac{1}{3}\sqrt{\frac{x}{2}} = \frac{1}{4}.$

108. $\frac{2}{3}\sqrt{\frac{x}{5}} = 1.$

109. $\frac{\sqrt{3x}}{3} = 1.$

110. $\frac{\sqrt{5y}}{5} = 2.$

111. $y^{-\frac{1}{2}} = 6.$

112. $z^{-\frac{1}{3}} = 3.$

113. $z^{-\frac{1}{4}} = 2.$

114. $\frac{1}{x^{\frac{1}{2}}} = \frac{1}{2}.$

115. $\frac{1}{x^{\frac{1}{3}}} = \frac{1}{3}.$

116. $x^{-\frac{1}{2}} = \frac{1}{5}.$

117. $y^{-\frac{1}{3}} = \frac{1}{4}.$

118. $z^{-\frac{1}{4}} = \frac{1}{2}$.

119. $x^{\frac{1}{2}} = 3^{\frac{1}{4}}$.

120. $\sqrt{x} = \sqrt{6}$.

121. $z^{\frac{1}{3}} = 2^{\frac{1}{6}}$.

122. $\sqrt{y} = \sqrt[3]{5}$.

123. $w^{\frac{2}{3}} = 5^{\frac{1}{3}}$.

124. $\sqrt[3]{x} = \sqrt{6}$.

125. $\sqrt{y} = 2\sqrt[3]{5}$.

126. $\sqrt{z} = 3\sqrt[3]{7}$.

SPECIAL EQUATIONS

13. We will now consider certain special equations, the solutions of which depend upon the solutions of quadratic or of linear equations.

Certain equations containing two different powers, x^{2n} and x^n, of an unknown quantity or expression x, one power being the square of the other, may be reduced to the form $ax^{2n} + bx^n + c = 0$ which is quadratic with reference to x^n. Such equations may be solved by the methods employed for the solution of the standard quadratic equation.

E.g. The following equations are all in quadratic form, since in each case the power of the unknown appearing in the first term is the square of the power appearing in the second term:

$$x^4 - 25\,x^2 + 144 = 0,$$
$$x - 14\,x^{\frac{1}{2}} + 45 = 0,$$
$$x^{\frac{4}{3}} - 13\,x^{\frac{2}{3}} + 36 = 0,$$
$$x^{-2} - x^{-1} - 6 = 0.$$

Ex. 1. Solve $x^4 - 25\,x^2 + 144 = 0$. (1)

Factoring, $(x^2 - 16)(x^2 - 9) = 0$. (2)

This equation is equivalent to the set of two quadratic equations

$$x^2 - 16 = 0 \quad \text{and} \quad x^2 - 9 = 0. \tag{3}$$

Hence $x = \pm 4$ and $x = \pm 3$.

All of these values will be found by substitution to satisfy the given equation.

Ex. 2. Solve $x - 14\,x^{\frac{1}{2}} + 45 = 0$. (1)

Observe that x, which appears in the first term, is the square of $x^{\frac{1}{2}}$ which appears in the second term.

Factoring with reference to $x^{\frac{1}{2}}$, we obtain,

$$(x^{\frac{1}{2}} - 9)(x^{\frac{1}{2}} - 5) = 0. \tag{2}$$

This single equation is equivalent to the set of equations

$$x^{\frac{1}{2}} - 9 = 0, \quad \text{and} \quad x^{\frac{1}{2}} - 5 = 0. \qquad (3)$$

Hence, $$x^{\frac{1}{2}} = 9, \quad \text{and} \quad x^{\frac{1}{2}} = 5.$$

Squaring both members of each of the equations above,

$$(x^{\frac{1}{2}})^2 = 9^2, \quad \text{and} \quad (x^{\frac{1}{2}})^2 = 5^2.$$

Therefore $$x = 81, \quad \text{and} \quad x = 25.$$

These values will be found, by substitution, to satisfy the given equation.

Ex. 3. Solve $$x^{\frac{4}{3}} - 13\,x^{\frac{2}{3}} + 36 = 0. \qquad (1)$$

Factoring with reference to $x^{\frac{2}{3}}$, we have

$$(x^{\frac{2}{3}} - 9)(x^{\frac{2}{3}} - 4) = 0. \qquad (2)$$

Hence, $$x^{\frac{2}{3}} - 9 = 0, \quad \text{and} \quad x^{\frac{2}{3}} - 4 = 0. \qquad (3)$$

Therefore, $$x^{\frac{2}{3}} = 9, \quad \text{and} \quad x^{\frac{2}{3}} = 4.$$

To obtain x from $x^{\frac{2}{3}}$, we may raise $x^{\frac{2}{3}}$ to the third power and extract the square root of the result, or first extract the square root of $x^{\frac{2}{3}}$, and then find the third power of the result.

That is, $$(x^{\frac{2}{3}})^{\frac{3}{2}} = 9^{\frac{3}{2}} \quad \text{and} \quad (x^{\frac{2}{3}})^{\frac{3}{2}} = 4^{\frac{3}{2}}.$$

Therefore $$x = \pm\, 27, \quad \text{and} \quad x = \pm\, 8.$$

These values will be found, by substitution, to satisfy the given equation.

Ex. 4. Solve $$x^{-2} - x^{-1} - 6 = 0. \qquad (1)$$

Factoring with reference to x^{-1},

we have $$(x^{-1} - 3)(x^{-1} + 2) = 0. \qquad (2)$$

Hence, $$x^{-1} - 3 = 0, \quad \text{and} \quad x^{-1} + 2 = 0. \qquad (3)$$

We find that $$x^{-1} = 3, \quad \text{and} \quad x^{-1} = -\, 2. \qquad (4)$$

To obtain x from x^{-1}, we may write $(x^{-1})^{-1} \equiv x^{+1}$.

Hence, $$(x^{-1})^{-1} = 3^{-1}, \quad \text{and} \quad (x^{-1})^{-1} = -\, 2^{-1}.$$

Therefore, $$x = \tfrac{1}{3}, \quad \text{and} \quad x = -\, \tfrac{1}{2}.$$

Instead of proceeding as above, we may obtain the values as follows:

From $$x^{-1} = 3 \quad \text{and} \quad x^{-1} = -\, 2,$$

we obtain $$\frac{1}{x} = 3 \quad \text{and} \quad \frac{1}{x} = -\, 2. \qquad (5)$$

Hence, $$x = \tfrac{1}{3} \quad \text{and} \quad x = -\, \tfrac{1}{2}.$$

Verifying by substituting in (1), we have,

Substituting $\frac{1}{3}$,	Substituting $-\frac{1}{2}$,
$(\frac{1}{3})^{-2}-(\frac{1}{3})^{-1}-6=0$	$(-\frac{1}{2})^{-2}-(-\frac{1}{2})^{-1}-6=0$
$9\quad-\quad3\quad-6=0$	$4\quad+\quad2\quad-6=0$
$0=0.$	$0=0.$

14. If the solution of a given equation cannot be readily obtained by factoring, we may either resort to the method of completing the square, or we may use the formula.

Ex. 5. Solve $3x-8x^{\frac{1}{2}}+2=0$.

Observe that x in the first term is the square of $x^{\frac{1}{2}}$ in the second term.

Referring to the standard equation $ax^2+bx+c=0$, we find that x and $x^{\frac{1}{2}}$ in the given equation are represented by x^2 and x respectively in the standard equation; the coefficient 3 of x in the given equation is represented by the coefficient a of x^2 in the standard equation; -8 is represented by b, and 2 is represented by c.

Corresponding to the solution of the standard equation,

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a},$$

we may write

$$\begin{aligned}x^{\frac{1}{2}}&=\frac{-(-8)\pm\sqrt{(-8)^2-4\cdot3\cdot2}}{2\cdot3}\\&=\frac{+8\pm\sqrt{64-24}}{6}\\&=\frac{+8\pm2\sqrt{10}}{6}\\&=\frac{+4\pm\sqrt{10}}{3}.\end{aligned}$$

That is, $x^{\frac{1}{2}}=\dfrac{+4+\sqrt{10}}{3}$ and $x^{\frac{1}{2}}=\dfrac{+4-\sqrt{10}}{3}$.

Squaring both members of each of the equations above,

we have, $(x^{\frac{1}{2}})^2=\left(\dfrac{+4+\sqrt{10}}{3}\right)^2$ and $(x^{\frac{1}{2}})^2=\left(\dfrac{+4-\sqrt{10}}{3}\right)^2$.

Or, $x=\dfrac{26+8\sqrt{10}}{9}$, and $x=\dfrac{26-8\sqrt{10}}{9}$.

These *exact* values will be found by substitution to satisfy the given equation.

By extracting the square root of 10 to any required number of significant figures, and replacing $\sqrt{10}$ by the approximate value thus found, *approxi-*

mate values may be obtained for x, which are correct to any required number of significant figures.

Thus, $x = 5.6997+$, and $x = .0780+$. (See Chap. XXII. § 20, Ex. 2.)

EXERCISE XXIV. 3

Find one or more solutions of each of the following equations, verifying all integral and fractional solutions:

1. $x^4 - 10x^2 + 9 = 0$.
2. $x^4 - 20x^2 + 64 = 0$.
3. $x^4 - 74x^2 = -1225$.
4. $x^2(3x^2 + 19) = 124$.
5. $x^4 - (x + 2)^2 = 0$.
6. $x^4 = (x + 6)^2$.
7. $16(x^2 - 3) = x^4$.
8. $(x + 12)^2 - x^4 = 0$.
9. $2x^3 + 48 = x^6$.
10. $x^6 + 9x^3 + 8 = 0$.
11. $5x^{\frac{1}{2}} - 2x = -3$.
12. $x + 5x^{\frac{1}{2}} = 36$.
13. $3x^{\frac{1}{2}} - 5x = -36$.
14. $x^{\frac{1}{2}} + 4x^{\frac{1}{4}} = 21$.
15. $x^{\frac{1}{2}}(3x^{\frac{1}{2}} - 2) = 8$.
16. $2x^{\frac{2}{3}} - 2 = 3x^{\frac{1}{3}}$.
17. $4x^{\frac{2}{3}} - 12x^{\frac{1}{3}} + 5 = 0$.
18. $14x^{\frac{3}{2}} = 1107 - x^3$.
19. $3x^{\frac{3}{2}} - 4x^{\frac{3}{4}} = 160$.
20. $3x^{\frac{3}{2}} - 7 = 4x^{\frac{3}{4}}$.
21. $x^{\frac{6}{5}} + x^{\frac{3}{5}} = 756$.
22. $5 - 3x^{-1} = 2x^{-2}$.
23. $x^{-1} - x^{-\frac{1}{2}} = 6$.
24. $4x^{-2} - 32 + x^{-4} = 0$.
25. $20x^{-\frac{2}{5}} - x^{-\frac{4}{5}} = 64$.
26. $x^3 - 5x^2 = 14x$.

    Hint. Write in the form $x(x^2 - 5x - 14) = 0$. Then $x = 0$ is one solution.
27. $x(x^2 - 16) = 45(x - 4)$.
28. $x^{\frac{5}{2}} - 6x^{\frac{3}{2}} - 40x^{\frac{1}{2}} = 0$.

15. Occasionally the terms of an equation of degree higher than the second may be so grouped as to allow of the reduction of the equation to a form which is quadratic with respect to some definite group of terms containing the unknown.

If we let $f(x)$ represent a group of terms containing the unknown, x, we may represent an equation which is quadratic with reference to this group of terms by

$$a[f(x)]^2 + b[f(x)] + c = 0.$$

Ex. 1. Solve $(x^2 + 3x)^2 - 2(x^2 + 3x) - 8 = 0$. (1)

Observe that the equation is quadratic with respect to the group of terms $(x^2 + 3x)$.

Factoring with respect to this group of terms, we obtain,

$$[(x^2 + 3x) - 4][x^2 + 3x) + 2] = 0. \qquad (2)$$

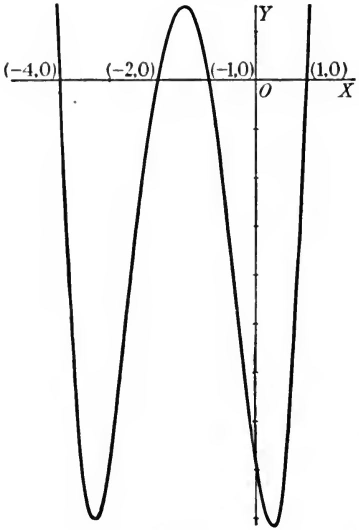


Fig. 1. $(x^2+3x)^2-2(x^2+3x)-8=y$.

Equation (2) is equivalent to the set of two equations obtained by writing the factors of its first member separately equal to zero.

Hence, $x^2 + 3x - 4 = 0, \qquad (3)$

and $x^2 + 3x + 2 = 0. \qquad (4)$

These equations in turn are equivalent to

$$(x + 4)(x - 1) = 0, \qquad (5)$$

and $(x + 2)(x + 1) = 0. \qquad (6)$

Hence, $x = - 4, x = 1,$

and $x = - 2, x = - 1.$

These values are all found by substitution to be roots of the given equation, which is of the fourth degree or biquadratic. (See Fig. 1.)

Ex. 2. Solve

$$x^4 - 12x^3 + 25x^2 + 66x - 80 = 0. \qquad (1)$$

We may obtain the term which is necessary to complete the square with respect to $x^4 - 12x^3$ by using $- 12x^3$ as a "finder term" with x^4, as follows:

$$\frac{-12x^3}{2x^2} \equiv - 6x.$$

Accordingly the complete trinomial square of which x^4 and $- 12x^3$ are the first two terms is

$$x^4 - 12x^3 + (- 6x)^2 \equiv x^4 - 12x^3 + 36x^2.$$

This may be constructed in the first member of (1) by adding $11x^2$ to $25x^2$.

Hence, $x^4 - 12x^3 + 25x^2 + 11x^2 - 11x^2 + 66x - 80 = 0. \qquad (2)$

Or, $x^4 - 12x^3 + 36x^2 - 11x^2 + 66x - 80 = 0. \qquad (3)$

Hence, $(x^2 - 6x)^2 - 11x^2 + 66x - 80 = 0. \qquad (4)$

To arrange the terms in quadratic form with respect to $x^2 - 6x$, we must so group them as to have both $(x^2 - 6x)^2$ and $(x^2 - 6x)$.

Grouping the terms $-11x^2$ and $66x$, we have

$$(x^2 - 6x)^2 - 11(x^2 - 6x) - 80 = 0. \tag{5}$$

We have derived from equation (1) the equivalent equation (5) in quadratic form.

Solving (5) by the method of factoring, we have

$$[(x^2 - 6x) - 16][(x^2 - 6x) + 5] = 0. \tag{6}$$

This equation is equivalent to the set of two equations

$$x^2 - 6x - 16 = 0, \quad (7) \qquad \text{and} \qquad x^2 - 6x + 5 = 0. \quad (8)$$

The solutions of these equations are found to be

$$x = 8,\ x = -2, \qquad \text{and} \qquad x = 5,\ x = 1.$$

All of the values are found by substitution to be solutions of the given equation of the fourth degree.

16. The introduction of an auxiliary letter in place of a term or group of terms containing the unknown often simplifies the process of solution of an equation.

Ex. 3. Solve

$$\frac{x^2+6}{x} + \frac{5x}{x^2+6} = 6. \tag{1}$$

Observe that in the given equation the expression $\frac{x^2+6}{x}$ and its reciprocal $\frac{x}{x^2+6}$ both appear.

If we let $\frac{x^2+6}{x} = y$, then its reciprocal may be written $\frac{x}{x^2+6} = \frac{1}{y}$.

For (1) we may substitute,

$$y + \frac{5}{y} = 6, \tag{2}$$

from which

$$y^2 - 6y + 5 = 0. \tag{3}$$

We find that

$$y = 5, \quad (4) \quad \text{and } y = 1. \quad (5)$$

Substituting these values for y in $\frac{x^2+6}{x} = y$ we shall obtain the two following equations, which are together equivalent to equation (1):

$$\frac{x^2+6}{x} = 5, \ (6) \qquad \text{and} \qquad \frac{x^2+6}{x} = 1. \ (7)$$

The solutions of these equations are found to be

$$x = 3,\ x = 2, \qquad \text{and} \qquad x = \frac{1 \pm \sqrt{-23}}{2}.$$

These values all satisfy the given equation.

Ex. 4. Solve

$$(x^2 + x + 2)(x^2 + x + 7) = 36. \tag{1}$$

If we let

$$x^2 + x + 2 = y,$$

then

$$x^2 + x + 7 = y + 5.$$

Substituting y and $y + 5$ for the polynomials in the given equation, we have

$$y(y+5) = 36. \qquad (2)$$

Or,

$$y^2 + 5y - 36 = 0. \qquad (3)$$

The solutions of this equation are found to be

$$y = -9, \quad (4) \qquad \text{and} \qquad y = 4. \quad (5)$$

Substituting these values for y in the assumed equation $x^2 + x + 2 = y$, we obtain the two following equations, which are together equivalent to equation (1):

$$x^2 + x + 2 = -9, \quad (6) \qquad \text{and} \qquad x^2 + x + 2 = 4. \quad (7)$$

The roots of these equations are found to be

$$x = \frac{-1 \pm \sqrt{-43}}{2}, \qquad \text{and} \qquad x = -2,\ x = +1.$$

These values will be found, by substitution, to satisfy the given equation.

Examples 3 and 4 may also be solved by the method employed for examples 1 and 2.

Exercise XXIV. 4

Solve the following equations, verifying all integral and fractional solutions:

1. $(x^2 - 3x)^2 - 8(x^2 - 3x) = 20.$
2. $(x^2 + x)^2 - 26(x^2 + x) + 120 = 0.$
3. $3x^2 + 2x + 1 = \dfrac{30}{3x^2 + 2x}.$
4. $x^2 - 4x - 26 + \dfrac{105}{x^2 - 4x} = 0.$
5. $x^4 + 2x^3 - 6x^2 - 7x - 60 = 0.$
6. $x^4 - 2x^3 + 6x^2 - 5x = 14.$
7. $x^4 + 6x^3 + 14x^2 + 15x + 6 = 0.$
8. $x^4 - 10x^3 + 14x^2 + 55x + 30 = 0.$
9. $(x^2 - x - 4)(x^2 - x - 3) - 6 = 0.$
10. $(x^2 + x + 1)(x^2 + x + 3) = 63.$
11. $\dfrac{x}{x^2 - 1} + \dfrac{x^2 - 1}{x} = \dfrac{73}{24}.$
12. $\dfrac{x^2 + 2}{x - 3} + \dfrac{35}{6} = \dfrac{x - 3}{x^2 + 2}.$
13. $\left(x + \dfrac{1}{x}\right)^2 + 4\left(x + \dfrac{1}{x}\right) = \dfrac{65}{4}.$

14. $\left(\frac{5}{x}+x\right)^2+\frac{5}{x}+x=42.$

15. $\left(\frac{x^2+1}{x}\right)^2-\frac{x^2+1}{x}=12.$

16. $\left(x-\frac{8}{x}\right)^2+3x-\frac{24}{x}=10.$

17. A **binomial equation** is an equation of the form $x^n+a=0$, in which n is a positive integer. Whenever the binomial x^n+a can be expressed as the product of two or more factors of the first or second degrees, the binomial equation $x^n+a=0$ may be solved by the methods employed for the solution of linear and quadratic equations.

Ex. 1. Find the three cube roots of + 1.

If x represents any one of the cube roots of + 1, we have

$$x^3=+1. \qquad (1)$$

By solving the binomial equation, we shall find the three cube roots desired.

Equation (1) is equivalent to $x^3-1=0$. (2)

Factoring, $(x-1)(x^2+x+1)=0$. (3)

This equation is equivalent to the set of two equations

$x-1=0$, (4) and $x^2+x+1=0$. (5)

The solutions of (4) and (5) are

$$x=1 \quad \text{and} \quad x=\frac{-1\pm\sqrt{-3}}{2}. \qquad \text{(See Fig. 2.)}$$

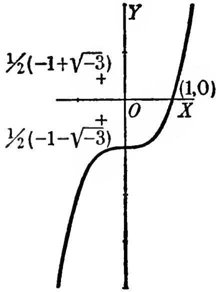


FIG. 2. $x^3-1=y$.

The positive real value $x=+1$ is the principal root of + 1. This is the root found by the arithmetic process for the extraction of the cube root.

The three values obtained above will all be found to satisfy the algebraic equation (1).

18. An equation in which the coefficients are the same, whether read in order forward or backward, is called a **reciprocal equation.**

Ex. 2. Solve the binomial equation $x^5-1=0$. (1)

Factoring, $(x-1)(x^4+x^3+x^2+x+1)=0$. (2)

This single equation is equivalent to the binomial equation

$x-1=0$, (3)

and the reciprocal equation $x^4+x^3+x^2+x+1=0$, (4), taken together.

The solution of (3) is $x=1$.

The solution of the reciprocal equation (4) may be obtained as follows:

Dividing each of the terms of (4) by x^2, we obtain

$$x^2+x+1+\frac{1}{x}+\frac{1}{x^2}=0, \tag{5}$$

or,

$$x^2+1+\frac{1}{x^2}+x+\frac{1}{x}=0. \tag{6}$$

The terms x^2 and $1/x^2$ suggest a trinomial square, $x^2+2+\frac{1}{x^2}$.

By adding 1 and also subtracting 1 in the first member of equation (6), we obtain,

$$x^2+2+\frac{1}{x^2}+x+\frac{1}{x}-1=0, \tag{7}$$

or,

$$\left(x+\frac{1}{x}\right)^2+\left(x+\frac{1}{x}\right)-1=0. \tag{8}$$

Equation (8) is quadratic with respect to $\left(x+\frac{1}{x}\right)$.

Accordingly, applying the formula, we obtain

$$x+\frac{1}{x}=\frac{-1\pm\sqrt{+5}}{2}. \tag{9}$$

Equation (9) is equivalent to the set of two separate equations

$$x+\frac{1}{x}=\frac{-1+\sqrt{5}}{2}, \quad (10) \qquad \text{and} \qquad x+\frac{1}{x}=\frac{-1-\sqrt{5}}{2}. \quad (11)$$

From these we derive

$$x^2-\left(\frac{-1+\sqrt{5}}{2}\right)x+1=0 \quad \text{and} \quad x^2-\left(\frac{-1-\sqrt{5}}{2}\right)x+1=0.$$

Hence, $x=\dfrac{-1+\sqrt{5}\pm\sqrt{-10-2\sqrt{5}}}{4}$ and $x=\dfrac{-1-\sqrt{5}\pm\sqrt{-10+2\sqrt{5}}}{4}$.

These four complex values, taken together with the real value, $x=1$, obtained from equation (3), are the five fifth-roots of $+1$, which are the solutions of the given equation $x^5-1=0$.

EXERCISE XXIV. 5

Solve the following equations, verifying all integral and fractional solutions :

1. $x^3-27=0$.
2. $x^3-64=0$.
3. $x^6=1$.
4. $x^3+1=0$.

5. $x^3 + 8 = 0$.

6. $x^6 - 64 = 0$.

7. $(x - 3)^3 = 8$.

8. $(x - 4)^4 - 81 = 0$.

9. $6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0$.

10. $10x^4 - 77x^3 + 150x^2 - 77x + 10 = 0$.

11. $6x^4 - 49x^3 + 86x^2 - 49x + 6 = 0$.

12. $6x^4 - 25x^3 + 38x^2 - 25x + 6 = 0$.

13. $12x^4 - 91x^3 + 194x^2 - 91x + 12 = 0$.

14. $15x^3 - 49x^2 + 49x - 15 = 0$.

15. $8x^5 - 46x^4 + 47x^3 + 47x^2 - 46x + 8 = 0$.

PROBLEMS IN PHYSICS

19. The Simple Pendulum. If a simple pendulum swings through an arc the extremities of which are A and B, the motion of the pendulum from A to B or from B to A is called an *oscillation* or a *vibration.*

20. The *time of vibration* is the time required for the pendulum to make a *single vibration,* that is, to move from A to B, or from B to A.

21. The distance from the lowest point of the arc AB to either extremity, A or B, that is, one-half of the arc AB, is called the *amplitude* of vibration.

22. When the amplitude of vibration is very small, an *approximate* value of the time of vibration, t, expressed in seconds, of a pendulum of length l, expressed in feet, is found by the formula

$$t = \pi\sqrt{\frac{l}{g}}.$$

In the following examples the approximate value 22/7 may be taken for the numerical constant π.

EXERCISE XXIV. 6

Solve each of the following problems relating to the simple pendulum :

1. Find the length in feet of a pendulum which vibrates once in a second at a place at which $g = 32.16$.

2. Find the value of g at a certain place if a pendulum which is 10 feet in length makes 20 vibrations in 35 seconds.

3. If a certain pendulum vibrates once in a second, find the time required for a pendulum which is twice as long to vibrate once.

4. Find the length of a pendulum which makes 80 vibrations per minute at a place at which the value of g is 32.16.

5. Find the length of a pendulum which vibrates once per second at a place at which the value of g is 32.19.

6. If at a certain place a pendulum 39 inches in length vibrates once in a second, find the length of a pendulum which at the same place will make one vibration in one minute.

7. If a ball suspended by a fine wire makes 88 vibrations in 15 minutes, find the length of the wire.

8. If a pendulum which is 39.1 inches in length vibrates once in a second at a certain place, find the length of a pendulum which will vibrate once in 5 seconds.

EXERCISE XXIV. 7. **Review**

1. If $a = 1$, $b = 3$, and $c = 2$, find the value of

$$(a+b)(b+c)(c+a) + a^b + b^c + c^a.$$

2. Factor $1 + 10x - 11x^2$.

3. Factor $x^2(y+1) + y^2(x+1) + x + y + 2xy$.

4. Factor $a^3 - x^3 - a(a^2 - x^2) + x(a-x)^2$.

5. Find the prime factors of $(a - a^2)^3 + (a^2 - 1)^3 + (1 - a)^3$.

Simplify the following expressions :

6. $\left(\frac{1}{x^2} + \frac{1}{x} + 1\right)(x^2 - x + 1)$.

7. $\left(\frac{a}{x} - \frac{b}{y}\right)\left(\frac{x}{a} - \frac{y}{b}\right)$.

8. $\left(x^5 - \frac{1}{32}\right) \div \left(x - \frac{1}{2}\right)$.

9. $\frac{(a^4 - b^4)(a^3 - b^3)}{(a^6 - b^6)(a - b)}$.

10. $\left(\frac{a-1}{a+1}\right)^2 - \left(\frac{a-2}{a+2}\right)^2$.

11. $(a^2 - b^2 - c^2 - 2bc) \div \frac{a+b+c}{a+b-c}$.

12. $\frac{a^2 - b^2}{\frac{1}{a} - \frac{1}{b}}$.

13. $(x^{-1} + y^{-1}) \div (x^{-\frac{1}{3}} + y^{-\frac{1}{3}})$.

14. $$\frac{27^{-2} \cdot 9 \times \frac{1}{3^{-1}} - 3^{-3}}{27^2 \cdot 3^3}.$$

15. Express $(-a - b) \div (-a^{-1} - b^{-1})$ with the minimum number of minus signs.

16. Find the value of

$$\frac{x+2m}{2n-x}+\frac{x-2m}{2n+x}+\frac{4mn}{x^2-4n^2}, \text{ if } x=\frac{mn}{m+n}.$$

17. Show that $(a^2-b^2)^2 \equiv a^3-ab+b^3$, if $a+b=1$.

Simplify each of the following expressions :

18. $(3-\sqrt{-2})(2-\sqrt{-3})$.

19. $ab+\sqrt{ab}+(a-\sqrt{b})(\sqrt{a}-b)$.

20. $(\sqrt{-3}-\sqrt{-2})^2+(\sqrt{3}-\sqrt{2})^2$.

21. $(\sqrt{5}-\sqrt{7}+2)(\sqrt{5}+\sqrt{7}-2)$.

22. $(\sqrt{7}+\sqrt{5}-\sqrt{3})(\sqrt{7}-\sqrt{5}+\sqrt{3}.)$

23. $(\sqrt{11}-\sqrt{6}+5)(\sqrt{11}-\sqrt{6}-5)$.

24. $\left(\frac{\sqrt{a}}{\sqrt{b}}-\frac{\sqrt{b}}{\sqrt{a}}\right) \div \left(\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}\right)$.

CHAPTER XXV

SYSTEMS OF SIMULTANEOUS EQUATIONS INVOLVING QUADRATIC EQUATIONS

Systems of Two Equations Containing Two Unknowns

1. The most general form for an equation of the second degree containing two unknowns, x and y, is

$$ax^2 + 2\,hxy + by^2 + 2\,gx + 2fy + c = 0,$$

in which a, b, c, f, g, and h all represent real known numbers.

If one or more of these letters be given the value zero, the terms of which they are the coefficients disappear, and we have special types of quadratic equations containing two unknowns, x and y, such as the following:

If f, g, be zero, we have $ax^2 + 2\,hxy + by^2 + c = 0$. (i.)
f, g, h, $ax^2 + by^2 + c = 0$. (ii.)
b, f, g, $ax^2 + 2\,hxy + c = 0$. (iii.)
a, f, g, $2\,hxy + by^2 + c = 0$. (iv.)
b, g, h, $ax^2 + 2fy + c = 0$. (v.)
a, f, h, $by^2 + 2\,gx + c = 0$. (vi.)
etc. etc.

2. In this chapter we shall obtain the solutions of certain systems of simultaneous equations, containing one or more equations or the second degree, of the general types shown above.

3. In Chapter XVII, we found that there exists a definite set of values of the unknowns which satisfies all of the equations of a given set simultaneously, provided that the given equations are independent and consistent, and that the number of equations is equal to the number of different unknowns appearing in the system.

We shall find, whenever one at least of the given equations composing a system is of the second or higher degree, that there can be found more than one set of values which satisfies all of the equations at the same time.

4. There are many systems consisting of two equations of the second or higher degrees with reference to two unknowns which cannot be solved by means of quadratic or linear equations.

5. Whenever a system consists of one equation of the second degree and another of the first degree with reference to two unknowns, say x and y, the equation of the second degree having the form either of the general equation

$$ax^2 + 2\,hxy + by^2 + 2\,gx + 2fy + c = 0,$$

or of one of the special forms $ax^2 + by^2 + c = 0$ (see § 1), we may always make the solution of the system depend upon the solution of equations of either the second or of the first degree.

6. *From the equation of the first degree, we may express the value of one of the unknowns, say y, in terms of the other, x; on substituting this expressed value for the same unknown, y, wherever it appears in the equation of the second degree, we shall derive an equation of the second degree containing but one unknown, x, called the* ***x-eliminant*** *of the system; this x-eliminant may be solved by the methods already shown for the solution of quadratic equations containing one unknown.*

From the given system is thus derived an equivalent system consisting of the given equation of the first degree with reference to both of the unknowns, and, as the case may be, either the x-eliminant or the y-eliminant of the system, which contains but one of the given unknowns.

The number of solutions of the particular eliminant employed depends upon its degree with reference to the unknown contained in it, and *by substituting the solutions of the eliminant separately in the remaining original equation of the first degree, we shall, for each value of the unknown substituted, say x, obtain a corresponding value for the remaining unknown, y.*

The number of sets of values thus obtained is the same as the degree of the "eliminant" equation.

If the eliminant be of the second degree, the number of solutions of the given system for finite values of the unknowns is two; if it be of the third degree, three; etc.

7. We shall speak of the *number* of sets of values which satisfy all of the equations of a given system simultaneously, as the **Order of the System.**

8. We will state, without proof, the following **Principles:**

(i.) *The number of roots of an integral equation containing one unknown is equal to the degree of the equation with reference to that unknown, and these roots may be equal or unequal, rational or irrational, real or complex, according to circumstances.*

(ii.) *The number of sets of values which satisfy all of the equations of a given determinate system cannot be greater than, and is in general equal to, the product of the degrees of the separate equations of which the system is composed.*

E. g. Since the equations $2x + y = 9$ and $2x^2 + y^2 = 33$ are of the first and second degrees respectively, they may be spoken of as constituting a 1–2 system of the second order.

The product of the degrees, 1 and 2, of the equations leads us to expect that there are two sets of values which satisfy the equations simultaneously. These sets are found to be, $x = 2$, $y = 5$, and $x = 4$, $y = 1$.

9. Since the x-eliminant or the y-eliminant of a 1–3 system containing two unknowns, x and y, is of the third degree with reference to either x or y, it follows that, to solve a system of the third order, we must solve an equation of the third degree, such as $ax^3 + bx^2 + cx + d = 0$, containing one unknown. Also, since the eliminant of a 2–2 system is of the fourth degree, it follows that to solve a system of the fourth order, we must solve an equation of the fourth degree, such as $ax^4 + bx^3 + cx^2 + dx + e = 0$, containing one unknown.

10. Except in certain very special cases, the solutions of systems of the third or higher orders cannot be made to depend upon the solutions of equations of either the second or of the first degree.

11. It should be observed that, although the solution of a given system in which quadratic or higher equations appear leads in general to the solution of an equation of higher degree than the second (that is, to the solution of the eliminant equation), the solutions of a great number of systems of simultaneous equations may be made to depend upon the solutions of quadratic equations.

12. We will now consider certain methods which may be applied to obtain the solutions of systems of two simultaneous equations containing two unknowns.

I. Elimination by Substitution

13. If one equation of a system of two simultaneous equations is of the first degree, and the other equation is of the second or higher degree, the solution of the system may in certain cases be made to depend upon the solution of a quadratic equation.

Ex. 1. Solve the system

$$\left.\begin{aligned} 3x^2 + y^2 &= 84, \quad (1) \\ 3x + 5y &= 42. \quad (2) \end{aligned}\right\} \text{I. Given System.}$$

Since the equations are of the second and first degrees respectively, this may be classed as a 2-1 system, and accordingly we may expect to find $2 \cdot 1$ or 2 sets of values for x and y which satisfy the equations simultaneously.

From equation (2) we may derive an equivalent equation in which the value of y is expressed in terms of x as follows:

$$y = \frac{42 - 3x}{5}. \qquad (3)$$

Substituting this expressed value in place of y in the first equation, we obtain the x-eliminant,

$$3x^2 + \left(\frac{42 - 3x}{5}\right)^2 = 84. \qquad (4)$$

Or,

$$(x + 1)(x - 4) = 0. \qquad (5)$$

The original system is equivalent to the derived system

$$\left.\begin{aligned} y &= \frac{42 - 3x}{5}, \quad (3) \\ (x + 1)(x - 4) &= 0. \quad (5) \end{aligned}\right\} \text{II. Equivalent Derived System.}$$

The solutions of the x-eliminant (5) are

$$x = -1, \quad \text{and} \quad x = 4.$$

Substituting these values for x in the remaining equation (3) of System II., we obtain:

Substituting -1 for x,

$$y = \frac{42 + 3}{5}$$

$$y = 9.$$

Substituting 4 for x,

$$y = \frac{42 - 12}{5}$$

$$y = 6.$$

The two solutions of the given equations are thus found to be

$$\left.\begin{array}{l} x = -1, \\ y = 9, \end{array}\right\} \text{ and } \left.\begin{array}{l} x = 4, \\ y = 6. \end{array}\right\}$$ These two groups of equations form an equivalent System III.

By substitution we find that the given equations are satisfied by these sets of values.

Graphical Interpretation.

14. The algebraic problem of finding the common solutions of a system of equations containing two unknowns suggests the graphical problem of finding the points of intersection of the graphs representing the given equations.

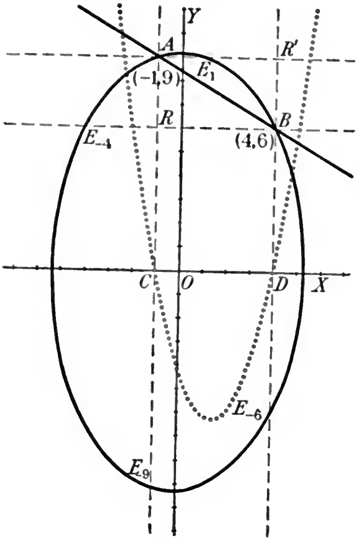


Fig 1.

In Fig. 1 the straight line *AB*, which is a portion of the graph of the linear equation $3x + 5y = 42$ (see Ex. 1, § 13), cuts the ellipse which is the graph of the quadratic equation $3x^2 + y^2 = 84$ (see Ex. 1, § 13), in two points, *A* and *B*.

By writing the first member of the *x*-eliminant (5) equal to *y*, we obtain the equation of which the graph is the parabola, a portion of which is shown in dotted line.

The distinction between the solution of a single equation containing one unknown, and the solution of a system consisting of several equations containing several unknowns, should be carefully noted.

The solutions of a single equation containing one unknown, that is, the values of its roots, may be taken as locating the points at which the graph of the equation crosses the axis of *X*.

Referring to Fig. 1, it may be seen that the solutions of the given System I. or of the equivalent derived System III.,

$$\left.\begin{array}{l} x = -1, \\ y = 9, \end{array}\right\} \text{ and } \left.\begin{array}{l} x = 4, \\ y = 6, \end{array}\right\}$$ if taken as coördinates,

serve to locate the intersections A and B of the ellipse and the oblique straight line which are the graphs of the given equations.

By eliminating y from the given equations (1) and (2) we derive the x-eliminant (5), which is a conditional equation expressing the relations existing between the x-values sought, without for the moment referring to the corresponding y-values.

In Fig. 1 it will be seen that the result of this elimination graphically is to locate, by means of the intersections of the parabola with the axis of X, two points C and D the distances of which from the origin O are equal to the x-coördinates of the points A and B respectively.

The roots of the x-eliminant, therefore, gave us these values,

$$x = -1, x = 4.$$

The y-eliminant, $y^2 - 15y + 54 = 0$ (not shown in the figure), would in a corresponding way express the relations existing between the y-values sought. Hence the roots of the y-eliminant, $y^2 - 15y + 54 = 0$, would thus give these values, $y = 9$ and $y = 6$.

If instead of substituting the values $x = -1$ and $x = 4$, obtained from the solution of the x-eliminant $x^2 - 3x - 4 = 0$ (5) in the given linear equation $3x + 5y = 42$ (2), we had substituted these same values in the quadratic equation $3x^2 + y^2 = 84$ (1), we would have obtained, corresponding to each value of x substituted, two values for y instead of one as before. Hence in addition to the values previously found we would have obtained the extra sets of values

$$\left.\begin{array}{l} x = -1, \\ y = -9, \end{array}\right\} \text{ and } \left.\begin{array}{l} x = 4, \\ y = -6. \end{array}\right\}$$

By examining Fig. 1, it appears that these values serve to locate extra points on the ellipse, marked E_{-9} and E_{-6}, which do not lie also upon the given straight line AB.

Accordingly these sets of values cannot be accepted as being solutions of the given system.

By substitution it will be found that these sets of values do not satisfy the given linear equation (2).

Hence the system composed of the solutions of the x-eliminant and the given quadratic equation is not equivalent to the original system.

Similarly, the system composed of the solutions of the y-eliminant $y^2 - 15y + 54 = 0$ and the given quadratic equation is not equivalent to the original system, for the solutions of this system bring in the extra solutions

$$\left.\begin{array}{l} x = 1, \\ y = 9, \end{array}\right\} \text{ and } \left.\begin{array}{l} x = -4, \\ y = 6. \end{array}\right\}$$

These extra sets of values may be taken as locating the extra points E_1 and E_{-4} which lie neither upon the oblique line AB which is the graph of the given linear equation $3x + 5y = 42$, nor upon the ellipse which is the graph of the quadratic equation $3x^2 + y^2 = 84$.

Accordingly these sets of values must be rejected as not being solutions of the given system.

The vertical dotted lines which are the graphs of the root values $x = -1$ and $x = 4$ of the x-eliminant, by their intersections with the oblique straight line, locate the points A and B respectively of the line AB, and no other points.

The horizontal dotted lines which are the graphs of the root values $y = 9$ and $y = 6$ of the y-eliminant $y^2 - 15y + 54 = 0$, by their intersections with the oblique straight line also locate the points A and B, and no other points.

It follows that the derived system composed of the solutions of either the x-eliminant or the y-eliminant, if taken together with the given equation of the first degree, is equivalent to the given system.

Furthermore, it will be seen, by referring to Fig. 1, that the horizontal and vertical dotted lines intersect in the four points A, B, R and R'.

The sets of values $x = -1$, $y = 6$, and $x = 4$, $y = 9$, corresponding to the coördinates of the points R and R', are not solutions of the given system, and must accordingly be rejected.

Thus, it may be seen that *to solve a given system it is not sufficient simply to obtain a certain number of different values, as when solving a single equation, but it is necessary also to arrange properly in sets the different values found, in such a way that on substitution the values of each set will satisfy all of the equations of a given system.*

It will be observed that the dotted straight lines which are the graphs of $x = -1$, $y = 9$, intersect at A, while the dotted lines which are the graphs of $x = 4$, $y = 6$, pass through B.

From the reasoning above, it appears that the two systems of equations

$$\left.\begin{aligned} x &= -1, \\ y &= 9, \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} x &= 4, \\ y &= 6, \end{aligned}\right\}$$

taken together, are equivalent to the given System I.

Ex. 2. Solve the system

$$\left.\begin{aligned} 3x^2 - 2xy - y^2 &= 64, \quad (1) \\ x - 3y &= 2. \quad (2) \end{aligned}\right\} \text{I. Given System.}$$

Since this is a 2–1 system, we may expect to find two sets of values for x and y.

Substituting in the first equation the expressed value of x in terms of y from the second equation, we obtain as the y-eliminant,

$$3(2 + 3y)^2 - 2(2 + 3y)y - y^2 = 64. \qquad (3)$$

Or,

$$(y - 1)(5y + 13) = 0. \qquad (4)$$

By the principles of equivalence, the system composed of the solutions of this eliminant, taken together with the linear equation (2), is equivalent to the given system.

That is,

$$\left.\begin{aligned} (y - 1)(5y + 13) &= 0, \quad (4) \\ x - 3y &= 2. \quad (2) \end{aligned}\right\} \text{II. Equivalent Derived System.}$$

The solutions of equation (4) are

$$y = 1, \qquad \text{and} \qquad y = -\tfrac{13}{5}.$$

Substituting these values in the remaining equation of System II., we obtain:

Substituting 1 for y,

$$x - 3 \cdot 1 = 2$$
$$x = 5.$$

Substituting $-\frac{13}{5}$ for y,

$$x - 3(-\tfrac{13}{5}) = 2$$
$$x = -\tfrac{29}{5}.$$

The following sets of values are solutions of the given system of equations, and by substitution are found to satisfy both equations:

$$\left.\begin{aligned} x &= 5, \\ y &= 1, \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} x &= -\tfrac{29}{5}, \\ y &= -\tfrac{13}{5}. \end{aligned}\right\}$$

EXERCISE XXV. 1

Solve each of the following systems of equations, rejecting all sets of values which do not satisfy both of the given equations:

1. $x^2 + y^2 = 200,$
   $x = 7y.$

2. $x - y = -2,$
   $x^2 + y^2 = 10.$

3. $x^2 + y^2 = 34,$
   $x + 2y = 13.$

4. $xy = 104.$
   $x - y = 5.$

5. $xy = 14,$
   $4x - y = 26.$

6. $x^2 - y^2 = 40,$
   $2x + y = 17.$

7. $x^2 + y^2 = 17,$
$x - 3y = 1.$

8. $x^2 + 4y^2 = 32,$
$5x + 6y = 8.$

9. $x^2 + 2y^2 = 73,$
$3x - y = 3.$

10. $5x^2 - xy = 15,$
$2x + 3y = 36.$

11. $x + 8y = xy,$
$x - y = 5.$

12. $x^2 + xy + y^2 = 7,$
$x + 4y = -1.$

13. $x - 5y = \frac{1}{2},$
$3x - 2y + 4y^2 = 9.$

14. $x + y + 2xy = \frac{25}{6},$
$5x - 2y = \frac{1}{3}.$

15. $2x^2 + 3xy + 4y^2 = 64,$
$x + y = -2.$

16. $2x - 3y = 2,$
$4x^2 - 3xy + y^2 = 44.$

17. $\frac{x+y}{y} = 6,$
$xy = 45.$

18. $\frac{x}{y} + \frac{y}{x} = 2,$
$6x - 5y = 1.$

19. $\frac{1}{x} - \frac{1}{y} = -\frac{1}{2},$
$x - 3y = -1.$

20. $\frac{x}{y} + \frac{y}{x} = \frac{5}{2},$
$x - y = -2.$

21. $\frac{x}{4} + \frac{y}{5} = 6,$
$\frac{4}{x} + \frac{5}{y} = \frac{6}{5}.$

22. $x - y - \frac{5}{6} = 0,$
$\frac{1}{x} + \frac{1}{y} - 5 = 0.$

23. $\frac{1}{x} + \frac{y}{2} = 2,$
$\frac{1}{y} + \frac{x}{2} = -\frac{1}{3}.$

24. $\frac{x+1}{y+1} = \frac{6}{5},$
$\frac{x^2+y}{x+y^2} = \frac{65}{46}.$

25. $\frac{x}{x+y} + \frac{y}{x-y} = \frac{61}{11},$
$2x + 3y = 54.$

II. Reduction of Systems of Equations by Factoring

15. We have seen that, *if the factors of the first member of an equation, the second member of which is zero, be separately equated to zero, the system composed of the entire group of equations thus formed is equivalent to the given single equation.* (See Chap. XII. § 48.)

E. g. If a given equation be represented by $A \cdot B \cdot C = 0$, in which A, B, and C are rational and integral with reference to certain unknowns, then the system composed of the separate equations $A = 0$, $B = 0$, and $C = 0$ is equivalent to the given single equation $A \cdot B \cdot C = 0$.

16. From this principle it follows that, *if a single determinate system of equations be represented by* $A \cdot B \cdot C = 0$ *and* $D = 0$, *in which* A, B, C, *and* D *represent expressions which are rational and integral with reference to certain unknowns, the single system*

$$\left.\begin{aligned} A \cdot B \cdot C &= 0, \\ D &= 0, \end{aligned}\right\} \text{I. Given System.}$$

is equivalent to the group of separate derived systems

$$\left.\begin{aligned} A &= 0, \\ D &= 0. \end{aligned}\right\} \text{(i.)} \qquad \left.\begin{aligned} B &= 0, \\ D &= 0. \end{aligned}\right\} \text{(ii.)} \qquad \left.\begin{aligned} C &= 0, \\ D &= 0. \end{aligned}\right\} \text{(iii.)}$$

(The following proof may be omitted when the chapter is read for the first time.)

For, any solution of the given system must reduce D to zero and also reduce to zero either one or all of the factors A, B, or C.

Hence, every solution of the given system must be a solution of at least one of the derived systems.

Any solution of (i.) must reduce D to zero and also A to zero, and accordingly must reduce to zero the product $A \cdot B \cdot C$.

Hence every solution of the derived system (i.) is also a solution of the given System I.

Similarly, every solution of any one of the derived systems (i.), (ii.) or (iii.) is also a solution of the given System I.

Accordingly the original single system is equivalent to the group of derived systems.

17. The application of this principle is not affected by the number of factors in the first member of any particular equation the second member of which is zero, or by the number of such equations.

E. g. Let it be required to separate the single system of equations

$$\left.\begin{aligned} A \cdot B &= 0, \\ C \cdot D &= 0, \end{aligned}\right\} \text{I. Given System.}$$

into a group of separate systems of equations which, taken together, are equivalent to the given system.

The two following derived systems of equations are equivalent to the given system :

$$\left.\begin{aligned} A &= 0, \\ C \cdot D &= 0, \end{aligned}\right\} \qquad \left.\begin{aligned} B &= 0, \\ C \cdot D &= 0. \end{aligned}\right\}$$

Since each of these derived systems can be further separated, we shall obtain finally

$$\left.\begin{array}{l}A=0,\\C=0.\end{array}\right\}\text{(i.)}\quad\left.\begin{array}{l}A=0,\\D=0.\end{array}\right\}\text{(ii.)}\quad\left.\begin{array}{l}B=0,\\C=0.\end{array}\right\}\text{(iii.)}\quad\left.\begin{array}{l}B=0,\\D=0.\end{array}\right\}\text{(iv.)}$$

These systems of equations, taken together, are equivalent to the given system of equations.

Ex. 1. Solve the system of equations

$$\left.\begin{array}{rl}2x^2+3xy+y^2=0, & (1)\\ x^2-3x=10. & (2)\end{array}\right\}\text{I.}\quad\text{Given System.}$$

Since this is a 2–2 system, we shall expect to obtain $2\cdot 2$ or 4 solutions.

Writing the given equations so that their second members shall be zero, and factoring the resulting first members, we obtain the equivalent system,

$$\left.\begin{array}{rl}(2x+y)(x+y)=0, & (3)\\ (x+2)(x-5)=0. & (4)\end{array}\right\}\text{II.}\quad\begin{array}{c}\text{Equivalent}\\\text{Derived System.}\end{array}$$

By the principle under consideration, the given system is equivalent to the following group of separate derived systems taken together:

$$\left.\begin{array}{r}2x+y=0,\\x+2=0.\end{array}\right\}\text{(i.)}\quad\left.\begin{array}{r}2x+y=0,\\x-5=0.\end{array}\right\}\text{(ii.)}\quad\left.\begin{array}{r}x+y=0,\\x+2=0.\end{array}\right\}\text{(iii.)}\quad\left.\begin{array}{r}x+y=0,\\x-5=0.\end{array}\right\}\text{(iv.)}$$

We have thus reduced the solution of the given system of quadratic equations to the solution of four systems of simultaneous linear equations.

The solutions of these separate systems are found to be

$$\left.\begin{array}{l}x=-2,\\y=4.\end{array}\right\}\text{(i.)}\quad\left.\begin{array}{l}x=5,\\y=-10.\end{array}\right\}\text{(ii.)}$$

$$\left.\begin{array}{l}x=-2,\\y=2.\end{array}\right\}\text{(iii.)}\quad\left.\begin{array}{l}y=5,\\y=-5.\end{array}\right\}\text{(iv.)}$$

By substitution, each of these sets of values is found to be a solution of the the given system.

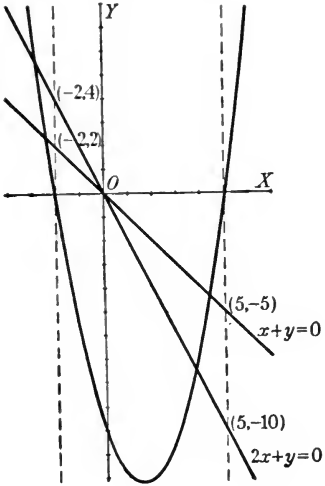


FIG. 2.

Graphical Interpretation

18. In Fig. 2 a portion of the graph of the equation $2x^2+3xy+y^2=0$ (See Ex. 1, § 17) is represented by the two oblique straight lines passing through the origin O. The graph of the equation $x^2-3x-10=y$ (see Ex. 1, § 17) is the parabola.

It should be observed that, since $x^2-3x-10=0$ contains the single unknown x, this equation may be treated as the x-eliminant of the given system. Hence our graphical problem becomes that of finding points on the oblique lines the x-coördinates of which are equal to the root values of the equation of which the graph is the parabola.

Accordingly, the solutions of this particular system of equations may be taken as locating points on the oblique lines, but not as locating the points of intersection of the parabola with the straight lines. (Compare with Fig. 1, § 14.)

Ex. 2. Solve the system of equations

$$\left.\begin{array}{rl} x^2 - 2xy + y^2 = 25, & (1)\\ 2x^2 = 7xy. & (2)\end{array}\right\}\text{ I. Given System.}$$

Since this system of equations is of the fourth order, we may look for four solutions.

Transposing all terms to the first members, and factoring, we obtain the equivalent system

$$\left.\begin{array}{rl} (x - y + 5)(x - y - 5) = 0, & (3)\\ x(2x - 7y) = 0. & (4)\end{array}\right\}\text{ II. }\begin{array}{c}\text{Equivalent}\\ \text{Derived System.}\end{array}$$

This single system is equivalent to the entire group of separate systems :

$$\left.\begin{array}{r} x - y + 5 = 0,\\ x = 0.\end{array}\right\}\text{(i.)}\qquad \left.\begin{array}{r} x - y + 5 = 0,\\ 2x - 7y = 0.\end{array}\right\}\text{(ii.)}$$

$$\left.\begin{array}{r} x - y - 5 = 0,\\ x = 0.\end{array}\right\}\text{(iii.)}\qquad \left.\begin{array}{r} x - y - 5 = 0,\\ 2x - 7y = 0.\end{array}\right\}\text{(iv.)}$$

The solutions of these systems of equations are found to be

$$\left.\begin{array}{l} x = 0,\\ y = 5.\end{array}\right\}\text{(i.)}\quad \left.\begin{array}{l} x = -7,\\ y = -2.\end{array}\right\}\text{(ii.)}\quad \left.\begin{array}{l} x = 0,\\ y = -5.\end{array}\right\}\text{(iii.)}\quad \left.\begin{array}{l} x = 7,\\ y = 2.\end{array}\right\}\text{(iv.)}$$

By substitution, these sets of values are all found to satisfy the given equations.

EXERCISE XXV. 2

Reduce each of the following systems of equations to equivalent groups of separate systems of equations, and solve :

1. $(x - 5)(y - 9) = 0,$
   $x + y = 10.$

2. $5x^2 - xy = 0,$
   $4x - y = 1.$

3. $(x - 2)(x + y - 3) = 0,$
   $(x - y - 4)(y - 5) = 0.$

4. $(x - y)(x + y - 1) = 0,$
   $(x + 1)(x + 2) = 0.$

5. $x^2 + 3xy = 18y^2,$
   $x + y = 2.$

6. $x^2 + 3y^2 = 12,$
   $x^2 - 2xy = 3y^2.$

7. $(x - y)^2 - 9 = 0,$
   $(x + y)^2 = 25.$

8. $x^2 - 2xy = 15y^2,$
   $x^2 - y^2 + 1 = 2x.$

9. $xy - 6y + 5x = 30,$
   $x + y = 9.$

10. $xy + x - y = 25,$
    $x(x - y) = 0.$

11. $x^2 - xy = 66,$
$5x^2 - 16xy + 11y^2 = 0.$

12. $x^2 - y^2 = x - y,$
$x^2 - 4xy = 4x - 16y.$

13. $x^2 + xy + x - y = 72,$
$3x^2 - 2xy - y^2 = 0.$

14. $4x^2 - 3xy + 5x - y = 11,$
$x^2 + xy = 0.$

III. Systems of Two Homogeneous Equations of the Second Degree Containing Two Unknowns

19. An equation, one member of which is a homogeneous function of x and y and the other member of which is either a homogeneous function of x and y or a known number, is said to be **homogeneous** with respect to the unknowns, x and y, appearing in it.

E. g.
$$x^2 + xy = y^2, \qquad x^3 + x^2y + xy^2 + y^3 = 0,$$
$$x^2 - 4xy + 3y^2 = 5x - y, \qquad x^2 + xy + 6y^2 = 8.$$

20. If the equations of which a system is composed are homogeneous with respect to the unknowns appearing in them, the system is called a **homogeneous system.**

21. The solutions of every system of two equations of the second degree, which are homogeneous with reference to two unknowns, can be obtained.

22. If the first members of the equations of a homogeneous system containing two unknowns, x and y, are of the second degree, while the second members are either known numbers or homogeneous functions of the same degree with reference to the unknowns, and if these second members differ only by a numerical factor, we may obtain the solution by factoring.

Solution by Factoring

We may represent two equations the first members of which are homogeneous with reference to two unknowns, x and y, and the second members of which are known numbers, by

$$a_1x^2 + b_1xy + c_1y^2 = d_1, \qquad (1)$$
$$a_2x^2 + b_2xy + c_2y^2 = d_2. \qquad (2)$$

In these equations a_1, a_2, b_1, b_2, etc., represent different real known numbers.

The known terms d_1 and d_2 may be eliminated from the two equations as follows:

Multiplying all of the terms of the first equation by d_2, and those of the second equation by d_1, we obtain the equivalent equations

$$a_1d_2x^2 + b_1d_2xy + c_1d_2y^2 = d_1d_2, \quad (3)$$
$$a_2d_1x^2 + b_2d_1xy + c_2d_1y^2 = d_2d_1. \quad (4)$$

By subtraction,

$$(a_1d_2 - a_2d_1)x^2 + (b_1d_2 - b_2d_1)xy + (c_1d_2 - c_2d_1)y^2 = 0. \quad (5)$$

Representing the known expressions in the different parentheses by the letters a, b, and c respectively, it appears that the derived equation (5) has the form

$$ax^2 + bxy + cy^2 = 0. \quad (6)$$

Equation (6), taken with either of the original equations (1) or (2), forms a system of equations which is equivalent to the given system. The factors of the first member of equation (6) may be obtained either by inspection or by applying the general quadratic formula with respect to either x or y as an unknown. Then the solutions of the derived system of equations may be obtained by the method of factoring.

Ex. 1. Solve the system of homogeneous equations

$$\left.\begin{aligned} 2x^2 - xy + 5y^2 &= 20, \quad (1) \\ x^2 + xy + 3y^2 &= 15. \quad (2) \end{aligned}\right\} \text{I. Given System.}$$

Since the given system of equations is of the fourth order, we may expect to find four solutions.

In preparation for the elimination of the known terms 20 and 15, we may derive the equivalent system

$$\left.\begin{aligned} 6x^2 - 3xy + 15y^2 &= 20 \cdot 3, \quad (3) \\ 4x^2 + 4xy + 12y^2 &= 15 \cdot 4. \quad (4) \end{aligned}\right\} \text{II. Equivalent Derived System.}$$

Subtracting the members of equation (4) from the corresponding members of equation (3) we obtain,

$$2x^2 - 7xy + 3y^2 = 0. \quad (5)$$

Or

$$(2x - y)(x - 3y) = 0. \quad (6)$$

Since the multipliers 3 and 4, used in the derivation of equations (3) and (4), are different from zero, it follows that equation (6), taken with either of the given equations (1) or (2), forms a system of equations which is equivalent to the given system.

As an equivalent system, we may take

$$\left.\begin{aligned} x^2 + xy + 3y^2 &= 15, \quad (2) \\ (2x - y)(x - 3y) &= 0. \quad (6) \end{aligned}\right\} \text{III. Equivalent Derived System}$$

This system may be resolved into the two separate derived systems which, taken together, are equivalent to System III.

$$\left.\begin{aligned} x^2 + xy + 3y^2 &= 15, \\ 2x - y &= 0. \end{aligned}\right\} \qquad \left.\begin{aligned} x^2 + xy + 3y^2 &= 15, \\ x - 3y &= 0. \end{aligned}\right\}$$

The solutions of these systems of equations are

$$\left.\begin{aligned} x &= 1, \\ y &= 2. \end{aligned}\right\}\text{(i.)} \quad \left.\begin{aligned} x &= -1, \\ y &= -2 \end{aligned}\right\}\text{(ii.)} \quad \left.\begin{aligned} x &= +3, \\ y &= +1. \end{aligned}\right\}\text{(iii.)} \quad \left.\begin{aligned} x &= -3, \\ y &= -1. \end{aligned}\right\}\text{(iv.)}$$

By substitution, these four sets of values are all found to satisfy both of the given equations.

Ex. 2. Solve the system of equations

$$\left.\begin{aligned} x^2 + 3y^2 &= -7x, \ (1) \\ 2x^2 + xy - 6y^2 &= 4x. \ (2) \end{aligned}\right\} \text{ I. Given System.}$$

Since this is a 2–2 system, we may look for four solutions.

Observe that the equations contain no known terms, and that the terms $-7x$ and $4x$, which are the only ones below the second degree, are similar, — that is, differ only by a numerical factor. Hence we may eliminate these terms and solve the system of equations by the method of factoring.

One solution of the system may be obtained immediately by inspection. It may be seen that if y be given the value zero, the first equation reduces to $x^2 + 7x = 0$, and the second equation reduces to $2x^2 - 4x = 0$. These equations have in common the solution $x = 0$, and no other. Hence, to the value $y = 0$, in either equation, corresponds the single value $x = 0$.

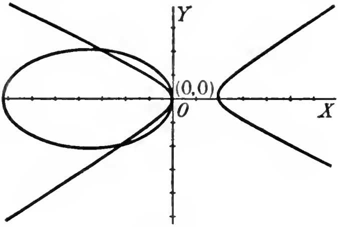


FIG. 3.

Accordingly, one solution of the given system of equations is $\left.\begin{aligned} x &= 0, \\ y &= 0. \end{aligned}\right\}$

The remaining solutions of the system may be obtained as follows by the method of factoring:

By eliminating the terms of the first degree from equations (1) and (2) we obtain the homogeneous equation

$$18x^2 + 7xy - 30y^2 = 0. \qquad (3)$$

Equation (3), taken together with either of the given equations, constitutes a system equivalent to the given system.

Accordingly we have

$$\left.\begin{aligned} 18x^2 + 7xy - 30y^2 &= 0, \quad (3) \\ x^2 + 3y^2 &= -7x. \quad (1) \end{aligned}\right\} \text{ II. Equivalent Derived System.}$$

Whenever the factors of an expression such as the first member of equation (3) are not readily obtained by inspection, we may proceed as follows:

Solving (3) as a quadratic with reference to x, y being for the moment taken as a known number, we have

$$x = \frac{-7y \pm \sqrt{49y^2 - 4(18)(-30y^2)}}{36} \qquad (4)$$

$$x = \frac{-7y \pm 47y}{36}. \qquad (5)$$

That is, $x = \frac{10y}{9}$, (6) and $x = -\frac{3y}{2}$. (7)

The factors of the first member of equation (3) may immediately be written from (6) and (7) which are the expressed values of x in terms of y.

Hence (3) becomes $(9x - 10y)(2x + 3y) = 0$. (8)

Accordingly the derived System II. may be written in the equivalent form,

$$\left.\begin{aligned} (9x - 10y)(2x + 3y) &= 0, && (8) \\ x^2 + 3y^2 &= -7x. && (1) \end{aligned}\right\} \text{III.} \quad \begin{array}{l}\text{Equivalent} \\ \text{Derived System.}\end{array}$$

The student should complete the process and obtain the remaining solutions of the given system. (See Fig. 3.)

Exercise XXV. 3

Solve the following systems of homogeneous equations:

1. $x^2 + xy = 6,$
   $xy + 6y^2 = 8.$

2. $x^2 + xy + 15 = 0,$
   $2x^2 + 3xy + y^2 = 10.$

3. $x^2 + 2xy = 39,$
   $x^2 + y^2 = 34.$

4. $x^2 - y^2 = 3,$
   $5x^2 - 4xy + 3y^2 = 15.$

5. $x^2 + xy + y^2 = 19,$
   $x^2 - xy + y^2 = 7.$

6. $5x^2 - 18xy + 16y^2 = 9,$
   $2x^2 - 5xy - 3y^2 = 12.$

7. $(2x + y)(2y + x) = 500.$
   $(x + y)(x - y) = 75.$

8. $13xy - 2x^2 - 18y^2 = 12,$
   $2x^2 - 5xy - 3y^2 = 16.$

9. $5x(x + 2y) + y^2 = 4,$
   $x^2 + 4y(x + 2y) = 16.$

10. $3x(x + 2y) + 5y^2 = 21,$
    $x^2 + 2y(x + 2y) = 28.$

11. $(2x + y)(2y + x) = \frac{5}{4},$
    $(x + y)(x - y) = \frac{3}{16}.$

12. $x + y = \frac{6}{x},$
    $x - y = \frac{1}{y}.$

Solution by Expressing the Value of One Unknown as a Multiple of the Other

23. The solution of a system of two equations which are of the second degree and homogeneous with reference to two unknowns, x and y, may be obtained by expressing the value of one unknown, as a multiple of the other.

24. *If by letting* $x = vy$, *we express the value of one of the unknowns,* x, *as a multiple of the other,* y, *we may substitute* vy *for* x *and obtain a derived system of equations in which* v *and* y *are to be regarded as the unknowns.*

By eliminating y *from the new system of equations thus obtained, we shall obtain a single equation in which* v *is the only unknown. This equation is the* v*-eliminant of the new system.*

The system of three equations consisting of the v*-eliminant, the assumed equation* $x = vy$, *and either of the equations obtained by substituting* vy *for* x *in one of the original equations, constitutes a system of equations which is equivalent to the given system of two equations.*

The values of v *found by solving the* v*-eliminant, when substituted in the remaining equations of the new system, determine the values of the unknowns* x *and* y.

25. It should be observed that, on condition that y is different from zero, we may from $x = vy$ obtain $\dfrac{x}{y} = v$.

Hence, for finite values of x, v increases in value indefinitely, — that is, it becomes infinite when y diminishes in value numerically, that is, when y approaches zero. Hence, whenever x is replaced by vy, *the solutions of which* $y = 0$ *is a part must, if they exist, be obtained separately.*

Ex. 1. Solve the system of homogeneous equations

$$\left.\begin{aligned} x^2 + 2y^2 &= 17, \quad (1) \\ xy - y^2 &= 2. \quad (2) \end{aligned}\right\} \text{I. Given System.}$$

Since this system is of the fourth order, we may expect to find four solutions.

In this particular example it will be noted that if y is given the value zero the resulting equations have no common solution with reference to x. Hence $y = 0$ is no part of a solution of the given system.

If we assume that $x = vy$, we may, by substituting vy for x in the given equations, obtain the equivalent system

$$\left.\begin{array}{rl} (vy)^2 + 2y^2 = 17, & (3)\\ (vy)y - y^2 = \ 2, & (4)\\ x = vy. & (5)\end{array}\right\}\text{ II. } \begin{array}{l}\text{Equivalent}\\ \text{Derived System.}\end{array}$$

To obtain the solutions of System II., we may proceed as follows:

From equation (3), From equation (4),

$$y^2 = \frac{17}{v^2+2}. \quad (6) \qquad\qquad y^2 = \frac{2}{v-1}. \quad (7)$$

Since the given equations are simultaneous, these "expressed" values for y^2 must be equal.

That is, $$\frac{17}{v^2+2} = \frac{2}{v-1}. \qquad (8)$$

Or, $$2v^2 - 17v + 21 = 0. \qquad (9)$$

The solutions of the v-eliminant (9) of the derived System II. are found to be $$v = \tfrac{3}{2} \text{ and } v = 7. \qquad (10)$$

Since neither of these values could have been introduced by the multipliers $v^2 + 2$ and $v - 1$, when deriving (9) from (8) they must also be roots of equation (8).

The system composed of the v-eliminant (9), the assumed equation $x = vy$ and either equation (6) or equation (7), constitutes a system of equations which is equivalent to System II.

$$\left.\begin{array}{rl} 2v^2 - 17v + 21 = \ 0, & (9)\\ y^2 = \dfrac{2}{v-1}, & (7)\\ x = vy. & (5)\end{array}\right\}\text{III. } \begin{array}{l}\text{Equivalent}\\ \text{Derived System.}\end{array}$$

Substituting the solutions of equation (9), $v = 3/2$ and $v = 7$, in equation (7), we obtain:

Substituting 3/2 for v, Substituting 7 for v,

$$y = \pm 2. \qquad\qquad y = \pm \tfrac{1}{3}\sqrt{3}.$$

To find the corresponding values of x, we may either substitute these values of y in the given equations (1) or (2), or we may substitute corresponding values of v and y in $x = vy$.

Substituting	Substituting	Substituting	Substituting
$\begin{cases} v = \frac{3}{2}, \\ y = 2. \end{cases}$	$\begin{cases} v = \frac{3}{2}, \\ y = -2. \end{cases}$	$\begin{cases} v = 7, \\ y = \frac{1}{3}\sqrt{3}. \end{cases}$	$\begin{cases} v = 7, \\ y = -\frac{1}{3}\sqrt{3}. \end{cases}$
We find that		We find that	
$x = (\frac{3}{2})\,2$	$x = \frac{3}{2}(-2)$	$x = 7(\frac{1}{3}\sqrt{3})$	$x = 7(-\frac{1}{3}\sqrt{3})$
$x = 3.$	$x = -3.$	$x = \frac{7}{3}\sqrt{3}.$	$x = -\frac{7}{3}\sqrt{3}.$

The four solutions of the given system are thus found to be

$$\left.\begin{aligned}x&=3,\\ y&=2.\end{aligned}\right\}\text{(i.)}\qquad \left.\begin{aligned}x&=-3,\\ y&=-2.\end{aligned}\right\}\text{(ii.)}\qquad \left.\begin{aligned}x&=\tfrac{7}{3}\sqrt{3},\\ y&=\tfrac{1}{3}\sqrt{3}.\end{aligned}\right\}\text{(iii).}\qquad \left.\begin{aligned}x&=-\tfrac{7}{3}\sqrt{3}.\\ y&=-\tfrac{1}{3}\sqrt{3}.\end{aligned}\right\}\text{(iv.)}$$

These four sets of values are found to satisfy the equations of the given system.

By extracting the square root of 3, we may obtain from (iii.) and (iv.) approximate values of x and y, correct to any required number of significant figures.

Ex. 2. Solve the system of homogeneous equations

$$\left.\begin{aligned}x^2+xy-y^2&=\quad 29, &(1)\\ 2x^2-xy-y^2&=-19. &(2)\end{aligned}\right\}\text{ I. Given System.}$$

Since this is a 2–2 system, we may expect to find four solutions.

By assigning the value zero to y, it may be seen that the corresponding values of x in the two equations are not equal. Accordingly, it follows that $y=0$ is no part of any solution of the given system.

Hence we may assume that $x=vy$, and substituting vy for x in the given equations, we obtain the equivalent derived system

$$\left.\begin{aligned}v^2y^2+vy^2-y^2&=\quad 29, &(3)\\ 2v^2y^2-vy^2-y^2&=-19, &(4)\\ x&=\quad vy. &(5)\end{aligned}\right\}\text{II. Equivalent Derived System.}$$

Observe that, by dividing the corresponding members of the two equations (3) and (4), the unknown y^2 may be eliminated, and we may immediately obtain the v-eliminant of System II.

$$\frac{v^2+v-1}{2v^2-v-1}=-\tfrac{29}{19}. \qquad (6)$$

Hence $77v^2-10v-48=0.$ (7)

The solutions of (7) are found to be

$$v=\tfrac{6}{7},\quad\text{and}\quad v=-\tfrac{8}{11}. \qquad (8)$$

The system composed of the v-eliminant (7), either one of the equations (3) or (4) in System II., and the assumed equation (5), constitutes a system equivalent to System II.

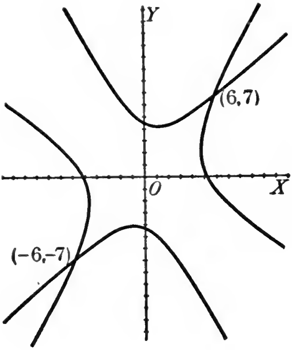


FIG. 4.

$$\left.\begin{aligned}77v^2-10v-48&=\ 0, &(7)\\ v^2y^2+vy^2-y^2&=29, &(3)\\ x&=vy. &(5)\end{aligned}\right\}\text{III. Equivalent Derived System.}$$

Substituting the solutions $v = \frac{6}{7}$ and $v = -\frac{8}{11}$ of (8) in (3), we obtain the values for y.

Substituting $\frac{6}{7}$ for v,
We find $y^2 = 49$.
Or, $y = \pm 7$.

Substituting $-\frac{8}{11}$ for v,
We find $-5y^2 = 121$.
Or, $y = \pm \frac{11}{5}\sqrt{-5}$.

The values of x may be found by substituting corresponding values of v and y in the assumed equation $x = vy$ (5), as follows:

Substituting $\begin{cases} v = \frac{6}{7}, \\ y = \pm 7. \end{cases}$ Substituting $\begin{cases} v = -\frac{8}{11}, \\ y = \pm \frac{11}{5}\sqrt{-5}. \end{cases}$

We find $x = \pm 6$ and $x = \mp \frac{8}{5}\sqrt{-5}$.

Accordingly the solutions of the given system of equations are:

$$\left.\begin{aligned} x &= +6, \\ y &= +7. \end{aligned}\right\} \text{(i.)} \qquad \left.\begin{aligned} x &= -\tfrac{8}{5}\sqrt{-5}, \\ y &= +\tfrac{11}{5}\sqrt{-5}. \end{aligned}\right\} \text{(iii.)}$$

$$\left.\begin{aligned} x &= -6, \\ y &= -7. \end{aligned}\right\} \text{(ii.)} \qquad \left.\begin{aligned} x &= +\tfrac{8}{5}\sqrt{-5}, \\ y &= -\tfrac{11}{5}\sqrt{-5}. \end{aligned}\right\} \text{(iv.)}$$

By substitution these values are all found to satisfy the given equations. Since we have found four sets of values, it appears that, in deriving the v-eliminant (7), no solutions were lost.

By referring to Fig. 4, it will be seen that the graphs of equations (1) and (2) intersect in but two points, the coördinates of which are $x = 6$, $y = 7$, and $x = -6$, $y = -7$.

We must accordingly interpret the imaginary values (iii.) and (iv.) of x and y as indicating that the graphs have no points of intersection the coördinates of which are these solutions.

EXERCISE XXV. 4

Solve the following systems of homogeneous equations:

1. $3x^2 + 2xy + 3y^2 = 88,$
   $2x^2 - 3xy + 2y^2 = 37.$
2. $x^2 + 4xy + y^2 = -11,$
   $7x^2 - 3y^2 = 51.$
3. $3x^2 - 2xy + y^2 = \frac{1}{4},$
   $2x^2 + xy + 3y^2 = \frac{41}{36}.$
4. $x^2 - xy + y^2 = 93,$
   $x^2 + 2xy = -40.$
5. $2xy + y^2 = 51,$
   $3x^2 - xy = 126.$
6. $3x^2 + 10xy + 3y^2 = -21,$
   $x^2 - y^2 = 5.$
7. $x^2 - 3xy = 10,$
   $5xy - 13y^2 = 33.$
8. $x^2 + 3xy - y^2 = 29,$
   $7x^2 - 2y^2 = 13.$
9. $3x(x - 3y) + y^2 = -11,$
   $x^2 + 2y(x - 3y) = 21.$
10. $2x^2 - 7y(x - y) = 23,$
    $3x(x - 2y) - 5y^2 = 3.$

IV. Reduction of Systems of Equations by Division

26. Representing by A, B, C, and D expressions which are integral with reference to two unknowns, x and y, it may be seen that *if the members $A \cdot C$ and $B \cdot D$ of one equation* (1) *of a System* I., *composed of two equations, contain as factors the corresponding members A and B of the remaining equation* (2) *of the system, then the given System* I. *is equivalent to the derived double system* (i.) *and* (ii.).

That is,

$$\left.\begin{array}{rl} A \cdot C = B \cdot D, & (1) \\ A = B, & (2) \end{array}\right\} \text{Given System.}$$

is equivalent to the double system

$$\left.\begin{array}{rl} C = D, & (3) \\ A = B, & (2) \end{array}\right\} \text{(i.)} \qquad \left.\begin{array}{rl} A = 0, & (4) \\ B = 0. & (5) \end{array}\right\} \text{(ii.)}$$

It should be observed that the derived equation $C = D$ (3) is obtained by dividing the members of equation (1) by the corresponding members of (2), while the remaining equation (2) of the given System I. is carried over unchanged into the derived system (i.).

The remaining system (ii.) is composed of the equations formed by equating to zero separately the factors A and B which are common to the corresponding members of equations (1) and (2) of the given system.

The Principle may be established as follows:

Substituting for B in (1) the equal value A, we obtain the equivalent equation $A \cdot C = A \cdot D$.

Or, $A \cdot C - A \cdot D = 0.$ (6)

Factoring, $A(C - D) = 0.$ (7)

Accordingly, from System I. we may obtain the equivalent system

$$\left.\begin{array}{rl} A(C - D) = 0, & (7) \\ A - B = 0. & (8) \end{array}\right\} \text{II.} \quad \begin{array}{l} \text{Equivalent} \\ \text{Derived System.} \end{array}$$

By the principle of § 16, this single system is equivalent to the derived double system

$$\left.\begin{array}{rl} C - D = 0, & (9) \\ A - B = 0, & (8) \end{array}\right\} \quad \text{and} \quad \left.\begin{array}{rl} A = 0, & (10) \\ A - B = 0. & (8) \end{array}\right\}$$

These systems in turn are equivalent to

$$\left.\begin{array}{rl} C = D, & (3) \\ A = B, & (2) \end{array}\right\} \text{(i.)} \quad \text{and} \quad \left.\begin{array}{rl} A = 0, & (4) \\ B = 0. & (5) \end{array}\right\} \text{(ii.)}$$

Ex. 1. Solve the system of equations

$$\left.\begin{aligned} 2x^2 - x &= y^2 - 1, \quad (1) \\ x &= y + 1. \quad (2) \end{aligned}\right\} \text{I.} \quad \text{Given System.}$$

Since the members of equation (2) are contained as factors in the corresponding members of equation (1), the given single System I. is equivalent to the derived double system

$$\left.\begin{aligned} 2x - 1 &= y - 1, \quad (3) \\ x &= y + 1, \quad (2) \end{aligned}\right\} \text{(i.)} \qquad \text{and} \qquad \left.\begin{aligned} x &= 0, \quad (4) \\ y + 1 &= 0. \quad (5) \end{aligned}\right\} \text{(ii.)}$$

Equation (3) is formed by dividing the members of equation (1) by the corresponding members of equation (2). Equations (4) and (5) are obtained by equating separately to zero the members of equation (2) which are contained as factors in the corresponding members of equation (1).

The solutions of the derived systems (i.) and (ii.) are

$$\left.\begin{aligned} x &= -1, \\ y &= -2, \end{aligned}\right\} \qquad \text{and} \qquad \left.\begin{aligned} x &= 0, \\ y &= -1. \end{aligned}\right\}$$

The number of sets of values thus obtained is equal to the order, 2, of the given system. By substitution these sets of values will be found to satisfy both of the given equations.

27. Whenever one of the expressions represented by A or B (see § 26) is a known number, it follows that the derived system (ii.) $A = 0$, (4), $B = 0$, (5), will have no finite solutions.

Ex. 2. Solve the system of equations

$$\left.\begin{aligned} x^3 - y^3 &= 26, \quad (1) \\ x - y &= 2. \quad (2) \end{aligned}\right\} \text{I.} \quad \text{Given System.}$$

Dividing the members of equation (1) by the corresponding members of equation (2), we may derive the equivalent system

$$\left.\begin{aligned} x^2 + xy + y^2 &= 13, \quad (3) \\ x - y &= 2. \quad (2) \end{aligned}\right\} \text{II.} \quad \begin{array}{l}\text{Equivalent} \\ \text{Derived System.}\end{array}$$

Observe that by separately equating to zero the factors $x - y$ and 2, which are common to the corresponding members of equations (1) and (2), we would have as one of the expected conditional equations a known number equal to zero, that is, $2 = 0$.

Hence the expected derived "double" system reduces to a single System II., equivalent to the one given.

Accordingly, although the given system of equations is of the third order, the number of finite solutions does not exceed the order of the derived equivalent System II., that is, there will be but two sets of values.

The solutions of the derived System II. are found to be

$$\left.\begin{aligned} x &= 3, \\ y &= 1, \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} x &= -1, \\ y &= -3, \end{aligned}\right\}$$

both of which satisfy each of the given equations.

EXERCISE XXV. 5

Solve each of the following systems of equations :

1. $x^3 = y(x + y),$
   $x^2 = x + y.$

2. $1 + y = x,$
   $1 + y^3 = x^3.$

3. $x(x - 3) = 4 - y^2,$
   $x = 2 + y.$

4. $x(y + 3) = 9y^2 - 1,$
   $x = 3y - 1.$

5. $x(x - a) = b^2 - y^2,$
   $x - a = b + y.$

6. $x^2 - y^2 = 77,$
   $x - y = 7.$

7. $x + y = 14,$
   $x^2 - y^2 = 56.$

8. $x^3 + y^3 = 72,$
   $x + y = 6.$

9. $x^3 - y^3 = 7,$
   $x - y = 2.$

10. $x^3 - y^3 = a^3 - b^3,$
    $x - y = a - b.$

11. $x^3 + y^3 = 91,$
    $x^2 - xy + y^2 = 13.$

12. $x^3 - y^3 = 19,$
    $x^2 + xy + y^2 = 19.$

13. $27x^3 - y^3 = 0,$
    $9x^2 + 3xy + y^2 = 243.$

14. $x^4 + x^2y^2 + y^4 = 21,$
    $x^2 - xy + y^2 = 3.$

V. Systems of Symmetric Equations

28. An equation is said to be **symmetric** with respect to the unknowns appearing in it, when its members remain unaltered in value if any two of its unknowns are interchanged.

29. The necessary and sufficient condition that an equation be symmetric with respect to two unknowns, x and y, is that the coefficients of like powers of the unknowns be equal.

E. g. The following equations are symmetric with respect to x and y :

$$\begin{aligned} x + y &= 5, & \qquad x^2 - xy + y^2 &= 7, \\ x^2 + y^2 &= 10, & \qquad x^2y^2 + xy + 1 &= 0. \end{aligned}$$

30. A system of two equations is said to be symmetric with respect to two unknowns, x and y, if the equations obtained by interchanging x and y are identical with those given.

E.g.
$$\left.\begin{array}{r}x^2+y=a,\\ y^2+x=a.\end{array}\right\}\qquad \left.\begin{array}{r}x^2=3x+5y,\\ y^2=3y+5x.\end{array}\right\}$$

31. It follows from the principles of symmetry that if any solution of a system of two symmetric equations containing two unknowns, x and y, be represented by $x=a$, $y=b$, then $x=b$, $y=a$, will also be a solution.

Ex. 1. Solve the system of symmetric equations

$$\left.\begin{array}{rl}x+y=4, & (1)\\ xy=3. & (2)\end{array}\right\}\text{ I. Given System.}$$

Observe that both equations are symmetric with respect to x and y, and that the first member of equation (1) is the sum of x and y.

By combining the given equations in such a way as to obtain the difference between x and y, it will be possible to make the solution of the given system of equations depend upon the solution of a system of two linear equations.

Squaring the members of (1), $\quad x^2+2xy+y^2=16. \quad (3)$

Multiplying members of (2) by 4, $\quad +4xy \quad =12. \quad (4)$

By subtraction, $\quad x^2-2xy+y^2= \ 4. \quad (5)$

The given System I. may be replaced by the following equivalent derived system of the same order:

$$\left.\begin{array}{rl}x+y=4, & (1)\\ x^2-2xy+y^2=4. & (5)\end{array}\right\}\text{ II. Equivalent Derived System.}$$

This derived system is equivalent to

$$\left.\begin{array}{rl}x+y=4, & (1)\\ (x-y)^2=4. & (6)\end{array}\right\}\text{ III. Equivalent Derived System.}$$

System III. is equivalent to the set of two derived systems,

$$\left.\begin{array}{r}x+y=4,\\ x-y=2,\end{array}\right\}\text{(i.)}\quad\text{and}\quad\left.\begin{array}{r}x+y=4,\\ x-y=-2.\end{array}\right\}\text{(ii.)}$$

The solutions of these systems are

$$\left.\begin{array}{r}x=3,\\ y=1,\end{array}\right\}\quad\text{and}\quad\left.\begin{array}{r}x=1,\\ y=3.\end{array}\right\}$$

Both of these sets of values satisfy each of the equations of the given system of the second order. (See Fig. 5.)

If with the derived equation (5) we had used the given equation of the second degree (2) instead of the equation of lower degree (1) to form the derived system, we would have passed from the given 1–2 system to a derived 2–2 system, and accordingly the two systems would not have been equivalent.

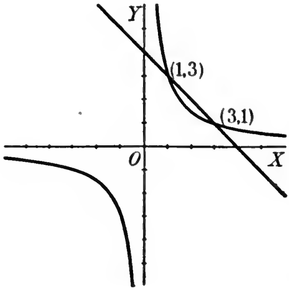


FIG. 5.

The derived 2–2 system would have been found to contain, besides the solutions of the given system of the second order, the two additional solutions

$$x = -3,\ y = -1, \text{ and } x = -1,\ y = -3.$$

These would have been introduced during the process of solution by squaring the members of equation (1).

Ex. 2. Solve the system of symmetric equations

$$\left.\begin{aligned} x^2 + y^2 &= 25, \ (1) \\ x + y &= \ \ 1. \ (2) \end{aligned}\right\} \text{I. Given System.}$$

Since this is a 2–1 system we may expect to obtain two sets of values which satisfy both equations.

We will first obtain the value of the difference $x - y$.

Squaring the members of equation (2) and subtracting from the corresponding members of equation (1), we obtain the equation $-2xy = 24$. (3)

Combining the corresponding members of equations (1) and (3) by addition, we obtain

$$x^2 - 2xy + y^2 = 49. \qquad (4)$$

The given system is equivalent to the derived system

$$\left.\begin{aligned} x^2 - 2xy + y^2 &= 49, \quad (4) \\ x + y &= \ \ 1. \quad (2) \end{aligned}\right\} \text{II. Equivalent Derived System.}$$

This single system is equivalent to the system

$$\left.\begin{aligned} (x - y)^2 &= 49, \quad (5) \\ x + y &= \ \ 1. \quad (2) \end{aligned}\right\} \text{III. Equivalent Derived System.}$$

This last system is equivalent to the set of two systems

$$\left.\begin{aligned} x - y &= +7, \\ x + y &= \ \ 1, \end{aligned}\right\} \text{(i.) and } \quad \left.\begin{aligned} x - y &= -7, \\ x + y &= \ \ 1. \end{aligned}\right\} \text{(ii.)}$$

The solutions of these systems are

$$\left.\begin{aligned} x &= \ \ 4, \\ y &= -3, \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} x &= -3, \\ y &= \ \ 4. \end{aligned}\right\}$$

These sets of values are found to satisfy each of the given equations. (See Fig. 6.)

Ex. 3. Solve the system of symmetric equations

$$\left.\begin{aligned} x^2+y^2&=100, \quad (1)\\ xy&=\ \ 48. \quad (2)\end{aligned}\right\}\text{ I. Given System.}$$

We may expect to obtain four solutions, since this is a 2–2 system.

To obtain the solutions we will find the sum and also the difference of the unknowns, as follows:

Multiplying the members of equation (2) by 2, then combining with equation (1) by addition and subtraction successively, we obtain

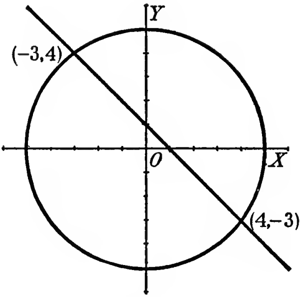


Fig. 6.

$$\left.\begin{aligned} x^2+2\,xy+y^2&=196, \quad (3)\\ x^2-2\,xy+y^2&=\ \ \ \ 4. \quad (4)\end{aligned}\right\}\text{ II. Equivalent Derived System.}$$

Equation (3) may be written in the form $(x+y)^2-196=0$, which is equivalent to $(x+y+14)(x+y-14)=0$, (5). Similarly, equation (4) may be written in the form

$$(x-y)^2-4=0,$$

which is equivalent to

$$(x-y+2)(x-y-2)=0, \quad (6).$$

Accordingly System II. is equivalent to the following derived system:

$$\left.\begin{aligned} (x+y+14)(x+y-14)&=0, \quad (5)\\ (x-y+\ \ 2)(x-y-\ \ 2)&=0. \quad (6)\end{aligned}\right\}\text{ III.}$$

Equivalent Derived System.

System III. is equivalent to the roup of systems of equations

$$\left.\begin{aligned} x+y&=14,\\ x-y&=\ \ 2.\end{aligned}\right\}\text{ (i.)}$$

$$\left.\begin{aligned} x+y&=\ \ 14,\\ x-y&=-2.\end{aligned}\right\}\text{ (ii.)}\qquad \left.\begin{aligned} x+y&=-14,\\ x-y&=\ \ \ \ 2.\end{aligned}\right\}\text{ (iii.)}\qquad \left.\begin{aligned} x+y&=-14,\\ x-y&=-\ \ 2.\end{aligned}\right\}\text{ (iv.)}$$

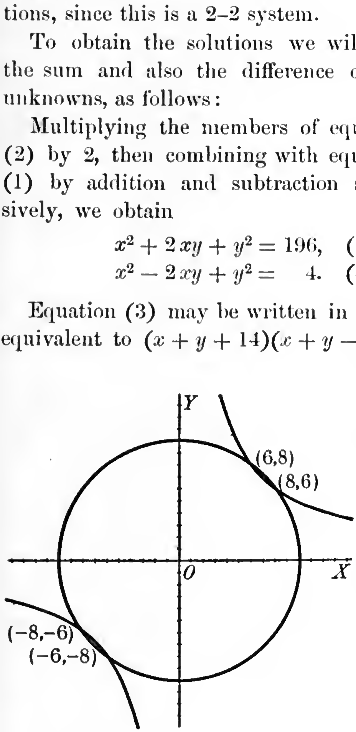


Fig. 7.

From these systems of equations we obtain the following solutions, which are numbered to correspond to the systems from which they are obtained. (See Fig. 7.)

$$\left.\begin{aligned}x&=8,\\ y&=6.\end{aligned}\right\}\text{(i.)}\qquad \left.\begin{aligned}x&=6,\\ y&=8.\end{aligned}\right\}\text{(ii.)}\qquad \left.\begin{aligned}x&=-6,\\ y&=-8.\end{aligned}\right\}\text{(iii.)}\qquad \left.\begin{aligned}x&=-8,\\ y&=-6.\end{aligned}\right\}\text{(iv.)}$$

By substitution these values are all found to satisfy the given equations.

Exercise XXV. 6

Solve the following systems of symmetric equations:

1. $x^2 + y^2 = 41,\quad xy = 20.$

2. $x^2 + y^2 = 61,\quad x + y = 11.$

3. $x^2 + y^2 = 53,\quad x + y = 9.$

4. $x^2 + y^2 = 73,\quad x + y = 11.$

5. $x + y = 15,\quad xy = 56.$

6. $x^3 + y^3 = 126,\quad x + y = 6.$

7. $x^2 + xy + y^2 = 73,\quad xy = 8.$

8. $x^2 - xy + y^2 = \frac{67}{36},\quad xy = \frac{1}{2}.$

9. $x^2 + xy + y^2 = 61,\quad x + y = 9.$

10. $x^2 + xy + y^2 = 37,\quad x + y = -7.$

11. $x^2 + y^2 = b,\quad x + y = a.$

12. $x + y = a,\quad xy = b.$

13. $\dfrac{1}{x} + \dfrac{1}{y} = 11.\quad \dfrac{1}{x^2} + \dfrac{1}{y^2} = 73.$

14. $\dfrac{1}{x} + \dfrac{1}{y} = -4,\quad \dfrac{1}{xy} = -45.$

15. $\dfrac{1}{x} + \dfrac{1}{y} = 1,\quad xy = -\dfrac{1}{132}.$

16. $\dfrac{1}{x^3} + \dfrac{1}{y^3} = 72,\quad \dfrac{1}{x} + \dfrac{1}{y} = 6.$

17. $\dfrac{1}{x^2} + \dfrac{1}{y^2} = 29,\quad xy = \dfrac{1}{10}.$

18. $xy(x + y) = 30,\quad \dfrac{x + y}{xy} = \dfrac{5}{6}.$

19. $x^4 + x^2y^2 + y^4 = 21,\quad x^2 - xy + y^2 = 3.$

20. $x^2 + xy + y^2 = \frac{13}{4}$,

$x^4 + x^2y^2 + y^4 = \frac{273}{16}$.

21. $x^2 + xy + y^2 = 84$,
$x + \sqrt{xy} + y = 14$.

22. $x^2 + xy + y^2 = 133$.
$x - \sqrt{xy} + y = 7$.

23. $(x + 1)(y + 1) = \frac{25}{6}$,
$xy = 1$.

24. $\frac{1}{x} + \frac{1}{y} = \frac{4}{3}$,

$\frac{1}{x+1} + \frac{1}{y+1} = \frac{3}{4}$.

Solutions by Special Devices

32. Certain systems of equations are of such special forms that special methods must be employed to obtain their solutions.

Ex. 1. Solve the symmetric system of equations

$$\left.\begin{aligned} x^2 &= 3x + 5y, \quad (1) \\ y^2 &= 3y + 5x. \quad (2) \end{aligned}\right\} \text{I. Given System.}$$

By adding the corresponding members of equations (1) and (2), we obtain $x^2 + y^2 = 8(x + y)$, (3) ; and by subtracting the members of equation (2) from the corresponding members of equation (1), we obtain $x^2 - y^2 = 2(y - x)$, (4).

Hence, the given system of equations is equivalent to the following derived system:

$$\left.\begin{aligned} x^2 + y^2 &= 8(x + y), \quad (3) \\ x^2 - y^2 &= 2(y - x). \quad (4) \end{aligned}\right\} \text{II. Equivalent Derived System.}$$

Transposing the terms of equation (4) to the first member and factoring, we obtain the equivalent derived equation

$$(x + y + 2)(x - y) = 0. \qquad (5)$$

Accordingly, equations (3) and (5) taken together constitute the following system of equations which is equivalent to the given system :

$$\left.\begin{aligned} x^2 + y^2 &= 8(x + y), \quad (3) \\ (x + y + 2)(x - y) &= 0. \quad (5) \end{aligned}\right\} \text{III. Equivalent Derived System.}$$

The solutions of System III. may be obtained by the method of § 16.

Ex. 2. Solve the symmetric system of equations

$$\left.\begin{aligned} x^2y^2 + xy - 12 &= 0, \quad (1) \\ x^2 + y^2 &= 10. \quad (2) \end{aligned}\right\} \text{I. Given System.}$$

Solve the first equation for xy as the unknown by factoring. Then, using the values thus found with equation (2), solve the two resulting symmetric systems.

Ex. 3. Solve the symmetric system of equations

$$\left.\begin{array}{r}x^2+y^2+x+y=74, \quad (1)\\ xy=\ \ 8. \quad (2)\end{array}\right\}\text{ I. Given System.}$$

By multiplying the members of equation (2) by 2 and adding the results thus obtained to the corresponding members of equation (1), we obtain the derived equation

$$(x+y)^2+x+y=90. \tag{3}$$

Hence the given system of equations is equivalent to the system

$$\left.\begin{array}{r}(x+y+10)(x+y-9)=0, \quad (4)\\ xy=8. \quad (5)\end{array}\right\}\text{ II. }\begin{array}{l}\text{Equivalent}\\ \quad\text{Derived System.}\end{array}$$

This system of equations is equivalent to the following set of two systems:

$$\left.\begin{array}{r}x+y+10=0,\\ xy=8.\end{array}\right\}\text{(i.)} \qquad \left.\begin{array}{r}x+y-9=0,\\ xy=8.\end{array}\right\}\text{(ii.)}$$

The solutions of these systems of equations may be obtained by applying the method of § 31.

33. Systems consisting of equations which become symmetric by changing the signs of one or more terms may be solved by the methods employed for the solution of systems of symmetric equations.

Ex. 4. Solve the system of equations

$$\left.\begin{array}{r}x-y=\ \ 6, \quad (1)\\ xy=16. \quad (2)\end{array}\right\}\text{ I. Given System.}$$

Multiplying the members of equation (2) by 4 and adding the results to the squares of the corresponding members of equation (1), we obtain the equation $(x+y)^2=100$, (3).

Accordingly, the given system of equations is equivalent to the system

$$\left.\begin{array}{r}(x+y)^2-100=0, \quad (3)\\ x-y=6. \quad (1)\end{array}\right\}\text{ II. }\begin{array}{l}\text{Equivalent}\\ \quad\text{Derived System.}\end{array}$$

This system of equations is equivalent to the following set of two systems of equations, the solutions of which should be obtained by the student:

$$\left.\begin{array}{r}x+y+10=0,\\ x-y=6.\end{array}\right\}\text{(i.)} \qquad \left.\begin{array}{r}x+y-10=0,\\ x-y=6.\end{array}\right\}\text{(ii.)}$$

34. A system of equations having the forms

$$\left.\begin{array}{r}ax+by=c,\\ dxy=e,\end{array}\right\}$$

may be solved by the methods employed for solving systems of symmetric equations.

Ex. 5. Solve the system of equations

$$\left.\begin{aligned} 3x+2y&=11, \quad (1)\\ xy&=3. \quad (2)\end{aligned}\right\}\text{ I. Given System.}$$

Since the first member of equation (1) is the binomial sum $3x+2y$ we proceed as follows to obtain an equation the first member of which is the binomial difference $3x-2y$.

By squaring both members of equation (1) we obtain

$$9x^2+12xy+4y^2=121. \qquad (3)$$

Since the square of the difference $3x-2y$ differs from the first member of equation (3), only in the sign of its "middle term" $12xy$, we may, by subtracting from the members of equation (3) the corresponding members of equation (2), each multiplied by 24, obtain

$$9x^2-12xy+4y^2=49. \qquad (4)$$

Equation (4) is equivalent to

$$(3x-2y+7)(3x-2y-7)=0. \qquad (5)$$

It follows that the given system of equations is equivalent to the following system:

$$\left.\begin{aligned} 3x+2y&=11, \quad (1)\\ (3x-2y+7)(3x-2y-7)&=0. \quad (5)\end{aligned}\right\}\text{ II. Equivalent Derived System.}$$

The solutions of this system of equations may be obtained by applying the method of § 16.

35. The solutions of a system of two symmetric equations containing two unknowns, x and y, may be obtained by solving the system of equations obtained by substituting particular functions of new unknowns for the given set of unknowns, x and y, and solving the resulting equations for the new unknowns.

It will sometimes be found convenient to substitute for the given unknowns, x and y, the sum and difference respectively of two other unknowns, r and s, that is, to let $x=r+s$ and $y=r-s$.

Solving the resulting system of equations, we obtain values for r and s which, when substituted in the assumed equations $x=r+s$ and $y=r-s$, determine the values of the unknowns, x and y.

Ex. 6. Solve the system of symmetric equations

$$\left.\begin{aligned} x^4 + y^4 &= 17, \quad (1) \\ x + y &= 3. \quad (2) \end{aligned}\right\} \text{I. Given System.}$$

If we let $\left.\begin{aligned} x &= r + s, \\ y &= r - s, \end{aligned}\right\}$ (3) we may substitute for the given system of equations (1) and (2) the equivalent derived system

$$\left.\begin{aligned} (r+s)^4 + (r-s)^4 &= 17, && (4) \\ 2r &= 3, && (5) \\ \left.\begin{aligned} x &= r + s, \\ y &= r - s. \end{aligned}\right\} & && (3) \end{aligned}\right\} \text{II. Equivalent Derived System.}$$

Eliminating r from equations (4) and (5), we obtain,

$$16s^4 + 216s^2 - 55 = 0. \qquad (6)$$

The system of equations consisting of equations (6) and (5), and the assumed equations (3) is equivalent to the given system.

The solutions of equation (6) are found to be

$$s = \pm\tfrac{1}{2}, \qquad \text{and} \qquad s = \pm\tfrac{1}{2}\sqrt{-55}.$$

Accordingly, using each of these values with the value of r from (5), we have the following pairs of values for r and s:

$$\left.\begin{aligned} r &= \tfrac{3}{2}, \\ s &= \tfrac{1}{2}. \end{aligned}\right\} \text{(i.)} \quad \left.\begin{aligned} r &= \tfrac{3}{2}, \\ s &= -\tfrac{1}{2}. \end{aligned}\right\} \text{(ii.)} \quad \left.\begin{aligned} r &= \tfrac{3}{2}, \\ s &= +\tfrac{1}{2}\sqrt{-55}. \end{aligned}\right\} \text{(iii.)} \quad \left.\begin{aligned} r &= \tfrac{3}{2}, \\ s &= -\tfrac{1}{2}\sqrt{-55}. \end{aligned}\right\} \text{(iv.)}$$

Substituting the sets of values (i.) and (ii.) for r and s in the assumed equations (3), we obtain the following sets of real values of x and y:

$$\left.\begin{aligned} x &= 2, \\ y &= 1, \end{aligned}\right\} \qquad \text{and} \qquad \left.\begin{aligned} x &= 1, \\ y &= 2. \end{aligned}\right\}$$

The values of x and y obtained by using the real and imaginary values for r and s in (iii.) and (iv.) are complex.

All of the values thus obtained will be found to satisfy the given equations.

Ex. 7. Solve the system of equations

$$\left.\begin{aligned} x^3 - y^3 &= 26, \quad (1) \\ x + y &= 4. \quad (2) \end{aligned}\right\} \text{I. Given System.}$$

Let $\qquad x = r + s, \qquad$ and $\qquad y = r - s. \quad$ (3)

Using the equations obtained by substituting $r + s$ for x and $r - s$ for y in equations (1) and (2) with the assumed equations $x = r + s$ and $y = r - s$, we obtain the following derived system of equations which is equivalent to the given system of equations:

$$\left.\begin{array}{rl} 3r^2s + s^3 = 13, & (4) \\ r = 2, & (5) \\ \left.\begin{array}{r} x = r + s, \\ y = r - s. \end{array}\right\} & (3) \end{array}\right\} \text{II.} \quad \begin{array}{l}\text{Equivalent} \\ \quad\text{Derived System.}\end{array}$$

From equations (4) and (5) we obtain, eliminating r, and transposing,

$$s^3 + 12s - 13 = 0. \tag{6}$$

By the Factor Theorem we find that $s - 1$ is one of the factors of the first member of equation (6).

Accordingly equation (6) is equivalent to

$$(s - 1)(s^2 + 5s + 13) = 0. \tag{7}$$

The solutions of (7) are found to be

$$s = 1, \qquad \text{and} \qquad s = \frac{-1 \pm \sqrt{-51}}{2}.$$

Using the value of r from (5), we obtain the following pairs of values for r and s:

$$\left.\begin{array}{l} r = 2, \\ s = 1. \end{array}\right\}\text{(i.)} \quad \left.\begin{array}{l} r = 2, \\ s = \dfrac{-1 + \sqrt{-51}}{2}. \end{array}\right\}\text{(ii.)} \quad \left.\begin{array}{l} r = 2, \\ s = \dfrac{-1 - \sqrt{-51}}{2}. \end{array}\right\}\text{(iii.)}$$

Substituting these different pairs of values for r and s in the assumed equations (3), we obtain pairs of values of x and y which are solutions of the given system of equations.

EXERCISE XXV. 7. Miscellaneous

Solve the following systems of equations:

1. $x = 6 - y,$
   $y^2 = -20 - xy.$

2. $x^2 - 9y^2 = 16,$
   $x + 3y = 8.$

3. $x + y = x^2,$
   $3y - x = y^2.$

4. $x^2 + y^2 = 178,$
   $x - y = 10.$

5. $(x + y)^2 = 81,$
   $(x - y)^2 = 9.$

6. $x^2 + 2xy + y^2 = 36,$
   $x^2 - 2xy + y^2 = 16.$

7. $(x - 1)(y + 1) = 39,$
   $x - y = 12.$

8. $x - 3y = 5,$
   $xy = 4x^2 - 2.$

9. $\dfrac{x}{y} = 3,$
   $xy = 12.$

10. $xy = 16,$
$\frac{y}{x} = 4.$

11. $xy = a^2,$
$\frac{x}{y} = b^2.$

12. $\frac{x+y}{y} = 5,$
$xy = 64.$

13. $\frac{x+y}{y} = a,$
$xy = b.$

14. $x + \frac{1}{y} = 1,$
$y + \frac{1}{x} = 4.$

15. $9x + \frac{1}{y} = 4 = 15x - \frac{1}{y}.$

16. $\frac{36x}{y} = \frac{25y}{x},$
$xy + x - y = 122.$

17. $\frac{1}{x} - \frac{1}{y} = 3,$
$\frac{1}{x-1} + \frac{1}{y-1} = -\frac{5}{3}.$

18. $\frac{1}{x^2} + \frac{1}{y^2} = \frac{50}{49},$
$\frac{1}{x} + \frac{1}{y} = \frac{8}{7}.$

19. $x^{-1} - y^{-1} = -1,$
$x^{-2} - y^{-2} = -5.$

20. $y(x+y) = x^2,$
$x(x-y) = y.$

21. $(x+1)(y+1) = 54,$
$x + y = 13.$

22. $x^2 + xy = 28,$
$xy + y^2 = -12.$

23. $x^2 - y^2 = 7,$
$xy = 12.$

24. $x(x+y) - 20 = 0,$
$y(y+x) - 16 = 0.$

25. $x^2 + xy = 42,$
$y(x-y) = 5.$

26. $\frac{x+y}{x-y} + \frac{x-y}{x+y} = \frac{10}{3},$
$x^2 + y^2 = 45.$

27. $x^4 + x^2y^2 + y^4 = 3,$
$x^2 - xy + y^2 = 1.$

28. $x^2 + xy + y^2 = 133,$
$x + \sqrt{xy} + y = 19.$

29. $3x^2y^2 - xy = 14,$
$x + y = -1.$

30. $x^2y^2 + xy = 20,$
$x + y = 5.$

31. $x^2 + y^2 = 189 - xy,$
$x + y = 9 + \sqrt{xy}.$

32. $x^3 + y^3 = 133,$
$x^2y + xy^2 = 70.$

33. $x^3 + y^3 = 65,$
$xy(x+y) = 20.$

34. $x^3 - y^3 = 26,$
$x^2y - xy^2 = 6.$

35. $x^3 - y^3 = 56,$
$x - y = \frac{16}{xy}.$

36. $\frac{x-y}{x} - \frac{y}{x+y} = 0,$
$x + y = 1.$

37. $y(x+y) + x(y+x) = 4\,xy,$
$y(x+y) + x + y = 24.$

38. $\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}} + \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}} = \frac{26}{5},$
$x^2 + y^2 = 97.$

39. $x + y\sqrt{x} = 40,$
$x^2 + xy^2 = 1312.$

40. $(5\,x - y)(y - 5\,x) = -1,$
$(5\,y + x)(y + 5\,x) = 189.$

41. $x^3 + y^3 = 224,$
$xy = 12.$

42. $x + y + 2\sqrt{x+y} = 24,$
$x^2 + y^2 = 130.$

43. $x^2 + y^2 - 7(x+y) = 8,$
$x + y + xy = -7.$

44. $\frac{y-1}{x-1} = \frac{5}{3},$
$\frac{y^2+y+1}{x^2+x+1} = \frac{43}{21}.$

45. $(x+y)(x^2+y^2) = -20,$
$(x-y)(x^2-y^2) = -32.$

46. $ax = by,$
$x^2 + y^2 = c.$

47. $px = qy,$
$(p+q)x - (p-q)\,y = r.$

48. $a\,(x+y) = b\,(x-y) = xy.$

49. $x^2 - y^2 = a,$
$x^4 - y^4 = b.$

50. $\frac{a^2}{x^2} + \frac{y^2}{b^2} = 6,$
$\frac{ab}{xy} = 1.$

51. $ax + by = 2\,ab,$
$\frac{a}{y} + \frac{b}{x} = 2.$

52. $bx + by = 2,$
$abxy = 1.$

53. $\frac{x+y}{y} = ax,$
$xy = b.$

54. $\frac{x-y}{\sqrt{x}-\sqrt{y}} = 8,$
$\sqrt{xy} = 12.$

55. $\frac{x-y}{\sqrt{x}-\sqrt{y}} = a,$
$\sqrt{xy} = b.$

56. $x^2 = 6\,x + 4\,y,$
$y^2 = 4\,x + 6\,y.$

57. $x^2 = cx + dy,$
$y^2 = dx + cy.$

58. $x^5 + y^5 = 33.$
$x + y = 3.$

59. $x^4 + y^4 = 97,$
$x + y = 5.$

60. $x^4 + y^4 = 97,$
$x - y = 1.$

61. $\frac{1}{x} - \frac{1}{y} = \frac{1}{x-y}$,
$\frac{1}{x^2} - \frac{1}{y^2} = \frac{1}{a^2}$.

62. $\frac{x^2}{y^2} + \frac{y^2}{x^2} + \frac{x}{y} - \frac{y}{x} = \frac{127}{36}$,
$xy = \frac{1}{6}$.

Simultaneous Equations Involving Decimal Fractions

Solve the following systems of equations and find approximate values of the unknowns which are correct to three places of decimals:

63. $x + y = 2.8$,
$xy = 1.87$.

64. $x + y = .051$,
$xy = .000518$.

65. $10\,x - 10\,y = .5$,
$10\,xy = .126$.

66. $.5\,x - .17 = .1\,y$,
$10\,xy = 1.2$.

67. $.001\,xy = 1.075$,
$.1\,x - .1\,y = 1.8$.

68. $x^2 + y^2 = .89$,
$10\,xy = 4$.

69. $100\,x^2 + 100\,y^2 = 65$,
$xy = .28$.

70. $.01\,x + .01\,y = .0015$,
$.1\,x^2 + .1\,y^2 = .00125$.

71. $x^2 + y^2 = 11.3$,
$x + y = 4.4$.

72. $x^2 + y^2 = 9.0625$,
$x + y = 3.25$.

73. $4\,x^2 - y^2 = 2.03$,
$x + y = 2$.

74. $.2\,x^2 - xy = -.742$,
$x + .1\,y = .82$.

75. $2.5\,x + .3\,y = 7$,
$.5\,xy = 6$.

76. $.7\,x + y = 8$,
$xy = 2.4$.

77. $x^2 + 10\,y^2 = 14.49$,
$x + y = 1.5$.

78. $5\,x^2 + y^2 = 9.2$,
$xy = .6$.

79. $x^3 - y^3 = .056$.
$x - y = .2$

80. $3\,x^2 - y^2 = 299.99$.
$xy = 1$.

SYSTEMS OF THREE OR MORE EQUATIONS CONTAINING THREE OR MORE UNKNOWNS

36. The solution of a system of three simultaneous equations containing three unknowns can be made to depend upon the solution of a quadratic equation only in exceptional cases.

There are certain systems of special equations the solutions of which can be made to depend upon the solutions of quadratic equations.

37. If a system of three or more simultaneous equations contains one and only one equation of the second degree with reference to the unknowns, the remaining equations being all of the first degree, the solution of the system can be made to depend upon the solution of a quadratic equation containing one unknown.

Ex. 1. Solve the system of equations

$$\left.\begin{aligned} x - y - 2z &= 0, \quad (1)\\ x + 2y + 3z &= 11, \quad (2)\\ x^2 + y^2 + z^2 &= 21. \quad (3)\end{aligned}\right\} \text{I. Given System.}$$

The unknowns x and y may be expressed in terms of the remaining unknown, z, as follows:

Employing equations (1) and (2), and eliminating y, we obtain

$$3x - z = 11. \qquad (4)$$

Hence, $$x = \frac{11 + z}{3}. \qquad (5)$$

From equations (1) and (2), eliminating x, by subtraction, we obtain

$$3y + 5z = 11. \qquad (6)$$

Hence, $$y = \frac{11 - 5z}{3}. \qquad (7)$$

Substituting these "expressed" values for x and y in equation (3), we obtain a quadratic equation containing z alone, from which the values of z are found to be 1 and $\frac{53}{35}$.

Substituting these values for z in equations (5) and (7), we obtain corresponding values of x and y.

Accordingly the solutions of the given system are found to be

$$\left.\begin{aligned} x &= 4,\\ y &= 2,\\ z &= 1,\end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} x &= \tfrac{146}{35},\\ y &= \tfrac{8}{7},\\ z &= \tfrac{53}{35}.\end{aligned}\right\}$$

These sets of values will be found to satisfy the given equations.

Ex. 2. Solve the system of equations

$$\left.\begin{aligned} xy &= 2, \ (1)\\ yz &= 4, \ (2)\\ zx &= 8. \ (3)\end{aligned}\right\} \text{I. Given System.}$$

We may obtain the following equation, the members of which are the continued products of the corresponding members of the three given equations:

$$x^2y^2z^2 = 64. \qquad (4)$$

From equation (4) we obtain

$$xyz = \pm 8. \qquad (5)$$

This result may be interpreted as representing the set of two equations, $xyz = +8$ and $xyz = -8$, which, taken together, are equivalent to equation (4).

Dividing the members of (5) by the corresponding members of equations (1), (2), and (3) respectively, we obtain the following results:

$$z = \pm 4, \qquad x = \pm 2, \qquad y = \pm 1.$$

Since the different members of all of the given equations are positive, and the first members contain two factors each, it follows that the signs of these factors must be like. Accordingly, we may arrange these values in sets, as follows:

$$\left.\begin{aligned} x &= 2, \\ y &= 1, \\ z &= 4, \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} x &= -2, \\ y &= -1, \\ z &= -4. \end{aligned}\right\}$$

These sets of values are found by substitution to satisfy the given equations.

Ex. 3. Solve the system of equations

$$\left.\begin{aligned} (x+y)(x+z) &= 4, & (1) \\ (y+z)(y+x) &= 16, & (2) \\ (z+x)(z+y) &= 36. & (3) \end{aligned}\right\} \text{I. Given System.}$$

Multiplying together the corresponding members of the given equations and taking the square roots of the results, we obtain

$$(x+y)(y+z)(z+x) = \pm 48. \qquad (4)$$

Using the members of the given equations as divisors with the corresponding members of (4), we obtain

$$y + z = \pm 12, \quad (5) \qquad x + y = \pm 3, \quad (6) \qquad z + x = \pm \tfrac{4}{3}, \quad (7)$$

from which the values of x, y, and z may be obtained.

Ex. 4. Solve the system of equations

$$\left.\begin{aligned} x^2 + 2yz &= 1, & (1) \\ y^2 + 2zx &= 1, & (2) \\ z^2 + 2xy &= 2. & (3) \end{aligned}\right\} \text{I. Given System.}$$

Adding the corresponding members of the given equations, we obtain,

$$x^2 + y^2 + z^2 + 2yz + 2zx + 2xy = 4. \qquad (4)$$

Hence,

$$[x + y + z + 2][x + y + z - 2] = 0. \qquad (5)$$

Subtracting the members of equation (2) from the corresponding members of (1), we obtain

$$x^2 - y^2 + 2\,yz - 2\,zx = 0. \tag{6}$$

Or,

$$(x - y)[x + y - 2\,z] = 0. \tag{7}$$

The system composed of equations (5) and (7), and any one of the given equations, such as (1), is equivalent to the given system of equations.

$$\left.\begin{array}{rl} [x + y + z + 2][x + y + z - 2] = 0, & (5) \\ (x - y)[x + y - 2\,z] = 0, & (7) \\ x^2 + 2\,yz = 1. & (1) \end{array}\right\} \text{II. Equivalent Derived System.}$$

Equating the factors of the first members of equations (5) and (7) separately to zero, and applying the method of § 16, we may separate the derived System II. into a group of four derived systems, which taken together are equivalent to System II.

Solving these systems separately, the solutions of the given system of equations may be obtained.

Ex. 5. Solve the system of equations

$$\left.\begin{array}{rl} x + xy + y = 3, & (1) \\ y + yz + z = 8, & (2) \\ x + xz + z = 15. & (3) \end{array}\right\} \text{I. Given System.}$$

From the first equation we may obtain the value of x, expressed in terms of y, as follows:

$$x(1 + y) + y = 3.$$

Hence,

$$x = \frac{3 - y}{1 + y}. \tag{4}$$

Substituting this value for x in equation (3), we obtain an equation containing y and z the members of which may be combined with those of equation (2) to obtain the values of y and z.

The values of x may be obtained by substituting in equation (4).

Ex. 6. Solve the system of equations

$$\left.\begin{array}{rl} x^2 - yz = a, & (1) \\ y^2 - zx = b, & (2) \\ z^2 - xy = c, & (3) \end{array}\right\} \text{I. Given System.}$$

If, from the squares of the members of equation (1) we subtract the product of the corresponding members of equations (2) and (3), we shall obtain equation (4).

Equations (5) and (6) may be obtained in a similar way.

$$x\,[x^3 + y^3 + z^3 - 3\,xyz] = a^2 - bc. \tag{4}$$

$$y\,[x^3 + y^3 + z^3 - 3\,xyz] = b^2 - ca. \tag{5}$$

$$z\,[x^3 + y^3 + z^3 - 3\,xyz] = c^2 - ab. \tag{6}$$

From equations (4) and (6) we obtain by division the value of the fraction x/z. Similarly, from equations (5) and (6) we obtain the value of the fraction y/z.

From the equations thus obtained we may find the expressed values of x and y in terms of z.

Substituting for x and y in equation (1) their expressed values thus found, we obtain the value of z in terms of the known numbers a, b, and c, in the form of a fraction having an irrational denominator,

$$z = \frac{\pm (c^2 - ab)}{\sqrt{a^3 + b^3 + c^3 - 3abc}}. \tag{7}$$

Either by substituting this value for z in the remaining equations, which may then be solved for x and y, or by repeating the process above with different pairs of equations, we obtain expressions of the same type as (7) for x and y.

EXERCISE XXV. 8

Find sets of values which satisfy each of the following systems of equations:

1. $$\begin{aligned} xy &= 30, \\ yz &= -60, \\ xz &= -50. \end{aligned}$$

2. $$\begin{aligned} yz &= a^2, \\ xz &= b^2, \\ xy &= c^2. \end{aligned}$$

3. $$\begin{aligned} x^2z &= 2, \\ y^2z &= 1, \\ z^2x &= 32. \end{aligned}$$

4. $$\begin{aligned} x^2yz &= a, \\ xy^2z &= b, \\ xyz^2 &= c. \end{aligned}$$

5. $$\begin{aligned} yz &= 2y + 4z, \\ zx &= 4z + x, \\ xy &= x + 2y. \end{aligned}$$

6. $$\begin{aligned} x(y + z) &= 5, \\ y(x + z) &= 8, \\ z(x + y) &= 9. \end{aligned}$$

7. $$\begin{aligned} yz &= bc, \\ \frac{x}{a} + \frac{y}{b} &= 1, \\ \frac{x}{a} + \frac{z}{c} &= 1. \end{aligned}$$

8. $$\begin{aligned} x^2 + y^2 &= 13, \\ x^2 + z^2 &= 34, \\ y^2 + z^2 &= 29. \end{aligned}$$

9. $$\begin{aligned} y + z &= \frac{1}{x}, \\ z + x &= \frac{1}{y}, \\ x + y &= \frac{1}{z}. \end{aligned}$$

10. $$\begin{aligned} xy^2z^2 &= -24, \\ \frac{yz^2}{x} &= \frac{4}{3}, \\ \frac{x^2y}{z} &= -\frac{9}{2}. \end{aligned}$$

11. $$\begin{aligned} \frac{xy}{x + y} &= \frac{4}{3}, \\ \frac{yz}{y + z} &= \frac{12}{5}, \\ \frac{zx}{z + x} &= \frac{3}{2}. \end{aligned}$$

12. $$\begin{aligned} \frac{xyz}{x + y} &= 2, \\ \frac{xyz}{x + z} &= \frac{3}{2}, \\ \frac{xyz}{y + z} &= \frac{6}{5}. \end{aligned}$$

13. $$\begin{aligned}\frac{xyz}{x+y} &= a,\\ \frac{xyz}{y+z} &= b,\\ \frac{xyz}{x+z} &= c.\end{aligned}$$

14. $$\begin{aligned}\frac{x^2+y^2}{xyz} &= \frac{5}{6},\\ \frac{y^2+z^2}{xyz} &= \frac{25}{12},\\ \frac{z^2+x^2}{xyz} &= \frac{17}{12}.\end{aligned}$$

15. $$\begin{aligned}(x+1)(y+1) &= 8,\\ (y+1)(z+1) &= 24,\\ (z+1)(x+1) &= 12.\end{aligned}$$

16. $$\begin{aligned}x(2-y) &= 16,\\ y(2-z) &= 9,\\ z(2-x) &= 4.\end{aligned}$$

17. $$\begin{aligned}x^2 &= yz,\\ x+y+z &= 21,\\ xyz &= 216.\end{aligned}$$

18. $$\begin{aligned}xy+xz+yz &= 3,\\ x-y &= 2,\\ y-z &= 1.\end{aligned}$$

19. $$\begin{aligned}x(x+y+z) &= 6.\\ y(x+y+z) &= 12,\\ z(x+y+z) &= 18.\end{aligned}$$

20. $$\begin{aligned}(y+z)(x+y+z) &= 6,\\ (z+x)(x+y+z) &= 8,\\ (x+y)(x+y+z) &= -6.\end{aligned}$$

21. $$\begin{aligned}xy &= c(x+y+z),\\ yz &= a(x+y+z),\\ xz &= b(x+y+z).\end{aligned}$$

22. $$\begin{aligned}(x+z)(x+y) &= a^2,\\ (x+y)(y+z) &= b^2,\\ (y+z)(x+z) &= c^2.\end{aligned}$$

23. $$\begin{aligned}x+y+z &= a,\\ xy &= b,\\ xyz &= c.\end{aligned}$$

24. $$\begin{aligned}x+y &= 3,\\ y+z &= 5,\\ z+w^2 &= 13.\\ x^2+z &= 8.\end{aligned}$$

25. $$\begin{aligned}x+y+z &= 9,\\ xy+yz+zx &= 26,\\ x^2+y^2-z^2 &= -3.\end{aligned}$$

26. $$\begin{aligned}x^2-(y-z)^2 &= 1,\\ y^2-(z-x)^2 &= 4,\\ z^2-(x-y)^2 &= 9.\end{aligned}$$

27. $$\begin{aligned}x(y+z)+3 &= 0,\\ y(z-3x)+27 &= 0,\\ z(3x-y) &= 0.\end{aligned}$$

28. $$\begin{aligned}x+xy+y &= 15,\\ y+yz+z &= 24,\\ x+xz+z &= 35.\end{aligned}$$

29. $$\begin{aligned}x^2-yz &= 2,\\ y^2-zx &= 4,\\ z^2-xy &= 1.\end{aligned}$$

30. $$\begin{aligned}x^2-yz &= 49,\\ y^2-zx &= 1,\\ z^2-xy &= 79.\end{aligned}$$

31. $$\begin{aligned}yz+x+y &= -9,\\ xz+y &= -5,\\ xy &= 2.\end{aligned}$$

32. $$\begin{aligned}x^2+xy+y^2 &= 3,\\ y^2+yz+z^2 &= 7,\\ z^2+zx+x^2 &= 7.\end{aligned}$$

33. $$x + y + z = \frac{13}{3},$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{13}{3},$$

$$xyz = 1.$$

EXERCISE XXV. 9

Solve the following problems employing conditional equations containing two or more unknown quantities:

1. Find two numbers the sum of which is 40, and the product of which is 256.

2. Find two numbers the difference of which is 15, and the sum of the squares of which is 293.

3. Find two numbers the difference of which is 6, and the product of which is 247.

4. Find two numbers the sum of which is 40, and the product of which is 391.

5. Find two numbers the product of which is 96, and the sum of the squares of which is 208.

6. Find two numbers the difference of the squares of which is 112, and the square of the difference of which is 64.

7. Find two numbers the sum of which is 10, and the sum of the cubes of which is 370.

8. The product of two numbers is 64, and the quotient obtained by dividing the greater number by the less is 4. Find the numbers.

9. If the sum of two numbers is divided by the less number, the quotient is 4; the product of the numbers is 27. Find the numbers.

10. Find two numbers the sum of which is 20, such that the sum of the quotients obtained by dividing each number by the other is 17/4.

11. Find two numbers such that, if each be increased by 1, the product is 124, and the product obtained by multiplying the first number by a number less by one than the second number is 60.

12. Find two numbers the sum of which is twice their difference, and the difference of the squares of which is 200.

13. The product of two numbers is 12, and the sum of their squares is five times the sum of the numbers. Find the numbers.

14. Find two numbers, of which the sum is 14, which are such that the product of the first and the reciprocal of the second, increased by the product of the second and the reciprocal of the first, is 25/12.

15. The sum of two numbers is 30, and the sum of the quotients resulting from dividing each number by the other is 82/9. Find the numbers.

16. The first of two numbers is ten times the reciprocal of the second, and the sum of the second number and ten times the reciprocal of the first is equal to the square of the second number. Find the numbers.

17. Find two fractions such that the sum of the first fraction and the reciprocal of the second is equal to 2, and the sum of the second fraction and the reciprocal of the first is 8/3.

18. Find two numbers the sum of which is 36, and half the product of which is equal to the cube of the less number.

19. Find a fraction the value of which is 3/4, and the product of the numerator and denominator of which is 48.

20. If a certain two-figure number, the sum of the figures of which is 12, be multiplied by the units' figure, the product is 375. What is the number?

21. A number expressed by two figures is equal to four times the sum of the figures. The number formed by writing the figures in reversed order exceeds three times the product of the figures by the square of the figure in tens' place of the given number. Find the number.

22. If it requires 240 rods of fence to enclose a rectangular field of 20 acres, what are the dimensions of the field?

23. A rectangular field contains 30 acres. By increasing its length by 40 rods and diminishing its width by 4 rods, the area is increased by 6 acres. What are its dimensions?

24. The length of the fence around a rectangular field is 274 yards, and the distance measured diagonally from corner to corner is 97 yards. What is the area?

25. Thirty-two yards of the fence about a rectangular field which is 184 yards long and 76 yards wide are destroyed. What must be the dimensions of a rectangular field in order that the length of fence remaining shall enclose the same area as before?

26. A property owner wishes to use the material from a stone wall enclosing a field, which has the form of a rectangle 80 rods long and 60 rods wide, to build another wall greater by 16 rods which shall enclose a second tract of land which has the form of a rectangle having the same area as the first. Find the dimensions of the second tract of land.

27. In widening a street, a strip of land 6 feet in width was removed from the entire frontage of a tract containing 28,800 square feet. By increasing the frontage of the reduced lot by 8 feet, the entire area became the same as before. Find the original dimensions of the land.

28. It is observed that, if a guy rope which is attached to a stake 7 feet

from the foot of a derrick were lengthened by 15 feet, it would reach to a stake 32 feet from the foot of the derrick. Find the height of the derrick and the length of the rope.

29. A tract of 10 acres of land is enclosed by a certain length of fence in such a way that there are two separate lots, each in the form of a square, so situated that the side of the smaller lot forms a part of the side of the larger lot. It is observed that the fence may be rebuilt to enclose a single lot in the form of a square containing $5\frac{5}{8}$ acres more than the original lots. Find the dimensions of the original lots.

30. At an entertainment $750 was realized from the sale of seats. For each reserved seat twenty-five cents more was charged than for an unreserved seat, but the sale of the unreserved seats yielded the same total amount as that of the reserved seats. Find the total number of seats sold, if the number of unreserved seats exceeded the number of reserved seats by 125.

31. Three stone crushers working together can crush a certain amount of stone in a week. The first machine has a capacity twice as great as that of the second, but working alone would require one week more than the third machine to perform the work. What time would each require, working alone?

32. The sum of a fraction and its reciprocal is equal to the numerator increased by the reciprocal of twice the denominator; and the difference between the reciprocal of the numerator and the reciprocal of the denominator is equal to the reciprocal of twice the denominator. What is the fraction?

33. Find a fraction such that, if its numerator be increased by 3 and its denominator diminished by 3, the result is the reciprocal of the fraction; but if the denominator be increased by 3 and the numerator diminished by 3, the result will be $1\frac{13}{20}$ less than the reciprocal of the fraction.

34. Find a fraction, the value of which is $2/3$, such that if the numerator be diminished by the reciprocal of the denominator and the denominator be increased by the reciprocal of the numerator, the value of the fraction will be multiplied by $149/151$.

35. Find two numbers the sum of the cubes of which is 133, and the sum of the squares of which diminished by their product is 19.

36. Find two numbers of which the sum multiplied by the product is equal to 30, and the sum of the cubes of which is 35.

37. A number is expressed by three figures, the sum of which is 10. The middle figure exceeds the sum of the other two by 2, and the sum of the squares of the separate figures is seven times the figure in tens' place, increased by the sum of the other two. Find the number.

38. Two boats leave simultaneously the opposite shores of a river which is $2\frac{1}{4}$ miles wide, and pass each other in 15 minutes. The faster boat completes the trip $6\frac{3}{4}$ minutes before the other reaches the opposite shore. Find the rates of the boats in miles per hour.

39. A train starts from a certain station to make a trip of 180 miles, travelling uniformly. Forty-five minutes later a faster train, also travelling uniformly, starts from the same station, and after travelling two hours and fifteen minutes reaches the station which the first train had passed thirty-six minutes previously. The speed of the second train is now increased by four miles an hour, with the result that the trains reach the terminus at the same time. Find the rates in miles per hour, at which they started.

40. Fifteen hours after an ocean steamship leaves the American shore to make a voyage of 3300 miles at a certain average uniform rate, a second ship starts for America from the opposite shore. The two ships meet after the second ship has been out $71\frac{3}{7}$ hours, and they complete their trips at the same instant. Find the rates of the ships in miles per hour.

41. An express train, an electric car, and an automobile all leave a given place for a certain destination. The express train travels 9 miles an hour faster than the electric car, and the automobile 13 miles an hour faster than the express. The express starts one hour after the electric and 39 minutes before the automobile. If all arrive at the end of their journeys at the same time, find the distance and the rates of travelling in miles per hour.

42. Sighting an enemy's war vessel at a distance of 10 miles, a submarine boat starts toward it, running on the surface at a certain uniform rate which exceeds its speed when submerged by 6 miles an hour. At a certain point in its course it dives beneath the surface, and when submerged at a distance of one-half mile from the battleship, a torpedo is discharged which travels at the rate of a mile in two minutes. It is observed from the shore that the time, measured from the instant the submarine starts until the explosion takes place, is exactly $44\frac{1}{3}$ minutes. Returning immediately after delivering its torpedo, and travelling the entire distance under water, the time required is one hour, three and one-third minutes. Find the rate of the submarine on the surface in miles per hour and also the distance from the starting-point of the spot at which it sank beneath the surface.

43. A torpedo boat, on being discovered $1\frac{1}{2}$ miles from port, immediately turns and tries to escape. One minute later a torpedo-boat destroyer is sent out and this overtakes it after a run of $9\frac{1}{2}$ miles. If the rates of the torpedo boat and destroyer could have been increased by 6 miles an hour and 2 miles an hour respectively, when the distance between the two boats was reduced to 1 mile the torpedo boat would have escaped to its squadron $10\frac{1}{2}$ miles away. Find the rates of the boats in miles per hour.

CHAPTER XXVI

RATIO, PROPORTION, AND VARIATION

I. Ratio

1. The **ratio** of one number to another of the same kind is the quotient obtained by dividing the first number by the second.

The ratio of a to b may be expressed by any symbol of division.

E. g. $a \div b$; $\frac{a}{b}$; a/b; or by $a : b$.

The ratio of 6 to 3 is $\frac{6}{3}$ or 2.

2. In a *ratio* $a : b$ (read, "the ratio of a to b"), a is called the **first term or antecedent** of the ratio, and b the **second term or consequent.**

3. Since a ratio has been defined as a fraction, it follows that all of the properties of fractions apply also to ratios.

E. g. $$\frac{am}{bm} \equiv \frac{a}{b}, \qquad \frac{12}{16} = \frac{3}{4}.$$

4. According as $a > b$ or $a < b$ the ratio $a : b$ is said to be a **ratio of greater inequality,** or a **ratio of less inequality.**

5. The values of two ratios may be compared by expressing them as fractions and then reducing the fractions to equivalent fractions having a common denominator.

E. g. Compare the values of the ratios $2 : 3$ and $7 : 8$.

We have $2 : 3 = \frac{2}{3} = \frac{16}{24}$; also $7 : 8 = \frac{7}{8} = \frac{21}{24}$.

Hence since $\frac{21}{24} > \frac{16}{24}$, it appears that $7 : 8 > 2 : 3$.

6. Two or more ratios are said to be **compounded** if their corresponding terms are multiplied together.

E. g. $acx : bdy$ is *compounded* of $a : b$, $c : d$, and $x : y$.

7. A **duplicate ratio** is the ratio formed by compounding two equal ratios.

E. g. $a^2 : b^2$ is the *duplicate* ratio of the ratio $a : b$.

8. A **triplicate ratio** is the ratio formed by compounding three equal ratios.

E. g. $a^3 : b^3$ is the *triplicate* ratio of the ratio $a : b$.

9. The **inverse ratio** of a ratio is obtained by interchanging the antecedent and the consequent.

E. g. $a : b$ and $b : a$ are *inverse* ratios.

10. Principle: *A ratio of greater inequality is diminished, and a ratio of less inequality is increased, by the addition of the same positive number to each of its terms.*

If a, b, and x are any positive numbers, the ratio $\frac{a}{b}$ is greater than or less than the ratio $\frac{a+x}{b+x}$ according as a is greater than or less than b.

For
$$\left|\frac{a}{b} - \frac{a+x}{b+x}\right| \equiv \left|\frac{x(a-b)}{b(b+x)}\right|. \qquad (1)$$

Since b and x are both positive, the denominator $b(b+x)$ is positive, and the value of the second member of (1) is positive or negative according as the factor $(a-b)$ of the numerator is positive or negative, — that is, according as a is greater than or less than b.

The value of $\frac{a}{b}$ is greater than or less than the value of $\frac{a+x}{b+x}$ according as the value of the difference $a-b$ is positive or negative.

In a similar manner it may be shown that a *ratio of greater inequality is increased, and a ratio of less inequality is diminished, by subtracting the same positive number from each of its terms.*

11. If two concrete quantities of the same kind can each be expressed in whole numbers in terms of some unit of measure, this common unit is called a **common measure** of the two given quantities, and the two given quantities are said to be **commensurable.** (See Chap. XVIII. § 8, and Chap. XX. § 1.)

12. The number which expresses the number of times a given unit is contained in a given quantity of the same kind is called the **numerical measure** of the quantity with respect to the given unit.

13. **The ratio of two concrete quantities** of the same kind, which can both be expressed in terms of the same unit by means of two rational numbers, is defined to be the ratio of their numerical measures.

E. g. The ratio of $2\frac{1}{3}$ feet to $3\frac{1}{5}$ feet is the ratio $\frac{2\frac{1}{3}}{3\frac{1}{5}}$, or $\frac{35}{48}$. By employing $\frac{1}{15}$ of a foot as a common unit of measure, we can express $2\frac{1}{3}$ feet and $3\frac{1}{5}$ feet in terms of this common unit by the numbers 35 and 48 respectively.

14. Two quantities of the same kind are said to be **incommensurable** if both cannot be expressed in *whole numbers* in terms of a common unit of measure.

The exact value of the ratio of two incommensurable quantities cannot be expressed in terms of a whole number, or of a fraction the numerator and denominator of which contain a finite number of figures.

An approximate value may be found which will differ from the true value of the ratio of two incommensurable quantities by less than any assignable value, however small.

(The following proof may be omitted when the chapter is read for the first time.)

Let A and B represent two incommensurable quantities of the same kind. It is always possible to find two whole numbers of which the ratio differs from the true value of the ratio of A to B by as small a value as we please.

For, if we separate the lesser of the two quantities, say B, into any integral number n of equal parts, then, since A and B are assumed to be incommensurable, it will follow that when A is divided by this unit B/n, there will be a remainder which is less than one of these nth parts of B.

Suppose that one of these nth parts of B is contained in A more than m times and less than $m + 1$ times.

Then the true value of A/B will lie between the *approximate values*

$$\frac{m}{n} \text{ and } \frac{m+1}{n}, \text{ that is, } \frac{m}{n} < \frac{A}{B} < \frac{m+1}{n}.$$

It follows that either approximate value, say m/n, differs from the true value of A/B by a value less than that by which it differs from the other approximate value, say $(m+1)/n$.

The difference $1/n$, between the approximate values m/n and $(m+1)/n$, can be made as small as we please by taking n great enough, but *it can never be made equal to zero.*

Accordingly, an approximate value may be found which will differ by less than any assignable value from the true value of the ratio of two given incommensurable quantities, A and B.

It should be observed that the commensurable ratios m/n and $(m+1)/n$, which approximate to the true value of the ratio of the incommensurable numbers A and B, define an incommensurable number.

Accordingly, the *fixed value* which is the ratio of two incommensurable quantities is called an **incommensurable ratio.**

E.g. The numbers $\sqrt{2}$ and 3 are *incommensurable* with respect to each other; hence their ratio $\frac{\sqrt{2}}{3}$ is an *incommensurable ratio.*

The incommensurable numbers $2\sqrt{3}$ and $5\sqrt{3}$ are *commensurable with respect to each other*, since their ratio $\frac{2\sqrt{3}}{5\sqrt{3}}$ is equal to $\frac{2}{5}$.

It may be shown by applying the Principles of Variables and Limits that *two incommensurable ratios are equal if their approximate values remain equal as the unit of measure is indefinitely diminished.*

EXERCISE XXVI. 1

Write each of the following ratios in simplest form:

1. $10 : 12$.
2. $25 : 30$.
3. $6a : 8a$.
4. $x^2 : xy$.
5. $abc : bcd$.
6. $7a^2 : 28ab$.
7. $\frac{1}{2} : \frac{1}{4}$.
8. $\frac{1}{3} : \frac{1}{2}$.
9. $\frac{3}{4} : \frac{4}{5}$.
10. $\frac{x}{y} : \frac{z}{w}$.
11. $\frac{a}{b} : \frac{b}{c}$.
12. $\frac{x^3}{y^3} : \frac{x^2}{y^2}$.

Find the ratio compounded of

13. $4 : 5$ and $10 : 6$.
14. $2 : 3$ and $4 : 9$.
15. $21 : 4$ and $10 : 7$.
16. $6 : 7$ and the duplicate of $2 : 3$.
17. $50 : 32$ and the triplicate of $4 : 5$.
18. The duplicate of $x^2 : y^2$ and the triplicate of $y : x$.

Which is the greater ratio:

19. $3:4$ or $4:5$? 20. $4:11$ or $2:5$? 21. $7:8$ or $23:24$?

Arrange in order of increasing values:

22. $5:7$, $6:8$, $9:14$, and $27:28$.

Which ratio is greater, (1) for x positive, (2) for x negative:

23. $2:3$ or $(2+x):(3+x)$? 24. $7:4$ or $(7-x):(4-x)$?

Find the value of the ratio $x:y$ in each of the following:

25. $2(x^2+y^2)=13(xy-y^2)$. 26. $10(x^2+y^2)=29xy$.

II. Proportion

15. A **proportion** is an expressed equality of two equal ratios.

E.g. The abstract numbers a, b, c, and d are said to be in proportion in the given order a, b, c, d, if $a:b=c:d$ (read "a is to b as c is to d").

16. The numbers a, b, c, and d are called the **terms** of the proportion $a:b=c:d$.

The first and fourth terms, a and d, are called the **extremes,** and the second and third terms, b and c, the **means,** of the proportion.

17. In a proportion $a:b=c:d$, the antecedents a and c of the equal ratios $a:b$ and $c:d$ are called the **antecedents of the proportion.**

18. Similarly, the consequents b and d of the equal ratios are called the **consequents of the proportion.**

19. In any proportion, $a:b=c:d$, the value of either of the equal ratios is called the **common ratio of the proportion.**

20. The symbol $::$ is frequently used instead of the equality sign between the equal ratios of a proportion.

E.g. $2:3::4:6$ instead of $2:3=4:6$.

21. The numbers $a, b, c, d, e, f, \cdots\cdots$, are said to be in **continued proportion** if $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=\frac{d}{e}=\frac{e}{f}=\cdots\cdots$, the consequent of any ratio being equal to the antecedent of the one following.

22. In a continued proportion, $a:b=b:c$, containing *three* numbers, a, b, and c, b is called the **mean proportional** between a and c, and c is called the **third proportional** to a and b.

23. In a proportion, $a : b = c : d$, containing *four* numbers, a, b, c, and d, the *fourth* number d is called the **fourth proportional** to the three numbers a, b, c, in the given order, a, b, c.

24. Proportions may be transformed according to the following

General Principles

(i.) *In any numerical proportion the product of the extremes is equal to the product of the means.*

That is, if $\boldsymbol{a : b = c : d}$, then $\boldsymbol{ad = bc}$.

For, multiplying both ratios of the proportion $a/b = c/d$ by bd we obtain $ad = bc$.

This principle may be used to determine whether or not four numbers are proportional in some specified order.

(ii.) *The mean proportional between any two numbers is equal to the square root of their product.*

That is, if $\boldsymbol{a : b = b : c}$, (1), then $\boldsymbol{b = \sqrt{ac}}$.

For, from (1) we have $b^2 = ac$. Hence $b = \sqrt{ac}$.

It should be observed that, since the two numbers which form either of the ratios of a given proportion must be numbers of the same kind, it is impossible that one should be positive and the other negative.

Accordingly, when a and c are both positive numbers, the positive sign only is taken for $\sqrt{ac}$ in the expression for the mean proportional, $b = \sqrt{ac}$. (Compare with Chap. XXVII. § 39.)

If, however, the numbers of which the mean proportional is to be found are both negative, for example -4 and -9, we have, representing the mean proportional by m, $\dfrac{-4}{m} = \dfrac{m}{-9}$.

Hence, $m = \sqrt{(-4)(-9)} = -6$, which is a negative number.

(iii.) *If the product of two numbers is equal to the product of two others, the numbers of either set may be made the extremes, and those of the other set may be made the means of a proportion.*

That is, if $\boldsymbol{ad = bc}$,

we obtain, dividing both members by bd, $a/b = c/d$.

Therefore either $\boldsymbol{a : b = c : d}$ (1) or $\boldsymbol{c : d = a : b}$. (2)

Similarly,

Dividing by ac, $d : c = b : a$ (3) or $b : a = d : c$. (4)

Dividing by ab, $d : b = c : a$ (5) or $c : a = d : b$. (6)

Dividing by cd, $a : c = b : d$ (7) or $b : d = a : c$. (8)

It appears that if a, b, c, and d form a proportion in any one of the eight orders given above, they also form a proportion when written in any one of the remaining orders.

From the results obtained above, it may be seen that the means of a proportion may be interchanged. Hence,

(iv.) *If four numbers or quantities of the same kind are in proportion, the terms may be rearranged by* ***alternation*** *; that is, the first term is to the third term as the second term is to the fourth term.*

That is, if $\boldsymbol{a : b = c : d}$, then $\boldsymbol{a : c = b : d}$.

It should be observed that the terms of a concrete proportion can be transformed by alternation only when the terms of both ratios are quantities of the same kind. For, if the quantities appearing in the terms of the first ratio are of a different kind from those appearing in the terms of the second ratio, the transformation by alternation will result in a proportion in which the terms of the first ratio are quantities of different kinds and the terms of the second ratio are also quantities of different kinds.

(v.) *In any proportion the terms may be rearranged by* ***inversion,*** *that is, the second term is to the first term as the fourth term is to the third term.*

That is, if $\boldsymbol{a : b = c : d}$, then $\boldsymbol{b : a = d : c}$.

For, expressing the proportion $a : b = c : d$ in the fractional notation, dividing unity by each member and simplifying, we obtain $b/a = d/c$.

(vi.) *In any proportion, the terms may be combined by* ***addition*** *; that is, the sum of the first and second terms is to either the first term or the second term as the sum of the third and fourth terms is to either the third term or the fourth term.*

That is, if $\boldsymbol{a : b = c : d}$, (1)

then $\boldsymbol{(a + b) : a = (c + d) : c}$, (2), and also $\boldsymbol{(a + b) : b = (c + d) : d}$. (3)

(3) may be derived as follows:

Employing the fractional notation for the ratios, we may express (1) as the equation

$$\frac{a}{b}=\frac{c}{d}.$$

Adding unity to each ratio,

$$\frac{a}{b}+1=\frac{c}{d}+1.$$

Hence, $\dfrac{a+b}{b}=\dfrac{c+d}{d}.$

(2) may be derived as follows:

Writing (1) by inversion, and expressing the result in fractional form, we obtain

$$\frac{b}{a}=\frac{d}{c}.$$

Adding unity to each ratio,

$$1+\frac{b}{a}=1+\frac{d}{c}.$$

Hence, $\dfrac{a+b}{a}=\dfrac{c+d}{c}.$

(vii.) *In any proportion the terms may be combined by* ***subtraction;*** *that is, the difference between the first term and the second term is to either the first term or the second term as the difference between the third term and fourth term is to either the third term or the fourth term.*

That is, if $\boldsymbol{a:b=c:d},$ (1)

then $(\boldsymbol{a-b}):\boldsymbol{a}=(\boldsymbol{c-d}):\boldsymbol{c},$ (2) and also $(\boldsymbol{a-b}):\boldsymbol{b}=(\boldsymbol{c-d}):\boldsymbol{d}.$ (3)

(3) may be obtained as follows:

Expressing the ratios of the given proportion by the fractional notation, we have

$$\frac{a}{b}=\frac{c}{d}.$$

Subtracting unity from each ratio, we have

$$\frac{a}{b}-1=\frac{c}{d}-1.$$

Hence, $\dfrac{a-b}{b}=\dfrac{c-d}{d}.$

(2) may be obtained as follows:

Writing (1) by inversion and expressing the result in fractional form, we have

$$\frac{b}{a}=\frac{d}{c}.$$

Subtracting both ratios from unity, we obtain

$$1-\frac{b}{a}=1-\frac{d}{c}.$$

Hence, $\dfrac{a-b}{a}=\dfrac{c-d}{c}.$

(viii.) *In any proportion the terms may be combined by* ***addition and subtraction;*** *that is, the sum of the first and second terms is to their difference as the sum of the third and fourth terms is to their difference.*

That is, if $\boldsymbol{a:b=c:d}$, (1), then $\boldsymbol{a+b:a-b=c+d:c-d}.$ (2)

The proportion (2) may be obtained by dividing the ratios obtained by applying (vi.) to (1) by the corresponding ratios obtained by applying (vii.) to (1).

(ix.) *In a series of equal ratios the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

That is, if $\boldsymbol{a : b = c : d = e : f = \cdots\cdots = m : n}$,

then $\boldsymbol{(a + c + e + \cdots\cdots + m) : (b + d + f + \cdots\cdots + n) =}$
$$\boldsymbol{a : b = c : d = e : f = \cdots\cdots = m : n},$$
provided that $(b + d + f + \cdots\cdots + n) \neq 0$.

Let r denote the value of each of the equal ratios.

Then from $\frac{a}{b} = r,\quad \frac{c}{d} = r,\quad \frac{e}{f} = r,\quad \cdots\cdots,\quad \frac{m}{n} = r,$

we have $a = br,\ c = dr,\ e = fr,\ \cdots\cdots,\ m = nr.$

By addition, $\boldsymbol{(a + c + e + \cdots\cdots + m) = (b + d + f + \cdots\cdots + n)r}$.

Then on condition that $(b + d + f + \cdots\cdots + n) \neq 0$,

we have $$\frac{a + c + e + \cdots\cdots + m}{b + d + f + \cdots\cdots + n} = r = \frac{a}{b} = \frac{c}{d} = \cdots\cdots = \frac{m}{n}.$$

(x.) *The products or the quotients of the corresponding terms of two proportions form a proportion.*

That is, if $\boldsymbol{a : b = c : d}$, (1), and $\boldsymbol{x : y = z : w}$, (2),

then $\boldsymbol{ax : by = cz : dw}$, (3), and also $\boldsymbol{a/x : b/y = c/z : d/w}$. (4)

Writing (1) and (2) in fractional form, and multiplying corresponding members of the equations, we obtain (3).

Similarly, (4) is obtained by division.

(xi.) *Like powers or like principal roots of the terms of a proportion are in proportion.*

That is, if $\boldsymbol{a : b = c : d}$, (1)

then $\boldsymbol{a^n : b^n = c^n : d^n}$, (2), and also $\boldsymbol{\sqrt[k]{a} : \sqrt[k]{b} = \sqrt[k]{c} : \sqrt[k]{d}}$. (3)

Let $a : b = r$ and $c : d = r$.

Then from $a = br$, (4) and $c = dr$, (5)

we have $a^n = b^n r^n$, and $c^n = d^n r^n$.

Hence $\frac{a^n}{b^n} = r^n$, and $\frac{c^n}{d^n} = r^n$.

Therefore $\boldsymbol{\frac{a^n}{b^n} = \frac{c^n}{d^n}}$. (6)

Similarly, from (4) and (5),

$$a^{\frac{1}{k}} = b^{\frac{1}{k}}r^{\frac{1}{k}}, \quad \text{and} \quad c^{\frac{1}{k}} = d^{\frac{1}{k}}r^{\frac{1}{k}},$$

and finally,

$$\frac{a^{\frac{1}{k}}}{b^{\frac{1}{k}}} = \frac{c^{\frac{1}{k}}}{d^{\frac{1}{k}}}. \quad (7)$$

Ex. 1. Find the mean proportional, x, between 4 and 25.

Let $4 : x = x : 25$. Then $x = \sqrt{4 \cdot 25} = 10$.

Ex. 2. Show that if $a : b = c : d$ it follows that

$$(a^2 + c^2) : (ab + cd) = (ab + cd) : (b^2 + d^2). \quad (1)$$

Let $a : b = r$ and $c : d = r$.

Then $a = br$ (2) and $c = dr$. (3)

Hence, $a^2 + c^2 \equiv b^2r^2 + d^2r^2$. (4)

Multiplying the members of (2) and (3) by b and d respectively and adding the corresponding members of the resulting equations, we have

$$ab + cd \equiv b^2r + d^2r. \quad (5)$$

From (4) and (5),

$$\frac{a^2 + c^2}{ab + cd} \equiv \frac{r^2(b^2 + d^2)}{r(b^2 + d^2)} \equiv r. \quad (6)$$

Similarly,

$$\frac{ab + cd}{b^2 + d^2} \equiv \frac{r(b^2 + d^2)}{b^2 + d^2} \equiv r. \quad (7)$$

Therefore, (1) follows from (6) and (7).

Ex. 3. Solve

$$\frac{\sqrt{x+6} - \sqrt{x-15}}{\sqrt{x+6} + \sqrt{x-15}} = \frac{3}{7}. \quad (1)$$

Applying (viii.) to (1),

$$\frac{2\sqrt{x+6}}{-2\sqrt{x-15}} = \frac{10}{-4} \quad (2)$$

$$\frac{x+6}{x-15} = \frac{25}{4}$$

$$4x + 24 = 25x - 375$$

$$x = 19.$$

Verifying by substituting 19 for x in (1), $\frac{3}{7} = \frac{3}{7}$.

EXERCISE XXVI. 2

Find the fourth proportional to

1. 2, 3, and 4.
2. 14, 15, and 16.
3. 15, 16, and 14.
4. 16, 15, and 14.

Find the mean proportional between

5. 2 and 8. 6. 1 and 4. 7. 12 and 6. 8. a and $\frac{1}{a}$.

Find the third proportional to

9. 2 and 8. 10. 5 and 10. 11. $\frac{a}{b}+\frac{b}{a}$ and $\frac{a}{b}$.

Construct proportions from the following products:

12. $a^2 = bc$. 13. $x^2 = 4 \cdot 40$. 14. $(a+b)(a-b) = c^2$.

If $a:b = c:d$ obtain each of the following proportions:

15. $a:b = \frac{1}{d}:\frac{1}{c}$. 16. $ac:bd = c^2:d^2$. 17. $a+c:b+d = a^2d:b^2c$.

18. $\frac{(a+b)^2}{(c+d)^2} = \frac{a^2}{c^2}$.

19. $\frac{(a+b)^2}{(c+d)^2} = \frac{4a^2-5b^2}{4c^2-5d^2}$.

20. $a:a+c = a+b:a+b+c+d$.

21. $a:b = \sqrt{a^2-c^2}:\sqrt{b^2-d^2}$.

22. $a^2-b^2:\frac{a^3}{a+b} = c^2-d^2:\frac{c^3}{c+d}$.

23. If $a:b = b:c$ show that

$$\frac{a+b}{a}:\frac{b-c}{b} = \frac{b+c}{b}:\frac{a-b}{a}.$$

24. What quantity must be added to the terms of $a^2 \div c^2$ to make it equal to $a:c$?

25. What expression must be subtracted from each of the following expressions in order that the remainders shall form a proportion?

$4a+b+c$, $5a+b+c$, $a+13b+c$, and $a+17b+c$.

Solve for x in each of the following proportions:

26. $2:21 = 3:x$.

27. $70:x = 14:2$.

28. $(m^2-n^2):(m-n) = x:1$.

29. $(x-5):3 = 5:12$.

Simplify the following equations by applying the Principles Governing Proportions, and then solve for x:

30. $\frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}-\sqrt{b}} = \frac{a}{b}$.

31. $\frac{x+\sqrt{x-1}}{x-\sqrt{x-1}} = \frac{21}{13}$.

32. $\frac{\sqrt{x+a}+\sqrt{x-a}}{\sqrt{x+a}-\sqrt{x-a}} = b$.

33. $\frac{\sqrt{a}+\sqrt{a+x}}{\sqrt{a}-\sqrt{a+x}} = \frac{\sqrt{c}+\sqrt{x-c}}{\sqrt{c}-\sqrt{x-c}}$.

34. Find three numbers in continued proportion whose sum is 14 and whose product is 64.

35. Divide $42.00 between two men so that their shares shall be in the ratio of 3 : 4.

36. Divide 44 into two parts such that the less, increased by one, shall be to the greater, decreased by one, as 5 : 6.

37. Two numbers are in the ratio of 4 : 5. If each is increased by 5 the sums will be in the ratio of 5 : 6. What are the numbers?

38. What number must be added to the numbers 3, 4, 7, and 9 in order that their sums shall form a proportion?

39. What number must be subtracted from 24, 27, 40, and 55, in order that the remainders shall form a proportion?

40. Find the ratio of the numerator of a fraction to its denominator if the value of the fraction remains unchanged when the numerator is increased by a and the denominator is increased by b?

The areas of two similar plane figures have the same ratio as the squares of any two corresponding dimensions.

41. The area of a triangle is 90 square inches and the base is 12 inches. What is the area of a similar triangle, provided that the base is 16 inches?

42. The area of the first of two similar polygons is 128 square inches, and the area of the second is 200 square inches. If one side of the first polygon is 8 inches, find the corresponding side of the second polygon.

The volumes of two similar solids have the same ratio as the cubes of any two corresponding dimensions.

43. The diameter of the first of two bottles which are of similar shape is three times that of the second. If the first holds 2 ounces, how much does the second hold?

44. If a sphere which is 2 inches in diameter weighs 5 lbs., what is the weight of a sphere of the same substance which is 3 inches in diameter?

Problems in Physics

25. The Inclined Plane. *If a body rests on a smooth inclined plane, the force (disregarding friction) which must be applied along the plane to hold the body in place against the action of the force of gravity has the same ratio to the weight of the body that the height of the plane has to the length of the plane.*

26. If F represents the force applied along an inclined plane, W the weight of the body, h the height of the plane, and l the length of the plane, we have

$$\frac{F}{W}=\frac{h}{l}.$$

The force represented by F which is applied along the plane is called the component of the weight W which is parallel to the plane.

EXERCISE XXVI. 3

Solve the following problems:

1. Find the force which must be exerted to draw a sled weighing 240 lbs. up a hill which is 300 feet long and 50 feet high.

2. What is the weight of a body if a force of 125 lbs., exerted along a smooth inclined plane which is 80 feet in length and 20 feet in height, prevents the body from sliding down the plane?

3. A boy who is able to exert a maximum force of 80 lbs. is able to keep a barrel from rolling down a plank which is 12 feet in length and the upper end of which is 3 feet from the ground. Find the weight of the barrel.

4. A porter who can exert a maximum force of 200 lbs. undertakes to roll a cask weighing 500 lbs. up a board which is 10 feet long. How high can the upper end of the board be placed without compelling the porter to allow the cask to roll down the board?

5. A car weighing 1200 lbs. is held at rest on a smooth inclined plane by a force of 30 lbs. applied parallel to the plane. If the length of the plane is 800 feet, find the height of the plane.

6. A boy is able to exert a maximum force of 80 lbs. How long an inclined plane must he use to push a truck weighing 320 lbs. up to a doorway which is $3\frac{1}{2}$ feet above the ground?

Boyle's Law. *The volume of a gas is (approximately) inversely proportional to the pressure, provided that the temperature remains constant.*

That is, representing the pressure by P_1 when the volume is V_1, and the pressure by P_2 when the volume is V_2, we have

$$\frac{V_1}{V_2}=\frac{P_2}{P_1}.$$

7. If, when confined with a pressure of 20 lbs. per square inch, a mass of gas occupies a volume of one cubic foot, find the volume of the gas when the pressure becomes 40 lbs. per square inch.

8. A gas bag containing 3 cubic feet of gas under a pressure of 18 lbs. per square inch must be subjected to what pressure to reduce the volume to half a cubic foot?

9. Six cubic feet of gas under a pressure of 45 lbs. per square inch will have what volume if the pressure is reduced to 15 lbs. per square inch?

10. A bladder holds 40 cubic inches of air under a pressure of 15 lbs.

per square inch. What is the size of the bladder when the pressure is reduced to 12 lbs. per square inch?

11. When under a pressure of 75 lbs. per square inch, the volume of a a mass of gas is 128 cubic inches. What is the pressure when the volume becomes 240 cubic inches?

12. If 100 cubic inches of air, at a pressure of 27 lbs. per square inch, be admitted to a vessel the volume of which is 450 cubic inches, what will be the pressure?

By *absolute temperature* expressed in degrees centigrade is meant the number of degrees above 0° C., plus 273° C.

Representing the absolute temperature by T, and the number of degrees above 0° C. by t, we have

$$T = t + 273.$$

Charles's Law. *The volume of a gas is directly proportional to the absolute temperature, provided that the pressure remains constant.*

Assuming that the pressure remains constant, we will represent by V_1 the volume of a mass of gas when the absolute temperature is T_1° C., and by V_2 the volume when the temperature is T_2° C. Then we have (approximately)

$$\frac{V_1}{V_2} = \frac{T_1}{T_2}.$$

13. The volume of a certain quantity of gas is 100 cubic centimeters at 0° C. At what temperature will the volume become 200 cubic centimeters, assuming that the pressure remains constant?

14. If the volume of a certain mass of gas is 500 cubic centimeters at 20° C., find the volume of the gas at 87° C., assuming that the pressure remains constant.

15. A mass of gas occupying a volume of 160 cubic centimeters at a temperature of 47° C. is cooled to a temperature of 17° C. Find the volume at the lower temperature.

16. A certain mass of gas occupying a volume of 90 cubic centimeters at 12° C. is raised in temperature to 50° C. Find the volume at the higher temperature.

Representing by V_1 the volume of a gas when the pressure is P_1 and the absolute temperature is T_1° C., and by V_2 the volume of the gas when the pressure is P_2 and the absolute temperature of T_2° C., it may be shown that the following relation is true:

$$\frac{V_1 P_1}{T_1} = \frac{V_2 P_2}{T_2}.$$

Since the barometer is commonly used to measure the pressure of a gas, it will be convenient to give the pressures of the gases in the following examples in terms of the height of a column of mercury.

17. Five hundred cubic centimeters of a gas at a temperature of 27° C are cooled to 2° C., and at the same time the external pressure upon the gas is changed from 74 centimeters of mercury to 76 centimeters of mercury. What does the volume of the gas become?

18. Fifty liters of gas are generated at a temperature of 12° C. and a pressure of 68 centimeters of mercury. Find the volume of the gas at 0° C. when the pressure is 76 centimeters of mercury.

19. The volume of a certain quantity of gas is found to be 300 cubic centimeters at a temperature of 0° C. and pressure of 75 centimeters of mercury. What must be the temperature of the gas in order that the volume may be 350 cubic centimeters when the pressure is 76 centimeters of mercury?

20. The volume of a certain mass of gas is found to be 784 cubic centimeters at a pressure of 75 centimeters of mercury and a temperature of 7° C. What must be the pressure in centimeters of mercury if the volume of the gas becomes 900 cubic centimeters when the temperature is 27° C.?

Linear Expansion of Solids. A solid expands when heated, and the increase in length of a substance in the form of a bar is approximately proportional to the original length of the bar and to the change in temperature, provided that the change in temperature is small.

The *coefficient of linear expansion* of a body is the ratio of the increase in length per degree rise in temperature to the length of the body at 0° C.

If k represents the coefficient of linear expansion of a bar the length of which at 0° C. is represented by l_0, and the length of which at t_1° C. is represented by l_1, we have

$$l_1 = l_0 + l_0kt_1 \equiv l_0(1 + kt_1).$$

If the length of the bar at t_2° C. is represented by l_2, we have

$$l_2 = l_0 + l_0kt_2 \equiv l_0(1 + kt_2).$$

Hence, from the relations above, we have the following proportion:

$$\frac{l_1}{l_2} = \frac{1 + kt_1}{1 + kt_2}.$$

21. An iron steam pipe is 75 feet in length at 0° C. What does its length become when steam at a temperature of 112° C. is passed through it, provided that the coefficient of linear expansion of iron is .000012?

22. The distance between two graduations on a brass bar is exactly one meter at 25° C. What is the distance between the graduations at 60° C., provided that the coefficient of linear expansion of brass is .000018?

23. A lightning rod which is made of copper is 40 feet in length when at a temperature of 0° C. Find the length of the rod when the temperature is 30° C., provided that the coefficient of linear expansion of copper is .000017.

24. What allowance should be made for expansion in a 1700-foot span of a steel bridge, assuming that the highest summer temperature is 40° C. and the lowest winter temperature is 20° C., provided that the coefficient of linear expansion of steel is .000011?

25. If a steel rule is exactly one foot in length at a temperature of 0° C., find the error in the rule, expressed as a fraction of an inch, at a temperature at 20° C., provided that the coefficient of linear expansion of steel is .000011.

III. VARIATION

27. One variable number or quantity is said to be a **function** of a second if a change in the value of the second produces in general a change in the value of the first. (See also Chap. IX. § 1.)

If the values of two variables are so related that the value of the first variable is regarded as depending upon the value which may be assigned to the second variable, then the first is called the **dependent variable** and the second the **independent variable.**

E. g. If the values of x and y be restricted to satisfy the conditional equation, $x = y + 7$, the value of either variable depends upon the value which may be assigned to the other.

If y be selected as the independent variable, then by assigning successively the values 0, 1, 2, 3, 4, etc., to y, the dependent variable x will assume successively the values 7, 8, 9, 10, 11, etc.

One Independent Variable

28. Two variable numbers or quantities are said **to vary directly** one as the other, if when the value of one is changed the value of the other is changed also, and *in the same ratio.*

E. g. The distance travelled in an hour by a person walking uniformly varies directly as the rate.

29. The symbol of direct variation, $\propto$, (which is read "varies directly as," or simply "varies as"), when placed between two variables denotes that their ratio is a constant number.

E.g. Representing some constant number by c, we may from $x \propto y$, obtain $\frac{x}{y} = c$.

30. As a result of the definition above, we have the following **Fundamental Principle:** *If the value of x depends on the value of y and y alone in such a way that if (x, y) and (x_1, y_1) represent two pairs of corresponding values, and if for every two such pairs we have $x : x_1 = y : y_1$, then it follows that x is a constant multiple of y.*

For, if
$$\frac{x}{x_1} = \frac{y}{y_1}, \tag{1}$$

it follows that
$$\frac{x}{y} = \frac{x_1}{y_1}. \tag{2}$$

If we let x_1 and y_1 denote any particular pair of corresponding values of x and y, while x and y denote any other pair of corresponding values, it appears that, if the ratio of all pairs of values are equal to the ratio of the same pair, x_1 and y_1, as in (2), we may denote the value of the ratio of this particular pair chosen for reference by some constant, c.

Hence (2) becomes $x/y = c$ (3), or $x = cy$ (4), in which c denotes some constant.

31. Since the ratio of any pair of corresponding values, x and y, is equal to the ratio of any other pair of corresponding values, it follows that any two pairs of corresponding values may be used to form a proportion.

Accordingly $x \propto y$ is often read, "x is proportional to y."

32. From $x = cy$, it follows that the value of the constant c may be obtained, provided that the value of x corresponding to any specified value of y is known.

Ex. 1. If $x \propto y$ and $x = 6$ when $y = 3$, find x when $y = 13$.

From $x \propto y$ we obtain the conditional equation $x = cy$, in which c denotes some constant.

Substituting 6 and 3 for x and y respectively, we obtain

$$6 = c \cdot 3 \quad \text{or} \quad c = 2. \qquad \text{Hence} \quad x = 2y.$$

Accordingly, when $y = 13$, we have $x = 2 \cdot 13$ or 26.

33. One variable number or quantity is said **to vary inversely as a second,** *or to be inversely proportional to a second*, when the first number varies as the reciprocal of the second.

We may indicate that x varies inversely as y, or is inversely proportional to y, by writing $x \propto \frac{1}{y}$.

From the notation $x \propto \frac{1}{y}$ we have $\frac{x}{\frac{1}{y}} = c$, in which c denotes some constant. Hence $xy = c$.

E.g. If eight men can do a given amount of work in 24 hours, by *reducing* the number of men to four, the time required to do the same work would be *increased* to 48 hours; two men would require 96 hours, and one man alone 192 hours.

It may be seen that the products 8×24, 4×48, 2×96, and 1×192, are all equal.

Two or More Independent Variables

34. One variable number or quantity is said **to vary as two others jointly,** *or to be proportional to two others jointly*, if it varies as the product of the other two.

We may indicate that x varies as y and z jointly by writing $x \propto yz$, from which it follows that $\frac{x}{yz} = c$, in which c denotes a constant.

E.g. The distance passed over by a body moving uniformly is proportional both to the rate and the time.

35. One variable number or quantity is said **to vary directly as a second and inversely as a third** if it varies directly as the second and inversely as the third *jointly*.

We may indicate that x varies directly as y and inversely as z by writing $x \propto \frac{y}{z}$, from which we obtain $\frac{x}{\frac{y}{z}} = c$, in which c denotes some constant. Accordingly $\frac{xz}{y} = c$.

E.g. The time required to complete a journey varies directly as the distance and inversely as the rate.

36. From the definitions above we obtain directly the following

General Principles

In the proofs of the following principles, x, y, z, etc., represent variables, and c, c_1, etc., represent constants.

(i.) ***If $x \propto y$ and $y \propto z$, then $x \propto z$.***

For, if $x \propto y$ then $\frac{x}{y} = c$, and if $y \propto z$, then $\frac{y}{z} = c_1$.

Therefore $\frac{x}{y} \times \frac{y}{z} = cc_1$, or $\frac{x}{z} = cc_1$. Hence, $x \propto z$.

(ii.) ***If $x \propto z$ and $y \propto w$, then $xy \propto zw$.***

For, if $x \propto z$, then $\frac{x}{z} = c$, and if $y \propto w$, then $\frac{y}{w} = c_1$.

Therefore, $\frac{x}{z} \times \frac{y}{w} = cc_1$, or $\frac{xy}{zw} = cc_1$; that is, $xy \propto zw$.

(iii.) ***If $x \propto yz$, then $x/y \propto z$, and $x/z \propto y$.***

For, if $x \propto yz$, then $\frac{x}{yz} = c$.

Hence $\frac{x \div y}{yz \div y} = c$, or $\frac{\frac{x}{y}}{z} = c$; that is, $x/y \propto z$.

Similarly $x/z \propto y$.

(iv.) ***If $x \propto z$ and $y \propto z$, then $x + y \propto z$ and $x - y \propto z$.***

For, if $x \propto z$, then $\frac{x}{z} = c$, and if $y \propto z$, then $\frac{y}{z} = c$.

Therefore $\frac{x}{z} + \frac{y}{z} = c + c_1$.

That is, $\frac{x + y}{z} = c + c_1$.

Hence, $x + y \propto z$.

Also, $\frac{x}{z} - \frac{y}{z} = c - c_1$.

That is, $\frac{x - y}{z} = c - c_1$.

Hence, $x - y \propto z$.

(v.) *If the value of x depends upon the values of both y and z, and on these alone, and if $x \propto y$ when z is constant and $x \propto z$ when y is constant, then $x \propto yz$ when both y and z vary.*

To establish this principle we will suppose that (x_1, y_1, z_1), (x, y_2, z_1), and (x_2, y_2, z_2) represent three sets of corresponding values of the variables x, y, and z. These values are such that in passing from the first set to the second set the value of z, represented by z_1, remains constant, and in passing from the second set to the third set the value of y, represented by y_2, remains constant.

Then from (x_1, y_1, z_1) and (x, y_2, z_1), since z_1 remains constant,

we have, $$\frac{x_1}{x} = \frac{y_1}{y_2}. \qquad (1)$$

Also from (x, y_2, z_1) and (x_2, y_2, z_2), since y_2 remains constant,

we have, $$\frac{x}{x_2} = \frac{z_1}{z_2}. \qquad (2)$$

Accordingly, multiplying together the corresponding members of (1) and (2), we have

$$\frac{x_1}{x_2} = \frac{y_1 z_1}{y_2 z_2}.$$

Hence, $$\frac{x_1}{y_1 z_1} = \frac{x_2}{y_2 z_2}.$$

Therefore the corresponding values of (x_1, y_1, z_1) and (x_2, y_2, z_2) are proportional; that is, $x \propto yz$.

The principle may be shown to apply when x depends for its value upon the values of three or more variables, $y, z, w, \cdots$.

Ex. 2. If $x \propto y$ and $x = 20$ when $y = 4$, find x when $y = 7$.

If $x \propto y$ we may assume $x = cy$, (1), in which c is a constant. Since this equation is satisfied when $x = 20$ and $y = 4$, we have $20 = 4c$. Therefore $c = 5$.

To find x when $y = 7$, we may substitute 5 for c and 7 for y in equation (1), and obtain $x = 35$.

Ex. 3. If w varies as x and y jointly, and $w = 42$ when $x = 2$ and $y = 3$, find w when $x = 4$ and $y = 5$.

From the given conditions we may assume that $w = cxy$, (1) in which c denotes some constant.

Substituting the given values for w, x and y in (1), we find that $c = 7$.

We may substitute 7, 4, and 5 for c, x, and y, respectively, and obtain $w = 140$.

EXERCISE XXVI. 4

1. If $x \propto y$ and when $y = 8$, $x = 56$, find x when $y = 1$.
2. If $x \propto 1/y$ and $x = 6$ when $y = 2$, find x when $y = 9$.
3. If $x \propto 1/y$ and $x = 1/2$ when $y = 16$, find y when $x = 2$.
4. If x varies jointly as y and z, and $x = 24$ when $y = 4$ and $z = 2$, find x when $y = 5$ and $z = 3$.

5. If x varies directly as y and inversely as z, and $x = 22$ when $y = 14$ and $z = 7$, find x when $y = 36$ and $z = 9$.

6. Find x when $y = 3$, if $x \propto 1/y^2$ and $x = 9$ when $y = 2$.

7. Find y when $x = 15$ and $w = 3$, if y varies as x and w jointly, and $y = 1$ when $x = 12$ and $w = 1/8$.

8. If $x \propto y$, show that $ax \propto ay$ when a is either a constant or a variable.

9. If $x \propto 1/y$ and $y \propto 1/z$ show that $x \propto z$.

The area of a circle varies as the square of its diameter.

10. If the area of a circle the radius of which is 14 feet is 616 square feet, find the area of a circle the radius of which is 18 feet.

11. Show that the area of a circle, the diameter of which is 10 inches, is equal to the sum of the areas of two circles, the diameters of which are 8 inches and 6 inches respectively.

12. The volume of a sphere varies as the cube of its radius, and the volume of a sphere of which the radius is 3 inches is $113\frac{1}{7}$ cubic inches. Find the volume of a sphere the radius of which is 5 inches.

13. Prove that the sum of the volumes of three spheres, the radii of which are 3, 4, and 5 inches respectively, is equivalent to the volume of a sphere the radius of which is 6 inches.

Problems in Physics

It has been found by experiment that the distance passed over by a falling body, moving freely and receiving no initial impulse, varies directly as the square of the time.

14. If a body falls 16 feet in 1 second, how far will it fall in 8 seconds?

15. A stone is dropped from the top of a cliff and strikes the bottom of the cliff in $3\frac{1}{4}$ seconds, nearly. What is the approximate height of the cliff?

16. From what height must a body fall from a state of rest to reach the earth after 10 seconds?

It has been found by experiment that the velocity acquired by a body falling freely from a state of rest varies directly as the time.

17. If the velocity of a falling body is 180 feet per second at the end of 5 seconds, what will be its velocity at the end of 9 seconds?

18. If the velocity of a falling body is 128 feet per second at the end of 4 seconds, what will be its velocity at the end of 7 seconds?

The intensity of illumination from a source of light varies inversely as the square of the distance from the source.

19. A candle is placed at a distance of 1 foot from a cardboard screen, and a second candle is placed at a distance of 7 feet from the screen on the other side. Compare the intensity of illumination on the two sides of the screen.

20. A gas jet which is 16 feet from a photometer, and a candle which is 4 feet from the photometer, are found to illuminate it equally. Compare the intensity of light from the two sources.

21. A "standard" 16-candle-power lamp, when placed at a distance of 51 centimeters from a screen, is found to illuminate it with the same intensity as an incandescent light placed at a distance of 49 centimeters from the screen. What is the candle power of the incandescent light?

When an elastic body is stretched, it is found that within the limits of perfect elasticity the elongations of the body are directly proportional to the forces producing them.

The elongation E produced by a stretching force F upon a substance in the form of a rod of diameter D and length L, varies directly as the force F, directly as the length L, and inversely as the cross section, — that is, inversely as the square of the diameter D.

That is, $$E \propto \frac{FL}{D^2}.$$

22. If a certain wire, 1/10 of an inch in diameter and 36 inches in length, stretches 3 inches under a force of 18 lbs., how much will it stretch under a force of 24 lbs?

23. A certain wire, the diameter of which is 1/10 of an inch and the length of which is 5 feet, is increased in length 4 inches by a force of 24 lbs. Find the length of a second wire of the same material and diameter if a force of 40 lbs. increases it in length by 7 inches.

24. If a wire which is 1/16 of an inch in diameter and 25 feet in length stretches 3 inches under a force of 15 lbs., how long is a wire the diameter of which is 1/20 of an inch, if a force of 40 lbs. produces an increase in length of 2 inches?

MENTAL EXERCISE XXVI. 5. **Review**

Solve each of the following equations:

1. $(x-6)^2 = 16.$
2. $(x-8)^2 = 4x^2.$
3. $x + \frac{1}{x} = 9 + \frac{1}{9}.$
4. $\sqrt{x} = 0.$
5. $\sqrt[3]{y} = 0.$
6. $\sqrt[n-1]{y} = n.$

7. Show that $x^2 - 1 = 0$, if $x - 1 = 0$.
8. Show that $x^2 - 4 = 0$, if $x - 2 = 0$.
9. Show that $x^2 - 9 = 0$, if $x + 3 = 0$.
10. Show that $a^2 - ab + b^2 = 0$, if $a + b = 0$.

Simplify each of the following:

11. $\dfrac{\sqrt{a+b}}{\sqrt{a}+\sqrt{b}}$.
12. $\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x-y}}$.
13. $(3+\sqrt{-2})(3-\sqrt{-2})$.
14. $(2+\sqrt{-3})(2-\sqrt{-3})$.
15. $(2\sqrt{-3}-3\sqrt{2})^2$.
16. $(5\sqrt{-2}-2\sqrt{-5})^2$.

Distinguish between

17. $-2 \div \sqrt{-4}$ and $-4 \div \sqrt{-2}$.
18. $-5 \div \sqrt{-25}$ and $-25 \div \sqrt{-5}$.

19. Simplify $\sqrt{250\,a^3}$; $\sqrt[3]{\dfrac{x^3y}{2\,z^2}}$; $\sqrt[6]{16\,a^6b^2}$.

Express the following as entire surds:

20. $2\sqrt[a]{a^2}$.
21. $a\sqrt[b]{b^a}$.
22. $3\,x\sqrt[3]{x}$.
23. $\dfrac{1}{2}\sqrt{3\,a}$.
24. $\dfrac{a}{3}\sqrt[3]{9\,b}$.
25. $\dfrac{1}{2\,c}\sqrt[4]{8\,c}$.

Simplify and express with positive exponents:

26. $x^2\left(\dfrac{x}{y}\right)^{-2}$.
27. $a^{-2}\left(\dfrac{b}{a}\right)^{-1}$.
28. $\left(\dfrac{a^{-1}}{b^{-2}}\right)^{-2}$.

Solve each of the following equations:

29. $\dfrac{1}{3}=\dfrac{64}{x^2}$.
30. $x^{\frac{2}{3}}=4$.
31. $x^{\frac{2}{3}}=9$.
32. $x^{\frac{3}{2}}=1$.
33. $x^{\frac{3}{2}}=-1$.
34. $\dfrac{\sqrt[3]{x}}{3}=1$.

35. Rationalize the denominators of $\dfrac{a^2-1}{\sqrt{a}+1}$ and $\dfrac{\sqrt{x}-1}{\sqrt{x}-1}$.

From each of the following conditional equations find the ratio of x to y:

36. $2\,x = 3\,y$.
37. $5\,x = 2\,y$.
38. $7\,y = 4\,x$.

39. $8x = y.$ 40. $x = \frac{y}{5}.$ 41. $x = \frac{6}{11}y.$

Distinguish between

42. $\frac{a}{2} + \frac{b}{2}$ and $a^{\frac{1}{2}} + b^{\frac{1}{2}}.$ 43. $(x - y)^{-1}$ and $x^{-1} - y^{-1}.$

Find the value of each of the following expressions:

44. $\left(\frac{1}{2}\right)^{-3}.$

45. $\frac{\frac{1}{3}}{\left(\frac{1}{2}\right)^{-1}}.$

46. $\left(\frac{1}{2}\right)^{-2}\left(\frac{1}{3}\right)^{-3}.$

47. $\left(\frac{1}{2^3}\right)^{-2}.$

48. $\left(\frac{1}{2}\right)^{-1} + \left(\frac{1}{3}\right)^{-1}.$

49. $\left(\frac{1}{2^{-2}}\right)^{-2}.$

50. $3^{-2} - 2^2.$

51. $2^{-2}.$

52. $-2^2.$

53. $2^{-2} - 2^2.$

54. $3^{-3} - 3.$

55. $(\sqrt{-2})^3 + (-\sqrt{2})^3.$

Show that

56. $12^3 \cdot 8^2 = 2^{12} \cdot 3^3.$ 57. $6^3 \cdot 18^5 = 2^8 \cdot 3^{13}.$

Simplify each of the following:

58. $a^{\frac{x}{y}} \div a^{\frac{y}{x}}.$

59. $x^{\frac{3}{2}} \div x^{\frac{2}{3}}.$

60. $a^{\frac{3}{4}} \div a^{-\frac{3}{4}}.$

61. $\frac{a^2}{\sqrt{a}}.$

62. $\frac{b^2}{\sqrt{b}}.$

63. $\frac{c^3}{\sqrt[3]{c^2}}.$

64. $\frac{d}{\sqrt[3]{d}}.$

65. $(a \div a^{\frac{1}{n}})^n.$

66. $\frac{7ab}{5a^0 + 2b^0}.$

67. $\frac{a}{b}\sqrt[n]{\frac{b^{n+1}}{a^{n-1}}}.$

68. $\left(\frac{5a^{\frac{1}{4}}}{3} + 2\right)\left(\frac{5a^{\frac{1}{4}}}{3} - 2\right).$

69. $(\sqrt{a} - \sqrt{-a})^2.$

70. $\frac{9a^{-2} - b^{-\frac{1}{3}}}{3a^{-1} - b^{-\frac{1}{6}}}.$

Find the mean proportional between

71. am^3 and $am.$ 72. b^3c and $bc^3.$

Simplify the following ratios:

73. $2/3 : 4/5$.

74. $3/7 : 3/8$.

75. $5/6 : 11/6$.

76. $3/4 : 4/3$.

77. $a/b : c/b$.

78. $x/y : z/x$.

79. $a/x : b/x$.

80. $m/n : n/q$.

Express the following proportions in the least numbers possible without altering the terms containing x:

81. $\frac{24}{40}=\frac{36}{x}$.

82. $\frac{24}{15}=\frac{6}{x}$.

83. $\frac{30}{4}=\frac{35}{x}$.

84. $\frac{12}{42}=\frac{x}{6}$.

85. $\frac{x}{10}=\frac{20}{75}$.

86. $\frac{4}{36}=\frac{x}{24}$.

87. $\frac{16}{15}=\frac{12}{x}$.

88. $\frac{x}{21}=\frac{8}{60}$.

89. $\frac{4}{90}=\frac{x}{10}$.

90. $\frac{4}{88}=\frac{x}{6}$.

91. $\frac{54}{24}=\frac{21}{x}$.

92. $\frac{8}{80}=\frac{x}{24}$.

CHAPTER XXVII

THE PROGRESSIONS

1. A SUCCESSION of numbers, each of which is formed according to some definite law, is called a **sequence.**

The successive numbers are called the **terms** of the sequence.

2. It should be understood that we cannot take numbers at random to form a sequence.

There must be some definite relation between the numbers chosen such that when the number of any particular term of the sequence is known, its value can be computed.

Hence it is possible to determine whether or not any specified number occurs in a given sequence.

E. g. $1, 2, 3, 4, \cdots\cdots, n, \cdots\cdots.$

$1, 4, 9, 16, \cdots\cdots, n^2, \cdots\cdots.$

$1/2, 2/3, 3/4, \cdots\cdots, n/(n+1), \cdots\cdots.$

3. A sequence of numbers, $a_1, a_2, a_3, a_4, \cdots\cdots, a_n, \cdots\cdots$, is said to be **given** or **known** if the value of any specified term is known or can be found when the position of the term in the sequence is given.

4. The law governing the formation of the successive terms may be such that any term after the first may be obtained by performing some definite operation upon the term which immediately precedes it.

E. g. In the sequence $5, 8, 11, 14, \cdots\cdots$, each term is obtained by adding 3 to the next preceding term.

In the sequence $2, 6, 18, 54, 162, \cdots\cdots$, each term is obtained by multiplying by 3 the term which immediately precedes it.

5. The law may be such that any term may be found when its location in the sequence is specified.

E. g. In the sequence $1, 4, 9, 16, \cdots\cdots, n^2 \cdots\cdots$, the seventh term is $7^2 = 49$; the tenth term is $10^2 = 100$; etc.

In the sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots\cdots, \frac{n}{n+1}, \cdots\cdots$, the ninth term is $\frac{9}{9+1} = \frac{9}{10}$; the twentieth term is $\frac{20}{21}$, etc.

6. A sequence is said to be **finite** if it contains a finite or limited number of terms, and **infinite** if it contains an infinite or unlimited number of terms.

I. Arithmetic Progression

7 An **arithmetic progression** (A. P.) is a sequence of numbers each of which, after the first, may be obtained by adding to the number which precedes it in the sequence a definite number called the **common difference.**

E. g. In the A. P. 4, 6, 8, 10, 12, 14, the common difference is 2;
In $-22, -12, -2, 8, 18$, the common difference is 10;
In $10, 7, 4, 1, -2, -5, -8$, the common difference is -3.

8. An arithmetic progression is said to be **increasing or decreasing** according as the common difference is positive or negative.

9. In order that the terms of the sequence, $a_1, a_2, a_3, a_4, \cdots\cdots$, a_n, shall form an arithmetic progression, it is necessary that

$$a_2 - a_1 = a_3 - a_2 = \cdots\cdots = a_n - a_{n-1}.$$

10. If a_1 represents the first term, d the *common difference* (that is, the difference between every two consecutive terms), and n the number of the place of any specified term in the progression, any arithmetic progression may be represented by the general expression

$$a_1,\ a_1 + d,\ a_1 + 2\,d,\ a_1 + 3\,d,\ \cdots\cdots,\ a_1 + (n-1)d.$$

11. The *n*th term. Observe that, since each term of the arithmetic progression after the first is obtained by adding d to the preceding term, the coefficient of d in any specified term is always less by unity than the number of the term.

Accordingly, in the nth term, $a_1 + (n-1)\,d$, the coefficient of d is $n-1$.

Representing the nth term by a_n, we have the formula

$$\boldsymbol{a_n = a_1 + (n-1)\,d.}$$

Ex. 1. Find the ninth term, a_9, of an A. P. the first term of which is 5 and common difference 3.

Substituting 5, 9, and 3 for a_1, n, and d respectively, in the formula above, we may write: $a_9 = 5 + (9 - 1)\,3 = 29$.

Ex. 2. Find the tenth term of the A. P. 11, 7, 3, $-1, \cdots\cdots$.

We have $\quad a_1 = 11,\ n = 10,\ d = -4.$

Therefore, $\quad a_{10} = 11 + (10 - 1)(-4) = -25.$

12. Since the formula $a_n = a_1 + (n - 1)d$ contains four quantities, a_n, a_1, n, and d, it follows that if any three of them are known the fourth may be found.

Ex. 3. Write the A. P. the first and fourteenth terms of which are 9 and 87 respectively.

We have $\quad a_1 = 9,\quad a_{14} = 87,\quad n = 14.$

Hence, substituting in $a_n = a_1 + (n - 1)d$, we have $87 = 9 + 13\,d$. Hence $d = 6$.

Hence, the required A. P. is 9, 15, 21, 27, $\cdots\cdots$, 81, 87.

13. From the principles relating to the solution of simultaneous linear equations, it follows that, if the values of any two of the quantites a_n, a_1, n, or d be unknown, two conditional equations are necessary to determine their values.

Hence, *an arithmetic progression may be written if any two of its terms are given.*

Ex. 4. Write the A. P. the fourth and thirteenth terms of which are 29 and 92 respectively.

We have $a_4 = 29$, and $a_{13} = 92$. Hence, when $n = 4$, the term a_n represents 29; and when $n = 13$, the term a_n represents 92.

Substituting these values in the formula $a_n = a_1 + (n - 1)d$,

we have $\quad 29 = a_1 + (4 - 1)d \quad (1),$

and also $\quad 92 = a_1 + (13 - 1)d \quad (2).$

Solving (1) and (2), we find that $a_1 = 8$ and $d = 7$.

Accordingly the required A. P. is 8, 15, 22, 29, 36, $\cdots\cdots$, 85, 92.

The required arithmetic progression may also be obtained by the following method:

Since the thirteenth term of the progression is the tenth term of the progression of which the given fourth term, 29, is the first term, we may reduce our problem to that of finding an A. P. the first term of which is 29 and the tenth term of which is 92. After determining this progression we may write the three necessary terms preceding the term 29, now considered as a first term.

We have $\quad 92 = 29 + 9\,d, \quad$ hence, $\quad d = 7.$

By writing three terms before the "first term," 29, and nine terms after 29, we obtain the same sequence as above.

Exercise XXVII. 1

1. Find the 12th and 15th terms of 2, 6, 10, $\cdots\cdots$.
2. Find the 10th and 16th terms of 3, 9, 15, $\cdots\cdots$.
3. Find the 17th and 11th terms of -8, -3, 2, $\cdots\cdots$.
4. Find the 7th and 13th terms of $2/3$, 1, $4/3$, $\cdots\cdots$.
5. Find the 20th and 40th terms of $2/15$, $-1/30$, $-1/5$, $\cdots\cdots$.
6. Find the 12th and 21st terms of -4, -13, -22, $\cdots\cdots$.
7. Find the 10th and 37th terms of $\frac{1}{2}$, 1, $1\frac{1}{2}$, $\cdots\cdots$.
8. Find the 8th and 13th terms of $\frac{a-1}{a}$, $\frac{a-2}{a}$, $\frac{a-3}{a}$, $\cdots\cdots$.
9. Find the 14th and 19th terms of $(a-3)$, $4a$, $(7a+3)$, $\cdots\cdots$.
10. Find the 17th and 31st terms of $(5x-4)$, $(2x-2)$, $-x$, $(-4x+2)$, $\cdots\cdots$.

Find the last term in each of the following arithmetic progressions:

11. 4, 8, 12, $\cdots\cdots$ to 36 terms.
12. 8, 2, -4, $\cdots\cdots$ to 93 terms.
13. -23, -17, -11, $\cdots\cdots$ to 100 terms.
14. 1, 1.3, 1.6, $\cdots\cdots$ to 21 terms.
15. $(a+5b)$, $(a+3b)$, $(a+b)$, $\cdots\cdots$ to 13 terms.

Arithmetic Means

14. If three numbers are in arithmetic progression, the one that lies between the other two is called the **arithmetic mean** of these two.

If a, A, b, be an arithmetic progression, A is called the arithmetic mean of a and b.

From the A. P. represented by a, A, b, we have by definition $A-a=b-A$.

Solving for A, we have $\boldsymbol{A=\frac{a+b}{2}}$.

That is, *the arithmetic mean of two numbers is one-half their sum.*

E. g. The arithmetic mean of 5 and 17 is 11.

15. All of the terms of an A. P. which lie between two specified terms are called the **arithmetic means** of these two terms.

E.g. In the A.P. 2, 3, 4, 5, 6, 7, 8, the five numbers 3, 4, 5, 6, and 7 are called the arithmetic means of 2 and 8.

In the A.P. 2, $3\frac{1}{5}$, $4\frac{2}{5}$, $5\frac{3}{5}$, $6\frac{4}{5}$, 8, the four numbers $3\frac{1}{5}$, $4\frac{2}{5}$, $5\frac{3}{5}$, and $6\frac{4}{5}$ are called the arithmetic means of 2 and 8.

In the A.P. 2, 4, 6, 8, the two numbers 4 and 6 are called the arithmetic means of 2 and 8.

In the A.P. 2, 5, 8, the single number 5 is called the arithmetic mean of 2 and 8.

Ex. 1. Insert 11 arithmetic means between 20 and 116.

The given numbers 20 and 116, taken together with 11 arithmetic means, form an A.P. containing 13 terms.

Hence, using the formula $a_n = a_1 + (n-1)d$, by substituting the values $a_{13} = 116$, $a_1 = 20$, and $n = 13$, we obtain $116 = 20 + 12d$. Hence $d = 8$.

Accordingly the required arithmetic means are

28, 36, 44, 52, 60, 68, 76, 84, 92, 100, 108.

16. The student should note the distinction between the arithmetic mean or **average** of n numbers, and the n arithmetic means inserted between two numbers.

E.g. The arithmetic mean or average of the five numbers 2, 6, 15, 50, 67 is $(2 + 6 + 15 + 50 + 67)/5 = 28$, but these numbers do not form an A.P., and cannot be regarded as being arithmetic means of any two numbers.

EXERCISE XXVII. 2

Find the arithmetic mean of

1. 26 and 32.
2. 17 and 11.
3. 8 and 47.
4. $3x$ and $11x$.
5. $x - y$ and $y - x$.
6. $\frac{a+b}{2}$ and $\frac{2}{a+b}$.
7. Find the thirty arithmetic means of 18 and 142.
8. Find the nineteen arithmetic means of -27 and 113.
9. Find the seventeen arithmetic means of 25 and 115.
10. Find the eight arithmetic means of 19 and 23.
11. Find the twelve arithmetic means of 2 and 3.
12. Find the fifteen arithmetic means of 2 and 9.

17. A **series** is the sum (or the limit of the sum) of a succession of numbers, each formed according to some common law.

The successive numbers are called the **terms** of the series.

E. g. If 2, 4, 6, 8, $\cdots\cdots$, $2n$ be a given sequence of numbers, the expression obtained by writing the sum of the successive terms of the sequence is called the series $2 + 4 + 6 + 8 + \cdots\cdots + 2n$.

18. A series is said to be **finite or infinite** according as the number of its terms is finite or infinite.

19. Although a succession of numbers forming a sequence might be described as a series of numbers, the word "series," as used in mathematics, has especial reference to addition.

Accordingly, we speak of the succession or *sequence of numbers* $a_1, a_2, a_3, a_4, \cdots\cdots, a_n$; but by writing the sum of these numbers we obtain the *series* $a_1 + a_2 + a_3 + a_4 + \cdots\cdots + a_n$.

20. **An arithmetic series** is a series the terms of which are in arithmetic progression.

Sum of the Terms of an Arithmetic Progression

21. The sum S_n of the terms of an arithmetic progression of n terms may be found as follows:

$$\begin{array}{l} S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \cdots\cdots + (a_n - 2d) + (a_n - d) + a_n, \\ S_n = a_n + (a_n - d) + (a_n - 2d) + \cdots\cdots + (a_1 + 2d) + (a_1 + d) + a_1. \\ \hline \end{array}$$

By Addition

$$2S_n = (a_1+a_n)+(a_1+a_n)+(a_1+a_n)+\cdots\cdots+(a_1+a_n)+(a_1+a_n)+(a_1+a_n)$$
$$= n(a_1 + a_n).$$

Hence

$$\boldsymbol{S_n = \frac{n}{2}(a_1 + a_n).} \qquad (1)$$

Expressing the last term, a_n, in terms of a_1, n, and d, by means of the formula $a_n = a_1 + (n - 1)d$, the expression above may be written

$$S_n = \frac{n}{2}[a_1 + a_1 + (n - 1)d].$$

$$\text{Or,} \quad \boldsymbol{S_n = \frac{n}{2}[2a_1 + (n - 1)d].} \qquad (2)$$

Ex. 1. Find the sum of 46 terms of the A. P. 11, 18, 25, $\cdots\cdots$.

Substituting the values $n = 46$, $a_1 = 11$, and $d = 7$, in the formula

$$S_n = \frac{n}{2}[2a_1 + (n - 1)d],$$

we obtain $S_{46} = \frac{46}{2}[2 \cdot 11 + 45 \cdot 7] = 7751.$

Ex. 2. Find the sum of the terms of the arithmetic series

$$5 + 8 + 11 + \cdots\cdots + 95.$$

We have $a_1 = 5$, $d = 3$, and $a_n = 95$.

To use either of the formulas for S_n, it is necessary to know the value of n, which may be found by means of the formula $a_n = a_1 + (n - 1)d$.

By substitution, $95 = 5 + (n - 1)3$. Hence $n = 31$.

Substituting in the formula, $S_n = \frac{n}{2}(a_1 + a_n)$,

we have $S_{31} = \frac{31}{2}(5 + 95) = 1550.$

22. The five numbers represented by a_n, a_1, d, n, and S_n are called the **elements of an arithmetic progression.**

Ex. 3. Write the A. P., having given the elements $S_{10} = 510$ and $a_{10} = 87$.

Substituting 510 for S_n, 87 for a_n and 10 for n in $S_n = \frac{n}{2}(a_1 + a_n)$, we obtain, $510 = 5(a_1 + 87)$. Hence $a_1 = 15$.

To write the required A. P. it is necessary to know d. Using the formula $a_n = a_1 + (n - 1)d$, we find that $d = 8$.

Hence the required A. P. is 15, 23, 31, 39, $\cdots\cdots$, 87.

Ex. 4. How many terms of the arithmetic series $-16 - 12 - 8 - 4 + 0 + 4 + \cdots\cdots$ must be taken to obtain the sum 72?

Substituting 72 for S_n, -16 for a_1, and 4 for d in $S_n = \frac{n}{2}[2a_1 + (n - 1)d]$,

we obtain $72 = \frac{n}{2}[-32 + (n - 1)4].$

Or, $72 = -16n + 2n^2 - 2n.$

Solving this quadratic equation for n, we obtain $n = 12$, and $n = -3$.

It will be found that the sum of the following twelve terms is 72:

$-16, -12, -8, -4, 0, 4, 8, 12, 16, 20, 24, 28.$

It may be observed that if, beginning with the last term 28, we count backward three terms, the sum is 72.

We have thus an interpretation for the negative value of n.

Ex. 5. How many terms of the A. P. 39, 34, 29, $\cdots\cdots$, must be taken in order that the sum shall be 168?

Substituting the values $a_1 = 39$, $d = -5$, and $S_n = 168$, in the formula $S_n = \frac{n}{2}[2a_1 + (n - 1)d]$, we obtain $336 = 78n - 5n^2 + 5n$, the solutions of which are found to be $n = 7$, and $n = 9\frac{3}{5}$.

It will be found that the sum of seven terms of the given arithmetic progression is 168.

The fractional value $n = 9\frac{3}{5}$ may be interpreted as meaning that the sum 168 is greater than the sum of nine terms and less than the sum of ten terms of the series.

In this particular arithmetic progression it will be found that 168 is the sum of nine terms increased by 3/5 of the common difference.

Exercise XXVII. 3

Find the sum of the terms of each of the following arithmetic progressions :

1. $3, 8, 13, \cdots\cdots, 33.$
2. $20, 18, 16, \cdots\cdots, 0.$
3. $25, 23, 21, \cdots\cdots, -15.$
4. $19, 32, 45, \cdots\cdots, 188.$
5. $\frac{1}{2}, \frac{3}{4}, 1, \cdots\cdots, 39\frac{3}{4}.$
6. $\frac{5}{6}, 0, -\frac{5}{6}, \cdots\cdots, -24\frac{1}{6}.$
7. $2/3, 14/15, 6/5, \cdots\cdots, 6.$
8. $7/12, 7/6, 7/4, \cdots\cdots, 7.$
9. $6, 10, 14, \cdots\cdots$ to 31 terms.
10. $-23, -27, -31, \cdots\cdots$ to 19 terms.
11. $-17, -6, 5, \cdots\cdots$ to 13 terms.
12. $-32, 23, 78, \cdots\cdots$ to 15 terms.
13. $7/2, 9/2, 11/2, \cdots\cdots$ to 16 terms.
14. $1/2, 0, -1/2, \cdots\cdots$ to 25 terms.
15. $3, 4\frac{1}{2}, 6, \cdots\cdots$ to 83 terms.
16. $\frac{a+1}{a}, \frac{a+2}{a}, \frac{a+3}{a}, \cdots\cdots$ to a terms.
17. $(c + d), (-c + 2d), (-3c + 3d), \cdots\cdots$ to 60 terms.

18. Find the sum of all of the even numbers from 20 to 80 inclusive.

19. If $a_1 = 22$, $d = 2$, and $S_n = 820$, find n.

20. Find d, knowing that $a_1 = 9$ and $a_{11} = 29$.

21. If $S_{11} = 66$ and $a_{11} = 23$, find a_6.

22. If $a_1 = 16$ and $d = -5$, find the terms the values of which lie between -70 and -100.

23. Find a_{20}, having given $a_6 = 2$ and $a_{15} = -2$.

24. Find the sum of the terms the values of which lie between 0 and 50 of the series the fourth term of which is 13 and the common difference of which is -7.

25. The product of three numbers in arithmetic progression is 48, and the first number is three times the last. Find the numbers.

26. Of three numbers which are in arithmetic progression the third is eleven times the first. Find the numbers if the sum of the three numbers is equal to the eighteenth term of the arithmetic progression $-18, -16, -14, \cdots\cdots$.

27. The sum of three numbers in arithmetic progression is 30 and the sum of their squares is 350. Find the numbers.

28. Find the number of terms in the arithmetic progression $1, 9, 17, \cdots\cdots$, the sum of which approximates most closely to 1000.

29. Find the sum of all of the multiples of 7 which lie between zero and 200.

30. Show that the sum of the first n odd numbers is equal to n^2.

Problems in Physics

31. A car starting from a state of rest moves down an inclined track, passing over distances of 1 foot the first second, 3 feet the second second, 5 feet the third second, etc. Find the distance passed over in one minute.

32. A ball starting from a state of rest rolls down an inclined board, passing over distances of 5 inches, 15 inches, 25 inches, etc., in successive seconds. Find the number of seconds required for the ball to pass over a distance of 15 feet.

33. If a ball, starting up an inclined plane, passes over 40 feet the first second, 36 feet the second second, 32 feet the third second, etc., find the number of seconds required by the ball to pass over a distance of 196 feet.

34. It is found that when a ball is thrown vertically upward the force of gravity diminishes the distance passed over in successive seconds by 32 feet per second (nearly). Find the distance passed over in 4 seconds by a ball which, when thrown vertically upward, rises to a height of 128 feet during the first second.

35. If the force of gravity increases the space passed over by a falling body in successive seconds by 32 feet per second, find the distance passed over in 6 seconds by a falling body which, when thrown downward, passes over a distance of 24 feet during the first second.

II. Harmonic Progression

23. A harmonic progression (H. P.) is a sequence of numbers the reciprocals of which are in arithmetic progression.

A **harmonic series** is a series of numbers the terms of which are in harmonic progression.

Principle: *If three numbers, represented by a, b, and c, are in harmonic progression, it follows that* $a : c = a - b : b - c$. (1)

For, if a, b, c are in harmonic progression, it follows by definition that $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$, is an arithmetic progression.

Hence, we have $$\frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}.$$

Hence, $$\frac{a-b}{ab} = \frac{b-c}{bc}.$$

Or $$c(a-b) = a(b-c).$$

That is, $$\boldsymbol{a : c = a - b : b - c.}$$

24. The numbers of any sequence are in harmonic progression if every three consecutive numbers are in harmonic progression.

25. When three numbers form a harmonic progression the middle number is called the **harmonic mean** of the other two.

E. g. The harmonic mean of 1/2 and 1/4 is 1/3.

26. The harmonic mean of two numbers, represented by a and b, may be found as follows:

Representing the harmonic mean by H, we have the harmonic progression, a, H, b.

Accordingly, $\frac{1}{a}, \frac{1}{H}, \frac{1}{b}$, must be an arithmetic progression.

Hence $$\frac{1}{H} - \frac{1}{a} = \frac{1}{b} - \frac{1}{H}.$$

Solving for H, we obtain $\boldsymbol{H = \frac{2ab}{a+b}}$.

Ex. 1. Find the harmonic mean of 4 and 12.

Substituting 4 for a and 12 for b in the formula $H = 2ab / (a + b)$, we obtain 6.

27. In any harmonic progression all of the terms lying between any two specified terms are called the **harmonic means** of these two terms.

E. g. 3/7, 3/8, 1/3, 3/10, 3/11 are harmonic means of 1/2 and 1/4.

28. Problems in harmonic progression are generally solved by obtaining the reciprocals of the terms and making use of the properties of the resulting arithmetic progression.

Ex. 2. Find the 12th term of the harmonic progression 1/4, 1/7, 1/10, 1/13, · · · · ·. The reciprocals of the terms of the given harmonic progression form the arithmetic progression, 4, 7, 10, 13, · · · · ·, the 12th term of which is found to be 37.

Accordingly, the required term of the given harmonic progression is 1/37.

29. There is no general formula for the sum of the terms of a harmonic progression.

EXERCISE XXVII. 4

1. Find the 8th term of 1/2, 1/3, 1/4, · · · · · · .

2. Find the 6th term of 1/50, 1/65, 1/80, · · · · · · .

3. Find the 17th term of 2, 3/2, 6/5, · · · · · · .

4. Find the 4th term of the H. P. the first term of which is 1/51 and the 13th term of which is 1/3.

5. Find the 17th and 18th terms of the H. P. the second and sixth terms of which are 1/11 and 1/27 respectively.

6. Find the H. P. in which the 6th term is 1/7 and the 11th term 1/13.

7. Find the H. P. in which the 37th term is 1/74 and the 13th term is 1/26.

8. Find the H. P. in which the 9th term is 1/4 and the 15th term is — 1/14.

9. Find the H. P. in which the third term is 5/6 and the sixth term is 1/3.

10. The first two terms in a harmonic progression are 14 and 7. Find the number of terms which lie between — 7 and — 2.

11. Find the harmonic mean of 3 and 9.

12. Find the harmonic mean of 1/7 and 1/8.

13. Find the harmonic mean of 1/20 and 1/30.

14. Find the two harmonic means of 3 and 10.

15. Insert 4 harmonic means between 5 and 15.

16. Insert 3 harmonic means between 1/4 and 1/324.

17. Insert 5 harmonic means between 1/7 and 1/22.

18. Insert 7 harmonic means between 1/9 and 1/65.

19. Find two numbers the sum of which is 20 and the harmonic mean of which is 15/2.

20. The difference between the arithmetic and harmonic means of two numbers is 99/10, and one of the numbers is four times the other. Find the numbers.

21. If $x + y$, $y + z$ and $z + x$ form a harmonic progression, show that y^2, x^2 and z^2 form an arithmetic progression.

III. GEOMETRIC PROGRESSION

30. A **geometric progression** (G. P.) is a sequence of numbers each of which, after the first, may be obtained by multiplying the number which precedes it in the sequence by some particular multiplier.

31. From this definition it follows that the quotient obtained by dividing any term in the progression, after the first, by the one which immediately precedes it, is the same for every two consecutive terms. The quotient thus obtained is called the **common ratio** and is usually denoted by r.

E. g. In the G. P. 2, 4, 8, 16, 32, the common ratio is 2.
In 81, 27, 9, 3, 1, 1/3, 1/9, the common ratio is 1/3.
In 100, – 20, 4, – 4/5, 4/25, – 4/125, the common ratio is – 1/5.

32. If a_1 represents the first term, r the common ratio, and n the number of the place of any specified term in the progression, any geometric progression may be represented by the general expression $a_1, a_1r, a_1r^2, a_1r^3, a_1r^4, \cdot \cdot \cdot \cdot, a_1r^{n-1}$.

33. Observe that in any specified term the exponent of the power to which r is raised is less by unity than the number of the term.

E. g. r^3 appears in the fourth term; r^9 in the tenth term, etc.

34. **The nth term.** It appears that a particular term, represented by a_n, may be calculated by means of the formula

$$a_n = a_1r^{n-1}.$$

That is, *in any geometric progression any term can be found by multiplying the first term by the common ratio raised to a power the exponent of which is equal to a number which is less by unity than the number of the required term.*

Ex. 1. Find the eighth term of the geometric progression 3, 6, 12, · · · · · .

The common ratio is obtained by dividing any one of the terms by the term immediately preceding it, for example $6 \div 3 = 2$.

Substituting 8 for n, 3 for a_1, and 2 for r in the formula $a_n = a_1r^{n-1}$,

we obtain $$a_8 = 3 \cdot 2^7 = 384.$$

Ex. 2. Find a_2, having given $a_7 = 12288$ and $a_5 = 768$.

Using the formula $a_n = a_1r^{n-1}$ we obtain two conditional equations in which a_1 and r may be regarded as unknowns.

$$12288 = a_1r^6, \ (1) \text{ and } 768 = a_1r^4. \quad (2)$$

Dividing the members of equation (1) by the corresponding members of equation (2), we obtain $16 = r^2$. Hence $r = \pm 4$.

Substituting for r in (2), we obtain $a_1 = 3$.

Substituting in $a_2 = a_1r$, we obtain the required second term.

$$a_2 = 3\ (\pm 4) = \pm\ 12.$$

35. A geometric progression is said to be **finite or infinite** according as the number of its terms is finite or infinite.

E. g. 1, 3, 9, 27, 81, is a finite G. P. the common ratio of which is 3.

1, 3, 9, 27, 81, · · · · · , is an infinite G. P. the common ratio of which is 3.

81, 27, 9, 3, 1, 1/3, 1/9, 1/27, · · · · · , is an infinite G. P. the common ratio of which is 1/3.

36. A geometric progression is said to be **increasing or decreasing** according as its successive terms increase or decrease.

The successive terms of a geometric progression increase or decrease according as the common ratio is greater than or less than unity.

37. Representing the first term of a geometric progression by a_1 and the common ratio by r it may be seen that the following general expression may be used to represent any geometric progression :

$$a_1,\ a_1r,\ a_1r^2,\ \cdot\cdot\cdot\cdot\cdot,\ a_1r^{n-1}.$$

The values of the different terms depend entirely upon the values which may be assigned to the letters a_1 and r, considered as independent variables.

Hence, in general, a geometric progression is completely determined when two independent conditions affecting the values of its terms are given.

EXERCISE XXVII. 5

Find the terms specified in each of the following geometric progressions :

1. The 8th and 9th terms of 2, 4, 8, 16, · · · · · .
2. The 9th and 11th terms of 20, 10, 5, · · · · · .
3. The 6th and 10th terms of 1/2, 1/4, 1/8, · · · · · .
4. The 6th and 12th terms of 1, 1/3, 1/9, · · · · · .
5. The 5th and 13th terms of 3, − 6, 12, · · · · · .
6. The 6th and 14th terms of 100, 50, 25, · · · · · .
7. The 10th and 20th terms of .1, .01, .001, · · · · · .
8. The 15th and 30th terms of m, m^2, m^3, · · · · · .
9. The 12th and 40th terms of $1/a^3$, $1/a^4$, $1/a^5$, · · · · · .
10. The 19th and 51st terms of 1, $1/c$, $1/c^2$, · · · · · .
11. The 11th and 14th terms of x, $-y$, y^2/x, · · · · · .
12. The cth and bth terms of a/b, a/bc, a/bc^2, · · · · · .

Geometric Means

38. When three numbers are in geometric progression, the middle number is called the **geometric mean** of the other two.

E. g. In the geometric progression 2, 4, 8, the geometric mean of 2 and 8 is 4; in the geometric progression 1, − 6, 36, the geometric mean of 1 and 36 is − 6.

39. The geometric mean of any two numbers, a and b, may be found as follows :

Denoting the geometric mean of a and b by G, we have the geometric progression a, G, b.

By the definition of a G. P. we have $G \div a = b \div G$.

Hence, $$\boldsymbol{G = \pm \sqrt{ab}}.$$

That is, *the geometric mean of two numbers is the square root of their product.*

It should be observed that, since the common ratio of a geometric progression may be either positive or negative, the successive terms of a geometric progression may be either all positive or all negative, or alternately positive and negative. Accordingly, the double sign ± should be employed before the radical sign in the expression for the geometric mean $G = \pm \sqrt{ab}$.

(Compare with Chapter XXVI. § 24 (ii.).)

Ex. 1. Find the geometric mean of 9 and 16.

Using the formula $G = \pm\sqrt{ab}$, we have $G = \pm\sqrt{9 \cdot 16} = \pm 12$.

40. In any geometric progression all of the terms lying between any two specified terms are called the **geometric means** of these two terms.

Ex. 2. Find the four geometric means of 1/32 and 32.

Taken together with the four geometric means, 1/32 and 32 may be regarded as the first and sixth terms, respectively, of a geometric progression.

Substituting 1/32 for a_1 and 32 for a_6 in the formula $a_n = a_1r^{n-1}$, we have

$$32 = \tfrac{1}{32}\, r^5$$

or

$$r^5 = 32 \cdot 32$$
$$= 2^{10}.$$

Hence

$$r = 4.$$

Writing the geometric progression the first term of which is 1/32 and common ratio 4, we obtain 1/32, 1/8, 1/2, 2, 8, 32.

Accordingly, the required geometric means are 1/8, 1/2, 2 and 8.

41. Representing the arithmetic mean of two numbers, a and b, by A, the geometric mean by G, and the harmonic mean by H, we have

$$A = \frac{a+b}{2}, \qquad (1)$$

$$G = \sqrt{ab}, \qquad (2)$$

$$H = \frac{2\,ab}{a+b}. \qquad (3)$$

From (1) and (2), $A - G = \dfrac{a+b}{2} - \sqrt{ab}$ (4)

$$= \frac{a - 2\sqrt{ab} + b}{2} \qquad (5)$$

$$= \frac{(\sqrt{a} - \sqrt{b})^2}{2}. \qquad (6)$$

The expression $(\sqrt{a} - \sqrt{b})^2/2$ is positive if a and b are real numbers and $a \neq b$.

It follows that A is greater than G for real values of a and b which are such that $a \neq b$.

From (1) and (3), $A \cdot H = \frac{a+b}{2} \cdot \frac{2ab}{a+b}.$ (7)

Or, $A \cdot H = ab.$ (8)

From (2), $G^2 = ab.$ (9)

From (9) and (8), $G^2 = A \cdot H.$ (10)

Hence, $\frac{A}{G} = \frac{G}{H}.$ (11)

That is, *the geometric mean of any two numbers is also the geometric mean of the arithmetic and harmonic means of the same numbers.*

Since $A > G$ it follows, from (11), that $G > H$.

Hence, for any positive numbers, $A > G > H$.

E. g. If $a = 1$ and $b = 49$, we have $A = 25$, $G = 7$, and $H = 1\frac{24}{25}$.

It should be observed that $25 > 7 > 1\frac{24}{25}$.

Exercise XXVII. 6

Find the geometric mean of:

1. 4 and 16.
2. 4 and 25.
3. 100 and 1.
4. 20 and 5.
5. 2 and 32.
6. 2 and 50.
7. 1/2 and 32.
8. 1/2 and 1/8.
9. Find the two geometric means of − 8 and 64.
10. Find the two geometric means of 3 and 192.
11. Find the two geometric means of 1/20 and 25/4.
12. Find three geometric means of 27/8 and 2/3.
13. Find three geometric means of 6 and 486.
14. Find the four geometric means of 1 and 1024.

42. A **geometric series** is a series of numbers the terms of which are in geometric progression.

43. A geometric series is said to be **increasing or decreasing** according as its terms form an increasing or decreasing geometric progression.

Sum of the Terms of a Geometric Progression

44. We can obtain an expression for the sum, S_n, of n terms of a given finite geometric progression, as follows:

$$S_n = a_1 + a_1r + a_1r^2 + a_1r^3 + \cdots\cdots + a_1r^{n-2} + a_1r^{n-1} \qquad (1)$$

$$rS_n = \qquad a_1r + a_1r^2 + a_1r^3 + \cdots\cdots + a_1r^{n-2} + a_1r^{n-1} + a_1r^n \qquad (2)$$

Hence, $$rS_n - S_n = a_1r^n - a_1. \qquad (3)$$

Or, $$S_n(r-1) = a_1(r^n - 1). \qquad (4)$$

Therefore, $$\mathbf{S_n = \frac{a_1(r^n-1)}{r-1}}. \qquad (5)$$

It should be observed that (2) is obtained by multiplying both members of (1) by r, and (3) results from subtracting the members of (1) from the corresponding members of (2).

45. When the first term a_1 and the nth term a_n are given, it is convenient to employ an alternative form for the sum S_n.

This may be obtained as follows:

The second member of the equation $S_n = \frac{a_1(r^n - 1)}{r - 1}$ may be expressed in the form $\frac{a_1r^n - a_1}{r - 1}$.

That is, $$S_n = \frac{a_1r^n - a_1}{r - 1}. \qquad (6)$$

The first term a_1r^n of the numerator may be transformed as follows: $$a_1r^n \equiv ra_1r^{n-1} \equiv a_nr.$$

Hence, we have, $$S_n = \frac{a_nr - a_1}{r - 1}. \qquad (7)$$

46. The sum of a finite number of terms of an arithmetic progression or of a geometric progression may be obtained by adding the consecutive terms as written.

When a formula is used it is unnecessary to add the terms separately. Hence, a formula is a labor-saving device.

Ex. 1. Find the sum of 7 terms of the geometric progression 5, 15, 45, $\cdots\cdots$.

Substituting 5 for a_1, 3 for r and 7 for n in the formula $S_n = \frac{a_1(r^n - 1)}{r - 1}$, we have $S_7 = \frac{5(3^7 - 1)}{3 - 1} = 5465$.

47. If $r = 1$, all the terms of a given geometric progression are equal, and hence the sum of n terms is na_1.

By taking n great enough, the sum na_1 can be made greater than any assignable number; hence it can be made infinitely great.

48. *By taking n great enough, the sum of n terms of a geometric progression in which r is numerically greater than unity may be made to exceed any assignable positive number.*

For, it appears from $S_n = \frac{a_1(r^n - 1)}{r - 1} \equiv \frac{a_1 r^n}{r - 1} - \frac{a_1}{r - 1}$ that if r has a value which is numerically greater than unity, the value of r^n, and accordingly of $a_1 r^n/(r - 1)$, may be made as great as we please by taking the value of n great enough.

Hence the value of S_n may be made greater than any assignable number, for $r > 1$.

49. The values represented by the five quantities a_n, a_1, r, n, and S_n are called the **elements** of a geometric progression.

It may be seen that n, representing the number of terms of a geometric progression, must be a positive integer, while the remaining elements, a_n, a_1, r, and S_n, may be positive or negative, integral or fractional.

EXERCISE XXVII. 7

Find the sum of the terms of each of the following progressions:

1. $2, 4, 8, \cdots\cdots 512$.
2. $20, -10, 5, \cdots\cdots$ to 6 terms.
3. $1/2, 1, 2, \cdots\cdots$ to 8 terms.
4. $1/2, 1/2^2, 1/2^3, \cdots\cdots$ to 9 terms.
5. $12, 4, 4/3, \cdots\cdots$ to 7 terms.
6. $1/5, 1/5^2, 1/5^3, \cdots\cdots$ to 10 terms.
7. $224, -168, 126, \cdots\cdots$ to 5 terms.
8. $6, -4, 8/3, \cdots\cdots$ to 11 terms.
9. $15, -1/3, 1/15, \cdots\cdots$ to 6 terms.
10. $2\sqrt{3}, 6\sqrt{6}, 36\sqrt{3}, \cdots\cdots$ to 8 terms.

Sum of the Terms of an Infinite Decreasing Geometric Progression

50. By changing the signs in both numerator and denominator of the formula for the sum of n terms of a finite geometric progression,

$$S_n = \frac{a_1(r^n - 1)}{r - 1},$$

we obtain
$$S_n = \frac{a_1}{1 - r} - \frac{a_1r^n}{1 - r}.$$

We shall for convenience of proof regard a_1 as being positive.

If r has a numerical value less than unity, the absolute value of a_1r^n and accordingly of $a_1r^n/(1 - r)$ will decrease as n increases in value.

Accordingly, by giving to n a value great enough, we may make the value of $a_1r^n/(1 - r)$ as small as we please, but we can never in this way make the value of the fraction zero.

As the value of the fraction $a_1r^n/(1 - r)$ diminishes, the sum S_n approaches more nearly the value of the first fraction $a_1/(1 - r)$, but it never becomes exactly equal to it, because $a_1r^n/(1 - r)$ can never become zero.

We may, by taking n great enough, make the sum *become and remain* as nearly equal to $a_1/(1 - r)$ as we please.

This is expressed by saying "the limit of the sum of an infinite number of terms of a decreasing geometric progression is $a_1/(1 - r)$."

Expressed in symbols, we have: $\lim S_\infty = \frac{a_1}{1 - r}$, which is read, "the limit of the sum of an infinite number of terms (of a given geometric progression) is equal to $a_1/(1 - r)$."

As an alternative form we have $S_\infty \doteq \frac{a_1}{1 - r}$, which is read, "the sum of an infinite number of terms (of a given geometric progression) approaches $\frac{a_1}{1 - r}$ as a limit."

Ex. 1. Find the sum of an infinite number of terms of the decreasing geometric progression 1, 1/2, 1/4, 1/8, · · · · · .

Substituting the value 1 for a_1, and 1/2 for r, in the formula

$$S_\infty \doteq \frac{a_1}{1 - r}, \quad \text{we obtain} \quad S_\infty \doteq 2.$$

Ex. 2. Find the sum of an infinite number of terms of 36, – 12, 4, – 4/3, · · · · · .

We have $a_1 = 36,\ r = -1/3.$

Hence from the formula, $$S_\infty \doteq \frac{a_1}{1-r},$$

we have $$S_\infty \doteq \frac{36}{1+1/3}.$$

Or, $$S_\infty \doteq 27.$$

51. By means of the formula for the sum of an infinite number of terms of a decreasing geometric progression we may obtain the **generating fraction** of a repeating decimal fraction, that is, the fraction which gives rise to a repeating decimal fraction if the numerator is divided by the denominator.

Ex. 3. Find the generating fraction of the repeating decimal fraction $.\dot{3}$.

It should be observed that the dot written above the 3 indicates that 3 is to be repeated indefinitely; that is, $.\dot{3} = .333333 +$.

We may write $.3333 \cdots\cdots = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \cdots\cdots$.

In this form the repeating decimal fraction appears as a decreasing geometric series the first term of which is 3/10, and the common ratio of which is 1/10.

Using the formula for the sum, $$S_\infty \doteq \frac{a_1}{1-r},$$

we have $$S_\infty \doteq \frac{3/10}{1-1/10}.$$

Or $$S_\infty \doteq \tfrac{1}{3}.$$

Ex. 4. Find the improper fraction which may be transformed into the repeating decimal fraction $3.2\dot{3}\dot{6}$.

We may write $3.2\dot{3}\dot{6} = 3.2 + .0\dot{3}\dot{6}$.

It should be understood that the dots above the 3 and 6 denote that 36 is to be repeated indefinitely, that is, $.0\dot{3}\dot{6} = .03636363636 +$.

Accordingly, we have $.0\dot{3}\dot{6} = \frac{36}{1000} + \frac{36}{100000} + \frac{36}{10000000} + \cdots\cdots,$ from which $a_1 = 36/1000$, and $r = 1/100$.

Using the formula $$S_\infty \doteq \frac{a_1}{1-r},$$

we obtain $$S_\infty \doteq \frac{36/1000}{1-1/100}.$$

Or, $$S_\infty \doteq \frac{2}{55}.$$

The required fraction may be obtained by finding the sum of 3.2 and 2/55, which is found to be 178/55.

The student should show, by dividing the numerator by the denominator, that the fraction 178/55 gives rise to the given repeating decimal fraction 3.23636+.

52. The process of finding the generating fraction corresponding to any given repeating decimal fraction is sometimes spoken of as *evaluating* the given repeating decimal fraction.

EXERCISE XXVII. 8

Find the sum of an infinite number of terms of each of the following series:

1. $1/2 + 1/4 + 1/8 + \cdots\cdots$.
2. $1 - 1/2 + 1/4 - \cdots\cdots$.
3. $2/3 + 2/9 + 2/27 + \cdots\cdots$.
4. $500 + 100 + 20 + \cdots\cdots$.
5. $5 - 1/2 + 1/20 - 1/200 + \cdots\cdots$.
6. $2 - 4 + 8 - 16 + \cdots\cdots$.
7. $1 - x + x^2 - x^3 + \cdots\cdots$, for $x < 1$.
8. $1 - 1/3 + 1/9 - \cdots\cdots$.
9. $1 + 1/x + 1/x^2 + \cdots\cdots$, for $x > 1$.

Evaluate the following:

10. $.\dot{3}$.
11. $.\dot{6}$.
12. $.\dot{2}\dot{5}$.
13. $.\dot{6}\dot{1}$.
14. $.\dot{1}2\dot{3}$.
15. $.2\dot{2}\dot{7}$.
16. $3.\dot{2}7\dot{9}$.
17. $7.5\dot{4}\dot{3}$.
18. $2.5\dot{6}\dot{4}$.

19. Find r, having given $a_2 = 6$ and $a_8 = 384$.

20. Find two numbers the difference of which is 48 and the geometric mean of which is 7.

21. Find S_{10}, having given $a_4 = 72$ and $a_7 = - 64/3$.

22. Find a_1, provided that $a_6 = 1/32$ and $a_8 = - 1/968$.

23. Find a_9, knowing that $a_3 = .008$ and $a_6 = .000064$.

24. The difference between two numbers is 70 and their arithmetic mean exceeds their geometric mean by 25. Find the numbers.

25. Find three numbers in geometric progression such that their sum shall be 14 and the sum of their squares 84.

26. The sum of the first four terms of a geometric progression is 15, and the sum of the next two terms is 48. Find the progression.

27. A number consists of three figures in geometric progression.

The sum of the figures is 7, and if 297 be added to the number the order of the figures will be reversed. Find the number.

28. A ball is thrown vertically upward to a height of 120 feet and after falling it rebounds one-third of the distance, and so on. Find the whole distance passed over by the ball before it comes to rest.

Mental Exercise XXVII. 9

Classify each of the following as Arithmetic, Harmonic, or Geometric Progressions:

1. 1, 4, 7, · · · · · .
2. 12, 16, 20, · · · · · .
3. 23, 17, 11, · · · · · .
4. 4, 12, 36, · · · · · .
5. 5, 20, 80, · · · · · .
6. $-15, -12, -9$, · · · · · .
7. $-7, -11, -15$, · · · · · .
8. $2, -4, 8$, · · · · · .
9. $-3, 6, -12$, · · · · · .
10. 2, 5/2, 3, · · · · · .
11. 2, 2/5, 2/9, · · · · · .
12. 1/3, 1/6, 1/9, · · · · · .
13. 3/7, 3/5, 3/2, · · · · · .
14. 1/3, 1/9, 1/27, · · · · · .
15. 1/20, 1/10, 1/5, · · · · · .
16. 1/4, 1/2, 1, · · · · · .
17. 1/4, 1/2, 3/4, · · · · · .
18. 1/3, 1, 3, · · · · · .
19. 1/5, 2, 20, · · · · · .
20. 15, 3, 3/5, · · · · · .
21. 21, 28, 35, · · · · · .
22. 1/2, 2/5, 1/3, · · · · · .
23. 1/2, 2/5, 8/25, · · · · · .
24. 1/2, 1/3, 1/4, · · · · · .
25. 1/2, 1/3, 1/6, · · · · · .
26. 2, 3, 6, · · · · · .
27. 6, 3, 2, · · · · · .
28. 1/2, 1/3, 2/9, · · · · · .
29. $-1, 1, 3$, · · · · · .
30. $-1, 1, 1/3$, · · · · · .
31. 1, 3, 9, · · · · · .
32. 1, 1/3, 1/9, · · · · · .
33. $5, 1, -3$, · · · · · .
34. 5, 1, 1/5, · · · · · .
35. 5, 1, 5/9, · · · · · .
36. 4, 6, 12, · · · · · .
37. 3, 5, 15, · · · · · .
38. 6, 9, 18, · · · · · .
39. 6, 4, 3, · · · · · .
40. 20, 15, 12, · · · · · .
41. $1, x, x^2$, · · · · · .
42. $x, 2x, 3x$, · · · · · .
43. a^2, a^4, a^6, · · · · · .
44. b^2, b^5, b^8, · · · · · .
45. c^2, c^6, c^{10}, · · · · · .
46. $b/2a, b/4a, b/6a$, · · · · · .
47. d, d^8, d^{15}, · · · · · .
48. a^3, a^3b, a^3b^2, · · · · · .
49. a, ab^2, ab^4, · · · · · .
50. $x/5, x/10, x/15$, · · · · · .
51. $1, 1/x, 1/x^2$, · · · · · .
52. $1/b^2, 1/b^4, 1/b^6$, · · · · · .
53. $1/x^2, 4/x^2, 7/x^2$, · · · · · .
54. $c, 1, 1/c$, · · · · · .

55. $a/2,\ a/3,\ a/4,\ \cdots\cdots.$
56. $a,\ b,\ b^2/a,\ \cdots\cdots.$
57. $3/x,\ 4/x,\ 5/x,\ \cdots\cdots.$
58. $x^{-1},\ x^{-2},\ x^{-3},\ \cdots\cdots.$
59. $a/bc,\ a/c,\ ab/c,\ \cdots\cdots.$
60. $x,\ -y,\ y^2/x,\ \cdots\cdots.$
61. $a,\ a/b,\ a/b^2,\ \cdots\cdots.$
62. $a,\ a+b,\ a+2b,\ \cdots\cdots.$
63. $x,\ x-y,\ x-2y,\ \cdots\cdots.$
64. $a+2, a-2, a-6,\ \cdots\cdots.$
65. $x-6,\ x-3,\ x,\ \cdots\cdots.$
66. $a-2b,\ a-b,\ a,\ \cdots\cdots.$
67. $a+2b,\ a,\ a-2b,\ \cdots\cdots.$
68. $m, 2m+n, 3m+2n,\ \cdots\cdots.$
69. $a^2, a^2-b^2, a^2-2b^2,\ \cdots\cdots.$
70. $a,\ a+\frac{1}{b},\ a+\frac{2}{b},\ \cdots\cdots.$
71. $\frac{a-1}{a},\ \frac{a-2}{a},\ \frac{a-3}{a},\ \cdots\cdots.$
72. $\frac{a-b}{c},\ \frac{a-b}{2c},\ \frac{a-b}{4c},\ \cdots\cdots.$
73. $a+b, a^2+ab, a^3+a^2b,\ \cdots\cdots.$
74. $\frac{a+b}{2},\ 1,\ \frac{2}{a+b},\ \cdots\cdots.$
75. $\frac{c}{c-d},\ 1,\ \frac{c-d}{c},\ \cdots\cdots.$
76. $\frac{6}{a-b},\ 6,\ 6a-6b,\ \cdots\cdots.$

EXERCISE XXVII. 10. **Review**

Simplify each of the following:

1. $\left(\frac{x^2}{y^2}+1+\frac{y^2}{x^2}\right)\left(\frac{x}{y}-\frac{y}{x}\right).$ 2. $\left(a^2+\frac{1}{a^2}\right)\left(a+\frac{1}{a}\right)\left(a-\frac{1}{a}\right).$

3. $\left(\frac{a^2+ab}{a^2+b^2}\right)\left(\frac{a^4-b^4}{ab+b^2}\right)\left(\frac{b}{a}\right).$

4. $\left(\frac{a}{1+\frac{1}{a}}+1-\frac{1}{a+1}\right)\div\left(\frac{a}{1-\frac{1}{a}}-a-\frac{1}{a-1}\right).$

5. $(a^{-2}-b^{-2})\div(a-b).$ 6. $\frac{a^{-1}-b^{-1}}{a^{-1}+b^{-1}}-\frac{a^{-1}+b^{-1}}{a^{-1}-b^{-1}}.$

7. $\left(\frac{x^b}{x^c}\right)^{b+c}\left(\frac{x^c}{x^a}\right)^{c+a}\left(\frac{x^a}{x^b}\right)^{a+b}.$

8. If $a=-1$, $b=-2$, $c=-3$, find the value of $\frac{a^a}{bc}+\frac{b^b}{ca}+\frac{c^c}{ab}$

9. Simplify $[(x+y)\sqrt{x-y}][(x-y)\sqrt{x+y}].$

10. Simplify $(\sqrt{x^2-y^2})(\sqrt{x+y})(\sqrt{x-y}).$

11. Simplify $(\sqrt{a}+\sqrt{b})(\sqrt[4]{a}+\sqrt[4]{b})(\sqrt[4]{a}-\sqrt[4]{b}).$

12. Show that $(a+3)(a+4)(a+5)(a+6)+1 \equiv (a^2+9a+19)^2$.

13. Show that $(x+2)(x+3)(x+4)(x+5)+1$ is a square.

14. Show that $\sqrt{2m+2\sqrt{m^2-n^2}} \equiv \sqrt{m+n}+\sqrt{m-n}$.

15. Show that $\dfrac{\sqrt{a-b}}{\sqrt{a}-\sqrt{b}} \equiv \dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{a-b}}$.

16. Show that $\dfrac{\sqrt[n]{\sqrt[n+1]{a}}}{\sqrt[n+1]{\sqrt[n+2]{a}}} \equiv \dfrac{\sqrt[n]{a}\,\sqrt[n+2]{a}}{\sqrt[n+1]{a^2}}$.

17. Rationalize the denominator of $\dfrac{5+3\sqrt{5}}{3+5\sqrt{3}}$.

Solve each of the following equations :

18. $x^6+7x^3=8$.

19. $\dfrac{1}{\sqrt{x}+1}+\dfrac{1}{\sqrt{x}-1}=\dfrac{1}{x-1}$.

20. $\sqrt{x+5}+\sqrt[4]{x+5}=12$.

21. $\sqrt{7x-6}-\sqrt{17x-2}+\sqrt{3x-2}=0$.

22. $\sqrt{3x+10}+\sqrt{5x}=\sqrt{19x+5}$.

23. $\dfrac{2(x-a)}{x-b}+\dfrac{3(x-b)}{x-a}=5$.

24. $\dfrac{x}{x+8}+\dfrac{5x-56}{(x-4)(x+8)}=\dfrac{x}{x+10}+\dfrac{7x+90}{(x-6)(x+10)}$.

25. $\dfrac{x}{x-3}-\dfrac{2(x+9)}{(x+5)(x-3)}=\dfrac{x}{x+6}+\dfrac{7x-6}{(x-2)(x+6)}$.

CHAPTER XXVIII

THE BINOMIAL THEOREM

1. In this chapter we shall consider the laws governing the expansion of a binomial to any real power, proving them for any positive integral exponent and applying them for positive or negative, integral or fractional exponents.

THE BINOMIAL THEOREM FOR POSITIVE INTEGRAL EXPONENTS

2. The student should obtain the following identities by actual multiplication :

Check. $a = b = 1$

$$\begin{array}{lr}
(a+b)^1 \equiv a^1 + b \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . & 2^1 = 2 \\
(a+b)^2 \equiv a^2 + 2ab + b^2 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . & 2^2 = 4 \\
(a+b)^3 \equiv a^3 + 3a^2b + 3ab^2 + b^3 \quad . \; . \; . \; . \; . \; . \; . & 2^3 = 8 \\
(a+b)^4 \equiv a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \quad . \; . \; . \; . & 2^4 = 16 \\
(a+b)^5 \equiv a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 \quad . & 2^5 = 32 \\
(a+b)^6 \equiv a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6 & 2^6 = 64
\end{array}$$

. . . .

By inspection of the identities above, we shall discover certain laws of coefficients and exponents which will be found to hold true for the expansion of any binomial raised to a power the exponent of which is any positive integer; and they may be shown to hold without exception for all powers, whether the exponents be positive or negative, integral or fractional, real or imaginary.

3. Observe that, in the expansion of a binomial to any power (obtained for the present by multiplication) :

(i.) *The number of terms is greater by unity than the index of the power of the binomial.*

E. g. In the expansion of $(a + b)^3$ there are four terms ; of $(a + b)^5$, six terms ; of $(a + b)^9$, ten terms.

(ii.) *The exponents of the first and last terms, and also the coefficients of the second term and the term next to the last, are equal to the index of the given binomial.*

E. g. In the expansion of $(a + b)^4$, the number 4 appears as exponent in a^4 and b^4; and as coefficient in $4\,a^3b$ and $4\,ab^3$.

(iii.) *The exponent of the first term, a, decreases by unity in successive terms, and the exponent of the second term, b, increases by unity in successive terms, appearing to the first power in the second term.*

E. g. In the expansion of $(a + b)^6$, a has the following exponents in the successive terms: 6, 5, 4, 3, 2, and 1; b, beginning with the second term, has the exponents 1, 2, 3, 4, 5, and 6.

The distribution of the exponents among the literal factors of the successive terms of the expansion of $(a + b)^9$ is indicated in the following table:

Terms	1	2	3	4	5	6	7	8	9	10
Powers of a	9	8	7	6	5	4	3	2	1	
b		1	2	3	4	5	6	7	8	9
Sum of Exponents	9	9	9	9	9	9	9	9	9	9

The exponents of the literal factors in the different terms will accordingly appear as follows:

$${}^{9},\quad {}^{8\ 1},\quad {}^{7\ 2},\quad {}^{6\ 3},\quad {}^{5\ 4},\quad {}^{4\ 5},\quad {}^{3\ 6},\quad {}^{2\ 7},\quad {}^{1\ 8},\quad {}^{9}.$$

Inserting the letters, we have the following powers of a and b:

$$a^9,\ a^8b^1,\ a^7b^2,\ a^6b^3,\ a^5b^4,\ a^4b^5,\ a^3b^6,\ a^2b^7,\ a^1b^8,\ b^9.$$

For the expansion of $(a + b)$ to any power, such as the nth, we have

$$a^n,\ a^{n-1}b,\ a^{n-2}b^2,\ a^{n-3}b^3,\ a^{n-4}b^4,\ a^{n-5}b^5,\ \cdots\cdots,\ a^2b^{n-2},\ ab^{n-1},\ b^n.$$

(iv.) *The sum of the exponents of a and b in any term of a given expansion is equal to the index of the power of the binomial.*

E. g. As indicated in the lowest row of the table for the exponents in the expansion of the ninth power of $(a + b)$ given above, the sum of the exponents of the literal factors in each term is 9.

(v.) *The coefficient of the first term of every expansion is* 1, *and that of the second term is equal to the index of the power to which the binomial is raised.*

E. g. In the expansion of $(a+b)^4$, the coefficient of the first term is 1, and that of the second term is 4.

We have the following multiplication and division rule for calculating the coefficients successively:

(vi.) *In the expansion of the binomial* $(a+b)$ *to any power, we may find the coefficient of any specified term from the coefficient and exponents of the preceding term by dividing the product of the coefficient and exponent of the first letter,* a, *in this preceding term, by a number which is greater by unity than the exponent of the second letter,* b.

E. g. In the expansion of $(a+b)^6$, we may obtain the coefficient 15 of the third term, $15\,a^4b^2$, from the coefficient and exponents of the second term, $6\,a^5b$, by multiplying the coefficient 6 by the exponent 5 and dividing the result by a number greater by unity than the exponent of b, — that is, by 2.

Similarly, we may obtain the coefficient 20 of the fourth term, $20\,a^3b^3$, from the coefficient and exponents of the third term, $15\,a^4b^2$, by performing the operations $15\times 4\div 3=20$.

To find the coefficient 15 of the fifth term, $15\,a^2b^4$, we use the coefficient and exponents of the fourth term as follows:

$$20\times 3\div 4=15.$$

The coefficient of the sixth term is $15\times 2\div 5=6$.

To obtain the coefficient of the seventh term, we write

$$6\times 1\div 6=1.$$

If we attempt to calculate the coefficient of another term after the seventh, we shall have $1\times 0\div 6=0$, and since the coefficient is zero, no such term exists.

In calculations such as these, it is convenient to perform the divisions before the multiplications.

E. g. In calculating the coefficients in the expansion of $(x+y)^{20}$ we may begin by writing $(x+y)^{20}\equiv x^{20}+20\,x^{19}y+\cdots\cdots$.

To calculate the coefficient of the third term, we may proceed as follows:

$$20\times 19\div 2=20\div 2\times 19=10\times 19=190.$$

Hence, the third term is $\quad 190\,x^{18}y^2.$

To obtain the coefficient of the fourth term, we may write

$$190 \times 18 \div 3 = 190 \times 6 = 1140.$$

Hence, the first four terms of the required expansion of $(x+y)^{20}$, are

$$(x+y)^{20} \equiv x^{20} + 20\,x^{19}y + 190\,x^{18}y^2 + 1140\,x^{17}y^3 + \cdots\cdots.$$

The student should complete the expansion.

(vii.) *The coefficients are repeated in inverse order after passing the middle term or terms of any expansion, so that after the coefficients of the terms in the first half of any particular expansion are calculated, the coefficients of the terms in the second half may immediately be written.*

(viii.) *If* $-b$ *is substituted for* b *throughout the expansion of* $(a+b)^n$, *the signs of all terms containing odd powers of* b *will be changed from* $+$ *to* $-$, *and those containing even powers of* b *will remain unchanged.*

E.g. $(a-b)^5 \equiv a^5 - 5\,a^4b + 10\,a^3b^2 - 10\,a^2b^3 + 5\,ab^4 - b^5.$

4. From the observations above we obtain the following rule for expanding any power of a given binomial (restricting the exponents, for the present, to positive integral numbers):

In the expansion of $(a+b)^n$, *the index,* n, *of the given binomial, the exponent,* n, *of the first term of the expansion (in which the first letter* a *appears alone), and the coefficient,* n, *of the second term, are equal.*

The exponent of the first letter, a, *decreases by unity in succeeding terms, while the exponent of the second letter,* b, *increases by unity in succeeding terms, beginning with unity in the second term.*

To find the coefficient of any term from the coefficient and exponents of the preceding term, divide the product of the coefficient and the exponent of the first letter, a, *in this preceding term, by a number greater by unity than the exponent of the second letter,* b, *in this term.*

The calculation of successive terms will stop when, in the computation, a term appears having a coefficient equal to zero.

If the given binomial is a sum, represented by $(a+b)$, *the signs of all of the terms in the expansion will be positive. If the given binomial is a difference, represented by* $(a-b)$, *the signs of all of the terms in the expansion will be alternately positive and negative, those terms being negative in which odd powers of* b *are found.*

5. If the given binomial is symmetric, the expansion of any power of the binomial will be symmetric. Hence, the coefficient of the first term, the coefficient of the second term, the coefficient of the third term, etc., will be respectively equal to the coefficient of the last term, the coefficient of the term next to the last, the coefficient of the term which is second from the last, etc.

Ex. 1. Expand $(2x + 3y)^5$.

We have,

$$\begin{aligned}(2x+3y)^5 &\equiv (2x)^5 + 5(2x)^4(3y) + 10(2x)^3(3y)^2 + 10(2x)^2(3y)^3 + 5\cdot 2x(3y)^4 + (3y)^5\\ &\equiv 32x^5 + 240x^4y + 720x^3y^2 + 1080x^2y^3 + 810xy^4 + 243y^5.\end{aligned}$$

Check. $x = y = 1$.

$3125 = 3125$.

Ex. 2. Expand $(m^2 - 4n^3)^3$.

We have, $$\begin{aligned}(m^2 - 4n^3)^3 &\equiv (m^2)^3 - 3(m^2)^2(4n^3) + 3(m^2)(4n^3)^2 - (4n^3)^3\\ &\equiv m^6 - 12m^4n^3 + 48m^2n^6 - 64n^9.\end{aligned}$$

Check. $m = 3$, $n = 1$.

$125 = 125$.

Proof of the Binomial Theorem for Positive Integral Exponents

6. We have found by actual multiplication that for positive integral values of n, equal to or less than 6,

$$(a+b)^n \equiv a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{2\cdot 3}a^{n-3}b^3 + \cdots\cdots \qquad (1)$$

It will now be shown that the formula above applies for all positive integral values of n.

Multiplying both members by $a + b$, we have

$$(a+b)^{n+1} \equiv a^{n+1} + \left.\begin{array}{r} n\\ +\ 1\end{array}\right| a^nb + \left.\begin{array}{r} \frac{n(n-1)}{2}\\ +\ n\end{array}\right| a^{n-1}b^2 + \left.\begin{array}{r} \frac{n(n-1)(n-2)}{2\cdot 3}\\ +\ \frac{n(n-1)}{2}\end{array}\right| a^{n-2}b^3 + \cdots\cdots, \qquad (2)$$

$$\equiv a^{n+1} + (n+1)a^nb + \frac{(n+1)n}{2}a^{n-1}b^2 + \frac{(n+1)n(n-1)}{2\cdot 3}a^{n-2}b^3 + \cdots\cdots \qquad (3)$$

It may be seen that, wherever n appears in (1), $n+1$ appears in (3). It follows that, if the theorem be true for any particular integral number represented by n, it will be true also for the next higher value, that is for the number represented by $n+1$.

Hence the laws which apply to the coefficients and exponents in the expansion of $(a+b)^n$ apply also to the coefficients and exponents in the expansion of $(a+b)^{n+1}$.

We have found by actual multiplication that they apply for the second power; hence, by the reasoning above, they must apply for the third power.

Since they hold for the third power, they must hold for the fourth power, and so on indefinitely.

Hence, the general formula (1) applies for all positive integral values of the exponent.

The method of reasoning employed in this proof is known as **mathematical induction.**

Exercise XXVIII. 1

Write the expansion of each of the following powers:

1. $(m+n)^3$.
2. $(t+n)^4$.
3. $(d+g)^5$.
4. $(a+c)^8$.
5. $(k+w)^{10}$.
6. $(c-f)^6$.
7. $(b-x)^7$.
8. $(h-y)^9$.
9. $(r-s)^4$.
10. $(p-q)^{11}$.
11. $(a+1)^4$.
12. $(b-1)^5$.
13. $(c-5)^3$.
14. $(d-9)^2$.
15. $(2+e)^6$.
16. $(3+f)^5$.
17. $(2-g)^6$.
18. $(1+v)^7$.
19. $(2x-y)^8$.
20. $(2h+w)^9$.
21. $(m+2n)^{10}$.

Write the first five terms of:

22. $(a+b)^{12}$.
23. $(b+c)^{13}$.
24. $(k+z)^{15}$.
25. $(d+r)^{16}$.
26. $(m+x)^{17}$.
27. $(v-y)^{14}$.

Write the first six terms of:

28. $(b-d)^{11}$.
29. $(s-x)^{20}$.
30. $(a-w)^{18}$.

Write the first and last three terms of:

31. $(c+d)^{19}$.
32. $(x+y)^{25}$.
33. $(p+q)^{14}$.

Write the expansion of each of the following:

34. $(5a+4b)^3$.
35. $(6a-7b)^3$.
36. $(3m-5n)^4$.
37. $(9c-11d)^2$.
38. $(2b-3x)^5$.
39. $(4x-y^2)^6$.

THE BINOMIAL THEOREM FOR NEGATIVE OR FRACTIONAL EXPONENTS

7. It may be proved, by means of certain principles, that the binomial theorem applies even when the exponent is a negative or a fractional number.

Without proving the theorem, we shall assume that for all positive or negative, integral or fractional values of the exponent, the formula is true in the following form for such values of the letters as make the first term of the given binomial greater than the second term:

$$(a+b)^n \equiv a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{2\cdot 3}a^{n-3}b^3 + \cdots\cdots.$$

It may be seen that, whenever n is negative or fractional, the numerator of the coefficient of the rth or general term,

$$\frac{n(n-1)(n-2)\cdots\cdots(n-r+2)}{1\cdot 2\cdot 3\cdots\cdots\cdots\cdots(r-1)},$$

can never become zero, and hence the number of terms for a particular expansion is unlimited.

Ex. 1. Write the expansion of $(2a^{\frac{1}{3}} - 3b^{-\frac{1}{2}})^3$.

We have,

$$(2a^{\frac{1}{3}} - 3b^{-\frac{1}{2}})^3 \equiv (2a^{\frac{1}{3}})^3 - 3(2a^{\frac{1}{3}})^2(3b^{-\frac{1}{2}}) + 3(2a^{\frac{1}{3}})(3b^{-\frac{1}{2}})^2 - (3b^{-\frac{1}{2}})^3.$$
$$\equiv 8a - 36a^{\frac{2}{3}}b^{-\frac{1}{2}} + 54a^{\frac{1}{3}}b^{-1} - 27b^{-\frac{3}{2}}.$$

Check. Let $a=8,\quad b=4.$

$-\frac{25}{8} = \frac{125}{8}.$

Ex. 2. Write the first four terms in the expansion of $(1-2x)^{-2}$.

We have,

$$(1-2x)^{-2} \equiv 1^{-2} - (-2)1^{-3}(2x) + \frac{(-2)(-3)}{2}1^{-4}(2x)^2 - \frac{(-2)(-3)(-4)}{2\cdot 3}1^{-5}(2x)^3 + \cdots\cdots$$
$$\equiv 1 + 4x + 12x^2 + 32x^3 + \cdots\cdots.$$

Since $(1-2x)^{-2} \equiv \frac{1}{(1-2x)^2} \equiv \frac{1}{1-4x+4x^2}$,

we can check the result by performing the division

$$\frac{1}{1-4x+4x^2} \equiv 1 + 4x + 12x^2 + 32x^3 + \cdots\cdots.$$

Ex. 3. Write the first four terms in the expansion of $(8 - x)^{-\frac{1}{3}}$.

We have,

$$(2^3 - x)^{-\frac{1}{3}} \equiv (2^3)^{-\frac{1}{3}} - (-\tfrac{1}{3})(2^3)^{-\frac{4}{3}}x + \frac{(-\frac{1}{3})(-\frac{1}{3}-1)}{2}(2^3)^{-\frac{7}{3}}x^2$$

$$- \frac{(-\frac{1}{3})(-\frac{1}{3}-1)(-\frac{1}{3}-2)}{2 \cdot 3}(2^3)^{-\frac{10}{3}}x^3 + \cdots\cdots$$

$$\equiv \frac{1}{2} + \frac{x}{48} + \frac{x^2}{576} + \frac{7x^3}{41472} + \cdots\cdots.$$

Since $(8 - x)^{-\frac{1}{3}} \equiv \frac{\sqrt[3]{(8-x)^2}}{8-x}$, it is possible to check the result by dividing $\sqrt[3]{64 - 16x + x^2}$ by $8 - x$.

Exercise XXVIII. 2

Write the expansion of each of the following:

1. $(a^{-\frac{1}{4}} + b^3)^2$.
2. $(3c^{-1} - d^{\frac{4}{5}})^2$.
3. $(5a^{\frac{1}{3}}b^{-2}c - 4x^{-\frac{3}{4}}y^{-6})^2$.
4. $(a^{-\frac{2}{3}} + b^4)^3$.
5. $\left(b^{-1} - \frac{c^{-2}}{2}\right)^4$.
6. $(m^{-2} + 2n)^4$.
7. $(c^{-2} + 3de^{\frac{1}{3}})^4$.
8. $(x^{-\frac{2}{3}} - y^{-\frac{3}{4}})^5$.
9. $(g^{\frac{1}{3}} - h^{\frac{1}{2}})^6$.
10. $(a^{-\frac{1}{2}} + b^{-1})^5$.

Expand to four terms each of the following:

11. $(1 + x)^{-1}$.
12. $(1 - a)^{-1}$.
13. $(1 - b)^{-2}$.
14. $(1 - c)^{-3}$.
15. $(1 - d)^{-4}$.
16. $(1 - e)^{-5}$.
17. $(1 - 2n)^{-3}$.
18. $(a - b^2)^{-5}$.
19. $(1 + x)^{\frac{1}{2}}$.
20. $(1 + y)^{\frac{1}{3}}$.
21. $(1 + z)^{\frac{1}{4}}$.
22. $(1 - m)^{-\frac{1}{2}}$.
23. $(4 - 3x)^{-\frac{1}{2}}$.
24. $(1 - 3r)^{-\frac{3}{2}}$.
25. $(1 + 4k)^{-\frac{4}{3}}$.
26. $(8 - 3x)^{\frac{2}{3}}$.

Selection of a Particular Term in the Expansion of $(a+b)^n$

8. Consider the following terms in the expansion of $(a+b)^n$, n being positive and integral.

$$(a+b)^n \equiv a^n + na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \cdots\cdots.$$

It should be observed that the denominator of the coefficient of any particular term consists of the product of the primary numbers 1, 2, 3, etc., up to a number which is less by unity than the number of the term.

E. g. In the fourth term the denominator is $1\cdot 2\cdot 3$.

Accordingly, the denominator of the coefficient of the rth term from the beginning must consist of the product $1\cdot 2\cdot 3\cdot 4\cdot 5\cdots\cdots(r-1)$.

9. The product of the successive primary numbers 1, 2, 3, 4, 5, $\cdots\cdots$, up to and including any specified number, f, is called **factorial *f***, and may be indicated by $\lfloor\underline{\;\;}$ or !, as shown in the following illustrations:

E. g. If f be 5,

$$\lfloor\underline{5} = 1\cdot 2\cdot 3\cdot 4\cdot 5 = 120.$$

$$4\,! = 1\cdot 2\cdot 3\cdot 4 \quad = \quad 24.$$

$$a\,! = 1\cdot 2\cdot 3\cdot 4\cdots\cdots(a-2)(a-1)a.$$

$$(r-1)\,! = 1\cdot 2\cdot 3\cdot 4\cdots\cdots(r-2)(r-1).$$

10. The denominator of the coefficient of any specified term in the expansion of a binomial to any power can be expressed by the factorial notation.

E. g. In particular, the first five terms in the expansion of $(a+b)^{10}$ may be written as follows:

$$(a+b)^{10}\equiv a^{10}+10a^9b+\frac{10\cdot 9}{1\cdot 2}a^8b^2+\frac{10\cdot 9\cdot 8}{1\cdot 2\cdot 3}a^7b^3+\frac{10\cdot 9\cdot 8\cdot 7}{1\cdot 2\cdot 3\cdot 4}a^6b^4+\cdots\cdots.$$

Or $$(a+b)^{10}\equiv a^{10}+10a^9b+\frac{10\cdot 9}{2\,!}a^8b^2+\frac{10\cdot 9\cdot 8}{3\,!}a^7b^3+\frac{10\cdot 9\cdot 8\cdot 7}{4\,!}a^6b^4+\cdots\cdots.$$

11. By examining any particular term in the expansion of $(a+b)^n$ (see § 8) it may be observed that:

(i.) *The exponent of the second letter, b, is equal to the last factor in the denominator of the coefficient.*

E. g. In the fourth term the exponent of the second letter b is 3, which is the last factor in the denominator of the coefficient.

In the rth term, the last factor in the denominator is $r - 1$, and the exponent of the second letter b is $r - 1$.

(ii.) *The exponent of the first letter, a, is equal to the difference between the index of the power to which the given binomial is raised and the exponent of the second letter, b, in the same term.*

E. g. In the expansion of $(a + b)^n$ the exponent of a in the rth term from the beginning is $n - (r - 1) \equiv n - r + 1$.

(iii.) *The first factor in the numerator of any coefficient is n which is the index of the power to which the given binomial is raised. The successive factors $n - 1$, $n - 2$, etc., decrease successively by unity, and the number of factors in the numerator of any coefficient is equal to the number of factors in the denominator.*

E. g. In the expansion of $(a + b)^{10}$ the first factor in the numerator of each numerical coefficient is 10, and in each coefficient the number of factors in the numerator is equal to the number of factors in the denominator. It should be observed that the last factor in each numerator is greater by unity than the exponent of the first letter, a, in the same term.

In particular, in the fourth term the last factor in the numerator is 8 which is greater than 7 by unity; in the fifth term the last factor in the numerator is 7 which is greater than 6 by unity.

Ex. 1. Write the 8th term in the expansion of $(a + b)^{10}$.

Since all of the terms in the expansion of a binomial sum are positive, the sign of the 8th term must be positive.

Since the exponent of the power to which the second letter b is raised is less by unity than the number of the term, the exponent of b in the 8th term must be 7. Accordingly, the coefficient of a, which is found by subtracting the exponent of b from the index of the power to which the given binomial is raised, must be $10 - 7 = 3$.

To obtain the numerical coefficient we may write a fraction the numerator of which consists of the product of the seven numbers $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4$, and the denominator of which is $7!$, that is, the product $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7$.

Hence the required 8th term is

$$+\frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} a^3 b^7.$$

12. It may be seen that the rth term or general term in the expansion of $(a + b)^n$ may be written as follows :

$$\frac{n(n-1)(n-2)(n-3)\cdots\cdots(n-r+2)}{1\cdot 2\cdot 3\cdot 4\cdots\cdots\cdots(r-1)}\,a^{n-r-1}b^{r-1}.$$

13. It should be observed that in the expansion of $(a - b)^n$ the sign of every term is negative in which the exponent of the power to which $-b$ is raised is odd, and the sign of every term is positive in which the exponent of the power to which $-b$ is raised is even. Accordingly, since in the rth term the exponent of the power to which $-b$ is raised is $r - 1$, it follows that the sign of the term may be determined by the factor $(-1)^{r-1}$.

Hence, we have the following expression as the rth term or general term in the expansion of $(a - b)^n$:

$$(-1)^{r-1}\frac{n(n-1)(n-2)(n-3)\cdots\cdots(n-r+2)}{1\cdot 2\cdot 3\cdot 4\cdots\cdots\cdots(r-1)}\,a^{n-r+1}b^{r-1}.$$

E. g. In the expansion of $(x - y)^{20}$ the fifteenth term containing $(-y)^{14}$ is positive; the sixteenth term containing $(-y)^{15}$ is negative. In the expansion of $(m - n)^{81}$ the nineteenth term containing $(-n)^{18}$ is positive; the twenty-sixth term containing $(-n)^{25}$ is negative.

Ex. 2. Write the sixth term in the expansion of $(b - c)^{19}$.

We have $(-1)^{6-1}\dfrac{19\cdot 18\cdot 17\cdot 16\cdot 15}{1\cdot 2\cdot 3\cdot 4\cdot 5}\,b^{14}c^5 \equiv -11628\,b^{14}c^5$.

EXERCISE XXVIII. 3

Write the indicated terms in the expansions of the following powers of binomials :

1. The 5th term of $(a + b)^{17}$.
2. The 8th term of $(b + c)^{20}$.
3. The 7th term of $(c + d)^{15}$.
4. The 9th term of $(x + y)^{18}$.
5. The 6th term of $(m + n)^{21}$.
6. The 10th term of $(r + s)^{23}$.
7. The 15th term of $(a - b)^{25}$.
8. The 11th term of $(p - q)^{16}$.
9. The 10th term of $(a - y)^{14}$.
10. The 4th term of $(g - h)^{26}$.
11. The 13th term of $(c - w)^{19}$.
12. The 20th term of $(z + w)^{22}$.
13. The 16th term of $(a + b)^{24}$.
14. The 11th term of $(b - c)^{11}$.

Expand each of the following powers of binomials:

15. $(a^2 + b)^3$.
16. $(c^3 - d)^3$.
17. $(x^2 - 2)^4$.
18. $(y^3 - 1)^5$.
19. $(z^4 - 1)^6$.
20. $(2a + b^2)^7$.
21. $(2a^3 + 3b^2)^4$.
22. $(3m^5 + 5n^3)^4$.

14. The formulas of §§ 12, 13, for the general or rth term in the expansion of $(a + b)^n$ and $(a - b)^n$ may be used also when the exponent n is negative or fractional.

Ex. 1. Find the prime factors of the coefficient of $a^{20}b^{-11}$ in the expansion of $\left(a - \frac{1}{2b}\right)^{31}$.

It may be seen that a^{20} and b^{-11} appear in the term of the expansion in which a^{20} and $-\left(\frac{1}{2b}\right)^{11}$ are found.

Since the exponent of the power to which $\left(-\frac{1}{2b}\right)$ is raised is less by unity than the number of the term, it appears that we are required to write the 12th term.

Since the given binomial is a difference and in the 12th term the factor $\left(-\frac{1}{2b}\right)$ is raised to the 11th power, it may be seen that the sign of the term is minus.

Hence, we may write

$$-\frac{31 \cdot 30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11} a^{20}\left(\frac{1}{2b}\right)^{11} \equiv$$

$$-(\tfrac{1}{2})^{11} \cdot 3^3 \cdot 5 \cdot 7 \cdot 13 \cdot 23 \cdot 29 \cdot 31\, a^{20}b^{-11}.$$

Ex. 2. Write the term containing x^{30} in the expansion of

$$\left(2x^3 - \frac{3}{x^{\frac{1}{2}}}\right)^{17}.$$

Referring to the general formula for the rth term in the expansion of $(a - b)^n$, § 13, it may be seen that x^3 appearing in the first term, $2x^3$, of the given binomial, is to be raised to the power represented by $(n - r + 1)$ in the formula for a general term, and that $x^{-\frac{1}{2}}$ appearing in the second term, $\left(-\frac{3}{x^{\frac{1}{2}}}\right)$, of the given binomial, is to be raised to the power represented by $r - 1$.

When reduced to simplest form, the rth term must contain x^{30}.

Hence, observing that $n = 17$, we may find r as follows:

$$(x^3)^{17-r+1} \cdot (x^{-\frac{1}{2}})^{r-1} = x^{30}$$

$$x^{51-3r+3-\frac{r}{2}+\frac{1}{2}} = x^{30}$$

Hence $$x^{\frac{109-7r}{2}} = x^{30}.$$

Since the bases are equal we can form a conditional equation of which the members are the exponents.

That is, $$\frac{109-7r}{2} = 30.$$

Hence, $$r = 7.$$

Accordingly, we are required to find the 7th term, which may be written as follows:

$$+\frac{17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}(2x^3)^{11}\left(\frac{3}{x^{\frac{1}{2}}}\right)^6 \equiv 2^{14} \cdot 3^6 \cdot 7 \cdot 13 \cdot 17 x^{33} \cdot \frac{1}{x^3}$$
$$\equiv 16384 \cdot 729 \cdot 1547 x^{30}$$
$$\equiv 18477268992 x^{30}.$$

EXERCISE XXVIII. 4

Write the specified terms in the expansions of the following powers of binomials:

1. The 4th term of $(a^{-2} + b^{\frac{1}{3}})^{10}$.
2. The 6th term of $(b^{-\frac{1}{4}} + 2c^{\frac{1}{5}})^{20}$.
3. The 8th term of $(m^{-\frac{1}{8}} + 4n^{-3})^{16}$.
4. The 13th term of $(x - x^{-\frac{2}{3}}y)^{20}$.
5. The term of $(x^4 + 2x^{-\frac{1}{4}})^{12}$ which contains x^{14}.
6. The middle term of $(a^{-3} + 2a^{\frac{1}{2}}b)^{18}$.
7. The term of $(x^{\frac{1}{2}} - y^{\frac{1}{3}})^{15}$ which contains x^3y^3.
8. The term of $(3x + 2x^{-\frac{1}{3}})^{20}$ which is free from x.
9. The 5th term of $(1 - 2x)^{\frac{1}{2}}$.
10. The 7th term of $(1 - x^2)^{\frac{3}{2}}$.
11. The 5th term of $(a^{-4} - b^{-3})^{-2}$.
12. The 6th term of $(a^{\frac{3}{2}} - b^{-1}c^2)^{-8}$.

15. The coefficients appearing in the expansion of $(a+b)^n$, called **binomial coefficients,** may be denoted by the following abbreviations:

$$n=\frac{n}{1}=\binom{n}{1}, \quad \frac{n\,(n-1)}{1\cdot 2}=\binom{n}{2}, \quad \frac{n\,(n-1)(n-2)}{1\cdot 2\cdot 3}=\binom{n}{3},$$

$$\frac{n\,(n-1)(n-2)\cdot\cdots\cdot(n-r+2)}{1\cdot 2\cdot 3\cdot\cdots\cdot(r-1)}=\binom{n}{r-1}.$$

Accordingly, we may write

$$(a+b)^n\equiv a^n+\binom{n}{1}a^{n-1}b+\binom{n}{2}a^{n-2}b^2+\binom{n}{3}a^{n-3}b^3+\cdots+\binom{n}{r-1}a^{n-r+1}b^{r-1}+\cdots$$

It may be observed that in the notation above the upper number in each symbol is the exponent of the power to which the given binomial is raised, and the lower number indicates the number of factors appearing in both numerator and denominator of the numerical coefficient.

E.g. In the expansion of $(a+b)^{10}$ the coefficient of the 7th term is

$$\binom{10}{6}=\frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}=210.$$

In the expansion of $(a+b)^4$ the coefficient of the fifth term is $\binom{4}{4}=\frac{4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4}=1.$

16. The binomial theorem may be used to obtain any required root of a number, as illustrated in the following example:

Ex. 1. Find the square root of 11 correct to four places of decimals.

We may write $\sqrt{11}=\sqrt{3^2+2}=(3^2+2)^{\frac{1}{2}}.$

Hence, to find $\sqrt{11}$ we may expand $(3^2+2)^{\frac{1}{2}}$ as follows:

$$\begin{aligned}(3^2+2)^{\frac{1}{2}}&=(3^2)^{\frac{1}{2}}+\tfrac{1}{2}(3^2)^{-\frac{1}{2}}\cdot 2-\tfrac{1}{8}(3^2)^{-\frac{3}{2}}\cdot 2^2+\tfrac{1}{16}(3^2)^{-\frac{5}{2}}\cdot 2^3-\tfrac{5}{128}(3^2)^{-\frac{7}{2}}\cdot 2^4+\cdots\cdots,\\ &=3+\tfrac{1}{3}-\tfrac{1}{54}+\tfrac{1}{486}-\tfrac{5}{17496}+\cdots\cdots,\\ &=3+(.33333+)-(.01851+)+(.00205+)-(.00020+)+\cdots\cdots.\\ &=3.3166+\text{, correct to four places of decimals.}\end{aligned}$$

EXERCISE XXVIII. 5

Find to four places of decimals:

1. $\sqrt{26}$.
2. $\sqrt{50}$.
3. $\sqrt{102}$.
4. $\sqrt[3]{9}$.
5. $\sqrt[3]{26}$.
6. $\sqrt[3]{124}$.
7. $\sqrt[4]{17}$.
8. $\sqrt[5]{33}$.
9. $\sqrt[5]{626}$.
10. $\sqrt[6]{65}$.
11. $\sqrt[6]{730}$.
12. $\sqrt[7]{2188}$.

MENTAL EXERCISE XXVIII. 6. **Review.**

Solve each of the following equations:

1. $\frac{1}{a} = \frac{b^2}{x^2}$.

2. $\sqrt[n]{nx} = n$.

3. $(x + 4)^2 = 9x^2$.

4. $\sqrt{9x^2 - b^2} = 0$.

5. $\sqrt{x + b} = a$.

6. $\frac{x^2}{b} = a^2 b$.

7. $\frac{mx^2}{n} = mn$.

8. Show that $x^3 + b^3 = 0$ if $x + b = 0$.

Simplify the following:

9. $-\frac{y - 3}{\sqrt{y - 3}}$.

10. $\frac{2 - x}{\sqrt{x - 2}}$.

11. $(\sqrt{5} + \sqrt{-3})(\sqrt{5} - \sqrt{-3})$.

12. $(\sqrt{-5} + \sqrt{-3})(\sqrt{-5} - \sqrt{-3})$.

13. $(a^{\frac{1}{2}} + \frac{1}{2})^2$.

14. $(b^{\frac{1}{3}} - \frac{1}{3})^2$.

15. $(2x^{\frac{1}{2}} + \frac{1}{2})^2$.

16. $(3\sqrt{-3} - 3\sqrt{3})^2$.

17. $\frac{a\sqrt{b}}{b\sqrt{a}}$.

18. $\frac{xa^x}{ax^a}$.

19. $\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)\left(\frac{x^2yz}{xy + yz + zx}\right)$.

20. $-(-x\sqrt{-x})^4$.

Find the value of:

21. $(x^0 + y^0)^2$.

22. $\frac{6}{a^0 b^0}$.

23. $\frac{10}{2x^0 + 3y^0}$.

24. $(-\sqrt{2})^2 - (\sqrt{-2})^2$.

25. Show that $-3\sqrt[3]{-3} = 3\sqrt[3]{3}$ while $-2\sqrt{-2} \neq 2\sqrt{2}$.

26. Simplify $\frac{1}{a}\left(\frac{b}{a}\right)^{-1}$, and express the result with positive exponents.

27. Show that a, b, c, and d are proportional if $\frac{a}{b} : \frac{d}{c} = 1$.

28. Express $2\sqrt{\frac{1}{2}\sqrt{\frac{1}{2}}}$ in simplest form.

Simplify

29. $(n\sqrt[n]{n^{n-1}})(n-1)\sqrt[n]{n}$.
30. $\left(\frac{a^{-1}}{b^{-1}}+c^{-1}\right)\left(\frac{a^{-1}}{b^{-1}}-c^{-1}\right)$.

31. Find the geometric mean of a and $1/a$.

32. Find the mean proportional between a and $1/a$.

Classify each of the following progressions as being arithmetic harmonic, or geometric:

33. $2, 4, 6, \cdots\cdots$.
34. $2, 4, 8, \cdots\cdots$.
35. $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \cdots\cdots$.

Find the values of:

36. $2!\,3!$.
37. $\frac{3!}{2!}$.
38. $\frac{8!}{7!}$.

Show that the following identities are true:

39. $4! = 3!\,2!\,2!$.
41. $\frac{5!}{2} + 2 \cdot 3! = 3 \cdot 4!$.

40. $2 \cdot 3! - 3 \cdot 2! = 3!$.
42. $\frac{25}{5!} = \frac{1}{4!} + \frac{1}{3!}$.

EXERCISE XXVIII. 7. **Review**

Simplify each of the following expressions:

1. $6abc + (c+a)^3 + (b+c)^3 + (a+b)^3 - (a+b+c)^3$.

2. $(x+y+z)^2 + (x+y-z)^2 - (y+z-x)^2 - (z+x-y)^2$.

3. $$\frac{1}{x^2 - \dfrac{x^2-1}{x + \dfrac{1}{x+1}}}.$$

4. $$\left(a+\frac{1}{a}\right)^2 + \left(b+\frac{1}{b}\right)^2 + \left(ab+\frac{1}{ab}\right)^2 - \left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)\left(ab+\frac{1}{ab}\right).$$

5. $$\frac{a-b}{c^2-(a-b)^2} + \frac{b-c}{a^2-(b-c)^2} + \frac{c-a}{b^2-(c-a)^2}.$$

6. Show that $\left(\frac{x}{x-a}+\frac{x}{x-b}+\frac{x}{x-c}\right)-\left(\frac{a}{x-a}+\frac{b}{x-b}+\frac{c}{x-c}\right) \equiv 3$, if $x \neq a$, $x \neq b$, $x \neq c$.

7. Show that $[(a-b)^2 + (b-c)^2 + (c-a)^2]^2$
$$\equiv 2\,[(a-b)^4 + (b-c)^4 + (c-a)^4].$$

8. Simplify $\dfrac{x^{-2}-y^{-2}}{x^{-3}-y^{-3}}$, using the minimum number of negative signs.

9. Simplify $\dfrac{(x^a)^2}{x^{b+c}} \times \dfrac{(x^b)^2}{x^{c+a}} \times \dfrac{(x^c)^2}{x^{a+b}}$.

10. Simplify $\dfrac{4^n \cdot 2 \times \dfrac{1}{2^{-3n}} - 32^n}{2^{5n} \times 4}$.

11. Simplify $\left[\sqrt[5]{\dfrac{a^{-\frac{1}{2}}b^{\frac{1}{3}}}{a^2b^{-\frac{1}{2}}}} \times \sqrt[3]{\dfrac{a^{\frac{1}{2}}b^{-\frac{1}{2}}}{a^{-\frac{1}{4}}b}}\right]^{-12}$.

12. Show that $\dfrac{\left(a+\dfrac{1}{b}\right)^a\left(a-\dfrac{1}{b}\right)^b}{\left(b+\dfrac{1}{a}\right)^a\left(b-\dfrac{1}{a}\right)^b} \equiv \left(\dfrac{a}{b}\right)^{a+b}$.

13. Show that $\dfrac{\left(1+\dfrac{a}{b}\right)^m\left(1-\dfrac{b}{a}\right)^n}{\left(1+\dfrac{b}{a}\right)^n\left(1-\dfrac{a}{b}\right)^m} \equiv (-1)^m\left(\dfrac{a+b}{a-b}\right)^{m-n}$.

14. For what value of n is $x^{n+1}y^{\frac{n}{5}+1} - y^{2n}z^{n-\frac{1}{2}}$ a homogeneous binomial ?

15. For what value of n is $x^{n+4}y^{\frac{n}{3}} + y^nz^{\frac{n}{2}+3}$ a homogeneous binomial ?

16. Simplify $\dfrac{x^2+2x\sqrt{y}+y}{x+\sqrt{y}}$.

17. Square the complex number $-\frac{1}{2}-\frac{1}{2}\sqrt{-2}$.

18. Simplify $(\sqrt{m+n}+\sqrt{m-n})^6 - (\sqrt{m+n}-\sqrt{m-n})^6$.

19. Simplify $\dfrac{\left[\sqrt{\dfrac{3+\sqrt{-3}}{3}} - \sqrt{\dfrac{3-\sqrt{-3}}{3}}\right]^2}{3-2\sqrt{3}}$.

20. Rationalize the denominator of $\dfrac{\sqrt{30}}{\sqrt{5}+\sqrt{3}-\sqrt{2}}$.

Solve each of the following equations :

21. $\sqrt{x+2}+\sqrt{4x-3}-\sqrt{9x+1}=0$.

22. $\left(x+\dfrac{9}{x}\right)^2 - 4\left(x+\dfrac{9}{x}\right) = 60$.

23. $\dfrac{x}{x+7} + \dfrac{8x-77}{(x-12)(x+7)} = \dfrac{x}{x+10} + \dfrac{11x-70}{(x-8)(x+10)}$.

24. $\dfrac{x}{x-5} + \dfrac{8x-35}{(x-6)(x-5)} = \dfrac{x}{(x-9)} + \dfrac{4x-99}{(x-2)(x-9)}$.

25. $\dfrac{x}{x-1} - \dfrac{5}{(x+4)(x-1)} = \dfrac{x}{x-3} - \dfrac{2(x+9)}{(x+5)(x-3)}$.

26. $\dfrac{x}{x-2} + \dfrac{x-12}{(x+3)(x-2)} = \dfrac{x}{x-5} - \dfrac{2(x+5)}{(x-1)(x-5)}$.

27. $\dfrac{x}{x-3} - \dfrac{2x-15}{(x-6)(x-3)} = \dfrac{x}{x-4} - \dfrac{3x-16}{(x-5)(x-4)}$.

28. $\dfrac{x}{x-9} + \dfrac{3(2x-33)}{(x-4)(x-9)} = \dfrac{x}{x-15} - \dfrac{270}{(x+3)(x-15)}$.

Solve the following systems of equations:

29. $18x + 12y = 7xy,$
$12x + 18y = 8xy.$

30. $x = \dfrac{y+m}{2} + \dfrac{n}{3},$
$y = \dfrac{x+n}{2} + \dfrac{m}{3}.$

31. $x^2 + xy + y = 21,$
$x + xy + y^2 = 9.$

32. $x^2 - xy - y = 22,$
$x - xy + y^2 = -2.$

33. Expand $(3x^{-\frac{1}{4}} - 2y^3)^5$ by the binomial theorem.

34. Expand $(1 - \frac{1}{3}x^3)^6$ by the binomial theorem.

35. Find the middle term in the expansion of $\left(x - \dfrac{1}{x}\right)^8$.

36. Find the middle term in the expansion of $\left(x^2 + \dfrac{1}{x^2}\right)^{12}$.

37. Find the middle term in the expansion of $\left(\dfrac{a}{x} + \dfrac{x}{a}\right)^8$.

38. In the expansion of $\left(x + \dfrac{1}{x}\right)^{14}$ write the term which is free from x.

39. Find the term free from x in the expansion of $\left(x - \dfrac{1}{x}\right)^{18}$.

40. Find the coefficient of $a^{12}b^5$ in $(a+b)^{17}$.

INDEX.

Numbers Refer to Pages.

BOOKS THE BEST COMPANIONS

THIS BOOK IS DUE ON THE LAST DATE
STAMPED BELOW

AN INITIAL FINE OF 25 CENTS
WILL BE ASSESSED FOR FAILURE TO RETURN THIS BOOK ON THE DATE DUE. THE PENALTY WILL INCREASE TO 50 CENTS ON THE FOURTH DAY AND TO $1.00 ON THE SEVENTH DAY OVERDUE.

DEC 4 1946
4 Oct'50CS
27 Aug'52AL
AUG 26 1952 LU
27Sep'52ES
SEP 18 1952 LU
15JUL'55DS
JUL 14 1955 LU
20JAN'56GC
JAN 6 1956 LU
24Aug58FF

REC'D LD
AUG 18 1958
20Mar'59FC
REC'D LD
MAR 9 - 1959
OCT 26 1966 9
RECEIVED
OCT 26 '66 -4 PM
LOAN DEPT.

LD 21–100m-12,'43(8796s)

M306115

QA 154
W48

THE UNIVERSITY OF CALIFORNIA LIBRARY