Bobby Eagleston

A FIRST COURSE IN HIGHER ALGEBRA

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A FIRST COURSE

IN

HIGHER ALGEBRA

BY

HELEN A. MERRILL, PH.D.
PROFESSOR OF MATHEMATICS IN WELLESLEY COLLEGE

AND

CLARA E. SMITH, PH.D.
ASSOCIATE PROFESSOR OF MATHEMATICS IN WELLESLEY COLLEGE

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TO

JAMES PIERPONT

A WISE AND INSPIRING TEACHER

THIS ATTEMPT TO LEAD YOUNG STUDENTS
ALONG PATHS OF MATHEMATICAL KNOWLEDGE
WHICH HE HAS MADE PLEASANT FOR US
IS GRATEFULLY DEDICATED

PREFACE

This book is an outgrowth of the conviction of the authors that Higher Algebra, to be worthy of the name, must employ advanced methods, and that the method which chiefly marks advanced work in analysis is that of limits. In all but a few chapters the work is based upon limits, the proofs being made as rigorous as seems advisable for immature students, with occasional comment on points where the proof is not rigorous, or where theorems not yet proved are employed. It is our hope that there is nothing to be unlearned in later work.

The ordinary Algebra course in college covers a semester's work — about forty-five class appointments. It has been found by actual use that in this time Chapters III, IV, V, VI, VII, XII can be covered, while Chapter X has been taught in connection with a course in Trigonometry. The chapters on Rational and Irrational Numbers are intended for reference rather than for detailed study, while the chapters on Permutations, Combinations and Probability, Partial Fractions, Complex Numbers, and Integration may be substituted for other chapters as subjects of study, or serve for reference in later work.

No mathematical knowledge is presupposed beyond the usual course in Elementary Algebra, except that a knowledge of the meaning of the trigonometric tangent is assumed in § 105, while in Chapter XI anything beyond the simplest treatment of the complex number necessitates the introduction of trigonometric formulas, which are first employed in § 195.

The text has as its main ends to broaden the students' thought by introducing them as early as possible to some of the most beautiful and most fruitful methods in analysis, with a few of their results; to give an acquaintance with the simplest notions of the Calculus, which are of the utmost use not only in Analytic Geometry, but also in Physics and Mechanics, and which shorten and simplify the early work in the regular Calculus course; and to open up to students some vision of the possibilities of later work along lines here treated only briefly. The students who are interested to do more than the assigned work have been kept constantly in mind, and it is hoped that there is much here to arouse curiosity, and to lead such students to find for themselves what lies just beyond. If some few are helped to a new realization of the wide range, the elegance, the simplicity, and the charm of these results of human thought and endeavor, our book will serve its purpose.

WELLESLEY COLLEGE,
May, 1917.

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INTRODUCTION

The following topics from elementary algebra are inserted for reference.

1. Graphical representation. A one-to-one correspondence between real numbers and the points on a line may be established as follows: Select one point on the line as the zero point, or origin, and taking a convenient unit of measure, lay it off successively to the right of the origin, numbering the end points of the intervals 1, 2, 3 $\cdots$. Then lay off the unit successively to the left of the origin, numbering the end points -1, -2, -3 $\cdots$. (See the horizontal line in the figure.) Thus all positive and negative integers may be represented. Since each unit interval may be divided into any desired number of equal parts, a point may be found corresponding to any rational number; and it is assumed that there is a point on the line corresponding to every real number, and conversely. All the values which the variable x may assume may then be laid off on this line. If it is desired to represent the values of another variable y, a line is drawn through the origin, perpendicular to the first, and positive values of y are laid off above, and negative values of y below, the origin. These two lines are called the x- and y-axes respectively.

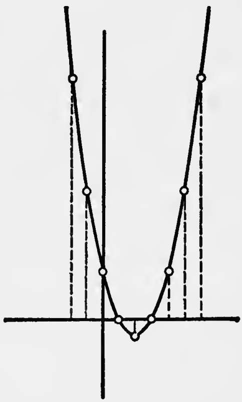
If an equation involves both x and y, as

(1) $$y = x^2 - 4x + 3$$

it is satisfied by an unlimited number of values of x and y; but there is also an unlimited number of values which do not satisfy it. The pairs of values which do satisfy the equation may be represented by a curve, called the **graph** of the equation, which is constructed as follows:

Take a convenient series of integral values for x, and arrange them in tabular form in order of magnitude. Substitute each of these values for x in equation (1), and write the value of y thus found in the y column, opposite the corresponding value of x. This is called **making a table of values.**

x	y
-2	15
-1	8
0	3
1	0
2	-1
3	0
4	3
5	8
6	15



Draw the x- and y-axes perpendicular to each other, and choose a convenient unit. The first value of x in the table is -2, hence a point is taken two units to the left of the origin, on the x-axis; and since the corresponding value of y is $+15$, a point is taken fifteen units above the point -2, which gives one point on the curve. The next pair of values is -1 and $+8$, which means a point is taken one unit to the left of the origin, and then one eight units above this for another point on the curve. So proceed until as many points as are desired have been located. Then connect these points in order by a smooth curve, which will be the **graph** of the equation. The operation of constructing the graph is called **plotting** the equation.

2. Roots of quadratic equations. When a quadratic equation is put in the form $ax^2 + bx + c = 0$, its roots are given by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

When $b^2 - 4ac > 0$, the roots are real and unequal.
When $b^2 - 4ac = 0$, the roots are real and equal.
When $b^2 - 4ac < 0$, the roots are imaginary.

3. Binomial theorem. When m is a positive integer,

$$\begin{aligned}(a+x)^m = a^m + ma^{m-1}x + \frac{m(m-1)}{2!}a^{m-2}x^2 \\ + \frac{m(m-1)(m-2)}{3!}a^{m-3}x^3 + \cdots \\ + \frac{m(m-1)(m-2)\cdots(m-n+2)}{(n-1)!}a^{m-n+1}x^{n-1} \\ + \cdots + max^{m-1} + x^m\end{aligned}$$

A study of the form of the binomial theorem makes clear the following characteristics:

1. *Exponents.* The first term a of the binomial appears in the first term of the expansion with the exponent of the binomial m, and in each succeeding term with an exponent less by 1. The second term x of the binomial appears for the first time in the second term of the expansion with the exponent 1; and in each succeeding term its exponent is increased by 1. It follows that in every term the sum of the two exponents is equal to m.

2. *Coefficients.* The coefficient of each term after the first may be obtained by the following rule: Multiply the coefficient of the preceding term by the exponent of a, and divide the product by the exponent of x increased by 1.

3. *Key number of term.* The number of factors in the numerator of any term, the number whose factorial occurs in the denominator, the exponent of x, and the number subtracted from m to form the exponent of a are always the same number, viz. $n-1$.

4. Arithmetical progression. In an arithmetical series,

$$l = a + (n-1)d; \quad s = \frac{n}{2}(a+l) = \frac{n}{2}[2a + (n-1)d]$$

5. Geometrical progression. In a geometrical series,

$$l = ar^{n-1}; \quad s = \frac{rl - a}{r-1} = \frac{a(r^n - 1)}{r-1}$$

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CHAPTER I

RATIONAL NUMBERS

Die ganzen Zahlen hat Gott gemacht, alles andere ist Menschenwerk.

— L. Kronecker.

6. Equality and inequality of numbers. It is assumed at the outset of any algebraic discussion that counting is a possible process. The number which results from counting is a **positive integer.**

If a one-to-one correspondence can be set up between a group of objects A, whose number is a, and a group B, whose number is b, the number a is said to be **equal** to the number b, and the fact is expressed by writing $a = b$.

If for every object in A there is a corresponding object in B, but the converse is not true, the number a is said to be less than the number b, written $a < b$; or the number b is said to be greater than the number a, written $b > a$.

If there is a one-to-one correspondence between the objects of A and those of B, and also between the objects of B and those of C, it is evidently possible to establish such a relation between the objects of A and those of C. Hence if $a = b$ and $b = c$, it follows that

$a = c$. Similarly, if $a < b$ and $b = c$, $a < c$; or if $a > b$ and $b = c$, $a > c$. Hence the principle that in any relation of equality or inequality equals may be substituted for equals.

If a and b are unequal the fact may be expressed by writing $a \neq b$.

7. Addition. If the result of counting the objects of A and then those of B as one group be c, since the total number of objects is the same as if the two groups were counted separately, it is said that c is the result of **adding** the numbers a and b, and the fact is written $a + b = c$. Since it is immaterial which group is counted first, $a + b = b + a$. This is the **commutative law** of addition. Similarly if three groups are counted and the corresponding numbers added, $(a + b) + c = a + (b + c) = a + b + c$, which is the **associative law** of addition.

8. Subtraction. If a is a number greater than b, the question arises: What number must be added to b in order to produce a? The problem may be stated thus: Determine x so that $b + x = a$. It is evident that when $a > b$ there is one and only one number x which satisfies this relation. The number is called $a - b$. Hence subtraction is defined as a process which is the inverse of addition.

9. The number zero. It was seen in the preceding section that when $a > b$, the symbol $a - b$ defines a unique positive integer. The symbol $a - a$ is evidently not one of the numbers introduced by counting, but it is called **zero**, and the symbol 0 is used to denote it. Assuming that it is subject to the commutative and associative laws, its other properties can be easily investigated. It is evident that zero is less than any positive integer, and that

$$a + 0 = a - 0 = a$$

10. Negative integers. The symbol $a - b$, where $a < b$, evidently does not belong to the set of positive integers, but may be defined as follows: If $b - a = c$, a positive integer, $a - b = - c$, a negative integer. Assuming that the commutative and associative laws hold for negative integers, their other properties are easily determined. The totality of positive and negative integers and zero form the system of integral numbers,

$$\cdots - 4, - 3, - 2, - 1, 0, 1, 2, 3, 4 \cdots$$

a system which has no first nor last number. In this system both addition and subtraction are always possible and unique.

The positive number obtained by omitting the sign before a number is called the **numerical** or **absolute value** of the number. This is indicated by enclosing it between vertical bars; that is, if a is any positive number

$$|-a| = |+a| = a.$$

11. Multiplication. If a groups consisting of b objects each be counted consecutively, the result is b taken a times. $b + b + b + \cdots (a \text{ times}) = ab$. This abbreviated addition is called **multiplication**, and is indicated by $a \times b$, $a \cdot b$, or, when no ambiguity can arise, simply ab. It is easily seen that counting a groups of b objects each, and counting b groups of a objects each give the same result, that is $ab = ba$, which is the **commutative law** of multiplication.

Since a groups of c objects each, and b groups of c

objects each may be counted together or separately with the same result, it follows that

$$\overbrace{c+c+\cdots+c}^{a \text{ times}}+\overbrace{c+c+\cdots+c}^{b \text{ times}}=ac+bc$$
$$=c(a+b)=(a+b)c$$

which is the **distributive law** of multiplication.

Similarly the counting of a groups of b subgroups of c objects each gives the same result as the counting of c groups of a subgroups of b objects each, or $a(bc)=c(ab)=(ab)c$ which is the **associative law** of multiplication.

If these three laws are assumed to hold for zero and negative integers, the necessary laws of operation on these numbers may be readily found. These include the well-known rules of combination that the product of two numbers is positive or negative according as the factors have like or unlike signs; that $a\cdot 0=0\cdot a=0$, and the following important theorem.

12. Theorem. *If the product of two or more factors is zero, at least one of the factors must be zero.*

For, let $ab=0$. If neither a nor b is zero, ab is a positive or negative number according as a and b have like or unlike signs, which contradicts the hypothesis that $ab=0$. Therefore either $a=0$ or $b=0$.

This theorem, when extended to all real and complex numbers, is of great value in the solution of equations, for if the expression, formed by transposing all the terms of an equation to one side, can be factored, the solution of the given equation can be reduced to the solution of two or more equations of lower degree, formed by setting each of the factors in turn equal to zero.

13. Division. As addition naturally leads to the inverse problem of subtraction, so multiplication suggests the question: By what number must a given number b be multiplied in order to produce another given number a? The problem may be stated thus: Determine x so that $bx = a$. If a is a multiple of b, x is uniquely determined as an integer defined by the relation $x = a \div b$.

The following facts are readily proved:

1. The law of signs for division follows that for multiplication; that is, The quotient is positive or negative, according as the dividend and divisor have like or unlike signs.

2. If the product of two numbers is divided by a third number, the result is the same as if either factor were divided by the number.

3. $a \div b \div c = a \div c \div b = a \div (bc)$.

4. The distributive law applies to the dividend but not to the divisor; that is, $(a+b) \div c = (a \div c) + (b \div c)$, but $a \div (b + c) \neq (a \div b) + (a \div c)$.

14. Case where the divisor is zero. In the relation $x = a \div b$, let the divisor $b = 0$. Then $0 \cdot x = a$, which by § 11 is impossible unless $a = 0$. Hence if the divisor is zero, the dividend must be zero also. Let $a = 0$. Then $0 \cdot x = 0$, a relation which is satisfied by any number whatever. Since division by zero, even when possible, is entirely indeterminate, it is rigidly excluded from mathematical work.

15. Case where the dividend is zero. Let $x = 0 \div b$. Then $bx = 0$. Since the case $b = 0$ has been excluded, $x = 0$, by § 12, and from considerations similar to those of § 14, x cannot be zero otherwise. Hence a quotient is zero when and only when the dividend is zero.

16. Fractions. If, in the relation $x = a \div b$, a is not a multiple of b, x is evidently not an integer. However, as when a similar problem arose in the case of subtraction, the symbol $a \div b$ or $\frac{a}{b}$ may be regarded as defining a new type of numbers called fractions, whose properties must be investigated. The sign of a fraction is determined by § 13, 1. From 2 of the same section it follows that $\frac{ab}{c} = \frac{a}{c} \cdot b = a \cdot \frac{b}{c}$; from 3 that $\frac{\frac{a}{b}}{c} = \frac{\frac{a}{c}}{b} = \frac{a}{bc}$; from 4 that $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$ but $\frac{a}{b+c} \neq \frac{a}{b} + \frac{a}{c}$ and $\frac{a+b}{c+d} \neq \frac{a}{c} + \frac{b}{d}$. Other laws of operation on fractions are readily deduced from the definition and laws of multiplication.

17. The totality of all integers and fractions forms the **system of rational numbers**, so called because they can be expressed as ratios. A fraction is evidently the ratio of its numerator to its denominator, and any integer may be expressed as a fraction by giving it the denominator unity, $a = \frac{a}{1}$. The system of rational numbers, like that of integers, has no first nor last number; but unlike that system it is **dense**; that is, between any two numbers of the system another number, and therefore an infinity of numbers, of the system may be found. No number has a **next** number.

18. Involution and evolution. The special case of multiplication where several equal factors are multiplied together is shortened by writing $\overbrace{a \cdot a \cdot a \cdots a}^{n \text{ factors}} = a^n$, and the process is called **involution.** Since it is a case of multiplication the result is a rational number.

Two inverse questions naturally arise. The problem of finding an exponent by which a given number must be affected so that the result shall equal another given number will be treated in Chapter X. The question what number raised to a given power will equal another given number may be considered in the form: Determine x so that $x^n = a$. When a is the product of n equal factors, x is easily determined as a rational number, but in general this is not the case. It is customary to write $x = a^{\frac{1}{n}}$, and annex these new numbers to the present system. Since the discussion of their properties involves the use of limits, it will be deferred until a later chapter. The symbol $a^{\frac{1}{n}}$ is called the nth root of a, and the process of determining its numerical value is called **evolution.**

19. Representation of rational numbers. Except in a few special cases, where Roman numerals are used, integers are represented by the **Arabic** or more properly **Hindu * system** of which ten is the base. One thousand three hundred and sixty-seven is in this sys-

* This system was in use in India in the year 669, and probably much earlier. There is some evidence to show that the Aryans obtained it from Thibet. The Arabs, in the course of commercial visits to India, learned of the system about 700, and brought it into Spain at the time of the Moorish conquest. It was introduced into Italy about 1200.

The work done by the ancient Greek mathematicians is the more remarkable when we consider the inadequate tools at their command for the representation of numbers.

tem $1 \cdot 10^3 + 3 \cdot 10^2 + 6 \cdot 10 + 7$, commonly written 1367, omitting the + signs and the powers of ten which are understood from the position of the digits which show how many multiples of each power are taken. The choice of ten as a base is purely arbitrary, and is probably due to the practice of primitive peoples of counting on their ten fingers.

In general, if m is the base of a system of numbers, a number of four digits would be written $am^3 + bm^2 + cm + d$. If seven were the base, the number 1367 would be written $3 \cdot 7^3 + 6 \cdot 7^2 + 6 \cdot 7 + 2$ or 3662. In this case only the digits up to 7 would be needed to represent all numbers. In general if m is the base, $m - 1$ digits are needed to represent all numbers.

It might seem that the simplest scale of notation would be the **dyadic** or **binary**, since 0 and 1 would be the only digits used, and arithmetical operations would be reduced to combining 1's. But this advantage would be more than counterbalanced by the great number of figures needed to represent even a moderately large number. For instance in the dyadic system 163 would be $2^7 + 2^5 + 2 + 1 = 10{,}100{,}011$.

20. The representation of fractional numbers as common fractions follows at once from their definition, but they may also be represented by the Hindu system, which gives the ordinary decimal fraction.

$$\tfrac{3}{8} = .375 = 3 \cdot 10^{-1} + 7 \cdot 10^{-2} + 5 \cdot 10^{-3}.$$

The representation may terminate as in the case just cited, where the common fraction is exactly represented by the decimal, or may continue indefinitely as in $\frac{1}{3} = .33333 \cdots$.

In the latter case, the decimal is *repeating or circulat-*

ing, that is, a single figure or group of figures is repeated an unlimited number of times. That this is always the case is seen from the fact that the decimal fraction is found by dividing the numerator of the common fraction by its denominator. Let the fraction be $\frac{m}{n}$. In dividing m by n there is at each step a remainder less than n, since, by hypothesis, the process does not terminate. At or before the $(n-1)$th step, a remainder will be found equal to one which has previously occurred, and from this point on there will be a repetition of the previous work. Hence if the decimal representation of a fraction is non-terminating, it is periodic, and the number of figures in the period is at least one less than the denominator of the common fraction from which the decimal is formed.

In this case the decimal is never exactly equal to the common fraction, though the difference can be made arbitrarily small by increasing the number of digits in the decimal.

It is easily shown that every repeating or circulating decimal represents a rational number. For

(1) $\qquad$ let $z = .1111111 \cdots$

(2) $\qquad 10z = 1.1111111 \cdots.$

Subtracting (1) from (2) $9z = 1$, $z = \frac{1}{9}$, a rational number.

Let

$$x = .343434 \cdots.$$

$$100x = 34.3434 \cdots.$$

$$99x = 34, \; x = \tfrac{34}{99}.$$

Any circulating decimal may be studied in a similar manner by multiplying by 10^n where n is the number of digits in the period.

As in the case of integers the choice of ten as a base is purely arbitrary. If seven were the base, $\frac{3}{8}$ would be

$$2 \cdot 7^{-1} + 4 \cdot 7^{-2} + 2 \cdot 7^{-3} + 4 \cdot 7^{-4} + \cdots.$$

In this case each period consists of two digits, the coefficients of the descending powers of 7 being 2, 4, 2, 4, $\cdots$ *ad infinitum*.

21. Continued fractions. An expression of the form

$$a + \cfrac{b}{c + \cfrac{d}{e + \cfrac{f}{g +}}}$$

is called a **continued fraction.** It is sufficient for the present purpose to consider only continued fractions of the form

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 +}}}$$

the numerators all being unity and the denominators positive integers. For compactness this is usually written

$$a_1 + \frac{1}{a_2 +} \frac{1}{a_3 +} \frac{1}{a_4} + \cdots.$$

A continued fraction may extend indefinitely, which case will be discussed in a later chapter; or it may terminate with one of its denominators, in which case it represents a rational number, since it may be changed to a common fraction by the ordinary process for simplifying complex fractions.

Any rational number may be expressed as a continued fraction as follows: Let $\frac{a}{b}$ be any rational number.

$\frac{a}{b} = a_1 + \frac{c}{b}$ where a_1 is the quotient and c the remainder in dividing a by b. $\frac{a}{b} = a_1 + \frac{1}{\frac{b}{c}} = a_1 + \frac{1}{a_2 + \frac{d}{c}}$, where a_2 is quotient and d remainder in dividing b by c. This process may be continued until the final remainder is zero.

Example.

$$\frac{53}{117} = \frac{1}{\frac{117}{53}} = \frac{1}{2 + \frac{11}{53}} = \frac{1}{2 + \frac{1}{\frac{53}{11}}} = \frac{1}{2 + \frac{1}{4 + \frac{9}{11}}}$$

$$= \frac{1}{2 + \frac{1}{4 + \frac{1}{\frac{11}{9}}}} = \frac{1}{2 + \frac{1}{4 + \frac{1}{1 + \frac{2}{9}}}} = \frac{1}{2 + \frac{1}{4 + \frac{1}{1 + \frac{1}{\frac{9}{2}}}}}$$

$$= \frac{1}{2 + \frac{1}{4 + \frac{1}{1 + \frac{1}{4 + \frac{1}{2}}}}} = \frac{1}{2 + \frac{1}{4 + \frac{1}{1 + \frac{1}{4 + \frac{1}{\frac{2}{1}}}}}}$$

$$= \frac{1}{2} \, \frac{1}{+ 4} \, \frac{1}{+ 1} \, \frac{1}{+ 4} \, \frac{1}{+ 2}.$$

22. An integer is **prime** if it has no divisor except itself and unity; **composite** if it has other integral divisors. Two integers are **mutually prime** when they have no common divisor greater than unity.

If one integer is not exactly divisible by another, there will be a remainder less than the divisor, and the following relation holds.

(1) Dividend = divisor × quotient + remainder.

23. Theorem. *If two numbers have the same factor, their sum and their difference will also have this factor.*

Let $a = \alpha c$ and $b = \beta c$.

Then $a \pm b = \alpha c \pm \beta c = c(\alpha \pm \beta)$ by the distributive law. Hence c is a factor of $a \pm b$.

24. Highest common divisor. Euclid* developed the following method, which is still in use, for finding the highest common divisor of two numbers. Let a be an integer greater than b. If a is divisible by b, then b is the divisor required. If this is not the case, § 22, (1) gives $a = b \cdot q_1 + r_1$ where q_1 is the quotient and the remainder r_1 is less than b. Dividing b by r_1 gives $b = r_1 \cdot q_2 + r_2$. The process of dividing the last divisor by the corresponding remainder is continued, with the result that $r_1 > r_2 > r_3 > \cdots > r_n$. If $r_n = 1$, the two numbers are mutually prime; but if $r_n = 0$, then r_{n-1} is the divisor sought.

$$\begin{aligned} a &= bq_1 + r_1 \\ b &= r_1 q_2 + r_2 \\ r_1 &= r_2 q_3 + r_3 \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ r_{n-3} &= r_{n-2} q_{n-1} + r_{n-1} \\ r_{n-2} &= r_{n-1} q_n + r_n. \end{aligned}$$

If $r_n = 0$, r_{n-2} is divisible by r_{n-1}. Then by § 23, $r_{n-2} q_{n-1} + r_{n-1}$ is divisible by r_{n-1}, that is, r_{n-3} also has the same divisor. Working backwards to the first relation, it is found that a and b are both divisible by r_{n-1}.

* Euclid's "Elements," written about 300 B.C., and used as a textbook for over 2000 years, being still employed in England, contains, besides the geometrical part, a treatise on theory of numbers, in which is found the rule for finding the highest common divisor, which bears Euclid's name.

25. Theorem. *If the sum of the digits of any number is divisible by $m-1$, where m is the base of the system, the number itself is divisible by $m-1$.*

Let the number be $am^3 + bm^2 + cm + d$
and let the sum of the digits be $a + b + c + d$

The remainder $a(m^3-1) + b(m^2-1) + c(m-1)$
is divisible by $m-1$. By hypothesis $a+b+c+d$ is divisible by $m-1$. Hence by § 23, the number $am^3 + bm^2 + cm + d$ is divisible by $m-1$.

Since in the usual notation $m = 10$, it follows that if the sum of the digits of any number is divisible by 9, the number itself is divisible by 9. Since 3 is a factor of 9, the same test holds for the divisibility of a number by 3.

26. Theorem. *If the difference between the sum of the digits in the even powers of m and the sum of the digits in the odd powers of m in any number where m is the base, be divisible by $m+1$, the number itself is divisible by $m+1$.*

Let the number be $am^3 + bm^2 + cm + d$
Let the difference between
the sums of the digits be $a - b + c - d$

The sum $a(m^3+1) + b(m^2-1) + c(m+1)$
is evidently divisible by $m+1$. Since the quantity $a-b+c-d$ has the factor $m+1$ by hypothesis, it follows from § 23 that the number $am^3 + bm^2 + cm + d$ is divisible by $m+1$.

In the decimal system of notation, this theorem becomes: If the difference between the sum of the digits in the units, hundreds, tens of thousands, etc. places, and the sum of the digits in the alternate places is divisible by 11, the number is divisible by 11.

27. Theorem. *If the last digit of any number is divisible by a factor of m, the number is divisible by that factor of m.*

The number $am^3 + bm^2 + cm + d$ is clearly divisible by any factor of m if d is thus divisible.

It follows that in the decimal system a number is divisible by 2 or 5 if its last digit is divisible by those numbers.

In a manner similar to the last theorem it may be proved that if a is a factor of m, any number is divisible by a^2 if its last two digits are divisible by a^2, by a^3 if its last three digits are divisible by a^3, etc.

In the decimal system this becomes: Any number is divisible by 4 or 25, when the number formed by its last two digits is divisible by 4 or 25; by 8 or 125 when the number formed by its last three digits is divisible by 8 or 125, etc.

28. Factorials. The product $1 \cdot 2 \cdot 3 \cdots n$ is of such frequent occurrence in mathematical problems, that it is convenient to abbreviate it. The product of all positive integers up to a certain one will be indicated by writing only the last factor with an exclamation point after it. Thus

$$5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5. \quad 17! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots 17.$$
$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots n.$$

Another abbreviation for such products, which is used in many American and English text-books, is $\underline{|4} = 1 \cdot 2 \cdot 3 \cdot 4$. Either abbreviation is read "factorial n," "factorial 4," as the case may be.

EXERCISES

1. What are the values of the following expressions?

$$\frac{8!}{7!},\ \frac{28!}{29!},\ \frac{n!}{(n-1)!},\ \frac{n!}{(n+1)!},\ \frac{(n+2)!}{3!\,n!}.$$

2. What is the shortest way of writing the following products? $5!\cdot 6\cdot 7\cdot 8,\ n!(n+1)(n+2).$

29. Factors of $n!$ The product $n! = 1\cdot 2\cdot 3 \cdots n$ may be resolved into its prime factors without evaluating it as follows: The symbol $\left[\frac{n}{k}\right]$ is used to denote the largest integer which is less than or equal to $\frac{n}{k}$ where $k = 1, 2, 3 \cdots$. The number of even factors in $n!$ is evidently $\left[\frac{n}{2}\right]$ which in this case is equal either to $\frac{n-1}{2}$ or $\frac{n}{2}$ according as n is odd or even. Taking the factor 2 out of each of the even factors of $n!$ is equivalent to removing the factor $2^{\left[\frac{n}{2}\right]}$ from $n!$ The quotient after dividing by $2^{\left[\frac{n}{2}\right]}$ evidently contains $\left[\frac{n}{4}\right]$ even factors, hence the factor $2^{\left[\frac{n}{4}\right]}$ may now be removed. Continuing this process it is found that $n!$ has the factor $2^{\left[\frac{n}{2}\right]+\left[\frac{n}{4}\right]+\left[\frac{n}{8}\right]+\cdots}$. The factors of $n!$ which are divisible by 3 are 3, 6, 9 ⋯ and their number is $\left[\frac{n}{3}\right]$.

Continuing as before, it is found that $n!$ has the factor $3^{\left[\frac{n}{3}\right]+\left[\frac{n}{9}\right]+\left[\frac{n}{27}\right]+\cdots}$. In general $n!$ has the factor $k^{\left[\frac{n}{k}\right]+\left[\frac{n}{k^2}\right]+\left[\frac{n}{k^3}\right]+\cdots}$ where k is any prime less than or equal to n.

Example. $15! = 2^{7+3+1} \cdot 3^{5+1} \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$
$= 2^{11} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13.$

In a similar manner it may be shown that the product $(a+1)(a+2)(a+3)\cdots(a+n)$ has the factor $k^{\left[\frac{n}{k}\right]+\left[\frac{n}{k^2}\right]+\left[\frac{n}{k^3}\right]+\cdots}$ where k is any prime less than or equal to n, and a is any integer. From this it follows that the product of n successive integers is always divisible by $n!$

30. Powers of integers. It is evident that any power of an odd number is odd, and any power of an even number is even.

31. Theorem. *The numbers n and n^{4m+1}, where m and n are any positive integers, have the same digit in the units' place.*

This will be proved by showing that $n^{4m+1} - n$ is divisible by 10 and will be first proved for $m = 1$.

$n^5 - n = n(n-1)(n+1)(n^2+1)$ is evidently divisible by 2. It remains to show that it is also divisible by 5. If any one of the factors n, $n-1$ or $n+1$ is a multiple of 5, the theorem is already proved. If none of them are multiples of 5 the number

$$n^5 - n = n(n-1)(n+1)(n^2+1)$$

must have one of the forms

$$(5r+1)(5r+2)(5r+3)[(5r+2)^2+1]$$

or $$(5r+2)(5r+3)(5r+4)[(5r+3)^2+1]$$

either of which is divisible by 5. Since $n^5 - n$ is a factor of $n^{4m+1} - n$ the latter number is also divisible by ten, and it follows that n and n^{4m+1} have the same digit in the units' place.

32. Theorem. *If m and n are positive integers, and n less than ten, the numbers* n^{2m} *and* $(10-n)^{2m}$ *have the same digit in the units' place.*

For, $(10-n)^{2m}-n^{2m}$ is clearly divisible by ten.

33. Theorem. *If m and n are positive integers, and n less than ten, the sum of the digit in the units' place of* n^{2m+1} *and the digit in the units' place of* $(10-n)^{2m+1}$ *is ten.*

For, $(10-n)^{2m+1}+n^{2m+1}$ is clearly divisible by ten.

34. Squares of integers. If the squares of integers up to 25 are known, the square of any integer up to 125 may be found by inspection by one of the following methods.

(1) If $25 < n \leq 75$, set $n = 50 \pm a$.

$$\begin{aligned} n^2 &= 2500 \pm 100a + a^2 \\ &= (25 \pm a)100 + a^2 \\ &= (n-25)100 + a^2. \end{aligned}$$

Example 1. $56^2 = (50+6)^2.\quad a = 6,\ n = 56.$

$\therefore\ 56^2 = 31 \cdot 100 + 36 = 3136.$

Example 2. $42^2 = (50-8)^2.\quad a = 8,\ n = 42.$

$\therefore\ 42^2 = 17 \cdot 100 + 64 = 1764.$

(2) If $75 \leq n \leq 125$, set $n = 100 \pm a$

$$\begin{aligned} n^2 &= 10000 \pm 200a + a^2 \\ &= (100 \pm 2a)100 + a^2 \\ &= (n \pm a)100 + a^2. \end{aligned}$$

Example 3. $116^2 = (100+16)^2.\quad a = 16,\ n = 116$

$\therefore\ 116^2 = 132 \cdot 100 + 256 = 132256.$

Example 4. $93^2 = (100-7)^2.\quad a = 7,\ n = 93.$

$\therefore\ 93^2 = 86 \cdot 100 + 49 = 8649.$

35. Powers of fractions. If $\frac{a}{b}$ is a fraction in its lowest terms, it is evident that any power of $\frac{a}{b}$ is also a fraction in its lowest terms.

If $\frac{a}{b} < 1$, $\left(\frac{a}{b}\right)^n < \frac{a}{b}$, where $n > 1$,

If $\frac{a}{b} > 1$, $\left(\frac{a}{b}\right)^n > \frac{a}{b}$, where $n > 1$.

CHAPTER II

PERMUTATIONS, COMBINATIONS, PROBABILITY

It is a truth very certain that, when it is not in our power to determine what is true, we ought to determine what is most probable.

— René Descartes.

36. Permutations. Each different arrangement of all or part of a group of objects is called a **permutation.**

37. Theorem. *The number of permutations of n things taken r at a time is*

$$n(n-1)(n-2)\cdots(n-r+1).$$

For, the first place may be filled in n different ways, the second in $n-1$ different ways, and so on, the rth place having $n-(r-1)=n-r+1$ possibilities in being filled. Hence the number of different ways in which r objects chosen from a set of n objects may be arranged is $n(n-1)(n-2)\cdots(n-r+1)$. This product is frequently symbolized by $P_{n,r}$.

Corollary. *The number of possible permutations of n things is n!.* This is proved by taking $r=n$ in the theorem.

Example. How many numbers of five figures each can be formed from the nine digits, no one being used twice?

$$n=9,\ r=5. \quad \therefore P_{n,r}=9\cdot 8\cdot 7\cdot 6\cdot 5=15120.$$

38. Theorem. *The number of permutations of n things when r of them are alike is* $\dfrac{n!}{r!}$.

If all were different, the number of permutations would be $n!$; but in this case $r!$ of the permutations effect no change. Hence the actual number is $\frac{n!}{r!}$.

Corollary. *The number of permutations of n things when r are of one kind, s of a second kind, and t of a third is* $\frac{n!}{r!\,s!\,t!}$.

Since $r+s+t$ must evidently be less than or equal to n, this number is an integer by § 29.

Example. How many arrangements can be made of twelve flags, two being red, three blue, three black, and four white?

The number is $\frac{12!}{2!\,3!\,3!\,4!} = 5 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 = 277200.$

EXERCISES

1. A railway signal has three arms, each of which can be placed in four positions. How many different signals can be made?

2. In how many ways can nine brown books and six green books be arranged on a shelf so that all of one color may be together?

3. A dozen rosebushes are to be planted in a border. Five are white, four are pink, and three are red. How many arrangements are possible?

39. Combinations. Each different subgroup which can be formed from a group of objects irrespective of their order is called a **combination.**

Illustration. Given four points A, B, C, D, no three of which are in a straight line. How many straight lines can be formed by joining them in pairs? Since from each of the four points a line can be drawn to each of the other three, it might at first appear that $3 \cdot 4 = 12$ lines would be possible. But when it is considered that the line formed by joining A to B is the same as the one formed by joining B to A, and so on, it is seen that there can only be $12 \div 2 = 6$ lines.

40. Theorem. *The number of combinations of n things taken r at a time is*

$$\frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} = \frac{P_{n,r}}{r!}.$$

By the corollary of § 37 the number of permutations of r things all taken together is $r!$ Hence the number of permutations in any given group is $r!$ times the number of combinations in that group. That is, the number of combinations is

$$\frac{P_{n,r}}{r!} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}.$$

This number, which by § 29 is an integer, is symbolized by $C_{n,r}$.

Then $P_{n,r} = r!\, C_{n,r}$.

Corollary. $C_{n,r} = C_{n,n-r}$.

For, multiplying both numerator and denominator of $C_{n,r}$ by $(n-r)!$ gives

$$\frac{n(n-1)\cdots(n-r+1)\cdot(n-r)!}{r!\,(n-r)!} = \frac{n!}{r!\,(n-r)!}$$

$$= \frac{n(n-1)(n-2)\cdots(r+1)}{(n-r)!}$$

$$= \frac{n(n-1)\cdots[n-(n-r)+1]}{(n-r)!} = C_{n,n-r}.$$

In some cases the computation of $C_{n, n-r}$ is simpler than that of $C_{n, r}$ and by virtue of this corollary either computation gives the same result.

Example. In how many ways could a crew of eight men be selected from a squad of twelve?

$$C_{12, 8} = C_{12, 4} = \frac{12 \cdot 11 \cdot 10 \cdot 9}{4\,!} = 495.$$

EXERCISES

1. If ten lines all intersect each other, how many triangles are formed by them?

2. In how many ways can a party of six people form partners?

3. If eight lines all intersect each other, how many points of intersection are there?

4. In how many ways can a span of horses be selected out of a stud of 6?

5. In how many ways can a baseball nine be chosen from a group of fifteen men?

41. Probability. The **probability** of an event is the ratio of the number of the chances it has of happening to the sum of all the chances, both for and against. Hence the probability of an event is always a fraction between zero and one. If the fraction is zero, the event is impossible; if it is $\frac{1}{2}$, the chances for and against are equal; and if it is unity, the event is certain. Since an event must either happen or not happen, the sum of the probabilities for and against is equal to unity. The **odds** in favor of an event are the ratio of all its chances of success to all its chances of failure. If an event can happen in a ways and fail in b ways, the probability of success is $\frac{a}{a+b}$, the probability of failure is $\frac{b}{a+b}$, while the odds in favor of success are $\frac{a}{b}$.

Example 1. If a bag contains ten balls of which three are white, the probability of drawing a white ball is $\frac{3}{10}$, while the odds in favor of drawing it are $\frac{3}{7}$.

Example 2. Out of a bag containing three white and four black balls, two balls are drawn. What is the probability (*a*) of both being white, (*b*) of both being black, (*c*) of one being white and one black?

The total number of ways in which two balls may be drawn is $C_{7,2} = \frac{7 \cdot 6}{2} = 21$. The number of ways in which two white balls may be drawn is $C_{3,2} = \frac{3 \cdot 2}{2} = 3$; and the number of ways in which two black balls may be drawn is $C_{4,2} = \frac{4 \cdot 3}{2} = 6$. Since one white ball may be drawn in three ways, and one black one in four ways, one white and one black may be drawn in $3 \cdot 4 = 12$ ways. Hence the probability of (*a*) is $\frac{3}{21} = \frac{1}{7}$, of (*b*) is $\frac{6}{21} = \frac{2}{7}$, of (*c*) is $\frac{12}{21} = \frac{4}{7}$.

42. Theorem. *The probability that two independent events will occur is equal to the product of their individual probabilities.*

For, if the first event has a chances of success out of a total of b ways in which it may occur either successfully or unsuccessfully, its probability is $\frac{a}{b}$. If the second event has c chances of success out of a total number of d ways in which it can happen, its probability is $\frac{c}{d}$. Each of the a successful chances of the first may combine with each of the c successful chances of the second, making the total number of chances of success for both $a \cdot c$. But since the total number of ways in which both events may occur is $b \cdot d$, the probability that both will occur successfully is $\frac{a \cdot c}{b \cdot d} = \frac{a}{b} \cdot \frac{c}{d}$.

Corollary. *The probability that any number of independent events will occur is equal to the product of their respective probabilities.*

43. It should be noted that the theorem of § 42 is true only when the two events are entirely independent. At first glance it might appear that case (c) of Example 2, § 41 would come under this theorem, but upon further reflection it will appear that the two events in that case are not independent.

When the probability of a second event depends on the occurrence of an earlier one, its probability is equal to the probability of the first multiplied by the probability of the second *after* the first has occurred. In case (c) of Example 2, § 41, the probability of first drawing a white ball is $\frac{3}{7}$. After this has occurred there are two white and four black balls remaining in the bag. Hence the probability that the second trial will produce a black ball is $\frac{4}{6} = \frac{2}{3}$. Then the total chance of drawing first a white and then a black ball is $\frac{3}{7} \cdot \frac{2}{3} = \frac{2}{7}$. In a similar way it may be found that the probability of drawing first a black and then a white ball is $\frac{2}{7}$. Hence the probability of drawing one white and one black ball in either order is $\frac{2}{7} + \frac{2}{7} = \frac{4}{7}$, the same conclusion which was reached by the former method.

Problems like the preceding, concerning games of chance, afford simple illustrations of the theory of probability; but the more important applications such as life and fire insurance present far more complicated problems. The question whether a certain man 25 years of age will be living at the end of a year is impossible to answer; but if it be asked "Out of 1000 men 25 years of age, how many will be living at the expiration of a year?" the event can be foretold with a fair degree of accuracy. If the question concerns 1,000,000 men, the answer will be much more exact, the degree of certainty increasing with the number of cases considered. In order to determine the expectation of life in any given case, large numbers of statistics must be collected, and mathematical methods employed which are beyond the scope of the present work.

For a more complete discussion of the topics of this chapter the student is referred to Chrystal's Algebra, Vol. II.

EXERCISES

1. From a Congressional committee consisting of 6 Republicans and 5 Democrats, a sub-committee of 3 is to be chosen by lot. What is the probability,

(*a*) that the committee will be composed of 3 Republicans;

(*b*) that it will be composed of 3 Democrats;

(*c*) that it will contain two Republicans and one Democrat;

(*d*) that it will contain one Republican and two Democrats?

2. Four Congressmen are to be elected in a state where two political parties are evenly balanced. What is the probability,

(*a*) that all four belong to one party;

(*b*) that three belong to one party and one to the other;

(*c*) that two belong to each party?

3. From a steamer which carried 150 passengers and a crew of 50, a person was lost overboard. What is the probability,

(*a*) that all the passengers are safe;

(*b*) that all the crew are safe;

(*c*) that a certain passenger, Mr. A., is safe?

4. An illiterate servant places 12 books on a shelf. What is the probability that three volumes of a set are together in their proper order?

5. Each student in a class of ten is likely to make a correct observation of the sun's transit, one time out of four. What is the probability that a given transit will be correctly observed by the entire class?

6. If four cards be drawn from a pack of 52, show that the probability of there being one from each of the four suits is $\frac{39}{51}\cdot\frac{26}{50}\cdot\frac{13}{49}$.

7. Show that the probability of a leap year containing 53 Sundays is $\frac{2}{7}$.

8. Two keys, either of which will unlock a certain door, are on a ring containing eight. What is the probability that the door can be unlocked without trying more than three keys?

9. A man wants a particular span of horses from a stud of six. If his groom brings him four horses chosen at random, what is the probability that the desired span will be among them?

10. A party of ten attending the opera are to occupy six seats in one group and four seats in another group.

(*a*) In how many different ways may the party break up?

(*b*) If the division is made at random, what is the probability of two given individuals, A and B, being in the same group?

11. In how many ways may a football team of 11 men and a crew of 8 be chosen from a squad of 25?

12. If 12 students are seated at random in a row, what is the probability that A and B are next each other?

13. How many distinct sounds may be made by ten keys of a piano if not more than three are struck at a time?

14. What is the probability of a player in a game of whist holding at least three aces?

15. Three roads lead from the town of A to the town of B, and four roads from the town of B to the town of C. By how many different routes may the journey from A to C be accomplished?

16. One purse contains six silver dollars and four quarters. Another contains two silver dollars and ten quarters. If a purse is chosen at random, and a coin extracted at random, what is the probability that the coin selected will be a dollar?

17. A company of 40 people is to be seated in four equal groups. In how many different ways can the groups be formed?

18. If 5 books are brought at random from a shelf of 20, what is the probability that three desired books will be among them?

CHAPTER III

DETERMINANTS

What is the theory of determinants? It is an algebra upon algebra; a calculus which enables us to combine and foretell the results of algebraic operations, in the same way as algebra itself enables us to dispense with the performance of the special operations of arithmetic. All analysis must ultimately clothe itself in this form.

—J. J. SYLVESTER.

44. In many problems in mathematics it is necessary to solve systems of simultaneous equations in two or more unknown quantities. If the equations are all of the first degree (linear), if the number of unknown quantities is small, and if the coefficients are numerical, the solution is not laborious; but as the number of unknown quantities increases, the work involved in the solution is enormously increased, and if the coefficients are literal, the results are long and awkward to handle.

Let it be required to solve the simultaneous equations

$$(1)\qquad \begin{cases} a_1x + b_1y = m_1, \\ a_2x + b_2y = m_2. \end{cases}$$

Multiply the first equation by b_2, the second by b_1, and take their difference

$$\begin{aligned} a_1b_2x + b_1b_2y &= m_1b_2 \\ a_2b_1x + b_1b_2y &= m_2b_1 \\ \hline (a_1b_2 - a_2b_1)x &= m_1b_2 - m_2b_1 \end{aligned}$$

Hence

$$(2)\qquad x = \frac{m_1b_2 - m_2b_1}{a_1b_2 - a_2b_1}.$$

The value of y is found by a similar process, or by substitution,

$$(3) \qquad y = \frac{a_1 m_2 - a_2 m_1}{a_1 b_2 - a_2 b_1}.$$

These results may serve as a rule for finding the values of x and y from two simple equations. The advantage of writing the equations in the form (1) is obvious.

By similar methods the following system of linear equations in three unknown quantities may be solved:

$$(4) \qquad \begin{cases} a_1 x + b_1 y + c_1 z = m_1, \\ a_2 x + b_2 y + c_2 z = m_2, \\ a_3 x + b_3 y + c_3 z = m_3, \end{cases}$$

giving as the value of x

$$x = \frac{m_1 b_2 c_3 - m_1 b_3 c_2 - m_2 b_1 c_3 + m_2 b_3 c_1 + m_3 b_1 c_2 - m_3 b_2 c_1}{a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_1 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1}$$

and for y and z similar results, which may again serve as a rule for solving any such system of equations.

45. A certain regularity of form, and the fact that similar expressions often occur in other mathematical work, led mathematicians as early as the seventeenth century to attempt to arrange these results in a compact and simple form, an effort which led to the introduction of **determinants.**

It will be observed that the number of different quantities occurring in each numerator and each denominator of the values of x and y in the preceding article is a perfect square. In accordance with a rule to be developed later, these quantities are arranged in a square between vertical bars, and this form is called a *determinant.*

Thus the following forms are determinants:

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad \begin{vmatrix} m_1 & b_1 & c_1 \\ m_2 & b_2 & c_2 \\ m_3 & b_3 & c_3 \end{vmatrix}.$$

The quantities which make up a determinant are called its *elements*.

The number of elements in any row or column indicates the *order* of the determinant. The above determinants are of the second and third orders.

In the term $a_1b_3c_2$ the factors are said to be arranged *in order of columns*. If the order were $a_1c_2b_3$ they would be arranged *in order of rows*. This result is brought about by *exchanging* b_3 and c_2. The term $a_2b_3c_1$ is arranged in order of columns; two exchanges are needed to bring the elements into order of rows; first exchange a_2 and c_1, then a_2 and b_3.

The symbol D will be used to represent a determinant.

46. A determinant of the *n*th order is a set of n^2 elements arranged in n rows and n columns between vertical bars, and having a value obtained as follows:

1. Form all possible products of n factors each, taking one and only one element from each row and each column.

2. If the factors of each product are arranged in order of columns (rows), find the number of exchanges of factors needed to bring them into the order of rows (columns), and give to each product the positive or negative sign according as this number of exchanges is even or odd. The algebraic sum of these products is the *value of the determinant.*

EXERCISES

Show that in accordance with this rule the values of the general determinants of the second and third orders are as follows:

1. $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1.$

2. $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1.$

Find the value of each of the following determinants:

3. $\begin{vmatrix} 4 & 2 \\ 3 & 5 \end{vmatrix}.$ **4.** $\begin{vmatrix} 9 & 8 \\ 2 & 1 \end{vmatrix}.$ **5.** $\begin{vmatrix} a & b \\ a & b \end{vmatrix}.$

6. In the expansion of the general determinant of the fourth order, what are the signs of the following terms: $a_3b_1c_4d_2$, $b_2c_1d_3a_4$, $c_2a_4b_3d_1$? How many terms contain the factor a_1? How many terms are there in the expansion?

7. How many terms does the general determinant of the *n*th order contain when expanded?

47. If the values of x and y in § 44, equations (2), (3) be written in terms of determinants, they afford a general rule for solving by determinants two equations in two unknown quantities.

$$x = \frac{\begin{vmatrix} m_1 & b_1 \\ m_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & m_1 \\ a_2 & m_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

In each case the denominator is the same, and is obtained by writing in order the coefficients of x and of y. This is known as *the determinant of the system.*

RULE. *In a system of linear equations in two unknown quantities the value of each unknown quantity may be expressed as the quotient of two determinants; the denominator is in each case the determinant of the system, while the numerator is obtained by replacing the coefficients of the quantity whose value is sought by the known terms in the right-hand members of the equations.*

Example. Solve for x and y the equations

$$4x + 3y + 6 = 0,$$
$$y - x + 5 = 0.$$

Rearrange the equations in the form of § 44 (1),

$$4x + 3y = -6,$$
$$x - y = 5.$$

By the above rule

$$x = \frac{\begin{vmatrix} -6 & 3 \\ 5 & -1 \end{vmatrix}}{\begin{vmatrix} 4 & 3 \\ 1 & -1 \end{vmatrix}} = \frac{-9}{-7} = \frac{9}{7},$$

$$y = \frac{\begin{vmatrix} 4 & -6 \\ 1 & 5 \end{vmatrix}}{\begin{vmatrix} 4 & 3 \\ 1 & -1 \end{vmatrix}} = \frac{26}{-7} = -\frac{26}{7}.$$

While the general determinant of any order is readily expanded by the method of § 46, it is a slow process to find the value of a numerical determinant. The following theorems lead to convenient methods of finding the value of any determinant. They are proved for determinants of the third order only, but the method is in each case applicable to determinants of any order.

48. Minors. The **minor of any element** of a determinant of the nth order is the determinant of the $(n-1)$th order formed by striking out the row and the column in which that element stands.

The minor of any element, as b_1, is represented by D_{b_1}, *i.e.* in the determinant of the third order, the minors of b_1 and c_3 are

$$D_{b_1} = \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}, \quad D_{c_3} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

49. Theorem. *If a determinant is transformed so as to read by columns as it formerly read by rows, the value of the determinant is unchanged.*

Let
$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad D' = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

By the rule for the expansion of a determinant in § 46 it is clear that D and D' have the same value.

It follows that any statement in regard to the rows of a determinant holds for the corresponding columns, and *vice versa.*

50. Theorem. *A determinant may be written as the algebraic sum of the elements of any row or column, each multiplied by its minor.*

Take the determinant of the third order

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1$$

and factor with respect to the elements of the first column.

$$D = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$
$$= a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$
$$= a_1D_{a_1} - a_2D_{a_2} + a_3D_{a_3}.$$

If the determinant be arranged with respect to the second row,

$$D = -a_2D_{a_2} + b_2D_{b_2} - c_2D_{c_2}$$

and so for any column or row.

By experiment it is found that each element times its minor occurs with the positive or negative sign according as the sum of the number of the row and of the column in which that element stands is even or odd. This gives the following *scheme of signs for the product of each element by its minor*, which is readily remembered:

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

51. Theorem. *If all the elements of a row or column are zero, the value of the determinant is zero.*

This may be proved either by developing according to the row or column in which the zeros stand, or by using § 46.

52. Theorem. *If two rows or columns are interchanged, the sign of the determinant is changed.*

$$\text{Let } D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$\text{and } D' = \begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} = a_3\begin{vmatrix} b_2 & c_2 \\ b_1 & c_1 \end{vmatrix} - a_2\begin{vmatrix} b_3 & c_3 \\ b_1 & c_1 \end{vmatrix} + a_1\begin{vmatrix} b_3 & c_3 \\ b_2 & c_2 \end{vmatrix}$$

Hence $D' = -D$.

53. Theorem. *If two rows or columns of a determinant are identical, the value of the determinant is zero.*

Let
$$D = \begin{vmatrix} a_1 & b_1 & a_1 \\ a_2 & b_2 & a_2 \\ a_3 & b_3 & a_3 \end{vmatrix}$$

and let D' represent the determinant obtained by interchanging the first and third columns. It is evident that $D = D'$; by § 52, $D = -D'$, and therefore $D = 0$.

NOTE. This theorem is also readily proved by expanding the determinant D according to the minors of the second column. In general the row or column that is in some respect distinguished from the others is the one to select for use in the expansion.

54. Theorem. *If each element of a row or column is multiplied by the same number, the determinant is multiplied by that number.*

Let D have the same meaning as in § 49

and let
$$D' = \begin{vmatrix} a_1 & b_1 & c_1 \\ na_2 & nb_2 & nc_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -na_2D_{a_2} + nb_2D_{b_2} - nc_2D_{c_2}$$
$$= n[-a_2D_{a_2} + b_2D_{b_2} - c_2D_{c_2}] = nD.$$

This theorem is often useful in simplifying a determinant. For instance,

$$\begin{vmatrix} 18 & -6 & 14 \\ 45 & 15 & -35 \\ -9 & 3 & 7 \end{vmatrix} = 9\begin{vmatrix} 2 & -6 & 14 \\ 5 & 15 & -35 \\ -1 & 3 & 7 \end{vmatrix}$$
$$= 9 \cdot 3 \begin{vmatrix} 2 & -2 & 14 \\ 5 & 5 & -35 \\ -1 & 1 & 7 \end{vmatrix}$$
$$= 9 \cdot 3 \cdot 2 \cdot 7 \cdot 5 \begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{vmatrix}$$

55. Theorem. *If the elements of any row or column are proportional to the corresponding elements of any other row or column, the value of the determinant is zero.*

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ ka_1 & kb_1 & kc_1 \end{vmatrix} = k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} = 0 \qquad \S\S\ 54,\ 53.$$

56. Theorem. *If each element of a row or column be multiplied by the minor of the corresponding element of any other row or column, following the rule of signs for the product of each element by its minor, the sum of these products is zero.*

Let the elements of the third column of D be multiplied by the minors of the corresponding elements of the second column, with their usual signs.

$$c_1D_{b_1} - c_2D_{b_2} + c_3D_{b_3} = \begin{vmatrix} a_1 & c_1 & c_1 \\ a_2 & c_2 & c_2 \\ a_3 & c_3 & c_3 \end{vmatrix} = 0. \qquad \S\ 53.$$

57. Theorem. *If each of the elements of a row or column is the sum of n numbers, the determinant may be written as the sum of n determinants.*

Let $n = 2$, a similar proof holds for larger values.

$$\begin{vmatrix} a_1 & b_1+k_1 & c_1 \\ a_2 & b_2+k_2 & c_2 \\ a_3 & b_3+k_3 & c_3 \end{vmatrix} = -(b_1+k_1)\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + (b_2+k_2)\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$$

$$-(b_3+k_3)\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = -b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + b_2\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - b_3\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

$$-k_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + k_2\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - k_3\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & k_1 & c_1 \\ a_2 & k_2 & c_2 \\ a_3 & k_3 & c_3 \end{vmatrix}$$

This theorem makes it possible to write any determinant as the sum of two determinants, one of which is zero.

$$\begin{vmatrix} 23 & 19 & 35 \\ 31 & 18 & 17 \\ 42 & 26 & 20 \end{vmatrix} = \begin{vmatrix} 4+19 & 19 & 35 \\ 13+18 & 18 & 17 \\ 16+26 & 26 & 20 \end{vmatrix} = \begin{vmatrix} 4 & 19 & 35 \\ 13 & 18 & 17 \\ 16 & 26 & 20 \end{vmatrix} + \begin{vmatrix} 19 & 19 & 35 \\ 18 & 18 & 17 \\ 26 & 26 & 20 \end{vmatrix}$$

The last determinant is equal to zero.

58. Theorem. *If the elements of any row or column, multiplied by any number, be added to or subtracted from the corresponding elements of any other row or column, the value of the determinant is unchanged.*

It is left to the student to prove by the use of §§ 57, 55 that

$$\begin{vmatrix} a_1 + nc_1 & b_1 & c_1 \\ a_2 + nc_2 & b_2 & c_2 \\ a_3 + nc_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

59. The theorem of § 58 makes it possible to simplify a determinant by diminishing its elements or replacing some of them by zeros, thus making its expansion easier. By this method all but one of the elements of any row or column may be replaced by zeros, and if the determinant be then developed according to the minors of that row or column, one minor only will remain ; *i.e.* the order of the determinant is reduced by one. This will be illustrated by examples.

Example 1. Evaluate the determinant

$$D = \begin{vmatrix} 39 & 28 & 56 \\ 19 & 15 & 38 \\ 22 & 17 & 50 \end{vmatrix}$$

Multiply the second column by 2 and subtract the result from the third.

$$D=\begin{vmatrix}39 & 28 & 0\\ 19 & 15 & 8\\ 22 & 17 & 16\end{vmatrix}$$

Multiply the second row by 2 and subtract the result from the third.

$$D=\begin{vmatrix}39 & 28 & 0\\ 19 & 15 & 8\\ -16 & -13 & 0\end{vmatrix}=-8\begin{vmatrix}39 & 28\\ -16 & -13\end{vmatrix}=8\begin{vmatrix}39 & 28\\ 16 & 13\end{vmatrix}$$

If desired, these numbers may be further reduced by subtracting the second row, multiplied by 2, from the first.

$$D=8\begin{vmatrix}7 & 2\\ 16 & 13\end{vmatrix}=8(91-32)=472.$$

Example 2. Evaluate the determinant

$$D=\begin{vmatrix}17 & 20 & 23\\ 18 & 21 & 24\\ 19 & 22 & 25\end{vmatrix}$$

It is best merely to reduce the numbers, as there is no advantageous way of obtaining zeros at once. Subtract the first row from the second and from the third.

$$D=\begin{vmatrix}17 & 20 & 23\\ 1 & 1 & 1\\ 2 & 2 & 2\end{vmatrix}=0$$

Example 3. Evaluate the determinant

$$D=\begin{vmatrix}1 & 1 & 1\\ a & b & c\\ a^2 & b^2 & c^2\end{vmatrix}$$

Subtract the third column from each of the others.

$$D=\begin{vmatrix} 0 & 0 & 1 \\ a-c & b-c & c \\ a^2-c^2 & b^2-c^2 & c^2 \end{vmatrix}=\begin{vmatrix} a-c & b-c \\ a^2-c^2 & b^2-c^2 \end{vmatrix}$$

$$=(a-c)(b-c)\begin{vmatrix} 1 & 1 \\ a+c & b+c \end{vmatrix}=(a-c)(b-c)(b-a)$$

EXERCISES

Find the value of each of the following determinants:

1. $\begin{vmatrix} 13 & 3 & 23 \\ 30 & 7 & 53 \\ 39 & 9 & 70 \end{vmatrix}$

2. $\begin{vmatrix} 29 & 26 & 22 \\ 25 & 31 & 27 \\ 63 & 54 & 46 \end{vmatrix}$

3. $\begin{vmatrix} 2 & 3 & 4 \\ 3 & 6 & 10 \\ 4 & 10 & 20 \end{vmatrix}$

4. $\begin{vmatrix} 1 & a^3 & a^2 \\ a^3 & 1 & a \\ a^2 & a & 1 \end{vmatrix}$

5. $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$

6. $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$

7. $\begin{vmatrix} 67 & 19 & 21 \\ 39 & 14 & 15 \\ 81 & 27 & 26 \end{vmatrix}$

8. $\begin{vmatrix} 1 & 1 & 1 \\ 1 & a+1 & 1 \\ 1 & 1 & b+1 \end{vmatrix}$

9. $\begin{vmatrix} 1 & a & a^2 \\ a & a^2 & 1 \\ a^2 & 1 & a \end{vmatrix}$

10. $\begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix}$

11. $\begin{vmatrix} 1 & 2 & 3 & -1 \\ 2 & 1 & -1 & 3 \\ -1 & 1 & 2 & 1 \\ 3 & -2 & 1 & 2 \end{vmatrix}$

12. $\begin{vmatrix} 3 & 1 & 0 & -1 \\ 5 & -2 & 5 & 3 \\ 0 & 4 & -2 & 7 \\ 2 & 3 & 6 & 4 \end{vmatrix}$

13. If every element of a determinant of the nth order be multiplied by k, what is the effect upon the determinant?

14. Under what conditions can the sum of two determinants be readily expressed as a single determinant?

15. Show that the following equation is satisfied when $x = 2$.

$$\begin{vmatrix} x+2 & x^2 & 1 \\ x-1 & 1 & 5 \\ x+1 & 3 & 7 \end{vmatrix} = 0$$

60. The preceding theorems will now be used to establish a method of solving by determinants a system of three linear equations in three unknown quantities. The method employed is applicable to a system of n equations in n unknown quantities.

Let the given equations be

$$(1) \qquad \begin{cases} a_1x + b_1y + c_1z = m_1 \\ a_2x + b_2y + c_2z = m_2 \\ a_3x + b_3y + c_3z = m_3 \end{cases}$$

Multiply the first equation by D_{a_1}, the second by $-D_{a_2}$, the third by D_{a_3}, and add the three products.

$$\begin{aligned} a_1D_{a_1}x + b_1D_{a_1}y + c_1D_{a_1}z &= m_1D_{a_1} \\ -a_2D_{a_2}x - b_2D_{a_2}y - c_2D_{a_2}z &= -m_2D_{a_2} \\ a_3D_{a_3}x + b_3D_{a_3}y + c_3D_{a_3}z &= m_3D_{a_3} \end{aligned}$$

$$(a_1D_{a_1} - a_2D_{a_2} + a_3D_{a_3})x + (b_1D_{a_1} - b_2D_{a_2} + b_3D_{a_3})y$$
$$+ (c_1D_{a_1} - c_2D_{a_2} + c_3D_{a_3})z = m_1D_{a_1} - m_2D_{a_2} + m_3D_{a_3}$$

The coefficient of x is equal to D, the determinant of the system; the coefficients of y and z are equal to zero, by § 56; and the terms on the right-hand side of the equation are equal to the determinant of the system with each a replaced by m. Hence

(2) $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} x = \begin{vmatrix} m_1 & b_1 & c_1 \\ m_2 & b_2 & c_2 \\ m_3 & b_3 & c_3 \end{vmatrix}$$

and the value of x is

(3) $$x = \frac{\begin{vmatrix} m_1 & b_1 & c_1 \\ m_2 & b_2 & c_2 \\ m_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

In a similar way the values of y and z are found to be

$$y = \frac{\begin{vmatrix} a_1 & m_1 & c_1 \\ a_2 & m_2 & c_2 \\ a_3 & m_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \qquad z = \frac{\begin{vmatrix} a_1 & b_1 & m_1 \\ a_2 & b_2 & m_2 \\ a_3 & b_3 & m_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

These values of x, y and z when substituted in equation (1) satisfy the equation.

It will be observed that the rule of § 47 holds for systems of three or more linear equations in the same number of unknown quantities.

EXERCISES

Solve by determinants the following systems of equations :

1. $7y - 3x = 139$
$2x + 5y = 91$

2. $69y - 17x = 103$
$14x - 13y + 41 = 0$

3. $29x - 175 = 14y$
$87x - 56y = 497$

4. $ax - acy = b^2$
$bx + bcy = a^2$

5. $x - 2y + 3z = 4$
$x - 3y + 2z = 1$
$x + 4y + 6z = 16$

6. $2x - 3y + 5z + 22 = 0$
$4x + 5y - z = 0$
$5x - y + 11 = 0$

7. $2x - 4y + 3z = 1$
$3x + 5y - 2z = 5$
$5x - 7y + 6z = 4$

8. $x + y + z = 1$
$ax + by + cz = k$
$a^2x + b^2y + c^2z = k^2$

9. $ax + by = 0$
$by + cz = 0$
$cz + ax = 1$

10. $3x + 4y - 2z = 9$
$x + y - 3z = 8$
$2x - 3y + 2z = 5$

11. $2x - 2y + z = 9$
$x - 3y - 2z = 13$
$4x + y - 3z = 8$

12. $2x - 3y + 7z = 8$
$2z + y - 4x = 7$
$2y - 3x + z = 10$

13. $5x + 4y - z = 17$
$2x - 5y + 3z = -10$
$x + 2y - 3z = 13$

14. $2x + 25y + z = 7$
$3x + y + 5z = 4$
$x + 3y + 7z = 6$

15. $2x - 3y + 5z = 11$
$5x + 4y - 6z = -5$
$4x - 7y + 8z = 14$

16. $6x - 2y - z = 19$
$2y + 7z = -13$
$3z + 2x - 5y = 16$

17. $5x - 4z = 42$
$3z + 5y = 1$
$4y - 3x = -10$

18. $x + y + z = 1$
$2x - y + 3z = 2$
$2y - x - z = 3$

19. $5x - y = 30$
$7y - z = 56$
$2z - x = 20$

20. $2x+3y-5z=-13$
$x+2y+3z=18$
$4x-3y+z=9$

21. $5x-y+2z=11$
$3x+2y-z=-1$
$x+3y-2z=-7$

22. $x+2y+3z=7$
$2x-4y+z=12$
$4x-y-2z=9$

23. $u+x+y=7$
$x+y+z=8$
$y+z+u=9$
$z+u+x=10$

24. $u-2x=-1$
$3x+2y+z=2$
$4u+3y+z=0$
$3x+y-2z=6$

25. $3u-2y=2$
$5x-7z=11$
$2x+3y=39$
$4y+3z=41$

26. $u+x-y+2z=8$
$2u-x+y-3z=-9$
$3u+2x-y+z=10$
$4u-x+2y-z=4$

27. $2u-3x+y+z=-6$
$u+2x-3y+4z=11$
$3u+x-2y+2z=5$
$-u-2x+y-z=-7$

28. $u+w-x=5$
$w-x+y=2$
$x+y-z=3$
$u+x-y=5$
$u-y+z=3$

29. $5u-x=32$
$2w+z=0$
$3y-5z=-22$
$2z-x-w=2$
$3y+z-u=-17$

30. $u+w+x+y+z=12$
$u-w-x+y+z=4$
$u+w+x-y-z=6$
$-u+w-x+y+z=2$
$u-w-x-y-z=-2$

61. Cases may arise in which the number of equations given is unequal to the number of unknown quantities which they contain.

1. If there is given a system of n linear equations in $n+r$ unknown quantities, there is in general an unlimited number of solutions. For example, $3x+2y=7$ is an equation in two unknown quantities, from which for any value of x a corresponding value of y can be

found. Also the equations

$$a_1x + b_1y + c_1z = m_1$$
$$a_2x + b_2y + c_2z = m_2$$

are satisfied by an unlimited number of sets of values for x, y, and z, since for each value that may be assigned to z there is in general a pair of values for x and y. So, in the general case, to r of the quantities values may be assigned at pleasure, and the resulting n equations in n unknown quantities give in general one solution.

2. If there is given a system of n linear equations in $n - r$ unknown quantities, there is in general no solution. For from $n - r$ equations the values of the $n - r$ unknown quantities can in general be found, and only in exceptional cases will these values satisfy the remaining r equations. The case in which $r = 1$ is of importance, and will be discussed in § 64.

62. In §§ 47 and 60 it was assumed that the determinant of the system was not equal to zero. If, however, its value is zero, there is in general no solution, as division by zero is excluded. Consider the value of x in § 60, equation (2):

$$Dx = \begin{vmatrix} m_1 & b_1 & c_1 \\ m_2 & b_2 & c_2 \\ m_3 & b_3 & c_3 \end{vmatrix}$$

If D is equal to zero, this equation can be true only when the determinant in the right-hand member has the value zero. Hence it follows that when D, the determinant of the system, is equal to zero, two cases arise:

1. If one or more of the determinants which occur in the numerators of the values of the unknown quantities have values other than zero, there is no solution. The equations are inconsistent.

2. If each one of these determinants is equal to zero, the equations are not independent, *i.e.* one of them may be expressed as the sum of the other two, each multiplied by a constant factor. There is then an unlimited number of solutions, since the given equations are in reality fewer than the unknown quantities. The above statement may be proved as follows:

For brevity let the three equations § 60, (1), with all terms written in the left-hand members, be represented respectively by $P = 0$, $Q = 0$, $R = 0$, and let the minors by which they are to be multiplied be represented respectively by k_1, k_2, k_3. Then

$$k_1P - k_2Q + k_3R = 0,$$

and

$$P = \frac{k_2}{k_1}Q - \frac{k_3}{k_1}R.$$

As an illustration consider the set of equations

$$\begin{cases} x + 2y + 3z = 6 \\ 2x + 3y + 4z = 7 \\ 3x + 4y + 5z = 8 \end{cases}$$

which have the determinant of the system equal to zero, as well as the three determinants which occur in the numerators of the solution. According to the method followed in § 60, these equations are to be multiplied respectively by -1, 2, -1, and added. As the sum vanishes, it follows that 2 times the second equation is equal to the sum of the first and third, as is readily seen to be the case. The number of solutions is unlimited. For example, the equations are satisfied by the following sets of values: $(-4, 5, 0)$, $(-2, 1, 2)$, $(-1, -1, 3)$ etc.; but it may be noted that the substitution of $z = 1$ gives no corresponding values of x and y.

63. If the right-hand member of equations (1), § 60 is in each case zero, the equations are known as *linear homogeneous* equations. The discussion of such equations may be made to depend upon that of non-homogeneous equations.

Let there be given a set of n linear homogeneous equations in m unknown quantities. The equations are evidently satisfied if each unknown quantity is equal to zero, the so-called "*zero solution.*" Additional solutions may be sought as follows. Let each equation be divided through by some one of the unknown quantities, for example z (provided its value is not zero). The resulting equations may be regarded as a system of n non-homogeneous equations in the $m - 1$ unknown quantities $\frac{x}{z}$, $\frac{y}{z}$, etc., and the problem is thus reduced to the form considered in § 61.

The most important case arises when $m = n$. For simplicity let $n = 3$, and let the given equations be

$$a_1x + b_1y + c_1z = 0$$

$$a_2x + b_2y + c_2z = 0$$

$$a_3x + b_3y + c_3z = 0$$

Divide each equation through by z, and solve the first two equations for the unknown quantities $\frac{x}{z}$ and $\frac{y}{z}$.

This gives the values

$$\frac{x}{z}=\frac{\begin{vmatrix}-c_1 & b_1\\ -c_2 & b_2\end{vmatrix}}{\begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix}}=\frac{\begin{vmatrix}b_1 & c_1\\ b_2 & c_2\end{vmatrix}}{\begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix}},\quad \frac{y}{z}=\frac{-\begin{vmatrix}a_1 & c_1\\ a_2 & c_2\end{vmatrix}}{\begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix}}.$$

If the three equations are consistent, these values when substituted in the third equation must satisfy it.

$$a_3\begin{vmatrix}b_1 & c_1\\ b_2 & c_2\end{vmatrix}-b_3\begin{vmatrix}a_1 & c_1\\ a_2 & c_2\end{vmatrix}+c_3\begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix}=0$$

which may be written $\begin{vmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}=0$

That is, three linear homogeneous equations in three unknown quantities have a solution other than the zero solution only in case the determinant of the system is equal to zero.

64. It is now possible to discuss the case mentioned in § 61, 2. Under what condition does a system of n equations in $n-1$ unknown quantities have a solution? Let $n=3$, and let the given equations be

$$a_1x+b_1y+c_1=0$$
$$a_2x+b_2y+c_2=0$$
$$a_3x+b_3y+c_3=0$$

For x and y substitute $\frac{x'}{z}$ and $\frac{y'}{z}$, and clear of fractions, thus obtaining three equations in three unknown quantities. By § 62 there is a solution other than the zero solution only in case the determinant of the system is equal to zero. The values of x and y in the original equations are found by letting z take the value 1.

Hence three equations in two unknown quantities are consistent only when the determinant formed by taking in order the coefficients of the unknown quantities and the constant terms is equal to zero.

The student has probably learned in elementary graphical work that an equation of the first degree in x and y represents a straight line. The condition that three equations in two unknown quantities are consistent, is also the condition that the three lines represented by these equations pass through a common point, *i.e.*, the three straight lines $\begin{cases} a_1x + b_1y + c_1 = 0 \\ a_2x + b_2y + c_2 = 0 \\ a_3x + b_3y + c_3 = 0 \end{cases}$ are *copunctal* provided the determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$

EXERCISES

How many solutions have the following sets of equations?

1. $5x - 7y - 3 = 0$
$3x + 4y - 10 = 0$
$x - 5y + 3 = 0$

2. $2x - 3y + 5z = 7$
$x + 6y + 4z = 8$
$7x + 2y + 20z = 9$

3. $x - 2y + 5 = 0$
$2x + y - 5 = 0$
$2x - 4y + 1 = 0$

4. $2x - y + 1 = 0$
$5x + y - 8 = 0$
$x + 3y - 10 = 0$

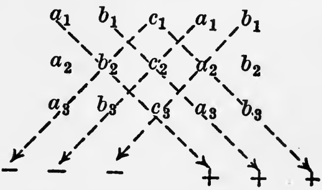
5. $2x - 3y + 4z = 0$
$7x + 2y - 3z = 0$
$4x - 5y + 6z = 0$

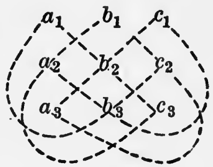
6. $x - 2y - 4z + u = 0$
$3x - 2y + z - 5u = 0$
$7x - 3y + 2z + 2u = 0$

7. $2x + y - 3z = 0$
$x + 4y + 2z = 0$
$3x + 5y - z = 0$

8. $x - 2y + 5z = 0$
$2x + y - 3z = 0$
$3x - y - 2z = 0$

65. The methods of this chapter may be employed to evaluate determinants of any order. A special method, applicable to a determinant of the third order only, may be mentioned. Write after the determinant to be evaluated its first two columns, and draw the diagonal lines as shown below. Take the product of the three numbers on each line, and give to the result the positive sign if the lines run downward to the right, otherwise the negative sign. The second diagram exhibits a similar mechanical method.





Many other properties of determinants, and methods of combining and using them in algebraic and geometric work, are discussed in more advanced books.

See Burnside and Panton: Theory of Equations, L. G. Weld: Determinants.

CHAPTER IV

VARIABLES AND THEIR LIMITS

That flower of modern mathematical thought — the notion of a function. — THOMAS J. MCCORMACK.

A limit is a peculiar and fundamental conception, the use of which cannot be superseded by any combination of other hypotheses and definitions. — WILLIAM WHEWELL.

66. Constants and variables. Elementary algebra is chiefly occupied with the consideration of quantities whose values remain fixed throughout the discussion of a given problem. Such quantities are called **constants.** Constants are of two classes, the first *numerical* or *absolute*, such as 3, $\sqrt{5}$, π, etc.; the second *arbitrary*, that is, a value may be assigned to each of them at pleasure; but once chosen, it must remain fixed throughout the discussion of a single problem. Such constants are usually denoted by the first letters of the alphabet, or by the last letters with accents, as a, b, x', x'', etc.

In practical life quantities frequently occur whose value is changing. In traveling, the distance traversed since the beginning of the journey is increasing. The time since any past event is constantly increasing, while the time before any future date is constantly decreasing. In many problems arising in the sciences, such quantities occur. A **variable** is a quantity whose value is changing, or one to which an unlimited number of values may be assigned. Variables are usually represented by the last letters of the alphabet.

A variable may change *continuously*, that is, it takes on successively every value between certain bounds, as in the cases mentioned above; or *discontinuously*, when it takes on only certain values. In either case it must change according to a fixed law.

Example. If x takes on the values 1, 2, 3, 4, ⋯ successively, it varies discontinuously, and the law is that it takes on all positive integral values.

EXERCISES

Determine the law of change for the following discontinuous variables:

1. $x = 1, 3, 5, 7, \cdots 2n-1 \cdots$
2. $x = 2, 4, 6, 8, \cdots 2n \cdots$
3. $x = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots \frac{1}{n} \cdots$
4. $x = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots \frac{1}{2^{n-1}} \cdots$
5. $x = 1, 4, 9, 16, \cdots n^2 \cdots$
6. $x = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots \frac{n}{n+1} \cdots$
7. $x = 2, \frac{3}{4}, \frac{4}{9}, \frac{5}{16}, \cdots \frac{n+1}{n^2} \cdots$

67. Functions. If two variables are connected by a relation, such that when one variable takes on a certain value, the value of the other variable is determined, it is said that the second variable is a **function** of the first.

Example 1. A is walking at a uniform rate of three miles per hour. Let x be the number of hours since he started, and y the number of miles he has walked. Then, since $y = 3x$, y is said to be a function of x; that is, y is a quantity whose value *depends* on the value of x. For any particular value of x, a corresponding definite value of y is determined.

Example 2. The distance between two towns is ten miles. B walks from one town to the other at a rate of x miles per hour. Let the time required for the walk be represented by y. Then $y = \frac{10}{x}$. Here also, y is a function of x, since its value is dependent on that of x.

Example 3. $y = x^2 - 3x + 2$. If any definite value is assigned to x, the value of y is completely determined. Hence y is a function of x.

A function is also called a *dependent variable*, while the other variable which changes independently, or to which values may be assigned at pleasure, is called the *independent variable*. Sometimes the relation between two variables is given in such a way that a choice is permitted which shall be the independent and which the dependent variable.

For example, if the relation $x + 2y = 7$ is given, either of the forms $y = \frac{7-x}{2}$, or $x = 7 - 2y$ may be used. The independent variable in the first case is x, and in the second case it is y; but once having chosen the independent variable, it may not be changed, except under conditions too complicated to be considered here.

A function may depend upon more than one independent variable. If a man walks at varying rates of speed, the distance he walks depends both upon his rate and upon the time. That is, the distance is a function of two variables, the rate and the time. If $z = x^2 - xy + 2xy^3$, z is a function of the two independent variables x and y.

68. An **algebraic function** is one which is formed by applying one or more of the six fundamental operations of algebra to the independent variable or variables a finite number of times.

69. It is often convenient to write $f(x)$ as an abbreviation for *function of x*. This may be used either as a general statement that the quantity so designated is a function of x, but the definite functional relation is not specified; or it may stand for a particular function, as

$$f(x) \equiv x^2 - 3x + 4$$

When this abbreviation is used, it must retain the same significance throughout the discussion of a problem. If several different functions are to be abbreviated, it may be done by prefixing different letters, or by using primes or subscripts.

(1)
$$\begin{aligned} f(x) &\equiv x^3 - 7x^2 + x - 3 \\ f'(x) &\equiv 3x^2 - 14x + 1 \\ f_1(x) &\equiv x^2 + 2x + 3 \\ g(x) &\equiv x^4 + x^2 - 2 \end{aligned}$$

These should be read "f function of x," "f prime function of x," "f sub one function of x," "g function of x." If $f(x)$ is defined as in (1), the expression $f(2)$ means that 2 is substituted for x in (1). That is,

$$f(2) \equiv 2^3 - 7 \cdot 2^2 + 2 - 3 \equiv -21$$

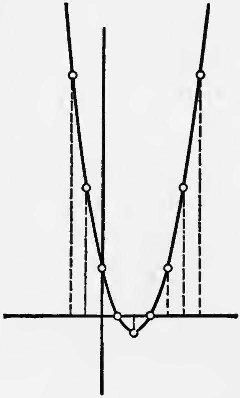
Let the student find $f(0)$, $f'(1)$, $f_1(3)$, $g(-2)$

70. The changes in the value of a function may be illustrated by a graph.*

Let $$y = x^2 - 4x + 3$$

Make a table of values, and plot.

x	y
-2	15
-1	8
0	3
1	0
2	-1
3	0
4	3
5	8
6	15



Since x is the independent variable, it may be made to change in any way desired. For simplicity let it always increase, as is done in the arrangement of the table of values; but when the manner of change for x has been chosen, the manner of change for y is uniquely determined. Corresponding to every value taken for x, there is one and only one value for y. In the graph, let x move continuously along the x-axis from left to right. Then y will move continuously along the curve from left to right, and the position of y, corresponding to any given position of x, may be found by drawing a line through x parallel to the y-axis.

71. Limits. If in accordance with its law of change a variable x takes on a series of values, such that the dif-

* See § 1.

ference between x and a constant l becomes and remains less in numerical value than any arbitrarily chosen positive number k, however small, then l is called the **limit** of x, which is expressed by writing

$$\lim x = l$$

According to this definition the condition that l is the limit of x may also be expressed by writing

$$| l - x | < k$$

where k represents a positive number chosen small at pleasure.

Whether the variable ever becomes exactly equal to its limit is immaterial. It is also immaterial whether the difference between the variable and its limit is positive or negative, since only its numerical value is considered. Thus the condition that l is the limit of x may be written either

$$l - x < k \text{ or } x - l < k$$

where k represents a positive number chosen small at pleasure. In all the following work, it is assumed that the larger quantity is taken as the minuend.

Example 1. Let $x = 1.414213 \cdots$

Then $\lim x = \sqrt{2}$

For, choosing $k = .000001$, $\sqrt{2} - x < k$

if x is computed to at least six decimal places. If k were chosen a smaller number, the condition $\sqrt{2} - x < k$ may still be satisfied by computing x to a sufficient number of decimal places.

Example 2. Let $x = .3333 \cdots$

Then $\lim x = \frac{1}{3}$

For, choosing k as small as may be desired, the condition $\frac{1}{3} - x < k$ can be satisfied by taking a sufficient number of decimal places for x.

72. The following principles concerning limits are immediate consequences of the definition:

1. *A variable can have only one limit.*

2. *If two variables are always equal, and one has a limit, the other has the same limit.*

3. *If two variables are always equal, and each has a limit, the limits are equal.*

For, given $x = y$, $\lim x = l_1$, $\lim y = l_2$. By 2, $\lim x = l_2$. $\therefore$ by 1, $l_1 = l_2$.

73. The most important problems in the theory of limits arise in connection with the limits of functions. If y be given as a certain function of x, where x has a definite limit, has y a corresponding definite limit, and if so, how can it be found?

As an example consider the function

$$y = \frac{1}{x-1}$$

whose graph is given on page 65. It is evident from this graph that when x has the limit 1, y becomes indefinitely large; but that when x becomes indefinitely large with either a positive or a negative sign, y has the limit zero. For a general discussion of the limits of functions, certain theorems are necessary which will be proved in the following sections.

74. Infinitesimals. A variable whose limit is zero is called an **infinitesimal.** An infinitesimal is then a variable whose numerical value becomes and remains less than any fixed positive number however small. If

$\lim v = 0$, then $v < k$, where k has been chosen in advance as small as desired. A constant, no matter how small it may be, is not an infinitesimal.

75. The following properties of infinitesimals are readily established :

1. *The difference between a variable and its limit is an infinitesimal.*

For, if $\lim x = l$, $x - l < k$. $\therefore \lim (x - l) = 0$.

2. *Conversely, if the difference between a variable and a constant is an infinitesimal, the constant is the limit of the variable.*

For, if $\lim (x - l) = 0$, by definition $|x - l| < k$, which is the condition that l is the limit of x.

3. *The sum of any given number of infinitesimals is an infinitesimal.*

For, if v_1, v_2, v_3 are infinitesimals, they can be taken small at pleasure. Hence

$$v_1 < \frac{k}{3}$$

$$v_2 < \frac{k}{3}$$

$$v_3 < \frac{k}{3}$$

Adding, $$v_1 + v_2 + v_3 < k$$

But since k was chosen in advance, small at pleasure,

$$\lim (v_1 + v_2 + v_3) = 0$$

The proof is readily extended to any desired number of infinitesimals.

4. *The product of an infinitesimal and any constant c is an infinitesimal.*

For, by definition, $v < \frac{k}{c}$. Then $cv < k$, where k is small at pleasure. $\therefore \lim cv = 0$.

This evidently covers the case of the quotient of an infinitesimal divided by a constant, since $\frac{v}{c} = \frac{1}{c} \cdot v$.

5. *The product of any given number of infinitesimals is an infinitesimal.*

For $v_1 < \sqrt[3]{k}$, $v_2 < \sqrt[3]{k}$, $v_3 < \sqrt[3]{k}$. $\therefore v_1v_2v_3 < k$ and similarly for any desired number of infinitesimals.

6. *If an infinitesimal be divided by a variable whose limit is not zero, the quotient is an infinitesimal.*

For, if $\lim x = l \neq 0$, it is possible to choose a positive number a which is numerically less than any value which x can take on when it is in the neighborhood of its limit. Then, since $x > a$, $\frac{v}{x} < \frac{v}{a}$.

Since $\frac{v}{a}$ is an infinitesimal by 4, $\frac{v}{x}$, which is less than $\frac{v}{a}$, is also an infinitesimal.

7. *If a constant be divided by an infinitesimal, the quotient is a variable which becomes indefinitely large.*

For, let $\frac{c}{v} = x$. Since c remains fixed, it is evident that as v becomes indefinitely small, x must become indefinitely large.

8. *If a variable whose limit is not zero be divided by an infinitesimal, the quotient is a variable which becomes indefinitely large.*

For, if $\lim x = l \neq 0$, it is possible, as in 6, to take $x > a$. $\therefore \frac{x}{v} > \frac{a}{v}$. Since by 7, $\frac{a}{v}$ can be made indefinitely large, the same must be true of $\frac{x}{v}$.

76. Theorem. *The limit of the algebraic sum of a constant and a variable is the corresponding sum of the constant and the limit of the variable.*

Given $\lim x = a$, to prove $\lim (x + c) = a + c$. $(a + c) - (x + c) = a - x$, which by § 75, 1, is an infinitesimal. Since the difference between the constant $a + c$ and the variable $x + c$ is an infinitesimal, $a + c$ is the limit of $x + c$ by § 75, 2.

77. Theorem. *The limit of the algebraic sum of a given number of variables is the corresponding algebraic sum of their limits.*

Given $\lim x = a$, $\lim y = b$, $\lim z = c$, to prove

$$\lim (x + y + z) = a + b + c$$

By § 75, 1, $x - a$, $y - b$, and $z - c$ are infinitesimals. Hence by § 75, 3, their sum

$$x - a + y - b + z - c = x + y + z - (a + b + c)$$

is an infinitesimal. Since the difference between the variable $x + y + z$ and the constant $a + b + c$ is an infinitesimal, $a + b + c$ is the limit of $x + y + z$, by § 75, 2. A similar proof may be given for any desired number of variables.

78. Theorem. *The limit of the product of a constant and a variable is the product of the constant and the limit of the variable.*

Let the student supply the proof by showing that if $\lim x = a$, $\lim cx = ca$.

79. Theorem. *The limit of the product of two variables is the product of their limits.*

Given $\lim x = a$, $\lim y = b$, to prove $\lim (xy) = ab$. By § 75, 1, $x - a$ and $y - b$ are infinitesimals. Hence by § 75, 5, $(x-a)(y-b)$ is an infinitesimal. But

$$(x-a)(y-b) = xy - ay - bx + ab$$
$$= xy - ab + ab - ay - bx + ab$$

Transposing,

$$xy - ab = (x-a)(y-b) - ab + ay + bx - ab$$
$$= (x-a)(y-b) + a(y-b) + b(x-a)$$

The quantity on the right is an infinitesimal by § 75, 3, 4. Therefore $xy - ab$ is an infinitesimal by § 72, 2, and $\lim xy = ab$ by § 75, 2.

80. Theorem. *The limit of the quotient of two variables is the quotient of their limits, provided the limit of the divisor is not zero.*

Given $\lim x = a$, $\lim y = b \neq 0$, to prove $\lim \frac{x}{y} = \frac{a}{b}$.

$$\frac{x}{y} - \frac{a}{b} = \frac{bx - ay}{by} = \frac{bx - ab + ab - ay}{by} = \frac{b(x-a) + a(b-y)}{by}$$

But $\lim \left[\frac{b(x-a) + a(b-y)}{by}\right] = 0$, by § 75, 1, 3, 4, 6, and § 78.

$\therefore \lim \left[\frac{x}{y} - \frac{a}{b}\right] = 0$, by § 72, 2, and $\lim \frac{x}{y} = \frac{a}{b}$, by § 75, 2

The student should note that the case when the limit of the divisor is zero is covered by § 75, 8, unless the dividend is also an infinitesimal, a case which requires further investigation.

81. Infinity. If a variable, in accordance with its law of change, becomes and remains greater than any arbitrarily chosen number M, however large, its limit is said to be **infinity**, and the fact is expressed by writing

$$\lim x = \infty$$

In accordance with this notation the theorem of § 75, 7, might be briefly expressed thus

$$\lim_{v=0} \frac{c}{v} = \infty$$

which is read "When the limit of v is zero, the limit of $\frac{c}{v}$ is infinity." It should, however, be noted that the word limit cannot here have the technical significance explained in § 71, since it would be absurd to say that the difference between any variable and infinity can be made arbitrarily small. The expression $\lim x = \infty$ is merely a convenient way of saying that x becomes indefinitely large.

The theorem of § 75, 7 is still further abbreviated in some mathematical works to read $\frac{c}{0} = \infty$. This, however, does not mean that c is divided by zero, but is a convenient way of stating the fact that the value of a fraction can be made indefinitely large by making the denominator indefinitely small. This usage is, however, to be deprecated.

82. Theorem. *If a constant be divided by a variable which becomes indefinitely large, the quotient is an infinitesimal.*

For, it is evident that if x becomes indefinitely large, the fraction $\frac{c}{x}$ can be made less than any arbitrarily small positive number k.

This fact is sometimes abbreviated by writing $\frac{c}{\infty} = 0$, but may be more properly stated thus:

$$\lim_{x=\infty} \frac{c}{x} = 0$$

which is read "When the limit of x is infinity, the limit of $\frac{c}{x}$ is zero."

83. Theorem. *If a variable which has a finite limit be divided by a variable which becomes indefinitely large, the quotient is an infinitesimal.*

For, if x has a finite limit, it is possible to find a number c greater than any value which x takes on; $\therefore$ since $x < c$, $\frac{x}{y} < \frac{c}{y}$. If y becomes indefinitely large, $\frac{c}{y}$ is an infinitesimal by § 82; $\therefore$ $\frac{x}{y}$ is an infinitesimal.

The student should read and explain the meaning of the expressions in the following examples:

1. $\lim\limits_{x=\infty} (cx) = \infty$

2. $\lim\limits_{x=\infty} \left(\frac{x}{c}\right) = \infty$

3. $\lim\limits_{x=\infty} (a^x) = 0$, when $a < 1$

4. $\lim\limits_{x=\infty} (a^x) = \infty$, when $a > 1$

5. $\lim\limits_{x=\infty} (a^{-x}) = \infty$, when $a < 1$

6. $\lim\limits_{x=\infty} (a^{-x}) = 0$, when $a > 1$

84. It will be seen by an application of the theorems of §§ 76–80 that in order to find the limit of any function, when the limit of the independent variable is a finite number, it is necessary only to substitute the limit for the variable in the function.

Example 1. $\lim\limits_{x=1} \left(\frac{x^2 + 3x + 2}{x^2 - 5x + 6}\right) = 3$ by use of the theorems of §§ 76–80, a result which is also obtained by the substitution of 1 for x in the function.

If, however, x has the limit 2, a value for which the denominator vanishes, the fraction becomes indefinitely large by § 75, 8. Then

$$\lim_{x=2}\left(\frac{x^2+3x+2}{x^2-5x+6}\right)=\infty$$

although the fraction has no meaning when $x=2$.

If now it is asked what is the value of the fraction when x increases indefinitely, the method of substitution gives no result, since infinity is not a number. In this case the expression must be put in a form where x occurs only in the denominators of fractions, after which § 82 may be applied. This is accomplished by dividing both numerator and denominator by the highest power of x that occurs, in this case by x^2. Then

$$\lim_{x=\infty}\left[\frac{x^2+3x+2}{x^2-5x+6}\right]=\lim_{x=\infty}\left[\frac{1+\dfrac{3}{x}+\dfrac{2}{x^2}}{1-\dfrac{5}{x}+\dfrac{6}{x^2}}\right]=1$$

Example 2. Find $\lim_{x=1}\left[\dfrac{x^2-3x+2}{x^2-1}\right]$.

Here when x has the limit 1, both numerator and denominator have the limit zero, and there is no theorem which directly applies. When this occurs, each case requires individual consideration. Here it may be seen

that $$\frac{x^2-3x+2}{x^2-1}=\frac{x-2}{x+1}.$$

Since the limit of the second fraction is $-\frac{1}{2}$ when x has the limit 1, the first fraction must have the same limit by § 72, 2.

EXERCISES

Find the following limits and state the theorems used:

1. $\lim_{x=\infty} \frac{x-1}{x}$

2. $\lim_{x=\infty} \frac{2x^2-5x+7}{3x^2+4x-1}$

3. $\lim_{x=\infty} \frac{(x+1)^2}{(x-1)^2}$

4. $\lim_{x=\infty} \frac{x^3-5x^2+2}{x^4+2x^2+3}$

5. $\lim_{x=\infty} \frac{x^3+2x-3}{x^2-5x+6}$

6. $\lim_{x=1} \frac{(x-1)^2}{x^2-1}$

7. $\lim_{x=2} \frac{x^2+x-1}{x^2-3x+2}$

8. $\lim_{x=0} \frac{x^3-4x+3}{x^2+2x-7}$

9. $\lim_{n=\infty} n\left(\frac{2a+(n-1)d}{2}\right)$

10. $\lim_{n=\infty} \frac{ar^n-a}{r-1}$ when $r>1$

11. $\lim_{n=\infty} \frac{ar^n-a}{r-1}$ when $r<1$

12. $\lim_{x=2} \frac{x^2-3x+2}{x^2+x-6}$

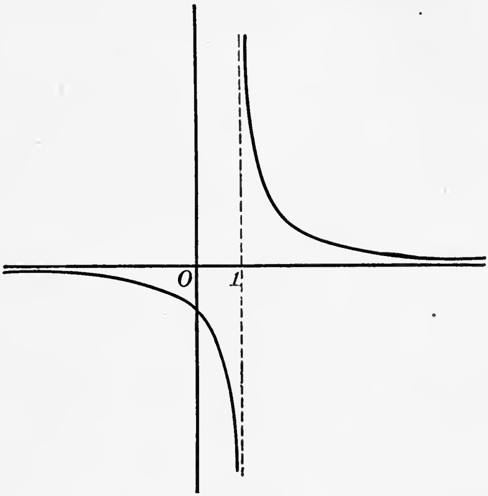
13. $\lim_{n=\infty} \frac{2n^2-n+7}{3n^2+2n-1}$

14. Show why the proof of § 75, 3 does not hold when the number of infinitesimals is indefinitely increased.

85. Continuity of functions. In the examples of functions given in the preceding section, it may be seen that a very small change in the independent variable produces a correspondingly small change in the function. The graph of such a function is evidently a continuous curve, that is, a curve which consists of only one part or branch, with no breaks in it. Such functions are called *continuous*. That all functions are not continuous for every value of the independent variable may be seen by considering the comparatively simple function

$$y = \frac{1}{x-1}$$

In plotting this function, it is found that $x = 1$ must be omitted from the table of values, since the substitution of unity for x makes the denominator zero, and division



by zero is inadmissible. Hence the function has no definite value for $x = 1$, and of course cannot be plotted at that point. It can, however, be plotted for all values of x greater than or less than unity. Hence the curve must consist of two parts breaking in two at $x = 1$. If a value a very little less than unity be substituted for x, a large negative value for y is obtained; while if a value a very little greater than unity be substituted for x, a large positive value for y is obtained. Hence in the neighborhood of the point $x = 1$, it is not true that a small change in the value of the variable makes only a small change in the value of the function.

A more precise definition of a continuous function may be given as follows: $f(x)$ is said to be a **continuous**

function of the independent variable x, if, however small the number k may be taken, the difference between $f(b)$ and $f(c)$ is numerically less than k, for any pair of points b and c, which lie sufficiently near together.

86. Theorem. *A polynomial in one variable is a continuous function of that variable.*

For, let the polynomial be

$$(1)\quad f(x) \equiv a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$$

Let b and c be any two values of x, whose difference is h, $c = b + h$. Substituting these values in (1),

$$f(b) \equiv a_0b^n + a_1b^{n-1} + a_2b^{n-2} + \cdots + a_{n-1}b + a_n$$
$$f(c) \equiv f(b+h) = a_0(b+h)^n + a_1(b+h)^{n-1} + \cdots$$
$$+ a_{n-1}(b+h) + a_n$$

Expand the terms of the last expression by the binomial theorem, and subtract $f(b)$ from it. It will be found that the difference between $f(b)$ and $f(c)$ contains h in every term. Now let h have the limit zero, that is, consider the points b and c as becoming indefinitely near together. Thus h is an infinitesimal, and by § 75, 3, 4, 5 each term containing it is an infinitesimal, and the difference $f(c) - f(b)$ is an infinitesimal. Hence this difference can be made as small as is desired, by making the difference between b and c sufficiently small.

CHAPTER V

DIFFERENTIATION OF ALGEBRAIC FUNCTIONS

Every one versed in the matter will agree that even the elements of a scientific study of nature can be understood only by those who have a knowledge of at least the elements of differential and integral calculus. — FELIX KLEIN.

87. In the preceding chapter, the problem of finding the limit of a function, when the independent variable on which it depends has a certain definite limit, has been investigated. The present chapter deals with the problem of finding the rate of change of a function compared with the rate of change of the variable on which it depends, — a problem which is of great importance in the study of many sciences.* In all the following discussion it will be assumed for convenience that the independent variable x is increasing at a uniform rate of speed.

An inspection of the graph of $y = x^2 - 4x + 3$ in § 70 shows that when x is increasing uniformly, *i.e.* moving from left to right along the x-axis at a uniform rate, the change in y is not uniform. First y decreases and then increases; first it decreases rapidly, then more and more slowly; it begins to increase at a slow rate, which grows more and more rapid as x increases. The present problem consists in finding an expression which, for any value of x, will show how the function is behav-

* It was while investigating problems of this kind that Sir Isaac Newton (1642–1727), about 1665, discovered the principles of the Infinitesimal Calculus, one of the most beautiful as well as most powerful tools in mathematics.

ing, whether it is increasing or decreasing as x increases, and how fast it changes in proportion to the change in x. In order to do this, the difference between two values of y will be compared with the difference between the two corresponding values of x.

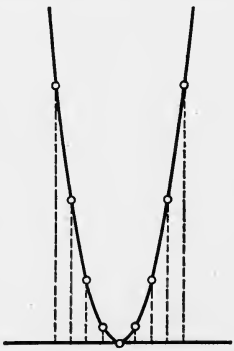
88. Increments. The difference between any two values of a variable is called an **increment** of the variable. An *increment* of y is then to be compared with the corresponding *increment* of x. A common abbreviation for "increment of x" is formed by writing the Greek letter delta before x, thus Δx. For "increment of y" Δy is written, etc. It should be noted that Δx does not mean Δ *times* x. Δ and x together form one symbol and cannot be separated.

89. Ratio of increments. As an example of the comparison of the change in the function with the change in the independent variable, consider the simple function

$$y = x^2$$

The table of values and the graph show many of the same characteristics as those of the function plotted in § 70.

x	y
-4	16
-3	9
-2	4
-1	1
0	0
1	1
2	4
3	9
4	16



Consider the changes between $x = 2$ and $x = 3$. The increment of x is 1. Since $y = 9$ when $x = 3$, and $y = 4$ when $x = 2$, the corresponding increment of y is 5. Then the increment of y divided by the increment of x is 5, which tells that between $x = 2$ and $x = 3$, y has increased five times as fast as x.

Now consider the interval between $x = 1$ and $x = 2$. Here $\Delta x = 1$, $\Delta y = 3$, $\frac{\Delta y}{\Delta x} = 3$; that is, in this interval y has increased three times as fast as x. An inspection of the graph, however, shows that this is only the *average* rate of increase in the interval, and, in reality, the rate for the first part of the interval is less, and for the last part more than this average. More precision is necessary. It is required to know how the rate of change in y compares with the rate of change in x, at the very beginning of the interval, that is, just as x is passing the point 1. It is evident that if the ratio of smaller increments is considered, a more accurate result can be obtained.

The results of taking Δx, and hence the corresponding Δy, smaller and smaller may be tabulated as follows:

Initial Value of x	New Value of x	Δx	Initial Value of y	New Value of y	Δy	$\frac{\Delta y}{\Delta x}$
1	2.	1.	1	4.	3.	3.
1	1.5	.5	1	2.25	1.25	2.5
1	1.4	.4	1	1.96	.96	2.4
1	1.3	.3	1	1.69	.69	2.3
1	1.2	.2	1	1.44	.44	2.2
1	1.1	.1	1	1.21	.21	2.1
1	1.01	.01	1	1.0201	.0201	2.01
1	1.001	.001	1	1.002001	.002001	2.001

From this table it is seen that as Δx becomes smaller and smaller, Δy also becomes smaller and smaller; in fact, when Δx has the limit zero, Δy also has the limit zero. Hence if the limit of $\frac{\Delta y}{\Delta x}$ is sought, the case occurs which is mentioned in Example 2, § 84, to which none of the theorems on limits applies, but which needs individual consideration. An inspection of the last column of the table leads to the conclusion that

$$\lim_{\Delta x=0}\left[\frac{\Delta y}{\Delta x}\right]=2$$

In order to make this work more general, instead of considering an interval beginning at $x=1$, an interval may be considered beginning at $x=x'$, where x' represents any one of the particular values which x may take on; x' is therefore constant. If x' be substituted for x in the function, the corresponding particular value of y, y' is obtained. Thus

$$y' = x'^2$$

Now consider a larger value of x, $x' + \Delta x$. This also substituted in the function gives a corresponding value of y, $y' + \Delta y$. Then

$$\begin{aligned} y' + \Delta y &= (x' + \Delta x)^2 = x'^2 + 2\,x'\Delta x + \Delta x^2 \\ y' &= x'^2 \end{aligned}$$

Subtracting, $\Delta y = 2\,x'\Delta x + \Delta x^2$

Dividing by Δx, $\frac{\Delta y}{\Delta x} = 2\,x' + \Delta x$

That is, $2\,x' + \Delta x$ is the ratio of the rates of change of the two variables, y and x, in an interval which begins at x', and whose length is Δx. But in order to determine more accurately what happens at the beginning of the interval when $x = x'$, let Δx be taken smaller and smaller; that is, let it have the limit zero. Then since the function $y = x^2$ is continuous by § 86, Δy has the limit zero when Δx has the limit zero, and there is no theorem on limits which directly applies to the fraction $\frac{\Delta y}{\Delta x}$.

But since

$$\lim_{\Delta x = 0} (2x' + \Delta x) = 2x'$$

then by § 72, 2

$$\lim_{\Delta x = 0} \left[\frac{\Delta y}{\Delta x}\right] = 2x'$$

Since x' was *any* particular value of x, this is true for *all* values of x, and the prime may be dropped.

$$\lim_{\Delta x = 0} \left[\frac{\Delta y}{\Delta x}\right] = 2x$$

Consider the meaning of this limit for various values of x.

Let $x = -2$, then $2x = -4$. Since this is negative, the function must be changing in the opposite way from x, at $x = -2$. That is, the function is decreasing, and moreover it is decreasing four times as fast as x increases. Let $x = 0$, then $2x = 0$, which says that the function is not changing at all at $x = 0$, is neither increasing nor decreasing, which may also be seen from the graph.

When $x = 3$, $2x = 6$, which tells that when $x = 3$ the function is increasing six times as fast as x. Compare this result with the graph.

90. Derivatives. The limit of the fraction $\frac{\Delta y}{\Delta x}$ when Δx has the limit zero, whatever its value may be in any particular case, is called the **derivative** of the function y with respect to x. It has been already seen that since both numerator and denominator of this fraction are infinitesimals,* no theorem directly applies in finding its limit. If, however, the limit exists, its value may be determined in any particular problem, by finding another expression for $\frac{\Delta y}{\Delta x}$ to which the theorems on limits may be applied, and then making use of § 72, 2, as was done in the example of the preceding section. If in any particular case the ratio of the increments is a constant, the derivative is the ratio itself.

91. The **derivative** of a function with respect to the variable on which it depends is an expression found in the manner just described, which shows the *rate of change* of the function compared with that of the independent variable for any value of the variable.

92. Theorem. *The derivative of a function for any particular value of the independent variable is a positive or a negative number, according as the function itself is increasing or decreasing for that value of the variable.*

For, the increment of the function and the increment of the variable must have the same or different signs according as the derivative is positive or negative; and since the independent variable is assumed to be increasing, the function must be increasing or

* If, for any particular value of x, the function is discontinuous, as is the function $\frac{1}{x-1}$ at the point $x = 1$, then Δy is not an infinitesimal and the function has no derivative at that point. All functions discussed in this chapter are continuous, except fractions for such values as make the denominator zero.

decreasing according as the increments have the same or opposite signs.

93. The process of finding the derivative is called **differentiation.** In the last section the function $y = x^2$ was *differentiated* with respect to x. The limit which was found was the *derivative* of x^2 with respect to x. The most common abbreviation for "the derivative with respect to x" is $\frac{d}{dx}$. Thus, $\frac{d}{dx}y$ means "the derivative of y with respect to x"; $\frac{d}{dx}(x^2)$, "the derivative of x^2 with respect to x"; $\frac{d}{dx}(f(x))$, "the derivative of function x with respect to x." It should be noted that the symbol $\frac{d}{dx}$ is not a fraction, although it is written in fractional form. Another abbreviation for "the derivative of y with respect to x," which is sometimes used, is $D_x y$. It is frequently indicated that the function $f(x)$ has been differentiated by adding a prime; that is, $f'(x)$ has the same meaning as $\frac{d}{dx}(f(x))$.

Example. Differentiate the function $y = x^2 - 4x + 3$. Let x' and $x' + \Delta x$ be two particular values of x; and y' and $y' + \Delta y$ the corresponding values of y. Then by substitution,

$$\begin{aligned}
y' + \Delta y &= (x' + \Delta x)^2 - 4(x' + \Delta x) + 3 \\
&= x'^2 + 2x'\Delta x + \Delta x^2 - 4x' - 4\Delta x + 3 \\
y' &= x'^2 \qquad\qquad\qquad\quad - 4x' \qquad\quad + 3 \\
\hline
\Delta y &= \qquad 2x'\Delta x + \Delta x^2 \qquad - 4\Delta x \\
\frac{\Delta y}{\Delta x} &= 2x' + \Delta x - 4
\end{aligned}$$

Since the quantity on the right-hand side has the limit $2x' - 4$, when Δx has the limit zero,

$$\lim_{\Delta x = 0}\left[\frac{\Delta y}{\Delta x}\right] = 2x' - 4 \quad \text{by § 72, 2.}$$

Since x' was any particular value of x, this is true for all values, and

$$\frac{d}{dx}y = 2x - 4$$

This may also be written

$$\frac{d}{dx}(x^2 - 4x + 3) = 2x - 4$$

Let this result be compared with the graph of the function in § 70. When $x = 2$, $\frac{d}{dx}y = 0$, which says that at the point $x = 2$, the function is neither increasing nor decreasing. For values of x less than 2, $\frac{d}{dx}y$ is negative, which means that for such values the function is decreasing, while for x greater than 2, $\frac{d}{dx}y$ is positive and increases as x increases, which shows that for x greater than 2, the function is increasing, and the larger x becomes, the more rapid is the increase. An inspection of the graph shows the same results.

EXERCISES

Find the derivatives of the following functions and also draw their graphs. Compare the two results.

1. $y = x^2 + x - 2$
2. $y = x^3 + 1$
3. $y = x^3 - 3x$
4. $y = \frac{x^2}{2} - x + \frac{1}{3}$
5. $y = x^3 - 3x^2 + 2x + 1$

6. $y = x^4 - 8x^2$

7. $y = \frac{1}{x}$

8. $y = \frac{x-1}{x}$

9. $y = \frac{1}{1+x}$

10. $y = x^2 - 2x$

11. $y = 3x^3$

12. $y = \frac{1-x^2}{x}$

13. $y = x^2 - 2x + 3$

14. $y = \frac{1}{x^2}$

94. Rules for differentiation. The derivative of any function may be found by the method of the preceding section; but in the case of complicated functions the labor is tedious. By use of the following theorems, the work can be much more expeditiously performed.

Theorem. *The derivative of any constant is zero.*

For, since the value of a constant is always the same, its increment is zero. Hence the ratio of the increment to any other increment is zero, and the derivative is zero.

Corollary. *Two functions which differ only by a constant have the same derivative and conversely.*

As an illustration consider the functions on page 206, which differ only by a constant. Their graphs are the same except in position. It is evident that at any point the functions have the same rate of change.

95. Theorem. *The derivative of any variable with respect to itself is unity.*

For, since the function and the independent variable are equal, their increments are equal. Hence their ratio is unity, and the derivative is unity.

96. Theorem. *The derivatives of equal functions are equal.*

For, since the functions are equal, their increments are equal. Dividing each by the increment of the independent variable, the ratios are equal. Hence if one ratio has a limit, the other has the same limit.

97. Theorem. *The derivative of the algebraic sum of several functions is the corresponding sum of their derivatives.*

(1) Let $y = u + v + w + a$, where u, v, and w are functions of x, and a is a constant. Let x assume in succession the constant values x' and $x' + \Delta x$. The functions which depend upon x must then assume corresponding constant values, and for these two values of x, equation (1) becomes in turn

$$\begin{array}{l} y' + \Delta y = u' + \Delta u + v' + \Delta v + w' + \Delta w + a \\ y' \qquad\quad = u' \qquad\quad + v' \qquad\quad + w' \qquad\quad + a \\ \hline \qquad \Delta y = \qquad \Delta u \qquad + \Delta v \qquad + \Delta w \end{array}$$

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x} + \frac{\Delta w}{\Delta x}$$

$$\lim_{\Delta x = 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x = 0} \left[\frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x} + \frac{\Delta w}{\Delta x}\right] \quad \text{by § 72, 2}$$

$$= \lim_{\Delta x = 0} \frac{\Delta u}{\Delta x} + \lim_{\Delta x = 0} \frac{\Delta v}{\Delta x} + \lim_{\Delta x = 0} \frac{\Delta w}{\Delta x} \quad \text{by § 77.}$$

Since by definition

$$\lim_{\Delta x = 0} \frac{\Delta y}{\Delta x} = \frac{d}{dx} y, \quad \lim_{\Delta x = 0} \frac{\Delta u}{\Delta x} = \frac{d}{dx} u, \quad \lim_{\Delta x = 0} \frac{\Delta v}{\Delta x} = \frac{d}{dx} v,$$

$\lim_{\Delta x = 0} \frac{\Delta w}{\Delta x} = \frac{d}{dx} w$, it follows that

$$\frac{d}{dx} y = \frac{d}{dx} u + \frac{d}{dx} v + \frac{d}{dx} w$$

98. Theorem. *The derivative of the product of a constant and a function is the product of the constant and the derivative of the function.*

Let the student supply the proof by finding the derivative of $y = az$, where z is a function of x.

99. Theorem. *The derivative of the product of two functions is the first times the derivative of the second, plus the second times the derivative of the first.*

Let $y = uv$, where the functions and their particular values are defined as in the proof of § 97. Then

$$\begin{aligned} y' + \Delta y &= (u' + \Delta u)(v' + \Delta v) \\ &= u'v' + u'\Delta v + v'\Delta u + \Delta u \Delta v \\ y' &= u'v' \\ \hline \Delta y &= \quad u'\Delta v + v'\Delta u + \Delta u \Delta v \end{aligned}$$

$$\frac{\Delta y}{\Delta x} = u'\frac{\Delta v}{\Delta x} + v'\frac{\Delta u}{\Delta x} + \Delta u\frac{\Delta v}{\Delta x}$$

$$\lim_{\Delta x = 0}\left[\frac{\Delta y}{\Delta x}\right] = u' \cdot \lim_{\Delta x = 0}\left[\frac{\Delta v}{\Delta x}\right] + v' \cdot \lim_{\Delta x = 0}\left[\frac{\Delta u}{\Delta x}\right]$$

$$+ \lim_{\Delta x = 0} \Delta u \cdot \lim_{\Delta x = 0}\left[\frac{\Delta v}{\Delta x}\right] \text{ by use of §§ 72, 2; 77; 78; 79.}$$

Since all the functions under consideration are continuous, Δu has the limit zero when Δx has the limit zero. Hence by definition

$$\frac{d}{dx}y = u'\frac{d}{dx}v + v'\frac{d}{dx}u$$

or since u' and v' were values of u and v corresponding to *any* value of x,

$$\frac{d}{dx}y = u\frac{d}{dx}v + v\frac{d}{dx}u$$

100. Theorem. *The derivative of the product of a given number of functions is the sum of all the terms that can be formed by multiplying the derivative of each factor by all the other factors.*

For $$\frac{d}{dx}(uvwz) = \frac{d}{dx}[(uv)(wz)]$$

$$= uv\frac{d}{dx}(wz) + (wz)\frac{d}{dx}(uv) \text{ by § 99}$$

$$= uv\left(w\frac{d}{dx}z + z\frac{d}{dx}w\right) + wz\left(u\frac{d}{dx}v + v\frac{d}{dx}u\right) \text{ by § 99}$$

$$= uvw\frac{d}{dx}z + uvz\frac{d}{dx}w + uwz\frac{d}{dx}v + vwz\frac{d}{dx}u$$

A similar proof may be given for the product of any desired number of functions.

101. Theorem. *The derivative of a fraction is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.*

Let $y = \frac{u}{v}$, and let the particular values of the variables be defined as in § 97. Then

$$y' = \frac{u'}{v'},\ y' + \Delta y = \frac{u' + \Delta u}{v' + \Delta v}$$

$$\Delta y = \frac{u' + \Delta u}{v' + \Delta v} - \frac{u'}{v'} = \frac{u'v' + v'\Delta u - u'v' - u'\Delta v}{v'(v' + \Delta v)} = \frac{v'\Delta u - u'\Delta v}{v'(v' + \Delta v)}$$

$$\frac{\Delta y}{\Delta x} = \frac{v'\dfrac{\Delta u}{\Delta x} - u'\dfrac{\Delta v}{\Delta x}}{v'(v' + \Delta v)}$$

$$\lim_{\Delta x=0}\frac{\Delta y}{\Delta x}=\frac{\lim\limits_{\Delta x=0}\left[v'\frac{\Delta u}{\Delta x}-u'\frac{\Delta v}{\Delta x}\right]}{\lim\limits_{\Delta x=0}[v'^2+v'\Delta v]}$$ by §§ 72, 2, and 80, provided $v'\neq 0$

$$=\frac{v'\frac{d}{dx}u-u'\frac{d}{dx}v}{v'^{\,2}}$$ by §§ 77 and 78 and definition of derivative,

or since u' and v' were values of u and v corresponding to *any* value of x,

$$\frac{d}{dx}y=\frac{v\frac{d}{dx}u-u\frac{d}{dx}v}{v^2}$$

102. Special cases. If the numerator of a fraction is a constant, that is, if the fraction has the form $\frac{c}{v}$, an application of § 101 gives

$$\frac{d}{dx}\left(\frac{c}{v}\right)=\frac{-c\frac{d}{dx}v}{v^2}$$

If the denominator is a constant, that is, if the fraction has the form $\frac{u}{c}$,

$$\frac{d}{dx}\left(\frac{u}{c}\right)=\frac{c\frac{d}{dx}u}{c^2}=\frac{\frac{d}{dx}u}{c}$$

a result which may also be obtained from § 98 by writing

$$\frac{u}{c}=\frac{1}{c}\cdot u$$

103. Theorem. *The derivative of a function with a constant exponent is the exponent times the function with its exponent diminished by one, times the derivative of the function.*

Case 1. When the exponent is a positive integer:

Let $y = z^m$. Then when x takes on the values x' and $x' + \Delta x$,

$$y' = z'^m,$$

$$y' + \Delta y = (z' + \Delta z)^m$$

$$= z'^m + mz'^{m-1}\Delta z + \frac{m(m-1)}{2} z'^{m-2}\Delta z^2 + \cdots$$

by use of the binomial theorem for positive integral exponents. Subtracting,

$$\Delta y = mz'^{m-1}\Delta z + \frac{m(m-1)}{2} z'^{m-2}\Delta z^2$$

$$+ \frac{m(m-1)(m-2)}{3!} z'^{m-3}\Delta z^3 + \cdots$$

$$\frac{\Delta y}{\Delta x} = mz'^{m-1}\frac{\Delta z}{\Delta x} + \frac{m(m-1)}{2} z'^{m-2}\frac{\Delta z}{\Delta x}\Delta z$$

$$+ \frac{m(m-1)(m-2)}{3!} z'^{m-3}\frac{\Delta z}{\Delta x}\Delta z^2 + \cdots$$

Since all the terms on the right-hand side after the first consist of the product of a constant, the ratio of two increments, and some power of the infinitesimal Δz, they are themselves infinitesimals by § 75, 4, 5, and it follows by § 75, 3 and definition that

$$\lim_{\Delta x = 0} \frac{\Delta y}{\Delta x} = mz'^{m-1}\frac{d}{dx}z, \text{ or, dropping primes,}$$

$$\frac{d}{dx}y = mz^{m-1}\frac{d}{dx}z$$

Since the binomial theorem has not yet been proved for fractional and negative exponents, those cases must be investigated separately.

Case 2. When the exponent is a positive fraction:

The fact that the function $z^{\frac{m}{n}}$ has a derivative is established in works on Calculus. Granting the existence of this derivative, the fact that its form is given by the last theorem may be shown as follows:

(1) Let $$y = z^{\frac{m}{n}}$$

Raising both sides to the nth power,

$$y^n = z^m$$

By § 96 $$\frac{d}{dx}(y^n) = \frac{d}{dx}(z^m)$$

By Case 1 $$ny^{n-1}\frac{d}{dx}y = mz^{m-1}\frac{d}{dx}z$$

Solving for $\frac{d}{dx}y$, $$\frac{d}{dx}y = \frac{mz^{m-1}\frac{d}{dx}z}{ny^{n-1}}.$$

Since the derivative is to be expressed in terms of z, substitute for y^{n-1} its value from (1), $y^{n-1} = z^{\frac{m(n-1)}{n}} = z^{m-\frac{m}{n}}$.

$$\therefore \frac{d}{dx}y = \frac{mz^{m-1}\frac{d}{dx}z}{nz^{m-\frac{m}{n}}} = \frac{m}{n}z^{\frac{m}{n}-1}\frac{d}{dx}z$$

Case 3. When the exponent is negative:

Let $$y = z^{-m} = \frac{1}{z^m}$$

By § 102 and Case 1

$$\frac{d}{dx}y = \frac{d}{dx}\left(\frac{1}{z^m}\right) = \frac{-\frac{d}{dx}(z^m)}{(z^m)^2} = \frac{-mz^{m-1}\frac{d}{dx}z}{z^{2m}} = -mz^{-m-1}\frac{d}{dx}z$$

Hence the theorem holds in all three cases.

Examples. Differentiate the following:

1 $$y = x^3 - 5x + 7$$

$$\frac{d}{dx}y = \frac{d}{dx}(x^3) - \frac{d}{dx}(5x) + \frac{d}{dx}(7) \text{ by § 97}$$

$$= 3x^2 - 5 \text{ by §§ 103, 98, 95, and 94}$$

2 $$y = \sqrt{x^2 - c} = (x^2 - c)^{\frac{1}{2}}$$

$$\frac{d}{dx}y = \frac{1}{2}(x^2 - c)^{-\frac{1}{2}}\frac{d}{dx}(x^2 - c) \text{ by § 103}$$

$$= \tfrac{1}{2}(x^2 - c)^{-\frac{1}{2}} \cdot 2x \text{ by §§ 97, 103, 95, and 94}$$

$$= \frac{x}{\sqrt{x^2 - c}}$$

3 $$f(x) = (x^2 - 7)(x^3 + 2x)$$

$$f'(x) = (x^2 - 7)\frac{d}{dx}(x^3 + 2x) + (x^3 + 2x)\frac{d}{dx}(x^2 - 7)$$

by § 99

$$= (x^2 - 7)(3x^2 + 2) + (x^3 + 2x) \cdot 2x$$

by §§ 97, 103, 95, and 94

$$= 3x^4 - 19x^2 - 14 + 2x^4 + 4x^2 = 5x^4 - 15x^2 - 14$$

4 $$f(x) = \frac{x^2 - 1}{(x + 2)^2}$$

$$f'(x) = \frac{(x + 2)^2\frac{d}{dx}(x^2 - 1) - (x^2 - 1)\frac{d}{dx}(x + 2)^2}{(x + 2)^4}$$

by § 101

$$= \frac{(x + 2)^2 \cdot 2x - (x^2 - 1)2(x + 2)}{(x + 2)^4}$$

by §§ 97, 103, 95, and 94

$$= \frac{(x + 2)2x - 2(x^2 - 1)}{(x + 2)^3} = \frac{2(2x + 1)}{(x + 2)^3}$$

EXERCISES

Differentiate the following functions with respect to x:

1. $y = x(1-x)^2$
2. $y = x^3(1-x^2)^2$
3. $f(x) = (1+x^2)(1-3x^2)$
4. $y = \frac{cx}{1+x^2}$
5. $f(x) = \frac{1-x^2}{1+x^2}$
6. $f(x) = \left(1+\frac{1}{x}\right)^{\frac{1}{2}}$
7. $f(x) = x^{\frac{2}{3}} - 6x^{\frac{1}{3}} + 5$
8. $y = (a+x^2)^{\frac{2}{3}}$
9. $y = x\sqrt{1+x^2}$
10. $y = (a+x)\sqrt{a-x}$
11. $y = \frac{2x^4}{a^2-x^2}$
12. $f(x) = \frac{3x^2}{(a-x)^2}$
13. $f(x) = \frac{(a+x)^{\frac{3}{2}}}{(a-x)^{\frac{1}{2}}}$
14. $f(x) = \frac{bx^2}{(1+x^3)}$
15. $f(x) = \frac{3x^2+2}{2x^2-5}$
16. $f(x) = \frac{ax+x^2}{(a^2-x^2)^2}$
17. $y = \frac{x^n}{1-x^n}$
18. $y = x(1-x)(1+x)^2$
19. $y = (a-x)^{\frac{1}{2}}(a+x)^{\frac{3}{2}}$
20. $y = \frac{\sqrt{1+x^2}}{x}$
21. $y = \frac{x}{\sqrt{1+x^2}}$
22. $y = \frac{a}{\sqrt{a^2-x^2}}$
23. $y = \frac{1}{\sqrt{1+x^2}+x}$
24. Prove Case (1) of § 103 by use of § 100.

104. Successive differentiation. As may be seen in the preceding exercises, the derivative of a function of x is, in general, itself a function of x, and can be differentiated. The derivative of a derivative is called the **second derivative**, and is written $f''(x)$. The derivative of the second derivative is called the **third derivative**, written $f'''(x)$, and so on. The second derivative of a function y may also be written $\frac{d^2}{dx^2}y$, the third derivative $\frac{d^3}{dx^3}y$, etc.

Example 1.

$$f(x) = x^3 + 3x^2 + 4x + 7$$
$$f'(x) = 3x^2 + 6x + 4$$
$$f''(x) = 6x + 6$$
$$f'''(x) = 6$$
$$f^{\text{IV}}(x) = 0$$

In certain functions it is easy to write down a general expression for the nth derivative $f^{(n)}(x)$, where n is any positive integer.

Example 2.

$$f(x) = \frac{1}{x}$$
$$f'(x) = -\frac{1}{x^2}$$
$$f''(x) = \frac{2}{x^3}$$
$$f'''(x) = \frac{-2\cdot 3}{x^4}$$
$$f^{\text{IV}}(x) = \frac{2\cdot 3\cdot 4}{x^5}$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$f^{(n)}(x) = \frac{(-1)^n 2\cdot 3\cdot 4\cdots n}{x^{n+1}}$$

Example 3.

$$f(x) = A_0 + A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \cdots$$
$$f'(x) = A_1 + 2A_2x + 3A_3x^2 + 4A_4x^3 + 5A_5x^4 + \cdots$$
$$f''(x) = 2A_2 + 2\cdot 3A_3x + 3\cdot 4A_4x^2 + 4\cdot 5A_5x^3 + \cdots$$

$$f'''(x) = 3!\,A_3 + 4!\,A_4x + 3\cdot 4\cdot 5\,A_5x^2 + \cdots$$
$$f^{\text{IV}}(x) = 4!\,A_4 + 5!\,A_5x + \cdots$$
$$f^{\text{V}}(x) = 5!\,A_5 + \cdots$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$f^{(n)}(x) = n!\,A_n + \cdots$$

Example 4.

$$f(x) = (a + 2x)^m$$
$$f'(x) = m(a + 2x)^{m-1}2$$
$$f''(x) = m(m-1)(a + 2x)^{m-2}2^2$$
$$f'''(x) = m(m-1)(m-2)(a + 2x)^{m-3}2^3$$
$$f^{\text{IV}}(x) = m(m-1)(m-2)(m-3)(a + 2x)^{m-4}2^4$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$f^{(n)}(x) = m(m-1)(m-2)\cdots(m-n+1)(a+2x)^{m-n}2^n$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$f^{(m)}(x) = m!\,2^m$ if m is a positive integer.

EXERCISES

1. How many successive derivatives has

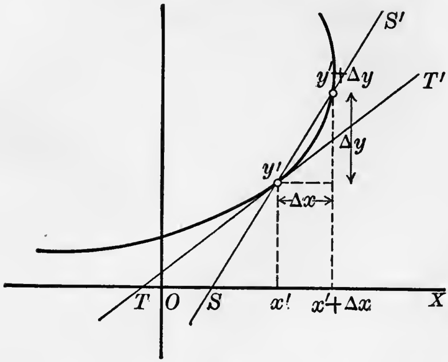
$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n?$$

2. Write four successive derivatives and the nth derivative of each of the following:

(*a*) $\dfrac{1}{1+x}$ (*c*) $(a+x)^m$

(*b*) $(1-x)^{-m}$ (*d*) $\dfrac{1+x}{1-x}$

105. Geometric interpretation of the derivative of a function. In the accompanying figure, let the curve represent the graph of $y = f(x)$, and let the secant SS' be drawn through the points y' and $y' + \Delta y$ of the curve, which correspond to the points x' and $x' + \Delta x$ on



the x-axis. Let a line be drawn through the point y' parallel to the x-axis. Then it is evident that the ratio $\frac{\Delta y}{\Delta x}$ is equal to the tangent of the angle $S'SX$. When Δx approaches zero as a limit, Δy also approaches zero as a limit, and the secant SS' approaches, as a limiting position, the line TT', which is tangent to the curve at the point y'. Then

$$\frac{d}{dx}y = \lim_{\Delta x=0} \frac{\Delta y}{\Delta x} = \tan T'TX$$

The trigonometric tangent of the angle which the geometric tangent to a curve at any point makes with the x-axis is called the **slope** of the curve at that point. Hence the value of the derivative of a function at any point is equal to the slope of the graph of the function at that point.*

106. Applications of the derivative. A **maximum value** of a continuous function is one that is greater than any other value in the immediate neighborhood. A **minimum value** is one that is less than any other value in the neighborhood. The graph of $x^3 - 3x^2 + 2$ on page 206 shows that the function has a maximum at $x = 0$ and a minimum at $x = 2$. At such points the

* While studying the slope of curves, Leibnitz (1646–1716), a German mathematician, discovered the principles of Differential Calculus, almost simultaneously with Newton.

function is neither increasing nor decreasing, and the derivative is equal to zero. If a tangent to the curve be drawn at such a point, it will be parallel to the x-axis. It does not necessarily follow that when the derivative is zero, the function will have a maximum or a minimum. The derivative of x^3+1 (see Ex. 2, page 74) is zero when $x=0$, but this is a so-called **point of inflection**, and not a maximum or minimum of the function. If, at a certain point, the derivative is zero, it can be determined whether the point is a maximum or a minimum, by investigating the sign of the derivative for values of x a little less and a little greater than the value which makes the derivative vanish. If the derivative is negative for a value a little less, and positive for a value a little greater, than the critical value, the point in question is a minimum, since, the derivative being first negative and then positive, the function must first decrease and then increase. If, on the other hand, the derivative is first positive and then negative, the critical value is a maximum; while if the derivative has the same sign on both sides of the critical point, it is neither a maximum nor a minimum, but a point of inflection. Frequently the conditions of a problem will show whether the point for which the derivative vanishes is a maximum or a minimum.

Example 1. It is required to make an open box from a piece of cardboard 12 inches square, by cutting a square from each corner in such a way that the contents of the box shall be as great as possible.

Let x be the side of the square which is cut from each corner. Then the volume of the completed box will be

$$V=(12-2x)^2x=144x-48x^2+4x^3$$

The problem requires that x shall be determined in such a way that V shall have its maximum value. If V be plotted as a function of x, this maximum may be roughly estimated from the graph. For convenience in plotting, the values of V have been divided by 16, thus flattening the graph without altering its essential features.

x	V
0	0
1	100
2	128
3	108
4	64
5	20
6	0

The graph points to a maximum in the vicinity of $x=2$, but the problem may be accurately solved by finding the derivative of V with respect to x

$$\frac{d}{dx}V = 144 - 96\,x + 12\,x^2$$

Setting this derivative equal to zero, and solving the resulting equation, 2 and 6 are found to be values of x which make the derivative vanish. It is evident that $x=2$ gives the desired maximum, that is, the required dimensions of the box are $8 \times 8 \times 2$ inches.

Example 2. What are the most economical dimensions for a cylindrical tin cup, which is required to have a capacity of 8π cubic inches?

The area of a cylindrical surface is $2\pi rh$, where r is the radius of the cylinder and h its altitude. The area of the base of the cylinder is πr^2. Hence the total surface of tin in the cup, or the function whose minimum is required, is

$$S = 2\pi rh + \pi r^2$$

This is a function of two variables, but one may be eliminated, since the volume of a cylinder is $\pi r^2 h$, which is given equal to 8π. Since

$$\pi r^2 h = 8\pi,\ h = \frac{8}{r^2}$$

Then
$$S = 2\pi rh + \pi r^2 = \frac{16\pi}{r} + \pi r^2$$

$$\frac{d}{dr}S = -\frac{16\pi}{r^2} + 2\pi r$$

Equating the derivative to zero and solving for r gives $r = 2$, which is evidently the value which makes S a minimum. By substitution the corresponding value of h is found to be 2. Hence, in order to require the smallest amount of tin, the radius and the height of the cup should each be 2 inches.

Example 3. What are the dimensions of the largest package which can be forwarded by parcel post, the government restriction being that the combined length and girth of the package shall not exceed 6 feet?

It can be shown that the girth of a package is less in proportion to its bulk if it is circular rather than rectangular. It is therefore assumed that the package is cylindrical in shape, but its length and radius are to be determined so the contents shall be a maximum. If the radius is r, the greatest value of the length l

under the restriction imposed is $l = 6 - 2\pi r$. Then the function whose maximum is required is

$$V = \pi r^2(6 - 2\pi r) = 6\pi r^2 - 2\pi^2 r^3$$

$$\frac{d}{dr}V = 12\pi r - 6\pi^2 r^2$$

Equating this derivative to zero, and solving for r, $r = 0$ or $\frac{2}{\pi}$. Evidently the first value gives a minimum and the second a maximum. If the radius is $\frac{2}{\pi}$, the circumference is 4 feet and the length 2 feet.

It may be interesting to note that this package has a greater volume than that of a cube 15 inches on each edge, which is unmailable under the present law.

EXERCISES

1. A box having a square base is to be sent by parcel post. Find its height and the edge of its base, so that its volume shall be the greatest allowed by law.

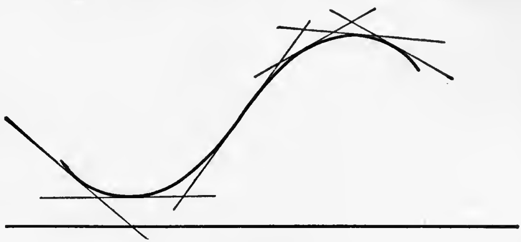
2. What are the most economical proportions for a cylindrical tin can whose volume is 16π cubic inches?

3. A person who can row 4 miles per hour, and walk 5 miles per hour, is on an island 3 miles from the nearest point on shore, and wishes to reach a place 10 miles from that point along the shore as quickly as possible. Where should he land?

4. A rectangular pen containing 8 square rods is to be constructed against a wall already built. What must be the dimensions of the rectangle in order that the least possible amount of additional wall may be required?

5. A conical tent is required to have a given volume V. What proportion must the radius of the base bear to the altitude in order that the least amount of canvas may be needed? Given $V = \frac{\pi r^2 a}{3}$, $S = \pi r l$, where $l = \sqrt{a^2 + r^2}$.

107. Geometric interpretation of the second derivative. In the same manner that the derivative of a function shows the rate of change of the function, so the second derivative shows the rate of change of the first derivative. As a point moves along a curve, the direction of the tangent to the curve at the moving point is, in general, constantly changing. From the considerations of § 105 it follows that the second derivative is an expression which shows the rate of change of the slope of the curve, that is, the change in the direction of the tangent as the point of tangency moves



along the curve. An examination of the accompanying figure makes it evident that when the slope of a curve is increasing, the tangent is below the curve, and the curve is concave upwards. When the slope of the curve is decreasing, the tangent is above the curve, and the curve is concave downwards. The point where the slope stops increasing and begins to decrease, or conversely, is called a **point of inflection** of the curve.

At such a point the tangent crosses the curve instead of being wholly above or below. It follows from these statements that when the second derivative is positive, the curve is concave upwards, when it is zero, the curve has a point of inflection, and when it is negative, the curve is concave downwards.

It has been previously shown that the first derivative gives many details about the shape of a curve which would escape observation in merely plotting a table of values. It is now seen that the second derivative gives still more accurate information on the subject. In § 106 it was found that when the first derivative vanishes, the function has either a maximum, a minimum, or a point of inflection. If the distinction between these three cases is not easily made otherwise, it may be determined by finding the second derivative. The critical point will be a minimum, a maximum, or a point of inflection according as the second derivative is positive, negative, or zero at that point.

108. Physical illustration of the second derivative. If a particle x is moving in a straight line, its distance from its initial position is a function of the time. Its rate of change, or its velocity at any point, is given by the derivative $\frac{d}{dt}x$ or $v = \frac{d}{dt}x$. The changes in the velocity, or the acceleration, are given by

$$\frac{d}{dt}v = \frac{d}{dt}\left[\frac{d}{dt}x\right] = \frac{d^2}{dt^2}x$$

For further work in the subject of this chapter, the student is referred to any good textbook in Calculus. A particularly simple and interesting one is "Calculus Made Easy," by F. R. S., published by Macmillan.

CHAPTER VI

CONVERGENCY OF SERIES

The ability to grasp together and to hold in steady view at once a multitude of ideas, to transcend the individuals, and compounding their forces, to seize the resultant meaning of them all, . . . these are the powers that mathematical activity in its higher rôles demands.

—CASSIUS J. KEYSER.

109. A **series** is an expression consisting of a succession of terms formed according to a fixed law, and connected by the signs + and −.

The terms may be constant or variable. The series here employed will be written in some such form as

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots$$

in which u_1 means the first term, u_2 the second term, and, in general, u_n the nth term.

The *law* of the series may be given in various ways. If four or five successive terms are given, the law can often be determined by inspection, as for example in the series

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} \cdots$$

Or the relation between consecutive terms may be given, as in the case of an arithmetical or geometrical progression, or a formula which holds for every term, as $u_n = \dfrac{1}{n^2}$, in which case the series reads

$$1 + \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{16} + \tfrac{1}{25} + \cdots$$

EXERCISES

1. Determine the law of each of the following series:

$$5 + 8 + 11 + 14 + \cdots$$
$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots$$
$$x + 2x^2 + 4x^3 + 8x^4 + \cdots$$

2. Write the first five terms of the series which have for their nth terms

$$\frac{x^{n+1}}{n^2+1}, \quad \frac{x^n}{n(n+1)}, \quad \frac{x^{2n+1}}{n^3}$$

110. The symbol S_n is used to denote the sum of the first n terms of a series. Thus in the series

$$u_1 + u_2 + u_3 + \cdots$$
$$S_1 = u_1$$
$$S_2 = u_1 + u_2$$
$$S_3 = u_1 + u_2 + u_3$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$S_n = u_1 + u_2 + u_3 + \cdots + u_n$$

Further, S_{2n} indicates the sum of an even number of terms, S_{2n+1} the sum of an odd number of terms.

111. Of all the various forms in which functions occur, no one is so important for theoretical or practical purposes as series. Their use in arithmetic is illustrated by the process of changing a common fraction into a decimal, a decimal number being really a series:

$$\tfrac{1}{3} = .33333 \cdots = \tfrac{3}{10} + \tfrac{3}{100} + \tfrac{3}{1000} + \cdots$$

or by the extraction of a root,

$$\sqrt{5} = 2.236 \cdots = 2 + \tfrac{2}{10} + \tfrac{3}{100} + \tfrac{6}{1000} + \cdots$$

In algebra series have been obtained by division, by the binomial formula, etc.

In more advanced work series are needed for many numerical computations, some of which will be met with later in this book, and there are many subjects in mathematics which are built up on series as a foundation.

Write as a series each of the following forms: $\frac{1}{8}$, $\frac{7}{11}$, $\sqrt{3}$, $(a+x)^{10}$, $\frac{1}{1-x}$, $\frac{2-x}{x^2+3x^3}$.

112. The series developed in § 111 may be divided into two classes:

1. Those in which there is a definite number of terms, *i.e.* there is a last term. These are called **finite series.**

2. Those in which no limit is set to the number of terms, *i.e.* there is no last term, but, however many terms may have been formed, there are always others, formed in accordance with the same law. These are called **infinite series.**

A finite series is merely a polynomial, with the properties of a polynomial. An infinite series has new properties, which do not belong to polynomials.

113. The sum of a finite series of any number of terms is readily found by the process of addition, the terms being combined in any convenient order. But it is obviously impossible to find the sum of the terms of an infinite series. The sum of a limited number of terms may be found, or, in the case of numerical terms, the sum may in many cases be found, correct to some specified number of decimal places.

For instance, in extracting the cube root of 2 it is impossible to find its value exactly, though it may be found correct to any desired number of decimal places.

EXERCISES

1. Find the value of S_5 when $u_n = \frac{1}{n}$.

2. Find the sum to ten terms of $1 + 2 + 3 + 4 + \cdots$.

3. Find the sum of 6 terms of the series

$$1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

expressing each term as a three-place decimal.

4. Can the value of

$$1 + x + x^2 + x^3 + \cdots$$

be found correct to three decimal places

when $$x = \tfrac{1}{3},\ x = 3,\ x = -1?$$

114. In attempting to find the sum of the terms of an infinite series one of three results is sure to be obtained as the number of the terms is increased:

1. The sum can be made to differ from some fixed number by less than any chosen number, however small.

2. The sum oscillates between two or more numbers, *i.e.* it is indeterminate.

3. The sum becomes greater than any number that may be chosen.

These results may be summed up as follows, letting S_n represent the sum of the first n terms of the series:

1. $\lim\limits_{n=\infty} S_n = S$, a finite number.

2. $\lim\limits_{n=\infty} S_n$ does not exist.

3. $\lim\limits_{n=\infty} S_n = \infty$.

115. The preceding considerations lead to the following definitions:

1. An infinite series is **convergent** if the sum of n terms, as n is indefinitely increased, has a fixed finite limit. This limit is called its **sum**.

2. An infinite series is **divergent** if, as the number of terms increases, their sum becomes either indeterminate or greater than any assigned quantity, however large.

The student should note that the word *sum* in Case 1 is used in a new sense. If, for instance,

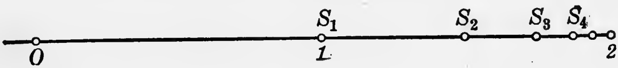
$$(1)\qquad S = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} = 1\tfrac{7}{8}$$

this is a true sum. But if

$$(2)\qquad S = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots = 2$$

this means that S can be made to differ from 2 by a number small at pleasure, *i.e.* not it, but its limit, is 2.

The sum of one, two, three ⋯ terms of the last series may be represented geometrically as follows:



Each new point is obtained by moving to the right of the preceding point one half the distance between it and the point 2. How many terms must be combined in order that their sum may differ from 2 by less than $\frac{1}{8}$? by less than $\frac{1}{100}$? by less than $\frac{1}{1000}$? Can the number of terms be found corresponding to any fraction that may be chosen?

As an illustration of a divergent series take

$$1 + 2 + 3 + 4 + \cdots$$

Can enough terms be combined to make the sum greater than 100? greater than 1000? greater than any number that may be selected?

As a second illustration consider the series

$$1-1+1-1+\cdots$$

If any even number of terms be combined, the sum is 0, if an odd number, the sum is 1, *i.e.* the sum is indeterminate, and the limit of the sum does not exist. This result may be expressed as follows: S_{2n} is always 0, S_{2n+1} is always 1, therefore $\lim_{n=\infty} S_n$ does not exist.

The latter kind of series is called an **oscillating** series. A series can oscillate only when it contains an unlimited number of both positive and negative terms.

116. When a series contains a variable, it may be, and generally is, convergent for some values of the variable and divergent for others. The reason for this is that the question concerns not a single series, but several series, equal in number to the number of values which the variable assumes.

Consider the series

$$1+x+x^2+x^3+\cdots$$

If $x=\frac{1}{2}$, the series becomes

$$1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\cdots$$

which is convergent.

If $x=3$, the series becomes

$$1+3+9+27+\cdots$$

which is divergent.

The above series may be obtained from the fraction $\dfrac{1}{1-x}$ by division, *i.e.*

(1) $$\frac{1}{1-x}=1+x+x^2+x^3+\cdots$$

If $x = \frac{1}{2}$, the first member becomes 2, and the second member

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots$$

The right-hand member approximates more and more nearly to the left as the number of terms is increased. If $x = 3$, the first member becomes $-\frac{1}{2}$ and the second member

$$1 + 3 + 9 + 27 + \cdots$$

The right-hand member diverges more and more from the left as the number of terms is increased.

117. The preceding examples illustrate one very important fact in regard to convergent series — that they alone give a true expression of the function from which they are developed. The consequence is that before making use of a series as the representation of a particular function, there must be some means of showing that it is convergent.

There is no one final and absolute test, applicable to all series, to determine whether or not they are convergent. For the more elementary series simple tests which are readily applied are sufficient; for others more elaborate tests are necessary. Tests sufficient for all the series used in this work will now be developed.

118. In examining a series for convergence the following statements, which are readily verified, are often useful.

1. In testing a series, any finite number of terms at the beginning may be disregarded. The original series is convergent or divergent according as the resulting series is convergent or divergent.

The advantage of this statement lies in the fact that the early terms of a series may fail to conform to a certain test to which the later terms conform. Suppose, for example, that a test requires that the terms grow smaller, and that the series to be tested is

$$1+5+\frac{5^2}{2!}+\frac{5^3}{3!}+\cdots$$

The first terms are increasing, but after the sixth term they begin to decrease, and meet the test. If the series beginning with the sixth term is convergent, the original series is convergent. For this reason the words *from a certain term on* may be inserted in every test for convergence.

2. A series is divergent if each term is in numerical value either equal to or greater than the preceding term.

3. A series is divergent if its nth term does not have for its limit zero as n is indefinitely increased.

4. If each term of a convergent (divergent) series is multiplied by the same number, the resulting series is convergent (divergent).

119. Theorem. *A series of positive terms is convergent if the sum of n terms can never exceed some fixed quantity, however great n may be.*

This is proved by showing that the sum cannot be infinite or indeterminate, and then applying § 114.

This theorem may also be stated in the following form, which will later be found useful:

If in an infinite set of numbers S_1, S_2, $S_3 \cdots$, each is greater than the preceding, but can never exceed a certain fixed quantity M, these numbers have a limit which is either equal to or less than M.

120. Theorem. *If a series of positive terms is convergent, the series formed by changing the signs of some or all of its terms is also convergent.*

1. Let all the terms of the second series be negative. Since by hypothesis $\lim_{n=\infty} S_n = S$, it follows that $\lim_{n=\infty} (-S_n) = -S$.

2. Let any finite number of terms be negative: The series then converges to a smaller limit, but the fact of its convergence is unaltered, as, after a certain number of terms, all the signs are positive. A similar argument holds if all but a finite number of the terms are negative.

3. Let the number of both positive and negative terms be unlimited. Each set of terms forms an infinite series, which must be convergent, as otherwise the sum of the two cannot be convergent. Let the series in question be written

$$(a_1 + a_2 + a_3 + \cdots) - (b_1 + b_2 + b_3 + \cdots)$$

and let $$\lim_{n=\infty} (a_1 + a_2 + a_3 + \cdots) = A$$

and $$\lim_{n=\infty} (b_1 + b_2 + b_3 + \cdots) = B$$

It follows that

$$\lim_{n=\infty} [(a_1 + a_2 + a_3 + \cdots) - (b_1 + b_2 + b_3 + \cdots)] = A - B$$

and the series is convergent.

121. If a formula for the sum of the first n terms of a series can be found, it is a simple matter to determine whether or not the series is convergent.

Example 1. Let the series be an arithmetical progression, *i.e.* one which obeys the law $u_{n+1} - u_n = d$ for every value of n. The formula for the sum of n terms is $S_n = n\left(\frac{2a + (n-1)d}{2}\right)$. By increasing n, S_n can be made to exceed any number that may be chosen, whatever the values of the constants a and d. Hence an infinite series in which the terms are in arithmetical progression is always divergent.

Example 2. Let the series be a geometric progression, *i.e.* one which obeys the law $\frac{u_{n+1}}{u_n} = r$ for every value of n. The formula for the sum of n terms is $S_n = \frac{ar^n - a}{r-1}$, which was shown on page 64, Exercises 10, 11, to have a finite limit, as n increases, only when the numerical value of r is less than 1.

The results may be summarized thus:

$$\text{A geometric series is } \begin{cases} \text{convergent if } |r| < 1, \\ \text{divergent if } |r| \geqq 1. \end{cases}$$

In general, an exact expression for the sum of n terms of a series cannot be found, so that the preceding method, though most satisfactory when applicable, can be used for testing only a very small number of series.

EXERCISES

Determine whether the following series are convergent or divergent; if convergent, determine the sum.

1. $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots$, the sum of n terms is $\frac{n}{n+1}$.

2. $\frac{1}{1\cdot 4} + \frac{1}{4\cdot 7} + \frac{1}{7\cdot 10} + \cdots$, the sum of n terms is $\frac{n}{3n+1}$.

3. $1 + \frac{3}{4} + \frac{9}{16} + \frac{27}{64} + \cdots$

4. $1 - 2 + 3 - 4 + \cdots$

5. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots$

6. $1 - \frac{5}{4} + \frac{25}{16} - \frac{125}{64} + \cdots$

122. Comparison tests.

Theorem 1. *If one of two series of positive terms is known to be convergent, and each term of the second series is equal to or less than the corresponding term of the first, the second series is also convergent.*

Since the terms of the second series are positive, the sum of n terms must increase with n, but will always remain less than the sum of the first series. Hence by § 119 the second series is convergent.

Corollary. *The second series is convergent whatever the signs of its terms may be.* § 120.

Theorem 2. *If one of two series of positive terms is known to be divergent, and each term of the second series is either equal to or greater than the corresponding term of the first, the second series is also divergent.*

However large M may be, the sum of n terms of the first series may be made greater than M by taking n sufficiently great. Hence the sum of n terms of the second series is greater than M for sufficiently large values of n, and the series is divergent. § 115, 2.

123. The successful employment of these tests depends upon an acquaintance with various convergent and divergent series, which may be used as the basis of comparison. The most useful series for this purpose are the geometric series,

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots$$

and the two series,

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots$$

$$1 + \frac{1}{2^c} + \frac{1}{3^c} + \frac{1}{4^c} + \cdots$$

which are known as the **harmonic** and the **general harmonic** series respectively.

Theorem 1. *The harmonic series is divergent.*

Let $$S = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots$$

and let the terms after the first two be grouped in sets of two, four, eight, etc., a process which may be continued indefinitely.

$$S = 1 + \tfrac{1}{2} + (\tfrac{1}{3} + \tfrac{1}{4}) + (\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}) + \cdots$$

In each parenthesis each fraction before the last is greater than the last, and if each were equal to the last, their sum would be $\frac{1}{2}$. Hence it follows that the value of each parenthetical expression is greater than $\frac{1}{2}$. If then $\frac{1}{2}$ be substituted for each parenthetical expression,

$$S > 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \cdots$$

By taking a sufficient number of terms the value of S may be made to exceed any chosen value. Hence the series is divergent.

This series is an example of what is known as a slowly diverging series. Though the above proof shows that, however large M may be, the sum of the terms of the series may be made to exceed M by using a sufficiently large number of terms, it can be proved that the sum of a million terms amounts to less than 15.

Theorem 2. *The general harmonic series is convergent or divergent according as the exponent c is greater or less than* 1.

Let $$S = 1 + \frac{1}{2^c} + \frac{1}{3^c} + \frac{1}{4^c} + \cdots$$

Case 1. Let c be greater than 1, and let the terms after the first be grouped in sets of two, four, eight, etc.

$$S = 1 + \left(\frac{1}{2^c} + \frac{1}{3^c}\right) + \left(\frac{1}{4^c} + \frac{1}{5^c} + \frac{1}{6^c} + \frac{1}{7^c}\right) + \cdots$$

In each parenthesis the fractions following the first are smaller than the first, and if each were equal to the first, the values of these parenthetical expressions would become $\frac{2}{2^c}, \frac{4}{4^c}$, etc., *i.e.* this would be a geometric series with its ratio r equal to $\frac{2}{2^c}$. Since c is greater than 1, r is less than 1, and, as the geometric series is convergent by § 121, Example 2, the original series is also convergent. § 122, Theorem 1.

Case 2. Let c be less than 1; each term of the series is then greater than the corresponding term of the harmonic series, hence the series is divergent. § 122, Theorem 2.

Example 1. Test for convergence the series

$$1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots$$

The following series is known to be convergent:

$$1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots$$

But from the third term on each term of the first series is less than the corresponding term of the second series, and the fact that this relation always holds is proved by comparing the nth terms of the two, for

$$\frac{1}{n!}<\frac{1}{2^{n-1}}$$

Example 2. Test for convergence the series

$$\frac{3}{1\cdot 2}+\frac{4}{2\cdot 3}+\frac{5}{3\cdot 4}+\frac{6}{4\cdot 5}+\cdots$$

Each term of this series is greater than the corresponding term of the harmonic series, as is readily seen by comparing the nth terms:

$$\frac{n+2}{n(n+1)} > \frac{1}{n} \times \frac{n+1}{n+1}$$

It is important to notice that the comparison of several terms at the beginning of two series is not sufficient to determine the question of convergence. In order to show that *every* term of one series is less than the corresponding term of the other, the nth terms must be compared, as is done in the two preceding examples. In the series

$$3 + \frac{3^2}{2!} + \frac{3^3}{3!} + \frac{3^4}{4!} + \cdots$$

each of the first eight terms is greater than the corresponding term of the harmonic series, yet it is convergent.

124. A series in which the terms are alternately positive and negative is called an **alternating series.**

Test for alternating series. *If in an alternating series the terms decrease numerically, and have for limit zero, the series is convergent.*

Let the series be

$$u_1 - u_2 + u_3 - u_4 + \cdots$$

where u_1, u_2, $u_3 \cdots$ are positive, $u_1 > u_2 > u_3 \cdots$, and $\lim_{n=\infty} u_n = 0$.

The sum of the series to any even number of terms may be written in the two following ways:

(1) $S_{2n} = (u_1 - u_2) + (u_3 - u_4) + \cdots + (u_{2n-1} - u_{2n})$

(2) $S_{2n} = u_1 - (u_2 - u_3) - (u_4 - u_5) - \cdots$
$- (u_{2n-2} - u_{2n-1}) - u_{2n}$

Equation (1) shows that this sum is positive, equation (2) that it must be less than u_1. Hence by § 119, S_{2n} has a limit, which may be represented by S. Since

$$S_{2n+1} = S_{2n} + u_{2n+1}$$

it follows that

$$\lim_{n=\infty} S_{2n+1} = \lim_{n=\infty} (S_{2n} + u_{2n+1}) = \lim_{n=\infty} S_{2n} + \lim_{n=\infty} u_{2n+1} = S$$

That is, the limit of the sum of n terms is the same whether n is odd or even, and the series is therefore convergent.

Example. The following series is convergent

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots$$

125. A geometric progression is a very special type of series in that the ratio of any term to the preceding is the same at every point in the series. In most series the ratio of any term to the preceding varies with the position of the term. Take for example the series

$$\tfrac{1}{2} + \tfrac{2}{3} + \tfrac{3}{4} + \tfrac{4}{5} + \cdots$$

The ratio of the second term to the first is $\frac{4}{3}$, of the third to the second is $\frac{9}{8}$, and, in general, the ratio of the $(n+1)$th term to the nth is $\dfrac{(n+1)^2}{n(n+2)}$. If $n = 6$, it is seen that the sixth term must be multiplied by $\frac{49}{48}$ to produce the seventh, and, in general, the number by which each term is multiplied to give the next term changes with the term required.

Since a series is convergent when $\dfrac{u_{n+1}}{u_n}$ is a constant quantity less than 1 (§ 121, Ex. 2), it is evident from § 122, Theorem 1, that a series in which $\dfrac{u_{n+1}}{u_n}$ is a variable less than 1 and decreasing as n increases is also convergent.

Suppose, for instance, that the second term of a series is obtained by multiplying the first by $\frac{1}{2}$, the third by multiplying the second by $\frac{1}{3}$, and, in general, the $(n+1)$th by multiplying the nth by $\frac{1}{n+1}$. The resulting series is convergent.

Write the series just described, taking 1 *as the first term.*

The number by which any given term must be multiplied to give the next term is readily found by forming the ratio of the $(n+1)$th term to the nth. If for all values of n this ratio is not merely less than unity, but does not come indefinitely near to it, the series is convergent.

126. The preceding considerations will serve to make clear the meaning of the following test.

The Cauchy* Ratio Test. *An infinite series of positive terms is convergent if the ratio of every term to the preceding term is less than a number which is itself less than unity.*

Let the series be

(1) $$u_1 + u_2 + u_3 + u_4 + \cdots$$

and by hypothesis $\frac{u_2}{u_1} < k, \frac{u_3}{u_2} < k \cdots \frac{u_{n+1}}{u_n} < k, k < 1.$

The given series may be written in the form

$$u_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_1} + \cdots + \frac{u_n}{u_1} + \cdots\right)$$

* Augustin Louis Cauchy (1789–1857), one of the most famous of French mathematicians, was among the first to treat scientifically the subject of infinite series. His most important work was done in analysis, but he wrote several memoirs on geometry and physics.

(2) $$= u_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2} \times \frac{u_2}{u_1} + \frac{u_4}{u_3} \times \frac{u_3}{u_2} \times \frac{u_2}{u_1} + \cdots\right)$$

Compare (2) with the series

(3) $$u_1(1 + k + k^2 + \cdots + k^{n-1} + \cdots)$$

which is convergent, since $k < 1$. Each term of (2) after the first is, according to hypothesis, less than the corresponding term of (3). Hence the given series is convergent, by § 122.

Corollary. *If a series of positive terms is found by this test to be convergent, it will be convergent whatever the signs of the terms, by* § 120.

127. In order to apply the Cauchy ratio test to a given series the following steps are necessary:

1. Write the nth and the $(n+1)$th terms.

2. Find the numerical value of their ratio, *i.e.* $\left|\frac{u_{n+1}}{u_n}\right|$.

3. Determine whether or not this ratio, as n increases, becomes and remains less than some fixed proper fraction.

The determination of the value of the ratio is often possible by inspection only. If, for example, $\frac{u_{n+1}}{u_n} = \frac{1}{n+5}$, it is evident that for all admissible values of n this ratio is less than $\frac{1}{5}$. In case the question is not settled so easily, find the limit of $\frac{u_{n+1}}{u_n}$ as n is indefinitely increased. If this limit is less than 1, it is always possible to insert a fraction between it and 1, and the ratio under discussion will be less than this fraction.

If the ratio is always equal to or greater than 1, the series is divergent by § 118, 2.

If the ratio is less than 1, but assumes values indefinitely near to 1 as n increases, no conclusion can be drawn as to the convergence of the series.

The preceding results may be summed up as follows:

$$\text{Let } r = \left|\frac{u_{n+1}}{u_n}\right|, \text{ and } l = \lim_{n=\infty} r = \lim_{n=\infty} \left|\frac{u_{n+1}}{u_n}\right|$$

If $l < 1$, the series is convergent.
If $l > 1$, the series is divergent.
If $l = 1$ and $r > 1$, the series is divergent.
If $l = 1$ and $r < 1$, the test fails.

Example. Test for convergence the series

$$2x + \frac{2^2x^2}{2} + \frac{2^3x^3}{3} + \frac{2^4x^4}{4} + \cdots + \frac{2^nx^n}{n} + \frac{2^{n+1}x^{n+1}}{n+1} + \cdots$$

$$\text{Here } r = \frac{u_{n+1}}{u_n} = \frac{2^{n+1}x^{n+1}}{n+1} \times \frac{n}{2^nx^n} = \frac{n}{n+1} 2x$$

$$l = \lim_{n=\infty} \frac{n}{n+1} 2x = 2x$$

The series is convergent for all values of x which make l numerically less than 1, *i.e.* when x is numerically less than $\frac{1}{2}$, and it is divergent when x is numerically greater than $\frac{1}{2}$.

It is necessary to consider the values $x = \pm\frac{1}{2}$, for which the test fails.

If $x = \frac{1}{2}$, the series becomes

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots$$

which is divergent by § 123, Theorem 1.

If $x = -\frac{1}{2}$, the series becomes

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots$$

which is convergent by § 124.

128. The foregoing tests are sufficient to determine the convergence or divergence of the most important elementary series. Even when the Cauchy ratio test fails, a comparison test will in most cases settle the question. In the case of series which converge or diverge very slowly, *i.e.* in which the ratio of each term to the preceding comes very near to unity, more elaborate and delicate tests are needed, which may be found in more advanced works on series. See Osgood: Introduction to Infinite Series.

EXERCISES

Determine whether the following series are convergent or divergent:

1. $1+\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}+\frac{1}{4\sqrt{4}}+\cdots$

2. $\frac{x^2}{3\cdot 3}+\frac{x^3}{3^2\cdot 4}+\frac{x^4}{3^3\cdot 5}+\cdots$

3. $1-\frac{x^2}{\sqrt{2}}+\frac{x^4}{\sqrt{4}}-\frac{x^6}{\sqrt{6}}+\cdots$

4. $1+\frac{x}{2}+\frac{x^2}{5}+\frac{x^3}{10}+\frac{x^4}{17}+\cdots$

5. $1+\frac{x}{2}+\frac{x^2}{2\cdot 2^2}+\frac{x^3}{3\cdot 2^3}+\cdots$

6. $1+2x+3x^2+4x^3+\cdots$

7. $1+2^2x+3^2x^2+4^2x^3+\cdots$

8. $1+\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\cdots$

9. $\frac{x}{1\cdot 2}+\frac{x^2}{2\cdot 3}+\frac{x^3}{3\cdot 4}+\cdots$

10. $1 + 10 + \frac{10^2}{2!} + \frac{10^3}{3!} + \cdots$

11. $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

12. $1 + \frac{x}{2} + \frac{x^2}{3^2} + \frac{x^3}{4^3} + \frac{x^4}{5^4} + \cdots$

13. $1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \cdots$

14. $\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \cdots$

15. $\frac{1}{3} + \frac{2x}{3^2} + \frac{3x^2}{3^3} + \frac{4x^3}{3^4} + \cdots$

16. $1 + \frac{4x}{2} + \frac{4^2x^2}{3} + \frac{4^3x^3}{4} + \cdots$

17. $\frac{1}{20} + \frac{2!}{20^2} + \frac{3!}{20^3} + \frac{4!}{20^4} + \cdots$

18. $\frac{1}{3} + \frac{1 \cdot 3}{3 \cdot 6} + \frac{1 \cdot 3 \cdot 5}{3 \cdot 6 \cdot 9} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{3 \cdot 6 \cdot 9 \cdot 12} + \cdots$

19. $1!\,x + 2!\,x^2 + 3!\,x^3 + \cdots$

20. If Cauchy's test shows that the series

$$a_0 + a_1x + a_2x^2 + \cdots$$

is convergent when $|x| < c$, show that for the same values of x the following series is also convergent:

$$a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots$$

an equation which is true for unlimited no. of values for x is identity. An equation which is true for limited no. of values for x is equation condition.

CHAPTER VII

DEVELOPMENT OF FUNCTIONS IN SERIES

The idea that series of powers are as serviceable for algebra as for arithmetic was first worked out by Newton, and in the theory of functions of a complex variable, as it now stands, the theory of such series is the solid foundation for the whole structure.

— HARKNESS AND MORLEY: *Introduction to Analytic Functions.*

129. Identities and equations of condition. An equation which is true for an unlimited number of values of the unknown quantity which it contains is called an **identical equation**, or an **identity**.

An equation which is true for only a limited number of values of the unknown quantity which it contains is called an **equation of condition.**

As illustrations consider the following equations:

(1) $$(x-3)^2 = x^2 - 6x + 9$$

Whatever values may be chosen for x, the two members of the equation will be equal, *i.e.* this is an identity.

(2) $$2x^2 - 14x = x^2 - 6x + 9$$

The two members of the equation are equal only when 9 and -1 are substituted for x, *i.e.* this is an equation of condition.

It follows that, when an equation is known to be an identity, any values whatever may be substituted for the unknown quantities which it contains.

The fact that an equation is an identity is often indicated by the sign $\equiv$, thus

$$(1-x)^3 \equiv 1 - 3x + 3x^2 - x^3$$

EXERCISES

For how many values of x are the following equations true?

1. $x^2 + 1 = \dfrac{x^3 + 1}{x + 1} + x$

2. $5x^2 + 7x - 3 = 5(x + 2)(x - 4) + 9$

3. $3x^2 + 2 = 3(x - 1)(x + 1) + 5$

4. $\dfrac{1}{1 - x^2} = 1 + x^2 + x^4 + x^6 + \dfrac{x^8}{1 - x^2}$

5. $\dfrac{1}{1 - x^3} = 1 + x^3 + x^6 + x^9 + \cdots$

130. In § 111 some methods are mentioned by which series are obtained in arithmetic and algebra. An additional method of developing functions into series is known as **the method of undetermined coefficients.** In this method the possibility of expanding the function into a series of integral ascending powers of the variable with constant coefficients is assumed (if not already proved), and means are devised for finding the values of the coefficients, which are at first unknown.

The theorem upon which this method depends is the following:

131. Theorem. *If two series containing the same variable are identically equal to each other, the coefficients of the same power of the variable in these two series are equal.*

Let the given identity be

$$(1)\quad A_0 + A_1x + A_2x^2 + A_3x^3 + \cdots \equiv B_0 + B_1x + B_2x^2 + B_3x^3 + \cdots$$

Since this is an identity, it is true for every value of x which renders both series convergent. Since 0 is clearly such a value, let $x = 0$, then

(2) $$A_0 \equiv B_0$$

Differentiate both members of (1).*

(3) $$A_1 + 2A_2x + 3A_3x^2 + \cdots \equiv B_1 + 2B_2x + 3B_3x^2 + \cdots$$

As before, let $x = 0$, then

(4) $$A_1 \equiv B_1$$

By continuing this process any two corresponding coefficients may be proved equal.

132. The use of this theorem in developing a given fraction into a series is best shown by examples.

Example 1. Develop into a series of ascending powers of x the fraction $\dfrac{2 - 3x + 4x^3}{2x^2 - 3x + 1}$

Since the result is to contain ascending powers of x, both numerator and denominator must be so arranged. Assume that the fraction may be developed into a series of the following form:

(1) $$\frac{2 - 3x + 4x^3}{1 - 3x + 2x^2} = A_0 + A_1x + A_2x^2 + A_3x^3 + A_4x^4 + \cdots$$

To find the values of the undetermined coefficients A_0, A_1, $A_2 \cdots$ clear of fractions. The result may be arranged in this compact form:

* An infinite series does not fall under the theorem of § 97, but it can be proved that, if a series in the form of either member of equation (1) is convergent for certain values of x, its derivatives are also convergent for those same values, and that these derivatives may be formed by differentiating the series term by term. See Example 20, page 112.

$$
\begin{array}{rl|l|l|l|l}
2-3x+4x^3 = & A_0 + A_1 & x + A_2 & x^2 + A_3 & x^3 + A_4 & x^4 + \cdots \\
 & -3A_0 & -3A_1 & -3A_2 & -3A_3 & \\
 & & +2A_0 & +2A_1 & +2A_2 &
\end{array}
$$

The first member may be regarded as an infinite series in which each power of x after the third has the coefficient 0. Since the coefficients of the same powers of x must be equal,

$$
\begin{array}{ll}
A_0 = 2 & \\
A_1 - 3A_0 = -3 & \therefore A_1 = 3 \\
A_2 - 3A_1 + 2A_0 = 0 & \therefore A_2 = 5 \\
A_3 - 3A_2 + 2A_1 = 4 & \therefore A_3 = 13 \\
A_4 - 3A_3 + 2A_2 = 0 & \therefore A_4 = 29
\end{array}
$$

Substituting these values in equation (1),

$$
(2) \quad \frac{2 - 3x - 4x^3}{1 - 3x + 2x^2} = 2 + 3x + 5x^2 + 13x^3 + 29x^4 + \cdots
$$

which is a true relation for every value of x for which the second member is convergent, but there are difficulties in the way of testing for convergence a series developed by this method, as an expression for the nth term cannot easily be found which does not involve one or more of the preceding terms. The law of this series is given by $A_n = 3A_{n-1} - 2A_{n-2}$.

It is possible to verify the form of the series by division.

If either numerator or denominator contains a monomial factor, it is best to remove this factor and to develop the resulting fraction into a series, which is to be multiplied or divided by this factor according as it was removed from the numerator or the denominator.

Example 2. Develop into a series of ascending powers of x the fraction $\dfrac{3 - 5x^2}{2x^2 + x^3 - 3x^4}$.

The fraction may be written $\frac{1}{x^2} \cdot \frac{3-5x^2}{2+x-3x^2}$.

Assume the series

$$\frac{3-5x^2}{2+x-3x^2} = A_0 + A_1x + A_2x^2 + A_3x^3 + \cdots$$

and proceed as in Example 1.

$$3 - 5x^2 = \begin{array}{l|l|l|l} 2A_0 + 2A_1 & x + 2A_2 & x^2 + 2A_3 & x^3 + \cdots \\ \quad\;\; + A_0 & \;\; + A_1 & \;\; + A_2 & \\ & - 3A_0 & - 3A_1 & \end{array}$$

$$\begin{array}{ll} 2A_0 = 3 & \therefore A_0 = \frac{3}{2} \\ 2A_1 + A_0 = 0 & \therefore A_1 = -\frac{3}{4} \\ 2A_2 + A_1 - 3A_0 = -5 & \therefore A_2 = \frac{1}{8} \\ 2A_3 + A_2 - 3A_1 = 0 & \therefore A_3 = -\frac{19}{16} \end{array}$$

Hence the given fraction may be written

$$\frac{3-5x^2}{2x^2+x^3-3x^4} = \frac{1}{x^2}\left(\frac{3}{2} - \frac{3}{4}x + \frac{1}{8}x^2 - \frac{19}{16}x^3 + \cdots\right)$$

$$= \frac{3}{2x^2} - \frac{3}{4x} + \frac{1}{8} - \frac{19x}{16} + \cdots$$

EXERCISES

Develop into series by the method of undetermined coefficients the fractions

1. $\frac{1+2x-3x^2}{1-3x+x^2}$
2. $\frac{2+5x}{1-3x}$
3. $\frac{x-x^2}{1-5x^2+2x^3}$
4. $\frac{1}{3x+x^2}$,
5. $\frac{1-2x^2-x^3}{x^2+x^3-x^4}$
6. $\frac{1-x+2x^2}{x^2-3x^3}$
7. $\frac{1+x}{2x-x^2}$
8. $\frac{x}{(3-x)^2}$
9. $\frac{2-x}{1-2x+3x^2}$

10. If the series $1 + a_1x + a_2x^2 + a_3x^3 + \cdots$ is identical with the series obtained by differentiating it term by term, determine the values of its coefficients.

133. Maclaurin's Formula. Undetermined coefficients will now be employed to develop into a series any function which can be represented in the form of a series in positive integral ascending powers of the variable, with constant coefficients.

Let $f(x)$ represent the function to be developed, and assume that its successive derivatives can be found, and that it can be developed into a series of the form

$$(1) \qquad f(x) = A_0 + A_1x + A_2x^2 + A_3x^3 + A_4x^4 + \cdots$$

Let the successive derivatives be formed: *

$$f'(x) = A_1 + 2A_2x + 3A_3x^2 + 4A_4x^3 + \cdots$$
$$f''(x) = 2A_2 + 2\cdot 3A_3x + 3\cdot 4\,A_4x^2 + \cdots$$
$$f'''(x) = 2\cdot 3A_3 + 2\cdot 3\cdot 4A_4x + \cdots$$

and in general, if k is any positive integer,

$$f^k(x) = k!\,A_k + \cdots$$

Since each of these equations is an identity for every value of x which makes the series convergent, and since 0 is clearly such a value, 0 may be substituted for x, giving the following results, from which the values of the coefficients A_0, A_1, $\cdots$ may be found:

$$f(0) = A_0 \qquad\qquad A_0 = f(0)$$
$$f'(0) = A_1 \qquad\qquad A_1 = f'(0)$$
$$f''(0) = 2!\,A_2 \qquad\qquad A_2 = \frac{1}{2!}f''(0)$$
$$f'''(0) = 3!\,A_3 \qquad\qquad A_3 = \frac{1}{3!}f'''(0)$$

. . . .

$$f^{n-1}(0) = (n-1)!\,A_{n-1} \qquad\qquad A_{n-1} = \frac{1}{(n-1)!}f^{n-1}(0)$$

* See footnote, page 115.

If now these values be substituted in equation (1), the resulting formula is known as Maclaurin's* Formula:

$$f(x)=f(0)+f'(0)x+f''(0)\frac{x^2}{2!}+f'''(0)\frac{x^3}{3!}$$
$$+\cdots+f^{n-1}(0)\frac{x^{n-1}}{(n-1)!}+\cdots$$

134. Maclaurin's formula makes it possible to develop into a series any function of x whose derivatives can be formed, and this series will represent the given function for all values of x for which it is convergent. In later work new functions will be added to those discussed in Chapter V, but at present the only forms which can be used are the fraction and a variable affected with a constant exponent.

For the fraction no additional method is needed, since two have already been found, but as an illustration of the use of the formula one example is given.

Example 1. Develop by the use of Maclaurin's formula $\frac{1+x}{(1-x)^2}$.

Let

$$f(x)=\frac{1+x}{(1-x)^2} \qquad f(0)=1$$

$$f'(x)=\frac{(1-x)^2-(1+x)2(1-x)(-1)}{(1-x)^4}$$

$$=\frac{3+x}{(1-x)^3} \qquad f'(0)=3$$

* Colin Maclaurin (1698–1746) was professor of mathematics at Aberdeen when nineteen years old. The formula known by his name was published by him in 1742 with the same mode of derivation here employed. The result had been published twelve years earlier by James Stirling.

$$f''(x)=\frac{10+2x}{(1-x)^4} \qquad f''(0)=10$$

$$f'''(x)=\frac{42+6x}{(1-x)^5} \qquad f'''(0)=42$$

$$f^{\text{IV}}(x)=\frac{216+24x}{(1-x)^6} \qquad f^{\text{IV}}(0)=216$$

Substitute these values in Maclaurin's formula,

$$\frac{1+x}{(1-x)^2}=1+3x+5x^2+7x^3+9x^4+\cdots$$

Example 2. Develop by the use of Maclaurin's formula $(a+2x)^{\frac{3}{2}}$.

Let

$$f(x)=(a+2x)^{\frac{3}{2}} \qquad f(0)=a^{\frac{3}{2}}$$

$$f'(x)=\tfrac{3}{2}(a+2x)^{\frac{1}{2}}\cdot 2=3(a+2x)^{\frac{1}{2}} \qquad f'(0)=3a^{\frac{1}{2}}$$

$$f''(x)=3\cdot\tfrac{1}{2}(a+2x)^{-\frac{1}{2}}\cdot 2=3(a+2x)^{-\frac{1}{2}} \quad f''(0)=3a^{-\frac{1}{2}}$$

$$f'''(x)=-3\cdot\tfrac{1}{2}(a+2x)^{-\frac{3}{2}}\cdot 2=-3(a+2x)^{-\frac{3}{2}}$$

$$f'''(0)=-3a^{-\frac{3}{2}}$$

$$f^{\text{IV}}(x)=3\cdot\tfrac{3}{2}(a+2x)^{-\frac{5}{2}}\cdot 2=9(a+2x)^{-\frac{5}{2}}$$

$$f^{\text{IV}}(0)=9a^{-\frac{5}{2}}$$

Substitute these values in Maclaurin's formula,

$$(a+2x)^{\frac{3}{2}}=a^{\frac{3}{2}}+3a^{\frac{1}{2}}x+\frac{3}{2!}a^{-\frac{1}{2}}x^2-\frac{3}{3!}a^{-\frac{3}{2}}x^3+\frac{9}{4!}a^{-\frac{5}{2}}x^4+\cdots$$

135. Maclaurin's formula will now be used to develop into a series a binomial affected with an exponent, in the form $(a+x)^m$, m being positive or negative, fractional or integral.

Let

$$f(x) = (a+x)^m \qquad f(0) = a^m$$

$$f'(x) = m(a+x)^{m-1} \qquad f'(0) = ma^{m-1}$$

$$f''(x) = m(m-1)(a+x)^{m-2} \qquad f''(0) = m(m-1)a^{m-2}$$

$$f'''(x) = m(m-1)(m-2)(a+x)^{m-3}$$

$$f'''(0) = m(m-1)(m-2)a^{m-3}$$

. . . .

$$f^{n-1}(x) = m(m-1)\cdots(m-n+2)(a+x)^{m-n+1}$$

$$f^{n-1}(0) = m(m-1)\cdots(m-n+2)a^{m-n+1}$$

(See § 103)

These values are to be substituted in Maclaurin's formula, giving as a result the **Binomial Theorem.**

$$(a+x)^m = a^m + ma^{m-1}x + \frac{m(m-1)}{2!}a^{m-2}x^2 + \frac{m(m-1)(m-2)}{3!}a^{m-3}x^3 + \cdots$$

$$\cdots + \frac{m(m-1)\cdots(m-n+2)}{(n-1)!}a^{m-n+1}x^{n-1} + \cdots$$

This is an identity for every value of x for which the second member is convergent. It follows that the binomial theorem, proved in elementary algebra for position integral exponents, holds for any rational exponent.

The form of this theorem was discovered by Sir Isaac Newton prior to 1676, but the first rigorous proof was given by Abel, a Norwegian mathematician (1802–1829).

136. It is necessary to determine the values of x which render this series convergent. To obtain most easily the value of r for the Cauchy ratio test, note that the ratio of each term to the preceding consists of the last factor of the numerator times x divided by the last factor of the denominator times a.

This gives $$r = \left|\frac{u_{u+1}}{u_n}\right| = \left|\frac{m-n+1}{n}\frac{x}{a}\right|$$

Hence $$l = \lim_{n=\infty}\left|\frac{m-n+1}{n}\frac{x}{a}\right| = \left|\frac{x}{a}\right|$$

and every value of x which is numerically less than a renders l numerically less than 1. This leads to the following conclusions:

If $|x| < |a|$, the series is convergent.
If $|x| > |a|$, the series is divergent.
If $|x| = |a|$, the test fails.

The substitution of $\pm a$ for x in the binomial theorem gives

$$(a+a)^m = a^m + ma^{m-1}a + \frac{m(m-1)}{2!}a^{m-2}a^2 + \cdots$$

$$= a^m\left[1 + m + \frac{m(m-1)}{2!} + \cdots\right]$$

$$(a-a)^m = a^m - ma^{m-1}a + \frac{m(m-1)}{2!}a^{m-2}a^2 - \cdots$$

$$= a^m\left[1 - m + \frac{m(m-1)}{2!} - \cdots\right]$$

Since the first members are respectively $(2a)^m$ and 0, and it is not necessary to expand such numbers into a series for any calculation, the question as to when these series give a true approximation of the first member need not be considered here.

Example 1. Develop by the binomial theorem

$$(c-3y)^{\frac{2}{3}}$$

Here the a of the formula is replaced by c, x by $-3y$, m by $\frac{2}{3}$. When these substitutions are made, the formula becomes

$$(c-3y)^{\frac{2}{3}} = c^{\frac{2}{3}} + \frac{2}{\cancel{3}}c^{-\frac{1}{3}}(-\cancel{3}y) + \frac{\frac{\cancel{2}}{\cancel{3}}\cdot -\frac{1}{\cancel{3}}}{\cancel{2}}c^{-\frac{4}{3}}(-\cancel{3}y)^2$$

$$+\frac{\frac{\cancel{2}}{\cancel{3}}\cdot -\frac{1}{\cancel{3}}\cdot -\frac{4}{\cancel{3}}}{\cancel{2}\cdot 3}c^{-\frac{7}{3}}(-\cancel{3}y)^3$$

$$+\frac{\frac{\cancel{2}}{\cancel{3}}\cdot -\frac{1}{\cancel{3}}\cdot -\frac{\cancel{4}}{\cancel{3}}\cdot -\frac{7}{\cancel{3}}}{\cancel{2}\cdot 3\cdot \cancel{4}}c^{-\frac{10}{3}}(-\cancel{3}y)^4 + \cdots$$

$$= c^{\frac{2}{3}} - 2c^{-\frac{1}{3}}y - c^{-\frac{4}{3}}y^2 - \frac{4}{3}c^{-\frac{7}{3}}y^3 - \frac{7}{3}c^{-\frac{10}{3}}y^4 - \cdots$$

Example 2. Find the eighth term of the expansion of $\sqrt{b^2 + y^{\frac{1}{2}}}$.

In the nth term substitute as follows:

$$a = b^2,\ x = y^{\frac{1}{2}},\ m = \tfrac{1}{2},\ n = 8$$

Then $$u_8 = \frac{\frac{1}{2}\cdot -\frac{1}{2}\cdot -\frac{3}{2}\cdot -\frac{5}{2}\cdot -\frac{7}{2}\cdot -\frac{9}{2}\cdot -\frac{11}{2}}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}(b^2)^{-\frac{13}{2}}(y^{\frac{1}{2}})^7$$

$$= \frac{33}{2048}b^{-13}y^{\frac{7}{2}}$$

EXERCISES

1. Write the nth and the $(n+1)$th terms of the binomial theorem, and find their ratio.

2. How many terms are there in the expansion of $(a+x)^m$ when m is a positive integer? a negative integer? a fraction? Why?

Expand to five terms the following functions:

3. $(1+2x)^{-\frac{1}{2}}$

4. $\dfrac{1}{(a-y)^3}$

5. $(x-2y^{\frac{1}{2}})^{\frac{3}{2}}$

6. $(a+bx^{-1})^{\frac{1}{2}}$

Expand to five terms the following functions, and write the nth term of each both by inspection and by use of the formula:

7. $(1+x)^{-1}$

8. $(1+x)^{-2}$

9. $(1+x)^{-3}$

10. $(1+x)^{-4}$

11. Find the fifth term of the expansion of $(c^{\frac{1}{2}}d^{\frac{2}{3}}-x^{-2})^{\frac{1}{2}}$.

12. Find the eighth term of the expansion of

$$\frac{1}{(a^{-1}-x^{-2})^3}.$$

13. Expand to five terms the two expressions $(1+4)^{\frac{1}{2}}$ and $(4+1)^{\frac{1}{2}}$, combine the terms, and compare the results.

14. Expand to five terms $\left(1+\dfrac{1}{n}\right)^n$. Assuming that the theorems of §§ 77 and 79 hold for an unlimited number of variables, find the limit of the right-hand member as n is indefinitely increased.

CHAPTER VIII

PARTIAL FRACTIONS

Mathematics is the queen of the sciences, and arithmetic the queen of mathematics. — GAUSS.

137. An especial use to which undetermined coefficients may advantageously be put is the breaking up of a fraction into the sum of two or more fractions. The converse problem has heretofore been met — to find a single fraction which shall be equal to the sum of two or more given fractions. It is clear that this new problem can be solved only when the denominator of the given fraction can be factored, and that these factors must be the denominators of the separate fractions into which the given one is resolved. These separate fractions are called **partial fractions.**

NOTE. The corresponding problem in arithmetic was one which forced itself very early upon the attention of mathematicians. As the methods of writing fractions, or indeed any numbers, were unsatisfactory, a fraction proved to be most easily handled when its numerator was unity. The oldest mathematical treatise in existence — the papyrus of Ahmes, dating from 1700 B.C., and probably a transcript of a far earlier work — gives rules for breaking up a fraction into the sum of fractions with each numerator unity. For instance, it is proved in one example that

$$\frac{2}{29} = \frac{1}{24} + \frac{1}{58} + \frac{1}{174} + \frac{1}{232}$$

138. The method by which an algebraic fraction is reduced to the sum of simpler fractions will be illustrated by examples. It is assumed that the numerator is of lower degree than the denominator. If this is not the case, the fraction should be changed to the required form by division.

Example. It is required to express $\frac{2x-1}{x^2-3x+2}$ as the sum of fractions.

Since the factors of the denominator are $x-1$ and $x-2$, the general form of the fractions of which this is composed is readily determined.

(1) $$\frac{2x-1}{x^2-3x+2}=\frac{A}{x-1}+\frac{B}{x-2}$$

Clear of fractions,

(2) $$2x-1=A(x-2)+B(x-1)$$

At this point either of two methods may be followed.

(*a*) Equate the coefficients of the like powers of x

$$A+B=2$$
$$-2A-B=-1$$

Hence $A=-1$, $B=3$, values which are to be substituted in (1), giving

$$\frac{2x-1}{x^2-3x+2}=-\frac{1}{x-1}+\frac{3}{x-2}$$

(*b*) Substitute in equation (2) values of x which make some of the terms vanish. Let $x=2$, the resulting equation is $B=3$; let $x=1$, the resulting equation is $-A=1$.

139. If a factor occurs more than once in the denominator, it may, but need not necessarily, account for more than one partial fraction, as the following examples show:

(1) $$\frac{x^2-3x+12}{(x-2)^2(x+3)}=\frac{2}{(x-2)^2}-\frac{1}{5(x-2)}+\frac{6}{5(x+3)}$$

(2) $$\frac{x^2-2x+10}{(x-2)^2(x+3)}=\frac{2}{(x-2)^2}+\frac{1}{x+3}$$

Since it is impossible to tell by inspection whether a multiple factor accounts for one or more than one simple fraction, the maximum number must be assumed; if any do not occur in the final result, their numerators will be equal to zero. Thus, if it had been assumed that the fractions in (2) had the form $\frac{A}{(x-2)^2}+\frac{B}{x-2}+\frac{C}{x+3}$, B would have proved to be equal to zero.

Example. Resolve $\frac{x+2}{(x-1)^3(x+1)}$ into partial fractions.

$$\frac{x+2}{(x-1)^3(x+1)}=\frac{A}{(x-1)^3}+\frac{B}{(x-1)^2}+\frac{C}{x-1}+\frac{D}{x+1},$$

Clear of fractions.

$$x+2=A(x+1)+B(x-1)(x+1)+C(x-1)^2(x+1) + D(x-1)^3$$

On expanding, and equating coefficients of the like powers of x,

$$A=\tfrac{3}{2},\ B=-\tfrac{1}{4},\ C=\tfrac{1}{8},\ D=-\tfrac{1}{8}$$

Therefore $$\frac{x+2}{(x-1)^3(x+1)}=\frac{3}{2(x-1)^3}-\frac{1}{4(x-1)^2} +\frac{1}{8(x-1)}-\frac{1}{8(x+1)}$$

The other method suggested on page 126 may be followed. Substitute for x successively the values ± 1, 0, 2. The resulting equations are easily solved.

$$3=2A,\ 1=-8D,\ 2=A-B+C-D,$$
$$4=3A+3B+3C+D$$

140. In the two examples just considered the factors of the denominator were all real. A quadratic factor may occur in the denominator, which has no real linear factors, and must therefore appear in the denominator of one of the partial fractions, the numerator of which may be, and hence must be assumed to be, of the first degree in x.

Example. Resolve $\dfrac{x+2}{(x^2+1)(x-1)}$ into partial fractions. Assume the partial fractions as follows:

$$(1) \qquad \frac{x+2}{(x^2+1)(x-1)}=\frac{Ax+B}{x^2+1}+\frac{C}{x-1}$$

and follow the usual procedure.

$$x+2=(Ax+B)(x-1)+C(x^2+1)$$

The values of A, B, and C may be formed either by equating the coefficients of the like powers of x, or by letting x assume successively the values 0, ± 1. These values, when substituted in equation (1), give

$$\frac{x+2}{(x^2+1)(x-1)}=-\frac{3x+1}{2(x^2+1)}+\frac{3}{2(x-1)}$$

141. If a quadratic factor appears more than once in the denominator of the fraction, the general principle used in the example of § 139 is followed. For instance, if the factor x^2+1 had occurred three times in the denominator of the fraction just considered, the following form would have been assumed:

$$\frac{x+2}{(x^2+1)^3(x-1)}=\frac{Ax+B}{(x^2+1)^3}+\frac{Cx+D}{(x^2+1)^2}+\frac{Ex+F}{x^2+1}+\frac{G}{x-1}$$

142. The cases already discussed are those which occur oftenest in the Integral Calculus, where this method finds its chief application. Other cases which may arise are readily handled by means of the suggestions already given.

It has been merely assumed in all this work that a fraction whose denominator is composed of real linear or quadratic factors can be expressed as the sum of partial fractions. In any given example these partial fractions, when found, may be added, and the assumption thus justified, but the proof that fractions in general can be so decomposed is left for more advanced books.

EXERCISES

Separate the following fractions into partial fractions:

1. $\dfrac{1}{x^2-1}$

2. $\dfrac{24-x}{x^2+x-12}$

3. $\dfrac{16-3x}{(x+1)(x^2-9)}$

4. $\dfrac{x^2+x-5}{(x-2)^2(x+3)}$

5. $\dfrac{3x^2-5x+2}{x^3-2x^2+3x-6}$

6. $\dfrac{1}{2ax-x^2}$

7. $\dfrac{1}{ax^2-x^3}$

8. $\dfrac{1}{1+x^3}$

9. $\dfrac{2x^2+3}{x^4+x^2+1}$

10. $\dfrac{1}{1-x^4}$

11. $\dfrac{x^2}{x^3+4x^2+5x+2}$

12. $\dfrac{2x^4-x^3-7x^2-2x+2}{(x^2+1)^2(x-2)}$

13. $\dfrac{3x^3-6x}{x^4-5x^2+4}$

14. $\dfrac{6x-9}{x^2-x-2}$

15. $\dfrac{1}{x^2(1-x)}$

16. $\dfrac{x^2}{(x+1)(x+2)^2}$

17. $\dfrac{1}{x^4(1+x^2)}$

CHAPTER IX

IRRATIONAL NUMBERS

If it is true, as Whewell says, that the essence of the triumphs of science and its progress consists in that it enables us to consider as evident and necessary views which our ancestors held to be unintelligible, and were unable to comprehend, then the extension of the number system to include the irrational and the imaginary is the greatest forward step which pure mathematics has ever taken.

— HERMANN HANKEL.

143. Preliminary considerations.

1. In § 17 it is stated that the system of rational numbers is dense, *i.e.* that between any two numbers an unlimited set of numbers can be inserted. For example, between 0 and 1 lies an infinite number of fractions, any desired number of which may be written as follows. Take all proper fractions in lowest terms with denominators successively 2, 3, 4, 5 ⋯,

$$\tfrac{1}{2};\ \tfrac{1}{3}, \tfrac{2}{3};\ \tfrac{1}{4}, \tfrac{3}{4};\ \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{4}{5};\ \tfrac{1}{6}, \tfrac{5}{6};\ \cdots$$

This process may be continued indefinitely, every fraction thus obtained lying between 0 and 1. Can there be other numbers between 0 and 1? Does the fact that a system of numbers is dense imply that no numbers can be added to it?

2. In § 20 it was proved that every fraction when expressed in decimal form yields either a terminating or a repeating decimal, and in the same section the converse was established — that every terminating or

repeating decimal can be expressed as a fraction. This suggests the question: Can there be such a thing as a non-terminating, non-repeating decimal, and if so, what kind of a number does it represent?

3. In § 18 it was shown that the process of evolution can in general not be carried out with rational numbers, that is, in the majority of instances there are no rational numbers which, when multiplied by themselves a certain number of times, produce a given integer or fraction.

In these three ways, therefore, the question of the possibility of numbers other than integers and fractions has arisen, and it is the object of this chapter to answer these questions.

144. The question proposed in 2, if answered in the affirmative, evidently affords an answer to the question of 1. It may well seem to the student that every decimal, if continued to a sufficient number of places, will prove to be a terminating or a repeating decimal,* but non-terminating, non-repeating decimal numbers can be formed in many ways.

For example, such a number may be formed from a repeating decimal by inserting in the first period any chosen digit, in the next period the same digit twice, in the third period the same digit three times, etc. The

* The belief that every decimal number must ultimately terminate or repeat its figures lay at the foundation of the efforts continued for centuries to find an exact expression for the value of π. One computer carried its value to over 700 decimal places, finding no trace of a repetition of figures. It was only in the latter half of the 18th century that the fact was definitely established that there can never be a regular repetition of the figures. But even to-day there are "circle-squarers," ignorant of this fact, who are working to obtain an exact value for π (evidently obtainable if given by a repeating decimal), and even publishing their discoveries of such a value.

resulting number is clearly non-periodic. Take, for example, the repeating decimals .333 ..., .4747 ..., and make from them the non-repeating decimals .313113111311113 ..., .487488748887 .... None of these numbers are rational, yet it is clear that they lie between 0 and 1, and that other such numbers can be formed at pleasure. Such numbers are called **irrational**, not at all because they are unreasonable or unpractical, for they are in constant use, but, just as the numbers which can be written as ratios are called rational, so those that are incapable of such representation are called irrational.*

145. The need of irrational numbers was shown by Euclid, to whom is due the following *proof that there is no rational number whose square is equal to* 2.

Suppose that there is a number $\frac{a}{b}$, where a and b are integers and mutually prime, whose square is 2, *i.e.* $\frac{a^2}{b^2}=2$, whence $a^2=2\,b^2$. Since a^2 is an even number, a is also even, and may be represented by $2\,c$, whence $4\,c^2=2\,b^2$, or $2\,c^2=b^2$. It follows that b^2, and therefore b, is an even number. But this contradicts the hypothesis that a and b have no common factor, and it follows that there is no rational number whose square is 2.

It is, however, customary to represent the process of extracting a square root by the symbol $\sqrt{\ }$, and so the

* The word used by Euclid to express the relation between two numbers was λόγος (*logos*), so that it is conceivable that if these ideas had come into Northern Europe directly from the Greek instead of through the medium of Latin, the corresponding terms to-day might be *logical* and *illogical* numbers.

symbol $\sqrt{2}$ is used to represent a number whose square is 2. This representation is so familiar that it is well to note that as a *number* it is meaningless until it has been shown that a value may be assigned to it.

146. Although there is no rational number whose square is 2, there are methods by which a rational number can be found whose square differs from 2 by less than any pre-assigned number, however small. Such a number clearly lies between 1 and 2, since $1^2 < 2 < 2^2$. Square the numbers 1.1, 1.2, 1.3 ... until a square appears which is greater than 2. These squares are 1.21, 1.44, 1.69, 1.96, 2.25. Hence it follows that

$$(1.4)^2 < 2 < (1.5)^2$$

Take now the squares of the numbers 1.41, 1.42, 1.43, ... until a square greater than 2 is found. The resulting numbers are

$$1.9881 < 2 < 2.0164$$

or

$$(1.41)^2 < 2 < (1.42)^2$$

Successive figures may thus be determined one by one, giving the following results:

$$1.99998998241 < 2 < 2.0000182084$$

or

$$(1.41421)^2 < 2 < (1.41422)^2$$

This process may be carried as far as is desired, the numbers between which 2 lies coming nearer together at each step, so that the difference between either one of these numbers and 2 can be made less than any assignable quantity, *i.e.* either set of numbers has 2 as a limit. Let the rational numbers 1, 1.4, 1.41, 1.414, ... be represented by a_1, a_2, a_3, a_4, ... Since $\lim_{n=\infty} a_n^2 = 2$, the symbol $\sqrt{2}$ is given a numerical meaning, viz.

$\sqrt{2} = \lim\limits_{n=\infty} a_n$. The sequence of numbers $a_1, a_2, a_3, a_4, \cdots$ is said to *define* the irrational number $\sqrt{2}$, and any one of them, a_n, gives the square root of 2 approximately, correct to $n - 1$ decimal places.

In the same general way the square roots of all rational numbers may be found, then the cube roots, the fourth roots, etc., giving rise to an infinite number of irrational numbers.

147. A similar method is used to determine the value of π, the ratio of the circumference of a circle to its diameter. The circumference of the circle lies between the perimeters of regular inscribed and circumscribed polygons, and by increasing the number of the sides of these polygons the difference between their perimeters may be made less than any assignable quantity, thus giving the circumference of the circle correct to any desired number of decimal places.

148. By some such methods sequences of rational numbers may be found which *define* various kinds of irrational numbers, any one of the set representing it correctly to the number of decimal places which it contains, each one of the set representing it more exactly than those that have come before it. The irrational number is positive or negative according as the rational numbers in the sequence which defines it are positive or negative.

From the definition of irrational numbers methods of combining them are readily deduced. Let two irrational numbers be defined by the sequences $a_1, a_2, a_3, \cdots$ and $b_1, b_2, b_3, \cdots$. The sum of these two numbers is defined by the sequence $a_1 + b_1,\ a_2 + b_2,\ a_3 + b_3, \cdots$, their difference by $a_1 - b_1,\ a_2 - b_2,\ a_3 - b_3, \cdots$; their product by $a_1b_1, a_2b_2, a_3b_3, \cdots$; their quotient by

$$\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \cdots.$$

For example, $\sqrt{2}$ and $\sqrt{3}$ are defined by the sequences 1, 1.4, 1.41, 1.414, 1.4142, 1.41421, ... and 1, 1.7, 1.73, 1.732, 1.73205, ... Their sum, $\sqrt{2}+\sqrt{3}$, is defined by the sequence, 2, 3.1, 3.14, 3.146, 3.1462, 3.14626, ...

The nth power of the irrational number defined by the sequence $a_1, a_2, a_3, \cdots$ is the number defined by the sequence $a_1^n, a_2^n, a_3^n, \cdots$. The nth root of an irrational number involves the discussion of a double limit, *i.e.* the question of the meaning of a sequence of irrational numbers, which cannot be taken up in this book. It is sufficient for the present to call attention to the fact that all that has been said about operations on irrational numbers amounts in actual practice to this — that their approximate values are to be used precisely as if they were rational numbers. It is clear that the irrational numbers which occur in any problem should be written with the same number of decimal places, and that the number of decimal places that can be depended upon in the result is ordinarily less than that which is used in representing the numbers. Any root of an irrational number is to be found as if the number were rational, but with proper care as to the number of decimal places. For example, if such a number is written with six decimal places, not more than three places can be relied on in its square root.

149. A meaning has now been given to the **algebraic combination** of irrational numbers, *i.e.* to any combination which involves only a finite number of repetitions of the six fundamental algebraic processes —

addition, subtraction, multiplication, division, involution, evolution, with the use of rational exponents only.

150. The meaning of an **irrational exponent** is as follows. Let c be any rational number, and a an irrational number defined by the sequence $a_1, a_2, a_3, \cdots$; then c^a is defined by the sequence $c^{a_1}, c^{a_2}, c^{a_3}, \cdots$, or $c^a = \lim\limits_{n=\infty} c^{a_n}$. As an illustration consider the value of $5^{\sqrt{2}}$. The value of $\sqrt{2}$, correct to six decimal places, has been found to be 1.414212. If now $5^{\frac{14}{10}}, 5^{\frac{141}{100}}, 5^{\frac{1414}{1000}}, \ldots$ be calculated, the following results are obtained, correct to four places.

$$5^{\frac{14}{10}} = 9.5183$$
$$5^{\frac{141}{100}} = 9.6727$$
$$5^{\frac{1414}{1000}} = 9.7352$$
$$5^{\frac{14142}{10000}} = 9.7383$$
$$5^{\frac{141421}{100000}} = 9.7384$$

By carrying the process far enough it is possible to make two consecutive numbers of this set differ from each other by less than any assignable number. The *limit* which they approach is the *value* of $5^{\sqrt{2}}$; it is approximately 9.7385.

As a further illustration of an incommensurable exponent let it be required to find the exponent with which 10 must be affected in order to produce 6. If the values of $10^{\frac{1}{10}}, 10^{\frac{2}{10}}, 10^{\frac{3}{10}}, \ldots$ be calculated, it is found that 6 lies between $10^{\frac{7}{10}}$ and $10^{\frac{8}{10}}$; further, that it lies between $10^{\frac{77}{100}}$ and $10^{\frac{78}{100}}$, etc. A sequence is thus found which defines the required number, viz. .7, .77, .778, .7781, .77815, $\cdots$. This means that if 10 is affected with the exponents $\frac{7}{10}$, $\frac{77}{100}$, $\frac{778}{1000}$, $\cdots$, the resulting numbers will ultimately differ from 6 by as small a decimal as may be chosen. These numbers are as follows, correct to four places:

$$10^{\frac{7}{10}} = 5.0119$$
$$10^{\frac{77}{100}} = 5.8884$$
$$10^{\frac{778}{1000}} = 5.9979$$
$$10^{\frac{7781}{10000}} = 5.9990$$
$$10^{\frac{77815}{100000}} = 5.9999$$

If an irrational number is to be affected with an irrational exponent, it is customary to express both numbers approximately, and to treat them as rational numbers. For example, $\pi^{\sqrt{2}}$ is approximately $3.1416^{1.4142}$ or 5.0472.

151. An **algebraic equation** is an equation in which the variables occur in algebraic combinations only (§ 149). Every algebraic equation in one variable may by suitable transformations be put in the form

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0,$$

where the a's are constants, and n a positive integer.

Every algebraic combination of rational numbers is a root of an algebraic equation with rational coefficients. For example, $\sqrt[3]{6}$ is a root of the equation $x^3 - 6 = 0$; $\sqrt{3 - \sqrt[3]{5}}$ is a root of the equation $x^6 - 9x^4 + 27x^2 - 32 = 0$, as is proved by freeing of radicals the equation $x = \sqrt{3 - \sqrt[3]{5}}$.

The converse, however, is not true, that is, not every algebraic equation with rational coefficients has roots that can be expressed as an algebraic combination of rational numbers. If the degree of the equation is less than 5, its roots are capable of such representation (see §§ 241, 242). But the roots of equations of higher degree cannot in general be so expressed. They may, however, be calculated to any desired degree of accuracy. For example, the roots of an equation of the third degree are calculated to three decimal places in § 237 by methods which are applicable to an equation of any degree.

Thus, in addition to the irrational numbers given by roots of rational numbers and by algebraic combinations of such roots, is found another unlimited set of irrational numbers, roots of equations of the fifth and higher degrees. All such numbers are called **algebraic numbers**.

152. This, however, by no means exhausts the supply of irrational numbers, as entirely different types will be found in the study of logarithms and of trigonometric functions. It may be remarked briefly that nearly all logarithms of rational numbers are irrational, and nearly all trigonometric functions (sine, cosine, tangent, etc.) are irrational, and that they can rarely be expressed as algebraic combinations of rational numbers. Such numbers are called **transcendental numbers**. They can never occur as roots of algebraic equations. Among the most famous transcendental numbers are the two which are ordinarily represented by the letters π and e.* In view of the many ways in which irrational numbers may arise, it is not hard to believe that they are more numerous than the rational numbers.

* For the meaning of e see § 167.

153. Continued fractions. Like rational numbers irrational numbers may be written as continued fractions of the form studied in § 21,

$$a_1 + \frac{1}{a_2 +}\,\frac{1}{a_3 +}\,\cdots$$

in which the a's are integers, and the connecting signs positive. In this case, however, the fraction does not terminate. The method of procedure is exactly like that used for rational numbers.

Let b represent any irrational number, and let a_1 be the greatest integer contained in it.

(1) $$b = a_1 + \frac{1}{b_1}$$

where $b_1 > 1$ and is irrational.

Let a_2 be the greatest integer contained in b_1,

(2) $$b_1 = a_2 + \frac{1}{b_2}$$

where $b_2 > 1$ and is irrational.

Continuing this process,

$$b_2 = a_3 + \frac{1}{b_3}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$b_{n-1} = a_n + \frac{1}{b_n}$$

Substitute these results in (1).

$$b = a_1 + \frac{1}{a_2 +}\,\frac{1}{a_3 +}\,\cdots$$

154. The continued fractions thus obtained fall into two classes:

1. The fraction is *periodic*, *i.e.* the successive denominators recur after a certain point in regular order.

2. The fraction is *non-periodic*, *i.e.* there is no regular repetition of values in the denominators.

In the first case it may be proved that the continued fraction always represents a *quadratic surd*, *i.e.* a number of the form $a+\sqrt{b}$. In the second case the fraction represents a more complicated type of irrational number.

155. The irrational number under discussion is represented approximately by each of the continued fractions $a_1+\frac{1}{a_2}$, $a_1+\frac{1}{a_2+}\frac{1}{a_3}$, $\cdots$ but is equal to the limit which they approach as the number of partial quotients is increased. The question of the convergence of the fraction is one that must be investigated in any extended study of continued fractions. It can be proved that every continued fraction of this type has a definite finite value.

Example 1. Express as a continued fraction $\sqrt{3}$.

$$(1)\quad \sqrt{3}=1+(\sqrt{3}-1)=1+\cfrac{1}{\cfrac{\sqrt{3}+1}{2}}$$

$$\frac{\sqrt{3}+1}{2}=1+\frac{\sqrt{3}-1}{2}=1+\cfrac{1}{\cfrac{2}{\sqrt{3}-1}}=1+\cfrac{1}{\cfrac{\not{2}(\sqrt{3}+1)}{\not{2}}}$$

$$\sqrt{3}+1=2+(\sqrt{3}-1)=2+\cfrac{1}{\cfrac{\sqrt{3}+1}{2}}$$

Since the last denominator is the same as in (1), the form of the fraction is determined:

$$\sqrt{3}=1+\frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{2+}\cdots$$

Example 2. Express as a continued fraction $\frac{1+3\sqrt{5}}{2}$.

(1) $$\frac{1+3\sqrt{5}}{2}=3+\frac{3\sqrt{5}-5}{2}$$

$$=3+\frac{1}{\frac{2}{3\sqrt{5}-5}}=3+\frac{1}{\frac{3\sqrt{5}+5}{10}}$$

(2) $$\frac{3\sqrt{5}+5}{10}=1+\frac{3\sqrt{5}-5}{10}$$

$$=1+\frac{1}{\frac{10}{3\sqrt{5}-5}}=1+\frac{1}{\frac{3\sqrt{5}+5}{2}}$$

(3) $$\frac{3\sqrt{5}+5}{2}=5+\frac{3\sqrt{5}-5}{2}$$

$$=5+\frac{1}{\frac{2}{3\sqrt{5}-5}}=5+\frac{1}{\frac{3\sqrt{5}+5}{10}}$$

The periodicity of the fraction is thus determined:

$$\frac{1+3\sqrt{5}}{2}=3+\frac{1}{1+}\frac{1}{5+}\frac{1}{1+}\frac{1}{5+}\cdots$$

156. By a similar method the value of π may be written as a continued fraction:

$$\pi=3+\frac{1}{7+}\frac{1}{15+}\frac{1}{1+}\frac{1}{292}+\cdots$$

The value of e may be obtained from a continued fraction interesting for its regularity:

$$\frac{e-1}{2}=1+\frac{1}{6+}\frac{1}{10+}\frac{1}{14+}\frac{1}{18+}\cdots$$

157. The totality of rational and irrational numbers forms the **system of real numbers.** This system is not only dense, but also **continuous**; *i.e.* as a variable changes from one real value to another, there is no value ever assumed which does not belong to this system. If, with some fixed length assumed as the unit, distances corresponding to all rational numbers are marked off on a straight line, each measured from a fixed point, then, whatever the length of the unit may be, there are gaps in the line, though invisible even to the most powerful microscope. For example, if one of these distances measured from the fixed point is used as the side of a square, there is no one of all these distances which represents exactly its diagonal. This, however, is not the case with *real* numbers; the result of every such geometric construction has its corresponding segment on the line.

For further information the student is referred to Fine: The Number System of Algebra, Dedekind: Essays on Numbers.

CHAPTER X

LOGARITHMS

Seeing there is nothing (right well-beloved Students of the Mathematics) that is so troublesome to mathematical practice, nor that doth more molest and hinder calculators than the multiplications, divisions, square and cubical extractions of great numbers, which besides the tedious expense of time are for the most part subject to many slippery errors, I began therefore to consider in my mind by what certain and ready art I might remove those hindrances. . . . Which secret invention, being (as all other good things are) so much the better as it shall be the more common, I thought good heretofore to set forth in Latin for the public use of mathematicians.

—JOHN NAPIER'S preface to the English translation of the *Descriptio*.

158. In the equation $a^b = c$, numerical values may be assigned to any two of the three quantities a, b, c, and the corresponding value of the third may be investigated. If values are assigned to a and b, the finding of c is merely a question of affecting a number with an exponent. If b and c are given, the equation to be solved is of the form $x^b = c$, which gives $x = c^{\frac{1}{b}}$. A new question arises when a and c are given, and it is required to solve the equation $a^x = c$, a question which may be formulated thus: With what exponent must a number a be affected in order to produce a given number c?

A little reflection shows that, in previous study involving exponents, attention has ordinarily been focused on the quantity affected with the exponent, or on the result of the operation, rather than on the exponent, which is now to be the special subject of study.

The exponent with which a given fixed number called the **base** must be affected in order to produce a given number is called the **logarithm** of the number.

Let a be the base, then if $a^m = p$, m is the logarithm of p to the base a, or briefly, $m = \log_a p$. The same fact is expressed by these two forms. The first may be called the *exponential form*, the second the *logarithmic form*.

EXERCISES

1. Express the following in logarithmic form:

$$2^5 = 32,\ 8^{-\frac{1}{3}} = \tfrac{1}{2},\ c^{m+n} = d$$

2. Express the following in exponential form:

$$\log_2 x = 10,\ \log_{100} 10 = \tfrac{1}{2},\ \log_c (mn) = a + b$$

3. If the base is 8, find the logarithms of 1, 2, 4, 32, $\frac{1}{2}$.

4. If the base is 25, find the logarithms of 625, 125, 1, .2, .04, $\frac{1}{625}$.

5. If the base is 10, find the logarithms of 1, 100, 10000, .001, .0001. Between what numbers does the logarithm of 20 lie? of 156? of 78425? of 1.53?

6. Which of the bases used in the above examples seems the most practical? Why?

7. What two special numbers cannot be used as the base of a system of logarithms? What one general class of numbers seems not practical for this purpose?

8. If the base is 2, 3, a, what numbers have as logarithms 0, 1, 2, 3, 4?

9. If a series of logarithms to the same base are in arithmetical progression, what is the relation between the corresponding numbers?

159. These examples serve to make clear the following facts:

1. Whatever base may be employed, the logarithm of 1 is 0, and the logarithm of the base is 1, *i.e.* $\log_a 1 = 0$, $\log_a a = 1$.

2. If two numbers are equal, their logarithms to the same base are equal.

3. The logarithms of two reciprocal numbers differ only in sign.

4. If the base is greater than 1, all numbers greater than 1 have positive logarithms, all numbers less than 1 have negative logarithms.

5. Whatever base may be selected, the logarithms of the majority of numbers are neither integers nor common fractions.

The last statement (5) shows that the majority of logarithms are incommensurable or irrational numbers. For an explanation of the meaning of an irrational exponent see § 150.

160. Theorem. *The logarithm of the product of two numbers is the sum of the logarithms of the numbers.*

Let the two numbers be h and k, and let their logarithms to the base a be m and n respectively. That is,

$$\log_a h = m, \quad \log_a k = n$$

Changing these to the exponential form,

$$a^m = h, \; a^n = k$$

By multiplication

$$a^{m+n} = hk$$

or

$$\log_a (hk) = m + n = \log_a h + \log_a k$$

Corollary. *This theorem may be extended to the case of three or more factors.*

161. Theorem. *The logarithm of the quotient of two numbers is the logarithm of the dividend minus the logarithm of the divisor.*

As in Theorem 1,

$$a^m = h,\ a^n = k$$

By division

$$a^{m-n} = \frac{h}{k}$$

or

$$\log_a \frac{h}{k} = m - n = \log_a h - \log_a k$$

NOTE. This theorem may be deduced from the theorem of § 160 by the aid of § 159, 3.

162. Theorem. *The logarithm of a number affected with an exponent is equal to the exponent multiplied by the logarithm of the number.*

Let $\log_a h = m$, and let c be any number, positive or negative, fractional or integral.

Then $$a^m = h$$

and $$a^{cm} = h^c,$$

that is, $$\log_a h^c = cm = c \log_a h$$

How may this theorem be stated when c is a positive integer? a fraction with numerator 1?

Show that when c is a positive integer the theorem may be proved directly from § 160.

163. These theorems show that if the logarithms of all numbers to some one base can be found, arithmetical processes can be greatly shortened, as by their use addition takes the place of multiplication, subtraction the place of division, multiplication the place of raising to a power, and division the place of extracting a root.*

As an illustration describe a method of working the following examples if there were a means of finding the logarithm of any number, and also the number corresponding to any logarithm.

1. Multiply 983.74 by 8736.9.
2. Divide .00396 by 78.159.
3. Raise 43 to the seventeenth power.
4. Extract the eleventh root of 8.

164. The use of logarithms in solving exponential equations.

An **exponential equation** is one in which the unknown quantity occurs in the exponent. If in such equations the only processes involved in either member are multiplication, division, involution, and evolution, the preceding theorems often make it possible to reduce them to ordinary algebraic forms.

Example 1. Solve the equation $a^{2x+1}b^{3x} = c^{5x}$.

Take the logarithm of each side of the equation to any base,

$$(2x + 1) \log a + 3x \log b = 5x \log c$$

This is a simple equation in x, which gives the solution

$$x = \frac{\log a}{5 \log c - 2 \log a - 3 \log b}$$

* As a time-saving device there is probably no mathematical invention or discovery comparable to logarithms, except the Arabic notation, which makes it possible for children to perform readily calculations which in earlier times, *e.g.* in Greece and Rome, were long, difficult, and, for the majority of even well-educated persons, impossible.

It was the intolerable burden of long computations, especially in astronomy, which led to the invention of logarithms, for the true mathematician does not enjoy long and tedious computations, but prefers to exercise his head for the sake of saving his fingers.

Example 2. Solve the equation $\frac{5^{x^2}}{25^{x+1}} = 10$. Take the logarithm of each side of the equation to the base 10

$$x^2 \log_{10} 5 - (x+1) \log_{10} 25 = 1$$

or

$$x^2 \log_{10} 5 - 2(x+1) \log_{10} 5 = 1$$

The ordinary method of solving a quadratic gives the result

$$x = 1 \pm \sqrt{\frac{1}{\log_{10} 5} + 3}$$

EXERCISES

In the following examples 1 to 8 write the logarithm of the first member of the equation in terms of the logarithms of the quantities occurring in the second member:

1. $A = \frac{1}{2} ba$
2. $C = 2 \pi r$
3. $V = \frac{4}{3} \pi R^3$
4. $A = \sqrt{s(s-a)(s-b)(s-c)}$
5. $x = \frac{a^{\frac{1}{2}} b^{\frac{2}{3}} c}{d^2}$
6. $y = \sqrt[10]{\frac{57 \times 101}{83}}$
7. $b = \sqrt{c^2 - a^2}$
8. $y = \frac{x^{\frac{2}{3}}}{(1-x^2)^{\frac{3}{4}}}$

9. Prove that $\log_b a \cdot \log_a b = 1$.

10. Solve for n each of the following formulas for a geometric progression: $s = \frac{ar^n - a}{r-1}$, $l = ar^{n-1}$

Solve for x each of the following equations:

11. $2^{x^2} \times 4^x = 8$

12. $256^x = .5$

13. $729^{x^x} = 3^{24}$

14. $\dfrac{2^x}{3^{2x}} = 6$

15. $25^{x^2} = \dfrac{5^{5x}}{20}$

16. $a^{2x+3}c^{x-1} = b^{x+2}$

17. $\log_2 x^3 = 12$

18. $\log(x + \sqrt{x}) + \log(x - \sqrt{x}) = \log 4 + \log x^2$

19. $\log_x 18 = \frac{3}{2}$

165. There are two systems of logarithms in actual use, one introduced by Napier, which is used for theoretical work, the other introduced by Briggs, which is used in computations.

In 1614 John Napier, a Scotch mathematician, published a small volume entitled **Mirifici Logarithmorum Canonis Descriptio,*** which in scientific importance is ranked below Newton's *Principia* only. This book formed the foundation for what have since been known as Napierian logarithms, also called natural logarithms. The base is an incommensurable number lying between 2 and 3.

Soon after the publication of this book, Henry Briggs, professor of geometry at Gresham College, London, visited Napier, and suggested some changes, notably the substitution of 10 for the base. In 1617 Briggs published a sixteen-page tract entitled **Logarithmorum Chilias Prima**, in which 10 was used as the base, and

* In the summer of 1914 the tercentenary of the publication of this book was celebrated in Edinburgh, the home of Napier. For an account of the celebration, portrait of Napier, facsimile of pages of his book, etc., see the *Napier Tercentenary Memorial Volume.* For an account of the Napier relics and of mathematical instruments exhibited in connection with the celebration see Horsburgh: *Handbook of the Exhibition of Napier Relics.*

the logarithms of integers from 1 to 1000 to fourteen decimal places were given.

The following sections give a method of computing the logarithms of numbers to any desired base.

166. In calculating logarithms both Briggs and Napier made use of the fact proved in Ex. 9, page 144, that if a set of logarithms is in arithmetical progression, the corresponding numbers are in geometrical progression. Briggs' method was the following. Since $\log_{10} 1 = 0$, and $\log_{10} 10 = 1$, the geometric mean between 1 and 10 has as logarithm the arithmetic mean between 0 and 1, *i.e.* $\log \sqrt{10} = \frac{1}{2}$. Similarly the geometric mean between 1 and $\sqrt{10}$ has as logarithm the arithmetic mean between 0 and $\frac{1}{2}$, or $\frac{1}{4}$. By continuing this process the following table may be formed.

$\log_{10} 10 = 1$	or	$\log_{10} 10 = 1$
$\log_{10} \sqrt{10} = \frac{1}{2}$		$\log_{10} 3.162277 = .5$
$\log_{10} \sqrt[4]{10} = \frac{1}{4}$		$\log_{10} 1.778279 = .25$
$\log_{10} \sqrt[8]{10} = \frac{1}{8}$		$\log_{10} 1.333521 = .125$
. . . . .		. . . .

In constructing his table Briggs continued this process fifty-four times, *i.e.* he found $\sqrt[2^{54}]{10}$.* From these results he devised a method of obtaining the logarithm of any desired number.

167. The following is in general the method employed by Napier. He formed the arithmetical progression $0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n} \cdots$, and then the geometric progression

$$1,\ 1+\frac{1}{n},\ \left(1+\frac{1}{n}\right)^2,\ \left(1+\frac{1}{n}\right)^3 \cdots$$

* 2^{54} is greater than 18015000000000000.

as the numbers corresponding to these logarithms. The relation between these numbers and their logarithms may be thus shown:

$$(1)\begin{cases}\text{Logarithm:} \\ \qquad 0,\ \frac{1}{n},\quad \frac{2}{n},\quad \frac{3}{n},\cdots \quad \frac{n}{n}\cdots \\ \text{Number: } 1,\ 1+\frac{1}{n},\ \left(1+\frac{1}{n}\right)^2,\ \left(1+\frac{1}{n}\right)^3,\cdots\left(1+\frac{1}{n}\right)^n\cdots\end{cases}$$

Since 1 is always the logarithm of the base, it follows that $\left(1+\frac{1}{n}\right)^n$ is the base of this system.

If the interval from 0 to 1 is divided into 10 equal parts, *i.e.* if $n = 10$,
(1) becomes

$$(2)\begin{cases}\text{Logarithm} \\ 0,\ \frac{1}{10},\ \frac{2}{10},\ \frac{3}{10},\ \frac{4}{10},\ \frac{5}{10},\ \frac{6}{10},\ \frac{7}{10},\ \frac{8}{10},\ \frac{9}{10},\ 1 \\ \text{Number} \\ 1,\ 1.1,\ (1.1)^2,\ (1.1)^3,\ (1.1)^4,\ (1.1)^5,\ (1.1)^6,\ (1.1)^7,\ (1.1)^8,\ (1.1)^9,\ (1.1)^{10} \\ 1,\ 1.1,\ 1.21,\ 1.331,\ 1.464,\ 1.611,\ 1.772,\ 1.949,\ 2.144,\ 2.358,\ 2.594\end{cases}$$

correct to three decimal places. The base is 2.594, and to this base $\log 1.1 = .1$, $\log 1.21 = .2$, etc.

If the interval is divided into 100 equal parts, *i.e.* if $n = 100$, the base is 2.704; if into 1000 parts, the base is 2.717, and the greater the number of divisions of the interval, the more accurately the logarithms can be found. Napier himself let n be 10^7, *i.e.* his base* was really $(1.0000001)^{10000000}$, which is equal to 2.7182, correct to four places. This process makes it possible to find the logarithms of ten million numbers from 1 to 2.7182.

* The word *base*, or its equivalent, was not used by Napier, as his definition of a logarithm was quite different from that used to-day, although it led directly to the modern definition.

In later years this base, which is represented by e, was defined as the limit of $\left(1+\frac{1}{n}\right)^n$ as n is indefinitely increased; *i.e.* the base of the Napierian system of logarithms is the quantity $e = \lim_{n=\infty}\left(1+\frac{1}{n}\right)^n$.*

168. It will now be proved that $\left(1+\frac{1}{n}\right)^n$ has a limit less than 3 as n is indefinitely increased, using for this purpose the theorem at the close of § 119. If it can be shown that as n increases $\left(1+\frac{1}{n}\right)^n$ increases, and that it can never exceed some fixed number, it will follow that the variable has a limit.

In the identity

$$(1)\qquad \frac{a^{m+1}-b^{m+1}}{a-b} = a^m + a^{m-1}b + a^{m-2}b^2 + \cdots + ab^{m-1} + b^m$$

let a be greater than b. The second member is then increased by substituting a for b, and consequently,

$$(2)\qquad \frac{a^{m+1}-b^{m+1}}{a-b} < (m+1)a^m$$

or, on multiplying both members by the positive number $a-b$,

$$(3)\qquad a^{m+1} - b^{m+1} < ma^{m+1} + a^{m+1} - ma^mb - a^mb$$

Transposing and combining,

$$(4)\qquad ma^mb + a^mb - ma^{m+1} < b^{m+1}$$

$$(5)\qquad a^m(mb + b - ma) < b^{m+1}$$

* In this section n assumes integral values only as it increases. It is, however, true that $\lim_{x=\infty}\left(1+\frac{1}{x}\right)^x$ is the same whether x assumes integral values only or changes continuously.

Since the only condition thus far imposed upon a and b is $a > b$, let $a = 1 + \frac{1}{m}$, $b = 1 + \frac{1}{m+1}$, and substitute these values in (5)

$$\left(1+\frac{1}{m}\right)^m\left(m+\frac{m}{m+1}+1+\frac{1}{m+1}-m-1\right)<\left(1+\frac{1}{m+1}\right)^{m+1}$$

This gives, when simplified,

$$(6) \qquad \left(1+\frac{1}{m}\right)^m<\left(1+\frac{1}{m+1}\right)^{m+1}$$

that is, as the exponent increases, $\left(1+\frac{1}{n}\right)^n$ increases.

In order to show that $\left(1+\frac{1}{n}\right)^n$ cannot exceed the value 3, expand $\left(1+\frac{1}{n}\right)^n$ by the binomial formula.

$$\left(1+\frac{1}{n}\right)^n=1+n\frac{1}{n}+\frac{n(n-1)}{2\,!}\frac{1}{n^2}+\frac{n(n-1)(n-2)}{3\,!}\frac{1}{n^3}+$$
$$\frac{n(n-1)(n-2)(n-3)}{4\,!}\frac{1}{n^4}+\cdots$$

$$(7) \qquad =1+1+\frac{1-\frac{1}{n}}{2\,!}+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{3\,!}+$$
$$\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)}{4\,!}+\cdots$$

This series is convergent for all values of n greater than 1, by § 136. Further, each term is positive, and is increased if 1 is substituted for each of the factors

$$1-\frac{1}{n},\ 1-\frac{2}{n},\ \cdots$$

Hence

$$\left(1+\frac{1}{n}\right)^n < 2 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

Again, since $\frac{1}{3!} < \frac{1}{2^2}, \frac{1}{4!} < \frac{1}{2^3}, \cdots$

$$\left(1+\frac{1}{n}\right)^n < 2 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$$

That is, however much n may be increased,

$$\left(1+\frac{1}{n}\right)^n < 3$$

Therefore $\left(1+\frac{1}{n}\right)^n$ has a limit which is less than 3, and it is this limit which is represented by e, and which serves as the Napierian base.

169. If in equation (7) of the previous section n is indefinitely increased, and the limit of each term is taken, the equation becomes

$$e = \lim_{n=\infty}\left(1+\frac{1}{n}\right)^n = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

It should be noted that an infinite series does not fall under the rule of § 77. It has, however, been proved that the above expression has a limit, and it is true that this limit is given, correct to any desired number of decimal places, by combining a sufficient number of terms of this series, a fact which is borne out by the discussion in § 167.

The value of e, correct to six decimal places, is readily computed as follows:

$$
\begin{array}{r|l}
 & 2.0000000 \\
3 & .5000000 = \frac{1}{2!} \\
4 & .1666667 = \frac{1}{3!} \\
5 & .0416667 = \frac{1}{4!} \\
6 & .0083333 = \frac{1}{5!} \\
7 & .0013889 = \frac{1}{6!} \\
8 & .0001984 = \frac{1}{7!} \\
9 & .0000248 = \frac{1}{8!} \\
10 & .0000028 = \frac{1}{9!} \\
 & .0000003 = \frac{1}{10!} \\
\hline
 & 2.7182819
\end{array}
$$

This is correct to six places, that is, $e = 2.718281+$.

In order to calculate logarithms to the base e what is known as the *logarithmic series* is needed. This is obtained by developing by Maclaurin's formula $\log_e(1+x)$, and for this purpose it is necessary to find the derivative of a logarithmic function.

170. Theorem. *The derivative of the Napierian logarithm of any function is the derivative of the function divided by the function.*

Let $y = \log_e z$, where z is a function of x. When x assumes successively the constant values x' and $x' + \Delta x$,

y and z assume corresponding constant values, thus giving the two equations

(1) $$y' = \log_e z'$$

(2) $$y' + \Delta y = \log_e(z' + \Delta z)$$

Hence $$\Delta y = \log_e(z' + \Delta z) - \log_e z'$$

Apply to the right hand member the formula

$\log a - \log b = \log\frac{a}{b}$

$$\Delta y = \log_e\frac{z' + \Delta z}{z'} = \log_e\left(1 + \frac{\Delta z}{z'}\right)$$

$$\frac{\Delta y}{\Delta} = \frac{1}{\Delta x}\log_e\left(1 + \frac{\Delta z}{z'}\right)$$

(3) $$= \frac{1}{z'}\frac{\Delta z}{\Delta x}\cdot\frac{z'}{\Delta z}\log_e\left(1 + \frac{\Delta z}{z'}\right)$$

Apply now the formula $c \log a = \log a^c$

(4) $$\frac{\Delta y}{\Delta x} = \frac{1}{z'}\frac{\Delta z}{\Delta x}\log_e\left(1 + \frac{\Delta z}{z'}\right)^{\frac{z'}{\Delta z}}$$

Let $\frac{z'}{\Delta z} = u$; then for all values of z except zero,

$$\lim_{\Delta x = 0} u = \lim_{\Delta x = 0}\frac{z'}{\Delta z} = \infty$$

Also $$\lim_{\Delta x = 0}\left(1 + \frac{\Delta z}{z'}\right)^{\frac{z'}{\Delta z}} = \lim_{u = \infty}\left(1 + \frac{1}{u}\right)^u = e$$

Therefore equation (4) becomes, on taking the limit of each member,

(5) $$\frac{d}{dx}\log_e z = \frac{1}{z'}\frac{d}{dx}z \log_e e = \frac{\frac{d}{dx}z}{z'}$$

Hence, in general, for any value of x which does not make z vanish,

$$\frac{d}{dx}\log_e z = \frac{\frac{d}{dx}z}{z}.$$

NOTE. In this proof the following theorem, not yet proved, has been employed: *The limit of the logarithm of a variable is the logarithm of the limit of the variable.*

171. The Logarithmic Series.

This is obtained by using Maclaurin's formula,

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \cdots$$
$$+ f^{n-1}(0)\frac{x^{n-1}}{(n-1)!} + \cdots$$

Let $f(x) = \log_e(1+x)$, form the successive derivatives, and find the corresponding functions of 0.

$$f(x) = \log_e(1+x) \qquad f(0) = \log_e 1 = 0$$

$$f'(x) = \frac{1}{1+x} = (1+x)^{-1} \text{ (§ 170)} \qquad f'(0) = 1$$

$$f''(x) = -(1+x)^{-2} \text{ (§ 103)} \qquad f''(0) = -1$$

$$f'''(x) = 2(1+x)^{-3} \qquad f'''(0) = 2$$

$$f^{\text{IV}}(x) = -3!\,(1+x)^{-4} \qquad f^{\text{IV}}(0) = -3!$$

$$\cdots\cdots\cdots \qquad \cdots\cdots\cdots$$

$$f^n(x) = (-1)^{n-1}(n-1)!\,(1+x)^{-n}$$
$$f^n(0) = (-1)^{n-1}(n-1)!$$

These functions of 0 are to be substituted in Maclaurin's formula:

$$\log_e(1+x) = x - \frac{x^2}{2!} + \frac{2\,x^3}{3!} - \frac{3!\,x^4}{4!} + \cdots$$

$$+ \frac{(-1)^{n-1}(n-1)!\,x^n}{n!} + \cdots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{n-1}x^n}{n} + \cdots$$

which is the **Napierian logarithmic series.**

NOTE. The reason for using $\log_e(1+x)$ for this expansion rather than $\log_e x$, which may seem the natural form to use, is clear if we let $f(x) = \log_e x$, and form $f'(0)$.

172. In order to determine the values of x for which the logarithmic series holds true, it must be tested for convergence.

Here $$r = \left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{n+1}}{n+1}\cdot\frac{n}{x^n}\right| = \left|\frac{n}{n+1}x\right|$$

$$l = \lim_{n=\infty}\left|\frac{n}{n+1}x\right| = |x|$$

Hence the series is convergent when $|x| < 1$, divergent when $|x| > 1$, and when $|x| = 1$, the Cauchy test fails.

If $x = 1$, the series becomes

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots$$

which is convergent by § 124.

If $x = -1$, the series becomes

$$-1 - \tfrac{1}{2} - \tfrac{1}{3} - \tfrac{1}{4} - \cdots$$

which is divergent by § 123, Theorem 1.

The series can therefore be used to find the logarithms of numbers between 0 and 2 only, and for these numbers its very slow rate of convergence makes its use undesirable.

For instance, $\log_e 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$
over one hundred terms of which must be used in order to compute $\log_e 2$ correct to two decimal places. For both these reasons the series is transformed into a new series which can readily be used for calculating the values of any logarithms.

173. In the logarithmic series substitute $-x$ for x.

(1) $$\log_e (1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$$

(2) $$\log_e (1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots$$

The difference of these two equations gives

$$\log_e \frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right)$$

by substituting for the difference of two logarithms the logarithm of the quotient of the corresponding numbers.

This equation, since it is only (1) in a changed form, is true only when x is numerically less than 1, so that, on substituting $\frac{1}{2z+1}$ for x, the resulting equation will be true for every positive value of z. This gives the series

$$\log_e \frac{1 + \frac{1}{2z+1}}{1 - \frac{1}{2z+1}} = 2\left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \cdots\right)$$

or since

$$\log_e \frac{1+\dfrac{1}{2x+1}}{1-\dfrac{1}{2x+1}} = \log_e \frac{z+1}{z} = \log_e (z+1) - \log_e z$$

$$(3) \qquad \log_e (z+1) = \log_e z + 2\left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \cdots\right),$$

which is the desired series.

By substituting 1 for z in this series $\log_e 2$ is found,

$$\log_e 2 = 2\left(\frac{1}{3} + \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} + \cdots\right)$$

and, in general, the Napierian logarithm of any number n can be found if the logarithm of $n-1$ is known.

Example 1. Find the value of $\log_e 2$. The work may be conveniently arranged as follows:

3	1.0000000	
9	$.3333333 = \frac{1}{3}$	$\frac{1}{3} = .3333333$
9	$.0370370 = \frac{1}{3^3}$	$\frac{1}{3.3^3} = .0123457$
9	$.0041152 = \frac{1}{3^5}$	$\frac{1}{5.3^5} = .0008230$
9	$.0004572 = \frac{1}{3^7}$	$\frac{1}{7.3^7} = .0000653$
9	$.0000508 = \frac{1}{3^9}$	$\frac{1}{9.3^9} = .0000056$
	$.0000056 = \frac{1}{3^{11}}$	$\frac{1}{11.3^{11}} = .0000005$
		.3465734
		2
		$\log_e 2 = .6931468$

Example 2. Find the value of $\log_e 3$.

In equation (3) of § 173 let $z = 2$.

$$\log_e 3 = \log_e 2 + 2\left(\frac{1}{5} + \frac{1}{3.5^3} + \frac{1}{5.5^5} + \cdots\right)$$

5	1.0000000	
5	$.2000000 = \frac{1}{5}$	$\frac{1}{5} = .2000000$
5	.0400000	
5	$.0080000 = \frac{1}{5^3}$	$\frac{1}{3.5^3} = .0026667$
5	.0016000	
5	$.0003200 = \frac{1}{5^5}$	$\frac{1}{5.5^5} = .0000640$
5	.0000640	
5	$.0000128 = \frac{1}{5^7}$	$\frac{1}{7.5^7} = .0000018$
5	.0000026	
	$.0000005 = \frac{1}{5^9}$	.2027325
		2
		.4054650
		$\log_e 2 = .6931468$
		$\log_e 3 = 1.0986118$

174. Series (3) of § 173 has the advantage of converging very rapidly, that is, not many terms are needed to insure a moderate degree of accuracy, and the larger the values of z that are used, the fewer terms are needed for any required degree of accuracy.

This may be illustrated by calculating $\log_e 13$, $\log_e 12$ being known.

$$\log_e 13 = \log_e 12 + 2\left(\frac{1}{25} + \frac{1}{3.25^3} + \frac{1}{5.25^5} + \cdots\right)$$

25	1.000000	
25	$.040000 = \frac{1}{25}$	$\frac{1}{25} = .040000$
25	$.001600 = \frac{1}{25^2}$	
25	$.000064 = \frac{1}{25^3}$	$\frac{1}{3.25^3} = .000021$
	$.000003 = \frac{1}{25^4}$	$.040021$
		2
		$.080042$
		$\log_e 12 = 2.484904$
		$\log_e 13 = 2.564946$

Only two terms of the series are needed to give a result correct to five decimal places.

175. The work of computation has now been carried far enough to show the possibility of making a table of logarithms for as many positive integers as may be desired, and from these the logarithms of common fractions and of roots of integers and fractions can be obtained, so that the approximate value of the logarithm of any number to the base e may now be found.

EXERCISES

1. How may the logarithm of 4 be obtained from the series in z? Without the use of the series show that

$$\log_e 4 = 1.386294$$

2. Find the Napierian logarithms of 5, 6, 7, 8, 9, 10.

3. In order to find the logarithms of all positive integers less than 25, in how many cases is it necessary to carry out computations similar to those in § 173?

4. How may the logarithms of fractions be found, *e.g.* $\frac{1}{2}$, $\frac{3}{5}$, $2\frac{1}{3}$?

5. Find the values of $\log_e \sqrt{3}$, $\log_e 1\frac{1}{2}$.

6. Find the values of $\log_e \sqrt{125}$, $\log_e \sqrt[3]{.036}$.

176. In order to find the logarithms of numbers to the base 10 the following theorem is needed:

Theorem. *The logarithms of any number in two different systems are in a fixed ratio,* i.e. *for every value of x the ratio $\frac{\log_a x}{\log_b x}$ is a constant.*

Let $\qquad \log_b c = n$

Then $\qquad b^n = c$

Since the logarithms of equal numbers are equal,

$$\log_a b^n = \log_a c$$

or $\qquad n \log_a b = \log_a c$

Substitute for n its equal, $\log_b c$

$$\log_b c \cdot \log_a b = \log_a c$$

Hence $\qquad \frac{\log_a c}{\log_b c} = \log_a b$

a result which is independent of c.

It follows that if the logarithms of numbers to any one base, a, are known, the logarithms to any other base, b, can be found readily, since from the last equation

$$\log_b c = \frac{1}{\log_a b} \log_a c$$

177. The number $\frac{1}{\log_a b}$ is called the **modulus** of the b system with respect to the a system. If the base of the a system is not specified, the Napierian base is always assumed. Thus the modulus of the common system of logarithms is $\frac{1}{\log_e 10}$. Since $\log_e 10$ is 2.302585 (page 162) the value of this modulus may be computed easily. It is ordinarily represented by M, and its value correct to seven decimal places is

$$M = .4342944$$

Since $$\log_{10} c = \frac{1}{\log_e 10} \log_e c = M \log_e c,$$

it follows that the common logarithm of any number is found by multiplying its Napierian logarithm by the value of M.

Exercise. Show that $\log_{10} 2 = .30103$, $\log_{10} 3 = .47712$.

178. It is hoped that the foregoing work will help the student to appreciate the labor that has gone into the construction of tables of logarithms. Long and toilsome as it is, it is far shorter than the work which had to be done by the early compilers of logarithmic tables, who had at their command no such means as the modern computer has in infinite series. But the years spent by a few men in this dreary work seem worth while in comparison with the saving of time thus effected for thousands of computers.

179. The results obtained by these and other methods of computing logarithms are arranged in various forms, the effort in every case being to make it as simple and rapid a process as possible to find the logarithm of a given number, or the number corresponding to a given logarithm. Special devices are introduced to facilitate the necessary work, and modern tables are extremely easy to use after a little practice.

As full instructions for using the tables are ordinarily included in each book, only a few general principles are given here.

In all practical work common logarithms are used, and consequently it is usual to omit the subscript which indicates the base. Thus $\log 6$ means $\log_{10} 6$.

Each logarithm consists of two parts, the integral part, which is called the **characteristic**, and the decimal part, which is called the **mantissa**.

180. The following example serves to illustrate three theorems now to be proved:

Given $\log 423.86 = 2.62722$

then $$\log 42.386 = \log \frac{423.86}{10} = \log 423.86 - \log 10$$
$$= 1.62722$$

Also $$\log 423860 = \log (423.86 \times 1000) = \log 423.86 + \log 1000 = 5.62722$$

Similarly,
$$\log 4.2386 = 0.62722$$

If the number is less than 1, its logarithm may be written in two different ways. Thus

(1) $$\log .42386 = \log \frac{4.2386}{10} = .62722 - 1$$

This may also be written

(2) $$\log .42386 = - .37278$$

The better plan is to keep form (1), but to write the integral part as the characteristic, with the minus sign above it to show that it alone is negative.

$$\log .42386 = \bar{1}.62722$$

Similarly,
$$\log .042386 = \bar{2}.62722$$
$$\log .0042386 = \bar{3}.62722$$

If this method is followed, mantissas are always positive numbers, and are combined according to the laws of arithmetic, while characteristics may be either positive or negative.

181. From the illustrations just given the following theorems are readily deduced:

Theorem 1. *If two numbers differ only in the position of the decimal point, the mantissas of their logarithms are the same.*

Theorem 2. *If a number is greater than unity, its logarithm has a characteristic less by* 1 *than the number of digits in the integral part of the number.*

For, if a number lies between 1 and 10, its logarithm lies between 0 and 1; if between 10 and 100, its logarithm lies between 1 and 2; and, in general, if a number lies between 10^n and 10^{n+1}, its logarithm lies between n and $n+1$.

Theorem 3. *If a number is less than unity, its logarithm has a negative characteristic, which is numerically greater by* 1 *than the number of zeros immediately following the decimal point.*

For, $\log .1 = \log 10^{-1}$, $\log .01 = \log 10^{-2}$, etc. Hence if a number lies between 1 and .1, the characteristic of its logarithm is -1; if between .1 and .01, the characteristic of its logarithm is -2; and, in general, if a number lies between 10^{-n} and $10^{-(n+1)}$, the characteristic of its logarithm is $-(n+1)$.

182. Some questions which arise in using negative characteristics are best answered by working a few examples.

Example 1. Raise to the seventh power .3159

$$\log .3159^7 = 7 \log .3159$$
$$\log .3159 = \bar{1}.49955$$
$$\underline{\qquad\qquad 7}$$
$$\log .3159^7 = \bar{4}.49685$$

Therefore $.3159^7 = .0003139$

Example 2. Extract the cube root of .3

$$\log \sqrt[3]{.3} = \tfrac{1}{3} \log .3$$
$$\log .3 = \bar{1}.47712$$
$$\tfrac{1}{3} \log .3 = \tfrac{1}{3}(\bar{1}.47712)$$
$$= \tfrac{1}{3}(-3 + 2.47712)$$
$$= \bar{1}.82571$$

Therefore $\sqrt[3]{.3} = .66943$

EXERCISES

1. If $\log 85 = 1.92942$, find $\log \sqrt{85}$, $\log \sqrt[3]{.085}$, $\log .0085^{100}$.

2. If $\log 2 = .30103$, and $\log 3 = .47712$, how many digits are there in 12^{10}?

3. Which is greater, 2^{35} or 3^{25}?

4. To what power must 6 be raised in order that the result may be greater than 1000000?

5. What is the number of integral places in $\sqrt[3]{50^{25}}$?

6. How many zeros are between the decimal point and the first significant figure of $(\frac{1}{3})^{1000}$?

7. By multiplying the Napierian logarithmic series (§ 171) by M obtain a series for $\log_{10}(1 + x)$.

8. By multiplying the series for $\log_e(z + 1)$ (§ 173) by M obtain a series for computing common logarithms.

9. Find the modulus of the system of logarithms which has 5 as its base.

10. What is the largest number that can be written with the use of three digits only? *

11. Draw the graph of $y = \log x$.

* For an interesting discussion of this number see *School Science and Mathematics* for 1914, page 451.

CHAPTER XI

COMPLEX NUMBERS

The imaginary calculus is one of the master keys to physical science. Those realms of the inconceivable afford in many places our only mode of passage to the domains of positive knowledge.

—THOMAS HILL.

183. A complex number is a quantity of the form $a+bi$, in which a and b are real numbers, rational or irrational, and i represents the positive square root of -1, *i.e.* $i^2=-1$.

The system of complex numbers includes both real numbers (when $b=0$) and pure imaginary numbers (when $a=0$). Real numbers and imaginary numbers have but one number in common, viz. zero. If then $a+bi=0$, it follows that $a=0$ and $b=0$.

184. Complex numbers belong to a new class, and in introducing any such numbers it is possible to define at pleasure the operations that may be carried out on them. It is, however, essential for convenience in using new numbers or functions that they shall conform to general laws already established; and without going into detail in regard to the laws governing operations on complex numbers, it will be assumed that they are combined precisely as in the case of real numbers. For example

$$(a+bk)(c+dk)=ac+(ad+bc)k+bdk^2$$

Similarly $$(a+bi)(c+di)=ac+(ad+bc)i+bdi^2$$

$$=ac-bd+(ad+bc)i$$

With this assumption the following theorems may then be proved.

185. Theorem. *The algebraic sum of two complex numbers is a complex number.*

Let the two numbers be $a+bi$ and $c+di$, and combine them according to the ordinary laws of addition. Their sum is $a+c+(b+d)i$, a complex number.

186. Theorem. *The product of two complex numbers is a complex number.*

Let the two numbers be $a+bi$ and $c+di$, and combine them according to the ordinary laws of multiplication. The product is $ac-bd+(ad+bc)i$, a complex number.

Corollary. *Any power of a complex number is a complex number.*

187. Theorem. *The quotient of two complex numbers is a complex number.*

Let the quotient be represented by $\frac{a+bi}{c+di}$, and multiply both numerator and denominator by $c-di$. The result is $\frac{ac+bd}{c^2+d^2}+\frac{bc-ad}{c^2+d^2}i$, a complex number.

Two complex numbers are said to be **conjugate** when they differ only in the sign of the term containing i; thus $a+bi$ and $a-bi$ are conjugate.

188. Theorem. *If the sum and the product of two complex numbers are real, the numbers are either real or conjugate.*

Let the numbers be $a+bi$ and $c+di$.

Then by hypothesis

$$a + c + (b + d)i \text{ is real} \quad \therefore\ b + d = 0 \qquad (1)$$

and $$ac - bd + (ad + bc)i \text{ is real} \quad \therefore\ ad + bc = 0 \qquad (2)$$

From (1) $b = -d$, which, substituted in (2), gives $ad - cd = 0$. Hence either $d = 0$, in which case $b = 0$, and both numbers are real; or $a = c$, in which case the numbers are conjugate.

189. The **modulus** of a complex number is the positive square root of the sum of the squares of its real term and the coefficient of i. Instead of the word modulus **absolute value** is often used. Thus the modulus of $a + bi$ is $\sqrt{a^2 + b^2}$, of $-3 + 6i$ is $\sqrt{45}$.

What is the modulus of $a - bi$, $2 - i\sqrt{3}$, $3i$, -4, $1 + i$? If the number is real, has the term *absolute value* here the same meaning as in § 10?

190. Theorem. *The modulus of the sum of two numbers is equal to or less than the sum of their moduli.*

Let the two numbers be $a + bi$ and $c + di$. It is to be proved that

$$\sqrt{(a + c)^2 + (b + d)^2} \overline{\gtrless} \sqrt{a^2 + b^2} + \sqrt{c^2 + d^2} \qquad (1)$$

By algebraic processes this expression may be transformed as follows:

$$a^2 + 2ac + c^2 + b^2 + 2bd + d^2$$
$$\overline{\gtrless}\ a^2 + b^2 + 2\sqrt{(a^2 + b^2)(c^2 + d^2)} + c^2 + d^2$$
$$ac + bd \overline{\gtrless} \sqrt{(a^2 + b^2)(c^2 + d^2)}$$
$$a^2c^2 + 2abcd + b^2d^2 \overline{\gtrless} a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2$$
$$2abcd \overline{\gtrless} a^2d^2 + b^2c^2$$
$$0 \overline{\gtrless} (ad - bc)^2$$

Since this last statement is evidently true, and (1) may be derived from it, it follows that (1) is true.

191. Theorem. *The modulus of the product of two complex numbers is equal to the product of their moduli.*

Let the two numbers be $a+bi$ and $c+di$. To prove that the modulus of $ac-bd+(ad+bc)i$ is equal to the product of the moduli of $a+bi$ and $c+di$.

$$\sqrt{(ac-bd)^2+(ad+bc)^2}=\sqrt{a^2c^2+b^2d^2+a^2d^2+b^2c^2}$$
$$=\sqrt{(a^2+b^2)(c^2+d^2)}=\sqrt{a^2+b^2}\sqrt{c^2+d^2}$$

192. Theorem. *The modulus of the quotient of two complex numbers is equal to the quotient of their moduli.*

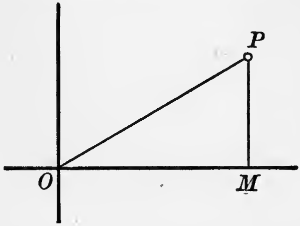
Let the two numbers be $a+bi$ and $c+di$. To prove that the modulus of $\frac{ac+bd}{c^2+d^2}+\frac{bc-ad}{c^2+d^2}i$ is the quotient of the moduli of $a+bi$ and $c+di$

$$\sqrt{\left(\frac{ac+bd}{c^2+d^2}\right)^2+\left(\frac{bc-ad}{c^2+d^2}\right)^2}=\sqrt{\frac{(a^2+b^2)(c^2+d^2)}{(c^2+d^2)^2}}=\frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}}$$

193. The fact that real numbers can be represented by points on a straight line has often proved of service. For many years no corresponding means of representing complex numbers was known, and this was one reason why they were called imaginary or non-existent. Early in the nineteenth century a method of representing them was published * which has come into general use among mathematicians and physicists, and has helped to make these numbers seem no more imaginary than the negative or irrational numbers. According to this method, a complex number, $a+bi$, is represented by that point in the plane which has as rectangular coördinates a and b.

* Jean Robert Argand (1768–1822) published in 1806 a pamphlet containing geometrical representations of complex numbers, and giving rules for their geometrical addition and subtraction.

There is then established a *one-to-one-relationship* between the points of the plane and all complex numbers, that is, to every point corresponds a complex number, and to every complex number a point.



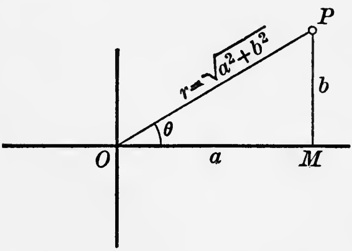
Given the point P, corresponding to the number $a + bi$, where is the point corresponding to

$$a - bi, \quad -a + bi, \quad -a - bi?$$

194. If the point P corresponds to the number $a+bi$, the distance $OP = \sqrt{a^2+b^2}$, *i.e.* OP represents the modulus of the number. The modulus or absolute value of a number is therefore represented by the distance of the corresponding point from the origin whether the number be real or complex.

The angle MOP is called the **amplitude** or **argument** of the complex number. The position of the point, and therefore the value of the number, is fixed if its modulus and amplitude are known.

195. Let the point P represent the number $a + bi$ and



let $\angle MOP = \theta$,

and $\sqrt{a^2+b^2} = r$.

Then $\sin\theta = \dfrac{b}{\sqrt{a^2+b^2}}$,

$$\cos\theta = \frac{a}{\sqrt{a^2+b^2}},$$

and since

$$a + bi = \sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}} + \frac{bi}{\sqrt{a^2+b^2}}\right),$$

another very useful *form for a complex number* is $r(\cos\theta + i\sin\theta)$. The number is then said to be written **in polar coördinates.**

Express in terms of polar coördinates $4 + 3i$, $2 - 3i$, $2i$, 4.

196. Theorem. *If two complex numbers are equal, their moduli are equal, and their amplitudes are equal or differ by a multiple of* 2π.

For, if $r(\cos\theta + i\sin\theta) = r'(\cos\theta' + i\sin\theta')$ (1)

then $r\cos\theta = r'\cos\theta'$, and $r\sin\theta = r'\sin\theta'$ (2)

The sum of the squares of equations (2) gives

$$r^2(\sin^2\theta + \cos^2\theta) = r'^2(\sin^2\theta' + \cos^2\theta')$$

whence

$$r = r'$$

But if $r = r'$, equations (2) can be true only when θ and θ' differ by a multiple of 2π.

197. Theorem. *The modulus of the product of two complex numbers is the product of the two moduli, and the amplitude of the product is the sum of the two amplitudes.*

Let the two numbers be written in polar coördinates,

$$c_1 = r_1(\cos\theta_1 + i\sin\theta_1),\ c_2 = r_2(\cos\theta_2 + i\sin\theta_2).$$

Then

$$\begin{aligned} c_1c_2 &= r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1r_2[\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 \\ &\qquad + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \\ &= r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \end{aligned}$$

Corollary. *The modulus of the nth power of a complex number is found by raising the modulus of the number to the nth power, the amplitude by multiplying its amplitude by n.*

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

198. Theorem. *The modulus of the quotient of two complex numbers is the quotient of their moduli, and its amplitude is the difference of their amplitudes.*

Choose the numbers as in the previous section.

Then $$\frac{c_1}{c_2}=\frac{r_1}{r_2}\frac{\cos\theta_1+i\sin\theta_1}{\cos\theta_2+i\sin\theta_2}\times\frac{\cos\theta_2-i\sin\theta_2}{\cos\theta_2-i\sin\theta_2}=$$

$$\frac{r_1}{r_2}\frac{\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2+i(\sin\theta_1\cos\theta_2-\cos\theta_1\sin\theta_2)}{\cos^2\theta_2+\sin^2\theta_2}$$

$$=\frac{r_1}{r_2}[\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)].$$

199. The formula given in § 197, Corollary, is a statement of what is known as **de Moivre's* theorem,** which is there established for positive integral values of n. It will now be shown to hold when the exponent is a negative integer, that is,

$$[r(\cos\theta+i\sin\theta)]^{-n}=r^{-n}[\cos(-n\theta)+i\sin(-n\theta)]$$

$$[r(\cos\theta+i\sin\theta)]^{-n}=\frac{1}{r^n(\cos n\theta+i\sin n\theta)}$$

$$=\frac{\cos n\theta-i\sin n\theta}{r^n(\cos^2 n\theta+\sin^2 n\theta)}$$

$$=r^{-n}[\cos(-n\theta)+i\sin(-n\theta)]$$

200. De Moivre's theorem will now be proved to hold for any root of a complex number, that is,

$$[r(\cos\theta+i\sin\theta)]^{\frac{1}{n}}=r^{\frac{1}{n}}\left(\cos\frac{\theta}{n}+i\sin\frac{\theta}{n}\right)$$

Assume that the nth root of the complex number $r(\cos\theta+i\sin\theta)$ is of the form $r'(\cos\theta'+i\sin\theta')$; then

$$r'(\cos\theta'+i\sin\theta')=[r(\cos\theta+i\sin\theta)]^{\frac{1}{n}}$$

or $$r'^n(\cos n\theta'+i\sin n\theta')=r(\cos\theta+i\sin\theta)$$

* Abraham de Moivre (1667–1754) is noted chiefly for his work in complex numbers and in probabilities.

It follows from § 196 that

$$r'^n = r, \text{ and } n\theta' = \theta + 2k\pi, \quad k = 0, 1, 2, \ldots$$

Hence $$r' = r^{\frac{1}{n}} \text{ and } \theta' = \frac{\theta + 2k\pi}{n}$$

For any given value of r the value of r' can always be found either exactly, or, by the aid of logarithms, approximately; θ' has an infinity of values, corresponding to the values of k. Denote these by $\theta'_1, \theta'_2, \theta'_3 \cdots$.

Then $$\theta'_1 = \frac{\theta}{n}, \quad \theta'_2 = \frac{\theta + 2\pi}{n}, \quad \theta'_3 = \frac{\theta + 4\pi}{n} \cdots$$

For each value of θ', $r'(\cos\theta' + i\sin\theta')$ has a different value until k reaches the value n, when $\theta'_{n+1} = \frac{\theta}{n} + 2\pi$, $\sin\theta'_{n+1} = \sin\theta'_1$, $\cos\theta'_{n+1} = \cos\theta'_1$. Hence there are n and only n distinct nth roots of a complex number.

Example. Find the fifth roots of 2.

$2 = 2(\cos 0° + i\sin 0°) = 2(\cos 360° + i\sin 360°)$
$= 2(\cos 720° + i\sin 720°) = 2(\cos 1080° + i\sin 1080°)$
$= 2(\cos 1440° + i\sin 1440°)$.

Hence the fifth roots of 2 are

$2^{\frac{1}{5}}$, $2^{\frac{1}{5}}(\cos 72° + i\sin 72°)$, $2^{\frac{1}{5}}(\cos 144° + i\sin 144°)$,
$2^{\frac{1}{5}}(\cos 216° + i\sin 216°)$, $2^{\frac{1}{5}}(\cos 288° + i\sin 288°)$.

The numerical values of these functions may be found in a table of natural functions.

It is left for the student to show that de Moivre's theorem holds for any positive or negative fraction, and hence for every rational value of n.

201. Binomial equations, that is, equations of the form $x^n \pm a = 0$ (n a positive integer) are readily solved by the use of § 200.

Example. Solve the equation $x^3 - 8 = 0$.

$$x = \sqrt[3]{8} = \sqrt[3]{8(\cos 2\,k\pi + i \sin 2\,k\pi} \qquad k = 0,\ 1,\ 2.$$

This gives three distinct values of x,

$$x = 2,\ 2\left(\cos\frac{2\,\pi}{3} + i\sin\frac{2\,\pi}{3}\right),\ 2\left(\cos\frac{4\,\pi}{3} + i\sin\frac{4\,\pi}{3}\right),$$

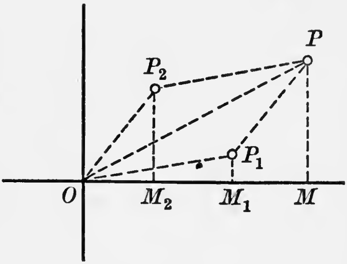
or, on substituting the numerical values of these functions,

$$x = 2,\ -\sqrt{3} + i,\ -\sqrt{3} - i$$

The one real root is the one which is ordinarily known as the cube root of 8.

In the following sections the geometric significance of operations on complex numbers will be considered.

202. Addition. Let the two numbers $a_1 + b_1 i$, $a_2 + b_2 i$ be represented by the points P_1 and P_2. It is required to find the point P corresponding to this sum. Draw P_1P parallel and equal to OP_2, and join P_2P. Then $OM = OM_1 + OM_2 = a_1 + a_2$, and $PM = P_1M_1 + P_2M_2 = b_1 + b_2$.



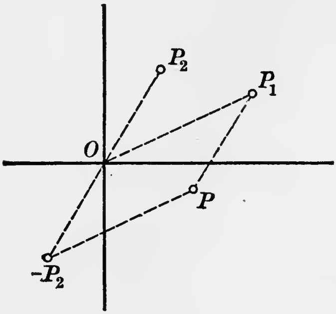
Hence the point P represents the number $a_1 + a_2 + (b_1 + b_2)i$, which is the sum of the given numbers, and the following rule is established.

To find the point representing the sum of two numbers which correspond to two given points, complete the parallelogram three of whose vertices are the two given points and the origin. The vertex opposite the origin is the required point.

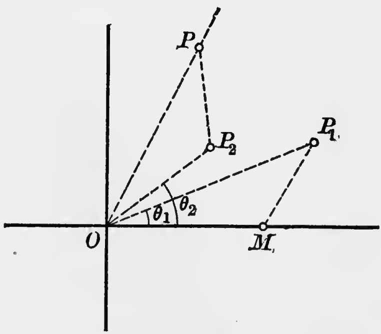
N

The figure shows clearly that the modulus of the sum of two numbers is never greater than the sum of their moduli.

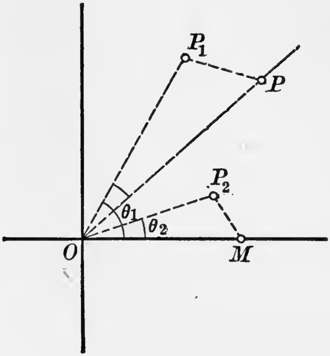
203. Subtraction. This is most readily handled by changing the sign of the number to be subtracted, and adding it to the other number. Thus in the figure, $-P_2$ is added to P_1 by the method just developed, $P = P_1 - P_2$. Note that the modulus of the difference is represented by the actual distance between the two points.



204. Multiplication. It is best to use polar coördinates. Let P_1 and P_2 represent the two numbers $r_1(\cos\theta_1 + i\sin\theta_1)$ and $r_2(\cos\theta_2 + i\sin\theta_2)$. The point which represents their product has as modulus r_1r_2, as amplitude $\theta_1 + \theta_2$, § 197. Draw the line OP making the angle MOP equal to $\theta_1 + \theta_2$, and let OM be the unit of measure. Join MP_1, and from P_2 draw a line making the angle OP_2P equal to OMP_1. In the similar triangles OP_1M and OPP_2, $r : r_2 = r_1 : 1$, hence $r = r_1r_2$ and P is the required point.

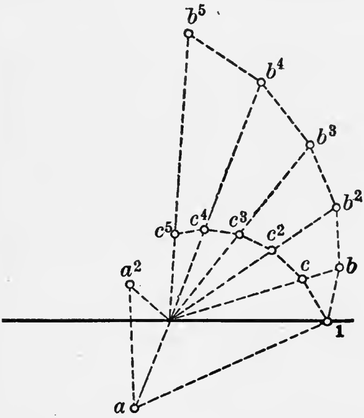


205. Division. As in § 204 use polar coördinates. Construct the angle P_1OP equal to θ_2, and from P_1 draw a line making the angle OP_1P equal to OP_2M. Then $r:1=r_1:r_2$, hence $r=\frac{r_1}{r_2}$, and P is the required point.



206. Involution. This is an extension of multiplication. Given the point P corresponding to

$$c=r(\cos\theta+i\sin\theta).$$

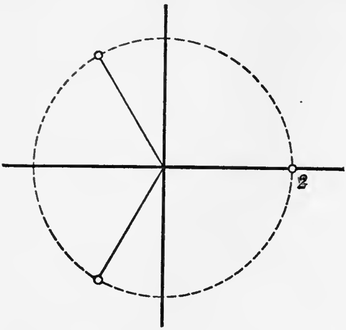


To find P' corresponding to

$$c^n=[r(\cos\theta+i\sin\theta)]^n$$

the amplitude is to be multiplied by n, and the successive powers of r constructed. The accompanying figure illustrates the square of a, a number with negative amplitude, and successive powers of b and c, numbers with the same amplitude, and with moduli respectively greater and less than unity.

207. Evolution. Given the point P corresponding to $c=r(\cos\theta+i\sin\theta)$. To find P' corresponding to $\sqrt[n]{c}$, the amplitude of P is to be divided into n equal parts,



and the nth root of its modulus is to be taken. The figure illustrates the three cube roots of 8. The operations may in any case be carried out approximately, using protractor and logarithmic tables. The nth roots of any number will lie on a circle about the origin, and divide its circumference into equal arcs.

EXERCISES

Locate the points corresponding to the following numbers:

1. $(2+3i)(-1+i)$, $\dfrac{1+2i}{3-i}$.

2. $1+i$, $(1+i)^2$, $(1+i)^3$, $(1+i)^4$.

3. i, i^2, i^3, i^4.

4. $\sqrt[4]{i}$, $\sqrt[3]{1+8i}$, $\sqrt[4]{1}$, $\sqrt[4]{-1}$, $\sqrt[6]{4}$, $\sqrt[8]{-4}$.

5. If a number be multiplied by i, how is the position of the corresponding point changed?

6. Represent graphically the roots of the equations $x^2-6x+13=0$, $x^4-1=0$, $x^6-1=0$.

7. Represent graphically the roots of the equation $x^4+4x^2+13=0$, and locate the point which corresponds to the sum of the roots.

8. Represent graphically any number and its reciprocal.

9. If two numbers are conjugate, what is the position of the corresponding points?

10. Represent graphically the roots of the equation $x^8 - 1 = 0$, and show that the sum of the roots is zero, and their product -1.

208. The foundation of all work in numbers is the positive integer.* Beginning with positive integers alone, it has been found necessary to introduce other numbers in order that the six algebraic operations may always be possible. Every inverse operation was found to require the introduction of one or more new kinds of numbers, subtraction leading to negative numbers, division to fractional numbers, evolution to irrational and complex numbers. With the introduction of complex numbers what is called the **Number System of Algebra** is now complete, and it is sufficient for the solving of all algebraic equations. Furthermore, all mathematical operations that may be met with later may be reduced to a combination of the six fundamental operations, repeated a finite or an infinite number of times, and in either case this system of numbers is adequate.

209. The preceding sections give the most important elementary facts in regard to complex numbers and their combinations. A few sections are added to show the possibilities of further work with these new numbers, and to indicate some of the many lines along which such work has developed.

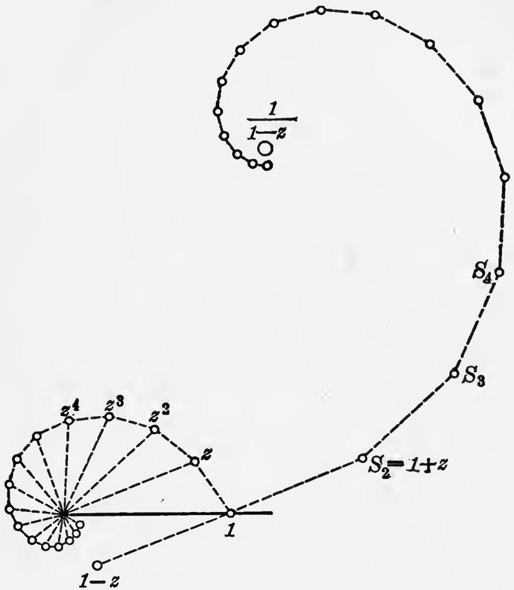
In the functions thus far studied the variable x has assumed real values only. If a variable in accordance with its law of change can assume a set of complex

* The student may at this point appreciate the statement of Kronecker, "Die ganzen Zahlen hat Gott gemacht, alles andere ist Menschenwerk."

values, it is called a **complex variable.** In order to indicate the complex character of such a variable it is ordinarily written in the form $x + iy$, though for convenience the single letter z is often used to represent this form. Thus $z = x + iy$ is a complex variable, and if x and y assume all real values, z assumes all complex values. Geometrically this means that if x and y traverse all points on the coördinate axes, z traverses every point in the plane, which is therefore often called "**the plane of complex numbers.**"

A little consideration shows that there is a marked difference between the graphical representations of functions of real and of complex variables, in that a plane is sufficient for the former, while in the latter case z alone requires the entire plane. Take for example the equation $y = x^3$. As x moves along the x-axis there is always one corresponding value of y, and these corresponding values of x and y determine a point in the plane. But in the equation $w = z^3$, z may move over the entire plane; for each value there is a corresponding complex value of w, which also requires two dimensions for its representation. The difficulty is ordinarily met by using two planes, in one of which z moves, while w traces the corresponding path in the other plane.

210. In any of the functions already studied in connection with real numbers complex numbers may be used, and the properties of the new functions may be studied. Thus series may be written with complex coefficients, or containing a complex variable, and the conditions of convergency may be investigated. The graphical representation of such a series illustrates the fact that the complex variable often leads to more interesting results than the real variable.

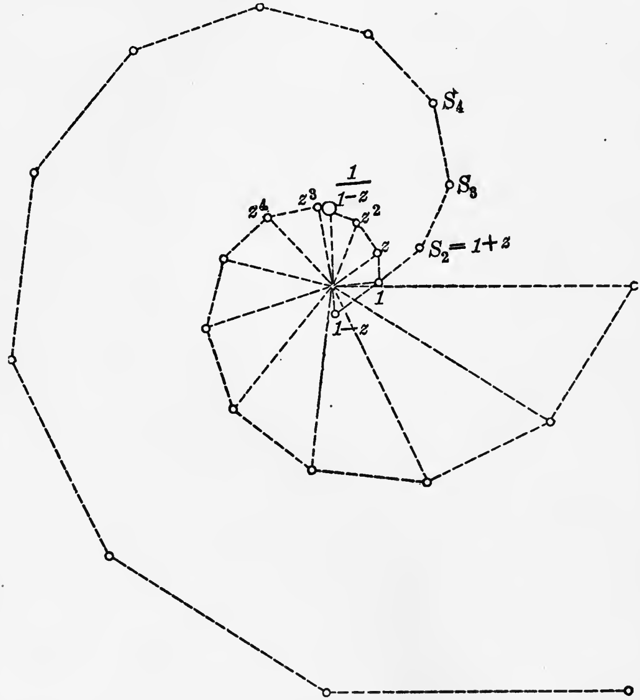


For example, let it be required to locate the points S_n which represent the sum to n terms of the series

$$1 + z + z^2 + z^3 + \cdots$$

Choose any point z with modulus less than 1, and locate the points z^2, $z^3 \cdots$. The points $1 + z$, $1 + z + z^2$, $1 + z + z^2 + z^3$, $\cdots$ are then to be located, and, by continuing this process far enough, the convergence of the series is clearly indicated, as the broken line tends to approach a fixed point. It is known that if the variable is real, and less in absolute value than unity, this series represents the fraction $\frac{1}{1-z}$. The point corresponding to this fraction is readily located, and is evidently the point toward which the series converges.

If on the other hand z be chosen with modulus greater than unity, the broken line will move ever farther from the origin, and from the point corresponding to $\frac{1}{1-z}$, illustrating the divergence of the series for such values of the variable.



211. Questions may well arise at this point as to the meaning of such expressions as $\sin z$, $\tan z$, e^z, $\log z$, where z is complex. These questions cannot be answered here, but it may interest the student to know that they have been answered, and that the answers lead the way to new and wide fields of mathematical knowledge.

For further work in this subject see Chrystal: Algebra, Fine: The Number System of Algebra, Pierpont: Functions of a Complex Variable.

CHAPTER XII

THEORY OF EQUATIONS

They that are ignorant of algebra cannot imagine the wonders in this kind that are to be done by it, and what further improvements and helps advantageous to other parts of knowledge the sagacious mind of man may yet find out, it is not easy to determine. — JOHN LOCKE.

212. Higher equations. In elementary algebra general methods are explained for solving equations of the first and second degrees containing one unknown quantity. Any equation of the first degree may be put in the form

$$ax + b = 0, \text{ whose root is } -\frac{b}{a}$$

Any equation of the second degree may be put in the form

$$ax^2 + bx + c = 0, \text{ whose roots are } \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Substitution of the coefficients of any given equation of first or second degree, in the corresponding formula, gives immediately a solution of the equation in terms of its coefficients.

The object of this chapter is the study of the equation*

(1) $$f(x) \equiv a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$$

where the coefficients are integers, and the exponents

* Equation (1) is not the most general form of an algebraic equation; but any algebraic equation may be put in this form, by a finite number of operations for clearing of fractions and removing radicals.

are positive integers. Equation (1) may evidently be put in the form

$$(2) \qquad x^n + b_1x^{n-1} + \cdots + b_{n-1}x + b_n = 0$$

where the coefficients are rational, but not necessarily integers.

Many attempts have been made to find general formulas which express the roots of such equations. It is now known that it is impossible to represent the roots of a general equation, whose degree is above the fourth, in terms of the coefficients, combined by ordinary algebraic operations. General solutions of the equations of the third and fourth degrees have been found; but, however perfect they may be in theory, great difficulties arise when they are applied to certain equations. It thus appears that the general solution of equations, whose degree is above the second, is frequently attended with difficulty, if it is not entirely impossible by elementary methods.

If, however, the inquiries are limited to equations of forms (1) and (2) having numerical coefficients, and to the *real* roots of such equations, it will be found that, whatever the degree of $f(x)$, it is always possible to determine the *real* roots; *exactly* if they are rational; or, if they are irrational, to find an approximate value, which is correct to any required number of decimal places.

The problem of this chapter will then be limited to the determination, either exact or approximate, of the *real* roots of *numerical* equations of any degree. Imaginary roots will be found only when the form of the equation is such that it can be solved by the quadratic formula.

213. Remainder theorem. *If $f(x)$ is divided by $x-a$, the remainder is $f(a)$.*

For, letting $q(x)$ represent the quotient obtained by dividing $f(x)$ by $x-a$, and letting r represent the remainder, principles of division give the identity

(1) $$f(x) \equiv q(x)(x-a)+r$$

Since the remainder must be of lower degree than the divisor, r does not contain x. Substituting a for x in (1),

(2) $$f(a) \equiv q(a)(a-a)+r \equiv r$$

Corollary. *If a is a root of the equation $f(x)=0$, then $f(x)$ is divisible by $x-a$. Conversely, if $f(x)$ is divisible by $x-a$, then a is a root of the equation.*

For, if a is a root of the equation, $f(a) \equiv 0$, and from (2) $r=0$. Then it follows from (1), that $f(x)$ is divisible by $x-a$. Conversely, if $f(x)$ is divisible by $x-a$, $r=0$. Hence from (2), $f(a)=0$, and a is a root of the equation.

From this corollary, it is evident that when one or more roots of an equation are known, the equation may be simplified by dividing $f(x)$ by the factors corresponding to the known roots. Thus the degree of the equation may be reduced, and the problem of finding the remaining roots rendered much easier. This is called **depressing** the degree of the equation, and should always be done whenever possible.

Example. The equation $x^4-4x^3-x+4=0$ has the roots 1 and 4. Find the other roots.

By the corollary $(x-1)(x-4)=x^2-5x+4$ is a factor of x^4-4x^3-x+4. Therefore

$$x^4-4x^3-x+4=(x-1)(x-4)(x^2+x+1)$$

The depressed equation $x^2+x+1=0$ may be solved by the quadratic formula, and its roots $\frac{-1\pm\sqrt{-3}}{2}$ will be the remaining roots of the original equation.

214. Number of roots. It will be assumed that every algebraic equation has at least one root. This is already known for equations of the first and second degrees; but the proof for the general case is too complicated to be given here.

Theorem. *Every equation of degree n has n and only n roots.*

Let the equation be

$$f(x)\equiv x^n+b_1x^{n-1}+b_2x^{n-2}+\cdots+b_{n-1}x+b_n=0$$

and let r_1 be the root whose existence is assumed. Then by § 213, Corollary, $f(x)$ is divisible by $x-r_1$,

$$f(x)\equiv(x-r_1)q_1(x),$$

where $q_1(x)$ is an algebraic function of degree $n-1$.

By the preceding assumption $q_1(x)=0$ must have a root, r_2. Hence $q_1(x)$ is divisible by $x-r_2$,

$$f(x)=(x-r_1)(x-r_2)q_2(x),$$

where $q_2(x)$ is of degree $n-2$. It is evident that this process may be performed $n-1$ times, and since $q_{n-1}(x)$ will be of first degree, it can be continued no further. Hence

$$f(x)\equiv(x-r_1)(x-r_2)\cdots(x-r_n)$$

Evidently the substitution of any of the quantities $r_1\cdots r_n$ for x will cause $f(x)$ to vanish, and it will not vanish by substituting any other quantity. Hence $f(x)=0$ has n and only n roots. These roots, however, may not all be real; and it may happen that some of them are equal.

It appears from the foregoing that an equation may be written having any given quantities as roots.

Example. Write the equation whose roots are 1, 3, -2.

By the preceding theorem $f(x)$ must be of third degree, and by the preceding section, its factors must be $x-1$, $x-3$, $x+2$. Hence the required equation is

$$(x-1)(x-3)(x+2)=x^3-2x^2-5x+6=0$$

215. Imaginary roots. In determining how many of the n roots of an equation are real, the following theorem concerning imaginary roots is often useful:

Theorem. *Imaginary roots occur in conjugate pairs; that is, if $a+b\sqrt{-1}$ is a root of an equation, $a-b\sqrt{-1}$ is also a root.*

For, if $a+b\sqrt{-1}$ is a root of $f(x)=0$, $x-(a+b\sqrt{-1})$ must be a factor of $f(x)$ by § 213, Corollary. Then if it can be shown that $x-(a-b\sqrt{-1})$ is also a factor of $f(x)$, by the same corollary $a-b\sqrt{-1}$ must be a root of the equation. If $f(x)$ is divided by

$$[x-(a+b\sqrt{-1})][x-(a-b\sqrt{-1})]=(x-a)^2+b^2,$$

principles of division give the identity

$$(1) \qquad f(x)\equiv[(x-a)^2+b^2]q(x)+R$$

Since R must be of lower degree than the divisor, its most general form is R_1x+R_2 which gives (1) the form

$$(2) \qquad f(x)\equiv[(x-a)^2+b^2]q(x)+R_1x+R_2$$

Since this is an identity it is true for any value of x. In place of x it is convenient to substitute $a+b\sqrt{-1}$, a quantity which by hypothesis makes the left-hand side vanish. Then (2) becomes

$$(3)\quad 0=[(a+b\sqrt{-1}-a)^2+b^2]q(a+b\sqrt{-1}) + R_1a+R_1b\sqrt{-1}+R_2$$

Since the quantity in brackets is zero, (3) becomes

$$(4)\qquad R_1a+R_1b\sqrt{-1}+R_2=0$$

By § 183, the real and the imaginary parts must vanish separately. Hence $R_1b\sqrt{-1}=0$, and since $b\neq 0$, $R_1=0$. Substitution of this value in (4) gives $R_2=0$. Since the remainder in (1) is zero, $f(x)$ is divisible by $(x-a)^2+b^2$, and therefore by its factor $x-(a-b\sqrt{-1})$. Therefore $a-b\sqrt{-1}$ is a root of $f(x)=0$.

Example. Write an equation, two of whose roots are $1-\sqrt{-3}$ and 4. Since $1-\sqrt{-3}$ is a root, $1+\sqrt{-3}$ is also a root, and an equation of the third degree satisfies the requirements.

$$[x-(1-\sqrt{-3})][x-(1+\sqrt{-3})][x-4] = x^3-6x^2+12x-16=0$$

EXERCISES

1. Prove that if $a+\sqrt{b}$ is a root of $f(x)=0$, then $a-\sqrt{b}$ is also a root.

2. One root of $x^3-4x^2+7x-6=0$ is $1+\sqrt{-2}$. Find the other roots.

3. Can an equation of odd degree have all of its roots imaginary? An equation of even degree?

4. One root of $x^4+2x^3-7x^2-14x-6=0$ is $1+\sqrt{3}$. Find the other roots.

5. Write the equations of lowest degree with rational coefficients which have the following roots:

(*a*) $1-\sqrt{2}, 3$; (*b*) $2, 1, -3$; (*c*) $2-\sqrt{-3}, 1+\sqrt{-2}$;
(*d*) $2-\sqrt{-1}, 1+\sqrt{3}$; (*e*) $-2, 1, 1+\sqrt{-3}$.

216. Relations between the coefficients and roots of an equation. In § 214 it was seen that

$$(1)\qquad f(x) \equiv x^n + b_1x^{n-1} + b_2x^{n-2} + \cdots + b_{n-1}x + b_n$$
$$\equiv (x - r_1)(x - r_2) \cdots (x - r_n)$$

where $r_1 \cdots r_n$ are the roots of $f(x) = 0$. It is evident that by equating coefficients of like powers of x in the identity (1), the following relations between the roots and coefficients of an equation may be deduced.

Theorem. *When the coefficient of the highest power of x is unity, the coefficient of the next lower power is the sum of all the roots with the sign changed.*

$$b_1 = -(r_1 + r_2 + \cdots + r_n)$$

The known term is the product of all the roots, with the sign changed or not, according as the degree of the equation is odd or even.

$$b_n = (-1)^n r_1 r_2 \cdots r_n$$

In general, the coefficient of x^{n-p} is the sum of all the products that can be formed by taking p roots at a time with the sign changed or not, according as p is odd or even.

When $n = 3$,

$$f(x) \equiv (x - r_1)(x - r_2)(x - r_3)$$
$$\equiv x^3 - (r_1 + r_2 + r_3)x^2 + (r_1r_2 + r_1r_3 + r_2r_3)x - r_1r_2r_3 = 0$$

Corollary 1. *When the coefficient of the highest power of x is unity, all the roots are factors of the known term.*

Corollary 2. *If the known term is lacking, the equation has at least one root zero.*

Corollary 3. *If the coefficient of x^{n-1} in $f(x) = 0$, where $f(x)$ is of form (1), be multiplied by m, the coefficient of x^{n-2} by m^2, and so on, the resulting equation will have the roots $mr_1, mr_2, \cdots mr_n$.*

Corollary 4. *The negative roots of the equation $f(x)=0$ will be the positive roots of the equation $f(-x)=0$.*

The theorem of this section affords a second method of writing down an equation whose roots are given. It also furnishes an easier method than substitution for testing the correctness of a solution.

Example. The sum of two roots of the equation

$$x^3 - 8x^2 + 13x - 6 = 0$$

is 2. Find all the roots.

By the theorem, the sum of all the roots is 8. Hence the third root must be 6. Dividing by $x-6$, the equation is depressed to

$$x^2 - 2x + 1 = 0$$

whose roots are evidently 1 and 1. The roots of the original equation are then 1, 1, and 6. The product of these is 6, and changing the sign, since the equation is of odd degree, gives the known term, which checks the solution.

217. Integral roots. Theorem. *If the coefficient of the highest power of x is unity, and all the other coefficients integers, all the rational roots of the equation are integers.*

For, suppose $f(x)=0$ has a fractional root $\frac{c}{d}$, where c and d have no common divisor, then

$$\left(\frac{c}{d}\right)^n + b_1\left(\frac{c}{d}\right)^{n-1} + b_2\left(\frac{c}{d}\right)^{n-2} + \cdots + b_{n-1}\left(\frac{c}{d}\right) + b_n \equiv 0$$

Transposing the first term to the right-hand side,

$$b_1\left(\frac{c}{d}\right)^{n-1} + b_2\left(\frac{c}{d}\right)^{n-2} + \cdots + b_{n-1}\left(\frac{c}{d}\right) + b_n \equiv -\left(\frac{c}{d}\right)^n$$

Multiplying both sides by d^{n-1}, the lowest common denominator of the fractions in the left-hand member,

$$b_1c^{n-1} + b_2dc^{n-2} + \cdots + b_{n-1}d^{n-2}c + b_nd^{n-1} \equiv -\frac{c^n}{d}$$

Since c and d have no common divisor, $\frac{c^n}{d}$ is a fraction, while the left-hand side is an integer. But since it is impossible for an integer to be equal to a fraction, the assumption that $\frac{c}{d}$ is a root of the equation is false.

This theorem, with Corollary 1 of the preceding section, gives a method of finding all the rational roots of an equation whose first coefficient is unity, and the other coefficients integers. Since all the rational roots are integers and factors of the known term, the equation may be tested for rational roots by dividing $f(x)$ successively by $x - d_1$, $x - d_2$, etc., where $d_1, d_2, \cdots$ are the factors of the known term. If $f(x)$ is divisible by any of these binomials, the corresponding d is a root of the equation by § 213, Corollary.

218. Descartes's * Rule of Signs. (1) *The equation $f(x) = 0$ can have no more positive roots than there are changes of sign in the coefficients of $f(x)$.* (2) *It can have no more negative roots than there are changes of sign in the coefficients of $f(-x)$.*

* René Descartes (1596–1650), the celebrated French mathematician and philosopher, published a treatise on universal science in 1637. The third section of the treatise was devoted to Analytical Geometry, which was a great advance over any previous work in that line. Our modern method of plotting equations is due to him. In the third book of this section on geometry, he stated this rule for finding the limit of the positive and negative roots of an equation, which has since been known by his name.

Inspection shows that the first statement of the rule is true for equations of the first and second degrees. For, if $f(x)$ is of first degree, the equation has the form

$$a_0x + a_1 = 0$$

If both coefficients have the same sign, there is evidently no positive root. The equation can have at most one positive root, and in this case the coefficients must have different signs.

If $f(x)$ is of second degree, the equation has the form

$$a_0x^2 + a_1x + a_2 = 0$$

If all the coefficients have the same sign, it is evident there can be no factor of form $x - r_1$, when r_1 is positive. Hence there is no positive root. If there is one change of sign in the coefficients, since a_0 is always taken positive, a_2 must be negative, and a_1 may be either positive or negative. Then the real factors must have the form $(x - r_1)(x + r_2)$, and there is one positive root. If there are two changes of sign in the coefficients, the middle term must be negative, and the others positive. Hence the factors have the form $(x - r_1)(x - r_2)$, and there are at most two positive roots.

It will now be shown that if the left-hand member of an equation be multiplied by $x - a$, where a is a positive number, the resulting equation will have at least one more change of sign than the original one. Suppose the sequence of signs in the original polynomial to be

$$+ - - + - + + + -$$

In multiplying by a binomial whose sequence of signs is $+ -$, the following signs occur at the various steps in the operation:

$$\begin{array}{r} + - - + - + + + - \\ + - \\ \hline + - - + - + + + - \\ - + + - + - - - + \\ \hline + - \pm + - + \pm \pm - + \end{array}$$

The original polynomial had five changes of sign, while the resulting one has at least six, even when the ambiguous signs are so chosen as to give the smallest possible number of changes. A little reflection shows that this will always be the case, and hence the number of changes of sign must be at least as great as the number of positive roots. A more complete proof will be given later. The second part of the rule follows from Corollary 4, § 216.

Example 1. $f(x) = x^3 - 2x^2 + 7x - 5 = 0.$

Here $f(x)$ has three changes of sign, while $f(-x)$ has none. Hence, by Descartes's rule, the equation may have three positive roots, but no negative root. There is therefore no further necessity for seeking negative roots. It should be noted that Descartes's rule does not say there *will* be three positive roots. Some of them may turn out to be imaginary.

Example 2. $f(x) = x^4 + 6x^2 + 7 = 0.$

Since neither $f(x)$ nor $f(-x)$ has any changes of sign, there can be no real root. All the roots are imaginary.

219. Synthetic division. In seeking for the rational roots of the equation $f(x) = 0$, by the method of trial divisors mentioned in § 217, it is desirable to reduce the labor of long division to a minimum. This is accomplished by **synthetic division,** which is best illustrated by an example.

Let it be required to divide $x^3 - 6x^2 + 11x - 5$ by $x - 2$. By long division,

$$
\begin{array}{rrrr|l}
x^3 & - 6x^2 & + 11x & - 5 & \underline{x - 2} \\
\underline{x^3} & \underline{- 2x^2} & & & x^2 - 4x + 3 \\
 & - 4x^2 & + 11x & & \\
 & \underline{- 4x^2} & \underline{+ 8x} & & \\
 & & 3x & - 5 & \\
 & & \underline{3x} & \underline{- 6} & \\
 & & & 1 &
\end{array}
$$

Since both dividend and divisor are arranged according to descending powers of x, it is unnecessary to write the x's; and the first number of each partial product may also be omitted, since it is a mere repetition of the number directly above it. Hence the work may without ambiguity be written as follows:

$$
\begin{array}{rrrr|l}
1 & - 6 & + 11 & - 5 & \underline{1 - 2} \\
 & \underline{- 2} & & & 1 - 4 + 3 \\
 & - 4 & + 11 & & \\
 & & \underline{+ 8} & & \\
 & & + 3 & - 5 & \\
 & & & \underline{- 6} & \\
 & & & 1 &
\end{array}
$$

In subtraction it is customary to change the signs of the subtrahend and add. Instead change all the signs of all the partial products, by changing the sign before the second term of the divisor, and then add each partial product to the corresponding number in the dividend. It is unnecessary to write the coefficient of the first term in the divisor, since in this work it is always unity. It is also unnecessary to write the coefficients of the dividend more than once. To save space the remaining figures may be brought up into one line; thus:

$$\begin{array}{rrrr|l} 1 & -6 & +11 & -5 & \underline{2} \\ & +2 & -8 & +6 & \\ \hline 1 & -4 & +3 & +1 & \end{array}$$

It should be observed that the last number on the lowest line is the remainder, and the other numbers on that line are the coefficients of the quotient, which may be written out in full, by remembering that since the divisor is of first degree, the quotient must be of degree one less than the dividend. Hence in this case the quotient is $x^2 - 4x + 3$, with remainder 1.

It should be noted that when any term of an equation is lacking, zero must be supplied for the missing coefficient.

Example 1. $x^3 - 2x^2 - 5x + 6 = 0.$

By Descartes's rule there can be no more than two positive roots and one negative root. By § 216, Corollary 1, the only possible rational roots are ± 1, ± 2, ± 3, ± 6. $f(x)$ may be tested for these roots by synthetic division. It will be noted that the number written at the right in the process of synthetic division is the number which is being tested as a root.

$$\begin{array}{rrrr|l} 1 & -2 & -5 & +6 & \underline{1} \\ & +1 & -1 & -6 & \\ \hline 1 & -1 & -6 & 0 & \end{array}$$

Since the remainder is zero, $f(x)$ is divisible by $x - 1$, and 1 is a root of the equation. The quotient is $x^2 - x - 6$, which set equal to zero gives the depressed equation, readily solved by factoring.

$$x^2 - x - 6 = (x - 3)(x + 2) = 0$$

The roots of the depressed equation are -2 and 3; the roots of the original equation are -2, 1, and 3. As a check it should be noted that the sum of the three roots is 2 and their product -6.

Example 2. $x^4 - 3x^3 - 8x + 24 = 0$

By Descartes's rule there may be two positive but no negative roots. Hence it is necessary to test only the positive factors of 24 for roots.

$$\begin{array}{l|l} 1-3+0-\ 8+24 & 1 \\ \ \ +1-2-\ 2-10 & \\ \hline 1-2-2-10+14 & \end{array} \qquad \begin{array}{l|l} 1-3+0-\ 8+24 & 2 \\ \ \ +2-2-\ 4-24 & \\ \hline 1-1-2-12\quad 0 & \end{array}$$

1 is not a root. 2 is a root.

The work is now transferred to the depressed equation. Since every root of the depressed equation is a root of the original equation, it is useless to try 1, which has already been tested. However, 2 may be a root of the depressed equation, and hence a root twice of the original equation.

$$\begin{array}{l|l} 1-1-2-12 & 2 \\ \ \ +2+2\quad 0 & \\ \hline 1+1\quad 0-12 & \end{array} \qquad \begin{array}{l|l} 1-1-2-12 & 3 \\ \ \ +3+6+12 & \\ \hline 1+2+4\quad 0 & \end{array}$$

2 is not a root. 3 is a root.

Solving the depressed equation $x^2 + 2x + 4 = 0$ by the quadratic formula, the two imaginary roots are obtained.

$$x = \frac{-2 \pm \sqrt{4-16}}{2} = -1 \pm \sqrt{-3}$$

CHECK. $2+3+(-1+\sqrt{-3})+(-1-\sqrt{-3})=3$

$2 \cdot 3(-1+\sqrt{-3})(-1-\sqrt{-3})=24$

Example 3. $x^4 + 9x^3 + 16x^2 - 36x - 80 = 0$

By Descartes's rule, there may be one positive and three negative roots. Since there is a greater chance of finding negative roots, the negative factors of 80 should be tested first.

$$\begin{array}{l} 1+9+16-36-80 \,\underline{|-1} \text{ is not a root.} \\ \underline{-1-8-8+44} \\ 1+8+8-44-36 \end{array}$$

$$\begin{array}{l} 1+9+16-36-80 \,\underline{|-2} \text{ is a root.} \\ \underline{-2-14-4+80} \\ 1+7+2-400 \end{array}$$

$$\begin{array}{l} 1+7+2-40 \,\underline{|-2} \text{ is not a root of the depressed equation.} \\ \underline{-2-10+16} \\ 1+5-8-24 \end{array}$$

$$\begin{array}{l} 1+7+2-40 \,\underline{|-4} \text{ is a root.} \\ \underline{-4-12+40} \\ 1+3-10 \end{array}$$

$x^2+3x-10=(x+5)(x-2)=0 \quad \therefore -5$ and 2 are roots.

Roots are $-5, -4, -2, 2$.

CHECK. $-5-4-2+2=-9,$

$$(-5)(-4)(-2)2=-80$$

220. The labor of testing an equation for rational roots by synthetic division may frequently be lessened by observing the following principle :

If, in testing for a positive root, the remainder and all the coefficients of the quotient are positive, it is useless to test any larger number. If, in testing for a negative root, the coefficients of the quotient and the remainder are alternately positive and negative, it is useless to test any larger negative number.

For, it is readily seen that if all signs are positive, a larger positive number would simply cause a larger positive remainder to pile up; while, if the signs are alternately positive and negative, a larger negative number will lead to a numerically larger remainder.

Hence in the search for rational roots, all positive factors of the constant term should be tested in order, until either all the positive roots which are possible under Descartes's rule have been found, or all the resulting signs are positive; and all negative factors of the constant term should be tested in order, until either all the negative roots possible by Descartes's rule have been found, or the resulting signs are positive and negative alternately.

Another useful application of synthetic division occurs in forming the table of values necessary for plotting a function.

EXERCISES

Solve the following equations:

1. $x^4 - 15x^2 - 10x + 24 = 0$
2. $x^3 + 2x^2 - 11x - 12 = 0$
3. $x^3 + 2x - 12 = 0$
4. $x^4 - 4x^3 - 8x + 32 = 0$
5. $x^3 - 7x^2 + 11x - 5 = 0$
6. $x^3 - 18x - 35 = 0$
7. $x^5 + 5x^4 - 15x^3 - 85x^2 - 26x + 120 = 0$
8. $x^4 + 15x^3 + 70x^2 + 120x + 64 = 0$
9. $x^3 - 26x^2 + 156x - 216 = 0$
10. $x^3 - 27x - 54 = 0$
11. $x^5 - 8x^4 + 8x^3 + 40x^2 - 32x = 0$
12. $x^5 - 2x^4 - 15x^3 + 8x^2 + 68x + 48 = 0$

13. $x^4 - 6x^3 + 13x^2 - 30x + 40 = 0$
14. $x^4 + 6x^3 + 7x^2 - 4x = 0$
15. $x^4 - 5x^3 - 4x^2 + 38x - 24 = 0$
16. $x^4 - 4x^2 + x + 2 = 0$
17. $x^3 + x^2 - 37x + 35 = 0$
18. $x^4 - 12x^2 + 13x - 12 = 0$
19. $x^5 + 4x^4 - 8x^3 - 36x^2 - 13x + 12 = 0$
20. $x^4 - 5x^2 - 12x = 0$
21. $x^3 - 8x^2 + 13x - 6 = 0$
22. $x^4 - 35x^2 + 90x - 56 = 0$
23. $x^4 - 2x^3 - 18x^2 + x + 70 = 0$
24. $x^4 - 5x^3 - 7x^2 + 41x - 30 = 0$
25. $x^4 - 7x^2 + 6x = 0$
26. $x^3 + 8x^2 + 17x + 10 = 0$
27. $x^3 - 12x^2 + 41x - 42 = 0$
28. $x^3 - 10x^2 + 28x - 16 = 0$
29. $x^3 - 4x^2 + 5x - 20 = 0$
30. $x^5 - 43x^3 - 84x^2 + 222x + 504 = 0$

221. Fractional roots. It has been seen in § 217 that if the first coefficient of an equation is unity, and the other coefficients integers, all the rational roots are integers. If, however, the equation has the form

$$a_0x^n + a_1x^{n-1} + \cdots + a_n = 0, \text{ where } a_0 \neq 1$$

there may be fractional roots. In order to find the rational roots of such an equation, an auxiliary equation is formed, whose roots are multiples of the roots of the given equation, and whose first coefficient is unity, while the other coefficients are integers. The given equation is first divided through by a_0 in order to make the first coefficient unity; then the auxiliary equation is written down by Corollary 3, § 216, and the multiplier m determined so that the coefficients are integers. The auxiliary equation may then be solved by the method of the preceding section, after which the roots must be divided by m, in order to obtain the roots of the original equation.

Example 1. $$18x^3 - 3x^2 - 7x + 2 = 0$$

Dividing by 18, $$x^3 - \frac{x^2}{6} - \frac{7x}{18} + \frac{1}{9} = 0$$

By Corollary 3, § 216, the auxiliary equation is

$$x^3 - \frac{mx^2}{6} - \frac{7m^2x}{18} + \frac{m^3}{9} = 0$$

Evidently, if $m = 6$, the coefficients will be integers.

$$x^3 - x^2 - 14x + 24 = 0$$

By the methods of the preceding section, the roots of this auxiliary equation are -4, 2, and 3. Hence, dividing by m, the roots of the given equation are $-\frac{2}{3}$, $\frac{1}{3}$, and $\frac{1}{2}$.

Example 2. $$100x^3 - 19x + 3 = 0$$

$$x^3 - \frac{19x}{100} + \frac{3}{100} = 0$$

The auxiliary equation is

$$x^3 - \frac{19m^2x}{100} + \frac{3m^3}{100} = 0$$

Taking $m = 10$, $$x^3 - 19x + 30 = 0$$

The roots of the auxiliary equation are -5, 2, 3. Hence the roots of the original equation are $-\frac{1}{2}$, $\frac{1}{5}$, $\frac{3}{10}$.

NOTE. If an equation of the preceding type has any integral roots, it is possible to find them by synthetic division without transforming the equation; but since in most cases an auxiliary equation is needed to determine the fractional roots, it is generally advisable to form it at the outset.

EXERCISES

Solve the following equations :

1. $4x^3 - 16x^2 + 19x - 6 = 0$
2. $3x^3 - 8x^2 + 13x - 6 = 0$
3. $2x^3 + x - 3 = 0$
4. $4x^5 - 4x^4 - 15x^3 + 4x^2 + 17x + 6 = 0$
5. $3x^3 - 26x^2 + 52x - 24 = 0$
6. $6x^4 + 19x^3 - x^2 - 11x + 3 = 0$
7. $8x^3 - 2x^2 - 4x + 1 = 0$
8. $4x^3 + 16x^2 - 9x - 36 = 0$
9. $12x^3 - 52x^2 + 23x + 42 = 0$
10. $9x^4 + 15x^3 - 143x^2 + 41x + 30 = 0$
11. $4x^3 - 10x^2 + 7x - 1 = 0$
12. $4x^4 - 12x^3 + 13x^2 - 15x + 10 = 0$
13. $8x^4 - 16x^3 + 4x^2 + 5x - 1 = 0$
14. $9x^3 - 18x^2 + 11x - 2 = 0$
15. $4x^3 - 9x + 14 = 0$
16. $5x^3 - 7x^2 + 2 = 0$

222. Graph of $f(x)$. The problem of finding the rational roots of any numerical equation is now completely solved. Before discussing irrational roots, the graphical interpretation of the work already done will be considered. Since a root of the equation $f(x) = 0$ is a number which substituted in the left-hand member makes it equal to zero, then all the points where the graph of $f(x)$ crosses the x-axis represent roots of the equation; and since all such points correspond to real numbers, these crossings indicate real roots. It follows from § 214 that the graph of an equation of nth degree cannot cross the x-axis more than n times. Since some of the roots may be imaginary, the curve may cross the x-axis less than n times. This may be illustrated by plotting the three equations

(1) $$x^3 - 3x^2 + 2 = 0$$

(2) $$x^3 - 3x^2 + 4 = 0$$

(3) $$x^3 - 3x^2 + 6 = 0$$

which differ only in the constant term. The table of values for the graph of equation (1) should be formed by use of the theorem of § 213, performing the division synthetically. It is evident that the values of $f(x)$ in equation (2) will be 2 greater, and in equation (3), 4 greater, than the corresponding values in equation (1).

$$\begin{array}{rrrr|r} 1 & -3 & +0 & +2 & -2 \\ & -2 & +10 & -20 & \\ \hline 1 & -5 & +10 & -18 & \end{array} \qquad \begin{array}{rrrr|r} 1 & -3 & +0 & +2 & 2 \\ & +2 & -2 & -4 & \\ \hline 1 & -1 & -2 & -2 & \end{array}$$

$$\begin{array}{rrrr|r} 1 & -3 & +0 & +2 & -1 \\ & -1 & +4 & -4 & \\ \hline 1 & -4 & +4 & -2 & \end{array} \qquad \begin{array}{rrrr|r} 1 & -3 & +0 & +2 & 3 \\ & +3 & +0 & +0 & \\ \hline 1 & +0 & +0 & +2 & \end{array}$$

$$\begin{array}{rrrr|r} 1 & -3 & +0 & +2 & 1 \\ & +1 & -2 & -2 & \\ \hline 1 & -2 & -2 & 0 & \end{array} \qquad \begin{array}{rrrr|r} 1 & -3 & +0 & +2 & 4 \\ & +4 & +4 & +16 & \\ \hline 1 & +1 & +4 & +18 & \end{array}$$

$x^3 - 3x^2 + 2 = 0$

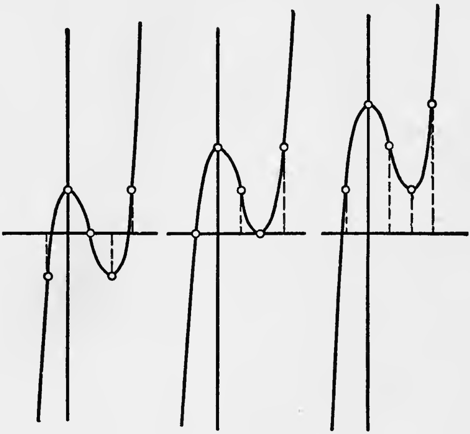
x	y
-2	-18
-1	-2
0	2
1	0
2	-2
3	2
4	18

$x^3 - 3x^2 + 4 = 0$

x	y
-2	-16
-1	0
0	4
1	2
2	0
3	4
4	20

$x^2 - 3x^2 + 6 = 0$

x	y
-2	-14
-1	2
0	6
1	4
2	2
3	6
4	22



It will be noted that the curves are identical in form, the second being two units higher, and the third, four units higher than the first. The first curve crosses the axis three times, indicating the existence of the one negative and two positive roots which Descartes's rule shows are possible for each of the three equations. As the curve is raised, the two positive roots come nearer together until, as in the second figure, the curve, instead of cutting the axis on the right-hand side, merely touches it, showing that in equation (2) the two positive roots are equal. When the curve is raised still further, it ceases to have any points in common with the axis on the right-hand side, showing that the positive roots have been replaced by two imaginary roots. In general, a bend of the curve which does not intersect the x-axis indicates a pair of imaginary roots. It is not, however, true that a bend of the curve can be found corresponding to every pair of imaginary roots.

223. Theorem. *If $f(a)$ and $f(b)$ have unlike signs, an odd number of real roots of the equation $f(x)=0$ lie between a and b. If $f(a)$ and $f(b)$ have like signs, either an even number of real roots or no roots at all lie between a and b.*

For if a and b are connected by a continuous curve, when they lie on opposite sides of the axis the portion of curve between the points must cut the axis an odd number of times. If they lie on the same side of the axis, the portion of curve between them either does not cut the axis at all or cuts it an even number of times.

224. Form of the graph. It is evident that if $f(x)$ is of odd degree, and a sufficiently large negative number be substituted for x, a negative value for $f(x)$ will result; while if a sufficiently large positive value be substituted for x, a positive value will result. Hence the graph of the equation will be a curve which begins below the x-axis on the left, and ends above it on the right. It must therefore cross the x-axis at least once. Moreover, if the constant term of the equation is positive, the curve is above the x-axis at $x=0$, and hence must cross the negative part of the x-axis at least once; while if the constant term is negative, the curve is below the x-axis at $x=0$, and must cross the positive part of the x-axis at least once. This leads to the following theorem:

Theorem. *Every equation of odd degree has at least one real root, whose sign is opposite to that of the constant term.*

The first part of the theorem also follows from the theorem of § 215.

225. If $f(x)$ is of even degree, it is evident that the substitution of a sufficiently large number, either positive or negative, will make $f(x)$ positive. Hence the graph of an equation of even degree will both begin and end above the x-axis. If the constant term of the equation is negative, the curve will be below the x-axis at $x = 0$. Hence in this case the curve must cross the positive and also the negative part of the x-axis at least once. If the constant term is positive, the curve may not cross the x-axis at all. This leads to the following theorem:

Theorem. *If the constant term of an equation of even degree is negative, the equation has at least one positive and one negative root. If the constant term is positive, the roots may all be imaginary.*

EXERCISES

Without solving the following equations, state four facts concerning the roots of each.

1. $x^5 - \frac{9}{2}x^4 + 7x - 4 = 0$

2. $x^6 + 2x^4 - 3x - 5 = 0$

3. $x^4 + 2x^3 + 5x^2 + x = 0$

4. $4x^3 - 3x^2 + 2x - 1 = 0$

What information can be gained concerning the roots of the following equations by drawing their graphs?

5. $x^3 - 2x - 5 = 0$

6. $x^3 - 5x^2 + 2 = 0$

by Newtons method – **7.** $x^4 - 9x^2 + 4x + 10 = 0$

8. $x^4 + x^3 + x^2 + x + 1 = 0$

9. $x^4 - 2x^3 + x^2 - 3 = 0$

10. $x^4 - 3x^3 + 2x = 0$

11. The equation $x^3 - 7x^2 - 15x + c = 0$ has the root 2. Find the value of c.

12. The equation $x^3 - 3x^2 + 2x + b = 0$ has two roots numerically equal, but of opposite signs. Find the value of b, and all the roots of the equation.

13. The equation $x^3 - ax^2 + 3x - 2 = 0$ has two roots which are reciprocals of each other. Find the value of a, and all the roots.

226. Extent of the table of values. In plotting an equation, the question arises: How far is it necessary to carry the table of values? The answer is similar to that given to the question: How far should the test for rational roots by trial division be carried? and a similar explanation holds.

Positive values should be substituted for x, until all the positive roots possible under Descartes's rule have been located by change of sign, or until all the resulting signs in the partial remainders are positive. Negative values should be substituted until all the negative roots possible under Descartes's rule have been located, or until the signs of the partial remainders are alternately positive and negative.

227. Incommensurable roots. If an application of the methods of §§ 219–221 shows that an equation has no rational roots, or if, when all the factors corresponding to rational roots of the equation have been removed, the depressed equation is of a degree above the second, the problem remains to determine whether the undiscovered roots are real, and, if so, to find their approximate location. The theorem of § 223 will often decide this question. Hence the first step in the search for incommensurable roots is to plot the equation. If the number of times the curve crosses the axis is the same as the possible number of real roots shown by Descartes's rule, it has been determined between which two integers each of the real roots lies. This method, however, is sometimes inconclusive, as is shown by the following example:

Example: $x^3 - 7x + 7 = 0$

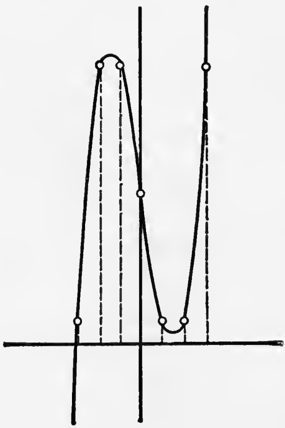
$$\begin{array}{rrrr|l} 1 & +0 & -7 & +7 & \underline{-1} \\ & -1 & +1 & +6 & \\ \hline 1 & -1 & -6 & +13 & \end{array} \qquad \begin{array}{rrrr|l} 1 & +0 & -7 & +7 & \underline{1} \\ & +1 & +1 & -6 & \\ \hline 1 & +1 & -6 & +1 & \end{array}$$

$$\begin{array}{rrrr|l} 1 & +0 & -7 & +7 & \underline{-2} \\ & -2 & +4 & +6 & \\ \hline 1 & -2 & -3 & +13 & \end{array} \qquad \begin{array}{rrrr|l} 1 & +0 & -7 & +7 & \underline{2} \\ & +2 & +4 & -6 & \\ \hline 1 & +2 & -3 & +1 & \end{array}$$

$$\begin{array}{rrrr|l} 1 & +0 & -7 & +7 & \underline{-3} \\ & -3 & +9 & -6 & \\ \hline 1 & -3 & +2 & +1 & \end{array} \qquad \begin{array}{rrrr|l} 1 & +0 & -7 & +7 & \underline{3} \\ & -3 & +9 & +6 & \\ \hline 1 & +3 & +2 & +13 & \end{array}$$

$$\begin{array}{rrrr|l} 1 & +0 & -7 & +7 & \underline{-4} \\ & -4 & +16 & -36 & \\ \hline 1 & -4 & +9 & -29 & \end{array}$$

x	y
-4	-29
-3	$+1$
-2	$+13$
-1	$+13$
0	$+7$
1	$+1$
2	$+1$
3	$+13$



The graph locates the real negative root, which must exist according to Theorem § 224, between -4 and -3; but leaves unsettled the question whether the two positive roots, which are possible by Descartes's rule, really exist or not. The graph, as drawn, does not cross the axis on the right-hand side; but it must be remembered that only points corresponding to integral values of x have been definitely located, and that the intervening values have merely been filled in so as to form a smooth curve. It is quite possible that between 1 and 2 the curve may drop so as to cross the axis twice. One way of testing this would be to plot the curve for intervening fractional values, but even this might not be conclusive; for, no matter how great the number of points plotted, if they were all above the axis, the possibility would still remain that the curve might drop across the axis between two of them.

In the following sections a process known as Sturm's* method will be developed, by which it can be definitely determined how many real roots of an equation lie in any given interval; but, since Sturm's method is frequently laborious in application, it should not be applied until other processes are known to be inconclusive.

228. Theorem. *When the equation $f(x) = 0$ has equal roots, $f'(x)$ also vanishes for these roots; each root occurs once more in $f(x) = 0$ than in $f'(x) = 0$; and for any root which occurs in $f(x)$ but once, $f'(x)$ does not vanish.*

For, if a is a root of $f(x) = 0$ m times, then by § 213, Corollary,

$$f(x) \equiv (x - a)^m q(x)$$

* This method was perfected in 1829 by Jacques Charles François Sturm (1803–1855).

Differentiating,

$$f'(x) \equiv (x-a)^m q'(x) + m(x-a)^{m-1} q(x)$$

Hence $f'(x)$ is divisible by $(x-a)^{m-1}$, and it follows that $f'(x)=0$ has the root a, $m-1$ times. If m be taken equal to unity, the last statement of the theorem follows.

Corollary 1. *If $f(x)$ and $f'(x)$ have a common divisor, each linear factor of their highest common divisor will occur in $f(x)$ once more than in the divisor itself. If $f(x)$ and $f'(x)$ have no common divisor, the equation $f(x)=0$ has no equal roots.*

Corollary 2. *If a is a root of $f(x)=0$ m times, then the first $m-1$ derivatives of $f(x)$ vanish for $x=a$.*

229. Theorem. *If a is a root of $f(x)=0$, and h a sufficiently small positive number, then $f(a-h)$ and $f'(a-h)$ have unlike signs, while $f(a+h)$ and $f'(a+h)$ have like signs.*

For, developing these four functions by Maclaurin's formula, considering h a variable,

$$f(a+h) \equiv f(a) + f'(a)h + \frac{f''(a)h^2}{2!} + \frac{f'''(a)h^3}{3!} + \cdots$$

$$\equiv h\left[f'(a) + \frac{f''(a)h}{2!} + \frac{f'''(a)h^2}{3!} + \cdots\right]$$

$$f'(a+h) \equiv f'(a) + f''(a)h + \frac{f'''(a)h^2}{2!} + \frac{f^{\text{IV}}(a)h^3}{3!} + \cdots$$

$$f(a-h) \equiv f(a) - f'(a)h + \frac{f''(a)h^2}{2!} - \frac{f'''(a)h^3}{3!} + \cdots$$

$$\equiv -h\left[f'(a) - \frac{f''(a)h}{2!} + \frac{f'''(a)h^2}{3!} - \cdots\right]$$

$$f'(a-h) \equiv f'(a) - f''(a)h + \frac{f'''(a)h^2}{2!} - \frac{f^{\text{IV}}(a)h^3}{3!} + \cdots$$

Since a is a root of $f(x) = 0$, $f(a)$ vanishes, and if h is sufficiently small, each expression will have the sign of the first term, which proves the theorem.

Corollary. *If a is a root of $f(x) = 0$ m times, then the signs of $f(x)$ and its first $m - 1$ derivatives are alternately positive and negative for values of x a very little less than a; while for values a very little greater than a, they all have the sign of the mth derivative, which is the first one which does not vanish for $x = a$.*

This follows from Corollary 2, § 228.

230. Theorem.* *If two numbers, a and b, of which b is the greater, be substituted for x in $f(x)$ and its successive derivatives, and the number of variations of sign in each set of resulting numbers noted, the difference in the number of variations in the two sets will be at least as great as the number of real roots of $f(x) = 0$ between a and b.*

For the only changes of sign that can occur as x increases from a to b are when x passes through a root of $f(x)$ or one of its derivatives. By § 229 whenever x passes through a single root of the equation, one variation of sign is lost; and when it passes through an m-fold root, by the corollary, m variations are lost. When x passes through a root of a derivative, which is not a root of the two adjacent derivatives, there is a loss of two variations in case the adjacent functions have like signs, or no loss of variations in case the adjacent functions have unlike signs. When x passes

* This is part of the Fourier-Budan theorem, which was proved by Jean Baptiste Joseph Fourier (1768–1830) in a work published after his death. A theorem which was practically the same had been stated by Budan in 1807, but his proof was unsatisfactory.

through a root of a derivative, which is also a root of several immediately following, there will be a loss of variations by the corollary of § 229. Hence, as x increases from a to b, there is never any gain in the number of variations, but one variation is lost for every root of $f(x) = 0$ passed through by x, and there may be other losses of variations. Therefore the loss in the number of variations of sign is at least as great as the number of roots of $f(x) = 0$ which lie between a and b.

Corollary. Descartes's Rule of Signs. See § 218.

$$f(x) \equiv a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$$

Take $a=0$, $b=+\infty$, then $f(0) = a_n$, $f'(0) = a_{n-1} \cdots$ $f^{(n-1)}(0) = a_1, f^{(n)}(0) = a_0$

Hence the variation of sign when 0 is substituted for x in $f(x)$ and its successive derivatives is the same as the variations of sign in the coefficients of $f(x)$. When ∞ is substituted for x, $f(x)$ and its derivatives all take the sign of the first term, which is positive. Hence the number of variations lost as x increases from 0 to $+\infty$ is the number of variations in the signs of the coefficients.

231. Equal roots. Corollary 1 of § 228 gives a method whereby any equation may be tested for equal roots.

Example 1.

$$f(x) \equiv x^6 - 2x^5 + x^4 - 4x^3 - x^2 - 2x - 1 = 0$$
$$f'(x) \equiv 6x^5 - 10x^4 + 4x^3 - 12x^2 - 2x - 2$$

The highest common divisor should be sought by the method of Euclid.* For simplicity take the numerical

* See § 24.

factor 2 out of $f'(x)$, and multiply $f(x)$ by 3 in order to avoid fractions.

$$
\begin{array}{r|l}
 & 3x^5 - 5x^4 + 2x^3 - 6x^2 - x - 1 \\ \hline
3x^6 - 6x^5 + 3x^4 - 12x^3 - 3x^2 - 6x - 3 & x - 1 \\
\underline{3x^6 - 5x^5 + 2x^4 - 6x^3 - x^2 - x} & \\
- x^5 + x^4 - 6x^3 - 2x^2 - 5x - 3 & \\
\underline{3} & \\
- 3x^5 + 3x^4 - 18x^3 - 6x^2 - 15x - 9 & \\
\underline{- 3x^5 + 5x^4 - 2x^3 + 6x^2 + x + 1} & \\
- 2x^4 - 16x^3 - 12x^2 - 16x - 10 &
\end{array}
$$

Discard the factor -2.

$$
\begin{array}{r|l}
 & x^4 + 8x^3 + 6x^2 + 8x + 5 \\ \hline
3x^5 - 5x^4 + 2x^3 - 6x^2 - x - 1 & 3x - 29 \\
\underline{3x^5 + 24x^4 + 18x^3 + 24x^2 + 15x} & \\
- 29x^4 - 16x^3 - 30x^2 - 16x - 1 & \\
\underline{- 29x^4 - 232x^3 - 174x^2 - 232x - 145} & \\
216x^3 + 144x^2 + 216x + 144 &
\end{array}
$$

Discard the factor 72.

$$
\begin{array}{r|l}
3x^4 + 24x^3 + 18x^2 + 24x + 15 & 3x^3 + 2x^2 + 3x + 2 \\ \hline
\underline{3x^4 + 2x^3 + 3x^2 + 2x} & x + 22 \\
22x^3 + 15x^2 + 22x + 15 & \\
\underline{3} & \\
66x^3 + 45x^2 + 66x + 45 & \\
\underline{66x^3 + 44x^2 + 66x + 44} & \\
x^2 + 1 &
\end{array}
$$

$$
\begin{array}{r|l}
3x^3 + 2x^2 + 3x + 2 & x^2 + 1 \\ \hline
\underline{3x^3 + 3x} & 3x + 2 \\
2x^2 + 2 & \\
\underline{2x^2 + 2} &
\end{array}
$$

$x^2 + 1$ is the highest common factor of $f(x)$ and $f'(x)$. Hence $(x^2+1)^2 = x^4 + 2x^2 + 1$ is a factor of $f(x)$. By division, the other factor is found to be $x^2 - 2x - 1$.

$$\begin{aligned} f(x) &\equiv x^6 - 2x^5 + x^4 - 4x^3 - x^2 - 2x - 1 \\ &\equiv (x^2+1)^2(x^2 - 2x - 1) = 0. \end{aligned}$$

Setting each factor equal to zero, and solving by the quadratic formula, the roots of $f(x)$ are $1 \pm \sqrt{2}$, $\pm\sqrt{-1}$, $\pm\sqrt{-1}$.

Example 2. Test for equal roots the equation

$$\begin{aligned} f(x) &\equiv x^3 - 7x + 7 = 0 \\ f'(x) &\equiv 3x^2 - 7 \end{aligned}$$

$$\begin{array}{r|l} x^3 - 7x + 7 & \\ 3 & \\ \hline 3x^3 - 21x + 21 & \underline{3x^2 - 7} \\ 3x^3 - 7x & x \\ \hline -14x + 21 & \text{Discard the factor } -7. \end{array}$$

$$\begin{array}{r|l} 3x^2 \qquad - 7 & \\ 2 & \\ \hline 6x^2 \qquad -14 & \underline{2x - 3} \\ 6x^2 - 9x & 3x + 9 \\ \hline 9x - 14 & \\ 2 & \\ \hline 18x - 28 & \\ 18x - 27 & \\ \hline -1 & \end{array}$$

There is no common divisor, hence the equation $f(x) = 0$ has no equal roots, by § 228, Corollary 1.

232. Sturm's Method. If, in the application of the test for equal roots, it is found, as in Example 2 of the last section, that $f(x)$ and $f'(x)$ have no common divisor, write down the set of functions

$$f(x), f'(x), f_2(x), f_3(x) \cdots f_n(x)$$

where $f_2(x) \cdots f_n(x)$ are the remainders which occur in the Euclidean process of finding the highest common divisor, *each with its sign changed.* The function $f(x)$, and the n auxiliary functions are called **Sturm's functions** for the equation $f(x)=0$, and the set has the following characteristic properties:

Properties of Sturm's functions. 1. *No two successive functions vanish for the same value of x.*

For, in any problem in division

$$\text{Dividend} = \text{Divisor} \times \text{Quotient} + \text{Remainder}$$

Since the auxiliary functions $f_2 \cdots f_n$ are the remainders which occur in applying Euclid's rule for finding the highest common divisor, with their signs changed, the following identities may be obtained:

(1) $$f(x) \equiv f'(x)q_1(x) - f_2(x)$$

(2) $$f'(x) \equiv f_2(x)q_2(x) - f_3(x)$$

(3) $$f_2(x) \equiv f_3(x)q_3(x) - f_4(x)$$

. . . .

$$f_{n-2}(x) \equiv f_{n-1}(x)q_{n-1}(x) - f_n(x)$$

Suppose two successive functions, as $f_3(x)$ and $f_4(x)$, vanished for $x = a$. Then they must both contain the factor $x - a$. From relation (3), any common factor of $f_3(x)$ and $f_4(x)$ must also be a factor of $f_2(x)$; but if $x - a$ is a factor of $f_2(x)$ and $f_3(x)$, from relation (2) it must be a factor of $f'(x)$; and similarly relation (1) shows that any common factor of $f'(x)$ and $f_2(x)$ must be a factor of $f(x)$. If, however, $f(x)$ and $f'(x)$ have a common factor, $f(x)$ has equal roots, and in this case there is no need for using Sturm's functions.

2. *If one of the auxiliary functions vanishes, the preceding and following functions have opposite signs.*

For, suppose $f'(x)$ vanishes for $x=a$. If a be substituted for x in the first of the identities used in the proof of property 1,

$$f(a)=-f_2(a)$$

If $f_2(x)$ vanishes for $x=b$, substituting b for x in identity (2),

$$f'(b)=-f_3(b)$$

and so on.

3. *The sign of the last function, $f_n(x)$, is always the same, since it is evidently a constant.*

233. Sturm's Theorem. *If two numbers, a and b, of which b is the greater, be substituted for x in the Sturmian functions of an equation $f(x)=0$, which has no equal roots, and the number of variations of sign in each case noted, the difference in the number of variations of the two sets will equal the number of real roots of $f(x)=0$, which lie between a and b.*

It is evident that a change in the number of variations can occur only when some function vanishes, since none of these functions can change its sign without vanishing. Hence the theorem will be proved by showing that, when $f(x)$ vanishes, a variation is lost at the very beginning of the set of signs, and that the vanishing of any other function can do no more than shift the position of some of the variations without changing their number.

These facts may be easily seen by considering the arrangement of signs for a value of x a little less than the root, such as $r-h$, and for a value a little greater than the root, such as $r+h$, which will be similar to those exhibited in the following tables:

	1			2			3			4		
	$r-h$	r	$r+h$	$r-h$	r	$r+h$	$r-h$	r	$r+h$	$r-h$	r	$r+h$
$f(x)$	$-$	0	$+$	$-$	0	$+$	$+$	0	$-$	$+$	0	$-$
$f'(x)$	$+$	$+$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$-$	$-$	$-$
$f_2(x)$	$-$	0	$+$	$+$	0	$-$	$-$	0	$+$	$+$	0	$-$
$f_3(x)$	$-$	$-$	$-$	$-$	$-$	$-$	$+$	$+$	$+$	$+$	$+$	$+$
. . .	. .	.	.	. .	.	.	. .	.	.	. .	.	.

In tables 1 and 2, $f(x)$ is increasing as it passes through the root r, hence $f'(x)$ is positive by § 92. In tables 3 and 4, $f(x)$ is decreasing, hence by the same theorem $f'(x)$ is negative. $f'(x)$ cannot change its sign in the interval from $r-h$ to $r+h$, since by property 1 it cannot vanish at the same time as $f(x)$.

If an auxiliary function, such as $f_2(x)$, vanishes, it may be either increasing, as in tables 1 and 3, or decreasing, as in tables 2 and 4. By property 1, $f_3(x)$ cannot change its sign in the interval from $r-h$ to $r+h$, and, by property 2, $f_3(x)$ must have its sign opposite to that of $f'(x)$. It will be seen by an inspection of the tables that, whenever $f(x)$ passes through a root, one variation of sign is lost; and when $f_2(x)$ passes through a root, the position of the variations is changed, but not their number. Hence the difference in the number of variations of signs found in the $r-h$ and $r+h$ columns will indicate the number of real roots of the equation $f(x)=0$ which occur between $r-h$ and $r+h$.

Let the student construct tables with different arrangements of sign by letting other auxiliary functions vanish; and show that unless $f(x)$ changes sign the number of variations of sign is unaltered.

Example. Sturm's method will now be applied to the equation

$$x^3 - 7x + 7 = 0$$

which was plotted in § 227 and which the application of preceding methods left still unsolved. The Sturmian functions may be immediately written down from the work done in testing the equation for equal roots in the preceding section. It is evident from the graph that if positive roots exist, they must lie between 1 and 2. Hence these numbers may be taken as the a and b of the theorem.

	1	2
$f(x) \equiv x^3 - 7x + 7$	+	+
$f'(x) \equiv 3x^2 - 7$	−	+
$f_2(x) \equiv 2x - 3$	−	+
$f_3(x) \equiv 1$	+	+

Since there are two variations for $x = 1$, and no variations for $x = 2$, by Sturm's theorem the equation has two real roots between 1 and 2, and the graph should be amended so that the curve will drop below the x-axis.

234. Practical hints for the application of Sturm's Theorem. 1. The process of finding the auxiliary Sturmian functions $f_2(x) \cdots f_n(x)$ differs from the ordinary rule for finding the highest common divisor of $f(x)$ and $f'(x)$ only in the fact that multiplication or division by any negative number is excluded, and that the sign of the remainder must always be changed before it is used as the next divisor. At any step in the work, however, a positive numerical factor may be discarded, or such a factor may be introduced, when it is necessary in order to avoid fractions in the quotient.

2. The labor of performing the last division may be

avoided by noting that it is unnecessary to determine the numerical value of $f_n(x)$, since only its sign is significant. By the second Sturmian property, this sign will be opposite to that which $f_{n-2}(x)$ assumes for the value of x which causes $f_{n-1}(x)$ to vanish. As an illustration of this, observe the Sturmian functions for the equation treated in the last section. Here

$$f_{n-1}(x) \equiv f_2(x) \equiv 2x - 3$$

which vanishes for $x = \frac{3}{2}$. This number substituted in $f'(x)$ gives it a negative value, which shows that $f_3(x)$ must be positive. Hence, whenever it is easier to solve $f_{n-1}(x) = 0$, and substitute this value in $f_{n-2}(x)$, than to perform the last division, the former method may be substituted for the latter.

3. The labor of performing the last two divisions may be avoided in case the function of second degree $f_{n-2}(x)$ has either equal or imaginary roots; since, in these cases, the sign of $f_{n-2}(x)$ can never change, and hence there can be no change in the number of variations of sign beyond that. For, if $f_{n-2}(x)$ and $f_n(x)$ both have fixed signs, these signs must be unlike, because the intervening function $f_{n-1}(x)$, being linear, must have a real root. When x passes through this root, $f_{n-2}(x)$ and $f_n(x)$ must have opposite signs (§ 232, 2), and since both signs are fixed, they must be unlike always. Therefore a change in the sign of $f_{n-1}(x)$ will not affect the number of variations.

Hence it is advisable, when $f_{n-2}(x)$ is found, to form the quantity $b^2 - 4ac$. If this is negative or zero, it is unnecessary to compute $f_{n-1}(x)$ and $f_n(x)$.

4. In the example worked out in the last section, it was evident from the graph where the positive roots, if

they exist at all, must lie. Frequently, however, the location of the roots is not so obvious, and in this case the interval from the lower limit of the roots to zero should be tested if Descartes's rule shows the possibility of negative roots; and the interval from zero to the upper limit of the roots, if Descartes's rule shows the possibility of positive roots; these limits being determined by § 226.

If it is desired to know merely the number of positive or of negative roots, without locating them more precisely, the intervals from $-\infty$ to 0 and from 0 to $+\infty$ may be tested.

If the leading coefficients of the Sturmian functions are all positive, the roots of the equation are all real, and conversely. For, if the leading coefficients are positive, the substitution of a sufficiently large negative number will make the functions of even degree positive and those of odd degree negative, causing n variations in sign; while the substitution of a sufficiently large positive number will make all the functions positive. Hence there is a loss of n variations in sign and all the roots are real. If any of the leading coefficients are negative, there must be some imaginary roots.

5. Sturm's theorem does not require that the coefficient of the first term of an equation to which it is applied shall be unity; but in practice this is likely to be the case, since it would be inexpedient to perform the labor of computing the Sturmian functions, until it is certain that the equation has no rational roots.

235. Summary of preceding methods. By the methods already described it is possible to find all the rational roots of a numerical equation of any degree, and the integral part of any irrational root. Before describing a method for computing irrational roots to any required number of decimal places, a summary of the work already done is presented.

In solving any equation of degree above the second, first apply Descartes's rule to determine the possible number of positive and negative roots (§ 218).

If the coefficient of the highest power of x is not unity, the other coefficients being integers, transform to an equation where these conditions are satisfied, by multiplying the roots by a suitable factor (§ 221). In this case it should not be forgotten that after the auxiliary equation is solved, the roots of the original equation must be found by division.

Third, test for integral roots by synthetic division.

Fourth, if the equation has not been depressed to a quadratic, plot the equation, or if some factors have been removed, plot the depressed equation.

If all the roots which Descartes's rule shows are possible have not been found by the preceding methods, apply the rule for finding the highest common divisor to $f(x)$ and $f'(x)$, changing the sign of each remainder before performing the next division. If they have such a divisor, each linear factor which it contains will be a factor of $f(x)$ to one higher power than it occurs in the divisor. This makes it possible to factor $f(x)$ and determine the roots (§ 231).

If $f(x)$ and $f'(x)$ have no common divisor, form the Sturmian functions, and determine the number and location of the real roots by Sturm's theorem.

EXERCISES

Find all real roots of the following equations, exactly if they are rational, and giving the integral part of each irrational root:

1. $x^3 + 2x^2 - 5x - 6 = 0$
2. $x^3 - 6x^2 + 9x + 1 = 0$
3. $x^3 - 2x^2 - 5x + 5 = 0$
4. $4x^4 + 8x^3 - x^2 - 8x - 3 = 0$
5. $x^3 - 3x^2 - 6x + 5 = 0$
6. $x^3 - 6x^2 + 5x + 13 = 0$
7. $x^4 - 4x^3 + 2x^2 + 4x + 1 = 0$
8. $x^4 - x^3 - 3x^2 - 9x + 12 = 0$
9. $x^3 - x^2 + 2x - 1 = 0$
10. $x^3 + 2x^2 - 3x - 5 = 0$
11. $x^4 + 2x^3 - 3x^2 - 9x - 6 = 0$
12. $x^4 - 5x^3 + 5x^2 + 5x - 6 = 0$
13. $x^3 + 3x^2 - 4x + 1 = 0$
14. $x^4 - x^3 - 2x^2 + x = 0$
15. $x^4 - 4x^3 - x + 2 = 0$
16. $9x^3 - 3x - 2 = 0$
17. $x^5 - 17x^3 - 12x^2 + 52x + 48 = 0$
18. $x^4 - 3x^2 + 3x - 1 = 0$
19. $x^4 + 2x^3 - 4x^2 - 3x + 2 = 0$
20. $x^4 - 4x^3 + 5x^2 - 2x - 12 = 0$
21. $x^4 - 2x^3 - 3x^2 + 12x - 12 = 0$

22. $x^4 + 2x^3 - 7x^2 - 8x + 13 = 0$

23. $x^5 - 6x^3 - 2x^2 + 8x + 8 = 0$

24. $x^3 - 6x + 6 = 0$

25. $x^4 - 2x^3 - 6x^2 + 10x + 5 = 0$

26. $x^3 + 3x^2 - 2x - 9 = 0$

27. $x^3 - 2x^2 - x + 3 = 0$

28. $x^4 - x^3 - 7x^2 - x + 3 = 0$

29. $x^4 - 4x^3 + x^2 + 6x + 2 = 0$

30. $x^5 - 10x^4 + 34x^3 - 44x^2 + 17x - 2 = 0$

31. $3x^3 - 6x^2 + 12x - 1 = 0$

32. $x^3 - 3x^2 - 4x + 13 = 0$

33. $4x^3 - 2x^2 + 3x - 1 = 0$

34. $27x^3 - 27x^2 - 9x + 11 = 0$

35. $8x^4 - 6x^2 - 6x - 1 = 0$

36. $x^4 + 7x^3 + 9x^2 - 27x - 54 = 0$

37. $x^4 - 6x^3 + 5x^2 + 14x - 4 = 0$

38. $x^4 - 4x^3 + 8x + 4 = 0$

236. Diminishing the roots of an equation. Given the equation

(1) $f(x) \equiv x^n + b_1x^{n-1} + \cdots + b_{n-1}x + b_n = 0$, or in another form

(2) $f(x) \equiv (x - r_1)(x - r_2) \cdots (x - r_n) = 0$

The present problem is to write an equation whose roots shall be equal to those of $f(x)$ diminished by a certain constant a. Taking $f(x)$ in form (2), the required equation is evidently

$$[x - (r_1 - a)][x - (r_2 - a)] \cdots [x - (r_n - a)] = 0$$

or, rearranging and removing the inner parentheses,

$$(x+a-r_1)(x+a-r_2)\cdots(x+a-r_n)=0$$

which is equation (2) with $x+a$ written in place of x. Since (1) and (2) are equivalent, the required equation may also be obtained by writing $x+a$ for x in (1), which gives

$$f(x+a)\equiv(x+a)^n+b_1(x+a)^{n-1}+\cdots$$
$$+b_{n-1}(x+a)+b_n=0$$

Expanding each term by the binomial theorem, and arranging according to powers of x, $f(x+a)$ is obtained in the form

(3) $$f(x+a)\equiv x^n+c_1x^{n-1}+\cdots+c_{n-1}x+c_n=0$$

The problem now is to find an easy method of computing the c's, which are functions of a and the coefficients of the original equation. Evidently, if $x-a$ is written in place of x in (3), the original equation is obtained; but it is now in the form

(4) $$f(x)\equiv(x-a)^n+c_1(x-a)^{n-1}+\cdots$$
$$+c_{n-1}(x-a)+c_n=0$$

It is evident from (4) that if $f(x)$ is divided by $x-a$, the remainder will be c_n. Also, if the quotient of this division, which is

$$(x-a)^{n-1}+c_1(x-a)^{n-2}+\cdots+c_{n-1}$$

is divided by $x-a$, the remainder will be c_{n-1}. Continuing this process, the remainders at each step will be the coefficients of the required equation in reverse order. Synthetic division is the most convenient method of performing the operation.

Example. Write an equation whose roots shall be less by 2 than those of

$$x^4 - 4x^3 - 3x + 23 = 0$$

$$\begin{array}{rrrrrl}
1 & -4 & +0 & -3 & +23 & \underline{|2} \\
 & +2 & -4 & -8 & -22 & \\
\hline
1 & -2 & -4 & -11 & |+1 & \\
 & +2 & +0 & -8 & & \\
\hline
1 & +0 & -4 & |-19 & & \\
 & +2 & +4 & & & \\
\hline
1 & +2 & |\;0 & & & \\
 & +2 & & & & \\
\hline
1 & +4 & & & &
\end{array}$$

The required equation is

$$x^4 + 4x^3 - 19x + 1 = 0$$

If these two equations are plotted, it will be found that the curves have precisely the same form, but the graph of the second will be two units farther to the left than that of the first.

237. Horner's Method* for approximate calculation of irrational roots. This method is best illustrated by an example. Consider the equation

$$(1) \qquad x^3 + 10x - 13 = 0$$

By Descartes's rule, there are no negative roots and but one positive root, which a graph will show to lie between 1 and 2. It is possible to compute the value of

* This method was presented in 1819, by an Englishman named William George Horner (1786–1837). It was improved and amplified by Augustus de Morgan (1806–1871), Professor in the University of London from 1828 to 1867. He thought so highly of the method that he wished it included in all arithmetics. One of his pupils computed the positive root of $x^3 - 2x - 5 = 0$ to 103 decimal places. Another pupil tried to outdo this record by computing it to 150 decimal places, but found he had made an error in the 76th figure.

this root to any required degree of exactness by plotting the equation for the values 1.1, 1.2, 1.3 ⋯ and determining between which tenths the curve crosses the axis. Having found that it crosses between 1.1 and 1.2, the equation may be plotted for 1.11, 1.12, 1.13 ⋯ which shows between which hundredths it crosses, and so on. This method, however, is extremely tedious, and Horner's method, based on the same principle, seeks to reduce the labor to a minimum. The process is carried on as follows:

Having determined that the required root lies between 1 and 2, a new equation is written whose roots are those of equation (1) decreased by 1, using the method of the preceding section.

$$\begin{array}{rrrr|l}
1 & +0 & +10 & -13 & \underline{1} \\
 & +1 & +1 & +11 & \\
\hline
 & +1 & +11 & -2 & \\
 & +1 & +2 & & \\
\hline
1 & +2 & +13 & & \\
 & +1 & & & \\
\hline
1 & +3 & & &
\end{array}$$

The new equation is

(2) $$x^3 + 3x^2 + 13x - 2 = 0$$

and its positive root lies between 0 and 1. The advantage of using this equation lies in the fact that in determining between which tenths the root lies, it is necessary to use only one figure, which makes the work of substitution much easier. If the graph of the equation has been carefully drawn, it can be seen that the curve crosses the axis much nearer 0 than 1; and this will help in reducing the number of trial divisions needed to determine between which tenths the root actually lies.

$$\begin{array}{rrrr|l} 1+ & 3. & +13. & -2. & \underline{.1} \\ & +\ .1+ & .31 & +1.331 & \\ \hline 1+ & 3.1+ & 13.31- & .669 & \end{array} \qquad \begin{array}{rrrr|l} 1+ & 3. & +13. & -2. & \underline{.2} \\ & +\ .2+ & .64 & +2.728 & \\ \hline 1+ & 3.2+ & 13.64+ & .728 & \end{array}$$

The change of sign shows that the root of equation (2) lies between .1 and .2; that is, the root of equation (1) is between 1.1 and 1.2. In order to find a second decimal place, an equation is formed whose roots are those of (2) decreased by .1.

$$\begin{array}{rrrr|l} 1+ & 3. & +13. & -2. & \underline{.1} \\ & +\ .1 & +\ \ .31 & +1.331 & \\ \hline 1+ & 3.1 & +13.31 & -\ \ .669 & \\ & +\ .1 & +\ \ .32 & & \\ \hline 1+ & 3.2 & +13.63 & & \\ & +\ .1 & & & \\ \hline 1+ & 3.3 & & & \end{array}$$

The new equation is

(3) $$x^3+3.3x^2+13.63x-.669=0$$

and its root is between 0 and .1. Evidently the graph will not help in estimating between which hundredths the root lies; but when a root is as small as this, its approximate value may be found by solving for x the last two terms of the equation. Set

$$13.63x-.669=0$$

$$x=\frac{.669}{13.63}=.04+$$

which makes it probable that the root is between .04 and .05. It must be remembered, however, that this is only an estimate and needs the following verification:

$$\begin{array}{rrrr|l} 1+ & 3.3 & +13.63 & -.669 & \underline{.04} \\ & +\ \ .04+ & .1336 & +.550544 & \\ \hline 1+ & 3.34+ & 13.7636- & .118456 & \end{array}$$

$$\begin{array}{rrrr|l} 1 + 3.3 & + 13.63 & - .669 & & \underline{.05} \\ + \ .05 + & .1675 + & .689875 & & \\ \hline 1 + 3.35 + & 13.7975 + & .020875 & & \end{array}$$

This shows that equation (1) has a root between 1.14 and 1.15. To get a third decimal place, the process is repeated. Write an equation whose roots are those of (3) decreased by .04.

$$\begin{array}{rrrl} 1 + 3.3 & + 13.63 & - .669 & \underline{.04} \\ + \ .04 & + \ .1336 & + .550544 & \\ \hline 1 + 3.34 & + 13.7636 & - .118456 & \\ + \ .04 & + \ .1352 & & \\ \hline 1 + 3.38 & + 13.8988 & & \\ + \ .04 & & & \\ \hline 1 + 3.42 & & & \end{array}$$

The new equation is

(4) $$x^3 + 3.42\,x^2 + 13.8988\,x - .118456 = 0$$

and its root is between 0 and .01. The thousandths between which it lies may be estimated by the preceding method. Set

$$13.8988\,x - .118456 = 0$$

$$x = \frac{.118456}{13.8988} = .008 +$$

This estimate is verified as follows:

$$\begin{array}{rrrrl} 1 + 3.42 & + 13.8988 & - .118456 & & \underline{.008} \\ + \ .008 + & .027424 + & .111409792 & & \\ \hline 1 + 3.428 + & 13.926224 - & .007046208 & & \end{array}$$

$$\begin{array}{rrrrl} 1 + 3.42 & + 13.8988 & - .118456 & & \underline{.009} \\ + \ .009 + & .030861 + & .125366949 & & \\ \hline 1 + 3.429 + & 13.929661 + & .006910949 & & \end{array}$$

This shows that equation (1) has a root between 1.148 and 1.149; that is, the value of the root correct to three decimal places is 1.148.

The essential features of the work may be concisely arranged as follows. The broken lines show the end of each process for finding an equation with diminished roots, and the numbers in heavy type are the coefficients of the successive equations.

1	+ 0	+ 10	− 13	1.148
	+ 1	+ 1	+ 11	
	+ 1	+ 11	− **2.**	
	+ 1	+ 2	+ 1.331	
	+ 2	+ **13.**	− **.669**	
	+ 1	+ .31	+ .550544	
	+ **3.**	+ 13.31	− **.118456**	
	+ .1	+ .32	+ .111409792	
	+ 3.1	+ **3.63**	− .007046208	
	+ .1	+ .1336		
	+ 3.2	+ 13.7636		
	+ .1	+ .1352		
	+ **3.3**	+ **13.8988**		
	+ .04	+ .027424		
	+ 3.34	+ 13.926224		
	+ .04			
	+ 3.38			
	+ .04			
	+ **3.42**			
	+ .008			
	+ 3.428			

Example 2. Find, correct to three decimal places, the real roots of

(1) $$x^3 - 7x + 7 = 0$$

From the graph in § 227, this equation has a negative root between − 3 and − 4; and it was shown in § 233 that there are two positive roots between 1 and 2. As the first step in finding the approximate value of the positive roots a new equation is formed whose roots are those of (1) decreased by 1.

1	+0	−7	+7	1
	+1	+1	−6	
1	+1	−6	+1	
	+1	+2		
1	+2	−4		
	+1			
1	+3			

The new equation is

(2) $$x^3 + 3x^2 - 4x + 1 = 0$$

and it has one root between .3 and .4, and another between .6 and .7. The condensed form of the work for finding the first root is as follows:

1 + 3.	− 4.	+ 1.	.356
.3	+ .99	− .903	
3.3	− 3.01	+ **.097**	
.3	+ 1.08	− .086625	
3.6	**− 1.93**	+ **.010375**	
.3	+ .1975	− .009048984	
3.9	− 1.7325	+ .001326016	
.05	+ .2		
3.95	**− 1.5325**		
.05	+ .024336		
4.00	− 1.508164		
.05			
4.05			
.006			
4.056			

1 + 3.	− 4.	+ 1.		.4
+ .4	+ 1.36	− 1.056		
1 + 3.4	− 2.64	− .056		

1 + 3.9	+ 1.93	+ .097		.06
.06 +	.2376	− .101544		
1 + 3.96	− 1.6924	− .004544		

1 + 4.05	− 1.5325	+ .010375		.007
.007 +	.028399	− .010528707		
1 + 4.057	− 1.504101	− .000153707		

The following is the work for finding the second root:

1 + 3.	− 4.	+ 1.	.692
.6	+ 2.16	− 1.104	
3.6	− 1.84	− **.104**	
.6	+ 2.52	+ .100809	
4.2	+ **.68**	− **.003191**	
.6	+ .4401	+ .003156888	
4.8	+ 1.1201	− .000034112	
.09	+ .4482		
4.89	+ **1.5683**		
.09	+ .010144		
4.98	+ 1.578444		
.09			
5.07			
.002			
5.072			

1 + 3.	− 4.	+ 1.		.7
.7 +	2.59 −	.987		
1 + 3.7	− 1.41 +	.013		

1 + 5.07	+ 1.5683	− .003191		.003
.003 +	.015219 +	.004750557		
1 + 5.073	+ 1.583519 +	.001559557		

The positive roots of equation (1) are 1.356 and 1.692.

In finding the approximate value of the negative root, use is made of Corollary 4, § 216. Since $f(x) \equiv x^3 - 7x + 7 = 0$ has a root between -3 and -4,

$f(-x) \equiv x^3 - 7x - 7 = 0$ has a root between 3 and 4, which may be found by the preceding method.

1 + 0	− 7	− 7	3.048
3	+ 9	+ 6	
3	+ 2	− **1.**	
3	+ 18	+ .814464	
6	+ **20.**	− **.185536**	
3	+ .3616	+ .166382592	
9.	+ 20.3616	− .019153408	
.04	+ .3632		
9.04	+ **20.7248**		
.04	+ .073024		
9.08	+ 20.797824		
.04			
9.12			
.008			
9.128			

1	+ 9.	+ 20.	− 1.	.1
	.1 +	.91 +	2.091	
1	+ 9.1	+ 20.91	+ 1.091	

1	+ 9.	+ 20.	− 1.	.05
	.05 +	.4525 +	1.022625	
1	+ 9.05	+ 20.4525	+ .022625	

1	+ 9.12	+ 20.7248	− .185536	.009
	.009 +	.082161 +	.187262649	
1	+ 9.129	+ 20.806961	+ .001726649	

Hence 3.048 is a root of $f(-x) = 0$, and -3.048 is a root of equation (1).

EXERCISES

Compute to three decimal places the real roots of the equations in § 235, Exercises 2, 3, 5, 6, 9.

238. Reciprocal equations. The equation $a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$ may be transformed into an equation whose roots are the reciprocals of those of the given equation by substituting $\frac{1}{y}$ for x. The equation then becomes

$$a_0\left(\frac{1}{y}\right)^n + a_1\left(\frac{1}{y}\right)^{n-1} + \cdots + a_{n-1}\left(\frac{1}{y}\right) + a_n = 0$$

or
$$a_n y^n + a_{n-1}y^{n-1} + \cdots + a_1 y + a_0 = 0$$

which is the required equation, since it is satisfied by the reciprocal of any root of the original equation. Hence it appears that the required equation may be obtained by reversing the order of the coefficients in the original equation.

It follows that if the coefficients of an equation are the same except perhaps for sign, whether read forwards or backwards, the roots of this equation occur in pairs of the form r and $\frac{1}{r}$; and it is called a **reciprocal equation.** An example of such an equation is

$$12x^3 - 37x^2 + 37x - 12 = 0$$

The roots are $\frac{3}{4}$, $\frac{4}{3}$ and 1, 1 being its own reciprocal.

239. Infinite roots. If in the equation $a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$, $a_n = 0$, one root of the equation is zero. If a_{n-1} is also zero, two roots are zero and so on. It follows that in the equation $a_nx^n + a_{n-1}x^{n-1} + \cdots a_1x + a_0 = 0$, if $\lim a_n = 0$, one root of the equation becomes indefinitely large. If also $\lim a_{n-1} = 0$, two roots become indefinitely large; and so on.

240. Removal of the second term of an equation. Any term of an equation after the first may be removed by a properly chosen transformation. If it be desired to remove the second term from an equation whose first coefficient is unity, and its second is b_1, set $x = y - \frac{b_1}{n}$. Since b_1 is numerically equal to the sum of the roots, $\frac{b_1}{n}$ is the arithmetic mean of the roots, and if $\frac{b_1}{n}$ be subtracted from each root, the sum will evidently be zero. Hence $x = y - \frac{b_1}{n}$ is the transformation which will change the equation $x^n + b_1x^{n-1} + b_2x^{n-2} + \cdots + b_n = 0$ into a new equation whose second term is lacking. The transformation may be effected by the method of § 236.

241. Solution of a cubic equation.* Since any equation may be transformed into one whose second term is lacking, it is sufficient to consider the solution of equations of the type

(1) $$x^3 + ax + b = 0$$

This equation must now be transformed into one which can be solved as a quadratic.

(2) Let $$x = y^{\frac{1}{3}} + z^{\frac{1}{3}}$$

Then $$x^3 = y + 3\,y^{\frac{2}{3}}z^{\frac{1}{3}} + 3\,y^{\frac{1}{3}}z^{\frac{2}{3}} + z$$
$$= y + z + 3\,y^{\frac{1}{3}}z^{\frac{1}{3}}\,(y^{\frac{1}{3}} + z^{\frac{1}{3}})$$

or $$x^3 - 3\,y^{\frac{1}{3}}z^{\frac{1}{3}}(y^{\frac{1}{3}} + z^{\frac{1}{3}}) - (y + z) = 0$$

Substituting from (2),

(3) $$x^3 - 3\,y^{\frac{1}{3}}z^{\frac{1}{3}}x - (y + z) = 0$$

Comparing the coefficients of (1) and (3)

$$y^{\frac{1}{3}}z^{\frac{1}{3}} = -\frac{a}{3}. \quad \therefore yz = -\frac{a^3}{27}, \; y + z = -b$$

* This solution of the cubic equation was first published in 1545 by Cardan, an Italian mathematician. He however did not discover it, but obtained it by fraud from Tartaglia (1500–1559), a Venetian mathematician.

(4) Solving, $y = \frac{1}{2}\left[-b + \sqrt{b^2 + \frac{4a^3}{27}}\right]$,

$$z = \frac{1}{2}\left[-b - \sqrt{b^2 + \frac{4a^3}{27}}\right]$$

(5) Then $x = y^{\frac{1}{3}} - \frac{a}{3y^{\frac{1}{3}}}$ where the value of y is given by (4).

Since $y^{\frac{1}{3}}$ has three values, these give the three values for x, and if z had been used in (5) no different values would be obtained, since the equations for determining y and z are symmetric.

This solution is perfect in theory, but is of little practical value in the actual solution of numerical equations. Since imaginary roots occur in pairs, a cubic equation must either have all its roots real, or have one real with two imaginary roots. In case all the roots are real, the quantity $b^2 + \frac{4a^3}{27}$ which occurs under the radical in the solution, will be negative,* so that in order to solve the equation it becomes necessary to extract the cube root of a complex number, for which there is no general arithmetical method, though the approximate value of the roots may be found by Trigonometry (see § 200).

When the equation has a pair of imaginary roots, $b^2 + \frac{4a^3}{27}$ is positive and the computation can be carried out, but usually the equation can be solved by simpler methods.

Example. Given the equation $x^3 - 6x^2 + 3x - 18 = 0$. Transform the equation by diminishing the

* This can be proved in a study of the nature of the roots of a cubic equation.

roots by 2, following the method of § 236. The transformed equation is

$$x^3 - 9x - 28 = 0$$

Setting $x = y^{\frac{1}{3}} + z^{\frac{1}{3}}$ it is found that $y^{\frac{1}{3}}z^{\frac{1}{3}} = 3$, $y + z = 28$, from which

$$y = 1,\ z = 27. \quad y^{\frac{1}{3}} = 1,\ z^{\frac{1}{3}} = 3. \quad x = 4$$

The other value of x may be found by using the imaginary cube roots of y and z, but more simply by depressing the equation, which gives $x = -2 \pm \sqrt{-3}$. Since these roots are 2 less than those of the original equation, the latter roots are 6 and $\pm\sqrt{-3}$.

242. Solution of a biquadratic equation. The following solution is due to Descartes. Since it depends on Cardan's solution of the cubic, it fails when that fails. As in the case of the cubic, the second term can always be removed, and so it is sufficient to consider equations of the form

(1) $$x^4 + ax^2 + bx + c = 0$$

It is possible to break this up into quadratic factors, where the coefficients of x are equal and of opposite sign, since the original function contains no term in x^3. That is,

$$x^4 + ax^2 + bx + c = (x^2 + dx + f)(x^2 - dx + g)$$
$$= x^4 + (g + f - d^2)x^2 + (dg - df)x + fg$$

Equating coefficients,

(2) $$g + f - d^2 = a,\ d(g - f) = b,\ fg = c$$

Eliminating f and g,

(3) $$d^6 + 2ad^4 + (a^2 - 4c)d^2 - b^2 = 0$$

This equation is a cubic in d^2, and if its solution is possible, the solution of the biquadratic can be completed. Since b^2 has the minus sign, (3) has a positive root which may be called r^2. Then r is a real value of d. Let the other roots be s^2 and t^2. Then

(4) $$r^2 + s^2 + t^2 = -2a, \quad rst = b.$$

Eliminating g from the first two equations of (2) gives

$$f = \frac{a + d^2 - \dfrac{b}{d}}{2} = \frac{\dfrac{-(r^2 + s^2 + t^2)}{2} + r^2 - st}{2} = \frac{r^2 - (s+t)^2}{4}$$

by substitution from (4). Also

$$g = \frac{a + d^2 + \dfrac{b}{d}}{2} = \frac{\dfrac{-(r^2 + s^2 + t^2)}{2} + r^2 + st}{2} = \frac{r^2 - (s-t)^2}{4}$$

Since r, s, and t are roots of the cubic (3), if this is soluble, the quadratic factors of (1) are determined.

Example. Given the equation

$$x^4 - 10x^2 - 20x - 16 = 0$$

Here $\quad a = -10, \; b = -20, \; c = -16$

The auxiliary equation in d is

$$d^6 - 20d^4 + 164d^2 - 400 = 0 \quad d^2 = 4 \text{ and } 8 \pm 6\sqrt{-1}$$

are the solutions.

$$r = \pm 2, \; s = \pm(3 + \sqrt{-1}), \; r = \pm(3 - \sqrt{-1})$$

Then $\quad f = -8, \quad g = 2$

Hence

$$x^4 - 10x^2 - 20x - 16 = (x^2 - 2x - 8)(x^2 + 2x + 2) = 0$$

The roots are 4, -2, and $-1 \pm \sqrt{-1}$.

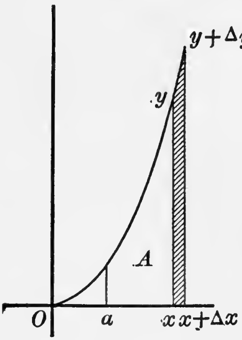
For further treatment of the subject of this chapter, the student is referred to Cajori's Theory of Equations, and Burnside and Panton's Theory of Equations.

CHAPTER XIII

ELEMENTARY THEORY OF INTEGRATION

Common integration is only the *memory of differentiation*. The different artifices by which integration is effected are changes not from the known to the unknown, but from forms in which memory will not serve to those in which it will. — A. DE MORGAN.

243. Area bounded by a curve. The problem of finding the area bounded by the x-axis, two lines parallel to the y-axis, and a curve whose equation is given, leads to the subject of integration. Let A represent the area bounded by the x-axis, the lines through the fixed point a and the variable point x parallel to the y-axis, and the curve $y = x^2$. A is evidently a function of x since its area depends on the position of the variable point x. Let another line be drawn parallel to the y-axis through the point $x + \Delta x$ and let the area of the shaded strip whose width is Δx be called ΔA. Since the average height of this strip lies between y and $y + \Delta y$,



$$y\Delta x \leqq \Delta A \leqq (y + \Delta y)\Delta x$$

Dividing this inequality by the positive quantity Δx gives

$$y \leqq \frac{\Delta A}{\Delta x} \leqq y + \Delta y$$

Then $$\lim_{\Delta x=0} y \leqq \lim_{\Delta x=0} \frac{\Delta A}{\Delta x} \leqq \lim_{\Delta x=0} (y + \Delta y)$$

It is evident that when Δx has the limit zero, ΔA and Δy will also have the limit zero. It is also evident that $\frac{\Delta A}{\Delta x}$ is a function whose value can be made arbitrarily near to the particular value of y which corresponds to the value of x that has been chosen. Since by definition (§ 90) $\lim_{\Delta x=0} \frac{\Delta A}{\Delta x} = \frac{d}{dx} A$, the following relations hold:

$$\lim_{\Delta x=0} \frac{\Delta A}{\Delta x} = y = x^2 \text{ or } \frac{d}{dx} A = x^2$$

This says that A, the area under consideration, is a function whose derivative is equal to x^2. By § 103 $\frac{d}{dx} x^3 = 3x^2$. $\therefore \frac{x^3}{3}$ is a function whose derivative is x^2. However by § 94, Corollary, $\frac{x^3}{3} + c$, where c is any constant, is also a function whose derivative is x^2. Hence $A = \frac{x^3}{3} + c$. In order to determine the numerical value of c, the value of A must be known for some particular value of x. It is evident from the figure that when

$$x = a, \quad A = 0$$

Hence $$0 = \frac{a^3}{3} + c \text{ or } c = -\frac{a^3}{3}$$

Then $$A = \frac{x^3}{3} - \frac{a^3}{3}$$

If $a = 1$, and $x = 4$, $A = \frac{4^3}{3} - \frac{1}{3} = 21$

244. A more general problem may be stated thus: To find the area A bounded by the x-axis, the two lines parallel to the y-axis through the points a and x, and the curve $y=f(x)$. As before, it is found that

$$\lim_{\Delta x=0}\frac{\Delta A}{\Delta x}=\frac{d}{dx}A=y=f(x)$$

that is, A is a function of x, whose derivative is $f(x)$. Let $F(x)$ be a function whose derivative is known to be $f(x)$. Then $A=F(x)+c$. Since $A=0$ when $x=a$, $c=-F(a)$, and $A=F(x)-F(a)$, or if x has the particular value b, $A=F(b)-F(a)$. These considerations lead to the following theorem.

Theorem. *If $F(x)$ is a function whose derivative is $f(x)$, the area bounded by the x-axis, the lines $x=a$, and $x=b$, and the curve $y=f(x)$, is given by $F(b)-F(a)$.*

If $f(x)$ is the derivative of $F(x)$, $F(x)$ is called the **primitive** function of $f(x)$.

245. Integration. The process of finding a function whose derivative is given is called **integration.** It is the inverse of differentiation, and like most inverse processes, the ease with which it can be performed depends upon the recollection of the results of the direct process, differentiation, just as facility in performing division depends on the recollection of results of previous multiplication, *i.e.* the multiplication table.

The operation of integration is denoted by writing the **integral sign** $\int$ in front of the function to be integrated. If $\frac{d}{dx}F(x)=f(x)$, then $\int f(x)=F(x)$. In Calculus this is usually written $\int f(x)dx=F(x)$ in order to indicate that x is the variable with respect to which the integration is performed. The equation should be read "The integral of $f(x)$ with respect to x equals $F(x)$." The symbol of integration $\int$ was originally a long S standing for the word *sum*, since integration may be defined not only as the inverse of differentiation, but as the limit of a sum, which is a very important notion in Calculus. The problem of finding areas, already mentioned, is only one of a multitude which may be solved by integration, and which are treated in works on Calculus.

246. Constant of integration. As was seen in § 243, a function is not fully determined when its derivative is known. The derivative simply gives the rate of change in the primitive function compared with the change in the independent variable. In order to determine completely the primitive function, its value at some one point must be known, together with its derivative or law of change. In the problem of § 243, since the value of the function A was known when $x = a$ and the derivative of A was also known, the value of A could be determined for any value of x.

Since $f'(x)$ is the derivative not only of $f(x)$, but also of every function having the form $f(x) + c$, where c is any constant,

$$\int f'(x)dx = f(x) + c.$$

The constant c is called the **constant of integration.** Since the value of the integral is indefinite unless c is known, $\int f'(x)dx$ is called the **indefinite integral** of $f'(x)$ with respect to x. If, however, as in the example of § 243, it is known that for a certain value of x, as $x = a$, $\int f'(x)dx = 0$, the constant c is determined by this initial condition, and is equal to $-f(a)$. This is indicated by writing

$$\int_a^x f'(x)dx = f(x) - f(a).$$

The expression on the left is called a **definite integral** and is read "The integral from a to x of $f'(x)$ with respect to x." If in particular $x = b$, the relation becomes

$$\int_a^b f'(x)dx = f(b) - f(a),$$

and gives the area bounded by the x-axis, the lines $x = a$ and $x = b$, and the curve $y = f(x)$.

The evaluation of a definite integral has two distinct steps. (a) Find the indefinite integral of the function under the integral sign, that is, find the primitive function whose derivative is the given function. (b) In this result, substitute first the upper and then the lower bound of the integral and subtract the last result from the first.

Example. $\int_1^4 x^2dx = \frac{x^3}{3}\Big]_1^4 = \frac{64}{3} - \frac{1}{3} = 21.$

247. Integration of elementary forms. Since integration is the inverse of differentiation, the fundamental formulas for the operation may be easily deduced from the rules for differentiation.

1. *The integral of the algebraic sum of several functions is the corresponding sum of their integrals.*

This may be deduced from § 97.

2. *The integral of the product of a constant and a function is the product of the constant and the integral of the function.*

This may be deduced from § 98.

3. *The integral of unity is the variable of integration,* or $\int dx = x$.

This may be deduced from § 95.

4. *The integral of the product of a function affected by a constant exponent, and the derivative of the function is the function, with its exponent increased by unity, divided by the new exponent.* That is,

$$\int y^n \frac{d}{dx} y \cdot dx = \frac{y^{n+1}}{n+1}$$

This may be deduced from § 103.

5. *The integral of a fraction whose numerator is the derivative of the denominator is the Napierian logarithm of the denominator.* That is

$$\int \frac{f'(x)}{f(x)} dx = \log_e f(x)$$

This may be deduced from § 170.

Example 1. $\int \left[\frac{a}{\sqrt{x}} + b - c\sqrt{x} + \sqrt[3]{x^2} \right] dx$

$= a\int x^{-\frac{1}{2}} dx + b\int dx - c\int x^{\frac{1}{2}} dx + \int x^{\frac{2}{3}} dx$, by 1, 2,

$= 2\,ax^{\frac{1}{2}} + bx - \frac{2}{3}\,cx^{\frac{3}{2}} + \frac{3}{5}\,x^{\frac{5}{3}} + c$, by 3, 4.

Example 2. $\int [1 + bx^2]^{\frac{1}{2}} x\,dx = \frac{1}{2\,b} \int [1 + bx^2]^{\frac{1}{2}} 2\,bx\,dx$

$= \frac{1}{2\,b} \frac{(1 + bx^2)^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{(1 + bx^2)^{\frac{3}{2}}}{3\,b} + c$, by 4.

Example 3. $\displaystyle\int\frac{x\,dx}{1+ax^2}=\frac{1}{2a}\int\frac{2ax\,dx}{1+ax^2}$

$$=\frac{1}{2a}\log_e(1+ax^2)+c, \text{ by } 5.$$

Example 4. $\displaystyle\int_0^2\frac{x\,dx}{9-2x^2}=-\frac{1}{4}\int_0^2\frac{-4x\,dx}{9-2x^2}$

$$=-\frac{1}{4}\log_e(9-2x^2)\Big]_0^2$$

$$=-\frac{1}{4}\Big[\log_e 1-\log_e 9\Big]$$

$$=\frac{1}{2}\log_e 3$$

Example 5. $\displaystyle\int_0^1(7-3x^2x\,dx)$

$$=-\tfrac{1}{6}\int_0^1(7-3x^2)(-6x)dx=-\tfrac{1}{12}(7-3x^2)^2\Big]_0^1$$

$$=-\tfrac{1}{3}-(-\tfrac{7}{12})=\tfrac{3}{12}=\tfrac{1}{4}.$$

Example 6. $\displaystyle\int_0^1\frac{(2x+3)dx^*}{x^2+3x+2}=\int_0^1\frac{dx}{x+1}+\int_0^1\frac{dx}{x+2}$

$$=\log(x+1)+\log(x+2)\Big]_0^1=\log(x^2+3x+2)\Big]_0^1$$

$$=\log 6-\log 2=\log\tfrac{6}{2}=\log 3.$$

* See Chapter VIII.

EXERCISES

Explain the steps in the following integrations and verify the result by differentiation.

1. $\displaystyle\int[4x^3-x^2+2x-5]dx=x^4-\frac{x^3}{3}+x^2-5x+c$

2. $\int x(a^2 - x^2)^{\frac{1}{2}} dx = -\frac{1}{3}(a^2 - x^2)^{\frac{3}{2}} + c$

3. $\int \frac{x\,dx}{\sqrt{1 + x^2}} = \sqrt{1 + x^2} + c$

4. $\int^b x^3\,dx = \frac{1}{4}(b^4 - a^4)$

5. $\int \frac{x\,dx}{1 - x^2} = \frac{1}{2}\int \frac{dx}{1 + x} + \frac{1}{2}\int \frac{dx}{1 - x} = \log_e \sqrt{\frac{1 + x}{1 - x}} + c$

6. $\int \frac{xdx\,*}{x^2 + 6\,x + 8} = \log_e \frac{(x + 4)^2}{x + 2} + c$

7. $\int_0^1 \sqrt{x}\,dx = \frac{2}{3}$

8. $\int_1^4 \frac{dx}{x^{\frac{3}{2}}} = 1$

9. $\int_0^2 \frac{dx*}{1 + 3\,x + 2\,x^2} = \log \frac{5}{3}$

Further treatment of the subjects discussed in this chapter may be found in any good work on Calculus.

* See Chapter VIII.

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