FIRST PRINCIPLES OF ALGEBRA

SLAUGHT AND LENNES

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FIRST PRINCIPLES OF ALGEBRA

Complete Course

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PREFACE

The First Principles of Algebra includes three parts, the first and second of which constitute an Elementary Course, designed solely for the first year, and the third an Advanced Course for complete review and further study. Each part is designed to cover the work of one half year, and the complete book will amply meet the entrance requirements of any college or technical school.

The Elementary Course has two chief aims:

(1) To provide a gradual and natural introduction to the symbols and processes of algebra.

(2) To give vital purpose to the study of algebra by using it to do interesting and valuable things.

Each of these aims leads to the same order of topics, which, however, differs somewhat from the conventional order.

If it is admitted that there should be a gradually increasing complexity of forms to be manipulated, it follows that factoring and complicated work in fractions have no proper place in the first half year. This book is arranged so that factoring may begin with the second semester and complicated fractions may come still later. Simple fractions are treated in Chapter V.

The pupil is introduced to the algebraic notation by recalling and stating in terms of letters certain rules of arithmetic with which he is already familiar. The simplicity of the algebraic formulas, compared with the arithmetical statement of rules known to the pupil, cannot fail to impress him with the usefulness and power of the subject which he is about to study. This impression will be deepened when, in Chapter VI, rules which caused considerable trouble in arithmetic are derived with the utmost ease by algebraic processes.

The study of equations and their uses is naturally the main topic of the Elementary Course. This topic is therefore developed early, simple simultaneous equations being completed in Part One. It is recognized that abstract equations will appear of little or no value to the pupil unless he finds uses for them. Hence frequent lists of problems are provided for translation into equations and for solution.

Many of these problems involve valuable mathematical concepts, such as the dimensions and areas of rectangles and triangles, the constitution of the decimal system (digit problems), the fundamental relations connecting velocity, distance, and time, properties of the lever, and so forth. There are also numerous artificial problems involving relations imposed upon abstract numbers or upon interesting informational data.

In this way, during the first half year, *algebra is made to appeal to the higher and more useful types of interest,* and not merely to the instinct for solving puzzles, which must be the case if the greater part of this time is spent on factoring and in manipulating complicated fractions.

The work of the second half year is intended to begin with special products and factors, long division being introduced in this connection. Factoring is at once applied to the solution of quadratic equations and problems depending on them.

In the chapters on radicals and quadratic equations, simple facts from geometry, which the pupil learned in arithmetic, afford the basis for a large number of problems that are of legitimate interest both on their own account, and because of the valuable geometrical relations involved.

The more complicated algebraic fractions, which serve little purpose in a first course except to afford drill in algebraic manipulation, are placed at about the middle of Part Two and are followed by chapters on Ratio, Variation, and Proportion, Equations involving Algebraic Fractions, and a General Review.

For the development of skill in algebraic manipulation it is not sufficient to solve a certain number of exercises when an operation is first introduced. To fix each operation in the learner's mind, there must be recurring drills extending over a considerable period of time. These are amply provided for in this book. The fundamental operations on integral and simple fractional expressions, the solution of simple equations, and the representation of given conditions in algebraic symbols are constantly reviewed in the numerous lists of "drill exercises," many of which may be solved mentally. Factoring is practised almost daily throughout the second half year.

The principles of algebra used in the Elementary Course are enunciated in a small number of short rules — eighteen in all. The purpose of these rules is to furnish, in simple form, a codification of those operations of algebra which require special emphasis. Such a codification has several important advantages:

By constant reference to these few fundamental statements they become an organic, and hence a permanent, part of the learner's mental equipment.

By their systematic use he is made to realize that the processes of algebra, which seem so multifarious and heterogeneous, are, in reality, few and simple.

Such a body of principles furnishes a ready means for the correction of erroneous notions, a constant incitement to effective review, and a definite basis upon which to proceed at each stage of progress.

In the Advanced Course the development is based upon the following important considerations:

(1) The pupil has had a one year's course in algebra, involving constant application of its elementary processes to the solution of concrete problems. This has invested the processes themselves with an interest which now makes them a proper object of study for their own sake.

(2) The pupil has, moreover, developed in intellectual maturity and is, therefore, able to comprehend processes of reasoning with abstract numbers which were entirely beyond his reach in the first year's course.

In consequence of these considerations, the treatment throughout is from a more mature point of view than in the Elementary Course. Relatively greater space and emphasis are given to the manipulation of standard algebraic forms, such as the student is likely to meet in later work in mathematics and physics, and especially such as were too complicated for the Elementary Course.

Attention is called to the following special features of the Advanced Course:

The clear and simple treatment of equivalent equations in Chapter III.

The discussion by formula, as well as by graph, of inconsistent and dependent systems of linear equations.

The unusually complete treatment of factoring and the clear and simple exposition of the general process of finding the Highest Common Factor, in Chapter V.

The careful discrimination in stating and applying the theorems on powers and roots in Chapter VI.

The unique treatment of quadratic equations in Chapter VII, giving a lucid exposition in concrete and graphical form of distinct, coincident, and imaginary roots.

The concise treatment of radical expressions in Chapter X, and especially — an innovation much needed in this connection — the rich collection of problems, in the solution of which radicals are applied.

The authors gratefully acknowledge the receipt of many helpful suggestions from teachers who have used their High School Algebra.

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TABLE OF CONTENTS

ELEMENTARY COURSE

PART ONE

PART TWO

ADVANCED COURSE

PART THREE

LIST OF PRINCIPLES

FIRST PRINCIPLES OF ALGEBRA

ELEMENTARY COURSE

PART ONE

CHAPTER I

INTRODUCTION TO ALGEBRA

LETTERS USED TO REPRESENT NUMBERS

1. Algebra, like arithmetic, deals with numbers. In arithmetic numbers are represented by means of the Arabic **numerals** 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. In algebra **letters,** as well as these numerals, are used to represent numbers.

2. One advantage in using letters to represent numbers is in abbreviating the rules of arithmetic.

Examples. What is the area of a blackboard whose length is 12 feet and whose width is 3 feet? What is the area of a city lot which is 25 feet wide and 120 feet deep?

In arithmetic we multiply the length by the width and the product is the area. In algebra we abbreviate by letting l represent the *length*, w the *width*, and a the *area*.

The rule then stands as follows:

$$l \times w = a$$

This is called a **formula** and covers all possible cases.

It is to be noted that l and w represent the **number of units** in the length and width respectively and a the **number of square units** in the area.

3. The signs of operation, $+$, $-$, $\times$, $\div$, and the sign of equality, $=$, are used in algebra with the same meaning that they have in arithmetic. However, the sign of multiplication is usually omitted between letters, and between an Arabic figure and a letter; but a sign omitted between two Arabic figures always means addition. Multiplication is also indicated by a point written above the line.

Thus, $2 \times l \times w$ may be written $2 \cdot l \cdot w$ or $2\,lw$.

But 25 means $20 + 5$, not $2 \cdot 5$.

Division may be indicated by a fraction, as in arithmetic.

Thus, $2 \div 3$ is written $\frac{2}{3}$ and $a \div b$ is written $\frac{a}{b}$.

Historical Note. The Arabs brought our present system of numerals to Europe when they invaded Spain in the eighth century of our era. It is now known that these numerals (including the zero) are really of Hindu origin, and that they were invented between the years 200 A.D. and 650 A.D. Of all mathematical inventions no one has contributed more to general intelligence or to practical welfare than this one. To appreciate this, one has only to try to multiply two numbers such as 589 and 642 when expressed in the Roman notation; that is, to multiply DLXXXIX by DCXLII.

It was about the year 1500 A.D. that our present symbols indicating addition and subtraction first appeared in a book by a German named Johann Widemann. The sign $\times$ for multiplication was first used about 50 years later by an Englishman, William Oughtred. About the same time the sign $=$ was first used by Recorde, also an Englishman; but the sign $\div$ for division does not appear until 1659 when it was used by a German, J. H. Rohn.

4. In the following exercises, note the simplicity with which the rules of arithmetic may be stated by means of letters. The **systematic use of letters to represent numbers** is one of the chief points of difference between algebra and arithmetic.

EXERCISES

1. A box is 6 inches long, 4 inches wide, and 3 inches high. How many cubic inches does it hold?

2. How many cubic feet of air in a schoolroom which is 35 feet long, 25 feet wide, and 15 feet high?

3. Given the length, width, and height of a rectangular solid, how do you find its volume? State this rule as a formula, using l, w, h, and v for the *number* of units in the length, width, height, and volume, respectively.

4. By a rule of arithmetic the product of $\frac{2}{3}$ and $\frac{5}{7}$ is $\frac{2 \cdot 5}{3 \cdot 7}$. What is the rule for finding the product of any two fractions? This rule may be stated as a formula as follows:

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$$

in which $\frac{a}{b}$ and $\frac{c}{d}$ are any two fractions.

5. By a rule of arithmetic $\frac{3}{4} \div \frac{5}{7} = \frac{3}{4} \times \frac{7}{5} = \frac{3 \cdot 7}{4 \cdot 5}$. What is the rule for dividing one fraction by another? Using $\frac{a}{b}$ and $\frac{c}{d}$ to represent any two fractions, state this rule as a formula.

6. By arithmetic 8 per cent of 90 means $90 \times .08 = 7.2$, in which 90 is called the **base**, .08 the **rate**, and 7.2 the **percentage**.

In like manner find 6 per cent of 125; 5 per cent of 350; 3 per cent of 80; and 10 per cent of 4.9. In each case how is the percentage found? State the rule.

State this rule as a formula, letting b represent the base, r the rate and p the percentage.

7. By a rule of arithmetic the interest on \$ 500 for 3 years at 6 per cent is $500 \times .06 \times 3 = 9$, in which 500 is the **principal**, .06 the **rate**, and 3 the **time**.

8. Find the interest on \$ 900 for 7 years at 4 per cent, and on \$ 1250 for 5 years at 5 per cent.

9. Given any principal, rate, and time, how do you find the interest?

State this rule as a formula, using p for principal, r for rate, t for time and i for interest.

ALGEBRAIC OPERATIONS

5. In algebra numbers are added, subtracted, multiplied, and divided as in arithmetic. There are, however, certain ways of carrying out these operations which are essential in algebra though not used extensively in arithmetic. Some of these we now proceed to study.

6. **Definitions.** Any combination of Arabic numerals, letters, and signs of operation, used for the purpose of representing numbers, is called a **number expression.**

E.g. 38, $18r$, $5n + 8n$, $\frac{3a-2}{b}$ are number expressions.

The expression $\frac{3a-2}{b}$ is said to be **written in symbols,** that is, by means of numerals, letters, and signs of operation.

This number expression written in words would be "three times the number a diminished by 2 and the result divided by b."

EXERCISES

1. If a and b are numbers, express in symbols their sum and also their product.

2. If m and n are numbers, write in symbols m divided by n; also m minus n.

3. If p and q are numbers, write the sum of 5 times p and 3 times q.

4. If a, b, and c are numbers, write in symbols that a multiplied by b equals c; also that c divided by a equals b.

If $a = 3$, $b = 5$, $c = 7$, find the value of each of the following number expressions:

5. $a + b + c$.
6. $a + b - c$.
7. $2a - b + c$.
8. $ab - 2c$.
9. $16a - bc$.
10. $bc - 3a$.
11. $\frac{b+c}{a}$.
12. $\frac{4a-b}{c}$.
13. $3abc - 2b + 3c$.
14. $\frac{2a+3b}{c} + 4b$.
15. $\frac{3c-2a}{3b}$.

7. **Definition.** If a number is the product of two or more numbers, these are together called the **cofactors** of the given number, and any one of them is called a **factor**.

E.g. 3 and 4 are cofactors of 12, as are also 2 and 6, 1 and 12. 1, 2, 3, 4, 6, and 12 are factors of 12. a, b, c are factors of abc.

8. **Definition.** If a number is the product of two factors, then either of these factors is called the **coefficient** of the other in that product.

E.g. In $2 \cdot 3$, 2 is the coefficient of 3, and 3 is the coefficient of 2. In $9\,rt$, 9 is the coefficient of rt, r is the coefficient of $9\,t$, and t is the coefficient of $9\,r$.

In such expressions as $9rt$ the factor represented by Arabic figures is usually regarded as the coefficient.

ADDITION AND SUBTRACTION OF NUMBERS HAVING A COMMON FACTOR

9. It is sometimes necessary to add number expressions like $5\,n$ and $8\,n$ without first assigning a definite value to n.

It seems clear that 5 times any number and 8 times the same number make 13 times that number. That is,

$$5\,n + 8\,n = 13\,n.$$

Determine whether $5\,n + 8\,n = 13\,n$ when $n = 3$; also when $n = 9$; when $n = 12$. Test this for still other values of n.

Note that the sum of $5 \cdot 3$ and $8 \cdot 3$ **may be** found by adding the coefficients of the common factor 3, while the sum of $5\,n$ and $8\,n$ **must be** found in this way.

In this manner perform the following additions:

1. $8\,x + 7\,x + 16\,x + 2\,x$.
2. $13\,n + 8\,n + 7\,n + 9\,n$.
3. $3\,a + a + 2\,a + 6\,a$.
4. $7\,b + 8\,b + b + 6\,b$.
5. $8\,t + 7\,t + 5\,t$.
6. $3\,r + 5\,r + 11\,r$.

Test the correctness of the result in each of the above by letting $x = 2$, $n = 1$, $a = 4$, $b = 3$, $t = 6$, and $r = 5$.

10. Numbers having a common factor may be subtracted in a similar manner.

Thus, from	$64 = 8 \cdot 8$	From	$84 = 12 \cdot 7$	From	$17\,n$
subtract	$48 = 6 \cdot 8$	subtract	$49 = 7 \cdot 7$	subtract	$6\,n$
Remainder	$16 = 2 \cdot 8$		$35 = 5 \cdot 7$		$11\,n$

In this way perform the following subtractions:

1. $8 \cdot 7 - 3 \cdot 7$.
2. $6 \cdot 99 - 5 \cdot 99$.
3. $6\,n - 2\,n$.
4. $8\,a - 3\,a$.
5. $10\,b - 4\,b$.
6. $7\,a - 4\,a$.
7. $23\,x - 16\,x$.
8. $15\,n - 3\,n$.
9. $14\,a - 8\,a$.
10. $12\,b - 9\,b$.
11. $8\,t - 2\,t$.
12. $19\,r - 11\,r$.

Test these results by giving the same values to the letters as were used on page 5. Try also other values.

It is evident that any two numbers having a common factor may be added or subtracted in this manner.

These examples illustrate

Principle I

11. Rule. *To find the sum or difference of two numbers having a common factor, add or subtract the coefficients of the common factor and multiply the result by the common factor.*

12. Definition. The substitution of special values for letters in a number expression in order to test the correctness of an operation is called a **check**.

EXERCISES

Perform the following indicated operations and check the results in the first eight by assigning values to the letters:

1. $68\,t - 11\,t$.
2. $15\,n + 25\,n - 18\,n$.
3. $70\,x - 15\,x + 7\,x - 23\,x$.
4. $18\,k - 3\,k - 2\,k + 6\,k$.
5. $20\,n - 6\,n + 2\,n$.
6. $5\,t + 20\,t - 3\,t$.
7. $8\,s - 3\,s + 20\,s$.
8. $6\,a - 4\,a + 3\,a - 2\,a$.

9. $7\,ab - 3\,ab + 2\,ab$.
10. $11\,rs - 2\,rs - 5\,rs$.
11. $34\,xy - 18\,xy - 4\,xy$.
12. $12\,mn - 6\,mn - 3mn$.
13. $17\,st + 3\,st - 12\,st$.
14. $12\,abc - 2\,abc - 6\,abc$.
15. $42\,xy + 6\,xy - 35\,xy$.
16. $29\,rst - 18\,rst - 6\,rst$.
17. $32\,ac - 17\,ac + 2\,ac$.
18. $91\,a - 81\,a + 2\,a$.
19. $16\,x + 24\,x + 8\,x - 40\,x$.
20. $5\,y + 31\,y - 9\,y - 21\,y$.
21. $63\,c - 47\,c - 8\,c + 7c$.
22. $16\,t - 11\,t - 2\,t + 3\,t$.
23. $12\,xy - 9\,xy + 8\,xy$.
24. $39\,ab - 27\,ab - 8\,ab$.

13. Definitions. Two number expressions representing the same number, when connected by the sign $=$, form an **equality**.

The expressions thus connected are called the **members** of the equality and are distinguished as the **right** and **left** members.

Equalities, such as $5\,n + 8\,n = 13\,n$, in which the letters may be *any numbers whatever*, are called **identities**. Not all equalities are of this kind. For example, $n + 3 = 5$ is true only when $n = 2$.

When it is desired to emphasize that an equality is an identity, the sign $\equiv$ is used. That is, $3\,a + 5\,a \equiv 8\,a$.

14. Definitions. A **parenthesis** is used to indicate that some operation is to be extended over the whole number expression inclosed by it. Thus $2(x + y)$ means that the *sum* of x and y is to be multiplied by 2, while $2\,x + y$ means that x alone is to be multiplied by 2.

Instead of a parenthesis a **bracket** [ ], or a **brace** { }, may be used with the same meaning. Any such symbols are called **symbols of aggregation.**

E.g. $2(x + y)$, $2[x + y]$or $2\{x + y\}$ all mean the same thing.

Historical Note. The parenthesis ( ) was first used with its present meaning by an Englishman, A. Girard, in a book on "Arithmetic," published in the year 1629. The bracket and brace are of later origin, as is also the sign $\equiv$ to denote identity.

MULTIPLICATION OF THE SUM OR DIFFERENCE OF TWO NUMBERS

15. The sum of two or more numbers may be multiplied by another number in two ways, as shown in the following examples:

(1) $4(2+7)= 4\cdot 9=36,$

or $4(2+7)=(4\cdot 2)+(4\cdot 7)=8+28=36.$

(2) $3(3+8+9)=(3\cdot 20)=60,$

or $3(3+8+9)=(3\cdot 3)+(3\cdot 8)+(3\cdot 9)=9+24+27=60.$

In each case the same result is obtained *whether we first add the numbers in the parenthesis and then multiply the sum, or first multiply these numbers one by one and then add the products.*

In case the numbers are represented by letters, the second process only is available.

E.g. $3(a+b)=3a+3b$ and $m(r+s)=mr+ms$.

Multiply each of the following in two ways where possible:

1. $3(2+7)$.	4. $3(a+6)$.	7. $x(3+7+10)$.
2. $5(3+4+5)$.	5. $11(h+k)$.	8. $15(x+y+z)$.
3. $8(5+9+7)$.	6. $4(a+b+c)$.	9. $20(m+n+p)$.

16. The difference of two numbers in Arabic figures may likewise be multiplied by a given number in either of two ways.

E.g. $6(8-3)=6\cdot 5=30,$

or $6(8-3)=(6\cdot 8)-(6\cdot 3)=48-18=30.$

In the case of numbers represented by letters evidently the second process only is available.

E.g. $6(r-t)=6r-6t$ and $a(c-d)=ac-ad$.

Perform as many as possible of the following multiplications in two ways:

1. $7(9-2)$.	4. $17(18-11)$.	7. $5(x-1)$.	10. $m(r-s)$.
2. $12(17-7)$.	5. $9(a-2)$.	8. $3(y-2)$.	11. $x(y-z)$.
3. $5(12-8)$.	6. $8(h-4)$.	9. $a(c-d)$.	12. $t(u-v)$.

The foregoing examples illustrate

Principle II

17. Rule. *To multiply the sum or difference of two numbers by a given number, multiply each of the numbers separately by the given number, and add or subtract the products.*

EXERCISES

1. Multiply $5+7+11$ by 3 without first adding, and then check by performing the addition before multiplying.

2. Multiply $m+n$ by 4 and check for $m=5$, $n=7$.

$$4(m+n)=4\,m+4\,n$$

Check. $4(5+7)=4\cdot 12=48$, also

$$4(5+7)=(4\cdot 5)+(4\cdot 7)=20+28=48.$$

3. Multiply $x+y$ by r and check for $x=2$, $y=4$, $r=6$.

4. Multiply $r+s$ by k and check for $r=4$, $s=5$, $k=6$.

5. Multiply $a+b+c$ by m and check for $a=3$, $b=2$, $c=1$, $m=4$.

6. Multiply $m-n+2$ by c and check for $m=5$, $n=2$ and $c=3$.

7. Multiply $a-b-c$ by d and check for $a=10$, $b=3$, $c=4$ and $d=8$.

8. Multiply $r+s+9-t$ by c and check for $r=1$, $s=8$, $t=3$, and $c=2$.

Find the following products:

9. $8(13-5)$.
10. $2(12+41-36)$.
11. $9(a+8-b)$.
12. $3(a+x+y-1)$.
13. $3(a+b-c)$.
14. $a(18-7)$.
15. $7(3+8+9-a)$.
16. $5(7+a-b)$.
17. $3(x+2-y)$.
18. $8(a+b-c+d)$.
19. $8(9-a+b+3)$.
20. $a(3+b-c)$.
21. $c(m-n+p)$.
22. $19(8-x-y+z)$.

DIVISION OF THE SUM OR DIFFERENCE OF TWO NUMBERS

18. In dividing the sum or difference of two numbers by a given number, when these are represented by Arabic figures, the process may be carried out in two ways. Thus,

(1) $(12 + 8) \div 2 = 20 \div 2 = 10,$

or $(12 + 8) \div 2 = (12 \div 2) + (8 \div 2) = 6 + 4 = 10.$

(2) $(20 - 12) \div 4 = 8 \div 4 = 2,$

or $(20 - 12) \div 4 = (20 \div 4) - (12 \div 4) = 5 - 3 = 2.$

Describe each of these two ways of dividing a sum or difference by a number. How do the results compare?

If the numbers in the dividend are represented by letters, the division can usually be carried out only in the second manner shown above.

E.g. $(r + t) \div 5 = (r \div 5) + (t \div 5)$, or, $\frac{r+t}{5} = \frac{r}{5} + \frac{t}{5}.$

In either case this is read: r *plus* t *divided by* 5 *equals* r *divided by* 5 *plus* t *divided by* 5.

In this manner perform each of the following divisions in two ways when possible and check the results:

1. $(16 + 12) \div 4.$
2. $(20 - 10) \div 5.$
3. $(x - y) \div 6.$
4. $(x + z) \div 3.$
5. $(m - r) \div a.$
6. $(m + r) \div b.$

These examples illustrate

Principle III

19. Rule. *To divide the sum or difference of two numbers by a given number divide each number separately and add or subtract the quotients.*

EXERCISES

1. Divide $72 + 56$ by 8 without first adding.
2. Divide $144 - 36$ by 12 without first subtracting.
3. Divide $r + t$ by 5 and check the quotient when $r = 15$, $t = 25$; also when $r = 60$, $t = 75$.

4. Multiply $7 + 9$ by 3 without first adding 7 and 9.

5. Multiply $25 - 8$ by 5 without first subtracting.

6. Find the product of 12 and $a + b$, checking the result when $a = 5$, $b = 7$.

Perform the following indicated operations:

7. $3(a + b + c + d)$. Check for $a = 1$, $b = 2$, $c = 3$, $d = 4$.

8. $7(r - s + t - x)$. Check for $r = t = 5$, $s = x = 4$.

9. $(m + n + r) \div 4$. Check for $m = 64$, $n = 32$, $r = 8$.

10. $(x + y + z) \div 5$. Check for $x = 100$, $y = 50$, and $z = 25$.

11. $74\,rs - 67\,rs - 2\,rs - 3\,rs$.

12. $(63 - 35 - 14 + 21) \div 7$.

13. $a(4 - d + b + c + 3)$.

14. $(21 - x - y + 3 + c)k$.

15. $49\,pq + 18\,pq - 62\,pq + 3\,pq$.

16. $13\,xyz + 3\,xyz - 8\,xyz - 7\,xyz$.

17. $(q + r + s + t - a - b) \div c$.

18. $35\,lm - 33\,lm + 7\,lm - 3\,lm - 2\,lm$.

19. $k(l + m + n + r - s - t)$.

20. $a(c + d - e + f - g)$.

21. $27\,abc - 19\,abc - 4\,abc + 8\,abc$.

22. $(a + r + s - t - q) \div 3$.

23. $(12 - x + y - z) \div 6$.

24. For what values of a, b, c, d are the following equalities true?

(a) $ab + ac + ad = a(b + c + d)$.

(b) $ab + ac - ad = a(b + c - d)$.

(c) $\dfrac{a + b + c}{d} = \dfrac{a}{d} + \dfrac{b}{d} + \dfrac{c}{d}$.

Historical Note. The fundamental character of Principles II and III was not fully appreciated until the first part of the last century. Principle II states what is called the **Distributive Law of Multiplication** with respect to addition and subtraction. That is, the multiplier is *distributed* over the multiplicand. The name was first used by a Frenchman F. J. Servois, in a paper published in 1814. Principle III states the same law for division. Compare notes on pages 57, 67.

MULTIPLICATION OF A PRODUCT

20. In arithmetic to multiply a product like $3 \cdot 5$ by 2, we first multiply 3 by 5 and this result by 2.

Thus, $$2 \cdot (3 \cdot 5) = 2 \cdot 15 = 30.$$

We cannot, however, multiply $3n$ by 2 in this way. But it is obvious that twice $3n$ is $3n + 3n = 6n$.

Thus, if $n = 5$, $$2 \cdot (3 \cdot 5) = 6 \cdot 5.$$

Hence in multiplying the product $3 \cdot 5$ by 2 we may multiply the factor 3 *only*. Would you get the same result if you multiplied the factor 5 only?

Example. Find the product $2(3 \cdot 4 \cdot 5)$ in as many ways as possible.

Solution:

$2(3 \cdot 4 \cdot 5) = 2 \cdot 60 = 120$, and $2(3 \cdot 4 \cdot 5) = 3 \cdot 8 \cdot 5 = 120$,
Also $2(3 \cdot 4 \cdot 5) = 6 \cdot 4 \cdot 5 = 120$, and $2(3 \cdot 4 \cdot 5) = 3 \cdot 4 \cdot 10 = 120$.

In like manner find the following products:

1. $5(3 \cdot 7)$. **2.** $8(2 \cdot 3 \cdot 4)$. **3.** $9(2 \cdot 5)$. **4.** $6(5 \cdot 3)$.
5. $3(5 \cdot 2)$. **6.** $4(7 \cdot 2 \cdot 3)$. **7.** $7(2 \cdot 4 \cdot 3)$. **8.** $5(2 \cdot 5 \cdot 3)$.

These examples illustrate

Principle IV

21. Rule. *To multiply the product of several factors by a given number, multiply any one of the factors by that number.*

22. Principles IV and II should be carefully contrasted, as in the following example:

$$2(2 \cdot 3 \cdot 5) = 4 \cdot 3 \cdot 5 = 2 \cdot 6 \cdot 5 = 2 \cdot 3 \cdot 10,$$

but $$2(2 + 3 + 5) = 4 + 6 + 10.$$

In multiplying the product of several numbers we operate upon any one of them, but in multiplying the sum or difference of numbers we operate upon each of them.

EXERCISES

Multiply as many as possible of the following in two or more ways. Check where letters are involved.

1. $7(3 \cdot 4 \cdot 5)$.	5. $3(5xy)$.	9. $15(7ab)$.
2. $8(7 \cdot 2 \cdot 3)$.	6. $3(8xyz)$.	10. $3(4mn)$.
3. $9(2 \cdot 3 \cdot 4)$.	7. $4(19 \cdot 25)$.	11. $7(2xy)$.
4. $5(2ab)$.	8. $5(7abc)$.	12. $2(16rs)$.

DIVISION OF A PRODUCT

23. Division of the product of several factors by a given number may be performed in various ways:

E.g. $(4 \cdot 6 \cdot 10) \div 2 = 240 \div 2 = 120.$

Also $(4 \cdot 6 \cdot 10) \div 2 = 2 \cdot 6 \cdot 10 = 120,$

$(4 \cdot 6 \cdot 10) \div 2 = 4 \cdot 3 \cdot 10 = 120,$

and $(4 \cdot 6 \cdot 10) \div 2 = 4 \cdot 6 \cdot 5 = 120.$

Note that in each case only one factor is divided.

Perform each of the following divisions in more than one way where possible:

1. $(5 \cdot 8 \cdot 3) \div 2$.	4. $(11 \cdot 20 \cdot 16) \div 4$.	7. $(10 \cdot 35 \cdot 3) \div 5$.
2. $20abc \div 4$.	5. $14xyz \div 7$.	8. $14xyz \div x$.
3. $12abc \div 3$.	6. $12abc \div c$.	9. $(12 \cdot 40 \cdot 13) \div 8$.

These examples illustrate

Principle V

24. **Rule.** *To divide the product of several factors by a given number divide any one of the factors by that number.*

25. Principle V is already known in arithmetic in the process called cancellation.

Thus, in the fraction $\frac{2 \cdot 6 \cdot 9}{3}$, 3 may be canceled out of either 6 or 9, giving $\frac{2 \cdot 6 \cdot 9}{3} = 2 \cdot 2 \cdot 9$ or $2 \cdot 6 \cdot 3$.

EXERCISES

1. Contrast Principles III and V.

By Principle V, $(4 \cdot 6 \cdot 8) \div 2 = 2 \cdot 6 \cdot 8 = 4 \cdot 3 \cdot 8 = 4 \cdot 6 \cdot 4$.
By Principle III, $(4 + 6 + 8) \div 2 = 2 + 3 + 4$.

That is *in dividing the product of several numbers by another number we operate upon any one of them, but in dividing their sum or difference we operate upon each of them.*

Perform the following indicated operations:

2. $5\,abc \div a$.

3. $15\,xy \div 3$.

4. $5(2\,x + 3\,y)$.

5. $(33\,x - 44\,y) \div 11$.

6. $(78\,s - 39\,t) \div 13$.

7. $8(3\,a + 2\,y)$.

8. $15(a - b + c) \div 5$.

9. $3\,a(7 - c + b) \div a$.

10. $5\,bc(d - e + 3) \div bc$.

11. $19(3\,a - 6\,b + 9\,c) \div 3$.

12. $13(8 - 4\,b + 12\,a) \div 4$.

13. $14(7 - 7\,m + 14\,n) \div 7$.

14. $12\,a(3\,b - 3\,c + 9) \div 3$.

15. $24(16\,b - 8\,c + 24\,d) \div 8$.

16. Divide $7\,a \cdot 14\,b \cdot 21\,c$ by 7 in three different ways.

17. Add $5\,a$, $\frac{12\,a}{2}$, $\frac{21\,a}{3}$, and $\frac{18\,a}{6}$, using Principles V and I.

18. From $\frac{28\,xy}{4}$ subtract $\frac{21\,xy}{7}$, using Principles V and I.

19. From $\frac{14\,a}{2} + \frac{10\,a}{5}$ subtract $\frac{6\,a}{3}$.

20. Find the sum of $\frac{16\,x}{8}$, $\frac{20\,x}{5}$, $\frac{16\,x}{4}$, $7\,x$, and $3\,x$.

21. Find the sum of $\frac{100\,rs}{10}$, $\frac{90\,rs}{9}$, and $\frac{25\,rs}{5}$.

22. From $25\,xy$ subtract $\frac{13\,xyz}{z}$.

23. Add $\frac{8\,abc}{c} + \frac{18\,ab}{3} + \frac{7\,abd}{d} + \frac{abe}{e}$.

24. Add $\frac{18\,a}{6} + \frac{5\,ba}{b} + \frac{4\,xa}{x} + \frac{am}{m}$.

ORDER OF INDICATED OPERATIONS

26. In a succession of indicated operations the final result depends in some cases upon the *order* in which the steps are taken:

Thus, $6 + 3 \cdot 8$ would give $9 \cdot 8 = 72$, if the addition were performed first; and would give $6 + 24 = 30$, if the multiplication were performed first.

Similarly, $24 \div 2 \cdot 3$ would give $12 \cdot 3 = 36$, if the division were performed first; and would give $24 \div 6 = 4$, if the multiplication were performed first.

27. However, *when only additions and subtractions are involved and no symbols of aggregation occur, the result is the same no matter in what order the operations are performed.*

Test this statement by performing each of the following indicated operations in several different orders:

1. $8 - 3 + 14 - 4$.
2. $13 + 4 - 3 + 2$.
3. $7 - 6 + 1 + 3$.
4. $15 - 12 + 4 - 2$.

A similar statement holds *in some cases* when only multiplications and divisions are involved.

Thus, $12 \times 6 \div 3 = 24$ no matter in which order the operations are performed, while $24 \div 2 \cdot 3$ may equal 4 or 36.

In case of doubt symbols of aggregation should be used to show the order intended.

Thus, $24 \div (2 \cdot 3) = 4$, while $(24 \div 2) \cdot 3 = 36$.

And $18 \div (6 \div 2) = 6$, while $(18 \div 6) \div 2 = \frac{3}{2}$.

The following rule is in accordance with universal custom:

28. *In an expression involving additions, subtractions, multiplications, and divisions without any symbols of aggregation, all multiplications and divisions are to be performed before any additions or subtractions.*

E.g. $5 + 3 \cdot 4 - 8 \div 2 = 5 + 12 - 4 = 13$, and $8 - 2 \div 2 - 2 \times 2 + 3 = 8 - 1 - 4 + 3 = 6$.

EXERCISES

Perform the following indicated operations:

1. $20 + 5 - 3 + 4 - 2.$
2. $12 \div 3 - 2 + 5.$
3. $(28 \cdot 4) \div 2 + 10.$
4. $28(4 \div 2) + 10.$
5. $(28 \div 4) \cdot 2 + 10.$
6. $5 \cdot 3 + 2 \cdot 7 - 19.$
7. $26 \div 2 - 6 \cdot 2.$
8. $18 - 2 \cdot 6 - 8 \div 4 + 20.$

DRILL EXERCISES

NOTE. — The pages headed "Drill Exercises" are intended to afford general practice in performing algebraic operations. Each such page consists of miscellaneous exercises on the various topics that precede.

If $a = 5$, $b = 3$, $c = 2$, find the numerical value of each of the following:

1. $ac + bc$ and $(a + b)c.$
2. $ac - bc$ and $(a - b)c.$
3. $abc, a \times (bc)$ and $b \times (ac).$
4. $a(b + c)$ and $ab + ac.$
5. $a(b - c)$ and $ab - ac.$
6. $\frac{ab}{c}, \frac{a}{c} \times b$ and $a \times \frac{b}{c}.$
7. $\frac{a+b}{c}$ and $\frac{a}{c} + \frac{b}{c}.$
8. $\frac{a-b}{c}$ and $\frac{a}{c} - \frac{b}{c}.$

Perform the following indicated operations and state what principle is used in each case:

9. $3\,ax + 7\,ax - 4\,ax.$
10. $7\,by + 4\,by - 9\,by.$
11. $3\,y - 2\,y + y.$
12. $c(4\,a + 2\,b).$
13. $5(16\,a - 3\,b).$
14. $a(11 - b).$
15. $(24\,x + 9\,y) \div 3.$
16. $(6\,a - 4\,b) \div 2.$
17. $(3\,xy + 2\,bx) \div x.$
18. $(7\,abc + 2\,abd) \div ab.$
19. $2\,a(7\,x + 3\,y - 4).$
20. $3\,x(2\,a - 5\,b + c).$

What principles are used in the following operations?

21. $3 \cdot 5\,ab = 15\,ab.$
22. $16\,xy \div x = 16\,y.$
23. $3\,t + 15\,t = 18\,t.$
24. $78\,h - 41\,h = 37\,h.$

29. Importance of the Principles. The five principles studied in this chapter, together with others which will be introduced when needed, will be found of increasing importance as we proceed. Your success in the further study of algebra will depend in no small degree upon the clearness with which you understand their real significance. The most effective way to master them is by means of simple numerical examples such as were used in introducing each one. Make a list of these principles in abbreviated form for yourself and note how frequently your own errors and those of your classmates are due to direct violations of one or more of them.

REVIEW QUESTIONS

1. How would $3 \cdot 5$ and $7 \cdot 5$ be added in arithmetic? Why cannot $3n$ and $7n$ be added in the same manner? State in full the principle by which $3n$ and $7n$ are added. In this example what number is represented by n? Test the identity $3n + 7n = 10n$ by substituting any convenient value for n.

2. What kind of numbers may be added by Principle I? Have the numbers ac and bc a common factor? What is it? What is the coefficient of this common factor in each? What is the sum of these coefficients? Is the equality $ac + bc = (a + b)c$ true no matter what numbers are represented by a, b, and c? When this can be said of an equality, what is it called?

3. How is $5 \cdot 9$ subtracted from $11 \cdot 9$ in arithmetic? In what different manner may this operation be performed? Why is it sometimes necessary to perform subtraction in the second way? In the identity $31x - 12x = 19x$, what number is represented by x? Test the equality by substituting any convenient number for x. Is this equality true for every value of x?

Principle I may be conveniently abbreviated as follows:

$$\begin{cases} ac + bc = (a + b)c, \\ ac - bc = (a - b)c. \end{cases}$$

4. How is $11+3$ multiplied by 4 in arithmetic? In what different way may this operation be performed? Why is it sometimes necessary to multiply in the second way? State in full the principle by which $a+8$ is multiplied by 7.

Principle II may be abbreviated thus:

$$\begin{cases} c(a+b)=ca+cb, \\ c(a-b)=ca-cb. \end{cases}$$

Notice that the identities in Principle II are the same as those in I read in reverse order.

5. How is $12+18$ divided by 6 in arithmetic? In what different way may this division be performed? Why is it sometimes necessary to perform division in the second way? State in full the principle used in performing the operation $(6x+9y)\div 3$. How do you divide $(6x-9y)$ by 3?

Principle III may be abbreviated thus:

$$\left\{\frac{a+b}{k}=\frac{a}{k}+\frac{b}{k}, \quad \frac{a-b}{k}=\frac{a}{k}-\frac{b}{k}.\right.$$

6. How is the product $2 \cdot 3 \cdot 5$ multiplied by 4 in arithmetic? In what different way may this multiplication be performed? Why is it ever performed in the second way? What are the factors of the number ab? How is the product of two numbers multiplied by another number? Should *both* factors be multiplied by the number or only one? Is it permissible to multiply *either one* we choose? Principle IV is abbreviated thus:

$$k \times (ab) = (ka) \times b = a \times (kb).$$

7. Divide $2 \cdot 4 \cdot 6 \cdot 20$ by 2 without first performing the multiplication indicated in $2 \cdot 4 \cdot 6 \cdot 20$. Do this in several ways and show that all the quotients obtained are equal. State in full the principle used.

Principle V is abbreviated thus:

$$(ab) \div k = \frac{a}{k} \times b = a \times \frac{b}{k}.$$

8. Contrast Principles II and IV; also III and V.

Historical Note. The oldest mathematical manuscript in existence is a papyrus now in the British Museum which was written by Ahmes, an Egyptian priest, possibly as early as 3000 B.C. This contains, besides work in arithmetic, some very simple problems involving the beginnings of algebra. We have no knowledge that the early Egyptians advanced beyond these simplest rudiments.

Singularly enough, the Greeks failed to make many important contributions to algebra, though they greatly advanced the sister science geometry. Diophantus, who lived in Alexandria about 320 A.D., is the only Greek whose contribution to algebra is worthy of note. Much of his work, however, is not of a kind which can be included in an elementary book such as this.

These beginnings in Egypt and Greece were so insignificant that the Hindus may be said to have created algebra. Most of the material contained in beginners' books such as the present one was known to the Hindus by the year 600 A.D. The Arabs learned algebra from the Hindus and brought it to Europe in the eighth century. It was not until the twelfth century, however, that the Arabian books on algebra were translated into Latin and thus became accessible to Europeans.

The Arabs used very few symbols. Number expressions were written out fully in words. Thus, instead of $x^2 + 10x = 39$ they wrote *A square and ten of its roots are equal to thirty-nine.* In this respect they took a step backward, for the Hindus, from whom the Arabs learned much of their algebra, had made considerable use of symbols. As we have seen (page 2) it was not until the middle of the seventeenth century that the present symbols came into general use.

Letters were used now and then from the earliest time as abbreviations for words representing numbers. But the systematic use of letters for this purpose began with the great French algebraist, François Vieta (1540–1607).

It must not be supposed, however, that the use of these symbols was adopted suddenly. It is a curious fact of history that people cling to the old, even though the new is vastly superior. It was only after René Descartes (1596–1650) had used these symbols in his works that they came to be used in texts on algebra (see also page 50).

CHAPTER II

EQUATIONS AND PROBLEMS

The principles developed in the last chapter will now be used in the solution of equations and problems.

SOLUTION OF EQUATIONS

30. Definitions. Equalities in which letters are used as number symbols are of two kinds, namely:

(1) Identities, such as $3(5x + 6y) = 15x + 18y$, which holds for all values which may be assigned to x and y. See § 13.

(2) Equalities, such as $3x = 18$, which holds if, and only if, $x = 6$.

The equality $3x = 18$ is said to be **satisfied** by $x = 6$, because this value of x reduces both members to the same number, 18.

Ex. 1. Is $x + 4 = 9$ satisfied by $x = 4$? by $x = 5$? by $x = 6$?

Ex. 2. Is $7x + 9 = 3x + 25$ satisfied by $x = 2$? by $x = 3$? by $x = 4$?

Ex. 3. Is $\frac{x}{4} + 3 = 6$ satisfied by $x = 4$? by $x = 8$? by $x = 12$?

Ex. 4. Is $\frac{x+1}{3} + 2x - 1 = 2x$ satisfied by $x = 2$? by $x = 4$? by $x = 1$?

31. An equality which is satisfied only when certain particular values are given to one or more of its letters is called a **conditional equality** with respect to those letters.

E.g. $3x + 5 = 35$ is an equality only on the condition that $x = 10$. $x + y = 10$ is an equality for certain pairs of values of x and y like 1 and 9, 2 and 8, 3 and 7, 5 and 5, but certainly not for all values of x and y; for instance, not for $x = 3$ and $y = 8$.

32. A conditional equality is called an **equation**, and a letter whose particular value is sought to satisfy an equation is called an **unknown** in that equation.

The equations at present considered contain only one unknown.

33. **To solve an equation** in one unknown is to find the value or values of the unknown which satisfy it. Such a value of the unknown is called a **root** or **solution** of the equation.

The following examples illustrate methods used in solving equations.

Ex. 1. Solve the equation $x - 5 = 9$. (1)

Solution. This equation states that 9 is 5 less than the number x, that is, if 9 be increased by 5, the result is x.

Hence, $x = 14$ is the solution of equation (1).

This result may be obtained by adding 5 *to each member of equation* (1), thus

$$x + 5 - 5 = 9 + 5,$$

or

$$x = 14. \qquad (2)$$

Check. Substitute $x = 14$ in equation (1) and get

$$14 - 5 = 9.$$

Hence the equation is satisfied by $x = 14$.

Ex. 2. Solve the equation $x + 7 = 12$. (1)

Solution. This equation states that 12 is 7 more than the number x, that is, if 12 be diminished by 7, the result is x.

Hence, $x = 5$ is the solution of equation (1).

This result may be obtained by subtracting 7 *from both members of* (1), thus

$$x + 7 - 7 = 12 - 7,$$

or

$$x = 5. \qquad (2)$$

Check. Substitute $x = 5$ in (1) and get

$$5 + 7 = 12.$$

Hence the equation is satisfied by $x = 5$.

Ex. 3. Solve the equation $\frac{1}{3}x = 7$. (1)

Solution. This equation states that one-third of the number x is 7, that is, x is three times 7, or 21.

Hence $x = 21$ is the solution of equation (1).

This result may be obtained by multiplying both members of (1) *by* 3, thus

$$3 \cdot \tfrac{1}{3}x = 3 \cdot 7,$$

or

$$x = 21. \qquad (2)$$

Check. Substitute $x = 21$ in (1) and get

$$\tfrac{1}{3} \cdot 21 = 7.$$

Hence the equation is satisfied by $x = 21$.

Ex. 4. Solve the equation $5x = 30$. (1)

Solution. This equation states that 5 times the number x is 30, that is, x is one-fifth of 30, or 6.

Hence, $x = 6$ is the solution of the equation.

This result may be obtained by dividing both members of equation (1) *by* 5, thus

$$\frac{5x}{5} = \frac{30}{5},$$

or

$$x = 6. \qquad (2)$$

Check. Substitute $x = 6$ in equation (1) and get

$$5 \cdot 6 = 30.$$

Hence the equation is satisfied by $x = 6$.

34. The above examples illustrate four ways of operating upon both members of an equation, so as to produce in each case a new equation *satisfied by the same value of the unknown as the original equation.*

Each of these operations *changes the value of both members, but changes them both alike.*

In the next example, equations (2) and (3) show two ways *of changing the form, but not the value, of one member alone,* and thus producing a new equation satisfied by the same value of the unknown as the original equation.

Ex. 5. Solve the equation

$$w + 2(w + 5) = 58. \quad (1)$$

By Principle II, $w + 2w + 10 = 58.$ (2)

By Principle I, $3w + 10 = 58.$ (3)

Subtracting 10 from both members, $3w = 48.$ (4)

Dividing both members by 3, $w = 16.$ (5)

Check. Putting $w = 16$ in (1), $16 + 2(16 + 5) = 16 + 2 \cdot 21 = 58.$

By substitution verify that $w = 16$ also satisfies equations (2), (3), and (4).

The operations involved in passing from (1) to (2) and (2) to (3) are called **form changes** which leave the value of the left member unaltered.

All the operations involved in Principles I to V are *form changes* of this character. See the list at the end of Chapter I. There are other form changes which will be considered as need arises.

35. The members of an equation may be likened to the **scale-pans of a common balance** in which are placed objects of uniform weight, say tenpenny nails. The scales balance only when the weights are the same in both pans; that is, when the number of nails is the same.

If now the scales are in balance, they will remain so under two kinds of changes in the weights:

(*a*) When the number of nails in the two pans is increased or diminished by the same amount; corresponding to **like changes in value of both members of an equation.**

(*b*) When the number in each pan is left unaltered but the nails are rearranged in groups or piles in any manner; corresponding to **form changes** on either member of an equation.

The equation, then, is like a balance, and its members are to be operated upon only in such ways as to preserve the balance.

EXERCISES

Solve the following equations and explain each step involved, as in the above illustrative examples.

1. $7x = 42.$

2. $11n = 77.$

3. $\frac{1}{3}y = 8.$

4. $\frac{1}{5}x = 18.$

5. $\frac{1}{6}n = 20.$

6. $6w = 28.$

7. $x - 3 = 7.$

8. $2n + 5 = 15.$

9. $3y + 8 = 17.$

10. $x - 4 = 12.$

11. $2y - 3 = 15.$

12. $4w - 5 = 21.$

13. $x + 3x + 4x = 16.$

14. $5n - 2n + 3n = 40.$

15. $6y + 8y - 3y = 33.$

16. $5x + 7 + 2x = 14.$

17. $6y + 8y - 5 = 23.$

18. $3w - 2w - 2 = 4.$

The foregoing examples illustrate

Principle VI

36. **Rule.** *An equation may be changed into another equation such that any value of the unknown which satisfies one also satisfies the other, by means of any of the following operations:*

(1) *Adding the same number to both members;*

(2) *Subtracting the same number from both members;*

(3) *Multiplying both members by any number not zero;*

(4) *Dividing both members by any number not zero;*

(5) *Changing the form of either member in any way which leaves its value unaltered.*

The operations under Principle VI are hereafter referred to in detail by means of the initial letters, *A* for *addition*, *S* for *subtraction*, *M* for *multiplication*, *D* for *division*, and *F* for *form changes* which leave the value of a member unaltered.

Historical Note. The first Arabic algebra known to us dates from the first half of the ninth century. It bore the name Al-gebr walmukâbâla. The word *al-gebr*, from which the word *algebra* is derived, means "restoration" and refers to the fact that the same member may be added to or subtracted from each member of an equation. The word "walmukâbâla" means the process of simplification or *form changes*, such as have just been described. Thus it appears that the Arabs regarded the solution of equations as the main business of algebra.

DIRECTIONS FOR WRITTEN WORK

37. In solving an equation the successive steps should be written as in the following:

Ex. 1. $25(n+1)+6(4n-3)=50+31n+2(3-n)-9.$ (1)

By F, using Principle II, we obtain from (1)

$$25n+25+24n-18=50+31n+6-2n-9. \quad (2)$$

By F, using Principle I, we obtain from (2)

$$49n+7=29n+47. \quad (3)$$

Subtracting 7 and $29n$ from each member of (3) and using Principle I, we have $20n=40.$ (4)

Dividing each member of (4) by 20,

$$n=2. \quad (5)$$

Check. Substitute $n=2$ in equation (1).

For convenience this work can be abbreviated as follows:

	$25(n+1)+6(4n-3)=50+31n+2(3-n)-9.$	(1)
By F, II,	$25n+25+24n-18=50+31n+6-2n-9.$	(2)
By F, I,	$49n+7=29n+47.$	(3)
By $S\mid 7, 29n$,	$20n=40.$	(4)
By $D\mid 20$,	$n=2.$	(5)

$S\mid 7, 29n$ means that 7 and $29n$ are each to be subtracted from both members of the preceding equation. $D\mid 20$ means that the members of the preceding equation are to be divided by 20.

Similarly, in case we wish to indicate that 6 is to be added to each member of an equation, we would write $A\mid 6$, and if each member is to be multiplied by 8, we would write $M\mid 8$. It is important that the nature of each step be recorded in some such manner.

Ex. 2. $17n+4(2+n)-6=5(4+n)-5+3n.$ (1)

By F, II,	$17n+8+4n-6=20+5n-5+3n.$	(2)
By F, I,	$21n+2=15+8n.$	(3)
By $S\mid 2, 8n$,	$21n-8n=15-2.$	(4)
By F, I,	$13n=13.$	(5)
By $D\mid 13$,	$n=1.$	(6)

Check. Substitute $n=1$ in equation (1).

38. By use of Principle VI, a term may be **transposed** from one member of an equation to the other by changing its sign.

E.g. in deriving equation (4) from (3) in the last solution, $8n$ was subtracted from both sides by mentally *dropping* it on the right and *indicating* its subtraction on the left. Likewise when 2 is subtracted from both sides it *disappears* on the left and *appears* on the right with the *opposite* sign. Each of these indicated subtractions might have been performed *mentally*, thus writing equation (5) directly from (3).

After a little practice this shorter process of *transposing terms* and *combining similar terms mentally* should always be used.

39. An **equation may be translated into a problem.** For example, the equation $21x + 2 = 8x + 15$ may be interpreted as follows: *Find a number such that 21 times the number plus 2 is 15 greater than 8 times the number.*

EXERCISES

Solve the following equations, putting the work in a form similar to the above and checking each result. Translate the first five into problems.

1. $13x + 40 - x = 88.$
2. $3x + 9 + 2x + 6 = 18 + 4x.$
3. $5x + 3 - x = x + 18.$
4. $13y + 12 + 5y = 32 + 8y.$
5. $4m + 6m + 4 = 9m + 6.$
6. $7m + 18 + 3m = 12 + 2m + 38.$
7. $3y + 4 + 2y + 6 = y + 7 + y + 3 + 30.$
8. $5x + 3 + 2x + 3 = 2x + 5 + 3x + 3 + x.$
9. $2x + 4x + 9 - x + 6 = 20 + 2x + 5 + x.$
10. $18 + 6m + 30 + 4m = 4m + 8 + 12 + 3m + 3 - m + 29.$
11. $y + 72 + 45y = 106 + 12y.$
12. $42x + 56 = 20x + 122.$
13. $6x + 8 + x + 4 + 5x = 7x + 32 - x - 20.$

14. $32x+4+7x=66+3x+5x.$

15. $12m+3-3m=38+2m.$

16. $15m+3-2m+7=3m+60.$

17. $a+7+3a=2a+45.$

18. $5b+30+6b=3b+150.$

19. $3c+18+14c=6c+51.$

20. $17x+4+3x=7x+30.$

21. $7(m+6)+10m=42+5m+24.$

22. $6x+4(4x+2)+3(2x+7)=85.$

23. $8+7(6+6n)+2n=2(4n+5)+18n+49.$

24. $5(9x+3)+4(3x+2)=18x+36.$

25. $7(3x+2)+13=5(2-x)+43.$

26. $15+3(3+x)+2(2+6x)=3(3x+4)+28.$

27. $11(x+5)+3(3x-1)=7(x+2)+4x+47.$

SOLUTION OF PROBLEMS

40. One great object in the study of algebra is to **simplify the solution of problems.** This is done by using letters to represent the unknown numbers, by stating the problem in the form of an equation, and by arranging the successive steps of the solution in an orderly manner.

41. Illustrative Problem. 1. The shortest railway route from Chicago to New York is 912 miles. How long does it take a train averaging 38 miles an hour to make the journey?

Solution. Let t be the number of hours required. Then $38t$ is the distance traveled in t hours. But 912 miles is the given distance traveled.

Hence $\quad 38t=912. \quad (1)$

By $D|38 \quad t=24. \quad (2)$

Check. Substitute $t=24$ in equation (1) and get

$$38\cdot 24\equiv 912.$$

Hence $t=24$ satisfies the conditions of the problem.

42. Illustrative Problem. 2. For how many years must \$850 be invested at 5% simple interest in order to yield \$255?

Solution. From arithmetic, we have

$$\textit{principal} \times \textit{rate} \times \textit{time} = \textit{interest},$$

or $prt = i.$ See page 3, Ex. 7.

Hence, from the conditions of this problem,

$$850 \times .05 \times t = 255, \quad (1)$$

or $$42.5\,t = 255.$$

By $D \mid 42.5$ $$t = 6. \quad (2)$$

Check. Substitute $t = 6$ in (1) and find

$$42.5 \times 6 \equiv 255.$$

Hence 6 years satisfies the conditions of the problem.

43. Illustrative Problem. 3. A boy, an apprentice, and a master workman have the understanding that the apprentice shall receive twice as much as the boy and the master workman five times as much as the boy. How much does each get, if the total amount received for a piece of work is \$104.

Solution. Let n represent the number of dollars received by the boy.

Then, $2n$ is the number of dollars received by the apprentice,

and $5n$ is the number of dollars received by the master workman.

Hence, $n + 2n + 5n$ and 104 are number expressions, each representing the total amount received.

Therefore, $$n + 2n + 5n = 104. \quad (1)$$

By Principle I, $$8n = 104. \quad (2)$$

By $D \mid 8$, $$n = 13. \quad (3)$$

Hence, the amount received by the boy is 13 dollars, by the apprentice 26 dollars, and by the master workman 65 dollars.

Check. By the conditions of the problem the sum of the amounts obtained should be \$104; the apprentice should receive twice as much as the boy and the master workman five times as much as the boy. That is, we should have

$$13 + 26 + 65 = 104, \quad 26 = 2 \cdot 13 \text{ and } 65 = 5 \cdot 13.$$

PROBLEMS

Solve the following problems by means of equations and check each result by verifying that it satisfies the conditions of the problem.

1. Five times a certain number equals 80. What is the number? Use n for the number.

2. Twelve times a number equals 132. What is the number?

3. A tank holds 750 gallons. How long will it take a pipe discharging 15 gallons per minute to fill the tank?

4. The cost of paving a block on a certain street was $7 per front foot. How long was the block, if the total cost was $4620?

5. A city lot sold for $7500. What was the frontage, if the selling price was $225 per front foot?

6. An encyclopedia contains 18,000 pages. How many volumes are there, if they average 750 pages to the volume?

7. For how many years must $3500 be invested at 6% simple interest to yield $2205?

8. At what rate must $2500 be invested for 3 years in order to yield $412.50?

Suggestion. By the conditions of the problem $2500 \times r \times 3 = 412.50$.

9. At what rate must 6800 be invested for 7 years in order to yield $2380?

10. How many dollars must be invested for 5 years at $4\frac{1}{2}\%$ simple interest to yield $351?

Suggestion. By the conditions of the problem $p \times .04\frac{1}{2} \times 5 = 351$.

11. How many dollars must be invested for 6 years at $4\frac{3}{4}\%$ simple interest to yield $2422.50?

12. A cut in an embankment is 500 yards long and 4 yards deep. How wide is it if 18,760 cubic yards are removed?

Suggestion. From arithmetic, length × width × height = volume or $lwh = v$.

13. How deep is a rectangular cistern which holds 500 cubic feet of water, if it is 6 feet wide and 8 feet long?

14. How long is a box containing 2240 cubic inches, if its width is 14 inches and its depth 10 inches?

15. If n is a number, how do you represent 10 times that number?

16. If n is a number, how do you represent that number plus 3 times itself?

17. If n is a number, how do you represent 5 times that number plus 3 times the number plus 8 times the number?

18. The greater of two numbers is 5 times the less, and their sum is 180. What are the numbers?

19. A number increased by twice itself, 4 times itself, and 6 times itself, becomes 429. What is the number?

20. A father is 3 times as old as his son, and the sum of their ages is 48 years. How old is each?

21. In a company there are 39 persons. The number of children is twice the number of grown people. How many are there of each?

22. How many dollars will amount to $620 in 4 years at 6% simple interest?

Solution. From arithmetic we have

$$amount = principal + interest,$$

or

$$a = p + i = p + prt.$$

Hence, by the conditions of the problem,

$$620 = p + .06 \times 4 \times p.$$

By Principle II,

$$620 = p(1 + .24).$$

By D,

$$p = \frac{620}{1.24} = 500.$$

23. Find what principal invested for 6 years at $4\frac{1}{2}$% simple interest will amount to $1270.

24. Find what principal invested for 12 years at $5\frac{1}{2}$% simple interest will amount to $4150.

ALGEBRAIC REPRESENTATION OF NUMBERS

Skill in translating problems into equations depends upon attention to the following points:

(1) *Read and understand* clearly the statement of the problem, as it is given in words.

(2) *Select the unknown number*, and represent it by a suitable letter, say the initial letter of a word which will keep its meaning in mind. If there are more unknown numbers than one, try to express the others in terms of the one first selected.

(3) Find two number expressions which, according to the problem, represent the same number, and set them equal to each other, *thus forming an equation.*

Special care is needed in expressing the various numbers involved in a problem in terms of a single unknown number, as illustrated in the following exercises.

1. If n is a number, represent in symbols a number 7 greater than n; 5 less than n; 8 times as great as n; one-third as great as n.

2. Write in symbols n increased by k; n decreased by k; n multiplied by k; n divided by k.

3. If the sum of two numbers is 10 and one of them is x, what is the other number?

4. If two numbers differ by 6 and the smaller is x, what is the other number?

5. If two numbers differ by 6 and the greater is x, what is the other number?

6. If A has m dollars, and B has 15 dollars more than A, how do you represent B's money? If C's money is twice B's, how do you represent C's money?

7. If n is an integer, that is, a whole number, how do you represent the next higher integer? The second higher? The next lower? The second lower?

8. What is the value of $2n$, for $n = 1, 2, 3, 4, 5, 6$, etc.? If n is any integer, the number represented by $2n$ is an *even* integer.

9. If $2n$ is any even integer, represent the next higher even integer.

10. Represent each of four consecutive even integers, the smallest of which is $2n$.

11. What is the value of $2n + 1$, for $n = 1, 2, 3, 4, 5$, etc.? If n is any integer, the number represented by $2n + 1$ is an *odd* integer.

12. If $2n + 1$ is any odd integer, represent the next higher odd integer.

13. Represent four consecutive odd integers, the smallest of which is $2n + 1$.

14. If x is a number, express in terms of x a number 5 less than 3 times x; also a number 5 times as great as x diminished by 3.

15. The length of a rectangle is 3 feet greater than its width. If w is the width, how do you represent its length?

16. If w and l are the width and length respectively of a rectangle, how do you represent its perimeter? (The perimeter of a rectangle means the sum of the lengths of its four sides.)

17. Express the perimeter of a rectangle in terms of its width w, if its length is 10 inches greater than its width.

18. Express the perimeter of a rectangle in terms of its length l, if the length is 6 inches greater than the width.

PROBLEMS

Check each solution by finding whether the result satisfies the conditions stated in the problem:

1. Four times a certain number plus 3 times the number minus 5 times the number equals 48. What is the number?

2. One number is 4 times another, and their difference is 9. What are the numbers?

3. Find a number such that when 4 times the number is subtracted from 12 times the number, the remainder is 496.

4. Thirty-nine times a certain number, plus 19 times the number, minus 56 times the number, plus 22 times the number, equals 12. Find the number.

5. There are three numbers whose sum is 80. The second is 3 times the first, and the third twice the second. What are the numbers?

6. There are three numbers such that the second is 11 times the first and the third is 27 times the first. The difference between the second and the third is 64. Find the numbers.

7. There are three numbers such that the second is 8 times the first and the third is 3 times the second. If the second is subtracted from the third, the remainder is 48. Find the numbers.

8. The number of representatives and senators together in the United States Congress, according to the new apportionment, is 531. The number of representatives is 51 more than 4 times the number of senators. Find the number of each.

9. The area of Illinois is 6750 square miles more than 10 times that of Connecticut. The sum of their areas is 61,640 square miles. Find the area of each state.

10. The sum of the horse powers of the steamships *Olympic* and *Mauretania* is 116 thousand. The *Mauretania* has 22 thousand horse-power less than twice that of the *Olympic*. What is the horse-power of each ship?

11. It is twice as far from Boston to Quebec as from Boston to Albany and 3 times as far from Boston to Jacksonville, Florida, as from Boston to Quebec. How far is it from Boston to each of the other three cities, the sum of the distances being 1818 miles?

12. Find three consecutive integers whose sum is 144.

13. Find four consecutive integers such that twice the first plus the last equals 48.

14. Find three consecutive even integers whose sum is 54.

15. Find three consecutive even integers such that 3 times the first is 12 greater than the third.

16. Find two consecutive integers such that 3 times the first plus 7 times the second equals 217.

17. Find two consecutive integers such that 7 times the first plus 4 times the second equals 664.

18. Find four consecutive odd integers such that 7 times the first equals 5 times the last.

19. The perimeter of a square is 64 inches. Find the length of a side.

20. A rectangle is 4 inches longer than it is wide. Find its length and width if the perimeter is 40 inches.

21. A rectangle is twice as long as wide. Find its dimensions if the perimeter exceeds the length by 60.

22. The length of a rectangle is $1\frac{1}{2}$ times as great as its width. Find its dimensions if the perimeter exceeds the width by 40 inches.

23. The width of a rectangle is $\frac{3}{4}$ of its length and the perimeter exceeds the length by 50 inches. Find its dimensions.

24. How much must be invested at 6 % interest to amount to $2650 at the end of one year?

25. How much must be invested at 5 % simple interest to amount to $2025 at the end of 7 years?

26. At what rate of interest per year must $800 be invested to amount to $1000 in 5 years?

27. Pikes Peak is 3282 feet higher than Mt. Ætna, and Mt. Everest is 708 feet more than twice as high as Pikes Peak. The sum of the altitudes of Mt. Ætna and Mt. Everest is 39,867 feet. Find the altitude of each of the three mountains.

28. The melting point of iron is 450 degrees centigrade higher than 5 times that of tin. Three times the number of degrees at which iron melts plus 7 times the number at which tin melts equals 6410. Find the melting point of each metal.

REVIEW QUESTIONS

1. Define equality; equation; identity. State in detail how the equation and the identity differ. Give an example of each.

2. What value of x satisfies the equation $x + 4 = 9$? What value of x will satisfy the equation obtained by adding 7 to each member of this equation? by adding 12? 24?

3. If 4 be added to the first member of the equation $x + 4 = 9$, and 6 to the second member, what value of x will satisfy the equation thus obtained?

4. If the *same* number is added to each member of an equation, is the resulting equation satisfied by the same value of the unknown as the first equation?

5. If *different* numbers are added to the members of an equation, is the resulting equation satisfied by the same value of the unknown as the first equation?

6. If the *same* number is subtracted from each member of an equation, is the resulting equation satisfied by the same value of the unknown as the first equation? Illustrate by an example.

7. If *different* numbers are subtracted from the members of an equation, is the resulting equation satisfied by the same value of the unknown as the first equation? Illustrate by an example.

8. Ask and answer questions, similar to the two preceding, about the effect of multiplying both members of an equation by the *same or different* numbers. Illustrate each by examples.

9. Ask and answer similar questions on the effect of dividing the members of an equation by the same or different numbers. Illustrate each by an example.

10. *The process of solving an equation consists in obtaining from it other equations which are satisfied by the same number as the original equation.*

Hence what operations may be performed in solving an equation?

11. State Principle VI in full.

CHAPTER III

POSITIVE AND NEGATIVE NUMBERS

44. Thus far the numbers used have been precisely the same as in arithmetic, though their representation by means of letters and some of the methods used in operating upon them are peculiar to algebra.

We now proceed to the study of a **new kind of number.**

Examples. What is the highest temperature you have ever seen recorded on the thermometer? the lowest?

In answering these questions you not only give certain numbers, but you attach to each a certain **quality**. The temperature is **above** zero or **below** zero, that is, the degrees on the thermometer are measured in **opposite** directions from a starting point which is labeled zero.

45. It has been found exceedingly useful in mathematics to extend the number system of arithmetic so as to make it apply directly to cases like this. The opposite qualities involved are designated by the words **positive** and **negative.**

It is commonly agreed to call *above zero* positive and *below zero* negative. Likewise, distances measured to the **right** from a zero point are called positive, and those to the left, negative. See § 48.

46. Definitions. The signs $^+$ and $^-$ stand respectively for the words *positive* and *negative*, and numbers marked with these signs are called **positive and negative numbers** respectively.

Thus, 5° above zero is written $^+5°$, and 15° below zero is written $^-15°$.

When no sign of quality is written, the positive sign is understood.

E.g. $^+5°$ is usually written 5°.

Positive and negative numbers are sometimes called **signed numbers**, because each such number consists of a numerical part, together with a **sign of quality** expressed or understood.

The numerical part of a signed number is called its **absolute value.**

Thus, the absolute value of $^{+}3$ and also of $^{-}3$ is 3.

47. The **integers of arithmetic** may be arranged in a series beginning at zero and extending indefinitely toward the right.

Thus, $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \cdots$

The integers of **algebra** may be arranged in a series beginning at zero and extending indefinitely both to the right and the left.

Thus, $\cdots\ ^{-}5,\ ^{-}4,\ ^{-}3,\ ^{-}2,\ ^{-}1,\ 0,\ ^{+}1,\ ^{+}2,\ ^{+}3,\ ^{+}4,\ ^{+}5, \cdots.$

48. One of the most extensive uses of signed numbers is for marking the points on a straight line. This will also appear later in connection with the graph in Chapter VII.

On an unlimited straight line call some starting point zero, and lay off from this point equal divisions of the line indefinitely both to the right and to the left, as shown in the figure.

In order to describe the position of any one of these division points, we require not only an integer of arithmetic, to specify *how far* the given point is from the point marked zero, but also a **sign of quality** to indicate on which side of this point it is.

E.g. $^{+}7$ marks the division point 7 units to the right of zero, and $^{-}5$ marks the point 5 units to the left of zero. Such a diagram is called the **scale of signed numbers.**

Fractions would of course be pictured at points between the integral division points, on the right or the left of the scale, according as the fractions are positive or negative.

ADDITION OF SIGNED NUMBERS

49. In arithmetic two numbers are added by starting with one and counting *forward* the number of units in the other.

E.g. To add 3 to 5 we start with 5 and count 6, 7, 8.

In algebra two signed numbers are added in the same manner except that the *direction,* forward or backward, in which we count, is determined by the sign, $^+$ or $^-$, of the number which we are adding.

Thus, to add $^+5$ to $^+7$, begin at 7 to the right of the zero point in the scale of signed numbers and count 5 more toward the right, arriving at $^+12$. Thus, $^+7 + {}^+5 = {}^+12$, which is read: *Positive* 7 *plus positive* 5 *equals positive* 12.

To add $^-5$ to $^-7$, begin at 7 to the left and count 5 more toward the left, arriving at $^-12$. Thus, $^-7 + {}^-5 = {}^-12$, read: *Negative* 7 *plus negative* 5 *equals negative* 12.

To add $^-5$ to $^+7$, begin at 7 to the right and count 5 toward the left, arriving at $^+2$. Thus, $^+7 + {}^-5 = {}^+2$, read: *Positive* 7 *plus negative* 5 *equals positive* 2.

To add $^+5$ to $^-7$, begin at 7 to the left and count 5 toward the right, arriving at $^-2$. Thus, $^-7 + {}^+5 = {}^-2$, read: *Negative* 7 *plus positive* 5 *equals negative* 2.

The addition of positive and negative numbers is further explained in the following:

50. Illustrative Problems. 1. If a man gains \$1500 and then loses \$800, what is the net result? *Answer,* \$700 gain.

In this case the result is obtained by subtracting 800 from 1500. Yet this is not really a problem in subtraction but in addition. That is, we are not asking for the difference between \$1500 gain and \$800 loss, but for the *net result* when the gain and the loss are taken together, or the **sum** of the profit and the loss. Hence, we say \$1500 gain + \$800 loss = \$700 gain, or using positive and negative signs,

$$^+1500 + {}^-800 = {}^+700.$$

2. The assets of a commercial house are \$250,000, and the liabilities are \$275,000. What is the net financial status of the house? *Answer*, \$25,000 net liabilities.

Thus,

\$250,000 assets + \$275,000 liabilities = \$25,000 net liabilities.

Or $^{+}250{,}000 + {}^{-}275{,}000 = {}^{-}25{,}000.$

3. The thermometer rises 18 degrees and then falls 28 degrees. What direct change in temperature would produce the same result? *Answer*, 10 degrees fall.

Thus, 18° rise + 28° fall = 10° fall.

Or $^{+}18 + {}^{-}28 = {}^{-}10.$

4. A man travels 700 miles east and then 400 miles west. What direct journey would bring him to the same final destination? *Answer*, 300 miles east.

Thus, 700 miles east + 400 miles west = 300 miles east.

Or $^{+}700 + {}^{-}400 = {}^{+}300.$

EXERCISES

Perform the following additions

1. $^{-}3 + {}^{+}4.$
2. $^{-}7 + {}^{-}3.$
3. $^{+}8 + {}^{-}16.$
4. $^{-}5 + {}^{+}5.$
5. $^{+}10 + {}^{-}10.$
6. $^{+}8 + {}^{-}4.$
7. $^{-}8 + {}^{+}17.$
8. $^{+}5\frac{1}{2} + {}^{-}3\frac{1}{4}.$
9. $^{-}20 + {}^{-}10.$
10. $^{+}12 + {}^{-}12.$
11. $^{-}8 + {}^{+}8.$
12. $^{-}9 + {}^{-}10.$

The preceding exercises illustrate

Principle VII

51. **Rule.** *To add two numbers with like signs, find the sum of their absolute values, and prefix to this their common sign.*

To add two numbers with opposite signs, find the difference of their absolute values, and prefix to this the sign of that one whose absolute value is the greater.

In case their absolute values are equal, their sum is zero.

52. **Definition.** The sum of two signed numbers thus obtained is called their **algebraic sum.**

Hereafter *addition* will mean finding the *algebraic sum.*

53. Signed numbers find application in any situation where *opposite qualities* of the kind here considered are present. Besides those already mentioned, other instances occur in the applications below:

APPLICATIONS OF SIGNED NUMBERS

1. A balloon which exerts an upward pull of 460 pounds is attached to a car weighing 175 pounds. What is the net upward or downward pull? Express this as a problem in addition, using positive and negative numbers.

Solution. 460 lb. upward pull plus 175 lb. downward pull equals 285 lb. net upward pull. Using positive numbers to represent upward pull and negative numbers to represent downward pull, this equation becomes

$$^{+}460 + {}^{-}175 = {}^{+}285.$$

In each of the following translate the solution into the language of algebra by means of signed numbers as in Ex. 1.

2. A 450-pound weight is attached to a balloon which exerts an upward pull of 600 pounds. What is the net upward or downward pull?

3. A man's property amounts to \$45,000 and his debts to \$52,000. What is his net debt or property?

4. The assets of a bankrupt firm amount to \$245,000 and the liabilities to \$325,000. What are the net assets or liabilities?

5. A man can row a boat at the rate of 6 miles per hour. How fast can he proceed against a stream flowing at the rate of $2\frac{1}{2}$ miles per hour? 7 miles per hour?

6. A steamer which can make 12 miles per hour in still water is running against a current flowing 15 miles per hour. How fast and in what direction does the steamer move?

7. A dove which can fly 40 miles per hour in calm weather is flying against a hurricane blowing at the rate of 60 miles per hour. How fast and in what direction is the dove moving?

8. If of two partners, one loses $1400 and the other gains $3700, what is the net result to the firm?

9. A man's income is $2400 and his expenses $1500 per year. What is the net result for the year?

10. A man loses $800 and then loses $600 more. What is the combined loss? Indicate the result as the sum of two negative numbers.

AVERAGES OF SIGNED NUMBERS

54. Half the sum of two numbers is called their average. Thus 6 is the average of 4 and 8. Similarly, the average of three numbers is one-third of their sum, and in general the average of n numbers is the sum of the numbers divided by n.

Find the average of each of the following sets:

1. 10, 12, 14, 16, 18.
2. 5, 9, 20, 30, 3.
3. 7, 9, 11, 13, 15.
4. 7, 10, 21, 29, 30.

The average gain or loss per year for a given number of years is the *algebraic sum* of the yearly gains and losses divided by the number of years.

Illustrative Problem. A man lost $400 the first year, gained $300 the second, and gained $1000 the third. What was the average loss or gain?

Solution. $$\frac{{}^-400 + {}^+300 + {}^+1000}{3} = \frac{{}^+900}{3} = {}^+300.$$

That is, the average gain is $300.

PROBLEMS

1. Find the average of $1800 loss, $3100 loss, $6800 gain, $10,800 loss, and $31,700 gain.

2. Find the average of $180 gain, $360 loss, $480 loss, $100 gain, $700 gain, $400 gain, $1300 loss, $300 gain, $4840 gain, and $12,000 gain.

Find the average yearly temperatures at the following places, the monthly averages having been recorded as given below:

3. For New York City: $^+29°$, $^+33°$, $^+39°$, $^+46°$, $^+53°$, $^+63°$, $^+67°$, $^+67°$, $^+61°$, $^+52°$, $^+47°$, $^+41°$.

4. For St. Vincent, Minnesota: $^-5°$, $0°$, $^+15°$, $^+35°$, $^+55°$, $^+60°$, $^+66°$, $^+63°$, $^+55°$, $^+40°$, $^+22°$, $^+5°$.

5. For Nerchinsk, Siberia: $^-23°$, $^-13°$, $^-10°$, $^+35°$, $^+55°$, $^+70°$, $^+70°$, $^+64°$, $^+50°$, $^+30°$, $^+5°$, $^-15°$.

SUBTRACTION OF SIGNED NUMBERS

55. Definition. **Subtraction** is the process of finding the number which added to a given number called **subtrahend** produces another given number called **minuend.** The number thus found is called the **difference** or **remainder.**

E.g. We say $8 - 5 = 3$, because $5 + 3 = 8$.

Signed numbers may be subtracted by a simple application of this definition.

Ex. 1. $^+8 - {}^+5 = {}^+3$, because $^+5 + {}^+3 = {}^+8$.

Ex. 2. $^-8 - {}^-5 = {}^-3$, because $^-5 + {}^-3 = {}^-8$.

Ex. 3. $^-8 - {}^+5 = {}^-13$, because $^+5 + {}^-13 = {}^-8$.

Ex. 4. $^+8 - {}^-5 = {}^+13$, because $^-5 + {}^+13 = {}^+8$.

Signed numbers may be subtracted by counting on the number scale in a direction opposite to that indicated by the sign of the subtrahend.

Thus to subtract $^+5$ from $^+8$ begin with $^+8$ and count 5 units to the left, and to subtract $^-5$ from $^+8$ begin with $^+8$ and count 5 units to the right.

56. A short rule for subtraction. Since $^+8 - {}^-5 = {}^+13$, and since $^+8 + {}^+5 = {}^+13$, it follows that *subtracting* $^-5$ from $^+8$ gives the same result as *adding* $^+5$ to $^+8$. Similarly, $^-8 - {}^-5 = {}^-3$ and $^-8 + {}^+5 = {}^-3$.

Hence, *subtracting a negative* number is equivalent to *adding a positive* number of the same absolute value.

Since $^{+}8 - {}^{+}5 = {}^{+}3$, and since $^{+}8 + {}^{-}5 = {}^{+}3$, it follows that subtracting $^{+}5$ from $^{+}8$ gives the same result as *adding* $^{-}5$ to $^{+}8$. Similarly, $^{-}8 - {}^{+}5 = {}^{-}13$ and $^{-}8 + {}^{-}5 = {}^{-}13$.

Hence, *subtracting a positive* number is equivalent to *adding a negative* number of the same absolute value.

These statements are illustrated by such facts as: *Removing a debt* is equivalent to *adding property* and *removing property* is equivalent to *adding debt.*

Perform the following subtractions by changing the sign of the subtrahend and adding:

1. $^{-}5 - {}^{-}2$.	6. $^{-}32 - {}^{+}34$.	11. $^{-}16 - {}^{+}4$.
2. $^{-}4 - {}^{+}1$.	7. $^{-}52 - {}^{-}32$.	12. $^{+}13 - {}^{-}20$.
3. $^{-}5 - {}^{+}2$.	8. $^{-}16 - {}^{-}12$.	13. $^{-}8 - {}^{-}7$.
4. $^{+}3 - {}^{-}5$.	9. $^{+}37 - {}^{+}50$.	14. $^{-}19 - {}^{+}14$.
5. $^{+}57 - {}^{-}32$.	10. $^{-}23 - {}^{+}57$.	15. $^{-}24 - {}^{-}19$.

The preceding exercises illustrate

Principle VIII

57. Rule. *To subtract one signed number from another signed number, change the sign of the subtrahend and then add it to the minuend.*

The change in the sign of the subtrahend may be made *mentally* without rewriting the problem. The results are to be checked by showing that the difference added to the subtrahend equals the minuend.

58. Subtraction always possible. In arithmetic subtraction is possible only when the subtrahend is less than or equal to the minuend.

E.g. In arithmetic we cannot subtract 5 from 2 since there is no positive number which added to 5 gives 2.

However, by means of negative numbers we can as easily perform the subtraction, 2 minus 5, as 5 minus 2.

Thus, $2 - 5 = {}^{-}3$, since $^{-}3 + 5 = 2$.

EXERCISES

Perform the following subtractions:

1. $^-10 - {}^-5$.
2. $^-15 - {}^+5$.
3. $^+20 - {}^-15$.
4. $^+11 - {}^+3$.
5. $^-11 - {}^+5$.
6. $^-17 - {}^-20$.
7. $^+6 - {}^-14$.
8. $^+7 - {}^-9$.
9. $^-11 - {}^+6$.
10. $^-21 - {}^-6$.
11. $^+93 - {}^+22$.
12. $^+17 - {}^-13$.
13. $^-78 - {}^-37$.
14. $^+57 - {}^+84$.
15. $^-48 - {}^-31$.
16. $^-39 - {}^-95$.
17. $^-91 - \ 3$.
18. $^-38 - {}^+74$.

59. **Double use of the Signs + and −.** In § 46, we agreed that when no sign of quality is written, the sign $^+$ is understood. Hence we may write:

$$^+8 + {}^+5 = 8 + 5. \quad (1)$$

$$^+8 - {}^+5 = 8 - 5. \quad (2)$$

By Principle VII, we have

$$^+8 + {}^-5 = {}^+8 - {}^+5 = 8 - 5. \quad (3)$$

By Principle VIII, we have

$$^+8 - {}^-5 = {}^+8 + {}^+5 = 8 + 5. \quad (4)$$

These examples show how we may dispense with the special signs of quality $^+$, or $^-$, as follows:

1. Positive numbers are written without any sign indicating quality except where special emphasis is desired, in which case the sign + is used.

2. A negative number when standing alone is preceded by the sign −. Thus $^-5$ is written -5.

3. When a negative number is combined with other numbers, its quality is indicated by the sign − with a parenthesis inclosing it.

Thus $8 + {}^-5$ is written $8 + (-5)$,

and $8 - {}^-5$ is written $8 - (-5)$.

But in such cases it is customary to apply Principles VII and VIII and write at once $8 - 5$ instead of $8 + (-5)$ and $8 + 5$ instead of $8 - (-5)$.

60. It is clear from the examples in this chapter that signed numbers are needed to represent actual conditions in life, as in case of the thermometer. While from now on such numbers will be distinguished, in accordance with universal custom, by the signs, +, −, it should be understood that each of these signs is thus made to represent either one of two entirely different things, namely, an *operation* or a *quality*.

However, after we acquire some understanding of the matter, this double use of the signs seldom leads to any confusion, since we can always tell from the context which use is meant. For example, in $5-3$, the sign − means *subtraction*, while in $x=-3$ it means *negative*.

But for the sake of avoiding confusion at the outset, and to make clear that a negative number is not necessarily a *subtrahend* and that a positive number is not necessarily an *addend*, we have up to this time used the special signs $^+$, $^-$, which could be readily distinguished from the signs of addition and subtraction, +, −.

EXERCISES

Perform the following indicated operations:

1. $9-(-4)$. **3.** $-10+6$. **5.** $5-8-14$.

2. $-10+(-3)$. **4.** $12-4$. **6.** $-7+8-18$.

7. Find the value of $a+b$ if (1) $a=4$, $b=-5$; (2) $a=-2$, $b=-7$; (3) $a=-6$, $b=8$; (4) $a=6$, $b=10$.

8. Find the value of $a-b$ if (1) $a=8$, $b=8$; (2) $a=-3$ $b=-7$; (3) $a=4$, $b=-9$; (4) $a=-3$, $b=6$.

Solve the following equations:

9. $x+8=4$. *Suggestion.* Subtract 8 from each member.

10. $x+3=7$. **14.** $-4+x=-9$. **18.** $-35+x=17$.

11. $x-9=1$. **15.** $-5+x=4$. **19.** $17+x=-35$.

12. $3+x=0$. **16.** $-5+x=12$. **20.** $x-14=-18$.

13. $x+13=7$. **17.** $-9+x=-18$. **21.** $x-25=16$.

MULTIPLICATION OF SIGNED NUMBERS

61. The multiplication of signed numbers is illustrated by the following problems:

Illustrative Problem. A balloonist, just before starting, makes the following preparations: (*a*) He adds 9000 cubic feet of gas with a lifting power of 75 pounds per thousand cubic feet. (*b*) He takes on 8 bags of sand, each weighing 15 pounds. How does each of these operations affect the buoyancy of the balloon?

Solution. (*a*) A lifting power of 75 lb. is indicated by $+75$, and adding such a power 9 times is indicated by $+9$. Hence, $+9 \cdot (+75) = +675$, or 675 the total lifting power added.

(*b*) A weight of 15 lb. is indicated by -15, and adding 8 such weights is indicated by $+8$. Since the total weight added is 120 lb., we have $+8 \cdot (-15) = -120$.

Illustrative Problem. During the course of his journey this balloonist opens the valve and allows 2000 cubic feet of gas to escape, and later throws overboard 4 bags of sand. What effect does each of these operations produce on the balloon?

Solution. (*a*) The gas, being a lifting power, is positive, but the removal of 2000 cubic feet of it is indicated by -2, and the result is a depression of the balloon by 150 lb.; that is, $-2 \cdot (+75) = -150$.

(*b*) The removal of 4 weights is indicated by -4, but the weights themselves have the negative quality of downward pull. Hence to remove 4 weights of 15 lb. each is equivalent to increasing the buoyancy of the balloon by 60 lb.; that is, $-4 \cdot (-15) = +60 = 60$.

62. These illustrations of multiplying signed numbers are natural extensions of the process of multiplication in arithmetic.

E.g. Just as $3 \cdot 4 = 4+4+4 = 12$, so $3 \cdot (-4) = -4 + (-4) + (-4) = -12$, and since $3 \cdot 4$ is the same as $+3 \cdot (+4)$, we write $+3 \cdot +4 = +12 = 12$.

Again, just as we take the multiplicand *additively* when the multiplier is a positive integer, so we take it *subtractively* when the multiplier is a negative integer.

E.g. $-3 \cdot (+4)$ means to subtract $+4$ three times; that is, to subtract $+12$. But to subtract $+12$ is the same as to add -12. Hence, $-3 \cdot (+4) = (-12)$. Again, $-3 \cdot (-4)$ means to subtract -4 three times; that is, to subtract -12. But to subtract -12 is the same as to add $+12$. Hence, $-3 \cdot (-4) = +12 = 12$.

EXERCISES AND PROBLEMS

Explain the following indicated multiplications and find the product in each case:

1. $-3 \cdot (-10)$.	4. $-7 \cdot (-8)$.	7. $-5 \cdot (-48)$.
2. $10 \cdot (-3)$.	5. $12 \cdot (-21)$.	8. $-25 \cdot (-16)$.
3. $50 \cdot (-5)$.	6. $-27 \cdot (-6)$.	9. $-8 \cdot 34$.

10. A man gained $212 each month for 5 months, then lost $175 per month for 3 months. Express his net gain or loss as the sum of two products.

11. A raft is made of cork and iron. What effects are produced upon its floating qualities by the following changes? (*a*) Adding 4 braces, each weighing (under water) 5 pounds. (*b*) Removing 3 pieces of cork, each capable of sustaining 3 pounds. (*c*) Adding 10 pieces of cork, each capable of sustaining 7 pounds.

The preceding exercises illustrate

Principle IX

63. Rule. *If two numbers have the same sign, their product is positive; if they have opposite signs, their product is negative.*

In applying this principle observe that the sign of the product is obtained quite independently of the absolute value of the two factors.

E.g. $\frac{3}{4} \cdot (-5) = -(\frac{15}{4}) = -3\frac{3}{4}$; $-12 \cdot (-3.5) = +42 = 42$.

64. The product of several signed numbers is found as illustrated in the following:

$-2 \cdot 5 \cdot (-3) \cdot (-4) \cdot 6 = -10 \cdot (-3) \cdot (-4) \cdot 6 = 30 \cdot (-4) \cdot 6 = -120 \cdot 6 = -720$. That is, the first two factors are multiplied together, then this product by the next factor, and so on, until all the factors are multiplied.

Since the product of all positive factors is positive, the final sign depends upon the number of negative factors. If this number is *even*, the product is positive; if it is *odd*, the product is negative.

E.g. If there are 5 negative factors, the product is negative; if there are 6, it is positive.

In the following exercises determine the sign of the product before finding its absolute value.

EXERCISES

1. $-4 \cdot 3 \cdot (-6) \cdot (-7)$.
2. $-2 \cdot (-3) \cdot (-5) \cdot 3$.
3. $-5 \cdot [-3 + (-7)]$.
4. $-5 \cdot (-4) \cdot 3 \cdot (-2)$.
5. $8 \cdot (-9) \cdot (-1) \cdot (-2)$.
6. $-50 \cdot (-20) \cdot (-30) \cdot (-40)$.
7. $-2 \cdot (-3) \cdot (-4) \cdot m \cdot (-n) \cdot p \cdot (-q)$.

DIVISION OF SIGNED NUMBERS

65. In arithmetic we test the correctness of division by showing that the quotient multiplied by the divisor equals the dividend.

E.g. $27 \div 9 = 3$, because $9 \cdot 3 = 27$.

Hence **division** may be defined as the process of finding one of two factors when their product and the other factor are given.

The given product is the **dividend**, the given factor the **divisor**, and the factor to be found is the **quotient**.

This definition also applies to the division of signed numbers. In dividing signed numbers, however, we must determine the *sign* of the quotient as well as its *absolute value*.

E.g. $-42 \div (+6) = -7$, because $-7 \cdot (+6) = -42$;
also $-42 \div (-6) = +7$, because $+7 \cdot (-6) = -42$.

So in every case the test is:

$$\textit{Quotient} \times \textit{Divisor} = \textit{Dividend}.$$

In like manner perform the following, and check as above:

1. $-25 \div 5$.
2. $-ab \div a$.
3. $5\,xy \div (-x)$.
4. $-9\,rs \div 3$.
5. $75\,y \div (-15)$.
6. $-121\,x \div 11$.

The preceding exercises illustrate

Principle X

66. **Rule.** *The quotient of two signed numbers is positive if the dividend and divisor have like signs, negative if they have opposite signs.*

EXERCISES

Perform the following indicated divisions, and check by multiplying quotient by divisor.

1. $\dfrac{-28}{7}$.
2. $\dfrac{-42}{-6}$.
3. $\dfrac{51}{-17}$.
4. $\dfrac{-21}{-3}$.
5. $\dfrac{-75}{5}$.
6. $\dfrac{-16}{-1}$.
7. $\dfrac{-49}{1}$.
8. $\dfrac{64}{-1}$.
9. $\dfrac{4 \cdot (-9)}{-3}$.
10. $\dfrac{-3 \cdot 8}{-4}$.
11. $\dfrac{100 \cdot (-99)}{-25}$.
12. $\dfrac{3 \cdot (-4) \cdot (-6) \cdot 8}{-3}$.

13. A man lost \$300, \$500, and \$700 during three consecutive months. Express his average monthly loss as a quotient.

14. During five consecutive days the maximum temperature was $-5°, -8°, -10°, -4°, -6°$ respectively. Find the average of these high marks.

15. A trader lost \$250 in each of three months and gained \$75 during each of the four succeeding months. Find the average gain or loss for the seven months.

67. While Principles I–V were studied in connection with unsigned, or arithmetic numbers only, it is now very important to note that they all apply to *signed* numbers as well.

In the statement of these principles the word *number* will from now on be understood to refer either to the ordinary numbers of arithmetic or to the signed numbers, as occasion may require. It should also be noticed that the numbers of arithmetic are used as freely in algebra as in arithmetic. It is only when we wish to distinguish them from negative numbers that they are called positive numbers.

The **number system of algebra,** so far as we have studied it consists of the *numbers of arithmetic* together with the *negative numbers.*

Historical Note. The Hindus appear to have had quite clear notions of a purely "negative number" as distinct from a number to be subtracted. They recognized the difference between positive and negative numbers by attaching to one the idea of debt and to the other that of assets, or by letting them represent distances in opposite directions. The Arabs, however, failed to understand the negative numbers and did not include them in the algebra which they brought to Europe. (See page 19.) With unimportant exceptions, until the beginning of the seventeenth century, mathematicians dealt exclusively with positive numbers.

The negative numbers were brought permanently into mathematics by René Descartes. (See pages 19 and 114.) Trying to number all the points of a complete straight line, Descartes was compelled to start at a point and **number in both directions.** Then it became convenient to distinguish the numbers on the two sides of this starting-point as positive and negative, respectively.

Sir Isaac Newton (1642–1727) was the first to let a letter stand for any number, negative as well as positive. In such a formula as $a(b+c) =$

$ab + ac$, the predecessors of Newton would restrict the letters to represent any **positive** numbers, while Newton regarded the letters as representing *any numbers whatever* either positive or negative. This was of very great importance, since it greatly reduced the number of formulas required.

Negative numbers appeared "absurd" or "fictitious" to mathematicians until they hit upon a **visual** or **graphical representation of them.** Cajori in his history of elementary mathematics says: "Omit all illustrations by lines, thermometers, etc., and negative numbers will be as obscure to modern students as they were to the early algebraists." From the experience of the early mathematicians it would appear that if the pupil wishes to really understand positive and negative numbers, he must study with care applications such as are given in the first part of this chapter.

INTERPRETATION AND USE OF NEGATIVE NUMBERS

68. In solving a problem, a negative result may have a natural interpretation or it may indicate that the conditions of the problem are impossible.

A similar statement holds in reference to fractional answers in arithmetic. For example, if we say there are twice as many girls as boys in a schoolroom and 35 pupils in all, the number of boys would be $35 \div 3 = 11\frac{2}{3}$, which indicates that the conditions of the problem are impossible.

69. Illustrative Problem. The crews on three steamers together number 94 men. The second has 40 more than the first, and the third 20 more than the second. How many men in each crew?

Solution. Let n = number of men in first crew.

Then, $n + 40$ = number of men in second crew,

and $n + 40 + 20$ = number of men in third crew.

Hence, $n + n + 40 + n + 40 + 20 = 94,$

and $3n + 100 = 94.$

$$3n = -6.$$

$$n = -2.$$

Here the negative result indicates that the conditions of the problem are *impossible*.

70. Illustrative Problem. A real estate agent gained $8400 on four transactions. On the first he gained $6400, on the second he lost $2100, on the third he gained $5000. Did he lose or gain on the fourth transaction?

Solution. Since we do not know whether he gained or lost on that transaction, we represent the unknown number by n, which may be positive or negative, as will be determined by the solution of the problem.

Thus we have $6400 + (-2100) + 5000 + n = 8400.$ (1)

Hence, by VII, *F*, $9300 + n = 8400.$ (2)

By S, $n = 8400 - 9300.$ (3)

By VIII, $n = -900.$ (4)

In this case the negative result indicates that there was a *loss* on the fourth transaction.

PROBLEMS

In the following problems give the solutions in full and state all principles used, together with the interpretation of the results:

1. A man gains $2100 during one year. During the first three months he loses $125 per month, then gains $500 per month during the next five months. What is the gain or loss per month during the remaining four months?

2. A man rowing against a swift current goes 9 miles in 5 hours. The second hour he goes one mile less than the first, the third two miles more than the second, and the fourth and fifth each one mile more than the third hour. How many miles did he go during each of the five hours?

3. There are three trees the sum of whose heights is 108 feet. The second is 40 feet taller than the first, and the third is 30 feet taller than the second. How tall is each tree?

Find the average yearly temperature at each of the following places, the average monthly temperatures being as here given:

4. Port Conger, off the northwest coast of Greenland: $-37°$, $-43°$, $-32°$, $-15°$, $14°$, $18°$, $35°$, $34°$, $25°$, $4°$, $-17°$, $-30°$.

5. Franz Joseph's Land: $-20°$, $-20°$, $-10°$, 0°, 15°, 30°, 35°, 30°, 20°, 10°, 0°, $-10°$.

6. North Central Siberia: $-60°$, $-50°$, $-30°$, 0°, 15°, 40°, 40°, 35°, 30°, 0°, $-30°$, $-50°$.

7. A merchant gained an average of $2800 per year for 5 years. The first year he gained $3000, the second $1500, the third $4000, and the fourth $2400. Did he gain or lose and how much during the fifth year?

8. A certain business shows an average gain of $4000 per year for 6 years. During the first 5 years the results were: $8000 loss, $10,000 gain, $7000 gain, $3000 gain, and $12,000 gain. Find the loss or gain during the sixth year.

9. A commercial house averaged $15,000 gain for 6 years. What was the loss or gain the first year if the remaining years show: $8000 gain, $24,000 gain, $2000 loss, $20,000 gain, and $50,000 gain, respectively?

REVIEW QUESTIONS

1. Name several pairs of opposite qualities all of which are conveniently described by the words *positive* and *negative*. What symbols are used to replace these words when applied to numbers?

2. When loss is added to profit, is the profit increased or decreased? What algebraic symbols may be used to distinguish the numbers representing profit and loss?

3. On the number scale indicate what is meant by $+2$; by -2. Indicate what is meant by the sign + in $5+2$; by the sign − in $5-2$; by the sign − in $x=-2$.

4. Why do we call positive and negative numbers signed numbers? What is meant by the absolute value of a number?

5. State Principle VII in full.

6. How is the correctness of subtraction tested in arithmetic? Is the same test applicable to subtraction in algebra?

7. Illustrate the subtraction of positive and negative numbers by an example involving profit and loss.

8. Show by counting on the number scale that the result of subtraction gives the *distance* from subtrahend to minuend and that the sign of the remainder shows the *direction* from subtrahend toward the minuend. For example, use $8-(-5)$ and $-8-(+5)$ to illustrate this.

9. How do negative numbers make subtraction possible in cases where it is impossible in arithmetic?

10. What is a convenient rule for subtracting signed numbers? State Principle VIII.

11. Write an equation whose solution is a negative number.

12. Give an example in which positive and negative numbers are multiplied. State Principle IX.

13. Define division. How do we obtain the law of signs in division? State Principle X. What is the test of the correctness of division?

14. Explain how one set of signs + and − can be used to indicate both quality and operation.

15. By means of Principles VII, VIII, IX, and X, simplify the expressions, $a+b$, $a-b$, $a \cdot b$, $\frac{a}{b}$, after substituting in each various positive and negative values of a and b.

16. Add Principles VII, VIII, IX, and X to the list which you made in Chapters I and II. It is absolutely necessary that you remember the rules stated in these principles. Any short phrases that will assist you in this are of value. For instance the following:

VII. *In addition, positive and negative numbers tend to cancel each other. The common sign or the sign of the numerically greater is the sign of the result.*

VIII. *In subtraction, change the sign of the subtrahend and add.*

IX. *In multiplication, like signs give + and unlike signs —.*

X. *In division, like signs give + and unlike signs give —.*

DRILL EXERCISES

In solving the following equations perform the required additions or subtractions by the direct method suggested in § 38.

1. $5x+7-2x=2x+9$.
2. $3n+2(n+4)=4n+14$.
3. $8+6x+3(2+x)=5x+26$.
4. $4(x+3)+2(3x+1)=5(x+2)+2(x+3)+19$.
5. $7(x+1)+3(2x+3)=4(x+5)+7(x+2)-8$.
6. $7(2x-3)+5(4x-1)=3(x+1)+2$.

Perform the following indicated operations:

7. $-8-(-9)-(+7)+8$.
8. $16+(-18)-(+2)+4$.
9. $12x+(-7x)-(-3x)+2x$.
10. $34n+(-30n)-(-7n)$.
11. $8\cdot(-3)\cdot(-1)$.
12. $[48\div(-6)]\div(-1)$.
13. $3\cdot(-2)(-3)(-1)$.
14. $(-12)\div[(-2)\div(-2)]$.
15. $(-42)\cdot(-2)\div(-7)$.
16. $[(7ab)\div(-a)]\cdot(-3)$.
17. What is meant by the average of several numbers?
18. Find the average of 20, 16, 8, 4, 0, -8, -12.
19. Find the average of $-8-32+14+26-40$.

Solve the following:

20. $x+6=4$.
21. $3x+12=6$.
22. $8+x=4$.
23. $3x-8=-16$.
24. $2(x+6)=3(x+5)$.
25. $2x+4=3x+8$.
26. If $-n$ represents a negative integer, how do you represent the next integer to the right on the number scale? the next to the left?
27. If $-2n$ represents a negative even integer, how do you represent the next even integer to the right? the next to the left?

CHAPTER IV

POLYNOMIALS

71. We have found that the solution of problems leads us to build **number expressions** out of single number symbols.

E.g. If x is a number representing my age in years, then $2(x - 10)$ is double the number representing my age 10 years ago.

Also $2[(x - 10) + (x + 15)]$ is the number representing twice the sum of my ages 10 years ago and 15 years hence.

Number expressions are now to be studied more in detail.

72. Definition. A number expression composed of parts connected by the signs $+$ and $-$ is called a **polynomial.** Each of the parts thus connected, together with the sign preceding it is called a **term.**

E.g. $5a - 3xy - \frac{2}{3}rt + 99$ is a polynomial whose terms are $5a$, $-3xy$, $-\frac{2}{3}rt$, and $+99$. The sign $+$ is understood before $5a$.

Likewise $3x + \frac{2}{x} + \frac{4}{y}$ is a polynomial whose terms are $3x$, $\frac{2}{x}$, and $\frac{4}{y}$.

NOTE. — The word *polynomial* is sometimes used in a more restricted sense in higher mathematics.

73. Definitions. A polynomial of two terms is called a **binomial**; one of three terms is called a **trinomial.** A term taken by itself is called a **monomial.**

E.g. $5a - 3xy$ is a binomial; $5a - 3xy - \frac{2}{3}rt$ is a trinomial whose terms are the monomials $5a$, $-3xy$, $-\frac{2}{3}rt$.

According to this definition $x + (b + c)$ may be called a binomial though it is equivalent to the trinomial $x + b + c$.

In this case x is called a **simple** term and $(b + c)$ a **compound** term. Likewise we may call $3t + 4x - 5(a + b)y$ a trinomial having the simple terms $3t$, $+4x$, and the compound term $-5(a + b)y$.

74. Definition. Two terms which have a factor in common are called **similar with respect to that factor.**

E.g. $5a$ and $-3a$ are similar with respect to a; $-3xy$ and $-7x$ are similar with respect to x; $5a$ and $-5b$ are similar with respect to 5; $7abc$ and $-\frac{2}{3}abc$ are similar with respect to abc.

Similar terms may be combined by Principle I.

E.g. $5a-3a=(5-3)a=2a$; $-3xy-7x=-x(3y+7)$; $5a-5b=5(a-b)$.

EXERCISES

Select the common factor and combine the similar terms in each of the following:

1. $7x-5x+4x$.
2. $3a-2a+4a$.
3. $7a+2a-5x+7x$.
4. $3x-2x+4x+2x$.
5. $8r+3r+2r-5r$.
6. $9t-3t+4t-3t$.
7. $ax+bx+cx$.
8. $3ab-2bc+5bd$.
9. $7ax+3bx+12cx$.
10. $5ax+3ax-2ax$.
11. $2ar+2br-2cr$.
12. $11rs-2st+4as$.
13. $6ab+7ac-ad$.
14. $4xy-3yz-5wy$.

ADDITION AND SUBTRACTION OF POLYNOMIALS

Addition of Polynomials. In adding polynomials we use

Principle XI

75. Rule. *If two or more terms are to be added, they may be arranged and combined in any desired order.*

The truth of this principle may be seen from simple examples:

Thus, $2+3+5=3+2+5=2+(3+5)=(3+2)+5=13$.

Also, $8+(-2)+6=-2+8+6=-2+(8+6)=12$.

Make other examples to illustrate this principle.

Historical Note. The fundamental character of Principle XI was first recognized about one hundred years ago. The principle as here given combines in one statement two laws of algebra: (1) **the associative law,** first so called by F. S. Servois (1814); (2) **the commutative law,** first so called by Sir William Hamilton.

76. In adding polynomials the work may be conveniently arranged by placing the terms in columns, each column consisting of terms which are similar. This is permissible by Principle XI.

Ex. Add $5x - 6y + 4z + 5at$, $-3x + 11y - 16z - 9bt$, and $-7y + 8z$.

Arranging as suggested and applying Principles I and VII we have

$$\begin{array}{rrrl} 5x - & 6y + & 4z + & 5at \\ -3x + & 11y - & 16z - & 9bt \\ - & 7y + & 8z & \\ \hline 2x - & 2y - & 4z + & (5a - 9b)t \end{array}$$

$5x$ and $-3x$ are similar with respect to their common factor x. Hence, by Principle I we add the other factors 5 and -3, obtaining $(5-3)x = 2x$.

Likewise we add $+5at$ and $-9bt$ with respect to the common factor t, obtaining $(5a - 9b)t$. In the second column the sum is $(-6 + 11 - 7)y = -2y$, and in the third column the sum is $(+4 - 16 + 8)z = -4z$.

Check by giving convenient values to x, y, z, t, a, and b.

EXERCISES IN ADDITION

1. Add $7b - 3c + 2d$; $-2b + 8c - 13d$.

2. Add $6x - 3y + 4t - 7z$; $x - 5y - 3t$; $4x - 4y + 8t$; $8x + 2y - 3t$.

3. Add $7a - 4x + 12z$; $8a - 3x + 2z$; $2a + 4x - 3z$; $5a - 2x - 4z$.

4. Add $5ac + 3bc - 4c + 8b$; $2b + 3c - 2bc - 3ac$; $4b + 4c + bc - ac$; $2bc + 4ac + c$; $3b - 4c$.

5. Add $16xy - 13cd$; $15ab - 2xy$; $34cd - 3xy + 2ab$; $14cd - 3xy - 2ab$.

6. Add $34ax + 4by - 3z$; $2by + 5z$; $3ax - 7by + 5z$; $7ax + 4by - az$.

7. Add $3b + 4cd - 2ae$; $ab - 3cd + 3ae$; $3cd - 2ab$; $4cd - 5ae + 7ab$.

8. Add $7ax - 13by + 5$; $9ax + 8by - 4$; $3by - 12ax$; $4ax + 7by - 9$.

9. Add $5ab - 3 \cdot 67 + 5(x-1)$; $5 \cdot 67 + 3ab - 2(x-1)$; $3(x-1) - 4 \cdot 67 + 2ab$.

10. Add $11(c-9) + 3(x+y) + 21wu$; $-71wu - 5(x+y) - 13(c-9)$.

11. Add $5(a+b) - 3(c-d)$; $3(c-d) - 8(a+b)$; $-2(a+b)$; $13(c-d) - 4(a+b)$.

12. Add $3 + 4(c-d) - 5(a-b-c)$; $4(a-b-c) + 5(c-d)$; $3(a-b-c) - 9(c-d) + 12$.

13. Add $(a-b) - 3(c-d) + 4(a-b)$; $5(a-b) + 4(c-d) + 7(c-d) - 9(a-b)$.

14. Add $7(x-y) - 4(x+y) + 4 \cdot 7$; $9(x+y) + 3(x-y) - 9 \cdot 7$; $6(x-y) + 2 \cdot 7 - 3(x+y)$.

15. Add $3(x-5) + 4(c+b) + 3(x-y)$; $8(c+b) - 5(x-y) + 8(x-5)$; $7(c+b) - 4(x-y)$; $3(x-y) + (x-5)$.

16. Add $16(a+b-c) - 3(x-y) + 2(a-b)$; $2(x-y) - 3(a-b) + (a+b-c)$; $7(a-b) + 4(x-y) - 8(a+b-c)$.

17. Add $6(a-b) - 5(x+y) + 7(x-z) - 4abc$; $7(x-z) - 9(x+y) + (a-b) + 2abc$; $11(a-b) + 10abc + 3(x-z) + 8(x+y)$.

77. Subtraction of Polynomials. Since subtraction is performed by adding the subtrahend, with its sign changed, to the minuend, Principle XI permits us to arrange the terms in any desired order as in addition.

This is illustrated as follows:

From $15ab - 17xy + 11rt$ subtract $-5ab + 4xy - 5nt$.

Arranging as on page 58 and applying Principles I and VIII:

$$\begin{array}{r} 15ab - 17xy + 11rt \\ \underline{-5ab + 4xy - 5nt} \\ 20ab - 21xy + t(11r + 5n) \end{array}$$

As suggested in § 57, it is sufficient to change the signs of the subtrahend *mentally*, rather than to rewrite them before adding to the minuend.

EXERCISES IN SUBTRACTION

1. From $9x + 3y - 11z$ subtract $-5x + 8y - 3z$.
2. From $12ab - 3cd + 12xy$ subtract $3ab + 2cd - 11xy$.
3. From $9xc + 4ad - 3cz + 5y$ subtract $3y - 3ad + 5cz$.
4. From $13t + 5mx - 5cv$ subtract $2t - 4mx - 3cv$.
5. From $3v - 2w + 5mn - 4xz$ subtract $-v + 5w - 3mn$.
6. From $31b + 4xy + 16ax - 4$ subtract $8b - 5xy - 3ax$.
7. From $4 - 3a - 5xz - 3vy - x$ subtract $7a + 2xz + 4vy$.
8. From $8xy - 3x + 4y$ subtract $-2xy + 13w + 4x - 2y$.
9. From $2ab - 5 + 7v + 13abc$ subtract $3ab + v + 8abc$.
10. From $8cxa - 4yb - 3yc$ subtract $4cxa + 2yb + 4yc - 49$.
11. From $31 \cdot 45 - 7xy$ subtract $12 \cdot 45 + 9xy$.
12. From $3abc - 4 + 2(x + y) - 3xy$ subtract $28 + 4xy - 3(x + y) + 8abc$.
13. From $21 + 9(xy - z) + 3(a + b)$ subtract $8(xy - z) - 8(a + b) + 15$.
14. From $5ax - 3by + 4ax + 5by$ subtract $5by + 3ax + 7by$.
15. From $15 \cdot 48 + 8ab + 49x$ subtract $7 \cdot 48 - 9ab - 14x$.
16. From $19(r - 5s) + 13(5x - 4) + 7(x - y)$ subtract $17(5x - 4) - 5(x - y) - 11(r - 5s)$.
17. From $30 + 14(x - 5yz) - 13(5y - z)$ subtract $32 + 8(5y - z) - 7(x - 5yz)$.
18. From $a(b + c) + 4(m + n) - 16c$ subtract $9(m + n) + 31c - d(b + c)$.
19. From $5(7x - 4) + 3(5y - 3x) + 35$ subtract $56 - 9(7x - 4) + 8(5y - 3x)$.
20. From $(3a + 9b - 12c) \div 3$ subtract $(6a - 12b - 18c) \div 6$.
21. From $(axy + ayz - axz) \div a$ subtract $y(x + z) - 2xz$.

EXERCISES IN ADDITION AND SUBTRACTION

1. Add $5x - 3y - 7r + 8t$, $-7x + 18y - 4r - 7t$, $-20x - 24y + 18r - 15t$, and $13x + 15y + 11r + 6t$.

Check the sum by substituting $x = 1$, $y = 1$, $r = 1$, $t = 1$.

2. Add $17a - 9b$, $3c + 14a$, $b - 3a$, $a - 17c$, and $a - 3b + 4c$. Check for $a = 1$, $b = 2$, $c = 3$.

3. Add $2x + 3y - t$, $-6y + 8t$, $-x + y - t$, $-4t + 7x$, and $3y$. Check for $x = 2$, $y = 3$, $t = 1$.

4. Add $17r + 4s - t$, $2t + 3u$, $2r - 3s + 4t$, $5u - 6t$, $7r - 3s + 8u$, and $8r - 2t + 6u$. Check by putting each letter equal to 1; also equal to 2.

5. Add $3h + 2t + 4u$ and $h + 3t + 3u$. Check by putting $h = 100$, $t = 10$, $u = 1$; *i.e.* $324 + 133 = 457$.

6. Add $4h + 3t + u$ and $3h + 2t + 7u$. Check as in 5.

7. Write 247, 323, 647, 239, and 41, as number expressions like those in Exs. 5 and 6 and then add them.

8. Add 647, 391, 276, and 444 as in Ex. 7.

9. Add $4t - u$, $5t - u$, $6t - u$, $7t - u$, and $8t - u$. Check for $t = 10$, $u = 1$; also $t = 1$, $u = 1$.

10. Simplify: $3xyz - 2xyz + 5xyz - 4xyz + xyz - xyz$.

11. Subtract $5a - 3b + 6c$ from $-8a + 7b - 11c$ and check.

12. From $7xy + 8xz + 9yz$ take $17xy - 19xz - 20yz$.

13. From $6x - 3y$ take $8y - 3z$.

14. From $3p - 4q + 8r$ take $7p - 11r + 11q$.

15. From $2x - 3y$ take $5x + 7y + 2a - 3b$.

16. From the sum of $18abc - 27xyz + 13rst$ and $-11abc + 16xyz - 52rst$ take $67rst - 39abc$.

17. To the difference between the subtrahend $15x - 18y + 27z$ and the minuend $117x + 97y - 81z$ add $4x - 6y + 3z$.

18. Add $11(x - y) + 15(a - b)$ and $-20(x - y) - 37(a - b)$ and from the sum subtract $135(x - y) - 213(a - b)$.

NUMBER EXPRESSIONS IN PARENTHESES

78. The sign + before a parenthesis means that each term within is to be added to what precedes, and the sign − means that each term within is to be subtracted from what precedes.

By Principle VII, $a+(+b)=a+b$ and $a+(-b)=a-b$; and by Principle VIII, $a-(+b)=a-b$ and $a-(-b)=a+b$. Hence, we have

Principle XII

79. Rule. *A parenthesis preceded by the sign + may be removed without further change.*

A parenthesis preceded by the sign − may be removed by changing the sign of each term within it.

Note that in each case the sign preceding the parenthesis is also removed after the operation indicated by it has been carried out, and that if no sign is written before the first term in the parenthesis, the sign + is understood.

Remove the parentheses and simplify the following:

Ex. 1. $3a+(a-b+4)-(2a+3b-2)$
$$=3a+a-b+4-2a-3b+2=2a-4b+6.$$

Ex. 2. $5(3x+y)-4(2x-3y+2)$
$$=15x+5y-8x+12y-8=7x+17y-16.$$

In Ex. 2 we multiply the terms within the first parenthesis by 5 and those in the second by 4 and then remove the parentheses by Principle XII.

EXERCISES

Remove the symbols of aggregation and simplify:

1. $(3x-2y)-(4x+3y-2)$.
2. $x-y-2z-(3x+2y-7z)$.
3. $3(a+b+c)-2(a-b+c)$.
4. $8(5x-y+2z)-11(3x+y-z)$.
5. $5(7x-4y)+9(x-y)-3(2x+3y)$.
6. $8(r-s)+(2r+s)-(r-2s)$.
7. $11t+(2t-1)-(1-3t)$.

8. $9(r-s)-3(r+s)+2(2r-s)$.
9. $3(5x-7y)-(4x-3y+2)-5y$.
10. $5x-(8-4x+7y)+(5x+3)-(5y+3x-99)$.
11. $-(3a+5b-7c)+(8a-4c)-(9c-4b+4a)-91a$.
12. $7-(4-4c+2d-2a)+31c-(4-2a-5d)-(-8c)$.
13. $(41ab-21c+4)-(36c+15-78ab)+(13c-90ab-8)$.
14. $9by-(4c-8by-13)-2c-16-(34by-12c+8by)$.
15. $6mn+(-9m-7n+14)-8n+(13mn-17m)+34mn$.
16. $34ax-(-17ax+42)+8x-(14a+24ax-7)$.
17. $19-(+2-7a-4b+11ab)-(-2b+8ab+4a)$.
18. $41by-(4b-13y+17by)-(-5b-17by+13y)$.
19. $39rs-20s-19r-(7rs+8s-19r)-(15r-5s-56)$
20. $x-\{8x-(2y-3x)+(2x-4y)\}$.

Suggestion. First remove the parentheses, then the brace.

21. $a+[a-(b+c)-2c]$.
22. $a-\{-(a-b)+(3a-2b)\}$.
23. $2x-3(x-1)-[x-2(2x-1)]$.
24. $a-\{a+(b-c)-2(a+b+c)\}$.

80. By the converse of Principle XII terms may be inclosed in a parenthesis with or without change of sign, according as the sign $-$ or $+$ precedes.

E.g. $a+b-c=a+(b-c)$ and $a-b+c=a-(b-c)$.

EXERCISES

In each of the following, place the last three terms in a parenthesis.

1. $x-y+2-5$.
2. $5a+3b-c+d$.
3. $m-n+p-q$.
4. $5a-3b+2c-d$.
5. $7m-4n-3p-q$.
6. $8+4b-3c-d$.
7. $5xy-5x+3y-2$.
8. $9ax+3by-4cd+2$.
9. $a-3b+d-5c+8$.
10. $13-7a-3b+9c$.
11. $19x-3c+4e-18d$.
12. $21ax-13bx+6dx-8e$.

PROBLEMS

See suggestions on solving problems on page 31. Check each result by showing that it satisfies the conditions of the problem.

1. The sum of two numbers is 16. Seven times one is 8 less than 5 times the other. What are the numbers?

2. From a certain number a there is subtracted 3 times the remainder when 8 is subtracted from $2a$. Express the result in terms of a.

3. A man invested a certain sum of money at 5% simple interest. The amount $3\frac{3}{4}$ years later was \$950. What was the investment?

4. A man bought a tract of coal land and sold it a month later for \$93,840. If his gain was at the rate of 24% per annum, what did he pay for the land?

5. The melting point of copper is 250 degrees (Centigrade) lower than 4 times that of lead. Ten times the number of degrees at which lead melts minus twice the number at which copper melts equals 1152. What is the melting point of each metal?

6. The Nile is 100 miles more than twice as long as the Danube. Ten times the length of the Danube minus 4 times the length of the Nile equals 3400 miles. How long is each river?

7. The Ganges River is 1800 miles shorter than the Amazon, and the Orinoco is 300 miles shorter than the Ganges. The sum of their lengths is 6900 miles. How long is each?

8. Lead weighs 259 pounds more per cubic foot than cast iron, and 166 pounds more than bronze; while a cubic foot of bronze weighs 807 pounds less than 3 cubic feet of iron. Find the weight per cubic foot of each metal.

9. The world's gold production in 1908 was 29 million dollars less than 3 times that of 1893, and the production in 1900 was 59 million less than twice that of 1893. The production of 1900 and 1908 together amounted to 697 million. How much was produced each year?

PROBLEMS ON THE ARRANGEMENT AND VALUE OF DIGITS

81. If we speak of the number whose 3 digits, in order from left to right, are 5, 3, and 8, we mean $538 = 500 + 30 + 8$. Likewise, the number whose three digits are h, t, and u is written $100\,h + 10\,t + u$.

Hence, when letters stand for the digits of numbers written in the decimal notation, care must be taken to multiply each letter by 10, 100, 1000, etc., according to the position it occupies.

Illustrative Problem. A number is composed of two digits whose sum is 6. If the order of the digits is reversed, we obtain a number which is 18 greater than the first number. What is the number?

Solution. Let $x =$ the digit in tens' place.

Then, $6 - x =$ the digit in units' place.

Hence, the number is $10\,x + 6 - x$. Reversing the order of the digits, we have as the new number $10(6 - x) + x$.

Hence, $10(6 - x) + x = 18 + 10\,x + 6 - x$.

In each of the examples 1 to 8 below there is a number composed of two digits.

1. The digit in units' place is 2 greater than the digit in tens' place. If 4 is added to the number, it is then equal to 5 times the sum of the digits. What is the number?

2. The digit in tens' place is 3 greater than the digit in units' place. The number is one more than 8 times the sum of the digits. What is the number?

3. The sum of the digits is 9. If the order of the digits is reversed, we obtain a number which is equal to 12 times the remainder when the units' digit is taken from the tens' digit. What is the number?

4. The sum of the digits is 12. If the order of digits is reversed, the number is increased by 18. Find the number.

5. The tens' digit is 2 less than its units' digit. The number is 1 less than 5 times the sum of its digits. What is the number?

6. The digit in units' place is 4 less than that in tens' place. If the order of the digits is reversed, we obtain a number which is 3 less than 4 times the sum of the digits. What is the number?

7. The digit in units' place is 2 less than twice the digit in tens' place. If the order of the digits is reversed, the number is unchanged. What is the number?

8. The digit in tens' place is 12 less than 5 times the digit in units' place. If the order of the digits is reversed, the number is equal to 4 times the sum of the digits. What is the number?

9. A number is composed of three digits. The digit in units' place is 3 greater than the digit in tens' place, which in turn is 2 greater than the digit in hundreds' place. The number is equal to 96 plus 4 times the sum of the digits. What is the number?

DRILL EXERCISES

1. Add $3x+4y-3z$, $5x-2y-z$, and $3y-5x+7z$.
2. From $15a+4b-13bc$ subtract $3a-8b+2bc$.
3. Subtract $7x-5y-7a$ from $6x+5y+3a$.
4. From $5x-4y-9z$ subtract $3x-8y+2z$.
5. Add $5a+3b-2c$ and $11a-7b+8c$.
6. Add $11axy+13x-14y$, $2y-4x$, and $3y+x-8axy$.
7. $(5x-3b)+(2x+b)-(4x-2b-x+5b)$.
8. Add $19b+3c$, $2b-7c$, $2c-14b$, and $c+8b$.
9. $-(a-3b-c)-(2c-a-5b)+(a-c+b)$.
10. Subtract $2x+4y+z$ from $13x-3y-5z+8$.
11. $5(x-7)+3(14-x)+60=1-10x$.
12. $13(1-x)-6(2x-5)=80+12x$.
13. Add $7x-3y-4$, $5x+2y+5$, and $3y-8x-6$.
14. Add $13a+4b-9c$, $2c-8b-16a$, and $8a-5b-8c$.
15. $8x-[2x+3(x-1)-(2x-3)]$.
16. From $17b-4a-2c-19$ subtract $8c-5a-8b+4$.

17. $3-(3-2+6+8-3)+8-(9-3+8)$.

18. $3(4-x)-2(5-6x)=8x+4$.

19. $12+(2a-3c-4b)-(3b-c-a-8)$.

20. $5x-(3x-2+2y+x)+13y-(6-3x+4)$.

21. Add $y-20$, $4y+6$, $2y+4x-13$, and $2x-8y-40$.

22. Subtract $16-x+2z-4y$ from $3x-5z-8y$.

23. $19+(2x-7)-(31-4x-8-2x)=5x+7$.

24. $16+5x-(8x+9-4x+17)=8x-3$.

25. $6x-3-(4x+8-9x)-(5x-2)=x+11$.

MULTIPLICATION OF POLYNOMIALS

The following principle is useful in **multiplying one monomial by another.**

82. Principle XIII. *To obtain the product of two or more factors, these may be arranged and multiplied in any desired order.*

The truth of this principle is clear from examples such as:

$$2 \cdot 3 \cdot 5 = 2 \cdot 5 \cdot 3 = 5 \cdot 3 \cdot 2 = 5 \cdot (3 \cdot 2) = 2 \cdot (3 \cdot 5) = 30.$$

Historical Note. Principle XIII like Principle XI states two fundamental laws of algebra: (1) *the associative law of factors* first so called by F. S. Servois; (2) *the commutative law of factors*, first so called by Sir William Hamilton.

83. In multiplying algebraic expressions, the same factor frequently occurs more than once in the same term.

Thus we may have $5 \cdot 5$ or $a \cdot a$. These are written 5^2 and a^2 respectively, and read *5 square* and *a square.*

In these expressions the 2 is called an **exponent** and shows that the number above which it is written is to be used *twice as a factor.*

This is a convenient way of abbreviating written expressions.

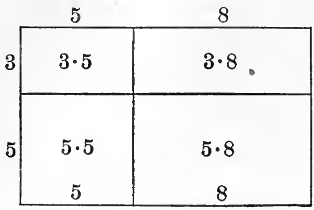
E.g. $5a \cdot a = 5a^2$, $5a \cdot 3a = (5 \cdot 3) \cdot (a \cdot a) = 15a^2$.

$7x \cdot 7x = 7 \cdot 7xx = 7^2x^2$, $ay \cdot ay = aayy = a^2y^2$.

84. The product of two binomials, such as $5+8$ and $5+3$, may be obtained in two ways:

(1) $(5+3)(5+8)=8\cdot 13=104.$

(2) $(5+3)(5+8)=5(5+8)+3(5+8)=5^2+5\cdot 8+3\cdot 5+3\cdot 8=104.$



The second method is illustrated by the accompanying figure in which $5+8$ is the length of a rectangle and $5+3$ is its width. The total area is the product $(5+3)\cdot(5+8)$ and is composed of the four small areas, 5^2, $5\cdot 8$, $3\cdot 5$, and $3\cdot 8$.

The second method here used for multiplying $(5+3)(5+8)$ is the only one available when the terms of the binomials cannot be combined.

Thus

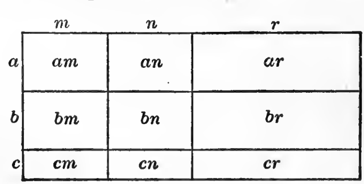
$$(x+4)(x+6)=x(x+6)+4(x+6)=x^2+6x+4x+4\cdot 6$$
$$=x^2+10x+24,$$

and $(a+b)(m+n)=a(m+n)+b(m+n)=am+an+bm+bn.$

Hence, to multiply two binomials, *multiply each term of one by every term of the other and add the products.*

85. In a manner similar to that just illustrated we may multiply two trinomials.

E.g. The product of $a+b+c$ and $m+n+r$, in which the letters represent any positive numbers, may represent the area of a rectangle, divided into small rectangles as follows:



Hence, the product is:

$(a+b+c)(m+n+r) = am + bm + cm + an + bn + cn + ar + br + cr,$

in which *each term of one trinomial is multiplied by every term of the other and the products are added.*

Evidently the same process is applicable to the product of two such polynomials each containing any number of terms.

EXERCISES AND PROBLEMS

Find the following products:

1. $(x+1)(x+2)$.
2. $(x+3)(x+5)$.
3. $(u+7)(u+4)$.
4. $(a+8)(a+3)$.
5. $(t+3)(t+7)$.
6. $(y+9)(y+2)$.
7. $(8+1)(8+7)$.
8. $(s+5)(s+3)$.
9. $(a+b)(c+d)$.
10. $(x+4)(x+3)$.
11. $(x+y+z)(a+b+c)$.
12. $(2x+3)(x+2)$.
13. $(5+x)(6+x)$.
14. $(5s+1)(s+5)$.
15. $(x+7)(3x+4)$.
16. $(a+4)(3a+1)$.
17. $(3+x)(2+5x)$.
18. $(a+b)(3a+7b)$.
19. $(x+y)(2x+3y)$.
20. $(7x+4)(x+8)$.

21. A rectangle is 7 feet longer than it is wide. If its length is increased by 3 feet and its width increased by 2 feet, its area is increased by 60 square feet. What are its dimensions?

22. A field is 10 rods longer than it is wide. If its length is increased by 10 rods and its width increased by 5 rods, the area is increased by 640 square rods. What are the dimensions of the field?

23. A farmer has a plan for a granary which is to be 12 feet longer than wide. He finds that if the length is increased 8 feet and the width increased 2 feet, the floor space will be increased by 160 square feet. What are the dimensions?

24. If the length of a rectangular flower bed is increased 3 feet and its width increased 1 foot, its area will be increased by 19 square feet. What are its present dimensions, if its length is 4 feet greater than its width?

86. Polynomials with negative terms. The polynomials multiplied in the foregoing exercises contain positive terms only. The same process is applicable to polynomials containing negative terms, as is seen in the following examples:

Ex. 1. Find the product of $(7-4)$ and $(3+5)$. This product, written out term by term, would give

$$[7+(-4)](3+5)=7\cdot 3+7\cdot 5+(-4)\cdot 3+(-4)\cdot 5$$
$$=21+35-12-20=24.$$

Also $(7-4)(3+5)=3\cdot 8=24$.

Ex. 2. Multiply $7-4$ and $8-3$.

$$(7-4)(8-3)=[7+(-4)][8+(-3)]$$
$$=7\cdot 8+7\cdot(-3)+(-4)\cdot 8+(-4)(-3)$$
$$=56-21-32+12=15.$$

Also $(7-4)(8-3)=3\cdot 5=15$.

Similarly, $(x+5)(x-2)=x^2+5x-2x-10=x^2-3x-10$,

and $(x-3)(x-5)=x^2-3x-5x+15=x^2-8x+15$.

EXERCISES

Perform the following indicated operations:

1. $(x-5)(x-3)$.
2. $(x-3)(x+4)$.
3. $(a-6)(a-1)$.
4. $(u+5)(u-3)$.
5. $(b+2)(b-7)$.
6. $(3-b)(4+b)$.
7. $(3+x)(7-3x)$.
8. $(n-4)(3-n)$.
9. $(a-b)(c+d)$.
10. $(a-b)(c-d)$.
11. $(x-4)(x-5)$.
12. $(a+b-c)(m-n)$.
13. $(a-b)(7a+3b)$.
14. $(5-y)(5x+3y)$.
15. $(2a-3b+c)(m+n)$.
16. $(v-t)(7v-5t)$.

The preceding exercises illustrate

Principle XIV

87. Rule. *The product of two polynomials is found by multiplying each term of one by every term of the other, and adding these products.*

88. If there are similar terms in either polynomial, these should be added first, thus putting each polynomial in as simple form as possible.

E.g. $(3x + 2 - 2x)(4x + 3 - 3x) = (x + 2)(x + 3)$
$= x^2 + 2x + 3x + 6 = x^2 + 5x + 6.$

It should be observed that Principle XIV involves a repeated application of Principle II. Thus

$$(a + b)(c + d) = (a + b)c + (a + b)d = ac + bc + ad + bd.$$

To simplify the process of combining similar terms in the product, it may be found convenient to arrange the work as in the following examples:

Ex. 1. Multiply $3x - 2$ by $2x - 5$.

Solution.

$$\begin{array}{rrr} 3x & -2 & \\ 2x & -5 & \\ \hline 6x^2 & -4x & \\ & -15x & +10 \\ \hline 6x^2 & -19x & +10 \end{array}$$

Ex. 2. Multiply $3x - 2y + 2$ by $4x - 3y - 2$.

Solution.

$$\begin{array}{rrrrrr} 3x & -2y & +2 & & & \\ 4x & -3y & -2 & & & \\ \hline 12x^2 & -8xy & +8x & & & \\ & -9xy & & +6y^2 & -6y & \\ & & -6x & & +4y & -4 \\ \hline 12x^2 & -17xy & +2x & +6y^2 & -2y & -4 \end{array}$$

EXERCISES AND PROBLEMS

Perform the following indicated operations:

In each case simplify the expressions within the parentheses as much as possible before multiplying:

1. $(x-7)(3x+4)$. **2.** $(x-2)(9x+4)$. **3.** $(a-x)(9x+4a)$.

4. $(5x + 3y - 4x - 2y)(6y + 3x - 2y + y)$.

5. $(13a - b - 12a)(2b - 3a)$.

6. $(xy - 5xy + 4)(8y - 3 - 7y)$.

7. $(11b + 3a)(2b - 3b + 5)$.

8. $(6 - 4x + 3x)(7x + y - 3x + 1)$.

9. $(x - y + 3)(5x - 3y + 5)$. 10. $(a - 13n)(a - n + 8)$.

11. $(x - 2 + y)(4y - 3x)$.

12. $(11b - a - 10b)(6a - 3b - 2a)$.

13. $(7 + y - x)(2y + x - 1)$. 14. $(5x + 3y - 1)(x - 2)$.

15. $(-7a - 1 + 8a)(5a - 8 - 3a)$.

Solve the following equations and check the results:

16. $(x + 2)(x + 3) = (x - 3)(x + 10) + 10$.

17. $(5x - 4)(6 - x) - 97 = (x - 1)(6 - 5x)$.

18. $(3n - 1)(18 - n) = (n + 6)(16 - 3n)$.

19. $(7 - a)(9a - 8) = 31 + (36 - 9a)(a + 2)$.

20. $(4a + 4)(a - 3) = (4a + 1)(a + 7) - 13a + 221$.

21. $(n + 6)(3n - 4) - 14 = (n + 8)(3n - 3)$.

22. $(8n + 6)(10 - n) + 150 = (1 - n)(8n + 3)$.

23. $(a - 1)(13 - 6a) = (6a - 3)(8 - a) - 21$.

24. $(7x - 13)(6 - x) - (x + 4)(3 - 7x) = 70$.

25. Find two numbers whose difference is 6 and whose product is 180 greater than the square of the smaller.

26. There are four consecutive *even* integers such that the product of the first and second is 40 less than the product of the third and fourth. What are the numbers?

27. There are four consecutive integers such that the product of the first and third is 223 less than the product of the second and fourth. What are the numbers?

28. Find four numbers such that the second is 5 greater than the first, the third 5 greater than the second, and the fourth 5 greater than the third. The product of the first and second is 250 less than the product of the third and fourth.

29. A club makes an equal assessment on its members each year to raise a certain fixed sum. One year each member pays a number of dollars equal to the number of members of the club less 175. The following year, when the club has 50 more members, each member pays $5 less than the preceding year. What was the membership of the club the first year and how much did each pay?

PROBLEMS ON RECTANGLES AND TRIANGLES

30. A rectangle is 10 inches longer than wide. Express its area in terms of the width w. If the width is increased by 4 and the length by 6 inches, express the area in terms of w.

31. A rectangle is 8 inches longer than wide. Express its area in terms of the width w after the width is increased 4 inches and the length decreased 10 inches.

32. A rectangle is 5 feet longer than it is wide. If it were 3 feet wider and 2 feet shorter, it would contain 15 square feet more. Find the dimensions of the rectangle.

33. A rectangle is 6 feet longer and 4 feet narrower than a square of equal area. Find the side of the square and the sides of the rectangle.

If b is the base of a triangle, h its altitude (height), and a its area, then *area* $= \frac{1}{2}$(*base* × *altitude*);

i.e.
$$a = \frac{bh}{2}.$$

34. The base of a triangle is 2 inches less than its altitude a. Express the area in terms of a.

35. The altitude of a triangle is 7 greater than its base b. If the altitude is decreased by 8 and the base by 6, express its area in terms of b.

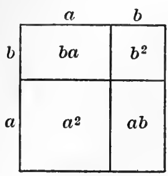
36. The altitude of a triangle is 16 inches less than the base. If the altitude is increased by 3 inches and the base by 2 inches, the area is increased by 52 square inches. Find the base and altitude of the triangle.

SQUARES OF BINOMIALS

89. Just as x^2 is written instead of $x \cdot x$, so $(a+b)^2$ is written instead of $(a+b)(a+b)$. The square of a binomial is found by multiplying the binomial by itself as in § 84.

E.g. $(a+b)(a+b) = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2$.

Hence, $$(\boldsymbol{a} + \boldsymbol{b})^2 = \boldsymbol{a}^2 + 2\,\boldsymbol{ab} + \boldsymbol{b}^2. \qquad \text{I}$$



This product is illustrated in the accompanying figure, and is evidently a special case of the type exhibited in the figures, page 68.

Translated into words this identity is: *The square of the sum of any two numbers is equal to the square of the first plus twice the product of the two numbers plus the square of the second.*

By formula I we may square any binomial sum.

E.g. $(3x + 2y)^2 = (3x)^2 + 2 \cdot (3x)(2y) + (2y)^2 = 9x^2 + 12xy + 4y^2$.

EXERCISES

Find the following products. Read the first four at sight.

1. $(a+2)^2$.
2. $(x+3)^2$.
3. $(m+n)^2$.
4. $(5+b)^2$.
5. $(2a+b)^2$.
6. $(1+3a)^2$.
7. $(2a+3b)^2$.
8. $(2x+1)^2$.
9. $(3y+2)^2$.
10. $(c+4b)^2$.
11. $(5x+4)^2$.
12. $(3a+7c)^2$.
13. $(1+8a)^2$.
14. $(x+11y)^2$.
15. $(a+9b)^2$.

90. Similarly, we obtain the square of the difference of two numbers: $$(\boldsymbol{a} - \boldsymbol{b})^2 = \boldsymbol{a}^2 - 2\,\boldsymbol{ab} + \boldsymbol{b}^2. \qquad \text{II}$$

Translate this identity into words.

Ex. By use of formula II, find the square of $a - 3b$.

Solution. $(a - 3b)^2 = a^2 + 2 \cdot a(-3b) + (-3b)^2 = a^2 - 6ab + 9b^2$.

While these squares are ordinary products of binomials and may be found by Principle XIV, they are of special importance and should be studied until they can be given from memory at any time.

EXERCISES

Find the following products. Read the first five at sight:

1. $(a-3)^2$.
2. $(b-4)^2$.
3. $(c-d)^2$.
4. $(x-7)^2$.
5. $(3-n)^2$.
6. $(1-2b)^2$.
7. $(3c-1)^2$.
8. $(2y-3)^2$.
9. $(3a-2b)^2$.
10. $(c-3b)^2$.
11. $(2x-3y)^2$.
12. $(7c-3a)^2$.
13. $(x-4y)^2$.
14. $(1-4z)^2$.
15. $(a-7b)^2$.

There are two other special products which should be memorized; namely, III and IV below.

91. Examples. Find the products,

$$(x+2)(x-2),\ (x+5)(x-5),\ (x+9)(x-9).$$

In each of these *one factor is the sum of two numbers and the other is the difference between the same numbers, while the product is the difference of the squares of these numbers.* This is expressed by the formula

$$(x+a)(x-a)=x^2-a^2. \qquad \text{III}$$

By means of this formula the product of the sum and difference of any two number expressions may be found.

E.g. $(3a+2b)(3a-2b)=(3a)^2-(2b)^2=9a^2-4b^2$.

Verify this by performing the multiplication.

EXERCISES

Read the following products:

1. $(a+1)(a-1)$.
2. $(a+3)(a-3)$.
3. $(k-b)(k+b)$.
4. $(3-x)(3+x)$.
5. $(2a+3b)(2a-3b)$.
6. $(a+2b)(a-2b)$.
7. $(2b-1)(2b+1)$.
8. $(1+3x)(1-3x)$.
9. $(1-7y)(1+7y)$.
10. $(a-4b)(a+4b)$.
11. $(6a-3b)(6a+3b)$.
12. $(7-9a)(7+9a)$.
13. $(2c+1)(2c-1)$.
14. $(3a+b)(3a-b)$.
15. $(5k+3h)(5k-3h)$.
16. $(9m+3n)(9m-3n)$.

92. Examples. Find the products:

$$(x+2)(x+3) \qquad (x+4)(x-7)$$
$$(x+5)(x-2) \qquad (x-5)(x-3)$$

Each of these products when simplified consists of three terms, of which the first is x^2, the last is the product of the second terms of the factors, and the coefficient of x in the middle term is the algebraic sum of the second terms of the factors.

This is expressed by the formula:

$$(x+a)(x+b)=x^2+(a+b)x+ab. \qquad \text{IV}$$

Verify this by performing the multiplication.

EXERCISES

Write the following products. Also try to read them at sight.

1. $(x+7)(x+3)$.
2. $(x+9)(x+6)$.
3. $(y+6)(y-2)$.
4. $(y-8)(y+3)$.
5. $(c-4)(c-2)$.
6. $(a-8)(a+10)$.
7. $(a+7)(a+6)$.
8. $(a-7)(a+6)$.
9. $(ab+3)(ab+7)$.
10. $(ab-5)(ab-3)$.

In the formula $(x+a)(x+b)=x^2+(a+b)x+ab$, replace a and b by the following values and simplify the results:

11. $a=5$, $b=3$.
12. $a=8$, $b=-7$.
13. $a=6$, $b=-11$.
14. $a=-5$, $b=-7$.

15. Find the square of 42 by writing it as a binomial, $40+2$.

16. Square the following numbers by writing each as a binomial sum: 51, 53, 93, 91, 102, 202, 301.

17. Find the square of 29 by writing it as a binomial, $30-1$.

18. Square the following numbers by first writing each as a binomial difference: 28, 38, 89, 77, 99, 198, 499, 998, 999.

19. Find the product of 41 and 39, first indicating the product thus, $(40+1)(40-1)$.

20. Find the following products by writing each pair of factors as the sum and difference of two numbers:

(1) $62 \cdot 58$.	(3) $53 \cdot 47$.	(5) $17 \cdot 13$.
(2) $27 \cdot 33$.	(4) $102 \cdot 98$.	(6) $99 \cdot 101$.

EQUATIONS AND PROBLEMS

Solve the following equations, verifying except where the answer is given.

1. $(a+4)^2+(a-1)(2a+5)=(a+4)(3a+2)$.

2. $(a-1)(3a-1)-(a+1)^2-2a^2-18$.

3. $(6-a)^2+(a-3)(2a-5)=(3a+1)(a-3)+84$.

4. $(7a-18)(a+4)-(a-1)^2=6(a+2)^2-79$.

5. $(2b-30)(b-1)-5b^2=6b-3(b+5)^2+65$.

6. $(5-c)^2+(7-c)^2+(9-c)^2=(c-1)(3c-58)-93$.

7. $(5c-3)(2+c)-4(c-1)^2=(c+1)^2+54$.

8. $(8-4c)(5-c)=(c+1)^2+(c+3)(3c-8)+218$.

9. $(y-1)^2+4(y+1)^2+(1-y)(5y+6)=15y-29$.

10. $x(x+3)+(x+1)(x+2)=2x(x+5)+2$.

11. $x^2=(x-3)(x+6)-12$.

12. $(5+5x)(3-x)+2(x+1)^2+3(x+1)(x-7)=17(x+1)$.

13. $(8+3x)(4-x)+(x-1)(x-2)+2(x+5)^2=105$.

14. $(5-b)(6b+5)+4(b-3)^2=20-2(b+1)^2+3+16b$.
Answer $3\frac{7}{11}$.

15. There is a square field such that if its dimensions are increased by 5 rods, its area is increased 625 square rods. How large is the field?

Suggestion: If a side of the original field is w, then its area is w^2, and the area of the enlarged field is $(w+5)^2$.

16. A rectangle is 9 feet longer than it is wide. A square whose side is 3 feet longer than the width of the rectangle is equal to the rectangle in area. What are the dimensions of the rectangle?

17. A boy has a certain number of pennies which he attempts to arrange in a solid square. With a certain number on each side of the square he has 10 left over. Making each side of the square one larger, he lacks 7 of completing it. How many pennies has he?

18. A room is 7 feet longer than it is wide. A square room whose side is 3 feet greater than the width of the first room is equal to it in area. What are the dimensions of the first room?

19. Find two consecutive integers whose squares differ by 51.

20. Find two consecutive integers whose squares differ by 97.

21. Find two consecutive integers whose squares differ by a. Show from the form of the equation obtained that a must be an *odd* integer.

22. There are four consecutive integers such that the sum of the squares of the last two exceeds the sum of the squares of the first two by 20. What are the numbers?

23. Two square pieces of land require together 360 rods of fence? If the difference in the area of the pieces is 900 square rods, how large is each piece? (*Hint:* $x^2 - (90 - x)^2 = 900$.)

24. There is a square such that if one side is increased by 12 feet and the other side decreased by 8 feet the resulting rectangle will have the same area as the square. Find the side of the square.

25. A regiment was drawn up in a solid square. After 50 men had been removed the officer attempted to draw up the square by putting one man less on each side, when he found he had 9 men left over. How many men in the regiment?

26. There is a rectangle whose length exceeds its width by 11 rods. A square whose side is 5 rods greater than the width of the rectangle is equal to it in area. What are the dimensions of the rectangle?

REVIEW QUESTIONS

1. What is a polynomial? a term? How are polynomials classified? What are similar terms? By what principle are similar terms added? By what principle are they subtracted?

2. In adding or subtracting polynomials how may the terms be arranged for convenience? State the principle on which this is based?

3. What is the principle for removing a parenthesis when preceded by the sign +? By the sign −? How may Principle XII be used for inclosing terms within a parenthesis?

4. In finding the product of two or more numbers how may the factors be arranged for convenience? State the principle on which this is based.

5. Make a diagram to show how to multiply $(7+4)$ by $(11+8)$ without first uniting the terms of the binomials. Multiply $(a+b)$ by $(c+d)$ in the same manner.

Multiply $(12-3)$ by $(9-7)$ in two ways and compare results. State the principle by which two polynomials are multiplied.

6. Describe a convenient manner of arranging the work in multiplying polynomials. What kind of terms in the product are placed in the same column? Find the product of $7x - 3y + 1$ and $2x - 8y - 3$, arranging your work this way.

7. State in words what is the square of the binomial $x+a$; of the binomial $x-a$.

8. Translate into words the formula $(x+a)(x-a)=x^2-a^2$.

9. Translate into words the formula

$$(x+a)(x+b)=x^2+(a+b)x+ab.$$

Interpret this formula for various positive and negative values of a and b.

10. Complete your list of Principles stated in symbols up to and including Principle XIV.

DRILL EXERCISES

1. $(x-1)(2x-2)+(x-5)^2=(3-x)(24-3x)-7.$

2. From $3-4a-5c+8x^2$ subtract $2x^2-2a-4c+8.$

3. $(4ab-6ac-5ad)(b-c+d).$

4. $(17x+3)(x-1)+8=(2-x)(6-17x)+19.$

5. $5-(a+b-c-d+8)+(3+a+c-d)-5.$

6. Add $6a+9$, $8a-13$, $46a-8$, and $6-54a.$

7. $(a-2)(6a-4)+2(a-1)^2=(6-a)(30-8a)+4.$

8. From $6(a+2)+3(c+4)-2(b-d)$
subtract $2(a+2)-2(c+4)+3(b-d).$

9. Add $12a^2b^2c+8ax$, $6ax-8a^2b^2c$, and $2ax+3a^2b^2c.$

10. Add $5xy^2+3x^2y+4xy$, $2x^2y-6xy^2-3xy$, and $4xy.$

11. Add $6ab-3c-2a$, $2c-4ab-5a$, $5c-a+ab$, and $3+5a-2c-3ab.$

12. $(n-4)(6-3n)-(6-n)^2-10=-4n(n-4).$

13. From $35ab-8x-9z+13$ subtract $16ab-4z+5x+8.$

14. $(n+2)^2+(n-1)^2+(n+1)^2=3n(n+2)+60n+130.$

15. Subtract $5a-8x-6y$ from $13x+14y-15z-4a.$

16. From $9y-4x-6z-3b$ subtract $8-9y-3x-2z.$

17. $2x+4-6(5x-8-7x)+2-4x=6(2-3x)-42.$

18. $-(7+4x-8-2x)+4-2x=6x+25.$

19. $(a-1+b-c-d)(4a+5b+3c-2d).$

20. $(4ax-3ay+5az-8)(x+y-z+2).$

21. $(3a-2b+4c)(2a+3b-c).$

22. $7-(3a-2b-4a)+b+2a-(3b-2a-a).$

23. $8x+(5y-5)+(2y-1)-(13y+8x-17).$

CHAPTER V

SIMPLE FRACTIONS

93. Definitions. In arithmetic a fraction such as $\frac{2}{3}$ is usually regarded as 2 of the 3 equal parts of a unit.

However, a fraction such as $\frac{5}{3\frac{1}{2}}$ cannot be regarded in this way, since a unit cannot be divided into $3\frac{1}{2}$ equal parts. $\frac{5}{3\frac{1}{2}}$ indicates that 5 is to be divided by $3\frac{1}{2}$; *i.e.* $\frac{5}{3\frac{1}{2}} = 5 \div 3\frac{1}{2}$.

In algebra any fraction is usually regarded as an **indicated division** in which the numerator is the dividend and the denominator is the divisor.

Thus, $\frac{a}{b}$ is understood to mean $a \div b$.

The numerator and denominator are together called the **terms of the fraction.**

Operations on algebraic fractions are performed in accordance with the same rules as in arithmetic.

EXERCISES

Supply the missing numerator in each of the following:

1. $\frac{1}{2} = \frac{\ }{4}$.
2. $\frac{3}{4} = \frac{\ }{12}$.
3. $\frac{1}{a} = \frac{\ }{ka}$.
4. $\frac{a}{b} = \frac{\ }{cb}$.
5. $\frac{m}{n} = \frac{\ }{3n}$.
6. $\frac{1}{a} = \frac{\ }{-2a}$.
7. $\frac{1}{a-1} = \frac{\ }{1-a}$.
8. $\frac{1}{a+1} = \frac{\ }{(a+1)(a-1)}$.
9. $\frac{1}{b+2} = \frac{\ }{(b+3)(b+2)}$.

The preceding examples illustrate

Principle XV

94. **Rule.** *Both terms of a fraction may be multiplied or divided by the same number without changing its value.*

95. **Definition.** The **lowest common multiple** (L. C. M.) of two or more numbers is the least number which contains as factors all the factors of these numbers.

E.g. 12 is the L. C. M. of 4 and 6. abc is the L. C. M. of ab, bc, and ac. $(a+b)(a+1)$ is the L. C. M. of $(a+b)$ and $(a+1)$.

EXERCISES

Find the L. C. M. of each of the following:

1. 3, 5.
2. 3, 4, 6.
3. 6, 48, 24.
4. a, b.
5. a, bk.
6. ab, bc, ac.
7. $3a$, $2b$, $4c$.
8. $a+1$, $a-1$.
9. $b+2$, $b+3$.
10. $1-a$, $1-2a$.
11. $2x+3$, $x-4$.
12. $m+3$, $m-5$.

REDUCTION TO A COMMON DENOMINATOR

96. **Examples.** 1. Reduce $\frac{1}{2}$ and $\frac{1}{3}$ to a common denominator. How is the common denominator related to the denominators 2 and 3?

2. Reduce $\frac{1}{a}$ and $\frac{1}{b}$ to a common denominator. How is the common denominator related to the denominators a and b?

3. Reduce $\frac{2}{a+1}$, $\frac{3}{(a+1)(a+2)}$, $\frac{4}{a+2}$ to a common denominator.

Solution. The required denominator is the L. C. M. of the given denominators; that is, $(a+1)(a+2)$.

Hence, $\frac{2}{a+1}=\frac{2(a+2)}{(a+1)(a+2)}$ and $\frac{4}{a+2}=\frac{4(a+1)}{(a+1)(a+2)}$.

$\frac{3}{(a+1)(a+2)}$ already has the required denominator.

EXERCISES

Reduce each of the following to a common denominator:

1. $\frac{1}{a}, \frac{1}{b}$.

2. $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$.

3. $\frac{1}{a}, \frac{1}{-a}$.

4. $\frac{1}{ab}, \frac{1}{ac}$.

5. $\frac{a}{bc}, \frac{b}{ac}$.

6. $\frac{a}{xy}, \frac{b}{xz}$.

7. $\frac{1}{ab}, \frac{1}{bc}, \frac{1}{ac}$.

8. $\frac{a}{bc}, \frac{b}{ac}, \frac{c}{ab}$.

9. $\frac{m}{xy}, \frac{n}{xz}, \frac{s}{yz}$.

10. $\frac{1}{x+1}, \frac{1}{x+2}$.

11. $\frac{2}{a-3}, \frac{1}{a+4}$.

12. $\frac{a}{a-b}, \frac{c}{c-d}$.

13. $\frac{1}{1-k}, \frac{1}{k-1}$.

14. $\frac{2}{x+2}, \frac{3}{x-2}$.

15. $\frac{a}{a-b}, \frac{b}{a+b}$.

16. $\frac{3}{n-4}, \frac{2}{(n-1)(n-4)}$.

17. $\frac{1}{x-1}, \frac{1}{(x-1)(x-2)}$.

18. $\frac{2}{a-b}, \frac{3}{(a-b)(c-d)}$.

ADDITION AND SUBTRACTION

97. Examples. 1. Add $\frac{3}{4}$ and $\frac{5}{6}$. How is the numerator of the sum found after the fractions have been reduced to a common denominator.

2. Add $\frac{1}{a+1}$ and $\frac{2}{a+3}$.

Solution. $\frac{1}{a+1} = \frac{a+3}{(a+1)(a+3)}$; $\frac{2}{a+3} = \frac{2(a+1)}{(a+1)(a+3)}$.

Hence, $\frac{1}{a+1} + \frac{2}{a+3} = \frac{a+3}{(a+1)(a+3)} + \frac{2(a+1)}{(a+1)(a+3)}$

$$= \frac{a+3+2(a+1)}{(a+1)(a+3)} = \frac{3a+4}{(a+1)(a+3)}.$$

EXERCISES

Perform the following additions:

1. $\frac{1}{a}+\frac{1}{b}$.
2. $\frac{1}{ab}+\frac{1}{ac}$.
3. $\frac{1}{a+1}+\frac{1}{a-1}$.
4. $\frac{2}{x+1}+\frac{3}{x+2}$.
5. $\frac{2}{a}+\frac{1}{-a}$.
6. $\frac{1}{1-k}+\frac{1}{k-1}$.
7. $\frac{3}{n-4}+\frac{2}{(n-1)(n-4)}$.
8. $\frac{2}{x+2}+\frac{3}{x-2}$.

98. **Examples.** 1. From $\frac{5}{7}$ subtract $\frac{2}{3}$. How is the numerator of the difference found after the fractions have been reduced to a common denominator?

2. From $\frac{3}{x-4}$ subtract $\frac{2}{x+3}$.

Solution:

$$\frac{3}{x-4}-\frac{2}{x+3}=\frac{3(x+3)}{(x-4)(x+3)}-\frac{2(x-4)}{(x-4)(x-3)}$$

$$=\frac{3(x+3)-2(x-4)}{(x-4)(x-3)}=\frac{x+17}{(x-4)(x-3)}.$$

EXERCISES

Perform the following subtractions:

1. $\frac{1}{a}-\frac{1}{b}$.
2. $\frac{1}{ab}-\frac{1}{ac}$.
3. $\frac{1}{a+1}-\frac{1}{a-1}$.
4. $\frac{2}{x+1}-\frac{3}{x+2}$.
5. $\frac{2}{-a}-\frac{1}{a}$.
6. $\frac{1}{1-k}-\frac{1}{k-1}$.
7. $\frac{2}{x+2}-\frac{3}{x-2}$.
8. $\frac{3}{n-4}-\frac{2}{(n-1)(n-4)}$.

EXERCISES

Perform the following additions and subtractions:

1. $\frac{1}{x-2}-\frac{1}{x+1}$.

2. $\frac{1}{a+4}-\frac{2}{a-3}$.

3. $\frac{1}{x-1}-\frac{1}{1-x}$.

4. $\frac{1}{x}-\frac{1}{y}$.

5. $\frac{a}{a-b}-\frac{b}{a+b}$.

6. $\frac{3}{k-1}-\frac{2}{k+1}$.

7. $\frac{4}{c-2}-\frac{3}{(c-1)(c-2)}$.

8. $\frac{1}{a}-\frac{1}{b}+\frac{1}{c}$.

9. $\frac{1}{ab}+\frac{1}{bc}-\frac{1}{abc}$.

10. $\frac{3}{a+b}-\frac{2}{a-b}+\frac{1}{a}$.

11. $\frac{a}{1-a}+\frac{1}{a-1}+\frac{1}{a}$.

12. $\frac{3}{x-y}-\frac{1}{y}+\frac{1}{x}$.

13. $\frac{x}{x-y}+\frac{x}{y}-\frac{1}{y}$.

14. $\frac{1}{x}+\frac{a}{y}-\frac{b}{z}$.

15. $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}$.

16. $\frac{a}{(x-1)(x+3)}-\frac{b}{x+3}$.

17. $\frac{1}{x+y}+\frac{1}{x-y}+\frac{1}{(x+y)(x-y)}$.

18. $\frac{1}{a+1}+\frac{1}{a-1}-\frac{1}{(a+1)(a-1)}$.

MULTIPLICATION AND DIVISION

99. **Examples.** 1. Multiply $\frac{3}{7}$ by 5, also $\frac{a}{b}$ by c.

2. Multiply $\frac{5}{8}$ by 4, also $\frac{a}{bc}$ by c.

3. Multiply $\frac{2}{3}$ by $\frac{5}{7}$, also $\frac{a}{b}$ by $\frac{c}{d}$.

4. Multiply $\frac{4(x+3)}{(x-1)(x+2)}$ by $\frac{3(x-1)}{4(x+3)(x+2)}$.

Solution. $\frac{\cancel{4}\cancel{(x+3)}}{\cancel{(x-1)}(x+2)}\times\frac{3\cancel{(x-1)}}{\cancel{4}\cancel{(x+3)}(x-2)}=\frac{3}{(x+2)(x-2)}=\frac{3}{x^2-4}$.

The first step is to cancel all possible factors.

EXERCISES

Perform the following multiplications:

1. $\frac{ab}{c} \times \frac{c}{a}$.
2. $\frac{ab}{cx} \times \frac{c^2x^2}{a}$.
3. $\frac{abc}{xy} \times \frac{bx}{ac}$.
4. $\frac{cx^2}{by^2} \times \frac{cy}{dx}$.
5. $\frac{mxy}{nyc} \times \frac{ny}{mx}$.
6. $\frac{rst}{xy} \times \frac{xy^2}{r^2s}$.
7. $\frac{3}{n+8} \times (n+1)$.
8. $\frac{2}{(a-2)(a+3)} \times 3(a-2)$.
9. $\frac{3c}{(3-c)(3+c)} \times \frac{(c-3)(c+3)}{6cd}$.
10. $\frac{3a+b}{2+3x} \times \frac{(3x+2)(2a-b)}{(a+b)(3a+b)}$.
11. $\frac{5ab}{2cd} \times \frac{4bc}{10a}$.
12. $\frac{(a+2b)(a+2b)}{a-2b} \times \frac{(2a-b)(a-2b)}{(3a+b)(a+2b)}$.

100. Examples. 1. Divide $\frac{4}{7}$ by 2, also $\frac{ab}{c}$ by a.

2. Divide $\frac{3}{5}$ by 7, also $\frac{a}{b}$ by c.

3. Divide $\frac{3}{4}$ by $\frac{5}{7}$, also $\frac{a}{b}$ by $\frac{c}{d}$.

4. Divide $\frac{ab}{(x-1)(x-2)}$ by $\frac{b}{x-1}$.

Solution. $$\frac{ab}{(x-1)(x-2)} \div \frac{b}{x-1} = \frac{ab}{(x-1)(x-2)} \times \frac{x-1}{b} = \frac{a}{x-2}.$$

After inverting the terms of the divisor cancel all possible factors.

EXERCISES

Perform the following divisions:

1. $\frac{ab}{c} \div c$.
2. $\frac{ab}{c} \div \frac{bx}{c}$.
3. $\frac{abc}{xy} \div \frac{bc}{ax}$.
4. $\frac{ax^2}{by^2} \div \frac{ax}{cy}$.
5. $\frac{mxy}{my} \div \frac{ny}{mx}$.
6. $\frac{rst}{xy} \div \frac{r^2s}{x^2y}$.

7. $\frac{abc}{x-1} \div ab.$

8. $\frac{xy}{a-b} \div (a+b).$

9. $\frac{a+1}{a-2} \div \frac{3(a+1)}{a+4}.$

10. $\frac{24\,ab}{25\,c} \div \frac{16\,ab}{15}.$

11. $\frac{4\,k^2}{-9\,abc^2} \div -\frac{2\,k}{ab}.$

12. $\frac{ab^2}{(x+y)^2} \div \frac{ab}{x+y}.$

13. $\frac{6a-3}{5\,x} \div \frac{2\,a-3}{15\,x^2}.$

14. $\frac{(2+x)(2-x)}{3\,x(x+1)} \div \frac{(2-x)^2}{x(x-2)}.$

NOTE. — A full treatment of fractions will be found toward the end of this course. To give such a treatment here would necessitate a complete study of factoring and consequently the postponement of much important work on the solution of equations and problems which naturally belongs early in the course. The treatment given in this chapter is based directly on *Arithmetic* and is sufficient for the solution of all problems depending on simple equations which naturally occur in an elementary course.

EQUATIONS INVOLVING FRACTIONS

Solve $n + \frac{n}{2} + \frac{n}{3} = 88.$ (1)

First Solution. The coefficients of n are 1, $\frac{1}{2}$, and $\frac{1}{3}$.

Applying Principle I, $(1+\frac{1}{2}+\frac{1}{3})n = 1\frac{5}{6}\,n = 88.$ (2)

By $D \mid 1\frac{5}{6}$, $n = 88 \div 1\frac{5}{6} = 48.$ (3)

Second Solution. Multiply both members of (1) by 6.

That is, by $M \mid 6$, $6\,n + \frac{6\,n}{2} + \frac{6\,n}{3} = 528.$ (2)

By *F*, V, $6\,n + 3\,n + 2\,n = 528.$ (3)

By *F*, I, $11\,n = 528.$ (4)

By $D \mid 11$, $n = 48$, as before. (5)

The object is to multiply both members of the equation by such a number as will cancel each denominator. Hence the multiplier must contain each denominator as a factor.

Evidently 12 or 18 might have been chosen for this purpose, but not 8 or 10. 6 is the smallest number which will cancel both 2 and 3.

101. The process explained in the second solution above is called **clearing of fractions.**

As another illustration solve the equation

$$\frac{4}{x+1} - \frac{1}{x-1} = \frac{4}{(x+1)(x-1)}. \qquad (1)$$

Here the lowest multiple available is $(x+1)(x-1)$.

Hence, $$\frac{4(x+1)(x-1)}{x+1} - \frac{(x+1)(x-1)}{x-1} = \frac{4(x+1)(x-1)}{(x+1)(x-1)}. \qquad (2)$$

By F, V, each denominator is now cancelled,

$$4(x-1) - (x+1) = 4. \qquad (3)$$

By F, II, XII, $$4x - 4 - x - 1 = 4. \qquad (4)$$

By F, I and $A \mid 5$, $$3x = 9. \qquad (5)$$

By $D \mid 3$, $$x = 3. \qquad (6)$$

Check. Substitute $x = 3$ in (1) and get

$$1 - \tfrac{1}{2} = \tfrac{4}{8} \text{ or } \tfrac{1}{2} = \tfrac{1}{2}.$$

After a little practice step (2) should be performed mentally and equation (3) should be derived immediately from (1).

EXERCISES AND PROBLEMS

Solve the following equations, indicating the principles used at each step. Check each solution by substituting the value obtained in the original equation. Translate the first four into problems.

For instance, from the equation in Ex. 2: Find a number such that when increased by its half, its third, and its fourth, the sum is 25.

1. $\frac{n}{2} + \frac{n}{3} = 5.$

2. $n + \frac{n}{2} + \frac{n}{3} + \frac{n}{4} = 25.$

3. $\frac{n}{4} + 4n - \frac{5n}{3} = \frac{3n}{2} + 26.$

4. $\frac{n}{2} + \frac{n}{3} - \frac{n}{4} + \frac{n}{10} = 82.$

5. $7x + \frac{4x}{7} + \frac{3x}{5} + 23 = \frac{x}{5} + \frac{4x}{10} + 5x + 113$

6. $4n + \frac{6n}{7} = \frac{3n+2}{2} + 46.$

7. $12+\frac{4(9x+6)}{3}-\frac{2(3+11x)}{5}=\frac{5(4x+4)}{3}+14.$

8. $13b+7-\frac{9b+8}{7}=\frac{2b+9}{5}+38.$

9. $\frac{5a+7}{2}+\frac{2a+4}{3}=\frac{3a+9}{4}+5.$

10. $\frac{17a-5}{3}-\frac{10a+2}{4}=\frac{5a+7}{2}-5.$

11. $\frac{5x+3}{2}+\frac{5(2x+10)}{5}=2x+24.$

12. $\frac{3}{x}-\frac{1}{2}=\frac{1}{x}.$

13. $\frac{1}{3x}+\frac{1}{2x}=\frac{1}{6}.$

14. $\frac{1}{4x}+\frac{1}{3x}=\frac{5}{6x}+\frac{1}{2}.$

15. $\frac{1}{x}+\frac{2}{x-1}=\frac{8}{x(x-1)}.$

16. $\frac{4}{x+1}-\frac{3}{x-1}=\frac{-2}{(x+1)(x-1)}.$

17. $\frac{1}{x-1}+\frac{1}{x+1}=\frac{4}{(x-1)(x+1)}.$

18. The sum of two numbers is 12, and the first number is $\frac{1}{3}$ as great as the second. What are the numbers?

19. The smaller of two numbers is $\frac{3}{8}$ of the larger. If their sum is 66, what are the numbers?

20. Find two consecutive integers such that $\frac{5}{7}$ of the first minus $\frac{3}{11}$ of the second equals 9.

21. Find three consecutive integers such that $\frac{1}{3}$ of the first plus the second minus $\frac{1}{2}$ the third equals 5.

22. Find three consecutive integers such that $\frac{1}{2}$ of the first plus $\frac{1}{3}$ of the second minus $\frac{1}{10}$ of the third equals 28.

23. There are three numbers such that the second is 4 more than 9 times the first, and the third is 2 more than 6 times the first. If $\frac{1}{8}$ of the third is subtracted from $\frac{1}{7}$ of the second, the remainder is 3. Find the numbers.

24. There are three numbers such that the second is 2 more than 9 times the first and the third is 5 more than 11 times the first. The remainder when $\frac{1}{3}$ of the third is substracted from $\frac{1}{2}$ of the second is one. Find the numbers.

25. What number must be subtracted from both the numerator and the denominator of the fraction $\frac{5}{7}$ in order to make the result equal to $\frac{1}{2}$.

26. What number must be subtracted from both numerator and denominator of the fraction $\frac{5}{3}$ in order that the fraction may be increased three-fold? *Ans.* $2\frac{1}{2}$.

27. What number added to both numerator and denominator of the fraction $\frac{3}{4}$ will double the fraction?

28. Find a number of two digits in which the tens' digit is 3 greater than the units' digit, and such that if the number is divided by the sum of the digits, the quotient is 7.

29. In a number of two digits the units' digit exceeds the tens' digit by 4, and when the number is divided by the sum of its digits the quotient is 4. Find the number.

30. There are two numbers whose sum is 40, such that if the greater is divided by their difference the quotient is 3.

31. The shortest railway route from Boston to Chicago is 166 miles more than 4 times that from Boston to New York; and the shortest route from Boston to Atlanta is 196 miles less than 6 times that from Boston to New York. The distance from Boston to Chicago is 481 miles more than $\frac{1}{2}$ the distance from Boston to Atlanta. Find each of the three distances.

REVIEW QUESTIONS

1. What principle is used in reducing fractions to a common denominator?

2. State from arithmetic the rules for multiplying or dividing a fraction by an integer; by another fraction. State these as formulas by using letters.

3. What is the effect on an equation if both members are multiplied by the same number? How may this be used to clear an equation of fractions?

4. Complete your list of principles stated in symbolic form up to and including Principle XV.

DRILL EXERCISES

1. $\frac{a}{3}+\frac{a+7}{4}-\frac{a-3}{3}=\frac{a+227}{5}-1.$

2. $(a-2-3c-8+2b)(6-a-c-b+8).$

3. $\frac{a-1}{2}+\frac{a+1}{2}+\frac{a-3}{12}=2+a.$

4. $a^2b-(3b-8a^2-7)+3ab^2-(4ab^2+8-2a^2).$

5. $\frac{a+1}{4}+\frac{a-3}{4}+\frac{a-7}{4}=2a-26.$

6. $\frac{n+1}{3}+\frac{n+3}{4}+\frac{n-1}{4}=\frac{n+13}{3}+\frac{n-2}{3}.$

7. $16ax+4-(8-8ax-a)-(12ax-13-ax).$

8. Add $15ax^2+3bc^2$, $2bc^2-7ax^2$, and $5+2ax^2-5bc^2$.

9. Add $16-7ab-2a^2+5ab$, $4a^2-2ab$, and $5ab-8$.

10. Add $51x^2y-35+12a^2$, $41-17a^2-57x^2y$, and $3x^2y$.

11. Add $35b^2-13c^2$, $8c^2-3b^2c^2$, and $6b^2-8c^2-9b^2c^2$.

12. Add $19-2x+3x^2b+2b$, $4x^2b+x+5b+8$, and $4x$.

13. $\frac{x}{4}+\frac{x}{8}+\frac{x}{16}-\frac{x}{2}+\frac{x}{32}=-2.$

14. $\frac{y-3}{4}-\frac{y+9}{10}=\frac{y-11}{4}-2.$

15. $\frac{y-4}{3}+\frac{y+2}{3}+\frac{y+8}{3}=2y-20.$

16. $\frac{y}{2}+\frac{y+20}{4}+\frac{y+5}{5}=25.$

17. $\frac{y}{3}-\frac{y+20}{5}+\frac{y-5}{5}+\frac{y-10}{2}=15.$

18. $\frac{x}{x-1}-\frac{x}{x+1}=\frac{4}{(x-1)(x+1)}.$

19. $-\frac{x-1}{x+2}+\frac{x+1}{x-3}=\frac{27}{(x+2)(x-3)}.$

20. $\frac{2x-1}{x+2}-\frac{4x-1}{2x-3}=\frac{-10}{(x+2)(2x-3)}.$

CHAPTER VI

LITERAL EQUATIONS AND THEIR USES

102. Some of the advantages of algebra over arithmetic in solving problems have been pointed out in the preceding chapters. For instance, the brevity and simplicity of statement secured through the use of letters to represent numbers; the translation of problems into equations; and the clear and logical solution of these equations, step by step.

Another advantage is set forth in the present chapter; namely, the opportunity offered in Algebra to *summarize the solution of a whole class of problems* by solving what is called a **literal equation,** thus obtaining a **formula** which may be used in solving other problems.

For example, in arithmetic we solved many problems obtaining the **interest** when the **principal, rate,** and **time** were given. We now see that all of these can be summarized in the one literal equation

$$i = prt.$$

Furthermore, the rules for obtaining the principal, the rate, or the time may now be derived directly from this equation by Principle VI, thus obtaining:

$$p = \frac{i}{rt}, \quad r = \frac{i}{pt}, \quad t = \frac{i}{pr}.$$

Translate each of these formulas into a rule of arithmetic.

103. Definition. The process of deriving $p = \frac{i}{rt}$ from $i = prt$ is called **solving** the equation $i = prt$ for p in terms of i, r, and t, or simply **solving the equation for** p.

104. In arithmetic a problem is said to be solved when a *numerical answer* is obtained which satisfies the conditions given. The solutions thus far found in algebra have, for the most part, been of this sort.

It is customary, however, to say that a problem has been solved in the *algebraic sense* when a formula is found which gives complete directions for deriving the numerical answer.

Thus, $p = \frac{i}{rt}$ is a solution for the principal since it states precisely how to find the principal in terms of interest, rate, and time.

105. It is thus seen that from the literal equation $i = prt$ we obtain the complete solution of every problem which calls for any one of these four numbers in terms of the other three.

In modern times machines are extensively used for computation. The algebraic solution of a literal equation *gets the problem ready for the computing machine*, that is, it gets the **formula** which the computer must use.

LITERAL EQUATIONS USED IN SOLVING PROBLEMS

I. PROBLEMS FROM ARITHMETIC

1. If \$700 is invested at 5 % simple interest, what is the amount at the end of 5 years? This problem calls for the amount, which is the sum of principal and interest.

If a = amount, then $\boldsymbol{a = p + i = p + prt = p(1 + rt)}$.

Applying this formula, $a = 700(1 + \frac{5}{100} \cdot 5) = 875$.

2. Solve the equation $a = p + prt$ for p in terms a, r, and t. State a rule of arithmetic represented by this solution and make a problem which can be solved by means of it.

3. Solve $a = p + prt$ for r in terms of a, p, and t. State a rule of arithmetic represented by this solution and make a problem which can be solved by means of it.

4. Solve $a=p+prt$ for t in terms of a, p, and r. State a rule of arithmetic represented by this solution and make a problem which can be solved by means of it.

5. A real estate dealer sold a house and lot for \$7500, for which he received a commission of 4%. What was his profit?

Solution. Letting c = commission, b = base, and r = rate, we have

$$c=br.$$

Applying this formula, $c = br = 7500 \cdot \frac{4}{100} = 75 \cdot 4 = 300$.

6. Solve the equation $c=br$ for b in terms of c and r.

7. Solve $c=br$ for r in terms of c and b.

8. State the rules of arithmetic represented by the solutions in Exs. 6 and 7, and make problems to be solved by these rules.

9. How much must I remit to my broker in order that he may buy \$600 worth of bonds and reserve 5% commission?

I must send him both base and commission. Calling this the amount and representing it by a, we have

$$a=b+c=b+br=b(1+r).$$

Hence, $a=600(1+\frac{5}{100})=630$.

10. Solve $a=b+br$ for b and translate the result into words.

11. Solve $a=b+br$ for r and translate the result into words.

12. A dealer sold berries for \$18.95, and after deducting a commission of 2% sent the balance to the truck gardener. How much did he remit?

The sum he sent was the difference between the base and the commission; calling this d, we have

$$d=b-c=b-br=b(1-r).$$

Hence, in this case $d=18.95(1-\frac{2}{100})=18.57$.

13. Solve the equation $d=b(1-r)$ for b in terms of d and r.

14. Solve the equation $d=b-br$ for r in terms of b and d.

15. State the rules of arithmetic represented by the solutions in Exs. 13 and 14 and make problems to be solved by these rules.

II. PROBLEMS INVOLVING MOTION

106. In scientific language the distance passed over by a moving body is called the **space**, and the number of units of space traversed is represented by s. The **rate** of uniform motion, that is, the number of units of space traversed in each unit of time, is called the **velocity**, and is represented by v. The number of units of time occupied is represented by t.

Ex. 1. If a train runs 40 miles per hour, how far does it run in 5 hours?

Ex. 2. At a certain temperature sound travels 1080 feet per second. How far does it travel in 5 seconds?

In each of these examples the space passed over is found by multiplying the velocity by the time. Using the symbols s, v, and t, we have

$$s = vt. \qquad (1)$$

1. Solve the equation $s = vt$ for t in terms of s and v, and for v in terms of s and t.

Translate each of these formulas into words.

It is to be understood in all problems here considered that the velocity remains the same throughout the period of motion; *e.g.* sound travels just as far in any one second as in any other second of its passage.

2. If sound travels 1080 feet per second, how far does it travel in 6 seconds?

3. If a transcontinental train averages 35 miles per hour, how far does it travel in $2\frac{1}{2}$ days? (Given $v = 35$, $t = 2\frac{1}{2} \cdot 24$, to find s.)

4. A hound runs 23 yards per second and a hare 21 yards per second. If the hound starts 79 yards behind the hare, how long will it require to overtake the hare?

If t is the number of seconds required, then by formula (1) during this time the hound runs $23\,t$ yards and the hare runs $21\,t$ yards. Since the hound must run 79 yards farther than the hare, we have: $23\,t = 21\,t + 79$.

5. An ocean liner making 21 knots an hour leaves port when a freight boat making 8 knots an hour is already 1240 knots out. In how long a time will the liner overtake the freight?

6. A motor boat starts $7\frac{2}{3}$ miles behind a sailboat and runs 11 miles per hour while the sailboat makes $6\frac{1}{2}$ miles per hour. How long will it require the motor boat to overtake the sailboat?

7. A freight train running 25 miles an hour is 200 miles ahead of an express train running 45 miles an hour. How long before the express will overtake the freight?

8. A bicyclist averaging 12 miles an hour is 52 miles ahead of an automobile running 20 miles an hour. How soon will the automobile overtake him?

9. A and B run a mile race. A runs 18 feet per second and B $17\frac{1}{2}$ feet per second. B has a start of 30 yards. In how many seconds will A overtake B? Which will win the race?

If in each of the examples 4 to 9 we call the velocity of the first moving object v_1 (read *v one*) and that of the second v_2 (read *v two*), then the distance traveled by the first in the required time t is v_1t, and that traveled by the second is v_2t.

Then if n is the distance which the second must go in order to overtake the first, we have

$$v_2t = v_1t + n. \quad (2)$$

The solution of (2) for t gives the time required in each problem for the second to overtake the first.

Equation (2) summarizes the solution of all problems like those from 4 to 9.

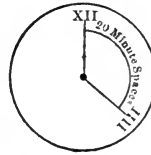
It is important that formulas (1) and (2) be clearly understood, since they are used very often in problems on motion.

10. A fleet, making 11 knots per hour, is 1240 knots from port when a cruiser, making 19 knots per hour, starts out to overtake it. How long will it require?

Use formula (2).

11. In how many minutes does the minute hand of a clock gain 15 minute spaces on the hour hand?

Using one minute space for the unit of distance and 1 minute as the unit of time, the rates are 1 and $\frac{1}{12}$ respectively, since the hour hand goes $\frac{1}{12}$ of a minute space in 1 minute. Letting t be the number of minutes required, we have, using formula (2), $1 \cdot t = \frac{1}{12} t + 15$.



12. In how many minutes after 4 o'clock will the hour and minute hands be together? (Here the minute hand must gain 20 minute spaces.) *Ans.* $21\frac{9}{11}$ min.

13. At what time between 5 and 6 o'clock is the minute hand 15 minute spaces behind the hour hand? At what time is it 15 minute spaces ahead?

Since, at 5 o'clock, it is 25 minute spaces behind the hour hand, in the first case it must gain $25 - 15 = 10$ minute spaces, and in the second case it must gain $25 + 15 = 40$ minute spaces. Make a diagram as in the preceding problem to show both cases.

14. At what time between 9 and 10 o'clock is the minute hand of a clock 30 minute spaces behind the hour hand? At what time are they together?

In each case, starting at 9 o'clock, how much has the minute hand to gain?

15. A fast freight leaves Chicago for New York at 8.30 A.M., averaging 32 miles per hour. At 2.30 P.M. a limited express leaves Chicago over the same road, averaging 55 miles per hour. In how many hours will the express overtake the freight?

If the express requires t hours to overtake the freight, the latter had been on the way $t + 6$ hours. Then the distance covered by the express is $55\,t$, and the distance covered by the freight is $32\ (t + 6)$. As these must be equal, we have $55\,t = 32(t + 6)$.

16. In a century bicycle race one rider averages $19\frac{1}{2}$ miles per hour, while another, starting 40 minutes later, averages $22\frac{1}{4}$ miles per hour. In how long a time will the latter overtake the former?

III. PROBLEMS INVOLVING THE LEVER

107. Two boys, A and B, play at teeter. They find that the teeter board will balance when equal products are obtained by multiplying the weight of each by his distance from the point of support.

Thus, if B weighs 80 pounds and is 5 feet from the point of support, then A, who weighs 100 pounds, must be 4 feet from this point, since $80 \times 5 = 100 \times 4$.

The teeter board is a certain kind of lever; the point of support is called the **fulcrum**.

In each of the following problems make a diagram similar to the above figure:

1. A and B weigh 90 and 105 pounds respectively. If A is seated 7 feet from the fulcrum, how far is B from that point?

2. Using the same weights as in the preceding problem, if B is $6\frac{1}{2}$ feet from the fulcrum, how far is A from that point?

3. A and B are 5 and 7 feet respectively from the fulcrum. If B weighs 75 pounds, how much does A weigh?

4. A and B weigh 100 and 110 pounds respectively. A places a stone on the board with him so that they balance when B is 6 feet from the fulcrum and A $5\frac{1}{2}$ feet from this point. How heavy is the stone?

5. If the distances from the boys to the fulcrum are respectively d_1 and d_2, and their weights w_1 and w_2, then

$$d_1 w_1 = d_2 w_2. \qquad (1)$$

This equation is a statement in the language of algebra of a very important law of nature. The law is the result of a very large number of careful experiments. It is a universal custom among scientific men so far as possible to express laws of nature by means of literal equations of this sort.

If any three of the four numbers d_1, w_1, d_2, w_2, are given, the fourth may be found by means of the equation $d_1 w_1 = d_2 w_2$.

6. Solve $d_1w_1 = d_2w_2$ for each of the four numbers involved in terms of the other three.

7. A and B are seated at the opposite ends of a 13-foot teeter board. Using the weights of problem 1, where must the fulcrum be located so that they shall balance?

If the fulcrum is the distance d from A, then it is $(13 - d)$ from B. Hence, $90\,d = 105(13 - d)$.

8. A, who weighs 75 pounds, sits 7 feet from the fulcrum, and B, who weighs 105 pounds, sits on the other side. At what distance from the fulcrum should B sit in order to make a balance?

9. A and B together weigh $212\frac{1}{2}$ pounds. They balance when A is 6 feet, and B $6\frac{3}{4}$ feet, from the fulcrum. Find the weight of each.

10. A lever 9 feet long carries weights of 17 and 32 pounds at its ends. Where should the fulcrum be placed so as to make the lever balance?

11. A lever of unknown length is balanced when weights of 30 and 45 pounds are placed on it at opposite ends. Find the length of the lever, if the smaller weight is two feet farther from the fulcrum than the greater.

SUGGESTION. Let x be the distance from the greater weight to the fulcrum.

EXERCISES

Solve the following for each letter in terms of the others:

1. $F = 32 + \frac{9}{5}C$.

2. $l = a + (n - 1)d$.

3. $s = \frac{n}{2}(a + l)$.

4. $s = \frac{a - rl}{1 - r}$.

Solve each of the following for x:

5. $ax + 3b = cx + d$.

6. $(a - x)(b + x) = x(b - x)$.

7. $(x + a)(x + b) = (x - c)^2$.

8. $\frac{ax + b}{c} - \frac{bx + c}{d} = 1$.

9. $\frac{x}{b} + \frac{x}{a - b} = \frac{a}{a + b}$.

10. $\frac{x+1}{x-1}=\frac{a+b}{a-b}$.

11. $\frac{x}{a}+\frac{x}{b}+\frac{x}{c}=1$.

12. $\frac{a+bx}{a+b}=\frac{c+dx}{c+d}$.

13. $\frac{a+b}{1+x}=\frac{a-b}{1-x}$.

14. $\frac{3\,ax}{a-b}-2\,a=5\,x$.

15. $\frac{ax-1}{bx}=\frac{1+bx}{ax}+1$.

16. $\frac{x-a}{a-b}-\frac{x+a}{a+b}=\frac{2\,x}{a+b}$.

17. $\frac{a+x}{a}-\frac{b+x}{b}-\frac{c+x}{c}=1$.

18. $\frac{x+1}{x-1}+\frac{x-1}{x+1}=\frac{2\,x(x+1)}{(x+1)(x-1)}$.

19. $\frac{x+m}{x-m}+\frac{x-m}{x+m}=\frac{2\,x(x+1)}{(x+m)(x-m)}$.

20. $\frac{x+b}{x-a}+\frac{x-c}{x+a}=\frac{2\,x(x+a)}{(x-a)(x+a)}$.

21. $\frac{b}{x-b}+\frac{a}{x+a}=\frac{a}{(x+a)(x-b)}$.

REVIEW QUESTIONS

1. Make a list of all the rules for interest which are derived from $i=prt$.

2. Make a list of all the rules for percentage which are derived from $a=p+prt$.

3. Make a list of the rules for commission which may be obtained from $c=br$, $a=b+c=b+br$, $d=b-c=b-br$.

4. What problems on motion in this chapter belong to the class whose solutions are summarized by the solutions of the equation,

$$v_2t=v_1t+n.$$

5. State fully the meaning of the equation $w_1d_1=w_2d_2$ in connection with the lever.

6. What is the difference in meaning between the solution of a special problem in arithmetic and that of a problem involving a literal equation in algebra?

DRILL EXERCISES

1. Subtract $2a - 6x^2b - 3x + 21$ from $19 - 2x + 3x^2b - 7a$.

2. From $6a - 45 + 8b - 3c + 82cb$ subtract $7b + 18 + 6c$.

3. $(17 + 2a - 3b - 4c)(2 - a + b - c)$.

4. $(13c - 4d + 8e - 3)(c - d)$.

5. $(4xy - 2y - 3x - 2)(y - x + y + 5)$.

6. $(9ax - 3x - 5a - 2x + 4)(5 - x)$.

7. $(9x - 3)(4 - x) + (x - 3)^2 = -8(x + 2)^2 + 94$.

8. $(x + 1)^2 + (x + 2)^2 + (x + 3)^2 = (3x - 1)(x + 12) - 43$.

9. $(2x + 5)(x - 7) - (x - 1)^2 = (x + 1)(x + 2) - 28$.

10. $3(5 - x)^2 - (2x - 1)(x - 1) = (x - 7)(x + 10) + 17x + 50$.

11. $(32 + x)(4x - 1) + (5 - x)^2 + (x - 1)^2 = 6(x + 1)^2 + 194$.

12. $(2x - 7)(5 - x) - (2 - 5x)(1 - x) = -x(7x - 34) - 17$.

13. $\frac{x+8}{2} - \frac{x-9}{12} + \frac{x-17}{6} = \frac{4x-7}{2} + \frac{2x+6}{3} + \frac{5-31x}{12}$.

14. $\frac{3x-1}{6} - \frac{3x+3}{3} + \frac{x-1}{2} = \frac{x+5}{6} + 4x - \frac{20}{3}$.

15. $\frac{x-b}{a} + \frac{x+a}{b} = 2$.

16. $\frac{3x-16}{2} + \frac{21}{x-8} = \frac{6x-11}{4}$.

17. If two numbers differ by d and if the greater of the numbers is x, how do you represent the other?

18. A father is 3 times as old now as his son was 7 years ago. If the son's age now is represented by x, how is the father's age represented?

19. A picture inside the frame is w inches wide and $w + 6$ inches long. The frame is 4 inches wide. Express the area of the frame in terms of w.

20. A picture inside the frame is w inches wide and l inches long. The frame is a inches wide. Express the area of the frame in terms of a, w, and l.

CHAPTER VII

GRAPHIC REPRESENTATION

108. Graphic Representation of Statistics. A graphic representation of the temperatures recorded on a certain day is shown on the next page. The readings were as follows:

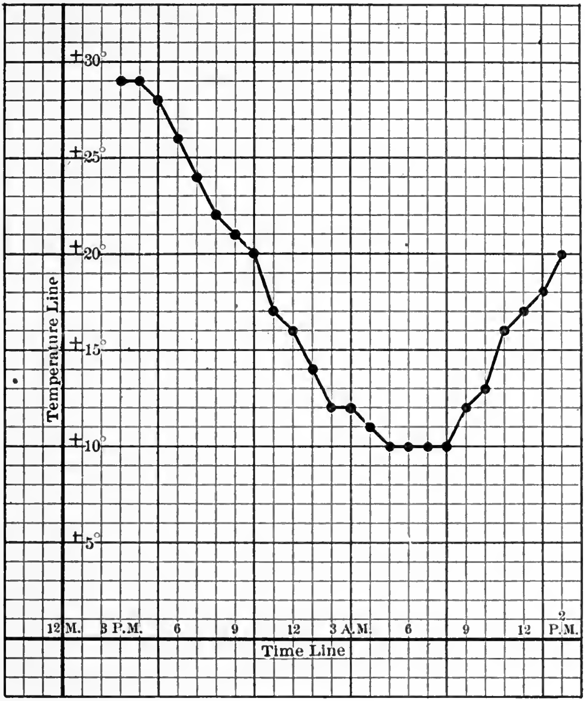
3 P.M.	29°	9 P.M.	21°	3 A.M.	12°	9 A.M.	12°
4 P.M.	29°	10 P.M.	20°	4 A.M.	11°	10 A.M.	13°
5 P.M.	28°	11 P.M.	17°	5 A.M.	10°	11 A.M.	16°
6 P.M.	26°	12 M'T.	16°	6 A.M.	10°	12 Noon	17°
7 P.M.	24°	1 A.M.	14°	7 A.M.	10°	1 P.M.	18°
8 P.M.	22°	2 A.M.	12°	8 A.M.	10°	2 P.M.	20°

In the graph each heavy dot represents the temperature at a certain hour. The distance of a dot to the right of the heavy vertical line indicates the hour of the day counted from noon, and its distance above the heavy horizontal line indicates the thermometer reading at that hour. The lines joining these dots complete the picture representing the gradual changes of temperature from hour to hour.

Graphs of this kind are used in commercial houses to represent variations of sales, fluctuations of prices, etc. They are used by the historian to represent changes in population, fluctuations in mineral productions, etc. In algebra they are used in solving problems and in helping to understand many difficult processes. In the succeeding exercises the cross-ruled paper is essential.

Make a graphic representation of the tables of data on the opposite page:

In each case the number to be represented by one space on the cross-ruled paper should be chosen so as to make the graph go conveniently on a sheet. Thus in Ex. 1 let one small horizontal space represent two years and one vertical space a million of population; and in Ex. 3 let one horizontal space represent one year and one large vertical space one hundred thousand of population.



EXERCISES

1. The population of the United States as given by the census reports from 1800 to 1910:

1800 . . 4.3 (million)	1840 . . 17.1	1880 . . 50.2
1810 . . 7.2	1850 . . 23.2	1890 . . 62.6
1820 . . 9.6	1860 . . 31.4	1900 . . 76.3
1830 . . 12.9	1870 . . 38.6	1910 . . 92.0

2. The population of the boroughs now constituting greater New York City:

1800 . . 79 (thousand)	1840 . . 391	1880 . . 1912
1810 . . 119	1850 . . 696	1890 . . 2507
1820 . . 152	1860 . . 1175	1900 . . 3427
1830 . . 242	1870 . . 1478	1910 . . 4767

3. The population of Chicago since 1850:

1850 . . 30 (thousand)	1880 . . 503	1900 . . 1698
1860 . . 109	1890 . . 1100	1910 . . 2185
1870 . . 306		

4. Observe the weather reports in a daily paper and make a graph representing the hourly change of temperature for twenty-four hours.

5. From your own state, city, or town obtain data which you can represent by means of graphs.

GRAPHIC REPRESENTATION OF MOTION

109. A useful picture of the distance traversed by a moving body can be made by a graph similar to the preceding.

E.g. Suppose a man is walking 3 miles per hour. We mark units of time from the starting point to the right along the horizontal reference line, and indicate miles traveled by the number of units measured vertically upward from this line. (See the figure on the opposite page.)

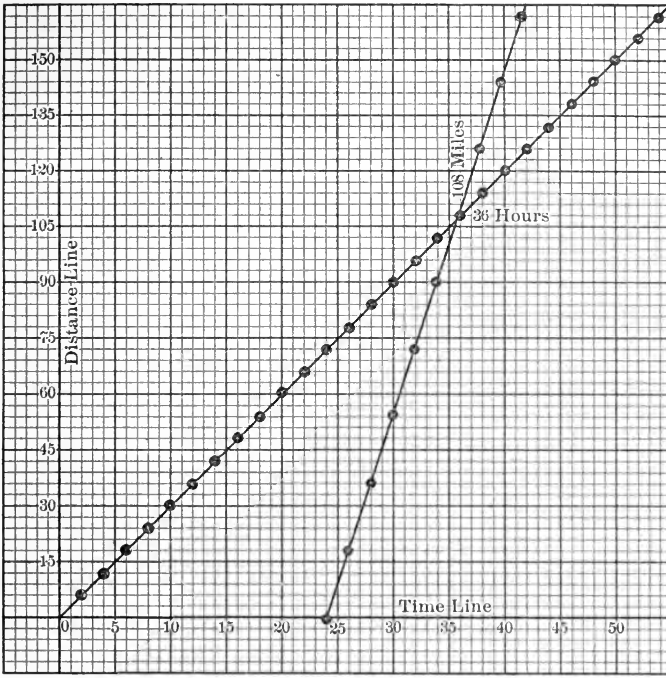
In the figure each horizontal space represents 1 hour, and each vertical space 3 miles. Then in 1 hour he goes 3 miles; in 5 hours, 15 miles; in 10 hours, 30 miles; etc. The dots representing the distances are found to lie on a straight line.

The graph shows at a glance the answers to such questions as: How many miles does he travel in 4 hours? in 13 hours? How long does it take him to go 18 miles? 23 miles?

Again, suppose 24 hours later a second man starts out on a bicycle to overtake the first man, and travels 9 miles an hour. The line drawn from the 24-hour point shows the distance the wheelman travels in any number of hours, counting from his time of starting. The points marked in this line show how far he has gone in 1, 2, 3, 4, 5, 6 hours, etc., namely, 9, 18, 27, 36, 45, 54, etc.

The point where these two lines intersect shows in how many hours after starting the pedestrian is overtaken, and also how far he has gone.

In like manner solve the following by means of graphs, and in each case suggest other questions which may be answered from the graph:



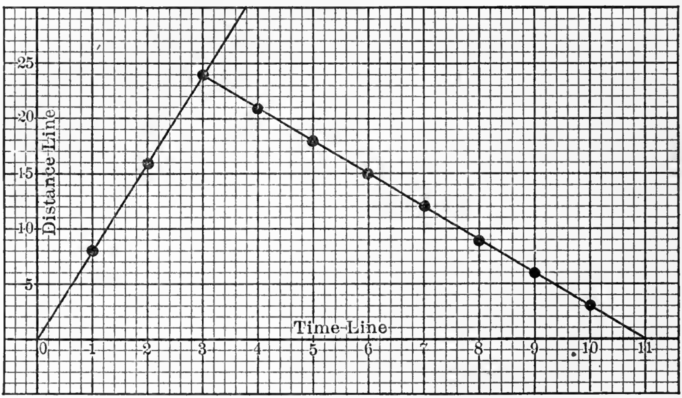
1. A starts for a town 12 miles distant, walking 3 miles per hour. $1\frac{1}{2}$ hours later B starts for the same place, driving $7\frac{1}{2}$ miles per hour. When does B overtake A? Where is A when B reaches town?

2. In a mile race A runs 6 yards per second, and B 5 yards per second. B has a start of 250 yards. Who will win the race? How far in the lead is the winner at the finish?

3. In a century bicycle race A averages 17 miles per hour; B, who starts 20 minutes later, averages 19 miles per hour. In how many hours will B overtake A? Who will win the race and where will the loser be when the winner finishes?

Let one vertical small space represent 2 miles, and one horizontal space represent 6 minutes.

110. Illustrative Problem. A man rides a bicycle into the country at the rate of 8 miles per hour. After riding a certain distance the wheel breaks down, and he walks back at the rate of 3 miles per hour. How far does he go before the accident, if he reaches home 11 hours after starting?



In this graph each large horizontal space represents 1 hour, and each small vertical space represents 1 mile. The problem is solved as follows:

(1) Construct the line representing the outward journey at the rate of 8 miles per hour, extending this line indefinitely.

(2) Beginning at the point corresponding to 11 hours, find the points representing his position at each preceding hour. The line connecting these points represents the homeward journey at the rate of 3 miles an hour. Extend this line until it meets the first line. The point where the lines meet represents 3 hours and 24 miles, which is the answer required in the problem.

PROBLEMS

Solve the following problems by means of graphs. In each case prepare a list of questions which may be answered from the graph.

1. A man rows 18 miles per hour down a river, and 2 miles per hour returning. How far down the river can he go if he wishes to return in 10 hours?

2. A man goes from Chicago to Milwaukee on a train running $42\frac{1}{2}$ miles per hour, and returns immediately on a steamer going 17 miles per hour. Find the distance, if the round trip requires 7 hours.

3. A pleasure trip from New York to Atlanta by steamer, and return by rail, occupied 77 hours. Find the distance, if the rate going was 16 miles per hour and returning 40 miles per hour.

Let one small horizontal space represent one hour and one small vertical space 16 miles.

4. A invests \$1000 at 5 %, and B invests \$ 5000 at 4 %. In how many years will the *amount* (principal and interest) of A's investment equal the *interest* on B's investment?

Let one large horizontal space represent one year and one small vertical space \$ 50. Then the line representing A's *amount* starts at the point marked \$1000, and rises one small vertical space each year. The line representing B's *interest* starts at the zero point and rises four small vertical spaces each year.

5. In how many years will the *interest* on \$ 6000 equal the *amount* on \$ 2000 if both are invested at 5 % ?

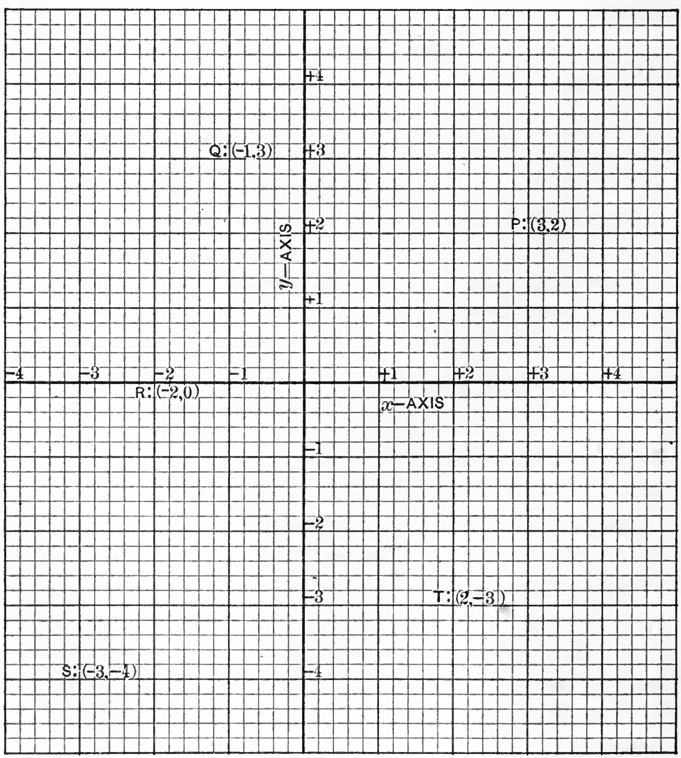
6. A invests \$ 500 at 6 % and B invests \$ 1000 at 5 %. In how many years will A's interest differ by \$ 300 from B's?

Let one small vertical space represent \$ 20. In this case both lines start from the zero point. Find the point on one line which is three large spaces *vertically* above the corresponding point on the other line.

7. A freight train starts from New York to Boston averaging 30 miles per hour. Three hours later a passenger train, averaging 50 miles per hour, starts from New York in the same direction. How long will it require the latter to overtake the freight?

GRAPHIC REPRESENTATION OF EQUATIONS

111. In all the graphs thus far constructed two lines at right angles to each other have been used as reference lines. These lines are called **axes.** The location of a point in the plane of such a pair of axes is completely described by giving its distance and direction from each of the axes. The direction to the right of the vertical axis is denoted by a positive sign, and to the left, by a negative sign; while direction upward from the horizontal axis is positive, and downward, negative.



The horizontal line is usually called the **x-axis** and the vertical line the **y-axis.** The perpendicular distance of any point P from the y-axis is called the **abscissa** of the point, and its distance from the x-axis is called its **ordinate.** The abscissa and ordinate of a point are together called its **coördinates.**

E.g. the abscissa of point P in the above figure is 3 and its ordinate 2, or we may say the coördinates of P are 3 and 2, and indicate it thus: $P:(3, 2)$, writing the abscissa first. In like manner for the other points we write $Q:(-1, 3)$, $R:(-2, 0)$, $S:(-3, -4)$, and $T:(2, -3)$.

We see that in this manner every point in the plane corresponds to a pair of numbers, and that every pair of numbers corresponds to a point. This scheme of locating points by two reference lines is already familiar to the pupil in geography, where cities are located by latitude and longitude; that is, by degrees north or south of the equator and east or west of the meridian of Greenwich.

EXERCISES

1. With any convenient scale, locate the following points: $(2, 6)$, $(-3, 5)$, $(0, 1)$, $(1, 0)$, $(0, 0)$, $(0, -1)$, $(0, -5)$, $(-5, 0)$, $(2\frac{1}{2}, 5\frac{1}{3})$, $(-4, -8)$, $(3, -10)$, $(-10, 3)$.

2. Locate the following series of points and then see if a straight line can be drawn through them: $(0, 0)$, $(1, 1)$, $(2, 2)$, $(3, 3)$, $(4, 4)$, $(-1, -1)$, $(-2, -2)$, $(-3, -3)$. Name still other points lying on the same line.

3. Locate the following and connect them by a line: $(1, 0)$, $(1, 2)$, $(1, 3)$, $(1, 4)$, $(1, 5)$, $(1, -2)$, $(1, -3)$, $(1, -4)$, $(1, -5)$. Name other points in this line.

4. Draw the line every one of whose points has its horizontal distance -2, also the line every one of whose points has its vertical distance $+3$.

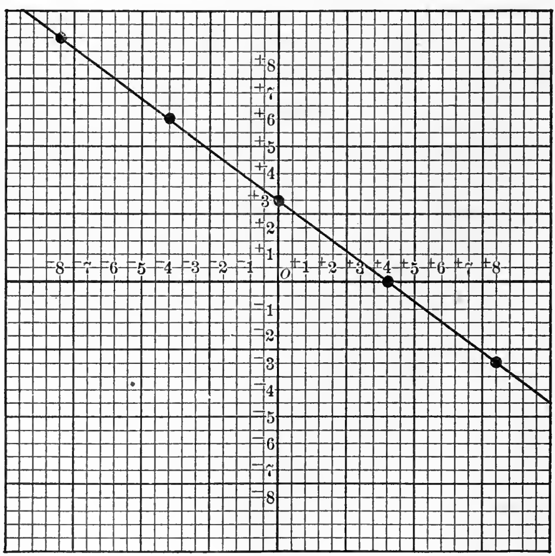
5. Locate the following points and see if a straight line can be passed through them: $(1, 0)$, $(0, 1)$, $(2, -1)$, $(3, -2)$, $(4, -3)$, $(-1, +2)$, $(-2, 3)$, $(-3, 4)$, $(-4, 5)$, $(\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{4}, \frac{3}{4})$, $(\frac{2}{3}, \frac{1}{3})$. Can you name other points on this line?

112. In the preceding exercises, in certain cases, a series of points has been found to **lie on a straight line**, as in examples 6, 7, and 8. Evidently this could not happen unless the points were located according to some definite scheme or law.

Illustrative Problem. Locate a series of points whose coördinates are values of x and y which satisfy: $3x + 4y = 12$.

We see that $x = 0$, $y = 3$, also $x = 4$, $y = 0$ are pairs of such values. Evidently as many pairs of values as we please may be found by giving any value to x and then solving the equation to find the corresponding value of y. A table may thus be constructed as follows:

Let	$x = 0,\ 4,\ \ 8,\ \ 12,\ -4,\ -8$, etc.
Then find	$y = 3,\ 0,\ -3,\ -6,\ \ 6,\ \ 9$, etc.



These pairs of values for x and y correspond to the points as plotted in the figure, and they are found to lie on a straight line. This line is called the **graph of the equation.**

Let the student find other pairs of numbers which satisfy this equation, and see if the corresponding points lie on this line. Also find the numbers which correspond to any chosen point on this line and see whether they satisfy the equation.

113. We may think of the point whose coördinates are x and y as moving along the line in the graph. Then both x and y will be constantly changing or **varying,** but subject to the relation $3x + 4y = 12$. x and y are therefore called **variables,** and the fixed relation according to which they vary is called a **functional relation.**

One of the graphs on page 105 represents the progress of a man walking 3 miles per hour. As the time varies the point moves along the line, and the distance is seen to vary as the time varies; that is, s varies with t according to the relation $s = 3t$. The variable s is said to be a **function of the variable t** because it is connected with t by a definite relation.

In the equation $s = 3t$ the variable s *increases* as the variable t increases. This may also be seen directly from the graph. In $3x + 4y = 12$, y is seen to *decrease*, as x *increases*.

114. Definitions. An equation is said to be of the **first degree in x and y** if it contains each of these letters in such a way that neither x nor y is multiplied by itself or by the other.

E.g. $13x - 5y = 14$ is of the first degree, while $2xy - x = 5$ and $3x - 5y^2 = 13$ are not of the first degree in x and y.

Every equation of the first degree in two variables has for its graph a straight line; hence such an equation is commonly called a **linear equation.**

115. To graph an equation of the first degree it is only necessary to find two points on the graph and draw a straight line through them.

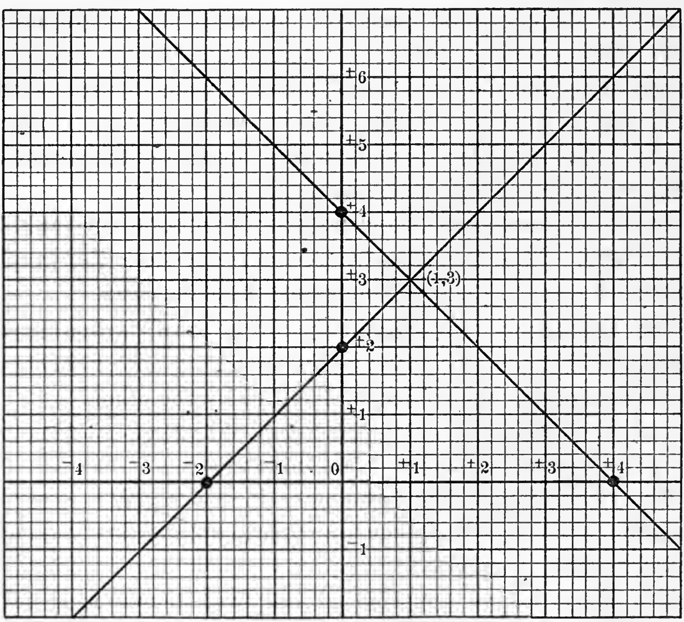
E.g. In graphing the equation $x - y = 5$, we choose $x = 0$ and find $y = -5$, and choose $y = 0$ and find $x = 5$ and plot the points $(0, -5)$ and $(5, 0)$. The line through these points is the one required.

EXERCISES

Construct the graph for each of the following equations. From each graph tell whether y increases or decreases as x increases.

1. $3x + 2y = 1$.	5. $5 - 2y = 6x$.	9. $3x - 4y = -7$.
2. $5x - 3y = -3$.	6. $3x + 5y = -15$.	10. $3x - 4y = -12$.
3. $7x + 10y = 2$.	7. $2x - y = 0$.	11. $7y = 9x - 63$.
4. $x + 2y = 0$.	8. $3x - 4y = 7$.	12. $x = 5y + 3$.

116. Illustrative Problem. Graph on the same axes the two equations $x + y = 4$ and $y - x = 2$.



Solution. The two graphs are found to intersect in the point (1, 3). Since the point lies on both lines, its coördinates should satisfy both equations, as indeed they do. Since these lines have only one point in common, there is no other pair of numbers which, when substituted for the variables x and y, can satisfy both equations.

Hence, $x=1$, $y=3$, which is written (1, 3), is called the **solution of this pair of equations.**

117. Definition. These two equations are called **independent** because their graphs are distinct. They are called **simultaneous** because there is at least one pair of values of x and y which satisfy both.

118. Since two straight lines intersect in but one point, it follows that two linear equations which are independent and simultaneous have one and only one solution. The solving of two simultaneous equations may be regarded as *finding the coördinates of the point where their graphs meet.* This may be done by constructing the graphs or by other methods of solution which are considered in the next chapter.

EXERCISES

Graph the following and thus find the solution of each pair of equations:

1. $\begin{cases} 2x-3y=25, \\ x+y=5. \end{cases}$

2. $\begin{cases} 5x+6y=7, \\ 2x-y=-4. \end{cases}$

3. $\begin{cases} 5x+3y=8, \\ 2x-y=-10. \end{cases}$

4. $\begin{cases} 6x+8y=16, \\ 2x-3y=11. \end{cases}$

5. $\begin{cases} 3x-4y=1, \\ 2x-7y=5. \end{cases}$

6. $\begin{cases} y+3x=7, \\ 2y+x=-6. \end{cases}$

7. $\begin{cases} x-2y=2, \\ 2x-y=-2. \end{cases}$

8. $\begin{cases} 5x-7y=21, \\ x-4y=-1. \end{cases}$

9. $\begin{cases} 5x+2y=8, \\ 2x-3y=-12. \end{cases}$

10. $\begin{cases} x+3y=-6, \\ 2x-4y=-12. \end{cases}$

119. Not all pairs of equations in two unknowns are **independent.**

E.g. If we attempt to plot $x+y=1$ and $2x+2y=2$ the graphs will be found to **coincide.** Such equations are called **dependent**, since one can be derived from the other.

Not all pairs of equations are **simultaneous.**

E.g. If we attempt to plot $x + y = 1$ and $x + y = 2$ the graphs will be found to be parallel and hence they have no point in common. Such equations are called **contradictory,** since it is impossible for $x + y$ to equal both 1 and 2 at the same time.

Historical Note. The representation of equations by means of lines is due to René Descartes (1596–1650). (See also page 50.) This must be regarded as one of the greatest contributions of all time to mathematics. Not only is it possible to represent straight lines by equations as we have here but a very large number of curves of different kinds may be so represented. The points where two curves meet are thus found by solving the equations which represent them. This enables us to use the operations of algebra in solving a large range of problems pertaining to lines and curves.

REVIEW QUESTIONS

1. How may a point in a plane be located by reference to two fixed lines? What are these fixed lines called? What names are given to the distances from the point to the fixed lines? Why are negative numbers needed in order to locate all points in this manner?

2. Draw a pair of axes in a plane and locate the following points: (5, 0), (– 2, 0), (0, 3), (0, – 1), (0, 0).

3. How many pairs of numbers can be found which satisfy the equation $x - 2y = 6$? State five such pairs and plot the corresponding points. How are these points situated with respect to each other? What can you say of all points corresponding to pairs of numbers which satisfy this equation? What is meant by the graph of an equation?

4. How many pairs of numbers will simultaneously satisfy the two equations $3x + 2y = 7$ and $x + y = 3$? Show by means of a graph that your answer is correct.

5. Is the graph representing the equation $x - y = 3$ limited in either direction? If negative numbers could not be used, would the graph of this equation be limited in either direction?

6. If negative numbers could not be used, how would the graph of $x + y = 3$ be limited?

DRILL EXERCISES

1. From $41a^2 + 7ab - 5c^2 - 9ab$ subtract $8ab + 7c^2 + 50a^2$.
2. From $15 + 2x - 9xy - 3y$ subtract $5xy - 4x - 2y$.
3. Subtract $12 + 8x - 4a - 6c - 18abc$ from $5x - 2a + 3c$.
4. $2 - (71x + 42y - 15x - 64) - (5 - 91x - 2y - 13xy)$.
5. $(31 - 2x + 3y - 5)(x + y)$.
6. $8x + (13 + 18x - 6) - (5 - 6x) = 16x + 10$.
7. $5 - (7 - 4x + 2y - 4b) - (8x - 6y - 9 + 2b) + 8a$.
8. $8x - (-3 - 2 - 4 - 7) + 5x + (2 + 6 + 4) - (-3x + 2)$.
9. $5y + 2x - (6 - 4x - 5x) - 3y - (4x - 2y) - (-7y + 8)$.
10. $35y - (41x - 16 - 12y) + 5x + (-6 + 46y - 18x)$.
11. $(x - 1)^2 - (x - 8)(2x - 1) = -x^2 + 98$.
12. $(7 + x)(x - 4) + (1 - x)^2 = -23 + 2x^2$.
13. $(12 - 4x)(2 - x) - 4(1 + x)^2 = 5x + 119$.
14. $(x - 17)(59 - 2x) - (1 - x)^2 = (6 - 3x)(x - 2) + 384$.
15. $(3x - 2) + (x - 1)^2 + (x - 2)^2 = 2(x - 1)(x - 2) + 5$.
16. $(6 - 3x)(2 + x) + 16(x - 1)^2 = 13(x + 4)^2 + 364$.
17. $\dfrac{1}{x - b} + \dfrac{1}{x - a} = \dfrac{2}{x - c}$.
18. $\dfrac{7}{x - a} - \dfrac{3}{x + a} = \dfrac{4}{x + c}$.
19. Solve for each letter in terms of the others:

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{c}.$$

20. The sum of two numbers is s and one of them is x. How do you represent the other?

21. The difference between two numbers is d and one of them is x. How do you represent the other?

22. How do you represent the area of a rectangle whose width and length are w and l feet respectively? How do you represent the area of a rectangle which is a feet wider and b feet longer than this one?

CHAPTER VIII

SIMULTANEOUS LINEAR EQUATIONS

120. Simultaneous linear equations have already been considered in connection with graphs. In this chapter such equations will be solved by the algebraic method called **elimination.** The process consists in combining the equations in such a way as *to get rid of one* of the unknown numbers.

ELIMINATION BY ADDITION OR SUBTRACTION

121. Illustrative Examples. Solve

$$\begin{cases} x+2y=7, & (1) \\ 3x-2y=5. & (2) \end{cases}$$

Adding the members of these equations, $+2y$ and $-2y$ cancel.

Hence, $4x=12$, and $x=3$.

Substituting in (1), $3+2y=7$, and $y=2$.

Verify this by drawing the graphs, and also by substituting $x=3$, $y=2$, in (1) and (2).

If one of the variables does not already have the same coefficient in both equations, we proceed as follows:

Solve the equations

$$\begin{cases} 7x+3y=4-y+4x, & (1) \\ 3x-y=4y-2-x. & (2) \end{cases}$$

From (1) by A, S,	$3x+4y=4.$	(3)
From (2) by A, S,	$4x-5y=-2.$	(4)
(3) · 4	$12x+16y=16.$	(5)
(4) · 3	$12x-15y=-6.$	(6)
(5) − (6)	$31y=22.$	(7)
$D\|31$	$y=\frac{22}{31}.$	(8)
Substituting in (3),	$x=\frac{12}{31}.$	(9)

Hence, the solution is $\frac{12}{31}$, $\frac{22}{31}$.

(3) · 4 means that the members of equation (3) were multiplied by 4. (5) – (6) means that the members of equation (6) were subtracted from those of (5).

The process used in the solution just given is called **elimination by addition or subtraction.**

Make a rule for solving a pair of equations by this process.

EXERCISES

Solve the following pairs of equations by addition or subtraction. Substitute the results in the given equations in each case to test the accuracy of the solution.

1. $\begin{cases} 2x+3y=22, \\ x-y=1. \end{cases}$

2. $\begin{cases} 5x-2y=21, \\ x-y=6. \end{cases}$

3. $\begin{cases} 6x+30=8y, \\ 3y+17=2-3x. \end{cases}$

4. $\begin{cases} 8x-4y=12x, \\ 4x+2y=3+4y. \end{cases}$

5. $\begin{cases} x+6y=2x-16, \\ 3x-2y=24. \end{cases}$

6. $\begin{cases} 5x+10y=-7, \\ 2x+5y=-2. \end{cases}$

7. $\begin{cases} 5x+3y=-2, \\ 3x+2y=-1. \end{cases}$

8. $\begin{cases} 3a+7b=7, \\ 5a+3b=29. \end{cases}$

9. $\begin{cases} r=3s-19, \\ s=3r-23. \end{cases}$

10. $\begin{cases} 2p=5q-16, \\ 7q=-3p+5. \end{cases}$

11. $\begin{cases} 7m=2n-3, \\ 19n=6m+89. \end{cases}$

12. $\begin{cases} 15k=10-20l, \\ 25k-30l=80. \end{cases}$

13. $\begin{cases} 6c+15d=-6, \\ 21d-8c=-74. \end{cases}$

14. $\begin{cases} 2x-3y=4, \\ 2y-3x=-21. \end{cases}$

15. $\begin{cases} u+v=27, \\ \frac{2}{3}v=19-\frac{3}{4}u. \end{cases}$

16. $\begin{cases} 7a=1+10y, \\ 16y=10a-1. \end{cases}$

17. $\begin{cases} 28x+14y=23, \\ 14x-14y=1. \end{cases}$

18. $\begin{cases} 5x+2y=x+18, \\ 2x+3y=3x+27. \end{cases}$

19. $\begin{cases} 7y-x=x-17, \\ 2y+3x=38. \end{cases}$

20. $\begin{cases} 6x+2y=-2, \\ x-4y=-35. \end{cases}$

ELIMINATION BY SUBSTITUTION

122. Illustrative Problem. Solve the equations:

$$\begin{cases} 2x+3y=13. & (1) \\ 5x-6y=-8. & (2) \end{cases}$$

From (1) by S and D, $y=\frac{13-2x}{3}$. (3)

Substituting in (2), $5x-\frac{6(13-2x)}{3}=-8$. (4)

By F, V, $5x-2(13-2x)=-8$. (5)

By IV, VII, $5x-26+4x=-8$. (6)

By I, A, $9x=18$. (7)

By D, $x=2$. (8)

From (3)' $y=\frac{13-2\cdot 2}{3}=3$. (9)

Verify this by substituting these values of x and y in (1) and (2).

The process here used is called **elimination by substitution.**

Make a rule for solving a pair of equations by this process.

This process is usually convenient when one of the variables has a coefficient unity in one equation and not in the others, since in that case no fractions are introduced by the substitution.

EXERCISES

1. $\begin{cases} x+2y=4, \\ 2x+y=5. \end{cases}$

2. $\begin{cases} 3x-y=5, \\ 5x+2y=23. \end{cases}$

3. $\begin{cases} 2x+y=3, \\ 3x-7y=30. \end{cases}$

4. $\begin{cases} 5y+x=7, \\ 5x-3y=4-2x+7. \end{cases}$

5. $\begin{cases} 5x+8y=-1, \\ 6y-x=4y-7. \end{cases}$

6. $\begin{cases} x-y=37, \\ 2x+3y=31x+13y. \end{cases}$

7. $\begin{cases} 2x-y=y+6, \\ x+2y=4y+3. \end{cases}$

8. $\begin{cases} 5x+3y=0, \\ 2x+y=1. \end{cases}$

9. $\begin{cases} 3x-2y=3, \\ 2x+3y=6x-1. \end{cases}$

10. $\begin{cases} 5x-3y=0, \\ 2x-6y=-x. \end{cases}$

EXERCISES

Solve by either method of elimination:

1. $\begin{cases} x + y = 4, \\ x - y = 10. \end{cases}$

2. $\begin{cases} x - y = -3, \\ x + 4y = 12. \end{cases}$

3. $\begin{cases} 2x + 3y = 5, \\ 7x - 5y = 33. \end{cases}$

4. $\begin{cases} 3x - 4y = 8, \\ 4x + 3y = -6. \end{cases}$

5. $\begin{cases} 2x - 4y = 8, \\ 3x + 2y = 4. \end{cases}$

6. $\begin{cases} 3x - 4y = 8, \\ 2x + 3y = 11. \end{cases}$

7. $\begin{cases} 4y - 2x = 3, \\ 2y + 5x = 6. \end{cases}$

8. $\begin{cases} 3x - y = 2x - 1, \\ 12x + y = 14. \end{cases}$

9. $\begin{cases} 2x + 3y = 5, \\ 6x + 14y = 0. \end{cases}$

10. $\begin{cases} 4x + 3y = 5, \\ 7x - 2y = 74. \end{cases}$

11. $\begin{cases} 6y + 2x = 11, \\ 3y + 12x = 18. \end{cases}$

12. $\begin{cases} 4y + 9x = -5, \\ x + y = -5. \end{cases}$

13. $\begin{cases} 3y + 5x = 12 + 2x, \\ 17x - y = 4y - 20. \end{cases}$

14. $\begin{cases} 6 + x + y = 2x - 1, \\ 3y + x = 6y + 9. \end{cases}$

15. $\begin{cases} y + 5x = 2x + 5, \\ 2y - 3x = 19. \end{cases}$

16. $\begin{cases} 6x + 2y = 23, \\ 10x - 5y = 21. \end{cases}$

Ans. $x = 3\frac{7}{50}$, $y = 2\frac{2}{25}$.

17. $\begin{cases} 3x - 7y = 15, \\ 5x + 4y = 11. \end{cases}$

Ans. $x = \frac{137}{47}$, $y = -\frac{42}{47}$.

18. $\begin{cases} 7x - 4y = 3, \\ 5x + 8y = 6. \end{cases}$

19. $\begin{cases} 12y - 10x = -6, \\ 7y + x = 99. \end{cases}$

20. $\begin{cases} 7x - 3y = -7, \\ 5y - 9x = 1. \end{cases}$

21. $\begin{cases} 7x + 4y = 3, \\ 2x + 3y = 25. \end{cases}$

22. $\begin{cases} 34x + 70y = 4, \\ 5x - 8y = -36. \end{cases}$

23. $\begin{cases} 7x + 9y = 8, \\ 2x - 3y = 21, \end{cases}$

24. $\begin{cases} 8x + 4y = 49, \\ 5x - 8y = 28, \end{cases}$

123. The equations thus far given have for the most part been written in a standard form, $ax + by = c$, in which all the terms containing x are collected, likewise those containing y, and those which contain neither variable. When the equations are not given in this form, they should be so reduced at the outset, as in the following.

Example: Solve
$$\begin{cases} \dfrac{7y-4}{5} + \dfrac{2x-3}{2} = \frac{3}{2}. & (1) \\ \dfrac{5x-2}{3} + \dfrac{2y+1}{5} = 2. & (2) \end{cases}$$

Solution. (1) · 10,	$14y - 8 + 10x - 15 = 15.$	(3)
Transposing in (3),	$14y + 10x = 38.$	(4)
(4) ÷ 2,	$7y + 5x = 19.$	(5)
(2) · 15,	$25x - 10 + 6y + 3 = 30.$	(6)
Transposing in (6),	$25x + 6y = 37.$	(7)
(5) · 5,	$25x + 35y = 95.$	(8)
(8) – (7),	$29y = 58.$	(9)
(9) ÷ 29,	$y = 2.$	(10)
Substituting in (7),	$x = 1.$	

Check. Substitute $x = 1$, $y = 2$ in (1) and (2) and see that each is satisfied.

EXERCISES

After reducing each of the following pairs of equations to the standard form, solve by means of either process of elimination:

1. $\begin{cases} x - 14 = 7y, \\ 6y + 1 = x. \end{cases}$

2. $\begin{cases} 16x - 3y = 7x, \\ 4y = 7x + 5. \end{cases}$

3. $\begin{cases} r + 1 = -4s, \\ 2s = 13 - 5r. \end{cases}$

4. $\begin{cases} m = \dfrac{1-8n}{5}, \\ 3m + 5n = 1. \end{cases}$

5. $\begin{cases} m - n = 16, \\ 3m = 8 - 2n. \end{cases}$

6. $\begin{cases} \dfrac{7x-15}{3}=y, \\ 2x-y=3. \end{cases}$

7. $\begin{cases} \dfrac{x-3}{5y}=-2, \\ x+7y=6. \end{cases}$

8. $\begin{cases} 3x-5=-y, \\ 8y+76=5x. \end{cases}$

9. $\begin{cases} a+4b=14, \\ 3a-b=14. \end{cases}$

Ans. $a=5\frac{5}{13}$, $b=2\frac{2}{13}$.

10. $\begin{cases} \dfrac{x+y}{2}+\dfrac{x-y}{2}=10, \\ 2x-y=16. \end{cases}$

11. $\begin{cases} \dfrac{x-y}{5}+\dfrac{x+y}{3}=8, \\ \dfrac{2x-y}{2}-\dfrac{3y-x}{4}=12\frac{1}{2}. \end{cases}$

12. $\begin{cases} \dfrac{7m+8}{5}-\dfrac{7n-1}{4}=-2, \\ \dfrac{2m-4}{4}+\dfrac{n-1}{3}=-\frac{1}{3}. \end{cases}$

13. $\begin{cases} \dfrac{x-3}{4}+\dfrac{y+8}{5}=2, \\ \dfrac{x+7}{2}+\dfrac{2y-4}{7}=5. \end{cases}$

14. $\begin{cases} \dfrac{8a-3}{9}+\dfrac{5b-2}{3}=13, \\ \dfrac{2a+7}{5}-\dfrac{3b+10}{10}=-3\frac{4}{5}. \end{cases}$

15. $\begin{cases} \dfrac{7y-4}{5}+\dfrac{2x-3}{2}=7, \\ \dfrac{6x-3}{5}+\dfrac{2y+1}{5}=7. \end{cases}$

Ans. $x=5\frac{3}{16}$, $y=2\frac{15}{16}$.

16. $\begin{cases} \dfrac{3y+7}{2}-\dfrac{5x-7}{3}=10, \\ \dfrac{2x-4}{3}-\dfrac{2y-1}{4}=-2\frac{1}{4}. \end{cases}$

17. $\begin{cases} \dfrac{5+3p}{7}-\dfrac{5q-2}{4}=-2, \\ 6p+8q=108. \end{cases}$

18. $\begin{cases} 3x-2y=4, \\ \dfrac{2x-1}{5}-\dfrac{7y-4}{3}=-19. \end{cases}$

19. $\begin{cases} 5x+7y=89\frac{1}{2}, \\ \dfrac{2x-4}{3}+\dfrac{6y-1}{5}=13\frac{1}{3}. \end{cases}$

20. $\begin{cases} 32x-9y=299, \\ \dfrac{2x-5}{7}-\dfrac{3y-1}{2}=-16. \end{cases}$

SIMULTANEOUS LITERAL EQUATIONS

Examples. 1. The sum of two numbers is 35 and their difference is 5. What are the numbers?

Let x represent one number and y the other. Form two equations and solve them.

2. The sum of two numbers is 48 and their difference is 24. What are the numbers?

3. The sum of two numbers is $41\frac{1}{2}$ and their difference is $23\frac{1}{2}$. What are the numbers?

4. The sum of two numbers is 8590 and their difference is 3480. What are the numbers?

5. If the sum of two numbers is s and their difference is d, find the number.

Solution. Let x represent one of the numbers and y the other.

Then, $$x + y = s, \quad (1)$$
$$x - y = d. \quad (2)$$

Solving, we get $$x = \frac{s+d}{2} = \frac{s}{2} + \frac{d}{2},$$

and $$y = \frac{s-d}{2} = \frac{s}{2} - \frac{d}{2}.$$

Translated into words these results are:

Given the sum of any two numbers and their difference. Then one of the numbers is half their sum plus half their difference and the other is half their sum minus half their difference.

6. Test this general solution by applying to the following:

$s = 48, \quad d = 24.$ $\qquad s = 40, d = 52.$

$s = 8590, d = 348.$ $\qquad s = 38, d = 50.$

In the following x and y are the unknowns. Solve for them in terms of the other letters.

7. $\begin{cases} ax + by = 1, \\ cx - dy = 1. \end{cases}$

8. $\begin{cases} 2x - 3y = c - d, \\ 3x - 2y = c + d. \end{cases}$

9. $\begin{cases} mx + 2ny = k, \\ 3 + 2mx = +ny. \end{cases}$

10. $\begin{cases} x + y = a, \\ \dfrac{x}{a+b} - \dfrac{y}{a-b} = 1. \end{cases}$

DRILL EXERCISES

1. $(x-2)^2-(x-1)(x+2)=6-5x$.

2. $(x-2)(x+2)+(3x-1)(2-x)=(x-2)(5-2x)$.

3. $(x-3)^2+(2x+5)^2=(5x-3)(x+5)-7$.

4. $(2-x)^2-(2x-1)^2=(-3x+1)(4+x)-4$.

5. $\begin{cases} 2x-3y-1=0, \\ 5x+2y=12. \end{cases}$

In the following x and y are the unknowns. Solve for them in terms of the other letters.

6. $\begin{cases} x+y=a, \\ x-y=b. \end{cases}$

7. $\begin{cases} ax+by=c, \\ ax-by=d. \end{cases}$

8. $\begin{cases} \frac{x}{a}+\frac{y}{b}=c, \\ \frac{x}{a}-\frac{y}{b}=d. \end{cases}$

9. $(x-a)^2+(x-b)^2=2(x-c)^2$.

10. $ax-a(b-x)+ac=3ab$.

11. $\frac{2x-a}{x+b}-\frac{2x-b}{2x+a}=1$.

12. $\frac{3}{x+a}+\frac{4}{x-b}=\frac{7}{x-a}$.

13. $2ax-4by+2cx-(cx-3ax-4by)$.

14. Find the value of $\frac{x^2-3cx+8c}{c^2+ab-15}$ for $x=1$, $a=2$, $b=3$, $c=4$.

15. A father is now twice the age of his son. If x represents the son's age now, express twice the sum of their ages 5 years ago.

16. One number is 3 less than 4 times another number. If x represents one of the numbers express one third the sum of the numbers.

17. If h is the digit in hundreds' place, t the digit in tens' place, and u the digit in units' place of a certain number, express the number obtained by inverting the order of the digits of the given number.

18. h, t, and u are the digits in hundreds', tens', and units' places of a number. Express the number obtained by increasing each digit by 2.

124. Many problems may be solved, using either one or two unknowns.

Ex. Find two numbers, whose sum is 20, such that when one of them is subtracted from twice the other, the remainder is 16.

(*a*) *Using one unknown.* Let x represent one number. Then $20-x$ is the other number, and the equation is $2x-(20-x)=16$.

(*b*) *Using two unknowns.* Let x and y represent the two numbers.

Then, $$\begin{cases} x+y=20, \\ 2x-y=16. \end{cases}$$

The translation of problems into equations is usually easier when more than one unknown is permitted. This is due to the fact that in this case each of the given relations between the numbers is put down as a *separate equation.*

PROBLEMS INVOLVING TWO VARIABLES

1. If w and l are the width and length of a rectangle, express its area and also its perimeter in terms of w and l.

2. If the width of the rectangle in the preceding is increased by 10 and its length by 20, express its new perimeter and also its new area in terms of w and l.

3. If x and y represent the ages of a father and son respectively, represent the sum of their ages 5 years ago in terms of x and y; also the difference of their ages 8 years hence.

4. If a number consisting of two digits is increased by 15 by changing the order of its digits, which is greater, the digit in tens' or in units' place?

5. If a number is decreased by changing the order of its digits, which is greater, the digit in tens' or in units' place?

In solving the following problems, use two variables in each case:

6. A rectangular field is 32 rods longer than it is wide. The length of the fence around it is 308 rods. Find the dimensions of the field.

7. Find two numbers such that 7 times the first plus 4 times the second equals 37; while 3 times the first plus 9 times the second equals 45.

8. A certain sum of money was invested at 5% interest and another sum at 6%, the two investments yielding $980 per annum. If the first sum had been invested at 6% and the second at 5%, the annual income would be $1000. Find each sum invested.

9. The combined distance from the sun to Jupiter and from the sun to Saturn is 1369 million miles. Saturn is 403 million miles farther from the sun than Jupiter. Find the distance from the sun to each planet.

10. Find two numbers such that 7 times the first plus 9 times the second equals 116, and 8 times the first minus 4 times the second equals 4.

11. The sum of two numbers is 108. 8 times one of the numbers is 9 greater than the other number. Find the numbers.

12. Two investments of $24,000 and $16,000 respectively yield a combined income of $840. The rate of interest on the larger investment is 1% greater than that on the other. Find the two rates of interest.

13. A father is twice as old as his son. Twenty years ago the father was six times as old as his son. How old is each now?

14. If the length of a rectangle is increased by 3 feet and its width decreased by 1 foot, its area is increased by 3 square feet. If the length is increased by 4 feet and the width decreased by 2 feet, the area is decreased by 3 square feet. What are the dimensions of the rectangle?

Note that if w and l are the original width and length of the rectangle, the term lw will cancel out of both equations.

15. A steamer on the Mississippi makes 6 miles per hour going against the current and $19\frac{1}{2}$ miles per hour going with the current. What is the rate of the current and at what rate can the steamer go in still water?

16. A starts at 7 A.M. for a walk in the country. At 10 A.M. B starts on horseback to overtake A, which he does at 1 P.M. If the rate of B had been two miles per hour less, he would have overtaken A at 4 P.M. At what rate does each travel?

17. A camping party sends a messenger with mail to the nearest post office at 5 A.M. At 8 A.M. another messenger is sent out to overtake the first, which he does in $2\frac{1}{4}$ hours. If the second messenger travels 5 miles per hour faster than the first, what is the rate of each?

18. There are two numbers such that 3 times the greater is 18 times their difference, and 4 times the smaller is 4 less than twice the sum of the two. What are the numbers?

19. A picture is 3 inches longer than it is wide. The frame 4 inches wide has an area of 360 square inches. What are the dimensions of the picture?

20. The difference between 2 sides of a rectangular wheat field is 30 rods. A farmer cuts a strip 5 rods wide around the field, and finds the area of this strip to be $7\frac{1}{2}$ acres. What are the dimensions of the field?

21. The sum of the length and width of a certain field is 260 rods. If 20 rods are added to the length and 10 rods to the width, the area will be increased by 3800 square rods. What are the dimensions of the field?

22. In a number consisting of two digits the sum of the digits is 12. If the order of the digits is reversed, the number is increased by 36. What is the number?

23. A bird attempting to fly against the wind is blown backward at the rate of $7\frac{1}{2}$ miles per hour. Flying with a wind $\frac{1}{4}$ as strong, the bird makes 48 miles an hour. Find the rate of the wind and the rate at which the bird can fly in calm weather.

24. There is a number whose two digits differ by 2. If the digit in units' place is multiplied by 3 and the digit in tens' place is multiplied by 2, the number is increased by 44. Find the number, the tens' digit being the larger.

25. In a number consisting of two digits one digit is equal to twice their difference. If the order of the digits is reversed, the number is increased by 18. Find the number.

26. If the length of a rectangle is doubled and 8 inches added to the width, the area of the resulting rectangle is 180 square inches greater than twice the original area. If the length and width of the rectangle differ by 10, what are its dimensions?

27. There is a number consisting of three digits, those in tens' and units' places being the same. The digit in hundreds' place is 4 times that in units' place. If the order of the digits is reversed, the number is decreased by 594. What is the number?

28. A man rowing against a tidal current drifts back $2\frac{1}{2}$ miles per hour. Rowing with this current, he can make $14\frac{1}{2}$ miles per hour. How fast does he row in still water and how swift is the current?

29. Flying against a wind a bird makes 28 miles per hour, and flying with a wind whose velocity is $2\frac{3}{5}$ times as great, the bird makes 46 miles per hour. What is the velocity of the wind and at what rate does the bird fly in calm weather?

30. A freight train leaves Chicago for St. Paul at 11 A.M. At 3 and 5 P.M. respectively of the same day two passenger trains leave Chicago over the same road. The first overtakes the freight at 7 P.M. the same day, and the other, which runs 10 miles per hour slower, at 3 A.M. the next day. What is the speed of each?

31. Two boys, A and B, trying to determine their respective weights, find that they balance on a teeter board when B is 6 feet and A 5 feet from the fulcrum. If B places a 30-pound weight on the board beside him, they balance when B is 4 and A 5 feet from the fulcrum. How heavy is each boy?

32. \$10,000 and \$8000 are invested at different rates of interest, yielding together an annual income of \$820. If the first investment were \$12,000 and the second \$6000, the yearly income would be \$840. Find the rates of interest.

SIMULTANEOUS EQUATIONS IN THREE VARIABLES

125. Illustrative Problem. Three men were discussing their ages and found that the sum of their ages was 90 years. If the age of the first were doubled and that of the second trebled, the aggregate of the three ages would then be 170. If the ages of the second and third were each doubled, the sum of the three would be 160. Find the age of each.

Solution. Let x, y, and z represent the number of years in their ages in the order named.

Then, $$x + y + z = 90, \quad (1)$$

$$2x + 3y + z = 170, \quad (2)$$

and $$x + 2y + 2z = 160. \quad (3)$$

If we subtract (1) from (2), we obtain a new equation from which z is eliminated.

I.e. $$x + 2y = 80. \quad (4)$$

Again, multiplying (2) by 2 and subtracting (3),

$$3x + 4y = 180. \quad (5)$$

(4) and (5) are two equations in the two variables x and y. Solving these by eliminating y, we find $x = 20$. (6)

Substituting $x = 20$ in (4), $y = 30$. (7)

Substituting x and y in (1), $z = 40$. (8)

Check by showing that the values of x, y, and z satisfy the original equations and the conditions of the problem.

The values of x, y, and z as thus found constitute the **solution of the given system** of equations.

Evidently x could have been eliminated first, using (1), (2), and (1), (3), giving a new set of two equations in y and z. Let the student find the solution in this manner.

Also find the solution by first eliminating y, using (1), (2), and (2), (3), getting two equations in x and z, from which the values of x and z can be found.

126. Definition. An equation is said to be of the **first degree** in x, y, and z if it contains these letters in such a way that no one of them is multiplied by itself or by one of the others (§ 115).

The fact that the solutions are found to be the same, no matter in what order the equations are combined, indicates that *a system of three independent and simultaneous equations of the first degree in three variables has one and only one solution.*

As in the case of two equations, each should be first reduced to a standard form in which all the terms containing a given variable are collected and united and all fractions removed. Make a rule for solving a system of three equations each of the first degree in three variables.

EXERCISES

Solve the following systems, and check the results:

1. $\begin{cases} 2x - y + z = 18, \\ x - 2y + 3z = 10, \\ 3x + y - 4z = 20. \end{cases}$

2. $\begin{cases} 5x - 3y + z = 15, \\ x + 3y - z = 3, \\ 2x - y + z = 8. \end{cases}$

3. $\begin{cases} 4x + 2y + z = 13, \\ x - y + z = 4, \\ x + 2y - z = 1. \end{cases}$

4. $\begin{cases} 6x + 4y - 4z = -4, \\ 4x - 2y + 8z = 0, \\ x + y + z = 4. \end{cases}$

5. $\begin{cases} x + 2y + 3z = 5, \\ 4x - 3y - z = 5, \\ x + y + z = 2. \end{cases}$

6. $\begin{cases} 2x - 8y + 3z = 2, \\ x - 4y + 5z = 1, \\ 3x - 10y - z = 5. \end{cases}$

7. $\begin{cases} x + y + z = 1, \\ x + 3y + 2z = 8, \\ 2x + 8y - 3z = 15. \end{cases}$

8. $\begin{cases} 2x - 3y + z = 5, \\ 3x + 2y - z = 5, \\ x + y + z = 3. \end{cases}$

9. $\begin{cases} x + y + z = 6, \\ 3x - 2y - z = 13, \\ 2x - y + 3z = 26. \end{cases}$

10. $\begin{cases} x + y + z = 6, \\ 4x - y - z = -1, \\ 2x + y - 3z = -6. \end{cases}$

11. $\begin{cases} 2x - 3y - 4z = 17, \\ 4x - 4y + 2z = -10, \\ 7x + 7y + 5z = 17. \end{cases}$

12. $\begin{cases} x + y + z = 0, \\ 5x + 3y + 4z = -1, \\ 2x - 7y + 6z = 21. \end{cases}$

PROBLEMS INVOLVING THREE VARIABLES

127. Illustrative Problem. A broker invested a total of \$15,000 in the street railway bonds of three cities, the first investment yielding 3 %, the second $3\frac{1}{2}$ %, and the third 4 %, thus securing an income of \$535 per year. If the second investment was one-half the sum of the other two, what was the amount of each?

Solution. Suppose x dollars were invested at 3 %, y dollars at $3\frac{1}{2}$%, and z dollars at 4 %.

Then, $x + y + z = 15000,$ (1)

$.03x + .035y + .04z = 535,$ (2)

and $x + z = 2y.$ (3)

From (3), $x - 2y + z = 0.$ (4)

Subtract (4) from (1), $3y = 15000,$ (5)

and $y = 5000.$ (6)

From (1), by M, $.035x + .035y + .035z = 525.$ (7)

Subtract (7) from (2), $-.005x + .005z = 10.$ (8)

Divide (8) by .005, $-x + z = 2000.$ (9)

Substitute (6) in (4), $x + z = 10000.$ (10)

Add (9) and (10), $2z = 12000.$ (11)

$z = 6000.$ (12)

Substitute (6) and (12) in (1), $x = 4000.$ (13)

Hence, \$4000, \$5000, and \$6000 were the sums invested.

Solve the following problems, using three unknowns:

1. The sum of three angles, A, B, and C of a triangle is 180 degrees. $\frac{1}{3}$ of $A + \frac{1}{4}$ of $B + \frac{1}{5}$ of C is 48 degrees, while $\frac{1}{6}$ of $A + \frac{1}{8}$ of $B + \frac{1}{4}$ of C is 30 degrees. How many degrees in each angle?

2. The combined weight of 1 cubic foot each of compact limestone, granite, and marble is 535 pounds. 1 cubic foot of limestone, 2 of granite, and 3 of marble weigh together 1041 pounds, while 1 cubic foot of limestone and 1 of granite together weigh 195 pounds more than 1 cubic foot of marble. Find the weight per cubic foot of each kind of stone.

3. A number is composed of 3 digits whose sum is 7. If the digits in tens' and hundreds' places are interchanged, the number is increased by 180; and if the order of the digits is reversed, the number is decreased by 99. What is the number?

4. The sum of the angles A, B, and C of a triangle is 180 degrees. If B is subtracted from C, the remainder is $\frac{1}{3}$ of A, and when C is subtracted from twice A, the remainder is 4 times B. How many degrees in each angle?

5. The sum of the three sides a, b, c of a certain triangle is 35, and twice a is 5 less than the sum of b and c, and twice c is 4 more than the sum of a and b. What is the length of each side?

6. If x is the number of seconds in the Eastern intercollegiate record for a mile run, y the number in the Western intercollegiate record, and z the number in the world's record, then

$$\begin{cases} x + y + z = 768.97, \\ -x + 2y + z = 518.95, \\ 2x - y + z = 502.75. \end{cases}$$

7. If x is the number of seconds in the Eastern intercollegiate record for a half mile run, y the number in the Western intercollegiate record, and z the number in the world's record, then

$$\begin{cases} 2x + 3y + z = 692.9, \\ 3x + 2y + 2z = 804.6, \\ 2x - y + z = 226.5. \end{cases}$$

8. If $x =$ number of seconds in the world's mile trotting record in 1806, $y =$ number of seconds in the world's record in

1885, and z = number of seconds in the world's record in 1911, then

$$\begin{cases} x+y+z=426.25, \\ 2x+4y+6z=1584, \\ -x+y+2z=186.75. \end{cases}$$

9. Diophantus of Alexandria (see page 19) gives the following problem: "Find three numbers such that the sum of each pair is a given number." Solve this problem if the given numbers are 20, 30, 40 (the numbers actually given by Diophantus).

Diophantus remarks that half the sum of the three given numbers must be greater than each singly. This is to prevent negative numbers in the results.

10. Solve the preceding problem when the given numbers are a, b, c. That is, solve the system

$$\begin{aligned} x+y&=a, \\ y+z&=b, \\ z+x&=c. \end{aligned}$$

It is interesting to note that Diophantus states his problem in words in its general form but he solves it for a special case, viz. for $x + y = 20$, $y + z = 30$, $z + x = 40$. The Greeks did not use letters to represent numbers in general. Hence they had no formulas such as we now have.

REVIEW QUESTIONS

11. Describe elimination by the process of substitution; also by the process of addition or subtraction. Under what conditions is one or the other of these methods preferable?

12. Describe the solution of a system of three linear equations in three unknowns. Is it immaterial which of the three variables is eliminated first?

13. Can you find a definite solution for two equations each containing three unknowns? Illustrate this by means of the equations $4x - 3y - z = 5$ and $x + y + z = 2$.

DRILL EXERCISES

1. $(1-3x)^2+(2x+1)^2=5x^2+(2x+6)(4x+27)$.

2. $(1+2x)(2-3x)+(x-4)(x+4)=(x+14)(18-5x)-1$.

3. $2(x+5)(x-5)=(x-5)(x+1)+(x-2)^2-1$.

4. $\begin{cases}\dfrac{x-1}{2}+\dfrac{y+2}{3}=3\frac{1}{6}, \\ \dfrac{x+1}{3}-\dfrac{y-1}{4}=\frac{7}{6}.\end{cases}$

5. $\begin{cases}\dfrac{x-a}{2}+\dfrac{y-b}{3}=1, \\ \dfrac{x+a}{3}+\dfrac{y+b}{2}=1.\end{cases}$

6. $\begin{cases}2ax-3by=c, \\ 2ax+3by=d.\end{cases}$

7. $\begin{cases}x+y+z=6, \\ 2x-y+z=3, \\ 3x+2y-2z=1.\end{cases}$

8. $\begin{cases}x-y+z=4, \\ x-2y+4z=6, \\ 2x+y-3z=10.\end{cases}$

9. $\begin{cases}x+y+z=0, \\ 2x-4y+z=-3, \\ 3x+2y+4z=3.\end{cases}$

Solve the following equations for x:

10. $(x-a)(x-a)=-x(a-x)+(a+b)x$.

11. $\dfrac{a}{b}=\dfrac{x}{c-x}$.

12. $\dfrac{4a}{3x}+\dfrac{4a}{x}=\dfrac{3}{2}+\dfrac{5a}{6x}$.

13. $\dfrac{2a-3b}{3x}+\dfrac{b}{x}+\dfrac{3a}{2x}=\dfrac{13a}{6}$.

14. Express the average of the numbers 3, 8, -9, 12; also of the numbers $3a$, b, $2c$, $-5b$.

15. Express the sum of the squares of four consecutive even integers of which $2n$ is the smallest.

16. Express the sum of the squares of four consecutive odd integers of which $2n+1$ is the greatest.

17. A wall is l feet long and h feet high. There are three windows in it k feet wide and m feet high. By how much does the area of the wall exceed that of the three windows? Express the result in terms of l, h, k, and m.

PART TWO

CHAPTER IX

PRODUCTS, QUOTIENTS, AND FACTORS

The principles and processes studied in Part One enabled us to *solve equations* of the first degree involving one or more unknowns, and to *solve all problems* which we could translate into such equations.

In Part Two we begin with factoring which will lead to the *solution of equations* of the second degree and thereby to the *solution of problems* still more difficult.

128. Repeated Factors. Number expressions containing repeated factors have already been considered in Chapter IV. $x \cdot x$ was written x^2 and called *the square of* x, or x *square;* similarly, $(a+b)(a+b)$ was written $(a+b)^2$ and read *the square of the binomial* $a+b$ or $a+b$ *squared.*

129. Definitions. Any number written over and to the right of a number expression is called an **index** or **exponent.** If an exponent is a *positive integer*, it shows how many times the expression is to be taken as a factor.

A product consisting entirely of equal factors is called a **power** of the repeated factor. The repeated factor is called the **base** of the power.

E.g. x^3 means $x \cdot x \cdot x$ and is read *the third power of* x or x *cube;* x^5 means $x \cdot x \cdot x \cdot x \cdot x$, and is read *the fifth power of* x or briefly x *fifth.* $(x-y)^3 = (x-y)(x-y)(x-y)$ and is read $x-y$ *cubed* or *the cube of the binomial* $x-y$.

The first power of x is written without an exponent. Thus x means x^1; 2 means 2^1 etc.

130. Notice the difference between a *coefficient* and an *exponent.* A coefficient is a *factor,* while an exponent shows how many times some number is used as a factor.

E.g. $5a = a + a + a + a + a$, while $a^5 = a \cdot a \cdot a \cdot a \cdot a$.

EXERCISES

Perform the following indicated multiplications:

1. $2^3, 2^4, 2^5$.
2. $3^2, 3^3, 3^4$.
3. $4^2, 4^3, 4^4$.
4. $5^2, 5^3, 5^4$.
5. $6^2, 6^3$.
6. $(a+b)^2$.
7. $(c-d)^2$.
8. $98^2 = (100-2)^2$.
9. $(a+b+c)^2$.
10. $(a+b-c)^2$.
11. $(3-a)^2$.
12. $(3-b-c)^2$.

PRODUCT OF TWO POWERS OF THE SAME BASE

131. In the case of factors expressed in Arabic figures multiplications like the following may be carried out in either of two ways.

E.g. $3^2 \cdot 3^4 = 9 \cdot 81 = 729$,

or $3^2 \cdot 3^4 = (3 \cdot 3)(3 \cdot 3 \cdot 3 \cdot 3) = 3^{2+4} = 3^6 = 729$.

But with literal factors the second process only is possible.

E.g. $a^2 \cdot a^4 = (a \cdot a)(a \cdot a \cdot a \cdot a) = a^{2+4} = a^6$.

EXERCISES

In the following exercises carry out each indicated multiplication in two ways in case this is possible:

1. $5 \cdot 5^2$.
2. $5^2 \cdot 5^4$.
3. $6^2 \cdot 6^2$.
4. $7 \cdot 7^3$.
5. $a^2 \cdot a^3$.
6. $x^3 \cdot x^2$.
7. $x^7 \cdot x^4$.
8. $t^3 \cdot t^4$.
9. $t^2 \cdot t^3 \cdot t^4$.
10. $2^3 \cdot 2^2 \cdot 2^4$.
11. $3 \cdot 3^2 \cdot 3^3$.
12. $2^2 \cdot 2^3 \cdot 2^2 \cdot 2$.

132. Illustrative Problem. To multiply 2^k by 2^n, k and n being any two positive integers.

Solution. 2^k means $2 \cdot 2 \cdot 2 \cdot 2$, etc., to k factors,

and 2^n means $2 \cdot 2 \cdot 2 \cdot 2$, etc., to n factors,

Hence, $2^k \cdot 2^n = (2 \cdot 2 \cdot 2 \cdots$ to k factors$)$ $(2 \cdot 2 \cdots$ to n factors$)$

$= 2 \cdot 2 \cdot 2 \cdot 2 \cdots$ to $k + n$ factors in all.

That is, $2^k \cdot 2^n = 2^{k+n}$.

The preceding examples illustrate

Principle XVI

133. Rule. *The product of two powers of the same base is found by adding the exponents of the factors and making this sum the exponent of the common base.*

Exponents are added in multiplication only when they apply to the same base.

E.g. $2^3 \cdot 3^2 = 8 \cdot 9 = 72$ cannot be found by adding the exponents.

EXERCISES

Perform the following indicated multiplications by means of Principle XVI:

1. $2^3 \cdot 2^7$.
2. $a^3 \cdot a^7$.
3. $3^4 \cdot 3^5$.
4. $x^4 \cdot x^5$.
5. $3^k \cdot 3^n$.
6. $x^k \cdot x^n$.
7. $4^a \cdot 4^b$.
8. $3^{2a} \cdot 3^{2b}$.
9. $5^{2+n} \cdot 5^{2-n}$.
10. $a^m \cdot a^n$.
11. $c^x \cdot c^{2-x}$.
12. $x^{ac} \cdot x^{bc}$.

Perform the following multiplications by means of Principles II and XVI.

13. $2^3(2^2 + 2^4)$.
14. $2^2(3 \cdot 2^4 + 5 \cdot 2^3)$.
15. $4^4(3 \cdot 4^3 - 5 \cdot 4^2)$.
16. $a^2(a^3b - ab^3)$.
17. $x^{2a}(4\,x^a + 3\,x^{3a})$.
18. $r^s(5\,r^{2s} - 3\,r^s)$.

Multiply the following:

19. $a^2m^2 - b^2m^3$ by m^4.
20. $4 \cdot 3^2 - 5 \cdot 7 \cdot 2$ by 2^4.
21. $4\,x^2 - 3\,x^3 + 6\,x^4$ by x^2.
22. $5\,x - 3\,x^2 + 2\,x^4$ by x^4.
23. $3\,y^2 + 4\,y^4 - y^3$ by y.
24. $7\,x^4 - 5\,x^3 - 2\,x$ by x^3.
25. $3\,a^2b^4 + 2\,a^2b - 4\,a^2b^2$ by a^3.
26. $4\,a^{2b} - a^{3c} + a^{2d}$ by a^{2b}.

PRODUCTS OF MONOMIALS

134. In multiplying together monomials like $3\,a^2bc$ and $2\,ab^2cd$ it is convenient to arrange the factors so that the same letters are associated together and likewise the numerical coefficients. This we are permitted to do by Principle XIII.

Thus, $3\,a^2bc \times 2\,ab^2cd = (3 \cdot 2)(a^2 \cdot a)(b \cdot b^2)(c \cdot c)d = 6\,a^3b^3c^2d.$

135. Notice that in the product *the exponent of each letter is the sum of the exponents of this letter in the factors and the numerical coefficient is the product of the numerical coefficients of the factors.*

E.g. $(2\,a^2b^2)(5\,a^4b^3c) = 10\,a^{4+2}b^{2+3}c = 10\,a^6b^5c.$

This is a convenient rule for finding the product of two or more monomials.

EXERCISES

Multiply:

1. $3\,ab$ by $5\,a^2b^2$.
2. $4\,x^3$ by $3\,xy^2$.
3. $2\,xyz$ by $3\,x^2yz$.
4. $6\,k^2$ by $7\,ak^3$.
5. $3\,x^4y^2$ by $4\,xy^3$.
6. $5\,a^2b^3c$ by ab^4c.
7. $2\,x^3b^4c^2$ by $5\,xb^4c$.
8. $3\,a^3b^4c$ by abd^4.
9. $2\,x(4 + 7\,x^4 - 3\,x^2y)$.
10. $4\,yx\,(3\,y^2x^3 - y^3x^2 + y^4x^4)$.
11. $5\,a^2b^2\,(a^3 - b^3 + a^2b^2)$.
12. $4\,x^4y^3(3\,ax - 4\,by + 2\,xy)$.
13. $(a + b)^2(a - b)$.
14. $(a - b)^2(a + b)$.
15. $(a + b)^2(a - b)^2$.
16. $(a^3 + a^2b + ab^2 + b^3)(a - b)$.
17. $(a + b - c)(a + b + c)$.
18. $(3\,x - 2\,y - 1)(2\,x + y)$.
19. $(1 + a + a^2)(1 - a)$.
20. $(1 - a + a^2 - a^3)(1 + a)$.
21. $(a + b)(a - b)(a^2 + b^2)$.
22. $(a + b)(a^2 - ab + b^2)$.
23. $(a - b)(a^2 + ab + b^2)$.
24. $(x^2 + x + 1)(x^2 - x + 1)$.
25. $(x + y)(x^3 - x^2y + xy^2 - y^3)$.
26. $(x - y)(x^3 + x^2y + xy^2 + y^3)$.
27. $(x^2 - xy + y^2)(x^2 + xy + y^2)$.
28. $(x^2 + 2\,xy + y^2)(x - y)^2$.
29. $(x^2 - y^2)(x^4 + x^2y^2 + y^4)$.
30. $(x^2 + y^2)(x^4 - x^2y^2 + y^4)$.
31. $(x^2 + y^2)(x + y)(x - y)$.
32. $(x - y)(x^2 + xy + y^2)(x^3 + y^3)$.

QUOTIENT OF TWO POWERS OF THE SAME BASE

136. Illustrative Problem. To divide x^6 by x^4.

Since by § 65 the quotient times the divisor equals the dividend, we seek an expression which multiplied by x^4 equals x^6.

Since by Principle XVI two powers of the same base are multiplied by adding their exponents, the expression sought must be that power of x whose exponent added to 4 equals 6. Hence the exponent of the quotient is $6 - 4 = 2$. That is, $x^6 \div x^4 = x^{6-4} = x^2$.

EXERCISES

Perform the following indicated divisions:

1. $2^4 \div 2^2$.
2. $2^3 \div 2^2$.
3. $2^4 \div 2$.
4. $3^3 \div 3^2$.
5. $3^4 \div 3$.
6. $3^4 \div 3^2$.
7. $9^{11} \div 9^2$.
8. $5^{13} \div 5^{12}$.
9. $7^{24} \div 7^{22}$.
10. $8^3 \div 8$.
11. $6^4 \div 6^2$.
12. $a^3 \div a^2$.
13. $a^4 \div a^3$.
14. $m^4 \div m^2$.
15. $x^4 \div x^2$.
16. $t^{14} \div t^4$.
17. $m^3 \div m$.
18. $n^6 \div n^2$.
19. $(20)^4 \div (20)$.
20. $(101)^{14} \div (101)^{13}$.
21. $(41)^7 \div (41)^6$.

The preceding exercises illustrate

Principle XVII

137. Rule. *The quotient of two powers of the same base is a power of that base whose exponent is the exponent of the dividend minus that of the divisor.*

For the present only those cases are considered in which the exponent of the dividend is greater than or equal to that of the divisor.

Notice that Principle XVII does not apply to powers of different bases.

E.g. $3^7 \div 2^4$ does not equal any integral base to the power, $7 - 4$. This division can be performed only by first multiplying out both dividend and divisor.

EXERCISES

Perform the following indicated divisions by means of Principle XVII:

1. $2^7 \div 2^3$.
2. $a^7 \div a^3$.
3. $3^4 \div 3^2$.
4. $x^4 \div x^2$.
5. $3^{3n} \div 3^{2n}$.
6. $x^{4n} \div x^{2n}$.
7. $3^{2a-1} \div 3^{a-2}$.
8. $5^{n+5} \div 5^{n+2}$.
9. $x^{a+4} \div x^{a+2}$.
10. $t^{4a} \div t^a$.
11. $x^{2a+b} \div x^{a+b}$.
12. $w^{2x} \div w^x$.
13. $(17)^{14} \div (17)^{13}$.
14. $5x^6 \div 2x^2$.
15. $(12)^4 \div (12)^3$.

In the following use Principles III, V, and XVII:

16. $(3 \cdot 2^4 + 5 \cdot 2^3) \div 2^2$.
17. $(3 \cdot 4^3 - 5 \cdot 4^4) \div 4^2$.
18. $(a^3b - a^4b^2) \div a^2$.
19. $(4x^3 + 3x^4) \div x^2$.
20. $(a^2m^4 - b^2m^3) \div m^3$.
21. $(4 x^2 - 5x^3 + x^4) \div x^2$.
22. $(3a^3 + 9a^4 - 2a^5) \div a^3$.
23. $(12x^3y - 11x^2y^2 + 5x^4) \div x^2$.
24. $(x^{3m+4} + x^{2m+3} - 5x^{m+2}) \div x^{m+1}$.
25. $(y^{3a+b} + y^{2a-b} - y^a) \div y^a$.

138. The process of division by subtracting exponents leads in certain cases to strange results.

Thus, according to this process, $x^4 \div x^4 = x^{4-4} = x^0$, which is as yet without meaning, since an exponent has been defined only when it is a *positive integer*. The exponent zero cannot indicate, as in the case of a positive integer, how many times the base is used as a factor. We know, however, that $x^4 \div x^4 = 1$, since any number divided by itself equals unity. Hence if we use the symbol x^0 it must be interpreted to mean 1, no matter what number x represents.

Again by this process, $x^2 \div x^4 = x^{2-4} = x^{-2}$, which is as yet without meaning, since negative exponents have not been defined. But we know that $x^2 \div x^4 = \frac{x^2}{x^4} = \frac{1}{x^2}$ by Principle XV. Hence if we use the symbol x^{-2}, it must be interpreted to mean $\frac{1}{x^2}$. Negative exponents and also fractional exponents are considered in detail in the Advanced Course.

DIVISION OF MONOMIALS

139. In dealing with the quotient of two monomials the indicated division may be written in the form of a fraction and the factors common to dividend and divisor may be cancelled by Principle XV as in the following examples:

(1) $$15\,a^3b^2c \div 3\,a^2bx^2y = \frac{15\,a^3b^2c}{3\,a^2bx^2y} = \frac{5\,abc}{x^2y}.$$

(2) $$12\,a^2x \div 3\,ax = \frac{12\,a^2x}{3\,ax} = \frac{4\,a}{1} = 4\,a.$$

EXERCISES

Divide:

1. $4 \cdot 7 \cdot 9$ by $2 \cdot 3$.
2. $12 \cdot 8 \cdot 20$ by $2 \cdot 4 \cdot 5$.
3. $6\,x^3y^2z$ by $2\,xyz$.
4. $6^4 \cdot 3^4 \cdot x^3$ by $6^2 \cdot 5^2 \cdot x^2$.
5. $12\,x^{12}y^{13}$ by $4\,xy^3z$.
6. $5\,a^4b^{11}c$ by ab^4c^2.
7. $10\,x^4b^{14}c^3$ by $2\,xb^4c$.
8. $36\,x^4y^3$ by $6\,x^3y^5$.
9. $4\,x^2y^3 - 3\,x^3y^2$ by x^2y^2.
10. $18\,x^4y^4 - 12\,x^3y^3 + 6\,x^2y^2$ by $6\,x^2y^2$.
11. $49\,a^4 + 21\,a^3 - 7\,a$ by $7\,a$.
12. $12\,ax^4y^3 - 16\,a^2x^3y^2 + 8\,a^3xy$ by $4\,axy$.
13. $2\,x^{3a} + 4\,x^{4a} - 8\,x^{2a}$ by $2\,x^a$.
14. $6\,x^{2n+1} + 12\,x^{3n+1} - 10\,x^{n+1}$ by $2\,x^{n+1}$.
15. $4\,x^{13} - 6\,x^{11}b - 10\,x^4c$ by $2\,x^4$.
16. $10\,a^3b^2 - a^2b^3 + 15\,a^4b^4$ by $5\,a^2b^2$.

FACTORS OF NUMBER EXPRESSIONS

140. Factors are of great importance in arithmetic. For instance, the multiplication table consists of pairs of factors whose products are committed to memory for constant use. Likewise in algebra the factors of certain special forms of number expressions are so important that they must be known at sight.

Definition. An expression containing no fractions is said to be **prime** if it has no factors in the integral form except itself and 1.

Thus 2, 3, x, $x+2$, a^2+b^2, are prime expressions.

MONOMIAL FACTORS

141. If the terms of a polynomial contain a **common monomial factor,** the polynomial may be divided by the monomial, and the quotient and the divisor are the factors of the polynomial.

E.g. $a^2 - ab = a(a-b)$, $5xy - 3x^2y + 4x^3y = xy(5-3x+4x^2)$, and $6x^{3a}y^{4b} + 8x^{4a+3}y^{6b} - 12x^{5a}y^{8b} = 2x^{3a}y^{4b}(3 + 4x^{a+3}y^{2b} - 6x^{2a}y^{4b})$.

Observe that factoring the various terms of a polynomial does not factor the polynomial.

E.g. $a^2 + ax + ab + by$ is not factored by writing it

$$a(a+x)+b(a+y).$$

Likewise $10a^2bc - 15ab^2c + 20abc^2$ is not factored, although *each term* is in the factored form. But if $10a^2bc - 15ab^2c + 20abc^2$ is written in the form $5abc(2a - 3b + 4c)$, it is then factored.

Note that removing a common factor from the terms of a polynomial is nothing more than the application of Principle I.

EXERCISES

Factor the following polynomials:

1. $7a^2 + 14ab + 21a$.
2. $13a^4b - 16a^3b^4 - 2a^3b^2$.
3. $15xy^4 - 20x^3y + x^2y^2$.
4. $9v^3w^4 + 21v^4w^3 - 18v^2w^2$.
5. $12a^4b^3 - 8a^3b^4 - 6a^2b^2$.
6. $11a^4x^2 - 44a^3x^4 + 33ax$.
7. $72x^4b^3a - 36x^4b^2a^2 - 48x^3b^3a^4$.
8. $84x^9b^7y^4 + 18x^3b^5y^4 + 12x^2b^3y^2$.
9. $17a^4b^3c^2 + 51a^3b^4c^3 - 34a^2b^2c^4$.
10. $38a^{12}b^{14}c^4 - 76a^{11}b^{12}c^3 - 76a^{13}b^{11}c^2$.
11. $4x^{2a}y^{3b} + 6x^{3a}y^{2b} - 8x^{5a}y^{4b}$.
12. $3a^{4n+2}b^{3n+4} + 6a^{6n+4}b^{3n+3} - 12a^{5n+3}b^{5n+6}$.

TRINOMIAL SQUARES

142. In §§ 89 and 90 we found by multiplication:

$$(a+b)^2 = a^2 + 2\,ab + b^2, \qquad (1)$$

and

$$(a-b)^2 = a^2 - 2\,ab + b^2. \qquad (2)$$

By means of these formulas we may square the sum or difference of any two number expressions.

E.g. $(3\,x+7)^2 = (3\,x)^2 + 2(3\,x)\cdot 7 + 7^2 = 9\,x^2 + 42\,x + 49.$

$(a+b+c)^2 = [(a+b)+c]^2 = (a+b)^2 + 2(a+b)c + c^2.$

$[(5+r)-(s-t)]^2 = (5+r)^2 - 2(5+r)(s-t) + (s-t)^2.$

The last expressions may now be reduced by performing the indicated operations.

In this manner write the squares of the following binomials. Read as many as possible at sight.

1. $t+8.$
2. $r-12.$
3. $3\,x-5\,y.$
4. $6\,a-7.$
5. $m^5+3\,n.$
6. $7\,x^2+3\,y.$
7. $3\,y^3-2\,z^2.$
8. $8\,m^2-7\,n^2.$
9. $5\,a^3-3\,b^3.$
10. $(a-3)-2(b+c).$
11. $(x-y)+2(z+y^2).$
12. $(3\,a-2\,b)+5.$
13. $7\,x-(4\,r-s).$
14. $(m^2-3)-(m^3+n).$

143. The binomial $a+b$ is one of the two equal factors of $a^2+2\,ab+b^2$, and is called the **square root** of this trinomial.

Likewise $a-b$ is the square root of $a^2-2\,ab+b^2$.

In each case a is the square root of a^2 and b of b^2. Hence $2\,ab$ *is twice the product of the square roots of* a^2 *and* b^2.

From the squares obtained in the last article, we learn to distinguish whether any given trinomial is a perfect square, as in the following examples:

1. $x^2+4\,x+4$ is in the form of (1), since x^2 and 4 are squares each with the sign +, and $4\,x$ is twice the product of the square roots of x^2 and 4. Hence

$$x^2+4\,x+4 = x^2+2(2\,x)+2^2 = (x+2)(x+2) = (x+2)^2.$$

2. $x^2 - 4x + 4$ is in the form of (2), since it differs from (1) only in the sign of the middle term. Thus

$$x^2 - 4x + 4 = x^2 - 2(2x) + 2^2 = (x-2)(x-2) = (x-2)^2.$$

3. $16x^2 + 9y^2 + 12xy$ is not in the form (1) since $12xy$ is not equal to $2(4x)(3y)$.

Definition. A trinomial which is the product of two equal factors is called a **trinomial square.**

EXERCISES

1. State the facts concerning a trinomial which make it a perfect square.

Determine whether the following are trinomial squares, and if so indicate the two equal factors.

2. $x^2 + 2xy + y^2$.
3. $x^2 - 2xy + y^2$.
4. $x^4 + 2x^2y^2 + y^4$.
5. $x^4 - 2x^2y^2 + y^4$.
6. $m^2 + n^2 - 2mn$.
7. $r^2 + s^2 + 2rs$.
8. $4x^2 - 8xy + 4y^2$.
9. $a^6 + b^6 + 2a^3b^3$.
10. $a^8 + b^8 - 2a^4b^4$.
11. $64 + t^2 - 16t$.
12. $16 + x^2 - 8x$.
13. $9 - 6y + y^2$.
14. $25x^2 + 16y^2 + 40xy$.
15. $4m^2 + n^2 + 2mn$.
16. $100 + s^2 + 20s$.
17. $64 + 49 + 112$.
18. $16a^2 + 25b^2 - 50ab$.
19. $16a^2 + 25b^2 + 40ab$.
20. $81 - 270b + 225b^2$.
21. $121 + 4a^2b^4 - 22ab^2$.

144. From the foregoing examples we see that *a trinomial is a perfect square if it contains two terms which are squares each with the sign +, while the third term, whose sign is either + or −, is twice the product of the square roots of the other two Then the square root of the trinomial is the sum or the difference of these square roots according as the sign of the third term is + or −.*

Since on multiplying we find $(a-b)^2$ and $(b-a)^2$ give the same result, we may write the factors of $a^2 - 2ab + b^2$ either $(a-b)(a-b)$ or $(b-a)(b-a)$. That is, the square root of $(a-b)^2$ is either $a-b$ or $-(a-b)$. Likewise the square root of a^2 is either $+a$ or $-a$. See page 176.

EXERCISES

Factor the following. If any one of the trinomials is found not to be a square, show where it is lacking.

1. $9 + 2 \cdot 3 \cdot 4 + 16$.
2. $x^2 + 4y^2 + 4xy$.
3. $9x^2 + 18xy + 9y^2$.
4. $4x^2 + 4xy + y^2$.
5. $4x^2 + 8xy + 4y^2$.
6. $25x^2 + 12xy + 4y^2$.
7. $16x^2 + 16xy + 4y^2$.
8. $9r^2 + 36rs + 25s^2$.
9. $16x^6 + 8x^3y + y^2$.
10. $4x^8 + 12x^4a^3 + 9a^4$.
11. $a^{10} + 6a^5b + 9b^2$.
12. $(a+1)^2 + 2(a+1)b + b^2$.
13. $(x+3)^2 + 4(x+3)y + 4y^2$.
14. $x^6 + 12x^3 + 36$.
15. $a^4 + 18a^2 + 12$.
16. $121 + 4x^8 - 44x^4$.
17. $16x^4 + 64y^4 - 64x^2y^2$.
18. $81a^2 - 216a + 144$.
19. $4a^2 + 8ab^2 + 4b^2$.
20. $9b^4 + 18b^2c^4 + 9c^8$.
21. $4x^2 + 4y^2 - 8xy$.
22. $9a^2 - 16ab + 4b^2$.
23. $9x^4 - 24a^2b + 16b^2$.
24. $25 + 49x^2 - 70x$.
25. $-30ab^2 + 9a^2 + 25b^4$.
26. $16a^2 - 24ab + 9b^2$.
27. $36x^2 - 84x + 49$.
28. $25 - 90 + 81$.
29. $64x^2 - 32x + 9$.
30. $(3+a)^2 + b^2 - 2b(3+a)$

THE DIFFERENCE OF TWO SQUARES

145. In § 91 we found by multiplication,

$$(a+b)(a-b) = a^2 - b^2.$$

By means of this formula the product of the sum and difference of any two number expressions may be found.

E.g.
$$\begin{aligned}(x+y-z)(x+y+z) &= [(x+y)-z][(x+y)+z]\\ &= (x+y)^2 - z^2.\end{aligned}$$

In this manner form the following products. Verify the first ten by actual multiplication. Read at sight the first seven.

1. $(4a+5b)(4a-5b)$.
2. $(24x+12y)(24x-12y)$.
3. $(16a^2b^3-3c)(16a^2b^3+3c)$.
4. $(5-6t^2)(5+6t^2)$.
5. $(3x-2y)(3x+2y)$.
6. $(x^3-y^3)(x^3+y^3)$.

7. $(3x^4 - 5x^4)(3x^4 + 5y^4)$.
8. $[x+(y-z)][x-(y-z)]$.
9. $(x^m + y^n)(x^m - y^n)$.
10. $[c-(a-b)][c+(a-b)]$.
11. $[x-(y+z)][x+(y+z)]$.
12. $(a+b+c)(a-b-c)$.
13. $(a+b+c)(a-b-c)$.
14. $(a-b+c)(a-b-c)$.
15. $(r-y-z)(r-y+z)$.
16. $(a+b+c)(a+b-c)$.

146. From the preceding examples we see that *every binomial which is the difference between two perfect squares is composed of two binomial factors; namely, the sum and the difference of the square roots of these squares.*

E.g. $16x^2 - 9y^2$ is the difference between two squares $(4x)^2$ and $(3y)^2$. Hence we have

$$16x^2 - 9y^2 = (4x)^2 - (3y)^2 = (4x+3y)(4x-3y).$$

Again, $(a-3b)^2-(2a+b)^2=[(a-3b)+(2a+b)][(a-3b)-(2a+b)]$
$= (a-3b+2a+b)(a-3b-2a-b)$
$= (3a-2b)(-a-4b)$.

EXERCISES

Factor each of the following. Read as many as possible at sight.

1. $x^2 - 4y^2$.
2. $9x^2 - 36y^2$.
3. $x^4 - b^2$.
4. $4x^2 - 9b^8$.
5. $16a^4 - 9b^4$.
6. $64 - b^2$.
7. $1 - b^2$.
8. $a^2 - 1$.
9. $1 - 9x^4$.
10. $4 - 36a^2$.
11. $1 - 64a^8$.
12. $144x^2b^4 - 1$.
13. $256a^4b^6 - c^2$.
14. $1 - (x+y)^2$.
15. $4 - (x-2y)^2$.
16. $16a - 25ab^2$.
17. $49x - 4xy^2$.
18. $225 - 64x^2y^4$.
19. $576a - 144ay^2$.
20. $5^8 - 3^8$.
21. $x^4 - 81y^2$.

147. It is important to determine whether a given expression can be written as the difference of two squares.

E.g. $a^2 + b^2 + 2\,ab - c^2 = (a + b)^2 - c^2 = (a + b + c)(a + b - c)$.

Also, $c^2 - a^2 + 2\,ab - b^2 = c^2 - (a^2 - 2\,ab + b^2) = c^2 - (a - b)^2$
$= (c - a + b)(c + a - b)$.

EXERCISES

In the following determine whether each is the difference of two squares, and if so, factor it accordingly.

1. $x^2 - (y - z)^2$.
2. $(x - y)^2 - z^2$.
3. $a^2 + b^2 - 2\,ab - 4$.
4. $x^2 + y^2 - 2\,xy - z^2$.
5. $4\,a^2b^2 - (a^2 + b^2 - c^2)^2$.
6. $a^2 - (b^2 + c^2 + 2\,bc)$.
7. $(2\,a - 5)^2 - (3\,a + 1)^2$.
8. $(3\,x^2 - y)^2 - (x + y)^2$.
9. $(3\,a - 2\,b)^2 - (8\,a + 5\,b)^2$.
10. $(3\,m - 4)^2 - (2\,m + 3)^2$.
11. $(2\,r + s)^2 - (3\,r - s)^2$.
12. $81 - (a + b + c)^2$.
13. $x^4 + x^2 + 1 - 4\,a^2$.
14. $a^2 - (x + 2\,y)^2$.
15. $9\,x^2 - (a - b)^2$.
16. $25\,m^2 - (3\,r + 2\,s)^2$.
17. $4\,c^2 - (4\,a^2 + 12\,ab - 9\,b^2)$.
18. $9\,a^2 + 6\,ab + b^2 - c^2$.
19. $16\,x^2 - a^2 + 4\,ab - 4\,b^2$.
20. $a^2 - 2\,ab + b^2 - c^2$.
21. $c^2 - (a^2 + 2\,ab + b^2)$.
22. $c^2 - (a^2 - 2\,ab + b^2)$.
23. $(a + b)^2 - (a - b)^2$.
24. $a^2 + 4\,ab + b^2 - x^2$.
25. $a^2 + 4\,ab + 4\,b^2 - x^2$.
26. $9\,a^2 + 16\,b^2 - 32\,ab - x^4$.

DRILL EXERCISES

1. $16c-(41-7c)+(15-8c)$.

2. $-(5a-3c)-(2c-8a)+3a$.

3. $-(-12x-7y-15x)-(-9y+8x+3y)$.

4. $(19x+4y-32x-17x)-12x-(49y+18x-70x)$.

5. $17a-3-(7a-2)+(6a-5)$.

6. $7(m+6)+10m=42-8(2m+2)+181$.

7. $20-3(x-4)+2x=2x+17$.

8. $\dfrac{x}{x-2}+\dfrac{4x}{x+3}=5$.

9. $\dfrac{x+a}{a+b}+\dfrac{x+b}{a-b}=1$.

10. $\begin{cases} x+2y=4, \\ 2x+y=-1. \end{cases}$

11. $\begin{cases} 5x+9y=19, \\ 3x-y=5. \end{cases}$

12. $\begin{cases} 3x-7y=-11, \\ 2x+y=4. \end{cases}$

13. $\begin{cases} 5x-3y=4-2x+7y, \\ 5y+x=7. \end{cases}$

14. $\begin{cases} x-3y+z=10, \\ 2x+y-z=1, \\ 3x-2y+5z=31. \end{cases}$

15. $\begin{cases} x+y+z=1, \\ 3x+4y-z=1, \\ -2x-y+3z=5. \end{cases}$

16. Represent by a graph the progress of a train going 40 miles per hour. Is the distance an increasing or decreasing variable as the time increases?

17. Represent by a graph the equation: $2x-3y=5$. Does y increase or decrease as x increases?

18. Graph $x+2y=6$ and state whether y increases or decreases as x increases.

19. How do you represent a fraction whose numerator is 3 less than twice that of $\frac{m}{n}$ and whose denominator is equal to 3 times the sum obtained by adding 2 to the denominator of this fraction?

20. There are two numbers such that if one half their product is divided by twice their sum the result is 12 times their difference. Write an equation representing this relation between the numbers.

DIVISION BY A POLYNOMIAL

148. The simplest case of division by a polynomial is that in which the dividend can be resolved into two factors, one being the given polynomial divisor and the other a monomial.

E.g. To divide $4x^3 + 4x^2y$ by $x + y$, factor the dividend, and we have
$$4x^2(x + y) \div (x + y) = 4x^2.$$

In case the dividend cannot be factored in this manner, then, if the division is possible, the quotient must be a polynomial. The process of finding the quotient under such circumstances is best shown by studying a particular case.

Illustrative Example 1. Consider the product

$$(x^2 + 2xy + y^2)(x + y) = x^2(x + y) + 2xy(x + y) + y^2(x + y).$$

The products, $x^2(x + y)$, $2xy(x + y)$, and $y^2(x + y)$ are called **partial products**, and their sum, $x^3 + 3x^2y + 3xy^2 + y^3$, the **complete product.**

In dividing $x^3 + 3x^2y + 3xy^2 + y^3$ by $x + y$ the quotient must be such a polynomial that when its terms are multiplied by $x + y$ the results are these partial products, which in the solution are called 1st, 2d, and 3d products.

The work may be arranged as follows:

Dividend or product:	$x^3 + 3x^2y + 3xy^2 + y^3$	$x + y$, divisor.
1st product, $x^2(x + y)$:	$x^3 + x^2y$	$x^2 + 2xy + y^2$,
Dividend minus 1st product:	$2x^2y + 3xy^2 + y^3$	[quotient.
2d product, $2xy(x + y)$:	$2x^2y + 2xy^2$	
Dividend minus 1st and 2d products:	$xy^2 + y^3$	
3d product, $y^2(x + y)$:	$xy^2 + y^3$	
Dividend minus 1st, 2d, and 3d products:	0	

Explanation. Since the dividend or product contains the term x^3, and since one of the factors, the divisor, contains the term x, the other factor, the quotient, must contain the term x^2. Multiplying this term of the quotient by the divisor, we obtain the first partial product, $x^3 + x^2y$.

Subtracting the first partial product from the whole product $x^3 + 3x^2y + 3xy^2 + y^3$, the remainder is $2x^2y + 3xy^2 + y^3$, which is the product of the divisor and that part of the quotient which has not yet been found. Since this product contains the term $2x^2y$ and the divisor contains the term x, the quotient must contain the term $2xy$. The product of $2xy$ and $x + y$ is the second partial product.

In like manner the third partial product is $xy^2 + y^3$.

Subtracting the third partial product the remainder is zero. Hence the sum of the three partial products thus obtained is equal to the whole product, and it follows that $x^2 + 2xy + y^2$ is the required quotient.

149. Problems in division may be checked by substituting any convenient values for the letters. For example, in this case, $x = 1$, $y = 1$, reduces the dividend to 8, the divisor to 2, and the quotient to 4, which verifies the correctness of the result.

Since division by zero is impossible (see Advanced Course), care must be taken not to select such values for the letters as will reduce the divisor to zero.

Illustrative Example 2. Divide $2x^4 + x^3 - 7x^2 + 5x - 1$ by $x^2 + 2x - 1$.

Solution.

		[divisor.
Dividend or product:	$2x^4 + x^3 - 7x^2 + 5x - 1$	$x^2 + 2x - 1$,
1st product, $2x^2(x^2 + 2x - 1)$:	$2x^4 + 4x^3 - 2x^2$	$2x^2 - 3x + 1$,
Dividend minus 1st product:	$-3x^3 - 5x^2 + 5x - 1$	[quotient.
2d product, $-3x(x^2 + 2x - 1)$:	$-3x^3 - 6x^2 + 3x$	
Dividend minus 1st and 2d products:	$+x^2 + 2x - 1$	
3d product, $1 \cdot (x^2 + 2x - 1)$:	$x^2 + 2x - 1$	
Dividend minus 1st, 2d, and 3d products:	0	

Check. Substitute $x = 2$ in dividend, divisor, and quotient.

Illustrative Example 3. Divide $20a^2-4+18a^4+18a-19a^3$ by $2a^2-3a+4$.

Solution. Arranging dividend and divisor according to the descending powers of a, we have

		[divisor.
Dividend or product:	$18a^4-19a^3+20a^2+18a-4$	$2a^2-3a+4$,
1st product:	$18a^4-27a^3+36a^2$	$9a^2+4a-2$,
Dividend minus 1st product:	$8a^3-16a^2+18a-4$	[quotient.
2d product:	$8a^3-12a^2+16a$	
Dividend minus 1st and 2d products:	$-4a^2+2a-4$	
3d product:	$-4a^2+6a-8$	
Dividend minus all products:	$-4a+4$	

Since $2a^2$ is not contained in $-4a$, the division ends and $-4a+4$ is the remainder. As in arithmetic we write this as the numerator of a fraction whose denominator is the divisor. Hence, the complete result is $9a^2+4a-2+\dfrac{4-4a}{2a^2-3a+4}$.

Check. Substitute $a=1$ in dividend, divisor, and quotient.

150. From a consideration of the preceding examples the process of dividing by a polynomial is described as follows:

1. Arrange the terms of dividend and divisor according to descending (or ascending) powers of some common letter.

2. Divide the first term of the dividend by the first term of the divisor. This quotient is the first term of the quotient.

3. Multiply the first term of the quotient by the divisor and subtract the product from the dividend.

4. Divide the first term of this remainder by the first term of the divisor, obtaining the second term of the quotient. Multiply the divisor by the second term of the quotient and subtract, obtaining a second remainder.

5. Continue in this manner until the last remainder is zero, or until a remainder is found whose first term does not contain as a factor the first term of the divisor. In case no remainder is zero, the division is not exact.

EXERCISES

Check the result in each case, being careful to substitute such numbers for the letters as do not make the divisor zero.

Divide the following:

1. $a^2 + 2ab + b^2$ by $a + b$.
2. $a^2 - 2ab + b^2$ by $a - b$.
3. $a^3 - 3a^2b + 3ab^2 - b^3$ by $a - b$.
4. $2x^3 + 2x^2y - 4x^2 - x - 4xy - y$ by $x + y$.
5. $x^3 + xy^2 - x^2y - y^3$ by $x - y$.
6. $x^3 + 4x^2 + x - 6$ by $x + 3$.
7. $x^3 + 4x^2 + x - 6$ by $x - 1$.
8. $x^4 - 6x^3 + 2x^2 - 3x + 6$ by $x - 1$.
9. $x^3 + 3x^2y + 3xy^2 + y^3$ by $x^2 + 2xy + y^2$.
10. $x^3 - 8x^2 + 75$ by $x - 5$.
11. $2a^3 + 19a^2b + 3ab^2$ by $2a + b$.
12. $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$ by $x - y$.
13. $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$ by $x^2 + 2xy + y^2$.
14. $x^4 + x^3y + xy^3 + y^4$ by $x + y$.
15. $x^4 + x^2y^2 + y^4$ by $x^2 - xy + y^2$.
16. $x^4 - y^4$ by $x - y$.
17. $a^3 + b^3 + c^3 - 3abc$ by $a + b + c$.
18. $2x^4 + 11x^3 - 26x^2 + 14x + 3$ by $x^2 + 7x - 3$.
19. $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$ by $x^2 + 2xy + y^2$.
20. $x^5 - x^4 - 27x^3 + 10x^2 - 30x - 200$ by $x^2 - 4x - 10$.
21. $3x^2 - 4xy + 8xz - 4y^2 + 6yz + 3z^2$ by $x - 2y + 3z$.
22. $9r^2s^2 - 4r^2t^2 + 4rst^2 - s^2t^2$ by $3rs - 2rt + st$.
23. $9a^2b^2 + 16x^2 - 4a^2 - 36b^2x^2$ by $3ab + 6bx - 2a - 4x$.
24. $x^3 + x^2y + x^2z - xyz - y^2z - yz^2$ by $x^2 - yz$.
25. $a^5 + a^4b + a^3 - a^3b^2 - 2ab^2 + b^3$ by $a^2 + ab - b^2$.
26. $a^3 + b^3 + 3ab - 1$ by $a + b - 1$.
27. $a^{3k} - 3a^{2k}b^k + 3a^kb^{2k} - b^{3k}$ by $a^k - b^k$.

THE SUM OF TWO CUBES

151. Example. Divide $a^3 + b^3$ by $a + b$.

Since the quotient multiplied by the divisor equals the dividend we have, $$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

This shows that *every binomial which is the sum of two cubes is the product of two factors, one of which is the sum of the numbers, and the other is the sum of their squares minus their product.*

E.g. (1) $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$.

(2) $8a^3 + 27b^3 = (2a)^3 + (3b)^3$
$= (2a + 3b)(4a^2 - 2a \cdot 3b + 9b^2)$.

(3) $x^6 + y^6 = (x^2)^3 + (y^2)^3 = (x^2 + y^2)(x^4 - x^2y^2 + y^4)$.

Notice the difference between the trinomial $x^2 - xy + y^2$ and the trinomial square $x^2 - 2xy + y^2$.

EXERCISES

Determine whether each of the following is the sum of two cubes, and if so, find the factors. Check the results by multiplication.

1. $x^3 + y^3$.
2. $a^3 + 8b^3$.
3. $27a^3 + b^3$.
4. $8a^3 + 1$.
5. $1 + 64x^3$.
6. $2^3 + 3^3$.
7. $125 + 729$.
8. $1 + 125x^6$.
9. $27x^6 + 1$.
10. $8a^3 + 27b^3$.
11. $8a^3 + 64b^3$.
12. $w^3x^6 + x^9a^3$.
13. $1 + 8a^3b^3$.
14. $64x^3 + 343$.
15. $1 + a^3$.
16. $a^3 + 9b^3$.
17. $125x^3 + y^6$.
18. $1 + x^8$.
19. $64x^2 + 27y^3$.
20. $8^3 + 10^3$.
21. $1 + 729x^6$.
22. $x^6 + y^{12}$.
23. $a^9 + b^3$.
24. $27r^3 + 125s^3$.
25. $x^9 + 27y^6$.
26. $64 + a^9$.
27. $a^3b^6 + x^3y^9$.

28. Find whether $x^3 + y^3$ is exactly divisible by $x - y$; by $x^2 - y^2$; by $x^2 + y^2$. What binomial is a divisor of the sum of two cubes? What is the quotient?

THE DIFFERENCE OF TWO CUBES

152. Example. Divide $a^3 - b^3$ by $a - b$ and obtain

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

This shows that the *difference of the cubes of two numbers is the product of two factors, one of which is the difference of the numbers, and the other the sum of the squares of the numbers plus their product.*

E.g. (1) $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$.

(2) $8a^3 - 64b^3 = (2a)^3 - (4b)^3$

$= (2a - 4b)(4a^2 + 2a \cdot 4b + 16b^2)$.

(3) $a^9 - b^9 = (a^3)^3 - (b^3)^3 = (a^3 - b^3)(a^6 + a^3b^3 + b^6)$.

Notice the difference between the factor $x^2 + xy + y^2$, and the trinomial square $x^2 + 2xy + y^2$.

EXERCISES

Determine whether each of the following is the difference of two cubes, and if so, find the factors. Check the results by multiplication.

1. $r^3 - s^3$.	9. $27 - 125a^3$.	17. $8x^4 - y^3$.
2. $8a^3 - b^3$.	10. $x^6 - y^6$.	18. $64a^3 - 27b^3$.
3. $a^3 - 8b^3$.	11. $1 - x^6$.	19. $1 - 729x^6$.
4. $8a^3 - 8b^3$.	12. $x^9 - 8$.	20. $x^6 - y^{12}$.
5. $3^3 - 2^3$.	13. $1 - 125x^3$.	21. $27r^3 - 125s^3$.
6. $1 - a^{10}$.	14. $8 - 27x^3$.	22. $125x^6 - 81y^2$.
7. $1 - 8a^3$.	15. $27x^3 - 64$.	23. $729 - 16x^3$.
8. $64x^3 - y^3$.	16. $2^3x^3 - 1$.	24. $125a^6 - 64b^9$.

25. Also factor Exs. 10, 11, and 20 as the difference of two squares.

26. Find whether $x^3 - y^3$ is exactly divisible by $x + y$, by $x^2 + y^2$, by $x^2 - y^2$. What binomial is a divisor of the difference of two cubes? What is the quotient?

TRINOMIALS OF THE FORM $x^2+(a+b)x+ab$

153. In § 92 were found such products as

$$(1)\quad (x+5)(x+2)=x^2+7x+10.$$
$$(2)\quad (x-5)(x-2)=x^2-7x+10.$$
$$(3)\quad (x+5)(x-2)=x^2+3x-10.$$
$$(4)\quad (x-5)(x+2)=x^2-3x-10.$$

All these are included in the form

$$(x+a)(x+b)=x^2+(a+b)x+ab,$$

in which the *coefficient of x is the algebraic sum of a and b and the last term is their product.*

154. It is possible to recognize such products at sight, and thus to find the factors by inspection.

Illustrative Examples. Determine whether the following trinomials are of the kind just considered:

1. $x^2+7x+12$. The question is whether two numbers can be found such that their sum is $+7$ and their product 12. The numbers 3 and 4 answer these conditions. Hence,

$$x^2+7x+12=(x+3)(x+4).$$

2. $x^2-5x-14$. Since the product of the numbers sought is -14, one number must have the sign $-$ and the other $+$; and since their sum is -5, the one having the greater absolute value must have the sign $-$. Hence, the numbers are -7 and $+2$, and we have $x^2-5x-14=(x-7)(x+2)$.

3. $x^2-7x+12=(x-3)(x-4)$. Since $(-3)(-4)=+12$ and $(-3)+(-4)=-7$.

4. $x^2+4x-12=(x+6)(x-2)$. Since $(+6)(-2)=-12$ and $(+6)+(-2)=+4$.

It is to be noted that it is not always possible to find integers to fulfill these two conditions.

E.g. Given $x^2 + 5x + 3$. By inspection, it is easily seen that there are no two integers such that their sum is $+5$ and their product $+3$.

Tell the steps to be taken in deciding whether x^2+ax+b has two binomial factors.

EXERCISES

Determine whether each of the following trinomials can be factored by inspection, and if so, find the factors.

1. x^2+3x+2.
2. x^2+x-6.
3. x^2-x-6.
4. x^2-6x+8.
5. x^2+6x+8.
6. x^2-3x-8.
7. x^2+2x-8.
8. $a^2-4a-32$.
9. $a^2+4a-32$.
10. $b^2+15b+56$.
11. $b^2+8b+15$.
12. b^2-b-56.
13. b^2+b-56.
14. $c^2-3c-15$.
15. $x^2-15x+56$.
16. $x^2+15x-54$.
17. $x^2-14x-95$.
18. $y^2+21y+98$.
19. $y^2-7y-98$.
20. $x^2-19x+78$.
21. x^4+18x^2+77.
22. x^4-5a^2-104.
23. $a^2+32a+240$.
24. a^4-11a^2+28.
25. a^4-11a^2-60.
26. $a^2-14a-51$.
27. $a^2-3a-54$.
28. x^4-8x^2-32.
29. a^6-3a^3-154.
30. $x^2-10x+25$.
31. $a^2b^6-13ab^3-30$.
32. $x^2-17xyz+72y^2z^2$.
33. $r^8+6r^4s-91s^2$.
34. $a^4c^4+9a^2c^2-162$.
35. $a^2+11a-210$.
36. $m^4+4m^2n^2+4n^2$.
37. $s^2t^2-15st-54$.
38. $a^2b^2-27ab+26$.
39. $l^2+13l+42$.
40. $x^2y^2-11xy-180$.
41. $9a^2+24a+16$.
42. $81a^2-99a+30$.
43. $g^2+26g+133$.
44. $x^2+5xy-84y^2$.
45. $r^2+3r-154$.
46. $u^2+38uv+165v^2$.
47. $(a+b)^2-19(a+b)+88$.
48. $(x-y)^2-14(x-y)+40$.
49. $(r-s)^2-17(r-s)+60$.
50. $x^2+(a+b)x+ab$.

TRINOMIALS OF THE FORM ax^2+bx+c

155. Find the product of $2x+5$ and $3x+2$.

$$\begin{array}{l} 2x+5 \\ 3x+2 \\ \hline 6x^2+15x \\ \qquad\quad 4x+10 \\ \hline 6x^2+19x+10 \end{array}$$

The products $3x \cdot 2x$ and $2 \cdot 5$ are called *end products* and $2 \cdot 2x$ and $5 \cdot 3x$ are called *cross products.* In this case the cross products are similar with respect to x and are added. Hence the *final product is a trinomial two of whose terms are the end products and the third term is the sum of the cross products.*

This fact enables us to write such products at once.

E.g. $(3a+4)(5a-7)=15a^2-a-28$.

In this case $15a^2$ is the first end product and -28 the second, while $-a$ is the sum of the two cross products, $20a$ and $-21a$.

EXERCISES

In this manner obtain the following products:

1. $(2a+3)(a+3)$.
2. $(4a-1)(3a+2)$.
3. $(2x+5)(x-7)$.
4. $(7r+8)(3r-6)$.
5. $(2x+8)(9x-4)$.
6. $(3m-1)(4m+3)$.
7. $(5s-7)(2s-4)$.
8. $(2x-1)(7x+4)$.
9. $(4n-9)(5n-7)$.
10. $(8y-1)(5y+11)$.
11. $(t-5)(t+4)$.
12. $(5x-y)(2x-3y)$.
13. $(3x-2y)(x+3y)$.
14. $(4a-3y)(a+y)$.
15. $(3r-2s)(2r+s)$.
16. $(5m-n)(2m+n)$.
17. $(5a+3x)(3a-4x)$.
18. $(4a-5b)(a+3b)$.
19. $(3a+5b)(a-b)$.
20. $(3c-7d)(2c+3d)$.
21. $(2a-3b)(3a+2b)$.
22. $(6x-5y)(2x+3y)$.
23. $(5a+3b)(2a-5b)$.
24. $(7m+5n)(3m+4n)$.

156. Trinomials in the form of the above products may sometimes be factored by inspection.

Ex. 1. Factor: $5x^2+16x+3$.

If this is the product of two binomials they must be such that the end products are $5x^2$ and 3 and the sum of the cross products $16x$.

One pair of binomials having the required end products is $5x+3$ and $x+1$, others are $5x-1$ and $x-3$, $5x+1$ and $x+3$, and $5x-3$ and $x-1$.

It is convenient to write down these possible pairs of factors as follows, as if arranged for multiplication:

$5x+3$	$5x-1$	$5x+1$	$5x-3$
$x+1$	$x-3$	$x+3$	$x-1$

The sum of the cross products in the first pair is $8x$, in the second pair $-16x$, in the third pair $16x$, and in the fourth $-8x$. Since $16x$ is the middle term required, the factors are $5x+1$ and $x+3$.

Ex. 2. Factor: $6a^2-5a-4$.

In this case, as in the one preceding, there are several pairs of binomials whose end products are $6a^2$ and -4, such as $2a-2$ and $3a+2$, $6a-1$ and $a+4$, etc. By trial we find that among these $3a-4$ and $2a+1$ is the only pair the sum of whose cross products is $-5a$. Hence $6a^2-5a-4=(3a-4)(2a+1)$.

In this manner factor the following:

1. $3x^2+5x+2$.
2. $9a^2+9a+2$.
3. $2x^2+11x+12$.
4. $9x^2+36x+32$.
5. $2x^2-x-28$.
6. $12s^2+11s+2$.
7. $6t^2+7t-3$.
8. $6x^2-x-2$.
9. $5r^2+18r-8$.
10. $14a^2-39a+10$.
11. $5x^2+26x-24$.
12. $2x^2-5x+2$.
13. $2m^2-m-3$.
14. $7c^2-3c-4$.
15. $5x^4+9x^2-18$.
16. $7a^4+123a^2-54$.
17. $6c^2-19c+15$.
18. $3a^2-21a+30$.
19. $6d^2+4d-2$.
20. $20a^2-a-99$.
21. $12c^2+25c+12$.
22. $8+6a-5a^2$.
23. $15-5x-10x^2$.
24. $6-5x-4x^2$.
25. $3h^2-13h+14$.
26. $15r^2-r-2$.
27. $2t^2+11t+5$.
28. $10-5x-15x^2$.
29. $5x^2-33x+18$.
30. $20-9x-20x^2$.

FACTORS FOUND BY GROUPING

157. Another method of general application is here applied to polynomials of four terms.

Ex. 1. Find the factors of $ax + ay + bx + by$.

By Principle I, the first two terms may be added and also the last two.

Thus, $ax + ay + bx + by = a(x + y) + b(x + y)$.

These two compound terms have a *common factor*, $(x + y)$, and may be added with respect to this factor by Principle I.

Thus, $a(x + y) + b(x + y) = (a + b)(x + y)$.

Hence, $ax + ay + bx + by = (a + b)(x + y)$.

Ex. 2. Factor $ax - ay - bx + by$.

Combining the first two terms with respect to a and the second two with respect to $-b$, we have,

$$ax - ay - bx + by = a(x - y) - b(x - y).$$

Again combining with respect to the factor $x - y$,

$$ax - ay - bx + by = (a - b)(x - y).$$

The success of this method depends upon the possibility of so grouping and combining the terms as to reveal a *common binomial factor.*

EXERCISES

Factor the following:

1. $ab^2 + ac^2 - db^2 - dc^2$.
2. $6\,ms - 15\,nt + 9\,ns - 10\,mt$.
3. $8\,ax - 10\,ay + 4\,bx - 5\,by$.
4. $2\,a^2 + 3\,ak - 14\,an - 21\,nk$.
5. $ac + bc + ad + bd$.
6. $ax^2 - bx^2 - ay^2 + by^2$.
7. $8\,ac - 20\,ad - 6\,bc + 15\,bd$.
8. $2\,ax - 6\,bx + 3\,by - ay$.
9. $5 + 4\,a - 15\,c - 12\,ac$.
10. $15\,b - 6 - 20\,bc + 8\,c$.
11. $2\,n^2 - cn + 2\,nd - cd$.
12. $5\,ax - 15\,ay - 3\,bx + 9\,by$.
13. $3\,xa - 12\,xc - a + 4\,c$.
14. $3\,xy - 4\,mn + 6\,my - 2\,xn$.
15. $7\,mn + 7\,mr - 2\,n^2 - 2\,nr$.
16. $a - 1 + a^3 - a^2$.
17. $3\,s + 2 + 6\,s^4 + 4\,s^3$.
18. $as^2 - 3\,bst - ast + 3\,bt^2$.
19. $3\,mn + 6\,m^2 - 2\,am - an$.
20. $2\,ar + 2\,as + 2\,br + 2\,bs$.

SQUARES OF POLYNOMIALS

158. Example. By multiplication find the square of $a+b+c$, and reduce the result to simplest form.

How many terms are there in the product? How many are squares? How many are of the type $2\,ab$?

From this we get the following rule:

The square of a trinomial consists of the sum of the squares of its terms plus twice the product of each term by each succeeding term.

In symbols this is

$$(a+b+c)^2 = a^2+b^2+c^2+2\,ab+2\,ac+2\,bc.$$

EXERCISES

1. How may the above rule be applied to find the square of $a-b+c$, of $a+b-c$, of $a-b+c$?

Suggestion. $a-b+c=a+(-b)+c$.

2. Find the square of $(2\,a+b-3\,c)$.

Suggestion. $(2\,a+b-3\,c)^2=(2\,a)^2+b^2+(-3\,c)^2+2(2\,a)b+2(2\,a)(-3\,c)+2\cdot b(-3\,c)$. This should now be simplified.

Find the squares of the following:

3. $a+2\,b+3\,c$.	6. $2\,x+3\,y-z$.	9. $5\,a+2\,b-3\,c$.
4. $2\,a+b+4\,c$.	7. $4\,x-y+2$.	10. $a-b-2\,c$.
5. $x-3\,y-z$.	8. $3\,a-b+c$.	11. $x-3\,y-z$.

12. By multiplication find the square of $a+b+c+d$, and reduce to simplest form. Study this product and try to make a rule for squaring any polynomial.

159. In the preceding exercises, the square of each trinomial consists of six terms, namely *three squared terms and three sums of cross products.*

It follows that the **square root of such a polynomial** may be found by taking the square roots of the three squared terms and *determining their signs by trial* in such a way as to give the proper cross products.

Ex. Find the square root of

$$4x^2 - 12xy - 16xz + 9y^2 + 24yz + 16z^2.$$

Solution. The terms $4x^2$, $9y^2$, and $16z^2$ are all squares. The square root of $4x^2$ is either $+2x$ or $-2x$, that of $9y^2$ is either $+3y$ or $-3y$, that of $16z^2$ is $+4z$ or $-4z$. Hence if the given polynomial is a perfect square, the terms $-12xy$, $-16xz$, and $24yz$ must be the sums of the cross products of $2x$, $3y$, and $4z$ each taken with the proper sign. By inspection we soon find that the sign of $2x$ must be $+$, and that of $3y$ and $4z$ each $-$.

Hence the required square root is $2x - 3y - 4z$.

Is every polynomial of six terms the square of a trinomial? In order to be such a square, how many squared terms must there be? What is the sign of each squared term and why? How are the signs of the square roots of these squared terms determined?

EXERCISES

In each of the following determine whether the polynomial is a perfect square and if so find its square root:

1. $x^2 + y^2 + z^2 - 2xy + 2xz - 2yz$.
2. $a^2 - 8ab + 16b^2 - 2ac + c^2 + 8bc$.
3. $9x^2 + 4y^2 + z^2 - 12xy + 6xz - 4yz$.
4. $y^4 - 4y^2 - 8xy^2 + 16x + 16x^2 + 4$.
5. $a^2 + a^2b^2 - 2a^2b + 2abc - ab^2c + b^2c^2$.
6. $x^6 - 4x^5 + 4x^4 + 6x^3 - 12x^2 + 9$.
7. $x^2 + 16x^2y^2 + 289 + 8x^2y + 34x + 136xy$.
8. $9a^4 + 4a^2 + 256 - 12a^3 - 96a^2 + 64a$.
9. $16x^6 + 4y^2 + 1 - 16x^3y + 8x^3 - 4y$.
10. $25 + 49x^2 + 4x^4 - 70x - 20x^2 + 28x^3$.
11. $a^4b^4 - 2a^3b^2 + a^2 - 2a^2b^3 + 2ab + b^2$.

Besides the methods of factoring which have been applied to the types of expressions thus far considered, there are various other processes which will be considered in the Advanced Course.

SUMMARY OF FACTORING

1. Tell how to decide whether a polynomial has a monomial factor. Give an example of such a polynomial and find its factors.

2. Tell how to decide whether a binomial is the difference of two squares. Give such a binomial and find its factors.

3. Tell how to decide whether a trinomial is a perfect square. Distinguish two kinds. Give a trinomial square of each kind and factor it.

4. Give a rule for factoring the sum of two cubes.

5. Give a rule for factoring the difference of two cubes.

6. Tell how to decide whether a trinomial of the form $x^2 + bx + c$ is the product of two binomials.

7. Tell how to decide whether a trinomial of the form $ax^2 + bx + c$ is the product of two binomials.

8. Tell how to factor a polynomial by grouping. Give a polynomial which can be factored in this way and find its factors.

9. Describe the square of a trinomial. Tell how to decide whether a polynomial is a perfect square.

MISCELLANEOUS EXERCISES

Classify the following expressions according to the foregoing types for factoring, and find the factors:

1. $x^2 + 5x + 6$.
2. $1 - x^3$.
3. $x^2 + 11x + 30$.
4. $4x^2 + 9y^2 + 12xy$.
5. $4x^2 + 9y^2 - 12xy$.
6. $5x^2 + 4ax + 7xy$.
7. $2n^2 - 6nc - 3ny + 9cy$.
8. $4x^2 - y^2$
9. $a^3 + b^3$.
10. $9x^2 + y^4 + 6xy^2$.
11. $2y^2a^3 + 4ya^2 - 8ya$.
12. $x^2 + 7x + 6$.

13. $9x^2 + 36y^4 + 36xy^2$.
14. $9y - 9z - 2xy + 2xz$.
15. $a^3 - 1$.
16. $a^4 + b^4 + 2a^2b^2$.
17. $a^4 - 25$.
18. $27a^3 - 125$.
19. $4a^2 + 4ab + ab^2$.
20. $4a^2 + 9x^4 - 12ax^2$.
21. $1 + x^3$.
22. $2x^2 + 5x + 3$.
23. $36 + 4x^6 + 24x^3$.
24. $(x-1)^2 - (x+1)^2$.
25. $8 + 64a^6$.
26. $ac - ax - 4bc + 4bx$.
27. $27 - 216a^3$.
28. $3^3 + 6^3a^3$.
29. $25(x+1)^2 - 4$.
30. $5cx - 10c + 4dx - 8d$.
31. $4(x+2)^2 + y^2 + 4(x+2)y$.
32. $ra + 2rh - 5sa - 10sh$.
33. $-2a^2b + a^4 + b^2$.
34. $2ha - hb + 6a - 3b$.
35. $3(a+1)^3 + 4(a+1)^2 + a + 1$.
36. $(x+a)^2 - (x-a)^2$.
37. $15m^2 + 224m - 15$.
38. $3x^2 + 27x + 42$.
39. $x^4 + 49a^2 + 14ax^2$.
40. $27a^6 - a^3x^3$.
41. $3^3a^6 + a^3x^3$.
42. $8bd - 40be + 3cd - 15ce$.
43. $x^2 - 11x + 30$.
44. $(x+2)^2 - 4(x-2)^2$.
45. $x^4 + 9y^2 - 6yx^2$.
46. $4a^2 - 7ca^2 - 4d^2 + 7cd^2$.
47. $a^2 + 15a - 16$.
48. $18 - 27c + 16b - 24cb$.
49. $4 - (a^2 + b^2 - 2ab)^2$.
50. $10r + 3bs - 6br - 5s$.
51. $25 + 64x^6 + 80x^3$.
52. $1000 - x^3$.
53. $10^3 + x^3$.
54. $8a^3 + a^3b^3 + b^2a^2$.
55. $100 - 49x^4$.
56. $100 + 625 + 500$.
57. $a^2 - 17a + 72$.
58. $a^2 + 17a + 72$.
59. $a^2 + 16b^2 - 8ab$.
60. $x^6 - y^9$.
61. $24a^2 + 37a - 72$.
62. $x^4 + 15x^2 - 100$.
63. $9^6 + 8^3$.
64. $9x^4 + 16y^2 + 24x^2y$.
65. $1 - 1000$.
66. $16a^2b^2 + 24ab + 36b^3$.
67. $64 + 8$.
68. $16a^2b^2 + 9a^2c^2 + 24a^2bc$.

69. $a^2 + 4b^2 + 4ab - 4x^2$.
70. $a^3b^6 + c^3$.
71. $5x^3 + 10x^3y^2 + 30x^3y^4$.
72. $16a^2c^2 + 4c^2x^2 + 16ac^2x$.
73. $a^6y^3 - z^3$.
74. $x^4 - 7x^2 - 120$.
75. $9a^4b^2 - 12a^3b + 4a^2$.
76. $8ab + 27ab^7$.
77. $x^4 + 4x^2 + 4 - x^6$.
78. $1 - 125a^3b^6$.
79. $16 + 16ab + 4a^2b^2$.
80. $64a^3 + 8a^2b^3$.
81. $65r^2 + 8r - 1$.
82. $a^2 - 13a - 140$.
83. $x^6 + 17x^3 + 30$.
84. $25 - (a^4 - 2a^2b^3 + b^6)$.
85. $36a^2 - 29ab + 5b^2$.
86. $a^2 - a - 380$.
87. $24a^4c^4 + a^6 + 144c^8a^2$.
88. $9x^2 + 4y^4 - 12xy^2 - 16$.
89. $81 + 100x^8 - 180x^4$.
90. $a^4 + 27a^2 + 180$.
91. $a^4 + 3a^2 - 180$.
92. $a^4 - 3a^2 - 180$.
93. $144 - (a^4 + b^2 - 2a^2b)$.
94. $81a^2b^4 + 49c^2 - 126ab^2c$.
95. $12s^2 - 23st + 10t^2$.
96. $36x^4 + 12x^2y^4 + y^8$.
97. $16x^6 + 9y^4 + 24x^3y^2 - 49$.
98. $y^2 + 35y + 300$.
99. $5y^2 - 80y + 300$.
100. $39x^4 - 16x^2 + 1$.
101. $ac - bc + ad - bd$.
102. $625 - (31 - 4a^2)^2$.
103. $z^3 + ya - y^3z^3 - ay^4$.
104. $x^4 + 2x^2 + 1 - x^2$.
105. $60x^2 + 7xy - y^2$.
106. $x^2 - 20xy + 75y^2$.
107. $x^2 - 17x - 60$.
108. $36a^2b^4 + c^2b^4 + 12ab^4c$.
109. $4a^2 + 9b^4 + 12ab^2 - 16a^4$.
110. $100 - (16x^2 + y^6 - 8xy^3)$.
111. $6rd - 15re + 22cd - 55ce$.
112. $-112a^2c^3 + 49a^4 + 64c^6$.
113. $4x^2 + 9y^2 + z^2 - 12xy - 4xz + 6yz$.
114. $a^2b^2 + a^2c^2 + b^2c^2 - 2a^2bc + 2acb^2 - 2abc^2$.
115. $4a^2 - 12ab + 4ac + 9b^2 + c^2 - 6bc$.
116. $a^6 + 10a^3 + 9a^2 + 25 - 6a^4 - 30a$.

REVIEW QUESTIONS

1. Define a positive integral exponent. Explain the difference between an exponent and a coefficient.

2. Show how the multiplication of monomials may depend upon Principle XIII.

3. Under what circumstances are exponents added in multiplication? State Principle XVI. Use this principle to show that $(a^2)^3 = a^6$, $(a^3)^4 = a^{12}$.

4. Under what circumstances are exponents subtracted in division? State Principle XVII.

5. Show how the division of monomials may depend upon Principles XV and XVII.

6. What is meant by factoring? Is the following expression factored? $x(a+b)+y(a+b)$. Why?

7. What are the characteristics of a trinomial square?
Are the following trinomials squares? If not, state what is lacking. $x + xy + y^2$; $x^4 + x^2y^2 + y^4$; $a^2 - 2\,ab - b^2$; $4\,a^2 + 4\,ab + 4\,b^2$.

8. What are the factors of the difference of two squares?
Factor $x^6 - y^6$ as the difference of two squares.

9. What are the factors of the sum of two cubes?
Factor $x^6 + y^6$ as the sum of two cubes.

10. What are factors of the difference of two cubes?
Factor $x^6 - y^6$ as the difference of two cubes.

11. Explain how to factor a trinomial by inspecting the end products and cross products of two binomials.

12. By means of the following examples explain the process of factoring by grouping.

$$x^2 + ax + bx + ab;\ x^3 - x - 3\,x^2 + 3.$$

13. What are the characteristics of the square of a trinomial? Of the square of *any* polynomial?

14. State Principles XVI and XVII as formulas and add them to your list.

DRILL EXERCISES

1. Find the average of the following temperatures: 7 A.M., $-4°$; 8 A.M., $-2°$; 9 A.M., $-1°$; 10 A.M.. $+1°$; 11 A.M., $+5°$; 12 M., $+7°$.

2. $7x+(8x+4)\div 2=4x+9$.

3. $6x+4(4x+2)=85-3(2x+7)$.

4. $8+7(6+6n)+2n=2(4n+5)+18n+49$.

5. $5(9x+3)+6x=24x-4(3x+2)+36$.

6. $\frac{7(12x+8)}{4}+13+5x-6=47$.

7. $\frac{a-b}{2x-1}=\frac{4c}{(2x+1)(2x-1)}-\frac{a-b}{2x+1}$.

8. $\frac{x+2}{a}=\frac{x-1}{b}+\frac{3x+2}{ab}$.

9. $\begin{cases} 6y-x=7+4y, \\ 5x+8y=1. \end{cases}$

10. $\begin{cases} x-y=37, \\ 2x+3y=314+13y. \end{cases}$

11. $\begin{cases} 2x-3y=y+6, \\ x-2y=4y+3. \end{cases}$

12. $\begin{cases} 5x+3y=0, \\ 2x+y=1. \end{cases}$

13. $\begin{cases} 2x+3y=6x-1, \\ 3x-2y=3. \end{cases}$

14. $\begin{cases} 5x-3y=0, \\ 2x+2-6y=2-x. \end{cases}$

15. $\begin{cases} x+2y+2z=3, \\ 3x-4y+z=19, \\ -2x+6y+32=0. \end{cases}$

16. $\begin{cases} 2x+5y+7z=7, \\ 3x-9y-2z=23, \\ -x+3y+3z=-10. \end{cases}$

17. Graph the equations: $x-4y=7$ and $2x+y=4$.

18. The older of two sisters is now 8 years less than twice as old as the other. If x represents the age of the younger sister, represent in symbols twice the sum of their ages 7 years ago.

19. A rear wheel of a wagon has a circumference 4 feet greater than that of a front wheel. If x represents the number of feet in the circumference of the rear wheel, represent in symbols the number of revolutions each wheel must make to go one mile.

CHAPTER X

EQUATIONS SOLVED BY FACTORING

160. Illustrative Problem. There are two consecutive integers the sum of whose squares is 61. What are the numbers?

Solution. If $x =$ one of the numbers, then $x + 1$ is the other.

Hence, $$x^2 + (x + 1)^2 = 61 \quad (1)$$

By *F*, $$x^2 + x^2 + 2x + 1 = 61 \quad (2)$$

By *F, I, S*, $$2x^2 + 2x = 60 \quad (3)$$

By *D*, $$x^2 + x = 30 \quad (4)$$

Equation (4) differs from any which we have studied heretofore in that it contains the square of the unknown number, after all possible reductions have been made.

161. Definition. Equations which involve the second but no higher degree of the unknown number are called **quadratic equations.**

One method of solving quadratic equations is now to be considered.

By *S*, equation (4) above may be written

$$x^2 + x - 30 = 0. \quad (5)$$

Factoring the left member,

$$(x + 6)(x - 5) = 0. \quad (6)$$

This equation is satisfied by $x = 5$ since $(5 + 6)(5 - 5) = 11 \cdot 0 = 0$, and also by $x = 6$ since $(-6 + 6)(-6 - 5) = 0 \cdot (-11) = 0$.

Test by substitution whether 5 and $5 + 1 = 6$ satisfy the conditions of the problem; also whether -6 and $-6 + 1 = -5$ satisfy it.

It thus appears that equation (4) has two roots, namely, 5 and -6. The two pairs of corresponding integers 5, 6 and -6, -5 each satisfy the conditions of the problem.

EXERCISES

1. If one of two factors is zero, what is the product? Does it matter what the other factor is?

2. Find a value of x which makes $(x-3)(x+2)$ equal to zero. Does this value of x make both factors equal to zero? Is it necessary that both factors should be made equal to zero?

3. Find a value of x which satisfies the equation

$$(x-7)(x^2+2x-3)=0;$$

also one which satisfies $(x+8)(x^3+x+4)=0$.

Suggestion. Find a value of x which makes the first factor zero in each case.

4. Find two values of x which satisfy $(x-3)(x+4)=0$, also two which satisfy $(x+8)(x-3)=0$.

5. Find two values of x which satisfy $5x(x+7)=0$. Does $x=0$ satisfy this equation?

6. Find two values of x which satisfy $(3x-2)(2x+5)=0$.

162. The method of solution suggested by the foregoing examples consists of three steps:

(1) *Transform the equation so that all terms are collected in one member, with similar terms united, leaving the other member zero.* This can always be done by Principle VI. It is convenient to make the *right* member zero.

(2) *Factor the expression on the left.*

(3) *Find the values of x which make each of these factors zero.* This is readily done by setting each factor equal to zero and solving it for the unknown.

EXERCISES

Find two solutions for each of the following quadratic equations:

1. $x^2-3x+2=0$.

2. $x^2+7x=30$.

3. $a^2-11a=-30$.

4. $a^2+13a=30$.

5. $a^2+10a+8=-3a-34$.

6. $a^2+3a=10a+18$.

7. $a^2+10a=-24-4a$.

8. $2x^2-6x=-40+12x$.

9. $3x + x^2 = 20x - 72$.

10. $17x + 30 = -x^2 - 40$.

11. $7x^2 + 2x = 30x - 21$.

12. $11x + 3x^2 = 20$.

13. $16 - 5x + x^2 = -2x^2 - 20x - 2$.

14. $x^2 - 16 = 0$. 15. $x^2 - 1 = 0$. 16. $x^2 - x = 0$. 17. $x^2 + x = 0$.

18. $4x^2 = 25$. 19. $x^2 + 4x + 4 = 0$. 20. $x^2 + 8x + 16 = 0$.

21. $x^2 + 12x + 6 = 5x - 4$.

22. $2x^2 - 7x = 60 + 7x$.

23. $60x + 4x^2 + 144 = 8x$.

24. $18x = 63 - x^2$.

25. $24x^2 = 12x + 12$. 27. $22x + x^2 = 363$. 29. $2x^2 = 2 - 3x$.

26. $2x = 63 - x^2$. 28. $3x^2 + 7x = 6$. 30. $x - 2 = -3x^2$.

163. It is sometimes possible by the above process to solve equations in which the exponent of the highest power of the unknown is greater than 2.

Ex. 1. Solve the equation:

$$x^3 + 30x = 11x^2. \quad (1)$$

By S, $$x^3 - 11x^2 + 30x = 0. \quad (2)$$

By § 141, $$x(x^2 - 11x + 30) = 0. \quad (3)$$

By § 154, $$x(x - 5)(x - 6) = 0. \quad (4)$$

(4) is satisfied if $x = 0$, if $x - 5 = 0$, and if $x - 6 = 0$.
Hence the roots are $x = 0$, $x = 5$, $x = 6$.

Ex. 2. $x(x + 1)(x - 2)(x + 3) = 0$.

Any value of x which makes one of these factors zero reduces the product to zero and hence satisfies the equation. Hence the roots of the equation are found from

$$x = 0,\ x + 1 = 0,\ x - 2 = 0, \text{ and } x + 3 = 0.$$

That is, $x = 0$, $x = -1$, $x = 2$, $x = -3$ are the values of x which satisfy the equation.

Notice that this process is applicable only when one member of the equation is *zero* and the other member is *factored.*

Solve the following equations by factoring:

1. $x^3 - x^2 = 6x$.

2. $5x = 4x^2 + x^3$.

3. $x^3 - 25x = 0$.

4. $x^3 - 3x^2 = -2x$.

5. $3x^3 = 15x^2 + 42x.$
6. $5x^3 + 315x = 80x^2.$
7. $x^2 + ax + bx + ab = 0.$
8. $x^2 - ax - bx + ab = 0.$
9. $4(x-2)^2 - (x+3)^2 = 0.$
10. $x^2 - ax + bx - ab = 0.$
11. $x^2 + ax - bx - ab = 0.$
12. $9(x+2)^2 - 4(x-3)^2 = 0.$
13. $x^3 - x - 3x^2 + 3 = 0.$
14. $x^3 - 4x - 8x^2 + 32 = 0.$

PROBLEMS SOLVED BY FACTORING

164. Illustrative Problem. The paving of a square court costs 40 ¢ per square yard and the fence around it costs \$ 1.50 per linear yard. If the total cost of the pavement and the fence is \$ 100, what is the size of the court?

Solution. Let $x =$ the length of one side in yards.

Then $40x^2 =$ cost in cents of paving the court,

and $150 \cdot 4x = 600x =$ cost of the fence in cents.

Hence, $40x^2 + 600x = 10000.$ (1)

By *D*, $x^2 + 15x = 250.$ (2)

By *S*, $x^2 + 15x - 250 = 0.$ (3)

Factoring, $(x - 10)(x + 25) = 0.$ (4)

Whence, $x = 10$, and also $x = -25.$ (5)

It is clear that the length of a side of the court cannot be -25 yards. Hence 10 is the only one of these two results which has a meaning in this problem.

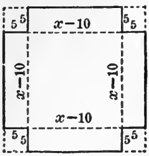
It happens frequently when a quadratic equation is used to solve a problem that one of the two numbers which satisfy this equation will not satisfy the conditions of the problem.

PROBLEMS

In each of the following problems find all the solutions possible for the equations and then determine whether or not each solution has a reasonable interpretation in the problem.

1. The dimensions of a picture inside the frame are 12 by 16 inches. What is the width of the frame if its area is 288 square inches?

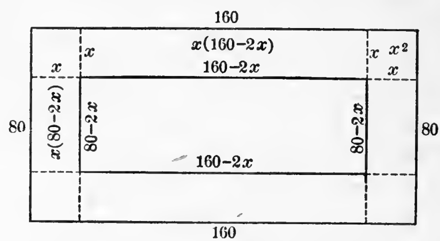
2. An open box is made from a square piece of tin by cutting out a 5-inch square from each corner and turning up the sides. How large is the original square if the box contains 180 cubic inches?



If $x =$ length of a side of the tin, then the volume of the box is: $5(x-10)(x-10)=180$. (See the figure.)

3. A rectangular piece of tin is 4 inches longer than it is wide. An open box containing 840 cubic inches is made by cutting a 6-inch square from each corner and turning up the ends and sides. What are the dimensions of the box?

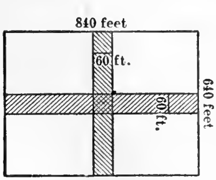
4. A farmer has a rectangular wheat field 160 rods long by 80 rods wide. In cutting the grain, he cuts a strip of equal width around the field. How many acres has he cut when the width of the strip is 8 rods?



5. How wide is the strip around the field of problem 5, if it contains $27\frac{1}{2}$ acres?

6. In the northwest a farmer using a steam plow starts plowing around a rectangular field 640 by 320 rods. If the strip plowed the first day lacks 16 square rods of being 24 acres, how wide is it?

7. A rectangular piece of ground 840 by 640 feet is divided into 4 city blocks by two streets 60 feet wide running through it at right angles. How many square feet are contained in the streets?



8. A farmer lays out two roads through the middle of his farm, one running lengthwise of the farm and the other crosswise. How wide are the roads if the farm is 320 by 240 rods, and the area of the roads is 1671 square rods?

QUADRATIC AND LINEAR EQUATIONS

165. When two simultaneous equations are given, one quadratic and one linear, they may be solved by the process of substitution, which was used (§ 122) in the case of two linear equations.

Illustrative Example. Solve the equations:

$$\begin{cases} x^2 - y^2 = -16. & (1) \\ x - 3y = -12. & (2) \end{cases}$$

From (2) by S, $x = 3y - 12.$ (3)

Substituting (3) in (1), $(3y - 12)^2 - y^2 = -16.$ (4)

From (4) by F, $9y^2 - 72y + 144 - y^2 = -16.$ (5)

From (5) by F, A, $8y^2 - 72y + 160 = 0.$ (6)

By D, $y^2 - 9y + 20 = 0.$ (7)

Factoring, $(y - 5)(y - 4) = 0.$ (8)

Hence, $y = 5$, and $y = 4.$ (9)

Substitute $y = 5$ in (3) and find $x = 3.$

Substitute $y = 4$ in (3) and find $x = 0.$

Therefore (1) and (3) are satisfied by the two pairs of values,

$$x = 3,\ y = 5, \text{ and } x = 0,\ y = 4.$$

Check by substituting these pairs of values in (1) and (2).

EXERCISES

In the manner just illustrated solve the following:

1. $\begin{cases} x + 2y = 8, \\ 5x^2 + 12y^2 = 128. \end{cases}$

2. $\begin{cases} x + y = 1, \\ x^2 + y^2 = 1. \end{cases}$

3. $\begin{cases} 2x - y = 6, \\ 4x^2 + 5y^2 = 36. \end{cases}$

4. $\begin{cases} x + 3y = 6, \\ x^2 + 3y^2 = 12. \end{cases}$

5. $\begin{cases} x - 2y = -2, \\ x^2 - 6y^2 = 10. \end{cases}$

6. $\begin{cases} 8x = 16y = -120, \\ 7x^2 + 2y^2 = 585. \end{cases}$

7. $\begin{cases} 7x + 9y = 88, \\ 7x^2 + 9y^2 = 736. \end{cases}$

8. $\begin{cases} x - y = 6, \\ x^2 - 7y^2 = 36. \end{cases}$

9. $\begin{cases} 3x+2y=7, \\ 3x^2+8y^2=35. \end{cases}$

10. $\begin{cases} x-8y=-11, \\ 3x^2-16y^2=11. \end{cases}$

11. $\begin{cases} x-y=-7, \\ 4x^2+3y^2=147. \end{cases}$

12. $\begin{cases} x-y=2, \\ x^2-5y^2=4. \end{cases}$

13. $\begin{cases} x-y=1, \\ 3x^2-2y^2=-5. \end{cases}$

14. $\begin{cases} 5x-7y=-28. \\ 15x^2+49y^2=784. \end{cases}$

15. $\begin{cases} 6x-7y=18, \\ 36x^2-7y^2=324. \end{cases}$

16. $\begin{cases} x-9y=2, \\ x^2-45y^2=4. \end{cases}$

17. $\begin{cases} x+y=8, \\ 13x^2+3y^2=160. \end{cases}$

18. $\begin{cases} 2x-5y=-16, \\ 4x^2+15y^2=256. \end{cases}$

19. $\begin{cases} 7x+4y=7, \\ 49x^2-8y^2=49. \end{cases}$

20. $\begin{cases} x-3y=-12, \\ x^2-y^2=-16. \end{cases}$

166. If squares are constructed on the two sides, and also on the hypotenuse of a right-angled triangle, then the *sum of the squares on the sides is equal to the square on the hypotenuse.* This is proved in geometry, but may be verified by counting squares in the accompanying figure. This proposition was first discovered by the great philosopher and mathematician Pythagoras, who lived about 550 B.C. Hence it is called the Pythagorean proposition.

We now proceed to solve some problems by this proposition.

PROBLEMS

1. The sum of the sides about the right angle of a right triangle is 35 inches, and the hypotenuse is 25 inches. Find the sides of the triangle.

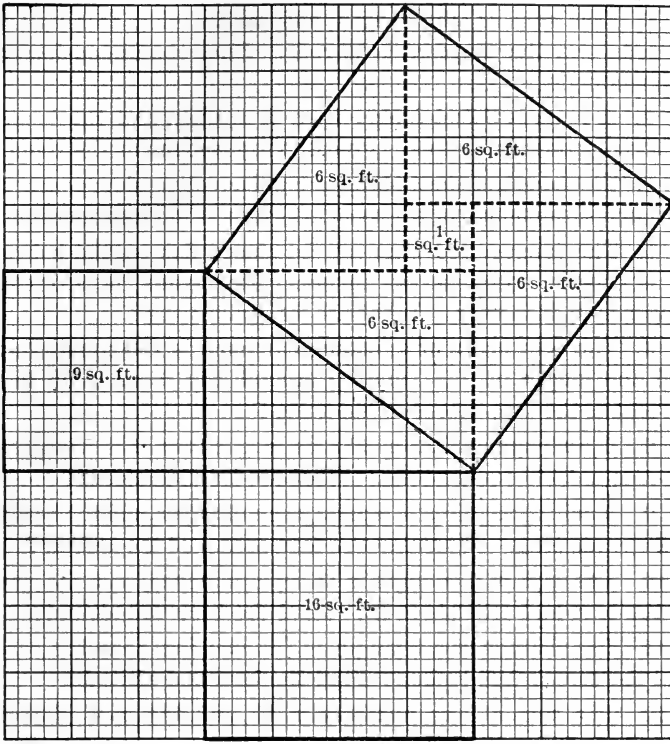
Solution. Let a = the length of one side in inches,

and b = the length of the other.

Then $a + b = 35,$ (1)

and $a^2 + b^2 = 25^2 = 625$ (Pythagorean proposition). (2)

From (1), $a = 35 - b.$



Substituting in (2), $(35 - b)^2 + b^2 = 625,$

or $1225 - 70\,b + b^2 + b^2 = 625,$

$$2\,b^2 - 70\,b + 600 = 0,$$

$$b^2 - 35\,b + 300 = 0,$$

Factoring, $(b - 20)(b - 15) = 0.$

Whence $b = 20$, and $b = 15$.

From (1), if $b = 20$, $a = 15$, and if $b = 15$, $a = 20$; that is, the sides of the triangle are 15 and 20.

2. The difference between the two sides of a right triangle is 2 feet, and the length of the hypotenuse is 10 feet. Find the two sides.

3. The sum of the length and width of a rectangle is 17 rods, and the diagonal is 13 rods. Find the dimensions of the rectangle.

4. A room is 3 feet longer than it is wide, and the length of the diagonal is 15 feet. Find the dimensions of the room.

5. The length of the molding around a rectangular room is 46 feet, and the diagonal of the room is 17 feet. Find its dimensions.

6. The longest rod that can be placed flat on the bottom of a certain trunk is 45 inches. The trunk is 9 inches longer than it is wide. What are its dimensions?

7. The floor space of a rectangular room is 180 square feet, and the length of the molding around the room is 56 feet. What are the dimensions of the room?

8. A rectangular field is 20 rods longer than it is wide, and its area is 2400 square rods. What are its dimensions?

9. A ceiling requires 24 square yards of paper, and the border is 20 yards long. What are the dimensions of the ceiling?

10. The area of a triangle is 18 square inches, and the sum of the base and altitude is 12. Find the base and altitude.

11. The altitude of a triangle is 7 inches less than the base, and the area is 130 inches. Find the base and altitude.

12. The sum of two numbers is 17, and the sum of their squares is 145. Find the numbers.

13. The difference of two numbers is 8, and the sum of their squares is 274. Find the numbers.

14. The difference of two numbers is 13, and the difference of their squares is 481. Find the numbers.

15. The sum of two numbers is 40, and the difference of their squares is 320. Find the numbers.

16. The sum of two numbers is 45, and their product is 450. Find the numbers.

17. The difference of two numbers is 32, and their product is 833. What are the numbers?

DRILL EXERCISES

1. $\frac{12(5+4x)}{6}-\frac{5(6+4x)}{2}+50=x+18.$

2. $15+\frac{21(3+x)}{7}+\frac{2(6+18x)}{3}=\frac{3(9x+12)}{3}+28.$

3. $\frac{11(5x+25)}{}+\frac{3(6x-2)}{2}=\frac{7(4x+8)}{4}+\frac{12x+36}{3}+35.$

4. $\frac{2(x+1)}{x-1}-3=-\frac{x-1}{x+1}.$

5. $\frac{x+a}{x-a}=5-\frac{4(x-a)}{x+a}.$

6. $\begin{cases}\frac{5x-3}{4}-\frac{2y-5}{2}=\frac{1}{2},\\ \frac{3x+5}{2}+\frac{y-10}{5}=6.\end{cases}$

7. $\begin{cases}\frac{2x-9}{3}-\frac{4y-2}{5}=-1,\\ \frac{7x+4}{2}+\frac{y+6}{4}=3.\end{cases}$

8. $\begin{cases}\frac{a+7}{3}+\frac{4b-1}{6}=\frac{1}{2},\\ \frac{4a+7}{3}+\frac{7b+3}{5}=-5.\end{cases}$

9. $\begin{cases}\frac{2a+5}{3}+\frac{3b-10}{4}=0,\\ \frac{3a+6}{8}-\frac{5-4b}{7}=2.\end{cases}$

10. $\begin{cases}\frac{x}{3}+\frac{y}{4}+\frac{z}{6}=6,\\ \frac{x}{2}+\frac{y}{8}-\frac{z}{12}=3,\\ \frac{x}{6}-\frac{y}{2}+\frac{z}{3}=1.\end{cases}$

11. $\begin{cases}x+y-z=35,\\ \frac{x}{3}+\frac{y}{5}+z=15,\\ \frac{x}{5}+\frac{y}{5}-z=3.\end{cases}$

12. If r represents the rate in miles per hour at which a train is moving, how far will it go in t hours? Another train runs 10 miles per hour faster. Express in symbols the sum of the distances which these two trains travel in t hours.

13. If r_1 represents the rate at which a river is flowing and r_2 the rate at which a steamer can go in still water, express in symbols the distance which the steamer can go in t hours: (*a*) down the river; (*b*) up the river.

CHAPTER XI

SQUARE ROOTS AND RADICALS

167. Definition. A **square root** of a number is one of its two equal factors.

Thus 3 is a square root of 9, since $3 \cdot 3 = 9$. Similarly $a + b$ is a square root of $a^2 + 2\,ab + b^2$.

It should be noted that every square has *two* square roots.

E.g. -3 is also a square root of 9 as well as $+3$, since $(-3) \cdot (-3) = 9$.

The positive square root of a number is indicated by the **radical sign** $\sqrt{}$ alone or preceded by the sign $+$. The negative square root is indicated by the radical sign preceded by the sign $-$.

Eg. $+\sqrt{9}$ or $\sqrt{9} = +3$ and not -3, and $-\sqrt{9} = -3$, and not $+3$.

The square root of any number is at once evident if we can resolve it into two equal groups of factors.

E.g.

$$\sqrt{576} = \sqrt{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3} = \sqrt{(2^3 \cdot 3)(2^3 \cdot 3)} = \sqrt{24 \cdot 24} = 24.$$

EXERCISES

Find by inspection the following indicated square roots:

1. $\sqrt{4}$.
2. $\sqrt{9}$.
3. $-\sqrt{16}$.
4. $\sqrt{25}$.
5. $\sqrt{36}$.
6. $-\sqrt{49}$.
7. $\sqrt{81}$.
8. $\sqrt{121}$.
9. $-\sqrt{169}$.
10. $\sqrt{225}$.
11. $\sqrt{196}$.
12. $-\sqrt{256}$.
13. $-\sqrt{576}$.
14. $\sqrt{400}$.
15. $\sqrt{324}$.
16. $\sqrt{289}$.
17. $-\sqrt{625}$.
18. $-\sqrt{900}$.
19. $\sqrt{10000}$.
20. $\sqrt{a^4}$.
21. $-\sqrt{x^6}$.
22. $-\sqrt{2^4}$.
23. $-\sqrt{5^{12}}$.
24. $-\sqrt{7^8}$.
25. $\sqrt{a^{12}}$.
26. $-\sqrt{3^{14}}$.
27. $\sqrt{a^{24}}$.
28. $\sqrt{3^4}$.

168. The square root of the product of several factors, each of which is a square, may be found in two ways if the factors are expressed in Arabic figures.

E.g. $\sqrt{4 \cdot 16 \cdot 25} = \sqrt{1600} = \sqrt{40 \cdot 40} = 40,$

or $\sqrt{4 \cdot 16 \cdot 25} = \sqrt{2^2 \cdot 4^2 \cdot 5^2} = 2 \cdot 4 \cdot 5 = 40.$

But with literal factors, the second process only is available.

E.g. $\sqrt{16\, a^2b^4c^2} = \sqrt{4^2a^2(b^2)^2c^2} = 4\, ab^2c.$

EXERCISES

Find the following indicated square roots:

1. $-\sqrt{2^2 \cdot 3^2}$.
2. $\sqrt{81 \cdot 121}$.
3. $\sqrt{49 \cdot 25 \cdot 169}$.
4. $-\sqrt{8^2 \cdot 5^2 \cdot 3^2}$.
5. $\sqrt{5^4 \cdot 3^2 \cdot 4^4}$.
6. $\sqrt{25 \cdot 36}$.
7. $-\sqrt{3^{12} \cdot 5^{14}}$.
8. $-\sqrt{2^{22} \cdot 3^{12}}$.
9. $\sqrt{16\, a^2b^2c^2}$.
10. $\sqrt{64\, a^4x^4}$.
11. $-\sqrt{4^4\, a^2b^4}$.
12. $\sqrt{3^2\, x^2y^2}$.
13. $\sqrt{9\, x^4y^{12}}$.
14. $-\sqrt{121\, a^2x^4}$.
15. $-\sqrt{7^4\, a^4b^2}$.
16. $\sqrt{625\, x^4y^2}$.
17. $\sqrt{1225\, a^{2r}}$.
18. $-\sqrt{36\, b^{4m}c^{2n}}$.

Notice that $\sqrt{9 + 16}$ is not equal to $\sqrt{9} + \sqrt{16}$.

The preceding exercises illustrate

Principle XVIII

169. Rule. *The square root of a product is obtained by finding the square root of each factor separately and then taking the product of these roots. That is,*

$$\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}.$$

In order that a factor may be a perfect square it must be a power whose exponent is *even.* Its square root is then a power of the same base whose exponent is equal to one-half the given exponent.

Thus, $\sqrt{x^6} = \sqrt{x^3 \cdot x^3} = x^3 = x^{6 \div 2}.$

Hence to find the square root of a monomial we divide the exponent of each factor by 2.

EXERCISES

Find the following square roots:

1. $-\sqrt{4\,a^6b^8}$.
2. $-\sqrt{3^2x^{12}y^{14}}$.
3. $\sqrt{5^2 \cdot 3^{22}\,t^{24}}$.
4. $\sqrt{121\,x^4y^{12}}$.
5. $-\sqrt{576\,a^2b^4}$.
6. $-\sqrt{10^4\,a^4b^4}$.
7. $\sqrt{5^4\,m^{14}}$.
8. $\sqrt{5^4 \cdot 3^8 \cdot 7^2}$.
9. $\sqrt{3^{14} \cdot 7^{12}\,a^4}$.
10. $-\sqrt{25\,a^2b^4c^{12}}$.
11. $\sqrt{81\,x^4y^8c^{10}}$.
12. $\sqrt{729\,a^6y^{10}z^{14}}$.
13. $-\sqrt{64 \cdot 625\,a^2b^4}$.
14. $\sqrt{256\,x^{2k}y^{4n}}$.
15. $\sqrt{3^{2x} \cdot 5^{2y}}$.

SQUARE ROOTS OF POLYNOMIALS

170. In § 159 the square roots of certain polynomials were found by inspection. **Any polynomial square** may be recognized and the square root found by that method, provided no similar terms have been combined in the square.

171. The case where some *terms of the square combine* is illustrated by the example:

$$(x^2 + x + 1)^2 = x^4 + x^2 + 1 + 2\,x^3 + 2\,x^2 + 2\,x = x^4 + 2\,x^3 + 3\,x^2 + 2\,x + 1.$$

Study the answers to the following questions. They are needed in finding the square root of such an expression:

1. What terms must be added to a^2 to make it the square of $a + b$?

2. What terms must be added to $a^2 + 2\,ab + b^2$ to make it the square of $a + b + c$?

3. What terms must be added to $a^2 + 2\,ab + b^2 + 2\,ac + 2\,bc + c^2$ to make it the square of $a + b + c + d$.

The above answers may be summarized as follows:

When a term is added to a polynomial, the square of the resulting polynomial is equal to the square of the original, plus twice the product of the new term and the sum of the original terms, plus the square of the new term.

This is expressed by the following formula: $\boldsymbol{(a + b + c + d)^2 = a^2 + 2\,ab + b^2 + 2(a + b)c + c^2 + 2(a + b + c)d + d^2}$.

172. Illustrative Example. Find the square root of

$$16x^6 - 24x^5 + 25x^4 - 52x^3 + 34x^2 - 20x + 25.$$

Solution

$$\begin{array}{lrr}
 & a \quad + b \quad + c + d & \\
\text{Square root} & 4x^3 - 3x^2 + 2x - 5 & \\
\hline
\text{Given square} & 16x^6 - 24x^5 + 25x^4 - 52x^3 + 34x^2 - 20x + 25 & (1) \\
a^2 = & 16x^6 & \\
\hline
 & -24x^5 + 25x^4 - 52x^3 + 34x^2 - 20x + 25 & (2) \\
2ab + b^2 = & -24x^5 + 9x^4 & (3) \\
\hline
 & 16x^4 - 52x^3 + 34x^2 - 20x + 25 & (4) \\
2(a+b)c + c^2 = & 16x^4 - 12x^3 + 4x^2 & (5) \\
\hline
 & -40x^3 + 30x^2 - 20x + 25 & (6) \\
2(a+b+c)d + d^2 = & -40x^3 + 30x^2 - 20x + 25 & (7) \\
\hline
\end{array}$$

Explanation. First arrange the terms according to increasing or decreasing powers of the letter involved, as in long division. The first term of the root is the square root of the first term of the polynomial. The square of this is subtracted, leaving (2).

The next term of the root, $-3x^2$, is found by dividing the first term of (2) by $2(4x^3) = 8x^3$, that is, $-24x^5 \div 8x^3 = -3x^2$, corresponding to $2ab \div 2a = b$.

We then subtract $2(4x^3)(-3x^2) + (-3x^2)^2 = -24x^5 + 9x^4$, corresponding to $2ab + b^2$, and this completes the subtraction of $(4x^3 - 3x^2)^2$.

Subtracting (5) completes the subtraction of $(4x^3 - 3x^2 + 2x)^2$, and subtracting (7) completes the subtraction of $(4x^3 - 3x^2 + 2x - 5)^2$.

At each step the next term of the root is found by dividing the first term of the remainder by twice the first term of the root. Thus from (4) the third term of the root is $16x^4 \div 2(4x^3) = 2x$, corresponding to $2ac \div 2a = c$.

Since the remainder is zero after the last subtraction, this shows that $(4x^3 - 3x^2 + 2x - 5)^2$ is exactly the given polynomial.

EXERCISES

Find the square roots of the following polynomials:

1. $x^4 + 2x^3 + 3x^2 + 2x + 1.$

2. $1 - 2a + 3a^2 - 2a^3 + a^4.$

3. $1 + 2b - b^2 - 2b^3 + b^4.$

4. $a^4 + 4a^3 + 6a^2 + 4a + 1.$

5. $c^4 - 4c^3 + 6c^2 - 4c + 1.$
6. $x^4 - 2x^3 + 5x^2 - 4x + 4.$
7. $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$
8. $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4.$
9. $a^4 + 53a^2 + 14a^3 + 28a + 4.$
10. $a^6 + 6a^5 + 15a^4 + 20a^3 + 15a^2 + 6a + 1.$
11. $a^6 - 6a^5 + 15a^4 - 20a^3 + 15a^2 - 6a + 1.$
12. $4x^6 - 12x^5 + 13x^4 - 14x^3 + 13x^2 - 4x + 4.$
13. $16a^6 + 24a^5 + 25a^4 + 20a^3 + 10a^2 + 4a + 1.$
14. $x^6y^6 + 2x^5y^5 + 3x^4y^4 + 4x^3y^3 + 3x^2y^2 + 2xy + 1.$
15. $1 + 2x + 3x^2 + 4x^3 + 5x^4 + 4x^5 + 3x^6 + 2x^7 + x^8.$

SQUARE ROOTS OF NUMBERS EXPRESSED IN ARABIC FIGURES

173. The square root of a number expressed in Arabic figures may be found by the process just used for polynomials.

Illustrative Example. Find the square root of 405769.

Solution.

Square Root	Square
$a + b + c$	
$600 + 30 + 7 = 637$	405769
$a^2 = \overline{600}^2$	360000
$2ab = 36000$	45769
$b^2 = 900$	
36900	36900
	8869
$2(a+b)c = 8820$	
$c^2 = 49$	
8869	8869
	0

Explanation. We see that the given number is greater than the square of 600 and less than the square of 700. Hence we take 600 as the first term of the root and subtract $\overline{600}^2$ from the number.

As in the case of the polynomial the second term of the root is found by dividing the remainder by twice the first term of the root, that is 45769 ÷ 1200. This gives a quotient greater than 30 and less than 40 and hence 30 is the next term of the root.

The third term of the root is found by dividing 8869 by 2 × 630, that is, by $2(a + b)$. In the case of a polynomial it is sufficient to divide by $2a$, but with arithmetical numbers the third term is usually more easily found if we divide by $2(a + b)$.

The remaining parts corresponding to $(a + b)^2$ and $(a + b + c)^2$ are now computed and subtracted exactly as in the case of a polynomial.

174. The first term of the root may be found as follows:

Separate the number into groups of two digits each from the decimal point toward the left. Take the square root of the largest square contained in the last group to the left and adjoin one zero for each remaining group.

Thus in the root of 87 23 56 the first term is 900, since 81 is the largest square in the left group and there are two other groups. In the root of 7 34 86.593, the first term is 200, since 4 is the largest square contained in the left group and there are two other groups

EXERCISES

Find the square root of each of the following:

1. 294,849.	**5.** 3481.	**9.** 100,489.	**13.** 357.21.
2. 37,636.	**6.** 7569.	**10.** 265.69.	**14.** 16,641.
3. 872,356.	**7.** 1849.	**11.** 87.4225.	**15.** 32,761.
4. 599,076.	**8.** 73,441.	**12.** 170,569.	**16.** 2332.89.

175. In case the number whose square root is to be found has no figure to the left of the decimal point, the first term of the root may be found by the following rule:

Separate the number into groups of two digits from the decimal point toward the right. If necessary, add a zero to get one complete group not all zeros. Take the square root of the largest square contained in the first group which is not all zeros and prefix to it as many zeros as there are groups preceding this one, thus locating the decimal point in the root.

Thus, in the root of .03 42 the first term is .1, since 1 is the largest square in 3. Similarly in the square root of .00 70 the first term is .08, since 64 is the largest square in 70 and there is one group preceding this one.

Illustrative Example. Find the square root of .06784.

Solution.

Square Root	Square
$a + b \ + c$	
$.2 + .06 + .0004 = .2604$	.06783
$a^2 =$	.04
$2\,ab = .024$	.02783
$b^2 = .0036$	
.0276	.0276
$2(a+b)\,c = .000208$	.00023
$c^2 = .00000016$	
.00020816	.00020816
	.00002194

Explanation. According to the rule, .2 is the first term of the root since 4 is the largest square in 6 and there is no group preceding .06. The process is the same as in the case of an integral square, but special care is now needed in handling the decimal points, which is done exactly as in operations upon decimals in arithmetic.

For instance, in finding the third term in this example, we divide .00023 by $2(.26) = .52$ and the quotient lies between .0004 and .0005. Hence $c = .0004$.

176. Evidently the process in this example may be carried on indefinitely. .2604 is an **approximation** to the square root of .06783. In fact, the square of .2604 differs from .06783 by only .00002184. .260 is the nearest approximation using three decimal places. If the fourth figure were 5, or any digit greater than 5, then .261 would be the nearest approximation using three decimal places. Hence, four places must be found in order to be sure of the nearest approximation to three places.

EXERCISES

Find the square roots of the following, correct to two decimal places:

1. 387.	7. 2.	13. .02.
2. 5276.	8. 3.	14. .003.
3. 2.92.	9. 5.	15. .5.
4. 27.29.	10. 7.	16. .005.
5. 51.	11. 8.	17. .307.
6. 3.824.	12. 11.	18. 200.002.

SQUARE ROOTS OF FRACTIONS

177. A **fraction is squared** by squaring its numerator and its denominator separately, since $\frac{a}{b} \times \frac{a}{b} = \frac{a^2}{b^2}$. Hence, to extract the square root of a fraction, we find the square root of its numerator and its denominator separately.

E.g. $\sqrt{\frac{16}{25}} = \frac{4}{5}$, since $\frac{4}{5} \times \frac{4}{5} = \frac{16}{25}$.

However, in approximating the square root of a fraction whose denominator is not a perfect square, the fraction may be reduced to a decimal before the root is approximated.

E.g. $\sqrt{\frac{2}{3}} = \sqrt{.666 \cdots} = .8165 \cdots.$

It is also sometimes convenient to make the **denominator of a fraction a perfect** square before approximating the root.

E.g. $\sqrt{\frac{2}{3}} = \sqrt{\frac{6}{9}} = \sqrt{\frac{1}{9} \cdot 6} = \sqrt{\frac{1}{9}} \cdot \sqrt{6} = \frac{1}{3}\sqrt{6} = \frac{2.4495}{3} = .8165 \cdots.$

Note that Principle XVIII is used in taking the step,

$$\sqrt{\tfrac{1}{9} \cdot 6} = \sqrt{\tfrac{1}{9}} \cdot \sqrt{6}.$$

It is clear that *any fraction can be changed into an equal fraction whose denominator is a perfect square by multiplying numerator and denominator by the proper number.*

E.g. $\frac{1}{2} = \frac{2}{4}$, $\frac{3}{5} = \frac{15}{25}$, $\frac{7}{8} = \frac{14}{16}$, etc.

178. **Example.** Find the square root of $\frac{2}{3}$ by finding the roots of 2 and 3 separately and then dividing to reduce to a decimal.

Compare this with the two methods given above and show in what respect they are simpler.

179. If it is required to approximate the value of $\frac{1}{\sqrt{5}}$, the simplest method is as follows:

$$\frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5}\sqrt{5}} = \frac{\sqrt{5}}{5}. \quad \text{Similarly, } \frac{3}{\sqrt{5}} = \frac{3\sqrt{5}}{\sqrt{5}\sqrt{5}} = \frac{3\sqrt{5}}{5}.$$

Note that $\sqrt{5}\,\sqrt{5} = 5$ by Principle XVIII read in reverse order.

Thus, $\sqrt{5}\,\sqrt{5} = \sqrt{25} = 5$.

EXERCISES

State three rules for finding the square root of a fraction.

Find approximately correct to two decimal places the following square roots, using the method which involves the least computation.

1. $\sqrt{\frac{3}{5}}$.
2. $\sqrt{\frac{4}{8}}$.
3. $\sqrt{\frac{5}{7}}$.
4. $\sqrt{\frac{11}{13}}$.
5. $\sqrt{\frac{5}{3}}$.
6. $\sqrt{\frac{4}{5}}$.
7. $\sqrt{\frac{31}{8}}$.
8. $\sqrt{\frac{42}{51}}$.
9. $\sqrt{\frac{6}{7}}$.
10. $\sqrt{\frac{5}{6}}$.
11. $\sqrt{\frac{25}{4}}$.
12. $\sqrt{\frac{16}{7}}$.
13. $\sqrt{\frac{1}{2}}$.
14. $\sqrt{\frac{3}{7}}$.
15. $\sqrt{\frac{1}{7}}$.
16. $\sqrt{\frac{2}{5}}$.
17. $\frac{1}{\sqrt{5}}$.
18. $\frac{5}{\sqrt{13}}$.
19. $\frac{3}{\sqrt{7}}$.
20. $\frac{7}{\sqrt{17}}$.
21. $\sqrt{\frac{3}{8}}$.

SIMPLIFYING RADICALS

180. Principle XVIII may also be used to advantage in approximating the square roots of certain integral numbers.

E.g. Suppose $\sqrt{2}$ has been computed, and $\sqrt{8}$ is desired. It is unnecessary to compute the $\sqrt{8}$ directly, for by XVIII,

$$\sqrt{8} = \sqrt{4 \cdot 2} = \sqrt{4} \cdot \sqrt{2} = 2\sqrt{2}.$$

This sort of simplification is possible whenever the number under the radical sign can be resolved into two factors, one of which is a perfect square.

E.g. Suppose $\sqrt{5}$ to have been computed, then

$$\sqrt{125} = \sqrt{25 \cdot 5} = \sqrt{25} \cdot \sqrt{5} = 5\sqrt{5}.$$

In like manner, $\sqrt{a^5b^3}$ may be written

$$\sqrt{a^4b^2 \cdot ab} = \sqrt{a^4b^2} \cdot \sqrt{ab} = a^2b\sqrt{ab}.$$

181. Definition. An expression in one of the forms $\sqrt{a^2b}$, $\frac{a}{\sqrt{b}}$, $\sqrt{\frac{a}{b}}$, is said to be **simplified** when it is reduced so that no radical occurs in a denominator and when the number under the radical sign is in the integral form and contains no factor which is a perfect square.

Such radical expressions may always be simplified by Principle XVIII.

E.g.

$$\sqrt{125} = \sqrt{25 \cdot 5} = 5\sqrt{5}$$

$$\sqrt{a^5b^3} = \sqrt{a^4b^2 \cdot ab} = a^2b\sqrt{ab}$$

$$\sqrt{\tfrac{1}{5}} = \sqrt{\tfrac{1}{25} \cdot 5} = \tfrac{1}{5}\sqrt{5}$$

$$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{\sqrt{3} \cdot \sqrt{3}} = \frac{\sqrt{3}}{3}$$

EXERCISES

Given $\sqrt{2} = 1.4142$, $\sqrt{3} = 1.7321$, $\sqrt{5} = 2.2361$, compute the following, correct to three places of decimals, without further extraction of roots:

1. $\sqrt{80}$.
2. $\sqrt{\frac{1}{3}}$.
3. $\sqrt{\frac{1}{5}}$.
4. $\sqrt{48}$.
5. $\sqrt{75}$.
6. $\sqrt{2 \cdot 3}$.
7. $\sqrt{72}$.
8. $\sqrt{98}$.
9. $\sqrt{363}$.
10. $\sqrt{125}$.
11. $\sqrt{27} + \sqrt{\frac{1}{3}}$.
12. $\sqrt{45} + \sqrt{\frac{1}{5}}$.
13. $\sqrt{50} - \sqrt{\frac{1}{2}} + \sqrt{8}$.
14. $\sqrt{48} + \sqrt{75} - \sqrt{3}$.
15. $\sqrt{32} + \sqrt{72} - \sqrt{18}$.

Simplify the following:

16. $\sqrt{32\,a^2b}$.
17. $\sqrt{81\,x^3b^2}$.
18. $\sqrt{50\,a^5b^4c^2}$.
19. $\sqrt{45\,x^3y^5b^3}$.
20. $\sqrt{63\,bc^5d^4}$.
21. $\sqrt{900\,ab^4c^5}$.
22. $\sqrt{500\,x^7a^3b}$.
23. $\sqrt{3\,x^2+6\,xy+3\,y^2}$.
24. $\sqrt{8\,x^2-12\,y^2}$.
25. $\sqrt{32\,a^2-64\,ab+32\,b^2}$.
26. $\sqrt{125\,x^2+250\,xy+125\,y^2}$.

27. Find approximately to two decimal places the sides of a square whose area is 120.

28. Approximate to two decimals the side of a square having an area equal to that of a rectangle whose sides are 15 and 20.

29. How many rods of fence are required to fence a square piece of land containing 50 acres, each acre containing 160 square rods?

30. A square checkerboard has an area of 324 square inches. What are its dimensions?

In adding or subtracting expressions containing radicals it is always best to first reduce each radical expression to its simplest form, since this often gives opportunity to combine terms which are similar with respect to some radical expression.

Ex. 1. $\sqrt{32}+\sqrt{72}-\sqrt{18}=4\sqrt{2}+6\sqrt{2}-3\sqrt{2}=7\sqrt{2}$ by Principles XVIII and I.

Ex. 2. $\sqrt{\frac{1}{3}}+\sqrt{12}-\sqrt{\frac{3}{4}}=\frac{1}{3}\sqrt{3}+2\sqrt{3}-\frac{1}{2}\sqrt{3}$
$$=(\tfrac{1}{3}+2-\tfrac{1}{2})\sqrt{3}=1\tfrac{5}{6}\sqrt{3}.$$

EXERCISES

Simplify each of the following as far as possible without approximating roots.

1. $\sqrt{27}+2\sqrt{48}-3\sqrt{75}$.
2. $\sqrt{20}+\sqrt{125}-\sqrt{180}$.
3. $3\sqrt{432}-4\sqrt{3}+\sqrt{147}$.
4. $3\sqrt{2450}-25\sqrt{2}+4\sqrt{13122}$.

5. $3y^2\sqrt{x^3z} + 2\sqrt{x^5z^3} - yz^4\sqrt{\frac{xz}{y^2}}$.

6. $\sqrt{4x^3y} + \sqrt{25xy^3} - x\sqrt{xy}$.

7. $\sqrt{ax^2 - bx^2} + \sqrt{4ar^2s^2 - 4br^2s^2}$.

8. $4\sqrt{\frac{3}{4}} - \frac{3}{4}\sqrt{\frac{3}{16}} - 3\sqrt{27}$.

9. $2\sqrt{\frac{5}{3}} + \sqrt{60} + \sqrt{\frac{3}{5}}$.

10. $5\sqrt{3} - 2\sqrt{48} + 7\sqrt{108}$.

11. $\sqrt{a^3 - a^2b} - \sqrt{ab^2 - b^3} - \sqrt{(a+b)(a^2 - b^2)}$.

12. $\sqrt{a} + 3\sqrt{2a} - 2\sqrt{3a} + \sqrt{4a} - \sqrt{8a} + \sqrt{12a}$.

13. $\sqrt{x^3 + 2x^2y + xy^2} - \sqrt{x^3 - 2x^2y + xy^2} - \sqrt{4xy^2}$.

14. $\sqrt{r - s} + \sqrt{16r - 16s} + \sqrt{rt^2 - st^2} - \sqrt{9(r - s)}$.

15. $\sqrt{(m-n)^2a} + \sqrt{(m+n)^2a} - \sqrt{am^2} + \sqrt{a(1-m)^2} - \sqrt{a}$.

16. $\sqrt{32x^2y^4} + \sqrt{162x^2y^4} - \sqrt{512x^2y^4} + \sqrt{1250x^2y^4}$.

EQUATIONS SOLVED BY SQUARE ROOTS

182. Since $2^2 = 4$ and also $(-2)^2 = 4$, it follows that the equation $x^2 = 4$ has two roots, namely $x = 2$ and $x = -2$. These are usually written $x = \pm 2$.

This solution is obtained by taking the square root of both sides, which is equivalent to dividing both sides by the same number.

This operation may now be added to those enumerated in Principle VI for the solution of equations.

EXERCISES

Find all roots of the following equations:

1. $x^2 = 9$.
2. $x^2 = 25$.
3. $x^2 = 16a^2$.
4. $x^2 = 49b^2$.
5. $x^2 = 81a^4b^2$.
6. $x^2 = 64am^6n^4a^2$.
7. $x^2 = 36r^2s^4$.
8. $x^2 = 81s^8r^6$.
9. $x^2 = 625a^4b^8$.
10. $x^2 = 72$.
11. $x^2 = 98$.
12. $x^2 = 80$.
13. $x^2 = 175$.
14. $x^2 = 49a^5$.
15. $x^2 = 36a^3b^5$.
16. $x^2 = 25(a - b)^2$.
17. $x^2 = 50(a + b)$.
18. $x^2 = 200(a + b)^3$.
19. $x^2 = 1250(a - b)^3c^5$.

DRILL EXERCISES

Using the formulas for $(a \pm b)^2$, obtain the squares of the following:

1. $(a+b)+(c-d)$.
2. $(a+3)-(b+c)$.
3. $(3a-2b)+5$.
4. $7x-(4r-s)$.
5. $(m^2-3)-2(m^3+n)$.
6. $3(2+y)-2(3+x)$.

Factor the following:

7. $(2-x)^2-2(2-x)(x-1)+(x-1)^2$.
8. $(2+y)^2+2(2+y)(1+x)+(1+x)^2$.
9. $(3a-2b)^2-10(3a-2b)+25$.
10. $(6a-b)^2+(2a+1)^2-2(6a-b)(2a+1)$.
11. $25(a+b)^2+50(a+b)(a-b)+25(a-b)^2$.
12. $x^2+12x(a+b+c)+36(a+b+c)^2$.
13. $49(m-3)^4+36(m+1)^6-84(m-3)^2(m+1)^3$.
14. $16(x-y)^2-16(x-y)(x+y)+4(x+y)^2$.
15. $-30(a+b)(a-b)^2+25(a-b)^4+9(a+b)^2$.

Using the formula for $(a+b)(a-b)$, write out the following products:

16. $[a+b+(c-d)][a+b-(c-d)]$.
17. $[x+y+(u+v)][x+y-(u+v)]$.
18. $[4x-(a-2b)][4x+(a-2b)]$.
19. $[a+2b-(x-y^2)][a+2b+(x-y^2)]$.
20. $(11b^3x-3bx^3)(11b^3x+3bx^3)$.

Factor the following:

21. $a^2+4ab+4b^2-(x^2-2xy+y^2)$.
22. $(3x-2)^2-(4x^2+9y^2-12xy)$.
23. $x^2+4xy+4y^2-(a^2+2ab+b^2)$.
24. $16x^4y^2-(4x^2+9y^2+12xy)$.
25. $(a+b)^2-(4a^2+9b^2-12bc)$.

APPLICATIONS OF SQUARE ROOT

183. Some of the most interesting and useful applications of the square root process are concerned with the sides and areas of triangles.

The fact that the sum of the squares on the two sides of a right triangle equals the square on the hypotenuse was used in Chapter X. (Pythagorean Proposition, page 172.)

If a and b are the lengths of the sides, and c the length of the hypotenuse, all measured in the same unit, this proposition says:

$$c^2 = a^2 + b^2. \qquad (1)$$

Hence, by S,

$$a^2 = c^2 - b^2, \qquad (2)$$

and

$$b^2 = c^2 - a^2. \qquad (3)$$

Taking the square root of both sides in each of these equations,

$$c = \sqrt{a^2 + b^2}. \qquad (4)$$

$$a = \sqrt{c^2 - b^2}. \qquad (5)$$

$$b = \sqrt{c^2 - a^2}. \qquad (6)$$

The negative square root is omitted here, as a negative length cannot apply to the side of a triangle. By these formulas, if any two sides of a right triangle are given, the other may be found.

E.g. if $a = 4, b = 3$, then, by (4),

$$c = \sqrt{4^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25} = 5.$$

If $c = 5, b = 3$, then, by (5),

$$a = \sqrt{5^2 - 3^2} = \sqrt{25 - 9} = \sqrt{16} = 4.$$

If $c = 5, a = 4$, then, by (6),

$$b = \sqrt{5^2 - 4^2} = \sqrt{25 - 16} = \sqrt{9} = 3.$$

Illustrative Problem. If the two sides of a right triangle are 8 and 12, find the hypotenuse correct to two decimal places.

Solution. We have $c = \sqrt{a^2 + b^2} = \sqrt{64 + 144} = \sqrt{208}$,

$$\sqrt{208} = \sqrt{16 \cdot 13} = \sqrt{16} \cdot \sqrt{13} = 4\sqrt{13} = 4(3.605) = 14.420.$$

PROBLEMS

In solving the following problems, simplify each expression under the radical sign before extracting the root. Find all results correct to two decimal places.

1. The sides about the right angle of a right triangle are each 15 inches. Find the hypotenuse.

2. The hypotenuse of a right triangle is 9 inches and one of the sides is 6 inches. Find the other side.

3. The hypotenuse of a right triangle is 25 feet and one of the sides is 15 feet. Find the other side.

4. The hypotenuse of a right triangle is 7 rods and one of the sides is 5 rods. Find the other side.

5. The hypotenuse of a right triangle is 12 inches and the two sides are equal. Find their length.

Let s be the length of one of the equal sides.

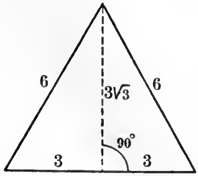
Then
$$s^2 + s^2 = 144.$$
$$2s^2 = 144.$$
$$s^2 = 72.$$
$$s = \sqrt{72} \doteq 6\sqrt{2} = 6 \times 1.414 = 8.484.$$

6. The hypotenuse of a right triangle is 30 feet and the sides are equal. Find their length.

7. The hypotenuse of a right triangle is h and the sides are equal. Find their length. Solve Exs. 5 and 6 by means of the formula here obtained.

8. The diagonal of a square is 8 feet. Find its area.

9. The diagonal of a square is d. Find an expression in terms of d representing its area.



10. The side of an equilateral triangle is 6 inches. Find the altitude.

A line drawn from a vertex of an equilateral triangle perpendicular to the base meets the base at its middle point. Hence this problem becomes: the hypotenuse of a right triangle is 6 and one side is 3. Find the remaining side.

11. The side of an equilateral triangle is 10. Find the altitude.

12. The side of an equilateral triangle is s. Find the altitude.

This is equivalent to finding a side of a right triangle whose hypotenuse is s, the other side being $\frac{s}{2}$. Let h equal altitude.

Then
$$h = \sqrt{s^2 - \left(\frac{s}{2}\right)^2} = \sqrt{s^2 - \frac{s^2}{4}}$$
$$= \sqrt{\frac{4\,s^2 - s^2}{4}} = \sqrt{\frac{3\,s^2}{4}} = \sqrt{\frac{s^2}{4}\cdot 3}$$
$$= \sqrt{\frac{s^2}{4}}\cdot\sqrt{3} = \frac{s}{2}\sqrt{3}.$$

This formula gives the altitude of any equilateral triangle in terms of the side. By means of this formula solve Exs. 11 and 12.

13. Find the altitude of an equilateral triangle whose side is $4\frac{1}{2}$. Substitute in the formula under Ex. 12.

14. Find the area of an equilateral triangle whose side is 5.

Since the area of a triangle is $\frac{1}{2}$ the product of the base and altitude, we first find the altitude by means of the formula under Ex. 12, and then multiply by $\frac{1}{2}$ the base.

15. Find the area of the equilateral triangle whose side is s. Show the result to be $\frac{s^2}{4}\sqrt{3}$.

16. If the area of an equilateral triangle is 16 square inches, find the length of the side.

Let s equal the length of the side. Then by the formula derived in Ex. 15, we have $16 = \frac{s^2}{4}\sqrt{3}$.

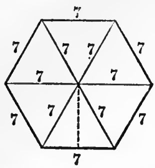
Hence (§§ 177–180), $s^2 = \frac{64}{\sqrt{3}} = \frac{64}{3}\sqrt{3} = 21.33 \times 1.732.$

17. The area of an equilateral triangle is 50 square inches. Find its side and altitude.

18. The area of an equilaterial triangle is a square inches. Find the side.

Solve the equation $a = \frac{s^2}{4}\sqrt{3}$ for s, and simplify the expression, finding $s^2 = \frac{4a}{\sqrt{3}}$, and $s = \sqrt{\frac{4a\sqrt{3}}{3}} = \frac{2}{3}\sqrt{3a\sqrt{3}}$.

19. The area of an equilateral triangle is 240 square inches. Find its side. (Substitute in the formula obtained in Ex. 18.)



20. Find the area of a regular hexagon whose side is 7.

A regular hexagon is composed of 6 equal equilateral triangles, whose sides are each equal to the side of the hexagon (see figure). Hence this problem may be solved by finding the area of an equilateral triangle whose side is 7, and multiplying the result by 6.

21. Find the area of a regular hexagon whose side is s. Solve Ex. 20 by substituting in the formula obtained here.

22. The area of a regular hexagon is 108 square inches. Find its side.

If the area of the hexagon is 108 square inches, the area of one of the equilateral triangles is 18 square inches. Hence this problem can be solved like Ex. 18.

23. The area of a regular hexagon is a square inches. Find its side. Solve Ex. 22 by substituting in the formula obtained here. *Ans.* $s = \frac{1}{3}\sqrt{2a\sqrt{3}}$.

24. Find the radius of a circle whose area is 9 square inches.

The area of a circle is found by squaring the radius and multiplying by 3.1416. The number 3.1416 is approximately the quotient obtained by dividing the length of the circumference by the diameter of the circle. This quotient is represented by the Greek letter π (pronounced pi). In this chapter we use $3\frac{1}{7}$ as an approximation to π. This differs from the real value of π by less than .0013, and hence is accurate enough for most purposes. If a represents the area of a circle, the above rule may be written

$$a = \pi r^2.$$

Hence if $a = 9$, $$r^2 = \frac{9}{\pi} = \frac{9}{3\frac{1}{7}} = \frac{63}{22} = 2.863,$$

and $$r = \sqrt{2.863}.$$

25. Find the radius of a circle whose area is 68 square feet.

26. Find the radius of a circle whose area is a square feet.

We have $$a = \pi r^2, \text{ or } r^2 = \frac{a}{\pi}.$$

Hence $$r = \sqrt{\frac{a}{\pi}} = \sqrt{\frac{a\pi}{\pi^2}} = \frac{1}{\pi}\sqrt{a\pi}.$$

In problems stated in terms of letters, the results, of course, cannot be reduced to a decimal. In such formulas it is best not to replace the letter π by any of its approximations.

FURTHER OPERATIONS ON RADICALS

184. The **radical sign** is used to indicate other roots than square roots by means of an *index* figure.

Thus, the *cube root of* 8, or one of its three equal factors, is written $\sqrt[3]{8} = 2$. The *fourth root* of 16, or one of its four equal factors, is written $\sqrt[4]{16} = 2$.

185. Definitions. Any expression which contains an indicated root is called a **radical expression.**

Integers and fractions of the form $\frac{m}{n}$ where m and n are integers are called **rational numbers.**

E.g. $2 + \sqrt[3]{5}$ is a radical expression. $5, \frac{2}{3}, \frac{75}{100}$, are rational numbers.

186. An expression which consists of a rational number under a radical sign, or one which can be reduced to this form, is called a **surd**, provided the whole expression is not reducible to a rational number.

E.g. $\sqrt{2}$ is a surd since it cannot be reduced to a rational number. $\sqrt[3]{4}, \sqrt[5]{3}$ are surds for the same reason. $\sqrt{9}$ is not a surd since $\sqrt{9} = 3$. $\sqrt{2 + \sqrt{2}}$ is not a surd since $2 + \sqrt{2}$ is not a rational number.

187. Surd expressions containing indicated square roots only, and in which no number is under more than one radical sign, are called **quadratic surds.**

E.g. $\sqrt{2}+\sqrt{3}$, $3+\sqrt{5}$, $\dfrac{3}{\sqrt{7}-\sqrt{5}}$, are quadratic surds, while $\sqrt{\sqrt{2}}$ and $\sqrt[3]{4}$ are not quadratic surds.

The following operations upon radicals deal only with quadratic surds.

MULTIPLICATION OF QUADRATIC SURDS

188. By Principle XVIII $\sqrt{ab}=\sqrt{a}\cdot\sqrt{b}$.

This equation read in the reverse order gives a rule for multiplying quadratic surds.

E.g. $\sqrt{2}\cdot\sqrt{8}=\sqrt{2\cdot 8}=\sqrt{16}=4$.

$\sqrt{3}\cdot\sqrt{15}=\sqrt{45}=\sqrt{9\cdot 5}=3\sqrt{5}$.

EXERCISES

Make a rule for multiplying quadratic surds.

Perform the following indicated operations and reduce each result to the simplest form.

1. $\sqrt{5}\cdot\sqrt{7}$.
2. $\sqrt{6}\cdot\sqrt{12}$.
3. $\sqrt{18}\cdot\sqrt{8}$.
4. $\sqrt{x^3}\cdot\sqrt{x^5}$.
5. $\sqrt{a^2b}\cdot\sqrt{ab^2}$.
6. $\sqrt{rs^3t}\cdot\sqrt{r^3s^3t^3}$.
7. $\sqrt{7}\cdot\sqrt{\frac{1}{7}}$.
8. $\sqrt{\frac{2}{3}}\cdot\sqrt{\frac{1}{3}}$.
9. $\sqrt{\frac{3}{5}}\cdot\sqrt{\frac{6}{5}}$.
10. $(\sqrt{2}+\sqrt{3})^2$.

Solution. $(\sqrt{2}+\sqrt{3})(\sqrt{2}+\sqrt{3})=(\sqrt{2})^2+2\sqrt{2}\sqrt{3}+(\sqrt{3})^2=4+2\sqrt{6}+3=7+2\sqrt{6}$.

11. $(\sqrt{2}-\sqrt{3})^2$.
12. $(1+\sqrt{2})^2$.
13. $(2-\sqrt{3})^2$.
14. $(1+\sqrt{2})(1-\sqrt{2})$.
15. $(\sqrt{2}+\sqrt{5})(\sqrt{2}-\sqrt{5})$.
16. $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})$.
17. $\sqrt{2}(2\sqrt{3}+3\sqrt{8}-5\sqrt{6})$.
18. $\sqrt{abc}(\sqrt{a}+\sqrt{b}+\sqrt{c})$.
19. $(3\sqrt{5}-2\sqrt{3})(3\sqrt{5}+2\sqrt{3})$.
20. $(1+\sqrt{2})(1-\sqrt{3}+\sqrt{5})$.

DIVISION OF QUADRATIC SURDS

189. When a divisor is a quadratic surd, it is convenient to indicate the division in the form of a fraction and then to reduce the denominators to the rational form as in the following example:

$$\frac{\sqrt{5}}{\sqrt{5}-\sqrt{2}}=\frac{\sqrt{5}(\sqrt{5}+\sqrt{2})}{(\sqrt{5}-\sqrt{2})(\sqrt{5}+\sqrt{2})}=\frac{5+\sqrt{10}}{5-2}=\frac{5+\sqrt{10}}{3}.$$

If it is desired to compute the approximate value of $\sqrt{5}\div(\sqrt{5}-\sqrt{2})$, it obviously requires less numerical work to use the form $\frac{5+\sqrt{10}}{3}$ rather than $\frac{\sqrt{5}}{\sqrt{5}-\sqrt{2}}$; since the latter involves the extraction of two square roots and a long division, while the former requires the extraction of only one root and a short division.

190. The process indicated in the above example is called **rationalizing the denominator**, and the factor by which the terms of the fraction are multiplied is called the **rationalizing factor.**

If the denominator is of the form $\sqrt{x}$ or $a\sqrt{x}$, then $\sqrt{x}$ is the rationalizing factor, since $\sqrt{x}\cdot\sqrt{x}=x$.

If the denominator is of the form $\sqrt{x}+\sqrt{y}$, then $\sqrt{x}-\sqrt{y}$ is the rationalizing factor, since $(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})=x-y$.

Give a rationalizing factor of each of the following:

(1) $\sqrt{3x}$, (2) $\sqrt{2x^3b}$, (3) $\sqrt{8b^3c^5}$, (4) $\sqrt{a}+\sqrt{b}$,

(5) $\sqrt{a}-\sqrt{b}$, (6) $\sqrt{3a}+\sqrt{2b^3}$, (7) $\sqrt{7}-\sqrt{27}$.

191. To rationalize the denominator of any fraction, it is first necessary to find an expression which multiplied by the denominator of the fraction gives a rational product, and then to multiply both terms of the fraction by this expression.

EXERCISES

Rationalize the denominators of each of the following:

1. $\dfrac{1}{\sqrt{x}}$.

2. $\dfrac{x+1}{\sqrt{x}}$.

3. $\dfrac{1}{\sqrt{x+1}}$.

4. $\dfrac{\sqrt{x-1}}{\sqrt{x+1}}$.

5. $\dfrac{\sqrt{x}+\sqrt{y}}{\sqrt{xy}}$.

6. $\dfrac{\sqrt{2}+\sqrt{3}}{\sqrt{2}}$.

7. $\dfrac{\sqrt{8}-\sqrt{3}}{\sqrt{3}}$.

8. $\dfrac{3\sqrt{6}+9\sqrt{2}}{3\sqrt{2}}$.

9. $\dfrac{1}{\sqrt{2}-1}$.

10. $\dfrac{1}{\sqrt{5}-\sqrt{3}}$.

11. $\dfrac{1}{\sqrt{3}+\sqrt{5}}$.

12. $\dfrac{2}{\sqrt{7}-\sqrt{3}}$.

13. $\dfrac{2}{\sqrt{a}-1}$.

14. $\dfrac{1}{\sqrt{a}+\sqrt{b}}$.

15. $\dfrac{\sqrt{a}}{\sqrt{a}-\sqrt{b}}$.

16. $\dfrac{\sqrt{b}}{\sqrt{a}+\sqrt{b}}$.

17. $\dfrac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}$.

18. $\dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}$.

19. $\dfrac{1}{a\sqrt{x}+b\sqrt{y}}$.

20. $\dfrac{a+b}{a\sqrt{x}-b\sqrt{y}}$.

EQUATIONS INVOLVING RADICALS

192. Illustrative Example. Solve for x the equation:

$$\sqrt{x-5}+\sqrt{x+1}=3 \qquad (1)$$

By $S\,|\sqrt{x+1}$ $\quad \sqrt{x-5}=3-\sqrt{x+1} \qquad (2)$

Squaring both sides $\quad x-5=9-6\sqrt{x+1}+x+1 \qquad (3)$

Simplifying $\quad 2\sqrt{x+1}=5 \qquad (4)$

Squaring both sides $\quad 4(x+1)=25 \qquad (5)$

Hence $\quad x=5\frac{1}{4} \qquad (6)$

Check. Substitute $\quad x=5\frac{1}{4}$ in (1)

If two radicals are involved or one radical and rational terms, it is best to get a radical alone on one side of the equation before squaring as in (2) and (4). Note that squaring both members of an equation is equivalent to multiplying both sides by the same number.

EXERCISES

Solve the following equations and check each result.

1. $\sqrt{x-5}=3.$
2. $x-2=\sqrt{x^2-4}.$
3. $\sqrt{x+6}=3\sqrt{x-2}.$
4. $\sqrt{x^2-2x+8}=x-4.$
5. $\sqrt{x^2+7x-4}=\sqrt{x^2+8x-5}.$
6. $\sqrt{x^2+5}+x=5.$
7. $\sqrt{4x^2-7}+2x=7.$
8. $\sqrt{x+1}-\sqrt{x-7}=2.$
9. $\sqrt{y+4}+\sqrt{y-1}=5.$
10. $\sqrt{x}+2=\sqrt{x+16}.$
11. $5-\sqrt{x}=\sqrt{x+5}.$
12. $\sqrt{x+7}=1+\sqrt{x+2}.$
13. $\dfrac{3(\sqrt{x}-2)}{\sqrt{x}-1}=\dfrac{\sqrt{x}+1}{\sqrt{x}+2}.$
14. $\dfrac{\sqrt{x-2}}{\sqrt{x-4}}=\dfrac{\sqrt{x+1}}{\sqrt{x-3}}.$
15. $\dfrac{1}{\sqrt{x}+1}+\dfrac{1}{\sqrt{x}-1}=\dfrac{4}{x-1}.$
16. $\dfrac{2-\sqrt{x}}{1+\sqrt{x}}=\dfrac{3+\sqrt{x}}{3-\sqrt{x}}.$

Suggestions. In Ex. 13 clear of fractions first; in 14 square both members and then clear of fractions; in 15 rationalize denominators.

REVIEW QUESTIONS

1. State Principle XVIII. Show by use of this principle how to find $\sqrt{28}$ having given $\sqrt{7}=2.696$.

2. Show how the value of the following may be approximated by finding only one square root.

$$5\sqrt{20}+2\sqrt{45}-3\sqrt{80}+2\sqrt{\tfrac{1}{5}}.$$

3. Write the square of $a+b+c+d$ in such a form as to derive from it the rule for finding the square root of a polynomial.

4. When is a radical expression said to be simplified? How is Principle XVIII used for this purpose?

5. Give examples of rational expressions, of surds, of quadratic surds. When is an expression said to be rationalized?

6. What principle is used in multiplying two quadratic surds? Find the product $(1+\sqrt{2})(2+3\sqrt{2})$.

7. How is division by a quadratic surd carried out? Simplify $2\sqrt{3}\div(\sqrt{2}-\sqrt{3})$; also $3\sqrt{5}\div(\sqrt{5}-\sqrt{2})$.

8. In solving an equation containing radicals, how are they removed (1) when only one radical expression is involved; (2) when two such expressions are involved?

9. State Principle XVIII in symbols and thus complete your list. Put them all on a single sheet for handy reference.

DRILL EXERCISES

1. Divide $a^3-12a^2+27a+40$ by $a-5$.

2. Divide $x^5-5x^4y+11x^3y^2-14x^2y^3+9xy^4-2y^5$ by $x^2-3xy+2y^2$.

3. Divide $x^4+x^2y^2+y^4$ by x^2-xy+y^2.

4. Divide $a^3+5a^2-2a-24$ by $a^2+7a+12$.

5. Divide $a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5$ by $a^2-2ab+b^2$.

6. Divide $x^5-5x^3y^2-5x^2y^3+y^5$ by $x^2-3xy+y^2$.

Factor:

7. a^3+8.	13. $1+64x^3$.	19. $1+125x^6$.
8. $27a^3+b^3$.	14. w^6+27a^6.	20. $27x^6-1$.
9. b^3-27.	15. w^6-8a^3.	21. $1-8x^3y^3$.
10. $8a^3-b^3$.	16. $27a^3-8b^3$.	22. $1+8x^3y^3$.
11. $1+64x^3$.	17. x^6+y^6.	23. $8x^3+27y^3$.
12. a^6-b^6.	18. $125a^3+b^3$.	24. $8x^3-27y^3$.

25. c^4-31c^2+220.

26. $ac+d^5a-b^4c-b^4d^5$.

27. $26+39n-22m-33mn$.

28. $12x^2+11x-56$.

29. $a^2+4ab+4b^2-(a^2-4ab+4b^2)$.

30. $(3x-1)^2-(x^2+4y^2-4xy)$.

31. $(x+3y)^2+(x-2y)^2+2(x+3y)(x-2y)$.

32. $16(a+b)^2-8(a-b)(a+b)+(a-b)^2$.

33. $256x^2-(49x^2+4y^4-28xy^2)$.

34. $(2x-a)^2+100(a-3x)^2+20(2x-a)(a-3x)$.

35. $-48(a-b)(a+b)+36(a-b)^2+16(a+b)^2$.

CHAPTER XII

QUADRATIC EQUATIONS

193. Equations of the form $x^2 + ax + b = 0$ have already been solved in cases where the left members could be factored by inspection. However, in a case like $x^2 + 5x + 3 = 0$, the factors of the left member cannot be found by any method thus far studied. It is therefore necessary to consider other methods for solving equations of this type.

194. As a preliminary step let us consider again the properties of a trinomial square.

How many terms of such a trinomial must be squares? How is the remaining term related to these? What term must be added to $a^2 + 2ab$ in order to make this a trinomial square?

The process of finding this third term is called **completing the square.** Since $2ab$ must be twice the product of the square roots of the squared terms, it follows that b *of the missing term may be found by dividing* $2ab$ *by twice* a.

EXERCISES

Complete the trinomial square in each of the following:

1. $x^2 + 2x$.
4. $x^2 + 8x$.
7. $(3x)^2 + 2(3x)$.
2. $x^2 + 4x$.
5. $x^2 + 3x$.
8. $(2x)^2 + 4(2x)$.
3. $x^2 + 6x$.
6. $x^2 + 5x$.
9. $16x^2 + 2(4x)$.

10. How do you complete the square in $a^2 - 2ab$? Is the rule different in this case?

11. Complete the square in each of the above exercises, first replacing the sign $+$ by $-$.

195. Solution of a quadratic by completing the square.

Ex. 1. Solve the equation:

$$x^2 + 6x + 4 = 0. \qquad (1)$$

By $S \mid 4$, $\qquad x^2 + 6x = -4. \qquad (2)$

Adding 3^2 to both members to complete the square,

$$x^2 + 6x + 3^2 = 3^2 - 4 = 5. \qquad (3)$$

Taking square root of both sides, $x + 3 = \pm\sqrt{5}. \qquad (4)$

Hence $\qquad x = -3 + \sqrt{5} = -3 + 2.24 = -.76,$

and . $\qquad x = -3 - \sqrt{5} = -3 - 2.24 = -5.24.$

Ex 2. Solve the equation:

$$x^2 - 12x + 42 = 56. \qquad (1)$$

By S, $\qquad x^2 - 12x = 14. \qquad (2)$

Completing the square, $x^2 - 12x + 36 = 14 + 36 = 50. \qquad (3)$

Taking square roots, $\qquad x - 6 = \pm\sqrt{50} = \pm 5\sqrt{2}. \qquad (4)$

By A, $\qquad x = 6 \pm 7.071. \qquad (5)$

Hence $\qquad x = 6 + 7.071 = 13.071,$

and also $\qquad x = 6 - 7.071 = -1.071.$

This process is called solving the quadratic equation by **completing the square.**

Make a rule for solving a quadratic equation by this process.

EXERCISES

In solving the following quadratic equations the result may in each case be reduced so that the number remaining under the radical sign shall be 2, 3, or 5. (§ 180.) Use $\sqrt{2}=1.414$, $\sqrt{3}=1.732$, $\sqrt{5}=2.236$.

1. $x^2 - 4x = 8.$	**6.** $x^2 - 12x = 12.$	**11.** $8 = x^2 + 4x.$
2. $x^2 = 3 - 6x.$	**7.** $x^2 - 8x = -14.$	**12.** $23 - 6x = x^2.$
3. $4x = 16 - x^2.$	**8.** $x^2 = 2x + 1.$	**13.** $7 + 2x = x^2.$
4. $x^2 + 6x = 9.$	**9.** $x^2 - 4x = 16.$	**14.** $25 - x^2 = 5x.$
5. $x^2 + 6x = 11.$	**10.** $x^2 = 24 + 4x.$	**15.** $x^2 + \frac{7}{3}x = 2.$
	16. $x^2 - \frac{3}{2}x = \frac{27}{2}.$	

196. The Hindu method of completing the square. In case the coefficient of x^2 is not unity, as in $3x^2+8x+4$, both members may be divided by this coefficient, and the solution is then like that of Exs. 1 and 2 above.

However, the following method is sometimes desirable:

$$3x^2 + 8x = 4. \tag{1}$$

Multiplying each member of (1) by $4 \cdot 3 = 12$,

$$36x^2 + 96x = 48. \tag{2}$$

Completing the square,

$$36x^2 + 96x + 8^2 = 48 + 64 = 112. \tag{3}$$

Taking square roots, $$6x + 8 = \pm\sqrt{112} = \pm 4\sqrt{7}. \tag{4}$$

Hence $$x = -\tfrac{4}{3} \pm 4\sqrt{7}. \tag{5}$$

The advantage of this form of solution is that fractions are avoided until the last step, and *the number added to complete the square is the square of the coefficient of x in the original equation.*

This is called the **Hindu method** of completing the square.

Note. Fractions would also be avoided in the above solution if equation (1) were multiplied by 3 instead of $4 \cdot 3$. This is the case only when the coefficient of x is an *even* number.

EXERCISES

In the solution of the following equations the roots which contain surds may be left in simplified radical form.

1. $2x^2+3x=2.$
2. $3x^2+5x=2.$
3. $3x=9-2x^2.$
4. $6x+1=-3x^2.$
5. $2x^2=5x+3.$
6. $4x=2x^2-1.$
7. $2x^2-3x=14.$
8. $3x^2=9+2x.$
9. $4x^2=2x+1.$
10. $6x-1=3x^2.$
11. $2x^2+4x=23.$
12. $3x^2-7=4x.$
13. $2x^2-5=3x.$
14. $4x^2=6x-1.$
15. $2x=1-5x^2.$
16. $3x-20=-2x^2.$
17. $2x+3x^2=9.$
18. $4x^2-1=3x.$
19. $4x=7-2x^2.$
20. $2x+1=5x^2.$
21. $3x^2+4x=7.$
22. $3x+9=2x^2.$
23. $2x-1=-4x^2.$
24. $5x^2+16x=-2.$

CHECKING RESULTS IN QUADRATIC EQUATIONS

197. Illustrative Examples. 1. Solving $x^2 - 7x + 12 = 0$, we get $x = 4$ and $x = 3$.

What is the *sum* of these roots? How does this compare with the coefficient of x? What is the *product* of these roots? How does this compare with the known term of the equation?

2. Solving $x^2 - 6x + 4 = 0$, we get $x = 3 + \sqrt{5}$ and $x = 3 - \sqrt{5}$.

What is the *sum* of these roots? How does this sum compare with the coefficient of x? What is the product of these roots? How does this *product* compare with the known term of the equation?

Solve the following equations. In each case compare the product of the roots with the known term and the sum of the roots with the coefficient of x.

1. $x^2 - 5x + 3 = 0$.
2. $x^2 + 3x + 2 = 0$.
3. $x^2 + 9x + 8 = 0$.
4. $x^2 - 4x - 8 = 0$.
5. $x^2 + 6x - 3 = 0$.
6. $x^2 - 8x = 6$.

198. These exercises are illustrations of a general rule for all quadratics written in the form $x^2 + px + q = 0$, in which the coefficient of the squared term is $+1$, and all terms are transposed to the left member; namely:

The sum of the roots is equal to the coefficient of x *with its sign changed and the product of the roots is equal to the known term.*

This may be used to check the results obtained in solving a quadratic. Note in particular that *before applying the test* you must put the equation into the specified form.

Note that in the radical form the product of the roots is *exactly* the known term, but when the roots are approximated in the decimal form, then their product is only *approximately* equal to the known term.

EXERCISES

Solve the following equations and check all results by means of § 198.

1. $4x^2+1=8x$.
2. $2x^2-3x=20$.
3. $2x^2-3=-5x$.
4. $3x^2+4x=8$.
5. $10-4x=5x^2$.
6. $1+4x^2=-6x$.
7. $5-3x=2x^2$.
8. $7+4x=2x^2$.
9. $6x^2+12x=2$.
10. $6x^2-12x=-2$.
11. $6x^2+12x=-2$.
12. $6x^2-12x=2$.
13. $3x^2+2x=5$.
14. $2+3x=2x^2$.
15. $8x+1=-4x^2$.
16. $8+4x=3x^2$.
17. $10+4x=5x^2$.
18. $2+5x=3x^2$.
19. $3x+14=2x^2$.
20. $3x^2-2x=5$.
21. $2x^2+4x=1$.
22. $4x^2+3x=1$.
23. $2x^2-4x=23$.
24. $2x^2-3x=1$.

SOLUTION OF THE QUADRATIC BY FORMULA

199. Solve the equation

$$ax^2+bx+c=0. \tag{1}$$

By *S*, *M*, $4a^2x^2+4abx=-4ac$. (2)

Completing the square, $4a^2x^2+4abx+b^2=b^2-4ac$. (3)

Taking square roots, $2ax+b=\pm\sqrt{b^2-4ac}$. (4)

By *S*, *D*, $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$. (5)

Calling the two values of x in the result x_1 and x_2 we have,

$$x_1=\frac{-b+\sqrt{b^2-4ac}}{2a};\quad x_2=\frac{-b-\sqrt{b^2-4ac}}{2a}.$$

Verify these results by means of § 198.

Any quadratic equation may be reduced to the form of (1) by simplifying and collecting the coefficients of x^2 and x. Hence any quadratic equation may be solved by substituting in the formulas just obtained.

EXAMPLE. Solve $2x^2 - 4x + 1 = 0.$

In this case $a = 2, b = -4, c = 1.$

Hence $x = \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 2 \cdot 1}}{2 \cdot 2}.$

From which $x_1 = \dfrac{2 + \sqrt{2}}{2}, \quad x_2 = \dfrac{2 - \sqrt{2}}{2}.$

Check the results by § 198.

200. A quadratic equation may be proposed for solution which has no roots expressible in terms of the numbers of arithmetic or algebra thus far studied.

EXAMPLE. Solve $x^2 + 4x = -8.$ (1)

By A, $x^2 + 4x + 4 = -4.$ (2)

Taking square roots, $x + 2 = \pm\sqrt{-4}.$ (3)

$\sqrt{-4}$ is unknown to us as a number symbol, since there is no number thus far considered whose square equals -4. (See Principle IX.) Such symbols are defined and used in the Advanced Course, and are called **imaginary numbers.** Any quadratic equation which gives rise to such a solution is to be interpreted as stating some condition not satisfied by any number so far studied.

EXERCISES

Leave the surds or imaginaries in simplified radical form.

1. $7 - 3x = 5x^2.$
2. $51x - 33 = 3x^2.$
3. $14x + 8 - x^2 = 52 - 3x^2.$
4. $12 - 51x = 36 + 6x^2.$
5. $5x + x^2 + 8 = 0.$
6. $5x^2 - 31x = -6.$
7. $x^2 - 8 + 3x = -15x.$
8. $11x^2 - 49x + 57 = 0.$
9. $3x^2 + 18 - 16x = 5.$
10. $37 - 4x^2 - 12x = 79 - 5x^2.$
11. $10x^2 + 41 + 7x = 44.$
12. $45 + 3x^2 - 85 - 2x = 0.$

MISCELLANEOUS QUADRATICS

Solve as many as possible of the following equations by factoring. When this is not convenient, use the formula of § 199, or complete the square independently in each case. It is important to check the results in the radical form as in § 198.

1. $x^2 + 11x = 210.$
2. $5x^2 - 3x = 4.$
3. $7x + 3x^2 - 18 = 0$
4. $2 = 5x + 7x^2.$
5. $6x - 11x^2 = -7.$
6. $-51 + 42x - 3x^2 = 0.$
7. $3x^2 + 3x = 2x + 4.$
8. $13 - 8x + 3x^2 = 0.$
9. $2x^2 + 11x = 32x - x^2 - 27.$
10. $176 + 3x - x^2 = 2x.$
11. $x^2 + 6x - 54 = 0.$
12. $5x^2 + 9x + 12 = 4x^2 + x.$
13. $2x^2 - 4x - 25 = 0.$
14. $7x^2 + 11x = 6.$
15. $2x^2 - 11x + 5 = 0.$
16. $2x^2 - 11x = 6.$
17. $25x - 95 = x^2.$
18. $11x^2 - 42x = 2.$
19. $x^2 - 8x - 4 = x - 22.$
20. $8x^2 + 5x = -8.$
21. $2x^2 + 3x - 3 = 12x + 2.$
22. $3x^2 - 7x = 10.$
23. $17x + 31 + 2x^2 = 0.$
24. $18 - 41x = 3 + x^2.$
25. $10x + 25 = 5 - 2x - x^2.$
26. $3x - 59 + x^2 = 0.$
27. $5x^2 + 7x - 6 = 0.$
28. $x^2 + 12 = 7x.$
29. $8x - 5x^2 = 2.$
30. $5x + 3x^2 - 22 = 0.$
31. $50 + 20x + x^2 = 5x.$
32. $x^2 + x + 4 = 0.$
33. $20x + 2x^2 + 42 = 33x + x^2.$
34. $17x - 3x^2 = -6.$
35. $8x + 5x^2 = -2.$
36. $10 + 15x + x^2 = 26x.$
37. $3x^2 - 2x - 7 = 0.$
38. $5x^2 - 9x - 18 = 0.$
39. $7x - 7x^2 + 24 = 0.$
40. $31 + 2x + x^2 = 0.$
41. $7x^2 + 7x - 5x^2 + 20 = x^2 - 2x + 2.$
42. $5x^2 + 3x - 7 = (x - 1)(x + 2).$
43. $(x - 3)^2 - (2x - 1)(2x + 1) + 7 = 0.$
44. $3x + (3x - 2)^2 = 4x^2 - 1.$
45. $9x^2 - (2x - 1)^2 = (x + 3)^2.$
46. $7x^2 = 5x - (x - 2)^2 + 7.$
47. $(3x - 2)(3x + 2) = (2x - 8)^2.$
48. $5x - 9x^2 + 8(x - x^2) = 4.$

DRILL EXERCISES

1. Divide $x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5$ by $x + y$.
2. Divide $x^6 - y^6$ by $x^2 + xy + y^2$.
3. Divide $x^8 - y^8$ by $x^3 + x^2y + xy^2 + y^3$.
4. Divide $x^9 + x^6y^6 + y^9$ by $x^3 + y^3$.

Find the square roots of:

5. $x^6 + 2x^4 + 2x^3 + x^2 + 2x + 1$.
6. $x^8 + 2x^6 + 3x^4 + 2x^2 + 1$.
7. $x^6 + 4x^5 + 10x^4 + 16x^3 + 17x^2 + 12x + 4$.
8. $4a^4 - 12a^3 - 7a^2 + 24a + 16$.

Rationalize the denominators of the following:

9. $\dfrac{3}{\sqrt{7} - \sqrt{4}}$.

10. $\dfrac{\sqrt{7} + \sqrt{2}}{\sqrt{7} - \sqrt{2}}$.

11. $\dfrac{a + b}{\sqrt{a} + \sqrt{b}}$.

12. $\dfrac{2 + \sqrt{3}}{4 - \sqrt{3}}$.

13. $\dfrac{\sqrt{8} + 7}{\sqrt{8} - 7}$.

14. $\dfrac{\frac{1}{3}\sqrt{2} - 1}{\frac{1}{3}\sqrt{2} + 1}$.

15. $\begin{cases} x + ay = c, \\ bx + y = d. \end{cases}$

16. $\begin{cases} ax + 3y = 2c, \\ bx - 2y = 3d. \end{cases}$

17. $\begin{cases} ax + by = 1, \\ cx + dy = 4. \end{cases}$

18. $\begin{cases} ax - by = c, \\ cx + dy = e. \end{cases}$

19. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = a, \\ \dfrac{1}{y} + \dfrac{1}{z} = b, \\ \dfrac{1}{z} + \dfrac{1}{x} = c. \end{cases}$

20. $\begin{cases} x + y + z = a, \\ 2x - 2y + 2z = b, \\ 3x - y - z = c. \end{cases}$

21. $\begin{cases} ax - y + bz = a, \\ x + ay - z = 1, \\ bx - y + az = b. \end{cases}$

22. If w and l are the length and width of a rectangle, express in symbols the length of its diagonal.

23. The lengths of the two sides of a right triangle are 16 and 24 respectively. Express the length of the hypotenuse in the simplest form without approximating a square root.

SYSTEMS INVOLVING QUADRATICS

201. The solution of two equations in two variables, one of which is linear and the other quadratic, can be reduced to the solution of a quadratic equation in one variable.

EXAMPLE. Solve
$$\begin{cases} x + y = 3, & (1) \\ 3x^2 - y^2 = 14. & (2) \end{cases}$$

From (1), $$y = 3 - x. \qquad (3)$$

Substituting in (2) and reducing,
$$2x^2 + 6x - 23 = 0. \qquad (4)$$

Substituting in the formula § 199,
$$x = \frac{-6 \pm \sqrt{36 - 4 \cdot 2(-23)}}{4} = \frac{-3 \pm \sqrt{55}}{2} \qquad (5)$$

Hence $$x_1 = 2.21 \text{ and } x_2 = -5.21.$$

Substituting these values of x in (1) we have as the approximate roots,
$$\left.\begin{matrix} x_1 = 2.21 \\ y_1 = 0.79 \end{matrix}\right\} \text{ and } \left.\begin{matrix} x_2 = -5.21 \\ y_2 = 8.21 \end{matrix}\right\}.$$

y_1 and y_2 are here used to designate the values of y, which correspond to x_1 and x_2 respectively.

EXERCISES

In the above manner solve the following systems of equations, finding in each case two pairs of roots. In the case of roots which are surds, find the approximate results to two places of decimals.

1. $\begin{cases} x - y = 1. \\ x^2 + y^2 = 13. \end{cases}$

2. $\begin{cases} x + y = 9. \\ x^2 + y^2 = 41. \end{cases}$

3. $\begin{cases} x + y = 13. \\ xy = 42. \end{cases}$

4. $\begin{cases} 3x - y = 5. \\ x^2 + y^2 = 25. \end{cases}$

5. $\begin{cases} x + 4y = 26. \\ x^2 - y^2 = 11. \end{cases}$

   Ans. $x_1 = 6, \ y_1 = 5.$
   $x_2 = -9\tfrac{7}{15}, \ y_2 = 8\tfrac{13}{15}.$

6. $\begin{cases} \dfrac{x^2}{9} - \dfrac{y^2}{4} = 3. \\ x - y = 4. \end{cases}$

7. $\begin{cases} x - y = 1. \\ \dfrac{x^2}{36} + \dfrac{y^2}{16} = \frac{1}{2}. \end{cases}$

8. $\begin{cases} 2x + y = 5. \\ 3x^2 - 5y^2 = 7. \end{cases}$

9. $\begin{cases} x - y = 3. \\ x^2 - 3y^2 = 13. \end{cases}$

10. $\begin{cases} x - 3y = 1. \\ y^2 + 2x^2 = 33. \end{cases}$

Ans. $x_1 = 4,\ y_1 = 1.$
$x_2 = -3\frac{17}{19},\ y_2 = -1\frac{12}{19}.$

11. $\begin{cases} 3x - 4y = 1. \\ x^2 - y^2 = 24. \end{cases}$

12. $\begin{cases} x + y = 4. \\ 2x^2 - 3xy + y^2 = 8. \end{cases}$

13. $\begin{cases} x - y = 1. \\ 4x^2 + 2xy - y^2 = 19. \end{cases}$

14. $\begin{cases} 5x + y = 12. \\ 2x^2 - 3xy + y^2 = 0. \end{cases}$

15. $\begin{cases} x - 2y = 3. \\ 2y^2 - x^2 = 4. \end{cases}$

Ans. $x_1 = -6.16,\ y_1 = -4.58.$
$x_2 = .16,\ y_2 = -1.42.$

16. $\begin{cases} x + y = 9. \\ x^2 - 2y^2 = -7. \end{cases}$

17. $\begin{cases} y - 2x = 5. \\ y^2 - 3xy = 16. \end{cases}$

Ans. $x_1 = 3.71,\ y_1 = 12.42.$
$x_2 = 1.21,\ y_2 = 2.58.$

18. $\begin{cases} 2y - 3x = 0. \\ y^2 + x^2 = 52. \end{cases}$

19. $\begin{cases} y - 2x = 5. \\ x^2 + y^2 = 40. \end{cases}$

20. $\begin{cases} x - 4y = 12. \\ 3x^2 + 2xy - 6y = 44. \end{cases}$

Ans. $x_1 = 4,\ y_1 = -2.$
$x_2 = -1\frac{6}{7},\ y_2 = -3\frac{13}{28}.$

PROBLEMS

In each problem find the two roots of the quadratic equation and determine whether both are applicable to the problem:

1. The area of a window is 2016 square inches and the perimeter of the frame is 180 inches. Find the dimensions of the window.

2. The area of a rectangular city block, including the sidewalk, is 19,200 square yards. The length of the sidewalk around the block when measured on the side next the street is 560 yards. Find the dimensions of the block.

3. A farmer starts to plow around a rectangular field which contains 48 acres. The length of the first furrow around the field is 376 rods. Find the dimensions of the field.

4. A rectangular blackboard contains 38 square feet and its perimeter is 27 feet. Find the dimensions of the board.

5. A park is 120 rods long and 80 rods wide. It is decided to double the area of the park, still keeping it rectangular, by adding strips of equal width to one end and one side. Find the width of the strips.

6. A fancy quilt is 72 inches long and 56 inches wide. It is decided to increase its area 10 square feet by adding a border. Find the width of the border.

7. A city block is 400 by 480 feet when measured to the outer edge of the sidewalk. At 4 cents per square foot it costs $416.64 to lay a sidewalk around the block. Find the width of the walk.

8. A farmer starts cutting grain around a field 120 rods long and 70 rods wide. How wide a strip must be cut to make 12 acres?

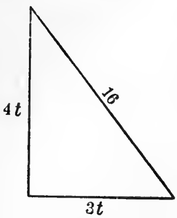
9. The sides of a right triangle are 6 and 8 inches respectively. How much must be added to each side so as to increase the hypotenuse 10 inches, it being understood that each side is increased by the same amount?

10. A rectangular lot is 16 by 12 rods. How wide a strip must be added to one end and one side to obtain a rectangular lot whose diagonal is 1 rod greater?

11. A picture is 15 inches by 20 inches. How wide a frame must be added to increase the diagonal 3 inches?

12. An athletic field is 800 feet long and 600 feet wide. The field is to be extended by the same amount in length and width so that the longest possible straight course (the diagonal) shall be increased by 100 feet. How much is the field extended in each direction? *Ans.* 71.36 feet.

13. A and B start from a certain cross roads at the same time, A going north 4 miles per hour and B going east 3 miles per hour. In how many hours will they be 16 miles apart, measuring in a straight line across country?



Let t equal the required number of hours.

Then $(4\,t)^2 + (3\,t)^2 = 16^2 = 256.$

$$16\,t^2 + 9\,t^2 = 256.$$

$$25\,t^2 = 256.$$

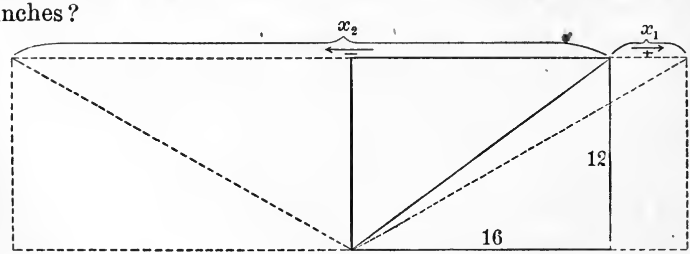
$$5\,t = \pm 16.$$

$$t = \pm 3\tfrac{1}{5}.$$

The solution $t = -3\frac{1}{5}$ may be interpreted as meaning that if the two men were traveling along these roads in the same direction before reaching the cross road, they would be 16 miles apart $3\frac{1}{5}$ hours before meeting.

14. In the preceding problem if A goes 5 miles per hour and B 4 miles per hour, in how many hours will they be 24 miles apart? *Ans.* 3.75.

15. A rectangle is 12 inches wide and 16 inches long. How much must be added to the length to increase the diagonal 4 inches?



Let x = number of inches to be added to the length. The diagonal of the original rectangle is $\sqrt{12^2 + 16^2} = 20$. Hence the diagonal of the required rectangle is 24.

Then $12^2 + (16 + x)^2 = 24^2,$

or $x^2 + 32\,x - 176 = 0.$

Solving, $x_1 = -16 + 12\sqrt{3} = 4.78,$

and $x_2 = -16 - 12\sqrt{3} = -36.78.$

The negative solution obtained here may be taken to mean that if the rectangle is extended in the *opposite* direction from the fixed corner, we shall get a rectangle which has the required diagonal. See the figure.

16. How much must the *width* of the rectangle in problem 15 be extended so as to increase the diagonal by 4?

17. A trunk 30 inches long is just large enough to permit an umbrella 36 inches long to lie diagonally on the bottom. How much must the length of the trunk be increased if it is to accommodate a gun 4 inches longer than the umbrella?

18. A rectangle is 21 inches long and 20 inches wide. The length of the rectangle is decreased twice as much as the width, thereby decreasing the length of the diagonal 4 inches. Find the dimensions of the new rectangle.

19. In a rectangular table cover 24 by 30 inches there are two strips of drawn work of equal width running at right angles through the center of the piece. What is the width of these strips if the drawn work covers one tenth of the whole piece?

20. A certain university campus is 100 rods long and 80 rods wide. There are two driveways running through the center of the campus at right angles to each other and parallel to the sides. What is the width of these driveways if their combined area is 356 square rods?

21. A farm is 320 rods long and 280 rods wide. There is a road 2 rods wide running around the boundary of the farm and lying entirely within it. There is also a road 2 rods wide running across the farm parallel to the ends. What is the area of the farm exclusive of the roads?

22. A rectangular park is 480 rods long and 360 rods wide. A walk is laid out completely around the park, and a drive through the length of the park parallel to the sides. What is the width of the walk if the drive is 3 times as wide as the walk and the combined area of the walk and the drive is 3110 square rods?

23. The sum of the sides of a right triangle is 18 and the length of the hypotenuse is 16. Find the length of each side.

24. The length of a fence around a rectangular athletic field is 1400 feet, and the longest straight track possible on the field is 500 feet. Find the dimensions of the field.

Using 100 feet for the unit of measure, the equations are

$$\begin{cases} x + y = 7, \\ x^2 + y^2 = 25. \end{cases}$$

25. The difference between the sides of a right triangle is 8 and the hypotenuse is 42. Find the lengths of the sides.

26. A room is 5 feet longer than it is wide, and the distance between two opposite corners is 25 feet. Find the length and width of the room.

27. One side of a right triangle is 8 feet, and the hypotenuse is 2 feet more than twice the other side. Find the length of the hypotenuse and of the remaining side.

28. A vacant corner lot has a 50-foot frontage on one street. What is the frontage on the other street if the distance between opposite corners along the diagonal is 110 feet less than twice this frontage.

29. The sum of the squares of two consecutive integers is 13,945. Find the numbers.

30. The product of two consecutive integers is 4422. Find the numbers.

31. A square piece of tin is made into an open box, containing 864 cubic inches, by cutting out a 6-inch square from each corner of the tin and then turning up the sides. Find the dimensions of the original piece of tin.

32. A rectangular piece of tin is 8 inches longer than it is wide. By cutting out a 7-inch square from each corner and turning up the sides, an open box containing 1260 cubic inches is formed. Find the dimensions of the original piece of tin.

33. By cutting out a square 8 inches on a side from each corner of a sheet of metal and turning up the sides, we obtain an open box such that the area of the sides and ends is 4 times the area of the bottom. Find the dimensions of the original sheet if it is twice as long as it is wide. *Ans.* 41.48 in. by 20.74 in.

34. An open box whose bottom is a square has a lateral area which is 400 square inches more than the area of the bottom. Find the other dimensions of the box if it is 10 inches high. (By lateral area is meant the sum of the areas of the four sides.)

35. A box whose bottom is 4 times as long as it is wide has a lateral area 600 square inches less than 4 times the area of the bottom. Find the dimensions of the bottom if the box is 6 inches high.

REVIEW QUESTIONS

1. Explain the method of solving a quadratic equation by factoring. Can you apply this method to solve the equation $(x+1)(x-2)=5$? Explain.

2. Explain the method of solving a quadratic equation by completing the square. What is the Hindu method and what are its advantages?

3. How many roots has a quadratic equation? Find the roots of $x^2+2x+5=0$. Are these roots in the form of any numbers thus far studied? What are such roots called? Solve $x^2+2x-5=0$. What are these roots called?

4. How are the roots of the equation $x^2+px+q=0$ related to p and q? How may this be used in checking the solutions?

5. Do both roots of a quadratic equation necessarily satisfy the conditions of the *problem* from which such an equation may be derived? In checking the solution of a problem is it sufficient to make the test alone in the equation derived from the problem?

DRILL EXERCISES

Simplify each of the following as much as possible without approximating roots:

1. $\sqrt{32}+\sqrt{72}-\sqrt{50}$.

2. $2a\sqrt{a^3b}-b\sqrt{b^3a}+\sqrt{ab}$.

3. $\sqrt{a^3+2a^2b+ab^2}+\sqrt{a^3-2a^2b+ab^2}-2\sqrt{a^3}$.

4. $\sqrt{a^3-a^2b}-\sqrt{b^2a-b^3}+\sqrt{a^3b^2-a^2b^3}$.

5. $\sqrt{(x+y)(x^2-y^2)}-\sqrt{(x+y)^2(x-y)}-\sqrt{x^3-x^2y}$.

Solve the following equations:

6. $\sqrt{x^2-8}+x=8$.

7. $\sqrt{5a-24}+4=\sqrt{5a}$.

8. $\sqrt{2a-1}=7-\sqrt{2a+6}$.

9. $\sqrt{x+2}=\sqrt{x-6}+2\sqrt{x-5}$.

Rationalize the denominators of the following:

10. $\dfrac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}}$.

11. $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$.

12. $\dfrac{a+\sqrt{b}}{a-\sqrt{b}}$.

13. $\dfrac{\sqrt{x-a}+b}{\sqrt{x+a}-c}$.

Solve:

14. $\begin{cases} 3ax-by=2, \\ 2x+3by=6. \end{cases}$

15. $\begin{cases} 2x-3by=c, \\ 2ax-5y=d. \end{cases}$

16. $\begin{cases} x-y-3z=-6, \\ 2x+y-z=11, \\ -x+3y+z=16. \end{cases}$

17. $\begin{cases} 2x-y+3z=20, \\ x+4y-z=-2, \\ 5x+y-6z=6. \end{cases}$

18. Divide $x^5-3x^4-18x^3+24x^2+52x-21$ by x^2+x-7.

19. Divide $6a^5+5a^4-60a^3+4a^2+71a+28$ by $3a^2-5a-4$.

Find the square roots of:

20. $16x^8-40x^6y^3+x^4y^6+30x^2y^9+9y^{12}$.

21. $4m^6-20m^5+41m^4+52m^3-14m^2-24m+9$.

CHAPTER XIII

ALGEBRAIC FRACTIONS

COMMON FACTORS

Simple algebraic fractions have already been studied in Chapter V, where they were treated exactly as fractions in arithmetic. This was sufficient for all our purposes up to this point. We now take up a **more formal** study of fractions.

202. If a number is a factor of each of two or more numbers, it is said to be a **common factor** of these numbers.

Thus, 8 is a common factor of 16 and 48, and 12 is a common factor of 12, 36, and 48.

If each of a given set of numbers is separated into prime factors, any common factor which they may have is at once apparent.

Illustrative Example. Find the common factors of

$10(x+y)^2(x-y)$, $5(x+y)(x^2-y^2)$, and $15(x+y)(x^3-y^3)$.

Factoring, $10(x+y)^2(x-y) = 2 \cdot 5(x+y)(x+y)(x-y)$.

$$5(x+y)(x^2-y^2) = 5(x+y)(x+y)(x-y).$$

$$15(x+y)(x^3-y^3) = 3 \cdot 5(x+y)(x-y)(x^2+xy+y^2).$$

The common *prime* factors are 5, $x+y$, and $x-y$. The other common factors, obtained by combining these, are $5(x+y)$, $5(x-y)$ and $5(x+y)(x-y)$. The last factor, $5(x+y)(x-y)$ is called the highest common factor.

The name *highest* instead of *greatest* is used in algebra referring to the number of prime factors which enter into it. Thus, x^2 is of higher degree than x, although if $x=\frac{1}{2}$, x^2 is not greater than x.

203. Definition. The product of all the common prime factors is called the **highest common factor**. This is usually abbreviated to H. C. F.

EXERCISES

Find the H. C. F. of the following sets of expressions:

1. $x - y,\ x^2 - y^2,\ x^2 - 2\,xy + y^2$.
2. $x^2 + 2\,x + 1,\ 3\,x + 6\,x^2 + 3\,x^3$.
3. $x^2 + 4\,x + 4,\ x^2 - 6\,x - 16$.
4. $x^2 - 8\,x + 16,\ x^2 + 10\,x - 56$.
5. $a^3 - b^3,\ a^2 - 2\,ab + b^2$.
6. $x^3 + y^3,\ x^2 - y^2,\ x^2 + 2\,xy + y^2$.
7. $x^2 - 7\,x + 12,\ ax - 3\,a - bx + 3\,b$.
8. $a^2 - 13\,a + 42,\ a^3 - 216,\ a^2 - a - 30$.
9. $27 + y^3,\ y^2 + 9\,y + 18,\ y^2 - 9$.
10. $b^2 + 7\,b - 30,\ b^2 + 11\,b - 42,\ b^2 - b - 6$.
11. $a^3 + 2\,a^2 + a,\ a^2 + a,\ a^3 + 5\,a^2 + 4\,a$.
12. $x^3 + y^3,\ x^3 + x^2y + xy^2 + y^3$.
13. $x^4 + 3\,x^3 + 2\,x^2,\ x^3 + x^2,\ x^4 + 7\,x^3 + 6\,x^2$.
14. $x^2 - 11\,x + 30,\ xz - 5\,z + x^2 - 5\,x$.
15. $m^3 - n^3,\ 2\,x^2m^2 + 2\,x^2mn + 2\,x^2n^2$.
16. $x^2 - 1,\ x^3 - 1,\ x^2 - 13\,x + 12$.
17. $1 - 64\,x^3,\ 1 - 16\,x^2,\ 5 - 2\,z - 20\,x + 8\,xz$.
18. $1 + 125\,a^3,\ 1 + 10\,a + 25\,a^2,\ 1 - 25\,a^2$.
19. $ac - ax + 3\,bc - 3\,bx,\ a^3 + 27\,b^3$.
20. $5\,c - 2,\ 5\,ac + 20\,c - 2\,a - 8$.
21. $4\,x^4 - x^2,\ 2\,x^4 + x^3 - x^2,\ 2\,x^4 - 3\,x^3 + x^2$.
22. $3\,a^3 - 3\,a,\ 3\,a^3 - 6\,a^2 + 3\,a,\ 6\,a^3 + 12\,a^2 - 15\,a$.
23. $6\,x - 10\,xy + 4\,xy^2,\ 18\,x - 8\,xy^2,\ 54\,x - 16\,xy^3$.
24. $3\,x^5 + 9\,x^4 - 3\,x^3,\ 5\,x^2y^2 + 15\,xy^2 - 5\,y^2,\ 7\,ax^2 + 21\,ax - 7\,a$.
25. $18\,x^3 - 57\,x^2 + 30\,x,\ 9\,x^3 - 15\,x^2 + 6\,x,\ 18\,x^3 - 39\,x^2 + 18\,x$.

COMMON MULTIPLES

204. A number is said to be a **multiple of any of its factors.** In particular any number is a multiple of itself and of one.

Thus, 18 is a multiple of 1, 2, 3, 6, 9, and 18, but not of 12. $3a^2x^2$ is a multiple of 3, $3x$, $3x^2$, etc.

Since a multiple of a number is divisible by that number, it must contain as a factor every factor of that number.

E.g., 108 is a multiple of 54 and contains as factors all the factors of 54, namely 3, 3, 3, and 2, and also 2, 6, 9, 18, and 54.

Definition. A number is a **common multiple** of two or more numbers if it is a multiple of each of them.

Thus, 18 is a common multiple of 6, 9, and 18. Evidently $3 \cdot 18$, $4 \cdot 18$, $5 \cdot 18$, etc. are also common multiples of 6, 9, and 18. Of all these common multiples 18 is called the *lowest* common multiple.

205. The process of finding the lowest common multiple of a set of expressions is shown as follows:

Illustrative Example. Find the lowest common multiple of

$$x^2 - y^2;\ x^2 + 2xy + y^2;\ \text{and } x^2 - 2xy + y^2.$$

Factoring,
$$x^2 - y^2 = (x - y)(x + y). \qquad (1)$$
$$x^2 + 2xy + y^2 = (x + y)(x + y). \qquad (2)$$
$$x^2 - 2xy + y^2 = (x - y)(x - y). \qquad (3)$$

In order that an expression may be a multiple of (1) it must contain the factors $x - y$ and $x + y$. To be a common multiple of (1) and (2) it must contain an additional factor $x + y$, that is, it must contain $(x - y)$, $(x + y)$, $(x + y)$. To be a common multiple of (1), (2), and (3) it must contain an additional factor $x - y$, that is, it must contain $(x - y)$, $(x + y)$, $(x + y)$, $(x - y)$. The product $(x - y)(x + y)(x + y)(x - y) = (x - y)^2(x + y)^2$ is called the lowest common multiple of (1), (2), and (3), since it is the common multiple which contains the smallest number of prime factors.

In general, the process may be described as follows: *to obtain the lowest common multiple of a set of expressions, factor each expression into prime factors; use all factors of the first expression together with those factors of the second which are not in the first, those of the third which are not in the first and second, etc.*

It is evident that in this manner we obtain a product which is a common multiple of the given expressions, but such that if any one of these factors is omitted, it will cease to be a multiple of some one of the expressions; that is, it will no longer be a common multiple of them *all*.

Thus, if in the example above either of the factors $x - y$ is omitted, the product will no longer be a multiple of $x^2 - 2xy + y^2$.

206. Definition. The **lowest common multiple** of a set of expressions is that common multiple which contains the smallest number of prime factors. The lowest common multiple is usually abbreviated to L. C. M.

EXERCISES

Find the L. C. M. of the following expressions:

1. $2 \cdot 3 \cdot 4;\ 3 \cdot 7 \cdot 8;\ 2^3 \cdot 3 \cdot 4.$
2. $5x^2y^4,\ 10x^3y,\ 25x^2y.$
3. $2ab,\ 6a^2,\ 4b^2c.$
4. $x^2 - y^2,\ x^2 - 2xy + y^2.$
5. $x - y,\ x + y,\ x^2 - y^2.$
6. $4 - x^2,\ 2 - x,\ 2 + x.$
7. $a^2 + 2ab + b^2,\ a^2 - 2ab + b^2.$
8. $x^2 + 3x + 2,\ x^2 - 4,\ x^2 - 1.$
9. $25x^2 - 1,\ 125x^3 - 1.$
10. $2x^2 - 7x + 6,\ 4x^2 - 11x + 6.$
11. $x^3 - y^3,\ x - y,\ x^2 + xy + y^2.$
12. $x^3 - y^3,\ x^3 + y^3,\ x^2 - y^2.$

13. $5x^2+7x-6,\ x^2-15x-34.$

14. $x^3+y^3,\ x^2-y^2,\ (x-y)^2.$

15. $3abc,\ a^2-4ac+4c^2,\ a-2c.$

16. $x^2-1,\ x+1,\ x^2+8x+7.$

17. $4x^3y-44x^2y+120xy,\ 3a^3x^2-22a^3x+35a^3.$

18. $x^2+2xy+y^2,\ 2ax^2-10ax+12a.$

19. $3bx^2-21bx+36b,\ x^2-5x+4.$

20. $5a^2b^2-5a^2c^2,\ b^2+2bc+b+c+c^2.$

21. $15c^2ax^2+16c^2ax+c^2a,\ 2cax^2+10cax+8ca.$

REDUCTION OF FRACTIONS TO LOWEST TERMS

207. By Principle XV any factor common to the numerator and denominator of a fraction may be cancelled. That is,

$$\frac{ak}{bk}=\frac{a}{b}.$$

Thus,
$$\frac{2\cdot\cancel{3}\cdot4\cdot5}{\cancel{3}\cdot7\cdot11}=\frac{2\cdot4\cdot5}{7\cdot11};\quad \frac{2^3\cdot\overset{3}{\cancel{3^2}}\cdot\cancel{4}\,x}{\underset{2}{\cancel{2^4}}\cdot\cancel{3}\cdot\underset{4}{\cancel{4^2}}}=\frac{3x}{2\cdot4}=\frac{3x}{8};$$

$$\frac{x^2-7x+12}{x^2-5x+6}=\frac{\cancel{(x-3)}(x-4)}{(x-2)\cancel{(x-3)}}=\frac{x-4}{x-2}.$$

If the terms of a fraction have no common factor, the fraction is said to be in its **lowest terms.**

EXERCISES

Reduce the following fractions to lowest terms:

1. $\dfrac{3\cdot9^2\cdot2^6}{2^4\cdot5^3\cdot9^4}.$

2. $\dfrac{4a^4b^{12}c^3}{8a^3b^4c^4}.$

3. $\dfrac{x^2y^3z^4}{xy^2z^3}.$

4. $\dfrac{a^4b^3}{a^2b^2}.$

5. $\dfrac{x^2+2xy+y^2}{x^2-y^2}.$

6. $\dfrac{x^2+7x-30}{x^2-7x+12}.$

7. $\dfrac{x^3-y^3}{2x^2-3xy+y^2}.$

8. $\dfrac{64-b^3}{16-8b+b^2}.$

9. $\dfrac{x^3+27z^3}{xy-5x+3yz-15z}.$

10. $\dfrac{1-216\,c^3}{x-4\,y-6\,cx+24\,cy}$.

11. $\dfrac{14\,bz-2\,bx+ax-7\,az}{x^2-49\,z^2}$.

12. $\dfrac{3\,a^2-29\,a+56}{63-9\,a-7\,m+ma}$.

13. $\dfrac{a(x-y)^3}{(x^2-y^2)(x-y)}$.

14. $\dfrac{x^3+27}{4\,x^2+24\,x+36}$.

15. $\dfrac{a^2-3\,a-3\,b+ab}{(a^2-b^2)(a-3)}$.

16. $\dfrac{2^7\cdot 3^5\cdot 5^4-2^6\cdot 3^4\cdot 5^7}{2^4\cdot 3^2\cdot 5^5-2^5\cdot 3^2\cdot 5^2}$.

17. $\dfrac{5\,x^3y^4-12\,x^2y^5+7\,x^3y^2}{6\,x^5y^2+3\,x^2y^2}$.

18. $\dfrac{5\,c+10\,b-bc-2\,b^2}{8\,c^3+64\,b^3}$.

19. $\dfrac{4\,x^4-28\,x^3+48\,x^2}{2\,x^4-8\,x^3+6\,x^2}$.

20. $\dfrac{9\,a^2b^4+18\,a^2b^3c+9\,a^2b^2c^2}{3\,ab^3-3\,abc^2}$.

21. $\dfrac{7\,xy^2-133\,xy+126\,x}{15\,xy^2-36\,xy+21\,x}$.

22. $\dfrac{20\,x^3+20\,x^2y+5\,xy^2}{60\,x^5-15\,x^3y^2}$.

23. $\dfrac{3\,ab^4-3\,ab^2c^2}{27\,a^3b^2+27\,a^3bc}$.

24. $\dfrac{4\,a^3-42\,a^2+20\,a}{2\,a^4b^6-20\,a^3b^6}$.

25. $\dfrac{(x-1)(x-2)(x-3)(x-4)}{(x-1)(x-3)(x-3)(x-4)}$.

26. $\dfrac{(x^2-y^2)(x^2+2\,xy+y^2)}{(x^2-2\,xy+y^2)(x+y)}$.

27. $\dfrac{(x^2-1)(x^2+1)(3\,x^2+3)}{3(x^4-1)}$.

REDUCTION OF FRACTIONS TO A COMMON DENOMINATOR

208. By the formula $\dfrac{a}{b}=\dfrac{ak}{bk}$ any factor may be introduced into the numerator and denominator of a fraction.

In this manner any fraction may be changed into an equal fraction whose denominator is any given multiple of the denominator of the given fraction.

E.g. $$\frac{3}{4}=\frac{3\cdot 5}{4\cdot 5},\quad \frac{a-b}{a+b}=\frac{(a-b)(a+b)}{(a+b)(a+b)}=\frac{a^2-b^2}{(a+b)^2}.$$

Any two or more fractions may therefore be changed into respectively equal fractions which shall have a *common denominator*, namely, a common multiple of the denominators of the given fractions.

Illustrative Example. Reduce $\frac{x-1}{x+1}, \frac{x+1}{x-1}, \frac{2x+3}{x^2-1}$ to fractions having a common denominator.

The L. C. M. of the denominators is $(x-1)(x+1)$. Multiply the numerator and denominator of each fraction by an expression which will make the denominator of each new fraction $(x-1)(x+1)$.

Thus, $$\frac{x-1}{x+1} = \frac{(x-1)(x-1)}{(x+1)(x-1)} = \frac{x^2-2x+1}{(x+1)(x-1)};$$

$$\frac{x+1}{x-1} = \frac{(x+1)(x+1)}{(x-1)(x+1)} = \frac{(x+1)^2}{(x+1)(x-1)};$$

$$\frac{2x+3}{x^2-1} = \frac{2x+3}{(x+1)(x-1)}.$$

It is best to *indicate* the multiplication in the common denominator, since this makes it more easily apparent by what expression the numerator and denominator of a fraction must be multiplied in order to reduce it to a fraction with the required denominator.

209. The Three Signs of a Fraction. It should be noticed that there are three signs in connection with a fraction: the sign of the fraction itself, the sign of the numerator, and the sign of the denominator. Any two of these signs may be changed simultaneously without changing the value of the fraction.

Thus, (1) $\frac{a}{b} = \frac{-a}{-b}$. (2) $\frac{a}{b} = -\frac{a}{-b}$. (3) $\frac{a}{b} = -\frac{-a}{b}$.

Show why each of these statements is true. What principles are involved in each case?

How may the sign of the numerator of a fraction be changed if it is in the form of a polynomial? If it is in the form of a product of several factors? Take care to distinguish these two cases.

This is useful in cases like the following:

Reduce $\frac{x+1}{1-x}, \frac{x}{x^2-1}$, and $\frac{1}{x+1}$ to fractions having a common denominator.

$$\frac{x+1}{1-x}=\frac{-x-1}{x-1}=\frac{(x+1)(-x-1)}{(x+1)(x-1)}=\frac{-x^2-2x-1}{x^2-1},$$

$$\frac{x}{x^2-1}=\frac{x}{x^2-1}, \text{ and } \frac{1}{x+1}=\frac{x-1}{(x+1)(x-1)}=\frac{x-1}{x^2-1}.$$

Apply the sign changes shown in (1), (2), and (3) above to each of the following fractions:

(1) $\frac{a}{b-c}$, (2) $\frac{a-b}{c-d}$, (3) $\frac{-a}{c+d}$, (4) $\frac{(a-b)(c-d)}{-x}$,

(5) $\frac{1}{(1-x)(x+2)}$, (6) $-\frac{b-a}{d-c}$, (7) $-\frac{-c}{(d-a)(b-a)}$.

EXERCISES

Reduce each of the following sets of fractions to equivalent fractions having a common denominator.

1. $\frac{x+3}{x-y}, \frac{4}{x^2-2xy+y^2}$.

2. $\frac{a-1}{a^2-b^2}, \frac{a+1}{a^2+2ab+b^2}$.

3. $\frac{3-x}{x^2-9x+20}, \frac{x+4}{7x^2-26x-8}$.

4. $\frac{2}{3-x}, \frac{x-1}{x+1}, \frac{x+1}{x-3}$.

5. $\frac{a+b}{b-a}, \frac{a-b}{(a+b)^2}, \frac{a}{a^2-b^2}$.

6. $\frac{1}{x^2-3x-4}, \frac{1}{x^2+3x+2}$.

7. $\frac{a+1}{a^2-2ab+b^2}, \frac{a-1}{a^2+2ab+b^2}$.

8. $\frac{1}{a^3-b^3}, \frac{1}{b-a}, \frac{1}{a^2+ab+b^2}$.

9. $\frac{a}{5a^2-4a-12}, \frac{b}{a^2+4a-12}, \frac{c}{a-2}$.

10. $\frac{x-2}{x^2-5x-6}, \frac{x+2}{x^2+12x-108}, \frac{x-1}{x^2+19x+18}$.

11. $\frac{mr}{m-1}, \frac{d}{1+n}$.

12. $\frac{Rr}{(R+r)(m-1)}, \frac{1}{R+r}$.

13. $\frac{a}{n-a}, \frac{b}{n-b}$.

14. $\frac{W(T-Q)}{w(Q-t)}, \frac{V}{w}.$

15. $\frac{V}{V-v}, \frac{V}{V+v}, \frac{1}{V^2-v^2}.$

16. $\frac{RA}{a-A}, \frac{1}{R+r}, \frac{1}{A-a}.$

17. $\frac{R}{x}, \frac{r}{y}, \frac{1}{x-y}, \frac{Rx}{x+y}.$

210. Since any number may be written as a fraction with the denominator 1, the above process may be used to reduce an integral expression to the form of a fraction having any desired denominator.

Thus, $3 = \frac{3 \cdot 5}{5}$; $x - y = \frac{(x-y)(x^2-1)}{x^2-1}$, etc.

It is sometimes convenient to reduce expressions, some of which are not fractions, to the form of fractions having a common denominator.

Illustrative Example. Reduce $5x, \frac{5x-1}{x^2-1}, \frac{2x-y}{x-1}$, to fractions having a common denominator. The lowest common denominator is x^2-1.

Thus, $5x = \frac{5x(x^2-1)}{x^2-1} = \frac{5x^3-5x}{x^2-1}$, $\frac{5x-1}{x^2-1} = \frac{5x-1}{x^2-1}$,

$$\frac{2x-y}{x-1} = \frac{(2x-y)(x+1)}{(x-1)(x+1)} = \frac{2x^2+2x-yx-y}{x^2-1}.$$

EXERCISES

Reduce the following expressions to fractions having a common denominator:

1. $5, x-y, \frac{5x-3}{x^2+2xy+y^2}.$

2. $\frac{3a-c}{x-y}, \frac{2b-c}{x+y}, 2c+2.$

3. $1+a+a^2, \frac{a+1}{a-1}.$

4. $x^2+xy+y^2, \frac{x+y}{x-y}.$

5. $x^2-xy+y^2, x^2-y^2, \frac{1}{x+y}.$

6. $x^4 + x^2y^2 + y^4, \frac{x}{x-y}, \frac{y}{x+y}.$

7. $3a - 2b - c, \frac{5}{a-b}, \frac{2}{b-c}.$

8. $x^4 - 1, x^2 - 1, \frac{x+1}{x-1}.$

9. $x^2 + 2xy + y^2, \frac{1}{x+y}, \frac{1}{1-x}.$

10. $x + y, x - y, \frac{x-y}{x^2+y^2}, \frac{x+1}{x-y}.$

11. $\frac{RA}{a-A}, r.$ 12. $\frac{RA}{a-A}, R - r.$ 13. $T, \frac{Q+t}{2}.$

14. $\frac{W(q-t)}{w}, sT, \frac{s(t-q)}{2}.$ 15. $H, \frac{hd}{D}.$

ADDITION AND SUBTRACTION OF FRACTIONS

211. By Principle III read in the reverse order:

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}, \text{ and } \frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}.$$

If fractions which are to be added or subtracted do not have a common denominator, they must be reduced to this form.

EXAMPLE. Add $\frac{a-b}{a+b}$ and $\frac{a^2+2ab+b^2}{a^2-2ab+b^2}.$

Reducing the fractions to the common denominator

$$(a-b)(a-b)(a+b),$$

we have

$$\frac{a-b}{a+b} = \frac{(a-b)(a-b)(a-b)}{(a+b)(a-b)(a-b)} = \frac{a^3 - 3a^2b + 3ab^2 - b^3}{(a+b)(a-b)(a-b)},$$

and $\frac{a^2+2ab+b^2}{a^2-2ab+b^2} = \frac{(a+b)(a^2+2ab+b^2)}{(a+b)(a-b)(a-b)} = \frac{a^3+3a^2b+3ab^2+b^3}{(a+b)(a-b)(a-b)}.$

Adding the numerators, we have $2a^3 + 6ab^2$; whence the sum of the fractions is

$$\frac{2a^3 + 6ab^2}{(a+b)(a-b)(a-b)}.$$

EXERCISES

Perform the following additions and subtractions:

1. $\frac{3}{4}+\frac{2}{7}+\frac{a+b}{3a-b}$.

2. $\frac{x-y}{(x+y)^2}-\frac{x-y}{x^2-y^2}$.

3. $\frac{x^2-9x+18}{x^2-13x+36}+\frac{x}{4-x}$.

4. $\frac{2}{3}+\frac{a}{a+b}+\frac{b}{a-b}$.

5. $\frac{3}{2^3\cdot 3^2}+\frac{5}{2^2\cdot 3^4}-\frac{2}{2^4\cdot 3^3}$.

6. $\frac{a^2-9b^2}{a^2+6b+9b^2}-\frac{a^2-6ab}{a^2-9b^2}$.

7. $\frac{x+y}{x-y}+\frac{x-y}{x+y}-2$.

8. $\frac{2x^2}{x^2-y^2}-\frac{y}{x-y}-\frac{y}{x+y}$.

9. $\frac{a+1}{a^2+a+1}-\frac{a-1}{a^2-a+1}$.

10. $\frac{b}{1-b}+\frac{b}{1+b^2}-\frac{b^2}{1-b^2}$.

11. $\frac{x+1}{x-2}+\frac{x+1}{x+2}+\frac{3x+2}{x^2-4}$.

12. $\frac{x-1}{x+1}-\frac{x+1}{x-1}+\frac{x^2-5}{x^2-1}$.

13. $\frac{y^2}{y^2-1}+\frac{y}{y+1}-\frac{y}{1-y}$.

14. $\frac{1}{x}-\frac{1}{y}-\frac{1}{x-y}+\frac{-1}{x+y}$.

15. $\frac{1}{2}-\frac{b}{b-a}+\frac{2a^2}{b^2-a^2}+\frac{-a}{b+a}$.

16. $\frac{x^2+4xy}{x^3+y^3}+\frac{1}{x+y}-\frac{x}{x^2-xy+y^2}.$

17. $\frac{1}{1-x^3}+\frac{1}{1-x}-\frac{1}{1+x+x^2}.$

18. $\frac{a-3}{a^2-3a+2}-\frac{a-1}{a^2-5a+6}+\frac{a}{a^2-4a+3}.$

19. $\frac{x}{x^2-5x-14}+\frac{2}{x-7}-\frac{x}{x^2-9x+14}.$

20. $\frac{a}{ac+ad-bc-bd}-\frac{b}{a^2-2ab+b^2}.$

21. $\frac{2}{a-5}+\frac{39}{a^2+3a-40}+\frac{3}{a+8}.$

22. $\frac{1}{y^2+8y+16}-\frac{1}{y(y+4)}+\frac{4}{y^2(y+4)}.$

23. $\frac{x}{x^2+4x-60}+\frac{x}{x^2-4x-12}.$

24. $\frac{a-c}{a^2-c^2}+\frac{c-a}{a^2+2ac+c^2}-\frac{2}{a-c}.$

25. $\frac{9}{x^2+7x-18}-\frac{8}{x^2+6x-16}.$

26. $\frac{a+2}{a^2-a-6}+\frac{a-4}{a^2-7a+12}-\frac{a+2}{a^2-2a-8}.$

27. $\frac{2}{x^2-11x+30}-\frac{1}{x^2-36}+\frac{1}{x^2-25}.$

28. $\frac{1}{(x-1)(x+2)}+\frac{1}{(x+2)(x-3)}+\frac{1}{(x-1)(3-x)}.$

29. $\frac{1}{(a-b)(b-c)}-\frac{1}{(b-a)(c-d)}+\frac{1}{(b-c)(c-d)}.$

30. $\frac{4}{a-3}-\frac{a-1}{a^2+3a+9}+\frac{a^2-38a-3}{a^3-27}.$

DRILL EXERCISES

Find the square root of each of the following:

1. $25a^2 + c^2 + 9b^2 - 10ac + 30ab - 6bc.$
2. $x^2 + y^2 + 4z^2 + 4v^2 - 2xy + 4xz - 4xv - 4yz + 4yv - 8zv.$
3. $x^6 - 2x^5 - x^4 + 3x^2 + 2x + 1.$
4. $x^4 - 6x^3 + 13x^2 - 12x + 4.$
5. Divide $6x^4 + x^3 + 12x^2 + 8$ by $2x^2 - x + 2.$
6. Divide $3x^6 + 4x^5 - x^4 + 6x^3 - 12x^2 + 8x - 12$ by $3x^2 - 2x + 3.$
7. Divide $3a^7 - 5a^5 + 8a^3 + 2a^2 - 18a + 12$ by $a^3 - a + 2.$
8. Divide $6a^4 + 10a^3 - 9a^2 + 11a - 6$ by $2a^2 + 4a - 3.$

To each of the following binomials add one squared term so as to make it a trinomial square:

9. $4a^2 + 8a.$
10. $9a^2 + 30a.$
11. $x^2 + 3x.$
12. $25x^2 - 7x.$
13. $7b^2 - 3b.$
14. $16b^2 - 7b.$
15. $3x^2 - 4x.$
16. $7x^2 - 11x.$
17. $8x^2 + 7x.$

Reduce the solution of each of the following to simplest form without approximating any roots:

18. $7x^2 = 27.$
19. $5x^2 = 108.$
20. $2x^2 = 3.$
21. $3a^2 = 343.$
22. $5x^2 = 8a^3b.$
23. $2x^2 = 27a(a + b)^2.$
24. $7ax^2 = 98.$
25. $2(a + b)x^2 = 300.$
26. $x^2 = \frac{162}{125ab}.$
27. $x^2 = \frac{128}{72cd^3}.$
28. $(a - b)x^2 = a + b.$

Solve each of the following and test results by the method given in § 198.

29. $x^2 - 7x + 9 = 0.$
30. $2x^2 - 5x + 2 = 0.$
31. $7x^2 + 18x - 3 = 0.$
32. $x^2 - 12x + 16 = 0.$
33. $x^2 + 2bx = 3c.$
34. $2x^2 - 5ax = a^2.$
35. $ax^2 + 2bx = 3c.$
36. $4a^2x^2 + 6bx = 3b^2.$

MULTIPLICATION AND DIVISION OF FRACTIONS

212. Since a fraction is an indicated quotient, and since multiplying the dividend or dividing the divisor multiplies the quotient, it follows that the *product of a fraction and an integral expression is obtained by multiplying the numerator or dividing the denominator by the integral expression.*

Thus, $$4 \cdot \frac{7}{8} = \frac{4 \cdot 7}{8} = \frac{7}{2} \text{ or } 4 \cdot \frac{7}{8} = \frac{7}{8 \div 4} = \frac{7}{2}.$$

That is, in general, $$a \cdot \frac{b}{c} = \frac{a \cdot b}{c} = \frac{ab}{c} = \frac{a}{c \div b}.$$

It is best to factor completely the expressions to be multiplied and to keep them in the factored form until all possible cancellations have been made.

EXERCISES

Find the following indicated products and reduce the fractions to the simplest form:

1. $(1-a) \times \frac{1+a+a^2}{a-1}$.
2. $(x^3-y^3) \times \frac{x+y}{x-y}$.
3. $(x^2-2\,xa+a^2) \times \frac{x+a}{x-a}$.
4. $\frac{3\,x-1}{x^2-5\,x+6} \times (x^2-11\,x+18)$.
5. $\frac{a^2-4\,a-3}{a^2-8\,a+16} \times (a^2-5\,a+4)$.
6. $(x^2+9\,x+18) \times \frac{x-5}{x^2-2\,x-15}$.
7. $(1-x^3) \times \frac{1-x}{1+x+x^2}$.
8. $(27\,a^3-1) \times \frac{a+1}{9\,a^2+3\,a+1}$.
9. $(a^2+ab+b^2) \times \frac{a-b}{a^3-b^3}$.
10. $(1-a+a^2) \times \frac{a+1}{a^3+1}$.
11. $\frac{x+1}{x^4-1} \times \frac{x^2-1}{(x+1)^2}$.
12. $\frac{a+b}{a^6-b^6} \times \frac{a^3+b^3}{a^2-b^2}$.

213. Since multiplying the divisor or dividing the dividend divides the quotient, it follows that *a fraction is divided by an integral expression by dividing the numerator or multiplying the denominator by the integral expression.*

Thus, $\frac{8}{9} \div 2 = \frac{8 \div 2}{9} = \frac{4}{9}$ or $\frac{8}{9} \div 2 = \frac{8}{9} \times \frac{1}{2} = \frac{8}{9 \times 2} = \frac{4}{9}$.

That is, in general, $\frac{a}{b} \div c = \frac{a}{b \cdot c} = \frac{a \div c}{b}$.

EXERCISES

Find the following indicated quotients and reduce the fractions to their lowest terms:

1. $\frac{x^3 - y^3}{x + y} \div (x^2 + xy + y^2)$.
2. $\frac{x^3 + y^3}{x - y} \div (x^2 - y^2)$.
3. $\frac{x^2 + 4x + 4}{x^2 - 2x + 1} \div (x^2 - 4)$.
4. $\frac{1 - 27x^3}{1 - 9x^2} \div (1 + 3x + 9x^2)$.
5. $\frac{x^2 + 2x - 35}{x^2 + 10x + 21} \div (x^2 - 4x - 5)$.
6. $\frac{x^2 - 16x + 39}{x^2 - 8x + 15} \div (x^2 - x - 156)$.
7. $\frac{ac + bc - ad - bd}{xy - 4x - 3y + 12} \div (cx - 3c - dx + 3d)$.
8. $\frac{x^2 + ax + bx + ab}{x^2 + ax - 3x - 3b} \div (x^2 + ax - 5x - 5a)$.
9. $\frac{mr + ms - nr - ns}{mx - m - nx + n} \div (3r - xr + 3s - xs)$.
10. $\frac{x^2 - 3x - 88}{x^2 - 9x - 22} \div (x^2 + 9x + 2)$.

TO MULTIPLY A FRACTION BY A FRACTION

214. To multiply a number by the quotient of two numbers is the same as to multiply by the dividend and then divide the product by the divisor.

Thus, $$\frac{4}{9}\cdot\frac{3}{2}=\left(\frac{4}{9}\cdot 3\right)\div 2=\frac{4}{3}\div 2=\frac{2}{3}.$$

In general, $$\frac{a}{b}\cdot\frac{c}{d}=\left(\frac{a}{b}\cdot c\right)\div d=\frac{ac}{b}\div d=\frac{ac}{bd}.$$

That is, the *product of two algebraic fractions is a fraction whose numerator is the product of the given numerators and whose denominator is the product of the given denominators.*

Illustrative Example. Multiply $\frac{x^2-1}{x^2-7x+10}$ by $\frac{x^2-3x+2}{x^2+2x+1}$ and reduce the resulting fraction to its lowest terms.

$$\frac{x^2-1}{x^2-7x+10}\times\frac{x^2-3x+2}{x^2+2x+1}=\frac{(x-1)\cancel{(x+1)}\cancel{(x-2)}(x-1)}{\cancel{(x-2)}(x-5)\cancel{(x+1)}(x+1)}$$

$$=\frac{(x-1)(x-1)}{(x-5)(x+1)}=\frac{x^2-2x+1}{x^2-4x-5}.$$

It is desirable to resolve each numerator and denominator into prime factors, and then cancel all common factors before performing any multiplication.

EXERCISES

Find the following indicated products and reduce each fraction to its lowest terms:

1. $\frac{3x^2y^2}{2yz^2}\times\frac{6az}{9x^3}$.

2. $\frac{5a(a-b)}{3c(a+b)}\times\frac{9(a+b)^2}{15(a^2-b^2)}$.

3. $\frac{12c^3b}{5(c^3-b^3)}\times\frac{35(c^2+cb+b^2)}{14c^2b^2}$.

4. $\frac{y^2+3y+2}{y^2-5y+6}\times\frac{y^2-7y+12}{y^2+8y+7}$.

5. $\frac{3^2 \cdot 4^3}{5^2 \cdot 2^4} \times \frac{10 \cdot 2}{3^4}$.

7. $\frac{x^2 - x}{x^2 - 1} \times \frac{2x^2 + 4x + 2}{3x^2 + 6x}$.

6. $\frac{3^2 \cdot 2^3}{5^2} \times \frac{5^4 \cdot 7^2}{3^4 \cdot 2^3} \times \frac{6 \cdot 3^2}{5^4 \cdot 7^3}$.

8. $\frac{a^2 - 10a + 16}{a^2 + 6a + 9} \times \frac{a + 3}{a^2 - 4}$.

9. $\frac{(x+y)^2 - z^2}{x^2 + xy - xz} \times \frac{x}{(x-z)^2 - y^2} \times \frac{(x-y)^2 - z^2}{xy - y^2 - yz}$.

10. $\frac{3(x+4)^2}{4(x+4)(x-7)} \times \frac{(x-7)^2}{3(x+4)(x-7)}$.

11. $\frac{a^3 + 5a^2 - 36a}{a^2 - 7a - 144} \times \frac{(a-16)(a-3)}{a(a-4)(a+2)}$.

12. $\frac{3a(a+7)(a-5)}{7b(a+3)(a+7)} \times \frac{b(a+3)(a+10)}{a(a-5)(a-10)}$.

13. $\frac{3t^2 - 2t - 1}{2t^2 + t - 1} \times \frac{2t^2 + 5t - 3}{3t^2 + 7t + 2} \times \frac{4t^2 + 10t + 4}{4t^2 - 2t - 2}$.

14. $\frac{6x^2 - 7x + 2}{10x^2 - 7x + 1} \times \frac{6x^2 - 5x - 1}{6x^2 + x - 1} \times \frac{10x^2 + 3x - 1}{5x^2 - 4x - 1}$.

15. $\frac{4b^2 - 17b + 4}{6b^2 - 7b + 2} \times \frac{10b^2 - 21b + 9}{5b^2 - 23b + 12} \times \frac{3b^2 - 5b + 2}{4b^2 - 5b + 1}$.

TO DIVIDE A FRACTION BY A FRACTION

215. To divide a number by the quotient of two numbers is the same as to divide by the dividend and then multiply the result by the divisor.

Thus, $$\frac{4}{9} \div \frac{2}{3} = \left(\frac{4}{9} \div 2\right) \times 3 = \frac{2}{9} \times 3 = \frac{2}{3}.$$

In general, $$\frac{a}{b} \div \frac{c}{d} = \left(\frac{a}{b} \div c\right) \times d = \left(\frac{a}{b \cdot c}\right) \times d = \frac{ad}{bc}.$$

That is, *a number is divided by a fraction by inverting the fraction and multiplying by the new fraction thus obtained.*

EXERCISES

Perform the following indicated divisions, and reduce the resulting fractions to their lowest terms:

1. $\dfrac{a^3+b^3}{a^2-9b^2}\div\dfrac{a+b}{a+3b}$.

2. $\dfrac{x^2+x-2}{x^2-3x}\div\dfrac{x^3+2x^2}{x^2+9x-36}$.

3. $\dfrac{x^2-6x-16}{x^2+4x-21}\div\dfrac{x^2+9x+14}{x^2-8x+15}$.

4. $\dfrac{x^2-1}{x^2-4x-5}\div\dfrac{x^2+2x-3}{x^2-25}$.

5. $\dfrac{x^2+9xy+18y^2}{x^2-9xy+20y^2}\div\dfrac{x^2+6xy+9y^2}{xy^2-4y^3}$.

6. $\dfrac{3a^4-9a^3-54a^2}{9a^3-117a^2+378a}\div\dfrac{a^3+8a^2+15a}{3a^2-33a+84}$.

7. $\dfrac{a^2-11a+30}{a^3-6a^2+9a}\times\dfrac{a^2-3a}{a^2-25}\div\dfrac{a^2-9}{a^2+2a-15}$.

8. $\dfrac{x^2-10x+21}{x^2+x-56}\div\dfrac{x^2-8x+15}{x^2+4x-32}$.

9. $\dfrac{a^2-b^2}{ab^2x}\times\dfrac{b(a-b)}{a^2+2ab+b^2}\div\dfrac{b(a+b)}{a^2-2ab+b^2}$.

10. $\dfrac{a+b}{ab}\times\dfrac{a^2-b^2}{3(a^2+b^2)}\div\dfrac{(a-b)^2(a+b)^2}{3a^3y+3ay^3}$.

11. $\dfrac{5x^4-5x^3}{7x^2-56x-63}\div\dfrac{x^4-9x^3+8x^2}{14x^2+14x-1260}$.

12. $\dfrac{8y^2(y+4)(y+5)}{2^4(y+5)(y-7)}\div\dfrac{y(y+4)(y+8)}{2^3(y-7)(y+11)}$.

13. $\dfrac{x^3(x-2)^2}{(x+2)^2}\times\dfrac{(x+2)(x-3)}{(x-2)(x-7)}\div\dfrac{x^2(x-3)(x-5)}{(x-5)(x-7)}$.

14. $\dfrac{a^2b^2(c+5)(c-4)}{(c-4)(c-8)}\times\dfrac{(c-8)(c+9)}{(c+4)(c+7)}\div\dfrac{ab(c+9)(c-1)}{(c+7)(c+1)}$.

15. $\dfrac{21x^2+23x-20}{10x^2-27x+5}\times\dfrac{6x^2-11x-10}{3x^2+2x-5}\div\dfrac{7x^2+17x-12}{5x^2+9x-2}$.

COMPLEX FRACTIONS

216. Sometimes fractions occur whose numerators or denominators, or both, contain fractions.

E.g.
$$\frac{1+\frac{1}{a}}{1-\frac{1}{a}} \text{ and } \frac{\frac{1}{x+1}+\frac{1}{x-1}}{\frac{1}{x-1}-\frac{1}{x+1}}.$$

Such fractions are called **complex fractions**. A complex fraction is said to be **simplified** when it is reduced to an equal fraction whose numerator and denominator are in the integral form.

Ex. 1.
$$\frac{1+\frac{1}{a}}{1-\frac{1}{a}}=\frac{\frac{a+1}{a}}{\frac{a-1}{a}}=\frac{a+1}{a}\div\frac{a-1}{a}=\frac{a+1}{a-1}.$$

This result may also be obtained directly by multiplying both terms of the given fraction by a, finding at once $\frac{a+1}{a-1}$.

Ex. 2.
$$\frac{\frac{1}{x+1}+\frac{1}{x-1}}{\frac{1}{x-1}-\frac{1}{x+1}}=\frac{\frac{x-1+x+1}{x^2-1}}{\frac{x+1-x+1}{x^2-1}}$$
$$=\frac{2x}{x^2-1}\times\frac{x^2-1}{2}=x.$$

By multiplying the terms of the given fraction by $(x+1)(x-1)$ we may also get directly $\frac{x-1+x+1}{x+1-x+1}=x$.

Make a rule for finding the expression by which numerator and denominator of a complex fraction may be multiplied so as to reduce it directly to a simple fraction. Apply this rule to each of the following.

Ex. 1. $\dfrac{\frac{1}{2}+\frac{1}{3}-\frac{1}{6}}{\frac{2}{3}+\frac{5}{6}-\frac{1}{2}}.$

Ex. 2. $\dfrac{\frac{1}{4}+\frac{1}{8}+\frac{1}{2}}{\frac{3}{2}-\frac{1}{4}-\frac{1}{8}}.$

EXERCISES

Reduce each of the following complex fractions to its simplest form:

1. $\dfrac{1+x}{1+\dfrac{1}{x}}$.

2. $\dfrac{1-\dfrac{a}{b}}{1+a}$.

3. $\dfrac{x+\dfrac{x}{2}}{x-\dfrac{x}{2}}$.

4. $\dfrac{\dfrac{m+n+1}{3}}{\dfrac{m-n-1}{2}}$.

5. $\dfrac{4+\dfrac{a+b}{2}}{4-\dfrac{a-b}{2}}$.

6. $\dfrac{\dfrac{x^2-y^2}{4}}{\dfrac{x+y}{2}}$.

7. $\dfrac{\dfrac{a^3-8b^3}{27}}{3a-2b}$.

8. $\dfrac{\dfrac{M}{D^3}}{\dfrac{1+d^2}{D^2}}$.

9. $\dfrac{\dfrac{H}{c}-\dfrac{hd}{cD}}{\dfrac{1+t}{c}}$.

10. $\dfrac{\dfrac{x}{x-3}-\dfrac{x}{x+3}}{\dfrac{x+2}{x-3}-\dfrac{x-2}{x+3}}$.

11. $\dfrac{\dfrac{1}{x-4}+\dfrac{1}{x-3}}{\dfrac{x}{x^2-7x+12}}$.

12. $\dfrac{1}{1+\dfrac{1}{1+x}}$.

13. $\dfrac{a}{a+\dfrac{a}{a+x}}$.

14. $\dfrac{\dfrac{x-3}{x+4}}{\dfrac{x+2}{x-4}} \div \dfrac{x-4}{x+2}$.

15. $\dfrac{\dfrac{2}{x-1}-\dfrac{1}{x-2}}{\dfrac{3}{x-3}-\dfrac{4}{x-1}}$.

16. $\dfrac{\dfrac{2}{x-3}+\dfrac{2x^2+2}{x-2x^3}}{\dfrac{1}{x-3}-\dfrac{1}{1-2x^2}}$.

DRILL EXERCISES

Solve the following systems of equations:

1. $\begin{cases} x^2 + y^2 = 25, \\ x - y = 4. \end{cases}$

2. $\begin{cases} 2x^2 - 3xy = 12, \\ x + 2y = 4. \end{cases}$

3. $\begin{cases} x^2 - y^2 = 8, \\ x - 2y = 3. \end{cases}$

4. $\begin{cases} xy - y^2 = 14, \\ x + y = 4. \end{cases}$

5. $\begin{cases} x^2 - xy = 8, \\ x + y = -2. \end{cases}$

6. $\begin{cases} 2x^2 - 3y^2 = 7, \\ x - y = 5. \end{cases}$

7. $\begin{cases} 3x - 7by = 2, \\ 2ax + 2y = 4. \end{cases}$

8. $\begin{cases} 6x + 3y = 1, \\ 5ax - 2by = c. \end{cases}$

9. $\begin{cases} \dfrac{7x-3}{2} - \dfrac{ay+1}{3} = 1, \\ \dfrac{bx+5}{4} + \dfrac{y-3}{3} = 6. \end{cases}$

10. $\begin{cases} \dfrac{2by}{3a} - 3x = 2, \\ 5y + \dfrac{2ax}{b} = 5. \end{cases}$

11. $\begin{cases} 2x - y - z = 8, \\ 3x + 2y + z = 24, \\ -x - 3y + 5z = 16. \end{cases}$

12. $\begin{cases} 8x - 3y - z = 0, \\ 2x + 2y + 3z = 10, \\ -x + y + 6z = 8. \end{cases}$

13. A picture inside the frame is 12 inches long and 8 inches wide. If the frame is a inches wide, express its area in terms of a.

14. If h is the length of the hypotenuse of a right triangle and a the length of one side, express the length of the third side in terms of h and a.

15. A rectangular piece of tin is w inches wide and l inches long. If a square a inches on a side is cut out of each corner, express in terms of w, l, and a the volume of a box formed by turning up the sides.

16. A farmer plows a strip a rods wide around a rectangular field w rods wide and l rods long. Express in terms of w, l, and a the area plowed.

CHAPTER XIV

RATIO, VARIATION, AND PROPORTION

217. Definitions. A fraction is often called a **ratio**. Thus $\frac{a}{b}$ may be read *the ratio of a to b*, and is also written $a : b$.

The numerator is called the **antecedent** of the ratio, and the denominator the **consequent**. The antecedent and consequent are called the **terms** of the ratio.

An equation, each of whose members is a ratio, is called a **proportion**.

Thus, $\frac{a}{b} = \frac{c}{d}$ is a proportion, and is also written $a : b = c : d$.

It is read *the ratio of a to b equals the ratio of c to d*, or briefly, *a is to b as c is to d*.

The four numbers a, b, c, and d are said to be **in proportion**, a and d are called the **extremes** of the proportion, and b and c the **means**.

218. If in the graph of $y = 2x$, or $\frac{y}{x} = 2$, we think of a point as moving along the line, we see that its coördinates x and y *vary*, but always so that their ratio is constant, namely 2.

Thus if x_1 and x_2 are any two values of x, and y_1 and y_2 are the corresponding values of y, given by the equation $y = 2x$, we have $\frac{y_1}{x_1} = 2$ and $\frac{y_2}{x_2} = 2$, and hence the proportion $\frac{y_1}{x_1} = \frac{y_2}{x_2}$.

219. Definition. When any two variables are connected in such a way that their ratio is constant, either one is said to **vary directly as the other.** This is usually written in the form

$$y = kx,$$

where k is a constant and it is read **y varies directly as x.**

In this case the variables x and y are said to be *proportional.* We also say that y is proportional to x or x is proportional to y.

Ex. 1. The formula for a moving body $s = vt$ states that if the velocity is constant, the distance varies as the time. Hence if t_1 and t_2 are two values of t, and s_1 and s_2 the corresponding values of s, we have the proportion

$$\frac{s_1}{t_1} = \frac{s_2}{t_2}.$$

In this case the **space** is said to be proportional to the **time.**

Ex. 2. From the formula for percentage, $p = br$ form a proportion, if r is fixed and p and b vary; also if b is fixed and p and r vary.

Here the **percentage** is proportional to the **base** if the rate is constant, and proportional to the **rate** if the base is constant.

Ex. 3. From the formula for interest, $i = prt$ form a proportion if p and r are fixed and i and t vary; if p and t are fixed and i and r vary; if r and t are fixed and i and p vary.

To what is i proportional if p and r are fixed? if r and t are fixed? if p and t are fixed?

EXERCISES

1. If y varies as x and $y = 3$ when $x = 7$, find y when $x = 12$.

Solution. We have the proportion $\frac{y_1}{x_1} = \frac{y_2}{x_2}$ in which are given $x_1 = 7$, $y_1 = 3$, $x_2 = 12$, to find y_2.

Substituting, we have $\frac{3}{7} = \frac{y_2}{12}$ or $y_2 = 5\frac{1}{7}$.

2. If s varies as t and if $s = 40$ when $t = 5$, find s when $t = 16$.

3. If percentage varies as the base and $p = 50$ when $b = 1000$, find b when $p = 175$.

IMPORTANT PROPERTIES OF A PROPORTION

220. The following examples show some of the properties of a proportion.

1. If, in the proportion $\frac{a}{b}=\frac{c}{d}$, both members of the equation be multiplied by bd, we have $ad=bc$.

That is: *If four numbers are in proportion, the product of the means equals the product of the extremes.*

2. Show that if $\frac{a}{b}=\frac{c}{d}$, then $\frac{b}{a}=\frac{d}{c}$.

Hint. Divide $1=1$ by the members of the given equation.

This process is called taking the proportion by **inversion.**

3. Show that if $\frac{a}{b}=\frac{c}{d}$, then $\frac{a}{c}=\frac{b}{d}$.

Hint. Multiply both members of the given equation by $\frac{b}{c}$.

To find this multiplier we inquire by what expression $\frac{a}{b}$ must be multiplied to give $\frac{a}{c}$.

This process is called taking the proportion by **alternation.**

4. Show that if $\frac{a}{b}=\frac{c}{d}$, then $\frac{a+b}{b}=\frac{c+d}{d}$.

Hint. Add 1 to both members of the given equation.

This process is called taking the proportion by **addition.**

5. Show that if $\frac{a}{b}=\frac{c}{d}$, then $\frac{a-b}{b}=\frac{c-d}{d}$.

Hint. Subtract 1 from each member of the given equation.

This process is called taking the proportion by **subtraction.**

6. Show that if $\frac{a}{b}=\frac{c}{d}$, then $\frac{a+b}{a-b}=\frac{c+d}{c-d}$.

Hint. Divide the members of the equation obtained under 4 by the members of the one obtained under 5.

This process is called taking a proportion by **addition and subtraction.**

7. Show that if $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, then $\frac{a+c+e}{b+d+f}=\frac{a}{b}$.

Let $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=k$; then $a=bk$, $c=dk$, $e=fk$.

Hence, $a+c+e=bk+dk+fk=(b+d+f)k$,

and $$\frac{a+c+e}{b+d+f}=k=\frac{a}{b}=\frac{c}{d}=\frac{e}{f}.$$

That is, *If several ratios are equal, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

EXERCISES

1. If $ad=bc$, show that $\frac{a}{b}=\frac{c}{d}$. *Hint.* Divide by bd.

2 If $ad=bc$, show that $\frac{a}{c}=\frac{b}{d}$.

3. If $ad=bc$, show that $\frac{d}{c}=\frac{b}{a}$.

4. If $ad=bc$, show that $\frac{d}{b}=\frac{c}{a}$.

5. If $\frac{a}{b}=\frac{c}{d}$, show that $\frac{a+b}{a}=\frac{c+d}{c}$.

6. If $\frac{a}{b}=\frac{c}{d}$, show that $\frac{a-b}{a}=\frac{c-d}{c}$.

7. If $\frac{a}{b}=\frac{c}{d}$, show that $\frac{a-b}{a+b}=\frac{c-d}{c+d}$.

8. If $\frac{a}{b}=\frac{c}{d}$, show that $\frac{a+b}{c+d}=\frac{a-b}{c-d}$.

9. If $\frac{a}{b}=\frac{c}{d}$, show that $\frac{a+b}{c+d}=\frac{a}{c}$.

10. If $\frac{a}{b}=\frac{c}{d}$, show that $\frac{a+c}{b+d}=\frac{a}{b}$.

11. If $\frac{a}{b}=\frac{c}{d}$, show that $\frac{a-b}{c-d}=\frac{a}{c}$.

12. If $\frac{a}{b}=\frac{c}{d}$, show that $\frac{a-c}{b-d}=\frac{a}{b}$.

13. Solve the equation $\frac{a}{b}=\frac{c}{d}$ for each letter in terms of all the others. If $a=3$, $b=5$, $c=8$, find d. If $b=7$, $c=9$, $d=3$, find a. If $c=13$, $d=2$, $a=5$, find b. If $d=50$, $a=3$, $b=-7$, find c.

14. If $\frac{a}{b}=\frac{c}{x}$, then x is said to be a **fourth proportional** to a, b, and c.

Find a fourth proportional to 3, 5, and 7; also to 9, 5, and 1, and to 3, -2, and -5.

15. If $\frac{a}{x}=\frac{x}{b}$, then x is called a **mean proportional** between a and b.

Solve the equation $\frac{a}{x}=\frac{x}{b}$ for x in terms of a and b. Show that there are two solutions, each of which is a mean proportional between a and b.

Find two mean proportionals between 4 and 9; also between 5 and 125, and between -4 and -36.

16. Which is the greater ratio, $\frac{5+3d}{5+4d}$ or $\frac{5+4d}{5+5d}$?

Hint. Reduce the fractions to a common denominator and compare numerators. (d is a positive number.)

17. Which is the greater ratio, $\frac{a+7b}{a+8b}$ or $\frac{a+9b}{a+10b}$?

18. Which is the greater ratio, $\frac{a}{b}$ or $\frac{a+c}{b+c}$, if b and c are positive, and a less than b? a equal to b? a greater than b?

19. Find two numbers in the ratio of 3 to 5 whose sum is 160.

20. Find two numbers in the ratio of 2 to 7 whose sum is -108.

21. Find two numbers in the ratio of 3 to -4 whose sum is -15.

22. What number added to each of the terms of the ratio $\frac{5}{7}$ makes it equal to $\frac{65}{75}$?

23. What number must be added to each term of the ratio $\frac{1}{11}$ to make it equal to the ratio $\frac{3}{5}$?

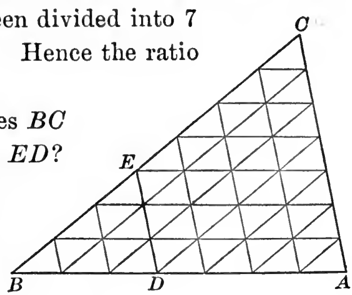
24. What number added to each of the numbers 3, 5, 7, 10, will make the sums in proportion, when taken in the given order?

25. Two numbers are in the ratio of 2 to 3, and the sum of their squares is 325. Find the numbers.

SIMILAR TRIANGLES

221. Triangles are called **similar** if they have the *same shape.*

Thus the triangles ABC and DBE of the figure are similar. Note that AB and DB have been divided into 7 and 3 equal parts, respectively. Hence the ratio of these sides is $\frac{7}{3}$.



What is the ratio of the sides BC and BE? Of the sides CA and ED? This is stated as follows: **The lengths of the pairs of corresponding sides of two similar triangles form a proportion.**

That is, we may write $\frac{AB}{DB} = \frac{CB}{EB} = \frac{CA}{ED}$.

Note that AB, BC, ⋯ represent the **lengths** of these sides.

EXERCISES

1. If in two similar triangles the sides in one are 11, 13, and 16, and that side in the other which corresponds to the one whose length is 11 is 33, find the other sides of the second triangle.

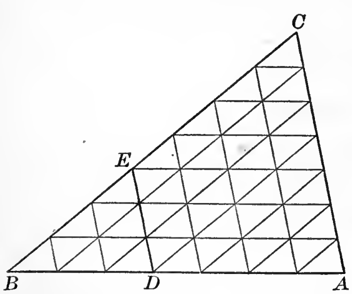
Solution. Let x represent the length of the side corresponding to the one whose length is 13. Then $\frac{x}{13}=\frac{33}{11}$, or $x = 39$.

In this manner find the remaining side.

2. If the sides of a triangle are 4, 6, and 10, and one side of a similar triangle is 9, find the remaining sides of the second triangle, if the given side corresponds to the side 4.

3. Solve Ex. 2 if the given side in the second triangle corresponds to 6.

4. Count the number of small triangles within each of the triangles ABC and DBE. Does the result illustrate the following statement:



The areas of similar triangles are in the same ratio as the squares of the lengths of their corresponding sides.

5. A triangular field has one side 15 rods long. Find the length of the corresponding side of a similar field whose area is 9 times as great.

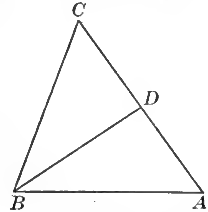
Suggestion. The equation is $\frac{x^2}{(15)^2}=\frac{9}{1}$.

6. A triangular field has the sides 15, 18, and 27 rods, respectively. Find the dimensions of a similar field having 4 times the area.

7. The areas of two similar triangles are 49 and 64, respectively. One side of the first is 12. Find the corresponding side of the second.

8. A triangular field, one of whose sides is 20 rods, has an area of 80 square rods. Find the area of a similar field whose corresponding side is 45 rods.

It is found in geometry that if a **line divides one angle of a triangle into two equal angles, it divides the opposite side into two parts which are in the same ratio as the other two sides of the triangle.**

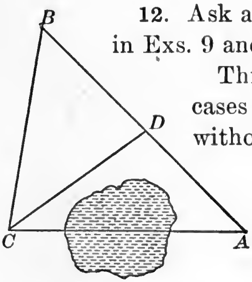


That is, in the figure $\frac{AD}{DC}=\frac{AB}{BC}$.

9. Knowing this fact, how many of the lines AD, DC, BC, and AB must you measure in order to find the rest of them ?

10. If in the figure $AD = 8$, $DC = 6$, and $BC = 12$, find AB.

11. If in the figure $BC = 14$, $AB = 18$, and $AD = 6$, find DC.



12. Ask and answer two more questions like those in Exs. 9 and 10.

This fact about geometry enables us in some cases to find the distance between two points without measuring it directly.

13. If in the figure CD divides the angle at C into two equal parts and if you know the lengths of AD, DB, and BC, show fully how to find the length of a straight line across the pond from C to A without measuring it directly.

14. Suppose it is found that $BC = 75$ rods, $BD = 50$ rods and $AD = 25$ rods. Compute AC.

15. Two corresponding sides of two similar triangles are in the ratio 13 : 14. Show that the perimeters (sum of the sides) of the triangles are in the same ratio.

16. The perimeters of two similar triangles are in the ratio 33 : 35. Two sides of the first triangle are 8 and 12. Find two sides of the second triangle corresponding to the given sides of the first.

DRILL EXERCISES

Simplify:

1. $\frac{x-1}{x-2}+\frac{x^2-3}{4-x^2}+\frac{x}{2+x}$.

2. $\frac{1}{(a-b)(b-c)}+\frac{1}{(c-b)(c-a)}-\frac{1}{(a-c)(b-a)}$.

3. $\left(a+2b-\frac{3ab+6b^2}{a+b}\right)\left(\frac{b^2+a^2+2ab}{a^2-3ab-4b^2}\right)\cdot\frac{1}{a+2b}$.

4. $\left(a-b+\frac{4ab}{a-b}\right)\div\left(\frac{(a+b)(b+2a)}{2a^2+3ab+b^2}\right)$.

5. $\dfrac{\frac{a+b}{a-b}-\frac{a-b}{a+b}}{\frac{a^2+b^2}{a^2-b^2}-\frac{a^2-b^2}{a^2+b^2}}$.

6. $\dfrac{\frac{x-y}{x}-\frac{x+y}{y}}{\frac{x-y}{y}+\frac{x+y}{x}}$.

7. $\left\{\frac{2}{x}-\frac{1}{a+x}+\frac{1}{a-x}\right\}\div\left\{\frac{a+x}{a-x}-\frac{a-x}{a+x}\right\}$.

8. $\left(2a-\frac{a^2-b^2}{a}\right)\left(3b+\frac{a^2+b^2}{b}\right)\div\left(\frac{a^2}{b^2}+5+\frac{4b^2}{a^2}\right)$.

9. What is the value of x^2-3x if $x=1+\sqrt{5}$? If $x=1-\sqrt{5}$?

10. What is the value of $3x^2-5x+6$ if $x=\frac{2-\sqrt{3}}{2}$? If $x=\frac{2+\sqrt{3}}{2}$?

11. What is the value of $5x^2+7x$ if $x=\frac{a+\sqrt{b}}{2}$? If $x=\frac{a-\sqrt{b}}{2}$?

Solve and check by the method of § 198:

12. $x^2-18x+4=0$.

13. $x^2-3ax+b=0$.

14. $x^2+9bx+c=0$.

15. $2x^2-7x=5$.

16. $3ax^2-7bx=3$.

17. $4a^2x^2-2bax=ca$.

18. The edge of a cube is a inches long. Express in terms of a (1) its volume, (2) the total area of surface, (3) the length of diagonal of one face, and (4) the length of a diagonal of the cube.

CHAPTER XV

EQUATIONS INVOLVING FRACTIONS

EQUATIONS INVOLVING ALGEBRAIC FRACTIONS

222. We have already seen, § 101, that in solving an equation involving fractions the first step is to multiply both members of the equation by a number which will cancel all the denominators. Evidently, any common multiple of the denominators is such a multiplier. For the sake of simplicity, and for reasons which are fully discussed in the Advanced Course, the *lowest common multiple is always used for this purpose.*

223. Illustrative Example. Solve the following equation:

$$\frac{2x-1}{x-1}+\frac{4}{x+1}-\frac{3x}{x^2-1}=2. \quad (1)$$

Solution. The L. C. M. of the denominators is x^2-1. In multiplying both members by x^2-1, we see that $x-1$ is canceled in the first fraction, $x+1$ in the second, and x^2-1 in the third,

giving $(2x-1)(x+1)+4(x-1)-3x=2(x^2-1). \quad (2)$

Solving, $2x^2+x-1+4x-4-3x=2x^2-2. \quad (3)$

$2x=3, \quad (4)$

and $x=\frac{3}{2}. \quad (5)$

Check by substituting $x=\frac{3}{2}$ in (1).

EXERCISES

Solve the following equations and check each solution by substituting in the original equation, except when the answer is given:

1. $\frac{3x-1}{x+1}-\frac{4x+3}{x-1}+\frac{x^2}{x^2-1}=-\frac{27}{x^2-1}+1.$

2. $\frac{3x+5}{x-9}+\frac{2x+1}{x+2}=\frac{x-1}{x^2-7x-18}$. *Ans.* $x=\frac{2}{5}$, or 1.

3. $\frac{3x-4}{x+5}-\frac{4x-1}{x+4}+\frac{x^2+44}{x^2+9x+20}=0.$

4. $\frac{x-4}{2x-10}-\frac{3x-15}{2x-6}=-\frac{3x^2-114}{4x^2-32x+60}.$

5. $\frac{6(x+4)}{x+5}-\frac{3(2x-1)}{x+1}=\frac{7}{2}$. *Ans.* $x=1$, or $-6\frac{1}{7}$.

6. $\frac{3x-4}{x-4}+\frac{5x-7}{2x-2}=\frac{9x^2-38}{2x^2-10x+8}$. *Ans.* $x=18\frac{1}{2}$, or 2.

7. $\frac{x+17}{x+5}-\frac{2(x+6)}{x+3}=-\frac{x-1}{x+3}.$

8. $\frac{x+2}{x-5}+\frac{3x-15}{x-3}=\frac{3x-21}{x-3}.$

9. $\frac{2x-3}{-4x}+\frac{3x+1}{x-2}=\frac{4x+17}{x-2}$. *Ans.* $x=\frac{-19\pm\sqrt{395}}{4}$.

10. $\frac{3x-2}{2x+3}=\frac{2x^2+15x+28}{2x^2+5x+3}+\frac{2x-1}{x+1}.$

11. $\frac{2x-3}{2x+2}-\frac{x-8}{5x+2}=\frac{x+2}{2x+2}.$

12. $\frac{20x^2+7x-3}{9x^2-1}-\frac{3x+1}{3x-1}=1.$

13. $\frac{7x^2+11x+4}{6x^2+13x+5}+\frac{x+3}{2x+1}=\frac{7x+11}{3x+5}.$

14. $\frac{3x+1}{5x-7}-\frac{x-3}{2x-7}=\frac{2x^2-10x+12}{10x^2-49x+49}.$

224. Sometimes it is best to *add fractions before multiplying* by the L. C. M. and in other cases to multiply by the L. C. M. of *part of the denominators first,* and, after simplifying, multiply by the L. C. M. of the remaining denominators.

Ex. 1. Solve $\frac{1}{x-2}-\frac{1}{x-1}=\frac{1}{x-4}-\frac{1}{x-3}$. (1)

Adding fractions on the right and left,

$$\frac{1}{(x-2)(x-1)}=\frac{1}{(x-4)(x-3)}. \quad (2)$$

Multiplying by L. C. M., $(x-4)(x-3)=(x-2)(x-1)$. (3)

Hence, $x=2\frac{1}{2}$. (4)

Check by substituting $x=2\frac{1}{2}$ in equation (1).

Ex. 2. Solve $\frac{4t-3}{16}-\frac{t-2}{4}=\frac{2t-2}{5t+2}$. (1)

Multiplying by 16, $4t-3-4t+8=\frac{32t-32}{5t+2}$. (2)

Hence, $5=\frac{32t-32}{5t+2}$. (3)

Multiplying by $5t+2$, $25t+10=32t-32$. (4)

Hence, $t=6$. (5)

Check by substituting $t=6$ in equation (1).

In Exs. 1 and 2, try to solve by clearing of fractions completely at the outset and see why the plan of solving here given is simpler.

EXERCISES

1. $\frac{3x+6}{5}-\frac{9x+3}{15}=\frac{x+7}{6x-8}+4$. *Ans.* $x=\frac{17}{19}$.

2. $\frac{7x+1}{12}-\frac{14x-22}{24}=\frac{11x+5}{8x-28}$.

3. $\frac{3x+4}{2}-\frac{12x+1}{8}=\frac{5x-1}{3x+2}$. *Ans.* $x=-7\frac{3}{5}$.

4. $\frac{7t+3}{5}-\frac{21t+9}{15}=\frac{17t-3}{3t+11}+2$. *Ans.* $x=-\frac{19}{23}$.

5. $\dfrac{11v-15}{10}-\dfrac{33v+15}{30}=\dfrac{5v+5}{v-5}.$

6. $\dfrac{x-1}{x-2}+\dfrac{x-2}{3-x}=\dfrac{x-3}{x-4}-\dfrac{x-4}{x-5}.$

7. $\dfrac{1}{x-1}-\dfrac{2}{2x+1}=\dfrac{1}{x-2}-\dfrac{4}{4x+1}.$

8. $\dfrac{x-2}{x-3}-\dfrac{x-3}{x-4}=\dfrac{x-4}{x-5}+\dfrac{x-5}{6-x}.$

9. $\dfrac{9}{x-7}-\dfrac{9}{x-2}=\dfrac{5}{x-8}-\dfrac{5}{x+1}.$

10. $\dfrac{x-1}{x-2}+\dfrac{x-2}{3-x}=\dfrac{x-3}{x-4}-\dfrac{x-5}{x-6}.$ *Ans.* $x=\pm 2\sqrt{3}.$

SIMULTANEOUS FRACTIONAL EQUATIONS

225. When pairs of fractional equations are given, each should be reduced to the integral form before eliminating, except in special cases like those in the second following illustrative example.

Ex. 1. Solve the equations:

$$\begin{cases} \dfrac{4}{x-y}+\dfrac{6}{x+y}=\dfrac{36}{x^2-y^2}. & (1) \\ \dfrac{3}{2x-y}-\dfrac{2}{x-3y}=\dfrac{-18}{(2x-y)(x-3y)}. & (2) \end{cases}$$

From (1) by *M*, $4(x+y)+6(x-y)=36.$ (3)

By *F*, *D*, $5x-y=18.$ (4)

From (2) by *M*, $3(x-3y)-2(2x-y)=-18.$ (5)

By *F*, *D*, $x+7y=18.$ (6)

From (4) by *M*, $35x-7y=126.$ (7)

Adding (6) and (7), $36x=144.$ (8)

By *D*, $x=4.$ (9)

Substitute $x=4$ in (6), $y=2.$ (10)

Check by substituting $x=4$, $y=2$ in (1) and (2).

Ex. 2. Solve the equations:

$$\begin{cases} \frac{2}{x}+\frac{3}{y}=2, & (1) \\ \frac{20}{x}-\frac{21}{y}=3. & (2) \end{cases}$$

In this case it is best to solve the equations for $\frac{1}{x}$ and $\frac{1}{y}$ instead of for x and y.

From (1) by M, $$\frac{14}{x}+\frac{21}{y}=14. \quad (3)$$

Adding (2) and (3), $$\frac{34}{x}=17. \quad (4)$$

Hence by D, $$\frac{1}{x}=\frac{1}{2}. \quad (5)$$

Substituting $\frac{1}{x}=\frac{1}{2}$ in (1), $$\frac{1}{y}=\frac{1}{3}. \quad (6)$$

From (5) and (6) by M, $$x=2,\ y=3. \quad (7)$$

In Ex. 2 try to solve by first clearing of fractions and see why the plan here used is simpler.

EXERCISES

Solve the following equations:

1. $$\begin{cases} \frac{2x-1}{x+1}-\frac{3y-1}{y+1}=\frac{-xy}{(x+1)(y+1)}, \\ \frac{x+2}{2y-1}+\frac{2x-1}{y+1}=\frac{5xy}{(2y-1)(y+1)}. \end{cases}$$

2. $$\begin{cases} \frac{3x+2}{3y-5}=\frac{x+1}{y-1}, \\ \frac{3x-2}{y+1}=\frac{3x-1}{y-1}-\frac{2}{(y-1)(y+1)}. \end{cases}$$

3. $$\begin{cases} \frac{5}{t}-\frac{6}{v}=2, \\ \frac{17}{v}+\frac{4}{t}=67. \end{cases}$$

4. $$\begin{cases} \frac{3}{x}+\frac{1}{y}=21, \\ \frac{7}{x}-\frac{9}{y}=-19. \end{cases}$$

5. $$\begin{cases} \frac{7}{a}-\frac{1}{b}=12\frac{1}{2}, \\ \frac{3}{a}+\frac{12}{b}=24. \end{cases}$$

6. $\begin{cases} \dfrac{3}{x} - \dfrac{2}{y} = -4, \\ \dfrac{6}{x} + \dfrac{11}{y} = 52. \end{cases}$ 7. $\begin{cases} \dfrac{12}{x} - \dfrac{10}{y} = 1, \\ \dfrac{9}{x} + \dfrac{2}{y} = 15. \end{cases}$ 8. $\begin{cases} \dfrac{a}{x} + \dfrac{b}{y} = 1, \\ \dfrac{c}{x} + \dfrac{d}{y} = 1. \end{cases}$

9. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = 16, \\ \dfrac{1}{y} + \dfrac{1}{z} = 14, \\ \dfrac{1}{z} + \dfrac{1}{x} = 12. \end{cases}$ 10. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = a, \\ \dfrac{1}{y} + \dfrac{1}{z} = b, \\ \dfrac{1}{z} + \dfrac{1}{x} = c. \end{cases}$

In Ex. 9 first add all three equations, and from half the sum subtract each equation separately. Likewise in Ex. 10.

PROBLEMS LEADING TO FRACTIONAL EQUATIONS

In solving the following problems use one or two unknowns as may be found most convenient.

1. There are two numbers whose sum is 51 such that if the greater is divided by their difference, the quotient is $3\frac{1}{3}$. Find the numbers.

2. Find two numbers whose sum is 91 such that if the greater is divided by their difference, the quotient is 7. Find the numbers.

3. There are two numbers whose sum is s such that if the greater is divided by their difference, the quotient is q. Find an expression in terms of s and q representing each number. Solve 1 and 2 by substituting in the formula just obtained.

4. What number must be subtracted from each term of the fraction $\frac{11}{17}$ so that the result shall be equal to $\frac{1}{2}$?

5. What number must be subtracted from each term of the fraction $\frac{21}{35}$ so that the result shall be equal to $\frac{2}{9}$?

6. What number must be subtracted from each term of the fraction $\dfrac{a}{b}$ so that the result shall be equal to $\dfrac{c}{d}$? Solve 4 and 5 by substituting in the formula obtained under 6.

7. What number must be added to each term of the fraction $\frac{1}{2}$ to obtain a fraction equal to $\frac{16}{17}$?

8. What number must be added to each term of the fraction $\frac{a}{b}$ to obtain a fraction equal to $\frac{c}{d}$?

9. There are two numbers whose difference is 153. If their sum is divided by the smaller, the quotient is equal to $\frac{47}{15}$.

10. There are two numbers whose difference is d. If their sum is divided by the smaller, the quotient is q. Find the numbers. Solve 9 by substituting in this formula.

11. Divide 548 into 2 parts, such that 7 times the first shall exceed 3 times the second by 474. *Ans.* One part is 211.8.

12. There are two numbers whose sum is 48 such that 3 times the first is 8 more than 5 times the second.

13. There are two numbers whose sum is s such that a times the first is b more than c times the second. Find both numbers.

14. What number must be subtracted from each of the numbers 12, 15, 19, and 25 in order that the remainders may form a proportion when taken in the order given?

15. What number must be added to each of the numbers 13, 21, 3, and 8 so that the sums shall be in proportion when taken in the order given?

16. What number must be added to each of the numbers a, b, c, d so that the sums shall be in proportion when taken in the given order?

17. What number must be subtracted from each of the numbers a, b, c, d so that the remainders shall be in proportion when taken in the given order?

Compare the results in 16 and 17 and explain the relation between them.

Solve Exs. 14 and 15 by substituting in the formulas obtained in 16 and 17.

18. There is a number composed of two digits whose sum is 11. If the number is divided by the difference between the digits, the quotient is $16\frac{3}{5}$. Find the number, the tens' digit being the larger.

19. There is a number composed of two digits whose sum is s. If the number is divided by the difference between the digits, the quotient is q. Find the number, the tens' digit being the larger.

20. Illustrative Problem. A can do a piece of work in 8 days, B can do it in 10 days. In how many days can they do it working together?

Since A can do the work in 8 days, in 1 day he can do $\frac{1}{8}$ of it, and since B can do it in 10 days, in 1 day he can do $\frac{1}{10}$ of it. If x is the number of days required when both work together, in 1 day they can do $\frac{1}{x}$ of it. Hence we have the equation,

$$\frac{1}{8}+\frac{1}{10}=\frac{1}{x}.$$

21. A can do a piece of work in 12 days and B can do it in 9 days. How long will it take both working together to do it?

22. A pipe can fill a cistern in 11 hours and another in 13 hours. How long will it require both pipes to fill it? *Ans.* $5\frac{23}{24}$ hours.

23. A can do a piece of work in a days and B can do it in b days. How long will it take both together to do it?

24. A cistern can be filled by one pipe in 20 minutes and by another in 30 minutes. How long will it take to fill the cistern when both are running together?

25. A pipe can fill a cistern in 12 hours, another in 10 hours, and a third can empty it in 8 hours. How long will it require to fill the cistern when they are all running?

26. A man can do a piece of work in 18 days, another in 21 days, a third in 24 days, and a fourth in 10 days. How long will it require them when all are working together?

Ans. $4\frac{52}{617}$ days.

27. A and B working together can do a piece of work in 12 days. B and C working together can do it in 13 days, and A and C working together can do it in 10 days. How long will it require each to do it when working alone?

Suggestion: Let $a =$ the fraction of the work A can do in one day, $b =$ the fraction of the work B can do in one day, and $c =$ the fraction of the work C can do in one day.

Then, $a + b = \frac{1}{12},\ b + c = \frac{1}{13},\ c + a = \frac{1}{10}.$

28. A and B working together can do a piece of work in l days. B and C can do it in m days and C and A can do it in n days. How long will it require each working alone?

29. The circumference of the rear wheel of a carriage is 1.8 feet more than that of the front wheel. In running one mile the front wheel makes 48 revolutions more than the rear wheel. Find the circumference of each wheel.

If x is the number of feet in the circumference of the front wheel, then $\frac{5280}{x}$ is the number of revolutions in going one mile.

30. The circumference of the rear wheel of a carriage is 1 foot more than that of the front wheel. In going one mile the two wheels together make 920 revolutions. Find the circumference of each.

31. The distance from Chicago to Minneapolis is 420 miles. By increasing the speed of a certain train 7 miles per hour the running time is decreased by 2 hours. Find the speed of the train.

If r is the original rate of the train, then $\frac{420}{r}$ is the running time.

32. The distance from New York to Buffalo is 442 miles. By decreasing the speed of a fast freight 8 miles per hour the running time is increased 4 hours. Find the speed of the freight.

33. A motor boat goes 10 miles per hour in still water. In 10 hours the boat goes 42 miles up a river and back again. What is rate of the current?

DRILL EXERCISES

Approximate the square roots of each of the following to two places of decimals:

1. 7.9482.
2. 4578.9.
3. 390.07.
4. 9.176.
5. .0048.
6. .04791.

Solve the following equations:

7. $\sqrt{2x^2-2}=x-3.$

8. $\sqrt{2x+7}=\sqrt{x}+3.$

9. $\dfrac{\sqrt{a}}{\sqrt{x-a}}-\dfrac{\sqrt{x+2a}}{2\sqrt{a}}=0.$

10. $\dfrac{\sqrt{5x+1}}{\sqrt{2x-3}}=\sqrt{\dfrac{6x-1}{3x-2}}.$

11. $\sqrt{x+2}=3-\sqrt{3-x}.$

12. $\dfrac{9x+8}{\sqrt{8x+1}}=\sqrt{10x+19}.$

Simplify the following:

13. $\dfrac{x-y}{x-2z}-\dfrac{x+y}{x+2z}+\dfrac{y-4z^2}{4z^2-x^2}.$

14. $\left(4-\dfrac{a^2+2ab+b^2}{ab}\right)\left(\dfrac{4ab}{a^2-2ab+b^2}+1\right).$

15. $\dfrac{a^3+27}{a^3-8}\div\dfrac{a^2b-3ab+9b}{a^2+2a+4}.$

16. $\left(y+\dfrac{xy}{y-x}\right)\left(y-\dfrac{xy}{x+y}\right)\left(\dfrac{y^2-x^2}{y^2+x^2}\right).$

17. $\left(\dfrac{x-2}{x-3}-\dfrac{x+3}{x+4}\right)\div\left(\dfrac{1}{x+1}+\dfrac{7}{x-3}\right).$

18. $\dfrac{\dfrac{1}{ab}+\dfrac{1}{ac}+\dfrac{1}{bc}}{\dfrac{(a-b-c)(a+b+c)}{ab}}.$

19. $\dfrac{\left(a-\dfrac{3}{a}\right)^2+16}{\left(a+\dfrac{3}{a}\right)^2+4}.$

20. $\dfrac{\dfrac{a}{b^2}}{1+\dfrac{a^2}{b^2}}+\dfrac{\dfrac{a}{(b-a)^2}}{\dfrac{b+a}{b-a}}.$

21. $\dfrac{2+\dfrac{1}{x-2}+x}{x+\dfrac{1}{x+2}-2}.$

CHAPTER XVI

GENERAL REVIEW

226. The purpose of this review is to reconsider some of the important topics of the course in order to show how they are *interrelated and unified by a few simple principles,* and to gain a little deeper insight into the *nature of algebraic processes and their uses.*

FIRST EXTENSION OF THE NUMBER SYSTEM

227. The number system of arithmetic consists of the integers including zero and the fractions of the form $\frac{m}{n}$, where m and n are integers (n not zero), together with certain indicated roots such as $\sqrt{2}$. In algebra we early encountered the *negative numbers.* These compelled us to give a distinct name to the numbers of arithmetic which were then called *positive numbers.* The principles according to which positive and negative numbers are added, subtracted, multiplied, and divided were studied in connection with concrete facts to which these numbers are naturally applicable.

EXERCISES

1. Do negative numbers apply to all things to which positive numbers apply? Can there be a negative number of books on a shelf?

2. Make a list of things to which both positive and negative numbers apply.

3. Show by concrete examples different from those in this book how Principle VII is obtained.

4. In arithmetic how did you find the difference in temperature between 8° above zero and 40° above zero? Between 8° above zero and 10° below zero? Note that in the latter case you *added* to find a *difference*. How does this compare with the subtraction of a negative number from a positive or a positive number from a negative? By concrete examples verify Principle VII. Show how a problem in subtraction may be changed into a problem in addition by means of Principle VIII.

5. Verify Principle IX by concrete examples.

6. How may the signs of the factors of a product be changed without changing the value of the product? Make all possible changes of sign in $(a-b)(b-c)(c-d)$ which will not change its value. Also make all possible changes of sign which will change the sign of the product.

7. Define division in terms of multiplication. Derive Principle X from Principle IX by means of the definition of division.

8. State how the signs involved in a fraction may be changed without changing the value of the fraction. Make all possible changes of signs which will not change the value of $\frac{a}{b}$, also of $-\frac{a}{b}$.

9. Make all possible changes of signs which will leave unchanged the value of the expression $\frac{(x-y)(y-z)(z-x)}{(a-b)(b-c)}$.

10. Reduce $\frac{a-b}{x-y}$, $\frac{b-c}{y-z}$, and $\frac{a+b}{(y-x)(z-y)}$ to fractions having a common denominator.

11. Reduce $\frac{a}{(a-b)(b-c)}$, $\frac{b}{(c-d)(b-a)}$, and $\frac{c}{(c-b)(d-c)}$ to fractions having a common denominator.

12. Recall some problem solved during the year, in which a negative result, while satisfying the equation, does not have meaning in the problem.

228. Rational and Irrational Numbers. The positive and negative integers and fractions, including zero, are called **rational numbers.** In attempting to solve the equation $x^2 = 2$ we get $x = \pm\sqrt{2}$ in which $\sqrt{2}$ stands for a number whose square is 2. It may be shown that no rational number satisfies this condition. Such a number is called an **irrational number.**

In this book the only irrational numbers encountered are irrational square roots called quadratic surds, whose values we learned to approximate to any desired degree of accuracy. Other forms of irrational numbers are found in more advanced work.

The essential property of a quadratic surd is stated in principle XVIII.

SECOND EXTENSION OF THE NUMBER SYSTEM

229. The equation $x^2 + 1 = 0$ or $x^2 = -1$ introduces still another kind of number, namely, $x = \pm\sqrt{-1}$. Such numbers, called **imaginaries**, were met in the solution of quadratic equations, but their study was postponed to the Advanced Course.

The **irrational and imaginary numbers** form important topics in higher courses in mathematics.

The extension of the number system to include negative numbers and imaginaries, besides the rational and irrational numbers of arithmetic, is **one of the chief distinctions between arithmetic and algebra.**

LITERAL EQUATIONS

230. Identities. There are two essentially different kinds of literal equations. One is the *identity* of which an example is $a(b+c) = ab + ac$.

If all the indicated operations in an algebraic identity are performed, the two members become exactly alike. Hence the chief use of identities is to formulate those operations on algebraic expressions which *change their form but not their value.*

EXERCISES

1. What values of the letters involved in an identity will satisfy it?

2. State in the form of identities as many as possible of the eighteen principles given in this book.

Determine which of the following are identities:

3. $(x+y)^2 = x^2 + 2xy + y^2$.
5. $(x+1)^2 = x^2 - 2$.
4. $(x-y)^2 = x^2 - 2xy + y^2$.
6. $a^2 + ab + b^2 = (a+b)^2 - ab$.
7. $a^2 - b^2 = (a-b)(a+b)$.
8. $(a-b)(a^2+ab+b^2) = a^3 - b^3$.
9. $(a+b)(a^2-ab+b^2) = a^3 + b^3$.
10. $(x+b)(x+a) = x^2 + (a+b)x + ab$.
11. $(x-a)(x+b) = x^2 - (a+b)x + ab$.
12. $(a+b-c)^2 = a^2 + b^2 + c^2 + 2ab - 2ac - 2bc$.

231. Equations of Condition. The other kind of literal equation is one which is not satisfied for all values of all the letters involved. These originate in a large variety of ways.

For example in $p = br$, which may be regarded as a definition of percentage, any value may be assigned to two of the letters but the third is thereby determined. Likewise in $i = prt$, which may be regarded as a definition of interest, any value may be given to three of the letters thereby fixing the fourth.

In the equation $w_1d_1 = w_2d_2$, which is the law of the lever and was established by experimentation, any three letters may be given arbitrarily, thereby fixing the fourth.

If in a literal equation numerical values are assigned to all the letters except one, then the value of this one is determined by the equation.

232. In general, every literal equation which is not an identity states a certain relation that must exist between the variables involved. In this sense such an equation is an **equation of condition.**

In using a literal equation it may become necessary to solve it for any one of the letters in terms of the others, and we thus have an important example of the solution of equations in one unknown.

EXERCISES

1. State as many rules of arithmetic as you can in the form of literal equations and indicate which of these are identities and which are equations of condition.

2. Indicate which of the equations of the preceding example can be derived from others.

3. Enumerate the various steps which may be used in the solution of an equation in one unknown.

4. Solve for x and y the simultaneous equations

$$\begin{cases} a_1x + b_1y = c_1, \\ a_2x + b_2y = c_2, \end{cases}$$

and reduce the results to the simplest form.

NOTE. — The letters a_1, a_2, b_1, b_2 are read a **one**, a **two**, b **one**, b **two**, etc. The numbers a_1 and a_2, etc., are entirely distinct so far as their values are concerned, the subscripts 1 and 2 being used to distinguish *two different coefficients of the same letter x.*

In solving these equations the steps involved are exactly the same as are used in solving any other pair of linear equations. The result obtained may, therefore, be used **as a formula** from which the solution of any two such equations may be obtained.

5. Solve equations 1 to 5 on page 119 by means of this formula.

6. What equations would have to be solved to obtain a formula for the solution of three equations in three unknowns?

7. What statements can you make about the relation of the equation $ax^2 + bx + c = 0$ to any other quadratic equation?

VARIABLES, GRAPHS, FUNCTIONALITY

233. We have seen that an equation like $s = vt$, where v is a constant, expresses a fixed relation between the variables s and t.

Thus, if $v = 8$, then $s = 8t$, and the variables change subject to the functional relation expressed by this equation.

The study of the dependence of one variable on other variables is to-day engaging the best effort of great men in all the sciences. The attempt is constantly made in each case to so determine the nature of such dependence that it may be expressed in the form of a *literal equation.*

The cases of functional relations between variables which have been studied in this book are but a very few of the simplest kind and are confined to *two variables only.* The graph is a convenient means of exhibiting a known relation between two variables.

EXERCISES

1. Write an equation of the first degree in two variables x and y in which y increases as x increases.

2. Construct a graph representing the equation of the preceding.

3. Write a similar equation in which y decreases as x increases.

4. Construct a graph representing the equation obtained in the preceding.

5. How many points do you need to locate on the graph of a linear equation before you can draw the whole graph?

6. Construct graphs of the equations $3x - 2y = 12$ and $2x = 5y + 10$.

7. What is the relation between the graphs of the equations $2x + 3y = 6$ and $2x + 3y = 12$?

SIMULTANEOUS EQUATIONS

234. We have seen that one equation in two variables may be regarded as stating a relation between these variables subject to which they are permitted to vary.

Thus in the equation $x + y = 5$, x and y may vary at will so long as their sum remains 5.

Such an equation is satisfied by an endless number of pairs of values. While this is true of each of two equations in two variables, there is only *one pair* of values which satisfy both equations, provided the equations are independent and simultaneous and both of the first degree. Hence two linear equations, each in two variables, completely determine these variables. This is most easily seen by means of the graph. See page 112.

The **solution** of a pair of simultaneous linear equations consists in finding this one pair of numbers which satisfy both equations.

EXERCISES

1. By means of an example explain elimination by addition and subtraction.

2. By means of an example explain elimination by substitution.

3. Under what conditions is elimination by substitution simpler than elimination by addition and subtraction? Under what conditions is it not?

FACTORING, QUADRATICS AND FRACTIONS

235. To be able to select at sight the **factors of simple algebraic expressions** is a prerequisite for successful and effective algebraic manipulation. It will be well to review the various types of factoring given in the text. You should now be able to read at sight the factors of many of the exercises in Chapter IX, and to solve at sight most of the equations on page 168, after reducing each to the type form $ax^2 + bx + c = 0$.

EXERCISES

1. Make a list of the various forms of binomials which are factorable, and show how to find the factors.

2. Give a similar treatment of trinomials.

3. What forms of polynomials of four terms are you able to factor?

4. Can you factor the following polynomials of five terms: (1) $a^2+2ab+b^2+a+b$? (2) $4x^2+12xy+9y^2+2x+3y$?

5. In the exercises on pages 143–146 and 152–155 find as many as you can of the results at sight.

6. In the exercises on pages 228–232 find as many as you can of the results at sight. You should at least be able to write the results without first writing in full all the factors of each polynomial.

7. What is the form of the square of a polynomial. Explain the method of finding the square root of a polynomial and also of a number expressed in arabic figures.

236. Factors of Quadratic Expressions. We have seen that certain quadratic equations may be solved by means of factoring.

For example, $x^2-7x+12=0$, in which $x-3$ and $x-4$ are the factors of the left member and 3 and 4 are the roots.

It is now possible to use the general solution of any quadratic equation $x^2+px+q=0$ for the purpose of factoring the trinomial x^2+px+q.

In $x^2-7x+12=0$ we observe that 4 is a root of the equation and that $x-4$ is a factor of the left member. Also 3 is a root and $x-3$ is a factor.

In like manner if r_1 and r_2 are the roots of the quadratic equation $x^2+px+q=0$, then $x-r_1$ and $x-r_2$ are the factors of x^2+px+q.

EXAMPLE: Factor $x^2 + 2x - 5$.

Solution: Solving the equation $x^2 + 2x - 5 = 0$ we get $x = -1 + \sqrt{6}$ and $x = -1 - \sqrt{6}$. Hence the factors of $x^2 + 2x - 5$ are $x - (-1 + \sqrt{6})$ and $x - (-1 - \sqrt{6})$.

That is, $x^2 + 2x - 5 = (x + 1 - \sqrt{6})(x + 1 + \sqrt{6})$.

Verify by multiplying the factors together to see if the product is $x^2 + 2x - 5$. The multiplication of the factors thus found may be used as a check on the solution of the quadratic.

EXERCISES

Factor by the above method the following quadratic expressions and verify the results by multiplying the factors:

1. $x^2 + 7x - 4$.	**4.** $x^2 - 8x + 1$.	**7.** $x^2 + ax - b$.
2. $x^2 + 12x + 13$.	**5.** $x^2 - 3x - 4$.	**8.** $x^2 - 2bx - 5b$.
3. $x^2 - 9x + 12$.	**6.** $x^2 - ax + b$.	**9.** $x^2 + 4ax + 2a$.

OPERATIONS AT SIGHT

237. You should now be sufficiently familiar with the simple algebraic operations to carry out many of them without writing down each step. This is illustrated in the following examples.

Ex. 1. Solve $45x - 14 + 7x = 13x + 7 + 8x + 41$.

Solution: $31x = 62$ and $x = 2$.

We add $45x$ and $7x$ and then subtract $13x$ and $8x$ (the latter are to be subtracted from both members of the equation) getting $31x$. Then we add 14, 7, and 41 (14 must be added to each member of the equation) getting 62. All this is done *mentally*.

In this manner solve Equations 1 to 20, pages 26, 27.

Ex. 2. Add the polynomials $3ax + 5by - 8b^2c^2$, $4b^2c^2 - 2ax - 7by$, and $4ax - 2by + 7b^2c^2$.

Solution: We pick out at once all terms having the factor ax and add them, then the terms having the factor by, and finally the terms having the factor b^2c^2 obtaining $5ax - 4by + 3b^2c^2$.

Polynomials may be subtracted in the same way.

In this manner solve Exs. 1 to 18, page 61.

Ex. 3. Solve $\begin{cases} 3x + 5y = 14, & (1) \\ 5x - 3y = 8. & (2) \end{cases}$

Solution: $25y + 9y = 70 - 24 = 46$ and $y = 46 \div 34 = \frac{23}{17}$.

We notice that to eliminate x we must multiply equation (1) by 5 and equation (2) by 3 and then subtract. Since this will eliminate x, we need not write down the terms containing it. Hence we get $5 \times 5y - 3(-3y) = 5 \times 14 - 3 \times 8$, or $34y = 46$.

In this manner solve Exs. 10 to 20, page 117.

The following operations should, for the most part, be done at sight, or with very little writing:

(1) Finding the products of any two binomials.

(2) Practically all the factoring in this book.

(3) Supplying the third term to complete any trinomial square, when two terms are given.

(4) Solving a quadratic by means of the formula.

(5) Most of the cancellations needed in dealing with fractions.

(6) Selection of the rationalizing factor for a quadratic surd and applying it.

Turn now to these various topics and consider each exercise a *challenge* to your mental skill.

Such a familiarity with the operations of algebra as is here indicated will greatly aid you, not only in your future work in mathematics, but also in the study of a science such as physics.

PROBLEMS

238. The **solution of problems by means of equations** is an important feature of algebra. It is a much more powerful method than arithmetic provides. Even though the operations be actually the same in both cases, yet the *equation* enables us to tabulate the data of a problem in such a way that they can be handled clearly and concisely and so that we may obtain the solution by means of a few standard operations.

239. Problems have in all ages held a peculiar interest for the human mind. Some are of interest because of the challenge they present to the mind as mere puzzles. Such is the following:

Ex. 1. A father is 9 years less than twice as old as his son and 12 years ago he was 19 years less than three times as old as the son. Find the age of each.

Other problems stated in the puzzle form attract attention because of the interesting facts revealed by their solutions. For example:

Ex. 2. The Fahrenheit reading at the temperature of liquid air is 128 degrees lower than the Centigrade reading. Find both the Centigrade and the Fahrenheit reading at this temperature.

Still other problems, while mere puzzles, are of value because they deal with important scientific or mathematical data. For example:

Ex. 3. A boatman trying to row up a river drifts back at the rate of $1\frac{1}{2}$ miles per hour, while he can row down the river at the rate of 12 miles per hour. What is the rate of the current?

Ex. 4. A beam is 12 feet long. It carries a 40-pound weight at one end, a 60-pound weight 3 feet from this end, and a 70-pound weight at the other end. Where is the fulcrum if the beam is balanced?

Ex. 5. There is a rectangle whose length is 60 feet more, and whose width is 20 feet less, than the side of a square of equal area. Find the dimensions of the square and the rectangle.

But by far the most important problems are those which it is necessary for mankind to solve in order to gain certain desired information. Such problems are called **real problems or practical problems.** For example:

Ex. 6. A farmer has a field 120 rods long and 60 rods wide. How wide a strip must he plow around it to make 5 acres?

Ex. 7. Find the average temperature for 12 hours from the following hourly observations; $-10°$; $-7°$; $-5°$; $-5°$; $-1°$; $+3°$; $+7°$; $+9°$; $+9°$; $+4°$; $+2°$; $-4°$.

Ex. 8. The earth and Mars were in conjunction. When are they next in conjunction if the earth's period is 365 days and that of Mars 687 days? (See figure, page 269.)

240. The development of every physical science depends to a large extent upon the solution of problems of this latter type, and algebra is one of the great tools for this purpose. It is not possible for you at this stage to apply algebra to the solution of many kinds of problems of this type because of your limited knowledge of the scientific principles involved in their statement. But if you have mastered the various processes of solving equations, if you are familiar with factoring and fractions, with square root and radicals, in short, with all the processes used in this course, you are thus in possession of the tools with which to solve any problems whose conditions you can translate into statements in the form of linear or quadratic equations.

SUPPLEMENTARY PROBLEMS ON CHAPTER II

1. The area of Louisiana is (nearly) 4 times that of Maryland, and the sum of their areas is 60,930 square miles. Find the (approximate) area of each state.

2. The horse power of a certain steam yacht is 12 times that of a motor boat. The sum of their horse-powers is 195. Find the horse power of each.

3. At a football game there were 2000 persons. The number of women was 3 times the number of children, and the number of men was 6 times the number of children. How many men, women, and children were there?

4. It is twice as far from New York to Syracuse as from New York to Albany, and it is 4 times as far from New York to Cleveland as from New York to Albany. The sum of the three distances is 1015 miles. Find each distance.

5. The altitude of Popocatepetl is 1716 feet less than that of Mt. Logan, and the altitude of Mt. St. Elias is 316 feet greater than that of Popocatepetl. Find the altitude of each mountain, the sum of their altitudes being 55,384 feet.

6. It is 4 times as far from New York City to Cincinnati as from New York to Baltimore. Twice the distance from New York to Cincinnati minus 5 times that from New York to Baltimore equals 567 miles. How far is it from New York to each of the other cities?

7. The melting temperature of glass is 276 degrees (Centigrade) higher than twice that of zinc. One-half the number of degrees at which glass melts plus 7 times the number at which zinc melts equals 3434. Find the melting point of each.

8. The melting temperature of nickel is 496 degrees (Centigrade) higher than that of silver. Three times the number of degrees at which nickel melts plus 2 times the number at which silver melts equals 6258. Find the melting point of each.

9. A cubic foot of nickel weighs 80 pounds more than one cubic foot of tin. Four cubic feet of nickel plus 2 cubic feet of tin weigh 3056 pounds. Find the weight per cubic foot of each metal.

10. A cubic foot of gold weighs 545 pounds more than a cubic foot of silver. One cubic foot of gold and one cubic foot of silver together weigh 1855 pounds. Find the weight per cubic foot of each metal.

11. A cubic foot of steel weighs 17 times as much as a cubic foot of yellow pine. The combined weight of 11 cubic feet of pine and 3 cubic feet of steel is 1773.2 pounds. Find the weight of one cubic foot of each.

12. The area of Great Britain is 7557 square miles more than 9 times that of the Netherlands, and the area of Japan is 42,065 square miles less than 15 times that of the Netherlands. One-third the area of Great Britain plus $\frac{1}{5}$ the area of Japan is 69,994 square miles. Find the area of each country.

13. The diameter of the earth is 1918 miles more than twice that of Mercury, and the diameter of Venus is 1700 miles more than twice that of Mercury. The diameter of the earth plus $\frac{1}{2}$ that of Venus equals 11,768 miles. Find the diameter of each planet.

14. The diameter of Jupiter is 500 miles more than 20 times that of Mars, and the diameter of Saturn is 4200 miles more than 16 times that of Mars. One-tenth the diameter of Jupiter plus $\frac{1}{2}$ that of Saturn is 45,150 miles. Find the diameter of each planet.

15. The diameter of Neptune is 2900 miles more than that of Uranus. The sum of their diameters is 66,700 miles. Find the diameter of each planet.

16. The money circulation of the United States in 1880 was 136 million dollars more than 3 times that in 1850. In 1910 it was 51 million more than 11 times that in 1850. The circulation of 1910 exceeded that of 1880 by 2147 million. Find the circulation in each year.

SUPPLEMENTARY PROBLEMS ON CHAPTER VI

1. A sparrow flies 135 feet per second and a hawk 149 feet per second. The hawk in pursuing the sparrow passes a certain point 7 seconds after the sparrow. In how many seconds from this time does the hawk overtake the sparrow?

2. A courier starts from a certain point, traveling v_1 miles per hour, and a hours later a second courier starts, going at the rate of v_2 miles per hour. In how long a time will the second overtake the first, supposing v_2 greater than v_1?

If the second courier requires t hours to overtake the first, the latter had been on the way $t + a$ hours. Thus the distance covered by the second courier is $v_2 t$ and by the first $v_1(t + a)$. As these numbers are equal we have

$$v_2 t = v_1(t + a).$$

This formula summarizes the solution of all problems like 3 and 4 below.

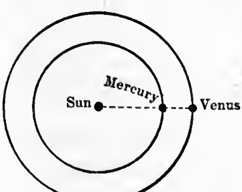
3. In an automobile race A drives his machine at an average rate of 53 miles per hour, while B, who starts $\frac{1}{4}$ hour later, averages 57 miles per hour. How long does it require B to overtake A? Use the above formula.

4. A freight steamer leaves New York for Liverpool, averaging $10\frac{1}{2}$ knots per hour, and is followed 4 days later by an ocean greyhound, averaging $25\frac{1}{2}$ knots per hour. In how long a time will the latter overtake the former?

5. One athlete makes a lap on an oval track in 26 seconds, another in 28 seconds. If they start together in the same direction, in how many seconds will the first gain one lap on the other? Two laps?

Let one lap be the unit of distance. Since the first covers one lap in 26 seconds, his rate per second is $\frac{1}{26}$. Likewise the rate of the other is $\frac{1}{28}$. If t is the required number of seconds, the distance covered by the first is $\frac{1}{26}t$ and by the second $\frac{1}{28}t$. If the first goes one lap farther than the second, the equation is $\frac{1}{26}t = \frac{1}{28}t + 1$.

6. Two automobiles are racing on a circular track. One makes the circuit in 31 minutes and the other in $38\frac{1}{2}$ minutes. In what time will the faster machine gain 1 lap on the slower?



7. The planet Mercury makes a circuit around the sun in 3 months and Venus in $7\frac{1}{2}$ months. Starting in conjunction, as in the figure, how long before they will again be in this position?

Note that the problem may be solved the same as if the two planets were moving in the same orbit at different rates.

8. Saturn goes around the sun in 29 years and Jupiter in 12 years. Starting in conjunction, how soon will they be in conjunction again?

9. Uranus makes the circuit of its orbit in 84 years and Neptune in 164 years. If they start in conjunction, how long before they will be in conjunction again? *Ans.* $172\frac{1}{5}$ years.

10. The hour hand of a watch makes one revolution in 12 hours and the minute hand in 1 hour. How long is it from the time when the hands are together until they are again together?

11. One object makes a complete circuit in a units of time and another in b units (of the same kind). In how many units of time will one overtake the other, supposing b to be greater than a?

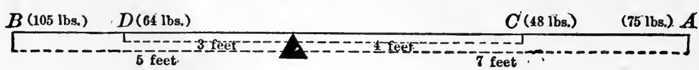
The solution of this problem summarizes the solution of all problems like those from 5 to 10.

12. At what times between 12 o'clock and 6 o'clock are the hands of a watch together? (Find the time required to gain one circuit, two circuits, etc.)

PROBLEMS INVOLVING THE LEVER

1. A teeter board is in balance when two boys, A and B, weighing 105 and 75 pounds, respectively, are seated at distances 5 and 7 feet from the fulcrum because $7 \cdot 75 = 5 \cdot 105$. If now two boys weighing 48 and 64 pounds are seated on the same board with the other boys, the teeter will again be in balance if their distances are 4 and 3 feet, because

$$7 \cdot 75 + 4 \cdot 48 = 5 \cdot 105 + 3 \cdot 64$$



The weight of the boy multiplied by his distance from the fulcrum is called his **leverage**. The sum of the leverages on the two sides must be the same. Hence, if two boys, weighing respectively, w_1 and w_2 pounds, are sitting at distances d_1 and d_2 on one side, and two boys, weighing w_3 and w_4 pounds, sitting at distances d_3, d_4 on the other side, then

$$\mathbf{w_1 d_1 + w_2 d_2 = w_3 d_3 + w_4 d_4}. \qquad (2)$$

2. If two boys weighing 75 and 90 pounds sit at distances of 3 and 5 feet respectively on one side and one weighing

82 pounds sits at 3 feet on the other side, where should a boy weighing 100 pounds sit to make the board balance?

3. A beam carries a weight of 240 pounds $7\frac{1}{2}$ feet from the fulcrum and a weight of 265 pounds at the opposite end which is 10 feet from the fulcrum. On which side and how far from the fulcrum should a weight of 170 pounds be placed so as to make the beam balance?

4. Two boys, *A* and *B*, having a 50-lb. weight and a teeter board, proceed to determine their respective weights as follows: They find that they balance when *B* is 9 feet and *A* 7 feet from the fulcrum. If *B* places the 50-lb. weight on the board beside him, they balance when *B* is 3 and *A* 4 feet from the fulcrum. How heavy is each boy?

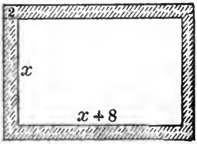
5. *C* is $6\frac{1}{2}$ feet from the point of support and balances *D*, who is at an unknown distance from this point. *C* places a 33-lb. weight beside himself on the board and, when $4\frac{2}{3}$ feet from the fulcrum, balances *D* who remains at the same point as before. *D*'s weight is 84 pounds. What is *C*'s weight, and how far is *D* from the fulcrum?

6. *E* weighs 95 pounds and *F* 110 pounds. They balance at certain unknown distances from the fulcrum. *E* then takes a 30-pound weight on the board, which compels *F* to move 3 feet farther from the fulcrum. How far from the fulcrum was each of the boys at first?

PROBLEMS INVOLVING GEOMETRY

1. A picture is 4 inches longer than it is wide. Another picture, which is 12 inches longer and 6 inches narrower, contains the same number of square inches. Find the dimensions of the pictures.

2. A picture, not including the frame, is 8 inches longer than it is wide. The area of the frame, which is 2 inches wide, is 176 square inches. Find the dimensions of the picture.

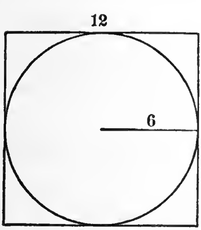


3. A picture, including the frame, is 10 inches longer than it is wide. The area of the frame, which is 3 inches wide, is 192 square inches. What are the dimensions of the picture?

4. The base of a triangle is 11 inches greater than its altitude. If the altitude and base are both decreased 7 inches, the area is decreased 119 square inches. Find the base and altitude of the triangle.

5. The base of a triangle is 3 inches less than its altitude. If the altitude and base are both increased by 5 inches, the area is increased by 155 square inches. Find the base and altitude of the triangle.

6. A square is inscribed in a circle and another circumscribed about it. The area of the strip inclosed by the two squares is 25 square inches. Find the radius of the circle.



7. Find the sum of the areas of a circle of radius 6 and the square circumscribed about the circle.

The area of the circle is $6^2\pi = 36\pi$, and the area of the square is $4 \cdot 6^2 = 4 \cdot 36$; *i.e.* the square contains 4 squares whose sides are 6. The sum of the areas is

$$4 \cdot 36 + 36\pi = (4 + \pi)36 = (4 + 3\tfrac{1}{7})36.$$

8. Find an expression for the sum of the areas of a circle of radius r and the circumscribed square. (Solve 7 by substituting in the formula here obtained.)

9. If the sum of the areas of a circle and the circumscribed square is 64, find the radius of the circle.

By the formula obtained under Ex. 8,

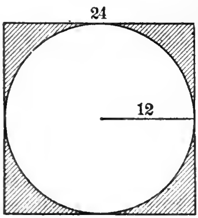
$$64 = (4 + \pi)r^2 = \tfrac{50}{7}r^2.$$

Hence, $$r = \sqrt{8.96} = 2.99.$$

10. If the sum of the areas of a circle and the circumscribed square is 640 square feet, find the radius of the circle.

11. The sum of the areas of a circle and the circumscribed square is a. Find an expression representing the radius of the circle. (Replace π by $3\frac{1}{7}$ before simplifying.)

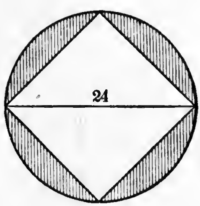
12. If the radius of a circle is 12, find the difference between the areas of the circle and the circumscribed square.



13. If the radius of a circle is r, find the difference between the areas of the circle and the circumscribed square. (Solve Ex. 12 by the use of the formula obtained here.)

14. If the radius of a circle is 16, find the area of the inscribed square. (This is the same problem as finding the area of a square whose diagonal is 32. See problems 8 and 9, page 190.)

15. If the radius of a circle is r, find an expression representing the area of the inscribed square. (This is problem 9, page 190.)

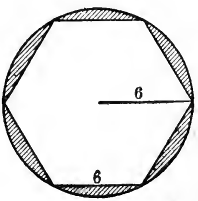


16. If the radius of a circle is 12, find the difference between the area of the circle and the area of the inscribed square.

17. If the radius of a circle is r, find an expression representing the difference between the areas of the circle and the inscribed square.

18. The radius of a circle is 10. Find the area of an inscribed hexagon. (See note, page 192.)

19. The radius of a circle is 6. Find the difference between the areas of the circle and the inscribed hexagon.



20. Find an expression representing the difference between the areas of a circle with radius r and the inscribed regular hexagon.

MISCELLANEOUS PROBLEMS

1. Divide the number 645 into two parts, such that 13 times the first part is 20 more than 6 times the other.

2. Divide the number a into two parts, such that b times the first part is c more than d times the second part.

3. The sum of three numbers is 98. The second is 7 greater than the first, and the third is 9 greater than the second. What are the numbers?

4. The sum of three numbers is s. The second is a greater than the first, and the third is b greater than the second. What are the numbers?

5. One boy runs around a circular track in 26 seconds, and another in 30 seconds. In how many seconds will they again be together, if they start at the same time and place and run in the same direction?

6. A bird flying with the wind goes 65 miles per hour, and flying against a wind twice as strong it goes 20 miles per hour. What is the rate of the wind in each case?

7. A steamer going with the tide makes 19 miles per hour, and going against a current $\frac{1}{2}$ as strong it makes 13 miles per hour. What is the speed of the steamer in still water?

8. Find the time between 4 and 5 o'clock when the hands of the clock are 30 minute spaces apart.

9. A man takes out a life insurance policy for which he pays in a single payment. Thirteen years later he dies and the company pays $ 12,600 to his estate. It was found that his investment yielded 2 % simple interest. How much did he pay for the policy?

10. After deducting a commission of 3 % for selling bonds, a broker forwarded $ 824.50. What was the selling price of the bonds?

11. A broker sold stocks for $ 1728 and remitted $ 1693.44 to his employer. What was the rate of his commission?

12. The difference between the areas of a circle and its circumscribed square is 12 square inches. Find the radius of the circle. (See problem 9, page 272.)

13. The difference between the areas of a circle and its inscribed square is 12 square inches. Find the radius of the circle.

14. The difference between the areas of a circle and the regular inscribed hexagon is 12 square inches. Find the radius of the circle.

15. The altitude of an equilateral triangle is 6. Find its side and also its area. Find the side and area, if the altitude is h.

16. The radius of a circle is 3 feet. Find the area of the regular circumscribed hexagon. Find the area if the radius is r feet.

17. The radius of a circle is r. Find the difference between the areas of the circle and the regular circumscribed hexagon.

18. The difference between the areas of a circle and the regular circumscribed hexagon is 9 square inches. Find the radius of the circle.

19. A circle is inscribed in a square and another circumscribed about it. The area of the ring formed by the two circles is 25 square inches. How long is the side of the square?

20. In a building there are at work 18 carpenters, 7 plumbers, 13 plasterers, and 6 hod carriers. Each plasterer gets $1.90 per day more than the hod carriers, the carpenters get 35 cents per day more than the plasterers, and the plumbers 50 cents per day more than the carpenters. If one day's wages of all the men amount to $183.45, how much does each get per day?

21. A train running 46 miles per hour leaves Chicago for New York at 7 A.M. Another train running 56 miles per hour leaves at 9.30 A.M. Find when the trains will be 15 miles apart. (Two answers.)

22. There is a number consisting of three digits, those in tens' and units' places being the same. The digit in hundreds' place is 4 times that in units' place. If the order of the digits is reversed, the number is decreased by 594. What is the number?

23. A hound pursuing a deer gains 400 yards in 25 minutes. If the deer runs 1300 yards a minute, how fast does the hound run? If the hound gains v_1 yards in t minutes and the deer runs v_2 yards per minute, find the speed of the hound.

24. A disabled steamer 240 knots from port is making only 4 knots an hour. By wireless telegraphy she signals a tug, which comes out to meet her at 17 knots an hour. In how long a time will they meet? If the steamer is s knots from port and making v_1 knots per hour, and if the tug makes v_2 knots per hour, find how long before they will meet.

25. A motor boat starts $7\frac{2}{3}$ miles behind a sailboat and runs 11 miles per hour while the sailboat makes $6\frac{1}{2}$ miles per hour. How far apart will they be after sailing $1\frac{1}{8}$ hours? If the motor boat starts s miles behind the sailboat and runs v_1 miles per hour, while the sailboat runs v_2 miles per hour, how far apart will they be in t hours?

26. An ocean liner making 21 knots an hour leaves port when a freight boat making 8 knots an hour is already 1240 knots out. In how long a time will the two boats be 280 knots apart? Is there more than one such position? If the liner makes v_1 knots per hour and the freight boat, which is s_1 knots out, makes v_2 knots per hour, how long before they will be s_2 knots apart?

27. A passenger train running 45 miles per hour leaves one terminal of a railroad at the same time that a freight running 18 miles per hour leaves the other. If the distance is 500 miles, in how many hours will they meet? If they meet in 8 hours, how long is the road? If the rates of the trains are v_1 and v_2 and the road is s miles long, find the time.

FIRST PRINCIPLES OF ALGEBRA

ADVANCED COURSE

PART THREE

CHAPTER I

FUNDAMENTAL LAWS

1. We have seen in the Elementary Course that **algebra**, like arithmetic, deals with **numbers** and with operations upon numbers. We now proceed to study in greater detail the laws that underlie these operations.

THE AXIOMS OF ADDITION AND SUBTRACTION

2. In performing the elementary operations of algebra we assume at the outset certain simple statements called **axioms.**

Definition. Two number expressions are said to be equal if they represent the same number.

Axiom I. *If equal numbers are added to equal numbers, the sums are equal numbers.*

That is, if $a = b$ and $c = d$, then $a + c = b + d$.

Axiom I implies that *two numbers have one and only one sum.* This fact is often referred to as the **uniqueness of addition.**

3. If $a = c$ and $b = c$ then $a = b$, since the given equations assert that a is the same number as b. Hence the usual statement: *If each of two numbers is equal to the same number, they are equal to each other.*

4. The sum of two numbers, as 6 and 8, may be found by adding 6 to 8 or 8 to 6, in either case obtaining 14 as the result.

This is a particular case of a general law for all numbers of algebra, which we state as

Axiom II. *The sum of two numbers is the same in whatever order they are added.*

This is expressed in symbols by the identity:

$$a + b = b + a. \qquad \text{[See § 75, E. C.*]}$$

Axiom II states what is called the **commutative law of addition**, since it asserts that numbers to be added may be *commuted* or interchanged in order.

Definition. Numbers which are to be added are called **addends.**

5. In adding three numbers such as 5, 6, and 7 we first add two of them and then add the third to this sum. It is immaterial whether we first add 5 and 6 and then add 7 to the sum, or first add 6 and 7 and then add 5 to the sum. This is a particular case of a general law for all numbers of algebra, which we state as

Axiom III. *The sum of three numbers is the same in whatever manner they are grouped.*

In symbols we have $\quad a + b + c = a + (b + c).$

When no symbols of grouping are used, we understand $a + b + c$ to mean that a and b are to be added first and then c is to be added to the sum.

Axiom III states what is called the **associative law of addition,** since it asserts that addends may be *associated* or grouped in any desired manner.

It is to be noted that an equality may be read in either direction. Thus $\quad a + b + c = a + (b + c)$ and $a + (b + c) = a + b + c$ are equivalent statements.

6. If any two numbers, such as 19 and 25, are given, then in arithmetic we can always find a number which added to the smaller gives the larger as a sum. That is, we can subtract the smaller number from the larger.

*E. C. means the Elementary Course.

In algebra, where negative numbers are used, any number may be subtracted from any other number.

That is: *For any pair of numbers* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *there is one and only one number* $\boldsymbol{c}$ *such that* $\boldsymbol{a} + \boldsymbol{c} = \boldsymbol{b}$.

The process of finding the number c when a and b are given is called **subtraction**. This operation is indicated thus, $b - a = c$, where b is the **minuend**, a the **subtrahend**, and c the **remainder**.

Since for a given minuend and a given subtrahend there is one and only one remainder, we have for all numbers of Algebra,

Axiom IV. *If equal numbers are subtracted from equal numbers, the remainders are equal numbers.*

Definitions. If $a + c = a$, then the number c is called **zero**, and is written 0. That is, $a + 0 = a$, or $a - a = 0$. Hence zero is the remainder when minuend and subtrahend are equal.

By definition of subtraction, the equality $b - a = c$ implies that c is a number such that $c + a = b$.

Adding a to each member of the equality $b - a = c$, we have $b - a + a = c + a$, which by hypothesis is equal to b. Hence *subtracting a number and then adding the same number gives as a result the original number operated upon.*

Axiom IV implies the **uniqueness of subtraction**.

THE AXIOMS OF MULTIPLICATION AND DIVISION

7. Axioms similar to those just given for addition and subtraction hold for multiplication and division.

Axiom V. *If equal numbers are multiplied by equal numbers, the products are equal numbers.*

This axiom implies the **uniqueness of multiplication**. That is, *two numbers have one and only one product.*

8. The product of 5 and 6 may be obtained by taking 5 six times, or by taking 6 five times. That is, $5 \cdot 6 = 6 \cdot 5$. This is a special case of a general law for all numbers of algebra, which we state as

Axiom VI. *The product of two numbers is the same in whatever order they are multiplied.*

In symbols we have $a \cdot b = b \cdot a.$

This axiom states what is called the **commutative law of factors** in multiplication.

9. The product of three numbers, such as 5, 6, and 7, may be obtained by multiplying 5 and 6, and this product by 7, or 6 and 7, and this product by 5. This is a special case of a general law for all numbers of algebra, which we state as

Axiom VII. *The product of three numbers is the same in whatever manner they are grouped.*

In symbols we have $abc = a(bc).$

The expression abc without symbols of grouping is understood to mean that the product of a and b is to be multiplied by c.

This axiom states what is called the **associative law of factors** in multiplication.

Principles IV and XIII of E. C. follow from Axioms VI and VII.

10. Another law for all numbers of algebra is stated as

Axiom VIII. *The product of the sum or difference of two numbers and a given number is equal to the result obtained by multiplying each number separately by the given number and then adding or subtracting the products.*

In symbols we have

$$a(b+c) = ab + ac \text{ and } a(b-c) = ab - ac.$$

Axiom VIII states what is called the **distributive law of multiplication.**

When these identities are read from left to right, they are equivalent to Principle II, E. C., and when read from right to left (see § 5) they are equivalent to Principle I, E. C. The form $a(b \pm c) = ab \pm ac$ is directly applicable to the multiplication of a polynomial by a monomial, and the form $ab \pm ac = a(b \pm c)$, to the addition and subtraction of monomials having a common factor.

11. Definitions. If $ac = b$, the process of finding c when a and b are given is called **division.** This operation is indicated thus: $b \div a = c$, or $\frac{b}{a} = c$, where b is the dividend, a the divisor, and c the quotient. For the case $a = 0$ see §§ 24, 25.

Axiom IX. *If equal numbers are divided by equal numbers (the divisors being different from zero) the quotients are equal numbers.*

Definition. If $a \cdot c = a$, $a \neq 0$,* then the number c is called **unity,** and is written 1. That is, $\frac{a}{a} = 1$. Hence unity is the quotient when dividend and divisor are equal.

By definition of division, the equality $\frac{b}{a} = c$ implies that c is a number such that $ac = b$.

Multiplying both sides of the equality $\frac{b}{a} = c$ by a, we have $a \cdot \frac{b}{a} = ac$, which by hypothesis equals b. Hence *dividing by a number and then multiplying by the same number gives as a result the original number operated upon.*

Axiom IX implies the **uniqueness of division.** That is: *For any two numbers, a and b, $a \neq 0$, there is one and only one number c such that $ac = b$, or $\frac{b}{a} = c$.*

12. Axioms I, IV (in case the subtrahend is not greater than the minuend), V, and IX underlie respectively the processes of addition, subtraction, multiplication, and division, from the very beginning in elementary arithmetic. Axioms II, III, VI, VII, and VIII are also fundamental in arithmetic, where they are usually assumed without formal statement.

E.g. Axiom VIII is used in long multiplication, such as 125×235, where we multiply 125 by 5, by 30, and by 200, and then add the products.

* The symbol $a \neq 0$ stands for the expression *a is not equal to zero.*

13. Negative Numbers. Axiom IV, in case the subtrahend is greater than the minuend, does not hold in arithmetic because of the absence of the negative number. This axiom therefore *brings the negative number into algebra.*

We now proceed to study the laws of operation upon this *enlarged number system.* In the Elementary Course concrete applications were used to show that certain rules of signs hold in operations upon positive and negative numbers. The same rules follow from the axioms just stated.

14. Definitions. If $a+b=0$, then b is said to be the **negative** of a and a the negative of b. If a is a positive number, that is, an ordinary number of arithmetic, then b is called a **negative number.** We denote the negative of a by $-a$. Hence, $a+(-a)=0$. a and $-a$ have the same **absolute value.**

If $a-b$ is positive, then a is said to be *greater than* b. This is written $a>b$. If $a-b$ is negative, then a is said to be *less than* b. This is written $a<b$. If $a-b=0$, then $a=b$. See § 6.

PRINCIPLES OF ADDITION AND SUBTRACTION

15. We now show that

$$\boldsymbol{a+(-b)=a-b.}$$ See § 51, E. C.

Let $$a+(-b)=x. \quad (1)$$

Adding b to each member of this equation we have, by Axiom I,

$$a+(-b)+b=x+b. \quad (2)$$

But by the associative law,

$$a+(-b)+b=a+[(-b)+b]=a. \quad (3)$$

Hence, $$a=x+b, \text{ or (§ 6), } a-b=x. \quad (4)$$

From (1) and (4), $$a+(-b)=a-b.$$

That is: *Adding a negative number is equivalent to subtracting this number with its sign changed.*

It follows that either of the symbols, $+(-b)$ and $-b$, may replace the other in any algebraic expression.

16. It is an immediate consequence of § 15 that *a parenthesis preceded by the plus sign may be removed without changing the sign of any term within it.* See § 79, E. C.

It also follows *that an expression may be inclosed in a parenthesis preceded by the plus sign without changing the sign of any of its terms.*

17. To show that $\boldsymbol{a-(-b)=a+b}$. See § 57, E. C.

Let $a-(-b)=x$. (1)

Adding $(-b)$ to both members (Ax. I),

$$a-(-b)+(-b)=x+(-b)=x-b. \quad (2)$$

But $a-(-b)+(-b)=a$. (3)

Hence, $a=x-b$ or $a+b=x$. (4)

From (1) and (4) we have $a-(-b)=a+b$.

That is: *Subtracting a negative number is equivalent to adding this number with its sign changed.*

It follows that either of the symbols $-(-b)$ and $+b$ may replace the other in any algebraic expression.

18. To show that

$$\boldsymbol{a-(b-c+d)=a-b+c-d}. \quad \text{See § 79, E. C.}$$

Let $a-(b-c+d)=x$. (1)

Then $a=x+(b-c+d)=x+b-c+d$. (2)

Adding c and subtracting b and d from each member,

we have $a-b+c-d=x$. (3)

From (1) and (3), $a-(b-c+d)=a-b+c-d$. (4)

That is: *A parenthesis preceded by the minus sign may be removed by changing the sign of each term within it.*

It also follows from equation (4), read from right to left, *that an expression may be inclosed in a parenthesis preceded by a minus sign, if the sign of each term within is changed.*

19. It follows by use of § 18 that

$$\boldsymbol{a-b=-(b-a)}.$$

For $a-b=-b+a=-(b-a)$.

20. It follows further by use of § 18 that

$$-a+(-b)=-(a+b).$$

For $\qquad -a+(-b)=-a-b=-(a+b).$

21. From the identities

$$a+(-b)\equiv a-b, \qquad § 15,$$

$$a-(-b)\equiv a+b, \qquad § 17,$$

$$a-b\equiv -(b-a), \qquad § 19,$$

$$-a+(-a)\equiv -(a+b), \qquad § 20,$$

it follows that addition and subtraction of positive and negative numbers are reducible to these operations *as found in arithmetic*, where all numbers added and subtracted are positive, and where the subtrahend is never greater than the minuend.

E.g. $\qquad 5+(-8)=5-8=-(8-5)=-3.$

$$5-(-8)=5+8=13.$$

$$-5-8=-(5+8)=-13.$$

PRINCIPLES OF MULTIPLICATION AND DIVISION

22. *To show that* $\mathbf{a}\cdot\mathbf{0}$ *or* $\mathbf{0}\cdot\mathbf{a}$ *equals* $\mathbf{0}$ *for all values of* $\mathbf{a}$.

By definition of zero, $\quad a\cdot 0=a(b-b).$

By Axiom VIII, $\quad a(b-b)=ab-ab.$

By definition of zero, $\quad ab-ab=0.$

Hence, $\quad a\cdot 0=0.$

By Axiom VII, $\quad a\cdot 0=0\cdot a=0.$

It follows that a product is zero if any one of its factors is zero; and if a product is zero, then at least one of its factors must be zero.

23. *To show that* $\dfrac{\mathbf{0}}{\mathbf{a}}=\mathbf{0}$, *provided* $\mathbf{a}$ *is not zero.*

Since by § 22, $0=a\cdot 0$, we have by the definition of division $\dfrac{0}{a}=0.$

24. *To show that* $\dfrac{\mathbf{0}}{\mathbf{0}}$ *represents any number whatever. That is,* $\dfrac{\mathbf{0}}{\mathbf{0}}=\mathbf{k}$, *for all values of* $\mathbf{k}$.

Since by § 22, $0 = 0 \cdot k$, we have by the definition of division $\frac{0}{0} = k$ for all values of k. Hence, $\frac{0}{0}$ does not represent any *definite* number.

25. *To show that there is no number* $\mathbf{k}$ *such that* $\frac{\mathbf{a}}{\mathbf{0}} = \mathbf{k}$, *provided* $\mathbf{a}$ *is not zero.*

If $\frac{a}{0} = k$, then by definition of division, $k \cdot 0 = a$. But by § 22, $k \cdot 0 = 0$ for all values of k. Hence, if a is not zero, k is *impossible.*

From §§ 24, 25, it follows that *division by zero is to be ruled out in all cases* unless special interpretation is given to the results thus obtained.

26. *To show that* $\mathbf{a(-b) = -ab}$. See § 63, E. C.

Let $$a(-b) = x. \quad (1)$$

Adding ab to both members,

$$a(-b) + ab = x + ab. \quad (2)$$

By Axiom VIII, $$a[(-b) + b] = x + ab, \quad (3)$$

or, $$a \cdot 0 = 0 = x + ab. \quad (4)$$

Hence (§ 14), $$x = -ab. \quad (5)$$

From (1) and (5), $$a(-b) = -ab.$$

That is, *the product of a positive and a negative number is negative.*

27. *To show that* $\mathbf{(-a)(-b) = ab}$. See § 63, E. C.

Let $$(-a)(-b) = x. \quad (1)$$

Adding $(-a)b$ to each member,

$$(-a)(-b) + (-a)b = x + (-a)b = x - ab. \quad (2)$$

By Axiom VIII, $$(-a)[(-b) + b] = x - ab, \quad (3)$$

or, $$0 = x - ab. \quad (4)$$

Hence (§ 14), $$ab = x. \quad (5)$$

From (1) and (5), $$(-a)(-b) = ab.$$

That is, *the product of two negative numbers is positive.*

28. *To show that if the signs of the dividend and divisor are alike, the quotient is positive; and if unlike, the quotient is negative.* See § 66, E. C.

If $a = bc$, then by the law of signs in multiplication, $-a = (-b)c$, $-a = b(-c)$, and $a = (-b)(-c)$. Hence, by definition of division, we have respectively:

$$\frac{a}{b} = c,\ \frac{-a}{-b} = c,\ \frac{-a}{b} = -c,\ \text{and}\ \frac{a}{-b} = -c.$$

29. *To show that* $\frac{a}{c} \cdot b = \frac{ab}{c}$.

Let $\qquad x = \frac{a}{c} \cdot b. \qquad (1)$

Then $\qquad cx = c \cdot \frac{a}{c} \cdot b = ab. \qquad (2)$

Dividing by c, $\qquad x = \frac{ab}{c}. \qquad (3)$

From (1) and (3), $\qquad \frac{a}{c} \cdot b = \frac{ab}{c}. \qquad (4)$

30. *To show that* $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$.

By § 29, $\qquad \frac{a+b}{c} = \frac{1 \cdot (a+b)}{c} = \frac{1}{c}(a+b). \qquad (1)$

By Axiom VIII, $\qquad \frac{1}{c}(a+b) = \frac{1}{c} \cdot a + \frac{1}{c} \cdot b = \frac{a}{c} + \frac{b}{c}. \qquad (2)$

Hence, $\qquad \frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}. \qquad (3)$

31. *To show that* $\frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}$.

By § 29 $\qquad \frac{a-b}{c} = \frac{1 \cdot (a-b)}{c} = \frac{1}{c}(a-b). \qquad (1)$

By Axiom VIII, $\qquad \frac{1}{c}(a-b) = \frac{1}{c} \cdot a - \frac{1}{c} \cdot b = \frac{a}{c} - \frac{b}{c}. \qquad (2)$

Hence, $\qquad \frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}. \qquad (3)$

CHAPTER II

FUNDAMENTAL OPERATIONS

32. The operations of addition, subtraction, multiplication, division, and finding powers and roots are called **algebraic operations.**

33. An **algebraic expression** is any combination of number symbols (Arabic figures or letters or both) by means of indicated algebraic operations.

E.g. 24, $3+7$, $9(b+c)$, $\frac{m+n}{k}$, $x^2+\sqrt{y}$, are algebraic expressions.

34. Any number symbol upon which an algebraic operation is to be performed is called an **operand.**

All the algebraic operations have been used in the Elementary Course. They are now to be considered in connection with the fundamental laws developed in the preceding chapter, and then applied to more complicated expressions. The finding of powers and roots will be extended to higher cases.

35. One of the two equal factors of an expression is called the **square root** of the expression; one of the three equal factors is called its **cube root**; one of the four equal factors, its **fourth root,** etc. A root is indicated by the **radical sign** and a number, called the **index** of the root, which is written within the sign. In the case of the square root, the index is omitted.

E.g. $\sqrt{4}$ is read *the square root of* 4; $\sqrt[3]{8}$ is read *the cube root of* 8; $\sqrt[4]{64}$ is read *the fourth root of* 64, etc.

36. A root which can be expressed in the *form* of an integer, or as the quotient of two integers, is said to be **rational,** while one which cannot be so expressed is **irrational.**

E.g. $\sqrt[3]{8}=2$, $\sqrt{a^2+2ab+b^2}=a+b$, and $\sqrt{\frac{9}{4}}=\frac{3}{2}$ are rational roots, while $\sqrt[3]{4}$ and $\sqrt{a^2+ab+b^2}$ are irrational roots.

An algebraic expression which involves a letter in an irrational root is said to be **irrational with respect to that letter;** otherwise the expression is rational with respect to the letter.

E.g. $a + b\sqrt{c}$ is rational with respect to a and b, and irrational with respect to c.

37. An expression is **fractional** with respect to a given letter if after reducing its fractions to their lowest terms the letter is still contained in a denominator.

E.g. $\frac{a}{c+d} + b$ is fractional with respect to c and d, but not with respect to a and b.

38. Order of Algebraic Operations. In a series of indicated operations where no parentheses or other symbols of aggregation occur, it is an established usage that the operations of finding powers and roots are to be performed first, then the operations of multiplication and division, and finally the operations of addition and subtraction.

E.g. $2 + 3 \cdot 4 + 5 \cdot \sqrt[3]{8} - 4^2 \div 8 = 2 + 3 \cdot 4 + 5 \cdot 2 - 16 \div 8$
$= 2 + 12 + 10 - 2 = 22.$

In cases where it is necessary to distinguish whether multiplication or division is to be performed first, parentheses are used.

E.g. In $6 \div 3 \times 2$, if the division comes first, it is written $(6 \div 3) \times 2 = 4$, and if the multiplication come first, it is written $6 \div (3 \times 2) = 1$.

ADDITION AND SUBTRACTION OF MONOMIALS

39. In accordance with § 10, the sum (or difference) of terms which are *similar with respect to a common factor* (§ 74, E. C.) is equal to the product of this common factor and the sum (or difference) of its coefficients.

Ex. 1. $8\,ax^2 + 9\,ax^2 - 3\,ax^2 = (8 + 9 - 3)\,ax^2 = 14\,ax^2.$

Ex. 2. $a\sqrt{x^2+y^2} + b\sqrt{x^2+y^2} = (a+b)\sqrt{x^2+y^2}.$

Ex. 3. $\frac{x(x-1)(x-2)}{1 \cdot 2 \cdot 3} + \frac{x(x-1)}{1 \cdot 2} = \left(\frac{x-2}{3} + 1\right)\frac{x(x-1)}{1 \cdot 2}$

$$= \frac{x+1}{3} \cdot \frac{x(x-1)}{1 \cdot 2} = \frac{(x+1)x(x-1)}{1 \cdot 2 \cdot 3}.$$

EXERCISES

Perform the following indicated operations:

1. $5x^4b^2 - 3x^4b^2 - 4x^4b^2 + 7x^4b^2$.

2. $3\sqrt{x^2-4} - 2\sqrt{x^2-4} + 2\sqrt{x^2-4} - 4\sqrt{x^2-4}$.

3. $ab^5c^4 - db^5c^4 + eb^5c^4 + fb^5c^4$.

4. $a^6x^4 + 5a^5x^4 - 5a^5x^5 - 3a^5x^4$.

5. $7x^3y^5 + 5x^4y^4 - 9x^4y^5 + 5x^3y^4$.

6. $2a^n + a^{n-1} + a^{n+1} = a^{n-1}(2a + 1 + a^2) = a^{n-1}(1 + a)^2$.

7. $n(n-1)(n-2)(n-3)(n-4) + n(n-1)(n-2)(n-3)$.

$n(n-1)(n-2)(n-3)$ is the common factor and $n - 4$ and 1 are the coefficients to be added.

8. $n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)$
$+ n(n-1)(n-2)(n-3)(n-4)(n-5)$.

9. $n(n-1)(n-2)(n-3) + (n-1)(n-2)(n-3)$.

10. $n(n-1)(n-2)(n-3)(n-4) + (n-1)(n-2)(n-3)$.

11. $(a-4)(b+3) + (2a-1)(b-2) + (a+3)(b+3)$.

First add $(a-4)(b+3)$ and $(a+3)(b+3)$.

12. $(x+2y)(x-2y) + (x-3y)(x-2y) + (2x-y)(x-y)$.

13. $(5a-3b)(a-b)(a+b) + (2b-4a)(a-b)(a+b)$
$+ (a-b)^2(2a-b)$.

14. $(7x^2+3y^2)(5x-y)(x+y) + (7x^2+3y^2)(x+y)(2y-4x)$
$+ (7x^2-3y^2)(x+y)^2$.

15. $2^3 \cdot 3^2 \cdot 5 + 2^4 \cdot 3 \cdot 5$.

The common factor is $2^3 \cdot 3 \cdot 5$. Hence the sum is

$$2^3 \cdot 3 \cdot 5(3+2) = 2^3 \cdot 3 \cdot 5^2.$$

16. $2 \cdot 3^4 \cdot 7 + 2^2 \cdot 3^3 \cdot 7^2 - 2^4 \cdot 3^3 \cdot 7$.

17. $3^4 \cdot 5^7 \cdot 13 + 3^5 \cdot 5^7 \cdot 13^2$.

18. $5^4 \cdot 7^3 \cdot 11 + 5^3 \cdot 7^2 \cdot 11 - 2^3 \cdot 3 \cdot 5^3 \cdot 7^2 \cdot 11.$

19. $3^{22} \cdot 7^{18} \cdot 13^{15} + 3^{21} \cdot 7^{17} \cdot 13^{15} + 3^{24} \cdot 7^{17} \cdot 13^{15}.$

20. $1 \cdot 2 \cdot 3 \cdots n + 1 \cdot 2 \cdot 3 \cdots n(n+1).$

The dots mean that the factors are to run on in the manner indicated up to the number n. The common factor in this case is $1 \cdot 2 \cdot 3 \cdots n$, and the coefficients to be added are 1 and $n+1$. Hence the sum is $1 \cdot 2 \cdot 3 \cdots n(n+2)$.

21. $1 \cdot 2 \cdot 3 \cdots n + 1 \cdot 2 \cdot 3 \cdots n(n+1)$
$+ 1 \cdot 2 \cdot 3 \cdots n(n+1)(n+2).$

22. $1 \cdot 2 \cdot 3 \cdots n + 3 \cdot 4 \cdot 5 \cdots n + 5 \cdot 6 \cdot 7 \cdots n.$

23. $n(n-1) \cdots (n-6) + n(n-1) \cdots (n-6)(n-7).$

24. $n(n-1) \cdots (n-r) + n(n-1) \cdots (n-r)(n-r-1).$

25. $na^nb + a^nb.$

26. $\frac{n(n-1)}{1 \cdot 2} a^{n-1}b^2 + na^{n-1}b^2.$

27. $\frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} a^{n-2}b^3 + \frac{n(n-1)}{1 \cdot 2} a^{n-2}b^3.$

The common factor is $\frac{n(n-1)}{1 \cdot 2} a^{n-2}b^3$ and the coefficients to be added are $\frac{n-2}{3}$ and 1.

28. $\frac{n(n-1)(n-2)(n-3)}{2 \cdot 3 \cdot 4} a^{n-3}b^4 + \frac{n(n-1)(n-2)}{2 \cdot 3} a^{n-3}b^4.$

29. $\frac{n(n-1)(n-2)(n-3)(n-4)}{2 \cdot 3 \cdot 4 \cdot 5} a^{n-4}b^5$
$+ \frac{n(n-1)(n-2)(n-3)}{2 \cdot 3 \cdot 4} a^{n-4}b^5.$

30. $\frac{n(n-1) \cdots (n-r+1)(n-r)}{2 \cdot 3 \cdots r(r+1)} a^{n-r}b^{r+1}$
$+ \frac{n(n-1) \cdots (n-r+1)}{2 \cdot 3 \cdots r} a^{n-r}b^{r+1}.$

ADDITION AND SUBTRACTION OF POLYNOMIALS

40. The **addition** of polynomials is illustrated by the following example.

Add $2a + 3b - 4c$ and $3a - 2b + 5c$.

The sum may be written thus:

$$(2a + 3b - 4c) + (3a - 2b + 5c).$$

By the associative law, § 5, and by § 16, we have,

$$2a + 3b - 4c + 3a - 2b + 5c.$$

By the commutative law, § 4, and by § 15, this becomes,

$$2a + 3a + 3b - 2b - 4c + 5c.$$

Again by the associative law, combining similar terms, we have,

$$5a + b + c.$$

From this example it is evident that several polynomials may be added by combining similar terms and then indicating the sum of these results.

For this purpose the polynomials are conveniently arranged so that similar terms shall be in the same column. Thus, in the above example,

$$\begin{array}{r} 2a + 3b - 4c \\ 3a - 2b + 5c \\ \hline 5a + b + c \end{array}$$

41. For **subtraction** the terms of the polynomials are arranged as for addition. The subtraction itself is then performed as in the case of monomials. See §§ 17–19.

Example. Subtract $4x - 2y + 6z$ from $3x + 6y - 3z$.

$$\begin{array}{r} 3x + 6y - 3z \\ 4x - 2y + 6z \\ \hline -x + 8y - 9z \end{array}$$

The steps are:

$$3x - 4x = -x;\ \ 6y - (-2y) = 8y;\ \ -3z - (+6z) = -9z.$$

EXERCISES

1. Add $8x^3-11x-7x^2$, $2x-6x^2+10$, $-5+4x^3+9x$, and $13x^2-5-12x^3$.

2. Add $5a^3-2a-12-10a^2$, $14-7a+a^2-9a^3$, $3a^2-13a^3+4-11a$, and $3-7a+10a^2+4a^3$.

3. From the sum of $9m^3-3m^2+4m-7$ and $3m^2-4m^3+2m+8$ subtract $4m^3-2m^2-4+8m$.

4. From the sum of $x^4-ax^3-a^2x^2-a^3x+2a^4$ and $3ax^3+7a^2x^2-5a^3x+2a^4$ subtract $3x^4+ax^3-3a^2x^2+a^3x-a^4$.

5. Add $37a-4b-17c+15d-6f-8h$ and $3c-31a+9b-5d-h-4f$.

6. Add $11q-10p-8n+3m$, $24m-17q+15p-13n$, $9n-6m-4q-7p-5n$, and $8q-4p-12m+18n$.

7. From the sum of $13a-15b-7c-11d$ and $7a-6b+8c+3d$ subtract the sum of $6d-5b-7c+2a$ and $5c-10d-28b+17a$.

8. Add $2^3\cdot 3^4x^3-2^5\cdot 3^2x^2+2^2\cdot 3^3\cdot 7x+2^2\cdot 3^2\cdot 5$, $2^2\cdot 3^3x^3-2^4\cdot 3^2\cdot 7x+2^4\cdot 3^3x^2-2^2\cdot 3^2\cdot 5^2$, and $2^3\cdot 3^3x^3-2^3\cdot 3^3\cdot 5+2^3\cdot 3^3x-2^4\cdot 3^4x^2$.

9. Add $(a+b-c)m+(a-b+c)n+(a-b-c)k$,
$(2a-3b+c)m+(b-3a+c)n+(4c+2b+a)k$,
and $(b-2c)m+(2a-2c+b)n+(2b-2a+c)k$.

10. From the sum of ax^3-bx^2+cx-d and bx^3+ax^2-dx+c subtract $(a-b)x^3+(c-a)x^2-(b+d)x-d+c$.

11. From $(m-n)(m-n)x^3+(n-m)^2x^2-(n+m)x+8$ subtract the sum of $n(m-n)x^3-4(n-m)^2x^2+(n+m)x-31$ and $2(n-m)^2x^2-m(m-n)x^3-2(n+m)x+25$.

12. Add $a^n+2a^{n+1}+a^{n+2}$ and $2a^n-4a^{n+1}+5a^{n+2}$ and from this sum subtract $7a^{n+1}-8a^n+a^{n+2}$.

REMOVAL OF PARENTHESES

42. By the principles of §§ 15–18, a parenthesis inclosing a polynomial may be removed with or without the change of sign of each term included, according as the sign $-$ or $+$ precedes the parenthesis.

In case an expression contains signs of aggregation, one within another, these may be removed *one at a time*, beginning with the *innermost*, as in the following example:

$$\begin{aligned} &a - \{b + c - [d - e + f - (g - h)]\} \\ =\ &a - \{b + c - [d - e + f - g + h]\} \\ =\ &a - \{b + c - d + e - f + g - h\} \\ =\ &a - b - c + d - e + f - g + h. \end{aligned}$$

Such involved signs of aggregation may also be removed *all at once*, beginning with *the outermost*, by observing the *number of minus signs* which affect each term, and calling the sign of any term $+$ if this number is *even*, $-$ if this number is *odd*.

Thus, in the above example, b and c are each affected by *one* minus sign, namely, the one preceding the brace. Hence we write, $a - b - c$.

d and f are each affected by *two* minus signs, namely the one before the brace and the one before the bracket, while e is affected by these two, and also by the one preceding it. Hence we write, $d - e + f$.

g is affected by the minus signs before the bracket, the brace, and the parenthesis, an *odd* number, while h is affected by these and also by the one preceding it, an *even* number. Hence we write $-g + h$.

By counting in this manner as we proceed from left to right, we give the final form at once, $a - b - c + d - e + f - g + h$.

EXERCISES

In removing the signs of aggregation in the following, either process just explained may be used. The second method is shorter and should be easily followed after a little practice.

1. $7 - \{-4 - (4 - [-7]) - (5 - [4 - 5] + 2)\}$.

2. $-[-(7-\{-4+9\}-13)-(12-3+[-7+2])]$.

3. $6-(-3-[-5+4]+\{7-3-(7-19)\}+8)$.

4. $5+[-(-\{-5-3+11\}-15)-3]+8$.

5. $4x-[3x-y-\{3x-y-(x-\overline{y-x})+x\}-3y]$.

The vinculum above $y-x$ has the same effect as a parenthesis, *i.e.* $-\overline{y-x}=-(y-x)$.

6. $3x^2-2y^2-(4x^2-\{3x^2-(y^2-2x^2)-3y^2\}-y^2+4x^2)$.

7. $7a-\{3a-[-2a-\overline{a+3}+a]-\overline{2a-5}\}$.

8. $l-(-2m-n-\{l-m\})-(5l-2n-[-3m+n])$.

9. $2d-[3d+\{2d-(e-5d)\}-(d+3e)]$.

10. $4y-(-2y-[-3y-\{-y-\overline{y-1}\}+2y])$.

11. $3x-[8x-(x-3)-\{-2x+6-\overline{8x-1}\}]$.

12. $x-(x-\{-4x-[5x-\overline{2x-5}]-[-x-\overline{x-3}]\})$.

13. $3x-\{y-[3y+2z]-(4x-[2y-3z]-\overline{3y-2z})+4x\}$.

14. $x-(-x-\{-3x-[x-\overline{2x+5}]-4\}-[2x-\overline{x-3}])$.

MULTIPLICATION OF MONOMIALS

43. In the elementary course we saw that $2^k\cdot 2^n=2^{k+n}$. More generally, if b is any number and k and n any positive integers, we have

$$\boldsymbol{b^k\cdot b^n=b^{k+n}.}$$

For by the definition of a positive integral exponent,

$$b\ =b\cdot b\cdot b\cdots\text{ to } k \text{ factors,}$$

and $\qquad b^n=b\cdot b\cdot b\cdots\text{ to } n \text{ factors.}$

Then, $\qquad b^k\cdot b^n=(b\cdot b\cdots\text{ to } k\text{ factors})(b\cdot b\cdots\text{ to } n\text{ factors})$

$$=b\cdot b\cdot b\cdots\text{ to } k+n \text{ factors.}$$

Hence, $\qquad b^k\cdot b^n=b^{k+n}.$

That is: *The product of two powers of the same base is a power of that base whose exponent is the sum of the exponents of the common base.*

44. In finding the product of two monomials, the factors may be *arranged* and *associated* in any manner, according to §§ 8, 9.

E.g. $(3ab^2) \times (5a^2b^3) = 3ab^2 \cdot 5a^2b^3$ § 9

$= 3 \cdot 5 \cdot a \cdot a^2 \cdot b^2 \cdot b^3$ § 8

$= (3 \cdot 5)(a \cdot a^2)(b^2 \cdot b^3)$ § 9

$= 15a^3b^5$ § 43

The factors in the product are arranged so as to associate those consisting of Arabic figures and also those which are powers of the same base. This arrangement and association of the factors is equivalent to multiplying either monomial by the factors of the other in succession. See § 134, E. C.

45. It is readily seen that a product is negative when it contains an *odd* number of *negative* factors; otherwise it is positive.

For by the commutative and associative laws of factors the negative factors may be grouped in *pairs*, each pair giving a *positive* product. If the number of negative factors is odd, there will be just one remaining, which makes the final product negative.

EXERCISES

Find the products of the following:

1. $2^3 \cdot 3^4 \cdot 4^7$, $2^7 \cdot 3^2 \cdot 4^2$.
2. $3 \cdot 2^4 \cdot 5^2$, $5 \cdot 2^2 \cdot 5$, $7 \cdot 2^3 \cdot 5^3$.
3. $2x^2y^3$, $5x^3y^2$, $2x^4y$.
4. $5xy$, $2x^3y$, $4xy^5$, x^2y^2.
5. $3a^5bc$, ab^2c, a^2bc^4, $4ab^5c$.
6. x^n, x^{n-1}, x^{n+1}, $2x^n$.
7. x^{m+n-1}, x^{m-n+1}, x^{2m}.
8. a^x, a^{3x-y}, a^{y-3x}.
9. a^nb^m, $a^{2n}b^{3m}$, $a^{1-3n}b^{2-4m}$.
10. $4ab^m$, $2a^3b^n$, $3a^6b^{2-m-n}$.
11. $2x^my^{m+n}$, $3x^{m-1}y^{2n-m+2}$.
12. $a^{d-2c+2}b^{m-3n}$, $a^{2c-d-1}b^{2-m+3n}$.
13. $3x^{a+3b}$, $2x^{a-2b}y^{c-3}$, $2x^{4-2a-b}y^{2c+3}$.
14. $a^{2x-3}b^{y+1}$, $a^{x+3}b^{y-1}$, $3a^3b^2$.
15. $3^{4a-2-2b} \cdot 2^{n+3-m}$, $3^{5-4a+2b} \cdot 2^{m+2-n}$.
16. x^{3x+1+y}, $x^{y-2x-1}y^{2x}$, y^{1-x}.
17. $7 \cdot 2^{3a-4}$, $3 \cdot 2^{5-2a}$, $5 \cdot 2^{3-a}$.
18. $2^{7x-1-4y}$, $3 \cdot 2^{1-5x-4y}$, $3^2 \cdot 2^{2-2x}$.
19. $3^{2-5m+3n} \cdot 2^{4a-3b}$, $3^{2-3n+6m} \cdot 2^{5+3b+5a}$.
20. $(1+a)^{7-3b+a} \cdot (1-a)^{2+a-b}$, $(1-a)^{b-a-1} \cdot (1+a)^{3b-a-6}$.

DIVISION OF MONOMIALS

46. In the Elementary Course we have seen that $x^6 \div x^4 = x^{6-4} = x^2$, etc. In general, if a is any number and m and k are any positive integers, of which m is the greater, then

$$a^m \div a^k = a^{m-k}.$$

For, since k and $m - k$ are both positive integers, we have, by § 43, $a^k a^{m-k} = a^{k+m-k} = a^m$. That is, a^{m-k} is the number which multiplied by a^k gives a product a^m, and hence by the definition of division,

$$a^m \div a^k = a^{m-k}.$$

Hence: *The quotient of two powers of the same base is a power of that base whose exponent is the exponent of the dividend minus that of the divisor.*

Under the proper interpretation of negative numbers used as exponents this principle also holds when $m < k$. This is considered in detail in § 177. We remark here that in case $m = k$, the dividend and the divisor are equal and the quotient is unity. Hence $a^m \div a^m = a^{m-m} = a^0 = 1$. See § 11.

47. We have seen in the earlier work that $\frac{4x}{6y} = \frac{2x}{3y}$, etc. In general, if a, b, and k are any number expressions:

$$\frac{ak}{bk} = \frac{a}{b}.$$

For, by definition of division, $a = \frac{a}{b} \cdot b.$ (1)

Multiplying both sides of (1) by k, $ak = \frac{a}{b} \cdot bk.$ (2)

Dividing both sides of (2) by bk, $\frac{ak}{bk} = \frac{a}{b}.$ (3)

Hence: *In dividing one algebraic expression by another, all factors common to dividend and divisor may be removed or canceled.*

EXERCISES

Divide:

1. $4 \cdot 2^4 \cdot 3^7 \cdot 5^2$ by $3 \cdot 2^3 \cdot 3^4 \cdot 5$.
2. $5 \cdot 3^7 \cdot 7^4 \cdot 13^5$ by $2 \cdot 3^5 \cdot 7^2 \cdot 13^2$.
3. $3x^7y^2z$ by $2x^3yz$.
4. $5a^5b^7c^8$ by $5a^4b^7c^4d^2$.
5. $x^{2n}y^mz^{3m}$ by $x^ny^mz^m$.
6. $a^{3n-5}y^{2n+3}$ by $a^{n+6}y^{2n+1}$.
7. $a^{c+3d+2}b^{d-2c+6}$ by a^{c+2d-4}.
8. $3^{a+2b-7} \cdot 5^{3b-2a+4}$ by $3^{b+a-8} \cdot 5^{2b-2a+3}$.
9. $a^{3+2m-3n}b^5c^{7-n}$ by $a^{2+m-4n}b^4c^{7-n}$.
10. $x^{4a-2b+1}y^{c-a+b}z^{3a+2b+c}$ by $z^{2b-c+3a}y^{a-c+b}x^{3a+b-2c}$.
11. $2^{3a-4+7b} \cdot 3^{3b-4c+6}$ by $2^{2a-5-7b} \cdot 3^{2b-6c+7}$.
12. $(x-2)^{3m+1-3n} \cdot (x+2)^{2m+2-3n}$ by $(x+2)^{1+2m-2n} \cdot (x-2)^{1-3n+2m}$.
13. $(x-y)^{5b-3c-1} \cdot (x+y)^{7c-2b+2}$ by $(x-y)^{-2-3c+5b} \cdot (x+y)^{-3-2b+7c}$.
14. $(a^2-b^2)^{8+4k+7b} \cdot (a^2-b^2)^{1-3k-5b}$ by $(a^2-b^2)^{4+k} \cdot (a^2-b^2)^{-2+2b}$.

MULTIPLICATION OF POLYNOMIALS

48. In § 87, E.C., we saw that *the product of two polynomials is equal to the sum of the products obtained by multiplying each term of one polynomial by every term of the other.*

This follows from the distributive law of multiplication, § 10. For by this law,

$$\begin{aligned}(m+n+k)(a+b+c) &= m(a+b+c)\\ &\quad + n(a+b+c)\\ &\quad + k(a+b+c).\end{aligned}$$

Applying the same law to each part, we have the product,

$$ma + mb + mc + na + nb + nc + ka + kb + kc.$$

EXERCISES

Find the following indicated products:

1. $(a+b)(a+b)$, i.e. $(a+b)^2$; also $(a-b)^2$.
2. $(a+b)(a+b)(a+b)$, i.e. $(a+b)^3$; also $(a-b)^3$.
3. $(a+b)(a+b)(a+b)(a+b)$, i.e. $(a+b)^4$; also $(a-b)^4$.
4. $(a^2+2ab+b^2)(a^2+2ab+b^2)(a+b)$.
5. $(a^2-2ab+b^2)(a^2-2ab+b^2)(a-b)$.

Note that Ex. 4 gives $(a+b)^5$ and Ex. 5 gives $(a-b)^5$.

6. $(a^2+2ab+b^2)^3$; also $(a+b)^6$.
7. $(a^2-2ab+b^2)^3$; also $(a-b)^6$.
8. Collect in a table the following products:

$(a+b)^2$, $(a-b)^2$, $(a+b)^3$, $(a-b)^3$, $(a+b)^4$.
$(a-b)^4$, $(a+b)^5$, $(a-b)^5$, $(a+b)^6$, $(a-b)^6$.

9. From the above table answer the following questions:

(*a*) How many terms in each product, compared with the exponent of the binomial?

(*b*) Tell how the signs occur in the various cases.

(*c*) How do the exponents of a proceed? of b?

(*d*) Make a table of the coefficients alone and memorize this.

E.g. For $(a+b)^5$, they are 1, 5, 10, 10, 5, 1.

10. From the above question make rules for finding the powers of a binomial up to the fifth. Apply these rules to the following examples:

(*a*) $(2x+3y)^2$.
(*b*) $(3x-5y)^2$.
(*c*) $\left(\frac{x}{2}-\frac{y}{3}\right)^2$.
(*d*) $(2x+3y)^3$.
(*e*) $(3a-b^2)^4$.
(*f*) $(\frac{1}{2}ax-\frac{2}{3}by)^3$.
(*g*) $(2m-n)^5$.
(*h*) $\left(\frac{a}{3}-\frac{b}{4}\right)^4$.
(*i*) $\left(2x-\frac{y}{2}\right)^5$.
(*j*) $\left(\frac{a}{2}-x^2\right)^3$.
(*k*) $(9x-2y)^2$.
(*l*) $(1-2x)^4$.
(*m*) $(1+3x)^5$.

11. $(a+b+c)^2$. 12. $(a+b-c)^2$. 13. $(a-b-c)^2$.

14. From Exs. 11–13 deduce a rule for squaring a trinomial, and apply it to the following:

(*a*) $(2x-3y+4z)^2$, (*b*) $(\frac{1}{2}m-\frac{1}{3}n+\frac{1}{4}r)^2$.

15. $(x+y+z+v)^2$. **16.** $(x-y+z-v)^2$.

17. From Exs. 15, 16 deduce a rule for squaring a polynomial, and apply it to the following:

(*a*) $(2x+y+z+3w)^2$.

(*b*) $(a^3+3a^2b+3ab^2+b^3)^2$.

(*c*) $(a^3-3a^2b+3ab^2-b^3)^2$.

18. $(a-b)(a^2+ab+b^2)$.

19. $(a+b)(a^2-ab+b^2)$.

20. $(a-b)(a^3+a^2b+ab^2+b^3)$.

21. $(a+b)(a^3-a^2b+ab^2-b^3)$.

22. $(a-b)(a^4+a^3b+a^2b^2+ab^3+b^4)$.

23. $(a+b)(a^4-a^3b+a^2b^2-ab^3+b^4)$.

24. $(a-b)(a^5+a^4b+a^3b^2+a^2b^3+ab^4+b^5)$.

25. $(a+b)(a^5-a^4b+a^3b^2-a^2b^3+ab^4-b^5)$.

26. $(1-r)(a+ar+ar^2+ar^3)$.

27. $(1-r)(a+ar+ar^2+ar^3+ar^4+ar^5)$.

DIVISION OF POLYNOMIALS

49. According to the distributive law of division, § 30, a *polynomial is divided by a monomial* by dividing each term separately by the monomial.

E.g. $\frac{ab+ac-ad}{a}=\frac{ab}{a}+\frac{ac}{a}-\frac{ad}{a}=b+c-d.$

A polynomial is divided by a polynomial by separating the dividend into polynomials, each of which is the product of the divisor and a monomial. Each of these monomial factors is a part of the quotient, their sum constituting the whole quotient. The parts of the dividend are found one by one as the work proceeds. See §§ 148–149, E.C. This is best shown by an example.

Dividend,	$a^4 + a^3 - 4a^2 + 5a - 3$	$a^2 + 2a - 3$, Divisor.
1st part of dividend:	$a^4 + 2a^3 - 3a^2$	$a^2 - a + 1$, Quotient.
	$-a^3 - a^2 + 5a - 3$	
2d part of dividend:	$-a^3 - 2a^2 + 3a$	
	$a^2 + 2a - 3$	
3d part of dividend:	$a^2 + 2a - 3$	
	0	

The three parts of the dividend are the products of the divisor and the three terms of the quotient. If after the successive subtraction of these parts of the dividend the remainder is zero, the division is exact. In case the division is not exact, there is a final remainder such that

$$\textit{Dividend} = \textit{Quotient} \times \textit{divisor} + \textit{Remainder}.$$

In symbols we have $\mathbf{D} = \mathbf{Q} \cdot \mathbf{d} + \mathbf{R}$.

EXERCISES

Divide:

1. $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$ by $x^2 + 2xy + y^2$.
2. $x^8 + x^4y^4 + y^8$ by $x^4 - x^2y^2 + y^4$.
3. $x^5 - y^5$ by $x - y$.
4. $x^8 - y^8$ by $x^3 + x^2y + xy^2 + y^3$.
5. $x^9 + y^9$ by $x^2 - xy + y^2$.
6. $x^8 - y^8$ by $x^2 - y^2$.
7. $a^6 + b^6$ by $a^2 + b^2$.
8. $x^{6a} - y^{6b}$ by $x^{2a} - y^{2b}$.
9. $a^{10} - a^5b^5 + b^{10}$ by $a^2 - ab + b^2$.
10. $2x^4 - 3x^3b + 6x^2b^2 - xb^3 + 6b^4$ by $x^2 - 2xb + 3b^2$.
11. $2x^6 - 5x^5 + 6x^4 - 6x^3 + 6x^2 - 4x + 1$ by $x^4 - x^3 + x^2 - x + 1$.
12. $26a^3b^3 + a^6 + 6b^6 - 5a^5b - 17ab^5 - 2a^4b^2 - a^2b^4$ by $a^2 - 3b^2 - 2ab$.
13. $x^4 + 2x^3 - 7x^2 - 8x + 12$ by $x^2 - 3x + 2$.
14. $4b^2 + 4ab + a^2 - 12bc - 6ac + 9c^2$ by $2b + a - 3c$.
15. $x^4 + 4xy^3 - 4xyz + 3y^4 + 2y^2z - z^2$ by $x^2 - 2xy + 3y^2 - z$.
16. $a^2b^2c + 3a^2b^3 - 3abc^3 - a^2c^3 + b^5 - 4b^3c^2 + 3ab^3c + 3bc^4 - 3a^2bc^2$ by $b^2 - c^2$.

CHAPTER III

INTEGRAL EQUATIONS OF THE FIRST DEGREE IN ONE UNKNOWN

50. When in an algebraic expression a letter is replaced by another number symbol, this is called a **substitution on that letter.**

E.g. In the expression, $2a + 5$, if a is replaced by 3, giving $2 \cdot 3 + 5$, this is a substitution on the letter a.

51. An equality containing a single letter is said to be satisfied by any substitution on that letter which reduces both members of the equality to the same number.

E.g. $4x + 8 = 24$ is satisfied by $x = 4$, since $4 \cdot 4 + 8 = 24$.

We notice, however, that the substitution *must not reduce the denominator of any fraction to zero.*

Thus $x = 2$ does *not* satisfy $\frac{x^2 - 4}{x - 2} = 8$ although it reduces the left member of the equation to $\frac{0}{0}$, which by § 24 equals 8 or *any other number whatever.*

On the other hand, $x = 6$ satisfies this equation, since

$$\frac{6^2 - 4}{6 - 2} = \frac{32}{4} = 8.$$

52. An equality in two or more letters is **satisfied** by any simultaneous substitutions on these letters which reduce both members to the same number.

E.g. $6a + 3b = 15$ is satisfied by $a = 2, b = 1$; $a = \frac{3}{2}, b = 2$; $a = 1, b = 3$, etc.

$\frac{x^2 - y^2}{x^2 + 2xy + y^2} = \frac{1}{2}$ is satisfied by $x = 3, y = 1$, but is *not* satisfied by any values of x and y such that $x = -y$, since these reduce the denominator (and also the numerator) to zero. See § 24.

53. An equality is said to be an **identity in all its letters**, or simply an **identity**, if it is satisfied by *every possible substitution* on these letters, not counting those which make any denominator zero.

If an equality is an identity, both members will be reduced to the same expression when all indicated operations are performed as far as possible.

The members of an identity are called **identical expressions.**

Thus in the identity $(a+b)^2 \equiv a^2 + 2\,ab + b^2$, performing the indicated operation in the first member reduces it to the same form as the second.

54. An equality which is not an identity is called an **equation of condition** or simply an **equation.**

The members of an equation *cannot* be reduced to the same expression by performing the indicated operations.

E.g. $(x-2)(x-3)=0$ cannot be so reduced. This is an equation which is satisfied by $x=2$ and $x=3$. See § 22.

55. In an *equation* containing several letters any one or more of them may be regarded as **unknown**, the remaining ones being considered **known.** Such an equation is said to be satisfied by any substitution on the *unknown* letters which reduces it to an *identity in the remaining letters.*

E.g. $x^2 - t^2 = sx + st$ is an equation in s, x, or t, or in any pair of these letters, or in all three of them.

As an equation in x it is satisfied by $x = s + t$, since this substitution reduces it to the identity in s and t,

$$s^2 + 2\,st \equiv s^2 + 2\,st.$$

As an equation in s it is satisfied by $s = x - t$, since this substitution reduces it to the identity in x and t,

$$x^2 - t^2 \equiv x^2 - t^2.$$

Any number expression which satisfies an equation in one unknown is called a **root of the equation.**

E.g. $s+t$ is a root of the equation $x^2 - t^2 = sx + st$, when x is the unknown, and $x - t$ is a root when s is the unknown.

56. An equation is **rational in a given letter** if every term in the equation is rational with respect to that letter.

An equation is **integral** in a given letter if every term is rational and integral in that letter.

57. The **degree** of a rational, integral equation in a given letter is the highest exponent of that letter in the equation.

In determining the degree of an equation according to this definition it is necessary that all indicated multiplications be performed as far as possible.

E.g. $(x-2)(x-3)=0$ is of the 2d degree in x, since it reduces to $x^2-5x+6=0$.

EQUIVALENT EQUATIONS

58. Two equations are said to be **equivalent** if every root of either is also a root of the other.

In the Elementary Course, § 36, we found that an equation may be changed into an equivalent equation by certain operations, which are now further considered in principles **1**, **2**, and **3** below:

59. Principle 1. *If one rational, integral equation is derived from another by performing the indicated operations, then the two equations are equivalent.*

This is evident, since in performing the indicated operations each expression is replaced by another identically equal to it. Hence any expression which satisfies the given equation must satisfy the other, and conversely.

E.g. $10x=50$ is equivalent to $3x+7x=50$, since $3x+7x\equiv 10x$; and $8(2x-3y)=2y-1$ is equivalent to $16x-24y=2y-1$, since $8(2x-3y)\equiv 16x-24y$.

60. Principle 2. *If any equation is derived from another by adding the same expression to each member, or by subtracting the same expression from each member, then the equations are equivalent.*

For simplicity consider an equation,

$$M=N, \qquad (1)$$

containing only one unknown, x. Add to each member an expression A, which may or may not contain x.

Then $$M + A = N + A \tag{2}$$

is easily seen to have the same roots as (1). For if a certain value of x makes M equal N, this will also make $M + A$ equal $N + A$, since any value whatever of x makes A equal A. Hence any number which is a root of (1) is also a root of (2).

Again, any value of x which makes $M + A = N + A$ will also make M equal N, since every value of x makes A equal A. Hence any number which is a root of (2) is also a root of (1). We have therefore shown that (1) and (2) are equivalent according to the definition, § 58.

This argument is based on axioms I and IV. By use of the same axioms we may show that $M - A = N - A$ and $M = N$ are equivalent equations.

61. It follows that *any equation can be reduced to an equivalent equation of the form* $\boldsymbol{R} = \mathbf{0}$.

For if an equation is in the form $M = N$, then by principle **2** it is equivalent to $M - N = N - N = 0$, which is in the form $R = 0$.

62. Principle 3. *If one equation is derived from another by multiplying or dividing each member by the same expression, then the equations are equivalent, provided the original equation is not multiplied or divided by zero or by an expression containing any of the unknowns of the equation.*

This principle follows from axioms V and IX by argument similar to that used in § 60. In this case, however, the expression A must not contain x and must not be zero, as was possible in principle **2**.

E.g. $x + 1 = 5$ and $(x - 1)(x + 1) = 5(x - 1)$ are not equivalent equations, since the first has only the root $x = 4$, while the second has in addition the root $x = 1$, as may be easily verified.

Similarly, $x - 5 = 7$ and $0 \cdot (x - 5) = 0 \cdot 7$ are not equivalent, since the first has only the root $x = 12$, while the second is satisfied by any number whatever.

63. The ordinary processes of solving equations depend upon principles **1**, **2**, and **3**, as is illustrated by the following examples:

Ex. 1.
$$(x+4)(x+5)=(x+2)(x+6). \qquad (1)$$
$$x^2+9x+20=x^2+8x+12. \qquad (2)$$
$$x=-8. \qquad (3)$$

By principle **1**, (1) and (2) are equivalent, and by principle **2**, (2) and (3) are equivalent. Hence (1) and (3) are equivalent. That is, -8 is the solution of (1).

Ex. 2.
$$\tfrac{2}{3}x+\tfrac{4}{3}=4. \qquad (1)$$
$$2x+4=12. \qquad (2)$$
$$2x=8. \qquad (3)$$
$$x=4. \qquad (4)$$

By principle **3**, (1) and (2) are equivalent. By principle **2**, (2) and (3) are equivalent. By **3**, (3) and (4) are equivalent. Hence (1) and (4) are equivalent and 4 is the solution of (1).

These principles are stated for equations, but they apply equally well to identities, inasmuch as the identities are changed into other identities by these operations.

64. If an *identity* is reduced to the form $R=0$, § 61, and all the indicated operations are performed, then it becomes $0=0$. See § 53. Conversely, if an equality may be reduced to the form $0=0$, it is an identity. This, therefore, is a *test* as to whether an equality is an identity.

E.g. $(x+4)^2=x^2+8x+16$ is an identity, since in $x^2+8x+16-x^2-8x-16=0$ all terms cancel, leaving $0=0$.

EXERCISES

In the following, determine which numbers or sets of numbers, if any, of those written to the right, satisfy the corresponding equation.

Remember that no substitution is legitimate which reduces any denominator to zero.

1. $4(x-1)(x-2)(x-3)=3(x-2)(x-3)$. 1, 2, 3, 4.

2. $\dfrac{x^2-16}{x+5}=(x-4)(x+6)$. 2, 4, 6.

3. $\dfrac{x+3}{\sqrt{x^2+7}}=x-\dfrac{3}{2}$. 2, 3, $\frac{1}{2}$.

4. $\dfrac{(x-3)(x-2)}{x^2-7x+10}=x^2-5x+6$. 2, 3, 0, -2.

5. $\dfrac{a^2+9a+20}{a^2+8a+16}=(a+4)(a-4)(a+5)$. 4, -4, 5, -5.

6. $3a+4b=12$. $\begin{cases}a=0,\\b=3.\end{cases}$ $\begin{cases}a=4,\\b=0.\end{cases}$ $\begin{cases}a=2,\\b=2.\end{cases}$

7. $\dfrac{369(a-b)}{a^2+b^2}=a+b$. $\begin{cases}a=0,\\b=0.\end{cases}$ $\begin{cases}a=1,\\b=1.\end{cases}$ $\begin{cases}a=5,\\b=4.\end{cases}$

8. $\dfrac{x^2-y^2}{x-y}=(x-2)(y-1)$. $\begin{cases}x=1,\\y=0.\end{cases}$ $\begin{cases}x=1,\\y=1.\end{cases}$ $\begin{cases}x=2,\\y=2.\end{cases}$

9. $\dfrac{u^3-v^3}{u^2-v^2}=(u^2+uv+v^2)(u-v)$. $\begin{cases}u=1,\\v=1.\end{cases}$ $\begin{cases}u=1,\\v=0.\end{cases}$ $\begin{cases}u=-1,\\v=0.\end{cases}$

10. $\dfrac{(r-s)(r+s)(r^2+s^2)}{r^3+rs^2-r^2s-s^3}=(r^2-s^2)(2r-3s)$. $r=1, s=1$;
$r=1, s=-1$; $r=2, s=-2$; $r=a, s=-a$.

11. $a+b+c=6$. $a=1, b=2, c=3$;
$a=3, b=3, c=0$; $a=10, b=0, c=-4$.

12. $\dfrac{a-b-c}{\sqrt{a^2+b^2+c^2}}=\dfrac{3a-2c+5b+2}{10}$. $a=8, b=0, c=6$;
$a=1, b=4, c=2$; $a=0, b=0, c=-4$.

13. $\dfrac{(a-b)(b-c)(c-a)}{ac-bc-a^2+ba}=(b-b)(c-a)$. $a=2, b=1, c=1$;
$a=3, b=2, c=3$; $a=6, b=6, c=0$.

14. $(x-z)(x-y)(y-z)=8\,xyz\,(x^2-y^2)(y^2-z^2)(z^2-x^2)$.
$x=1,\ y=0,\ z=1;\ x=1,\ y=2,\ z=3$.

15. $x^3+3\,x^2y+3\,xy^2+y^3=(x+y)^3$. $\begin{cases}x=1,\\ y=1.\end{cases}$ $\begin{cases}x=1,\\ y=2.\end{cases}$

16. Show by reducing the equality in Ex. 15 to the form $R=0$ that it is satisfied by any pair of values whatsoever for x and y, *e.g.*, for $x=348764$, $y=594021$. What kind of an equality is this?

Which of the following four equalities are identities?

17. $12(x+y)^2+17(x+y)-7=(3\,x+3\,y-1)(4\,x+4\,y+7)$.

18. $\dfrac{a^5-b^5}{a-b}=a^4+a^3b+a^2b^2+ab^3+b^4$.

19. $\dfrac{a^5+b^5}{a+b}=a^4-a^3b+a^2b^2-ab^3+b^4$.

20. $2(a-b)^2+5(a+b)+8\,ab=(2\,a+2\,b+1)(a+b+1)$.

Solve the following equations, and verify results in 21–25:

21. $(2\,a+3)(3\,a-2)=a^2+a\,(5\,a+3)$.

22. $6\,(b-4)^2=-5-(3-2\,b)^2-5\,(2+b)(7-2\,b)$.

23. $(y-3)^2+(y-4)^2-(y-2)^2-(y-3)^2=0$.

24. $(x-3)(3\,x+4)-(x-4)(x-2)=(2\,x+1)(x-6)$.

25. $2(3\,r-2)(4\,r+1)+(r-4)^3=(r+4)^3-2$.

26. $a^3-c+b^3c+abc=b$. (Solve for c.)

In the next three examples solve for y.

27. $(b-2)^2(b-y)-3\,by+(2\,b+1)(b-1)=3-2\,b$.

28. $ny\,(y+n)-(y+m)(y+n)(m+n)+my\,(y+m)=0$.

29. $(m+n)(n+b-y)+(n-m)(b-y)=n\,(m+b)$.

In the next eight examples solve for x.

30. $2(12-x)+3(5x-4)+2(16-x)=12(3+x)$.

31. $(b-a)x-(a+b)x+4a^2=0$.

32. $(x-a)(b-c)+(b-a)(x-c)-(a-c)(x-b)=0$.

33. $(x-3)(x-7)-(x-5)(x-2)+12=2(x-1)$.

34. $(a+b)^2+(x-b)(x-a)-(x+a)(x+b)=0$.

35. $\frac{7}{12}(5x-1)+\frac{5}{16}(2-3x)+\frac{1}{3}(4+x)=\frac{3}{8}(1+2x)-\frac{9}{16}$.

36. $a(x-b)-(a+b)(x+b-a)=b(x-a)+a^2-b^2$.

37. $(l-m)(x-n)+2l(m+n)=(l+m)(x+n)$.

Solve each of the following equations for each letter in terms of the others:

38. $l(W+w')=l'W'$.

39. $(v-n)d=(v-n_1)d_1$.

40. $m_2s_2(t_2-t)=(m+m_1)(t-t_1)$.

41. $(m+m_1)(t_1-t)=lm_2+m_2t$.

PROBLEMS

1. What number must be added to each of the numbers 2, 26, 10 in order that the product of the first two sums may equal the square of the last sum?

2. What number must be subtracted from each of the numbers 9, 12, 18 in order that the product of the first two remainders may equal the square of the last remainder?

3. What number must be added to each of the numbers a, b, c in order that the product of the first two sums may equal the square of the last?

Note that problem 1 is a special case of 3. Explain how 2 may also be made a special case of 3.

4. What number must be added to each of the numbers a, b, c, d in order that the product of the first two sums may equal the product of the last two?

5. State and solve a problem which is a special case of problem 4.

6. What number must be added to each of the numbers a, b, c, d in order that the sum of the squares of the first two sums may equal the sum of the squares of the last two?

7. State and solve a problem which is a special case of problem 6.

8. What number must be added to each of the numbers a, b, c, d in order that the sum of the squares of the first two sums may be k more than twice the product of the last two?

9. State and solve a problem which is a special case of problem 8.

10. The radius of a circle is increased by 3 feet, thereby increasing the area of the circle by 50 square feet. Find the radius of the original circle.

The area of a circle is πr^2. Use $3\frac{1}{7}$ for π.

11. The radius of a circle is decreased by 2 feet, thereby decreasing the area by 36 square feet. Find the radius of the original circle.

12. State and solve a general problem of which 10 is a special case.

13. State and solve a general problem of which 11 is a special case.

How may the problem stated under 12 be interpreted so as to include the one given under 13?

14. Each side of a square is increased by a feet, thereby increasing its area by b square feet. Find the side of the original square.

Interpret this problem if a and b are both negative numbers.

15. State and solve a problem which is a special case of 14, (1) when a and b are both positive, (2) when a and b are both negative.

16. Two opposite sides of a square are each increased by a feet and the other two by b feet, thereby producing a rectangle whose area is c square feet greater than that of the square. Find the side of the square.

Interpret this problem when a, b, and c are all negative numbers.

17. State and solve a problem which is a special case of 16, (1) when a, b, and c are all positive, (2) when a, b, and c are all negative.

18. A messenger starts for a distant point at 4 A.M., going 5 miles per hour. Four hours later another starts from the same place, going in the same direction at the rate of 9 miles per hour. When will they be together? When will they be 8 miles apart? How far apart will they be at 2 P.M.?

For a general explanation of problems on motion, see p. 95, E. C.

19. One object moves with a velocity of v_1 feet per second and another along the same path in the same direction with a velocity of v_2 feet. If they start together, how long will it require the latter to gain n feet on the former?

From formula (2), p. 96, E. C., we have $t = \dfrac{n}{v_2 - v_1}$.

Discussion. If $v_2 > v_1$ and $n > 0$, the value of t is positive, *i.e.* the objects will be in the required position some time *after* the time of starting.

If $v_2 < v_1$ and $n > 0$, the value of t is negative, which may be taken to mean that if the objects had been moving before the instant taken in the problem as the time of starting, then they would have been in the required position some time *earlier*.

If $v_2 = v_1$ and $n \neq 0$, the solution is impossible. See § 25. This means that the objects will never be in the required position. If $v_1 = v_2$ and $n = 0$, the solution is indeterminate. See § 24. This may be interpreted to mean that the objects are always in the required position.

20. State and solve a problem which is a special case of 19 under each of the conditions mentioned in the discussion.

21. At what time after 5 o'clock are the hands of a clock first in a straight line?

22. Saturn completes its journey about the sun in 29 years and Uranus in 84 years. How many years elapse from conjunction to conjunction? See figure, p. 269, E. C.

23. An object moves in a fixed path at the rate of v_1 feet per second, and another which starts a seconds later moves in the same path at the rate of v_2 feet per second. In how many seconds will the latter overtake the former?

24. In problem 23 how long before they will be d feet apart?

If in problem 24 d is zero, this problem is the same as 23. If d is not zero and a is zero, it is the same as problem 19.

25. A beam carries 3 weights, one at each end weighing 100 and 120 pounds respectively, and the third weighing 150 pounds 2 feet from its center, where the fulcrum is. What is the length of the beam if this arrangement makes it balance?

For a general explanation of problems involving the lever, see pp. 98 and 270, E. C.

26. A beam whose fulcrum is at its center is made to balance when weights of 60 and 80 pounds are placed at one end and 2 feet from that end respectively, and weights of 50 and 100 pounds are placed at the other end and 3 feet from it respectively. Find the length of the beam.

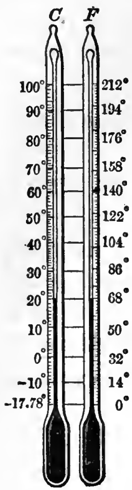
27. A man weighing 190 pounds is trying to pry up a rock by use of a plank 12 feet long and a block on which the plank rests as a fulcrum. Find the weight of the stone if he can just lift it when the fulcrum is 2 feet from the stone. How heavy a stone could he lift by putting the fulcrum 3 feet away from it? Four feet away from it?

28. A man can do a piece of work in 16 days, another in 18 days, and a third in 15 days. How many days will it require all to do it when working together?

29. A can do a piece of work in a days, B can do it in b days, C in c days, and D in d days. How long will it require all to do it when working together?

PROBLEMS ON THERMOMETER READINGS

There are two kinds of thermometers in use in this country, called the Fahrenheit and Centigrade, the former for common purposes, and the latter for scientific records and investigations. Hence it frequently becomes necessary to translate readings from one kind to the other.



The freezing and boiling points are two fixed temperatures by means of which the computations are made. On the Centigrade these are marked 0° and 100° respectively and on the Fahrenheit they are marked 32° and 212° respectively. See the cut. Hence between the two fixed points there are 100 degrees Centigrade and 180 degrees Fahrenheit.

That is, 100 degree spaces on the Centigrade correspond to 180 degree spaces on the Fahrenheit.

Hence 1° *Centigrade corresponds to* $\frac{9}{5}$° *Fahrenheit*, or 1° *Fahrenheit corresponds to* $\frac{5}{9}$° *Centigrade.*

All problems comparing the two thermometers are solved by reference to these fundamental relations.

1. If the temperature falls 15 degrees Centigrade, how many degrees Fahrenheit does it fall?

2. If the temperature rises 18 degrees Fahrenheit, how many degrees Centigrade does it rise?

3. Translate +25° Centigrade into Fahrenheit reading.

25° Centigrade equals $\frac{9}{5} \cdot 25° = 45°$ Fahrenheit.

45° above the freeezing point = 45° + 32° above 0° Fahrenheit. Hence, calling the Fahrenheit reading F, we have $F = 32 + \frac{9}{5} \cdot 25$.

4. Translate +14° Centigrade into Fahrenheit reading.

Reasoning as before, $F = 32 + \frac{9}{5} \cdot 14$.

From the two preceding problems we have the formula

$$F = 32 + \tfrac{9}{5} C. \quad (1)$$

Translate this into words, understanding that F and C stand for readings on the respective thermometers.

5. Solve the above equation for C in terms of F and find

$$C = \tfrac{5}{9}(F - 32). \quad (2)$$

6. Graph the equation $F = 32 + \frac{9}{5}C$, marking Centigrade readings on the horizontal axis and Fahrenheit on the vertical. From the graph answer the following questions:

7. Find the Fahrenheit reading when the Centigrade is zero; 10°; 20°; 30°; −5°; −15°.

8. Find Centigrade reading when the Fahrenheit is 32°; 50°; 68°; 86°; 41°; 59°; 14°; 0°.

It thus appears that the graph, so far as it extends, shows to the eye all the information contained in the equation

$$F = 32 + \tfrac{9}{5} C.$$

9. By substituting in the equation $F = 32 + \frac{9}{5}C$, find the Fahrenheit reading when the Centigrade is 60°; 100°; −40° or the freezing point of mercury.

10. In like manner find the Centigrade reading when the Fahrenheit is −40°; −20°; 98° or blood heat; 212° or the boiling point.

11. What is the temperature Centigrade when the sum of the Centigrade and Fahrenheit readings is 102°?

12. What is the temperature Fahrenheit when the sum of the Centigrade and Fahrenheit readings is zero?

13. What is the temperature Centigrade when the sum of the Centigrade and Fahrenheit readings is 140°?

14. What is the temperature in each reading when the Fahrenheit is 50° higher than the Centigrade?

CHAPTER IV

INTEGRAL LINEAR EQUATIONS IN TWO OR MORE UNKNOWNS

INDETERMINATE EQUATIONS

65. If a single equation contains two unknowns, an **unlimited number** of pairs of numbers may be found which satisfy the equation.

E.g. In the equation, $y = 2x + 1$, by assigning any value to x, a corresponding value of y may be found such that the two together satisfy the equation.

Thus, $x = -3$, $y = -5$; $x = 0$, $y = 1$; $x = 2$, $y = 5$, are pairs of numbers which satisfy this equation.

For this reason a single equation in two unknowns is called an **indeterminate** equation, and the unknowns are called **variables.** A **solution** of such an equation is any pair of numbers which satisfy it.

A picture or map of the real (see §§ 135, 195) solutions of an indeterminate equation in two variables may be made by means of the **graph** as explained in §§ 111, 112, E. C.

66. The **degree** of an equation in two or more letters is the sum of the exponents of those letters in that one of its terms in which this sum is greatest. See § 114, E. C.

E.g. $y = 2x + 1$ is of the *first degree* in x and y. $y^2 = 2x + y$ and $y = 2xy + 3$ are each of the *second degree* in x and y.

An equation of the first degree in two variables is called a **linear** equation, since it can be shown that the graph of every such equation is a straight line.

67. It is often important to determine those solutions of an indeterminate equation which are **positive integers**, and for this purpose the graph is especially useful.

Ex. 1. Find the positive integral solutions of the equation

$$3x + 7y = 42.$$

Solution. Graph the equation carefully on cross-ruled paper, finding it to cut the x-axis at $x = 14$ and the y-axis at $y = 6$.

Look now for the *corner points* of the unit squares through which this straight line passes. The coördinates of these points, if there are such points, are the solutions required. In this case the line passes through only one such point, namely the point (7, 3). Hence the solution sought is $x = 7$, $y = 3$.

Ex. 2. Find the *least* positive integers which satisfy

$$7x - 3y = 17.$$

Solution. This line cuts the x-axis at $x = 2\frac{3}{7}$ and the y-axis at $y = -5\frac{2}{3}$. On locating these points as accurately as possible, the line through them *seems* to cut the corner points (5, 6) and (8, 13). The coördinates of both these points satisfy the equation. Hence the solution sought is $x = 5$, $y = 6$.

EXERCISES

Solve in positive integers by means of graphs, and check:

1. $x + y = 7.$
2. $x + y = 3.$
3. $x - 27 = -9y.$
4. $7y - 112 = -4x.$
5. $90 - 5x = 9y.$
6. $5x = 29 - 3y.$
7. $140 - 7x - 10y = 0.$
8. $8 - 2x - y = 0.$

Solve in least positive integers, and check:

9. $7x = 3y + 21.$
10. $5x - 4y = 20.$
11. $4x = 9y - 36.$
12. $5x - 2y + 10 = 0.$

68. In the case of two indeterminate equations, each of the first degree in two variables, the coördinates of the point of intersection of their graphs form a solution of *both equations.*

Since these graphs are straight lines, they have *only one point* in common, and hence there is only *one solution* of the given pair of equations.

E.g. On graphing $x + y = 4$ and $y - x = 2$, the lines are found to intersect in the point (1, 3). Hence the solution of this pair of equations is

$$x = 1,\ y = 3.$$

EXERCISES

Graph the following and thus find the solution of each pair of equations. Check by substituting in the equations.

1. $\begin{cases} 3x - 2y = -2, \\ x + 7y = 30. \end{cases}$

2. $\begin{cases} x + y = 2, \\ 3x + 2y = 3. \end{cases}$

3. $\begin{cases} x - 4y = 1, \\ 2x - y = -5. \end{cases}$

4. $\begin{cases} x = -1, \\ 2x - 3y = 1. \end{cases}$

5. $\begin{cases} 4x = 2y + 6, \\ x - 5 = y - 1. \end{cases}$

6. $\begin{cases} x = y - 5, \\ 5y = x + 9. \end{cases}$

7. $\begin{cases} 8x = 7y, \\ x + 3 = 5y + 3. \end{cases}$

8. $\begin{cases} y = 1, \\ 3y + 4x = y. \end{cases}$

9. $\begin{cases} 2x - 4y = 4, \\ x - y = 6y - 3. \end{cases}$

10. $\begin{cases} x = 4, \\ y + x = 8. \end{cases}$

11. $\begin{cases} y = -3, \\ 3x + 2y = 3. \end{cases}$

12. $\begin{cases} x = -2, \\ y = 5. \end{cases}$

SOLUTION BY ELIMINATION

69. The solution of a pair of equations such as the foregoing may be obtained without the use of the graph by the process called **elimination**. See pages 116–121, E. C.

70. Elimination by **substitution** consists in expressing one variable in terms of the other in one equation and substituting this result in the other equation, thus obtaining an equation in which only one variable appears. See § 122, E. C.

71. Elimination by **addition or subtraction** consists in making the coefficients of one variable the same in the two equations (§ 62), so that when the members are added or subtracted this variable will not appear in the resulting equation. See § 121, E. C.

72. Elimination by **comparison** is a third method, which consists in expressing the same variable in terms of the other in each equation and equating these two expressions to each other.

As an example of elimination by comparison, solve

$$\begin{cases} 3y + x = 14, & (1) \\ 2y - 5x = -19. & (2) \end{cases}$$

From (1), $x = 14 - 3y.$ (3)

From (2), $x = \dfrac{19 + 2y}{5}.$ (4)

From (3) and (4), $14 - 3y = \dfrac{19 + 2y}{5}.$ (5)

Solving (5), $y = 3.$

Substituting in (1), $x = 5.$

Check by substituting $x = 5$, $y = 3$ in both (1) and (2).

In applying any method of elimination it is desirable first to reduce each equation to the standard form: $ax + by = c$. See § 123, E. C.

EXERCISES

Solve the following pairs of equations by one of the processes of elimination.

1. $\begin{cases} 3x + 2y = 118, \\ x + 5y = 191. \end{cases}$

2. $\begin{cases} 5x - 8\frac{1}{2} = 7y - 44, \\ 2x = y + \frac{5}{7}. \end{cases}$

3. $\begin{cases} 6x - 3y = 7, \\ 2x - 2y = 3. \end{cases}$

4. $\begin{cases} 3x + 7y - 341 = 7\frac{1}{2}y + 43\frac{1}{2}x, \\ 2\frac{1}{2}x + \frac{1}{2}y = 1. \end{cases}$

5. $\begin{cases} 5x - 11y - 2 = 4x, \\ 5x - 2y = 63. \end{cases}$

6. $\begin{cases} 3y + 40 = 2x + 14, \\ 9y - 347 = 5x - 420. \end{cases}$

7. $\begin{cases} 5y - 3x + 8 = 4y + 2x + 7, \\ 4x - 2y = 3y + 2. \end{cases}$

8. $\begin{cases} 6y - 5x = 5x + 14, \\ 3y - 2x - 6 = 5 + x. \end{cases}$

9. $\begin{cases}(x+5)(y+7)=(x+1)(y-9)+112,\\ 2x+10=3y+1.\end{cases}$

10. $\begin{cases}73-7y=5x,\\ 2y-3x=12.\end{cases}$

11. $\begin{cases}ax=by,\\ x+y=c.\end{cases}$

12. $\begin{cases}\frac{x}{a}=\frac{y}{b},\\ x+y=s.\end{cases}$

13. $\begin{cases}x+y=a,\\ x-y=b.\end{cases}$

14. $\begin{cases}ax+by=c,\\ fx+gy=h.\end{cases}$

15.* $\begin{cases}\frac{3}{x}-\frac{5}{y}=6,\\ \frac{2}{x}+\frac{3}{y}=2.\end{cases}$

16.* $\begin{cases}\frac{a}{x}+\frac{b}{y}=c,\\ \frac{f}{x}+\frac{g}{y}=h.\end{cases}$

SOLUTION BY FORMULA

73. We now proceed to a more general study of a pair of linear equations in two variables.

Ex. 1. Solve

$$\begin{cases}2x+3y=4, & (1)\\ 5x+6y=7. & (2)\end{cases}$$

Multiplying (1) by 5 and (2) by 2,

$$5\cdot 2x+5\cdot 3y=5\cdot 4, \qquad (3)$$

$$2\cdot 5x+2\cdot 6y=2\cdot 7. \qquad (4)$$

Subtracting (3) from (4), $(2\cdot 6-5\cdot 3)y=2\cdot 7-5\cdot 4.$ (5)

Solving for y, $$y=\frac{2\cdot 7-5\cdot 4}{2\cdot 6-5\cdot 3}=\frac{-6}{-3}=2. \qquad (6)$$

In like manner, solving for x by eliminating y,

we have $$x=\frac{4\cdot 6-7\cdot 3}{2\cdot 6-5\cdot 3}=\frac{3}{-3}=-1. \qquad (7)$$

Ex. 2. In this manner, solving,

$$\begin{cases}7x+9y=71,\\ 2x+3y=48,\end{cases}$$

we find $$x=\frac{71\cdot 3-48\cdot 9}{7\cdot 3-2\cdot 9} \text{ and } y=\frac{7\cdot 48-2\cdot 71}{7\cdot 3-2\cdot 9}.$$

* Let $\frac{1}{x}$ and $\frac{1}{y}$ be the unknowns.

In Ex. 2, the various coefficients are found to occupy the *same relative positions* in the expressions for x and y as the corresponding coefficients do in Ex. 1.

Show that this is also true in the following:

Ex. 3. $\begin{cases} 3x + 7y = 10, \\ 2x - 5y = 7. \end{cases}$ Ex. 4. $\begin{cases} 5x - 3y = 8, \\ 2x + 7y = 19. \end{cases}$

A convenient rule for reading directly the values of the unknowns in such a pair of equations may be made from the solution of the following:

Ex. 5. Solve $\begin{cases} a_1x + b_1y = c_1, \\ a_2x + b_2y = c_2. \end{cases}$

Eliminating first y and then x as in Ex. 1, we find:

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} \text{ and } y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}.$$

To remember these results, notice that the coefficients of x and y in the given equations stand in the form of a square, thus $\begin{matrix} a_1 & b_1 \\ a_2 & b_2 \end{matrix}$, and that the denominator in the expressions for both x and y is the *cross product* a_1b_2 minus the *cross product* a_2b_1. The numerator in the expression for x is read by replacing the a's in this square by the c's, *i.e.*, $\begin{matrix} c_1 & b_1 \\ c_2 & b_2 \end{matrix}$, and then reading the cross products as before. The numerator for y is read by replacing the b's by the c's, *i.e.*, $\begin{matrix} a_1 & c_1 \\ a_2 & c_2 \end{matrix}$, and then reading the cross products.

74. To indicate that the coefficients in a pair of equations are to be treated as just described, we write $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \equiv a_1b_2 - a_2b_1$ and call $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ a **determinant**. These are much used in higher algebra.

Since any pair of linear equations in two unknowns may be reduced to the standard form as given in Ex. 5, it follows that the values of x and y there obtained constitute a *formula for the solution of any pair of such equations.*

EXERCISES

Reduce the following pairs of equations to the standard form. Solve by use of the formula Exs. 1–7, and 12–14.

1. $\begin{cases} 3x + 4y = 10, \\ 4x + y = 9. \end{cases}$

2. $\begin{cases} 4x - 5y = -26, \\ 2x - 3y = -14. \end{cases}$

3. $\begin{cases} 6y - 17 = -5x, \\ 6x - 16 = -5y. \end{cases}$

4. $\begin{cases} \frac{1}{4}(x-3) = -\frac{1}{3}(y-2) + \frac{1}{2}x, \\ \frac{1}{2}(y-1) = x - \frac{1}{3}(x-2). \end{cases}$

5. $\begin{cases} 2x - y = 53, \\ 19x + 17y = 0. \end{cases}$

6. $\begin{cases} ax - by = 0, \\ x - y = c. \end{cases}$

7. $\begin{cases} mx + ny = p, \\ rx + sy = t. \end{cases}$

8. $\begin{cases} a(x+y) - b(x-y) = 2a, \\ a(x-y) - b(x+y) = 2b. \end{cases}$

9. $\begin{cases} (k+1)x + (k-2)y = 3a, \\ (k+3)x + (k-4)y = 7a. \end{cases}$

10. $\begin{cases} 2ax + 2by = 4a^2 + b^2, \\ x - 2y = 2a - b. \end{cases}$

11. $\begin{cases} (a+b)x - (a-b)y = 4ab, \\ (a-b)x + (a+b)y = 2a^2 - 2b^2. \end{cases}$

12. $\begin{cases} \frac{1}{2}(a-b) - \frac{1}{5}(a-3b) = b - 3, \\ \frac{3}{4}(a-b) + \frac{5}{6}(a+b) = 18. \end{cases}$

13. $\begin{cases} a(x+y) + b(x-y) = 2, \\ a^2(x+y) - b^2(x-y) = a - b. \end{cases}$

14. $\begin{cases} 7(x-5) = 3 - \dfrac{y}{2} - x, \\ \frac{1}{4}(x-y) + \frac{1}{2}y - \frac{5}{3}(x-1) = -1. \end{cases}$

15. $\begin{cases} mx + ny = m^3 + 2m^2n + n^3, \\ nx + my = m^3 + 2mn^2 + n^3. \end{cases}$

16. $\begin{cases} (m+n)x - (m-n)y = 2lm, \\ (m+l)x - (m-l)y = 2mn. \end{cases}$

17. $\begin{cases} (a-b)x + (a+b)y = 2a, \\ (a-b)x - (a+b)y = 2b. \end{cases}$

18. $\begin{cases} (h+k)x + (h-k)y = 2(h^2+k^2), \\ (h-k)x + (h+k)y = 2(h^2-k^2). \end{cases}$

19. $\begin{cases} (a-b)x + y(a^2+b^2) = (a+b)^2 + a + b - 2ab, \\ (b-a)x + y(a^2+b^2) = a + b - a^2 - b^2. \end{cases}$

INCONSISTENT AND DEPENDENT EQUATIONS

75. A pair of linear equations in two variables may be such that they either have no solution or have an unlimited number of solutions.

Ex. 1. Solve

$$\begin{cases} x - 2y = -2, & (1) \\ 3x - 6y = -12, & (2) \end{cases}$$

On graphing these equations they are found to represent two parallel lines. Since the lines have no point in common, it follows that the equations have no solution. See Fig. 1.

Attempting to solve them by means of the formula, § 73, we find:

$$x = \frac{(-2)(-6) - (-12)(-2)}{1(-6) - 3(-2)} = \frac{-12}{0},$$

and

$$y = \frac{1(-12) - 3(-2)}{1(-6) - 3(-2)} = \frac{-6}{0}.$$

But by § 25, $\frac{-12}{0}$ and $\frac{-6}{0}$ *are not numbers.* Hence, from this it follows that the given equations have no solution. In this case no solution is possible, and the equations are said to be **contradictory.**

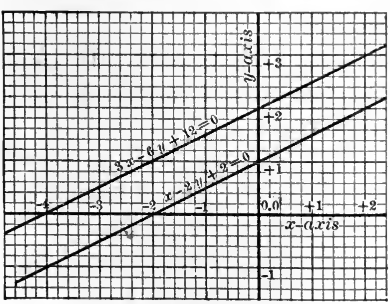


FIG. 1.

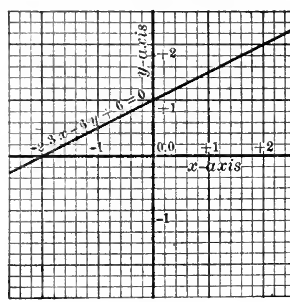


FIG. 2.

Ex. 2. Solve

$$\begin{cases} 3x - 6y = -6, & (1) \\ x - 2y = -2, & (2) \end{cases}$$

On graphing these equations, they are found to represent the *same line.* Hence every pair of numbers satisfying one equation must satisfy the other also. See Fig. 2.

Solving these equations by the formula, we find:

$$x = \frac{(-6)(-2)-(-2)(-6)}{3(-2)-1(-6)} = \frac{0}{0} \text{ and } y = \frac{3(-2)-1(-6)}{3(-2)-1(-6)} = \frac{0}{0}.$$

But by § 24, $\frac{0}{0}$ may represent *any number whatever*. Hence we may select for one of the unknowns any value we please and find from (1) or (2) a corresponding value for the other, but we may not select arbitrary values for *both* x and y.

In this case the solution is **indeterminate** and the equations are **dependent**; that is, one may be derived from the other.

Thus, (2) is derived from (1) by dividing both members by 3.

76. Two linear equations in two variables which have one and only one solution are called **independent and consistent.**

The cases in which such pairs of equations are *dependent* or *contradictory* are those in which the denominators of the expressions for x and y become *zero*. Hence, in order that such a pair of equations may have a *unique* solution, the denominator $a_1b_2 - a_2b_1$ of the formula, § 73, *must not reduce to zero*. This may be used as a test to determine whether a given pair of equations is independent and consistent.

EXERCISES

In the following, show both by the formula and by the graph which pairs of equations are independent and consistent, which dependent, and which contradictory.

1. $\begin{cases} 5x-3y=5, \\ 5x-3y=9. \end{cases}$

2. $\begin{cases} x-7+5y=y-x-2, \\ 5x+3y-4=3x-y+3. \end{cases}$

3. $\begin{cases} 7x-3y-4=2x-2, \\ x+y-3=2x-7. \end{cases}$

4. $\begin{cases} x-3y=6, \\ 5x-15y=18. \end{cases}$

5. $\begin{cases} 3y-4x-1=2x-5y+8, \\ 2y-5x+8=3x+y. \end{cases}$

6. $\begin{cases} 3x-6y+5=2x-5y+7, \\ 5x+3y-1=3x+5y+3. \end{cases}$

7. $\begin{cases} 2y+7x=2+6x, \\ 4x-3y=4+3x-5y. \end{cases}$

8. $\begin{cases} 5x-3=7y+8, \\ 2x+7=4y-9. \end{cases}$

9. $\begin{cases} 5x+2y=6+3x+5y, \\ 3x+y=18-3x+10y. \end{cases}$

10. $\begin{cases} 3x+4y=7+5y, \\ x-y=6-2x. \end{cases}$

SYSTEMS OF EQUATIONS IN MORE THAN TWO UNKNOWNS

77. If a single linear equation in three or more variables is given, there is no limit to the number of sets of values which satisfy it.

E.g. $3x + 2y + 4z = 24$ is satisfied by $x = 1$, $y = 3$, $z = 3\frac{3}{4}$; $x = 2$, $y = 2$, $z = 3\frac{1}{2}$; $x = 0$, $y = 0$, $z = 6$; etc.

If two linear equations in three or more variables are given, they have in general an unlimited number of solutions.

E.g. $3x + 2y + 4z = 24$ and $x + y + z = 6$ are both satisfied by $x = 2$, $y = -1$, $z = 5$; $x = 3$, $y = -1\frac{1}{2}$, $z = 4\frac{1}{2}$; etc.

But if a system of linear equations contains *as many equations as variables*, it has in general one and only one set of values which satisfy all the equations.

E.g. The system $\begin{cases} x + y + z = 6, \\ 3x - y + 2z = 7, \\ 2x + 3y - z = 5, \end{cases}$

is satisfied by $x = 1$, $y = 2$, $z = 3$, and by no other set of values.

It may happen, however, as in the case of two variables, that such a system is not *independent* and *consistent*.

Such cases frequently occur in higher work, and a general rule is there found for determining the nature of a system of linear equations *without solving them;* namely, by means of *determinants* (§ 73). In this book the only test used is the result of the solution itself as explained in the next paragraph.

78. An independent and consistent system of linear equations in three variables may be solved as follows:

From two of the equations, say the 1st and 2d, eliminate one of the variables, obtaining *one* equation in the remaining two variables.

From the 1st and 3d equations eliminate the same variable, obtaining a *second* equation in the remaining two variables.

Solve as usual the two equations thus found. Substitute the values of these two variables in one of the given equations, and thus find the value of the third variable.

The process of elimination by addition or subtraction is usually most convenient. See § 125, E. C.

EXERCISES

Solve each of the following systems and check by substituting in each equation:

1. $\begin{cases} 2x + 5y - 7z = 9, \\ 5x - y + 3z = 16, \\ 7x + 6y + z = 34. \end{cases}$

2. $\begin{cases} a + b + c = 9, \\ 8a + 4b + 2c = 36, \\ 27a + 9b + 3c = 93. \end{cases}$

3. $\begin{cases} 18l - 7m - 5n = 161, \\ 4\frac{2}{5}m - \frac{2}{3}l + n = 18, \\ 3\frac{1}{2}n + 2m + \frac{3}{4}l = 33. \end{cases}$

4. $\begin{cases} \frac{a}{3} + \frac{b}{6} + \frac{c}{9} = -2, \\ \frac{a}{6} + \frac{b}{9} + \frac{c}{12} = -4, \\ \frac{a}{9} + \frac{b}{12} + \frac{c}{15} = -4. \end{cases}$

5. $\begin{cases} 8z - 3y + x = -2, \\ 3x - 5y - 6z = -46, \\ y + 5x - 4z = -18. \end{cases}$

6.* $\begin{cases} \frac{3}{a} = \frac{2}{b}, \\ \frac{2}{a} + \frac{5}{b} - \frac{4}{c} = 17, \\ \frac{7}{a} - \frac{3}{b} + \frac{6}{c} = 8. \end{cases}$

7. $\begin{cases} x + 2y - 3z = -3, \\ 2x - 3y + z = 8, \\ 5x - 4y - 7z = -5. \end{cases}$

8. $\begin{cases} 2x + 3y - 7z = 19, \\ 5x + 8y + 11z = 24, \\ 7x + 11y + 4z = 43. \end{cases}$

Show that this system is not independent.

9. $\begin{cases} x + y = 16, \\ z + x = 22, \\ y + z = 28. \end{cases}$

10. $\begin{cases} x + 2y = 26, \\ 3x + 4z = 56, \\ 5y + 6z = 65. \end{cases}$

11. $\begin{cases} a + b + c = 5, \\ 3a - 5b + 7c = 79, \\ 9a - 11b = 91. \end{cases}$

12. $\begin{cases} l + m + n = 29\frac{1}{4}, \\ l + m - n = 18\frac{1}{4}, \\ l - m + n = 13\frac{3}{4}. \end{cases}$

* Use $\frac{1}{a}$, $\frac{1}{b}$, and $\frac{1}{c}$ as the unknowns.

13. $\begin{cases} l+m+n=a, \\ l+m-n=b, \\ l-m+n=c. \end{cases}$

14. $\begin{cases} ax+by=p, \\ cy+dz=q, \\ ex+fz=r. \end{cases}$

15. $\begin{cases} \frac{1}{a}+\frac{1}{b}=4, \\ \frac{1}{a}+\frac{1}{c}=3, \\ \frac{1}{b}+\frac{1}{c}=2. \end{cases}$

16. $\begin{cases} \frac{1}{x}+\frac{1}{y}=a, \\ \frac{1}{x}+\frac{1}{z}=b, \\ \frac{1}{y}+\frac{1}{z}=c. \end{cases}$

17. $\begin{cases} u+v+x+y=10, \\ 2u-3v+4x-5y=-12, \\ 3u+4v-5x+6y=20, \\ 4u+5v+6x-7y=4. \end{cases}$

18. Make a rule for solving a system of four or more linear equations in as many variables as equations.

PROBLEMS INVOLVING TWO OR MORE UNKNOWNS

1. A man invests a certain amount of money at 4% interest and another amount at 5%, thereby obtaining an annual income of $3100. If the first amount had been invested at 5% and the second at 4%, the income would have been $3200. Find each investment.

2. The relation between the readings of the Centigrade and the Fahrenheit thermometers is given by the equation $F = 32 + \frac{9}{5}C$. The Fahrenheit reading at the melting temperature of osmium is 2432 degrees higher than the Centigrade. Find the melting temperature in each scale.

In the Réaumur thermometer the freezing and boiling points are marked 0° and 80° respectively. Hence if C is the Centigrade reading and R the Réaumur reading, then $R = \frac{4}{5}C$.

3. What is the temperature Fahrenheit (*a*) if the Fahrenheit reading equals $\frac{1}{2}$ of the sum of the other two, (*b*) if the Centigrade reading equals $\frac{1}{2}$ of the Fahrenheit minus the Réaumur, (*c*) if the Réaumur is equal to the sum of the Fahrenheit and Centigrade?

4. Going with a current a steamer makes 19 miles per hour, while going against a current $\frac{3}{4}$ as strong the boat makes 5 miles per hour. Find the speed of each current and the boat.

5. There is a number consisting of 3 digits whose sum is 11. If the digits are written in reverse order, the resulting number is 594 less than the original number. Three times the tens' digit is one more than the sum of the hundreds' and the units' digit.

6. A certain kind of wine contains 20 % alcohol and another kind contains 28 %. How many gallons of each must be used to form 50 gallons of a mixture containing 21.6 % alcohol?

7. The area of a certain trapezoid of altitude 8 is 68. If 4 is added to the lower base and the upper base is doubled, the area is 108. Find both bases.

A trapezoid is a four-sided figure whose upper base, b_1, and lower base, b_2, are parallel, but the other two sides are not. If h is the perpendicular distance between the bases, then the area is $a = \frac{h}{2}(b_1 + b_2)$.

8. The Centigrade reading at the boiling point of alcohol is 96° lower than the Fahrenheit reading. Find both the Centigrade and the Fahrenheit reading at this temperature.

Use C and F as the unknowns. Then one of the equations is the formula connecting Fahrenheit and Centigrade readings obtained on page 313 and the other is $C + 96 = F$.

9. The Fahrenheit reading at the temperature of liquid air is 128 degrees lower than the Centigrade reading. Find both the Centigrade and the Fahrenheit reading at this temperature.

10. The Centigrade reading at the melting point of silver is 796° lower than the Fahrenheit reading. Find both Centigrade and Fahrenheit readings at this temperature.

11. The Fahrenheit reading at the melting point of gold is 992° higher than the Centigrade reading. Find both Centigrade and Fahrenheit readings at this temperature.

12. The upper base of a trapezoid is 6 and its area is 168. If $\frac{1}{3}$ the lower base is added to the upper, the area is 210. Find the altitude and the lower base.

13. A and B can do a piece of work in 18 days, B and C in 24 days, and C and A in 36 days. How long will it require each man, working alone, to do it, and how long will it require all working together?

14. A and B can do a piece of work in m days, B and C in n days, and C and A in p days. How long will it require each to do it working alone?

15. A beam resting on a fulcrum balances when it carries weights of 100 and 130 pounds at its respective ends. The beam will also balance if it carries weights of 80 and 110 pounds respectively 2 feet from the ends. Find the distance from the fulcrum to the ends of the beam.

16. A beam carries three weights, A, B, and C. A balance is obtained when A is 12 feet from the fulcrum, B 8 feet from the fulcrum (on the same side as A), and C 20 feet from the fulcrum (on the side opposite A). It also balances when the distance of A is 8 feet, B 10 feet, and C 18 feet. Find the weights B and C if A is 50 lbs.

17. At 0° Centigrade sound travels 1115 feet per second with the wind on a certain day, and 1065 feet per second against the wind. Find the velocity of sound in calm weather, and the velocity of the wind on this occasion.

18. If the velocities of sound in air, brass, and iron at 0° Centigrade are x, y, z meters per second respectively, then $3x+2y-z=2505$, $5x-2y+z=151$, and $x+y+z=8777$. Find the velocity in each.

19. If x, y, z are the Centigrade readings at the temperatures which liquefy hydrogen, nitrogen, and oxygen respectively, then $3x-8y+2z=440$, $-8x+2y+4z=903$, and $x+4y-6z=60$. Find each temperature in both Centigrade and Fahrenheit readings.

20. If x, y, z are the Centigrade readings at the freezing temperatures of hydrogen, nitrogen, and oxygen respectively, then we have $x + y - 3z = 199$, $2x - 5y + z = 328$, and $-4x + 2y + 2z = 156$. Find each temperature.

21. If x, y, z are respectively the melting point of carbon, the temperature of the hydrogen flame in air, and the temperature of this flame in pure oxygen, then $10x + 2y + z = 41{,}892$, $15x + y + 2z = 60{,}212$, and $7x + y + z = 29{,}368$. Find each.

22. Two boys carry a 120-pound weight by means of a pole, at a certain point of which the weight is hung. One boy holds the pole 5 ft. from the weight and the other 3 ft. from it. What proportion of the weight does each boy lift?

Solution. Let x and y be the required amounts, then $5x$ is the leverage of the first boy and $3y$ that of the second, and these must be equal as in the case of the teeter, p. 98, E. C. Hence we have

$$5x = 3y, \text{ and } x + y = 120.$$

Solving, we find $x = 45$, $y = 75$.

23. If, in problem 22, the boys lift P_1 and P_2 pounds respectively at distances d_1 and d_2, and w is the weight lifted, then

$$P_1 d_1 = P_2 d_2. \quad (1)$$

$$P_1 + P_2 = w. \quad (2)$$

Solve (1) and (2), (a) when P_1 and P_2 are unknown, (b) when P_1 and w are unknown, (c) when P_1 and d_2 are unknown.

24. A weight of 540 pounds is carried on a pole by two men at distances of 4 and 5 feet respectively. How much does each lift?

25. A weight of 470 pounds is carried by two men, one at a distance of 3 feet and the other lifting 200 pounds. At what distance is the latter?

26. Two men are carrying a weight on a pole at distances of 4 and 6 feet respectively. The former lifts 240 pounds. How many pounds are they carrying?

CHAPTER V

FACTORING

79. A rational integral expression is said to be **completely factored** when it cannot be further resolved into factors which are rational and integral. Such factors are called **prime factors.**

The simpler forms of factoring are given in the following outline.

A **monomial factor** of any expression is evident at sight, and its removal should be the first step in every case.

E.g. $$4ax^2 + 2a^2x = 2ax(2x + a).$$

FACTORS OF BINOMIALS

80. *The difference of two squares.*

E.g. $$4x^2 - 9z^4 = (2x + 3z^2)(2x - 3z^2).$$

81. *The difference of two cubes.*

E.g. $$8x^3 - 27y^3 = (2x - 3y)[(2x)^2 + (2x)(3y) + (3y)^2]$$
$$= (2x - 3y)(4x^2 + 6xy + 9y^2).$$

82. *The sum of two cubes.*

E.g. $$27x^3 + 64y^3 = (3x + 4y)[(3x)^2 - (3x)(4y) + (4y)^2]$$
$$= (3x + 4y)(9x^2 - 12xy + 16y^2).$$

FACTORS OF TRINOMIALS

83. *Trinomial squares.*

E.g. $$a^2 + 2ab + b^2 = (a + b)^2 = (a + b)(a + b),$$

and $$a^2 - 2ab + b^2 = (a - b)^2 = (a - b)(a - b).$$

84. *Trinomials of the form $x^2 + px + q$.*

E.g. $$x^2 + 3x - 10 = (x+5)(x-2).$$

A trinomial of this form has two binomial factors, $x + a$ and $x + b$, if two numbers a and b can be found whose product is q and whose algebraic sum is p.

85. *Trinomials of the form $mx^2 + nx + r$.*

E.g. $$6x^2 + 7x - 20 = (3x - 4)(2x + 5).$$

A trinomial of this form has two binomial factors of the type $ax + b$ and $cx + d$, if four numbers, a, b, c, d, can be found such that $ac = m$, $bd = r$, and $ad + bc = n$. See § 155, E. C.

86. *Trinomials which reduce to the difference of two squares.*

E.g. $$x^4 + x^2y^2 + y^4 = x^4 + 2x^2y^2 + y^4 - x^2y^2 = (x^2+y^2)^2 - x^2y^2$$
$$= (x^2 + y^2 - xy)(x^2 + y^2 + xy).$$

In this case x^2y^2 is both added to and subtracted from the expression, whereby it becomes the difference of two squares. Evidently the term added and subtracted must itself be a *square*, and hence the degree of the trinomial must be 4 or a multiple of 4, since the degree of the middle term is *half* that of the trinomial.

Ex. $$4a^8 - 16a^4b^4 + 9b^8 = 4a^8 - 12a^4b^4 + 9b^8 - 4a^4b^4$$
$$= (2a^4 - 3b^4)^2 - 4a^4b^4$$
$$= (2a^4 - 3b^4 + 2a^2b^2)(2a^4 - 3b^4 - 2a^2b^2).$$

EXERCISES ON BINOMIALS AND TRINOMIALS

Factor the following:

1. $a^3 + b^3$.
2. $a^3 - b^3$.
3. $(a+b)^3 - c^3$.
4. $(a+b)^3 + c^3$.
5. $7ax^2 - 56a^4x^5$.
6. $a^5 - ab^4$.
7. $121x^7 - 4xy^4$.
8. $\frac{1}{8}a^3 + \frac{1}{125}b^3$.
9. $\frac{1}{8}r^3 - \frac{9}{32}rs^2$.
10. $8r^4 - 27r$.
11. $(a+b)^2 - c^2$.
12. $c^2 - (a-b)^2$.
13. $5c^2 + 7cd - 6d^2$.
14. $x^4 - 3x^2y^2 + y^4$.
15. $4x^2 - 12xy + 9y^2$.
16. $x^2 + 11xz + 30z^2$.

17. $6x^2 - 5xy - 6y^2$.

18. $3a^2x^2y^4 - 69a^2xy^2 + 336a^2$.

19. $20a^2b^2 + 23abx - 21x^2$.

20. $a^4 + 2a^2b^2 + 9b^4$.

21. $48a^3x^4y - 75ay^5$.

22. $16a^4x^3y + 54ay^4$.

23. $x^4y^2 + 2x^2yz + z^2$.

24. $a^2 + 10a - 39$.

25. $8a^2y^3 - 48a^2y^2z + 72a^2yz^2$.

26. $4m^8 - 60m^4n^4 + 81n^8$.

27. $35a^{2k} - 6a^kb^k - 9b^{2k}$.

28. $(a+b)^2 - (c-d)^2$.

29. $72a^2x^2 - 19axy^2 - 40y^4$.

30. $4(a-3)^6 - 37b^2(a-3)^3 + 9b^4$.

31. $6(x+y)^2 + 5(x^2 - y^2) - 6(x-y)^2$.

32. $9(x-a)^2 - 24(x-a)(x+a) + 16(x+a)^2$.

33. $12(c+d)^2 - 7(c+d)(c-d) - 12(c-d)^2$.

34. $(a^2 + 5a - 3)^2 - 25(a^2 + 5a - 3) + 150$.

FACTORS OF POLYNOMIALS OF FOUR TERMS

A polynomial of four terms may be readily factored in case it is in any one of the forms given in the next three paragraphs.

87. *It may be the cube of a binomial.*

Ex. 1. $a^3 - 3a^2b + 3ab^2 - b^3 = (a-b)^3$.

Ex. 2. $8x^3 + 36x^2y + 54xy^2 + 27y^3$

$= (2x)^3 + 3(2x)^2(3y) + 3(2x)(3y)^2 + (3y)^3$

$= (2x + 3y)^3$. See Ex. 34, (*d*), p. 23.

88. *It may be resolvable into the difference of two squares.*

In this case three of the terms must form a trinomial square.

Ex. 1. $a^2 - c^2 + 2ab + b^2 = (a^2 + 2ab + b^2) - c^2$

$= (a+b)^2 - c^2 = (a+b+c)(a+b-c)$.

Ex. 2. $4x^2 + z^6 - 4x^4 - 1 = z^6 - 4x^4 + 4x^2 - 1$

$= z^6 - (4x^4 - 4x^2 + 1) = z^6 - (2x^2 - 1)^2$

$= (z^3 + 2x^2 - 1)(z^3 - 2x^2 + 1)$.

89. *A binomial factor may be shown by grouping the terms.*

In this case the terms are grouped by twos as in the following examples.

Ex. 1. $ax + ay + bx^2 + bxy = (ax + ay) + (bx^2 + bxy)$
$= a(x + y) + bx(x + y) = (a + bx)(x + y).$

Ex. 2. $ax + bx + a^2 - b^2 = (ax + bx) + (a^2 - b^2)$
$= x(a + b) + (a - b)(a + b) = (x + a - b)(a + b).$

EXERCISES

Factor the following polynomials:

1. $x^3 + 3x^2y + 3xy^2 + y^3.$
2. $8a^3 - 36a^2b + 54ab^2 - 27b^3.$
3. $4a^4 - 4a^2b^2 + b^4 - 16x^2.$
4. $2ad + 3bc + 2ac + 3bd.$
5. $27x^3 - 54x^2y + 36xy^2 - 8y^3.$
6. $36a^4 - 24a^3 + 24a - 16.$
7. $mnx^2 - mrx - rn^2x + r^2n.$
8. $a^2b^2 - a^2bc^2n - abn + an^2c^2.$
9. $2y^2 + 4by + 3cy + 6bc.$
10. $bcyz + c^2z^2 + bdy + dcz.$
11. $5a^2c + 12cd - 6ad - 10ac^2.$
12. $a^2 - b^2x^2 + acx^2 - bcx^3.$
13. $b^3c^2 - c^2y^3 - b^3y^2 + y^5.$
14. $a^{3k} - 2a^{2k}b^k - 2a^kb^{2k} + b^{3k}.$
15. $m^{a+b} + m^an^a + m^bn^b + n^{a+b}.$
16. $b^2y^3 - b(c - d)y^2 + d(by - c) + d^2.$

FACTORS OF POLYNOMIAL SQUARES

90. In § 158, E. C., squares of trinomials were factored by inspection. In the examples there considered there were no similar terms combined in the polynomial squares. When this is the case, it is easy to recognize at sight any polynomial square and hence to get its square root.

E.g. $a^2 + b^2 + c^2 + d^2 + 2ab + 2ac + 2ad + 2bc + 2bd + 2cd$ is the square of $a + b + c + d$.

It is also possible in some cases to recognize a polynomial square after some similar terms have been combined.

E.g. Examine $x^4 + 2x^3 + 3x^2 + 2x + 1$. We see that x^4 and 1 are squares. If this polynomial is a square, there must be another squared term which is combined with some other term. $3x^2$ is the only such term possible, and hence the other square must be x^2. Hence, we try x^4, x^2, and 1 for the squared terms and soon see that x^2+x+1 is the required square root.

EXERCISES

Factor the following:

1. $4x^4+9y^2+1-12xy+4x-6y.$

2. $\frac{1}{x^2}+\frac{1}{y^2}+1+\frac{2}{xy}+\frac{2}{x}+\frac{2}{y}.$

3. $\frac{a^2}{b^2}+\frac{c^2}{d^2}+4+\frac{2ac}{bd}-\frac{4a}{b}-\frac{4c}{d}.$

4. $x^4-2x^3+3x^2-2x+1.$

5. $x^4-2x^3-x^2+2x+1.$

6. $9a^6+12a^4-6a^3+4a^2-4a+1.$

7. $x^6-8x^5+22x^4-24x^3+9x^2.$

8. $x^6+8x^4-2x^2-4+\frac{1}{x^2}.$

9. $a^4-2a^3+a^2+2-\frac{2}{a}+\frac{1}{a^4}.$

10. $4x^2+9+\frac{1}{4x^2}+6x+2+\frac{3}{x}.$

11. $a^2+4b^2+9c^2+d^2-4ab+6ac-2ad-12bc+4bd-6cd.$

FACTORS FOUND BY GROUPING

91. The discovery of factors by the proper grouping of terms as in § 89 is of wide application. Polynomials of five, six, or more terms may frequently be thus resolved into factors.

Ex. 1. $a^2+2ab+b^2+5a+5b=(a+b)^2+(5a+5b)$
$=(a+b)(a+b)+5(a+b)=(a+b+5)(a+b).$

Ex. 2. $x^2-7x+6-ax+6a=x^2-7x+6-(ax-6a)$
$=(x-1)(x-6)-a(x-6)=(x-6)(x-1-a).$

Ex. 3. $a^2-2ab+b^2-x^2-2xy-y^2$
$=(a^2-2ab+b^2)-(x^2+2xy+y^2)$
$=(a-b)^2-(x-y)^2=(a-b+x-y)(a-b-x+y).$

Ex. 4. $ax^2 + ax - 6a + x^2 + 7x + 12$

$$= a(x^2 + x - 6) + (x^2 + 7x + 12)$$
$$= a(x+3)(x-2) + (x+3)(x+4)$$
$$= (x+3)[a(x-2) + x + 4] = (x+3)(ax - 2a + x + 4).$$

In some cases the grouping is effective only after a term has been separated into two parts.

Ex. 5. $2a^3 + 3a^2 + 3a + 1 = a^3 + (a^3 + 3a^2 + 3a + 1)$

$$= a^3 + (a+1)^3 = (a + a + 1)[a^2 - a(a+1) + (a+1)^2]$$
$$= (2a+1)(a^2 + a + 1).$$

As soon as the term $2a^3$ is separated into two terms the expression is shown to be the sum of two cubes.

Again, the grouping may be effective after a term has been both added and subtracted:

Ex. 6. $a^4 + b^4 = (a^4 + 2a^2b^2 + b^4) - 2a^2b^2$

$$= (a^2 + b^2)^2 - (ab\sqrt{2})^2$$
$$= (a^2 + b^2 + ab\sqrt{2})(a^2 + b^2 - ab\sqrt{2}).$$

In this case the factors are irrational as to one coefficient. Such factors are often useful in higher mathematical work.

EXERCISES

Factor the following:

1. $x^2 - 2xy + y^2 - ax + ay.$ **3.** $a^3 - b^3 - a^2 - ab - b^2.$

2. $a^2 - ab + b^2 + a^3 + b^3.$ **4.** $a^2 - 2ab + b^2 - x^2 + 2xy - y^2.$

5. $a^4 + 2a^3b - a^2c^2 + a^2b^2 - 2abc^2 - b^2c^2.$

6. $x^4 - y^4 + ax^2 + ay^2 - x^2 - y^2.$ **7.** $a^4 + a^2b^2 + b^4 + a^3 + b^3.$

In 7 group the first three and the last two terms.

8. $a^3 - 1 + 3x - 3x^2 + x^3.$ Group the last four terms.

9. $x^3 + x^2 + 3x + y^3 - y^2 + 3y.$

Group in pairs, the 1st and 4th, 2d and 5th, 3d and 6th terms.

10. $x^4 + x^3y - xy^3 - y^4 + x^3 - y^3.$

11. $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 - x^4.$

12. $x^4 + 4x^2z - 4y^2 + 4yw + 4z^2 - w^2$.

13. $2a^2 - 12b^2 + 3bd - 5ab - 9bc - 6ac + 2ad$.

Group the terms: $2a^2 - 5ab - 12b^2$.

14. $a^2 + ab - 4ac - 2b^2 + 4bc + 3ad - 3bd$.

15. $a^3 + 2 - 3a$. Group thus: $(a^3 - 1) + (3 - 3a)$.

16. $4a^2 + a - 8ax - x + 4x^2$.

17. $3a^2 - 8ab + 4b^2 + 2ac - 4bc$.

18. $x^8 + y^8$. Add and subtract $2x^4y^4$.

19. $a^6 + 2a^3b^3 + b^6 - 2a^4b - 2ab^4$.

20. $a^3 - 3a^2 + 4$. Group thus: $(a^3 - 2a^2) + (4 - a^2)$.

21. $a^2c - ac^2 - a^2b + ab^2 - b^2c + bc^2$.

22. $a^2b - a^2c + b^2c - ab^2 + ac^2 - bc^2$.

23. $3x^3 - x^2 - 4x + 2$. Add and subtract $-2x^2$.

24. $2x^3 - 11x^2 + 18x - 9$. Add and subtract $9x^2$.

FACTORS FOUND BY THE FACTOR THEOREM

92. It is possible to determine in advance whether a polynomial in x is divisible by a binomial of the form $x - a$.

E.g. In dividing $x^4 - 4x^3 + 7x^2 - 7x + 2$ by $x - 2$, the quotient is found to be $x^3 - 2x^2 + 3x - 1$.

Since *Quotient* × *Divisor* ≡ *Dividend*, we have

$$(x - 2)(x^3 - 2x^2 + 3x - 1) \equiv x^4 - 4x^3 + 7x^2 - 7x + 2.$$

As this is an *identity*, it holds for all values of x. For $x = 2$ the factor $(x - 2)$ is zero, and hence the left member is zero, § 22.

Hence for $x = 2$ the right member must also be zero. This is indeed the case, viz.:

$$2^4 - 4 \cdot 2^3 + 7 \cdot 2^2 - 7 \cdot 2 + 2 = 16 - 32 + 28 - 14 + 2 = 0.$$

Hence if $x - 2$ is a factor of $x^4 - 4x^3 + 7x^2 - 7x + 2$, the latter must reduce to zero for $x = 2$.

93. In general let D represent any polynomial in x. Suppose D has been divided by $x - a$ until the remainder no longer contains x. Then, calling the quotient Q and the remainder R, we have the identity

$$D \equiv Q(x - a) + R, \qquad (1)$$

which holds for all values of x.

The substitution of a for x in (1) does not affect R, reduces $Q(x - a)$ to zero, and may or may not reduce D to zero.

(1) If $x = a$ reduces D to zero, then $0 \equiv 0 + R$. Hence R is zero, and the division is exact. That is, $x - a$ is a factor of D.

(2) If $x = a$ does *not* reduce D to zero, then R is not zero, and the division is not exact. That is, $x - a$ is not a factor of D.

Hence: *If a polynomial in* x *reduces to zero when a particular number* a *is substituted for* x*, then* $x - a$ *is a factor of the polynomial, and if the substitution of* a *for* x *does not reduce the polynomial to zero, then* $x - a$ *is not a factor.*

This principle is often called the **factor theorem.**

In applying the factor theorem the trial divisor must always be written in the form $x - a$.

Ex. 1. Factor $x^4 + 6x^3 + 3x^2 + x + 3$.

If there is a factor of the form $x - a$, then the only possible values of a are the various divisors of 3, namely $+1$, -1, $+3$, -3.

To test the factor $x + 1$, we write it in the form $x - (-1)$ where $a = -1$. Substituting -1 for x in the polynomial, we have

$$1 - 6 + 3 - 1 + 3 = 0.$$

Hence $x + 1$ is a factor.

On substituting $+1$, $+3$, -3 for x successively, no one reduces the polynomial to zero. Hence $x - 1$, $x - 3$, $x + 3$ are not factors.

Ex. 2. Factor $3x^3 - x^2 - 4x + 2$.

If $x - a$ is a factor, then a must be a factor of $+2$. We therefore substitute, $+2$, -2, $+1$, -1 and find the expression becomes zero when $+1$ is substituted for x. Hence $x - 1$ is a factor. The other factor is found by division to be $3x^2 + 2x - 2$, which is prime.

Hence $$3x^3 - x^2 - 4x + 2 = (x - 1)(3x^2 + 2x - 2).$$

EXERCISES

Factor by means of the factor theorem:

1. $3x^3-2x^2+5x-6$.
2. $2x^3+3x^2-3x-4$.
3. $2x^3+x^2-12x+9$.
4. $x^3+9x^2+10x+2$.
5. a^3-3a+2.
6. m^3+5m^2+7m+3.
7. $x^4+3x^3-3x^2-7x+6$.
8. $3r^3+5r^2-7r-1$.
9. $2z^3+7z^2+4z+3$.
10. $a^3-6a^2+11a-6$.

11. Show by the factor theorem that x^k-a^k contains the factor $x-a$ if k is *any* integer.

12. Show that x^k-a^k contains the factor $x+a$ if k is any *even* integer.

13. Show that x^k+a^k contains the factor $x+a$ if k is any *odd* integer.

14. Show that x^k+a^k contains neither $x+a$ nor $x-a$ as a factor if k is an *even* integer.

MISCELLANEOUS EXERCISES ON FACTORING

1. $20a^3x^3y-45a^3xy^3$.
2. $24am^5n^2-375am^2n^5$.
3. $432ar^4s+54ars^4$.
4. $16x^2-72xy+81y^2$.
5. $162a^3b+252a^2b^2+98ab^3$.
6. $48x^5y-12x^3y-12x^2y+3y$.
7. $12a^2bx^2+8ab^2x^2+18a^2bxy+12ab^2xy$.
8. $18x^3y-39x^2y^2+18xy^3$.
9. $4x^2-9xy+6x-9y+4x+6$.
10. $6x^2-13xy+6y^2-3x+2y$.
11. $6x^4+15x^2y^2+9y^4$.
12. $16x^4+24x^2y^2+8y^4$.
13. $15x^4+24x^2y^2+9y^4$.
14. a^6+y^6.
15. $a^{12}+y^{12}$.
16. a^8-y^8.
17. $a^{16}-y^{16}$.
18. $a^8+a^4y^4+y^8$.
19. a^3+a-2.
20. $a^8-18a^4y^4+y^8$.
21. $a^{16}-6a^8y^8+y^{16}$.
22. x^3+4x^2+2x-1.
23. $3x^3+2x^2-7x+2$.

24. $a^8 - 3a^4y^4 + y^8$.

25. $a^3 + a^2 + a + 1$.

26. $a^3 + 9a^2 + 16a + 4$.

27. $2x^4 + x^3y + 2x^2y^2 + xy^3$.

28. $m^5 + m^4a + m^3a^2 + m^2a^3 + ma^4 + a^5$.

29. $(x-2)^3 - (y-z)^3$.

30. $a^6 + b^6 + 2ab(a^4 - a^2b^2 + b^4)$.

31. $x^5y^5 + x^4y^4z + x^3y^3z^2 + x^2y^2z^3 + xyz^4 + z^5$.

32. $8a^3 + 6ab(2a - 3b) - 27b^3$.

33. $a(x^3 + y^3) - ax(x^2 - y^2) - y^2(x + y)$.

34. $a^3 - b^3 + 3b^2c - 3bc^2 + c^3$.

35. $a^4 + 2a^3b - 2ab^2c - b^2c^2$.

36. $a^4 + 2a^3b + a^2b^2 - a^4b^2 - 2a^2b^2c - b^2c^2$.

SOLUTION OF EQUATIONS BY FACTORING

94. Many equations of higher degree than the first may be solved by factoring. (See §§ 160–163, E. C.)

Ex. 1. Solve $2x^3 - x^2 - 5x - 2 = 0$. (1)

Factoring the left member of the equation, we have

$$(x-2)(x+1)(2x+1) = 0. \quad (2)$$

A value of x which makes one factor zero makes the whole left member zero and so satisfies the equation. Hence $x = 2$, $x = -1$, $x = -\frac{1}{2}$ are roots of the equation.

To solve an equation by this method first reduce it to the form $A = 0$, and then factor the left member. Put each factor equal to zero and solve for x. The results thus obtained are roots of the original equation.

Ex. 2. Solve $x^3 - 12x^2 = 12 - 35x$. (1)

Transposing and factoring, $(x-4)(x^2 - 8x + 3) = 0$. (2)

Hence the roots of (1) are the roots of $x - 4 = 0$ and $x^2 - 8x + 3 = 0$. From $x - 4 = 0$, $x = 4$. The quadratic expression $x^2 - 8x + 3$ cannot be resolved into *rational* factors. See § 155.

EXERCISES

Solve each of the following equations by factoring, obtaining all roots which can be found by means of rational factors.

1. $x^3 + 3x^2 = 28x$.
2. $6x^3 + 8x + 5 = 19x^2$.
3. $x^4 + 12x^2 + 3 = 7x^3 + 9x$.
4. $x^3 = -2x^2 + 5x + 6$.
5. $x^3 - 4x^2 = 4x + 5$.
6. $2x^3 + 3x = 9x^2 - 14$.
7. $5x^3 + x^2 - 14x + 8 = 0$.
8. $2x^3 + x^2 = 8x - 3$.
9. $x^4 + 2x^3 + 12 = 7x^2 + 8x$.
10. $x^4 + x + 6 = x^3 + 7x^2$.

COMMON FACTORS AND MULTIPLES

95. If each of two or more expressions is resolved into prime factors, then their **Highest Common Factor** (H. C. F.) is at once evident as in the following example. See § 202, E. C.

Given (1) $x^4 - y^4 = (x^2 + y^2)(x + y)(x - y)$,

(2) $x^6 - y^6 = (x^3 + y^3)(x^3 - y^3)$

$= (x + y)(x^2 - xy + y^2)(x - y)(x^2 + xy + y^2)$.

Then $(x + y)(x - y) = x^2 - y^2$ is the H. C. F. of (1) and (2).

In case only one of the given expressions can be factored by inspection, it is usually possible to select those of its factors, if any, which will divide the other expressions and so to determine the H. C. F.

Ex. Find the H. C. F. of $6x^3 + 4x^2 - 3x - 2$,

and $2x^4 + 2x^3 + x^2 - x - 1$.

By grouping we find:

$$6x^3 + 4x^2 - 3x - 2 = 2x^2(3x + 2) - (3x + 2)$$
$$= (2x^2 - 1)(3x + 2).$$

The other expression cannot readily be factored by any of the methods thus far studied. However, if there is a common factor, it must be either $2x^2 - 1$ or $3x + 2$. We see at once that it cannot be $3x + 2$. (Why?) By actual division $2x^2 - 1$ is found to be a factor of $2x^4 + 2x^3 + x^2 - x - 1$. Hence $2x^2 - 1$ is the H. C. F.

96. The **Lowest Common Multiple** (L. C. M.) of two or more expressions is readily found if these are resolved into prime factors. See § 205, E. C.

Ex. 1. Given $6\,abx - 6\,aby = 2 \cdot 3\,ab(x-y),$ (1)

$$8\,a^2x + 8\,a^2y = 2^3a^2(x+y), \quad (2)$$

$$36\,b^3(x^2-y^2)(x+y) = 2^23^2b^3(x-y)(x+y)^2. \quad (3)$$

The L. C. M. is $2^3 \cdot 3^2\,a^2b^3\,(x-y)(x+y)^2$, since this contains all the factors of (1), all the factors of (2) not found in (1), and all the factors of (3) not found in (1) and (2), with no factors to spare.

In case only one of the given expressions can be factored by inspection, it may be found by actual division whether or not any of these factors will divide the other expressions.

Ex. 2. Find the L. C. M. of $6\,x^3 - x^2 + 4\,x + 3,$ (1)

and $6\,x^3 + 3\,x^2 - 10\,x - 5.$ (2)

(1) is not readily factored. Grouping by twos, the factors of (2) are $3\,x^2 - 5$ and $2\,x + 1$. Now $3\,x^2 - 5$ is not a factor of (1). (Why?) Dividing (1) by $2\,x + 1$ the quotient is $3\,x^2 - 2\,x + 3$.

Hence $6\,x^3 - x^2 + 4\,x + 3 = (2\,x+1)(3\,x^2 - 2\,x + 3),$

$$6\,x^3 + 3\,x^2 - 10\,x - 5 = (2\,x+1)(3\,x^2-5).$$

Hence the L. C. M. is $(2\,x+1)(3\,x^2-2\,x+3)(3\,x^2-5)$.

Ex. 3. Find the L. C. M. of $a^3 + 2\,a^2 - a - 2,$ (1)

and $10\,a^3 - 3\,a^2 + 4\,a + 1.$ (2)

By means of the factor theorem, $a - 1$, $a + 1$, and $a + 2$ are found to be factors of (1), but none of the numbers, $1, -1, -2$, when substituted for a in (2) will reduce it to zero. Hence (1) and (2) have no factors in common. The L. C. M. is therefore the product of the two expressions; viz. $(a+1)(a-1)(a+2)(10\,a^3 - 3\,a^2 + 4\,a + 1)$.

97. The H. C. F. of three expressions may be obtained by finding the H. C. F. of two, and then the H. C. F. of this result and the third expression. Similarly the L. C. M. of three expressions may be obtained by finding the L. C. M. of two of them, and then the L. C. M. of this result and the third expression.

This may be extended to any number of expressions.

EXERCISES

Find the H. C. F. and also the L. C. M. in each of the following:

1. $x^2+y^2,\ x^6+y^6.$
2. $x^2+xy+y^2,\ x^3-y^3.$
3. $x^2-5x-6,\ x^2-2x-3,\ x^2+19x+18.$
4. $x^4-6x^2+1,\ x^3+x^2-3x+1,\ x^3+3x^2+x-1.$
5. $162a^3b+252a^2b^2+9ab^3,\ 54a^3+42a^2b.$
6. $2x^3+x^2-8x+3,\ x^2+2x-1.$
7. $3r^3+5r^2-7r-1,\ 3r^2+8r+1.$
8. $a^3-3a^2+4,\ ax-ab-2x+2b.$
9. $a^6+2a^3b^3+b^6-2a^4b-2ab^4,\ a^3-2ab+b^3.$
10. $8a^3-36a^2b+54ab^2-27b^3,\ 4a^2-9b^2.$
11. $36a^4-9a^2-24a-16,\ 12a^3-6a^2-8a.$
12. $2y^2+4by+3cy+6bc,\ y^2-3by-10b^2$
13. $x^{16}-y^{16},\ x^8-y^8,\ x^4-y^4.$
14. $m^3+8m^2+7m,\ m^3+3m^2-m-3,\ m^3-7m-6.$

98. An **important principle** relating to common factors is illustrated by the following example:

Given $\qquad x^2+7x+10=(x+5)(x+2), \qquad (1)$

and $\qquad x^2-x-6=(x-3)(x+2). \qquad (2)$

Add (1) and (2), $\qquad 2x^2+6x+4=2(x+1)(x+2). \qquad (3)$

Subtract (2) from (1), $\qquad 8x+16=8(x+2). \qquad (4)$

We observe that $x+2$, which is a common factor of (1) and (2), is also a factor of their sum (3), and of their difference (4). This example is a special case of the following principle.

99. *A common factor of two expressions is also a factor of the sum or difference of any multiples of those expressions.*

For let A and B be any two expressions having the common factor f. Then if k and l are the remaining factors of A and B respectively,

$$A = fk \text{ and } B = fl.$$

Also let mA and nB be any multiples of A and B.

Then $mA = mfk$ and $nB = nfl$, from which we have:

$$mA \pm nB = mfk \pm nfl = f(mk \pm nl).$$

Hence f is a factor of $mA \pm nB$.

100. Another important principle is the following: *If **f** is a factor of **mA** ± **nB** and also of **A**, then **f** is a factor of **B**, provided **n** has no factor in common with **A**.*

For let f be a factor of $mA \pm nB$ and also of A, where mA and nB are integral multiples of the expressions A and B.

Then f must divide both mA and nB. We know it divides mA because it was given as a factor of A. If it divides nB, it must be a factor of either n or B. It cannot be a factor of n, for then A and n would have a factor in common contrary to agreement. Hence f is a factor of B.

101. By successive applications of the above principles it is possible to find the H.C.F. of any two integral expressions.

Ex. 1. Find the H.C.F. of $9x^4 - x^2 + 2x - 1$, (1)

and $27x^5 + 8x^2 - 3x + 1$. (2)

Multiplying (1) by $3x$ and subtracting from (2), we have

$$\begin{array}{r} 27x^5 + 8x^2 - 3x + 1 \\ \underline{27x^5 - 3x^3 + 6x^2 - 3x} \\ 3x^3 + 2x^2 + 1 \end{array} \qquad (3)$$

By § 99, any common factor of (1) and (2) is a factor of (3).

Calling expressions (1) and (2) B and A respectively of principle **2**, then (3) is $A - 3x \cdot B$; and since the multiplier, $3x$, has no factor in common with (2), it follows from the principle that any common factor of (3) and (2) is a factor of (1), and also that any common factor of (3) and (1) is a factor of (2). Hence (1) and (3) have the *same common factors*, that is, the same H. C. F. as (1) and (2). Therefore we proceed to obtain the H. C. F. of

$$9x^4 - x^2 + 2x - 1, \qquad (1)$$

and

$$3x^3 + 2x^2 + 1. \qquad (3)$$

Multiplying (3) by $3x$ and subtracting from (1) we have

$$-6x^3 - x^2 - x - 1. \qquad (4)$$

By argument similar to that above, (3) and (4) have the same H. C. F. as (1) and (3) and hence the same as (1) and (2). Multiplying (3) by 2 and adding to (4) we have,

$$3x^2 - x + 1. \qquad (5)$$

Then the H. C. F. of (5) and (3) is the same as that of (1) and (2). Multiplying (5) by x and subtracting from (3), we have

$$3x^2 - x + 1. \qquad (6)$$

Then the H. C. F. of (5) and (6) is the same as that of (1) and (2). But (5) and (6) are identical, that is, their H. C. F. is $3x^2 - x + 1$. Hence this is the H. C. F. of (1) and (2).

The work may be conveniently arranged thus :

$$\begin{array}{ll} (1) & 9x^4 \qquad - x^2 + 2x - 1 \\ & \underline{9x^4 + 6x^3 \qquad + 3x} \\ (4) & \quad -6x^3 - x^2 - x - 1 \\ & \quad \underline{6x^3 + 4x^2 \qquad + 2} \\ (5) & \qquad\quad 3x^2 - x + 1 \end{array} \qquad\qquad \begin{array}{rl} 27x^5 \qquad + 8x^2 - 3x + 1 & (2) \\ \underline{27x^5 - 3x^3 + 6x^2 - 3x} & \\ 3x^3 + 2x^2 \qquad + 1 & (3) \\ \underline{3x^3 - x^2 + x} & \\ 3x^2 - x + 1 & (6) \end{array}$$

The object at each step is to obtain a new expression of as low a degree as possible. For this purpose the highest powers are eliminated step by step by the method of addition or subtraction.

E.g. In Ex. 1, x^5 was eliminated first, then x^4, and then x^3.

By principles **1** and **2**, each new expression contains all the factors common to the given expressions. Hence, whenever an expression is reached which is *identical with the preceding one*, this is the H. C. F.

102. The process is further illustrated as follows:

Ex. 2. Find the H.C.F. of $2x^3-2x^2-3x-2$,

and $3x^3-x^2-2x-16$.

Arranging the work as in Ex. 1, we have

(1)	$2x^3-2x^2-3x-2$	$3x^3-x^2-2x-16$	(2)
$2\cdot$(1)	$4x^3-4x^2-6x-4$	$6x^3-2x^2-4x-32$	$2\cdot$(2)
$x\cdot$(3)	$4x^3+5x^2-26x$	$6x^3-6x^2-9x-6$	$3\cdot$(1)
(4)	$-9x^2+20x-4$	$4x^2+5x-26$	(3)
	$9x^2-18x$	$36x^2+45x-234$	$9\cdot$(3)
(7)	$2x-4$	$-36x^2+80x-16$	$4\cdot$(4)
(8)	$x-2$	$125x-250$	(5)
		$x-2$	(6)

Explanation. To eliminate x^3, we multiply (1) by 3 and (2) by 2 and subtract, obtaining (3).

To eliminate x^2 from (3), we need another expression of the second degree. To obtain this we multiply (1) by 2 and (3) by x and subtract, obtaining (4).

Using (4) and (3), we eliminate x^2, obtaining (5). Since (5) contains all factors common to (1) and (2), and since 125 is not such a factor, this is discarded without affecting the H.C.F., giving (6).

Multiplying (6) by 9 and adding to (4) we have (7). Discarding the factor 2 gives (8) which is identical with (6). Hence $x-2$ is the H.C.F. sought.

103. Any *monomial* factors should be removed from each expression at the outset. If there are such factors *common* to the given expressions, these form a part of the H.C.F.

When this is done, then any monomial factor of any one of the derived expressions may be at once discarded without affecting the H.C.F., as in (5) of the preceding example.

In this way also the hypothesis of principle **2** is always fulfilled; namely, that at every step the multiplier of one expression shall have no factor in common with the other expression.

Ex. 3. Find the H. C. F. of $3x^3 - 7x^2 + 3x - 2$,
and $$x^4 - x^3 - x^2 - x - 2.$$

$$\begin{array}{lrrl}
(1) & 3x^3 - 7x^2 + 3x - 2 & x^4 - x^3 - x^2 - x - 2 & (2) \\
4\cdot(1) & 12x^3 - 28x^2 + 12x - 8 & 3x^4 - 3x^3 - 3x^2 - 3x - 6 & 3\cdot(2) \\
3\cdot(3) & \underline{12x^3 - 18x^2 - 3x - 18} & \underline{3x^4 - 7x^3 + 3x^2 - 2x \quad\quad} & x\cdot(1) \\
(4) & -10x^2 + 15x + 10 & 4x^3 - 6x^2 - x - 6 & (3) \\
(5) & -5(2x+1)(x-2). & &
\end{array}$$

Explanation. To eliminate x^4, we multiply (1) by x and (2) by 3 and subtract, obtaining (3).

To eliminate x^3, we multiply (1) by 4 and (3) by 3 and subtract, obtaining (4).

At this point the work may be shortened by factoring (4) as in (5). We may now reject, not only the factor -5, but also $2x + 1$, which is a factor of neither (1) nor (2), since $2x$ does not divide the highest power of either expression. But $x - 2$ is seen to be a factor of (2), by §§ 91, 92, and hence it is a common factor of (2) and (4) and therefore of (1) and (2). Hence $x - 2$ is the H. C. F. sought.

EXERCISES

Find the H.C.F. of the following pairs of expressions:

1. $a^3 + 6a^2 + 6a + 5$, $a^3 + 4a^2 - 4a + 5$.
2. $x^4 - 2x^3 - 2x^2 + 5x - 2$, $x^4 - 4x^3 + 6x^2 - 5x + 2$.
3. $2x^3 - 9x^2 - 13x - 4$, $x^3 - 12x^2 + 31x + 28$.
4. $x^4 - 5x^2 + 3x - 2$, $x^4 - 3x^3 + 3x^2 - 3x + 2$.
5. $2x^3 - 9x^2 + 8x - 2$, $2x^3 + 5x^2 - 5x + 1$.
6. $3a^4 - 2a^3 + 10a^2 - 6a + 3$, $2a^4 + 3a^3 + 5a^2 + 9a - 3$.
7. $15x^4 + 19x^3 - 44x^2 - 15x + 9$,
   $15x^4 - 6x^3 + 51x^2 + 11x - 15$.
8. $r^5 + 2r^4 - 2r^3 - 8r^2 - 7r - 2$, $r^5 - 2r^4 - 2r^3 + 4r^2 + r - 2$.

104. The following principle enables us to find the L.C.M. of two expressions by means of the method which has just been used for finding the H. C. F.

The L. C. M. of two expressions is equal to the product of either expression and the quotient obtained by dividing the other by the H. C. F. of the two expressions.

For let A and B be two expressions whose H. C. F. is F so that $A = mF$ and $B = nF$. Hence the L. C. M. of A and B is mnF. But $mnF = m(nF) = mB$. Also $mnF = n(mF) = nA$. Therefore the L. C. M. is either mB or nA, where $m = A \div F$ and $n = B \div F$.

Ex. Find the L.C.M. of $9x^4 - x^2 + 2x - 1$, (1)

and $27x^5 + 8x^2 - 3x + 1$. (2)

The H. C. F. was found in § 101 to be $3x^2 - x + 1$.
Dividing (1) by $3x^2 - x + 1$ we have $3x^2 + x - 1$.

Hence the L. C. M. of (1) and (2) is

$$(27x^5 + 8x^2 - 3x + 1)(3x^2 + x - 1).$$

EXERCISES

Find the L. C. M. of each of the following sets.

1. $a^4 + a^3 + 2a^2 - a + 3,\ a^4 + 2a^3 + 2a^2 - a + 4$.
2. $a^3 - 6a^2 + 11a - 6,\ a^3 - 9a^2 + 26a - 24$.
3. $2a^3 + 3a^2b - 2ab^2 - 3b^3,\ 2a^4 - a^3b - 2a^2b^2 + 4ab^3 - 3b^4$.
4. $2a^3 - a^2b - 13ab^2 - 6b^3$,
   $2a^4 - 5a^3b - 11a^2b^2 + 20ab^3 + 12b^4$.
5. $4a^3 - 15a^2 - 5a - 3,\ 8a^4 - 34a^3 + 5a^2 - a + 3$,
   $2a^3 - 7a^2 + 11a - 4$.
6. $a^4 + a^2 + 1,\ a^3 + 2a^2 - 2a + 3$.
7. $2k^3 - k^2l - 13kl^2 + 5l^3,\ 3k^3 - 16k^2l + 24kl^2 - 7l^3$.
8. $12r^4 - 20r^3s - 15r^2s^2 + 35rs^3 - 12s^4$,
   $6r^3 - 7r^2s - 11rs^2 + 12s^3$.
9. $2a^3 - 7a^2 + 6a - 2,\ a^3 + 2a^2 - 13a + 10,\ a^3 + 6a^2 + 6a + 5$.
10. $x^3 - xy^2 + yx^2 - y^3,\ 2x^3 + x^2y + xy^2 + 2y^3$,
    $2x^3 + 3x^2y + 3xy^2 + 2y^3$.

PROBLEMS INVOLVING DENSITIES

105. If a cubic foot of a certain kind of rock weighs 2.5 times as much as a cubic foot of fresh water, the **density** of this rock is said to be 2.5.

A cubic centimeter of distilled water, which weighs one gram at a temperature 4° above zero, is used as a standard of comparison. We therefore say that the density of any substance is equal to the number of grams which a cubic centimeter of it weighs. That is, the weight of an object in grams is the product of its volume in cubic centimeters multiplied by its density.

Hence, if we represent the weight of an object in grams by w, its volume in cubic centimeters by v, and its density by d, we have the relation,

$$w = vd. \tag{1}$$

1. If 500 ccm. of alcohol, density .79, is mixed with 300 ccm. of distilled water, what is the density of the mixture?

The volume of the mixture is the sum of the volumes, and the weight of the mixture is the sum of the weights of the water and alcohol. Hence, from formula (1):

$$500 \times .79 + 300 \times 1 = d \times 800. \quad \text{To find } d.$$

2. How many cubic centimeters of cork, density .24, must be combined with 75 ccm. of steel, density 7.8, in order that the average density shall be equal to that of water, *i.e.* so that the combined mass will just float?

Let $v =$ volume of cork to be used. Then the total volume is $75 + v$, the total weight is $75 \times 7.8 + .24\,v$, and the density is 1. Hence, $75 \times 7.8 + .24\,v = 1 \cdot (75 + v)$. Solve this equation for v.

3. Brass is an alloy of copper and zinc. How many cubic centimeters of zinc, density 6.86, must be combined with 100 ccm. of copper, density 8.83, to form brass whose density is 8.31?

4. Coinage silver is an alloy of copper and silver. How many ccm. of copper, density 8.83, must be added to 10 ccm. of silver, density 10.57, to form coinage silver, whose density is 10.38?

5. The density of pure gold is 19.36 and of nickel 8.57. How many ccm. of nickel must be mixed with 10 ccm. of pure gold to form 14 karat gold whose density is 14.88?

6. How much mercury, density 13.6, must be added to 20 ccm. of gold, density 19.36, so that the density of the compound shall be 16.9?

7. What is the average density of 40 ccm. of water, density 1, and 180 ccm. of alcohol, density .79?

8. How many cubic centimeters of water must be mixed with 350 ccm. of alcohol, so that the density of the mixture shall be 97?

9. The density of copper is 8.83. 500 ccm. of copper is mixed with 700 ccm. of lead, whose density is 11.35. What is the density of the combined mass?

10. When 960 ccm. of iron, density 7.3, is fastened to 8400 ccm. of white pine, the combination just floats, *i.e.* has a density of 1. What is the density of white pine?

11. How many cubic centimeters of matter, density 4.20, must be added to 150 ccm. of density 8.10 so that the density of the compound shall be 5.4?

12. How many cubic centimeters of nitrogen, density 0.001255, must be mixed with 210 ccm. of oxygen, density 0.00143, to form air, whose density is 0.001292?

CHAPTER VI

POWERS AND ROOTS

106. Each of the operations thus far studied leads to a **single result.**

E.g. Two numbers have one and only one *sum*, § 2, and one and only one *product*, § 7.

When a number is subtracted from a given number, there is one and only one *remainder*, § 6.

When a number is divided by a given number, there is one and only one *quotient*, § 11.

We are now to study an operation which leads to **more than one result**; namely, the operation of finding roots.

Thus both 3 and -3 are square roots of 9, since $3 \cdot 3 = 9$, and also $(-3)(-3) = 9$. The two square roots of 9 are written $\pm\sqrt{9} = \pm 3$.

The operations of addition, subtraction, multiplication, and division are **possible** in all cases *except dividing by zero,* which is explicitly ruled out, §§ 24, 25.

Division is possible in general because *fractions* are admitted to the number system, and subtraction is possible in general because *negative numbers* are admitted. Thus $7 \div 3 = 2\frac{1}{3}$, $5 - 7 = -2$.

107. The operation of finding roots is not possible in all cases, unless other numbers besides positive and negative integers and fractions are admitted to the number system.

E.g. The number $\sqrt{2}$ is not an *integer* since $1^2 = 1$ and $2^2 = 4$.

Suppose $\sqrt{2} = \frac{a}{b}$ a fraction reduced to its lowest terms, so that a and b have no common factor. Then $\frac{a^2}{b^2} = 2$. But this is impossible, for if b^2 exactly divides a^2, then a and b must have factors in common. Hence $\sqrt{2}$ is not a *fraction.*

108. If a positive number is not the square of an integer or a fraction, a number may be found in terms of integers and fractions whose square differs from the given number by as little as we please. See p. 182, E. C.

E.g. 1.41, 1.414, 1.4141 are successive numbers whose squares differ by less and less from 2. In fact $(1.4141)^2$ differs from 2 by less than .0004, and by continuing the process by which these numbers are found, § 176, E. C., a number may be reached whose square differs from 2 by as little as we please.

1.41, 1.414, 1.4141, etc., are successive approximations to the number which we call *the square root of* 2, and which we represent by the symbol $\sqrt{2}$.

109. Definition. If a number is neither an integer nor a fraction, but if it can be *approximated* by means of integers and fractions to any specified degree of accuracy, then such a number is called an **irrational number.** See § 36.

E.g. $\sqrt{2}$, $\sqrt[3]{2}$, $\sqrt[3]{5}$, etc., are irrational numbers, whereas $\sqrt{4}$, $\sqrt[3]{8}$, are rational numbers.

It will be found also in higher work that there are other irrational numbers besides indicated roots. For instance, the number π, which is the ratio of the circumference to the diameter of a circle, is an irrational number though it is not an indicated root.

It is shown in higher algebra that irrational numbers correspond to definite points on the line of the number scale, § 48, E. C., just as integers and fractions do.

We, therefore, now enlarge the number system to include **irrational numbers** as well as integers and fractions.

The set of numbers consisting of all rational and irrational numbers is called the **real number system.**

110. Even with the number system as thus enlarged, it is still not possible to find roots in all cases. The exception is the **even root of a negative number.**

E.g. $\sqrt{-4}$ is neither $+2$ nor -2, since $(+2)^2 = +4$ and $(-2)^2 = +4$, and no approximation to this root can be found as in the case of $\sqrt{2}$.

111. Definition. The indicated *even* root of a negative number, or any expression containing such a root, is called an **imaginary number,** or more properly, a **complex number.** All other numbers are called **real numbers.**

E.g. $\sqrt{-4}$, $\sqrt[4]{-2}$, $1+\sqrt{-2}$, are *complex numbers,* while 5, $\sqrt[3]{2}$, $1+\sqrt{2}$ are *real numbers.*

Complex numbers cannot be pictured on the line which represents real numbers, but another kind of graphic representation of complex numbers is made in higher algebraic work, and such numbers form the basis of some of the most important investigations in advanced mathematics.

112. With the number system thus enlarged, by the admission of irrational and complex numbers, we have the following **fundamental definition.**

$$(\sqrt[k]{n})^k = n.$$

That is, a kth root of any number n is such a number that, if it be raised to the kth power, the result is n itself.

E.g. $(\sqrt[3]{2})^3 = 2$, $(\sqrt{4})^2 = 4$, $(\sqrt{-2})^2 = -2$.

The imaginary or complex **unit** is $\sqrt{-1}$. By the above definition we have

$$(\sqrt{-1})^2 = -1.$$

In operating upon complex numbers, they should first be expressed in terms of the **imaginary unit.**

E.g. $\sqrt{-2} = \sqrt{2} \cdot \sqrt{-1}$, $\sqrt{-16} = \sqrt{16} \cdot \sqrt{-1} = 4\sqrt{-1}$.

$$\sqrt{-4} \cdot \sqrt{-9} = (\sqrt{4} \cdot \sqrt{-1})(\sqrt{9}\ \sqrt{-1}) = 2 \cdot 3(\sqrt{-1})^2 = -6.$$

$$\sqrt{-4} + \sqrt{-9} = \sqrt{4} \cdot \sqrt{-1} + \sqrt{9} \cdot \sqrt{-1} = (2+3)\sqrt{-1} = 5\sqrt{-1}.$$

$$\frac{\sqrt{-16}}{\sqrt{-9}} = \frac{\sqrt{16} \cdot \sqrt{-1}}{\sqrt{9} \cdot \sqrt{-1}} = \frac{\sqrt{16}}{\sqrt{9}} = \frac{4}{3}.$$

113. By means of irrational and complex numbers it can be shown that every number has *two* square roots, *three* cube roots, *four* fourth roots, etc. See § 195, Exs. 17–20.

E.g. The square roots of 9 are $+3$ and -3. The square roots of -9 are $\pm\sqrt{-9} = \pm 3\sqrt{-1}$. The cube roots of 8 are $2, -1+\sqrt{-3}$ and $-1-\sqrt{-3}$. The fourth roots of 16 are $+2, -2, +2\sqrt{-1}$ and $-2\sqrt{-1}$.

Any positive real number has two real roots of *even* degree, one positive and one negative.

E.g. ± 2 are fourth roots of 16. The square roots of 3 are $\pm\sqrt{3}$.

Any real number, positive or negative, has one real root of *odd* degree, whose sign is the same as that of the number itself.

E.g. $\sqrt[3]{27} = 3$ and $\sqrt[5]{-32} = -2$.

114. The positive *even* root of a positive real number, or the real *odd* root of any real number, is called the **principal root.**

The positive square root of a negative real number is also sometimes called the *principal* imaginary root.

E.g. 2 is the principal square root of 4, 3 is the principal 4th root of 81; -4 is the principal cube root of -64; and $+\sqrt{-3}$ is the principal square root of -3.

Unless otherwise stated the radical sign is understood to indicate the *principal root.*

Since a number has more than one root it is necessary to limit certain theorems so as to make them apply to principal roots only.

Thus the square root of 2^4 is not 2^2 unless the *principal* square root of 2^4 is understood.

Again, the cube root of $2^3 \cdot 3^6$ is not equal to $2 \cdot 3^2$ unless the *principal* cube root is understood.

Also the cube root of $\frac{3^6}{4^3}$ is not $\frac{3^2}{4}$ unless it is specified that the *principal* cube root is the one taken.

Unless this restriction is understood it may be easily shown that the conclusions in §§ 119, 120, 121 are not true.

PRINCIPLES INVOLVING POWERS AND ROOTS

115. From § 43, $(2^3)^2 = (2^2)^3 = 2^6 = 64.$

In general, if n and k are any positive integers,

$$(b^k)^n = (b^n)^k = b^{nk}.$$

For
$$(b^k)^n = b^k \cdot b^k \cdot b^k \cdots \text{to } n \text{ factors}$$
$$= b^{k+k+k\cdots \text{ to } n \text{ terms}} = b^{nk}.$$

Likewise,
$$(b^n)^k = b^{nk}.$$

Hence: *The nth power of the kth power of any base is the nkth power of that base.*

116. Again, from §§ 43, 44, $(2^3 \cdot 3^2)^2 = 2^6 \cdot 3^4.$

In general, if k, r, and n are any positive integers,

$$(a^k b^r)^n = a^{nk} b^{nr}.$$

For
$$(a^k b^r)^n = (a^k b^r) \cdot (a^k b^r) \cdots \text{to } n \text{ factors}$$
$$= (a^k \cdot a^k \cdots \text{to } n \text{ factors})(b^r \cdot b^r \cdots \text{to } n \text{ factors})$$
$$= (a^k)^n \cdot (b^r)^n = a^{nk} b^{nr}.$$

Hence: *The nth power of the product of several factors is the product of the nth powers of those factors.*

117. From § 43, we have $\left(\frac{2^3}{3^2}\right)^2 = \frac{2^3}{3^2} \cdot \frac{2^3}{3^2} = \frac{2^6}{3^4}.$

In general,
$$\left(\frac{a^k}{b^r}\right)^n = \frac{a^{nk}}{b^{nr}}.$$

For we have
$$\left(\frac{a^k}{b^r}\right)^n = \frac{a^k}{b^r} \cdot \frac{a^k}{b^r} \cdot \frac{a^k}{b^r} \cdots \text{to } n \text{ factors}$$
$$= \frac{a^{nk}}{b^{nr}}.$$

Hence: *The nth power of the quotient of two numbers equals the quotient of the nth power of those numbers.*

118. It follows from §§ 115, 116, 117, that

Any positive integral power of a monomial is found by multiplying the exponents of the factors by the exponent of the power.

119. We may easily verify that $\sqrt{3^4} = 3^{4\div 2} = 3^2 = 9$.

In general, if k and r are positive integers and b any positive real number, we have:

$$\sqrt[r]{b^{kr}} = b^{kr\div r} = b^k.$$

For, from § 115, $(b^k)^r = b^{kr}$.

Hence by definition b^k is an rth root of b^{kr}, and since b^k is *real* and *positive*, it is the *principal* rth root of b^{kr} (§ 114).

That is, $\sqrt[r]{b^{kr}} = b^{kr\div r} = b^k$.

Hence: *The **r**th root of the **kr**th power of any positive real number is a power of that number whose exponent is* $\boldsymbol{kr \div r = k}$.

E.g. $\sqrt[4]{2^{12}} = 2^{12\div 4} = 2^3 = 8$. But it does *not* follow that

$$\sqrt[4]{(-2)^{12}} = (-2)^{12\div 4} = (-2)^3 = -8,$$

since $(-2)^{12} = (2)^{12}$, and hence $\sqrt[4]{(-2)^{12}} = \sqrt[4]{2^{12}} = +8$.

The corresponding principle holds when b is *negative* if r is *odd* and also when b is *negative* if k is *even*.

E.g. $\sqrt[3]{(-2)^6} = (-2)^{6\div 3} = (-2)^2 = 4$; $\sqrt[3]{(-2)^{15}} = (-2)^5 = -32$.

120. Another general principle, if a and b are any real numbers and r any positive integer, is

$$\sqrt[r]{ab} = \sqrt[r]{a} \cdot \sqrt[r]{b}.$$

For by § 116, $(\sqrt[r]{a} \cdot \sqrt[r]{b})^r = (\sqrt[r]{a})^r \cdot (\sqrt[r]{b})^r$,

And by § 112, $= a \cdot b$.

Hence $ab = (\sqrt[r]{a} \cdot \sqrt[r]{b})^r$.

Taking the principal rth root of both members,

we have $\sqrt[r]{ab} = \sqrt[r]{a} \cdot \sqrt[r]{b}$.

Hence: *The rth root of the product of two positive real numbers equals the product of the rth roots of the numbers.*

When r is *even* the corresponding principle does not hold if a and b are both *negative.*

For example, it is *not true* that $\sqrt{(-4)(-9)} = \sqrt{-4} \cdot \sqrt{-9}$.
For $\sqrt{(-4)(-9)} = \sqrt{36} = 6$; while $\sqrt{-4} \cdot \sqrt{-9}$
$= 2\sqrt{-1} \cdot 3\sqrt{-1} = 6(\sqrt{-1})^2 = -6$. See § 112.

121. Again, if a and b are any positive real numbers and r is any positive integer,

$$\sqrt[r]{\frac{a}{b}} = \frac{\sqrt[r]{a}}{\sqrt[r]{b}}.$$

For we have by §§ 117, 112, $\left(\frac{\sqrt[r]{a}}{\sqrt[r]{b}}\right)^r = \frac{(\sqrt[r]{a})^r}{(\sqrt[r]{b})^r} = \frac{a}{b}.$

Hence, taking the principal rth root of both members,

we have $$\sqrt[r]{\frac{a}{b}} = \frac{\sqrt[r]{a}}{\sqrt[r]{b}}.$$

That is: *The rth root of the quotient of two positive real numbers equals the quotient of the rth roots of the numbers.*

E.g. $$\sqrt{\frac{16}{25}} = \frac{\sqrt{16}}{\sqrt{25}} = \frac{4}{5}; \quad \sqrt[3]{\frac{-8}{27}} = \frac{\sqrt[3]{-8}}{\sqrt[3]{27}} = \frac{-2}{3} = -\frac{2}{3}.$$

The corresponding principle does *not* hold when r is *even* if a is positive and b is negative. Thus it is not true that

$$\sqrt{\frac{4}{-9}} = \frac{\sqrt{4}}{\sqrt{-9}} = \frac{2}{3\sqrt{-1}} = \frac{2\sqrt{-1}}{3(\sqrt{-1})^2} = \frac{2\sqrt{-1}}{-3} = -\frac{2}{3}\sqrt{-1}.$$

But we have $$\sqrt{\frac{4}{-9}} = \sqrt{\frac{4}{9}(-1)} = \frac{2}{3}\sqrt{-1}.$$

If r is odd, the principle holds for *all* real values of a and b.

122. From §§ 119, 120, 121, it follows that:

If a monomial is a perfect power of the **k***th degree, its* **k***th root may be found by dividing the exponent of each factor by the index of the root.*

In applying the above principle to the reduction of algebraic expressions containing letters, it is assumed that the values of the letters are such that the principles apply.

EXERCISES

Find the following indicated powers and roots, and reduce each expression to its simplest form:

1. $(a^3b^4c^5)^7$.

2. $(2^{a+b}\cdot 3^c\cdot 5^b)^{a-b}$.

3. $\left(\dfrac{a^3b^4c^5}{2^3\cdot 3^2\cdot 4^3}\right)^2$.

4. $(a^{x-y})^{x^2+xy+y^2}$.

5. $(x^4y^5z^{x+y})^{x-y}$.

6. $\left(\dfrac{5^2b^2mn}{3^7bc^4}\right)^3$.

7. $(3^5\cdot 4^5\cdot 2^5)^{a-b}$.

8. $\sqrt[a]{3^{2a}\cdot 2^a\cdot 5^{3a}}$.

9. $\sqrt[3]{\dfrac{-27\cdot 8\,a^6}{64\,c^9d^{6a}}}$.

10. $(a^{m+n-1}b^{m-n}c^{mn})^{m+n}$.

11. $(3^{a+4}\cdot 4^{b-7}\cdot 5^{c-1})^{abx}$.

12. $\sqrt[2a]{3^{6a}\cdot 4^{2a}\cdot 5^{8a}\cdot 7^{4a}}$.

13. $\sqrt[a-b]{3^{a^3-b^3}\cdot 4^{a-b}\cdot 5^{a^2-b^2}}$.

14. $\sqrt{64\cdot 25\cdot 256\cdot 625}$.

15. $\sqrt[3]{27\cdot 125\cdot 64\cdot 3^{6a}}$.

16. $(a-b)^{m-n}(b-c)^{m-n}(a+b)^{m-n}$.

17. $\sqrt{\dfrac{(a-b)^2(a^2+2\,ab+b^2)}{(a-b)^4(a+b)^2}}$.

18. $\sqrt{\dfrac{(4\,x^2+4\,x+1)(4\,x^2-4\,x+1)}{36\,x^4-12\,x^2+1}}$.

19. $\sqrt[3]{(-343)(-27)x^6(a+b)^{3a}}$.

20. $\sqrt[3]{\dfrac{(-8)(-27)(-125)a^{3m}b^{9n}}{(-1)(-512)(1000)x^{15a}y^{21}}}$.

ROOTS OF POLYNOMIALS

123. In the Elementary Course, pp. 178–180, it was shown that the process for finding the **square root** of a polynomial is obtained by studying the relation of the expressions, $a+b$, $a+b+c$, and $a+b+c+d$, to their respective squares.

The process for finding the **cube root** of a polynomial is obtained by studying the relation of the cube

$$a^3+3a^2b+3ab^2+b^3 \text{ or } a^3+b(3a^2+3ab+b^2)$$

to its cube root $a+b$.

An example will illustrate the process.

Ex. 1. Find the cube root of

$$27m^3+108m^2n+144mn^2+64n^3.$$

Given cube,		$27m^3+108m^2n+144mn^2+64n^3$	$3m+4n$, **cube root**
	$a^3=27m^3$		**1st partial product**
	$3a^2=27m^2$	$108m^2n+144mn^2+64n^3$,	**1st remainder**
	$3ab=36mn$		
	$b^2=16n^2$		
$3a^2+3ab+b^2=$	$27m^2+36mn+16n^2$	$108m^2n+144mn^2+64n^3$	$=b(3a^2+3ab+b^2)$
		0	

Explanation. The cube root of the first term, namely $3m$, is the first term of the root and corresponds to a of the formula. Cubing $3m$ gives $27m^3$ which is the a^3 of the formula.

Subtracting $27m^3$ leaves $108m^2n+144mn^2+64n^3$, which is the $b(3a^2+3ab+b^2)$ of the formula.

Since b is not yet known, we cannot find completely either factor of $b(3a^2+3ab+b^2)$, but since a has been found, we can get the first term of the factor $3a^2+3ab+b^2$; viz. $3a^2$ or $3(3m)^2=27m^2$, which is the partial divisor. Dividing $108m^2n$ by $27m^2$ we have $4n$, which is the b of the formula.

Then $3a^2+3ab+b^2=3(3m)^2+3(3m)(4n)+(4n)^2=27m^2+36mn+16n^2$ is the complete divisor. This expression is then multiplied by $b=4n$, giving $108m^2n+144mn^2+64n^3$, which corresponds to $b(3a^2+3ab+b^2)$ of the formula. On subtracting, the remainder is zero and the process ends. Hence, $3m+4n$ is the required root.

Ex. 2. Find the cube root of

$$33\,x^4 - 9\,x^5 + x^6 - 63\,x^3 + 66\,x^2 - 36\,x + 8.$$

We first arrange the terms with respect to the exponents of x.

$$\begin{array}{rl}
 & \underline{x^2 - 3\,x + 2, \text{ cube root}} \\
\text{Given cube,} & x^6 - 9\,x^5 + 33\,x^4 - 63\,x^3 + 66\,x^2 - 36\,x + 8 \\
a^3 = x^6 & \\
3\,a^2 = 3\,x^4 & \big|\, -9\,x^5 + 33\,x^4 - 63\,x^3 + 66\,x^2 - 36\,x + 8 \\
3\,a^2 + 3\,ab + b^2 = 3\,x^4 - 9\,x^3 + 9\,x^2 & \big|\, -9\,x^5 + 27\,x^4 - 27\,x^3 \\
3\,a'^2 = 3(x^2 - 3\,x)^2 = 3\,x^4 - 18\,x^3 + 27\,x^2 & \big|\, 6\,x^4 - 36\,x^3 + 66\,x^2 - 36\,x + 8 \\
3\,a'^2 + 3\,a'b' + b'^2 = 3\,x^4 - 18\,x^3 + 33\,x^2 - 18\,x + 4 & \big|\, 6\,x^4 - 36\,x^3 + 66\,x^2 - 36\,x + 8 \\
 & 0
\end{array}$$

The cube root of x^6, or x^2, is the first term of the root. The first partial divisor, which corresponds to $3\,a^2$ of the formula, is $3(x^2)^2 = 3\,x^4$. Dividing $-9\,x^5$ by $3x^4$ we have $-3\,x$, which is the second term of the quotient, corresponding to b of the formula.

After these two terms of the root have been found, we consider $x^2 - 3\,x$ as the a of the formula and call it a'. The new partial divisor is $3\,a'^2 = 3(x^2 - 3\,x)^2 = 3\,x^4 - 18\,x^3 + 27\,x^2$, and the new b, which we call b', is then found to be 2.

Substituting $x^2 - 3\,x$ for a' and 2 for b' in $3\,a'^2 + 3\,a'b' + b'^2$, we have $3\,x^4 - 18\,x^3 + 33\,x^2 - 18\,x + 4$, which is the complete divisor. On multiplying this expression by 2 and subtracting, the remainder is zero. Hence the root is $x^2 - 3\,x + 2$.

In case there are four terms in the root, the sum of the first three, when found as above, is regarded as the new a, called a''. The remaining term is the new b and is called b''. The process is then precisely the same as in the preceding step.

EXERCISES

Find the square roots of the following:

1. $m^2 + 4\,mn + 6\,ml + 4\,n^2 + 12\,ln + 9\,l^2$.

2. $4\,x^4 + 8\,ax^3 + 4\,a^2x^2 + 16\,b^2x^2 + 16\,ab^2x + 16\,b^4$.

3. $9\,a^2 - 6\,ab + 30\,ac + 6\,ad + b^2 - 10\,bc - 2\,bd + 25\,c^2 + 10\,cd + d^2$.

4. $9a^2 - 30ab - 3ab^2 + 25b^2 + 5b^3 + \frac{b^4}{4}.$

5. $\frac{4}{9}a^2x^4 - \frac{4}{3}abx^3z + \frac{8}{3}a^2bx^2z^2 + b^2x^2z^2 - 4ab^2xz^3 + 4a^2b^2z^4.$

6. $a^2 - 6ab + 10ac - 14ad + 9b^2 - 30bc + 42bd + 25c^2 - 70cd + 49d^2.$

7. $9a^6 - 24a^3b^4 - 18a^3c^5 + 6a^3d^2 + 16b^8 + 24b^4c^5 - 8b^4d^2 + 9c^{10} - 6c^5d^2 + d^4.$

8. $x^{10} - 8x^5w^5 + 16w^{10} - 4x^5y^3 + 16y^3w^5 + 4y^6 + 6x^5z^4 - 24z^4w^5 - 12y^3z^4 + 9z^8.$

Find the cube root of each of the following:

9. $x^3 - 3x^2y + 3xy^2 - y^3 + 3x^2z - 6xyz + 3y^2z + 3xz^2 - 3yz^2 + z^3.$

10. $a^3 + 3a^2b + 3a^2c + 3ab^2 + 6abc + 3ac^2 + b^3 + 3b^2c + 3bc^2 + c^3.$

11. $8x^6 - 36x^5 + 114x^4 - 207x^3 + 285x^2 - 225x + 125.$

12. $27z^6 - 54az^5 + 63a^2z^4 - 44a^3z^3 + 21a^4z^2 - 6a^5z + a^6.$

13. $1 - 9y^2 + 39y^4 - 99y^6 + 156y^8 - 144y^{10} + 64y^{12}.$

14. $125x^6 - 525x^5y + 60x^4y^2 + 1547x^3y^3 - 108x^2y^4 - 1701xy^5 - 729y^6.$

15. $64l^{12} - 576l^{10} + 2160l^8 - 4320l^6 + 4860l^4 - 2916l^2 + 729.$

16. $a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6.$

17. $a^9 - 9a^8b + 36a^7b^2 - 84a^6b^3 + 126a^5b^4 - 126a^4b^5 + 84a^3b^9 - 36a^2b^7 + 9ab^8 - b^9.$

18. $a^3 + 6a^2b - 3a^2c + 12ab^2 - 12abc + 3ac^2 + 8b^3 - 12b^2c + 6bc^2 - c^3.$

19. $343a^6 - 441a^5b + 777a^4b^2 - 531a^3b^3 + 444a^2b^4 - 144ab^5 + 64b^6.$

20. $a^{18} + 12a^{15} + 60a^{12} + 160a^9 + 240a^6 + 192a^3 + 64.$

21. $27l^{12} + 189l^{11} + 198l^{10} - 791l^9 - 594l^8 + 1701l^7 - 729l^6.$

ROOTS OF NUMBERS EXPRESSED IN ARABIC FIGURES

124. The cube root of a number expressed in Arabic figures, like the square root of such a number, may be found by the process used for polynomials. An example will illustrate.

Ex. 1. Find the cube root of 389,017.

In order to decide how many digits there are in the root, we observe that $10^3 = 1000$, $100^3 = 1{,}000{,}000$. Hence the root lies between 10 and 100, that is, it contains two digits. Since $70^3 = 343{,}000$ and $80^3 = 512{,}000$, it follows that 7 is the largest number possible in tens' place. The work is arranged as follows:

The given cube,	389 017	$70 + 3$, cube root.
$a^3 = 70^3 =$	343 000	1st partial product.
$3a^2 = 3 \cdot 70^2 = 14700$	46 017	1st remainder.
$3ab = 3 \cdot 70 \cdot 3 = 630$		
$b^2 = 3^2 = 9$		
$3a^2 + 3ab + b^2 = 15339$	46 017	$= b(3a^3 + 3ab + b^2)$.
	0	

Having decided as above that the a of the formula is 7 tens, we cube this and subtract, obtaining 46,017 as the remaining part of the power.

The first partial divisor, $3a^2 = 14{,}700$, is divided into 46,017, giving a quotient 3, which is the b of the formula. Hence the first complete divisor, $3a^2 + 3ab + 3b^2$, is 15,339 and the product, $b(3a^2 + 3ab + b^2)$, is 46,017. Since the remainder is zero, the process ends and 73 is the cube root sought.

125. The cube of any number from 1 to 9 contains one, two, or three digits; the cube of any number between 10 and 99 contains four, five, or six digits; the cube of any number between 100 and 999 contains seven, eight, or nine digits, etc. Hence it is evident that if the digits of a number are separated into groups of three figures each, counting from units' place toward the left, the number of groups thus formed is the same as the number of digits in the root.

Ex. 2. Find the cube root of 13,997,521.

$$
\begin{array}{rrl}
 & \text{The given cube, } 13\,997\,521 & \underline{|200 + 40 + 1 = 241, \text{ cube root.}} \\
a^3 = 200^3 = & 8\,000\,000 & \\
3\,a^2 = 120000 & \overline{|\,5\,997\,521} & \\
3\,ab = 24000 & | & \\
b^2 = \underline{1600} & | & \\
145600 & \underline{|\,5\,824\,000} & = b\,(3\,a^2 + 3\,ab + b^2) \\
3\,a'^2 = 172800 & |\,173\,521 & \\
3\,a'b' = 720 & | & \\
b'^2 = \underline{1} & | & \\
173521 & \underline{|\,173\,521} & = b'\,(3\,a'^2 + 3\,a'b' + b'^2). \\
 & 0 &
\end{array}
$$

Since the root contains three digits, the first one is the cube root of 8, the largest integral cube in 13.

The first partial divisor, $3 \cdot 200^2 = 120{,}000$, is completed by adding $3\,ab = 3 \cdot 200 \cdot 40 = 24{,}000$, and $b^2 = 1600$.

The second partial divisor, $3\,a'^2$, which stands for $3(200 + 40)^2 = 172{,}800$, is completed by adding $3\,a'b'$ which stands for $3 \cdot 240 \cdot 1 = 720$, and b'^2 which stands for 1, where a' represents the part of the root *already* found and b' the next digit to be found. At this step the remainder is zero and the root sought is 241.

EXERCISES

Find the square root of each of the following:

1. 58,081. **2.** 795,664. **3.** 11,641,744.

Find the cube root of each of the following:

4. 110,592. **7.** 205,379. **10.** 2,146,689.

5. 571,787. **8.** 31,855,013. **11.** 19,902,511.

6. 7,301,384. **9.** 5,929,741. **12.** 817,400,375.

126. Since the cube of a decimal fraction has three times as many places as the given decimal, it is evident that the cube root of a decimal fraction contains one decimal place for every three in the cube. Hence for the purpose of determining the places in the root, the decimal part of a cube should be divided into groups of three digits each, counting from the decimal point toward the right.

Ex. Approximate the cube root of 34.567 to two places of decimals.

	$34.567 \mid \underline{3 + .2 + .05 + .007} = 3.257$
$a^3 = 3^3 =$	27.000
$3a^2 = 3 \cdot 3^2 = 27.$	7.567
$3ab = 3 \cdot 3(.2) = 1.8$	
$b^2 = (.2)^2 = .04$	
28.84	$5.768 = b(3a^2 + 3ab + b^2)$
$3a'^2 = 3(3.2)^2 = 30.72$	1.799000
$3a'b' = 3(3.2)(.05) = .48$	
$b'^2 = (.05)^2 = .0025$	
31.2025	$1.560125 = b'(3a'^2 + 3a'b' + b'^2)$
$3a''^2 = 3(3.25)^2 = 31.6875$	$.238875000$
$3a''b'' = 3(3.25)(.007) = .06825$	
$b''^2 = (.007)^2 = .000049$	
31.755799	$.222290593 = b''(3a''^2 + 3a''b'' + b''^2)$
	$.016584407$

The decimal points are handled exactly as in arithmetic work.

Evidently the above process can be carried on indefinitely. 3.257 is an **approximation** to the cube root of 34.567. In fact the cube of 3.257 differs from 34.567 by less than the small fraction .017. The nearest approximation using two decimal places is 3.26. If the third decimal place were any digit less than 5, then 3.25 would be the nearest approximation using two decimal places. Hence three places must be found in order to be sure of the nearest approximation to two places.

EXERCISES

Approximate the cube root of each of the following to two places of decimals.

1. 21.4736.
2. 6.5428.
3. 58.
4. 2.
5. 3.
6. .003.
7. .3917.
8. .5.
9. .05.
10. 6410.37.
11. .004178.
12. 200.002.
13. 572.274.
14. 31.7246.
15. 54913.416.
16. Approximate the square root in Exs. 1, 2, 10, 11, and 15 of the above list.

PROBLEMS ON MOMENTUM

127. The force with which a moving body strikes another depends both upon its mass and upon its rate of motion. The product of the mass and velocity of a moving body is called its **momentum.** The mass of a body is proportional to its weight. Hence weight is often used instead of mass.

By careful experiment it has been found that when a moving body strikes a body at rest but free to move, the two will move on with a combined momentum equal to the momentum of the first body before the impact.

Thus, if a freight car, weighing 25 tons and moving at the rate of 12 miles per hour, strikes a standing car weighing 15 tons, the two will move on with the original momentum of $12 \cdot 25$. But as the combined weight is now $25 + 15$, the *rate* of motion has been decreased to $7\frac{1}{2}$ miles per hour, since $12 \cdot 25 = 7\frac{1}{2}(25 + 15)$.

Again, if a croquet ball weighing 8 ounces and moving 20 feet per second, strikes another weighing 7 ounces and starts it off at the rate of 18 feet per second, then if the diminished velocity of the first ball is called v, we have

$$8 \cdot 20 = 7 \cdot 18 + 8v,$$

and solving, $v = 4.25.$

This indicates that the first ball is nearly stopped, which coincides with common observation.

1. In a switch yard a car weighing 40 tons and moving 8 miles per hour strikes a standing car weighing 24 tons. What is the velocity of the two after impact?

2. A billiard ball weighing 6 ounces and moving 16 feet per second strikes another ball which it sends off at the rate of 10 feet per second. The rate of the first ball is reduced to 9 feet per second by the impact. What is the weight of the second ball?

3. A bowler uses a 16-ounce ball to take down the last pin. The ball sends the pin off at a velocity of 6 feet per second,

the weight of the pin being 48 ounces, while the velocity of the ball is reduced to 4 feet per second. With what velocity did the ball strike the pin?

4. In each of these problems we have considered the weight of two bodies, which we may call w_1 and w_2. If we call v_1 the velocity with which the first strikes the second, v_2 the velocity imparted to the second, and v_1' the resulting decreased velocity of the first, we have

$$w_1v_1 = w_1v_1' + w_2v_2. \qquad (1)$$

5. Solve the equation (1) for v_1 in terms of w_1, w_2, v_1', and v_2. Translate into words.

6. Solve the equation (1) above for w_1 in terms of v_1, v_1', w_2, and v_2. Translate into words.

7. Solve the equation (1) above for v_1' in terms of w_1, w_2, v_1, and v_2. Translate into words.

8. Solve equation (1) for w_2 in terms of w_1, v_1, v_1', v_2, and translate the result into words.

9. Solve equation (1) for v_2, in terms of w_1, w_2, v_1, v_1', and translate the result into words.

10. In the result of the last exercise substitute $w_1 = 50$, $w_2 = 40$, $v_1 = 10$, $v_1' = 2$, and find the value of v_2. Make a problem to fit this case.

11. In the result of problem 8 substitute $w_1 = 1000$, $v_1 = 75$, $v_1' = 25$, $v_2 = 50$, and find the value of w_2. Make a problem to fit this case.

12. In the result of problem 6 substitute $w_1 = 60$, $v_1' = 10$, $w_2 = 80$, $v_2 = 25$, and find the value of w_1. Make a problem to fit this case.

13. In the result of problem 7 substitute $w_1 = 250$, $w_2 = 125$, $v_1 = 50$, and $v_2 = 50$, and find the value of v_1'. Make a problem to fit this case.

CHAPTER VII

QUADRATIC EQUATIONS

EXPOSITION BY MEANS OF GRAPHS

128. We saw, § 65, that a single equation in two variables is satisfied by indefinitely many pairs of numbers. If such an equation is of the **first degree** in the two variables, the graph is in every case a *straight line.*

We are now to consider graphs of equations of the **second degree** in two variables. See § 66.

Ex. 1. Graph the equation $y = x^2$.

By giving various values to x and computing the corresponding values of y, we find pairs of numbers as follows which satisfy this equation:

$$\begin{cases} x = 0, \\ y = 0. \end{cases} \begin{cases} x = 1, \\ y = 1. \end{cases} \begin{cases} x = -1, \\ y = 1. \end{cases} \begin{cases} x = 2, \\ y = 4. \end{cases} \begin{cases} x = -2, \\ y = 4. \end{cases} \begin{cases} x = 3, \\ y = 9. \end{cases} \begin{cases} x = -3, \\ y = 9. \end{cases} \text{etc.}$$

These pairs of numbers correspond to points which lie on a curve as shown in Figure 3. (Use two squares for one unit on x-**axis.**)

By referring to the graph the curve is seen to be symmetrical with respect to the y-axis. This can be seen directly from the equation itself since x is involved only as a square and hence, if $y = x^2$ is satisfied by $x = a$, $y = b$, it must also be satisfied by $x = -a$, $y = b$.

It may easily be verified that no three points of this curve lie on a straight line. The curve is called a **parabola.**

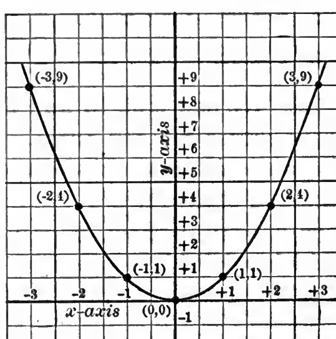


FIG. 3.

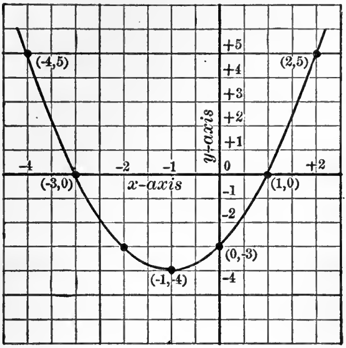


FIG. 4.

Ex. 2. Graph the equation $y = x^2 + 2x - 3$.

Each of the following pairs of numbers satisfies the equation:

$$\begin{cases} x = 0, \\ y = -3. \end{cases} \begin{cases} x = 1, \\ y = 0. \end{cases} \begin{cases} x = -1, \\ y = -4. \end{cases}$$

$$\begin{cases} x = 2, \\ y = 5. \end{cases} \begin{cases} x = -2, \\ y = -3. \end{cases} \begin{cases} x = -3, \\ y = 0. \end{cases}$$

$$\begin{cases} x = -4, \\ y = 5. \end{cases}$$

Plotting these points and drawing a smooth curve through them, we have the graph of the equation, as in Figure 4.

EXERCISES

In this manner graph each of the following:

1. $y = x^2 - 1$.
2. $y = x^2 + 4x$.
3. $y = x^2 + 3x - 4$.
4. $y = x^2 + 5x + 4$.
5. $y = x^2 - 7x + 6$.
6. $y = 3x^2 - 7x + 2$.
7. $y = 5x - x^2 - 4$.
8. $y = 4x - x^2 + 5$.
9. $y = x^2 + 5x - 6$.
10. $y = -x^2 + x$.
11. $y = 4x^2 - 3x - 1$.
12. $y = -4x^2 + 3x + 1$.

129. We now seek to find the points at which each of the above curves cuts the x-axis. The value of y for all points on the x-axis is zero. Hence we put $y = 0$, and try to solve the resulting equation.

Thus in Ex. 2 above, if $y = 0$, $x^2 + 2x - 3 = (x + 3)(x - 1) = 0$, which is satisfied by $x = 1$ and $x = -3$. Hence this curve cuts the x-axis in the two points $x = 1$, $y = 0$ and $x = -3$, $y = 0$, as shown in Figure 4.

Similarly in Ex. 1, if $y = 0$, $x^2 = 0$, and hence $x = 0$. Hence the curve meets the x-axis in the point $x = 0$, $y = 0$, as shown in Figure 3. On this point see § 131, Ex. 2.

EXERCISES

Find the points in which each of the twelve curves in the preceding list cuts the x-axis.

Notice that in every case the expression to the right of the equality sign can be factored, so that when $y = 0$ the resulting equation in x may be solved as in § 94.

Ex. 3. Plot the curve $y = x^2 + 4x + 2$ and find its intersection points with the x-axis.

We are not able to factor x^2+4x+2 by inspection. Hence we solve the equation $x^2 + 4x + 2 = 0$ by completing the square as in § 195, E. C., obtaining $x = -2 + \sqrt{2}$ and $x = -2 - \sqrt{2}$. Hence the curve cuts the x-axis in points whose abscissas are given by these values of x.

In making this graph, we first plot points corresponding to *integral* values of x, as before; then, in drawing the smooth curve through these, the intersections made with the x-axis are approximately the points on the number scale corresponding to the *incommensurable* numbers, $-2 + \sqrt{2}$ and $-2 - \sqrt{2}$. See § 109.

EXERCISES

In this manner, find the points at which each of the following curves cuts the x-axis, and plot the curves. For reduction of the results to simplest forms, see §§ 168, 169, E. C.

1. $y = x^2 + 5x + 3.$
2. $y = 3x^2 + 8x - 2.$
3. $y = 6x - 4x^2 + 5.$
4. $y = -4 - 2x + 5x^2.$
5. $y = 2x - 5x^2 + 8.$
6. $y = 5 + 8x - 3x^2.$
7. $y = 3 - 9x^2 - 11x.$
8. $y = -2 - 2x + x^2.$

130. Each of the foregoing exercises involves the solution of an equation of the general form $ax^2 + bx + c = 0$. Obviously, by solving this equation, we shall obtain a formula by means of which every equation of this type may be solved. See § 199, E. C.

The two values of x are:

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

EXERCISES

By means of this formula, find the solutions of each of the following equations:

1. $2x^2 - 3x - 4 = 0.$
2. $3x^2 + 2x - 1 = 0.$
3. $3x^2 - 2x - 1 = 0.$
4. $4x^2 + 6x + 1 = 0.$
5. $x^2 - 7x + 12 = 0.$
6. $5x^2 + 8x + 3 = 0.$
7. $5x^2 - 8x + 3 = 0.$
8. $5x^2 + 8x - 3 = 0.$
9. $5x^2 - 8x - 3 = 0.$
10. $2x - 3x^2 + 7 = 0.$
11. $3x - 9x^2 + 1 = 0.$
12. $7x^2 - 3x - 2 = 0.$
13. $6x^2 + 7x + 1 = 0.$
14. $4x^2 + 5x - 3 = 0.$
15. $4x^2 - 5x - 3 = 0.$
16. $8x^2 + 3x - 5 = 0.$
17. $7x^2 + x - 3 = 0.$
18. $7x^2 - x - 4 = 0.$
19. $x^2 - 2ax = 3b - a^2.$
20. $x^2 - 6ax = 49c^2 - 9a^2.$
21. $x^2 + \frac{a(a+b)}{3} = ax + \frac{(a+b)x}{3}.$
22. $-2x^2 - \frac{c-d}{2}x - 2c^2x = \frac{c^2(c-d)}{2}.$
23. $x^2 - \frac{mx}{2} + 2mn = 4nx.$
24. $x^2 - 2ax + 4ab = b^2 + 3a^2.$
25. $x^2 - abx + a^2b - ax = ab^2 - bx.$
26. $x^2 + 9 - c = 6x.$
27. $nx^2 + m^2n = mn^2x + mx.$
28. $2(a+1)x^2 - (a+1)^2x + 2(a+1) = 4x.$
29. $x^2 + 9cd + 3c = (3c + 3d + 1)x.$
30. $x^2 + 2a^2 + 3a - 2 = (3a+1)x.$

131. We now consider the intersections of other straight lines besides the x-axis with curves like those plotted above.

Ex. 1. Graph on the same axes the straight line, $y = -2$ and the curve, $y = x^2 + 2x - 3$.

This line is parallel to the x-axis and two units below it. It cuts the curve in the two points whose abscissas are $x_1 = -1 + \sqrt{2}$ and $x_2 = -1 - \sqrt{2}$, as found by substituting -2 for y in $y = x^2 + 2x - 3$ and solving the resulting equation in x by the formula, § 130.

Ex. 2. Graph on the same axes $y = -4$ and $y = x^2 + 2x - 3$.

This line *seems not to cut* the curve but to *touch* it at the point whose abscissa is $x = -1$.

Substituting and solving as before, we find,

$$x_1 = \frac{-2 + \sqrt{4-4}}{2} = \frac{-2+0}{2} = -1$$

and $$x_2 = \frac{-2 - \sqrt{4-4}}{2} = \frac{-2-0}{2} = -1.$$

In this case the two values of x are *equal*, and there is only *one* point common to the line and the curve. This is understood by thinking of the line $y = -2$, in the preceding example, as moved down to the position $y = -4$, whereupon the two values of x which were *distinct* now *coincide*.

132. From the formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, it is clear that the general equation, $ax^2 + bx + c = 0$ has two *distinct* solutions unless the expression $b^2 - 4ac$ reduces to zero, in which case the two values of x *coincide*, giving $x_1 = \dfrac{-b+0}{2a} = -\dfrac{b}{2a}$ and $x_2 = \dfrac{-b-0}{2a} = -\dfrac{b}{2a}$.

Ex. 1. In $2x^2 - 9x + 8 = 0$, determine without solving whether the two values of x are distinct or coincident.

In this case, $a = 2,\ b = -9,\ c = 8.$

Hence $b^2 - 4ac = 81 - 64 = 17.$

Hence the values of x are distinct.

Ex. 2. In $4x^2 - 12x + 9 = 0$, determine whether the values of x are distinct or coincident.

In this case, $b^2 - 4ac = 144 - 4 \cdot 4 \cdot 9 = 0$. Hence the values of x are *coincident.*

EXERCISES

In each of the following, determine without solving whether the two solutions are distinct or coincident:

1. $x^2-7x+4=0$.
2. $4x^2+28x+49=0$.
3. $9x^2+12x+4=0$.
4. $x^2+6x+9=0$.
5. $-x^2+9x+25=0$.
6. $6x^2-3x-1=0$.
7. $4x^2-16x+16=0$.
8. $8x^2-13=4x$.
9. $12x^2-18=24x$.
10. $16x^2-56x=-49$.

133. Definition. A line which cuts a curve in two *coincident points* is said to be **tangent to the curve.**

134. Problem. What is the value of a in $y=a$, if this line is tangent to the curve $y=x^2+5x+8$?

Substituting a for y and solving by means of the formula, we have

$$x=\frac{-5\pm\sqrt{25-4(8-a)}}{2}.$$

If the line is to be tangent to the curve, then the expression under the radical sign must be zero so that the two values of x may coincide. That is, $25-4(8-a)=0$, or $a=\frac{7}{4}$.

On plotting the curve, the line $y=\frac{7}{4}$ is found to be tangent to it.

EXERCISES

In the first 18 exercises on p. 368 obtain equations of curves by letting the left members equal y. Then find the equations of straight lines, $y=a$, which are tangent to these curves.

135. Problem. Find the intersection points of the curve $y=x^2+3x+5$ and the line $y=2\frac{1}{2}$.

Substituting for y and solving for x we have

$$x_1=\frac{-6+\sqrt{36-40}}{4}=\frac{-6+2\sqrt{-1}}{4}=\frac{-3+\sqrt{-1}}{2};$$

$$x_2=\frac{-6-\sqrt{36-40}}{4}=\frac{-6-2\sqrt{-1}}{4}=\frac{-3-\sqrt{-1}}{2}.$$

These results involve the **imaginary unit** already noticed in § 112. Numbers of the type $a + b\sqrt{-1}$ are discussed further in § 195. For the present we will regard such results as merely indicating that the conditions stated by the equations cannot be fulfilled by *real numbers.* This means that the curve and the line have *no point in common,* as is evident on constructing the graphs.

By proceeding as in § 134 we find that the line $y = \frac{11}{4}$ is *tangent* to the curve $y = x^2 + 3x + 5$. Clearly all lines $y = a$, in which $a > \frac{11}{4}$, are *above* this line and hence cut this curve in *two points.*

All such lines for which $a < \frac{11}{4}$ are *below* the line $y = \frac{11}{4}$ and hence do *not meet* the curve at all.

Solving $y = a$ and $y = x^2 + 3x + 5$ for x by first substituting a for y we have

$$x = \frac{-3 \pm \sqrt{4a - 11}}{2}.$$

If $a > \frac{11}{4}$ the number under the radical sign is *positive,* and there are *two real and distinct* values of x. Hence the line and the curve meet in two points.

If $a < \frac{11}{4}$, the number under the radical sign is *negative.* Consequently the values of x are *imaginary* and the line and the curve do not meet.

Hence we see that the conclusions obtained from the solution of the equations agree with those obtained from the graphs.

136. From the two preceding problems it appears that it is possible to determine the *relative* positions of the line and the curve *without completely solving* the equations. Namely, as soon as y is eliminated and the equation in x is reduced to the form $ax^2 + bx + c = 0$, we examine $b^2 - 4ac$ as follows:

(1) If $b^2 - 4ac > 0$, *i.e. positive,* then the line cuts the curve in two distinct points.

(2) If $b^2 - 4ac = 0$, then the line is tangent to the curve. See § 133.

(3) If $b^2 - 4ac < 0$, *i.e. negative,* then the line does not cut the curve.

137. Problem. Find the points of intersection of

$$y = x^2 + 3x + 13 \quad (1), \quad \text{and} \quad y + 3x = 7 \quad (2).$$

Eliminating y and reducing the resulting equation in x to the form $ax^2 + bx + c = 0$, we have $x^2 + 6x + 6 = 0$.

Solving, $x_1 = -3 + \sqrt{3}$, $x_2 = -3 - \sqrt{3}$.

Substituting these values of x in (2) and solving for y, we have

$$\begin{cases} x_1 = -3 + \sqrt{3} \\ y_1 = 16 - 3\sqrt{3} \end{cases} \text{and} \quad \begin{cases} x_2 = -3 - \sqrt{3} \\ y_2 = 16 + 3\sqrt{3} \end{cases}$$

which are the points in which the line meets the curve.

Here $b^2 - 4ac = 12$, which shows in advance that there are *two points* of intersection.

EXERCISES

In each of the following determine without graphing whether or not the line meets the curve, and in case it does, find the intersection points:

1. $\begin{cases} y = 2x^2 - 3x - 4, \\ y - x = 3. \end{cases}$

2. $\begin{cases} y = 2x^2 + 2x - 1, \\ 2y = x - 1. \end{cases}$

3. $\begin{cases} y = 3x^2 - 2x - 1, \\ 2x - y = 4. \end{cases}$

4. $\begin{cases} y = 4x^2 + 6x + 1, \\ x = y + 5. \end{cases}$

5. $\begin{cases} y = x^2 - 7x + 12, \\ 5x - y = -1. \end{cases}$

6. $\begin{cases} y = 5x^2 + 8x + 3, \\ 2y - 5x - 2 = 0. \end{cases}$

7. $\begin{cases} y = 5x^2 - 8x + 3, \\ 3 - x = 3y. \end{cases}$

8. $\begin{cases} y = -5x^2 + 8x - 3, \\ 2 - 4y - x = 0. \end{cases}$

9. $\begin{cases} y = -5x^2 - 8x - 3, \\ 5y - 3x = 8. \end{cases}$

10. $\begin{cases} y = 3x - 3x^2 + 7, \\ -5 - 3x + 2y = 0. \end{cases}$

138. Problem. Graph the equation $x^2 + y^2 = 25$.

Writing the equation in the form $y = \pm\sqrt{25 - x^2}$, and assigning values to x, we compute the corresponding values of y as follows:

$$\begin{cases} x = 0, \\ y = \pm 5. \end{cases} \begin{cases} x = \pm 5, \\ y = 0. \end{cases} \begin{cases} x = 3, \\ y = \pm 4. \end{cases} \begin{cases} x = -3, \\ y = \pm 4. \end{cases} \begin{cases} x = 4, \\ y = \pm 3. \end{cases} \begin{cases} x = -4, \\ y = \pm 3. \end{cases}$$

Evidently, for x greater than 5 in absolute value, the corresponding y's are *imaginary*, and for each x between -5 and $+5$ there are two y's equal in absolute value, but with opposite signs.

It seems apparent that these points lie on the circumference of a circle whose radius is 5, as shown in Figure 5. Indeed, if we consider any point x_1, y_1 on this circumference, it is evident that $x_1^2 + y_1^2 = 25$, since the sum of the squares on the sides of a right triangle is equal to the square on the hypotenuse. (See figure, p. 173, E. C.)

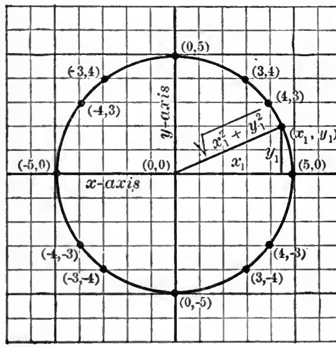


FIG. 5.

The equation $x^2 + y^2 = 25$ is, therefore, the equation of a circle with radius 5. Similarly, $x^2 + y^2 = r^2$ is the equation of a circle with center at the point (0, 0) and radius r.

139. Problem. Find the points of intersection of the circle $x^2 + y^2 = 25$ and the line $x + y = 7$.

Eliminating y from these equations, and reducing the equation in x to the form $ax^2 + bx + c = 0$, we have

$$x^2 - 7x + 12 = 0.$$

From which $$x_1 = 4, \quad x_2 = 3.$$

Substituting these values of x in $x + y = 7$, we have $y_1 = 3$, $y_2 = 4$. Hence $x_1 = 4$, $y_1 = 3$ and $x_2 = 3$, $y_2 = 4$ are the required points.

Verify this by graphing the two equations on the same axes.

140. Problem. Find the points of intersection of the circle $x^2 + y^2 = 25$ and the line $3x + 4y = 25$.

Eliminating y and solving for x, we find $x = \dfrac{6 \pm 0}{2} = 3$.

Hence $x_1 = x_2 = 3$, from which $y_1 = y_2 = 4$.

Since the two values of x coincide, and likewise the two values of y, the circumference and the line have but *one point* in common. Verify by graphing the line and the circle on the same axes.

141. Problem. Find the points of intersection of

$$x^2+y^2=25$$
$$\text{and } x+y=10.$$

Substituting for y and solving for x we have

$$x=\frac{20\pm\sqrt{400-600}}{4}=\frac{20\pm\sqrt{-200}}{4}$$
$$=\frac{20\pm10\sqrt{-2}}{4}=\frac{10\pm5\sqrt{-2}}{2}.$$

The imaginary values of x indicate that there is no intersection point. Verify by plotting.

EXERCISES

In each of the following determine by solving whether the line and the circumference meet, and in case they do, find the points of intersection. Verify each by constructing the graph.

1. $\begin{cases} x^2+y^2=16, \\ x+y=4. \end{cases}$

2. $\begin{cases} x^2+y^2=36, \\ 4x+y=6. \end{cases}$

3. $\begin{cases} x^2+y^2=25, \\ 2x+y=-5. \end{cases}$

4. $\begin{cases} x^2+y^2=20, \\ 2x+y=0. \end{cases}$

5. $\begin{cases} x^2+y^2=7, \\ x+y=8. \end{cases}$

6. $\begin{cases} x^2+y^2=8, \\ x-y=4. \end{cases}$

7. $\begin{cases} x^2+y^2=41, \\ x-3y=7. \end{cases}$

8. $\begin{cases} x^2+y^2=29, \\ 3x-7y=-29. \end{cases}$

9. $\begin{cases} x^2+y^2=12, \\ x-y=6. \end{cases}$

10. $\begin{cases} x^2+y^2=4, \\ 2x-3y=4. \end{cases}$

11. $\begin{cases} x^2+y^2=40, \\ x+2y=10. \end{cases}$

12. $\begin{cases} x^2+y^2=25, \\ x+y=9. \end{cases}$

142. Problem. Graph on the same axes the circle, $x^2+y^2=5^2$, and the lines, $3x+4y=20$, $3x+4y=25$, and $3x+4y=30$.

The first line *cuts* the circumference in two distinct points, the second seems to be *tangent* to it, and the third does *not meet* it. Observe that the three lines are parallel. See Figure 6.

In order to discuss the relative positions of such straight lines and the circumference of a circle, we solve the following equations simultaneously:

$$x^2 + y^2 = r^2 \qquad (1)$$

$$3x + 4y = c \qquad (2)$$

Eliminating y by substitution, and solving for x, we find

$$x = \frac{3c \pm 4\sqrt{25r^2 - c^2}}{25}. \qquad (3)$$

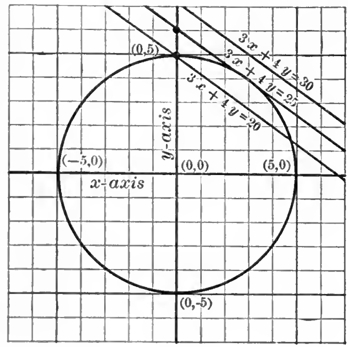


FIG. 6.

The two values of x from (3) are the abscissas of the points of intersection of the circumference (1) and the line (2).

These values of x are *real and distinct* if $25r^2 - c^2$ is *positive*, *real and coincident* if $25r^2 - c^2 = 0$, and *imaginary* if $25r^2 - c^2$ is *negative*.

Now $25r^2 - c^2$ is *positive* if $r = 5$, $c = 20$; *zero* if $r = 5$, $c = 25$; and *negative* if $r = 5$, $c = 30$.

Hence these results obtained from the solution of the equations agree with the facts observed in the graphs above.

143. Definition. Letters such as c and r in the above solution to which any arbitrary constant values may be assigned are called **parameters,** while x and y are the **unknowns** of the equations.

EXERCISES

Solve each of the following pairs of equations.

Give such values to the parameters involved that the line (a) may cut the curve in two distinct points, (b) may be tangent to the curve, (c) shall fail to meet the curve.

1. $\begin{cases} x^2 + y^2 = 4, \\ ax + 3y = 16. \end{cases}$

2. $\begin{cases} x^2 + y^2 = 16, \\ 2x + by = 12. \end{cases}$

3. $\begin{cases} x^2 + y^2 = 25, \\ 2x + 3y = c. \end{cases}$

4. $\begin{cases} y^2 = 8x, \\ 3x + 4y = c. \end{cases}$

5. $\begin{cases} 5y^2 = 2px, \\ x + y = 1. \end{cases}$

6. $\begin{cases} y = x^2 + mx + 4, \\ x + y = 4. \end{cases}$

7. $\begin{cases} y = mx^2 - x - 4, \\ x - 3y = 8. \end{cases}$

8. $\begin{cases} y = 2x^2 - 3x + 1, \\ 2x - by - 1 = 0. \end{cases}$

9. $\begin{cases} y = 3x^2 + mx, \\ x + y + 3 = 0. \end{cases}$

10. $\begin{cases} y = mx^2 + 2x, \\ 2y - x - 5 = 0. \end{cases}$

11. $\begin{cases} y = x^2 + 1, \\ ax + 2y = 10. \end{cases}$

12. $\begin{cases} x^2 + y^2 = 1, \\ ax + by = a. \end{cases}$

144. Problem. Graph the equation $\frac{x^2}{25} + \frac{y^2}{16} = 1.$

Writing the equation in the form $y = \pm \frac{4}{5}\sqrt{25 - x^2}$, and assigning values to x, we compute the corresponding values of y as follows:

$$\begin{cases} x = 0, \\ y = \pm 4, \end{cases} \quad \begin{cases} x = \pm 5, \\ y = 0, \end{cases} \quad \begin{cases} x = 1, \\ y = \pm 3.9, \end{cases} \quad \begin{cases} x = -1, \\ y = \pm 3.9. \end{cases}$$

$$\begin{cases} x = 2, \\ y = \pm 3.7, \end{cases} \quad \begin{cases} x = 3, \\ y = \pm 3.2, \end{cases} \quad \begin{cases} x = 4, \\ y = \pm 2.4. \end{cases}$$

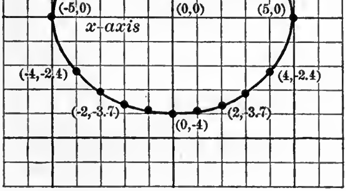


FIG. 7.

Evidently if x is greater than 5 in absolute value, the corresponding values of y are imaginary.

Plotting these points, they are found to lie on the curve shown in Figure 7. This curve is called an **ellipse**.

EXERCISES

Solve the following pairs of equations.

In this way determine whether the straight line and the curve intersect, and in case they do, determine the coördinates of the intersection points. Verify each by constructing the graphs.

1. $\begin{cases} \frac{x^2}{16} + \frac{y^2}{9} = 1, \\ 3x + 4y = 12. \end{cases}$

2. $\begin{cases} \frac{x^2}{49} + \frac{y^2}{16} = 1, \\ 2x - 7y = 8. \end{cases}$

3. $\begin{cases} x^2 + 4y^2 = 25, \\ 2x - y = 4. \end{cases}$

4. $\begin{cases} 3x^2 + 2y^2 = 11, \\ x - 3y = 7. \end{cases}$

5. $\begin{cases} \dfrac{x^2}{25} + \dfrac{y^2}{9} = 1, \\ x - y = 5. \end{cases}$

6. $\begin{cases} y = 2x^2 - 3x + 4, \\ y - 4x - 8 = 0. \end{cases}$

7. $\begin{cases} x^2 + y^2 = 16, \\ x + y = 7. \end{cases}$

8. $\begin{cases} \dfrac{x^2}{64} + \dfrac{y^2}{12} = 1, \\ 4y - 2x = 4. \end{cases}$

9. $\begin{cases} \dfrac{x^2}{36} + \dfrac{y^2}{45} = 1, \\ -5x + 6y = 10. \end{cases}$

10. $\begin{cases} \dfrac{x^2}{49} + \dfrac{y^2}{25} = 1, \\ x + y = 12. \end{cases}$

When arbitrary constants are introduced in the equations of a straight line and an ellipse, we may determine values for these constants so as to make the line cut the ellipse, touch it, or not cut it, as in the case of the circle, § 142.

EXERCISES

Solve each of the following pairs simultaneously.

Give such values to the constants that the line shall (*a*) cut the curve in two distinct points, (*b*) be a tangent to the curve, (*c*) have no point in common with the curve.

In case (*b*) is found very difficult, this may be omitted.

1. $\begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{16} = 1, \\ 4x + y = 2. \end{cases}$

2. $\begin{cases} \dfrac{x^2}{25} + \dfrac{y^2}{b^2} = 1, \\ x + 5y = 5. \end{cases}$

3. $\begin{cases} \dfrac{x^2}{25} + \dfrac{y^2}{16} = 1. \\ 4x - 5y = c. \end{cases}$

4. $\begin{cases} \dfrac{x^2}{16} + \dfrac{y^2}{25} = 1, \\ 5x - by = 20. \end{cases}$

5. $\begin{cases} \dfrac{x^2}{16} + \dfrac{y^2}{25} = 1, \\ ax + 4y = 20. \end{cases}$

6. $\begin{cases} \dfrac{x^2}{36} + \dfrac{y^2}{16} = 1, \\ ax + 6y - 24 = 0. \end{cases}$

7. $\begin{cases} \dfrac{x^2}{36} + \dfrac{y^2}{25} = 1, \\ 5x + by = 30. \end{cases}$

8. $\begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{4} = 1, \\ 2x - 2y = 5. \end{cases}$

9. $\begin{cases} x^2 + y^2 = r^2, \\ x - 3y = 4. \end{cases}$

10. $\begin{cases} 5x^2 + 3y^2 = 16. \\ hx - 3y = 8. \end{cases}$

11. $\begin{cases} x^2 + 7y^2 = 144, \\ x + by = 12. \end{cases}$

12. $\begin{cases} x^2 + 4y^2 = 1, \\ ax + by = a. \end{cases}$

SPECIAL METHODS OF SOLUTION

145. We have thus far solved simultaneously one equation of the second degree with one of the first degree. After substitution such cases reduce to the solution of an ordinary quadratic, namely, of the form, $ax^2+bx+c=0$.

While this is an effective general method, yet some important special forms of solution are shown in the following examples:

Ex. 1. Solve $\begin{cases} x^2+y^2=a, & (1) \\ x-y=b. & (2) \end{cases}$

Square both members of (2) and subtract from (1).

$$2xy=a-b^2. \tag{3}$$

Add (1) and (3). $x^2+2xy+y^2=2a-b^2.$ (4)

Hence $x+y=\pm\sqrt{2a-b^2}.$ (5)

From (2) and (5), adding and subtracting

$$\begin{cases} x_1=\dfrac{\sqrt{2a-b^2}+b}{2}, \\ y_1=\dfrac{\sqrt{2a-b^2}-b}{2}, \end{cases} \text{ and } \begin{cases} x_2=\dfrac{-\sqrt{2a-b^2}+b}{2}, \\ y^2=\dfrac{-\sqrt{2a-b^2}-b}{2}. \end{cases}$$

Ex. 2. Solve $\begin{cases} x^2-y^2=a, & (1) \\ x-y=b. & (2) \end{cases}$

From (1) $(x-y)(x+y)=a.$ (3)

Substituting b for $x-y$ in (3), $x+y=\dfrac{a}{b}.$ (4)

Then (2) and (4) may be solved as above.

Ex. 3. Solve $\begin{cases} x+y=a, & (1) \\ xy=b. & (2) \end{cases}$

Multiply (2) by 4, subtract from the square of (1), and get

$$x^2-2xy+y^2=a^2-4b, \tag{3}$$

whence, $x-y=\pm\sqrt{a^2-4b}.$ (4)

Then (1) and (4) may be solved as in Ex. 1.

The equations $\begin{cases} x - y = a, \\ xy = b, \end{cases}$

may be solved in a similar manner.

146. We are now to study the solution of a pair of equations each of the second degree. See § 66.

Consider
$$x^2 + y = a, \tag{1}$$
$$x + y^2 = b. \tag{2}$$

Solving (1) for y and substituting in (2) we have,

$$x + a^2 - 2\,ax^2 + x^4 = b,$$

which is of the fourth degree and cannot be solved by any methods thus far studied. There are, however, special cases in which two equations each of the second degree can be solved by a proper combination of methods already known.

147. Case I. *When only the squares of the unknowns enter the equations.*

Example. Solve $\begin{cases} a_1x^2 + b_1y^2 = c_1, \\ a_2x^2 + b_2y^2 = c_2. \end{cases}$

These equations are *linear* if x^2 and y^2 are regarded as the unknowns.

Solving for x^2 and y^2 as in § 73, we obtain,

$$x^2 = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}, \quad y^2 = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}.$$

Hence, taking square roots,

$$\begin{cases} x_1 = \sqrt{\dfrac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}}, \\ y_1 = \sqrt{\dfrac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}}, \end{cases} \qquad \begin{cases} x_3 = \sqrt{\dfrac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}}, \\ y_3 = -\sqrt{\dfrac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}}, \end{cases}$$

$$\begin{cases} x_2 = -\sqrt{\dfrac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}}, \\ y_2 = \sqrt{\dfrac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}}, \end{cases} \qquad \begin{cases} x_4 = -\sqrt{\dfrac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}}, \\ y_4 = -\sqrt{\dfrac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}}. \end{cases}$$

In this case there are four pairs of numbers which satisfy the two equations. This is in general true of two equations each of the second degree.

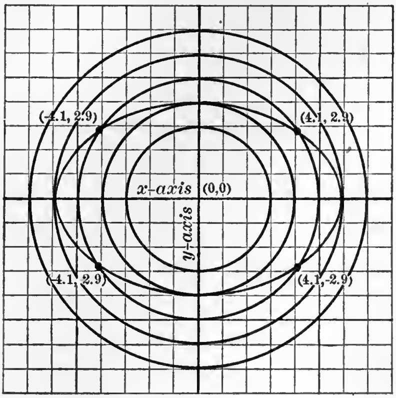


FIG. 8.

Example. Solve simultaneously, obtaining results to one decimal place:

$$\begin{cases} \dfrac{x^2}{36} + \dfrac{y^2}{16} = 1, & (1) \\ x^2 + y^2 = 25. & (2) \end{cases}$$

Clear (1) of fractions and proceed as above. Verify the solution by reference to the graph given in Figure 8.

EXERCISES

Solve simultaneously each of the following pairs of equations and interpret all the solutions in each case from the graph in Figure 8:

1. $\begin{cases} \dfrac{x^2}{36} + \dfrac{y^2}{16} = 1, \\ x^2 + y^2 = 36. \end{cases}$

2. $\begin{cases} \dfrac{x^2}{36} + \dfrac{y^2}{16} = 1, \\ x^2 + y^2 = 16. \end{cases}$

3. $\begin{cases} \dfrac{x^2}{36} + \dfrac{y^2}{16} = 1, \\ x^2 + y^2 = 49. \end{cases}$

4. $\begin{cases} \dfrac{x^2}{36} + \dfrac{y^2}{16} = 1, \\ x^2 + y^2 = 31. \end{cases}$

148. Problem. Graph the equation $\dfrac{x^2}{25} - \dfrac{y^2}{16} = 1.$

Writing the equation in the form $y = \pm \frac{4}{5}\sqrt{x^2 - 25}$, and assigning values to x, we compute the corresponding values of y exactly or approximately as follows:

$\begin{cases} x = \pm 5, \\ y = 0, \end{cases}$ $\begin{cases} x = 6\frac{1}{4}, \\ y = \pm 3, \end{cases}$ $\begin{cases} x = -6\frac{1}{4}, \\ y = \pm 3, \end{cases}$ $\begin{cases} x = 7, \\ y = \pm 3.9, \end{cases}$ $\begin{cases} x = -7, \\ y = \pm 3.9, \end{cases}$ $\begin{cases} x = 8, \\ y = \pm 5, \end{cases}$ $\begin{cases} x = -8, \\ y = \pm 5. \end{cases}$

Evidently when x is less than 5 in absolute value, y is imaginary, and as x increases beyond 8 in absolute value, y continually increases.

Plotting these points, they are found to lie on the curve as shown in Figure 9. This curve is called a **hyperbola**.

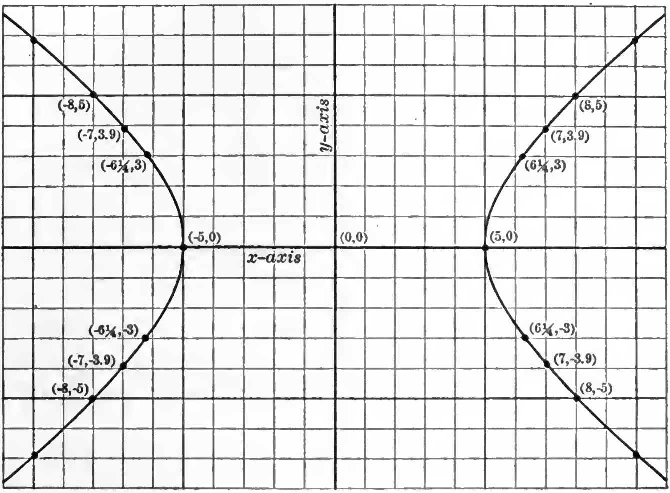


FIG. 9.

EXERCISES

Solve each of the following pairs of equations.

Construct a graph similar to the one in Figure 8 which shall contain the hyperbola $\frac{x^2}{25}-\frac{y^2}{16}=1$ and the circles given in Exs. 1, 2, and 3. Construct another graph containing the same hyperbola and the ellipses given in Exs. 4, 5, and 6. From these graphs interpret the solutions of each pair of equations.

1. $\begin{cases}\frac{x^2}{25}-\frac{y^2}{16}=1,\\ x^2+y^2=16.\end{cases}$ **2.** $\begin{cases}\frac{x^2}{25}-\frac{y^2}{16}=1,\\ x^2+y^2=25.\end{cases}$ **3.** $\begin{cases}\frac{x^2}{25}-\frac{y^2}{16}=1,\\ x^2+y^2=36.\end{cases}$

4. $\begin{cases} \dfrac{x^2}{25} - \dfrac{y^2}{16} = 1, \\ \dfrac{x^2}{36} + \dfrac{y^2}{16} = 1. \end{cases}$ 5. $\begin{cases} \dfrac{x^2}{25} - \dfrac{y^2}{16} = 1, \\ \dfrac{x^2}{25} + \dfrac{y^2}{16} = 1. \end{cases}$ 6. $\begin{cases} \dfrac{x^2}{25} - \dfrac{y^2}{16} = 1, \\ \dfrac{x^2}{16} + \dfrac{y^2}{9} = 1. \end{cases}$

7. Graph the equation $xy = 9$.

Graph $xy = 8$ on the same axes with each of the following:

8. $x^2 + y^2 = 16$. 9. $x^2 + y^2 = 25$. 10. $x^2 + y^2 = 4$.

11. $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$. 12. $\dfrac{x^2}{25} + \dfrac{y^2}{10.24} = 1$. 13. $\dfrac{x^2}{16} + \dfrac{y^2}{4} = 1$.

14. From those graphs in Exs. 8 to 13, in which the curves meet, determine as accurately as possible by measurement the coördinates of the points of intersection or tangency.

15. Solve simultaneously the pairs of equations given in Exs. 8 to 13, after studying the method explained in Ex. 1, § 150. Compare the results with those obtained from the graphs.

NOTE. In Ex. 11 clear of fractions and then apply § 150. This gives $4x + 5y = \pm 12\sqrt{5}$ and $4x - 5y = \pm 4\sqrt{5}$. Solve these simultaneously. Examples 12 and 13 may be solved in a similar manner.

149. Case II. *When all terms containing the unknowns are of the second degree in the unknowns.*

Example. Solve $\begin{cases} 2x^2 - 3xy + 4y^2 = 3, & (1) \\ 3x^2 - 4xy + 3y^2 = 2. & (2) \end{cases}$

Put $y = vx$ in (1) and (2), obtaining

$$\begin{cases} x^2(2 - 3v + 4v^2) = 3, & (3) \\ x^2(3 - 4v + 3v^2) = 2. & (4) \end{cases}$$

Hence from (3) and (4),

$$x^2 = \frac{3}{2 - 3v + 4v^2}, \text{ and also } x^2 = \frac{2}{3 - 4v + 3v^2}. \quad (5)$$

From (5) $$\frac{3}{2-3v+4v^2}=\frac{2}{3-4v+3v^2}, \qquad (6)$$

or $$v^2-6v+5=0. \qquad (7)$$

Hence $$v=1, \text{ and } v=5. \qquad (8)$$

From $y=vx$, $$y=x, \text{ and } y=5x. \qquad (9)$$

If $y=x$, then from (1) and (2),

$$\begin{cases} x=1, \\ y=1, \end{cases} \text{ and } \begin{cases} x=-1, \\ y=-1. \end{cases}$$

If $y=5x$, then from (1) and (2),

$$\begin{cases} x=\dfrac{1}{\sqrt{29}}, \\ y=\dfrac{5}{\sqrt{29}}, \end{cases} \text{ and } \begin{cases} x=-\dfrac{1}{\sqrt{29}}, \\ y=-\dfrac{5}{\sqrt{29}}. \end{cases}$$

Verify each of these four solutions by substituting in equations (1) and (2).

150. There are many other special forms of simultaneous equations which can be solved by proper combination of the methods thus far used. Also, many pairs of equations of a degree higher than the second in the two unknowns may be solved by means of quadratic equations.

The suggestions given in the following examples illustrate the devices in most common use.

The solution should in each case be completed by the student.

Ex. 1. Solve $$\begin{cases} x^2+y^2=58, & (1) \\ xy=21. & (2) \end{cases}$$

Adding twice (2) to (1) and taking square roots, we have

$$x+y=10, \text{ and } x+y=-10. \qquad (3)$$

Each of the equations (3) may now be solved simultaneously with (2), as in Ex. 3, p. 396.

Ex. 2. Solve
$$\begin{cases} \frac{1}{x}+\frac{1}{y}=5, & (1) \\ \frac{1}{x^2}+\frac{1}{y^2}=13. & (2) \end{cases}$$

Let $\frac{1}{x}=a$ and $\frac{1}{y}=b$. Then these equations reduce to
$$\begin{cases} a+b=5, & (3) \\ a^2+b^2=13. & (4) \end{cases}$$

(3) and (4) may then be solved as in Ex. 1, p. 378.

Ex. 3. Solve
$$\begin{cases} x^2+y^2+x+y=8, & (1) \\ xy=2. & (2) \end{cases}$$

Add twice (2) to (1), obtaining
$$x^2+2\,xy+y^2+x+y=12. \quad (3)$$

Let $x+y=a$. Then (3) reduces to
$$a^2+a=12,$$
or,
$$a=3,\ a=-4. \quad (4)$$
Hence
$$x+y=3, \text{ and } x+y=-4. \quad (5)$$
Now solve each equation in (5) simultaneously with (2).

Ex. 4. Solve
$$\begin{cases} x^4y^4+x^2y^2=272, & (1) \\ x^2+y^2=10. & (2) \end{cases}$$

In (1) substitute a for x^2y^2. Then
$$a^2+a=272, \text{ whence } a=16, \text{ and } -17.$$
Hence
$$xy=\pm\sqrt{16}=\pm 4, \text{ and } \pm\sqrt{-17}.$$

Each of these equations may now be solved simultaneously with (2), as in Ex. 1, p. 383.

Ex. 5. Solve
$$\begin{cases} x^3-y^3=117, & (1) \\ x-y=3. & (2) \end{cases}$$

By factoring, (1) becomes

$$(x-y)(x^2+xy+y^2)=117. \qquad (3)$$

Substituting 3 for $x-y$, we have

$$x^2+xy+y^2=39. \qquad (4)$$

(2) and (4) may now be solved by substitution as in §§ 140–144.

Ex. 6. Solve $\begin{cases} x^3+y^3=513, & (1) \\ x+y=9. & (2) \end{cases}$

Factor (1) and substitute 9 for $x+y$. Then proceed as in Ex. 5.

Ex. 7. Solve $\begin{cases} x^2y+xy^2=126, & (1) \\ x+y=9. & (2) \end{cases}$

Factoring (1) and substituting 9 for $x+y$, we have

$$xy=14. \qquad (3)$$

(2) and (3) may then be solved as in Ex. 3, p. 378.

Ex. 8. Solve $\begin{cases} x^3+y^3=54\,xy, & (1) \\ x+y=6. & (2) \end{cases}$

Factor (1) and substitute 6 for $x+y$, obtaining

$$x^2-xy+y^2=9\,xy. \qquad (3)$$

(2) and (3) may now be solved by substitution, as in §§ 140–144.

Ex. 9. Solve $\begin{cases} x^3-y^3=63, & (1) \\ x^2+xy+y^2=21. & (2) \end{cases}$

Factor (1) and substitute 21 for x^2+xy+y^2, then proceed as in Ex. 8.

Ex. 10. Solve $\begin{cases} x^3+y^3=243, & (1) \\ x^2y+xy^2=162. & (2) \end{cases}$

Multiply (2) by 3 and add to (1), obtaining a perfect cube. Taking cube roots, we have

$$x+y=9. \qquad (3)$$

(1) and (3) are now solved as in the preceding example.

Ex. 11. Solve $\begin{cases} x^4 + y^4 = 641, & (1) \\ x + y = 7. & (2) \end{cases}$

Raise (2) to the fourth power and subtract (1), obtaining

$$4x^3y + 6x^2y^2 + 4xy^3 = 1760. \quad (3)$$

Factoring, $2xy(2x^2 + 3xy + 2y^2) = 1760.$ (4)

Squaring (2) we have

$$2x^2 + 4xy + 2y^2 = 98, \quad (5)$$

or

$$2x^2 + 3xy + 2y^2 = 98 - xy. \quad (6)$$

Substituting (6) in (4), we have

$$2xy(98 - xy) = 1760, \quad (7)$$

or

$$x^2y^2 - 98xy + 880 = 0. \quad (8)$$

In (8) put $xy = a$, obtaining

$$a^2 - 98a + 880 = 0. \quad (9)$$

The solution of (9) gives two values for xy, each of which may now be combined with (2) as in Ex. 3, p. 378.

EXERCISES

Solve each of the following pairs of equations:

1. $\begin{cases} r^2 + rs + s^2 = 63, \\ r - s = 3. \end{cases}$

2. $\begin{cases} 3x^2 + 2y^2 = 35, \\ 2x^2 - 3y^2 = 6. \end{cases}$

3. $\begin{cases} 3x^2 + 2xy = 16, \\ 4x^2 - 3xy = 10. \end{cases}$

4. $\begin{cases} a^2 + ab + b^2 = 7, \\ a^2 - ab + b^2 = 19. \end{cases}$

5. $\begin{cases} x^2 + y^2 = 5, \\ xy = 2. \end{cases}$

6. $\begin{cases} x^2 + y^2 = 5, \\ x^2 + z^2 = 10, \\ y^2 + z^2 = 13. \end{cases}$

7. $\begin{cases} 3x - 4y = 0, \\ x^2 + y^2 = 4. \end{cases}$

8. $\begin{cases} x^2 + xy = 4, \\ y^2 + xy = 5. \end{cases}$

9. $\begin{cases} x^3 + y^3 = 91, \\ x + y = 7. \end{cases}$

10. $\begin{cases} x^2 + y^2 = 11, \\ x^2 - y^2 = 7. \end{cases}$

11. $\begin{cases} x^2 - 3xy = 0, \\ 5x^2 + 3y^2 = 9. \end{cases}$

12. $\begin{cases} \dfrac{1}{x^3} + \dfrac{1}{y^3} = 19, \\ \dfrac{1}{x} + \dfrac{1}{y} = 1. \end{cases}$

13. $\begin{cases} 3x - 2y = 6, \\ 3x^2 - 2xy + 4y^2 = 12. \end{cases}$

14. $\begin{cases} a + b + ab = 11, \\ (a + b)^2 + a^2b^2 = 61. \end{cases}$

15. $\begin{cases} \dfrac{1}{a^2} + \dfrac{1}{ab} + \dfrac{1}{b^2} = 49, \\ \dfrac{1}{a} + \dfrac{1}{b} = 8. \end{cases}$

16. $\begin{cases} 4a^2 - 2ab = b^2 - 16, \\ 5a^2 = 7ab - 36. \end{cases}$

17. $\begin{cases} 3x^2 - 9y^2 = 12, \\ 2x - 3y = 14. \end{cases}$

18. $\begin{cases} x^2 + xy + y^2 = 9, \\ x^2 + y^2 = 3. \end{cases}$

19. $\begin{cases} x^2 + y^2 + x + y = 18, \\ xy = 6. \end{cases}$

20. $\begin{cases} x^2 + y^2 + x - y = 36, \\ xy = 15. \end{cases}$

21. $\begin{cases} x^2 - 5xy + y^2 = -2, \\ x^2 + 7xy + y^2 = 22. \end{cases}$

22. $\begin{cases} a^2 + 6ab + b^2 = 124, \\ a + b = 8. \end{cases}$

23. $\begin{cases} a^2 - 3ab + 2b^2 = 0, \\ 2a^2 + ab - b^2 = 9. \end{cases}$

24. $\begin{cases} x^2 + y^2 + 2x + 2y = 27, \\ xy = -12. \end{cases}$

25. $\begin{cases} x^2 + y^2 - 5x - 5y = -4, \\ xy = 5. \end{cases}$

26. $\begin{cases} (7 + x)(6 + y) = 80, \\ x + y = 5. \end{cases}$

27. $\begin{cases} (x - 4)^2 + (y + 4)^2 = 100, \\ x + y = 14. \end{cases}$

28. $\begin{cases} xy + y + x = 17, \\ x^2y^2 + y^2 + x^2 = 129. \end{cases}$

29. $\begin{cases} b + a^2 = 5(a - b), \\ a + b^2 = 2(a - b). \end{cases}$

30. $\begin{cases} (13x)^2 + 2y^2 = 177, \\ (2y)^2 - 13x^2 = 3. \end{cases}$

31. $\begin{cases} \left(\dfrac{9}{x}\right)^2 = \left(\dfrac{25}{y}\right)^2 - 16, \\ \dfrac{9}{x^2} = \dfrac{25}{y^2}. \end{cases}$

32. $\begin{cases} x^2 + y^2 = 20, \\ 5x^2 - 3y^2 = 28. \end{cases}$

33. $\begin{cases} x^2 = -5 - 3xy, \\ 2xy = y^2 - 24. \end{cases}$

34. $\begin{cases} x + y + \sqrt{x + y} = 12, \\ x^3 + y^3 = 189. \end{cases}$

35. $\begin{cases} x^4 + x^2y^2 + y^4 = 133, \\ x^2 - xy + y^2 = 7. \end{cases}$

36. $\begin{cases} x + xy + y = 29, \\ x^2 + xy + y^2 = 61. \end{cases}$

37. $\begin{cases} 2x^2 - 5xy + 3x - 2y = 22, \\ 5xy + 7x - 8y - 2x^2 = 8. \end{cases}$

38. $\begin{cases} x + y = 74, \\ x^2 + y^2 = 3026. \end{cases}$

39. $\begin{cases} 7y^2 - 5x^2 + 20x + 13y = 29, \\ 5(x - 2)^2 - 7y^2 - 17y = -17. \end{cases}$

40. $\begin{cases}(3x+4y)(7x-2y)+3x+4y=44,\\(3x+4y)(7x-2y)-7x+2y=30.\end{cases}$

41. $\begin{cases}x+y=4,\\x^3+x^2y+xy^2+y^3=32.\end{cases}$

42. $\begin{cases}x^3-y^3=37,\\x-y=1.\end{cases}$

43. $\begin{cases}x^4+y^4=82,\\x+y=4.\end{cases}$

44. $\begin{cases}x^2+y^2-xy=80,\\x-y-xy=-8.\end{cases}$

45. $\begin{cases}8a+8b-ab-a^2=18,\\5a+5b-b^2-ab=24.\end{cases}$

46. $\begin{cases}x^3+x^2y+xy^2+y^3=120,\\x^3-x^2y+xy^2-y^3=40.\end{cases}$

47. $\begin{cases}2(x+4)^2-5(y-7)^2=75,\\7(x+4)^2+15(y-7)^2=1075.\end{cases}$

48. $\begin{cases}x^3+y^3=(a+b)(x-y),\\x^2-xy+y^2=a-b.\end{cases}$

HIGHER EQUATIONS INVOLVING QUADRATICS

151. An equation of a degree above the second may often be reduced to the solution of a quadratic after applying the factor theorem. See § 92.

Example. Solve $2x^3+x^2-10x+7=0.$ (1)

By the factor theorem, $x-1$ is found to be a factor,

giving $(x-1)(2x^2+3x-7)=0.$ (2)

Hence by § 22, $x-1=0$ and $2x^2+3x-7=0.$ (3)

From $x-1=0,$ $x=1.$ (4)

From $2x^2+3x-7=0,$ $x=\dfrac{-3\pm\sqrt{65}}{4}.$ (5)

Hence (4) and (5) give the three roots of (1).

EXERCISES

Solve each of the following equations:

1. $7x^3 - 11x^2 + 4x = 0.$
2. $3x^4 + x^3 + 2x^2 + 24x = 0.$
3. $3x^3 - 16x^2 + 23x - 6 = 0.$
4. $5x^3 + 2x^2 + 4x = -7.$
5. $28x^3 - 10x^2 - 44x = 6.$
6. $x^4 - 3x^3 + 3x^2 - x = 0.$
7. $4x^3 + 12x^2 - 3x - 9 = 0.$
8. $x^4 - 5x^3 + 2x^2 + 20x = 24.$
9. $6x^3 + 29x^2 - 19x = 16.$
10. $15x^4 + 49x^3 - 92x^2 + 28x = 0.$

EQUATIONS IN THE FORM OF QUADRATICS

152. If an equation of higher degree contains a certain expression and also the square of this expression, and involves the unknown in no other way, then the equation is a **quadratic in the given expression.**

Ex. 1. Solve $$x^4 + 7x^2 = 44. \quad (1)$$

This may be written, $(x^2)^2 + 7(x^2) = 44,$ (2)

which is a *quadratic in* x^2. Solving, we find

$$x^2 = 4 \text{ and } x^2 = -11. \quad (3)$$

Hence, $$x = \pm 2 \text{ and } x = \pm\sqrt{-11}. \quad (4)$$

Ex. 2. Solve $$x + 2 + 3\sqrt{x+2} = 18. \quad (1)$$

Since $x + 2$ is the square of $\sqrt{x+2}$, this is a quadratic in $\sqrt{x+2}$.

Solving we find $$\sqrt{x+2} = 3 \text{ and } \sqrt{x+2} = -6. \quad (2)$$

Hence $$x+2 = 9 \text{ and } x+2 = 36, \quad (3)$$

Whence $$x = 7 \text{ and } x = 34. \quad (4)$$

Ex. 3. Solve $(2x^2 - 1)^2 - 5(2x^2 - 1) - 14 = 0.$

First solve as a quadratic in $2x^2 - 1$ and then solve the two resulting quadratics in x.

Ex. 4. Solve $x^2-7x+40-2\sqrt{x^2-7x+69}=-26.$ (1)

Add 29 to each member, obtaining

$$x^2-7x+69-2\sqrt{x^2-7x+69}=3. \quad (2)$$

Solve (2) as a quadratic in $\sqrt{x^2-7x+69}$, obtaining

$$\sqrt{x^2-7x+69}=3 \text{ and } \sqrt{x^2-7x+69}=-1, \quad (3)$$

whence
$$x^2-7x+69=9 \text{ or } 1. \quad (4)$$

The solution of the two quadratics in (4) will give the four values of x satisfying (1).

EXERCISES

Solve the following equations:

1. $x^6+2x^3=80.$
2. $5x-4-2\sqrt{5x-4}=63.$
3. $(2-x+x^2)^2+x^2-x=18.$
4. $a^2-3a+4-3\sqrt{a^2-3a+4}=-2.$
5. $3a^6-7a^3-1998=0.$
6. $x^2-8x+16+6\sqrt{x^2-8x+16}=40.$
7. $\left(a+\frac{2}{a}\right)^2+4\left(a+\frac{2}{a}\right)=21.$
8. $a^8-97a^4+1296=0.$
9. $a^2-3a+4+\sqrt{a^2-3a+15}=19.$
10. $(5x-7+3x^2)^2+3x^2+5x-247=0.$
11. $\sqrt[3]{7x-6}-4\sqrt[6]{7x-6}+4=0.$

RELATIONS BETWEEN THE ROOTS AND THE COEFFICIENTS OF A QUADRATIC

153. If in the general quadratic, $ax^2+bx+c=0$, we divide both members by a and put $\frac{b}{a}=p$, $\frac{c}{a}=q$, we have $x^2+px+q=0$.

Solving, $x_1=\dfrac{-p+\sqrt{p^2-4q}}{2}$, and $x^2=\dfrac{-p-\sqrt{p^2-4q}}{2}$.

Adding x_1 and x_2, $$x_1 + x_2 = -\frac{2p}{2} = -p. \qquad (1)$$

Multiplying x_1 and x_2, $$x_1x_2 = \frac{p^2 - (p^2 - 4q)}{4} = q. \qquad (2)$$

Hence in a quadratic of the form $x^2 + px + q = 0$, the sum of the roots is $-p$, and the product of the roots is q.

The expression $p^2 - 4q = \frac{b^2}{a^2} - \frac{4c}{a} = \frac{b^2 - 4ac}{a^2}$.

Hence $p^2 - 4q$ is positive, negative, or zero, according as $b^2 - 4ac$ is positive, negative, or zero.

Hence, as found on pp. 387, 389, the roots of

$$ax^2 + bx + c = 0, \text{ or } x^2 + px + q = 0 \text{ are:}$$

real and distinct, if $b^2 - 4ac > 0$, *or* $p^2 - 4q > 0$, (3)

real and equal, if $b^2 - 4ac = 0$, *or* $p^2 - 4q = 0$, (4)

imaginary, if $b^2 - 4ac < 0$, *or* $p^2 - 4q < 0$. (5)

By means of (1) to (5), we may determine the character of the roots of a quadratic without solving it.

Ex. 1. Determine the character of the roots of

$$8x^2 - 3x - 9 = 0.$$

Since $b^2 - 4ac = 9 - 4 \cdot 8(-9) = 297 > 0$, the roots are real and distinct. Since $b^2 - 4ac$ is not a perfect square, the roots are irrational.

Since $q = -\frac{9}{8} = x_1x_2$, the roots have *opposite* signs.

Since $p = -\frac{3}{8}$ or $-p = \frac{3}{8} = x_1 + x_2$, the positive root is *greater* in absolute value.

Ex. 2. Examine $3x^2 + 5x + 2 = 0$.

Since $b^2 - 4ac = 25 - 4 \cdot 3 \cdot 2 = 1 > 0$, the roots are *real* and *distinct.*

Since $b^2 - 4ac$ is a perfect square, the roots are rational.

Since $q = \frac{2}{3} = x_1x_2$, the roots have the *same* sign.

Since $-p = -\frac{5}{3} = x_1 + x_2$, the roots are both *negative.*

Ex. 3. Examine $x^2 - 14x + 49 = 0$.

Since $p^2 - 4q = 196 - 4 \cdot 49 = 0$, the roots are real and coincident.

Ex. 4. Examine $x^2 - 7x + 15 = 0$.

Since $p^2 - 4q = 49 - 4 \cdot 15 = -11$, the roots are imaginary.

EXERCISES

Without solving, determine the character of the roots in each of the following:

1. $5x^2 - 4x - 5 = 0.$
2. $6x^2 + 4x + 2 = 0.$
3. $x^2 - 4x + 8 = 0.$
4. $2 + 2x^2 = 4x.$
5. $6x + 8x^2 = 9.$
6. $1 - a^2 = 3a.$
7. $6a - 30 = 3a^2.$
8. $6a^2 + 6 = 13a.$
9. $16m^2 + 4 = 16m.$
10. $25a^2 - 10a = 8.$
11. $20 - 13b - 15b^2 = 0.$
12. $10y^2 + 39y + 14 = 0.$
13. $3a^2 + 5a + 22 = 0.$
14. $3a^2 - 22a + 21 = 0.$
15. $5b^2 + 6b = 27.$
16. $6a - 17 = 11a^2.$

FORMATION OF EQUATIONS WHOSE ROOTS ARE GIVEN

154. Ex. 1. Form the equation whose roots are 7 and -4.

From (1) and (2), § 153, we have

$$x_1 + x_2 = -p = 7 + (-4) = 3. \quad \text{Hence } p = -3.$$

And $\quad x_1x_2 = q = 7(-4) = -28.$

Hence $\quad x^2 + px + q = 0$ becomes $x^2 - 3x - 28 = 0.$

In case the equation is to have more than two roots, we proceed as in the following example:

Ex. 2. Form the equation whose roots are 2, 3, and 5.

Recalling the solution by factoring, we may write the desired equation in the factored form as follows:

$$(x-2)(x-3)(x-5) = 0.$$

Obviously 2, 3, and 5, are the roots and the only roots of this equation. Hence the desired equation is:

$$(x-2)(x-3)(x-5) = x^3 - 10x^2 + 31x - 30 = 0.$$

EXERCISES

Form the equations whose roots are:

1. $3, -7$.
2. b, c.
3. $a, -b, -c$.
4. $5, -4, -2$.
5. $\sqrt{5}, -\sqrt{5}$.
6. $a-\sqrt{3}, a+\sqrt{3}$.
7. $-5, -6$.
8. $-b+k, -b-k$.
9. $\sqrt{-1}, -\sqrt{-1}$.
10. $a, -b$.
11. $8+\sqrt{3}, 8-\sqrt{3}$.
12. $2, 3, 4, 5$.
13. $3+2\sqrt{-1}, 3-2\sqrt{-1}$.
14. $5-\sqrt{-1}, 5+\sqrt{-1}$.
15. $1, \frac{1}{2}, \frac{1}{3}, 3$.
16. $\frac{-b+\sqrt{b^2-4ac}}{2a}, \frac{-b-\sqrt{b^2-4ac}}{2a}$.

155. An expression of the second degree in a single letter may be **resolved into factors**, each of the first degree in that letter, by solving a quadratic equation.

Ex. 1. Factor $6x^2-17x+5$.

This trinomial may be written, $6(x^2-\frac{17}{6}x+\frac{5}{6})$.

Solving the equation, $x^2-\frac{17}{6}x+\frac{5}{6}=0$, we find $x_1=\frac{1}{3}$ and $x_2=\frac{5}{2}$. Hence by the factor theorem, § 92, $x-\frac{1}{3}$ and $x-\frac{5}{2}$ are factors of $x^2-\frac{17}{6}x+\frac{5}{6}$. And finally

$$6(x^2-17x+5)=6(x-\tfrac{1}{3})(x-\tfrac{5}{2})=3(x-\tfrac{1}{3})\cdot 2(x-\tfrac{5}{2})$$
$$=(3x-1)(2x-5).$$

This process is not needed when the factors are *rational*, but it is applicable equally well when the factors are *irrational* or *imaginary*.

Ex. 2. Factor $3x^2+8x-7=3(x^2+\frac{8}{3}x-\frac{7}{3})$.

Solving the equation $x^2+\frac{8}{3}x-\frac{7}{3}=0$, we find,

$$x_1=\frac{-4+\sqrt{37}}{3} \text{ and } x_2=\frac{-4-\sqrt{37}}{3}.$$

Hence as above:

$$3x^2+8x-7=3\left[x-\frac{-4+\sqrt{37}}{3}\right]\left[x-\frac{-4-\sqrt{37}}{3}\right]$$
$$=3\left[x+\frac{4}{3}-\frac{\sqrt{37}}{3}\right]\left[x+\frac{4}{3}+\frac{\sqrt{37}}{3}\right].$$

EXERCISES

In exercises 1 to 16, p. 410, transpose all terms of each equation to the first member, and then factor this member.

PROBLEMS INVOLVING QUADRATIC EQUATIONS

In each of the following problems, interpret both solutions of the quadratic involved:

1. The area of a rectangle is 2400 square feet and its perimeter is 200 feet. Find the length of its sides.

2. The area of a rectangle is a square feet and its perimeter is $2b$ feet. Find the length of its sides. Solve 1 by substitution in the formula thus obtained.

3. A picture measured inside the frame is 18 by 24 inches. The area of the frame is 288 square inches. Find its width.

4. If in problem 3 the sides of the picture are a and b and the area of the frame c, find the width of the frame.

5. The sides a and b of a right triangle are increased by the same amount, thereby increasing the square on the hypotenuse by $2k$. Find by how much each side is increased.

Make a problem which is a special case of this and solve it by substitution in the formula just obtained.

6. The hypotenuse c and one side a are each increased by the same amount, thereby increasing the square on the other side by $2k$. Find how much was added to the hypotenuse.

Make a problem which is a special case of this and solve it by substituting in the formula just obtained.

7. A rectangular park is 80 by 120 rods. Two driveways of equal width, one parallel to the longer and one to the shorter side, run through the park. What is the width of the driveways if their combined area is 591 square rods?

8. If in problem 7 the park is a rods wide and b rods long and the area of the driveways is c square rods, find their width.

9. The diagonal of a rectangle is a and its perimeter $2b$. Find its sides.

Make a problem which is a special case of this and solve it by substituting in the formula just obtained.

10. If in problem 9 the difference between the length and width is b and the diagonal is a, find the sides. Show how one solution can be made to give the results for both problems 9 and 10.

11. Find two consecutive integers whose product is a.

Make a problem which is a special case of this and solve it by substituting in the formula just obtained.

What special property must a have in order that this problem may be possible. Answer this from the formula.

12. A rectangular sheet of tin, 12 by 16 inches, is made into an open box by cutting out a square from each corner and turning up the sides. Find the size of the square cut out if the volume of the box is 180 cubic inches.

The resulting equation is of the third degree. Solve it by factoring. See § 151. Obtain three results and determine which are applicable to the problem.

13. A square piece of tin is made into an open box containing a cubic inches, by cutting from each corner a square whose side is b inches and then turning up the sides. Find the dimensions of the original piece of tin.

14. A rectangular piece of tin is a inches longer than it is wide. By cutting from each corner a square whose side is b inches and turning up the sides, an open box containing c cubic inches is formed. Find the dimensions of the original piece of tin.

15. The hypotenuse of a right triangle is 20 inches longer than one side and 10 inches longer than the other. Find the dimensions of the triangle.

16. If in problem 15 the hypotenuse is a inches longer than one side and b inches longer than the other, find the dimensions of the triangle.

17. The area of a circle exceeds that of a square by 10 square inches, while the perimeter of the circle is 4 less than that of the square. Find the side of the square and the radius of the circle.

Use $3\frac{1}{7}$ as the value of π.

18. If in problem 17 the area of the circle exceeds that of the square by a square inches, while its perimeter is $2b$ inches less than that of the square, find the dimensions of the square and the circle.

Determine from this general solution under what conditions the problem is possible.

19. Find three consecutive integers such that the sum of their squares is a.

Make a problem which is a special case of this and solve it by means of the formula just obtained. From the formula discuss the cases, $a = 2$, $a = 5$, $a = 14$. Find another value of a for which the problem is possible.

20. The difference of the cubes of two consecutive integers is 397. Find the integers.

21. The upper base of a trapezoid is 8 and the lower base is 3 times the altitude. Find the altitude and the lower base if the area is 78.

See problem 7, p. 326.

22. The lower base of a trapezoid is 4 greater than twice the altitude, and the upper base is $\frac{1}{2}$ the lower base. Find the two bases and the altitude if the area is $52\frac{1}{2}$.

23. The lower base of a trapezoid is twice the upper, and its area is 72. If $\frac{1}{2}$ the altitude is added to the upper base, and the lower is increased by $\frac{1}{4}$ of itself, the area is then 120. Find the dimensions of the trapezoid.

24. The upper base of a trapezoid is equal to the altitude, and the area is 48. If the altitude is decreased by 4, and the upper base by 2, the area is then 14. Find the dimensions of the trapezoid.

25. The upper base of a trapezoid is 4 more than $\frac{1}{2}$ the lower base, and the area is 84. If the upper base is decreased by 5, and the lower is increased by $\frac{1}{2}$ the altitude, the area is 78. Find the dimensions of the trapezoid.

26. The area of an equilateral triangle multiplied by $\sqrt{3}$, plus 3 times its perimeter, equals 81. Find the side of the triangle.

See problem 15, p. 191, E. C.

27. The area of a regular hexagon multiplied by $\sqrt{3}$, minus twice its perimeter, is 504. Find the length of its side.

See problem 20, p. 192, E. C.

28. If a times the perimeter of a regular hexagon, plus $\sqrt{3}$ times its area, equals b, find its side.

29. The perimeter of a circle divided by π, plus $\sqrt{3}$ times the area of the inscribed regular hexagon, equals $122\frac{1}{2}$. Find the radius of the circle.

30. The area of a regular hexagon inscribed in a circle plus the perimeter of the circle is a. Find the radius of the circle.

31. One edge of a rectangular box is increased 6 inches, another 3 inches, and the third is decreased 4 inches, making a cube whose volume is 864 cubic inches greater than that of the original box. Find its dimensions.

32. Of two trains one runs 12 miles per hour faster than the other, and covers 144 miles in one hour less time. Find the speed of each train.

In a township the main roads run along the section lines, one half of the road on each side of the line.

33. Find the area included in the main roads of a township if they are 4 rods wide.

34. If the area included in the main roads of a township is 68,796 square rods, find the width of the roads.

35. Find the width of the roads in problem 34 if the area included in them is a square rods.

CHAPTER VIII

ALGEBRAIC FRACTIONS

156. An **algebraic fraction** is the indicated quotient of two algebraic expressions.

Thus $\frac{n}{d}$ means *n divided by d.*

From the definition of a fraction and § 11, it follows that the *product of a fraction and its denominator equals its numerator.*

That is,

$$d \cdot \frac{n}{d} = n.$$

REDUCTION OF FRACTIONS

157. The **form** of a fraction may be modified in various ways without changing its **value.** Any such transformation is called a **reduction of the fraction.**

The most important reductions are the following:

(*A*) *By manipulation of signs.*

E.g. $\frac{n}{d} = -\frac{-n}{d} = -\frac{n}{-d} = \frac{-n}{-d}$; $\frac{b-a}{c-d} = -\frac{a-b}{c-d} = \frac{a-b}{d-c}$.

(*B*) *To lowest terms.*

E.g. $$\frac{x^4+x^2+1}{x^6-1} = \frac{(x^2+x+1)(x^2-x+1)}{(x-1)(x^2+x+1)(x+1)(x^2-x+1)}$$
$$= \frac{1}{(x-1)(x+1)}.$$

(*C*) *To integral or mixed expressions.*

E.g. $\frac{2x^3+x^2+x+2}{x^2+1} = 2x+1+\frac{-x+1}{x^2+1} = 2x+1-\frac{x-1}{x^2+1}$.

(*D*) *To equivalent fractions having a common denominator.*

E.g. $\frac{2}{x+3}$ and $\frac{3}{x+2}$ become respectively $\frac{2(x+2)}{(x+3)(x+2)}$ and $\frac{3(x+3)}{(x+3)(x+2)}$; $a+1$ and $\frac{1}{a-1}$ become respectively $\frac{a^2-1}{a-1}$ and $\frac{1}{a-1}$.

158. These reductions are useful in connection with the various operations upon fractions. They depend upon the principles indicated below.

Reduction (*A*) is simply an application of the law of signs in division, § 28. It is often needed in connection with reduction (*D*). See § 159.

Reduction (*B*) depends upon the theorem, § 47, $\frac{ak}{bk}=\frac{a}{b}$, by which *a common factor may be removed from both terms of a fraction.* It is useful in keeping expressions *simplified.* This reduction is *complete* when numerator and denominator have been divided by their H. C. F. See §§ 95–102.

Reduction (*C*) is merely the process of performing the indicated division, the result being *integral* when the division is *exact*, otherwise a *mixed expression.*

In case there is a *remainder* after the division has been carried as far as possible, this part of the quotient can only be *indicated.*

Thus $$\frac{D}{d}=q+\frac{R}{d},$$

in which D is dividend, d is divisor, q is quotient, and R is remainder.

Reduction (*D*) depends upon the theorem of § 47, $\frac{a}{b}=\frac{ka}{kb}$, by which *a common factor is introduced into the terms of a fraction.*

A fraction is thus reduced to another fraction whose denominator is *any required multiple* of the given denominator.

If two or more fractions are to be reduced to equivalent fractions having a common denominator, this denominator must be a *common multiple* of the given denominators, and for simplicity the L. C. M. is used.

EXERCISES

Reduce the following so that the letters in each factor shall occur in alphabetical order, and no negative sign shall stand before a numerator or denominator, or before the first term of any factor.

1. $\frac{n-m}{b-a}$.

2. $-\frac{(b-a)(c-d)}{x(s-r-t)}$.

3. $\frac{-(x-y)}{(b-a)(c-d)}$.

4. $\frac{-(x-y)(z-y)}{-(b-a)(c-d)}$.

5. $\frac{r-s}{(a-b)(c-b)(c-a)}$.

6. $\frac{-a(c+b)}{b(c-a)}$.

7. $\frac{-(c-a)(d-c)}{(a-b)(b-c)}$.

8. $\frac{(b-a)(c-b)(c-a)}{(y-x)(y-z)(z-x)}$.

9. $-\frac{-1}{(a-b)(b-c)(c-a)}$.

10. $\frac{(c-b-a)(b-a-c)}{3(a-c)(b-c)(c-a)}$.

11. $\frac{(3c-2a)(4b-a)d}{(-a+b)(a-b)(c-a)}$.

12. $\frac{-(-r-s)(s-t)(t-r)}{(n-m)(-k-m-l)}$.

Reduce each of the following to lowest terms:

13. $\frac{a^4-b^4}{a^6-b^6}$.

14. $\frac{c^2-(a-b)^2}{(a+c)^2-b^2}$.

15. $\frac{7ax^2-56a^4x^5}{28x^2(1-64a^6x^6)}$.

16. $\frac{m^3+5m^2+7m+3}{m^2+4m+3}$.

17. $\frac{a^3-7a+6}{a^3-7a^2+14a-8}$.

18. $\frac{x^3+2x^2+2x+1}{x^4+x^3-x^2-2x-2}$.

19. $\frac{2x^3-x^2-8x-3}{2x^3-3x^2-7x+3}$.

20. $\frac{4x^3+8x^2-3x+5}{6x^3-5x^2+4x-1}$.

21. $\frac{x^2-xy+y^2+x-y+3}{x^3+y^3+x^2-y^2+3x+3y}$.

22. $\frac{a^4+a^2b^2+b^4+a^3+b^3}{a^2+ab+b^2+a+b}$.

23. $\dfrac{x^4+4x^3y+6x^2y^2+4xy^3+y^4-a^4}{x^2+2xy+y^2-a^2}$.

24. $\dfrac{x^2y-x^2z+y^2z-xy^2+xz^2-yz^2}{x^2-(y+z)x+yz}$.

25. $\dfrac{2x^4-x^3-20x^2+16x-3}{3x^4+5x^3-30x^2-41x+15}$.

26. $\dfrac{3a^3-8a^2b-5ab^2+6b^3}{a^3+a^2b-9ab^2-9b^3}$.

27. $\dfrac{2r^3+r^2s+rs^2+2s^3}{2r^4+r^3s+3r^2s^2+rs^3+2s^4}$.

Reduce each of the following to an integral or mixed expression:

28. $\dfrac{x^4+1}{x+1}$.

29. $\dfrac{x^5+1}{x+1}$.

30. $\dfrac{x^4}{x-1}$.

31. $\dfrac{a^3}{a^2+a+1}$.

32. $\dfrac{c^5}{c^3+c^2-c+1}$.

33. $\dfrac{x^2-x+1}{x^2+x+1}$.

34. $\dfrac{a^4+a^2b^2+b^4}{a-b}$.

35. $\dfrac{3a^3-3a^2+3a-1}{a-2}$.

36. $\dfrac{x^3-x^2-x+1}{x^3+x^2+x-1}$.

37. $\dfrac{4m^4-3m^3+3}{2m^2-2m+1}$.

Reduce each of the following sets of expressions to equivalent fractions having the lowest common denominator:

38. $\dfrac{1}{x^4-3x^2y^2+y^4}, \dfrac{1}{x^2-xy-y^2}, \dfrac{1}{x^2+xy-y^2}$.

39. $\dfrac{a+b}{5a^2c+12cd-6ad-10ac^2}, \dfrac{a}{5ac-6d}, \dfrac{b}{a-2c}$.

40. $\dfrac{x^2+y^2}{x^3+y^3+x^2-xy+y^2}, \dfrac{x+y-1}{x^2-xy+y^2}, \dfrac{x^2+xy+y^2}{x+y+1}$.

41. $\dfrac{x}{(a-b)(c-b)(c-a)}$, $\dfrac{y}{(a-b)(b-c)(a-c)}$, $\left[\dfrac{z}{(b-a)(b-c)(a-c)}\right.$.

42. $\dfrac{a}{b+c}$, $\dfrac{b}{c+a}$, $\dfrac{c}{a+b}$, d.

43. $\dfrac{b-c}{(a-c)(a-b)}$, $\dfrac{a-b}{(c-a)(b-c)}$, $\dfrac{c-a}{(b-a)(c-b)}$.

44. $\dfrac{m-n}{a^3-6a^2+11a-6}$, $\dfrac{a+2}{a^2-4a+3}$, $\dfrac{a+3}{a^2-3a+2}$.

If a, b, m are positive numbers, arrange each of the following sets in decreasing order. Verify the results by substituting convenient Arabic numbers for a, b, m.

Suggestion. Reduce the fractions in each set to equivalent fractions having a common denominator.

45. $\dfrac{a}{a+1}$, $\dfrac{2a}{a+2}$, $\dfrac{3a}{a+3}$. 46. $\dfrac{m}{2m+1}$, $\dfrac{2m}{3m+2}$, $\dfrac{3m}{4m+3}$.

47. $\dfrac{a+3b}{a+4b}$, $\dfrac{a+b}{a+2b}$, $\dfrac{a+4b}{a+5b}$.

48. Show that, for a different from zero, neither $\dfrac{n+a}{d+a}$ nor $\dfrac{n-a}{d-a}$ can equal $\dfrac{n}{d}$, unless $n=d$. State this result in words, and fix it in mind as an impossible reduction of a fraction.

ADDITION AND SUBTRACTION OF FRACTIONS

159. Fractions which have a common denominator are added or subtracted in accordance with the distributive law for division, §§ 30, 31.

That is,
$$\frac{a}{d}+\frac{b}{d}-\frac{c}{d}=\frac{a+b-c}{d}.$$

In order to add or subtract fractions not having a common denominator, they should first be reduced to equivalent fractions having a common denominator.

When several fractions are to be combined, it is sometimes best to take only part of them at a time. In any case it is advantageous to keep all expressions in the factored form as long as possible.

Ex. $$\frac{1}{(x-1)(x-2)}-\frac{1}{(2-x)(x-3)}+\frac{1}{(3-x)(4-x)}.$$

Taking the first two together, we have

$$\frac{1}{(x-1)(x-2)}+\frac{1}{(x-2)(x-3)}=\frac{2x-4}{(x-1)(x-2)(x-3)}=\frac{2}{(x-1)(x-3)}.$$

Taking this result with the third,

$$\frac{2}{(x-1)(x-3)}+\frac{1}{(x-3)(x-4)}=\frac{3x-9}{(x-1)(x-3)(x-4)}=\frac{3}{(x-1)(x-4)}.$$

If all are taken at once, the work should be carried out as follows: The numerator of the sum is

$$(x-3)(x-4)+(x-1)(x-4)+(x-1)(x-2).$$

Adding the first two terms with respect to $(x-4)$, we have

$$2(x-2)(x-4)+(x-1)(x-2).$$

Adding these with respect to $(x-2)$, we have $3(x-3)(x-2)$.

Hence the sum is $$\frac{3(x-3)(x-2)}{(x-1)(x-2)(x-3)(x-4)}=\frac{3}{(x-1)(x-4)}.$$

EXERCISES

Perform the following indicated additions and subtractions:

1. $\frac{2}{x-3}+\frac{3}{x-4}-\frac{4}{x-5}.$

2. $\frac{3}{4(x+3)}-\frac{5}{8(x+5)}-\frac{1}{8(x+1)}.$

3. $\frac{1}{2(x-1)}-\frac{4}{x-2}+\frac{7}{2(x-3)}.$

4. $\frac{1}{12(x+1)}-\frac{7}{3(x-2)}+\frac{13}{4(x-3)}.$

5. $\dfrac{2}{(x+1)^2}+\dfrac{3}{x+1}+\dfrac{4}{x-2}$.

6. $\dfrac{1}{3(x+1)}-\dfrac{x-2}{3(x^2-x+1)}$.

7. $\dfrac{5x+6}{x^2+x+1}-\dfrac{3x-4}{x^2-x+1}$.

8. $\dfrac{1}{5(x+2)}+\dfrac{4x-8}{5(x^2+1)}$.

9. $\dfrac{2}{(x-2)^2}-\dfrac{1}{x-2}+\dfrac{1}{x+1}$.

10. $\dfrac{2}{(x-2)^2}+\dfrac{1}{5(x-2)}-\dfrac{x+2}{5(x^2+1)}$.

11. $\dfrac{1}{(x-1)^2}+\dfrac{1}{(x-1)}-\dfrac{1}{(x^2-1)}$.

12. $\dfrac{1}{2(1-3x)^3}+\dfrac{3}{8(1-3x)^2}+\dfrac{3}{32(1-3x)}+\dfrac{1}{32(1+x)}$.

13. $\dfrac{1}{(1-a)(2-a)}-\dfrac{1}{(2-a)(a-3)}+\dfrac{2}{(3-a)(a-1)}$.

14. $\dfrac{xy}{(z-y)(x-z)}-\dfrac{yz}{(x-z)(x-y)}-\dfrac{xz}{(y-x)(y-z)}$.

15. $\dfrac{1}{a-1}-\dfrac{2a-5}{a^2-2a+1}-\dfrac{5a^2-3a-2}{(a-1)^3}$.

16. $\dfrac{1}{m^2+m+1}-\dfrac{1}{m^2-m+1}+\dfrac{2m+2}{m^4+m^2+1}$.

17. $\dfrac{1}{b^2-3b+2}+\dfrac{1}{b^2-5b+6}-\dfrac{2}{b^2-4b+3}$.

18. $\dfrac{r+s}{(r-t)(s-t)}-\dfrac{s+t}{(r-s)(t-r)}-\dfrac{r+t}{(t-s)(s-r)}$.

19. $\dfrac{p^2+q^2}{(p-q)(p+r)}+\dfrac{q^2-pr}{(q-r)(q-p)}+\dfrac{r^2+pq}{(r-q)(r+p)}$.

20. $\dfrac{3x^2+1}{5x^2-18x+9}-\dfrac{2x^2+2}{4x^2-11x-3}$.

MULTIPLICATION AND DIVISION OF FRACTIONS

160. *The product of two fractions is a fraction whose numerator is the product of the given numerators and whose denominator is the product of the given denominators.*

That is, $$\frac{a}{b} \cdot \frac{n}{d} = \frac{an}{bd}.$$

For let $$x = \frac{a}{b} \cdot \frac{n}{d}.$$

Then $$bdx = bd\left(\frac{a}{b} \cdot \frac{n}{d}\right). \qquad \S\ 7$$

$$bdx = b \cdot \frac{a}{b} \cdot d \cdot \frac{n}{d}. \qquad \S\ 8$$

$$bdx = an. \qquad \S\ 11$$

Hence, $$x = \frac{an}{bd}.$$

Therefore, $$\frac{a}{b} \cdot \frac{n}{d} = \frac{an}{bd}. \qquad \S\ 2$$

It follows that a fraction is raised to any power by raising numerator and denominator separately to that power.

For $\frac{a}{b} \cdot \frac{a}{b} = \frac{a^2}{b^2}$, $\frac{a}{b} \cdot \frac{a}{b} \cdot \frac{a}{b} = \frac{a^3}{b^3}$, etc.

A fraction multiplied by itself inverted equals + 1.

For $\frac{n}{d} \cdot \frac{d}{n} = \frac{nd}{nd} = +1$ and $-\frac{n}{d} \cdot \left(-\frac{d}{n}\right) = \frac{nd}{nd} = +1$.

161. **Definitions.** If the product of two numbers is 1, each is called the **reciprocal** of the other. Hence by § 160, the reciprocal of a fraction is the fraction inverted.

Also, since from $ab = 1$ we have $a = \frac{1}{b}$ and $b = \frac{1}{a}$, it follows that if two numbers are reciprocals of each other, then either one is the quotient obtained by dividing 1 by the other.

162. *To divide by any number is equivalent to multiplying by its reciprocal.*

For it is an immediate consequence of § 29 that

$$n \div d \text{ or } \frac{n}{d} = n \cdot \frac{1}{d}.$$

To divide a number by a fraction is equivalent to multiplying by the fraction inverted.

For by § 161 the reciprocal of the fraction is the fraction inverted.

A fraction is divided by an integer by multiplying its denominator or dividing its numerator by that integer.

For $$\frac{n}{d} \div a = \frac{n}{d} \cdot \frac{1}{a} = \frac{n}{ad},$$

and $$\frac{n}{d} \div a = \frac{n \div a}{d}, \text{ since } \frac{n}{ad} = \frac{n \div a}{d} \text{ by § 47.}$$

In multiplying and dividing fractions their terms should at once be put into *factored* forms.

When mixed expressions or sums of fractions are to be multiplied or divided, these operations are indicated by means of parentheses, and the additions or subtractions within the parentheses should be performed first, § 38.

Ex. Simplify $\left[\left(1 - a + \frac{2a^2}{1+a}\right) \div \left(\frac{1}{1+a} - \frac{1}{1-a}\right)\right] \cdot \frac{3a^3}{a^4 - 1}.$

Performing the indicated operations within the parentheses, we have

$$\left[\frac{1+a^2}{1+a} \div \frac{2a}{a^2-1}\right] \cdot \frac{3a^3}{a^4-1} = \frac{1+a^2}{1+a} \cdot \frac{a^2-1}{2a} \cdot \frac{3a^3}{(a^2-1)(a^2+1)} = \frac{3a^2}{2(a+1)}.$$

EXERCISES

Perform the following indicated operations and reduce each result to its simplest form.

1. $\frac{x^4 + x^2y^2 + y^4}{x^3 - y^3} \cdot \frac{x^2 - y^2}{x^3 + y^3}.$

2. $\dfrac{a^2-b^2x^2+acx^2-bcx^3}{36\,a^4-9\,a^2+24\,a-16} \div \dfrac{-ay^2-bxy^2-cx^2y^2}{20\,x^2-15\,ax^2-30\,a^2x^2}.$

3. $\dfrac{20\,r^2s^2+23\,rst-21\,t^2}{8\,m^2n^3-48\,m^2n^2y+72\,m^2ny^2} \times \dfrac{12\,mn^3-28\,mn^2y-24\,mny^2}{10\,r^2s^2+24\,rst-18\,t^2}.$

4. $\left(\dfrac{a}{b}-\dfrac{b}{a}\right)\left(\dfrac{a^2}{b^2}+\dfrac{b^2}{a^2}-\dfrac{a}{b}-\dfrac{b}{a}+1\right)\div\dfrac{a^5+b^5}{a-b}.$ $\left[\div\dfrac{3\,n+2\,y}{2\,s+3\,t}.\right.$

5. $\left(\dfrac{1}{x}+\dfrac{1}{y}\right)\left(\dfrac{1}{x}-\dfrac{1}{y}\right)\left(1-\dfrac{x-y}{x+y}\right)\left(2+\dfrac{2\,y}{x-y}\right).$

6. $\left(\dfrac{a}{a-b}-\dfrac{b}{a+b}\right)\left(\dfrac{1}{a^2}-\dfrac{1}{b^2}\right)\div\left(\dfrac{1}{a^2}+\dfrac{1}{b^2}\right).$

7. $\left(1+\dfrac{b}{a-b}\right)\left(1-\dfrac{b}{a+b}\right)\div\left(1+\dfrac{b^2}{a^2-b^2}\right).$

8. $\left(\dfrac{m+n}{m-n}-\dfrac{m-n}{m+n}\right)\left(m+n+\dfrac{2\,n^2}{m-n}\right)\div\left(\dfrac{m+n}{m-n}+\dfrac{m-n}{m+n}\right).$

9. $\left(\dfrac{x^2+y^2}{x^2-y^2}-\dfrac{x^2-y^2}{x^2+y^2}\right)\cdot\left(x^2+y^2+\dfrac{2\,x^2y^2+2\,y^4}{x^2-y^2}\right)\div\left(\dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}\right).$

10. $\left(\dfrac{x+y+z}{x+y}+\dfrac{z^2}{(x+y)^2}\right)\cdot\left(\dfrac{(x+y)^3}{(x+y)^3-z^3}\right)\cdot\left(1+\dfrac{z}{x+y}\right).$

11. $\dfrac{a^2+ab+b^2}{a^2-ab+b^2}\cdot\dfrac{a+b}{a^3-b^3}\cdot\left(a^2+\dfrac{b^3-a^2b}{a+b}\right).$

12. $\dfrac{m^2+mn}{m^2+n^2}\cdot\dfrac{m^3-mn^2-m^2+n^2}{m^3n-n^4}\cdot\dfrac{m^2n^2+mn^3+n^4}{m^4-2\,m^3+m^2}$
$\div\dfrac{m^3n+2\,m^2n^2+mn}{m^4-n^4}.$

13. $\left(xy^9+x^9y-\dfrac{2\,x^{11}y^3}{x^2y^2}\right)\div\left[\dfrac{x^2+y^2}{x^2}\cdot\left(\dfrac{1}{y^2}-\dfrac{1}{z^2}\right)-\dfrac{x^2+y^2}{y^2}\cdot\left(\dfrac{1}{x^2}-\dfrac{1}{z^2}\right)\right].$

COMPLEX FRACTIONS

163. A fraction which contains a fraction either in its numerator or in its denominator or in both is called **a complex fraction.**

Since every fraction is an indicated operation in division, any complex fraction may be simplified by performing the indicated division.

It is usually better, however, to remove all the minor denominators at once by multiplying both terms of the complex fraction by the least common multiple of all the minor denominators according to § 47.

For example, $$\frac{\frac{x}{3}+\frac{x}{2}}{\frac{2x^2}{3}-\frac{3}{2}}=\frac{\left(\frac{x}{3}+\frac{x}{2}\right)\cdot 6}{\left(\frac{2x^2}{3}-\frac{3}{2}\right)\cdot 6}=\frac{2x+3x}{4x^2-9}=\frac{5x}{4x^2-9}.$$

A complex fraction may contain another complex fraction in one of its terms.

E.g. $\dfrac{1}{a+\dfrac{a+1}{a+\dfrac{1}{a-1}}}$ has the complex fraction $\dfrac{a+1}{a+\dfrac{1}{a-1}}$ in its denominator. This latter fraction is first reduced by multiplying its numerator and denominator by $a-1$, giving

$$\frac{a+1}{a+\frac{1}{a-1}}=\frac{a^2-1}{a^2-a+1}.$$

Substituting this result in the given fraction, we have

$$\frac{1}{a+\frac{a+1}{a+\frac{1}{a-1}}}=\frac{1}{a+\frac{a^2-1}{a^2-a+1}}=\frac{a^2-a+1}{a^3+a-1}.$$

EXERCISES

Simplify each of the following,

1. $$\frac{\dfrac{m^2+mn}{m^2-n^2}}{\dfrac{m}{m-n}-\dfrac{n}{m+n}}.$$

2. $$\frac{\dfrac{a^4-b^4}{a^2-2ab+b^2}}{\dfrac{a^2+ab}{a-b}}.$$

3. $$\frac{\dfrac{x^5-3x^4y+3x^3y^2-x^2y^3}{x^3y-y^4}}{\dfrac{x^5-2x^4y+x^3y^2}{x^2y^2+xy^3+y^4}}.$$

4. $$\frac{\dfrac{1}{a+x}+\dfrac{1}{a-x}+\dfrac{2a}{a^2-x^2}}{\dfrac{1}{a+x}-\dfrac{1}{a-x}-\dfrac{2a}{a^2-x^2}}.$$

5. $$\frac{\dfrac{1}{a+x}+\dfrac{1}{a-x}+\dfrac{2a}{a^2+x^2}}{\dfrac{1}{a-x}-\dfrac{1}{a+x}+\dfrac{2x}{a^2+x^2}}.$$

6. $$\frac{m^2-mn+n^2-\dfrac{m^3-n^3}{m+n}}{m^2+mn+n^2+\dfrac{m^3+n^3}{m-n}}.$$

7. $$\frac{\dfrac{a-\dfrac{1}{a^2}}{a-2+\dfrac{1}{a}}}{\dfrac{a^2+1+\dfrac{1}{a^2}}{a-2+\dfrac{2}{a}-\dfrac{1}{a^2}}}.$$

8. $$\frac{1}{1-\dfrac{1}{1-\dfrac{1}{x}}}.$$

9. $$\frac{1}{1-\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{x}}}}$$

10. $$\frac{3}{3+\dfrac{3}{3+\dfrac{3}{3+\dfrac{3}{x}}}}.$$

EQUATIONS INVOLVING ALGEBRAIC FRACTIONS

164. In solving a fractional equation, it is usually convenient to **clear it of fractions**, that is, to transform it into an equivalent equation containing no fractions.

In case no denominator contains any unknown this may be done by multiplying both members by the L. C. M. of all the denominators, § 62.

When, however, the unknown appears in any denominator, multiplying by the L. C. M. of all the denominators *may or may not* introduce new roots, as shown in the following examples.

It may easily be shown, that multiplying an *integral* equation by any expression containing the unknown *always* introduces new roots.

Ex. 1. Solve $$\frac{2}{x-2}+\frac{1}{x-3}=2. \quad (1)$$

Clearing of fractions by multiplying by $(x-2)(x-3)$, and simplifying, we have $$2x^2-13x+20=(x-4)(2x-5)=0. \quad (2)$$

The roots of (2) are 4 and $2\frac{1}{2}$, both of which satisfy (1). Hence no new root was introduced by clearing of fractions.

Ex. 2. Solve $$\frac{1}{x-1}=\frac{1}{(x-1)(x-2)}. \quad (1)$$

Clearing of fractions, we have,

$$x-2=1. \quad (2)$$

The only root of (2) is $x=3$, which is also the only root of (1). Hence no new root was introduced.

Ex. 3. Solve $$\frac{x-2}{x^2-4}=1. \quad (1)$$

Clearing of fractions and simplifying, we have,

$$x^2-x-2=(x-2)(x+1)=0. \quad (2)$$

The roots of (2) are 2 and -1. Now $x=-1$ is a root of (1), but $x=2$ is *not*, since we are not permitted to make a substitution which reduces a denominator to zero, § 50. Hence a new root has been introduced and (1) and (2) are not equivalent.

If the fraction $\frac{x-2}{x^2-4}$ is first reduced to lowest terms, we have the equation

$$\frac{1}{x+2}=1. \quad (3)$$

Clearing of fractions, $$x+2=1. \quad (4)$$

Now (3) and (4) are equivalent, -1 being the only root of each.

Ex. 4. Solve $$\frac{4x}{x^2-1}-\frac{x+1}{x-1}=1. \qquad (1)$$

Clearing of fractions and simplifying,

$$x^2 - x = x(x-1) = 0. \qquad (2)$$

The roots of (2) are 0 and 1. $x = 0$ satisfies (1), but $x = 1$ does not, since it is not a permissible substitution in either fraction of (1). Hence a new root has been introduced.

165. Examples 3 and 4 illustrate the *only cases in which new roots can be introduced* by multiplying by the L. C. M. of the denominators.

This can be shown by proving certain theorems, the results of which are here used in the following directions for solving fractional equations:

(1) Reduce all fractions to their lowest terms.

(2) Multiply both members by the least common multiple of the denominators.

(3) Reject any root of the *resulting* equation which reduces any denominator of the *given* equation to zero. The remaining roots will then satisfy both equations, and hence are the solutions desired.

If when each fraction is in its lowest terms the given equation contains no two which have a factor common to their denominators, then *no new root* can enter the resulting equation and none need to be rejected. See Ex. 1 and Ex. 3 after being reduced.

If, however, any two or more denominators have some common factor $x - a$, then $x = a$ *may or may not be a new root* in the resulting equation, but in any case it is the only possible new root which can enter, and must be tested. Compare Exs. 2 and 4.

Ex. 5. Examine $$\frac{3x+7}{x^2+2x+11}+\frac{5x}{x^2+3x+2}-\frac{x+1}{x-1}=8.$$

Since each fraction is in its lowest terms and no two denominators contain a common factor, then clearing of fractions will give an equation equivalent to the given one.

Ex. 6. Examine $\frac{2x+3}{x^2+5x+6}-\frac{x-7}{x^2+3x+2}=4.$

Each fraction is in its lowest terms, but the two denominators have the factor $x+2$ in common. Hence $x=-2$ is the *only possible* new root which can enter the resulting integral equation, but on trial it is found not to be a root. Hence the two equations are equivalent.

EXERCISES

Determine whether each of the following when cleared of fractions produces an equivalent equation, and solve each.

1. $\frac{3x^3+3}{3x^2-7x+3}=x-7.$

2. $\frac{x^2+4x+4}{x^2-4}=2x+3.$

3. $\frac{3}{2x^2-x-1}+\frac{5}{x^2-1}+\frac{1}{x+1}=0.$

4. $\frac{2x}{2x-1}+\frac{x}{x+1}-\frac{3x}{x-1}=-1.$

5. $\frac{1}{3(x-1)}-\frac{1}{x^2-1}=\frac{1}{4}.$

6. $\frac{2a-1}{a}+\frac{1}{2}=\frac{3a}{3a-1}.$

7. $\frac{2}{x-a}+\frac{3}{x-b}=\frac{6}{x-c}.$

8. $\frac{1}{a-x}-\frac{1}{a+x}=-\frac{3+x^2}{a^2-x^2}.$

9. $a^2b-\frac{a+x}{b}=ab^2-\frac{b+x}{a}.$

10. $b=\frac{x-a}{1-ax}.$

11. $\frac{2}{x-10}+10=x+\frac{2}{10-x}.$

12. $\frac{6-x}{x-4}+\frac{x-4}{6-x}=\frac{5}{2}.$

13. $\frac{a}{2a-1}+\frac{24}{4a^2-1}=\frac{2(a-4)}{2a+1}-\frac{1}{9}.$

14. $\frac{a}{a-1}+\frac{a-1}{a}=\frac{a^2+a-1}{a^2-a}.$

15. $\frac{1}{a^2-4}-\frac{3}{2-a}=1+\frac{1}{3(a+2)}.$

16. $\dfrac{ax+b}{a+bx}+\dfrac{cx+d}{c+dx}=\dfrac{ax-b}{a-bx}+\dfrac{cx-d}{c-dx}.$

17. $\dfrac{(a-x)(x-b)}{(a-x)-(x-b)}=x.$ 18. $\dfrac{x+m-2n}{x+m+2n}=\dfrac{n+2m-2x}{n-2m+2x}.$

19. $\dfrac{(a-x)^2-(x-b)^2}{(a-x)(x-b)}=\dfrac{4ab}{a^2-b^2}.$

20. $\dfrac{1+3x}{5+7x}-\dfrac{9-11x}{5-7x}=14\cdot\dfrac{(2x-3)^2}{25-49x^2}$

21. $\dfrac{x+2a}{2b-x}+\dfrac{x-2a}{2b+x}=\dfrac{4ab}{4b^2-x^2}.$

22. $\dfrac{1}{x-2}+\dfrac{7x}{24(x+2)}=\dfrac{5}{x^2-4}.$

23. $\dfrac{x+a}{x-a}+\dfrac{x-a}{x+a}=\dfrac{2(a^2+1)}{(1+a)(1-a)}.$

24. $\dfrac{x-m}{x+m}=\dfrac{n-x}{n+x}.$ 25. $\dfrac{1}{x-a}-\dfrac{2a}{x^2-a^2}=b.$

26. $\dfrac{4}{3x+1}+\dfrac{4(3x-1)}{2x+1}=\dfrac{2x+1}{3x+1}.$

27. $\dfrac{2x+3}{2(2x-1)}+\dfrac{7-3x}{3x-4}+\dfrac{x-7}{2(x+1)}=0.$

28. $\dfrac{1}{a-b}+\dfrac{a-b}{x}=\dfrac{1}{a+b}+\dfrac{a+b}{x}.$

29. $\dfrac{1}{\dfrac{3(m+n)^2}{p^2x}-\dfrac{m+n}{p}}=\dfrac{p}{2(m+n)}.$

30. $\dfrac{y^2+2y-2}{y^2+5y+6}+\dfrac{y}{y+3}=\dfrac{y}{y+2}.$

31. $\dfrac{5}{2x+3}+\dfrac{7}{3x-4}=\dfrac{8x^2-13x-64}{6x^2+x-12}.$

32. $\dfrac{3a^2}{x^3-a^3}-\dfrac{1}{x-a}+\dfrac{a}{x^2+ax+a^2}=c.$

33. $\dfrac{1-2x}{3-4x}-\dfrac{5-6x}{7-8x}=\dfrac{8}{3}\cdot\dfrac{1-3x^2}{21-52x+32x^2}.$

34. $\dfrac{m-q}{x-n}+\dfrac{n-p}{x-q}=\dfrac{m-q}{x-p}+\dfrac{n-p}{x-m}.$

35. $\dfrac{3}{x-3}-\dfrac{2}{x-2}+\dfrac{8}{4x^2-20x+24}=0.$

36. $\dfrac{27}{x^3+27}-\dfrac{3}{x^2-3x+9}+\dfrac{1}{x+3}=0.$

37. $\dfrac{x-9}{x-5}-\dfrac{x-7}{x-2}-\dfrac{x-9}{x-4}=\dfrac{x-8}{x-5}-\dfrac{x-7}{x-4}-\dfrac{x-8}{x-2}.$

38. $3=\dfrac{(x+b-c)(x-b+c)}{(b+c+x)(b+c-x)}.$

PROBLEMS

1. Find a number such that if it is added to each term of the fraction $\frac{3}{8}$ and subtracted from each term of the fraction $\frac{13}{24}$ the results will be equal.

2. Make and solve a general problem of which 1 is a special case.

3. Three times one of two numbers is 4 times the other. If the sum of their squares is divided by the sum of the numbers, the quotient is $42\frac{6}{7}$ less than that obtained by dividing the sum of the squares by the difference of the numbers. Find the numbers.

4. The sum of two numbers less 2, divided by their difference, is 4, and the sum of their cubes divided by the difference of their squares is $1\frac{2}{5}$ times their sum. Find the numbers.

5. The circumference of the rear wheel of a carriage is 4 feet greater than that of the front wheel. In running one mile the front wheel makes 110 revolutions more than the rear wheel. Find the circumference of each wheel.

6. State and solve a general problem of which 5 is a special case, using b feet instead of one mile, letting the other numbers remain as they are in problem 5.

7. In going one mile the front wheel of a carriage makes 88 revolutions more than the rear wheel. If one foot is added to the circumference of the rear wheel, and 3 feet to that of the front wheel, the latter will make 22 revolutions more than the former. Find the circumference of each wheel.

8. State and solve a general problem of which 5 is a special case, using letters for all the numbers involved.

9. The circumference of the front wheel of a carriage is a feet, and that of the rear wheel b feet. In going a certain distance the front wheel makes n revolutions more than the rear wheel. Find the distance.

10. State and solve a problem which is a special case of problem 9, using the formula just obtained.

11. Find a number consisting of two digits whose sum, divided by their difference, is 4. This number divided by the sum of its digits is equal to twice the digit in units' place plus $\frac{1}{8}$ of the digit in tens' place, the tens' digit being the greater.

12. There is a fraction such that if 3 is added to each of its terms, the result is $\frac{4}{5}$, and if 3 is subtracted from each of its terms, the result is $\frac{1}{2}$. Find the fraction.

13. State and solve a general problem of which 12 is a special case.

14. A and B working together can do a piece of work in 6 days. A can do it alone in 5 days less than B. How long will it require each when working alone?

15. State and solve a general problem of which 14 is a special case.

16. On her second westward trip the *Mauretania* traveled 625 knots in a certain time. If her speed had been 5 knots less per hour, it would have required $6\frac{1}{4}$ hours longer to cover the same distance. Find her speed per hour.

17. By increasing the speed a miles per hour, it requires b hours less to go c miles. Find the original speed. Show how problem 16 may be solved by means of the formula thus obtained.

18. A train is to run d miles in a hours. After going c miles a dispatch is received requiring the train to reach its destination b hours earlier. What must be the speed of the train for the remainder of the journey?

19. A man can row a miles down stream and return in b hours. If his rate up stream is c miles per hour less than down stream, find the rate of the current, and the rate of the boat in still water.

20. State and solve a special case of problem 19.

21. A can do a piece of work in a days, B can do it in b days, and C in c days. How long will it require all working together to do it?

22. Three partners, A, B, and C, are to divide a profit of p dollars. A had put in a dollars for m months, B had put in b dollars for n months, and C c dollars for t months. What share of the profit does each get?

23. State and solve a problem which is a special case of the preceding problem.

CHAPTER IX

RATIO, VARIATION, AND PROPORTION

RATIO AND VARIATION

166. In many important applications fractions are called **ratios.**

E.g. $\frac{3}{5}$ is called the ratio of 3 to 5 and is sometimes written 3 : 5.

It is to be understood that a ratio is the *quotient of two numbers* and hence is itself a *number.* We sometimes speak of the ratio of two magnitudes of the same kind, meaning thereby that these magnitudes are expressed in terms of a common unit and a ratio formed from the resulting *numbers.*

E.g. If, on measuring, the heights of two trees are found to be 25 feet and 35 feet respectively, we say the *ratio of their heights* is $\frac{25}{35}$ or $\frac{5}{7}$.

167. Two magnitudes are said to be **incommensurable** if there is no common unit of measure which is contained exactly an integral number of times in each.

E.g. If a and d are the lengths of the side and the diagonal of a square, then $d^2 = a^2 + a^2$, § 166, E. C. Hence, $\frac{a^2}{d^2} = \frac{1}{2}$ or $\frac{a}{d} = \frac{1}{\sqrt{2}}$. But since $\sqrt{2}$ is neither an *integer* nor a *fraction* (§ 108), it follows that a and d have no common measure, that is, they are *incommensurable.*

168. In many problems, especially in Physics, magnitudes are considered which are constantly changing. Number expressions representing such magnitudes are called **variables,** while those which represent fixed magnitudes are **constants.**

E.g. Suppose a body is moving at a uniform rate of 5 ft. per second. If t is the *number* of seconds from the time of starting and s the *number* of feet passed over, then s and t are *variables.*

The variables s and t, in case of uniform motion, have a *fixed ratio;* namely, in this example, $s:t = 5$ for every pair of corresponding values of s and t throughout the period of motion.

169. When two variables are so related that for all pairs of corresponding values, their *ratio remains constant,* then each one is said to **vary directly as the other.**

E.g. If $s:t = k$ (a constant) then s varies directly as t, and t varies directly as s.

Variation is sometimes indicated by the symbol $\propto$. Thus $s \propto t$ means *s varies as t,* i.e. $\frac{s}{t} = k$ or $s = kt$.

170. When two variables are so related that for all pairs of corresponding values their *product remains constant,* then each one is said to **vary inversely as the other.**

E.g. Consider a rectangle whose area is A and whose base and altitude are b and h respectively. Then, $A = h \cdot b$.

If now the base is multiplied by 2, 3, 4, etc., while the altitude is divided by 2, 3, 4, etc., then the area will remain constant. Hence, b and h may both *vary* while A remains *constant.*

The relation $b \cdot h = A$ may be written $b = A \cdot \frac{1}{h}$ or $h = A \cdot \frac{1}{b}$. It may also be written $b : \frac{1}{h} = A$ or $h : \frac{1}{b} = A$, so that the ratio of either b or h to the *reciprocal* of the other is the constant A. For this reason one is said to *vary inversely as the other.*

171. If $y = kx^2$, k being constant and x and y variables, then y varies **directly as x^2.** If $y = \frac{k}{x^2}$, then y varies **inversely as x^2.** If $y = k \cdot wx$, then y **varies jointly as w and x.** If $y \propto wx$, then $y \propto w$ if x is constant and $y \propto x$ if w is constant. If $y = k \cdot \frac{w}{x}$, then y **varies directly as w and inversely as x.**

Example. The resistance offered by a wire to an electric current varies directly as its length and inversely as the area of its cross section.

If a wire $\frac{1}{8}$ in. in diameter has a resistance of r units per mile, find the resistance of a wire $\frac{1}{4}$ in. in diameter and 3 miles long.

Solution. Let R represent the resistance of a wire of length l and cross-section area $s = \pi \cdot (\text{radius})^2$. Then $R = k \cdot \frac{l}{s}$ where k is some constant. Since $R = r$ when $l = 1$ and $s = \pi(\frac{1}{16})^2$, we have

$$r = k \cdot \frac{1}{\frac{\pi}{256}} \text{ or } k = \frac{\pi r}{256}$$

Hence, when $l = 3$ and $s = \pi(\frac{1}{8})^2$, we have,

$$R = \frac{\pi r}{256} \cdot \frac{3}{\frac{\pi}{64}} = \frac{3}{4} r.$$

That is, the resistance of three miles of the second wire is $\frac{3}{4}$ the resistance *per mile* of the first wire.

PROBLEMS

1. If $z \propto w$, and if $z = 27$ when $w = 3$, find the value of z when $w = 4\frac{1}{3}$.

2. If z varies jointly as w and x, and if $z = 24$ when $w = 2$ and $x = 3$, find z when $w = 3\frac{1}{2}$ and $x = 7$.

3. If z varies inversely as w, and if $z = 11$ when $w = 3$, find z when $w = 66$.

4. If z varies directly as w and inversely as x, and if $z = 28$ when $w = 14$ and $x = 2$, find z when $w = 42$ and $x = 3$.

5. If z varies inversely as the square of w, and if $z = 3$ when $w = 2$, find z when $w = 6$.

6. If q varies directly as m and inversely as the square of d, and $q = 30$ when $m = 1$ and $d = \frac{6}{100}$, find q when $m = 3$ and $d = 600$.

7. If $y^2 \propto x^3$, and if $y = 16$ when $x = 4$, find y when $x = 9$.

8. The weight of a triangle cut from a steel plate of uniform thickness varies jointly as its base and altitude. Find the base when the altitude is 4 and the weight 72, if it is known that the weight is 60 when the altitude is 5 and base 6.

9. The weight of a circular piece of steel cut from a sheet of uniform thickness varies as the square of its radius. Find the weight of a piece whose radius is 13 ft., if a piece of radius 7 feet weighs 196 pounds.

10. If a body starts falling from rest, its velocity varies directly as the number of seconds during which it has fallen. If the velocity at the end of 3 seconds is 96.6 feet per second, find its velocity at the end of 7 seconds; of ten seconds.

11. If a body starts falling from rest, the total distance fallen varies directly as the square of the time during which it has fallen. If in 2 seconds it falls 64.4 feet, how far will it fall in 5 seconds? In 9 seconds?

12. The number of vibrations per second of a pendulum varies inversely as the square root of the length. If a pendulum 39.1 inches long vibrates once in each second, how long is a pendulum which vibrates 3 times in each second?

13. Illuminating gas in cities is forced through the pipes by subjecting it to pressure in the storage tanks. It is found that the volume of gas varies inversely as the pressure. A certain body of gas occupies 49,000 cu. ft. when under a pressure of 2 pounds per square inch. What space would it occupy under a pressure of $2\frac{4}{5}$ pounds per square inch?

14. The amount of heat received from a stove varies inversely as the square of the distance from it. A person sitting 15 feet from the stove moves up to 5 feet from it. How much will this increase the amount of heat received?

15. The weights of bodies of the same shape and of the same material vary as the cubes of corresponding dimensions. If a ball $3\frac{1}{4}$ inches in diameter weighs 14 oz., how much will a ball of the same material weigh whose diameter is $3\frac{1}{2}$ inches?

16. On the principle of problem 15, if a man 5 feet 9 inches tall weighs 165 pounds, what should be the weight of a man of similar build 6 feet tall?

PROPORTION

172. Definitions. The four numbers a, b, c, d are said to be **proportional** or to **form a proportion** if the ratio of a to b is equal to the ratio of c to d. That is, if $\frac{a}{b}=\frac{c}{d}$. This is also sometimes written $a:b::c:d$, and is read a is to b as c is to d.

The four numbers are called the **terms** of the proportion; the first and fourth are the **extremes**; the second and third the **means** of the proportion. The first and third are the **antecedents** of the ratios, the second and fourth the **consequents.**

If a, b, c, x are proportional, x is called the **fourth proportional** to a, b, c. If a, x, x, b are proportional, x is called a **mean proportional** to a and b, and b a **third proportional** to a and x.

173. If four numbers are proportional when taken in a *given order,* there are other orders in which they are also proportional.

E.g. If a, b, c, d are proportional in this order, they are also proportional in the following orders: a, c, b, d; b, a, d, c; b, d, a, c; c, a, d, b; c, d, a, b; d, c, b, a; and d, b, c, a.

Ex. 1. Write in the form of an equation the proportion corresponding to each set of four numbers given above, and show how each may be derived from $\frac{a}{b}=\frac{c}{d}$. See § 196, E. C.

Show first how to derive $\frac{a}{c}=\frac{b}{d}$ (1), and then $\frac{b}{a}=\frac{d}{c}$ (2).

Derive also $\frac{a+b}{a}=\frac{c+d}{c}$ (3), and $\frac{a-b}{a}=\frac{c-d}{c}$ (4).

In (1) the original proportion is said to be taken by **alternation,** and in (2) by **inversion**; in (3) by **composition,** and in (4) by **division.**

Ex. 2. From $\frac{a}{b}=\frac{c}{d}$ and (1), (2), (3), (4) obtain the following. See pp. 279–281, E. C.

$$\frac{a\pm b}{b}=\frac{c\pm d}{d}, \qquad \frac{a\pm b}{c\pm d}=\frac{a}{c}, \qquad \frac{a\pm c}{b\pm d}=\frac{a}{b}, \qquad \frac{a\pm b}{c+d}=\frac{a-b}{c\mp d}.$$

When the double sign occurs, the *upper* signs are to be read together and the *lower* signs together.

EXERCISES

1. What principles in the transformation of equations are involved (*a*) in taking a proportion by inversion, (*b*) by alternation, (*c*) by composition, (*d*) by division, (*e*) by composition and division?

2. From $\frac{a}{b}=\frac{c}{d}$ derive $\frac{c+d}{a+b}=\frac{d}{b}$ and state all the principles involved.

3. From $a:b::c:d$ show that $a^2:b^2::c^2:d^2$.

4. From $a:b::c:d$ and $m:n::r:s$ show that $am:bn::cr:ds$.

5. From $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\cdots=\frac{a_n}{b_n}$ show that

$$\frac{a_1+a_2+\cdots+a_n}{b_1+b_2+\cdots+b_n}=\frac{a_1}{b_1}=\frac{a_2}{b_2}=\cdots=\frac{a_n}{b_n}.$$

Suggestion. Let $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\cdots=\frac{a_n}{b_n}=k$.

6. If $\frac{a}{b}=\frac{c}{d}$ show that $\frac{ma-b}{mc-d}=\frac{a}{c}$.

7. If $\frac{x}{y}=\frac{z}{w}$ show that $\frac{x+ky}{x-ky}=\frac{z+kw}{z-kw}$.

8. If $\frac{a}{b}=\frac{c}{d}$ show that $\frac{ma+nb}{ma-nb}=\frac{mc+nd}{mc-nd}$.

9. (*a*) Find a fourth proportional to 17, 19, and 187.
(*b*) Find a mean proportional between 6 and 54.
(*c*) Find a third proportional to 27 and 189.

10. Find the unknown term in each of the following proportions:

$x:42::27:126$; $78:x::13:3$; $99:117::x:39$; $171:27::57:x$.

11. If s_1 and s_2 are two distances passed over by a body falling from rest in the time intervals t_1 and t_2, then $\frac{s_1}{s_2}=\frac{t_1^2}{t_2^2}$ (see problem 11, p. 420). If it is known that a body falls 144.9 feet in 3 seconds, how far will it fall in 8 seconds?

12. When a weight is attached to a spring balance the index is displaced a distance proportional to the weight. Thus, if d_1 and d_2 are displacements and w_1, w_2 the corresponding weights, then $\frac{d_1}{d_2}=\frac{w_1}{w_2}$. If a 2-pound weight displaces the index $\frac{1}{4}$ inch, how much will a 50-pound weight displace it?

13. The intensity of light is *inversely* proportional to the square of the distance from the source of the light. That is, if i_1 and i_2 are the measures of intensities at the distances d_1 and d_2, then $\frac{i_1}{i_2}=\frac{d_2^2}{d_1^2}$. If the intensity of a given light at a distance of 2 feet is 20 candle power, find the intensity at 5 feet?

14. If w_1 and w_2 are weights resting on the two ends of a beam, and if the distances from the fulcrum are d_1 and d_2 respectively, then the beam will balance when $\frac{w_1}{w_2}=\frac{d_2}{d_1}$. That is, the weights are *inversely* proportional to the distances.

If a stone weighing 850 pounds at a distance of 1 foot from the fulcrum is to be balanced by a 50-pound weight, where should the weight be applied?

15. Find where the fulcrum should be in order that two boys weighing 110 and 80 pounds respectively may balance on the ends of a 16-foot plank.

16. The weight of a body above the earth's surface varies inversely as the square of its distance from the earth's center. If an object weighs 2000 pounds at the earth's surface, what would be its weight if it were 12,000 miles above the center of the earth, the radius of the earth being 4000 miles?

CHAPTER X

EXPONENTS AND RADICALS

FRACTIONAL AND NEGATIVE EXPONENTS

174. The meaning heretofore attached to the word *exponent* cannot apply to a fractional or negative number.

E.g. Such an exponent as $\frac{2}{3}$ or -5 cannot indicate the *number of times* a base is used as a factor.

It is possible, however, to interpret fractional and negative exponents in such a way *that the laws of operations which govern positive integral exponents shall apply to these also.*

175. The laws for positive integral exponents are:

I.	$a^m \cdot a^n = a^{m+n}$.	§ 43
II.	$a^m \div a^n = a^{m-n}$.	§ 46
III.	$(a^m)^n = a^{mn}$.	§ 115
IV.	$(a^m \cdot b^n)^p = a^{mp}b^{np}$.	§ 116
V.	$(a^m \div s^n)^p = a^{mp} \div s^{np}$.	§ 117

176. Assuming Law I to hold for positive fractional exponents and letting r and s be positive integers, we determine as follows the meaning of $b^{\frac{r}{s}}$ (read *b exponent r divided by s*).

By definition, $\left(b^{\frac{r}{s}}\right)^s = b^{\frac{r}{s}} \cdot b^{\frac{r}{s}} \cdots$ to s factors,

which by Law I $\qquad = b^{\frac{r}{s}+\frac{r}{s}+\cdots\text{ to } s \text{ terms}} = b^{\frac{r}{s}\cdot s} = b^r$.

Hence, $b^{\frac{r}{s}}$ is one of the s equal factors of b^r.

That is, $b^{\frac{r}{s}} = \sqrt[s]{b^r}$, and in particular $b^{\frac{1}{s}} = \sqrt[s]{b}$. See § 114

Similarly, from $\left(b^{\frac{1}{s}}\right)^r = b^{\frac{1}{s}} \cdot b^{\frac{1}{s}} \cdots$ to r factors, $= b^{\frac{r}{s}}$,

we show that $$b^{\frac{r}{s}} = \left(b^{\frac{1}{s}}\right)^r = (\sqrt[s]{b}\,)^r.$$

Hence, $$b^{\frac{r}{s}} = \sqrt[s]{b^r} = (\sqrt[s]{b}\,)^r.$$ See § 119

Thus a positive fractional exponent means *a root of a power or a power of a root, the numerator indicating the power and the denominator indicating the root.*

E.g. $a^{\frac{2}{3}} = \sqrt[3]{a^2} = (\sqrt[3]{a}\,)^2$; $8^{\frac{2}{3}} = \sqrt[3]{64} = 4$, or $(\sqrt[3]{8}\,)^2 = 2^2 = 4$.

177. Assuming Law I to hold also for negative exponents, and letting t be a positive number, integral or fractional, we determine as follows the meaning of b^{-t} (*read b exponent negative t*).

By Law I, $$b^t \cdot b^{-t} = b^0 = 1.$$ § 46

Therefore, $$b^{-t} = \frac{1}{b^t}.$$ § 11

Hence a number with a negative exponent means *the same as the reciprocal of the number with a positive exponent of the same absolute value.*

E.g. $a^{-2} = \dfrac{1}{a^2}$; $4^{-\frac{3}{2}} = \dfrac{1}{4^{\frac{3}{2}}} = \dfrac{1}{2^3} = \dfrac{1}{8}$.

178. It thus appears that fractional and negative exponents simply provide *new ways of indicating operations already well known.* Sometimes one notation is more convenient and sometimes the other.

Fractional and negative exponents are also called *powers.*

E.g. $x^{\frac{2}{3}}$ may be read *x to the* $\frac{2}{3}$ *power*, and x^{-4} may be read *x to the* $-$ *4th power.*

The limitations as to principal roots and the sign of the base, imposed in theorems on powers and roots in Chapter VI, necessarily apply to the corresponding theorems in this chapter. See §§ 114–122.

In any algebraic expression, radical signs may now be replaced by fractional exponents, or fractional exponents by radical signs.

In a fraction, any *factor* may be changed from numerator to denominator, or from denominator to numerator, by changing the sign of its exponent.

Ex. 1. $\sqrt[3]{x^2} + 3\sqrt[5]{x^3} \cdot \sqrt{y} + 5\sqrt[7]{x^4}\sqrt[3]{y^2} \equiv x^{\frac{2}{3}} + 3x^{\frac{3}{5}}y^{\frac{1}{2}} + 5x^{\frac{4}{7}}y^{\frac{2}{3}}.$

Ex. 2. $\frac{ab}{x^2} = abx^{-2}$, since $abx^{-2} = ab \cdot \frac{1}{x^2} = \frac{ab}{x^2}.$

Ex. 3. $ab^{-3}c^2 = ac^2 \cdot \frac{1}{b^3} = \frac{ac^2}{b^3}.$

Ex. 4. $32^{-\frac{4}{5}} = \frac{1}{32^{\frac{4}{5}}} = \frac{1}{(\sqrt[5]{32})^4} = \frac{1}{2^4} = \frac{1}{16}.$

EXERCISES

(*a*) In the expressions containing radicals on p. 436, replace these by fractional exponents.

(*b*) Replace all positive fractional exponents on this page by radicals.

(*c*) Change all expressions containing negative exponents to equivalent expressions having only positive exponents.

179. Fractional and negative exponents have been defined so as to conform to Law I, §§ 176, 177. We now show that when so defined they also conform to Laws II, III, IV, and V.

To verify Law II. Since by Law I, $a^{m-n} \cdot a^n = a^m$, for m and n integral or fractional, positive or negative, it follows by § 11 that $a^m \div a^n = a^{m-n}$ for all rational exponents.

To verify Law III. Let r and s be positive integers, and let k be any positive or negative integer or fraction. Then we have:

(1) $(a^k)^{\frac{r}{s}} = \sqrt[s]{(a^k)^r} = \sqrt[s]{a^{kr}} = a^{\frac{rk}{s}} = a^{\frac{r}{s}\cdot k}$, by §§ 176, 115.

(2) $(a^k)^{-\frac{r}{s}} = \frac{1}{(a^k)^{\frac{r}{s}}} = \frac{1}{a^{\frac{r}{s}\cdot k}} = a^{-\frac{r}{s}\cdot k}$, by § 177 and (1).

Hence $(a^k)^n = a^{nk}$ for all rational values of n and k.

To verify Law IV. Let m and n be positive or negative integers or fractions, and let r and s be positive integers, then we have

(1) $(a^m b^n)^{\frac{r}{s}} = \sqrt[s]{(a^m b^n)^r} = \sqrt[s]{a^{mr} b^{nr}}$, by §§ 176, 115,

$= \sqrt[s]{a^{mr}} \cdot \sqrt[s]{b^{nr}} = a^{\frac{r}{s}\cdot m} \cdot b^{\frac{r}{s}\cdot n}$, by §§ 120, 176.

(2) $(a^m b^n)^{-\frac{r}{s}} = \dfrac{1}{(a^m b^n)^{\frac{r}{s}}} = \dfrac{1}{a^{\frac{r}{s}\cdot m} \cdot b^{\frac{r}{s}\cdot n}} = a^{-\frac{r}{s}\cdot m} \cdot b^{-\frac{r}{s}\cdot n}$, by § 177 and (1).

Hence $(a^m b^n)^p = a^{pm} b^{pn}$ for all rational values of m, n, and p.

To verify Law V. We have $\left(\dfrac{a^m}{b^n}\right)^p = \dfrac{a^{mp}}{b^{np}}$ for all rational values of m, n, and p, since

$$\left(\frac{a^m}{b^n}\right)^p = (a^m \cdot b^{-n})^p = a^{mp} \cdot b^{-np} = \frac{a^{mp}}{b^{np}}, \quad \text{§ 177 and Law IV.}$$

180. From Laws III, IV, and V, it follows that any monomial is affected with any exponent by multiplying the exponent of each factor of the monomial by the given exponent.

Ex. 1. $(a^{\frac{1}{2}} b^{-2} c^3)^{-\frac{2}{3}} = a^{-\frac{2}{3}\cdot\frac{1}{2}} b^{-\frac{2}{3}\cdot -2} c^{-\frac{2}{3}\cdot 3} = a^{-\frac{1}{3}} b^{\frac{4}{3}} c^{-2}$.

Ex. 2. $\left(\dfrac{3a^2x^6}{by^4}\right)^{-\frac{1}{2}} = \dfrac{3^{-\frac{1}{2}} a^{-1} x^{-3}}{b^{-\frac{1}{2}} y^{-2}} = \dfrac{b^{\frac{1}{2}} y^2}{3^{\frac{1}{2}} a x^3}$.

Ex. 3. $\left(\dfrac{8x^9}{27y^6}\right)^{-\frac{1}{3}} = \left(\dfrac{27y^6}{8x^9}\right)^{\frac{1}{3}} = \dfrac{27^{\frac{1}{3}} y^2}{8^{\frac{1}{3}} x^3} = \dfrac{3y^2}{2x^3}$.

EXERCISES

Perform the operations indicated by the exponents in each of the following, writing the results without negative exponents and in as simple form as possible:

1. $(\frac{16}{25})^{-\frac{1}{2}}$.
2. $(\frac{27}{64})^{-\frac{1}{3}}$.
3. $(\frac{25}{36})^{\frac{3}{2}}$.
4. $(27a^{-9})^{\frac{1}{3}}$.
5. $(x^{-\frac{2}{3}} y^{\frac{4}{3}})^{-\frac{3}{4}}$.
6. $25^{\frac{3}{2}}$.
7. $25^{-\frac{3}{2}}$.
8. 25^0.
9. $(\frac{27}{8})^{-\frac{2}{3}}$.
10. $(\frac{8}{27})^{\frac{2}{3}}$.
11. $(0.25)^{\frac{1}{2}}$.
12. $(0.25)^{-\frac{1}{2}}$.
13. $(.0009)^{\frac{3}{2}}$.
14. $(.027)^{\frac{1}{3}}$.
15. $(32a^{-5}b^{10})^{\frac{1}{5}}$.
16. $8^{\frac{1}{3}} \cdot 4^{-\frac{1}{2}}$.

17. $\left(\frac{a^{-8}}{16}\right)^{-\frac{1}{4}}$.

18. $(27\,x^6y^{-3}z^{-1})^{-\frac{1}{3}}$.

19. $(\frac{1}{32})^{-\frac{1}{5}}(\frac{1}{81})^{-\frac{3}{4}}$.

20. $\sqrt[3]{\frac{512}{729}}\cdot(\frac{256}{729})^{-\frac{1}{2}}$.

21. $(-\frac{243}{32})^{\frac{1}{5}}\div(\frac{16}{81})^{-\frac{1}{4}}$.

22. $\left(\frac{x^3y^{-4}}{x^{-2}y}\right)^3\left(\frac{x^{-3}y^2}{xy^{-1}}\right)^5$.

23. $\sqrt[4]{81\,a^{-4}b^8}(-27\,a^3b^{-6})^{-\frac{1}{3}}$.

24. $\left(\frac{m^{-1}n}{m^{\frac{1}{2}}n^{-\frac{2}{3}}}\right)^{-2}\div\left(\frac{m^{-3}}{n^{-1}}\right)^{-\frac{1}{3}}$.

25. $\left(\frac{a^3b^{-2}}{a^{-2}b^3}\right)^{\frac{1}{3}}\div\left(\frac{a^3b^{-3}}{a^{-3}b^3}\right)^{-1}$.

26. $\sqrt{16\,a^{-4}b^{-6}}\cdot\sqrt[3]{8\,a^3b^{-6}}$.

27. $(-2^{-2}a^{-3}b^{-6})^{-\frac{1}{3}}(-2^{-\frac{1}{3}}a^{-\frac{1}{2}}b^{-1})^2$.

28. $\left(\frac{81\,r^{-16}s^4t}{625\,r^4s^8t}\right)^{\frac{1}{4}}\left(\frac{9\,r^2s^{-4}t}{25\,r^{10}s^0t}\right)^{-\frac{1}{2}}$.

29. Prove Law III in detail for the following cases:

$$(1)\ (a^{\frac{m}{n}})^{-\frac{r}{s}},\quad (2)\ (a^{-\frac{m}{n}})^{\frac{r}{s}},\quad (3)\ (a^{-\frac{m}{n}})^{-\frac{r}{s}}.$$

30. Prove Law IV in detail for the following cases:

$$(1)\ (a^{-k}b^{-l})^{\frac{r}{s}},\quad (2)\ (a^{-k}b^{-l})^{-\frac{r}{s}}.$$

Multiply:

31. $x^{-2}+x^{-1}y^{-1}+y^{-2}$ by $x^{-1}-y^{-1}$.

32. $x^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}$ by $x^{\frac{1}{3}}+y^{\frac{1}{3}}$.

33. $x^{\frac{4}{5}}+x^{\frac{3}{5}}y^{\frac{1}{5}}+x^{\frac{2}{5}}y^{\frac{2}{5}}+x^{\frac{1}{5}}y^{\frac{8}{5}}+y^{\frac{4}{5}}$ by $x^{\frac{1}{5}}-y^{\frac{1}{5}}$.

34. $\sqrt[4]{a^3}+\sqrt[5]{b^2}$ by $\sqrt[4]{a^3}-\sqrt[5]{b^2}$.

35. $x^{\frac{4}{3}}+x^{\frac{2}{3}}y^{\frac{2}{3}}+y^{\frac{4}{3}}$ by $x^{\frac{2}{3}}-y^{\frac{2}{3}}$.

36. $x-3\,x^{\frac{2}{3}}y^{-\frac{1}{2}}+3\,x^{\frac{1}{3}}y^{-1}-y^{-\frac{3}{2}}$ by $x^{\frac{2}{3}}-2\,x^{\frac{1}{3}}y^{-\frac{1}{2}}+y^{-1}$.

37. $x^{\frac{3}{2}}+xy^{\frac{3}{4}}+x^{\frac{1}{2}}y^{\frac{3}{2}}+y^{\frac{9}{4}}$ by $x^{\frac{1}{2}}-y^{\frac{3}{4}}$.

Divide:

38. $x^2-x^{\frac{11}{6}}y+x^{\frac{3}{2}}y-x^{\frac{1}{2}}y^{\frac{1}{3}}+x^{\frac{1}{3}}y^{\frac{4}{3}}-y^{\frac{4}{3}}$ by $x^{\frac{1}{2}}-x^{\frac{1}{3}}y+y$.

39. $3\,a^{\frac{7}{4}}-ab^{\frac{2}{3}}+4\,ab^2-3\,a^{\frac{3}{4}}b+b^{\frac{5}{3}}-4\,b^3$ by $3\,a^{\frac{3}{4}}-b^{\frac{2}{3}}+4\,b^2$.

40. $x^2-3\,x^{\frac{5}{3}}+6\,x^{\frac{4}{3}}-7\,x+6\,x^{\frac{2}{3}}-3\,x^{\frac{1}{3}}+1$ by $x^{\frac{2}{3}}-x^{\frac{1}{3}}+1$.

41. $4\,x^{\frac{5}{4}}b^{-2}-17\,x^{\frac{1}{4}}b^2+16\,x^{-\frac{3}{4}}b^6$ by $2\,x^{\frac{1}{2}}-b^2-4\,x^{-\frac{1}{2}}b^4$.

Find the square root of:

42. $4x^2 - 4xy^{\frac{1}{3}} + 4xz^{-\frac{1}{2}} + y^{\frac{2}{3}} - 2y^{\frac{1}{3}}z^{-\frac{1}{2}} + z^{-1}$.

43. $a^{-\frac{2}{3}} - 2a^{-\frac{1}{3}}b^{\frac{2}{3}} + b^{\frac{4}{3}} + 2a^{-\frac{1}{3}}c^2 + c^4 - 2b^{\frac{2}{3}}c^2$.

44. $b^{-\frac{1}{2}} - 2b^{-\frac{1}{4}}c^{\frac{1}{3}} + c^{\frac{2}{3}} + 2b^{-\frac{1}{4}}d^{\frac{1}{5}} + 2b^{-\frac{1}{4}}e^{-\frac{1}{2}} - 2c^{\frac{1}{3}}d^{\frac{1}{5}} + d^{\frac{2}{5}}$
$+ 2d^{\frac{1}{5}}e^{-\frac{1}{2}} - 2c^{\frac{1}{3}}e^{-\frac{1}{2}} + e^{-1}$.

Find the cube root of:

45. $\frac{1}{8}a^3 - \frac{3}{2}a^2b^{\frac{1}{2}} + 6ab - 8b^{\frac{3}{2}}$.

46. $a^6 - 3a^5 + 5a^3 - 3a - 1$.

47. $a^{-1} + 3a^{-\frac{2}{3}}b^{\frac{2}{5}} + 3a^{-\frac{1}{3}}b^{\frac{4}{5}} + b^{\frac{6}{5}}$.

REDUCTION OF RADICAL EXPRESSIONS

181. An expression containing a root indicated by the radical sign or by a fractional exponent is called a **radical expression.** The expression whose root is indicated is the **radicand.**

E.g. $\sqrt[3]{5}$ and $(1+x)^{\frac{2}{3}}$ are radical expressions. In each case the **index** of the radical is 3.

The reduction of a radical expression consists in *changing its form without changing its value.*

Each reduction is based upon one or more of the Laws I to V, § 175, as extended in § 179.

182. To remove a factor from the radicand. This reduction is possible only when the radicand contains a factor which is a perfect power of the degree indicated by the index of the root, as shown in the following examples:

Ex. 1. $\sqrt{72} = \sqrt{36 \cdot 2} = \sqrt{36}\sqrt{2} = 6\sqrt{2}$.

Ex. 2. $(a^3x^2y^6)^{\frac{1}{3}} = (a^3y^6 \cdot x^2)^{\frac{1}{3}} = (a^3y^6)^{\frac{1}{3}} \cdot (x^2)^{\frac{1}{3}} = ay^2x^{\frac{2}{3}}$.

This reduction involves Law IV, and may be written in symbols thus:

$$\sqrt[r]{x^{kr}y} = \sqrt[r]{x^{kr}}\sqrt[r]{y} = x^k\sqrt[r]{y}.$$

EXERCISES

In the expressions on p. 436, remove factors from the radicands where possible.

In the case of negative fractional exponents, first reduce to equivalent expressions containing only positive exponents.

183. To introduce a factor into the radicand. This process simply retraces the steps of the foregoing reduction, and hence also involves Law IV.

Ex. 1. $6\sqrt{2} = \sqrt{6^2} \cdot \sqrt{2} = \sqrt{36 \cdot 2} = \sqrt{72}.$ See § 112

Ex. 2. $ay^2x^{\frac{2}{3}} = \sqrt[3]{(ay^2)^3} \cdot \sqrt[3]{x^2} = \sqrt[3]{(ay^2)^3x^2} = \sqrt[3]{a^3y^6x^2}.$

Ex. 3. $x\sqrt[r]{y} = \sqrt[r]{x^r}\,\sqrt[r]{y} = \sqrt[r]{x^ry}.$

EXERCISES

In the expressions on p. 436, introduce into the radicand any factor which appears as a coefficient of a radical.

184. To reduce a fractional radicand to the integral form. This reduction involves Law IV or Law V, and may always be accomplished.

Ex. 1. $\sqrt{\frac{3}{5}} = \sqrt{\frac{15}{25}} = \sqrt{\frac{1}{25} \cdot 15} = \frac{1}{5}\sqrt{15}.$ Law IV

Ex. 2. $\left(\frac{a-b}{a+b}\right)^{\frac{1}{2}} = \left(\frac{a^2-b^2}{(a+b)^2}\right)^{\frac{1}{2}} = \frac{(a^2-b^2)^{\frac{1}{2}}}{[(a+b)^2]^{\frac{1}{2}}} = \frac{(a^2-b^2)^{\frac{1}{2}}}{a+b}.$ Law V

Ex. 3. $\frac{\sqrt[3]{40}}{\sqrt[3]{5}} = \sqrt[3]{\frac{40}{5}} = \sqrt[3]{8} = 2.$

In symbols, we have $\sqrt[r]{\frac{a}{b}} = \sqrt[r]{\frac{ab^{r-1}}{b^r}} = \frac{\sqrt[r]{ab^{r-1}}}{\sqrt[r]{b^r}} = \frac{1}{b}\sqrt[r]{ab^{r-1}}.$

EXERCISES

In the expressions on p. 436, reduce each fractional radicand to the integral form.

In case negative exponents are involved, first reduce to equivalent expressions containing only positive exponents.

185. To reduce a radical to an equivalent radical of lower index. This reduction is effective when the radicand is a perfect power corresponding to *some factor of the index.*

Ex. 1. $\sqrt[6]{8}=8^{\frac{1}{6}}=8^{\left(\frac{1}{3\cdot 2}\right)}=(8^{\frac{1}{3}})^{\frac{1}{2}}=2^{\frac{1}{2}}=\sqrt{2}.$

Ex. 2. $\sqrt[4]{a^2+2ab+b^2}=\sqrt{\sqrt{a^2+2ab+b^2}}=\sqrt{a+b}.$

This reduction involves Law III as follows:

$$(x^{\frac{1}{r}})^{\frac{1}{s}}=(x^{\frac{1}{s}})^{\frac{1}{r}}=x^{\frac{1}{rs}},$$

from which we have

$$\sqrt[rs]{x}=\sqrt[r]{\sqrt[s]{x}}=\sqrt[s]{\sqrt[r]{x}}.\qquad \text{See § 114}$$

By this reduction a root whose index is a composite number is made to depend upon roots of lower degree.

E.g. A fourth root may be found by taking the square root twice; a sixth root, by taking a square root and then a cube root, etc. In the case of *literal* expressions this can be done only when the radicand is a perfect power of the degree indicated by the index of the root.

But when the radicand is expressed in Arabic figures, such roots may in any case be approximated as in § 127.

EXERCISES

In the expressions on p. 436, make the reduction above indicated where possible.

In the case of arithmetic radicands, approximate to two places of decimals such roots as can be made to depend upon square and cube roots.

186. To reduce a radical to an equivalent radical of higher index. This reduction is possible whenever the required index is a *multiple* of the given index. It is based on Law III as follows:

$$x^{\frac{r}{s}}=(x^{\frac{r}{s}})^{\frac{t}{t}}=x^{\frac{rt}{st}}.\qquad \text{See § 179}$$

Ex. 1. $\sqrt{a}=a^{\frac{1}{2}}=(a^{\frac{1}{2}})^{\frac{3}{3}}=a^{\frac{3}{6}}=\sqrt[6]{a^3}.$

Ex. 2. $\sqrt[3]{b}=b^{\frac{1}{3}}=b^{\frac{2}{6}}=\sqrt[6]{b^2}.$

Definition. Two radical expressions are said to be of the **same order** when their indicated roots have the *same index.*

By the above reduction two radicals of *different* orders may be changed to equivalent radicals of the *same* order, namely, a common multiple of the given indices.

E.g. $\sqrt{a}$ and $\sqrt[3]{b}$ in Exs. 1 and 2 above.

EXERCISES

In Exs. 3, 4, 6, 17, 18, 23, 28, 30, on p. 436, reduce the corresponding expressions in the first and second columns to equivalent radicals of the same order.

187. In general, radical expressions should be at once reduced so that the *order is as low* as possible and the *radicand is integral and as small* as possible. A radical is then said to be in its **simplest form.**

ADDITION AND SUBTRACTION OF RADICALS

188. Definition. Two or more radical expressions are said to be **similar** when they are of the same order and have the same radicands.

E.g. $3\sqrt{7}$ and $5\sqrt{7}$ are similar radicals as are also $a\sqrt[7]{x^4}$ and $bx^{\frac{4}{7}}$.

If two radicals can be reduced to similar radicals, they may be added or subtracted according to § 10.

Ex. 1. Find the sum of $\sqrt{8}$, $\sqrt{50}$, and $\sqrt{98}$.

By § 182, $\sqrt{8} = 2\sqrt{2}$, $\sqrt{50} = 5\sqrt{2}$, and $\sqrt{98} = 7\sqrt{2}$.

Hence $\sqrt{8} + \sqrt{50} + \sqrt{98} = 2\sqrt{2} + 5\sqrt{2} + 7\sqrt{2} = 14\sqrt{2}$.

Ex. 2. Simplify $\sqrt{\frac{1}{5}} - \sqrt{20} + \sqrt{3\frac{1}{5}}$.

By § 184, $\sqrt{\frac{1}{5}} = \frac{1}{5}\sqrt{5}$, $\sqrt{20} = 2\sqrt{5}$, $\sqrt{3\frac{1}{5}} = \sqrt{\frac{16}{5}} = 4\sqrt{\frac{1}{5}} = \frac{4}{5}\sqrt{5}$.

Hence $\sqrt{\frac{1}{5}} - \sqrt{20} + \sqrt{3\frac{1}{5}} = \frac{1}{5}\sqrt{5} - 2\sqrt{5} + \frac{4}{5}\sqrt{5} = -\sqrt{5}$.

If two radicals cannot be reduced to equivalent similar radicals, their sum can only be indicated.

E.g. The sum of $\sqrt{2}$ and $\sqrt[3]{5}$ is $\sqrt{2}+\sqrt[3]{5}$.

Observe, however, that

$$\sqrt{10}+\sqrt{6}=\sqrt{2}\cdot\sqrt{5}+\sqrt{2}\cdot\sqrt{3}=\sqrt{2}\,(\sqrt{5}+\sqrt{3}).$$

EXERCISES

(*a*) In Exs. 1, 2, 5, 7, 8, 19, 20, 21, p. 436, reduce each pair so as to involve similar radicals and add them.

(*b*) Perform the following indicated operations:

1. $\sqrt{28}+3\sqrt{7}-2\sqrt{63}$.
2. $\sqrt[3]{24}-\sqrt[3]{81}-\sqrt[3]{\tfrac{3}{125}}$.
3. $\sqrt[5]{a^6}+\sqrt[5]{a^{11}}-\sqrt[5]{32a}$.
4. $2\sqrt{48}-3\sqrt{12}+3\sqrt{\tfrac{1}{3}}$.
5. $\sqrt{\tfrac{1}{7}}+\sqrt{63}+5\sqrt{7}$.
6. $\sqrt{99}-11\sqrt{\tfrac{1}{11}}+\sqrt{44}$.
7. $2\sqrt{\tfrac{4}{7}}+3\sqrt{\tfrac{9}{7}}+\sqrt{175}$.
8. $\sqrt[4]{\tfrac{9}{625}}+6\sqrt{\tfrac{1}{3}}-\sqrt{12}$.
9. $\sqrt[6]{9}+\sqrt[9]{27}+\sqrt[3]{-24}$.
10. $(x^2+1)\sqrt{a^3+a^2b}-\sqrt{(a^2-b^2)(a-b)}$.

MULTIPLICATION OF RADICALS

189. Radicals of the *same order* are multiplied according to § 120 by multiplying the radicands. If they are not of the same order, they may be reduced to the same order according to § 186.

E.g. $\sqrt{a}\cdot\sqrt[3]{b}=a^{\frac{1}{2}}b^{\frac{1}{3}}=a^{\frac{3}{6}}b^{\frac{2}{6}}=\sqrt[6]{a^3}\cdot\sqrt[6]{b^2}=\sqrt[6]{a^3b^2}$.

In many cases this reduction is not desirable. Thus, $\sqrt[3]{x^2}\cdot\sqrt{y^3}$ is written $x^{\frac{2}{3}}y^{\frac{3}{2}}$ rather than $\sqrt[6]{x^4y^9}$.

Radicals are multiplied by adding exponents when they are reduced to the *same base* with fractional exponents, § 176.

E.g. $\sqrt[5]{x^2}\cdot\sqrt{x^3}=x^{\frac{2}{5}}\cdot x^{\frac{3}{2}}=x^{\frac{2}{5}+\frac{3}{2}}=x^{\frac{19}{10}}$.

190. The principles just enumerated are used in connection with § 10 in multiplying polynomials containing radicals.

Ex. 1.
$$\begin{array}{llll} 3\sqrt{2} & +2\sqrt{5} & & \\ 2\sqrt{2} & -3\sqrt{5} & & \\ \hline 6\cdot 2 & +4\sqrt{10} & & \\ & -9\sqrt{10} & -6\cdot 5 & \\ \hline 12 & -5\sqrt{10} & -30 & \end{array}$$

Ex. 2.
$$\begin{array}{llll} 3\sqrt{2} & +2\sqrt{5} & & \\ 3\sqrt{2} & -2\sqrt{5} & & \\ \hline 9\cdot 2 & +6\sqrt{10} & & \\ & -6\sqrt{10} & -4\cdot 5 & \\ \hline 18 & & -20 & \end{array}$$

Hence $(2\sqrt{2}+2\sqrt{5})(2\sqrt{2}-3\sqrt{5}) = -18-5\sqrt{10}$,

and $(3\sqrt{2}+2\sqrt{5})(3\sqrt{2}-2\sqrt{5}) = 18-20 = -2$.

EXERCISES

(*a*) In Exs. 21 to 38, p. 436, find the products of the corresponding expressions in the two columns.

(*b*) Find the following products:

1. $(3+\sqrt{11})(3-\sqrt{11})$.
2. $(3\sqrt{2}+4\sqrt{5})(4\sqrt{2}-5\sqrt{5})$.
3. $(2+\sqrt{3}+\sqrt{5})(3+\sqrt{3}-\sqrt{5})$.
4. $(3\sqrt{2}-2\sqrt{18}+2\sqrt{7})(2\sqrt{2}-\sqrt{18}-\sqrt{7})$.
5. $(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})(a^2+ab+b^2)$.
6. $(\sqrt{\sqrt{13}+3})(\sqrt{\sqrt{13}-3})$.
7. $(\sqrt{2+3\sqrt{5}})(\sqrt{2+3\sqrt{5}})$.
8. $(3a-2\sqrt{x})(4a+3\sqrt{x})$.
9. $(3\sqrt{3}+2\sqrt{6}-4\sqrt{8})(3\sqrt{3}-2\sqrt{6}+4\sqrt{8})$.
10. $(\sqrt{a}+\sqrt{b}-\sqrt{c})(\sqrt{a}-\sqrt{b}+\sqrt{c})$.

11. $(a - \sqrt{b} - \sqrt{c})(a + \sqrt{b} + \sqrt{c})$.

12. $(2\sqrt{\frac{2}{3}} + 3\sqrt{\frac{3}{6}} + 4\sqrt{\frac{3}{2}})(2\sqrt{\frac{2}{3}} - 5\sqrt{\frac{3}{6}})$.

13. $(\sqrt[3]{a^2} + \sqrt[3]{b^2})(\sqrt[6]{a^2} + \sqrt[3]{a^2b^2} + \sqrt[9]{b^3})$.

14. $(\sqrt[4]{x^3} - y^{3})^3$.

DIVISION OF RADICALS

191. Radicals are divided in accordance with Laws II and V. That is, the exponents are *subtracted* when the *bases* are the *same*, and the bases are *divided* when the *exponents* are the *same*. See §§ 179, 121.

Ex. 1. $\sqrt[5]{x^2} \div \sqrt{x^3} = x^{\frac{2}{5}} \div x^{\frac{3}{2}} = x^{\frac{2}{5}-\frac{3}{2}} = x^{-\frac{11}{10}}$.

Ex. 2. $x^{\frac{2}{3}} \div y^{\frac{2}{3}} = \left(\frac{x}{y}\right)^{\frac{2}{3}} = (xy^{-1})^{\frac{2}{3}} = \sqrt[3]{x^2y^{-2}}$.

Ex. 3. $\sqrt{a} \div \sqrt[3]{b} = a^{\frac{3}{6}} \div b^{\frac{2}{6}} = \left(\frac{a^3}{b^2}\right)^{\frac{1}{6}} = \sqrt[6]{a^3b^{-2}}$.

EXERCISES

(*a*) In each of the Exs. 1 to 20, on p. 436, divide the expression in the first column by that in the second.

(*b*) Perform the following divisions:

1. $(\sqrt{a^5} + 2\sqrt{a^5} - 3\sqrt{a}) \div 6\sqrt{a}$.

2. $(\sqrt{a} + \sqrt[3]{b} - c) \div \sqrt{c}$.

3. $(2\sqrt[3]{9} + 3\sqrt[3]{12} - 4\sqrt[3]{15}) \div \sqrt[3]{3}$.

4. $(4\sqrt[5]{7} - 8\sqrt[5]{21} + 6\sqrt[5]{42}) \div 2\sqrt[5]{7}$.

EXERCISES

1. $3\sqrt{45}$, $\quad 2\sqrt{125}$.
2. $a^{\frac{2}{3}}$, $\quad a^{\frac{5}{3}}$.
3. $x^{\frac{2}{5}}$, $\quad x^{\frac{5}{3}}$.
4. $3\sqrt{x^2y}$, $\quad 2\sqrt[3]{x^4}$.
5. $d\sqrt[3]{a^2b^5}$, $\quad c\sqrt[3]{a^5b^2}$.
6. $7\sqrt{(a+b)^3}$, $\quad 11\sqrt[3]{(a-b)^6}$.
7. $\sqrt[3]{a^4}$, $\quad a^{\frac{7}{3}}$.
8. $n\sqrt{m^5}$, $\quad m^{\frac{3}{2}}$.
9. $\sqrt{\frac{7}{2}}$, $\quad \sqrt{\frac{8}{7}}$.
10. $\sqrt{\frac{11}{72}}$, $\quad \sqrt{\frac{23}{54}}$.
11. $\sqrt{\frac{36}{27}}$, $\quad \sqrt{\frac{13}{18}}$.
12. $\dfrac{1}{b^{-\frac{1}{2}}}$, $\quad r\sqrt{b^3}$.
13. $a^{-2}b^3$, $\quad \dfrac{a^3}{b^4}$.
14. $\dfrac{3}{m^{-7}}$, $\quad \dfrac{n^{-\frac{1}{3}}m^{-1}}{4}$.
15. $\dfrac{3a^{-\frac{3}{2}}}{c^{-2}b^3}$, $\quad \dfrac{2a^4b^{-2}}{c^3}$.
16. $\sqrt[3]{\frac{3}{4}}$, $\quad \sqrt[3]{\frac{1}{4}}$.
17. $\sqrt[4]{\frac{3}{8}}$, $\quad \sqrt[6]{\frac{2}{9}}$.
18. $\sqrt[3]{\frac{3}{64}}$, $\quad \sqrt[6]{\frac{2}{7}}$.
19. $18^{\frac{1}{2}}$, $\quad \sqrt{32}$.
20. $\sqrt{12}$, $\quad 48^{\frac{1}{2}}$.
21. $3(50)^{\frac{1}{2}}$, $\quad 4(72)^{\frac{1}{2}}$.
22. $a^{\frac{3}{4}}$, $\quad a^{\frac{5}{3}}$.
23. $ax^{\frac{7}{2}}$, $\quad bx^{\frac{3}{4}}$.
24. $a^{\frac{2}{3}}b^{\frac{7}{5}}$, $\quad a^{\frac{3}{2}}b^{\frac{5}{7}}$.
25. $m^{\frac{3}{4}}n^{\frac{4}{5}}l^{\frac{7}{8}}$, $\quad m^{\frac{3}{2}}n^{\frac{2}{3}}l^{\frac{4}{5}}$.
26. $5\sqrt[5]{a^6b^4c^9}$, $\quad 3\sqrt[5]{a^3b^7c^{13}}$.
27. $\sqrt[2]{a^3}\sqrt[4]{b^{10}}$, $\quad \sqrt[3]{b^9}\sqrt[4]{a^6}$.
28. $8^{\frac{2}{3}}$, $\quad 16^{\frac{3}{2}}$.
29. $25^{-\frac{1}{2}}$, $\quad 125^{-\frac{4}{3}}$.
30. $9^{\frac{3}{2}}$, $\quad 8^{\frac{2}{3}}$.
31. $(\frac{1}{9})^{\frac{1}{3}}$, $\quad (\frac{1}{27})^{-\frac{1}{2}}$.
32. $\dfrac{5x^3y^5}{m^{-3}n^{-5}}$, $\quad \dfrac{x^3y^5z^2}{m^{-1}n^{-2}l^{-3}}$.
33. $\dfrac{3a^{-\frac{3}{2}}b^{\frac{3}{2}}}{4a^{-\frac{1}{2}}b^{-\frac{2}{3}}}$, $\quad \dfrac{4a^{-\frac{5}{8}}b^{-\frac{3}{2}}}{5a^{-\frac{1}{2}}b^{-\frac{1}{3}}}$.
34. $\dfrac{c^{-\frac{1}{2}}d^{-\frac{1}{3}}}{d^{-\frac{1}{4}}c^{-\frac{1}{5}}}$, $\quad \dfrac{d^{\frac{3}{4}}c^0}{d^{-\frac{1}{4}}c^{-1}}$.
35. $5(a+b)^{-\frac{3}{2}}$, $\quad 3(a+b)^{-\frac{2}{3}}$.
36. $(-64)^{\frac{5}{6}}$, $\quad -64^{\frac{5}{6}}$.
37. $(\frac{1}{27})^{\frac{1}{3}}$, $\quad (\frac{1}{16})^{-\frac{3}{4}}$.
38. $\dfrac{1}{b}\sqrt[3]{\dfrac{a^3}{b}}$, $\quad \frac{1}{3}\sqrt[3]{\frac{27}{5}}$.
39. $\sqrt[3]{4a^3-12a^2b+12ab^2-4b^3}$.
40. $\sqrt{(3a-2b)(9a^2-4b^2)}$.

192. Rationalizing the divisor. In case division by a radical expression cannot be carried out as in the foregoing examples, it is desirable to *rationalize* the denominator when possible.

Ex. 1. $\dfrac{\sqrt{2}}{\sqrt{5}}=\dfrac{\sqrt{2}\cdot\sqrt{5}}{\sqrt{5}\cdot\sqrt{5}}=\dfrac{\sqrt{10}}{5}.$

Ex. 2. $\dfrac{\sqrt{a}}{\sqrt{a}-\sqrt{b}}=\dfrac{\sqrt{a}(\sqrt{a}+\sqrt{b})}{(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})}=\dfrac{a+\sqrt{ab}}{a-b}.$

Evidently this is always possible when the divisor is a *monomial* or *binomial* radical expression of the second order.

The number by which numerator and denominator are multiplied is called the *rationalizing factor.*

For a monomial divisor, $\sqrt{x}$, it is $\sqrt{x}$ itself. For a binomial divisor, $\sqrt{x}\pm\sqrt{y}$, it is the same binomial with the opposite sign, $\sqrt{x}\mp\sqrt{y}$.

EXERCISES

Reduce each of the following to equivalent fractions having a rational denominator.

1. $\dfrac{3}{2-\sqrt{5}}.$

2. $\dfrac{7}{\sqrt{5}+\sqrt{3}}.$

3. $\dfrac{\sqrt{27}}{\sqrt{3}+\sqrt{11}}.$

4. $\dfrac{2-\sqrt{7}}{2+\sqrt{7}}.$

5. $\dfrac{\sqrt{2}-\sqrt{3}}{\sqrt{2}+\sqrt{3}}.$

6. $\dfrac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}.$

7. $\dfrac{3\sqrt{3}-2\sqrt{2}}{3\sqrt{3}+2\sqrt{2}}.$

8. $\dfrac{\sqrt{a^2+1}-\sqrt{a^2-1}}{\sqrt{a^2-1}+\sqrt{a^2+1}}.$

9. $\dfrac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}.$

10. $\dfrac{\sqrt{a-b}-\sqrt{a+b}}{\sqrt{a-b}+\sqrt{a+b}}.$

193. In finding the value of such an expression as $\frac{\sqrt{7}+\sqrt{3}}{\sqrt{7}-\sqrt{3}}$, the approximation of *two* square roots and division by a decimal fraction would be required. But $\frac{\sqrt{7}+\sqrt{3}}{\sqrt{7}-\sqrt{3}}$ equals $\frac{11+2\sqrt{21}}{4}$ which requires only *one* root and division by the integer 4.

EXERCISES

Find the approximate values of the following expressions to three places of decimals.

1. $\frac{3\sqrt{5}+4\sqrt{3}}{\sqrt{5}-\sqrt{3}}$.

2. $\frac{\sqrt{7}}{\sqrt{7}-\sqrt{2}}$.

3. $\frac{4\sqrt{3}}{\sqrt{3}-\sqrt{2}}$.

4. $\frac{11\sqrt{5}-3\sqrt{3}}{2\sqrt{5}+\sqrt{3}}$.

5. $\frac{7\sqrt{5}+3\sqrt{8}}{2\sqrt{5}-3\sqrt{2}}$.

6. $\frac{5\sqrt{19}-3\sqrt{7}}{3\sqrt{7}-\sqrt{19}}$.

7. $\frac{3\sqrt{2}-\sqrt{5}}{\sqrt{5}-6\sqrt{2}}$.

8. $\frac{5\sqrt{6}-7\sqrt{13}}{3\sqrt{13}-7\sqrt{6}}$.

194. Square root of a radical expression. A radical expression of the second order is sometimes a perfect square, and its **square root** may be written by inspection.

E.g. The square of $\sqrt{a} \pm \sqrt{b}$ is $a + b \pm 2\sqrt{ab}$. Hence if a radical expression can be put into the form $x \pm 2\sqrt{y}$, where x is the *sum* of two numbers a and b whose *product* is y, then $\sqrt{a} \pm \sqrt{b}$ is the *square root* of $x + 2\sqrt{y}$.

Example. Find the square root of $5 + \sqrt{24}$.

Since $5 + \sqrt{24} = 5 + 2\sqrt{6}$, in which 5 is the *sum* of 2 and 3, and 6 is their *product*, we have $\sqrt{5+\sqrt{24}} = \sqrt{2} + \sqrt{3}$.

EXERCISES

Find the square root of each of the following:

1. $3 - 2\sqrt{2}$.
2. $7 + \sqrt{40}$.
3. $8 - \sqrt{60}$.
4. $7 + 4\sqrt{3}$.
5. $24 - 6\sqrt{7}$.
6. $28 + 3\sqrt{12}$.

195. **Radical expressions involving imaginaries.** According to the definition, § 112, $(\sqrt{-1})^2 = -1$. Hence, $(\sqrt{-1})^3 = (\sqrt{-1})^2\sqrt{-1} = -\sqrt{-1}$ and $(\sqrt{-1})^4 = (\sqrt{-1})^2(\sqrt{-1})^2 = (-1)(-1) = +1$.

The following examples illustrate operations with radical expressions containing imaginaries.

Ex. 1. $\sqrt{-4} + \sqrt{-9} = \sqrt{4}\sqrt{-1} + \sqrt{9}\sqrt{-1}$

$$= (2 + 3)\sqrt{-1} = 5\sqrt{-1}.$$

Ex. 2. $\sqrt{-4} \cdot \sqrt{-9} = \sqrt{4} \cdot \sqrt{9} \cdot (\sqrt{-1})^2 = -2 \cdot 3 = -6.$

Ex. 3. $\dfrac{\sqrt{-4}}{\sqrt{-9}} = \dfrac{\sqrt{4}\sqrt{-1}}{\sqrt{9}\sqrt{-1}} = \dfrac{\sqrt{4}}{\sqrt{9}} = \dfrac{2}{3}.$

Ex. 4. $\sqrt{-2} \cdot \sqrt{-3} \cdot \sqrt{-6} = \sqrt{2} \cdot \sqrt{3} \cdot \sqrt{6} \cdot (\sqrt{-1})^3$

$$= -\sqrt{36}\sqrt{-1} = -6\sqrt{-1}.$$

Ex. 5. Simplify $(\frac{1}{2} + \frac{1}{2}\sqrt{-3})^3$.

We are to use $\frac{1}{2}(1 + \sqrt{3}\sqrt{-1})$ three times as a factor. Reserving $(\frac{1}{2})^3 = \frac{1}{8}$ as the final coefficient, we have,

$$\begin{array}{l}
1 + \sqrt{3}\sqrt{-1} \\
\underline{1 + \sqrt{3}\sqrt{-1}} \\
1 + \sqrt{3}\sqrt{-1} \\
\underline{\quad\ \sqrt{3}\sqrt{-1} - 3} \\
1 + 2\sqrt{3}\sqrt{-1} - 3
\end{array}
\qquad\qquad
\begin{array}{l}
-2 + 2\sqrt{3}\sqrt{-1} \\
\underline{\quad 1 + \sqrt{3}\sqrt{-1}} \\
-2 + 2\sqrt{3}\sqrt{-1} \\
\underline{\qquad\ - 2\sqrt{3}\sqrt{-1} - 6} \\
-2 \qquad\qquad\qquad - 6 = -8.
\end{array}$$

Hence $(\frac{1}{2} + \frac{1}{2}\sqrt{-3})^3 = \frac{1}{8}(-8) = -1.$

EXERCISES

Perform the following indicated operations.

1. $\sqrt{-16}+\sqrt{-9}+\sqrt{-25}$.

2. $\sqrt{-x^4}-\sqrt{-x^2}$.

3. $3+5\sqrt{-1}-2\sqrt{-1}$.

4. $(2+3\sqrt{-1})(3+2\sqrt{-1})$.

5. $(2+3\sqrt{-1})(2-3\sqrt{-1})$.

6. $(4+5\sqrt{-3})(4-5\sqrt{-3})$.

7. $(2\sqrt{2}-3\sqrt{-3})(3\sqrt{3}+2\sqrt{-2})$.

8. $(\sqrt{-3}+\sqrt{-2})(\sqrt{-3}-\sqrt{-2})$.

9. $(3\sqrt{5}+2\sqrt{-7})(2\sqrt{5}-3\sqrt{-7})$.

10. $(-\frac{1}{2}-\frac{1}{2}\sqrt{-3})(-\frac{1}{2}-\frac{1}{2}\sqrt{-3})^2$.

Rationalize the denominators of

11. $\dfrac{2}{1-\sqrt{-1}}$.

12. $\dfrac{3}{\sqrt{3}+\sqrt{-1}}$.

13. $\dfrac{1-\sqrt{-1}}{1+\sqrt{-1}}$.

14. $\dfrac{\sqrt{2}+\sqrt{-3}}{\sqrt{2}-\sqrt{-3}}$.

15. $\dfrac{5}{2-3\sqrt{-5}}$.

16. $\dfrac{x+y\sqrt{-1}}{x\sqrt{-1}-y}$.

17. Solve $x^4-1=0$ by factoring. Find four roots and verify each.

18. Solve $x^3+1=0$ by factoring and the quadratic formula. Find three roots and verify each.

19. Solve $x^3-1=0$ as in the preceding and verify each root.

20. Solve $x^6-1=0$ by factoring and the quadratic formula.

SOLUTION OF EQUATIONS CONTAINING RADICALS

196. Many equations containing radicals are reducible to equivalent rational equations of the first or second degree.

The method of solving such equations is shown in the following examples.

Ex. 1. Solve $$1+\sqrt{x}=\sqrt{3+x}. \quad (1)$$

Squaring and transposing, $2\sqrt{x}=2.$ (2)

Dividing by 2 and squaring, $x=1.$ (3)

Substituting in (1), $1+1=\sqrt{3+1}=2.$

Observe that only principal roots are used in this example.

If (1) is written $$1+\sqrt{x}=-\sqrt{3+x}, \quad (4)$$

then (2) and (3) follow as before, but $x=1$ does *not satisfy* (4). Indeed equation (4) has no root. This should not be confused with the fact that every integral, rational equation has at least one root.

Ex. 2. Solve $$\sqrt{x+5}=x-1. \quad (1)$$

Squaring and transposing, $x^2-3x-4=0.$ (2)

Solving, $x=4$ and $x=-1$.

$x=4$ satisfies (1) if the *principal* root in $\sqrt{x+5}$ is taken. $x=-1$ does not satisfy (1) as it stands but would if the *negative* root were taken.

Ex. 3. Solve $$\frac{\sqrt{4x+1}-\sqrt{3x-2}}{\sqrt{4x+1}+\sqrt{3x-2}}=\frac{1}{5}. \quad (1)$$

Clearing of fractions and combining similar radicals.

$$2\sqrt{4x+1}=3\sqrt{3x-2}. \quad (2)$$

Squaring and solving, we find $x=2$.

This value of x satisfies (1) when *all* the roots are taken *positive* and also when all are taken *negative*, but otherwise *not*.

Ex. 4. Solve $\sqrt{2x+3}=\dfrac{3x-1}{\sqrt{3x}-1}-1.$ (1)

The fraction in the second member should be reduced as follows:

$$\frac{3x-1}{\sqrt{3x}-1}=\frac{(\sqrt{3x}-1)(\sqrt{3x}+1)}{\sqrt{3x}-1}=\sqrt{3x}+1.$$

Hence, (1) reduces to $\sqrt{2x+3}=\sqrt{3x}+1-1=\sqrt{3x}.$ (2)

Solving, $x=3$, which satisfies (1).

If we clear (1) of fractions in the ordinary manner, we have

$$(\sqrt{3x}-1)\sqrt{2x+3}=-\sqrt{3x}+3x. \qquad (2')$$

Squaring both sides and transposing all rational terms to the second member,

$$2x\sqrt{3x}-6\sqrt{3x}=3x^2-8x-3. \qquad (3)$$

Factoring each member,

$$2(x-3)\sqrt{3x}=(x-3)(3x+1), \qquad (4)$$

which is satisfied by $x=3.$

Dividing each member by $x-3$, squaring and transposing, we have

$$9x^2-6x+1=(3x-1)^2=0, \qquad (5)$$

which is satisfied by $x=\frac{1}{3}.$

Equation (1) is *not* satisfied by $x=\frac{1}{3}$, since the fraction in the second member is reduced to $\frac{0}{0}$ by this substitution. See § 50. The root $x=\frac{1}{3}$ is *introduced by clearing of fractions without first reducing the fraction to its lowest terms*, $\sqrt{3x}-1$ being a factor of both the numerator and the denominator. See § 165.

Ex. 5. Solve $\dfrac{6-x}{\sqrt{6-x}}-\sqrt{3}=\dfrac{x-3}{\sqrt{x-3}}.$ (1)

Reducing the fractions by removing common factors, we have

$$\sqrt{6-x}-\sqrt{3}=\sqrt{x-3}. \qquad (2)$$

Squaring, transposing, and squaring again,

$$x^2-9x+18=0, \qquad (3)$$

whence $x=3,\ x=6.$

But neither of these is a root of (1). In this case (1) has *no* root.

197. In solving an equation containing radicals, we note the following:

(1) If a fraction of the form $\dfrac{a-b}{\sqrt{a}-\sqrt{b}}$ is involved, this should be reduced by dividing numerator and denominator by $\sqrt{a}-\sqrt{b}$ *before* clearing of fractions.

(2) After clearing of fractions, transpose terms so as to leave one radical alone in one member.

(3) Square both members, and if the resulting equation still contains radicals, transpose and square as before.

(4) In every case verify all results by substituting in the given equation. In case any value does not satisfy the given equation, determine whether the roots could be so taken that it would. See Ex. 3.

EXERCISES

Solve the following equations:

1. $\sqrt{x^2+7x-2}-\sqrt{x^2-3x+6}=2.$

2. $\sqrt{3y}-\sqrt{3y-7}=\dfrac{5}{\sqrt{3y-7}}.$

3. $\dfrac{by-1}{\sqrt{by}+1}=\dfrac{\sqrt{by}-1}{2}+4.$

4. $\sqrt{5x-19}+\sqrt{3x+4}=9.$

5. $\dfrac{\sqrt{x^2+a^2}-x}{\sqrt{x^2+a^2}+x}=2.$

6. $\sqrt{a+\sqrt{ax+x^2}}=\sqrt{a}-\sqrt{x}.$

7. $\dfrac{y-l}{\sqrt{y}+\sqrt{l}}=\dfrac{\sqrt{y}-\sqrt{l}}{3}+2\sqrt{l}.$

8. $\sqrt{5x}+1=1-\dfrac{5x-1}{\sqrt{5x}+1}.$

9. $\dfrac{4}{x}-\dfrac{\sqrt{4-x^2}}{x}=\sqrt{3}.$

10. $\dfrac{4+x+\sqrt{8x+x^2}}{4+x-\sqrt{8x+x^2}}=4.$

11. $\dfrac{a-x}{\sqrt{a-x}}+\dfrac{x+a}{\sqrt{x+a}}=\sqrt{a-b}.$

12. $\dfrac{x-a}{\sqrt{x}-\sqrt{a}}=\dfrac{\sqrt{x}+\sqrt{a}}{2}+2\sqrt{a}.$

13. $\dfrac{m-y}{\sqrt{m-y}}-\sqrt{m-n}=-\dfrac{y-n}{\sqrt{y-n}}.$

14. $2\sqrt{x-a}+3\sqrt{2x}=\dfrac{7a+5x}{\sqrt{x-a}}.$

15. $\sqrt{2x+7}+\sqrt{2x+14}=\sqrt{4x+35+2\sqrt{4x^2+42x-21}}.$

16. $\sqrt{x-3}+\sqrt{x+9}=\sqrt{x+18}+\sqrt{x-6}.$

17. $\sqrt{x+7}-\sqrt{x-1}=\sqrt{x+2}+\sqrt{x-2}.$

18. $a\sqrt{y+b}-c\sqrt{b-y}=\sqrt{b(a^2+c^2)}.$

19. $y\sqrt{y-c}-\sqrt{y^3+c^3}+c\sqrt{y+c}=0.$

20. $\dfrac{\sqrt{x}}{\sqrt{m}}+\dfrac{\sqrt{m}}{\sqrt{x}}=\dfrac{\sqrt{m}}{\sqrt{n}}+\dfrac{\sqrt{n}}{\sqrt{m}}.$

21. $\sqrt{14+\sqrt{x}}+\sqrt{6-\sqrt{x}}=\dfrac{12}{\sqrt{6-\sqrt{x}}}.$

22. $\sqrt{3x}+\sqrt{3x+13}=\dfrac{91}{\sqrt{3x+13}}.$

23. $\sqrt{6x+3}+\sqrt{x+3}=2x+3.$

24. $\sqrt{x-a}+\sqrt{b-x}=\sqrt{b-a}.$

25. $\dfrac{\sqrt{x-a}+\sqrt{x-b}}{\sqrt{x-a}-\sqrt{x-b}}=\sqrt{\dfrac{x-a}{x-b}}.$

26. $\sqrt{(x-1)(x-2)}+\sqrt{(x-3)(x-4)}=\sqrt{2}.$

27. $\sqrt{2x+2}+\sqrt{7+6x}=\sqrt{7x+72}.$

28. $\sqrt{2abx}+\sqrt{a^2-bx}=\sqrt{a^2+bx}.$

29. $\dfrac{a+x+\sqrt{a^2-x^2}}{a+x-\sqrt{a^2-x^2}}=\dfrac{c}{x}.$

30. $\sqrt{x^2-2x+4}+\sqrt{3x^2+6x+12}=2\sqrt{x^2+x+10}.$

PROBLEMS

1. Find the altitude drawn to the longest side of the triangle whose sides are 6, 7, 8.

HINT. See figure, p. 190, E. C. Calling x and $8 - x$ the segments of the base and h the altitude, set up and solve two equations involving x and h.

2. Find the area of a triangle whose sides are 15, 17, 20.

First find one altitude as in problem 1.

3. Find the area of a triangle whose base is 16 and whose sides are 10 and 14.

4. Find the altitude on a side a of a triangle two of whose sides are a and a third b.

A three-sided pyramid all of whose edges are equal is called a regular tetrahedron. In Figure 10 AB, AC, AD, BC, BD, CD are all equal.

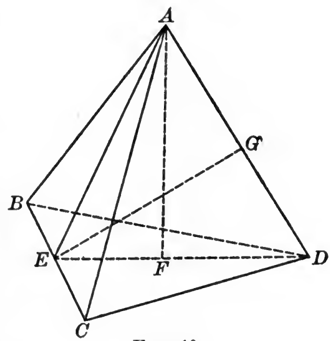


FIG. 10.

5. Find the altitude of a regular tetrahedron whose edges are each 6. Also the area of the base.

HINT. First find the altitudes AE and DE and then find the altitude of the triangle AED on the side DE, *i.e.* find AF.

6. Find the volume of a regular tetrahedron whose edges are each 10.

The volume of a tetrahedron is $\frac{1}{3}$ the product of the base and the altitude.

7. Find the volume of a regular tetrahedron whose edges are each a.

8. In Figure 10 find EG if the edges are each a.

9. If in Figure 10 EG is 12, compute the volume.

Use problem 8 to find the edge, then use problem 7 to find the volume.

10. Express the volume of the tetrahedron in terms of EG. That is if $EG = b$, find a general expression for the volume in terms of b.

11. If the altitude of a regular tetrahedron is 10, compute the edge accurately to two places of decimals.

12. Express the edge of a regular tetrahedron in terms of its altitude.

13. Express the volume of a regular tetrahedron in terms of its altitude.

14. Express the edge of a regular tetrahedron in terms of its volume.

15. Express the altitude of a regular tetrahedron in terms of its volume.

16. Express EG of Figure 10 in terms of the volume of the tetrahedron.

17. Find the edge of a regular tetrahedron such that its volume multiplied by $\sqrt{2}$, plus its entire surface multiplied by $\sqrt{3}$, is 144.

The resulting equation is of the third degree. Solve by factoring.

18. An electric light of 32 candle power is 25 feet from a lamp of 6 candle power. Where should a card be placed between them so as to receive the same amount of light from each?

Compare problem 13, p. 423. Compute result accurately to two places of decimals.

19. Where must the card be placed in problem 18 if the lamp is between the card and the electric light?

Notice that the roots of the equations in 18 and 19 are the same. Explain what this means.

20. State and solve a general problem of which 18 and 19 are special cases.

21. If the distance between the earth and the sun is 93 million miles, and if the mass of the sun is 300,000 times that of the earth, find two positions in which a particle would be equally attracted by the earth and the sun.

The gravitational attraction of one body upon another varies *inversely* as the square of the distance and directly as the product of the masses. Represent the mass of the earth by unity.

22. Find the volume of a pyramid whose altitude is 7 and whose base is a regular hexagon whose sides are 7.

The volume of a pyramid or a cone is $\frac{1}{3}$ the product of its base and its altitude.

23. If the volume of the pyramid in problem 22 were 100 cubic inches, what would be its altitude, a side of the base and the altitude being equal? Approximate the result to two places of decimals.

24. Express the altitude of the pyramid in problem 22 in terms of its volume, the altitude and the sides of the base being equal.

25. If in a right prism the altitude is equal to a side of the base, find the volume, the base being an equilateral triangle whose sides are a.

The volume of a right prism or cylinder equals the product of its base and its altitude.

26. Find the volume of the prism in problem 25 if its base is a regular hexagon whose side is a.

27. Express the side of the base of the prism in problem 25 in terms of its volume. State and solve a particular problem by means of the formula thus obtained.

28. Express the side of the base of the prism in problem 26 in terms of its volume. State and solve a particular problem by means of the formula thus obtained.

In Figures 11 and 12 the altitudes are each supposed to be three times the side a of the regular hexagonal bases.

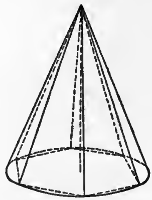


FIG. 11.

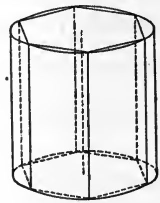


FIG. 12.

29. Express the difference between the volumes of the pyramid and the circumscribed cone in terms of a.

The volume of a cone equals $\frac{1}{3}$ the product of its base and altitude.

30. Express a in terms of the difference between the volumes of the cone and pyramid. State and solve a particular problem by means of the formula thus obtained.

31. Express the volume of the pyramid in terms of the difference between the areas of the bases of the cone and the pyramid. State a particular case and solve by means of the formula first obtained.

The lateral area of a right cylinder or prism equals the perimeter of the base multiplied by the altitude.

32. Express the difference of the lateral areas of the cylinder and the prism in terms of a.

The following four problems refer to Figure 12. In each case state a particular problem and solve by means of the formula obtained.

33. Express a in terms of the difference of the lateral areas.

34. Express the volume of the prism in terms of the difference of the perimeters of the bases.

35. Express the volume of the cylinder in terms of the difference of the lateral areas.

36. Express the sum of the volumes of the prism and cylinder in terms of the difference of the areas of the bases.

CHAPTER XI

LOGARITHMS

198. The operations of multiplication, division, and finding powers and roots are greatly shortened by the use of **logarithms.**

The logarithm of a number, in the system commonly used, is the *index of that power of* 10 *which equals the given number.*

Thus, 2 is the logarithm of 100 since $10^2 = 100$.

This is written $\log 100 = 2.$

Similarly $\log 1000 = 3$, since $10^3 = 1000$,

and $\log 10000 = 4$, since $10^4 = 10000$.

The logarithm of a number which is *not an exact* rational *power* of 10 is an irrational number and is written approximately as a decimal fraction.

Thus, $\log 74 = 1.8692$ since $10^{1.8692} = 74$ approximately.

In higher algebra it is shown that the laws for rational exponents (§ 179) hold also for irrational exponents.

199. The decimal part of a logarithm is called the **mantissa,** and the integral part the **characteristic.**

Since $10^0 = 1$, $10^1 = 10$, $10^2 = 100$, $10^3 = 1000$, etc., it follows that for all numbers between 1 and 10 the logarithm lies between 0 and 1, that is, the characteristic is 0. Likewise for numbers between 10 and 100 the characteristic is 1, for numbers between 100 and 1000 it is 2, etc.

200. Tables of logarithms (see p. 452) usually give the mantissas only, the characteristics being supplied, in the case of *whole* numbers, according to § 199, and in the case of decimal numbers, as shown in the examples given under § 201.

201. An important property of logarithms is illustrated by the following:

From the table of logarithms, p. 470, we have:

$$\log 376 = 2.5752, \text{ or } 376 = 10^{2.5752}. \qquad (1)$$

Dividing both members of (1) by 10 we have

$$37.6 = 10^{2.5752} \div 10^1 = 10^{2.5752-1} = 10^{1.5752}.$$

Hence, $\log 37.6 = 1.5752$,

Similarly, $\log 3.76 = 1.5752 - 1 = 0.5752$,

$$\log .376 = 0.5752 - 1, \text{ or } \bar{1}.5752,$$

$$\log .0376 = 0.5752 - 2, \text{ or } \bar{2}.5752,$$

where $\bar{1}$ and $\bar{2}$ are written for -1 and -2 to indicate that the characteristics are negative while the mantissas are positive.

Multiplying (1) by 10 gives

$$\log 3760 = 2.5752 + 1 = 3.5752,$$

and $\log 37600 = 2.5752 + 2 = 4.5752.$

Hence, if the decimal point of a number is moved a certain number of places to the *right* or to the *left*, the characteristic of the logarithm is *increased* or *decreased* by a corresponding number of units, the mantissa *remaining the same.*

From the table on pp. 452, 453, we may find the mantissas of logarithms for all integral numbers from 1 to 1000. In this table the logarithms are given to four places of decimals, which is sufficiently accurate for most practical purposes.

E.g. for log 4 the mantissa is the same as that for log 40 or for log 400.

To find log .0376 we find the mantissa corresponding to 376, and prefix the characteristic $\bar{2}$. See above.

Ex. 1. Find log 876.

Solution. Look down the column headed N to 87, then along this line to the column headed 6, where we find the number 9425, which is the mantissa. Hence $\log 876 = 2.9425$.

Ex. 2. Find log 3747.

Solution. As above we find $\log 3740 = 3.5729$,

and $\log 3750 = 3.5740$.

The difference between these logarithms is 11, which corresponds to a difference of 10 between the numbers. But 3740 and 3747 differ by 7. Hence, their logarithms should differ by $\frac{7}{10}$ of 11, *i.e.* by 8.1. Adding this to the logarithm of 3740, we have 3.5737, which is the required logarithm.

The assumption here made, that the logarithm varies directly as the number, is not quite, but very nearly, accurate, when the variation of the number is confined to a narrow range, as is here the case.

Ex. 3. Find the number whose logarithm is 2.3948.

Solution. Looking in the table, we find that the nearest *lower* logarithm is 2.3945 which corresponds to the number 248. See § 199.

The given mantissa is 3 greater than that of 248, while the mantissa of 249 is 17 greater. Hence the number corresponding to 2.3948 must be 248 plus $\frac{3}{17}$ or $.1\dot{7}6$. Hence, 248.18 is the required number, correct to 2 places of decimals.

Ex. 4. Find log .043.

Solution. Find log 43 and subtract 3 from the characteristic.

Ex. 5. Find the number whose logarithm is $\bar{4}.3949$.

Solution. Find the number whose logarithm is 0.3949, and move the decimal point 4 places to the left.

EXERCISES

Find the logarithms of the following numbers:

1. 491.	6. .541.	11. .006.	16. 79.31.
2. 73.5.	7. .051.	12. .1902.	17. 4.245.
3. 2485.	8. 8104.	13. .0104.	18. .0006.
4. 539.7.	9. 70349.	14. 2.176.	19. 3.817.
5. 53.27.	10. 439.26.	15. 8.094.	20. .1341.

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	7857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

Find the numbers corresponding to the following logarithms:

21. 1.3179.	26. 2.9900.	31. $\bar{1}.5972$.	36. 0.2468.
22. 3.0146.	27. 0.1731.	32. $\bar{1}.0011$.	37. 0.1357.
23. 0.7145.	28. 0.8974.	33. $\bar{2}.7947$.	38. $\bar{2}.0246$.
24. $\bar{1}.5983$.	29. 0.9171.	34. $\bar{2}.5432$.	39. $\bar{1}.1358$.
25. 2.0013.	30. 3.4015.	35. 0.5987.	40. $\bar{4}.0478$.

202. Products and powers may be found by means of logarithms, as shown by the following examples.

Ex. 1. Find the product $49 \times 134 \times .071 \times 349$.

Solution. From the table,

$$\log 49 = 1.6902 \text{ or } 49 = 10^{1.6902}.$$

$$\log 134 = 2.1271 \text{ or } 134 = 10^{2.1271}.$$

$$\log .071 = \bar{2}.8513 \text{ or } .071 = 10^{\bar{2}.8513}.$$

$$\log 349 = 2.5428 \text{ or } 349 = 10^{2.5428}.$$

Since powers of the same base are multiplied by *adding* exponents, § 176, we have $49 \times 134 \times .071 \times 349 = 10^{5.2114}$.

Hence $\log (49 \times 134 \times .071 \times 349) = 5.2114$.

The number corresponding to this logarithm, as found by the method used in Ex. 3 above, is 162704. By actual multiplication the product is found to be 162698.914 or 162699 which is the nearest approximation without decimals. Hence the product obtained by means of logarithms is 5 too large. This is an error of about $\frac{1}{32000}$ of the actual result and is therefore so small as to be negligible.

Ex. 2. Find $(1.05)^{20}$.

Solution. $\text{Log } 1.05 = 0.0212$ or $10^{0.0212} = 1.05$.

Hence $(1.05)^{20} = (10^{0.0212})^{20} = 10^{(0.0212)\cdot 20} = 10^{0.424}$,

or $\log (1.05)^{20} = 0.4240$.

Hence $(1.05)^{20}$ is the number corresponding to the logarithm 0.4240.

Since logarithms are **exponents** of the base 10, it follows from the laws of exponents (see § 198) that

(*a*) *The logarithm of the product of two or more numbers is the sum of the logarithms of the numbers.*

(*b*) *The logarithm of a power of a number is the logarithm of the number multiplied by the index of the power.*

That is,

$$\log(a \cdot b \cdot c) = \log a + \log b + \log c, \text{ and } \log a^n = n \log a.$$

EXERCISES

By means of the logarithms obtain the following products and powers:

1. $243 \times 76 \times .34$.
2. 823.68×370.
3. 216.83×2.03.
4. $57^2 \times (.71)^2$.
5. $510 \times (9.1)^3$.
6. $43.71 \times (21)^2$.
7. 5.93×10.02.
8. 486×3.45.
9. $(.02)^2 \times (0.8)$.
10. $(65)^2 \times (91)^3$.
11. $(84)^2 \times (75)^3$.
12. $(.960)^2(49)^2$.
13. $(49)^3 \times .19 \times 21^2$.
14. $.21084 \times (.53)^2$.
15. $7.865 \times (.013)^2$.
16. $(6.75)^3 \times (723)^2$.
17. $(1.46)^2 \times (61.2)^2$.
18. $(3.54)^3 \times (29.3)^2$.
19. $(4.132)^2 \times (5.184)^2$.
20. $1946 \times 398 \times .08$.

203. Quotients and roots may be found by means of logarithms, as shown by the following examples.

Ex. 1. Divide 379 by 793.

Solution. From the table,

$$\log 379 = 2.5786 \text{ or } 10^{2.5786} = 379.$$

$$\log 793 = 2.8993 \text{ or } 10^{2.8993} = 793.$$

Hence by the law of exponents for division, § 175,

$$379 \div 793 = 10^{2.5786 - 2.8993}.$$

Since in all operations with logarithms the mantissa is positive, write the first exponent $3.5786 - 1$ and then subtract 2.8993.

Hence $\log(379 \div 793) = .6793 - 1 = \bar{1}.6793$.

Hence the quotient is the number corresponding to this logarithm.

Ex. 2. By means of logarithms approximate $\sqrt[3]{42^2 \times 37^5}$.

By the methods used above we find

$$\log(42^2 \times 37^5) = 11.0874 \text{ or } 10^{11.0874} = 42^2 \times 37^5.$$

Hence $$\sqrt[3]{42^2 \times 37^5} = (10^{11.0874})^{\frac{1}{3}} = 10^{\frac{11.0874}{3}} = 10^{3.6958}.$$

That is, $$\log \sqrt[3]{42^2 \times 37^5} = 3.6958.$$

Hence the result sought is the number corresponding to this logarithm.

It follows from the laws of exponents (see § 198) that

(*a*) *The logarithm of a quotient equals the logarithm of the dividend minus the logarithm of the divisor.*

(*b*) *The logarithm of a root of a number is the logarithm of the number divided by the index of the root.*

That is

$$\log \frac{a}{b} = \log a - \log b \text{ and } \log \sqrt[n]{a} = \frac{\log a}{n}.$$

EXERCISES

By means of logarithms approximate the following quotients and roots:

1. $45.2 \div 8.9$.
2. $231.18 \div 4.2$.
3. $.04905 \div .327$.
4. $\sqrt{196 \times 256}$.
5. $\dfrac{5334 \times .02374}{27.43 \times 3.246}$.
6. $\sqrt[5]{69} \div \sqrt[3]{21}$.
7. $\sqrt[7]{15} \times \sqrt[8]{67}$.
8. $\sqrt[10]{211} \times \sqrt[11]{34.7}$.
9. $(5184)^{\frac{1}{2}} \div (38124)^{\frac{1}{3}}$.
10. $(6.75)^3 \div (2.132)^2$.
11. $\sqrt[9]{105} \div \sqrt[13]{76}$.
12. $(91125)^{\frac{1}{3}} \div (576)^{\frac{1}{3}}$.
13. $(3.040)^3 \div (.0065)^3$.
14. $(29.3)^{\frac{1}{3}} \div \sqrt{(3.47)^3}$.
15. $\sqrt[3]{39} \times \sqrt[3]{56} \times \sqrt[4]{87}$.
16. $\sqrt[3]{\dfrac{13^4 \times .31^2 \times 4.31^3}{\sqrt{71} \times \sqrt[3]{41} \times \sqrt{51}}}$.
17. $\sqrt[5]{\dfrac{4^9 \times .57^3 \times 42^3}{\sqrt[3]{5.2} \times \sqrt[5]{.83} \times \sqrt{23}}}$.
18. $\left(\dfrac{\sqrt[3]{54} \times \sqrt[4]{28} \times \sqrt[5]{7}}{\sqrt[2]{4^7} \times \sqrt[3]{7^4} \times (.003)^{\frac{3}{4}}}\right)^{\frac{4}{5}}$.

CHAPTER XII

PROGRESSIONS

ARITHMETIC PROGRESSIONS

204. An arithmetic progression is a series of numbers, such that any one after the first is obtained by adding a fixed number to the preceding. The fixed number is called the **common difference.**

The general form of an arithmetic progression is

$$a,\ a+d,\ a+2d,\ a+3d,\ \cdots,$$

where a is the first term and d the common difference.

E.g. 2, 5, 8, 11, 14, ⋯ is an arithmetic progression in which 2 is the first term and 3 the common difference. Written in the general form, it would be

$$2,\ 2+3,\ 2+2\cdot 3,\ 2+3\cdot 3,\ 2+4\cdot 3,\ \cdots.$$

205. If there are n terms in the progression, then the last term is $a+(n-1)d$. Indicating the last term by l, we have

$$l=a+(n-1)d. \qquad \text{I}$$

An arithmetic progression of n terms would then be written in general form, thus,

$$a,\ a+d,\ a+2d,\ \cdots,\ a+(n-2)d,\ a+(n-1)d.$$

EXERCISES

1. Solve I for each letter in terms of all the others.

In each of the following find the value of the letter not given, and write out the progression in each case.

2. $\begin{cases} a=2, \\ d=2, \\ n=7. \end{cases}$ **3.** $\begin{cases} a=3, \\ d=5, \\ l=43. \end{cases}$ **4.** $\begin{cases} a=1, \\ n=15, \\ l=15. \end{cases}$ **5.** $\begin{cases} a=7, \\ n=31, \\ l=91. \end{cases}$

6. $\begin{cases} a=4, \\ d=-3, \\ n=18. \end{cases}$ 8. $\begin{cases} a=3, \\ d=-5, \\ l=-32. \end{cases}$ 10. $\begin{cases} d=-5, \\ n=13, \\ l=-63. \end{cases}$ 12. $\begin{cases} a=11, \\ l=-39, \\ d=-5. \end{cases}$

7. $\begin{cases} a=-5, \\ d=4, \\ n=7. \end{cases}$ 9. $\begin{cases} d=7, \\ n=8, \\ l=24. \end{cases}$ 11. $\begin{cases} a=-3, \\ n=9, \\ l=-27. \end{cases}$ 13. $\begin{cases} a=x, \\ l=y, \\ n=z. \end{cases}$

206. The sum of an arithmetic progression of n terms may be obtained as follows:

Let s_n denote the sum of n terms of the progression. Then,

$$s_n = a + [a+d] + [a+2d] + \cdots + [a+(n-2)d] + [a+(n-1)d]. \quad (1)$$

This may also be written, reversing the order of the terms, thus,

$$s_n = [a+(n-1)d] + [a+(n-2)d] + \cdots + [a+2d] + [a+d] + a. \quad (2)$$

Adding (1) and (2), we have

$$2s_n = [2a+(n-1)d] + [2a+(n-2)d+d] + \cdots + [2a+(n-2)d+d] + [2a+(n-1)d].$$

The expression in each bracket is reducible to $2a+(n-1)d$, which may also be written $a+[a+(n-1)d] = a+l$, by § 205.

Since there are n of these expressions, each $a+l$, we have

$$2s_n = n(a+l).$$

Hence

$$\mathbf{s_n = \frac{n}{2}(a+l).} \qquad \text{II}$$

This formula for the sum of n terms involves a, l, and n, that is, the first term, the last term, and the number of terms.

207. In the two equations,

$$l = a + (n-1)d, \qquad \text{I}$$

$$s = \frac{n}{2}(a+l), \qquad \text{II}$$

there are five letters, namely, a, d, l, n, s. If any three of these are given, the equations I and II may be solved simultaneously to find the other two, considered as the *unknowns*.

The solution of problems in arithmetic progression by means of equations I and II is illustrated in the following examples:

Ex. 1. Given $n=11$, $l=23$, $s=143$. Find a and d.

Substituting the given values in I and II,

$$23 = a + (11 - 1)d. \quad (1)$$

$$143 = \tfrac{11}{2}(a + 23). \quad (2)$$

From (2), $a = 3$, which in (1) gives $d = 2$.

Ex. 2. Given $d=4$, $n=5$, $s=75$. Find a and l.

From I and II, $$l = a + (5 - 1)4, \quad (1)$$

$$75 = \frac{5}{2}(a + l). \quad (2)$$

Solving (1) and (2) simultaneously, we have $a = 7$, $l = 23$.

Ex. 3. Given $d=4$, $l=35$, $s=161$. Find a and n.

From I and II, $$35 = a + (n - 1)4, \quad (1)$$

$$161 = \frac{n}{2}(a + 35). \quad (2)$$

From (1) $$a = 39 - 4n,$$

which in (2) gives $$161 = \frac{n}{2}(74 - 4n) = 37n - 2n^2.$$

Hence $$n = \tfrac{23}{2}, \text{ or } 7.$$

Since an arithmetic progression must have an *integral* number of terms, only the second value is applicable to this problem.

Ex. 4. Given $d=2$, $l=11$, $s=35$. Find a and n.

Substituting in I and II, and solving for a and n, we have

$$a = 3,\ n = 5, \text{ and } a = -1,\ n = 7.$$

Hence there are two progressions,

$$-1, 1, 3, 5, 7, 9, 11,$$

and $$3, 5, 7, 9, 11,$$

each of which satisfies the given conditions.

EXERCISES

In each of the following obtain the values of the two letters not given.

If fractional or negative values of n are obtained, such a result indicates that the problem is impossible. This is also the case if an *imaginary* value is obtained for *any* letter. In each exercise interpret all the values found.

1. $\begin{cases} s=96, \\ l=19, \\ d=2. \end{cases}$ **4.** $\begin{cases} s=88, \\ l=-7, \\ d=-3. \end{cases}$ **7.** $\begin{cases} d=-1, \\ n=41, \\ l=-35. \end{cases}$ **10.** $\begin{cases} d=6, \\ l=49, \\ s=232. \end{cases}$

2. $\begin{cases} s=34, \\ l=14, \\ d=3. \end{cases}$ **5.** $\begin{cases} n=18, \\ a=4, \\ l=13. \end{cases}$ **8.** $\begin{cases} l=30, \\ s=162, \\ n=9. \end{cases}$ **11.** $\begin{cases} s=7, \\ d=1\frac{1}{2}, \\ l=7. \end{cases}$

3. $\begin{cases} a=7, \\ l=27, \\ s=187. \end{cases}$ **6.** $\begin{cases} n=14, \\ a=7, \\ s=14. \end{cases}$ **9.** $\begin{cases} a=30, \\ n=10, \\ s=120. \end{cases}$ **12.** $\begin{cases} s=14, \\ d=3, \\ l=4. \end{cases}$

In each of the following call the two letters specified the *unknowns* and solve for their values in terms of the remaining three letters supposed to be *known*.

13. a, d. **15.** a, n. **17.** d, l. **19.** d, s. **21.** l, s.

14. a, l. **16.** a, s. **18.** d, n. **20.** l, n. **22.** n, s.

208. Arithmetic means. The terms between the first and the last of an arithmetic progression are called **arithmetic means**.

Thus, in 2, 5, 8, 11, 14, 17, the four arithmetic means between 2 and 17 are 5, 8, 11, 14.

If the first and the last terms and the number of arithmetic means between them are given, then these means can be found.

For we have given a, l, and n. Hence d can be found and the whole series constructed.

Example. Insert 7 arithmetic means between 3 and 19.

In this progression $a = 3$, $l = 19$, and $n = 9$.

Hence from $l = a + (n - 1)d$ we find $d = 2$ and the required means are 5, 7, 9, 11, 13, 15, 17.

209. The case of *one* arithmetic mean is important. Let A be the arithmetic mean between a and l. Since a, A, l are in arithmetic progression, we have $A = a + d$, and $l = A + d$. Hence

$$A - l = a - A$$

or

$$A = \frac{a + l}{2}. \qquad \text{III}$$

EXERCISES AND PROBLEMS

1. Insert 5 arithmetic means between 5 and -7.

2. Insert 3 arithmetic means between -2 and 12.

3. Insert 8 arithmetic means between -3 and -5.

4. Insert 5 arithmetic means between -11 and 40.

5. Insert 15 arithmetic means between 1 and 2.

6. Insert 9 arithmetic means between $2\frac{3}{4}$ and $-1\frac{1}{2}$.

7. Find the arithmetic mean between 3 and 17.

8. Find the arithmetic mean between -4 and 16.

9. Find the tenth and eighteenth terms of the series 4, 7, 10, $\cdots$.

10. Find the fifteenth and twentieth terms of the series $-8, -4, 0, \cdots$.

11. The fifth term of an arithmetic progression is 13 and the thirtieth term is 49. Find the common difference.

12. Find the sum of all the integers from 1 to 100.

13. Find the sum of all the odd integers between 0 and 200.

14. Find the sum of all integers divisible by 6 between 1 and 500.

15. Show that $1 + 3 + 5 + \cdots + n = k^2$ where k is the number of terms.

16. In a potato race 40 potatoes are placed in a straight line one yard apart, the first potato being two yards from the basket. How far must a contestant travel in bringing them to the basket one at a time?

17. There are three numbers in arithmetic progression whose sum is 15. The product of the first and last is $3\frac{1}{5}$ times the second. Find the numbers.

18. There are four numbers in arithmetic progression whose sum is 20 and the sum of whose squares is 120. Find the numbers.

19. If a body falls from rest 16.08 feet the first second, 48.24 feet the second second, 80.40 the third, etc., how far will it fall in 10 seconds? 15 seconds? t seconds?

20. According to the law indicated in problem 19 in how many seconds will a body fall 1000 feet? s feet?

If a body is thrown downward with a velocity of v_0 feet per second, then the distance, s, it will fall in t seconds is v_0t feet plus the distance it would fall if starting from rest.

That is, $s = v_0t + \frac{1}{2}gt^2$, where $g = 32.16$.

21. In what time will a body fall 1000 feet if thrown downward with a velocity of 20 feet per second?

22. With what velocity must a body be thrown downward in order that it shall fall 360 feet in 3 seconds?

23. A stone is dropped into a well, and the sound of its striking the bottom is heard in 3 seconds. How deep is the well if sound travels 1080 feet per second?

A body thrown upward with a certain velocity will rise as far as it would have to fall to acquire this velocity. The velocity (neglecting the resistance of the atmosphere) of a body starting from rest is gt where $g = 32.16$ and t is the number of seconds.

24. A rifle bullet is shot directly upward with a velocity of 2000 feet per second. How high will it rise, and how long before it will reach the ground?

25. From a balloon 5800 feet above the earth, a body is thrown downward with a velocity of 40 feet per second. In how many seconds will it reach the ground?

26. If in Problem 25 the body is thrown upward at the rate of 40 feet per second, how long before it will reach the ground?

GEOMETRIC PROGRESSIONS

210. A **geometric progression** is a series of numbers in which any term after the first is obtained by multiplying the preceding term by a fixed number, called the **common ratio.**

The general form of a geometric progression is

$$a,\ ar,\ ar^2,\ ar^3,\ \cdots,\ ar^{n-1},$$

in which a is the first term, r the constant multiplier, or common ratio, and n the number of terms.

E.g. 3, 6, 12, 24, 48, is a geometric progression in which 3 is the first term, 2 is the common ratio, and 5 is the number of terms.

Written in the general form it would be $3,\ 3\cdot 2,\ 3\cdot 2^2,\ 3\cdot 2^3,\ 3\cdot 2^4$.

211. If l is the last or nth term of the series, then

$$l = ar^{n-1}. \qquad \text{I}$$

If any three of the four letters in I are given, the remaining one may be found by solving this equation.

EXERCISES

In each of the following find the value of the letter not given:

1. $\begin{cases} l=162, \\ r=3, \\ n=5. \end{cases}$ **4.** $\begin{cases} a=-1, \\ r=-2, \\ n=9. \end{cases}$ **7.** $\begin{cases} a=-\frac{1}{2}, \\ r=\frac{3}{2}, \\ n=6. \end{cases}$ **10.** $\begin{cases} l=32, \\ r=-2, \\ n=6. \end{cases}$

2. $\begin{cases} a=1, \\ r=2, \\ n=8. \end{cases}$ **5.** $\begin{cases} l=1024, \\ r=-2, \\ n=11. \end{cases}$ **8.** $\begin{cases} l=18, \\ r=\frac{1}{3}, \\ n=6. \end{cases}$ **11.** $\begin{cases} a=-2, \\ r=-\frac{3}{2}, \\ n=7. \end{cases}$

3. $\begin{cases} a=-4, \\ r=-3, \\ n=6. \end{cases}$ **6.** $\begin{cases} l=1024, \\ r=2, \\ n=11. \end{cases}$ **9.** $\begin{cases} l=-16, \\ r=-\frac{3}{4}, \\ n=5. \end{cases}$ **12.** $\begin{cases} a=3, \\ r=2, \\ l=1536. \end{cases}$

212. The **sum of n terms** of a geometric expression may be found as follows:

If s_n denotes the sum of n terms, then

$$s_n = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}. \quad (1)$$

Multiplying both members of (1) by r, we have

$$rs_n = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n. \quad (2)$$

Subtracting (1) from (2), and canceling terms, we have

$$rs_n - s_n = ar^n - a. \quad (3)$$

Solving (3) for s_n we have

$$s_n = \frac{ar^n - a}{r-1} = \frac{a(r^n - 1)}{r-1}. \quad \text{II.}$$

This formula for the sum of n terms of a geometric series involves only a, r, and n.

Since $ar^n = r \cdot ar^{n-1} = r \cdot l$, s^n may also be written:

$$s_n = \frac{rl - a}{r-1} = \frac{a - rl}{1-r}. \quad \text{III}$$

This formula involves only r, l, and a.

213. From equations I and II or I and III any two of the numbers a, l, r, s, and n can be found when the other three are given, as in the following examples.

Ex. 1. Given $n = 7$, $r = 2$, $s = 381$. Find a and l.

From I and III, $$l = a \cdot 2^6 = 64\,a, \quad (1)$$

$$381 = \frac{2\,l - a}{2-1} = 2\,l - a. \quad (2)$$

Substituting $l = 64\,a$ in (2), we obtain $a = 3$, and $l = 192$.

Ex. 2. Given $a = -3$, $l = -243$, $s = -183$. Find r and n.

From I and III, $$-243 = (-3)r^{n-1}, \quad (1)$$

$$-183 = \frac{-243\,r + 3}{r-1}. \quad (2)$$

From (2) $r = -3.$ (3)

From (1) $81 = (-3)^{n-1}.$ (4)

Since $(-3)^4 = 81$, we have $n - 1 = 4$ or $n = 5$.

EXERCISES

1. Solve II for a in terms of the remaining letters.

2. Solve III for each letter in terms of the remaining letters.

In each of the following find the terms represented by the interrogation points.

3. $\begin{cases} a=1, \\ r=3, \\ n=5, \\ s=? \end{cases}$ 4. $\begin{cases} s=635, \\ r=2, \\ n=7, \\ a=? \end{cases}$ 5. $\begin{cases} s=13, \\ r=\frac{2}{3}, \\ n=4, \\ a=? \end{cases}$ 6. $\begin{cases} l=-\frac{16}{81}, \\ s=? \\ n=5, \\ r=\frac{2}{3}. \end{cases}$

7. $\begin{cases} a=1, \\ s=\frac{25}{64}, \\ l=-\frac{27}{64}, \\ r=? \end{cases}$ 8. $\begin{cases} r=\frac{1}{6}, \\ n=5, \\ l=1296, \\ a=? \\ s=? \end{cases}$ 9. $\begin{cases} r=\frac{3}{2}, \\ n=8, \\ s=1050\frac{5}{6}, \\ l=? \\ a=? \end{cases}$ 10. $\begin{cases} a=\frac{9}{2}, \\ n=7, \\ l=\frac{32}{81}, \\ r=? \\ s=? \end{cases}$

214. Geometric means. The terms between the first and the last of a geometric progression are called **geometric means**.

Thus in 3, 6, 12, 24, 48, three geometric means between 3 and 48 are 6, 12 and 24.

If the first term, the last term, and the number of geometric means are given, the ratio may be found from I, and then the means may be inserted.

Example. Insert 4 geometric means between 2 and 64.

We have given $a = 2$, $l = 64$, $n = 4 + 2 = 6$, to find r.

From I, $64 = 2 \cdot r^{6-1}$ or $r^5 = 32$ and $r = 2$.

Hence, the series is 2, 4, 8, 16, 32, 64.

215. The case of *one* geometric mean is important. If G is the geometric mean between a and b, we have $\frac{G}{a} = \frac{b}{G}.$

Hence, $$\boldsymbol{G = \sqrt{ab}}.$$

216. Problem. In attempting to reduce $\frac{2}{3}$ to a decimal, we find by division .666 ⋯, the dots indicating that the process goes on indefinitely.

Conversely, we see that $.666 \cdots = \frac{6}{10} + \frac{6}{100} + \frac{6}{1000} + \cdots$, that is, a geometric progression in which $a = \frac{6}{10}$, $r = \frac{1}{10}$, and n is not fixed but goes on *increasing indefinitely.*

As n grows large, l grows small, and by taking n sufficiently large, l can be made as small as we please. Hence formula III, § 212, is to be interpreted in this case as follows:

$$s_n = \frac{a - rl}{1 - r} = \frac{\frac{6}{10} - \frac{l}{10}}{1 - \frac{1}{10}} = \frac{6 - l}{9},$$

in which l grows small indefinitely as n increases indefinitely, so that by taking n large enough s_n can be made to differ as little as we please from $\frac{6-0}{9} = \frac{6}{9} = \frac{2}{3}$.

In this case we say s_n **approaches** $\frac{2}{3}$ **as a limit** as n increases indefinitely.

Observe that this interpretation can apply only when the constant multiplier r is a proper fraction.

EXERCISES AND PROBLEMS

1. Insert 5 geometric means between 2 and 128.
2. Insert 7 geometric means between 1 and $\frac{1}{256}$.
3. Find the geometric mean between 8 and 18.
4. Find the geometric mean between $\frac{1}{12}$ and $\frac{1}{4}$.
5. Find the fraction which is the limit of .333 ⋯.
6. Find the fraction which is the limit of .1666 ⋯.
7. Find the fraction which is the limit of .08333 ⋯.
8. Find the 13th term of $-\frac{16}{3}, 4, -3 \cdots$.
9. Find the sum of 15 terms of the series $-243, 81, -27 \cdots$.
10. Find the limit of the sum $\frac{4}{3} + \frac{2}{3} + \frac{1}{3} + \frac{1}{6} + \cdots$, as the number of terms increases indefinitely.

Given	Find	Given	Find
11. a, r, n	l, s	15. a, n, l	s, r
12. a, r, s	l	16. r, n, l	s, a
13. r, n, s	l, a	17. r, l, s	a
14. a, r, l	s	18. a, l, s	r

19. The product of three terms of a geometric progression is 1000. Find the second term.

20. Four numbers are in geometric progression. The sum of the second and third is 18, and the sum of the first and fourth is 27. Find the numbers.

21. Find an arithmetic progression whose first term is 1 and whose first, second, fifth, and fourteenth terms are in geometric progression.

22. Three numbers whose sum is 27 are in arithmetic progression. If 1 is added to the first, 3 to the second, and 11 to the third the sums will be in geometric progression. Find the numbers.

23. To find the compound interest when the principal, the rate of interest, and the time are given.

Solution. Let p equal the number of dollars invested, r the rate of per cent of interest, t the number of years, and a the amount at the end of t years.

Then $a = p(1 + r)$ at the end of one year.

$a = p(1 + r)(1 + r) = p(1 + r)^2$ at the end of two years.

and $a = p(1 + r)^t$ at the end of t years.

That is, the amount for t years is the last term of a geometric progression in which p is the first term, $1 + r$ is the ratio, and $t + 1$ is the number of terms.

24. Show how to modify the solution given under problem 23 when the interest is compounded semiannually; quarterly.

25. Solve the equation $a = p(1 + r)^t$ for p and for r.

26. Solve $a = p(1+r)^t$ for t.

Solution. $\log a = \log p(1+r)^t = \log p + \log(1+r)^t$

$= \log p + t\log(1+r)$. (See § 202.) Hence $t = \frac{\log a - \log p}{\log(1+r)}$.

27. At what rate of interest compounded annually will $1200 amount to $1800 in 12 years?

28. At what rate of interest compounded semiannually will a sum double itself in 20 years? in 15 years? in 10 years?

29. In what time will $8000 amount to $13,500, the rate of interest being 3½% compounded annually?

30. In what time will a sum double itself at 3%, 4%, 5%, compounded semiannually?

The present value of a debt due at some future time is a sum such that, if invested at compound interest, the amount at the end of the time will equal the debt.

31. What is the present value of $2500 due in 4 years, money being worth 3½% interest compounded semiannually?

32. A man bequeathed $50,000 to his daughter, payable on her twenty-fifth birthday, with the provision that the present worth of the bequest should be paid in case she married before that time. If she married at 21, how much would she receive, interest being 4% per annum and compounded quarterly?

33. What is the rate of interest if the present worth of $24,000 due in 7 years is $19,500?

34. In how many years is $5000 due if its present worth is $3500, the rate of interest being 3¾% compounded annually?

HARMONIC PROGRESSIONS

217. A **harmonic progression** is a series whose terms are the reciprocals of the corresponding terms of an arithmetic progression.

E.g. $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9} \cdots$ is a harmonic progression whose terms are the reciprocals of the terms of the arithmetic progression 1, 3, 5, 7, 9 ⋯.

The name *harmonic* is given to such a series because musical strings of uniform size and tension, whose lengths are the reciprocals of the positive integers, *i.e.* $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ ..., vibrate in harmony.

The general form of the harmonic progression is

$$\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \cdots, \frac{1}{a+(n-1)d}. \qquad \text{I}$$

It follows that if a, b, c, d, e, ... are in harmonic progression, then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}, \frac{1}{e}$, ... are in arithmetic progression. Hence, all questions pertaining to a harmonic progression are best answered by first converting it into an arithmetic progression.

218. Harmonic means. The terms between the first and the last of a harmonic progression are called **harmonic means** between them.

Example. Insert five harmonic means between 30 and 3.

This is done by inserting five arithmetic means between $\frac{1}{30}$ and $\frac{1}{3}$. By the method of § 208 the arithmetic series is found to be $\frac{1}{30}, \frac{1}{12}, \frac{2}{15}, \frac{11}{60}, \frac{7}{30}, \frac{17}{60}, \frac{1}{3}$. Hence, the harmonic series is $30, 12, \frac{15}{2}, \frac{60}{11}, \frac{30}{7}, \frac{60}{17}, 3$.

219. The case of a single harmonic mean is important. Let a, H, l be in harmonic progression. Then $\frac{1}{a}, \frac{1}{H}, \frac{1}{l}$ are in arithmetic progression.

Hence, by § 209, $\frac{1}{H} = \frac{\frac{1}{a}+\frac{1}{l}}{2}$ or $H = \frac{2al}{a+l}$.

220. The arithmetic, geometric, and harmonic means between a and l are related as follows:

We have seen $\quad A = \frac{a+l}{2}, \; G = \sqrt{al}, \; H = \frac{2al}{a+l}$.

Hence, $\quad \frac{A}{G^2} = \frac{a+l}{2} \div al = \frac{a+l}{2al}$.

Therefore, $\quad \frac{A}{G^2} = \frac{1}{H}$, or $\frac{A}{G} = \frac{G}{H}$.

That is, G is a *mean proportional* between A and H. See § 172.

EXERCISES AND PROBLEMS

1. Insert three harmonic means between 22 and 11.

2. Insert six harmonic means between $\frac{1}{3}$ and $\frac{23}{6}$.

3. The first term of a harmonic progression is $\frac{1}{2}$ and the tenth term is $\frac{1}{20}$. Find the intervening terms.

4. Two consecutive terms of a harmonic progression are 5 and 6. Find the next two terms and also the two preceding terms.

5. If a, b, c are in harmonic progression, show that $a \div c = (a - b) \div (b - c)$.

6. Find the arithmetic, geometric, and harmonic means between:

(a) 16 and 36; (b) $m + n$ and $m - n$; (c) $\dfrac{1}{m+n}$ and $\dfrac{1}{m-n}$.

7. The harmonic mean between two numbers exceeds their arithmetic mean by 7, and one number is three times the other. Find the numbers.

8. If x, y, and z are in arithmetic progression, show that mx, my, and mz are also in arithmetic progression.

9. x, y, and z being in harmonic progression, show that $\dfrac{x}{x+y+z}$, $\dfrac{y}{x+y+z}$, and $\dfrac{z}{x+y+z}$ are in harmonic progression, and also that $\dfrac{x}{y+z}$, $\dfrac{y}{x+z}$, and $\dfrac{z}{x+y}$ are in harmonic progression.

10. The sum of three numbers in harmonic progression is 3, and the first is double the third. Find the numbers.

11. The geometric mean between two numbers is $\frac{1}{4}$ and the harmonic mean is $\frac{1}{5}$. Find the numbers.

12. Insert n harmonic means between the numbers a and b.

CHAPTER XIII

THE BINOMIAL FORMULA

221. In Chapter II the following products were obtained:

$$(a + b)^2 = a^2 + 2\,ab + b^2.$$

$$(a + b)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3.$$

$$(a + b)^4 = a^4 + 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 + b^4.$$

$$(a + b)^5 = a^5 + 5\,a^4b + 10\,a^3b^2 + 10\,a^2b^3 + 5\,ab^4 + b^5.$$

By a study of these the following facts may be observed:

1. Each product has one term more than the number of units in the exponent of the binomial.

2. The exponent of a in the *first* term is the same as the exponent of the binomial, and diminishes by unity in each *succeeding* term.
The exponent of b in the *last* term is the same as the exponent of the binomial, and diminishes by unity in each *preceding* term.

3. The sum of the exponents in each term is equal to the exponent of the binomial.

4. The coefficient of the first term is unity; of the second term, the same as the exponent of the binomial; and the coefficient of any other term may be found by multiplying the coefficient of the next preceding term by the exponent of a in that term and dividing this product by a number one greater than the exponent of b in that term.

5. The coefficients of any pair of terms equally distant from the ends are equal.

Statements 2 and 4 form a rule for writing out any power of a binomial up to the fifth. Let us find $(a + b)^6$.

Multiplying $(a+b)^5$ by $a+b$, we have

$$\begin{array}{r} (a+b)^5(a+b) = a^6 + 5a^5b + 10a^4b^2 + 10a^3b^3 + 5a^2b^4 + ab^5 \\ a^5b + 5a^4b^2 + 10a^3b^3 + 10a^2b^4 + 5ab^5 + b^6 \\ \hline \text{Hence } (a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6 \end{array}$$

From this it is seen that the rule holds also for $(a+b)^6$.

PROOF BY MATHEMATICAL INDUCTION

222. A proof that the above rule holds for *all positive integral powers* of a binomial may be made as follows:

First step. Write out the product as it *would be* for the nth power on the supposition that the rule holds.

Then the first term would be a^n and the last term b^n. The second terms from the ends would be $na^{n-1}b$ and nab^{n-1}. The third terms from the ends would be $\frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2$ and $\frac{n(n-1)}{1 \cdot 2}a^2b^{n-2}$. The fourth terms from the ends would be

$$\frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^{n-3}b^3 \text{ and } \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^3b^{n-3},$$

and so on, giving by the hypothesis,

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2 + \cdots + \frac{n(n-1)}{1 \cdot 2}a^2b^{n-2} + nab^{n-1} + b^n.$$

Second step. Multiply this expression by $a+b$ and see if the result can be so arranged as to conform to the same rule. Then, $(a+b)^n(a+b)$

$$\begin{array}{l} = a^{n+1} + na^nb + \frac{n(n-1)}{2}a^{n-1}b^2 + \cdots \qquad + na^2b^{n-1} + ab^n \\ \qquad a^nb \qquad + na^{n-1}b^2 + \cdots + \frac{n(n-1)}{1 \cdot 2}a^2b^{n-1} + nab^n + b^{n+1}. \\ \hline \end{array}$$

Hence adding,

$$(a+b)^{n+1} = a^{n+1} + (n+1)a^nb + \left[\frac{n(n-1)}{1 \cdot 2} + n\right]a^{n-1}b^2 + \cdots$$
$$+ \left[n + \frac{n(n-1)}{1 \cdot 2}\right]a^2b^{n-1} + (n+1)ab^n + b^{n+1}.$$

Combining the terms in brackets, we have,

$$(a+b)^{n+1} = a^{n+1} + (n+1)a^nb + \frac{(n+1)n}{1 \cdot 2}a^{n-1}b^2 + \cdots$$
$$+ \frac{(n+1)n}{1 \cdot 2}a^2b^{n-1} + (n+1)ab^n + b^{n+1}.$$

The last result shows that the rule holds for $(a+b)^{n+1}$ *if it holds for* $(a+b)^n$. That is, *if the rule holds for any positive integral exponent, it holds for the next higher integer.*

Third step. It was found above by *actual multiplication* that the rule does hold for $(a+b)^6$. Hence by the above argument we know that the rule holds for $(a+b)^7$.

Moreover, since we now know that the rule holds for $(a+b)^7$, we conclude by the same argument that it holds for $(a+b)^8$, and if for $(a+b)^8$, then for $(a+b)^9$, and so on.

Since this process of extending to higher powers can be carried on indefinitely, we conclude that the five statements in § 221 hold for all positive integral powers of a binomial.

The essence of this proof by **mathematical induction** consists in applying the *supposed* rule to the nth power and finding that the rule does hold for the $(n+1)$th power *if it holds for the nth power.*

223. The general term. According to the rule now known to hold for any positive integral exponent, we may write as many terms of the expansion of $(a+b)^n$ as may be desired, thus:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2$$

$$+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}a^{n-4}b^4 + \cdots. \qquad \text{I}$$

From this result, called the **binomial formula**, we see:

(1) The exponent of b in any term is one less than the number of that term, and the exponent of a is n minus the exponent of b. Hence the exponent of b in the $(k+1)$st term is k, and that of a is $n-k$.

(2) In the coefficient of any term the last factor in the denominator is the same as the exponent of b in that term, and the last factor in the numerator is one more than the exponent of a.

Hence the $(k+1)$st term, which is called the **general term** is

$$\frac{n(n-1)(n-2)(n-3)\cdots(n-k+1)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots k}a^{n-k}b^k. \qquad \text{II}$$

224. The process of writing out the power of a binomial is called **expanding the binomial**, and the result is called the **expansion of the binomial.**

Ex. 1. Expand $(x-y)^4$.

In this case $a = x$, $b = -y$, $n = 4$.

Hence substituting in formula I,

$$(x-y)^4 = x^4 + 4x^3(-y) + \frac{4(4-1)}{2}x^2(-y)^2 + \frac{4(4-1)(4-2)}{2\cdot 3}x(-y)^3 + \frac{4(4-1)(4-2)(4-3)}{2\cdot 3\cdot 4}(-y)^4 \quad (1)$$

$$= x^4 - 4x^3y + \frac{4\cdot 3}{2}x^2y^2 - \frac{4\cdot 3\cdot 2}{2\cdot 3}xy^3 + \frac{4\cdot 3\cdot 2\cdot 1}{2\cdot 3\cdot 4}y^4. \quad (2)$$

Hence $(x-y)^4 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$. (3)

Notice that this is precisely the same as the expansion of $(x+y)^4$ except that every other term beginning with the second is *negative*.

Ex. 2. Expand $(1-2y)^5$.

Here $a = 1$, $b = -2y$, $n = 5$.

Since the coefficients in the expansion of $(a+b)^5$ are 1, 5, 10, 10, 5, 1, we write at once,

$$(1-2y)^5 = 1^5 + 5\cdot 1^4\cdot(-2y) + 10\cdot 1^3\cdot(-2y)^2 + 10\cdot 1^2\cdot(-2y)^3 + 5\cdot 1\cdot(-2y)^4 + (-2y)^5$$
$$= 1 - 10y + 40y^2 - 80y^3 + 80y^4 - 32y^5.$$

Ex. 3. Expand $\left(\frac{1}{x}+\frac{y}{3}\right)^5$.

Remembering the coefficients just given, we write at once,

$$\left(\frac{1}{x}+\frac{y}{3}\right)^5 = \left(\frac{1}{x}\right)^5 + 5\left(\frac{1}{x}\right)^4\left(\frac{y}{3}\right) + 10\left(\frac{1}{x}\right)^3\left(\frac{y}{3}\right)^2 + 10\left(\frac{1}{x}\right)^2\left(\frac{y}{3}\right)^3 + 5\left(\frac{1}{x}\right)\left(\frac{y}{3}\right)^4 + \left(\frac{y}{3}\right)^5\left(\frac{1}{x}+\frac{y}{3}\right)^5$$

$$= \frac{1}{x^5} + \frac{5}{3}\frac{y}{x^4} + \frac{10}{9}\frac{y^2}{x^3} + \frac{10}{27}\frac{y^3}{x^2} + \frac{5}{81}\frac{y^4}{x} + \frac{y^5}{243}.$$

In a similar manner any positive integral power of a binomial may be written.

Ex. 4. Write the *sixth term* in the expansion of $(x-2y)^{10}$ without computing any other term.

From II, § 223, we know the $(k+1)$st term for the nth power of $a+b$, namely, $\frac{n(n-1)(n-2)\cdots(n-k+1)}{2\cdot 3\cdot 4\cdots k}a^{n-k}b^k.$

In this case $a=x$, $b=-2y$, $n=10$, $k+1=6$. Hence $k=5$. Substituting these particular values, we have

$$\frac{10(10-1)(10-2)\cdots(10-5+1)}{2\cdot 3\cdot 4\cdot 5}x^{10-5}(-2y)^5$$

$$=\frac{10\cdot 9\cdot 8\cdot 7\cdot 6}{2\cdot 3\cdot 4\cdot 5}x^5(-32y^5)$$

$$=-32\cdot 252\,x^5y^5=-8064\,x^5y^5.$$

EXERCISES

1. Make a list of the coefficients for each power of a binomial from the 2d to the 10th.

Expand the following:

2. $(x-y)^3$.

3. $(2x+3)^3$.

4. $(3x+2y)^4$.

5. $(3+y)^5$.

6. $(x^3+y)^6$.

7. $(x-y^3)^6$.

8. $(x^3-y^2)^7$.

9. $(x^{\frac{1}{2}}-y^{\frac{1}{2}})^4$.

10. $(x^{-1}+y^{-2})^5$.

11. $(a-b)^8$.

12. $(x+y)^9$.

13. $(m-n)^{10}$.

14. $(r^{\frac{1}{2}}+s^2)^4$.

15. $(c^{-2}-d^{-\frac{1}{2}})^5$.

16. $(\sqrt{a}-\sqrt{b})^6$.

17. $\left(\frac{x^2}{y}-\frac{y^2}{x}\right)^3$.

18. $\left(\frac{2x}{y^2}-y\sqrt{x}\right)^3$.

19. $\left(\frac{\sqrt{m}}{\sqrt[3]{n^2}}+\sqrt{\frac{y}{n}}\right)^4$.

20. $\left(\frac{c\sqrt[3]{c}}{\sqrt[5]{d^4}}-\frac{\sqrt[3]{d}}{c}\right)^7$.

21. $(2a^2x^{-2}-3by^{-3})^4$.

22. $(3xy^{-3}-x^{-3}y)^8$.

In each of the following find the term called for without finding any other term:

23. The 5th term of $(a+b)^{12}$.

24. The 7th term of $(3x-2y)^{11}$.

25. The 6th term of $(\sqrt{x}-\sqrt[3]{y})^{10}$.

26. The 9th term of $(x-y)^{25}$.

27. The 8th term of $(\frac{1}{2}m-\frac{1}{3}n)^{18}$.

28. The 7th term of $(a^2b-ab^2)^{30}$.

29. The 6th term of $(a-a^{-1})^{2k}$.

30. The 11th term of $(x^2y-x^{-2}y^{-1})^{3m}$.

31. The 5th term from each end of the expansion of $(a-b)^{20}$.

32. The 7th term from each end of $(a\sqrt{a}-b\sqrt{b})^{21}$.

33. Which term, counting from the beginning, has the same coefficient as the 7th term of $(a+b)^{10}$? Verify by finding both coefficients. How do the exponents differ in these terms?

34. What other term has the same coefficient as the 19th term of $(a+b)^{24}$? How do the exponents differ? Find in the shortest way the 21st term of $(a+b)^{25}$.

35. Find the 87th term of $(a+b)^{90}$.

36. Find the 53d term of $(a^{\frac{1}{2}}-b^{\frac{1}{3}})^{56}$.

37. What other term has the same coefficient as the 5th term in the expansion of $(x+y)^{19}$?

38. Expand $[(a+b)+c]^3$ by the binomial formula.

39. Expand $[1+(2x+3y)]^4$ by the binomial formula.

40. Expand $(2x-3y+4z)^3$ by the binomial formula.

41. Write the $(k+1)$st term of $(a+b)^n$. Write the $(n+1)$st term of $(a+b)^n$. Show that the next and also all succeeding terms after the $(n+1)$st term have zero coefficients, thus proving that there are exactly $n+1$ terms in the expansion.

INDEX

[References are to pages.]

[References are to pages.]

[References are to pages.]

[References are to pages.]

[References are to pages.]

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