# NEWTON'S PRINCIPIA

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# THE FIRST THREE SECTIONS

OF

# NEWTON'S PRINCIPIA,

## WITH AN APPENDIX;

AND

## THE NINTH AND ELEVENTH SECTIONS.

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*THE FIFTH EDITION,*

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# PREFACE TO THE FIFTH EDITION.

In the present Edition the first three sections of Newton's *Principia*, together with the Chapters headed 'Definitiones,' and 'Axiomata, sive Leges Motûs,' which form the Introduction, have been translated from the Latin edition of Le Seur and Jacquier, with omission of the Scholia at the end of these Chapters: notes have been added in small print in cases where the text seemed to require explanation. This part of the book is followed by an Appendix, consisting mainly of examples in illustration of the methods used in the Lemmas, and of three important propositions from the first book of the *Principia*. The ninth and eleventh sections are added with little alteration from the last edition of Evans.

A collection of Examples is given at the end, for which I am partly indebted to the kindness of Mr Hudson, Fellow of St John's College, partly to various University and College Examination Papers.

In the course of preparing this Edition I have received many useful suggestions from friends, and especially from Mr Besant, late Fellow of St John's College, which I am glad to take this opportunity of gratefully acknowledging.

P. T. MAIN.

ST JOHN'S COLLEGE,
*August* 29, 1871.

ERRATUM.

On page 93, last line but three,

$$\textit{for } \pi\sqrt{\frac{AC^3}{\mu}} \textit{ read } 2\pi\sqrt{\frac{AC^3}{\mu}}.$$

# NEWTON'S PRINCIPIA.

## DEFINITIONS.

DEF. 1. *Quantity of matter is the measure of it arising from its density and bulk conjointly.*

AIR, when its density is doubled, and it also occupies twice the space, is quadrupled in amount; in thrice the space is increased six-fold. The same is to be understood when snow or powders are condensed by compression or liquefaction. And the same rule applies to all bodies which are condensed in different manners by any causes whatever.

No account is taken here of the medium, if there be one, which penetrates freely the interstices between the parts of bodies.

This *quantity of matter* is, in what follows, sometimes called the body, or mass. It is known for each body by means of its weight; for it has been found, by very accurate experiments with pendulums, to be proportional to the weight.

*Note* 1. It is found by experiments with pendulums that the time of oscillation at any given place depends on the length of the pendulum, but not on the nature or amount of the substance of which it is composed.

From this it is inferred that the motion produced in a pendulum by the earth's attraction is produced equally on every part of the matter of it, so that the whole motion produced (Def. 2) is proportional to the quantity of matter in it: and for any two pendulums of the same length, since each oscillates in the same period, the whole motion produced in each is simply proportional to the quantity of matter.

This is also confirmed by experiments in tubes exhausted of air, in which a feather, a lump of lead, of gold, or of any other substance, are all found to occupy equal times in falling through equal spaces. The motion produced by gravity in every part of each body being thus seen to be the same, it follows that the whole quantity of motion which gravity produces or tends to produce in each—in other words, the weight—is proportional to the quantity of matter.

*Note* 2. The quantity of matter, or mass, of a body, is denoted by the symbol $M$.

*Note* 3. The density of a body may be defined as the quantity of matter it contains in a unit of volume.

If this be represented by $\rho$, and the volume by $V$, we have

$$M = V\rho.$$

Def. 2. *The quantity of Motion of a body is the Measure of it, arising from its velocity and the quantity of matter conjointly.*

The motion of the whole body is the sum of the motions of its several parts; and therefore in double the body with an equal velocity there is double, and with double the velocity quadruple the quantity of motion.

*Note.* Let $M$ be the mass of a body, $v$ its velocity, then $Mv$ is the *motion* of a body, i.e. the quantity of motion of the body.

Def. 3. *The innate force of matter is its power of resisting, whereby every body, so far as depends on itself, perseveres in its state, either of rest, or of uniform motion in a straight line.*

This is always proportional to the body, and differs in no respect from the inertia of the mass, except in the manner of viewing it. To the inertia of matter is due the difficulty of disturbing bodies from their state of rest or motion; on which account the innate force may be called by the very suggestive name, *force of inertia.*

A body, however, only exerts this force when a change is made in its state by another force impressed on it; and the exertion of it constitutes, from opposite points of view, both resistance and pressure: resistance, inasmuch as the body, in order to preserve its own state, opposes the impressed force; pressure, inasmuch as the body tries, by yielding with difficulty to the force of an opposing obstacle, to change the state of the obstacle.

People in general attribute resistance to bodies at rest and pressure to bodies in motion; but motion and rest, as ordinarily understood, are only distinguished one from another in appearance; and bodies are not always really at rest which are popularly supposed to be at rest.

*Note.* To produce a given change in the velocity of a body a force must be exerted on it which will be greater or less according to the mass of the body, and will in fact be proportional to the mass, so that a body of twice the mass will require twice the force to produce the same change in the velocity. It is in this sense that a body is said in this Definition to possess an innate power of resisting an attempt to move it, proportional to its mass.

Def. 4. *An impressed force is an action exerted on a body, tending to change its state either of rest or of uniform motion in a straight line.*

This force consists in the action only, and does not remain in the body after the action. For the body perseveres in each new state by the force of inertia alone.

An impressed force may arise in various ways, as from a blow, a pressure, a centripetal force.

*Note.* In modern works on Mechanics it is customary to restrict the use of the word force to impressed forces only; in place, therefore, of the expression, force of inertia, the word *inertia* alone is used.

Def. 5. *A centripetal force is one by which bodies are drawn, impelled, or in any other way tend from all parts towards some point as a centre.*

Of this kind is *gravity*, by which bodies tend to the centre of the earth; *magnetic force*, by which iron approaches a magnet; and that force, whatever it may be, by which the planets are perpetually drawn away from rectilinear motions, and forced to revolve in curves.

A stone whirled in a sling tries to get away from the hand by which it is whirled; and by its effort stretches the sling, and that the more powerfully the quicker it revolves; and as soon as it is released, it flies off. The force, which opposes this effort, by which the sling perpetually drags the stone back towards the hand and retains it in its orbit, since it is directed to the hand as the centre of its orbit, is called the *centripetal* force.

The same account applies to all bodies, which are made to move in an orbit. They all try to recede from the centres of their orbits; and unless there is some centripetal force, as it is called, opposing the effort to recede—by which the bodies may be kept back and retained in their orbits—they will go off in straight lines with uniform motion.

A projectile, if it were deprived of the force of gravity, would not be deflected towards the earth, but go off into space in a straight line; and with uniform motion, if the resistance of the air were withdrawn. By its gravity it is drawn away from a rectilinear course, and perpetually turned towards the earth, and more or less according to its gravity and the velocity of its motion. The less gravity it has for a given quantity of matter, or the greater the velocity with which it is projected, the less will it deviate from a rectilinear course, and the farther it will go. If a ball of lead, shot from a cannon by the force of gunpowder with a given velocity in a horizontal direction, went in a curve to a distance of two miles before falling to the ground; with double the velocity it would go twice as far, and with ten times the velocity ten times as far; provided the resistance of the air were removed. And by increasing the velocity the distance to which it is shot may be increased at pleasure, and the curvature of the path it describes be diminished, so that it may fall at a distance of ten, or twenty, or ninety degrees; or even go all round the earth, or lastly go right away into space, and proceed for ever with the motion with which it went off.

And for the same reason, that a projectile can be deflected by the force of gravity and go all round the earth, the moon may, either by the force of gravity, if it has weight, or by some other force urging it towards the earth, be always drawn from its rectilinear course earthwards, and deflected into its orbit; and without such a force the moon cannot be retained in its orbit. This force, if it were less than the proper amount, would not sufficiently deflect the moon from its rectilinear course; if greater than the proper amount, it would deflect it more than enough and turn it out of its orbit earthwards. In fact it is necessary that the force should be of the right amount; and it is for mathematicians to find the force by which a body can be exactly retained in any given orbit whatever with a given velocity; and conversely, to find the curvilinear path into which a body, moving from any given place with a given velocity, would be deflected by a given force.

The quantity of this centripetal force is of three kinds, absolute, accelerative, and motive.

Def. 6. *The absolute quantity of a centripetal force is a measure of it which is greater or less according to the efficacy of the cause which propagates it from the centre through the regions of space all round it.*

Just as *magnetic force* is greater in one magnet and less in another, according to the mass of the magnet or the intensity of its magnetism.

*Note.* This absolute quantity of a centripetal force arising from any cause, such as, for instance, the attraction of the earth, is usually measured by the efficacy of the cause at a unit of distance from the centre of the force,—in this case, the centre of the earth; and it is measured by the acceleration (Def. 7) which it is capable of producing at this distance.

Thus, if $\mu$ be the absolute quantity of a centripetal force, the force is such that at a unit of distance the acceleration produced by it would be represented by $\mu$.

The absolute quantity of a centripetal force is, for brevity, called the *absolute force.*

DEF. 7. *The accelerative quantity of a centripetal force is a measure of it proportional to the velocity which it generates in a given time.*

Just as the *power of the same magnet* is greater at a less distance, less at a greater. Or, as *gravitating force* is greater in valleys, less on the peaks of high mountains, and so (as it is proved to be) less the greater the distance from the earth; but at equal distances the same on all sides, because it accelerates equally all falling bodies (heavy or light, great or small).

*Note* 1. In treating of centripetal forces, where simply the *force* is spoken of in Newton's Principia, the *accelerative* quantity of the force is meant, unless otherwise stated.

The accelerative quantity of a centripetal force is usually called, for brevity, the *accelerating*, or *accelerative force.*

If a body is moving in a straight line to or from a centre of force, and the force *adds* a velocity $v$ in time $t$, it produces an accelerating effect equivalent to an addition (during the time $t$) of a velocity $\frac{v}{t}$ each unit of time, *on the average.*

A force which adds equal velocities in equal times is called a *uniform*, or *constant force.*

Thus $\frac{v}{t}$ is the measure of an accelerating force when the force may be considered to remain constant during the time $t$.

*Note* 2. If $t$ be taken sufficiently small $\frac{v}{t}$ may also be taken as the measure of a variable force, for a variable force may be considered constant during a very small time, the variation of the force bearing a ratio to the force itself which is smaller the smaller the time is, and which becomes indefinitely small when the time does.

Hence, if $v$ be the velocity generated in time $t$, the accelerating force is in all cases measured by the limit of $\frac{v}{t}$, when $t$ is indefinitely small.

The accelerating force is usually denoted by $f$; thus $f =$ limit of $\frac{v}{t}$.

*Note* 3. A force which acts to diminish the velocity of a body moving in a straight line is called a *retarding force*, and is measured, as to its retarding effect, by the velocity subtracted in a given time; thus a retarding force may be measured by $\frac{v}{t}$ if it is constant, $v$ being the velocity subtracted in time $t$; and by the limit of $\frac{v}{t}$ if it is variable.

*Note* 4. Forces are compared as to their accelerating or retarding effect by comparing the velocity, $v$, added or subtracted in equal indefinitely small intervals of time.

DEF. 8. *The motive quantity of a centripetal force is a measure of it proportional to the motion which it generates in a given time.*

Just as *weight* is greater in a greater mass, less in a less mass; and, in the same, is greater near the earth, less in remote space.

This quantity is the body's entire centripetency or tendency towards the centre of force, and (so to speak) its weight; and it is always known by the force equal and opposite to it, by which the fall of the body may be prevented.

These quantities of forces may for brevity be called *motive*, *accelerative*, and *absolute forces;* and, for the sake of distinctness, may be ascribed severally to the bodies which tend to the centre, to the positions of the bodies, and to the centre of forces: so that, in fact, the motive force is ascribed to the body, as if it were the effort of the whole composed of the efforts of all its parts; the accelerative force to the position of the body, as if there were diffused from the centre to all places around it some power efficacious towards moving bodies which are in those places: and the absolute force to the centre, as if at this point there were situated something which was the cause of motive forces being propagated through space in all directions; whether that cause be some central body (just as a magnet is at the centre of magnetic force, or the earth at the centre of gravitating force) or any other cause which is not ascertained. This is simply a mathematical conception; the physical causes and seats of the forces are not here considered.

The accelerative force is, then, to the motive force as the velocity generated is to the motion. For the quantity of motion arises from the velocity and the quantity of matter conjointly, and the motive force from the accelerative force and the quantity of the same matter; for the sum of the actions of the accelerative force on the several particles of a body is the motive force of the whole.

Hence, at the surface of the earth, where the accelerative gravity, or force of gravitation, is the same on all bodies, the motive gravity, or weight, varies as the body; but if we ascend into regions where the accelerative gravity is less, the weight will diminish equally, and will be always as the body and the accelerative gravity conjointly. Thus in regions where the accelerative gravity is half as great, the weight of a body half or a third as great will be a fourth or a sixth as great.

Moreover, we may in the same sense speak of attractions and impulses as accelerative and motive. But the words, attraction, impulse, tendency, of any body towards a centre may be used indifferently and promiscuously one for another; these forces being here considered not in a physical, but only in a mathematical sense. The reader should beware, in using words of this sort, of considering them as defining the kind or manner of the action, or their physical cause or reason; or of attributing to the centres (which are mathematical points) forces in a real physical sense, when it is said either that the centres attract, or that there are forces at the centres.

*Note* 1. By *motion* is meant, in this definition, and elsewhere, the quantity of motion as defined in Def. 2.

The word *body* is used here in the sense of Def. 1.

If $M$ be the mass of a body, $v$ the velocity generated in it by a force in time $t$, $Mv$ is the measure of the motion produced in it in time $t$; and $\frac{Mv}{t}$ is the measure of the average motive force during the time $t$; or of the actual motive force, if $t$ be taken indefinitely small.

*Note* 2. Forces are compared with each other as to their motive effect by comparing the motion, or $Mv$, produced in equal indefinitely small intervals of time.

# AXIOMS, OR LAWS OF MOTION.

LAW I. *Every body perseveres in its state of rest, or of uniform motion in a straight line, except in so far as it is compelled to change that state by forces impressed on it.*

Projectiles persevere in their motions, except in so far as they are retarded by the resistance of the air, and driven downwards by the force of gravity. A hoop, whose parts continually draw each other from their rectilinear motions by cohesion, ceases to roll only in consequence of its motion being retarded by the air. But the larger bodies of planets and comets, whose motions, both progressive and circular, take place in less resisting spaces, retain these motions longer.

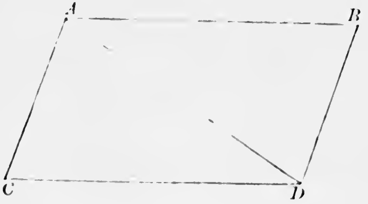
LAW II. *Change of motion is proportional to the moving force impressed, and takes place in the straight line in which that force is impressed.*

If a force produce any motion, twice the force will produce twice the motion, thrice the force three times the motion, whether it has been impressed all at once, or by successive gradations. And this motion (since it must always take place in the same direction as the force which produces it) is—if the body was originally in motion—added to its original motion if that motion was in the same direction, subtracted from it if in the opposite; or if in an inclined direction, is added to it in an inclined direction, and compounded with it, the position of the body being determined by the motion in each direction.

LAW III. *An action is always opposed by an equal reaction; or, the mutual actions of two bodies are always equal and act in opposite directions.*

Whatever presses or pulls something else, is pressed or pulled by it in the same degree. If a man presses a stone with his finger, his finger is also pressed by the stone. If a horse draws a stone tied to a rope, the horse will be (so to speak) drawn back equally towards the stone: for the rope being stretched at both ends will by the same attempt to relax itself urge the horse towards the stone and the stone towards the horse; and will impede the progress of one as much as it promotes the progress of the other. If a body impinge on another and by its force change the motion of the other in any way, the latter will in its turn (on account of the equality of the mutual pressure) undergo the same change of motion in a contrary direction. To these actions are equal the changes, not of velocities, but of motions; that is, in bodies not hindered in their motions by other forces. For the changes of velocities, which also take place in the same direction, are—since the motions are changed equally—reciprocally proportional to the bodies. This law holds also in Attractions.

COR. 1. *By the combined action of two forces a body will describe the diagonal of a parallelogram in the same time as the sides would be described by the body under the action of each force separately.*




If a body, by the action of the force *M* only, impressed at *A*, would move with uniform motion from *A* to

*B* in a given time, and, by the action of the force *N* only, impressed at the same point, would move from *A* to *C*; complete the parallelogram *ABDC*; then, by the action of both forces, the body will move in the same time on the diagonal from *A* to *D*.

For, since the force *N* acts in the direction of *AC* parallel to *BD*, this force, by Law II., will not alter the velocity of approach to the line *BD*, due to the other force.

Therefore the body will approach the line *BD* in the same time, whether the force *N* be impressed or not; and thus, at the end of the given time, it will be found somewhere on the line *BD*.

By the same reasoning, it will be found at the end of the same time to be somewhere on the line *CD*; and therefore it must necessarily be at the point *D* where these lines meet.

And it will go from *A* to *D* with uniform rectilinear motion, by Law I.

Cor. 2. *And hence follows at once the composition of a force* AD *out of other forces* AB *and* AC *acting in different directions; and conversely the resolution of any force* AD *into two others* AB *and* AC.

*Note* 1. In other words the parallelogram of forces follows at once from Corollary 1.

For by Law II. the change of motion of a body, or of the velocity of the body, is proportional to the moving force impressed.

Now, by Cor. 1, the resultant of *M* and *N* impresses on the body a velocity with which it would describe the straight line *AD* in the same time as the straight lines *AB*, *AC* would be described with the velocities impressed on the body by *M* and *N*.

Therefore *M*, *N* and the resultant of these, being, by Law II. proportional to the velocities impressed by them severally, are proportional to *AB*, *AC*, and *AD*; and they act respectively in these directions, by Law II.

And thus, if *AB*, *AC* be taken to represent the forces *M* and

$N$ in magnitude and direction, $AD$ will represent their resultant also in magnitude and direction.

As stated under Law II., it is immaterial in what manner the forces $M$ and $N$ act; if they be measured, as that Law directs, simply by the change of motion (Def. 2) produced.

*Note* 2. Force may be conceived to act in two ways; (1) impulsively, that is by instantaneously generating a change of motion of the body on which it acts: (2) continuously, that is, so as to generate a change of motion which shall be finite in any finite time, but indefinitely small in an indefinitely small time.

A force which is supposed to act in the first way is called an *impulsive force:* and a force which is supposed to act in the second way is called a *finite force.*

An impulsive force is measured, by Law II., by the change of motion produced; a finite force by the change of motion produced *in a given time* (Def. 8).

A finite force is said to be *constant*, or *uniform*, if the motion produced in any given interval of time is always the same. Thus a constant force produces in a given body always the same change of velocity in a given time. It is hence also called a uniformly accelerating force. A finite force is said to be *variable*, when the motion produced in a given time is not always the same. Thus, the force of gravitation is variable, the variation depending on the distance of the attracted from the attracting body.

*Note* 3. When force is constant, since equal changes of motion are produced in equal intervals of time, the change of motion produced in any time is proportional to the time; this is not the case with variable forces. But in either case the amount of force impressed *in any given time* is measured by the amount of motion produced in that time.

Forces are compared with one another by comparing the motions generated in the same given time; hence *constant* forces, since they generate equal amounts of motion in equal times, are to one another in a ratio *independent of the time in which they generate their motions.*

Again, during any indefinitely small period (Def. 7, Note 2), a *variable* force produces, in equal times, motions proportional to the times; therefore the motions produced by any two *variable* forces in given equal times are to one another ultimately in a ratio independent of the time in which the motions are generated, when the time is indefinitely diminished.

If therefore forces whether uniform or variable be compared among one another by comparing the motions produced by them in equal times *when those times are indefinitely diminished*, their measures will be independent of the time in which they generate their motions. For this reason forces are always estimated by comparing the motions produced by them *in equal indefinitely small intervals of time.*

COR. 3. *The quantity of motion, which is obtained by taking the sum of the motions which take place in the same direction, and the difference of the motions in opposite directions, is not changed by any action of bodies among one another.*

For an action and its opposite reaction are equal, by Law III.; and therefore by Law II. equal changes of motion are produced by them in opposite directions. Therefore, if the motion of two bodies take place in the same direction, whatever is added to the motion of the foremost, will be subtracted from the motion of the hindermost, so that the sum remains the same as before. And if the bodies meet there will be an equal loss of the motion of each, and therefore the difference of the motions taking place in opposite directions will remain the same.

COR. 4. *The common centre of gravity of two or more bodies does not change its state of motion or rest through the mutual actions of the bodies; and hence, in the absence of external actions or resistances, the common centre of gravity either is at rest or moves uniformly in a straight line.*

COR. 5. *Bodies inclosed in a given space have the same motions relatively to one another, whether that space be at rest, or be moving uniformly in a straight line without rotation.*

COR. 6. *If bodies are moving relatively to one another in any manner, and are urged by equal accelerating* *forces in parallel directions; they will all continue to move relatively to one another in the same manner as if they were not acted on by those forces.*

[The proofs of Cors. 4, 5, and 6, together with the Lemma on which they depend, are given in the Appendix.]

# SECTION I.

## *On the Method of Prime and Ultimate Ratios.*

### Lemma I.

*Quantities, and ratios of quantities, which tend constantly to equality during any finite time, and approach each other more nearly than for any assignable difference, become ultimately equal.*

If not, let them become ultimately unequal, and let their ultimate difference be $D$.

Therefore, they cannot approach each other more nearly than for the difference $D$; contrary to the hypothesis.

*Note* 1. In this Lemma, the *quantities*, and the *ratios of quantities*, are supposed to remain finite throughout.

When, as in succeeding Lemmas, quantities are concerned which become indefinitely small, or which become indefinitely great, another set of quantities is taken which bear constant ratios to the quantities with which we are concerned, and one at least of which remains finite.

The ratios of the quantities *inter se* is then the same for each set; and to *these ratios*, if they remain finite, the Lemma applies.

*Note* 2. If the quantities tend constantly to equality during a time which is *not* finite, they will not necessarily become equal in any finite time.

*Note* 3. *If* a, b, *and* c *be quantities, which at the end of a finite time vanish together, and the ratio of* a *to* b *be ultimately a ratio of equality, the ratio of* a + c *to* b + c *shall be ultimately a ratio of equality.*

For, take $A$, $B$, and $C$ always proportional to $a$, $b$, and $c$ respectively, and such that $A$ is ultimately finite;

then, ultimately, $A = B$;

therefore, ultimately, $A + C = B + C$;

therefore, ultimately, the ratio of $A + C$ to $B + C$ is one of equality.

But, by hypothesis, $\frac{a}{A} = \frac{b}{B} = \frac{c}{C}$ always; and each of these fractions is equal to $\frac{a+c}{A+C}$ and $\frac{b+c}{B+C}$; therefore

$$\frac{a+c}{A+C} = \frac{b+c}{B+C};$$

and

$$\frac{a+c}{b+c} = \frac{A+C}{B+C};$$

and this being true always, is true ultimately; therefore the ratio of $a + c$ to $b + c$ is ultimately a ratio of equality.

The ultimate value of a quantity or a ratio is called its *limit;* thus limit of $a + x$ (when $x$ vanishes) is $a$; this word is sometimes abbreviated into *lt.*, thus,

$$\text{lt. } \frac{1+x}{1-x} \text{ (when } x = 0) = 1.$$

*Note* 4. The ultimate value of the ratio of two quantities is often called the limiting ratio; as we shall frequently have occasion to speak of limiting ratios, we shall allude to them by the abbreviation L. R.

Thus, in the previous note, it is proved that, if the limiting ratio of $a$ to $b$ is one of equality, the limiting ratio of $a + c$ to $b + c$ is one of equality; or, if L.R. of $a$ to $b$ is one of equality, then L.R. of $a + c$ to $b + c$ is one of equality.

Again, a ratio of equality will for brevity be often designated simply by 1.

Thus, if L.R. of $a$ to $b$ is 1, then also L.R. of $a + c$ to $b + c$ is 1.

*Note* 5. Two quantities are said to be *equal* when their difference vanishes, and to be *ultimately equal* when their difference ultimately vanishes.

The ratio to one another of two equal quantities is a ratio of equality.

But the limiting ratio of two quantities which are ultimately equal is not necessarily a ratio of equality; for the quantities themselves may vanish, and their ultimate ratio may then be any whatever.

The student is therefore advised, in any reasoning in which vanishing quantities are concerned, not to say merely that two quantities are ultimately equal when he means that their ratio becomes a ratio of equality. If this distinction be not borne in mind, he will be apt sometimes to draw, from the fact that two quantities are ultimately equal, the inference that their ratio is one of equality, where such inference is not warranted.

In Note 3, we proved that ultimately

$$A + C = B + C,$$

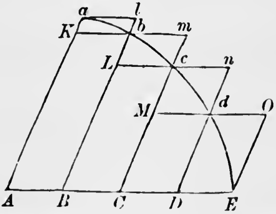
and we inferred that ultimately the ratio of $A + C$ to $B + C$ is a ratio of equality; this is correct, because their difference vanishes, but they themselves do not. We were not able to infer from the fact that $a + c$ is equal to $b + c$, that the ratio of $a + c$ to $b + c$ is one of equality, because both these quantities vanish.

In fact, we can infer, from two quantities being ultimately equal, that their limiting ratio is one of equality, only *when the quantities do not vanish.*

## Lemma II.

*If in any figure* AacE *bounded by the straight lines* Aa, AE, *and the curve* acE, *there be inscribed any number of parallelograms* Ab, Bc, Cd, ... *on equal bases* AB, BC, CD, &c., *and with sides* Bb, Cc, Dd, ... *parallel to the side* Aa *of the figure; and the parallelograms* aKbl, bLcm, cMdn, ... *be completed: then, if the breadth of these parallelograms be diminished and their number increased indefinitely, the ultimate ratios which the inscribed figure* AKbLcMdD, *the circumscribed figure* AalbmcndeE, *and the curvilinear figure* AabcdE *bear to one another are ratios of equality.*

For the difference of the inscribed and circumscribed figures is the sum of the parallelograms *Kl*, *Mm*, *Nn*, *Do*; which is (since all the bases are equal) the rectangle con-




tained by one base *Kb* and the sum of all the altitudes *Aa*. *i.e.* the parallelogram *ABla*.

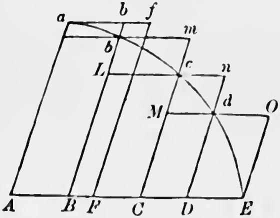
But this parallelogram, when its breadth is diminished indefinitely, becomes less than any assignable quantity.

Therefore by Lemma I., the inscribed and circumscribed figures, and *à fortiori* the curvilinear figure which is intermediate between them, are equal. Q. E. D.

### Lemma III.

*The same ultimate ratios are also ratios of equality. when the breadths* AB, BC, CD... *of the parallelograms are unequal, and are all diminished indefinitely.*

For let *AF* be equal to the greatest breadth, and let the parallelogram *FAaf* be completed.




This will be greater than the difference between the inscribed and circumscribed figures; but when its breadth

$AF$ is diminished indefinitely, it will become less than any assignable rectangle. Q. E. D.

COR. 1. Hence the ultimate sum of the vanishing parallelograms coincides in all respects with the curvilinear figure.

COR. 2. And *à fortiori* the rectilinear figure, bounded by the chords of the evanescent arcs *ab*, *bc*, *cd*..., coincides ultimately with the curvilinear figure.

COR. 3. As also does the circumscribed rectilinear figure, bounded by the tangents to the same arcs.

COR. 4. And consequently these ultimate figures (in respect of their perimeters $acE$) are not rectilinear but curvilinear limits of rectilinear figures.

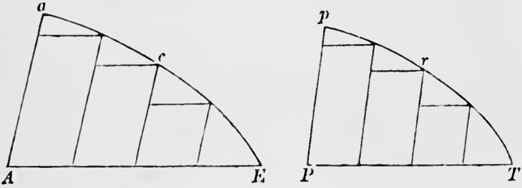
*Note.* By this is meant that the curvilinear perimeter of any curvilinear figure is identical with the limit of a rectilinear perimeter of a polygon; consisting either of an infinite number of sides which coincide ultimately with consecutive chords of indefinitely small arcs of the curve into which the polygon degenerates, or, which is the same thing when the limit is reached, of tangents to consecutive points of the curve; so that any proposition regarding the perimeter of such a polygon is true of the curve which is its limit.

This corollary gives implicitly a means of measuring an arc of a curve in terms of an indefinite number of indefinitely small straight lines, which may be either the series of consecutive chords or of consecutive tangents. It is tacitly assumed that these two measures would be ultimately the same. This is easily proved by Newton's method (*vide* Appendix, Lemma VII.).

### LEMMA IV.

*If in the two figures* AacE, PprT *there be inscribed two series of parallelograms, and there be the same number in each series, and if when the breadths are indefinitely diminished the ultimate ratios of the parallelograms in the one figure to the parallelograms in the other are the same, each to each; the two figures* AacE, PprT *are to each other in that ratio.*

For as the parallelograms are to each other, each to each, so (*componendo*) is the sum of all in one figure to the




sum of all in the other, and so therefore the one figure to the other; the former figure being to the former sum, and the latter figure to the latter sum (by Lemma III.), in a ratio of equality. Q. E. D.

COR. Hence if two quantities of any kind whatever be divided in any manner into the same number of parts; and those parts, when their number is increased and their size diminished indefinitely, have a given ratio to each other, the first part to the first, the second to the second, and the rest to the rest in their order, the whole shall be to each other in the same ratio. For if, in the figures of this Lemma, the parallelograms be taken to each other in the same ratios as the parts, the sums of the parts will be to each other always as the sums of the parallelograms; and therefore,—when the number of the parts and parallelograms is increased and their magnitude diminished indefinitely,—in the ultimate ratio of parallelogram to parallelogram, *i.e.* (by hypothesis) in the ultimate ratio of part to part.

*Note.* The proof of this Lemma requires to be amplified; for it assumes that for the purposes of the proof each parallelogram in one figure bears to the corresponding parallelogram in the other the same ratio, whereas this is not supposed to be true till the limit has been reached. We may, however, say that the sum of the parallelograms in the first figure, bears a less ratio to that in the other than it would if each parallelogram of the first figure were to each parallelogram in the second in the greatest of the ratios of the corresponding parallelograms, and a less ratio than if each parallelogram were to each in the least of these ratios; and thus, since these greatest and least ratios are both, by hypothesis, ultimately the same, we infer that the sum of one set of parallelograms is ultimately to the sum of the other set in that ratio.

## Lemma V.

*The homologous sides, both curvilinear and rectilinear, of similar figures are proportionals; and their areas are in the duplicate ratio of the sides.*

By Euclid VI. Def. 1, similar rectilinear figures have their homologous sides proportional.

Hence, *componendo*, the sum of any number of sides of one figure has the same ratio to the sum of the corresponding sides of the other that any side of one has to the corresponding side of the other. Again (Euclid VI. 20), the areas are in the duplicate ratio of the sides.

Now let the number of the sides be increased and their lengths diminished indefinitely; then the sum of the sides in each figure becomes (Lemma III. Cor. 4) the arc which is their curvilinear limit, the areas become curvilinear areas, and the similar rectilinear figures similar curvilinear figures.

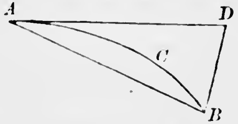
*Note* 1. No proof of this Lemma is given by Newton, the truth of it appearing at once from Lemma III. Cor. 4, and Euclid VI. 20, as indicated above.

*Note* 2. All lines which are proportional in similar rectilinear figures are proportional in the similar curvilinear figures which are their limits. And it appears, from Euclid VI. 20, that lines joining corresponding pairs of points in two similar rectilinear figures are all proportional to each other; this is therefore also true of similar curvilinear figures.

## Lemma VI.

*If any arc given in position be subtended by a chord* AD, *and be touched at a point* A *in the midst of continuous curvature by a straight line* AD *produced in either direction; then, if the points* A, B *move up to each other and finally coincide, the angle* BAD *between the chord and the tangent shall diminish indefinitely and ultimately vanish.*

For if that angle does not vanish the arc $ACB$ will make with the tangent $AD$ an angle equal to a rectilineal



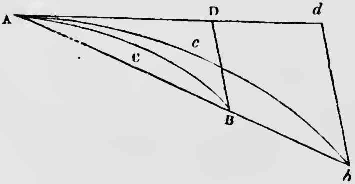

angle, and hence the curvature will not be continuous; contrary to the hypothesis.

*Note.* A straight line $AD$ touches a curve $ACB$ when it coincides with the limiting position of a straight line joining $A$ with a point of the curve near $A$ which moves up to and ultimately coincides with $A$. Thus the limiting position of $AB$ touches the curve at $A$; and if it does not coincide with $AD$, there are two straight lines touching the curve at $A$, so that the curve in passing through $A$ passes abruptly from contact with the limiting position of $AB$ to contact with $AD$; in this case, the curvature at $A$, that is, its rate of separation from the line that touches it at $A$, changes abruptly, and is therefore not continuous.

## Lemma VII.

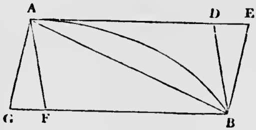
*If any arc given in position be subtended by a chord* AB, *and be touched at a point* A *in the midst of continuous curvature by a straight line* AD; *the ultimate ratio between the arc, chord, and tangent is a ratio of equality.*

For while the point $B$ moves up to $A$, suppose $AB$ and $AD$ to be produced to distant points $b$ and $d$, so that $bd$ is parallel to $BD$. At $Ab$ let an arc be described always similar to the arc $ACB$. Then, when the points $A$, $B$




coincide, the angle $dAb$ vanishes, by the preceding Lemma; and thus the straight lines $Ab$, $Ad$ and the intermediate arc $Acb$ coincide, and are therefore equal. Hence the straight lines $AB$, $AD$ (which are always proportional to $Ab$, $Ad$) and the intermediate arc $ACB$, will ultimately vanish in a ratio of equality. Q. E. D.

Cor. 1. Hence, if $BF$ be drawn through $B$ parallel to the tangent cutting any straight line $AF$ through $A$ in $F$, this line $BF$ will ultimately have to the vanishing arc




$ACB$ a ratio of equality, since—completing the parallelogram $AFBD$—it has always a ratio of equality to $AD$.

Cor. 2. And if through $B$ and $A$ be drawn several straight lines $BE$, $BD$, $AF$, $AG$, cutting the tangent $AD$ and the parallel to it $BF$; the ultimate ratio of all the abscissæ $AD$, $AE$, $BF$, $BG$, and of the chord and arc $AB$, will be a ratio of equality.

Cor. 3. And hence all these lines, in all reasoning on ultimate ratios, may be used one for another.

*Note* 1. By distant points $b$, $d$ are meant points taken so as to be always at a finite distance from $A$ (see note to Lemma I.).

*Note* 2. By the tangent $AD$ in this Lemma, is meant a part of the unlimited tangent at $A$ cut off by a line $BD$ which *always make a finite angle* with it. For in the proof it is assumed that the points $b$ and $d$ coincide in the limit, which they do not necessarily unless $bd$ makes always a finite angle with $Ad$.

*Note* 3. It is here assumed that $Acb$ is always touched by $Ad$. That it is so may be seen thus: by the properties of similar figures (*vide* Lemma V. Note 2) straight lines from $A$ to corresponding points on $ACB$, $Acb$ make the same angle with $AB$ or $Ab$, and are proportional to $AB$, $Ab$. Hence any straight line through $A$ meeting the two curves will be divided by them in the constant ratio of $AB$ to $Ab$. If then such a line be supposed to move up to and ultimately coincide with $AD$, the portion cut off by $ACB$ vanishes, since $AD$ touches $ACB$; therefore the portion cut off by $Acb$ vanishes, and consequently $AD$ touches $Acb$.

*Note* 4. These observations apply also to the two following Lemmas.

*Note* 5. In this and succeeding Lemmas Newton finds the limiting ratio of vanishing quantities by taking quantities always proportional to them, one of these new quantities, as $Ad$, remaining finite; he thus determines the ratios of quantities which vanish by means of the ratios of quantities which do not vanish.

The figure which the Lemma is concerned with may in fact be conceived to be magnified, the magnifying power applied being continually increased as the figure continually diminishes, so as to keep the image continually finite.

*Note* 6. *If one angle* A *of a triangle* ABD *continually diminishes and ultimately vanishes, the others remaining finite, the ratio of the sides including the angle which vanishes is in the limit a ratio of equality.*

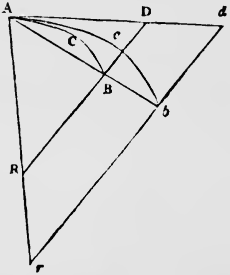
This proposition is incidentally proved in this Lemma.

*Note* 7. In this and succeeding Lemmas, the magnified lines and areas are proved to be equal, by Lemma I.; and hence the vanishing lines and areas, which are always proportional to them, vanish in a ratio of equality.

Thus, in this Lemma, we have $Ad$ equal to $Ab$, by Lemma I.; and therefore $AD$, $AB$ vanish in a ratio of equality.

### LEMMA VIII.

*If the straight lines* AR, BR *make with the chord* AB, *the arc* ACB *and the tangent* AD, *the three triangles* RAB, RACB, RAD; *then, if the points* A, B *move up to*




*each other, the evanescent triangles are ultimately similar, and their ratio is a ratio of equality.*

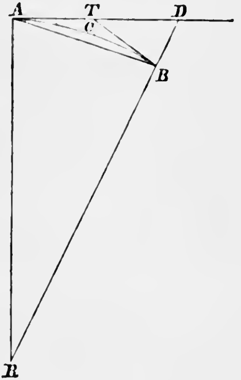
For, while the point *B* moves up to *A*, suppose *AB*, *AD*, *AR* to be produced to distant points *b*, *d*, *r* so that *rbd* is parallel to *RD*, and an arc *Acd* to be described always similar to *ACB*.

Then, when the points *A*, *B* coincide, the angle *bAd* will vanish, and therefore the triangles *rAb*, *rAcb*, *rAd* (which remain always finite) will coincide, and be, for that reason, similar and equal. Hence, also, *RAB*, *RACB*, *RAD*, which are always similar and equal to them, will be ultimately similar and equal to each other. Q.E.D.

COR. And hence these triangles, in all reasoning on ultimate ratios, may be used one for another.

*Note* 1. For the direct application of Lemma I. to the proof of this Lemma it is essential that the triangle *rAd* should be finite, and therefore that its angles, and the angles which are equal to them of the triangle *RAD*, should be finite. Hence *RD* moves up to *A* in such a manner as to make finite angles with *AB* and *AR*; and the points *R*, *D* ultimately coincide with *A*.

*Note* 2. The Lemma is also true if $R$ is fixed and $RD$ revolves about $R$ so as ultimately to coincide with $RA$, for then $r$ moves to an infinite distance, and therefore the triangles $rAb$, $rAcb$, $rAd$ are ultimately in a ratio of equality.




*Note* 3. Let $BT$, the tangent at $B$, meet the tangent at $A$ in $T$.

Then by using the construction and method of proof of this Lemma it is easy to see that the figure $ATBR$ is ultimately in a ratio of equality with the triangles $ABR$, $ADR$, and the area $ACBR$.

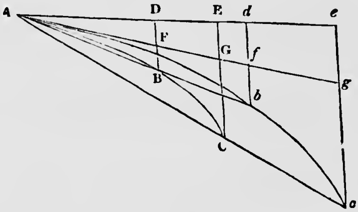
*Note* 4. In the case of this Lemma, and also in the case of Note 2, that is when $AR$, $BR$ make always finite angles with $AD$,

$$L \,.\, R \,.\, RB : RD = L \,.\, R \,.\, rb : rd = 1.$$

## Lemma IX.

*If the straight line* AE *and the curve* ABC, *given in position, cut one another at a given angle* A, *and ordinates* BD, CE *be drawn to the straight line, meeting the curve in* B, C; *then if the points* B, C *move up simultaneously to the point* A, *the areas of the triangles* ABD, ACE *will be to each other ultimately in the duplicate ratio of the sides.*

For while the points *B, C* are moving up to the point *A*, suppose *AD* to be produced to the distant points *d* and




*e*, so that *Ad, Ae* may be always proportional to *AD, AE*, and the ordinates *db, ec* to be drawn parallel to the ordinates *DB, EC* and meeting *AB, AC* produced in *b* and *c*.

And suppose a curve *Abc* to be drawn similar to *ABC*, and the straight line *Ag* touching both curves in *A*, and cutting the ordinates *DB, EC, db, ec* in *F, G, f, g*.

Then the length *Ae* being fixed, let the points *B, C*, coincide with *A*; thus, when the angle *cAg* vanishes, the curvilinear areas *Abd, Ace* will coincide with the rectilinear areas *Afd, Age*, and will therefore (by Lemma V.), be in the duplicate ratio of the sides *Ad, Ae*.

But to these areas the areas *ABD, ACE*, are always proportional, and to these sides the sides *AD, AE*.

Therefore also the areas *ABD, ACE* are ultimately in the duplicate ratio of the sides *AD, AE*. Q.E.D.

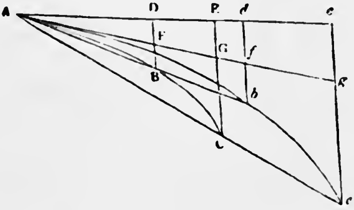
*Note* 1. In this Lemma the angle *A* which *AE* makes with the curve must be finite: for the proof will not hold if *AE* coincides in the limit with the tangent *AD*.

*Note* 2. Since *DF* is to *DB* as *df* to *db*; and that *df* is ultimately equal to *db*, because the angle *fAb* vanishes; therefore the ultimate ratio of *DF* to *DB* is one of equality.

### Lemma X.

*The spaces described by a body under the action of any finite force, whether that force be constant or either continually increasing or continually diminishing, are at the very beginning of the motion in the duplicate ratio of the times.*

Let the times be represented by the lines *AD*, *AE*, and the velocities generated by the ordinates *DB*, *EC*.




Then the spaces described with these velocities will be as the areas *ABD*, *ACE* described by these ordinates, that is at the very beginning of the motion (by Lemma IX.), in the duplicate ratio of the times *AD*, *AE*. Q. E. D.

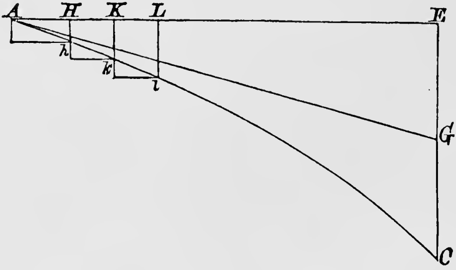
*Note* 1. The object of this Lemma is to determine in what manner the displacement of a body by any finite force will initially vary with the time during which its action is considered; or starting from any given moment, what amounts of displacement the body will experience due to the action of the force during indefinitely small intervals of time.

*Note* 2. By a finite force is meant a force which, if it remained constant for a finite time, would produce a finite change in the motion of a body; that is, would produce a finite change in its velocity. Thus in the figure, if the initial force had remained constant throughout the time *Ae*, it would have generated in the body the finite velocity *eg*: and since the velocity generated by a constant force is equal in equal times, the velocity generated is proportional to the time; the velocity generated in time $AD$ would thus have been $DF$; and since $eg$ is finite $DF$ bears always, and therefore ultimately, a finite ratio to $AD$.

Hence the ultimate ratio of the measure of the velocity generated by a finite force to the measure of the time is finite; so that $Ag$ makes a finite angle with $Aa$, and thus Lemma IX. is applicable.

*Note* 3. To shew that the space described in any time $AE$ by the action of the force is represented by the area $ACE$ we proceed as follows:

Suppose the time $AE$ divided into any number of equal intervals




$AH$, $HK$, $KL$, &c., and let parallelograms $Ah$, $Hk$, $Kl$, &c. on these bases, be inscribed in the figure $Ac$ as in Lemma II.

Then $Ah$, $Hk$, $Kl$, &c., are the spaces which would be described in the times $AH$, $HK$, $KL$, &c., by a succession of impulses which should cause the body to move during the successive intervals with the velocities $Hh$, $Kk$, $Ll$, &c.

Thus the space described in the time $AE$ is represented by the sum of the parallelograms.

And when these intervals are increased in number and diminished in magnitude without limit, the space described becomes by Lemma II. the area $AEC$; and the series of impulses then becomes a continuous force causing the body to have at each instant, as at end of time $AL$, the velocity represented by $Ll$ drawn perpendicular to $AL$ to meet the curve.

*Note 4. To find a measure for a force in terms of the space described from rest by a body under its action, and the time.*

By Def. 7. the accelerating force is proportional to the velocity generated in a given time: thus a uniform force is measured by the velocity generated in a unit of time; or, which is the same thing, by the ratio which the velocity generated in a given time bears to the time. In the same way a variable force may be measured provided the given time be taken indefinitely small (Def. 7, note 2).

Let $v$ be the velocity produced in time $t$ by the action of the force, $s$ the space described; then $\frac{v}{t}$, when $t$ is indefinitely diminished, is the accelerating measure of the force. Now in the figure to Lemma X.,

$$\text{L. R. of area } ABD \text{ to triangle } ABD = 1;$$

$$\text{or L. R. of space described from rest} : \frac{1}{2}vt = 1;$$

$$\therefore f = \text{accelerating measure of the force}$$

$$= \text{limit } \frac{v}{t} = \text{limit } \frac{vt}{t^2}$$

$$= \text{limit } \frac{2 \times \text{space from rest}}{(\text{time})^2}.$$

We have then $$f = \text{limit } \frac{v}{t},$$

and $$f = \text{limit } \frac{2s}{t^2}$$

$$= \text{limit } \frac{2s}{v^2} \cdot \frac{v^2}{t^2}$$

$$= \text{limit } \frac{2sf^2}{v^2};$$

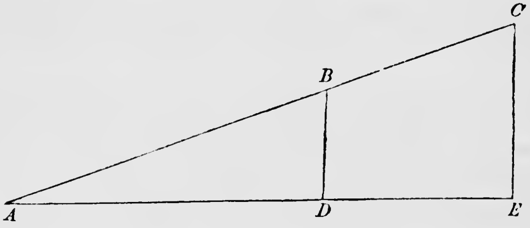
hence $$1 = \text{limit } \frac{2fs}{v^2},$$

and $$f = \text{limit } \frac{v^2}{2s}.$$

Thus we have three expressions for the force; viz. the limit of

$$\frac{v}{t}, \text{ of } \frac{2s}{t^2}, \text{ and of } \frac{v^2}{2s}.$$

*Note* 5. If the force is uniform, let the ordinates $BD$, $CE$



perpendicular to $AE$ represent the velocities produced by it in the times $AD$, $AE$.

Since the force is uniform $BD$ is proportional to $AD$; and the locus of $B$ is the straight line $ABC$.

Let $AD=t$ : $BD=v$ : $f$ the accelerating measure of the force :

$$\therefore v=ft.$$

Again, let $s$ be the space described from rest in time $t$;

then $\quad s = \text{triangle } ADB = \frac{1}{2}vt = \frac{1}{2}ft^2$ :

$$\therefore v^2 = f^2t^2 = 2fs :$$

thus for a uniform force $f$,

$$\frac{v}{t} = \frac{2s}{t^2} = \frac{v^2}{2s} = f.$$

Def. *When a body is moving under the action of a centripetal force, the space through which it would have to fall from rest in a straight line to the centre of force, (the force being supposed to remain constant,) in order to acquire a given velocity, is called the* space due to that velocity.

Thus if $f$ be the accelerating force, $v$ the velocity, the space due to the velocity $= \frac{v^2}{2f}$.

Cor. 1. Hence it is easily inferred that, when bodies which describe similar parts of similar curves in proportional times, are disturbed by any *equal* forces similarly applied to them, the errors so generated, measured by the distances of the bodies from those points on the similar figures which they would have reached without these forces, are as nearly as possible as the squares of the times in which they are generated.

Cor. 2. But the errors generated by *proportional* forces similarly applied at similar parts of similar figures, are as the forces and squares of the times conjointly.

Cor. 3. The same is true of any spaces whatever which bodies describe by the action of different forces. These are, at the very beginning of the motion, as the forces and the squares of the times conjointly.

Cor. 4. And therefore the forces are directly as the spaces described at the very beginning of the motion and inversely as the squares of the times.

Cor. 5. And the squares of the times are directly as the spaces described and inversely as the forces.

*Note* 1. By equal forces in Cor. 1 are meant forces capable of producing equal accelerations; for example, the forces exerted by a planet on the Earth and Moon when these are at the same distance from the planet are, in this sense, considered equal, because they produce equal accelerations in the two bodies. It is found in nature that bodies possess by their attractions a power of producing acceleration, which power depends on the mass of the attracting body and on its distance from the body attracted, but not on the mass of the latter body: this power is called by Newton the *accelerating force* of the attracting body, and is measured by the acceleration produced.

The *moving force* exerted on a body by this accelerating force is measured according to the second law of motion by the whole motion produced, that is, by the mass of the body moved and its acceleration conjointly.

*Note* 2. By the *error* in these corollaries is meant the distance of the point actually reached at the end of the time from the point which would have been reached if the disturbing force had not been acting. The change of motion by which the body has been brought to the former point instead of the latter is, by the second law of motion, always proportional to and in the direction of the moving force to which it is due.

If the force were, throughout the small time during which it is supposed to act, constant in direction, the straight line joining the points would be the path of the body due to the force, and is (since the Lemma is approximately true for small intervals of time) very nearly proportional to the square of the time.

If, however, the direction of the force during that time were not constant, the motion of the body which is due to the disturbing force would be in a curve joining the two points; the curvature of this curve from point to point corresponding to the change of direction of the force; but in a small interval of time this change of direction, and therefore the curvature of the curve, may be neglected, and the error, which is approximately proportional to the square of the time, is approximately the straight line joining the two points.

*Note* 3. By proportional forces in Cor. 2 are meant forces which produce proportional accelerations. The quantities of motion produced in equal times under the action of forces producing proportional accelerations, must be, by the second law of motion, proportional to the moving forces; the spaces described must therefore be in the ratio of the accelerating forces; and by the Lemma, the spaces due to the action of equal accelerating forces are proportional to the squares of the times.

Hence the spaces due to the action of different accelerating forces during unequal times are proportional to the accelerating forces and the squares of the times conjointly.

## Scholium.

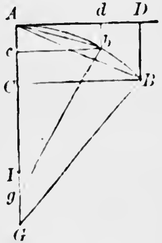
If varying quantities of different kinds be compared together, and one be said to be directly or inversely as another; the meaning is that the one increases or diminishes in the same ratio as the latter or as its reciprocal. And if any quantity be said to be directly or inversely as two or more others, the meaning is that the first increases or diminishes in the ratio which is compounded of the ratios in which the others or the reciprocals of the others increase or diminish. Thus, if $A$ be said to be directly as $B$ and $C$ and inversely as $D$: the meaning is that $A$ increases or diminishes in the same ratio as $B \times C \times \frac{1}{D}$; that is, that $A$ and $\frac{BC}{D}$ are to each other in a given ratio.

## Lemma XI.

*The vanishing subtense of the angle of contact, in all curves which have finite curvature at the point of contact, is ultimately in the duplicate ratio of the chord of the conterminous arc.*

Case 1. Let $AB$ be the arc, $AD$ its tangent, $AB$ the chord; and let the subtense $BD$ of the angle of contact be perpendicular to the tangent.

To the chord $AB$ and the tangent $AD$ draw perpendiculars $BG$, $AG$ meeting in $G$; and let the points $B$, $D$, $G$




move towards the points $d$, $b$, $g$, and $I$ be the ultimate intersection of the lines $BG$, $AG$ when the points $D$, $B$ move up to $A$.

It is evident that the distance $GI$ may be made less than any assignable length.

Now (from the nature of the circles through the points $ABG$, $Abg$) the square on $AB$ is equal to the rectangle under $AG$ and $BD$, and the square on $Ab$ is equal to the rectangle under $Ag$ and $bd$; and thus the ratio of the square on $AB$ to the square on $Ab$ is compounded of the ratios of $AG$ to $Ag$ and $BD$ to $bd$.

But since $GI$ may be made less than any assignable length, the ratio of $AG$ to $Ag$ may be made to differ from a ratio of equality by less than any assigned difference, and therefore the ratio of the square on $AB$ to the square on $Ab$

may be made to differ from the ratio of $BD$ to $bd$ by less than any assigned difference. Therefore, by Lemma I., the ultimate ratio of the square on $AB$ to the square on $Ab$ is the same as the ultimate ratio of $BD$ to $bd$. Q. E. D.

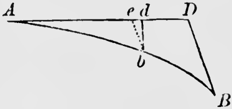
CASE 2. Now let $BD$ be inclined to $AD$ at any given angle; the ultimate ratio of $BD$ to $bd$ will always be the same as before (Note 2), and therefore the same as that of the square on $AB$ to the square on $Ab$. Q. E. D.

CASE 3. And even if the angle at $D$ were not given, but the straight line $BD$ were to converge to a given point or were drawn according to any other law whatever; still the angles at $D$, $d$, being constructed according to the same law, will always tend to equality and will approach each other more nearly than for any assigned difference, and will therefore by Lemma I. be ultimately equal, and consequently (Note 3) the lines $BD$, $bd$ will be ultimately to each other in the same ratio as before. Q. E. D.

*Note* 1. In Case 1 it is assumed in the proof that $I$ is at a finite distance from $A$, or that the circle which touches the curve at $A$ and passes through $B$ becomes neither indefinitely great nor indefinitely small when $B$ moves up to and coincides with $A$. This is implied in the phrase *finite curvature* in the enunciation.

*Note* 2. The ultimate ratio $BD$ to $bd$ is the same in Case 2 as in Case 1, because any parallel straight lines through $B$ and $b$ make with $BD$, $AD$, and with $bd$, $Ad$, a pair of similar triangles.

*Note* 3. To complete the proof of Case 3, draw a straight line $be$ always parallel to $BD$: then, since the angles at $D$ and $d$




are ultimately equal, the angles at $e$ and $d$ are ultimately equal, and their difference, the angle $dbe$, ultimately vanishes.

Therefore, by the proof of Lemma VII., the ultimate ratio of $bd$ to $be$ is one of equality: and thus the ultimate ratio of $BD$ to $bd$ is the same as that of $BD$ to $be$, that is, it is the ultimate ratio of the squares on $AB$ and $Ab$, by Case 2.

Cor. 1. Hence, when the tangents $AD$, $Ad$, the arcs $AB$, $Ab$, and the lines $BC$, $bc$, perpendicular to $AG$, become ultimately equal to the chords $AB$, $Ab$, their squares also will ultimately be as the subtenses $BD$, $bd$.

Cor. 2. The squares of the same lines are also as the sagittæ (Note 1) of the arcs, which bisect the chords and converge to a given point. For these sagittæ are as the subtenses $BD$, $bd$.

Cor. 3. Thus the sagitta is in the duplicate ratio of the time in which a body describes the arc with a given velocity.

Cor. 4. The rectilinear triangles $ADB$, $Adb$ are ultimately in the duplicate ratio of the sides $AD$, $Ad$, and in the sesquiplicate ratio of the sides $DB$, $db$; for they are in the ratio compounded of the ratios of $AD$ and $DB$ to $Ad$ and $db$. So also the triangles $ABC$, $Abc$ are ultimately in the triplicate ratio of the sides $BC$, $bc$. The sesquiplicate ratio—which is the ratio compounded of the simple and subduplicate ratios—is also called the subduplicate of the triplicate ratio.

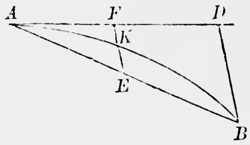
Cor. 5. And since $DB$, $db$ are ultimately parallel and in the duplicate ratio of $AD$, $Ad$, the curvilinear areas $ADB$, $Adb$ will ultimately (from the nature of the parabola, Note 2) be two-thirds of the rectilinear triangles $ADB$, $Adb$; and the segments $AB$, $Ab$ will be the third parts of the same triangles.

And hence these areas and these segments will be in the triplicate ratio of the tangents $AD$, $Ad$, and of the chords and arcs $AB$, $Ab$.

*Note* 1. Def. *By a sagitta of an arc is meant a straight line drawn at a finite angle to its chord from the middle point of it to meet the arc.*

The sagittæ of the arcs $AB$, $Ab$ are as the subtenses parallel to them.

For let $EKF$ be the sagitta parallel to $BD$ of the arc $AKB$:




so that $AB$ is twice $AE$, and therefore $AD$ twice $AF$, and $DB$ twice $EF$.

Then by Cor. 1 of the Lemma, since $AD$ is twice $AF$, $DB$ is ultimately to $KF$ in the ratio of 4 to 1; therefore the ultimate ratio of $EF$ to $KF$ is 2 to 1; therefore that of $EK$ to $EF$ is 1 to 2; and hence that of $EK$ to $DB$ is 1 to 4.

Thus the sagittæ vary as the subtenses parallel to them, and are therefore to each other as the squares of the arcs, chords, and tangents.

*Note* 2. Since the ultimate ratio of the square on $AD$ to the rectangle under $DB$ and $AI$ is a ratio of equality, if a parabola were described touching $AD$ at $A$, having $A$ for its vertex and a latus rectum equal to $AI$, the ordinates to the parabola and the curve for any the same abscissa $AD$ are ultimately in a ratio of equality: hence, by Lemma IV., the areas which $AD$, $DB$ include with the parabola and with the curve are ultimately in a ratio of equality.

That this area is two-thirds of the area of the triangle $ADB$ is proved thus. Complete the parallelogram $ADBC$ as in the figure to the Lemma. Then the curvilinear area $ABC$ is two-thirds of the parallelogram $ADBC$ (Appendix, Lemma II.); therefore the curvilinear area $ABD$ is one-third of this parallelogram, and thus, two-thirds of the triangle $ADB$.

## Scholium.

In all that precedes we suppose that the angle of contact (Note 1) is neither infinitely greater than the angles of contact which circles make with their tangents, nor infinitely less; that is, that the curvature (Note 2) at the point $A$ is neither infinitely small nor infinitely great, or that the distance $AI$ is finite. For $DB$ can be taken as $AD^3$, in which case no circle can be drawn through the point $A$ between the tangent $AD$ and the curve $AB$, and consequently the angle of contact will be infinitely less than in circles. And by similar reasoning, if $DB$ be made successively as $AD^4$, $AD^5$, $AD^6$, $AD^7$... there will be a series of angles of contact extending to infinity of which each is infinitely less than the preceding. Also if $DB$ be made successively as $AD^2$, $AD^{\frac{3}{2}}$, $AD^{\frac{4}{3}}$, $AD^{\frac{5}{4}}$, $AD^{\frac{6}{5}}$, $AD^{\frac{7}{6}}$... there will be another infinite series of angles of contact, of which the first is of the same kind as in circles, the second infinitely greater, and each infinitely greater than the preceding. And moreover, between any two of these angles can be inserted a series of intermediate angles extending to infinity in either direction, every one of which is infinitely greater or smaller than the preceding. Thus we may insert between the terms $AD^2$ and $AD^3$ the series $AD^{\frac{13}{6}}$, $AD^{\frac{11}{5}}$, $AD^{\frac{9}{4}}$, $AD^{\frac{7}{3}}$, $AD^{\frac{5}{2}}$, $AD^{\frac{8}{3}}$, $AD^{\frac{11}{4}}$, $AD^{\frac{14}{5}}$, $AD^{\frac{17}{6}}$... And again between any two angles of this series can be inserted a new series of angles, intermediate between them, and differing from them by infinite intervals. And in the nature of things there can be no limit to the process.

The properties proved of curved lines and areas enclosed by them are easily applied to curved surfaces and contents of solids. But these lemmas have been introduced to escape the tedious ad absurdum method of proof adopted by the old geometers. For the proofs are made shorter by the method of indivisibles; but as the hypothesis of indivisibles is somewhat harsh, and that method consequently must be considered somewhat ungeometrical, it has been thought better to reduce the demonstrations of the following propositions to the ultimate sums and ratios of vanishing quantities, and the prime sums and ratios of nascent quantities; and accordingly to give, as briefly as possible, demonstrations of these limits. For while we thus establish the same principles as by the method of indivisibles, we shall use them with greater safety. In what follows, then, if quantities are treated as consisting of small parts, or if small curve lines be considered straight, they are to be understood not as being indivisibles but evanescent divisibles, and their sums and ratios not as the sums and ratios of determinate parts but as the limits of sums and ratios; and the force of the proofs is made to depend on the method of the preceding Lemmas. It may be objected that there is no ultimate proportion of vanishing quantities; for, before they have vanished it is not ultimate, and when they have vanished there is no proportion. But by the same argument it might be contended that there is no ultimate velocity of a body arriving at a certain place, where its motion comes to an end; for the velocity before the body arrives at the place is not the ultimate velocity, and when it reaches it, there is no velocity. And the answer is easy: namely, that by the ultimate velocity is to be understood the velocity with which the body is moving, neither before it reaches the ultimate position where the motion ceases, nor afterwards, but at the moment it arrives there; that is, that very velocity with which the body reaches the ultimate position, and with which the motion ceases. And similarly we are to understand by the ultimate ratio of vanishing quantities not their ratio before they vanish nor after, but that with which they vanish. Similarly the prime ratio of nascent quantities is the ratio with which they begin to exist. And prime and ultimate sums are the sums with which they begin and cease to be (or to be increased or diminished). There is a limit which the velocity can attain at the end of the motion, but cannot exceed; this is the ultimate velocity. And similarly for the ratio of the limits of all quantities and proportions in their initial and final states. And since this limit is a certain and definite result it is strictly a geometrical problem to determine it. But any geometrical method may legitimately be used in determining and demonstrating other geometrical results.

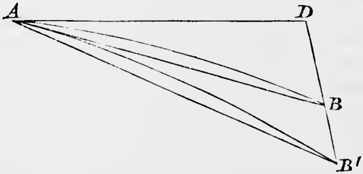
It may also be contended that, if ultimate ratios be given of vanishing quantities, it must be granted that they have ultimate magnitudes: and thus every quantity will consist of indivisibles, contrary to what Euclid has proved in reference to incommensurable quantities in the tenth book of the elements. But this objection rests on a false hypothesis. The ultimate ratios with which quantities vanish are in fact not the ratios of ultimate quantities, but limits which the ratios of quantities diminishing without limit continually approach, and which they can come nearer to than for any assigned difference, but never go beyond, and which they cannot reach before the quantities are indefinitely diminished. The matter will be more clearly understood in the case of quantities which are indefinitely great. If two quantities, whose difference is given, be increased indefinitely, their ultimate ratio will be given, namely a ratio of equality, and yet there is not given any ultimate or greatest quantities of which that is the ratio. In what follows, therefore, if ever, with a view to making any matter easier to comprehend, quantities are spoken of as being as small as possible or evanescent or ultimate, quantities of determinate magnitude are not meant, but quantities which are to be conceived as diminishing without limit.

*Note* 1. *Angle of contact.* Let $AD$ be the tangent at $A$ to a curve $AB$; $AB$ any small arc of the curve: then angle $BAD$


is the angle of contact of the arc $AB$. Draw $BD$ at right angles to $AD$. With centre $A$ describe an arc of a circle $BE$ meeting the tangent at $A$ in $E$: then the angle of contact of arc $AB$ with the tangent at $A$ = ratio of $BE$ to chord $AB$.

Let $A'D'$ be the tangent at $A'$ to any other curve $A'B'$; $B'E'$ an arc of a circle cutting the tangent at $A'$ in $E'$; draw $B'D'$ at right angles to $A'D'$: the angle of contact of $AB$ is to angle of contact of $A'B'$ as ratio of $BE$ to $B'E'$ is to ratio of chord $AB$ to chord $A'B'$. But the limiting ratio of $BE$ to $B'E'$ is equal to ratio of $BD$ to $B'D'$ (by Lemma VII. Cor. 1); and the limiting ratio of chord $AB$ to chord $A'B'$ is equal to the ratio of arc $AB$ to arc $A'B'$, and of tangent $AD$ to tangent $A'D'$. Hence, the limiting ratio of the angles of contact of two arcs of any curves is the ratio which the ratio of the subtenses bears to the ratios of the arcs, chords, or tangents: and thus, *the limiting ratio of the angles of contact of two arcs is that of the subtenses of equal arcs, chords, or tangents.*

*Note* 2. Let $AD$ be the common tangent at $A$ to any two curves $AB$, $AB'$: draw $DBB'$ at any finite angle to $AD$ meeting




the curves in $BB'$; join $AB$, $AB'$. Then the angles $BAD$, $B'AD$ are the angles of contact of the curves at $A$ with the tangent $AD$.

When $B$ and $B'$ move up to $A$, the limiting ratio of the angle of contact of $AB$ to that of $AB'$ or of angle $BAD$ to angle $B'AD$ may be finite, or zero, or infinite; if zero, the angle of contact of $AB$ is said to be infinitely smaller than that of $AB'$; if infinite, infinitely greater.

If $AB'$ be an arc of a circle, the angle of contact of $AB$ is said to be finite or infinitely small or infinitely great according as the limiting ratio of the angle $BAD$ to the angle $B'AD$ is finite, zero, or infinite.

*Note* 3. The curvatures of curves are compared by comparing at any points the angles of contact with the same or equal tangents. If the limiting ratio of these angles is one of equality the curvatures are said to be equal, or the curves are said to have the same curvature at those points. If the limiting ratio of the angle $BAD$ to $B'AD$ be less than unity, the curvature of $AB$ at $A$ is less than that of $AB'$; if greater than unity, the curvature of $AB$ is the greater.

If the limiting ratio of the angle of contact of $AB$ to that of $AB'$ be zero or infinite, the curvature of $AB$ at $A$ is said to be infinitely less or infinitely greater than that of $AB'$; or, if $AB'$ be a circle, to be infinitely small or infinitely great.

*Note* 4. DEF. *By the circle of curvature of a curve at any point is meant the circle which has the same curvature* (Note 3) *as the curve at that point.*

Thus in the Lemma the limiting position of the circle described about the triangle $ABG$ is the circle of curvature of the curve $AB$ at $A$.

The circle of curvature to a curve at $A$ is therefore the limiting position of the circle which has the same tangent as the curve at $A$ and passes through a point $B$ near $A$, when $B$ moves up to and coincides with $A$.

Or, the circle of curvature at $A$ is the circle which touches the curve at $A$, and the limiting ratio of the subtense of which for a given tangent $AD$ to the subtense $BD$ of the curve is one of equality.

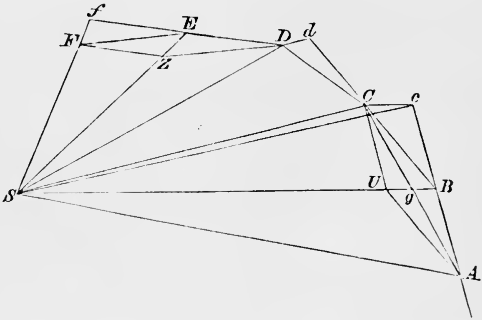
*Note* 5. DEF. *The diameter and radius of the circle of curvature at any point* P *of a curve are generally called the diameter and radius of curvature of the curve at that point; and a chord of the circle through* P *in any direction is called the chord of curvature at* P *in that direction.*

## SECTION II.

### *On finding Centripetal Forces.*

PROP. I. (THEOREM I.)

*When a body moves in an orbit under the action of forces tending to a fixed centre, the areas described by radii to that centre are in a fixed plane, and are proportional to the times of describing them.*




Suppose the time to be divided into equal parts, and in the first part the body to describe by its inertia the straight line $AB$; the body would, if unimpeded, proceed in the second part of the time straight on to $c$ (by Law 1), describing the line $Bc$ equal to $AB$; so that radii $AS$, $BS$, $cS$ being drawn to the centre, there would be formed the equal areas $ASB$, $BSc$.

But when the body comes to $B$, suppose a centripetal force to act on it by a single but great impulse, so as to make the body swerve from the straight line $Bc$ and go in the straight line $BC$. To $BS$ let a parallel $cC$ be drawn, meeting $BC$ in $C$: then at the end of the second part of the time, the body will (by Cor. 1 of the laws of motion) be found at $C$ in the same plane with the triangle $ASB$.

Join $SC$; then the triangle $SBC$ will, because $SB$, $Cc$ are parallel, be equal to the triangle $SBc$, and therefore to the triangle $SAB$.

By similar reasoning, if a centripetal force acts successively at $C$, $D$, $E$, &c. making the body describe in the several portions of time the several straight lines $CD$, $DE$, $EF$, &c. these will all lie in the same plane, and the triangle $SCD$ will be equal to the triangle $SBC$, and $SDE$ to $SCD$, and $SEF$ to $SDE$; therefore in equal times equal areas are described in a fixed plane: and *componendo*, any sums of these areas, as $SADS$, $SAFS$, are as the times of describing them.

Now let the number of triangles be increased and their width diminished indefinitely; then the limit of the perimeter $ADF$ will (by Lemma III. Cor. 4) be a curve; and thus the centripetal force by which the body is perpetually drawn away from the tangent to this curve will act incessantly; and the areas which are described, as $SADS$, $SAFS$, being always proportional to the times of describing them, will be proportional to the times in this case. Q.E.D.

*Note.* The motion of the body in the proof of this proposition is determined by impulses towards $S$ acting at the end of equal intervals of time in which the body describes $AB$, $BC$, $CD$, ......; for each one of these times it receives one impulse. Thus $Cc$ is the effect of the impulse corresponding to the time $t$ of describing $AB$: and the impulse is measured as to its accelerating quantity (Def. 7) by the velocity which it generates in a given time; also the accelerating effect of the force is to generate in an interval of time $t$ the velocity with which the body describes $Cc$ in time $t$.

Now velocity generated in time $t = \dfrac{Cc}{t}$,

∴ accelerative measure of the force

$$= \text{limit of } \frac{\text{velocity generated in time } t}{t}$$

$$= \text{limit of } \frac{Cc}{t^2}.$$

COR. 1. The velocity of a body attracted to a fixed centre is (at any point in its orbit), in non-resisting spaces, reciprocally as the perpendicular let fall from that centre on the straight line touching the orbit at that point.

For the velocities at the points $A$, $B$, $C$, $D$, $E$ are as the bases $AB$, $BC$, $CD$, $DE$, $EF$ of equal triangles; and these bases are reciprocally as the perpendiculars let fall upon them.

*Note.* If $v$ be the velocity at $A$, $p$ the perpendicular on $AB$ from $S$, $t$ the time of describing $AB$; then $AB = vt$; therefore $vpt = 2 \times$ area of triangle $SAB$.

Let now $h = 2 \times$ area described by the body about $S$ in a unit of time; then, since equal areas are described in equal times,

$ht = 2 \times$ area described in $t$ units of time:

$= 2 \times$ area of triangle $SAB$,

or $ht = vpt$;

therefore $h = vp$.

This equation is a symbolical expression of the law of equable description of areas.

COR. 2. If $AB$, $BC$ are chords of two arcs successively described in equal times in non-resisting spaces by the same body, and the parallelogram $ABCU$ be completed, and its diagonal $BU$, in the position which it ultimately takes when those axes are diminished indefinitely, be produced both ways; $BU$ will pass through the centre of forces.

*Note.* Draw $CU$ parallel to $ABc$, meeting $SB$ in $U$: join $AU$. Then since $CU$ is equal and parallel to $Bc$ which is equal to $AB$, $ABCU$ is a parallelogram (Euc. I. 33).

COR. 3. If $AB$, $BC$, $DE$, $EF$ be chords of arcs described in equal times in non-resisting spaces, and the parallelograms $ABCU$, $DEFZ$ be completed; the forces at $B$ and $E$ are to each other in the ultimate ratio of the diagonals $BU$, $EZ$ when those arcs are diminished indefinitely.

For the motions $BC$ and $EF$ of the body are compounded (by Cor. 1 of the Laws) of the motions $Bc$, $BU$ and $Ef$, $EZ$; but $BU$ and $EZ$, being equal to $Cc$ and $Ff$, were (in the proof of this proposition) described by the impulses at $B$ and $E$ of the centripetal force, and are therefore proportional to these impulses.

Cor. 4. The forces by which any bodies in non-resisting spaces are drawn away from rectilinear motions and deflected into curved orbits, are to one another as those sagittæ of arcs described in equal times which converge to the centre of forces, and bisect the chords when those arcs are diminished indefinitely.

For these sagittæ are the halves of the diagonals with which we had to do in the third Corollary.

*Note.* Let $AC$ meet $BU$ in $g$; then $Cg = gA$, and $Bg = \frac{1}{2} BU$: also accelerative measure of force at $B$

$$= \text{limit of } \frac{Cc}{t^2} \text{ (Prop. I. Note 1)},$$

$$= \text{limit of } \frac{2Bg}{t^2};$$ where $t$ is the time of describing $AB$.

Cor. 5. And therefore these forces are to the force of gravity as those sagittæ are to sagittæ drawn vertically to parabolic arcs which projectiles describe in the same time.

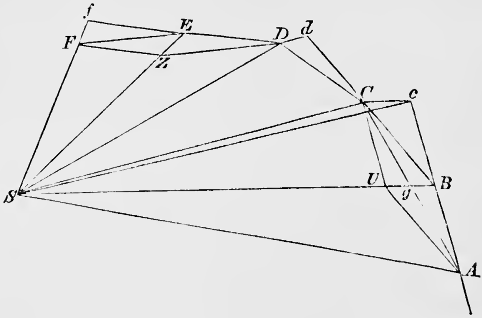
Cor. 6. The same things hold (by Cor. 5 of the Laws), when the planes in which the bodies move, together with the centres of forces which are in those planes, are not at rest, but in uniform motion in a straight line.

## Prop. II. (Theorem II.)

*Every body, which moves in a plane curve, and which —by a radius drawn either to a fixed point or to a point moving uniformly in a straight line—describes areas about that point proportional to the times, is acted on by a centripetal force tending to that point.*

Case 1. For every body which moves in a curve is deflected from a rectilinear course by some force acting on it (by Law 1).

And the force, by which a body is deflected from a rectilinear course, and compelled to describe the equal in-




definitely small triangles *SAB*, *SBC*, *SCD*, &c. about the fixed point *S* in equal times, acts at *B* in a line parallel to *cC* (by Euc. I. 40, and Law 2), that is, in the line *BS*; and at *C* in a line parallel to *dD*, that is, in the line *CS*; &c.

Therefore the force acts always in lines tending to the fixed point *S*.

Case 2. And, by the fifth Corollary of the Laws, this is true, whether the surface in which the body describes the curvilinear figure be at rest, or be moving together with the body, the figure described, and the point *S*, uniformly in a straight line.

Cor. 1. In non-resisting spaces or media, if the areas are not proportional to the times, the forces do not tend to the point in which the radii meet; but deviate from that direction; *in consequentiâ*, or towards the direction of motion, if the description of areas is accelerated; *in antecedentiâ*, if it is retarded.

Cor. 2. Also in resisting media, if the description of areas is accelerated, the directions of the forces deviate from the point in which the radii meet towards the direction of motion.

*Note.* In this proposition, as in Prop. I., the force is supposed to act by impulses at equal indefinitely small intervals of time, so that the body describes successively $AB$, $BC$, $CD$, &c.: also $Bc$ is equal to $AB$ as in Prop. I.; hence

$$\triangle SBc = \triangle SAB = \triangle SBC,$$

since equal areas are described in equal times; therefore by Euc. I. 40, $cC$ is parallel to $SB$; and by Law 2, $cC$ is the direction of the force at $B$.

### Scholium.

A body may be acted on by a centripetal force compounded of several forces. In this case the meaning of the proposition is, that the force which is compounded of them all tends to the point $S$. Moreover if any force acts continually in a direction perpendicular to the surface described, this will have the effect of deflecting the body from the plane of its motion: but it will neither increase nor diminish the amount of surface described, and may therefore be left out of consideration in compounding the forces.

### Prop. III. (Theorem III.)

*Every body, which describes areas proportional to the times about the centre of another body moving in any manner, by radii drawn to that centre, is acted on by a force compounded of the centripetal force tending to that other body, and of the whole accelerating force by which that other body is affected.*

Let $L$ be the first body, and $T$ the other: then (by Cor. 6 of the Laws), if by a new force equal and opposite to that ($Q$) by which the other body $T$ is acted on, both bodies be acted on in parallel directions, the first body $L$ will go on describing about the other $T$ the same areas as before: but the force by which the other $T$ was acted on will now be destroyed by a force equal and opposite to it.

Therefore (by Law 1) $T$ being now left to itself will either keep at rest or move uniformly in a straight line: and $L$ will proceed to describe areas proportional to the times about $T$ under the action of the difference of the forces, that is, of the force which remains [when *L*'s force is compounded with *Q* reversed]. Therefore (by Theorem II.) the difference of the forces tends to the other body *T* as centre [therefore *L*'s force is compounded of *Q*, which is *T*'s force, and of a centripetal force to *T* as centre]. Q. E. D.

*Note.* By the difference of two forces is here meant the resultant of one of them and of the other reversed; and in Cor. 1 of this proposition one force is said to be subtracted from another when, its direction being reversed, it is compounded with the other.

Cor. 1. Hence, if one body *L*, by radii drawn to another *T*, describes areas proportional to the times; and if from the whole force (whether it be a single force, or compounded of several according to the second corollary of the Laws), by which the first body *L* is acted on, be subtracted (by the same corollary of the Laws) the whole accelerating force by which the other body is acted on: all the remaining force by which the first body *L* is acted on tends to the other *T* as centre.

Cor. 2. And if those areas are very nearly proportional to the times, the remaining force tends very nearly to the other body *T*.

Cor. 3. And *vice versâ*, if the remaining force tends very nearly to the other body *T*, the areas will be very nearly proportional to the times.

Cor. 4. If a body *L* by radii drawn to another *T* describes areas which are very far from being proportional to the times, and the body *T* is either at rest or moving uniformly in a straight line: either there is no action of centripetal force tending to the body *T*, or it is merged in and compounded with very powerful actions of other forces; and the whole force compounded of them all (if there are more than one) is directed to another centre (either fixed or moving). The same holds when the other body (*T*) is affected with any motion whatever; provided that force be taken as the centripetal force which remains after subtracting from the whole force on *L* the whole force which acts on the other body *T*.

### Scholium.

Since the equable description of areas indicates the centre which that force tends to, by which the body is most affected and by which it is drawn away from rectilinear motion and retained in its orbit; we shall in what follows make use of the property of equable description of areas as a means of finding the centre about which any orbital motion in free space is performed.

### *Angular Velocity.*

When a body $P$ moves in an orbit, its angular velocity round any point $C$ is the rate at which $CP$ separates from any fixed line through $C$; and, if uniform, is measured by the angle described by $CP$ in a second of time.

Thus the angle described in $t$ seconds = angular velocity $\times\, t$;

and $$\text{angular velocity} = \frac{\text{angle described in } t \text{ seconds}}{t}.$$

If the angular velocity be not uniform, the fraction on the right-hand side will not be independent of $t$; but, if $t$ be an indefinitely small fraction of a second, the fraction will in general approach some limiting value; this limiting value of

$$\frac{\text{angle described in } t''}{t}$$

is called the angular velocity which the body has at the instant under consideration; and it is such that if the body moved round $C$ with such an angular velocity through an interval of time $t$, the angle it would describe is in a ratio of equality with the angle actually described in that interval, when $t$ is indefinitely diminished.

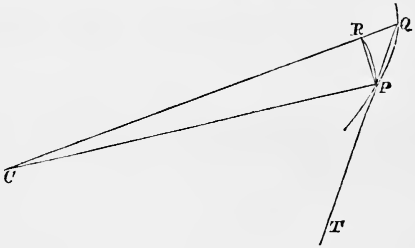
If therefore $PCQ$ be the angle described by a body in $t''$ after leaving $P$,

$$\text{the angular velocity} = \text{limit } \frac{\text{angle } PCQ}{t}.$$

The angular velocity at $P$ is, when variable, the angle which would be described by the body in one second after arriving at $P$, if it remained throughout that second of the same magnitude as at $P$.

PROP. *To find a relation between the angular velocity of a body about any point and its linear velocity.*

Let $C$ be the point; $PQ$ a small arc described by the body;




join $CP$, $CQ$: describe about $C$ an arc of a circle $PR$ cutting $CQ$ in $R$; join $PQ$, $PR$.

Let $v$ be the velocity, $\omega$ the angular velocity about $C$, and $PT$ the tangent, at $P$; and let $CP = r$, and angle $CPT = \phi$.

Then, the angle $CRP$, and therefore its supplement the angle $QRP$, is ultimately a right angle when $Q$ moves up to $P$;

$$\therefore \text{ limit } \frac{\text{arc } PR}{\text{arc } PQ} = \text{limit } \frac{PR}{PQ} = \text{limit } \frac{\sin PQR}{\sin PRQ}$$
$$= \sin CPT = \sin \phi;$$

but
$$\text{limit } \frac{\text{arc } PR}{\text{arc } PQ} = \text{limit } \frac{\frac{\text{arc } PR}{CP} \cdot CP}{\text{arc } PQ}$$
$$= \text{limit } \frac{\angle PCQ \times CP}{\text{arc } PQ} = \frac{\omega r}{v};$$

$$\therefore \omega r = v \sin \phi; \text{ the relation required.}$$

If $p$ be the perpendicular from $C$ on the tangent at $P$, $p = r \sin \phi$; hence the relation found above may be expressed in the form

$$\omega r^2 = pv.$$

COR. Since in an ellipse the focal distances make equal angles with the tangent at any point $P$, if $\omega_1$, $\omega_2$ are the angular velocities about $S$ and $S'$, we have

$$\omega_1 \times SP = \omega_2 \times S'P.$$

PROP. *If C be the centre of force about which P describes its orbit, the angular velocity* $\omega = \frac{h}{CP^2}$.

For, since $C$ is the centre of force,

$$h = vp \text{ (Prop. I. Cor. 1, Note)},$$
$$= \omega r^2.$$

DEF. When a body describes an orbit under the action of a centripetal force, its *mean angular velocity* about any point is the angular velocity with which, if it remained constant throughout a revolution, the body would describe its orbit in the same period as it actually does.

Thus, let $\Omega$ be the mean angular velocity of a body; $P$ its periodic time; then in time $P$ with uniform angular velocity $\Omega$ the body would describe an angle $2\pi$ about the point;

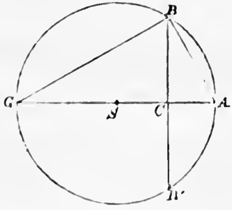
$$\therefore\ 2\pi = \Omega P;$$

$$\text{and } \Omega = \frac{2\pi}{P}.$$

DEF. The *versine* of a circular arc $BA$, centre $S$, is the intercept $AC$, cut off by the perpendicular from $B$ on the radius $AS$.

The versine of an arc $BA$ is, thus, the sagitta—through the centre $S$ of the circle—of an arc $BAB'$, which is twice $BA$.

Let $AG$ be the diameter of the circle: join $BG$,




then $$AC = \frac{(\text{chord } AB)^2}{AG};$$

or $$\text{versine} = \frac{(\text{chord})^2}{\text{diameter}}.$$

## Prop. IV. (Theorem IV.)

*The centripetal forces by which bodies describe circles with uniform motion tend to the centres of the circles; and are to one another directly as the squares of arcs described in equal times and inversely as the radii of the circles.*

These forces tend to the centres of the circles, by Prop. II. and Prop. I. Cor. 2; and are to one another as the versines of indefinitely small arcs described in equal times, by Prop. I. Cor. 4; that is, directly as the squares of those arcs and inversely as the diameters of the circles, by Lemma VII.

Therefore, as these arcs are as the arcs described in *any* equal times, and the diameters are as the radii; the forces are directly as the squares of any arcs described in equal times, and inversely as the radii of the circles. Q. E. D.

*Note.* By Prop. II. the forces tend to a point about which equal areas are described in equal times.

By Prop. I. Cor. 2, the point which has this property is found; and in this case it is found to be the centre of the circle.

That the centre of the circle is the point about which equal areas are described in equal times in the present case is sufficiently obvious; but Prop. I. Cor. 2 is here alluded to as giving a method generally applicable for finding the centre of forces.

Cor. 1. Since these arcs are as the velocities ($v$) of the bodies, the centripetal forces are directly as the squares of the velocities of the bodies, and inversely as the radii ($r$).

$$\left[\text{force} \propto \frac{v^2}{r}.\right]$$

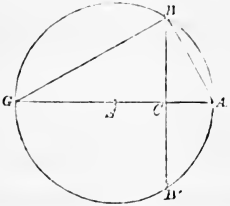
*Note.* Let $t$ be the time of describing an arc $AB$ of the circle $ABG$ about a centripetal force at the centre $S$; then the force is (Prop. I. Note)

$$=\text{limit of } \frac{2AC}{(\text{time of describing } AB)^2}$$

$$= \text{limit of } \frac{2 \times (\text{chord } AB)^2}{\text{diameter } AG \times t^2}$$

$$= 2 \text{ limit of } \frac{(\text{arc } AB)^2}{2AS \times t^2}$$

$$= \frac{(\text{velocity})^2}{\text{radius}}.$$




Or, if $v$ be the velocity, $r$ the radius, $f$ the accelerating force,

$$f = \frac{v^2}{r}.$$

Cor. 2. And, since the periodic times ($P$) are directly as the radii and inversely as the velocities; the centripetal forces are directly as the radii, and inversely as the squares of the periodic times $\left[\text{force} \propto \frac{v^2}{r}; \text{ and } P \propto \frac{r}{v}; \text{ therefore force} \propto \frac{1}{r}\left(\frac{r}{P}\right)^2 \propto \frac{r}{P^2}\right]$.

*Note.*
$$P = \frac{\text{circumference}}{\text{velocity}}$$
$$= \frac{2\pi r}{v}.$$

Cor. 3. Hence, if the periodic times are equal, and therefore the velocities are as the radii, the centripetal forces will be as the radii $\left[\text{for here force} \propto \frac{v^2}{r} \propto \frac{r^2}{r} \propto r\right]$.

And conversely [if the forces vary as the radii the periodic times are equal; for here $\frac{v^2}{r} \propto r$: therefore $v \propto r$; therefore $P \propto \frac{r}{v}$ is constant].

COR. 4. If the periodic times, and therefore also the velocities, are in the subduplicate ratio of the radii; the centripetal forces will be equal [for here $P \propto r^{\frac{1}{2}}$; therefore $\frac{r}{v} \propto r^{\frac{1}{2}}$, and $v \propto r^{\frac{1}{2}}$; therefore force $\propto \frac{v^2}{r}$ is constant]. And conversely [if the centripetal forces are equal the periodic times and the velocities will both be in the subduplicate ratio of the radii; for here $\frac{v^2}{r}$ is constant; therefore $v \propto r^{\frac{1}{2}}$; therefore $P \propto \frac{r}{v} \propto r^{\frac{1}{2}}$].

COR. 5. If the periodic times are as the radii and consequently the velocities equal, the centripetal forces will be inversely as the radii [for here $P \propto \frac{r}{v} \propto r$; therefore $v$ is constant; therefore force $\propto \frac{v^2}{r} \propto \frac{1}{r}$]. And conversely [if the forces are inversely as the radii the velocities will be equal, and the periodic times will be as the radii; for here $\frac{v^2}{r} \propto \frac{1}{r}$: therefore $v$ is constant and $P \propto \frac{r}{v} \propto r$].

COR. 6. If the periodic times are in the sesquiplicate ratio of the radii, and therefore the velocities reciprocally as the subduplicate ratio of the radii, the centripetal forces will be inversely as the squares of the radii [for here

$P \propto \frac{r}{v} \propto r^{\frac{3}{2}}$; therefore $v \propto \frac{1}{r^{\frac{1}{2}}}$; therefore force $\propto \frac{v^2}{r} \propto \frac{1}{r^2}$].

And conversely [if the forces are reciprocally as the squares of the radii the periodic times will be in the sesquiplicate ratio of the radii; for here $\frac{v^2}{r} \propto \frac{1}{r^2}$; therefore $v \propto \frac{1}{r^{\frac{1}{2}}}$; therefore periodic time $\propto \frac{r}{v} \propto r^{\frac{3}{2}}$].

COR. 7. And generally, if the periodic time be as $r^n$, and therefore the velocity reciprocally as $r^{n-1}$, the centripetal force will be reciprocally as $r^{2n-1}$ [for here $P \propto \frac{r}{v} \propto r^n$; therefore $v \propto \frac{1}{r^{n-1}}$; and force $\propto \frac{v^2}{r} \propto \frac{1}{r^{2n-1}}$].

And conversely [if the forces are reciprocally as $r^{2n-1}$, the periodic times will be as $r^n$; for here $\frac{v^2}{r} \propto \frac{1}{r^{2n-1}}$; therefore $v \propto \frac{1}{r^{n-1}}$; therefore $P \propto \frac{r}{v} \propto r^n$].

COR. 8. The same statements with respect to the times, velocities, and forces, with which bodies describe similar parts of *any similar figures* about centres of force similarly situated in those figures, follow by applying to these cases the demonstrations which have preceded. But in applying them we must substitute the uniform description of areas for uniform motion, and the distances of the bodies from the centres of forces for the radii.

COR. 9. By the method of proof used in this proposition it follows that the arc described in any time by a body revolving uniformly in a circle under the action of a given centripetal force, is a mean proportional between the diameter of the circle and the space through which the body would fall by the action of the same force during the same time.

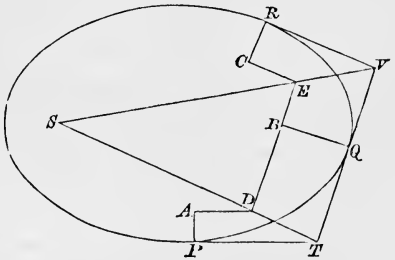
## Scholium.

The case of the sixth Corollary is that of the heavenly bodies (as Wren, Hook, and Halley inferred independently); and therefore questions relating to a centripetal force decreasing inversely as the square of the distance from the centre of force, is treated of at greater length in a subsequent part of the work.

Moreover by means of the preceding proposition and its corollaries we can determine the proportion of a centripetal force to any known force such as that of gravity. For if a body revolves by its own gravity in a circle concentric with the earth, this gravity is its centripetal force; and, by Cor. 9 of this proposition, given the space through which bodies fall by their gravity in a given time, the time of one revolution is known, and the arc described in any given time. And it is by propositions of this kind that Huyghens in his tract *De Horologio Oscillatorio* compared the force of gravity with the centrifugal forces of revolving bodies.

## Prop. V. (Problem I.)

*Given at any points the velocities with which a body describes any figure under forces all tending to one common centre, to find that centre.*



Let the figure described be touched at the points $P$, $Q$, $R$, by the straight lines $PT$, $TQV$, $VR$ which meet in $T$ and $V$. Draw $PA$, $QB$, $RC$ perpendicular to the tangents, and reciprocally proportional to the velocities of the body at the points $P$, $Q$, $R$ at which they are drawn: that is, so that $PA$ may be to $QB$ as the velocity at $Q$ to the velocity at $P$, and $QB$ to $RC$ as the velocity at $R$ to the velocity at $Q$; at the extremities $A$, $B$, $C$ of these perpendiculars draw $AD$, $DBE$, $EC$ at right angles to them and meeting in $D$ and $E$. Then the lines $TD$, $VE$ will meet in the required centre.

For the perpendiculars from the centre $S$ on the tangents $PT$, $QT$ are (by Prop. I. Cor. 1) reciprocally as the velocities of the body at $P$ and $Q$, and therefore, by construction, directly as the perpendiculars $AP$, $BQ$, that is, as the perpendiculars from the point $D$ on the tangents; hence it easily follows that $S$, $D$, $T$ are in one straight line.

And by similar reasoning the points $S$, $E$, $V$ may be shewn to be in one straight line: and therefore the centre $S$ is situated at the point of intersection of the straight lines $TD$, $VE$. Q. E. D.

*Note* 1. In this proposition the velocities at three points $P$, $Q$, $R$ of a given orbit are given; also the orbit being given the positions of the tangents $PT$, $TQV$, and $RV$ are known. Only *three* points are taken in the proof, that number being sufficient, as the proof shews.

*Note* 2. It is shewn in the proof that the perpendiculars from $S$ on $PT$, $QT$ are directly as $AP$, $BQ$; that is, directly as the perpendiculars from $D$ on $PT$, $QT$; and it is inferred that $S$, $D$, $T$ are in one straight line.

This is easily seen thus: the perpendiculars from $S$ and $D$ on $PT$ are to each other in the ratio of the distance of $S$ and $D$ from the point in which $SD$ produced meets $PT$; and these perpendiculars are, by what has been proved, as the perpendiculars from $S$ and $D$ on $QT$, that is, as the distances of $S$ and $D$ from the point in which $SD$ produced meets $QT$.

Hence $SD$ produced is divided by the point in which it meets $PT$ in the same ratio in which it is divided by the point in which it meets $QT$: and therefore $SD$ produced meets $PT$ and $QT$ in the same point, that is, in $T$.

## Prop. VI. (Theorem V.)

*If a body revolve in non-resisting space in any orbit about a fixed centre, and describe any indefinitely small arc in an indefinitely small time; and a sagitta of the arc be drawn to bisect the chord, and so as to pass through the centre of forces when produced; the centripetal force on the body while describing the arc will be directly as the sagitta, and inversely as the square of the time.*

For the sagitta of the arc described in a given [indefinitely small] time is proportional to the force (by Prop. I. Cor. 4); and as the time increases in any ratio, since the arc increases in the same ratio, the sagitta (by Lemma XI. Cors. 2 and 3) increases in the square of that ratio; therefore, the sagitta varies as the force and as the square of the time. Hence, dividing both sides by the square of the time, the force varies directly as the sagitta and inversely as the square of the time. Q. E. D.

This proposition is also easily proved by Lemma X. Cor. 4.

*Note* 1. In the proof of this proposition we must suppose the same construction and the same suppositions made as in Prop. I.; then, referring to the figure of Prop. I., we have, velocity added in each equal interval of time by the impulse to $S$, proportional to $Cc$, which is equal to $BV$; but half $BV$ is the sagitta of $ABC$, which is described in a given interval: thus, the sagittæ of arcs described in equal (indefinitely small) times are proportional to the forces. This is the substance of Prop. I. Cor. 4.

Again, in the same orbit described by the same body under the same forces, the sagitta of an indefinitely small arc through any given point is (by Lemma XI. Cor. 3) in the duplicate ratio of the time in which the body describes the arc with the velocity it then has at that point. Thus, in equal times the sagittæ vary as the forces: and with equal forces as the square of the time: and therefore, the sagittæ of arcs described in various indefinitely small times under various forces vary conjointly as the force and the square of the time.

*Note* 2. Again, as in Lemma X. Note 4, a force is measured by the ratio which the measure of the velocity produced by it in any given time bears to the measure of the time; if, then, $v'$ measure the velocity due to the impulse at $B$ (fig. Prop. I.), $t$ the time of describing $AB$, the force at $B$ is measured by $\frac{v'}{t}$.

But $v'$ is the velocity with which $Cc$ is described in time $t$; hence the measure of $v'$ is $\frac{Cc}{t}$;

$$\text{therefore, the measure of the force at } B \text{ is } \frac{Cc}{t^2},$$

$$\text{and this (Prop. I. Cor. 4)} = 2 \times \text{lt.} \frac{\text{sagitta of arc } ABC}{(\text{time of describing } AB)^2}.$$

*Note* 3. Let $Pu$ be the sagitta to $S$ of a small arc $QPQ'$: $QR'$, $Q'R'$ subtenses parallel to $SP$ (see figure on next page): then, by Lemma XI.,

$$\begin{aligned} \text{L. R. } QR : Q'R' &= \text{L. R. } QP^2 : Q'P^2 \\ &= \text{L. R. } Qu^2 : Q'u^2 \text{ (by Lemma VII. Cor. 1)}; \\ &= 1; \end{aligned}$$

therefore, since $Pu$ is intermediate between $QR$ and $Q'R'$,

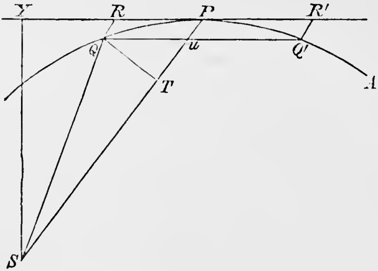
$$\text{L. R. } QR : Pu : QR = 1.$$

Hence, force at $P$

$$= 2 \times \text{lt.} \frac{Pu}{(\text{time of describing } PQ)^2} = 2 \times \text{lt.} \frac{QR}{(\text{time of describing } PQ)^2}$$

$$= 2 \text{ lt.} \frac{QR}{t^2}, \text{ where } t \text{ is the time of describing } PQ.$$

*Note* 4. By Note 3, since $t$ is the time of describing the area $PSQ$ about $S$, if any curves touch each other at $P$ and pass through $Q$, the forces to $S$ by which they may be described vary (when $Q$ moves up to $P$) inversely as the square of the times of describing equal areas. For $QR$ is the same for all the curves, and the areas $PSQ$ are ultimately in a ratio of equality.




Hence, since these curves have ultimately the same curvature at $P$, the forces to the same centre, at a point in which any curves touch each other and have the same curvature, are inversely as the square of the times of describing equal areas: that is, directly as the squares of areas described in equal times, or directly as $h^2$ (Prop. I. Cor. 1, Note).

Cor. 1. If a body $P$ revolving about a centre $S$ describe the curve $APQ$, and the straight line $PR$ touch the curve at any point $P$, and from any other point $Q$, $QR$ be drawn parallel to $SP$, and $QT$ perpendicular to $SP$, the centripetal force will be reciprocally as $\frac{SP^2 \times QT^2}{QR}$, that value of this fraction being always taken which it has in the limit when the points $P$ and $Q$ coincide.

For $QR$ is equal to the sagitta of an arc double of the arc $QP$, of which $P$ is the middle point; and twice the triangle $SQP$, or $SP \times QT$, is proportional to the time in which that double arc is described, and may therefore be written for that time.

*Note* 1. By Note 3 of the proposition, $QR$ is ultimately in a ratio of equality to the sagitta $Pu$ of an arc $QPQ'$, which is ultimately double of the arc $QP$.

*Note* 2. By Prop. I. Cor. 1, Note,

twice the triangle $SQP = ht$:

therefore $SP \times QT = ht$;

$$\text{and force at } P = 2\,\text{lt.}\ \frac{QR}{t^2},$$

$$\text{or } F = 2\text{lt.}\ \frac{QR \times h^2}{SP^2 \times QT^2}.$$

Cor. 2. For the same reason the centripetal force is reciprocally as the fraction $\dfrac{SY^2 \times QP^2}{QR}$, $SY$ being the perpendicular from the centre of forces on the tangent $PR$ to the orbit.

For the rectangles $SY \times QP$ and $SP \times QT$ are equal.

*Note.* L. R. $SY \times PR : SP \times QT =$ L. R. $\triangle SRP : \triangle SQP$;

therefore L. R. $SY \times PQ : SP \times QT = 1$;

$$\text{hence force at } P = 2\,\text{lt.}\ \frac{QR \times h^2}{SP^2 \times QT^2},$$

$$\text{thus } F = 2\,\text{lt.}\ \frac{QR \times h^2}{SY^2 \times QP^2}.$$

Cor. 3. If the orbit either is a circle, or touches or cuts a circle concentrically, that is touches or cuts the circle at the least possible angle; having the same curvature and the same radius of curvature at the point $P$; and if $PV$ be the chord of this circle drawn through the centre of forces, the centripetal force will be reciprocally as $SY^2 \times PV$.

For $PV$ is the limit of $\dfrac{QP^2}{QR}$.

*Note* 1. Let $AB$ be a small arc of a curve; and let a circle be described passing through $A$ and $B$, and having at $A$ the same tangent $AD$ as the curve: this circle touches or cuts the curve at the angle $BAD$.

Now let $B$ move up to and coincide with $A$: the limiting position of this circle will touch or cut the curve at the least possible angle; this limiting circle is concentric and coincides with the circle of curvature at $A$, and in this sense touches or cuts the curve concentrically.

*Note* 2. If on passing from $A$ to $B$ the curvature of the curve remains constant, $AB$ coincides with an arc of the circle of curvature; but if it diminishes, $AB$ falls between $AD$ and the circle of curvature; if the curvature increases from $A$ to $B$, the circle of curvature falls between $AD$ and $AB$.

In general, in any indefinitely small arc $BAB'$, the curvature is either continuously increasing or continuously diminishing from $B$ to $B'$; and thus, if the curvature increases from $B$ to $A$, it diminishes from $B'$ to $A$; and *vice versâ*. Hence, the circle of curvature, in general, falls between the curve and its tangent at one side of its point of contact, and inside the curve on the other side of the point. In this sense the circle of curvature in general *cuts* the curve at the point of contact.

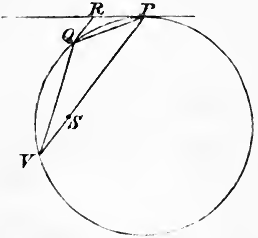
The only cases in which the circle of curvature *touches* the curve at the point of contact, in the sense of falling outside the curve on both sides, or inside the curve on both sides of the point, are the cases in which the curvature of the curve increases as we proceed *in both directions* from the point, or diminishes as we proceed in both directions; that is, at points of minimum or maximum curvature.

*Note* 3. Describe a circle touching the curve at $P$ and cutting it at $Q$, and meeting $PS$ produced in $V$: then $\angle QPR = \angle PVQ$ (Euc. III. 32); and $\angle RQP = \angle QPV$: thus $\triangle PQR$ is similar to $\triangle PVQ$;

therefore $$PV : PQ = PQ : PR;$$

hence $$PV = \frac{PQ^2}{PR};$$

and $PV$ is ultimately the chord through $S$ of the circle of curvature, or, in the words of the Corollary, of the circle which touches or cuts the curve concentrically.




$$\text{And force} = 2 \text{ lt. } \frac{QR \times h^2}{SY^2 \times QP^2} \text{ (Cor. 2)},$$

$$\text{thus } F = \frac{2h^2}{SY^2 \times PV}.$$

Cor. 4. Under the same circumstances (as in Cor. 3), the centripetal force varies directly as the square of the velocity and inversely as the chord $PV$.

For the velocity is inversely as the perpendicular $SY$, by Prop. I. Cor. 1.

*Note* 1. Thus force $= 2 \text{ lt. } \dfrac{h^2}{SY^2 \times PV}$ (Cor. 3, Note 3),

$$\text{or } F = \frac{2v^2}{PV} \text{ (Prop. I. Cor. 1, Note).}$$

*Note* 2. Let $s$ be the space through which the force $F$, if it remained constant, would be required to draw the body from rest in order to give it the velocity $v$; this space is called the *space due to the velocity* v.

Now $\qquad v^2 = 2Fs,$

and by Note 1, $\qquad v^2 = F\dfrac{PV}{2};$

therefore $\qquad s = \dfrac{PV}{4}:$

hence, *the space due to the velocity at any point of an orbit described about any centre of force is one-fourth of the chord of curvature through the centre of force.*

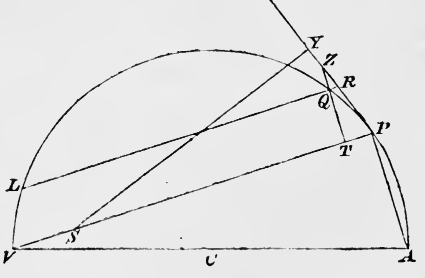
Cor. 5. Hence if any curvilinear figure $APQ$ be given, and also the point $S$ in it to which the centripetal force continually tends, we can find the law of the centripetal force by the continuous action of which any body $P$ may be drawn from its rectilinear course, and made to move in the perimeter of that figure, and to describe it as an orbit. In fact we have simply to find the value of $\frac{SP^2 \times QT^2}{QR}$ or of $SY^2 \times PV$, which are reciprocally proportional to this force. Examples of this will be given in the problems which follow.

*Note.* And by Cor. 1, Note 2, the measure of the force is given by 2 limit $\frac{QR \times h^2}{SP^2 \times QT^2}$, or, by Cor. 3, Note 3, by $\frac{2h^2}{SY^2 \times PV}$, where $PV$ is the chord of curvature through the centre of forces $S$.

## Prop. VII. (Problem II.)

*A body moves in the circumference of a circle: required the law of centripetal force tending to any given point.*

Let $VQPA$ be the circumference of the circle, $S$ the given point to which the force tends as its centre; $P$ a



point on the circumference at which the body has arrived, $Q$ a point very near to $P$ towards which the body is moving, and $PRZ$ the tangent to the circle at $P$.

Through the point $S$ draw the chord $PV$; draw the diameter $VA$ and join $AP$; draw $QT$ perpendicular to $SP$ meeting the tangent $PR$ in $Z$; and, lastly, through $Q$ draw $LR$ parallel to $SP$ and meeting the circle in $L$, and the tangent $PZ$ in $R$.

Then, from the similar triangles $ZQR$, $ZTP$, $VPA$,

$$RP^2(=RQ \times RL) : QT^2 = AV^2 : PV^2,$$

$$\therefore \frac{RQ \times RL \times PV^2}{AV^2} = QT^2.$$

Multiply both these equals by $\frac{SP^2}{QR}$, and, as the points $P$ and $Q$ coincide in the limit, write $PV$ for $RL$.

$$\therefore \frac{SP^2 \times PV^3}{AV^2} = \frac{SP^2 \times QT^2}{QR} \text{ in the limit.}$$

Therefore (by Prop. VI. Cors. 1 and 5) the centripetal force is inversely as $\frac{SP^2 \times PV^3}{AV^2}$; that is (since $AV$ is given), inversely as the square of the distance $SP$ and the cube of the chord $PV$. Q. E. I.

*Note* 1. The triangles $ZTP$, $VPA$ are similar by Euc. III. 32, as in Prop. VI. Cor. 3, Note 3.

*Note* 2. The law of the force is found in this proof by the expression for it in Prop. VI. Cor. 1.

The measure of the force is found by Prop. VI. Cor. 1, Note 2; thus,

$$\text{force at } P = 2 \text{ lt. } \frac{QR \times h^2}{SP^2 \times QT^2};$$

and $$\text{lt. } \frac{QR}{SP^2 \times QT^2} = \frac{AV^2}{SP^2 \times PV^3} \text{ (as proved in the Proposition);}$$

$$\text{hence force at } P = \frac{2AV^2 \times h^2}{SP^2 \times PV^3}.$$

*Note* 3. The measure of the force is often found to consist of two factors, one factor depending on the position of the body with respect to the centre of force, and the other factor constant for all positions of the body. The constant factor is usually denoted by $\mu$.

Thus in the above expression,

$$\text{the force at } P = \frac{\mu}{SP^2 \times PV^3};$$

$$\text{where } \mu = 2AV^2 \times h^2.$$

ANOTHER PROOF.

Draw $SY$ perpendicular to the tangent $PR$ produced; then from the similar triangles $SYP$, $VPA$,

$$AV : PV :: SP : SY,$$

$$\therefore \frac{SP \times PV}{AV} = SY;$$

$$\therefore \frac{SP^2 \times PV^3}{AV^2} = SY^2 \times PV.$$

Therefore (by Prop. VI. Cors. 3 and 5) the centripetal force is inversely as $\dfrac{SP^2 \times PV^3}{AV^2}$; that is, since $AV$ is given, inversely as $SP^2 \times PV^3$. Q. E. I.

*Note.* In this proof the law of force is found by Prop. VI. Cor. 3.

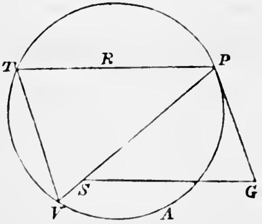
The expression for the force is found thus by Prop. VI. Cor. 3, Note 3.

$$\text{Force at } P = 2 \text{ lt. } \frac{h^2}{SY^2 \times PV}$$

$$= \frac{2AV^2 \times h^2}{SP^2 \times PV^3} \text{ (by this proof).}$$

COR. 1. Hence if the given point $S$, to which the centripetal force always tends, be situated on the circumference of the circle as at $V$, the centripetal force will be inversely as the fifth power of the distance $SP$.

Cor. 2. The force by which a body $P$ is made to revolve in the circle $APTV$ about a centre of forces $S$, is to the force by which the same body $P$ may be made to revolve in the same circle and in the same periodic time




about any other centre of forces $R$ as $RP^2 \times SP$ to $SG^3$; where $SG$ is a straight line drawn from the first centre of forces $S$ to the tangent $PG$ to the orbit, parallel to $RP$ the distance of the body from the other centre of forces.

For, by the proposition, the former force is to the latter as $RP^2 \times PT^3$ to $SP^2 \times PV^3$; that is, as $SP \times RP^2$ is to $\frac{SP^3 \times PV^3}{PT^3}$, or (since the triangles $PSG$, $TPV$ are similar), as $SP \times RP^2$ is to $SG^3$.

*Note* 1. On referring to the figure and proof of Prop. I., it is seen that the force at any point $B$ is known if $Cc$ is known, and the interval of time between successive impulses: or by Cor. 4, if the sagitta of an indefinitely small arc is known, and the time of describing the arc; or again, by Prop. VI. Note 3, if the subtense $QR$ be known, and the time $t$ of describing $PQ$: the force being in all cases proportional to the limit of $\frac{QR}{t^2}$. This expression is proportional to the force at any point of any orbit, so that the forces at different points of the same orbit and at any points of different orbits are all proportional to the limits of this expression in the different cases.

When, as in the Corollaries to Prop. VI., and in the succeeding applications of them, expressions are obtained by substituting for the time $t$ of describing $PQ$, the area $PSQ$ about

$S$, which is proved in Prop. I. to be, *in any given orbit* described about a fixed centre of forces, proportional to the time, it must be remembered that this proportionality is only true for the same orbit, described in the same manner by the same body ; in different orbits the areas described in equal times differ in general : and thus the expressions so obtained are only in general applicable for comparing the forces at different points of the same orbit.

If two bodies describe the same orbit in the same periodic time, they will describe any given fraction of that orbit in the same fraction of the periodic time; that is, they will describe any the same area in the same time ; hence the expressions obtained in the Corollaries to Prop. VI. are available for comparing at any points, the same or different, the forces with which bodies are acted on which describe *any the same orbit in the same periodic time.*

*Note* 2. By Prop. I. we have, in any orbit about a fixed centre of forces,

$$\frac{\text{area } PSQ}{t''} = \frac{\text{area described in } 1''}{1} = \frac{\text{area of orbit}}{\text{periodic time in seconds}};$$

or, if the periodic time in seconds be $P$, and $h = 2 \times$ area described in one second,

$$\frac{2 \text{ area } PSQ}{t} = h = \frac{2 \text{ area of orbit}}{P}.$$

Thus the expressions for the force in the Corollaries, found by substituting for $t$ the area $PSQ$, are available for comparing forces at points of the same or different orbits provided $h$ is the same ; or, provided the area of the orbit and the periodic time are the same (or proportional) in the two cases.

COR. 3. The force by which a body describes *any orbit* about a centre of forces $S$, is to the force by which the same body $P$ can be made to describe the same orbit in the same periodic time about any other centre of forces $R$, as $SP \times RP^2$—that is, the product of the distance of the body from the first centre of forces $S$, and of the square of the distance from the second centre of forces $R$—to the cube of $SG$, which is the line drawn from the first centre $S$ to the tangent $PG$ to the orbit, parallel to the distance $RP$ of the body from the second centre of forces.

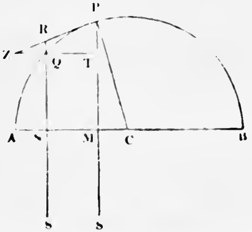
For the forces in this orbit at any point of it, $P$, are the same as in a circle of the same curvature.

*Note.* The proofs of Prop. VII. and of its Corollaries 1 and 2 apply to any orbit, the circle referred to in the proot and in the results being the circle which touches the curve at $P$ and passes through $Q$, or the limit of that circle when $Q$ moves up to $P$, that is, the circle of curvature at $P$.

For $SP$, $QT$ and $QR$ are the same for the curve and the circle, and, therefore, the limit of $\frac{SP^2 \times QT^2}{QR}$, on which (Cor. 2, Note 2) the expression for the force depends, is the same for both.

## Prop. VIII. (Problem III.)

*A body moves in a semicircle* PQA ; *to find the law of centripetal force tending to a point* S *which is so distant that all lines, as* PS, RS, *drawn to it may be considered parallel.*




From the centre $C$ of the semicircle draw the semi-diameter $CA$ cutting those parallels in $M$ and $N$; and join $CP$.

Because the triangles $CPM$, $PZT$ and $RZQ$ are similar,

$$CP^2 : PM^2 :: PR^2 : QT^2;$$

and (Euc. III. 36)

$$PR^2 = QR \times (RN + QN);$$

or, when the points $P$ and $Q$ coincide,

$$PR^2 = QR \times 2PM;$$

$$\therefore\ CP^2 : PM^2 :: QR \times 2PM : QT^2:$$

$$\therefore\ \frac{QT^2}{QR} = \frac{2PM^3}{CP^2},$$

and $$\frac{QT^2 \times SP^2}{QR} = \frac{2PM^3 \times SP^2}{CP^2}.$$

Therefore (by Prop. VI. Cors. 1 and 5), the centripetal force is inversely as $\frac{2PM^3 \times SP^2}{CP^2}$; that is $\left(\text{neglecting the constant ratio } \frac{2SP^2}{CP^2}\right)$, inversely as $PM^3$. Q. E. I.

The same result is also easily obtained by means of the previous proposition.

*Note* 1. The measure of the force is (Prop. VI. Cor. 1, Note 1)

$$2\ \text{lt.}\ \frac{QR \times h^2}{SP^2 \times QT^2};$$

which in the case of this Proposition is shewn to be equal to

$$\frac{h^2 \times CP^2}{PM^3 \times SP^2};$$

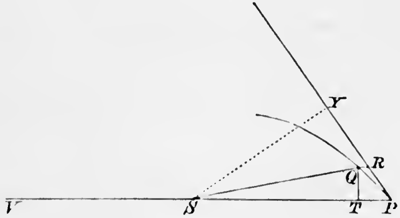
$$\text{thus force} = \frac{\mu}{PM^3};\ \text{where}\ \mu = \frac{h^2 \times CP^2}{SP^2}.$$

SCHOLIUM.

And by somewhat similar reasoning it can be shewn that a body may be made to describe an ellipse, or even a hyperbola or a parabola, by a centripetal force which varies inversely as the cube of an ordinate drawn towards a very distant centre of forces.

PROP. IX. (PROBLEM IV.)

*A body revolves in a spiral* PQS *which cuts all the radii* SP, SQ, &c. *in the same given angle—(an equiangular spiral); to find the law of centripetal force tending to the centre of the spiral.*




Given any indefinitely small angle $PSQ$, the form of the figure $SPRQT$ is given, since all its angles are given. Therefore the ratio $\frac{QT}{QR}$ is given: and $\frac{QT^2}{QR}$ is as $QT$, that is (since the form of the figure is given), as $SP$.

Now let the angle $PSQ$ be changed in any manner; then $QR$, which subtends the angle of contact $QPR$, will (by Lemma XI.) be changed in the duplicate ratio of $PR$, or of $QT$. Therefore $\frac{QT^2}{QR}$ will remain the same as before, that is, proportional to $SP$.

Therefore $\frac{QT^2 \times SP^2}{QR}$ varies as $SP^3$: and thus (by Prop. VI. Cors. 1 and 5) the centripetal force varies inversely as the cube of the distance $SP$. Q. E. I.

ANOTHER PROOF.

The perpendicular $SY$ on the tangent and the chord $PV$ of the circle concentrically cutting the spiral (that is, of the circle of curvature at $P$), are to the distance $SP$ in given ratios; and thus $SP^3$ varies as $SY^2 \times PV$, that is (by Prop. VI. Cors. 3 and 5), inversely as the centripetal force.

*Note* 1. DEF. *An equiangular spiral is a curve which cuts all radii from a certain point called the focus at the same angle: and this angle is called the angle of the spiral.*

Given any radius $SP$ of an equiangular spiral, the curve is known through $P$ for any distance in either direction, if the angle of the spiral is known: hence if $SQ$ be taken making a given angle with $SP$, the figure $SPRQT$ is completely known; and its angles (and the angles of all the triangles into which it can be divided) and the lengths of its sides depend only on $SP$, the angle of the spiral, and the angle $PSQ$. The angles of this figure are therefore known in terms of the angle of the spiral and the angle $PSQ$; and the sides in terms of $SP$ and those angles.

Hence, if in any the same equiangular spiral an angle $PSQ$ be given, all the angles of all the triangles into which the figure $SPRQT$ is divided are given, and the ratios of the sides to each other: the figure is therefore given in respect to its angles and the ratios of its sides, but not in the absolute magnitudes of the sides; this is what is meant in the Proposition by the form of the figure being given.

*Note* 2. The angles of the triangle $SPY$ are known, the angle of the spiral being known, and therefore the ratio of $SY$ to $SP$ is given.

Again, the circle touching $PY$ in $P$ and passing through $Q$ is given in magnitude and position if the angle $PSQ$, the angle of the spiral, and $SP$ are given: hence $PV$ is known in terms of $SP$ and those angles, and therefore the ratio of $PV$ to $SP$ is given in terms of the angle of the spiral when the angle $PSQ$ vanishes, that is when $PV$ becomes the chord through $S$ of the circle of curvature.

*Note* 3. To find the expression for the force to the focus by which an equiangular spiral is described.

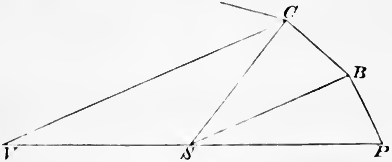
By Prop. VI. Cor. 3, Note 3, the force to $S = \frac{2h^2}{SY^2 \times PV}$;

let the angle of the spiral $= \alpha$; then $SY = SP \sin \alpha$;

$$\text{and force to } S = \frac{2h^2}{SP^2 \times PV \times \sin^2 \alpha}.$$

To find $PV$.

An equiangular spiral is the limit of a polygon, the lines joining whose angular points with $S$ form a number of similar triangles (Note 1).



Let $PSB$, $BSC$ be two successive triangles of the series into which the polygon is divided, so that $\angle PSB = \angle BSC$; and $\angle SPB = \angle SBC$: draw $CV$ parallel to $BS$ meeting $PS$ produced in $V$.

Then $$\angle SPB = \angle SBC;$$

$$\therefore \angle s\ SPB,\ SBP = \angle PBC;$$

and $$\angle PSB = \angle PVC;$$

$$\therefore \angle s\ SPB,\ SBP,\ PSB = \angle s\ PBC,\ PVC;$$

thus $\angle$s $PBC$, $PVC$ = 2 right angles;

therefore the circle through $P$, $B$, $C$ passes through $V$.

Now the limit of the circle through $P$, $B$, $C$ is the circle of curvature when $P$, $B$, $C$ coincide.

Again, because $SB$ makes equal angles with $SP$, $SC$; and $VC$ is parallel to $SB$;

$$\therefore \angle CVS = \angle VCS,$$

$$\therefore SV = SC = SP \text{ ultimately}.$$

Hence $PV$ the chord of curvature at $P$ through $S = 2SP$;

therefore force to $S = \dfrac{2h^2}{SP^2 \times PV \times \sin^2 \alpha}$

$$= \frac{h^2}{SP^3 \times \sin^2 \alpha}$$

$$= \frac{\mu}{SP^3},$$

where $\mu = \dfrac{h^2}{\sin^2 \alpha}$.

*Note* 4. In this case, and in all cases in which the variable factor in the expression for the force is some power of the distance of the body from the centre of force, the constant factor $\mu$ is the absolute force (Def. 6, Note).

## Lemma XII.

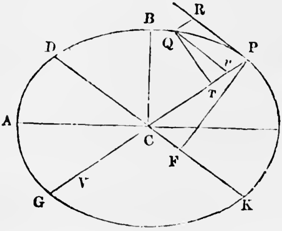
*All parallelograms described about any conjugate diameters of a given ellipse or hyperbola are equal to one another.*

(See Besant's *Conic Sections*, pp. 69 and 113.)

## Prop. X. (Problem V.)

*A body moves in an ellipse; required the centripetal force tending to the centre of the ellipse.*

Let $CA$, $CB$ be the semiaxes of the ellipse; $GP$, $DK$




any conjugate diameters; $PF$, $QT$ perpendiculars on those diameters; $Qv$ an ordinate to the diameter $GP$; then, completing the parallelogram $QvPR$, we have (Besant's *Conic Sections*, page 66, Prop. XX.),

$$Pv.vG : Qv^2 :: PC^2 : CD^2;$$

and, from the similar triangles $QvT$, $PCF$,

$$Qv^2 : QT^2 :: PC^2 : PF^2;$$

$\therefore$ componendo,

$$Pv.vG : QT^2 :: PC^4 : CD^2.PF^2,$$

$$\therefore vG : \frac{QT^2}{Pv} :: PC^2 : \frac{CD^2.PF^2}{PC^2}.$$

For $Pv$ put $QR$, and (by Lemma XII.) $BC.CA$ for $CD.PF$; and (as the points $P$ and $Q$ coincide) $2PC$ for $rG$, and we have

$$2PC : \frac{QT^2}{QR} :: PC^2 : \frac{BC^2.CA^2}{PC^2},$$

$$\therefore \frac{QT^2.PC^2}{QR} = \frac{2BC^2.CA^2}{PC}.$$

Therefore (by Prop. VI. Cor. 5) the centripetal force is inversely as $\frac{2BC^2.CA^2}{PC}$; that is (since $2BC^2.CA^2$ is given), inversely as $\frac{1}{PC}$, or directly as the distance $PC$.

Q. E. I.

*Note* 1. The expression for the force is

$$2 \text{ lt. } \frac{QR \times h^2}{CP^2 \times QT^2} = 2 \text{ lt. } \frac{PC \times h^2}{2BC^2 \times CA^2} \text{ (by the Proposition)}$$

$$= \mu CP;$$

where $$\mu = \frac{h^2}{BC^2 \times CA^2}.$$

*Note* 2. Since

$$\frac{h}{1} = \frac{2 \times \text{area of the ellipse}}{\text{periodic time}} \quad \text{(Prop. VII. Cor. 2, Note 2)}$$

$$= \frac{2\pi AC \times BC}{P},$$

$$\mu = \frac{h^2}{AC^2 \times BC^2}$$

$$= \frac{4\pi^2}{P^2};$$

and force $= \frac{4\pi^2}{P^2} CP$.

*Note* 3. The forces at different points of the same ellipse described about a centre of force in the centre vary as the distance of the body from the centre.

The forces at points of different ellipses described about centres of force in the centre vary as the distance directly, and the square of the periodic time inversely. Hence, in ellipses described about the centre *in the same periodic time*, the forces at all points vary as the distance.

Cor. 1. The force therefore varies as the distance of the body from the centre of the ellipse. And conversely, if the force vary as the distance the body will move in an ellipse whose centre is at the centre of the forces; or else in the circle which the ellipse may become in a particular case.

*Note* 1. Since the circle is a particular case of the ellipse, the forces to the centre by which bodies describe different circles vary as the radii of the circles directly, and as the squares of the periodic times inversely (see Note 3 to the Proposition); and, if the periodic times are equal, the forces to the centre by which different circles are described vary as the radii of the circles.

Cor. 2. The periodic times of all bodies which describe ellipses about the same centre of force in the centre of the ellipses will be equal.

For (by Prop. IV. Cors. 3 and 8) the periodic times are equal in similar ellipses; and in ellipses having the same major axis the periodic times are directly as the whole areas of the ellipses and inversely as the portions of the areas which are described in equal times; that is, directly as the minor axis, and inversely as the velocities at the extremity of the major axis; that is, directly as the minor axis, and inversely as the ordinates at the same point of the common major axis: or (since these ordinates are as the minor axes) in a ratio of equality.

*Note* 1. In this Corollary the periodic times in any two ellipses $A$ and $B$ about the centre are compared by comparing—(1) the periodic time in $A$, (2) the periodic time in $B$—with that in an ellipse similar to $A$, and having a major axis equal to that of $B$.

*Note* 2. If $P$ is the periodic time (in seconds) in an ellipse whose major axis is given, about a centre of force in the centre,

$$\frac{P}{2 \times \text{area of ellipse}} = \frac{1}{2 \times \text{area described in } 1''} = \frac{1}{h};$$

$$\therefore \quad P \propto \frac{\text{area of ellipse}}{h} \propto \frac{BC}{h}, \text{ since } AC \text{ is given.}$$

And if $V$ is the velocity at the extremity of the major axis,

$$h = V \times AC \propto V, \text{ since } AC \text{ is given};$$

and the velocity $V$ varies as the arc described in a given indefinitely small time, or as the ordinate (Lemma VII. Cor. 1) corresponding to a given indefinitely small abscissa from $A$; and such ordinates in different ellipses are as the minor axes.

Thus $h \propto BC$; and therefore $P$ is constant.

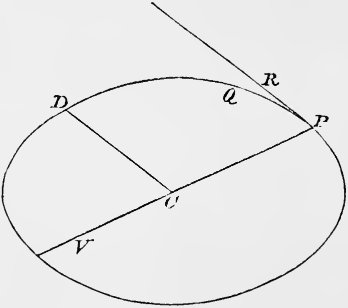
Or thus;—by Note 2 to the Prop. $P^2 = \frac{4\pi^2}{\mu}$; and therefore the periodic times are all equal if $\mu$ is given.

## Scholium.

If the ellipse is changed into a parabola by its centre being removed to an infinite distance, the body will move in this parabola; and the force tending to a centre at an infinite distance becomes constant. This is Galileo's theorem. And if the parabolic section of the cone, by varying the inclination of the plane, be changed into a hyperbola, the body will move in the perimeter of this hyperbola with a centrifugal in place of a centripetal force.

And just as in a circle or an ellipse, if the forces tend to the centre of figure—which is a point on the abscissa—and if the figure be altered by increasing or diminishing the ordinates in any given ratio, or by changing the angle of inclination of the ordinates to the abscissa, the forces are in all cases increased or diminished in the ratio of the distance from the centre, provided the periodic times remain unchanged; so, in any figure whatever, if the ordinates be increased or diminished in any given ratio, or their angle of inclination to the abscissa be changed, the periodic time in the new figure being the same as in the old; the forces tending to any point whatever on the abscissa as centre, will be changed, in the case of each ordinate, in the ratio of the distance from the centre.

*Note.* Let $PR$ be the tangent at $P$ to the small arc $PQ$ of any curve described about a centre of force $C$: draw $CD$ parallel




to $PR$, and $= \sqrt{\frac{1}{2}PV \times CP}$, where $PV$ is the chord of curvature at $P$ through $C$. Then $PV$ is also (Besant's *Conic Sections*, Art. 162, Cor. 3) the chord of curvature at $P$ of the ellipse whose centre is $C$, and of which $CP$, $CD$ are semi-conjugate diameters.

Now the forces at $P$ to $C$ by which the ellipse and curve may be described in equal periodic times are to each other (Prop. VI. Note 4) as the squares of the areas described in equal times, and therefore as the squares of the whole areas of the ellipse and the curve.

Now let the ordinates of the curve and ellipse be all changed in either of the ways mentioned in the Scholium: in each case the areas of the curve and ellipse are altered in the same ratio (see Lemma IV. Appendix); also the ellipse still becomes an ellipse with centre $C$. And since the limiting position of a secant through $P$ of the curve and ellipse is the same for both, being the common tangent at $P$, the limiting position of the corresponding secant to the altered curve and ellipse is the same for both, and is a tangent at the point ($p$) corresponding to $P$; and the limiting position of the circle touching at $P$, and passing through a point near $P$ of the curve or ellipse, is the same for both, and becomes in the altered figure an ellipse having the same curvature as each of the altered figures at the point of contact.

Thus the ellipse and curve have, after the alteration of the ordinates as in the Scholium, the ratio of their areas unaltered, and still have a common tangent and common curvature at the point $p$ corresponding to $P$.

Hence the forces at $p$ to $c$ by which the altered ellipse and curve are described in the same periodic time are to each other as the squares of their areas,—that is, as the squares of the areas of the original ellipse and curve, or, by what was proved, as the forces at $P$ to $C$ by which these curves are described in equal periodic times.

Therefore

force at $P$ in original curve : force at $P$ in original ellipse
= force at $p$ in altered curve : force at $p$ in altered ellipse.

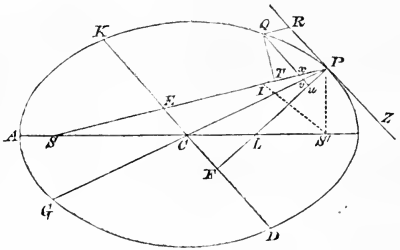
And hence the forces at $P$ and $p$ in the curves are to each other as the forces at $P$ and $p$ in the ellipses, or as $CP$ to $Cp$; the periodic times being equal.

# SECTION III.

## *On the Motion of Bodies in Excentric Conic Sections.*

### PROP. XI. (PROBLEM VI.)

*A body revolves in an ellipse; required the law of centripetal force tending to one of the foci.*




Let $S$ be the focus. Draw $SP$ cutting the conjugate diameter $DK$ of the ellipse in $E$, and its ordinate $Qv$ in $x$; and complete the parallelogram $QxPR$.

Then $PE$ is equal to the semi-axis major $AC$ (Besant's *Conics*, p. 58).

Draw $QT$ perpendicular to $SP$. Then the centripetal force to $S$ varies inversely as the limit of $\frac{SP^2 \times QT^2}{QR}$, when $Q$ and $P$ coincide (Prop. VI. Cors. 1 and 5).

Now $$QR : Pv = Px : Pv,$$
$$= PE : PC,$$
$$= AC : PC.$$

Again, if the normal at $P$ meet $DK$ in $F$,

$$QT^2 : Qx^2 = PF^2 : PE^2$$
$$= PF^2 : CA^2$$
$$= CB^2 : CD^2 \text{ (\textit{Conics}, p. 69)};$$

and $Qx^2 : Qv^2$ is a ratio of equality when $Q$ and $P$ coincide (Lemma VII. Cor. 2);

also $$Qv^2 : Gv \, . \, Pv = CD^2 : CP^2.$$

Therefore, compounding these ratios,

$$QT^2 : Gv \, . \, Pv = CB^2 : CP^2;$$

but $Gv \, . \, Pv : 2CP \, . \, Pv$ is a ratio of equality when $Q$ and $P$ coincide; hence ultimately

$$QT^2 . CP^2 = CB^2 \times 2CP \, . \, Pv;$$

therefore $$\frac{QT^2}{Pv} = \frac{2CB^2}{CP};$$

and, from above, $$\frac{Pv}{QR} = \frac{CP}{CA};$$

therefore $$\text{lt.}\ \frac{QT^2}{QR} = \frac{2CB^2}{CA} = L \ldots\ldots\ldots\ldots\ldots\ldots (a);$$

where $L$ is the latus rectum of the ellipse.

Thus $$\frac{SP^2 \times QT^2}{QR} = L \times SP^2:$$

and force varies inversely as $L \times SP^2$, or inversely as the square of the distance $SP$. Q. E. I.

*Note.* Hence, measure of the force to the focus of an ellipse

$$= 2\ \text{lt.}\ \frac{QR \times h^2}{SP^2 \times QT^2} \text{ (Prop. VI., Cor. 2, Note)}$$

$$= \frac{2h^2}{L} \times \frac{1}{SP^2}, \text{ from } (a)$$

$$= \frac{\mu}{SP^2}, \text{ where } \mu = \frac{2h^2}{L}.$$

## Another Proof.

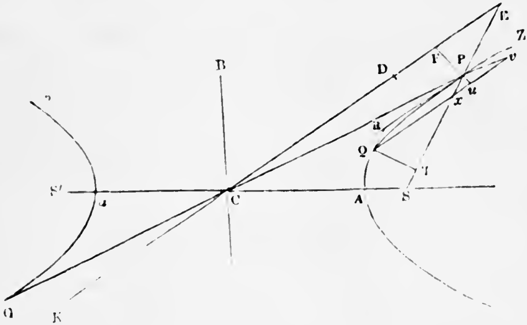
Since the force tending to the centre of the ellipse, by which the body $P$ can be made to revolve in the ellipse, varies (by Prop. X. Cor. 1) as the distance $CP$ of the body from the centre $C$, draw $CE$ parallel to the tangent $PR$.

Then the force, by which the same body $P$ can be made to revolve about any other point $S$, varies as $\frac{PE^3}{SP^2}$ (by Prop. VII. Cor. 3); that is, if $S$ be a focus of the ellipse, and $PE$ therefore constant, inversely as $SP^2$. Q. E. I.

We may proceed in the present case with the same ease as in Problem V. to extend it to the case of motion in a parabola or hyperbola; but on account of the importance of the problem, and of the use that will be made of it, it will be desirable to confirm the other cases by independent proofs.

### Prop. XII. (Problem VII.)

*A body moves in a hyperbola: required the law of centripetal force tending to one of the foci.*




Let $CA$, $CB$ be the semi-axes of the hyperbola: $PG$, $DK$ any conjugate diameters: $PF$ perpendicular to $DK$: and $Qv$ an ordinate to the diameter $GP$.

Draw $SP$ cutting the diameter $DK$ in $E$ and the ordinate $Qv$ in $x$, and complete the parallelogram $QRPx$.

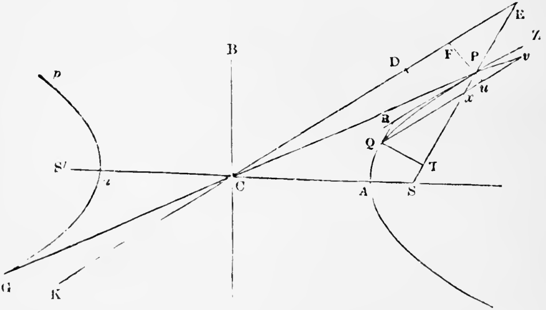
Then $PE$ is equal to the transverse semi-axis $AC$ (Besant's *Conics*, p. 99).

Draw $QT$ perpendicular to $SP$. Then the centripetal force to $S$ varies inversely as the limit of $\dfrac{SP^2 \times QT^2}{QR}$, when $Q$ and $P$ coincide (Prop. VI. Cors. 1 and 5).

Now,

$$QR : Pv = Px : Pv,$$
$$= PE : PC,$$
$$= AC : BC.$$

Again, $QT^2 : Qx^2 = PF^2 : PE^2,$
$= PF^2 : CA^2,$
$= CB^2 : CD^2$ (by Conics):




and $Qx^2 : Qv^2$ is a ratio of equality, when $Q$ and $P$ coincide; (Lemma VII. Cor. 2.)

also, $Qv^2 : Gv \times Pv = CD^2 : CP^2.$

Therefore, compounding these ratios,

$$QT^2 : Gv \times Pv = CB^2 : CP^2:$$

but $Gv \times Pv$ is in a ratio of equality with $2CP \times Pv$ when $Q$ and $P$ coincide;

$$\therefore QT^2 \times CP^2 = CB^2 \times 2CP \times Pv;$$

$$\therefore \frac{QT^2}{Pv} = \frac{2CB^2}{CP}:$$

and, from above, $\frac{Pv}{QR} = \frac{CP}{AC}$;

therefore $\text{lt.} \frac{QT^2}{QR} = \frac{2CB^2}{AC} = L$ ............... (a),

where $L$ is the latus rectum of the hyperbola:

$$\therefore \frac{SP^2 \times QT^2}{QR} = L \times SP^2;$$

therefore force varies inversely as $L \times SP^2$, or inversely as the square of the distance $SP$. Q. E. I.

*Note.* Hence the measure of the force to the focus of a hyperbola

$$= 2 \text{ lt. } \frac{QR \times h^2}{SP \times QT^2} = \frac{2h^2}{L} \times \frac{1}{SP^2}, \text{ from } (a)$$

$$= \frac{\mu}{SP^2}, \text{ where } \mu = \frac{2h^2}{L}.$$

Another Proof.

Find the force tending to the centre $C$ of the hyperbola; this will be found to be proportional to $CP$.

Hence (by Prop. VII. Cor. 3) the force to the focus $S$ will vary as $\frac{PE^3}{SP^2}$; that is, since $PE$ is constant, inversely as $SP^2$. Q. E. I.

In the same manner it may be shewn that if this centripetal force be changed into a centrifugal the body will move in the conjugate hyperbola.

Lemma XIII.

*The parameter of any diameter of a parabola is four times the focal distance of the vertex of that diameter.* (*Conics*, p. 32.)

Lemma XIV.

*The perpendicular from the focus of a parabola on any tangent is the mean proportional between the distances of the focus from the point of contact and the vertex of the parabola.* (*Conics*, p. 23.)

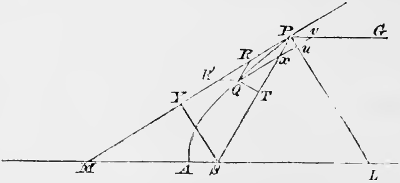
Cor. 1. $SP^2 : SY^2 = SP : SA$.

Cor. 2. Since $SA$ is constant $SY^2$ varies as $SP$.

Cor. 3. The perpendicular from the focus on any tangent meets it on the tangent at the vertex.

## Prop. XIII. (Problem VIII.)

*A body moves in a parabola: required the law of centripetal force to the focus.*




Let the tangent $PM$ at any point $P$ meet the axis of the parabola in $M$: draw $SY$ perpendicular to $PM$.

From a point $Q$ very near $P$, draw $QR$ parallel and $QT$ perpendicular to $SP$: and $Qv$ parallel to the tangent meeting the diameter $PG$ in $v$, and $SP$ in $x$.

Then, from the similar triangles $Pxv$, $SPM$, since $SP = SM$, $Px$ (or $QR$) $= Pv$.

Again, $Qv^2 = 4SP \times Pv$ (Lemma XIII.)

$$= 4SP \times QR.$$

And in the limit when the points $Q$, $P$ coincide, $Qv$ is to $Qx$ in a ratio of equality; therefore in the limit,

$$Qx^2 = 4SP \times QR.$$

And by similar triangles $QxT$, $SPY$,

$$QT^2 : Qx^2 = SY^2 : SP^2$$

$$= SA : SP \text{ (Lemma XIV. Cor. 1),}$$

or in the limit

$$QT^2 : 4SP \times QR = 4SA \times QR : 4SP \times QR;$$

$$\therefore \text{ lt. } \frac{QT^2}{QR} = 4SA \ldots\ldots\ldots\ldots\ldots\ldots\ldots (a);$$

$$\therefore \frac{SP^2 \times QT^2}{QR} = 4SA \times SP^2;$$

or, since $4SA$ is constant, the centripetal force (by Prop. VI. Cors. 1 and 5) varies inversely as the square of the distance $SP$. Q. E. I.

*Note.* Hence the measure of the force to the focus of a parabola

$$= 2 \text{ lt. } \frac{QR \times h^2}{SP^2 \times QT^2} = \frac{2h^2}{4SA} \times \frac{1}{SP^2}, \text{ from } (a)$$

$$= \frac{\mu}{SP^2},$$

$$\text{where } \mu = \frac{2h^2}{L}, \left(\text{or } \frac{h^2}{2SA}\right).$$

COR. 1. From the last three Propositions it follows that, if a body $P$ be moving at the point $P$ in any direction $PR$ with any velocity whatever, and is at the time under the action of a centripetal force varying inversely as the square of the distance, it will move in a conic section having a focus in the centre of force; and conversely.

For given a focus, a point of contact, and the position of the tangent at the point, a conic section can be described having a given curvature at that point. Now the curvature is given by the centripetal force and the velocity of the body being given: and there cannot be two orbits touching each other described with the same velocity and the same centripetal force.

[For the centripetal force is that which deflects the body towards the centre of force; and a body which is moving from a given point in a given direction with a given velocity, and is deflected in a given manner, has obviously its whole motion given, and can describe but one orbit.]

*Note.* Given the focus and three points of a conic the directrix can be found (Besant's *Conics*, p. 215, Prop. V.): hence by making the points move up to one another, we see that a conic can be described having a given focus, touching a given straight line at a given point, and having a given curvature at that point.

In the case of this Corollary, we have $v$ and $p$ given, and therefore $h$ (Prop. I., Cor. 1, Note); and by Prop. VI. the force at any point is known if $h$, $p$, and $PV$ are given; and hence conversely $PV$ (and therefore the curvature) is known when $h$, $p$, and the force at $P$ are known, as in the case of this Corollary.

If then a conic be described having the given centre of force as a focus, and passing through the given point, and having the given tangent and curvature at that point, it may by the present and preceding Propositions be described by a body under the action of the given force, varying inversely as the square of the distance.

Also, the motion of a body starting from a given point with a given velocity depends simply on the force at that point and the law of variation of the force, and is therefore completely determined if these are given.

Thus not more than one orbit can be described under the given conditions; and the body will therefore move in the conic section found above.

Cor. 2. If the velocity with which a body moves at $P$ be that with which a body can describe the very small straight line $PR$ in any given very small time, and the centripetal force is such as could move the body in the same time through the space $QR$, the body will move in some conic section, the latus rectum of which is the limit of $\frac{QT^2}{QR}$ when $PR$ and $QR$ are diminished indefinitely.

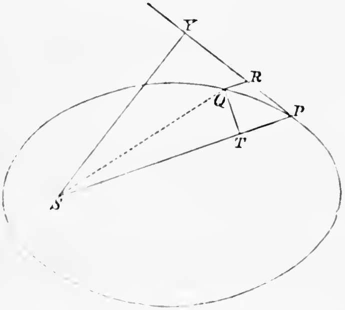
In this Corollary a circle is reckoned as a particular case of an ellipse; and the case of motion in a straight line to the centre is not included.

*Note.* The orbit which the body will dessribe has been proved in Cor. 1 to be a Conic Section.

And it has been proved for all the Conic Sections that lt. $\frac{QT^2}{QR} = L$, the latus rectum [see equations (a), Props. XI., XII., XIII.].

PROP. XIV. (THEOREM VI.)

*If any number of bodies revolve about a common centre, and the centripetal force to that centre vary inversely as the square of the distance from it; the latera recta of the orbits are as the squares of the areas described in equal times by radii drawn from the bodies to the centre.*




For (by Prop. XIII. Cor. 2) the latus rectum $L$ is equal to the limit of $\frac{QT^2}{QR}$ when $P$ and $Q$ coincide.

But the indefinitely small line $QR$ described *in a given time* in the direction $SP$, is in the limit as the centripetal force which produces it, *i.e.* (by hypothesis) inversely as $SP^2$.

Therefore $\frac{QT^2}{QR}$ varies as $QT^2 \times SP^2$; that is, the latus rectum $L$ as the square of the area $\frac{1}{2}QT \times SP$ described in a given time. Q. E. D.

*Note* 1. In the figure to the present Proposition, let $QR$ be the subtense of an arc $PQ$ of one of the orbits, described in any given indefinitely small interval of time $t$.

Then (Prop. VI., Note 3) the forces in the different orbits vary as lt. $\frac{QR}{t^2}$; or, since $t$ is given, as lt. $QR$.

*Note* 2. Since in each orbit the areas described are proportional to the times, the ratio of areas described in equal times in any two of the orbits is independent of the time; and therefore the ratio of areas described in the two orbits in a given indefinitely small time is equal to the ratio of areas described in any the same finite time, and is the ratio of $h : h'$, if $h$, $h'$ are twice the areas described in the two orbits in a unit of time.

Cor. Hence the whole area of an ellipse (also the rectangle contained by its axes, which is proportional to this area) varies in the ratio of the periodic time and in the subduplicate ratio of the latus rectum conjointly.

For the whole area is proportional to the product of the area $\frac{1}{2}QT \times SP$ described in any given time, and the periodic time.

*Note* 1. The area of an ellipse $= \pi AC \,.\, BC$ (Appendix, Lemma IV.).

*Note* 2. The whole area is equal to the product of the area described in a unit of time, and the periodic time (Prop. I.).

Also from Prop. XIV. the area described in a unit of time varies as square root (or in the subduplicate ratio) of the latus rectum.

*Note* 3. In any orbit about a fixed centre of forces, if $P$ be the periodic time,

$hP = 2 \times$ area described in time $P$ (Prop. I. Cor. 1, Note)

$= 2 \times$ area of the orbit;

also by the preceding Propositions, if the force vary inversely as the square of the distance, the orbits will be conic sections with the centre of force in the centre; and if $L$ be the latus rectum in one of these orbits, the absolute force $\mu = \frac{2h^2}{L}$.

$$\text{Thus, area of the orbit} = \frac{hP}{2} = \frac{P}{2}\sqrt{\frac{\mu L}{2}}.$$

## PROP. XV. (THEOREM VII.)

*If any number of bodies revolve about a common centre, and the centripetal force to that centre vary inversely as the square of the distance from it; the periodic times in the ellipses are to each other in the sesquiplicate ratio of the major axes.*

For $\because\ 4BC^2 = 2AC \times L,$

$$\therefore\ (2AC \times 2BC)^2 = 8AC^3 \times L;$$

$$\therefore\ 2AC \times 2BC \text{ varies as } (2AC)^{\frac{3}{2}} \times L^{\frac{1}{2}}:$$

but (by Prop. XIV. Cor.) the rectangle contained by the axes varies as $L^{\frac{1}{2}}$ and the periodic time conjointly: therefore the periodic time varies as $(2AC)^{\frac{3}{2}}$. Q. E. D.

*Note.* As in Prop. XIV. Cor., Note 3,

$$2 \times \text{area of orbit} = hP = \sqrt{\frac{\mu L}{2}}\,.\,P,$$

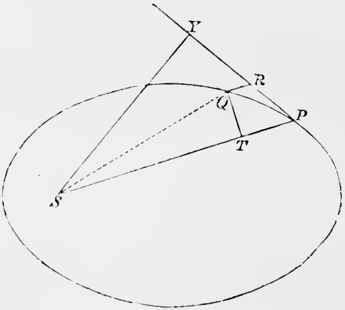
$$\text{or } 2\pi AC \times BC = \sqrt{\frac{\mu BC^4}{AC}}\,.\,P;$$

$$\text{therefore } P = 2\pi\sqrt{\frac{AC^3}{\mu}}.$$

COR. The periodic times in ellipses are the same as in circles whose diameters are equal to the major axes of the ellipses.

PROP. XVI. (THEOREM VIII.)

*Under the same conditions, if tangents be drawn to the orbits at any points, and perpendiculars from the common focus upon them; the velocities of the bodies at these points are inversely as the perpendiculars and directly in the subduplicate ratio of the latera recta.*




Draw $SY$ from the focus $S$ perpendicular to the tangent $PR$; then the velocity of the body at $P$ shall be inversely as $\dfrac{SY}{L^{\frac{1}{2}}}$.

For the velocity is as the arc $PQ$ described in a given indefinitely small time; that is (by Lemma VII.), as the tangent $PR$. And, since $QT$ is equal to the perpendicular from $R$ on $SP$,

$$RP : QT = SP : SY;$$

therefore $$PR = \frac{SP \times QT}{SY}:$$

or $PR$ varies inversely as $SY$ and directly as $SP \times QT$. And $SP \times QT$ varies as the area described in a given time, that is (by Prop. XIV.), in the subduplicate ratio of the latus rectum. Q. E. D.

*Note.* Let $v$ be the velocity at $P$,

then $v = \dfrac{h}{p}$ (Prop. I., Cor. 1, Note) $= \dfrac{\sqrt{\mu L}}{\sqrt{2} \times p}$.

Cor. 1. The latera recta are proportional to the product of the squares of the perpendiculars and the squares of the velocities. [$L \propto SY^2 \times (\text{velocity})^2$.]

*Note.* Since

$$\mu = \frac{2h^2}{L},$$

$$L = \frac{2v^2p^2}{\mu}.$$

Cor. 2. The velocities of bodies, at their greatest and least distances from the common focus, vary inversely as the distances and directly in the subduplicate ratio of the latera recta.

For the perpendiculars are in these cases the same as the distances.

Cor. 3. And thus the velocity in a conic section at the greatest or least distance from the focus is to the velocity in a circle at the same distance from the centre in the subduplicate ratio of the latus rectum to twice that distance.

*Note.* For the circle is an ellipse whose latus rectum is equal to its diameter, that is to twice the distance of the body.

Cor. 4. Bodies moving in ellipses about a common focus have velocities at their mean distances from the common focus equal to the velocities in circles at the same distances; that is (by Prop. IV. Cor. 6), inversely in the subduplicate ratio of the distances.

For the perpendiculars here are the semi-axes minor, and these are mean proportionals between the mean distances and the latera recta. Inverting this ratio and compounding with the direct ratio of the latera recta, we have the inverse subduplicate ratio of the distance.

*Note.* The mean distance, that is, the mean of the greatest and least distances, is the semi-axis major $AC$, and the points at this distance from the focus are the extremities of the minor axis.

Velocity at the point $B=\frac{h}{p}=\frac{h}{BC}$

$$=\frac{\sqrt{\mu L}}{\sqrt{2}\times BC}=\sqrt{\frac{\mu}{AC}}$$

$$=\sqrt{\frac{\mu}{AC^2}\times AC}$$

$=$velocity in circle radius $AC$

(Prop. IV., Cor. 1, Note).

Cor. 5. In the same figure, or in different figures whose latera recta are equal, the velocity of a body is inversely as the perpendicular from the focus on the tangent.

*Note.* For $v=\frac{h}{p}$, and this (Props. XI., XII., XIII., Notes), $=\frac{\sqrt{\mu L}}{\sqrt{2}\,.\,p}$, when force varies inversely as the square of the distance.

Cor. 6. In a parabola the velocity is inversely in the subduplicate ratio of the distance of the body from the focus; in an ellipse it varies more than in that ratio, in a hyperbola less.

For (by Lemma XIV. Cor. 2) the perpendicular from the focus on a tangent to a parabola is in the subduplicate ratio of the distance. In the hyperbola the perpendicular varies in a ratio less than this, and in an ellipse more.

*Note.* In any orbit $v=\frac{h}{p}$; thus in a *parabola*, since

$$p^2=SY^2=SP\,.\,SA,$$

$$v^2=\frac{h^2}{SP\,.\,SA}=\frac{\mu L}{2SP\,.\,SA}=\frac{2\mu}{SP}\quad\text{.................. (1)},$$

in an *ellipse*, since

$$\frac{SY}{SP}=\frac{S'Y'}{S'P}, \text{ and } SY\,.\,S'Y'=BC^2, \text{ and } SP+S'P=2AC,$$

therefore $$\frac{SY^2}{SP^2}=\frac{SY\,.\,S'Y'}{SP\,.\,S'P}=\frac{BC^2}{SP\,(2AC-SP)},$$

and $$SY^2 = \frac{SP}{2AC - SP} \cdot BC^2;$$

thus $$v^2 = \frac{h^2}{SY^2} = \frac{h^2}{BC^2} \cdot \frac{2AC - SP}{SP},$$

$$= \frac{\mu}{AC} \cdot \frac{2AC - SP}{SP}$$

$$= \frac{\mu L}{2BC^2} \cdot \frac{2AC - SP}{SP};$$

or $$v^2 = \frac{2\mu}{SP} - \frac{\mu}{AC} \quad \text{.........................} \quad (2).$$

in a hyperbola, since $S'P - SP = 2AC$,

$$\frac{SY^2}{SP^2} = \frac{SY \cdot S'Y'}{SP \cdot S'P} = \frac{BC^2}{SP\,(2AC + SP)},$$

and $$SY^2 = \frac{SP}{2AC + SP}\, BC^2;$$

therefore $$v^2 = \frac{h^2}{SY^2} = \frac{h^2}{BC^2} \cdot \frac{2AC + SP}{SP},$$

$$= \frac{\mu L}{2BC^2} \cdot \frac{2AC + SP}{SP},$$

$$= \frac{\mu}{AC} \cdot \frac{2AC + SP}{SP},$$

$$= \frac{2\mu}{SP} + \frac{\mu}{AC} \quad \text{................................} \quad (3).$$

From (1), (2), and (3), we see that the velocity is in all cases less the greater the distance, and *vice versâ*. Take any two points $P$, $Q$; and suppose them to be points on a parabola described about the centre of force at $S$, varying inversely as the square of the distance; next suppose them to be points on an ellipse; and lastly on a hyperbola, about the same centre of force. Then by (1), (2), and (3), the ratios of the squares of the velocities at $PQ$ in the three cases are respectively

$$\frac{SQ}{SP}; \quad \frac{SQ}{SP} \cdot \frac{2AC - SP}{2AC - SQ}; \text{ and } \frac{SQ}{SP} \cdot \frac{2AC + SP}{2AC + SQ}.$$

If now $SP$ be greater than $SQ$, $\frac{SQ}{SP}$ is less than unity, $\frac{2AC-SP}{2AC-SQ}$ is less than unity, $\frac{2AC+SP}{2AC+SQ}$ is greater than unity; thus, in the parabola the ratio of the square of the velocities is less than unity, in the ellipse is still less, and in the hyperbola it is nearer unity than in the parabola. In other words the inequality between the velocities at any two points is greater for the ellipse, and less for the hyperbola, than it is for the parabola.

The equations (1), (2), (3) give the velocity at any given point of a conic section described about a centre of force varying inversely as the square of the distance. The student should commit them to memory.

COR. 7. In a parabola the velocity of a body at any distance from the focus is to the velocity of a body revolving in a circle at the same distance from the centre as the square root of 2 is to 1: in an ellipse the ratio is less than this, in an hyperbola greater.

For, by Cor. 2 of this proposition, the velocity at the *vertex* of a parabola is in this ratio; and by Cors. 6 of this proposition and of Prop. IV., the same proportion obtains at all distances. Hence also, in a parabola the velocity is at every point equal to the velocity of a body revolving in a circle at half the distance, in an ellipse less, and in a hyperbola more.

*Note.* The velocity in any orbit $=\frac{h}{p}=\frac{\sqrt{\mu L}}{\sqrt{2}.p}$ : at the vertex $A$ of a parabola $p$ is the same—namely $SA$—both for a body describing the parabola and a circle radius $SA$; thus velocity at $A$ in the parabola and circle varies as the square root of the latus rectum of the parabola and circle, that is as $\sqrt{4SA}:\sqrt{2SA}$; or as $\sqrt{2}:1$.

Again, in a parabola, at $P$, $v^2=\frac{2\mu}{SP}$ (Cor. 6, Note); and in a circle, radius $SP$, $v^2=\frac{\mu}{SP^2}.SP$ (Prop. IV., Cor. 1, Note).

$$=\frac{\mu}{SP};$$

therefore at any point $P$, velocity in parabola : velocity in circle $= \sqrt{2} : 1$.

Cor. 8. The velocity of a body describing any conic section is to the velocity of a body describing a circle at the distance of the semi latus-rectum of the conic, as that distance is to the perpendicular from the focus on the tangent to the conic. This appears from Cor. 5.

*Note.* For the latus rectum of the circle is double its radius, that is, it is equal to the latus rectum of the conic section; and the velocities are inversely as the perpendiculars on the tangents; hence

Vel$^y$. in conic : vel$^y$. in circle = semi-latus rectum : perpendicular on tangent to the conic.

Cor. 9. Hence, since (by Prop. IV. Cor. 6) the velocity of a body describing this circle is to the velocity of a body revolving in any other circle inversely in the subduplicate ratio of the distances; it follows that the velocity of a body describing any conic section is to the velocity of a body describing a circle at the same distance, as a mean proportional between that common distance and the semi-latus rectum is to the perpendicular from the common focus on the tangent to the conic.

*Note.* (Vel$^y$.)$^2$ in any conic section at $P = \frac{h^2}{p^2} = \frac{\mu L}{2SY^2}$;

(Vel$^y$.)$^2$ in circle radius $SP = \frac{\mu}{SP^2} \times SP = \frac{\mu}{SP}$;

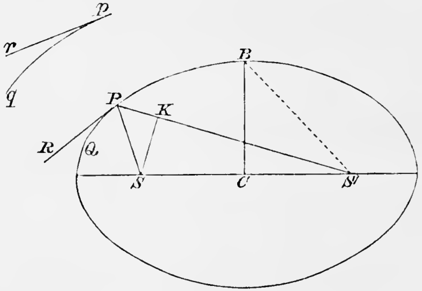
hence these two velocities are in the ratio $(SP \times L)^{\frac{1}{2}}$ to $SY$.

## Prop. XVII. (Problem IX.)

*Given that the centripetal force is inversely proportional to the square of the distance from the centre of forces, and the absolute quantity of that force is known;*

*required the path described by a body which moves from a given point in a given direction with a given velocity.*

Let the centripetal force tending to the point $S$ be that with which a body $p$ describes *any given* orbit $pq$, and suppose the velocity of the body at $p$ to be known. From the point $P$ let the body $P$ move in the direction $PR$,




from which it is deflected by the action of the centripetal force so as to describe some conic section $PQ$; the straight line $PR$ will therefore touch the orbit at $P$. Let $pr$ be the tangent at $p$ to the orbit $pq$.

Then, if from $S$ perpendiculars be let fall on the tangents at $P$ and $p$, the latus rectum of the required conic section $PQ$ will be to the latus rectum of the given orbit $pq$ (by Prop. XVI. Cor. 1) in a ratio composed of the duplicate ratio of the perpendiculars and the duplicate ratio of the velocities, and is therefore given. Let $L$ be the latus rectum of the conic section $PQ$: its focus $S$ is given; and the angle $RPS'$ being the supplement of $RPS$ is given, and therefore the line $PS'$, in which is the other focus $S'$, is given in position.

Draw $SK$ perpendicular to $PS'$; and let $BC$ be the semiaxis minor.

Then

$$SP^2 - 2PK \,.\, PS' + PS'^2 = SS'^2 = 4CS^2$$

$$= 4CA^2 - 4CB^2$$

$$= (SP + PS')^2 - L \times (SP + PS');$$

$$\therefore\ L \times (SP + PS') = 2SP \times PS' + 2PK \times PS'$$

$$= PS'(2SP + 2PK);$$

$$\therefore\ SP + PS' : PS' :: 2SP + 2PK : L;$$

thus $PS'$ is given in length as well as in direction.

If, now, the velocity of the body at $P$ be such that the latus rectum $L$ is less than $2SP + 2PK$, $PH$ lies on the same side of the tangent as $PS$; and therefore the figure will be an ellipse, and will be determined from the known foci $S$, $S'$, and the major axis $SP + PS'$.

If, however, the velocity is so great that the latus rectum $L$ is equal to $2SP + 2KP$, $PS$ will be infinite; and thus the figure will be a parabola whose axis is a straight line through $S$ parallel to $PK$, and the parabola is determined since the focus, axis, and one point are given.

If the velocity with which the body moves through the point $P$ be still greater $L$ will be greater than $2SP + 2PK$, and therefore the ratio $\dfrac{SP + PS'}{PS'}$ or $1 + \dfrac{SP}{PS'}$, being equal to the ratio $\dfrac{2SP + 2PK}{L}$, is less than unity: thus $PS'$ must be drawn in the opposite direction or on the other side of the tangent $PR$; hence, the tangent passing between the foci, the figure will be a hyperbola, whose major axis equals the difference between $SP$ and $PS'$, and which is therefore determined.

The figure formed in each of these cases is the required path. For, if the body revolve in the conic section thus found, it is proved in Props. XI., XII., XIII. that the centripetal force will be inversely as the square of the distance of the body from the centre of forces $S$; and therefore the path $PQ$ has been determined which the body will describe under the action of such a force starting from the given point $P$ in the given direction $PR$ with the given velocity. Q. E. F.

Cor. 1. Hence in any conic section, given a vertex $A'$, the latus rectum $L$, and one focus $S$, the other focus $S'$ is given by taking $A'S'$ to $A'S$ as the latus rectum to the difference between the latus rectum and $4A'S$.

For the proportion

$$SP+PS' : PS' :: 2SP+2KP : L$$

becomes in the case of this corollary,—since $K$ and $S$ coincide—

$$SA'+A'S' : A'S' :: 4A'S : L,$$

whence $$SA' : A'S' :: 4A'S-L : L.$$

Cor. 2. Hence, if the velocity of the body be given at the vertex $A'$, the orbit will be readily found, by taking for its latus rectum a line which is to $2A'S$ in the duplicate ratio of the given velocity to the velocity of a body describing a circle at distance $A'S$ (by Prop. XVI. Cor. 3); and then taking $A'S'$ to $A'S$ as the latus rectum is to the difference between the latus rectum and $4A'S$.

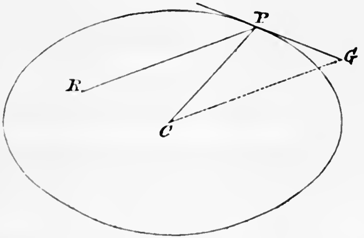
Cor. 3. Hence, if a body move in any conic section, and be disturbed from its orbit by any impulse; we can find the orbit which it will then proceed to describe.

For, compounding the undisturbed motion of the body in its orbit, with the motion generated by the impulse, we shall have the motion with which it will proceed from the given point in a given direction.

Cor. 4. And if that body be *continuously* disturbed by any force extraneously impressed, the path is known very approximately, by calculating the changes of motion produced by the force at certain points, and estimating the changes continually made at intermediate places by interpolation.

## Scholium.

If a body describe any given conic section whose centre is $C$ under the action of a centripetal force tending to any




point $R$ whatever, and the law of the centripetal force is required; draw $CG$ parallel to the radius $RP$, meeting the tangent $PG$ in $G$; then the force will (by Prop. X. Cor. 1, and Scholium, and Prop. VII. Cor 3) vary as $\frac{CG^3}{RP^2}$.

*Note.* By Prop. VII. Cor. 3, if the conic section be described in the same periodic time under forces to $R$ and $C$, force to $R$ : force to $C$ = $CG^3 : CP \cdot RP^2$; also, by Prop. X. Cor. 1, and Scholium, force to $C$ varies as $CP$;

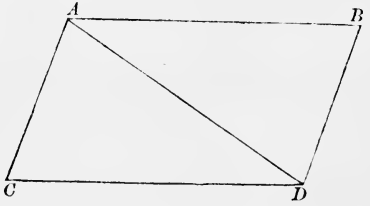
therefore, force to $R$ varies as $\frac{CG^3}{RP^2}$.

# APPENDIX.

## *On the Parallelogram of Forces.*

1. The parallelogram of forces, deduced from the second law of motion in Cor. 1 of the laws of motion, can be expressed conveniently in another form, thus ;

Since the resultant of forces, which act at a point and are represented in magnitude and direction by the sides $AB$ and $AC$ of a parallelogram, is represented in magnitude and direction by the diagonal $AD$ ; and since the side $BD$ of the triangle $ABD$ is of the same magnitude and in the same direction as $AC$;




therefore the forces represented by $AB$ and $BD$ have for resultant a force represented by $AD$; and thus the force which will with the forces $AB$, $BD$ form a system in equilibrium, may be represented by $DA$ ; or in other words, *forces which act at a point and are represented in mag-*

*nitude and direction by the sides* AB, BD, DA (*taken in order*) *of the triangle* ABD *form a system in equilibrium.*

This proposition is called the *triangle of forces.*

2. The following is yet another form in which the same proposition may be expressed, which is sometimes useful.

*If either* (1) *two forces and their resultant,* or (2) *three forces in equilibrium, act at a point and have directions parallel to the sides of a triangle* ABD, *their magnitudes shall be proportional to the same sides.*

(1) Let $AB$, $BD$ be the directions of two forces acting at a point, $AD$ that of their resultant; then the magnitudes shall be proportional to $AB$, $BD$, $AD$.

Complete the parallelogram $BC$: and represent the magnitude of the force in direction $AD$ by $AD$: then the forces in directions $AB$, $AC$, of which $AD$ is the resultant, are found by drawing $DB$ parallel to $AC$ and $DC$ parallel to $AB$, so as to form the parallelogram $BC$; they are therefore represented in magnitude (as well as in direction) by $AB$ and $AC$, or by $AB$ and $BD$.

Thus the magnitudes of the three forces are proportional to $AB$, $BD$, $AD$.

(2) Let $AB$, $BD$, $DA$ be parallel to the directions of three forces in equilibrium, the magnitudes of these forces shall be proportional to the sides $AB$, $BD$, $DA$.

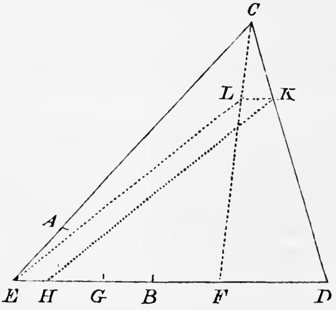
For since a force in direction parallel to $DA$ is in equilibrium with forces parallel to $AB$, $BD$, the resultant of these two forces must be in direction parallel to $AD$: and the proposition is therefore proved for the magnitude of the forces in this case in the same manner as in case (1).

3. By considering the proof of Cor. 1. of the laws we see that the parallelogram and triangle of forces may be applied, *mutatis mutandis,* to velocities.

### LEMMA XXIII.

*If two straight lines* AC, BD *be terminated at* A, B, *and have a given ratio to one another, and the straight line* CD, *joining the variable points* C, D, *be cut in a given ratio at* K: *the point* K *will lie on a straight line given in position.*

Let the straight lines $AC$, $BD$ meet in $E$. On $BE$ take $BG$ to $AE$ as $BD$ is to $AC$; thus $BG$ is given and




the point $G$ is given in position; take $DF$ always equal to the given length $EG$;

then from the construction, $GD = EF$;

therefore $EC : EF = EC : GD$

$= AC : BD$, a given ratio;

thus the triangle $EFC$ is given in form.

Divide $CF$ at $L$ so that

$CL : CF = CK : CD$, a given ratio,

then the ratio of $LF$ to $CF$ is given:

also the ratio of $EF$ to $CF$ is given:

therefore the ratio of $EF$ to $FL$ is given, and the triangle $EFL$ is given in form, and thus the point $L$ will lie always on a straight line $EL$ given in position.

Join $LK$; then the triangles $CLK$, $CFD$ are similar; also $FD$ is given ($=EG$); and the ratio of $LK$ to $FD$ is given: thus $LK$ is given.

Take $EH=LK$; then $ELKH$ is always a parallelogram. Thus $K$ lies always on the straight line $HK$, which passes through a fixed point $H$ and is parallel to $EL$, and is therefore given in position. Q. E. D.

COR. Since the triangles $ELF$, $ECF$ are given in form, the three straight lines $EF$, $EL$ and $EC$—that is, $GD$, $HK$ and $EC$—are to each other in given ratios.

*Note* 1. Thus if two bodies moving in the same plane are at the same instant at $A$ and $B$, and describe the indefinite straight lines $AC$, $BD$ with uniform velocities, and arrive simultaneously at $C$ and $D$, their centre of gravity $K$ will describe a straight line $HK$; and it will move with uniform velocity, since $HK$ is to $EC$ and $GD$, and therefore to $AC$ and $BD$, in constant ratios.

*Note* 2. If the bodies move in different planes, let $AC$, $BD$ be the projections on a given plane of the spaces described by them in a given time, and let $K$ be the projection of the centre of gravity; then $K$ divides $CD$ in a given ratio; hence, by the construction and proof of this Lemma, $K$ describes a straight line $HK$ with uniform velocity; thus the centre of gravity of the two bodies describes a path the projection of which on *any* plane is a straight line, and which is therefore itself a straight line; and it describes it with uniform velocity, since its projection $K$ describes $HK$ with uniform velocity.

We now proceed to the proofs of Cors. 4, 5, and 6 of the laws of motion.

COR. 4. *The common centre of gravity of two or more bodies does not change its state of motion or rest through the mutual actions of the bodies; and hence, in the absence of external actions or resistances, the common centre of gravity either is at rest or moves uniformly in a straight line.*

For if two points move uniformly in straight lines, and the distance between the points be divided in a given ratio, the point in which it is divided either is at rest or moves uniformly in a straight line (Lemma XXIII., Cor. and Notes).

If, therefore, any number of bodies move uniformly in straight lines, the common centre of gravity of any two either is at rest or in uniform motion in a straight line: for the line which joins the centres of these bodies which move uniformly in straight lines is divided by that centre in a given ratio;

similarly, also, the common centre of these two and any third either is at rest or moves uniformly in a straight line; for this point divides the distance between the common centre of the two bodies and the centre of the third in a given ratio;

in the same way the common centre of these three and any fourth body either is at rest or moves uniformly in a straight line: for it divides the distance between the common centre of the three and the centre of the fourth in a given ratio; and so on *in infinitum;*

therefore, in a system of bodies, which are acted upon neither by mutual actions nor by external forces, and each of which therefore moves uniformly in a straight line, the common centre of gravity of them all either remains at rest or moves uniformly in a straight line.

Moreover, in a system of two bodies mutually acting on each other, since the distances of their respective centres from their common centre of gravity are reciprocally as the bodies, the relative motions of these bodies to or from that centre will be equal; now the position of the centre of gravity is moved neither forward nor backward by equal motions of the two bodies in opposite directions, and hence it undergoes no change in its state of motion or rest through the mutual actions of the bodies.

Again, in a system of several bodies, since the common centre of gravity of any two which act on each other is unaffected in its state of motion or rest by that action; and since the common centre of gravity of the remainder is unaffected by this action, in which they are not concerned; since also the distance of these two centres is divided by the common centre of all the bodies into parts inversely proportional to the sum totals of the bodies whose centres they are, so that, those two centres maintaining their state of motion or rest, the common centre of all maintains its state also; it is evident that the common centre of all the bodies never changes its state as regards motion or rest in consequence of the mutual actions of pairs of them.

Now in a system such as we are considering, all the mutual actions of the bodies are either actions between the bodies two and two, or composed of such actions, and therefore never induce in the common centre of all the bodies a change in its state of motion or rest.

Therefore, since the centre of gravity of a system of bodies which do *not* act upon one another either remains at rest, or moves uniformly in some straight line; it will continue, *in spite of mutual actions* between the bodies composing it, either to remain for ever at rest, or to progress uniformly in a straight line; unless it be disturbed from that state by forces impressed on the system from without.

Thus, there is for a system of several bodies the same law as for a single body, in respect of persevering in its state of motion or rest: for the progressive motion, whether of a single body or of a system, must always be estimated by the motion of the centre of gravity.

COR. 5. *Bodies inclosed in a given space have the same motions relatively to one another, whether that space be at rest, or be moving uniformly in a straight line without rotation.*

For the differences of velocities in the same direction, and the sums of velocities in opposite directions, are initially the same in both cases (by hypothesis), and from these sums and differences of velocities arise the collisions and the impacts with which bodies strike one another.

Therefore, by Law II. the effects of the collisions will be the same in the two cases; and thus the motions of the bodies relatively to one another in the one case will remain equal to the motions relatively to one another in the other case.

The same thing is proved very clearly by experiment; thus, all motions on board a ship take place in the same manner, whether the ship be at rest or be moving uniformly in a straight line.

COR. 6. *If bodies are moving relatively to one another in any manner, and are urged by equal accelerating forces in parallel directions; they will all continue to move relatively to one another in the same manner as if they were not acted on by those forces.*

For these forces, being equally accelerating forces, act on the bodies in motion in proportion to the quantities of the bodies, and they act in parallel straight lines; therefore, by Law II. they move all the bodies equally—as regards velocity—and can never alter their positions and motions among one another.

The following propositions are added in illustration of Lemma I:—

1. *If four quantities of the same kind vanish together in such a manner that the ultimate ratio of the first to the second is equal to that of the third to the fourth; the ultimate ratio of the first to the third shall be equal to that of the second to the fourth; all the ratios being supposed to remain finite.*

Let $a$, $b$, $c$, $d$ be the four quantites: take $A$, $B$, $C$, $D$ always proportional to them, making one, say $A$, always finite: then, since all the ratios remain finite, the rest $B$, $C$, and $D$ remain finite. We have, therefore,

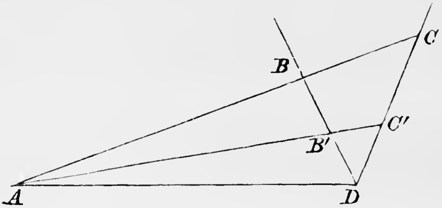
in the limit $A : B :: C : D$;

therefore in the limit $A : C :: B : D$;

therefore the ultimate ratio of $a$ to $c$ is equal to that of $b$ to $d$.

2. *Straight lines are drawn from a point* A *cutting two fixed straight lines* DB, DC. *If* B, C; B′, C′ *are the points in which they are cut by two of these straight lines, the limiting ratio of* DB *to* DB′ *is equal to that of* DC *to* DC′, *when* ABC, AB′C′ *move up to and coincide with* AD.

For, the ratio of $DB$ to $DC$ depends simply on the angles of the triangle $DBC$; and these angles approach a fixed and finite limit; therefore the limiting ratio of $DB$ to $DC$ is fixed, and is the same as that of $DB'$ to $DC'$:




therefore the limiting ratio of $DB$ to $DB'$ is the same as that of $DC$ to $DC'$.

The following propositions are added in illustration of Lemma II.

1. *To find the area of any portion of a plane curve referred to Cartesian co-ordinates.*

Let $ABDC$ be the area required, bounded by the arc $APB$, the ordinates $AC$, $BD$, and the line of abscissas $DC$.

Inscribe, as in the Lemma, $n$ parallelograms on $n$ equal bases, and let $PN$ be one of these.

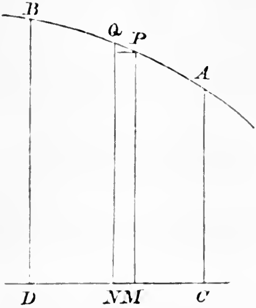
Then the area required is by the Lemma equal to the limit of the sum of the parallelograms when $n$ is indefinitely increased.

Let $\quad DC = a : AC = b_1 : BD = b_2 : \angle BDC = i$:

then area of parallelogram $PN = PM \times MN \times \sin i$

$$= PM \times \frac{a}{n} \times \sin i = \frac{PM}{n} \times a \sin i,$$

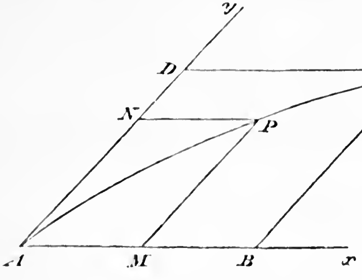
therefore, area required = limit of $a \sin i \Sigma \frac{PM}{n}$;




where $\Sigma \frac{PM}{n}$ indicates the sum obtained by adding all the $n$ values of $PM$ from $b_1$ to $b_2$, and dividing by $n$.

2. *To find the area of a portion of a parabola cut off by a diameter and one of its ordinates.*

Let $Ax$ be the diameter, $PM$ any ordinate; $CB$ the




ordinate bounding the area $ABC$ required : $AB = h$, $CB = k$.

Then, by a property of the parabola,

$$\frac{PM^2}{BC^2} = \frac{AM}{AB};$$

$$\therefore AM = \frac{PM^2}{k^2} h.$$

Let $Ay$ be the tangent at $A$ : and draw $PN$, $CD$ parallel to $Ax$ to meet $Ay$ : let $\angle yAx = i$, then $AM = PN$: and $PM = AN$:

therefore $$PN = \frac{AN^2}{k^2} h. \qquad \text{(1)}.$$

If, now, in the figure $ADCP$ be inscribed $n$ parallelograms on $n$ equal bases formed by dividing $AD$ into $n$ equal parts,

$$\text{the area } ADC = \text{limit of } \Sigma PN \times \frac{AD}{n} \times \sin i,$$

$$= \text{limit of } \Sigma \frac{AN^2}{k^2} h \times \frac{k}{n} \sin i,$$

$$= \text{limit of } \frac{h}{k} \sin i \, \Sigma \frac{AN^2}{n},$$

by taking all the values of $AN$ from $O$ to $AD$;

these values are $$\frac{AD}{n}, \frac{2AD}{n}, \frac{3AD}{n}, \ldots\ldots \frac{n \times AD}{n};$$

$$\therefore \Sigma \frac{AN^2}{n} = \Sigma \frac{AD^2 + 2^2AD^2 + 3^2AD^2 + \ldots + n^2AD^2}{n^3}$$

$$= AD^2 \times \Sigma \frac{1^2 + 2^2 + \ldots\ldots + n^2}{n^3}$$

$$= AD^2 \times \frac{1}{n^3} \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)$$

$$= AD^2 \left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right);$$

and the limit of this when $n$ is indefinitely increased

$$= \frac{1}{3} AD^2 = \frac{k^2}{3};$$

$$\therefore \text{ area } ADC = \frac{h}{k} \sin i \times \frac{k^2}{3}$$

$$= \frac{1}{3} hk \sin i$$

$= \frac{1}{3}$ of the parallelogram $ABCD$;

$\therefore$ parabolic area $ABC = \frac{2}{3}$ circumscribing parallelogram.

3. *To find the volume of a pyramid.*

Let $A$ be the area of the base of the pyramid; $h$ the perpendicular from the vertex on the base. Divide $h$ into $n$ equal parts, and through the $r$th point of division from the vertex draw a plane parallel to the base.

Then the area of the section of the pyramid thus made

$$= A \frac{\left(\frac{rh}{n}\right)^2}{h^2} = A \frac{r^2}{n^2};$$

on this area as a base describe a right prism, whose altitude $= \frac{h}{n}$;

the volume of this prism $= A \frac{r^2}{n^2} \cdot \frac{h}{n} = Ah \frac{r^2}{n^3}$;

therefore volume of the pyramid

$$= \text{limit of sum of all the prisms}$$

$$= Ah \times \text{limit } \Sigma \frac{r^2}{n^3}$$

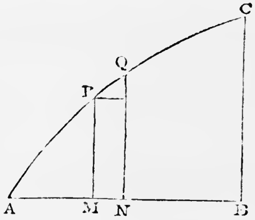
$$= Ah \times \text{limit of} \left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right)$$

$$= \frac{1}{3} Ah$$

$= \frac{1}{3}$ the volume of a right prism whose base and altitude are equal to those of the pyramid.

4. *The volume of a solid of revolution may be deduced by applying the method of Lemma II.*

Let $ABC$ be any plane curvilinear area, by the revolu-




tion of which about $AB$ the solid is generated, and let $CB$ be perpendicular to $AB$.

Divide $AB$ into $n$ equal parts, and inscribe in the figure $ABC$ $n$ rectangular parallelograms on these parts as bases.

Then, by the method of Lemma II. it may be shewn that the volume of the solid of revolution is the limit of the sum of the cylinders formed by the revolution of all these parallelograms—such as $PN$—about $AB$.

The volume of the cylinder described by $PN$

$$= \pi PM^2 . MN$$

$$= \pi PM^2 \times \frac{AB}{n};$$

then volume of revolution required

$$= \text{limit of } \Sigma \pi PM^2 \times \frac{AB}{n},$$

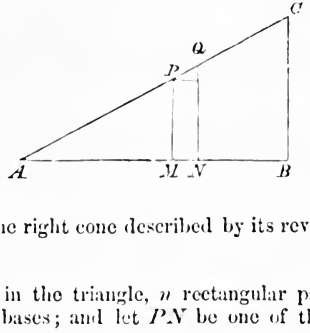
when $n$ is indefinitely increased,

$$= \pi AB \times \Sigma \frac{PM^2}{n}.$$

5. *To find the volume of a right cone.*

A right cone is the figure generated by the revolution of a right-angled triangle about one of the sides containing the right-angle.

Let $ABC$ be any right-angled triangle; to find the




volume of the right cone described by its revolution about $AB$.

Inscribe in the triangle, $n$ rectangular parallelograms on $n$ equal bases; and let $PN$ be one of them; so that $MN = \frac{1}{n} \times AB$;
then volume of cylinder described by $PN$

$$= \pi PM^2 \times \frac{AB}{n}$$

$$= \pi AM^2 \times \frac{BC^2}{AB^2} \times \frac{AB}{n}$$

$$= \frac{\pi BC^2}{AB} \times \frac{AM^2}{n};$$

∴ whole volume of revolution required

$$= \text{limit of } \Sigma \frac{\pi BC^2}{AB} \times \frac{AM^2}{n}$$

$$= \text{limit of } \frac{\pi BC^2}{AB} \Sigma \frac{AM^2}{n},$$

when $n$ is indefinitely increased.

Now the values of $AM$ are $\frac{AB}{n}, \frac{2AB}{n} \ldots\ldots \frac{nAB}{n}$;

$$\therefore \Sigma \frac{AM^2}{n} = \Sigma \frac{AB^2 + 2^2AB^2 + \ldots\ldots + n^2AB^2}{n^3}$$

$$= AB^2 \times \Sigma \frac{1^2 + 2^2 + \ldots\ldots + n^2}{n^3}$$

$$= AB^2\left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right);$$

$$= \frac{AB^2}{3}, \text{ in the limit :}$$

∴ volume of cone of revolution required

$$= \frac{\pi BC^2}{AB} \times \frac{AB^2}{3}$$

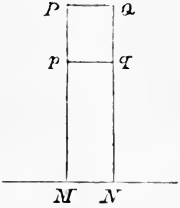
$$= \frac{1}{3}\pi BC^2 . AB$$

$$= \frac{1}{3} \text{ circumscribing cylinder.}$$

The following propositions are added in illustration of Lemma IV.:—

1. *If all the ordinates of any given curve be increased or diminished in any given ratio, the area of the curve is increased or diminished in the same ratio.*

Inscribe in the given curve a series of parallelograms as in Lemma IV., and let $PMNQ$ be one of the series, cor-




responding to a point $P$ of the curve: draw $pq$ parallel to $PQ$ and cutting the ordinate $PM$ in the given ratio.

Then $PMNQ$, $pMNQ$ are in the ratio of $PM$ to $pM$, that is, in the given ratio: hence the sum of the parallelograms $PMNQ$ is in the given ratio to the sum of the parallelograms $pMNq$.

Thus, by Lemma IV. the area of the given curve is increased or diminished in the given ratio.

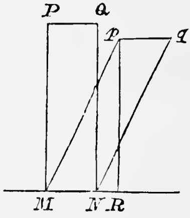
Ex. The area of an ellipse is to that of a circle on its major axis as diameter as the minor axis is to the major axis.

Hence, area of ellipse $= \dfrac{BC}{AC} \times \pi AC^2$

$= \pi AC \,.\, BC.$

2. *If all the ordinates of any given curve be turned through any given angle, the area of the curve is diminished in the ratio of the cosine of that angle.*

Inscribe a series of parallelograms in the given figure, as in Lemma IV.; and let $PMNQ$ be the parallelogram




corresponding to the point $P$ of the curve: let $PM$, $QN$ be turned through the given angle and come into the positions $pM$, $qN$: join $pq$; then $pq$ is parallel to $MN$ (Euc. I. 33).

Draw $pR$ perpendicular to $MN$.

Then $pMNq : PMNQ = pR \times MN : PM \times MN$;

hence L. R. $pMNq : PMNQ = pR : PM$

$= pR : pM$

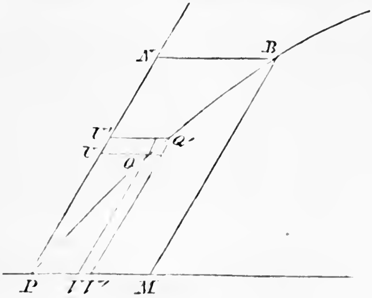
$= \cos pMP : 1.$

Thus, by Lemma IV. the area of the given curve is diminished in the ratio of the cosine of the angle $pMP$.

3. *To find the area of a portion of a parabola included between the curve, a diameter, and an ordinate to the diameter.*

Let $PMB$ be the area required; $BM$ being the ordinate, parallel to the tangent $PN$.

Take two points $Q$, $Q'$ near each other, and draw $QV$, $Q'V'$; $QU$, $Q'U'$; parallel to $PN$ and $PM$ respectively.




Then, since the (acute) angles at $U$ and $V$ are each equal to the angle at $P$, they are equal to each other;

therefore the area of the parallelogram $QU'$ is to area of the parallelogram $QV'$ as $QU \,.\, UU' : QV \,.\, VV'$.

Now, by the nature of the parabola,

$$QV^2 = 4SP \,.\, PV,$$

and

$$Q'V'^2 = 4SP \,.\, PV';$$

$$\therefore \; Q'V'^2 - QV^2 = 4SP \,.\, VV',$$

or

$$(Q'V' + QV)(Q'V' - QV) = 4SP \,.\, VV';$$

$$\therefore \; \frac{UU'}{VV'} = \frac{4SP}{Q'V' + QV};$$

$$\therefore \text{ lt. } \frac{UU'}{VV'} = \frac{4SP}{2QV};$$

$$\therefore \text{ lt. } \frac{QU \,.\, UU'}{QV \,.\, VV'} = \frac{4SP \,.\, QU}{2QV^2}$$

$$= \frac{4SP \,.\, PV}{2QV^2};$$

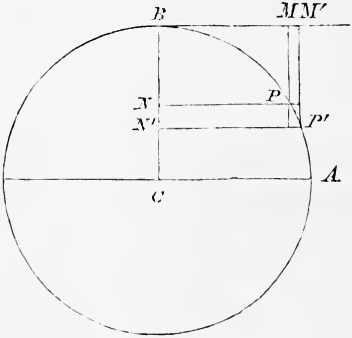
$$\therefore \text{ lt. } \frac{\text{parallelogram } QU'}{\text{parallelogram } QV'} = \frac{1}{2};$$

$$\therefore \text{ by Lemma IV, whole area } PBN = \frac{1}{2} \text{ area } PBM;$$

$$\therefore \text{ area } PMB = \frac{2}{3} \times \text{parallelogram } MN.$$

4. *To find the volume of a prolate spheroid;* i.e. *of the solid generated by the revolution of an ellipse about its major axis.*

Let $P$, $P'$ be two points near each other on the ellipse: and draw $PN$, $P'N'$, parallel to the major axis, meeting




the minor axis in $N$, $N'$; and $PM$, $P'M'$ parallel to the minor axis, meeting tangent at $B$ in $M$, $M'$.

Let $CA = a$, $CB = b$, $PN = x$, $CN = y$, $MM' = h$, $NN' = k$; thus $P'N' = x + h$, $CN' = y - k$;

then $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1;$$

and $$\frac{(x+h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1;$$

$$\therefore \quad \frac{2xh+h^2}{a^2}+\frac{-2ky+k^2}{b^2}=0\,;$$

$$\therefore \quad \frac{h\,(2x+h)}{k\,(2y-k)}=\frac{a^2}{b^2}\,;$$

$$\therefore \quad \text{lt.}\ \frac{2xh}{2yk}=\frac{a^2}{b^2}\,;$$

$$\therefore \quad \text{lt.}\ \frac{h}{k}=\frac{a^2y}{b^2x} \qquad \text{.................}(1)$$

Again, area described by revolution of $NN'$ about $CA$

$$=\pi CN^2-\pi CN'^2\,;$$

$\therefore$ volume described by revolution of parallelogram $PN'$

$$=\pi\,(CN^2-CN'^2)\,PN$$

$$=\pi\,(CN'+CN)\,PN\,.\,NN'$$

$$=\pi\,(2y+k)\,xk.$$

Also, area described by revolution of $PM$ (or of $BN$)

$$=\pi\,(CB^2-CN^2)$$

$$=\pi\,(b^2-y^2)\,;$$

$\therefore$ volume described by revolution of parallelogram $PM'$

$$=\pi\,(b^2-y^2)\,h\,;$$

$\therefore$ these volumes are in the ratio of $\dfrac{(2y+k)\,xk}{(b^2-y^2)\,h}$,

or of $\qquad \dfrac{(2y+k)\,x}{b^2-y^2}\cdot\dfrac{b^2x}{a^2y}$, by (1);

or, in the limit, of $\dfrac{2b^2x^2}{a^2(b^2-y^2)}$ (since $k$ vanishes);

and this $=2$, from the equation to the ellipse.

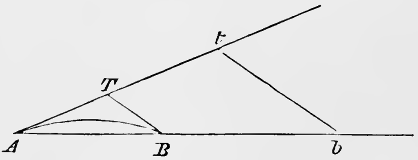
Hence by Lemma IV. the volume of the ellipsoid is twice the volume between it and the circumscribing cylinder;

∴ the volume of the ellipsoid is two-thirds of the circumscribing cylinder.

The following proposition was referred to in the note to Lemma III. Cor. 4.

*The limiting ratio of the sum of the series of chords joining consecutive points of a curve to the sum of the series of tangents at those points is one of equality.*

Let $AT$, $TB$ be the tangents at two points $A$ and




$B$, very near each other on any curve, $AB$ the chord. Produce $AB$ to a fixed distant point $b$.

As the points $A$ and $B$ move up to one another, draw $bt$ always parallel to $BT$, meeting $AT$ produced in $t$.

Then (by Lemma VI.) the angles $TAB$, $TBA$ diminish indefinitely and ultimately vanish: hence also the angles $tAb$, $tbA$, which are equal to them, ultimately vanish.

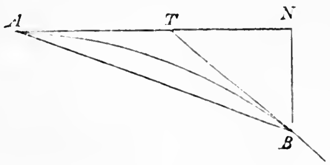
Thus ultimately $t$ lies on $Ab$, and therefore the ratio of $At+tb$ to $Ab$ is a ratio of equality: and by similar triangles $ATB$, $Atb$, the ratio of $AT+TB$ to $AB$ is always the same as that of $At+tb$ to $Ab$.

Hence, in the limit, the ratio of $AT+TB$ to $AB$ is one of equality: and this is true with respect to each chord.

Thus the limit of the ratio of the sum of the series of chords to the sum of the series of tangents is a ratio of equality.

The following propositions are added as illustrations of Lemmas VI. and VII.

1. AB *is the chord, and* AT, BT *the tangents at* A *and* B, *of an arc* AB, *of continuous curvature;* BN *is a*



*straight line meeting the tangent at* N. *If the angle* N *is always finite, the limiting ratio of the triangles* ATB, NTB *is one of equality.*

For, since the curvature of the curve is continuous,

L. R. $AT : TB = 1$ (page 130);

and since the angles $TNB$, $TBN$ remain finite when $BTN$ vanishes,

L. R. $TB : TN = 1$ (Lemma VII. Note 6);

$\therefore$ L. R. $\triangle ATB : \triangle NTB =$ L. R. $AT : NT$
$= 1.$

2. *To find the ultimate ratio of the segments of two equal chords of a curve, when the chords move up to each other and coincide.*

Let $AB$, $A'B'$ be two equal chords intersecting in $O$ (fig. next page).

On $OA'$ take $Om$ equal to $OA$: and on $OB$ take $On$ equal to $OB'$; join $AA'$, $BB'$.

Then $\qquad An = B'm;$

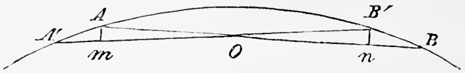
$$\therefore\ AB = An + nB = B'm + nB;$$

add $A'm$ to each;

$$\therefore\ AB + A'm = A'm + B'm + nB = A'B' + nB;$$

but $\qquad AB = A'B$ (by hypothesis),

$$\therefore\ A'm = Bn;$$




and by similar triangles $AOm$ $B'On$,

$$\frac{AO}{Am} = \frac{B'O}{B'n},$$

$$\therefore\ AO \times \frac{A'm}{Am} = B'O \times \frac{Bn}{B'n};$$

but $\qquad \dfrac{Am}{A'm} = \dfrac{\sin AA'm}{\sin A'Am};$

$$\therefore\ \text{lt.}\ \frac{Am}{A'm} = \tan AA'm$$

(since $\angle AmA'$ is ultimately a right angle);

$$\text{similarly lt.}\ \frac{B'n}{Bn} = \tan B'Bn;$$

$$\therefore\ \frac{AO}{\tan AA'm} = \frac{B'O}{\tan B'Bn}, \text{ ultimately};$$

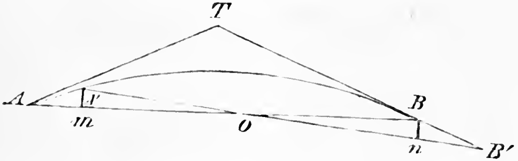
$$\therefore\ \text{lt.}\ \frac{AO}{B'O} = \text{lt.}\ \frac{AO}{BO} = \text{lt.}\ \frac{\tan AA'm}{\tan B'Bn};$$

and the angles $AA'm$, $B'Bn$ are ultimately the angles which $AB$ makes with the tangents to the curve at $A$, $B$. Calling these angles $\alpha$ and $\beta$, we have

$$\text{L. R. } AO : BO = \tan\alpha : \tan\beta.$$

3. *To find the ultimate ratio of the segments of two chords cutting off equal arcs of a curve.*

Let $AB$, $A'B'$ be two chords meeting in $O$, and




cutting off equal arcs, so that arc $AB$ = arc $A'B'$, and therefore arc $AA'$ = arc $BB'$.

Draw $A'm$, $Bn$ perpendicular to $AB$, $A'B'$ : join $AA'$, $BB'$.

Then L. R. chd $AA'$ : chd $BB'$ = L. R. arc $AA'$ : arc $BB'$;

$$\therefore \text{lt.} \frac{\text{chord } AA'}{\text{chord } BB'} = 1 ;$$

$$\therefore \text{lt.} \frac{A'm}{Bn} = \text{lt.} \frac{AA' \sin A'Am}{BB' \sin BB'n}$$

$$= \text{lt.} \frac{\sin A'Am}{\sin BB'n}.$$

Let the tangents at $A$ and $B$ meet in $T$;

then $$\text{lt.} \frac{A'm}{Bn} = \text{lt.} \frac{\sin A'Am}{\sin BB'n},$$

$$\therefore \text{lt.} \frac{A'O}{BO} = \frac{\sin TAm}{\sin TBm}$$

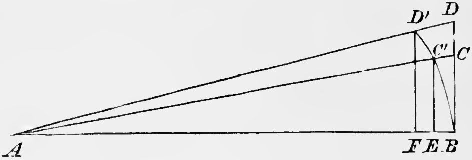
(when $A$, $A'$ and $B$, $B'$ coincide);

or $$\text{lt.} \frac{AO}{OB} = \frac{TB}{TA}$$

or $$\text{L. R. } AO : BO = TB : TA.$$

4. *The limiting ratio of two angles which vanish together is that of their sines, and tangents.*

Let $BAC$, $BAD$ be two angles which vanish together; draw $DCB$ perpendicular to $AB$ meeting $AC$, $AD$ in $C$




and $D$: with centre $A$ and radius $AB$ describe an arc of a circle $BC'D'$ cutting $AC$, $AD$ in $C'$, $D'$; and draw $C'E$, $D'F$ perpendicular to $AB$.

We have to shew that the limiting ratio of the angles $CAB$, $DAB$ when they vanish is equal to that of $C'E$ to $D'F$; and of $CB$ to $DB$.

Since the angles at $C$, $D$, $E$ and $F$ remain finite, we have by Lemma VII. Cor. 1, limiting ratio of $C'E$ to $D'F$ equal to that of $BC'$ to $BD'$;

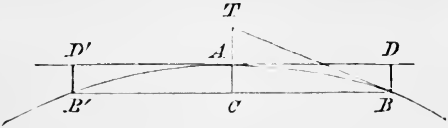
again, by the same Lemma, limiting ratio of $CB$ to $DB$ is equal to that of $C'B$ to $D'B$;

and arc $C'B$ is to $D'B$ as angle $C'AB$ to angle $D'AB$: therefore the limiting ratio of the sines and of the tangents is the same as that of the angles.

### *Remarks on Lemma XI.*

1. By this Lemma it is proved that subtenses of arcs of any curve in the neighbourhood of any given point vary as the squares of the arcs, provided the subtenses ultimately coincide in direction with any straight line through that point other than the tangent.

2. Let $BB'$, $DD'$ be the chord, and tangent at $A$, of any arc $BAB'$; $C$ a point on the chord; join $CA$, and




draw $BD$, $B'D'$ parallel to $CA$. Then $AC$ is intermediate in magnitude and position between $BD$, and $B'D'$.

By the Lemma, L. R. of $BD$ to $B'D'$ is L. R. of $AB^2$ to $AB'^2$; if then L. R. of $AB$ to $AB'$ is 1, L. R. of $BD$ to $B'D'$ is also 1: and therefore the limiting ratio of $AC$ to either $BD$ or $B'D'$ is a ratio of equality.

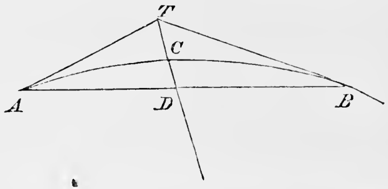
And conversely, if $BB'$ be drawn parallel to the tangent at $A$ (or if the limiting ratio of $BD$ to $B'D'$ be one of equality), the arc $BAB'$ is ultimately bisected in $A$.

Hence, by Lemma VII., *if either the arc* BAB′ *or the chord* BCB′ *be ultimately bisected by* AC, AC *is ultimately in a ratio of equality to the subtenses, drawn parallel to* AC *from* B *and* B′, *to the tangent at* A;
*and conversely, if either* BB′ *be drawn parallel to the tangent at* A, *or the limiting ratio of* BD *to* B′D′ *be one of equality, the chord, arc, and tangent, are ultimately bisected in* C *and* A.

3. Again, draw the tangent $BT$ meeting $CA$ produced in $T$; then $AT$, $BD$ being parallel subtenses of the same arc $AB$, their limiting ratio is one of equality, by Lemma XI.

*If, therefore, either* BB′ *be drawn parallel to the tangent at* A, *or the arc* BAB′ *be ultimately bisected in* A, *the limiting ratio of* TA *to* AC *is one of equality, or* TC *is ultimately bisected in* A.

4. *The chord* AB *of any arc* ACB *of continuous curvature makes angles with the tangents* TA, TB *at the*




*extremities, which are ultimately in a ratio of equality when the arc is indefinitely diminished.*

For, if these angles were ultimately in a ratio of inequality, the curvatures at $A$ and $B$ would be ultimately unequal (Lemma XI. Scholium Note 3), when $A$ and $B$ coincide, and the curvature of the arc would not be continuous.

5. Hence, *the limiting ratio of* TA *to* TB *is one of equality* (page 128).

*Also, the limiting ratio of* TA *or* TB *to the arc or chord* AB *is* 1 *to* 2 : this is seen by constantly magnifying the figure so as to keep $AB$ finite.

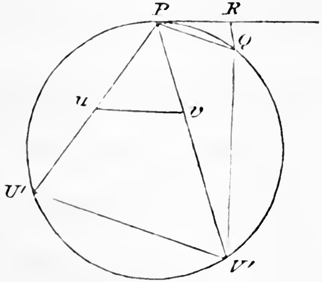
6. Draw any straight line $TCD$ meeting the arc in $C$ and the chord in a finite angle at $D$.

Then, since the angles $TAB$, $TBA$ continually diminish and ultimately vanish when $B$ and $A$ coincide, the limiting ratios of $AC$ to $AD$ and $AT$, and of $BC$ to $BD$ and $BT$, are ratios of equality by Lemma VII. Cor. 1 ; therefore the limiting ratios of $DA$ to $DB$, and of $CA$ to $CB$, are the same as that of $TA$ to $TB$, or are ratios of equality.

Therefore, the chord $AB$ and arc $ACB$ are ultimately bisected in $D$ and $C$; therefore (page 129, Art. 3), *the limiting ratio of* TC *to* CD *is one of equality.*

*On Curvature.*

1. *If* PV *be a chord of curvature at a point* P *of a curve,* PR *the tangent at* P, PQ *a chord,* QR *a subtense*




*parallel to* PV, *then, when* Q *moves up to and coincides with* P,

$$PV = \text{limit of } \frac{PQ^2}{QR}.$$

Describe a circle touching the curve at $P$ and passing through $Q$; the limiting position of this circle is (page 43, note 4) the circle of curvature at $P$. Let this circle meet $PV$ in $V'$; join $QV'$.

Then (Euc. III. 32) $\angle RPQ = \angle PV'Q$;

and, because $QR$ is parallel to $PV'$, $\angle PQR = \angle QPV'$;

therefore the triangles $RPQ$, $QV'P$ are similar;

$$\therefore\ PQ^2 = QR \,.\, PV';$$

$$\therefore\ PV = \text{limit of } PV' = \text{limit of } \frac{PQ^2}{QR}.$$

Also by Lemma VII. the ultimate ratio of $PQ$ to the arc and tangent of either the circle or the curve is one of equality;

$$\therefore PV = \text{limit of } \frac{(\text{tangent } PR)^2}{\text{subtense } QR \text{ parallel to } PV},$$

$$\text{or} \qquad = \text{limit of } \frac{(\text{arc } PQ)^2}{\text{subtense } QR \text{ parallel to } PV}.$$

2. *If the chords of curvature* PU, PV *at any point* P *of a curve meet any straight line parallel to the tangent at* P *in* u *and* v, *then*

$$\frac{PU}{PV} = \frac{Pv}{Pu}.$$

Let a circle touching the curve at $P$ and passing through $Q$ meet $PU$, $PV$ in $U'$, $V'$: join $U'V'$.

Then, $\angle PU'V' = \angle RPV'$ (Euc. III. 32)

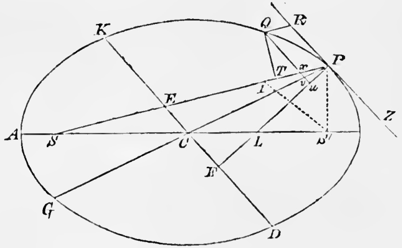
$$= \angle Pvu;$$

similarly $\angle PV'U' = \angle Puv$:

$\therefore$ triangles $Puv$, $PV'U'$ are similar;

$$\text{therefore } \frac{PU}{PV} = \text{limit of } \frac{PU'}{PV'} = \frac{Pv}{Pu}.$$

3. *The chord of curvature, through the centre, at any point* P *of an ellipse* $= \dfrac{2\text{CD}^2}{\text{CP}}$.



Let $PQ$ be a small arc of the ellipse, $QR$ parallel to the diameter $PCG$; draw $Qv$ parallel to the tangent $PR$ meeting $PC$ in $v$.

Then chord of curvature of the ellipse, through $C$,

$$= \text{limit of } \frac{PR^2}{QR}$$

$$= \text{limit of } \frac{Qv^2}{Pv};$$

but $Qv^2 : Pv \,.\, vG = CD^2 : CP^2$ (*Conics*, p. 66),

$\therefore$ chord of curvature through $C =$ limit of $\dfrac{CD^2}{CP^2} \cdot vG$;

and when $Q$ coincides with $P$, $Gv = 2CP$;

$\therefore$ chord of curvature through $C = \dfrac{2CD^2}{CP}$.

Cor. 1. *The chord of curvature through the focus*

$$= \frac{2CD^2}{CA}.$$

Join $P$ with the focus $S$, and let $PS$ meet $CD$ in $E$; then, since $CD$ is parallel to $PR$,

chord through $S$ : chord through $C = CP : PE$;

$\therefore$ chord through $S = \dfrac{2CD^2}{CP} \cdot \dfrac{CP}{PE}$

$= \dfrac{2CD^2}{CA}$ (*Conics*, p. 58).

Cor. 2. *The diameter of curvature of an ellipse at* P

$$= \frac{2CD^2}{PF}.$$

For since the diameter of curvature coincides in direction with the normal $PF$ to the ellipse,

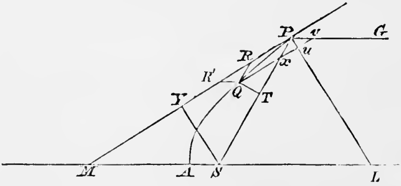
diameter of curvature : chord through $C = CP : PF$,

$$\therefore \text{ diameter of curvature} = \frac{2CD^2}{CP} \cdot \frac{CP}{PF}$$

$$= \frac{2CD^2}{PF}.$$

4. The same results may be obtained by precisely similar reasoning for the chord through the centre and focus, and the diameter, of curvature of a hyperbola.

5. *The chord of curvature of a parabola at* P *parallel to the axis* $= 4SP$.




Let $QR'$ be a subtense parallel to the axis; and $Qv$ an ordinate to the diameter through $P$; then chord of curvature parallel to the axis

$$= \text{limit of } \frac{PR'^2}{QR'}$$

$$= \text{limit of } \frac{Qv^2}{Pv}$$

$$= 4SP \text{ (Conics, p. 34).}$$

Cor. 1. *The chord of curvature of a parabola through the focus* $= 4SP$.

Let $Qv$ meet $SP$ in $x$;

then chord through $S$ : chord parallel to axis $= Pv : Px$; but $Pv$ and $Px$ make equal angles with $Qv$, which is parallel to the tangent at $P$;

$$\therefore Pv = Px;$$

$\therefore$ chord through $S$ = chord parallel to the axis

$$= 4SP.$$

Cor. 2. *The diameter of curvature of a parabola*

$$= \frac{4SP^2}{SY}.$$

Let the normal $PL$ meet $Qr$ in $u$;

$\therefore$ diameter of curvature : chord through $S = Px : Pu$;

$$\therefore \text{ diameter of curvature} = \frac{Px}{Pu} \cdot 4SP$$

$$= \frac{4SP^2}{SY},$$

by similar triangles $SYP$, $Pux$.

6. *The diameter of curvature at any point of a conic*

$$= \frac{8 \text{ normal}^3}{\text{latus rectum}^2}.$$

(1) For the ellipse and hyperbola, diameter of curvature $= \frac{2CD^2}{PF}$;

but $PF \cdot PL = BC^2$ (*Conics*, p. 62),

and $CD \cdot PF = AC \cdot BC$ (*Conics*, p. 69);

$$\therefore \text{ diameter of curvature} = \frac{2AC^2 \cdot BC^2}{PF^3}$$

$$= \frac{2AC^2 \cdot BC^2 \cdot PL^3}{BC^6}$$

$$= \frac{8PL^3 \cdot AC^2}{4BC^4}$$

$$= \frac{8 \text{ normal}^3}{\text{latus rectum}^2}.$$

(2) For the parabola,

$$\text{diameter of curvature} = \frac{4SP^2}{SY};$$

let $PY$ meet the axis in $M$; then $SP = SM$;

$$\therefore PY = YM;$$

$$\therefore PL = 2SY;$$

$$\therefore \text{diameter of curvature} = \frac{4SP^2}{SY}$$

$$= \frac{4SP^2 . SA^2}{SY . SA^2} = \frac{4SY^4}{SY . SA^2}$$

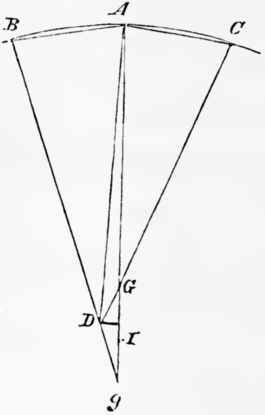
$$= \frac{4SY^3}{SA^2} = \frac{8SY^3}{2SA^2}$$

$$= \frac{PL^3}{2SA^2} = \frac{8PL^3}{16SA^2}$$

$$= \frac{8 \text{ normal}^3}{(\text{latus rectum})^2}.$$

*Note.* In the following propositions the curvatures of the curves in the neighbourhood of the points concerned are supposed to be continuous.

7. *The limit of the circle through three points* A, B, C,



*near one another on a curve, when these points coincide, is the circle of curvature.*

Draw the normal $Ag$ to the curve at $A$, and let $I$ be the extremity of the diameter of curvature at $A$; draw $CD$, $BD$ perpendicular to $AC$ and $AB$, meeting each other in $D$, and the normal to the curve at $A$ in $G$ and $g$.

Then, as in Lemma XI., the points $G$ and $g$ ultimately coincide with $I$;

therefore $Gg$ vanishes, and the angles $DGg$, $DgG$ vanish;

therefore the triangle $DGg$ vanishes, and $D$, $G$, $g$, and $I$ all ultimately coincide.

Now, since the angles $ABD$, $ACD$ are right angles, the circle through $A$, $B$, $C$ passes through $D$, and $AD$ is its diameter;

thus the diameter $AD$ of the circle through $A$, $B$, $C$, coincides ultimately with the diameter $AI$ of the circle of curvature, and therefore the circles coincide.

8. Hence, *the centre of the circle of curvature is the limiting position of the intersection of perpendiculars to two consecutive chords* BA, AC *through their middle points.*

For the distances of this point of intersection from $A$, $B$, and $C$ are equal, therefore the circle described with this point as centre at the distance of any one of them will pass through the others and will be the circle through the three points $A$, $B$, $C$.

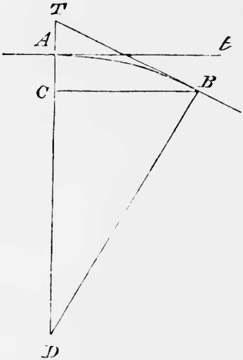
9. *The limiting position of the intersection of the normals at two near points* A *and* B *on a curve, when* A *and* B *coincide, is the centre of the circle of curvature.*

Let $AD$, $BD$ be the normals at $A$, $B$, meeting in $D$; draw $BC$ perpendicular to $AD$ or parallel to $At$ the tangent at $A$, and the tangent $BT$ at $B$ meeting $DA$ produced in $T$.

Then, by the similar right-angled triangles $TCB$, $TBD$, we have

$$TC \,.\, TD = TB^2;$$

therefore $$TD = \frac{TB^2}{TC}.$$




Now the limit of $TD$ is $AD$, when $B$ and $A$ coincide: and (page 129) since $BC$ is parallel to the tangent at $A$,

$TC$ is ultimately bisected at $A$:

therefore limit of $AD$ = limit of $\frac{TB^2}{2TA}$:

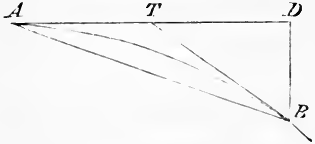
but the limit of $\frac{TB^2}{TA}$ is (page 132) the limit of the chord of curvature at $B$ parallel to $AD$, which is the diameter of curvature at $A$;

therefore, limit of $AD = \frac{1}{2}$ limit of $\frac{TB^2}{TA}$

= radius of curvature at $A$;

thus, the limiting position of $D$ is the centre of the circle of curvature at $A$.

10. *The radius of curvature of any arc* AB *is the limit of the arc divided by the circular measure of the exterior angle* BTD *between the tangents at its extremities.*




Draw $BD$ perpendicular to $AD$:

then, diameter of curvature = ultimate value of $\frac{AD^2}{DB}$ (p. 132);

and L. R. of $DB$ to $AD = \angle DAB$ (in circular measure);

therefore, diameter of curvature = limit of $\frac{AD}{\angle DAB}$,

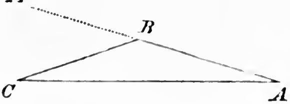
and (by Lemma VII.) radius of curvature = lt. $\frac{\text{arc } AB}{2 \times \angle DAB}$;

but, since L. R. of $\angle TAB$ to $\angle TBA$ is one of equality,

L. R. of $\angle$s $TAB + TBA$, that is of $\angle BTD$, to $2 \times \angle TAB$ is one of equality;

therefore the radius of curvature = limit of $\frac{\text{arc } AB}{\angle BTD}$.

11. *Required the radius of curvature* ($\rho$) *at any point of a curve considered as the limit of a polygon.*




Let $A$, $B$, $C$ be three points near each other on the curve; $a$, $b$, $c$ the sides of the triangle $ABC$; produce $AB$ to $A'$:

then, by trigonometry, diameter of the circle about $ABC$

$$= \frac{b}{\sin B}:$$

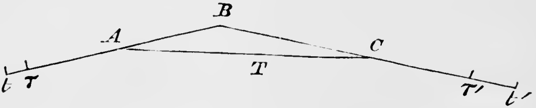
$\therefore$ (page 137) radius of curvature = limit of $\dfrac{AC}{2 \sin B}$ :

and, the L. R. of $\angle A'BC$ to $\sin B$ is unity (page 128):

$$\therefore \rho = \text{limit} \frac{AC}{2 \times \angle A'BC}.$$

COR. Therefore, L. R. $\angle A'BC : \dfrac{AC}{2\rho} = 1.$

12. *The limit of the circle touching three tangents to a curve, when these tangents coincide, is the circle of curvature.*




Let $AB$, $CB$, $CA$ be the three tangents: $t$, $t'$, $T$ their points of contact with the curve: and $\tau$, $\tau'$ the points of contact of $BA$, $BC$ with the circle touching $AB$, $BC$, $CA$;

then $\quad 2B\tau = B\tau + B\tau' = BA + BC + AC$ .............. (1),

also $\quad Bt + Bt' = BA + BC + At + Ct'$ .............. (2),

and since $Bt$, $Bt'$ touch the curve the limiting ratio of $Bt$ to $Bt'$ is one of equality (page 130); similarly, the limiting ratios of $At$ and $Ct'$ to $AT$ and $CT$ respectively are ratios of equality;

therefore, L. R. of $Bt + Bt'$ to $2Bt$ is one of equality;

and L. R. of $At + Ct'$ to $AT + CT$, or $AC$, is one of equality;

and therefore L. R. of $2Bt$ to $BA + BC + AC$ is the L. R. of $Bt + Bt'$ to $BA + BC + At + Ct'$, or is a ratio of equality, from (2);

thus from (1) the L. R. of $Bt$ to $B\tau$ is a ratio of equality;
similarly the L. R. of $Bt'$ to $B\tau'$ is a ratio of equality;
therefore the L. R. of $Bt + Bt'$ to $B\tau + B\tau'$ is one of equality.

But $Bt + Bt'$ is ultimately in a ratio of equality with the arc of the curve between $t$ and $t'$:

and $B\tau + B\tau'$ is in a ratio of equality with the arc of the circle between $\tau$ and $\tau'$:

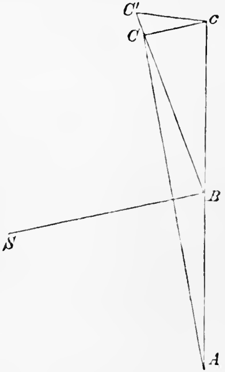
also the tangents at the extremities of these arcs are the same, namely $BAt$ and $BCt'$;

and thus the radius of curvature, being equal to the L. R. of the arc to the exterior angle between the tangents at its extremities (page 139), is the same for both the curve and circle.

Therefore the limit of the circle is the circle of curvature.

The following proposition is given as an illustration of the method of Section II. Prop. I.

*A body describes a curve under the action of any forces;*




*to find the accelerations at any point resolved in the directions of the tangent and normal at the point.*

Let the forces be supposed to act by impulses at indefinitely small intervals of time, $t$.

In one of these intervals let the body describe $AB$; then if no force acted the body would describe in the next interval $Bc$, in the direction of $AB$ produced, equal to $AB$.

But at $B$ let impulses act simultaneously on the body, causing it to move in the straight line $BC'$, and to describe $BC'$ in the same time in which it described $AB$.

On $BC'$ take $BC = AB - Bc$: join $cC$.

Then the effect of the impulses at $B$ is the same as that of two impulses, one in direction $BS$, parallel to $cC$, deflecting the body from $c$ to $C$, and the other in direction $BC'$, causing the body to describe the space $BC'$ instead of $BC$, in the interval.

The velocities due to these impulses are respectively $\frac{Cc}{t}$ and $\frac{CC'}{t}$; and they are added in an interval of time $t$;

$\therefore$ the accelerating effects of the impulses are respectively $\frac{Cc}{t^2}$ and $\frac{CC'}{t^2}$.

But L. R. $Cc : Bc \times \angle CBc = 1$ by Lemma VII.;

or $$\text{L. R. } Cc : Bc \times \frac{AC}{2\rho} = 1 \text{ (page 140)};$$

$$\therefore \text{limit } \frac{Cc}{t^2} = \text{limit } \frac{Bc \times AC}{2\rho \times t^2} = \text{limit } \frac{Bc \times Ac}{2\rho \times t^2}$$

$$= \text{limit } \frac{AB^2}{\rho \times t^2}$$

$$= \frac{(\text{velocity})^2}{\text{radius of curvature}} \quad \dots\dots\dots\dots\dots (1).$$

Again, $CC' = BC' - BC = BC' - AB$;

$$\therefore \frac{CC'}{t} = \frac{BC'}{t} - \frac{AB}{t}$$

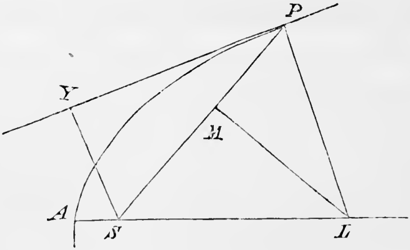
$$= \text{difference of velocities at } A \text{ and } B;$$

$$\therefore \text{limit } \frac{CC'}{t^2} = \text{rate of increase of velocity} \dots\dots\dots\dots (2).$$

And since $\angle CcB$ is ultimately a right angle, (1) and (2) are the accelerations of the body resolved normally to the curve and tangentially.

The following proposition is often found useful in estimating the effect on the motion of a body about a centre of force varying inversely as the square of the distance, produced by a disturbing force.

*A body describes a conic section under the action of a centre of force in one focus S; to resolve the velocity at*




*any point P into two components, one perpendicular to the radius vector SP, and the other perpendicular to the major axis.**

Draw $SY$ perpendicular to the tangent, $PL$ the normal and $LM$ perpendicular to $SP$: then the sides of the triangle $PSL$ are at right angles to the directions of the velocity and of its components, and are therefore proportional to them (page 105).

If then $v$ be the velocity, $v_1$ $v_2$ the components perpendicular to $SP$ and $SL$,

$$v_1 = \frac{SP}{PL} v = \frac{SP}{PL} \cdot \frac{h}{SY}$$

$$= \frac{h}{PL} \cdot \frac{SP}{SY} = \frac{h}{PL} \cdot \frac{PL}{PM} \text{ (by similar triangles)}$$

$$= \frac{h}{PM} = \frac{h}{l},$$

where $l$ is the semi-latus-rectum. (*Conics*, p. 16.)

* For the proof of this proposition here given I am indebted to Mr Besant.

Again, $$\frac{v_2}{v_1} = \frac{SL}{SP}$$

$$= e, \text{ the excentricity of the orbit :}$$

therefore $$v_2 = ev_1 = \frac{eh}{l}.$$

Hence, *the velocity at any point of a conic section described by a body under the action of a force to the focus may be resolved into two constant components, one* $\left(\frac{h}{l}\right)$ *perpendicular to the radius vector, and the other* $\left(\frac{eh}{l}\right)$ *perpendicular to the major axis.*

To find the velocity at any point of any orbit described by a body about any centre of force, the law of the force being known, we use the formula given in Prop. VI. Cor. 4, Note 1, from which we get

$$v^2 = F \times \frac{PV}{2},$$

where $v$ is the velocity at $P$, $F$ the force, and $PV$ the chord of curvature through the centre of force.

We will apply this formula to finding the velocities in the cases of Section II.

In Prop. VII. the orbit is a circle about any point; and (Prop. VII. Note 3)

$$F = \frac{\mu}{SP^2 \times PV^3};$$

$$\therefore v^2 = \frac{\mu}{2SP^2 \times PV^2};$$

and $$v = \frac{\sqrt{\mu}}{\sqrt{2} . SP \times PV}$$

In Prop. VIII. the orbit is a semicircle about a point infinitely distant; and (Prop. VIII., Note)

$$F=\frac{\mu}{PM^3};$$

also $$PV=2PM \text{ (fig. page 71)};$$

$$\therefore\ v^2=\frac{\mu}{PM^3}\times PM=\frac{\mu}{PM^2};$$

and $$v=\frac{\sqrt{\mu}}{PM}.$$

In Prop. IX. the orbit is an equiangular spiral, about the centre of the spiral; and (Prop. IX. Note 3)

$$F=\frac{\mu}{SP^3};$$

and $$PV=2SP;$$

$$\therefore\ v^2=\frac{\mu}{SP^2};$$

and $$v=\frac{\sqrt{\mu}}{SP}.$$

In Prop. X. the orbit is an ellipse about the centre; and (Prop. X. Note 1)

$$F=\mu CP;$$

also $$PV=\frac{2CD^2}{CP};$$

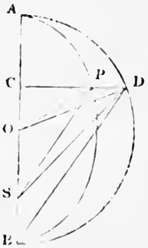
$$\therefore\ v^2=\mu CD^2;$$

and $$v=\sqrt{\mu}CD.$$

*Force varying as (distance)*$^{-2}$. *To find the time of motion and the velocity acquired by a body falling through a given space from rest* (Props. XXXIII. and XXXVI.).

Let $S$ be the centre of force, $A$ the point from which the body begins to fall;

$$\frac{\mu}{SP^2} = \text{force at distance } SP.$$




Let $APB$ be a semi-ellipse, focus $S$ and axis major $ASB$; $ADB$ a semicircle, whose diameter is $ASB$; and suppose a body revolving in the ellipse round the focus $S$ to come to $P$; bisect $AB$ in $O$, draw $DPC$ perpendicular to $AB$, and join $OP$, $OD$.

Then the time through $AP \propto$ area $ASP \propto$ area $ASD$; and this being true for all values of the axis minor will be true when it is diminished without limit, in which case the ellipse coincides with the axis major and the point $P$ with $C$, or the body is moving in the straight line $AC$; the point $B$ also coincides with $S$, since $AS \,.\, SB = (\frac{1}{2}\text{ axis minor})^2$; and since space due to velocity at $A = \frac{1}{4}$ chord of curvature at $A$ through $S = \frac{1}{4}$ latus rectum $= \dfrac{(\text{axis minor})^2}{4AB} = 0$, the body begins to move from rest at $A$.

Hence time from rest through $AC \propto$ area $ABD$,

$$\therefore \frac{\text{time through } AC}{\text{time through } AB \; (=\tfrac{1}{2} \text{ periodic time in ellipse})}$$

$$= \frac{\text{area } ABD}{\text{semicircle } ABD};$$

$$\therefore \text{time through } AC = \frac{\pi . AO^{\frac{3}{2}}}{\sqrt{\mu}} . \frac{\tfrac{1}{2} AO . (AD + CD)}{\tfrac{1}{2}\pi . AO^2}$$

$$= \sqrt{\frac{AS}{2\mu}} . (AD + CD).$$

Again, velocity at $P = \sqrt{\frac{\mu}{AO} \cdot \frac{HP}{SP}}$ (Prop. XVI.), and

when the ellipse coincides with the axis major,

$$\text{velocity at } C = \sqrt{\frac{2\mu}{AS} \cdot \frac{AB - BC}{BC}} = \sqrt{\frac{2\mu}{AS} \cdot \frac{AC}{SC}}.$$

Cor. Time through $AS = \sqrt{\frac{AS}{2\mu}} . \pi \frac{AS}{2} = \frac{\pi \left(\frac{AS}{2}\right)^{\frac{3}{2}}}{\sqrt{\mu}}$

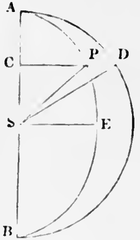
$= \frac{1}{2}$ per. time in an ellipse,

of which $AS$ is the axis major.

*Force varies as distance. To find the time of motion and the velocity acquired by a body in falling through a given space from rest.* (Prop. XXXVIII.)

Let $S$ be the centre of force, $A$ the place from which the body begins to fall: on $AB = 2AS$ describe a semi-ellipse $APB$, and a semicircle $ADB$, and let a body moving in the ellipse come to $P$. Draw $DPC$ perpendicular to $AB$, and join $SP$, $SD$.

Then time through $AP \propto$ area $ASP \propto$ area $ASD$, and this being true, whatever be the axis minor of the ellipse,




will be true when it is diminished without limit, in which case the body will be at $C$, having fallen from rest at $A$,

$$\therefore \text{ time through } AC \propto \text{ area } ASD;$$

$$\therefore \frac{\text{time through } AC}{\text{time through } AS \ (=\tfrac{1}{4} \text{ periodic time in a circle})}$$

$$= \frac{\text{sector } ASD}{\tfrac{1}{4} \text{ area of a circle}};$$

$$\therefore \text{ time through } AC = \frac{\pi}{2\sqrt{\mu}} \cdot \frac{\tfrac{1}{2} AS . AD}{\tfrac{1}{4} \pi AS^2}$$

$$= \frac{AD}{AS\sqrt{\mu}}.$$

Again, let $SE$ be the semi-axis minor,

then vel. at $P$ = semi-conjugate at $P$ . $\sqrt{\mu}$ (page 146)

$$= \sqrt{AS^2 + SE^2 - SP^2} . \sqrt{\mu},$$

$$\therefore \text{ vel. at } C = \sqrt{AS^2 - SC^2} . \sqrt{\mu}$$

$$= CD\sqrt{\mu}.$$

Cor. Time to centre of force $= \dfrac{\tfrac{1}{2}\pi AS}{AS\sqrt{\mu}} = \dfrac{1}{4} \dfrac{2\pi}{\sqrt{\mu}}$

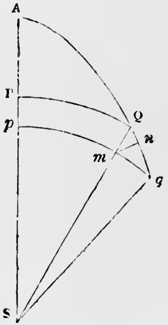
$= \tfrac{1}{4}$ per. time in an ellipse, force in centre.

Hence the times through all distances to the centre of force are equal.

Vel. acquired in falling through $AS = AS\sqrt{\mu}$.

*If the velocities of two bodies, one of which is falling directly towards a centre of force, and the other describing a curve about that centre, be equal at any equal distances, they will always be equal at equal distances.* (Prop. XL.)

Let $S$ be the centre of force, and let one of the bodies be moving in the straight line $APS$ and the other in the




curve $AQq$; with radii $SQ$, $Sq$ describe the circular arcs $QP$, $qp$: let $SQ$ cut $pq$ in $m$, and draw $mn$ perpendicular to $Qq$; and suppose the velocities of the bodies at $P$ and $Q$ to be equal.

Since the centripetal forces at $P$ and $Q$ are equal, $Pp$, $Qm$ may be taken to represent them: $Pp$ is wholly effective in accelerating $P$, but the effective part of $Qm$ is $Qn$, $nm$ being wholly employed in retaining the body in the curve. Also since the velocities at $P$ and $Q$ are equal, the times of describing $Pp$ and $Qq$, when the spaces are diminished indefinitely, are proportional to $Pp$ and $Qq$; hence

$$\text{force at } P : \text{force at } Q = Pp : Qn,$$

$$\text{and time through } Pp : \text{time through } Qq = Pp : Qq;$$

$\therefore$ velocity added in describing $Pp$ : velocity added in describing $Qq$

$$= Pp^2 : Qn \,.\, Qq = Qm^2 : Qn \,.\, Qq$$

$$= 1 : 1,$$

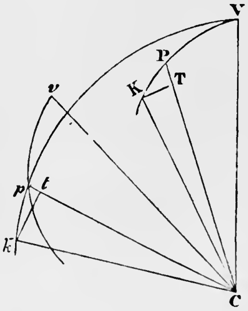
and the same may be shewn at all corresponding points equally distant from $S$. Therefore, *If the velocities, &c.*

## SECTION IX.

### *On the Position of the Apsides in Orbits very nearly circular.*

PROP. XLIII. *The orbit in which a body moves revolves round the centre of force with an angular velocity, which always bears a fixed ratio to that of the body; to shew that the body may be made to move in the revolving orbit in the same manner as in the orbit at rest by the action of a force tending to the same centre.*

Let $C$ be the centre of force, and when the body in the fixed orbit $VCP$ has described the arc $VP$, let $vCp$ be




the position of the revolving orbit, and $p$ that of the body moving in it; then $\angle vCp = \angle VCP$. Also let the angular velocity of the orbit be to that of $P$ as $G-F : F$.

The angles $VCv$, $VCP$ begin together at $V$, and their contemporary increments are as the angular velocities of

$Cr$ and $CP$, that is, as $G-F : F$, therefore the angles themselves are in that ratio, or

$$VCr : VCP \text{ (or } rCp) = G-F : F;$$

$$\therefore \text{ componendo } VCp : VCP = G : F;$$

hence, if the angle $VCp$ be always taken $= \frac{G}{F} \times$ angle $VCP$, and $Cp = CP$, $Vp$ the locus of $p$ will be the curve traced out in fixed space by a body $p$ moving in the revolving orbit in the same manner as $P$ in the fixed orbit.

Also the body may describe the orbit $Vp$ by the action of a force placed in $C$.

For let $PCK$, $pCk$ be the areas described by $CP$, $Cp$ in the same small increment of time; draw $KT$, $kt$ perpendicular to $CP$, $Cp$; then the contemporary increments of the areas, described by $p$ and $P$, are ultimately as

$$Cp . kt : CP . KT = CP^2 . \sin pCk : CP^2 . \sin PCK$$

$$= \angle pCk : \angle PCK = \angle^r \text{ vel. of } Cp : \angle^r \text{ vel. of } CP = G : F;$$

and the whole areas begin together at $V$, therefore they are themselves in the same ratio; hence area $VCp \propto$ area $VCP \propto$ the time (Prop. I.); and therefore (Prop. II.) a body may be made to move in the orbit $Vp$ by a proper centripetal force placed in $C$.

Def. An apse or apside is a point in an orbit at which the direction of the body's motion is perpendicular to the distance; and the angle between two consecutive apsidal distances is called the apsidal angle.

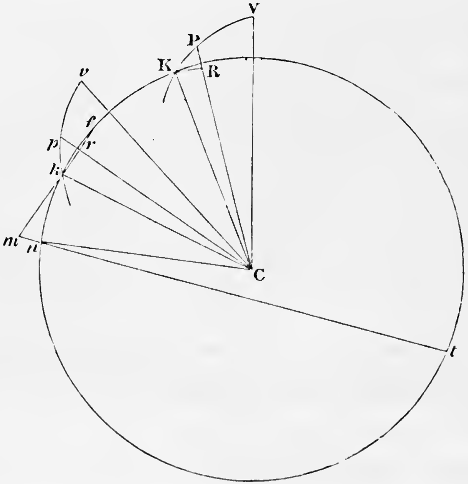
Cor. *If $a$ be the apsidal angle in the orbit* VP, *the corresponding apsidal angle in the orbit* Vp $= \frac{G}{F} a$.

For the motion of $p$ is compounded of two motions, one arising from the angular motion of the orbit, and therefore perpendicular to the distance, and the other the same as the motion of $P$ in the fixed orbit; hence when the latter body is at an apse, the whole motion of $p$ will be perpendicular to the distance, or $p$ will be at an apse; also the angles described in the same time in the orbits $Vp$ and $VP$ are always as $G : F$,

$$\therefore \text{ apsidal angle in orbit } Vp = \frac{G}{F}\alpha.$$

PROP. XLIV. *To find the difference of the forces, by which the bodies are retained in the fixed and revolving orbits.*

Let $P$ and $p$ be contemporary positions of the two bodies, $PK$ a small arc of the fixed orbit described in $t''$;




take $pk = PK$, and with radius $CK$ or $Ck$ describe the circle $Kk$;

draw $KR$, $kr$ perpendicular to $CP$, $Cp$ and in $rk$, produced if necessary, take $rm = \frac{G}{F} \cdot kr$. Let the velocities of $P$ and $p$ be each resolved into two, one central or in the direction of the distance, and the other transverse or perpendicular to it; then since $PK$ is very small, $PR$ and $RK$ may be taken to represent $P$'s central and transverse velocities respectively; and since the angular motion of the orbit affects only the transverse motion of $p$, $pr = PR$ will represent $p$'s central motion: also transverse vel. = angular vel. × dist.;

∴ transv. vel. of $p$ : transv. vel. of $P$ = ∠r vel. of $p$ : ∠r vel. of $P$ $= G : F$;

$$\therefore \text{ transverse vel. of } p = \frac{G}{F} \cdot KR = rm.$$

Hence, in consequence of the two motions $pr$, $rm$, $p$ will be at $m$, when $P$ is at $K$. But if we take

$$\angle VCn = \frac{G}{F} \angle VCK, \text{ and } Cn = CK,$$

$p$ must be at $n$, when $P$ is at $K$, in order that it may move in the manner required; join $mn$; then an additional force must have acted on $p$, sufficient to draw it through $mn$ in $t''$, and therefore the difference of the forces on $P$ and $p$

$$= 2 \lim. \frac{mn}{t^2} \text{ (Lem. X. Note 4, and Cor. 4).}$$

Let $mn$, $mr$ produced cut the circle again in $t$ and $f$,

then $$mn = \frac{mk \cdot mf}{mt}.$$

Now $$mr = \frac{G}{F} \cdot kr, \quad \therefore mk = \frac{G-F}{F} kr,$$

and $$mf = \frac{G+F}{F} \cdot kr; \quad \therefore mk \cdot mf = \frac{G^2 - F^2}{F^2} \cdot kr^2.$$

Let $h=2$ area described by $P$ in 1″,

$$\therefore h=2 \text{ lim. } \frac{\text{area } PCK}{t}=\text{lim.}\frac{CP \,.\, KR}{t}; \therefore \text{lim. } \frac{KR}{t}=\frac{h}{CP};$$

also $mt$ ultimately passes through $C$ and equals $2CP$;

$$\therefore 2 \,.\, \text{lim. } \frac{mn}{t^2}=2 \,.\, \text{lim. } \frac{G^2-F^2}{F^2} \,.\, \frac{kr^2}{t^2 \,.\, 2CP};$$

$$\therefore \text{force on } p - \text{force on } P=\frac{G^2-F^2}{F} \,.\, \frac{h^2}{CP^3}, \text{ and } \therefore \propto \frac{1}{CP^3}.$$

PROP. XLV. *The law of force in an orbit nearly circular being given, to find an approximate value of the apsidal angle.*

Let $\frac{1}{r^3}.fr$ be the force at any distance $r$, $a$ the greatest value of $r$, and $a-x$ any other value; then

$$\frac{1}{r^3}fr=\frac{1}{r^3}f(a-x),$$

which being expanded in a series ascending by powers of $x$

$$=\frac{1}{r^3}(fa-f'a \,.\, x+\&c.)=\frac{1}{r^3}(fa-f'a \,.\, x) \text{ very nearly,}$$

since $x$ is very small.

Let $VP$ (Fig. Prop. XLIII.) be an ellipse of small excentricity, $C$ the focus, $CV$ the greatest distance $=a$, $L$ the latus rectum, and let $\frac{F^2}{a^2}=$ force at $V$; then (Prop. XI. Note), if $h=2$ area described in 1″ by a body revolving in the ellipse round a centre of force in the focus,

$$F^2=\frac{2h^2}{L}=\frac{h^2}{a}, \text{ since } L=2a \text{ nearly; hence,}$$

$$\text{force on } p = \frac{F^2}{Cp^2} + \frac{G^2 - F^2}{F^2} \cdot \frac{h^2}{Cp^3} \text{ (Prop. XLIV.)}$$

$$= \frac{1}{Cp^3}\{F^2Cp + (G^2 - F^2)\,a\}$$

$$= \frac{1}{r^3}\{F^2(a - x) + (G^2 - F^2)\,a\} \text{ since } Cp = r \text{ or } a - x,$$

$$= \frac{1}{r^3}(G^2a - F^2x).$$

Now the values of $G$ and $F$ being indeterminate, this expression may be made equal to the above value of the force in the orbit, of which the apsidal angle is required, that is,

$$G^2a - F^2x = fa - f'ax,$$

from which equation, since it must hold true for the different values of $x$, we obtain

$$G^2a = fa, \text{ and } F^2 = f'a, \text{ and therefore, } \frac{G}{F} = \sqrt{\frac{fa}{af'a}}.$$

Now since the proposed orbit is nearly circular, (vel.)$^2$ at apsidal distance $(a)$ = force $\times\, a$ nearly (Prop. VI. Cor. 4), $= \frac{1}{a^2}.fa$, and since at an apse the velocity is wholly transverse, (vel.)$^2$ at $V$ in orbit $Vp = \frac{G^2}{F^2}.$(vel.)$^2$ at $V$ in orbit $VP$, $= \frac{G^2}{F^2}.\frac{F^2}{a} = \frac{G^2}{a}$ = (vel.)$^2$ in proposed orbit, since $G^2a = fa$. Since then in the orbit $Vp$, and in that of which the apsidal angle is required, the apsidal distances and the forces at equal distances, as well as the velocities at the apsidal distances, are equal, the orbits will be similar, and the apsidal angles equal; but the apsidal angle in the orbit $Vp$

$$= \frac{G}{F}.\,180^\circ \text{ (Prop. XLIII. Cor.)} = \sqrt{\frac{fa}{af'a}}\,180^\circ;$$

and therefore the apsidal angle required $\sqrt{\frac{fa}{af'a}}\,180^\circ$.

Ex. 1. Let the force $= \mu r^{n-3}$;

$$\therefore \text{ force} = \frac{\mu}{r^3} r^n = \frac{\mu}{r^3}(a-x)^n$$

$$= \frac{\mu}{r^3}(a^n - na^{n-1} . x), \text{ nearly};$$

$$\therefore fa = \mu a^n, f'a = \mu n a^{n-1};$$

$$\therefore \text{ apsidal angle} = \frac{180^0}{\sqrt{n}}.$$

Ex. 2. Let the force $= \dfrac{\mu r^m + \nu r^n}{r^3}$,

$$\therefore \text{ force} = \frac{1}{r^3}\{\mu (a-x)^m + \nu . (a-x)^n\}$$

$$= \frac{1}{r^3}\{\mu a^m + \nu a^n - (m\mu a^{m-1} + n\nu a^{n-1})\, x + \&c.\};$$

$$\therefore fa = \mu a^m + \nu a^n,$$

$$f'a = m\mu a^{m-1} + n\nu a^{n-1};$$

$$\therefore \text{ apsidal angle} = \sqrt{\left\{\frac{\mu a^m + \nu a^n}{m\mu a^m + n\nu a^n}\right\}} . 180^0.$$

If $a = 1$, apsidal angle $= \sqrt{\left\{\dfrac{\mu + \nu}{m\mu + n\nu}\right\}}\, 180^0$.

In this manner, as will be shewn in the next section, the motion of the moon's apsides might be found approximately, if the direction of the disturbing force of the sun upon the moon tended wholly to the earth's centre; but since this is not the case, their motion cannot be determined by the method here proposed.

## SECTION XI.

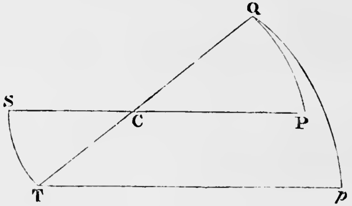
### *On the Motion of Bodies mutually attracting each other.*

The motion of a physical point, attracted to an immovable centre of force, has been explained in the preceding sections. We now proceed to consider the motions of mutually attracting bodies, of which the masses bear a finite ratio to each other. In this case the attracting body placed in the centre of force is no longer immovable, for by the third law of motion the actions of the attracting and attracted bodies are mutual and equal; so that if $M$ represent the mutual attraction of two bodies, whose masses are $S$ and $P$, the bodies themselves will be acted on by accelerating forces $\frac{M}{S}$ and $\frac{M}{P}$ respectively, and a motion will consequently be generated in each, the nature of which it is now proposed to investigate.

Prop. LVII. *Two bodies attracting each other describe similar figures about their centre of gravity, and about each other.*

Let $S$ and $P$ be the bodies, join $SP$ and take $SC : SP$ $P : S+P$, then $C$ is their centre of gravity. If $C$ be in motion, let a motion always equal and opposite to that of $C$ be applied to the system, then $C$ will continue at rest; and since the same motion applied to all the parts of a system produces no alteration in their relative motions, the relative orbits described by $S$ and $P$ about $C$ and about each other will not be affected.

Let $ST$ and $PQ$ be arcs described in the same time round $C$; then




$$TC : CQ = P : S = SC : CP,$$

$$\therefore\ TC : SC = CQ : CP,$$

and angle $SCT$ = angle $PCQ$; therefore $ST$ and $PQ$ are similar figures, and they are the figures described about the centre of gravity.

Again, draw $Tp$ parallel and equal to $SP$. To a spectator at $S$, who is insensible of his own motion and refers the whole motion to $P$, $P$ at first will be seen in the direction $SCP$ or $Tp$, and afterwards in the direction $TQ$, and will therefore appear to have described the angle $pTQ$ about $S$,

$$\text{and } SP \text{ (or } Tp) : CP = S + P : S = TQ : CQ\,;$$

$$\therefore\ Tp : TQ = CP : CQ,$$

and angle $pTQ$ = angle $PCQ$, therefore the curves $pQ$ and $PQ$ are similar; that is, the figure described by $P$ round $S$ in motion is similar to the figures described by $P$ and $S$ round their centre of gravity.

PROP. LVIII. *An orbit similar and equal to the apparent orbit of* P *round* S *in motion may be described round* S *fixed by the action of the same central force.*

Let $PQ$ and $ST$ be the similar orbits described by $P$ and $S$ round $C$, their centre of gravity. Take

$$Sp = SP, \quad \angle pSq = \angle PCQ,$$

and take $Sq$ such, that

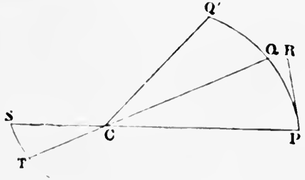
$$Sq : Sp = CQ : CP = TQ : SP;$$

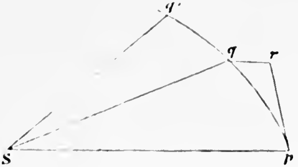
$$\therefore Sq = TQ,$$

and therefore $q$ traces out the apparent orbit of $P$. Draw the subtenses $QR$, $qr$, parallel to $CP$, $Sp$, and meeting the tangents at $P$, $p$ in $R$, $r$.

Let a body be projected from $p$ with a velocity $v$, which is to $V$ the velocity at $P$,

as $\sqrt{S+P} : \sqrt{S}$, as $\sqrt{Sp} : \sqrt{CP}$, as $\sqrt{pr} : \sqrt{PR}$




by similar figures, and let $T$, $t$ be the times of describing $PR$, $pr$; then ultimately

$$\frac{T}{t} = \frac{PR}{V} \div \frac{pr}{v} = \frac{PR}{pr} \cdot \sqrt{\frac{pr}{PR}} = \sqrt{\frac{PR}{pr}} = \sqrt{\frac{QR}{qr}}.$$

Also the force being the same,

$$\frac{\text{space through which } P \text{ is drawn in } T''}{\text{space through which } p \text{ is drawn in } t''} = \frac{T^2}{t^2} = \frac{QR}{qr}$$

ultimately,

but $RQ$ = space through which $P$ is drawn in $T''$,

$\therefore rq = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; p \; \ldots\ldots\ldots\ldots \; t''$;

and therefore $q$ is the place of the body at the end of $t''$; it will also continue in the curve, for the forces being equal and the orbits similar, the resolved parts of the forces in the directions of the tangents will be equal at all corresponding points in the arcs $PQ$, $pq$; hence the increments of the velocities continually generated, as the bodies describe the arcs, will be ultimately as the times of describing similar arcs, that is,

$$\text{as } T : t, \text{ as } \sqrt{S} : \sqrt{S+P};$$

$$\therefore \text{componendo, vel. at } q : \text{vel. at } Q = \sqrt{S+P} : \sqrt{S},$$

hence the body is under the same circumstances as at $p$, and will therefore continue in the curve.

COR. 1. *Two bodies, which attract each other with forces varying as the distance, describe similar ellipses about their centre of gravity and about each other as centres.*

For the orbits described about $C$ and about each other are similar to that described about $S$ fixed, which in this case is an ellipse, whose centre is $S$.

COR. 2. *Two bodies, which attract each other with forces varying inversely as the square of the distance, describe similar ellipses about their centre of gravity and about each other as foci.*

COR. 3. *Two bodies revolving round their centre of gravity describe round it areas proportional to the times.*

Let $PQ$, $PQ'$ be arcs respectively similar to $pq$, $pq'$, and let $T$, $T'$, $t$, $t'$, be the times of describing the four arcs respectively;

now
$$\frac{t}{T} = \frac{\sqrt{S+P}}{\sqrt{S}} = \frac{t'}{T'};$$

$$\therefore \frac{t}{t'} = \frac{T}{T'};$$

also by similar figures,

$$\frac{\text{area } PCQ}{\text{area } PCQ'} = \frac{\text{area } pSq}{\text{area } pSq'} = \frac{t}{t'} = \frac{T}{T'};$$

$\therefore$ area $PCQ \propto$ time of describing it.

PROP. LIX. *The periodic time of* P *round* S *at rest* : *that of* P *or* S *round* C = $\sqrt{S+P} : \sqrt{S}$.

For the orbits, being similar, may be divided into the same number of similar parts, as $pq$, $PQ$ in Prop. LVIII.;

and time of describing $pq$ : time of describing $PQ$

$$= \sqrt{S+P} : \sqrt{S},$$

and the same being true for the times of describing all the similar arcs, we have componendo

periodic time of $P$ round $S$ at rest : that of $P$ or $S$ round $C$

$$= \sqrt{S+P} : \sqrt{S}.$$

PROP. LX. *Force* $\propto$ *(dist.)*$^{-2}$. *If* (a) *be the axis major of the apparent orbit described by* P *round* S *in motion*, (a′) *that of an orbit described by* P *round* S *at rest in the same periodic time, then* a : a′ = $\sqrt[3]{S+P} : \sqrt[3]{S}$.

Let $p'q'$ be the ellipse, of which $a'$ is the axis major, that is, let $p'q'$ be an ellipse described by $P$ round $S$ at rest, in the same periodic time as that in which $P$ describes an ellipse round $S$ in motion, or as that in which $PQ$ is described; and let $pq$ be the apparent orbit described by $P$ round $S$ in motion; then, since the force in the two orbits is the same, periodic time in $p'q'$ : periodic time in $pq = a'^{\frac{3}{2}} : a^{\frac{3}{2}}$ (Prop. XV.); also by Prop. LIX.

$$\text{period. time in } pq : \text{period. time in } PQ = \sqrt{S+P} : \sqrt{S};$$

$$\therefore \text{period. time in } p'q' : \text{period. time in } PQ$$

$$= \sqrt{(S+P)\,a'^3} : \sqrt{Sa^3},$$

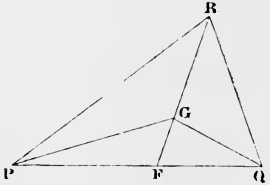
the first term of which proportion is equal to the second by the hypothesis,

$$\therefore (S+P)\,a'^3 = Sa^3;$$

$$\therefore a : a' = \sqrt[3]{S+P} : \sqrt[3]{S}.$$

PROP. LXIV. *To determine the motion of a system of bodies attracting each other with forces varying as the distance between their centres.*

Let $P$ and $Q$ be two bodies collected in their respective centres of gravity. Join $PQ$ and take $PF : PQ$




$= Q : P+Q$, then $F$ is the centre of gravity of $P$ and $Q$: and $(P+Q).PF = Q.PQ$ = force of $Q$ on $P$; but $(P+Q)PF$ = the force, which two bodies equal to $P$ and $Q$ placed at $F$ would exert on $P$, therefore $P$ is attracted in the same manner as if a body equal to the sum of the bodies were placed at $F$, and will therefore describe an ellipse round $F$ at rest as its centre. Similarly $Q$ will describe an ellipse round the same point as a centre, and in the same periodic time, since the absolute force $P+Q$ is the same in both cases.

Let $R$ be a third body, join $RP$, $RQ$, $RF$; the forces $R.PR$ and $R.QR$, which $R$ exerts on $P$ and $Q$, may be resolved respectively into $R.PF$, $R.FR$, and $R.QF$, $R.FR$; the force $R.FR$, being the same for either body, produces no disturbance in their relative motions, and therefore the bodies will move in the same manner with respect to each other, as if that force did not act. The other forces $R.PF$, $R.QF$, varying as the distance of $P$ and $Q$ from $F$, will not cause any perturbations in the orbits described by $P$ and $Q$ round $F$, and therefore these bodies will still describe ellipses round $F$, but since the absolute force is increased in the ratio of $P+Q+R$ : $P+Q$, the periodic time will be diminished in the ratio of $\sqrt{P+Q} : \sqrt{P+Q+R}$.

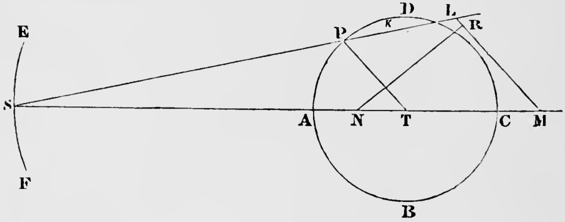
Again, in $FR$ take $FG : FR = R : P+Q+R$, and join $PG$, $QG$; then $G$ is the centre of gravity of $P$, $Q$, $R$; and $R.FR = (P+Q+R).FG$; hence the force which $R$ exerts on $P$ is equivalent to the forces $R.PF$ and $(P+Q+R).FG$, and the force which $Q$ exerts on $P$ is equal to $(P+Q).PF$; hence the whole force on $P$ is equal to $(P+Q+R)\ PF$ and $(P+Q+R)\ FG$, that is, to $(P+Q+R)\ PG$, and therefore $P$ will describe an ellipse round $G$ as a centre. Similarly $Q$ will describe an ellipse round the same point as a centre, and therefore $P$ and $Q$ describe ellipses round their common centre of gravity and round the centre of gravity of the system.

In the same manner it may be shewn that $P$ and $R$, and $Q$ and $R$, will describe ellipses round their common centres of gravity respectively, and round the centre of gravity of the system; and the same may be proved of any number of bodies.

PROP. LXVI. *Force* $\propto$ *(dist.)*$^{-2}$. *Two bodies* S *and* P *revolve round a third* T *in such a manner, that* P *describes the interior orbit: to shew that* P *will describe round* T *areas more nearly proportional to the times, and a figure more nearly resembling an ellipse, if* T *be acted on by the attractions of the other two, than if it were either not attracted by them at all, or attracted much more or much less.*

Let $PAB$, $ESF$ be the orbits of $P$ and $S$ respectively.

1. Let the orbits be in the same plane. Join $SP$, $PT$, $TS$, and in $SP$, produced if necessary, take $KS$ equal to the mean distance of $P$ from $S$, and let it represent the




accelerating force of attraction of $P$ to $S$ at that distance; take also $LS = \frac{KS^2}{PS^2} \cdot KS$, then $LS$ will represent the attraction of $P$ to $S$ at the distance $PS$. Draw $LM$ parallel to $PT$ meeting $ST$, produced if necessary, in $M$, and resolve $LS$ into the forces $LM$, $MS$.

$P$ is acted on by three forces, $LM$, $MS$ and its original gravitation to $T$, the last of which would cause it to describe areas proportional to the times and an ellipse, focus $T$: the force $LM$, acting in the direction $PT$, does not affect the equable description of areas, but since by composition with the attraction of $T$ on $P$ it forms a force not varying as $(\text{dist.})^{-2}$, it will disturb the elliptic form of $P$'s orbit; and the force $MS$, neither in the direction $PT$, nor varying as $(\text{dist.})^{-2}$, will disturb both the equable description of areas and the elliptic form of the orbit.

Let $NS$ represent the attraction of $S$ on $T$; then if $MS$ and $NS$ are equal, these equal forces, acting in parallel directions on $P$ and $T$, will not disturb the relative motions of the two bodies; but if they are unequal, the disturbing force on $P$ will be represented by their difference $MN$; hence the less $MN$ is, the smaller will be the disturbances produced: now since the distance of $P$ from $S$ is sometimes greater and sometimes less than that of $T$ from $S$, the mean attraction $KS$ of $P$ to $S$ differs less from $NS$, than if $T$ were attracted by a *much* greater or *much* less force; that is, the disturbing force $MN$ will be less, and therefore the equable description of areas and the elliptic form of $P$'s orbit will be less disturbed, if $T$ be attracted by $S$, than if it were not attracted by $S$ at all, or attracted much more or much less.

Def. The *Line of Nodes* is the straight line, in which the planes of the orbits of $P$ and $S$ intersect each other.

2. Let the orbits lie in different planes. The same construction being made, the force $LM$ acting in direction $PT$, which is in the plane of $P$'s orbit, produces the same effect as in the first case, and has no tendency to draw $P$ from the plane of its orbit. But $MN$, acting in a direction inclined to that plane, except when the line of nodes passes through $S$, not only produces the effects mentioned in the first case, but also tends to draw $P$ from the plane of its orbit; and this and the other perturbations depending on the magnitude of $MN$ will be least, when $MN$ is least, that is when $NS$ is equal or nearly equal to $KS$, as before.

Obs. In the proposition $P$ is supposed to describe an orbit round $T$ fixed; this cannot in reality be the case, as long as its magnitude bears a finite ratio to that of $T$; for, leaving out the consideration of the forces which $S$ exerts, the two bodies $P$ and $T$ describe orbits about their centre of gravity. The orbit here meant is the *apparent* orbit of $P$ to a spectator at $T$, that is, the orbit $pQ$ in Prop. 57. If, however, we suppose a force applied every instant to $P$ and $T$ equal and opposite to that which $P$ exerts on $T$, $T$ will remain at rest, and the gravitation of $P$ to $T$ will be the sum of the attractions of $T$ on $P$, and of $P$ on $T$, acting in the direction $PT$; so that the whole gravitation of $P$ to $T = \frac{P+T}{PT^2}$.

Prob. I. *To investigate expressions for the disturbing forces of S on P, on the supposition that P's orbit is circular, and coincident with the plane of S's orbit.*

Force of $S$ on $P$ represented by $LS = \frac{S}{SP^2}$,

∴ force of $S$ on $P$ in direction $PT$

$$=\frac{S}{SP^2}\cdot\frac{LM}{LS}=\frac{S}{SP^2}\cdot\frac{PT}{SP}=\frac{S.PT}{SP^3}\ \ldots\ldots\ldots\ldots\ldots (1),$$

this is called the *addititious* force, and is represented by $LM$.

Again, force of $S$ on $P$ in direction $TS$

$$=\frac{S}{SP^2}\cdot\frac{MS}{LS}=\frac{S}{SP^2}\cdot\frac{SP}{ST}=\frac{S.ST}{SP^3},$$

and force of $S$ on $T$ in direction $TS=\frac{S}{ST^2}$;

∴ disturbing force of $S$ on $P$ in direction $TS$

$$=S\left\{\frac{ST}{SP^3}-\frac{1}{ST^2}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

this is called the *ablatitious* force, and is represented by $MN$.

Draw $NR$ perpendicular to $LM$; then $MN$ is equivalent to $MR$, $RN$,

$$RN=MN\sin NMR=S.\left\{\frac{ST}{SP^3}-\frac{1}{ST^2}\right\}\sin PTS \ldots (3),$$

this force acts in the direction of the tangent at $P$, and is called the *tangential* force.

Similarly, $MR=S.\left\{\frac{ST}{SP^3}-\frac{1}{ST^2}\right\}\cos PTS$,

hence $LR=LM-MR=$

$$\frac{S.PT}{SP^3}-S.\left\{\frac{ST}{SP^3}-\frac{1}{ST^2}\right\}\cos PTS,\ \ldots\ldots\ldots (4),$$

this force, which is the resultant of the disturbing forces of $S$ on $P$ in direction $PT$, is called the *central disturbing* force.

Hence the gravitation of $P$ to $T$

$$=\frac{P+T}{PT^2}+S.\left\{\frac{PT}{SP^3}-\left(\frac{ST}{SP^3}-\frac{1}{ST^2}\right)\cos PTS\right\}.$$

PROB. II. *To find approximate expressions for the above disturbing forces, when* ST *is very great compared with* PT.

$$SP = \{ST^2 - 2ST \, . \, PT \cos PTS + PT^2\}^{\frac{1}{2}}$$

$$= ST \left\{1 - \frac{2PT}{ST} \cos PTS\right\}^{\frac{1}{2}} \text{ nearly ;}$$

$$\therefore \frac{1}{SP^3} = \frac{1}{ST^3}\left\{1 + \frac{3PT}{ST} \cos PTS\right\} \text{ nearly,}$$

$$\therefore \frac{ST}{SP^3} - \frac{1}{ST} = \frac{3PT}{ST^2} \cos PTS,$$

hence the ablatitious force

$$= \frac{3S \, . \, PT}{ST^3} \cos PTS,$$

the tangential force

$$= \frac{3S \, . \, PT}{ST^3} \cos PTS \, . \sin PTS = \frac{3S \, . \, PT}{2ST^3} \sin 2PTS.$$

the central disturbing force

$$= \frac{S \, . \, PT}{ST^3}\left\{1 + \frac{3PT}{ST} \cos PTS\right\} - \frac{3S \, . \, PT}{ST^3} \cos^2 PTS$$

$$= \frac{S \, . \, PT}{ST^3} \, . \{1 - 3 \cos^2 PTS\} \text{ nearly}$$

$$= -\frac{S \, . \, PT}{2ST^3}\{1 + 3 \cos 2PTS\}.$$

COR. 1. Let $F$ = the mean central disturbing force, or the force, which, acting uniformly for a whole revolution of $P$ round $T$, would produce the same effect as the variable central disturbing force; and let the four right angles through which $TP$ moves in one revolution be divided into $n$ equal angles; then

$$F = -\frac{S \, . \, PT}{2ST^3} \cdot \frac{1}{n}\left\{n + 3\left(\cos\frac{4\pi}{n} + \cos\frac{8\pi}{n} + \cos\frac{12\pi}{n} + \ldots + \cos\frac{4n\pi}{n}\right)\right\} \text{ when } n \text{ is infinite,}$$

$$= -\frac{S.PT}{2ST^3}\left\{1+\frac{3}{n}\cdot\frac{\cos\left(\frac{n+1}{n}\cdot 2\pi\right)\sin 2\pi}{\sin\frac{2\pi}{n}}\right\}$$

when $n$ is infinite,

$$= -\frac{S.PT}{2ST^2},$$

and therefore the mean central disturbing force is ablatitious, and diminishes the gravitation of $P$ to $T$.

DEF. 1. $P$ is said to be in *syzygy*, when its orthogonal projection on the plane of $S$'s orbit lies either in $ST$ or in $ST$ produced, and in *quadrature* when the projections lie in a line drawn through $T$ in the plane of $S$'s orbit perpendicular to $ST$.

In the first nine corollaries to the Proposition the planes of the two orbits are supposed to coincide, and therefore $P$ will be in syzygies at $A$ and $C$, when crossing the line $ST$ or $ST$ produced, and in quadratures at $B$ and $D$, 90° distant from $A$ or $C$. $S$ and $P$ move in the directions $ESF$, $DAB$. The distance $PS$ is supposed invariable, and so great as to be always considered parallel to $TS$. In the eighth and ninth corollaries the excentricity of $P$'s orbit is taken into account, but the expressions above obtained for the disturbing forces on the supposition that $P$'s orbit is circular, may, on account of the smallness of the excentricity, be applied without affecting the *general* correctness of the results deduced.

COR. 2. If the planes of the two orbits coincide, the central disturbing force $= -\frac{2S.PT}{ST^3}$ when $P$ is in syzygies, and $= \frac{S.PT}{ST^3}$ when $P$ is in quadratures; and is therefore ablatitious in the former case, and addititious in the latter.

DEF. 2. If the Earth, Moon and Sun be supposed to be represented by $T$, $P$, and $S$, the Moon is said to be in *perigee* when at the nearer, and in *apogee* when at the farther apse.

### Corollaries to the Proposition.

Cor. 1. What has been proved as to the disturbances caused by $S$ may be proved as to those produced by any other body revolving round $T$: hence if several bodies $P$, $S$, $R$, &c. revolve about another $T$, the motion of the innermost body $P$ will be least disturbed by the attractions of $S$, $R$, &c. when $T$ is attracted by the others in the same manner as they mutually attract each other.

Cor. 2. *The areas, described by* P *round* T *in the same given times, continually increase as* P *moves from quadrature to syzygy, and continually decrease from syzygy to quadrature.*

For the only part of the disturbing force which affects the equable description of areas is the tangential force, and it acts in consequentiâ from upper quadrature to syzygy, and in antecedentiâ from syzygy to lower quadrature.

Similarly the areas described in the same given times increase continually from lower quadrature to syzygy, and decrease from syzygy to upper quadrature.

Cor. 3. *The velocity of* P *is greatest in syzygies, and least in quadratures.*

Cor. 4. *If* P's *orbit be originally circular, the curvature of the disturbed orbit will be greatest in quadratures, and least in syzygies.*

For the radius of curvature in an orbit nearly circular $\propto \frac{(\text{vel.})^2}{\text{central force}}$, and therefore the curvature, which varies inversely as the radius, varies as $\frac{\text{force}}{(\text{vel.})^2}$. Now the force of $P$ to $T$ is greatest in quadratures, and least in syzygies, and the velocity of $P$ is least in the former case, and greatest in the latter; hence on both accounts the curvature is greatest in quadratures and least in syzygies.

Cor. 5. *Hence P's orbit, if it be originally circular, will assume the form of an oval, whose axis major passes through quadratures and axis minor through syzygies.*

Cor. 6. *To consider the effect produced by the disturbing forces on the periodic time of P round T.*

The tangential force accelerates and retards $P$'s motion equally in a whole revolution, and therefore does not affect the periodic time. But the central disturbing force in a whole revolution diminishes the gravitation of $P$ to $T$, and therefore increases the distance $PT$; hence the periodic time, which $\propto \frac{(\text{rad.})}{\sqrt{\text{absolute force}}}$, will from both these causes be increased by the action of the central disturbing force.

Obs. If $S$ approach towards the system $T$ and $P$, the central disturbing force, which varies inversely as $ST^3$, will be increased, and consequently the gravitation of $P$ to $T$ will be still more diminished, and the distance $PT$ increased: hence the periodic time will be still farther increased.

Cor. 7. *The orbit of P being supposed nearly circular, to consider the effect of the central disturbing force on the motion of its apsides during a whole revolution.*

Let $PT = r$, and let $\frac{\mu}{r^2}$ represent the force of $T$ on $P$; then if $\nu r$ represent the addititious force, when $P$ is in quadrature, $-2\nu r$ will represent the ablatitious force when $P$ is in syzygy; and therefore the whole attractions of $P$ to $T$ in quadrature and syzygy respectively will be $\frac{\mu}{r^2} + \nu r$, and $\frac{\mu}{r^2} - 2\nu r$. Hence if the force in quadratures prevailed for a whole revolution, the apsidal angle would $= \sqrt{\frac{\mu + \nu}{\mu + 4\nu}} \cdot 360^\circ$, which is less than $360^\circ$, or the apside would regrede; and if the force in syzygies prevailed for the same time, it would

$= \sqrt{\frac{\mu - 2\nu}{\mu - 8\nu}} . 360^\circ$, which is greater than 360°, or the apside would progrede. At any other point the apside will regrede or progrede, according as the disturbing force at that point increases or diminishes the gravitation of $P$ to $T$; but the gravitation is on the whole diminished by the central disturbing force, and therefore its tendency is to make the apsides progrede.

Obs. In investigating in this and the following Corollaries the effects produced on $P$'s orbit by the different disturbing forces, it is to be observed that only general results are obtained: the disturbing force may be supposed to act by impulses, its effects are then examined at the points where its action is most effective, and from these a general conclusion is drawn as to its effect in a whole revolution of $P$.

Cor. 8. *The orbit to* P *being supposed excentric, to consider the effect of the central disturbing force on the motion of its apsides.*

1. Let the apsidal line be in a syzygy; draw the tangent $Py$ in the direction of $P$'s motion. As $P$ approaches perigee, the central disturbing force being ablatitious*, tends to draw $P$ from $T$; hence the *acute* angle $TPy$ is increased by it, or $P$ arrives at an apse ($\pi$) sooner than it would have done in the undisturbed orbit; therefore the apsidal line regredes. For a short time after passing perigee, the disturbing force, being still ablatitious, tends to increase the *obtuse* angle $TPy$, so that $P$ appears to have proceeded from an apse ($\pi'$) still more distant than $\pi$; hence if the disturbing force now ceased acting, so that $P$ described an undisturbed ellipse, the apogee, found by producing $\pi' T$, will have regreded more than that found by producing $\pi T$, and therefore both before and after perigee, the tendency of the central disturbing force is to make the apsidal line regrede. As $P$ approaches near to

* In this and the remaining Corollaries, the central disturbing force is called ablatitious, when it acts in the direction $TP$, and therefore tends to diminish the gravitation of $P$ to $T$.

apogee, the disturbing force being still ablatitious increases the obtuse angle $TPy$, and $\therefore$ $P$ arrives at the apse later than it would otherwise have done, or the line of apses progredes; and in like manner as before it may be shewn to progrede still farther after $P$ leaves apogee; hence when $P$ is near apogee the line of apses is progressive. Now the disturbing force, varying as $PT$, is greater in the latter case than in the former, hence the progression of the apsidal line, when $P$ is near apogee, is greater than the regression, when $P$ is near perigee.

2. Let the apsidal line be in quadrature: then at the apsides the disturbing force is addititious; and it may be shewn as above, that when $P$ is near perigee, the apsidal line progredes, and regredes when $P$ is near apogee; and the regression in this case is greater than the progression; therefore since the whole motion of the apsides for other positions of $P$ is inconsiderable, in this position the apsidal line is regressive.

The apsidal line then progredes when in syzygy, and regredes in quadrature: but the progression exceeds the regression; for the former is due to the difference of the ablatitious forces at apogee and perigee, when the apsidal line is in syzygy, and the latter to the difference of the addititious forces at the same point, when that line is in quadrature, and the former difference equals twice the latter. As the line of apses by the actual motion of $S$ appears to revolve from syzygy to quadrature, the progression for the same reason exceeds the regression; hence during a whole revolution of $S$ the effect of the central disturbing force is to make the line of apses progrede.

Moreover, when the apsidal line is in syzygy and therefore progressive, it is moving in the same direction as $S$, and thus continues longer in syzygy than if $S$ were quiescent, and hence the progression is increased. When the apsidal line is in quadrature, the contrary takes place, and the regression is not so great as if $S$ were stationary. (Vid. Airy's *Gravitation.*)

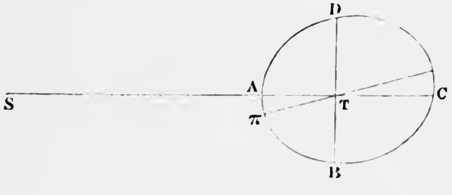
Cor. 9. *To consider the effect of the central disturbing force on the excentricity of* P*'s orbit.*

As $P$ moves from perigee to apogee, the ablatitious force tends to increase, and the addititious force tends to diminish the *obtuse* angle $TPy$, which the tangent $Py$ makes with $PT$; also the velocity at any point, and therefore the axis major, remains nearly unaltered; therefore in the former case the form of the orbit departs farther from, and in the latter approaches nearer to that of a circle; that is, the tendency of the ablatitious force is to increase, and that of the addititious to diminish the excentricity. As $P$ moves from apogee to perigee, the *acute* angle $TPy$ is increased by the former force, and diminished by the latter, that is, the excentricity is diminished by the ablatitious and increased by the addititious force.

1. When the line of apsides is in either syzygy or quadrature, the effects in either case of these disturbing forces separately, as $P$ moves from perigee to apogee, are equal and opposite to those produced by them during $P$'s motion from apogee to perigee; and therefore the excentricity of $P$'s orbit in either of these positions of the apsidal line is unaltered by the central disturbing force.

2. Let the perigee $\pi$ lie between lower quadrature and nearer syzygy.

At $A$ and $C$ the disturbing force is ablatitious, and at the former point $P$ is moving towards, and at the latter




from perigee; hence at $A$ the force tends to diminish, and at $C$ to increase the excentricity: but $TC$ is greater than $TA$, and $2\times$distance is a measure of the ablatitious force at these points, therefore the combined effects of the forces at $A$ and $C$ will increase the excentricity. A

*B* and *D* the force is addititious, and at *B*, *P* is moving from, and at *D* towards perigee, hence the tendency of the force at *B* is to diminish, and at *D* to increase the excentricity; but *TD* is greater than *TB*, and the distance is a measure of the addititious force at these points, therefore on the whole the forces at *B* and *D* increase the excentricity. Hence in this position of the apsidal line the excentricity is increased in a whole revolution.

3. By reasoning similar to the above it may be shewn, that as the perigee moves from syzygy to upper quadrature, the excentricity is continually decreasing; that it increases as it moves through *DC*, and decreases through *CB*; so that generally, *the excentricity continually increases as the apsidal line revolves from quadrature to syzygy, and decreases as that line revolves from syzygy to quadrature.*

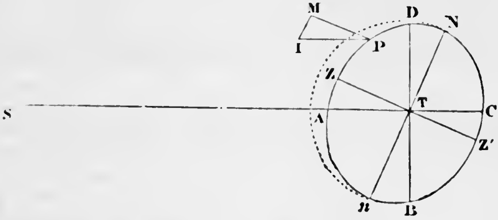
Cor. 10. *To consider the effects produced on the inclination of* P*'s orbit to that of* S *by the ablatitious force.*

Let *Nn* be the line of nodes: through *P* draw *PI* parallel to *TS* to represent the ablatitious force at *P*, *IM* perpendicular to the plane of *P*'s orbit, and join *PM*: the force *PI* may be resolved into the two *PM*, *MI*, of which the latter alone affects the inclination of the orbit; and since during *P*'s motion from upper to lower quadrature the ablatitious force acts in direction *TS* or *PI*, and through the remaining part of the orbit in direction *ST* or *IP*, the perpendicular force in the former case acts in direction *MI*, and in the latter in direction *IM*; hence the perpendicular force tends towards the plane of *S*'s orbit through *Dn* and *BN*, and from it through *nB* and *ND*; and similarly, whatever be the position of the nodal line, the perpendicular force tends towards the plane of *S*'s orbit, except when *P* is between quadrature and the nearer node.

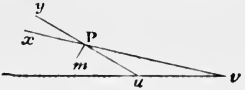
In the plane of *P*'s orbit, draw *ZTZ'* perpendicular to *Nn*.

1. When the nodes are in syzygy, since no part of the disturbing force acts out of the plane of *P*'s orbit, the inclination will not be affected by it.

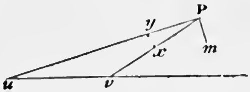
2. Let the node $N$ lie between upper quadrature and farther syzygy, and let the portion $NPn$ of its orbit be above the plane of that of $S$. From upper quadrature to




$Z$, $P$ is moving from the plane of $S$'s orbit; let $Py$ the tangent at $P$ (fig. 2) be produced backward to meet that plane in $u$, draw $Pm$ parallel to $MI$; then $Px$, the new

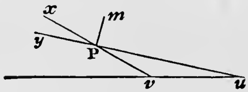



direction of $P$'s motion will fall between $Py$ and $Pm$, and when produced backwards will cut the plane in $v$ at a less angle than that in which $yP$ cuts it, and therefore the inclination of $P$'s orbit, the position of which is determined




by the point $T$ and the direction of $P$'s motion, is diminished. From $Z$ to $n$, $P$ is moving *towards* the plane of $S$'s orbit, and therefore, as appears from fig. 3, $Px$ will cut the plane at a greater angle than that at which $Py$ cuts it, or the inclination is increased.

From $n$ to lower quadrature $P$ is moving from the




plane, and the perpendicular force now tends from the plane, and therefore, as in fig. 4, the inclination is increased.

In a similar maner it may be shewn, that as $P$ moves from $B$ to $Z'$ the inclination is diminished, that it increases from $Z'$ to $N$, and also from $N$ to $D$; hence if $NTD = a$, the inclination in this position of the line of nodes is increased, while $P$ describes $180^0 + 2a^0$ and diminished through $180^0 - 2a^0$.

3. When the nodes are in quadrature the inclination is as much increased as it is diminished, and therefore at the end of one revolution it is unaffected by the ablatitious force.

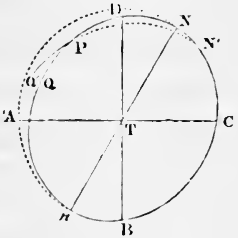
4. Let $N$ lie between $C$ and $B$ at an angular distance $(a)$ from $B$; then it may be shewn by reasoning similar to the above, that in this position the inclination is increased, while $P$ moves through $180^0 - 2a^0$, and diminished through $180^0 + 2a^0$.

As the node recedes (see Cor. 11) from quadrature to syzygy, the inclination is increased, and from syzygy to quadrature it is as much diminished, so that in a whole revolution of the nodes the inclination is neither increased nor diminished. The inclination is a maximum when the nodes are in syzygy, and a minimum when they are in quadrature; and least of all when the nodes are in quadrature and $P$ in syzygy.

Cor. 11. *To consider the effects produced on the motion of the Nodes by the ablatitious force.*

Let $P$ be the place of the body; resolve the ablatitious force at $P$ into two, one perpendicular to and the other in the plane of $P$'s orbit: and let $PQ$ be a small arc of the orbit which $P$ would describe, were there no perpendicular force; $PQ'$ a small arc of the disturbed orbit.

Then it is manifest that when $P$ is ascending from the node, $N'$ the node of $PQ'$ will lie behind or before $N$, that




is, the node will be retrograde or progressive, according as $Q'$ is at a less or greater distance from the plane of $S$'s orbit than $Q$, that is, according as the perpendicular force tends towards or from that plane; and the same is true of the node $n$, when $P$ is approaching that node. Now by what has been shewn in the first part of Cor. 10, the force tends always towards the plane, except between quadrature and the nearer node; hence the motion of the node is always retrograde, except when $P$ is moving between quadrature and the nearer node.

If $a$ be the angular distance of the node from quadrature, the node will be progressive while $P$ moves through $2a^0$, and retrograde through $360^0 - 2a^0$.

Since $a$ is less than 90 except at syzygy, the nodes in a whole revolution of $P$ regrede more than they progrede.

If the nodes be in quadratures, they will regrede during the whole revolution; when they are in syzygies, the disturbing force acting in the plane of $P$'s orbit, produces no effect upon the node, which therefore remains stationary; it will however pass out of syzygy by the motion of $S$, and become retrograde.

COR. 12. *The effects produced by the disturbing forces are greater, when* P *is in conjunction than when in opposition.*

For when $P$ is at nearer syzygy or in conjunction, the addititious force $=\frac{S.PT}{SA^3}$, and when at farther syzygy or in opposition, it $=\frac{S.PT}{SC^3}$; and $SA$ being less than $SC$, the former value is greater than the latter. Also in the former case the ablatitious force $=\frac{3S.PT}{SA^2}$, and in the latter it $=\frac{3S.BT}{SC^2}$, and therefore is greater in conjunction than in opposition. Hence, the effects produced by these forces will be greater in conjunction than in opposition.

COR. 13. The reasoning employed in this proposition is wholly independent of the magnitude of $S$; if therefore $S$ be so great, that the system of $P$ and $T$ revolves round $S$ fixed, the disturbing forces will be of the same *kind* as when $S$ moved round $T$ fixed; but since each varies as $S$, they will all be increased in the same ratio as that in which we suppose $S$ to be increased.

COR. 14. *If* S *and the distance* ST *vary, whilst the system of* P *and* T *remains the same, the angular error of* P *as seen from* T, *produced in a given time by the disturbing force of* S, *will vary inversely as the square of the periodic time of* T *round* S, *or directly as the cube of the apparent diameter of* S *as seen from* T.

For let $S'$ and $S'T$ be other values of $S$ and $ST$; then in any given position of $P$, since $PT$ is the same, the disturbing forces of $S$ on $P$ are to those of $S'$ as $\frac{S}{ST^3} : \frac{S'}{S'T^3}$, and therefore the linear errors produced by them in the same unit of time are in the same ratio, and $PT$ being given, the angular errors as seen from $T$ will be proportional to the linear errors; and the same being true of all corresponding angular errors, componendo, the angular errors generated in a given time will be as $\frac{S}{ST^3} : \frac{S'}{S'T^3}$, that is, by Prop. XV. as

$$\frac{1}{(\text{per}^{c}.\ \text{time})^2 \text{ of } T \text{ round } S} : \frac{1}{(\text{per}^{c}.\ \text{time})^2 \text{ of } T \text{ round } S'},$$

and therefore the angular error varies inversely as

$$(\text{per}^{c}.\ \text{time})^2 \text{ of } T \text{ round } S.$$

Also if $D$ = diameter of $S$, $S \propto D^3$, and therefore angular error $\propto \frac{D^3}{ST^3} \propto$ the cube of the apparent diameter of $S$ as seen from $T$.

COR. 15. *If there be two systems* P, T, S *and* P′, T′, S′, *such that* S : S′ = T : T′, *and* PT : ST = P′T′ : S′T′; *and if the orbits of* P *and* P′ *be similar and similarly situated, their periodic angular errors round* T *and* T′ *arising from the disturbing forces of* S *and* S′ *will be equal.*

The bodies $P$ and $P'$ at any two similarly situated points in each orbit, are similarly acted on by proportional forces, and therefore the linear errors, generated while they move through small similar parts of their orbits, will be similar and proportional, and will therefore be respectively as the diameters of the orbits; hence, the angular errors through those small parts will be equal; and this being true of the errors through all corresponding parts, the periodic angular errors will be equal.

COR. 16. *In any two systems* P, T, S *and* P′, T′, S′, *in which the orbits of* P *and* P′ *are similar and similarly situated, to compare the periodic angular errors round* T *and* T′.

Let $P$ and $p$ be the periodic times of $T$ round $S$ and of $P$ round $T$,

$P'$ and $p'$ .............................. $T'$ ......... $S'$ and of $P'$ round $T'$.

In $TS$, produced if necessary, place a body $s$ such that $s : S' = T : T'$, and at a distance $sT$ from $T$, such that $sT : PT = S'T' : P'T'$; $\therefore s = \frac{T}{T'}S'$, and $sT = \frac{PT}{P'T'}.S'T'$. Then by Cor. 15, the periodic angular errors in the system $P'$, $T'$, $S'$ equal the errors in the system $P$, $T$, $s$. Again, by Cor 14, in the systems $P$, $T$, $S$ and $P$, $T$, $s$ the angular errors in a given time, and therefore the periodic angular errors are

$$\text{as} \quad \frac{S}{ST^3} : \frac{s}{sT^3}, \qquad \text{or as} \; \frac{S}{ST^3} : \frac{S'}{S'T'^3} \cdot \frac{T}{T'} \cdot \frac{P'T'^3}{PT^3},$$

$$\text{as} \quad \frac{S}{ST^3} \cdot \frac{PT^3}{T} : \frac{S'}{S'T'^3} \cdot \frac{P'T'^3}{T'}, \qquad \text{or as} \; \frac{p^2}{P^2} : \frac{p'^2}{P'^2};$$

therefore the periodic angular errors in the systems $P$, $T$, $S$ and $P'$, $T'$, $S'$ are as $\frac{p^2}{P^2} : \frac{p'^2}{P'^2}$.

Hence, if the orbits of two satellites be similar, and equally inclined to the orbits of their primaries, the periodic angular errors in their orbits will vary directly as the squares of the periodic times of the satellites, and inversely as the squares of those of the primaries.

The errors here spoken of are the angular motions of the nodal line, apsidal line, &c.

Cor. 17. *To compare the mean addititious force with the force of* T *on* P.

Let $P$ be the periodic time of $T$ round $S$,

$p$ that of $P$ and $T$ round their centre of gravity;

therefore $\sqrt{\frac{P+T}{T}}.p =$ time in which $P$ would revolve round $T$ at rest at the same distance $TP$, by Prop. LIX.

Now, mean addititious force $= \frac{S.PT}{ST^3}$,

and force of $S$ on $T = \frac{S}{ST^2}$;

$\therefore$ mean addititious force : force of $S$ on $T = PT : ST$,

and by Prop. IV.

force of $S$ on $T$ : force of $T$ on $P = \frac{ST}{P^2} : \frac{PT}{p^2} \cdot \frac{T}{P+T}$;

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$\therefore$ mean addititious force : force of $T$ on $P = \frac{1}{P^2} : \frac{1}{p^2} \cdot \frac{T}{P+T}$.

The force of $T$ on $P$ here spoken of is that with which $T$ alone draws $P$, and this force is to that with which $P$ and $T$ are drawn towards each other as $T : P+T$; hence compounding this with the above proportion, we have

mean addit$^s$. force : force of $P$ and $T$ towards each other $= \frac{1}{P^2} : \frac{1}{p^2}$.

The formulæ which we subjoin have been proved in Sections II. and III., but are here collected together on account of their great use in examples on motion in conic sections about a force to the centre or a focus.

In all orbits,

$$h = vp,$$

and

$$v^2 = F\frac{PV}{2}.$$

In an ellipse or hyperbola about the centre,

$$F = \mu . CP = \mu r,$$

$$h = \sqrt{\mu} . ab,$$

$$v = \sqrt{\mu} . CD \left(\text{or } \frac{h}{p}\right);$$

in an ellipse about the centre,

$$P = \frac{2\pi}{\sqrt{\mu}}.$$

In an ellipse about the focus $S$,

$$F = \frac{\mu}{r^2},$$

$$h = \sqrt{\frac{\mu L}{2}},$$

$$v^2 = \frac{2\mu}{r} - \frac{\mu}{a} \left(\text{or } \frac{h}{p}\right);$$

in a hyperbola about the focus, the same formulæ except for $v$, which is

$$v^2 = \frac{2\mu}{r} + \frac{\mu}{a};$$

in an ellipse about the focus,

$$P = 2\pi\sqrt{\frac{a^3}{\mu}};$$

in a parabola about the focus,

$$F = \frac{\mu}{r^2},$$

$$h = \sqrt{\frac{\mu L}{2}},$$

$$v^2 = \frac{2\mu}{r}\left(\text{or } \frac{h}{p}\right);$$

$r$ being the distance of the body from the centre of force, $L$ the latus rectum, $a, b$ the semi-axes of the ellipse or hyperbola, and $\mu$ the absolute force.

In order to understand the relation which numerical results given by these formulæ have to numerical measures of space and time, consider any force $f$ whatever, and suppose it to act constantly on a body for time $t$ and to generate in that time a velocity with which a body would describe a space $s$ in an equal time $t$; then if $t$ were our unit of time $f$ would be numerically equal to $s$ (Def. 7, page 6); if not, the velocity generated is represented by $\frac{s}{t}$, and therefore the force by $\frac{\left(\frac{s}{t}\right)}{t}$ or by $\frac{s}{t^2}$.

Thus $\frac{s}{f}=t^2$; that is, $f$ is a linear space, and is so related to the space $s$ that the ratio of $s$ to $f$ is equal to the square of the ratio which the time $t$ of generating the velocity $\frac{s}{t}$ bears to the unit of time; or briefly, $\frac{s}{f}$ is the square of a time.

We will apply this consideration to the expressions above given for the forces in an ellipse about the centre, and about the focus.

(1) The orbit is an ellipse about the centre:

here $$F=\mu CP.$$

And if $s$ be any linear space,

$$\frac{s}{F} \text{ is the square of a time,}$$

or $$\frac{s}{\mu CP} \text{ is the square of a time;}$$

let $$\frac{s}{\mu CP}=t^2:$$

then $$\frac{1}{\sqrt{\mu}}=\sqrt{\frac{CP}{s}}\,.\,t,$$

or $\frac{1}{\sqrt{\mu}}$ is a constant time connected with the variable time $t$ by multiplying the latter by the variable ratio $\sqrt{\frac{CP}{s}}$.

The periodic time in the orbit must bear some definite numerical ratio to this time $\frac{1}{\sqrt{\mu}}$: or $P=\frac{c}{\sqrt{\mu}}$; where $c$ is some number. This number is (page 79) $2\pi$, or 6·18...

(2) The orbit is an ellipse about the focus $S$:

here $$F = \frac{\mu}{SP^2}.$$

And if $s$ be any linear space,

$\frac{s}{F}$ is the square of a time;

let then $$\frac{s}{\frac{\mu}{SP^2}} \equiv \frac{s \times SP^2}{\mu} = t^2,$$

therefore $\sqrt{\frac{s \times SP^2}{\mu}}$ is the variable time $t$:

thus $\mu$ must contain a factor expressing linear space in such a manner as to bear a ratio to $s \times SP^2$.

Let then $l$ be any given length, and let $\mu = \mu' l^3$;

therefore $$\sqrt{\frac{s \times SP^2}{\mu' l^3}} = t:$$

and $$\frac{1}{\sqrt{\mu'}} = \sqrt{\frac{l^3}{s \times SP^2}} \cdot t.$$

Thus $\frac{1}{\sqrt{\mu'}}$ $\left(\text{or } \sqrt{\frac{l^3}{\mu}}\right)$ is a time, which is constant,—because $l$ and $\mu$, and therefore $\frac{l^3}{\mu}$, are constant—and is connected with the variable time $t$ by multiplying the latter by the variable ratio $\sqrt{\frac{l^3}{s \times SP^2}}$.

The periodic time in the orbit must bear some definite numerical ratio to this time $\frac{1}{\sqrt{\mu'}}$ or $\sqrt{\frac{l^3}{\mu}}$: hence

$$P = c\sqrt{\frac{l^3}{\mu}},$$

where $c$ is a numerical constant: if $l$ be taken equal to $a$ the semi-axis-major of the ellipse, we have (page 93) $c = 2\pi$.

As another example, let a body describe an orbit about a centre of force varying as the inverse $n^{\text{th}}$ power of the distance.

Here $$F = \frac{\mu}{r^n};$$

and, if $s$ be any linear space,

$$\frac{s}{F} = t^2, \text{ where } t \text{ is a time};$$

$$\therefore \frac{sr^n}{\mu} = t^2:$$

let $$\mu = l^{n+1}\mu',$$

where $l$ is a constant length;

then $$\frac{1}{\sqrt{\mu'}} = \sqrt{\frac{l^{n+1}}{sr^n}}\,.\,t:$$

thus $$\frac{1}{\sqrt{\mu'}} \text{ is a constant time}:$$

hence $$P = \frac{c}{\sqrt{\mu'}} = c\sqrt{\frac{l^{n+1}}{\mu}},$$

where $\mu$ is the absolute force, $c$ a numerical constant, $l$ a constant length.

The following examples are added to be worked out by the processes employed in Newton's first three sections. In some of them the following proposition will be found useful; for the proof of which we refer the student to Todhunter's *Algebra*, Art. 666.

PROP. The limiting value of the fraction

$$\frac{1^p + 2^p + 3^p + \ldots + n^p}{n^{p+1}},$$

when $n$ is indefinitely increased, is $\frac{1}{p+1}$.

# EXAMPLES.

## SECTION I.

1. Two straight lines $AB$, $A'B'$ cut off the same area from a given oval curve; find their point of ultimate intersection when $A'B'$ moves up to coincidence with $AB$.

2. Two circles have their centres on a given curve and pass through a given point; find the limiting position of their chord of intersection as their centres move up to coincidence with a given point of the arc.

3. Circles of the same radius $R$ are described cutting a straight line $ABC$ in $AB$, $AC$ respectively; prove that when $C$ approaches indefinitely near to $B$ the line joining the centres of the circles is inclined to $ABC$ at an angle whose sine is $\frac{AB}{2R}$.

4. $AP$ is a chord of a given circle, $AQ$ a chord near $AP$; find the position of the point of ultimate intersection of circles described on $AP$, $AQ$ as diameters, when $Q$ approaches and ultimately coincides with $P$.

5. $AB$, $AE$ are two straight lines intersecting in $A$; $AE$ is divided into three equal parts in $C$, $D$; $S$ is any other point; $SC$, $SD$, $SE$ cut $AB$ in $M$, $M'$, $M''$. Prove that

$$\text{L. R. } \frac{MM'}{M'M''} = \frac{3}{1},$$

when $AE$ is indefinitely great.

6. In a right-angled triangle the altitude is constant, and the base variable: find the ultimate ratio of the increment of the base to the increment of the hypothenuse.

7. $A$ and $B$ are two fixed points, $CD$ is a fixed straight line, and $cd$ is another straight line moving, subject to the condition that the rectangle under the perpendiculars upon it from $A$ and $B$ is equal to the rectangle under the corresponding perpendiculars upon $CD$. If the lines $CD$ and $cd$ intersect in $P$, prove that, ultimately, when $CD$ and $cd$ coincide, the angle $APC$ will be equal to the angle $BPD$.

8. Prove that the surface generated by the revolution of a semicircle round its bounding diameter is to that generated by the revolution of the same semicircle round the tangent at the extremity of the diameter in the ratio of the length of the diameter to the length of the semicircle.

9. Two tangents are drawn to an ellipse; draw a third so that the area of the triangle formed shall be the least possible.

10. Two similar lumps of ice are melting, and the diminution of volume at any instant is proportional to the area of the surface of each lump. Do the volumes tend to equality?

11. Find the area of a portion of a circle cut off by the diameter $AB$ and a chord $AP$.

12. Find the area bounded by a portion of a hyperbola, two ordinates parallel to one asymptote, and the other asymptote.

13. There are $n$ curves of which the areas cut off by ordinates at any the same distances from their vertices are as $a_1 : a_2 \ldots : a_n$. Shew that if they revolve about their axes, the solids of equal heights so generated are as $a_1^2 : a_2^2 \ldots : a_n^2$.

14. Find by Lemma II. the area bounded by the curve $a^4y = x^5$, the ordinate $y = b$, and the axis of $x$.

15. Find the volume generated by the revolution of the area bounded by the curve $y^3 = ax^2$, the ordinate $y = b$, and the axis of $x$, about the same axis.

16. Find the position of the ordinate which bisects the area bounded by the curve $ay^3 = x^4$, the ordinate $y = b$, and the axis of $x$.

17. Find the area of the curve $r^2 = a^2 \cos \theta$.

18. Find the area of a loop of the curve $r^2 = a^2 \cos 3\theta$.

19. Apply Lemma IV. to shew that the volume of a right cone is one-third that of the cylinder on the same base and of the same altitude.

20. Assuming the area of a parabolic segment to be two-thirds of the circumscribing triangle, deduce the volume of a sphere by comparing it with a right cylinder having for its base the parabolic segment described upon a diameter for chord and latus rectum.

21. Compare the volume of the ring generated by revolution of a circle round a line in its own plane with the volume of a cylinder whose base is the circle and whose height is the distance of the axis of revolution from the centre of the circle.

22. If two bodies move so as to have their velocities at every instant in a constant ratio, shew that the spaces described by them will be in the same ratio.

23. Similar conterminous arcs which have a common tangent at one extremity have parallel tangents at the other extremity.

24. $SY$ is perpendicular to the tangent at $P$ to a curve from a fixed point $S$; and $SZ$ is the perpendicular from $S$ on the tangent at $Y$ to the locus of $Y$: shew that $SY$ is a mean proportional between $SP$ and $SZ$.

25. $AB$ the chord of an arc $ACB$ is bisected at right angles by $CD$; shew that, as $B$ approaches $A$, $C$ becomes the middle point of the arc.

26. Equilateral triangles are described about a given oval curve; shew how to draw a tangent at any point of the curve described by their vertices.

27. If $S$ be a fixed point $PT$, the tangent at $P$ to a curve, $PQ$ a small arc of the curve, $ST$ perpendicular to $SP$, $QR$ perpendicular to $SP$; shew that

$$\text{L. R. } QR : SQ - SP = ST : SP.$$

28. If a circle touch a parabola at the vertex, and their centre and focus coincide, and a straight line parallel to the axis cut the common tangent and the curves in $T, P, Q$; then when these points move up to coincidence at $A$,

$$\text{L. R. area } TAP : \text{area } PAQ = 1.$$

29. If the curve in Lemma X. be an arc of a parabola the axis of which is perpendicular to the straight line along which time is measured, prove that the accelerating effect of the force will vary as the distance from the axis of the parabola.

30. One circle rolls uniformly within another of twice its radius; prove that the resultant acceleration of a particle situated on the circumference of the rolling circle tends to the centre of the fixed circle, and varies as the distance from that centre.

31. An arc of continuous curvature $PQQ'$ is bisected in $Q$; $PT$ is the tangent at $P$; shew that, as $Q'$ approaches $P$, the ultimate ratio of the angle $Q'PT$ to the angle $QPT$ is two to one.

32. $APB$ is a semicircle and $PN$ is an ordinate to the diameter $AB$; if from $AP$, $AQ$ be cut off equal to $PN$, shew that the area enclosed by the curve traced out by $Q$ is one-fourth the area of the semicircle.

33. A fixed line intersects a curve in the point $P$; a point $Q$ in the line is joined with two fixed points $A$, $B$; $QA$, $QB$, meet the curve in $R$, $S$. Find the limiting ratio of $QR$ to $QS$, as $Q$ moves up to $P$.

34. The extremities of a straight line slide upon two given straight lines, so that the area of the triangle formed by the three straight lines is constant: find the limiting position of the chord of intersection of two consecutive positions of the circle described about that triangle.

35. A straight line of constant length moves with its extremities upon two given straight lines; find the limiting position of the chord of intersection of two consecutive positions of the circumscribing circle.

36. $AB$ is a diameter of a circle, $P$ a point contiguous to $A$, and the tangent at $P$ meets $BA$ produced in $T$: shew that ultimately the difference between $BA$, $BP$ is in a ratio of equality to $\frac{1}{2}TA$.

37. If $PQ$ be an arc of continuous curvature, and the tangent and normal at $Q$ meet the tangent at $P$ in $T$ and $N$, prove that L. R. $\Delta PQN : \Delta PQT = 2 : 1$.

38. The circle of curvature at a point of an ellipse passes through a vertex; find the abscissa of the point.

39. If the circle of curvature at one extremity of the major axis of an ellipse passes through the farther focus; find its excentricity.

40. The foci of all parabolas which have the same curvature as a given curve at a given point lie on a circle.

41. If the angle between the tangent and radius vector of a curve is a maximum or a minimum, the chord of curvature through $S = 2SP$.

42. Two curves of finite curvature touch each other at the point $P$, and from $T$, a fixed point in the common tangent, a secant is drawn cutting one curve in the points $A$, $B$, and the other in the points $A'$, $B'$, and the lines $AP$, $A'P$, $BP$, $B'P$ are drawn; shew that, if the secant moves up to and ultimately coincides with the tangent, the angles $APA'$, $BPB'$ will be ultimately in a ratio of equality.

43. In the curve in which the difference between the arc and the intercept of the normal on the axis of abscissæ is constant, the difference between the ordinate and the normal is also constant.

44. From a point on the circumference of a vertical circle a chord and tangent are drawn, the chord terminating in the lowest point, the tangent in the vertical diameter produced. Compare the velocities acquired by a heavy body falling down the chord and tangent when they are indefinitely diminished.

## SECTIONS II, III.

45. If a particle describe an ellipse about a focus $S$, the rate of description of areas round the other focus $S'$ varies as $S'Z^2$, where $S'Z$ is the perpendicular from $S'$ on the tangent.

46. A particle $P$ moves in space under the action of two centres of force $S$ and $S'$; shew that the projection of $P$ on a plane perpendicular to $SS'$ describes areas in equal times about the point where $SS'$ meets the plane.

47. If a particle acted on only by a central force, not necessarily continuous, move with constant velocity, prove that its path will be a straight line, a circle, or straight lines alternating with arcs of circles.

48. In a central orbit the velocity of the foot of the perpendicular from the centre on the tangent varies inversely as the length of the chord of curvature through the centre of force.

49. A point moves on the circumference of a circle; prove that the angular velocity about all points in the circumference is the same.

50. A body is describing an ellipse round a centre of force in one of the foci. Prove that the velocity of the point of intersection of the perpendicular from that focus upon the tangent at any point of the orbit is inversely proportional to the square upon the conjugate diameter.

51. A number of bodies describe different circles round the same centre of force varying as $\frac{1}{r^2}$, setting out together from the same radius vector; find the curve in which they will be situated when one of them has completed a revolution.

52. Find the time of revolution of a conical pendulum.

53. A body describing an orbit about a central force equal to $\frac{\mu}{SP \,.\, PV}$ has, at the distance of 27 feet from the centre, a velocity of 15 miles per hour; find the numerical value of $\mu$ (1) when a foot and a second are the units of space and time, (2) when a yard and a minute are the units.

54. If the angular velocity about the centre of force varies as the linear velocity, find the orbit.

55. Two equal circles are described in equal times under the action of forces to the centre and a point in the circumference respectively. Compare the absolute forces.

56. A body moves in an ellipse with uniform velocity under the action of two central forces at the foci: shew that the forces to the foci are always equal and vary inversely as the product of the focal distances.

57. If in an ellipse the velocity varies as the diameter parallel to the direction of motion directly and as the distance from the major axis inversely, shew that the centre of force is infinitely distant.

58. $T$ describes a circle uniformly about $C$; $P$ describes another circle uniformly about $T$ as centre in the same direction,

and their velocities are as $CT : PT$. Shew that $P$ describes a circle about $C$, and find the force on $P$ tending to $C$.

59. $T$ describes a circle uniformly about $C$; $P$ describes another circle uniformly about $T$ as centre in the opposite direction, and their velocities are as $CT : PT$. Shew that $P$ describes an ellipse about $C$, and hence that an ellipse may be described by a force to the centre which varies as the distance.

60. What is the acceleration of a point in the circumference of a circle which rotates uniformly while its centre describes a straight line? In what case would such a point describe a cycloid?

61. A parabola can be described by a body acted on by two forces, one a constant repulsive force to the focus, and the other a force inwards along the normal and varying inversely as the normal.

62. If a perfectly elastic particle describing an equiangular spiral impinge upon a hard plane, prove that after impact it will describe an equiangular spiral. How must the plane be placed that it may describe a similar spiral?

63. What impulse must be applied to a body moving in an equiangular spiral to make it proceed to describe a circle?

64. In an ellipse described about a centre of force at the centre the square of the velocity at the end of the equi-conjugate diameters is the arithmetic mean between the squares of the greatest and least velocities.

65. Two bodies are describing equal and similar concentric ellipses under the action of a force tending to their centre, the axis-major of the one being at right angles to the axis-major of the other; determine the time that each is within the orbit of the other, and find its limiting value as the excentricity is indefinitely diminished.

66. In an ellipse described about a force to the centre, shew that the time in which any given area will be swept out by the radius vector is independent of the excentricity of the ellipse if the area of the ellipse be given.

67. A particle describes an ellipse about a centre of force in the centre. Prove that the angular velocity of the particle about a focus is inversely proportional to the distance from that focus.

68. If a triangle be inscribed in an ellipse so that its centre of gravity coincides with the centre of the ellipse, prove that the velocities of a particle describing the ellipse under the action of a force to the centre when at the angular points $A$, $B$, $C$ will be proportional to the opposite sides of the triangle, and also that the times from $A$ to $B$, $B$ to $C$, $C$ to $A$ will be equal to each other.

69. Two particles describe concentric, similar, and similarly situated ellipses under the action of the same force tending to their common centre; shew that their centre of gravity moves in another concentric, similar and similarly situated ellipse.

70. An ellipse is described by a body under a force to the centre. If the accelerating effect at the extremity of the axis-major is equal to that of gravity, and the period is a day, find the number of miles in the axis-minor.

71. Two particles describe the same ellipse about the same force in the centre in the same periodic time: shew that their directions of motion at any time intersect on a similar ellipse.

72. A particle is describing an ellipse about the centre, and when it arrives at one extremity of the minor-axis the centre of force is suddenly transferred to the other; find the orbit subsequently described. What is the excentricity of the ellipse if the new orbit is a circle?

73. A body is describing an ellipse about a force to the centre; when at the extremity of an equi-conjugate diameter the direction of motion is suddenly changed the velocity remaining unaltered; find the position of the axes of the new orbit, and the condition that it may be a circle.

74. A body moves in a parabola about a centre of force in the focus. If the velocity be diminished in the ratio of $\sqrt{2} : 1$ when the body is at the extremity of the latus rectum, find the position and axes of the new orbit.

75. If any number of bodies describe parabolas about a common centre of force in the focus, the square of the time of passing from one extremity of the latus rectum to the other varies as the cube of the latus rectum.

76. A body describes an ellipse about a centre of force in a focus; prove that the velocity of the point of intersection of

the perpendicular from that focus upon the tangent at any point of the orbit varies inversely as the square of the conjugate diameter.

77. The ratio of the periodic times of two bodies revolving about two centres of force varying inversely as the square of the distance is $\frac{1}{13}$, and the ratio of the mean distances of the bodies from the centres of force about which they respectively revolve is $\frac{1}{400}$; compare the absolute forces at the two centres.

78. If $v$, $v'$ be the velocities of a body describing an ellipse about a centre of force at one focus, at the extremities of any focal chord, and $u$ that at the extremity of the latus rectum, then will $v^2$, $u^2$, $v'^2$ be in arithmetical progression.

79. The time of moving in an ellipse from $P$ to $D$, the extremities of equi-conjugate diameters, by a body acted on by a force to the focus, is one-fourth of the periodic time.

80. In an ellipse about the focus $S$, the velocity may be resolved into two equal velocities perpendicular respectively to $SP$ and $HP$, and each varying as $HP$.

81. A particle describes an ellipse about a centre of force in one of the foci; if lines be drawn always parallel to the direction of motion at a distance from the centre of force proportional to the velocity of the particle, these lines will touch a similar ellipse.

82. A particle moves in an ellipse about a centre of force in the focus $S$; when the particle is at $B$, the extremity of the minor-axis, the centre of force is changed to $R$ in $SB$, so that $RB$ is one-fifth of $SB$, and the absolute force is diminished to one-eighth of its original value; shew that the periodic time is unaltered, and that the new minor-axis is two-fifths of the old.

83. A body describes a hyperbola round a centre of force at the nearer focus. Find, when possible, the point in the orbit at which the angular velocities about the foci are in a given ratio, and state the limits between which the given ratio may vary.

84. A body describes a hyperbola under the action of a force tending to one focus. Prove that the rate at which areas are described by the moving particle about the centre of the hyperbola, is inversely proportional to the distance of the particle from the centre of force.

85. A body is moving in a given hyperbola under the action of a force tending to the nearer focus $S$; when it arrives at a point $P$, the force suddenly becomes repulsive, find the position and magnitude of the axes of the new orbit; shew that the difference of the squares of the excentricities of the new and old orbits varies inversely as $SP$.

86. A particle describing an ellipse about a force in the focus comes to the extremity of the major-axis nearer to the centre of force; find in what ratio the absolute force must be then suddenly diminished so that the particle may proceed to describe a hyperbola whose excentricity is the reciprocal of that of the ellipse.

87. Three tangents are drawn to a given orbit, described by a particle under the action of a central force, one of them being parallel to the external bisector of the angle between the other two. If the velocity at the point of contact of this tangent be a mean proportional between those at the other two points, prove that the centre of force will lie on the circumference of a certain circle.

88. A body describes a parabola under a force to the focus, and a straight line is drawn from the focus perpendicular to the tangent and proportional to the velocity; prove that its extremity describes a circle.

89. A perfectly elastic particle, describing a parabola about a centre of force in the focus and moving towards the vertex, impinges on a fixed plane passing through the latus rectum and perpendicular to the plane of the parabola; determine its subsequent motion.

90. A body moves in a parabola about a centre of force in the vertex; shew that the time of moving from any point to the vertex varies as the cube of the distance of the point from the axis of the parabola.

91. Two particles describe the same ellipse about a centre of force in one of the foci, starting simultaneously from opposite extremities of the transverse axis. When will they be moving with equal velocities?

92. Two bodies describe the same ellipse, one about a focus, the other about the centre; the forces are such that at the point where they are equal the velocities are also equal; prove that the periods are as $1 \pm e : 1$.

93. When a body arrives at $P$ in an ellipse about the focus $S$, the centre of force is suddenly transferred to the other focus, and the same orbit is described; shew that, if $\lambda$, $\lambda'$ are the absolute forces, $\lambda : \lambda' = SP^2 : S'P^2$.

94. A particle is describing an ellipse about the focus; when it comes to the extremity of the conjugate axis, the absolute force is diminished by one-third. Determine the position and dimensions of the new orbit, and prove that the distance between its focus and its centre is bisected by the conjugate axis of the original orbit.

95. If a body be moving in an ellipse under the action of a force $= \frac{\mu}{r^2}$, and if at any point the velocity $V$ and the absolute force $\mu$ be increased by very small increments such that the ratio of the increment of $V$ to $V$ is half that of the increment of $\mu$ to $\mu$, shew that the orbit will remain unaltered, but the periodic time will be changed.

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